Labor Economics Homework 1

Due:

1. Julia and VSCode

1. Go to `https://julialang.org/downloads/` and download the latest stable version of Julia.

2. Go to `https://code.visualstudio.com/download` and download Visual Studio Code.

3. Go to the extensions tab in VSCode and install the Julia extension.

Now you should be able to run Julia code in VSCode. To install packages, you can open the terminal and type `julia`; then type `]add PackageName` to install the package `PackageName`. For more details, see `https://julia.quantecon.org/intro.html`.

2. Methodology

Suppose that you are a restaurant owner and you are considering increasing the price. You know that the customers' preference can be represented by

$$u_i(x, y) = \frac{\epsilon_i}{1 - \epsilon_i} \log(x) + \log(y),$$

where x is the restaurant's food and y represents the other goods. $\epsilon_i \overset{iid}{\sim} Beta(\alpha, \beta)$ is an unobserved term, and you do not observe α and β either. Each customer i faces the budget constraint

$$px + y \leq w_i,$$

where the price of the other foods is normalized to 1. To simplify the problem, assume that $w_i = 1$ for all i.

1. Solve for the demand of x_i.

2. After surveying the customers, you obtain the data $\{x_i\}_{i=1}^N$. Write down the maximum likelihood estimator of α and β.

3. Suppose that you have estimated α and β. How much, on average, would the demand of x_i be if you set the price to p'?

Solution. 2.1.
The F.O.C. is

$$\begin{cases} \dfrac{\epsilon_i}{1-\epsilon_i}\dfrac{1}{x_i} = \lambda p, \\ \dfrac{1}{y_i} = \lambda, \\ px_i + y_i = 1. \end{cases}$$

Solving the system of equations yields

$$x_i = \frac{\epsilon_i}{p}.$$

This is the demand for the customer i. ∎

Solution. 2.2.
Let $f(\cdot;\alpha,\beta)$ be the density of $Beta(\alpha,\beta)$. The likelihood function is

$$L(\alpha,\beta) = \prod_{i=1}^N f(\epsilon_i;\alpha,\beta) = \prod_{i=1}^N f(px_i;\alpha,\beta).$$

The maximum likelihood estimator is

$$(\hat{\alpha},\hat{\beta}) = \arg\max_{\alpha,\beta} L(\alpha,\beta).$$

Solution. 2.3.
On average, the demand of x_i would be

$$\mathbb{E}[x_i] = \mathbb{E}\left[\frac{\epsilon_i}{p'}\right] = \frac{1}{p'}\frac{\alpha}{\alpha+\beta}.$$

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3. Robinson Crusoe's Coconuts

Robinson Crusoe finds z_0 units of coconut on the beach in the morning of day 0. Each day, he can either consume the coconuts or leave them for the next day. If he consumes c_t coconuts on day t, he will have $z_{t+1} = z_t - c_t$. In each day, the utility of consuming c_t coconuts is $u(c_t) = \sqrt{c_t}$. Furthermore, the coconuts will rot at the end of day T. Given the time discount factor $\beta \in (0,1)$, Crusoe's prblem is thus formulated as follows:

$$\max_{\{c_t\}_{t=0}^T} \sum_{t=0}^{T} \beta^t u(c_t) \quad \text{s.t.} \quad z_{t+1} = z_t - c_t.$$

1. Write down the value function $v(z,t)$ and the Bellman equation.

2. Write down the first-order condition of the problem.

3. Solve the optimal consumption c_0^* corresponding to z_0 analytically.

4. Now instead of the coconuts, Crusoe finds z_0 units of canned food. Since the canned food will not rot, the problem is formulated as

$$\max_{\{c_t\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad z_{t+1} = z_t - c_t.$$

   Write down the value function and the Bellman equation. Solve the optimal consumption c_0^* corresponding to z_0 analytically.

Solution. 3.1.
The value function is

$$v(z,t) = \max_{\{c_s\}_{s=t}^{T}} \sum_{s=t}^{T} \beta^s u(c_s)$$

with $z = \sum_{s=t}^{T} c_s$. The Bellman equation is

$$v(z,t) = \max_{c_t} \left\{ u(c_t) + \beta v(z - c_t, t+1) \right\}.$$

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Solution. 3.2.
The F.O.C. is

$$u'(c_t) = \beta u'(c_{t+1}) \quad \forall\, t = 0, 1, \dots, T-1.$$

Since $u(c_t) = \sqrt{c_t}$, the F.O.C. becomes

$$\frac{1}{2\sqrt{c_t}} = \beta \frac{1}{2\sqrt{c_{t+1}}}.$$

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Solution. 3.3.
Rearranging the F.O.C. yields

$$c_{t+1} = \beta^2 c_t.$$

Combining this with the budget constraint $z_0 = \sum_{s=0}^{T} c_s$, we have

$$z_0 = c_0 + \beta^2 c_0 + \cdots + \beta^{2T} c_0 = c_0 \sum_{s=0}^{T} \beta^{2s} = c_0 \frac{1 - \beta^{2(T+1)}}{1 - \beta^2}.$$

Thus

$$c_0^* = \frac{z_0(1-\beta^2)}{1 - \beta^{2(T+1)}}.$$

■

Solution. 3.4.
Taking the limit as $T \to \infty$, the value function is

$$v(z) = \max_{\{c_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^s u(c_s).$$

Observe that the value function doe not depend on t since today's problem is the same as tomorrow's.
The Bellman equation becomes

$$v(z) = \max_{c} \{u(c) + \beta v(z-c)\}.$$

The F.O.C. is the same as *3.3*. Thus the optimal consumption is

$$c_0^* = z_0(1-\beta^2)$$

by taking $T \to \infty$. ■

4. McCall's Job Search Model

A worker is finding a job. In each period, the worker receives a job offer with wage w_t. The wage is i.i.d. across periods in $Beta(\alpha, \beta)$. The worker can either accept the offer ($s_t = 1$) and work at the wage w_t forever or reject it ($s_t = 0$) and receives $\underline{c}$ as the unemployment benefit. Once the worker accepts an offer, he cannot search for another job or quit. Assume that the worker cannot borrow or save; thus $u(c_t) = w_t$. With the discount factor $\rho \in (0,1)$, the worker's problem is

$$\max_{\{s_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \rho^t (w_t s_t + (1-s_t)\underline{c}) \right] \quad \text{s.t. } \{s_t \mid s_t \in \{0,1\}\} \text{ is a non-decreasing sequence.}$$

1. Write down the Bellman equation.

2. Define the Bellman operator and verify that it is a contraction mapping on the space of bounded continuous functions defined on $[0,1]$ with the supremum norm.
   (Hint: $|\max\{a,b\} - \max\{a,c\}| \le |b-c|$ for real a, b, c.)

3. Given $\rho = 0.95$, $\underline{c} = 0.1$, $\alpha = 2$, and $\beta = 5$, solve the model numerically.

(Hint: You may use the linear interpolation and monte carlo simulation to approximate the expectation of v.)

4. Plot the value function and the optimal policy function as functions of w. What is the reservation wage?

5. Simulate the model for 1000 homogeneous agents. Plot the distribution of the time to find a job.

6. Plot the reservation wage as a function of $\underline{c} = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$ and repeat the simulation in problem 5 for different values of $\underline{c}$. Explain your results.

Solution. 4.1.
The Bellman equation is

$$v(w) = \max \left\{ \frac{w}{1-\rho}, \underline{c} + \rho \int v(w')F(dw') \right\},$$

where F is the CDF of $Beta(\alpha, \beta)$. ■

Solution. 4.2.
The Bellman operator is

$$(Tv)(w) = \max \left\{ \frac{w}{1-\rho}, \underline{c} + \rho \int v(w')F(dw') \right\}.$$

Now let v_1 and v_2 be two bounded continuous functions on $[0,1]$ and $\|\cdot\|$ denote the supremum norm. Then

$$\begin{aligned} \|Tv_1 - Tv_2\| &= \sup_{w\in[0,1]} \left| \max \left\{ \frac{w}{1-\rho}, \underline{c} + \rho \int v_1(w')F(dw') \right\} \right. \\ &\qquad \left. - \max \left\{ \frac{w}{1-\rho}, \underline{c} + \rho \int v_2(w')F(dw') \right\} \right| \\ &\leq \sup_{w\in[0,1]} \left| \rho \int v_1(w')F(dw') - \rho \int v_2(w')F(dw') \right| \\ &\leq \rho \int \sup_{w\in[0,1]} \left| v_1(w') - v_2(w') \right| F(dw') \\ &= \rho \int \|v_1 - v_2\| F(dw') = \rho \|v_1 - v_2\|. \end{aligned}$$

Thus T is a contraction mapping. ■

Solution. Alternative Approach for 4.2.
One may observe that there exists a "reservation wage", $\bar{w}$, where the worker is indifferent between accepting and rejecting the offer. That is,

$$\frac{\bar{w}}{1-\rho} = \underline{c} + \rho \int v(w')F(dw').$$

By the Bellman equation, substituting $v(w')$ yields

$$\frac{\bar{w}}{1-\rho} = \underline{c} + \rho \int \max\left\{\frac{w'}{1-\rho}, \frac{\bar{w}}{1-\rho}\right\} F(dw'),$$

which is equivalent to

$$\bar{w} = (1-\rho)\underline{c} + \rho \int \max\left\{w', \bar{w}\right\} F(dw').$$

Define the new operator U defined on $[0,1]$ as

$$U\bar{w} = (1-\rho)\underline{c} + \rho \int \max\left\{w', \bar{w}\right\} F(dw').$$

We claim that U is a contraction mapping.
Let $\bar{w}_1, \bar{w}_2 \in [0,1]$. Then

$$\begin{aligned}
|U\bar{w}_1 - U\bar{w}_2| &= \left|\rho \int \max\left\{w', \bar{w}_1\right\} F(dw') - \rho \int \max\left\{w', \bar{w}_2\right\} F(dw')\right| \\
&= \rho \left|\int \max\left\{w', \bar{w}_1\right\} - \max\left\{w', \bar{w}_2\right\} F(dw')\right| \\
&\leq \rho \int \left|\max\left\{w', \bar{w}_1\right\} - \max\left\{w', \bar{w}_2\right\}\right| F(dw') \\
&\leq \rho \int |\bar{w}_1 - \bar{w}_2| F(dw') = \rho |\bar{w}_1 - \bar{w}_2| \int F(dw') = \rho |\bar{w}_1 - \bar{w}_2| .
\end{aligned}$$

Thus U is a contraction mapping.
Once we obtain the reservation wage, the decision rule becomes obvious: accept the offer if $w \geq \bar{w}$ and reject it otherwise. We also implement this approach in the Julia code. ■

Solution. 4.3, 4.4, 4.5, 4.6.
See the Julia code file. ■