Labor Economics Homework 4

Due:

1. Occpational Choice Model

Suppose a worker i chooses an occupation $j \in \{0, 1, \dots, J\}$ to maximize the utility function

$$u_i(j) = \alpha_j + \epsilon_{ij},$$

where $\epsilon_{ij} \overset{iid}{\sim} T1EV$ across workers and occupations that is observed only by the individual but not us. α_j is the parameter that we want to estimate. We normalize $\alpha_0 = 0$.

1. What is the probability of observing a worker in occupation j?
2. Use the parameters $(\alpha_1, \alpha_2, \alpha_3) = (0.2, 0.3, -0.1)$ to simulate the data for 1000 workers and $J = 3$ occupations.
3. Derive the likelihood function for the parameters α_j.
4. Estimate the parameters α_j via MLE. Can you recover the true parameters?
5. Compute $\Pr(D_i = 1) / \Pr(D_i = 2)$ for the cases $J = 2$ and $J = 3$. What do you find?

Solution. 1.1.

By the *T1EV* assumption, we have

$$\Pr(D_i = j) = \Pr(\alpha_j + \epsilon_{ij} > \alpha_k + \epsilon_{ik} \text{ for all } k \neq j) = \frac{\exp(\alpha_j)}{\sum_{k=0}^{J} \exp(\alpha_k)}.$$

■

Solution. 1.2, 1.4.

See the Julia code file. ■

Solution. 1.3.
The likelihood function is

$$L(\alpha_1,\alpha_2,\alpha_3)=\prod_{i=1}^{1000}\prod_{j=0}^{J}\Pr(D_i=j)^{\mathbb{1}\{D_i=j\}}=\prod_{i=1}^{1000}\prod_{j=0}^{J}\left\{\frac{\exp(\alpha_j)}{\sum_{k=0}^{J}\exp(\alpha_k)}\right\}^{\mathbb{1}\{D_i=j\}}.$$

■

Solution. 1.5.
By 1.1, for $J=2$, we have

$$\frac{\Pr(D_i=1)}{\Pr(D_i=2)}=\frac{\exp(\alpha_1)}{\exp(\alpha_2)}=\exp(\alpha_1-\alpha_2).$$

For $J=3$, we have

$$\frac{\Pr(D_i=1)}{\Pr(D_i=2)}=\frac{\exp(\alpha_1)}{\exp(\alpha_2)}=\exp(\alpha_1-\alpha_2).$$

Notice that they are the same. This is called the IIA assumption. One should be careful when using the multinomial logit model despite its convenience. ■

2. Inventory Dynamics

Consider a firm has a warehouse to store its inventory with a capacity of S units. The time line in each period goes as follows: At the beginning of period t, the firm observes the inventory I_t and $\epsilon_{0t},\epsilon_{1t}$ but not the demand D_t. The firm then decides whether to restock ($s_t=1$) the inventory to S units or not ($s_t=0$). After the decision, the firm pays the storage cost c. Finally, the firm observes D_t and the profit flow is realized. The profit flow is

$$\pi(s_t,I_t,D_t)=\begin{cases}pD_t-cS-h(S-I_t) & \text{if } s_t=1,\\ p\min\{D_t,I_t\}-cI_t & \text{if } s_t=0,\end{cases}$$

where I_t is the inventory level at the beginning of period t, p is the price of the product, c is the cost of holding one unit of inventory. h is the cost of restocking with

$$h(x)=\theta_0+\theta_1 x.$$

Let $\epsilon_t(s)$ be random term iid across choice and time following *T1EV* distribution. We normalize it as

$$\epsilon_t(s) = -\gamma + s\epsilon_{1t} + (1-s)\epsilon_{0t}.$$

Remark that ϵ_t is observed by the firm but not us. The law of motion for the inventory level is

$$I_{t+1} = \begin{cases} S - D_t & \text{if } s_t = 1, \\ \max\{0, I_t - D_t\} & \text{if } s_t = 0. \end{cases}$$

1. Assume that $D_t \overset{iid}{\sim} BetaBinomial(\alpha, \beta, S)$, $S = 20$. Generate the transition array Q for the inventory level I_t.

2. Let the discount factor be $\rho \in (0,1)$. Write down the Bellman equation for the firm's problem.

3. Given the parameters $(p, c, \theta_0, \theta_1, \alpha, \beta, \rho) = (1.3, 1.0, 3.0, 0.7, 5.0, 2.0, 0.95)$, solve the firm's policy function using the value function iteration.

4. Given the same parameters, solve the firm's policy function using the policy function iteration.
   (Hint: In T_σ step, $v_\sigma = (I - \rho Q_\sigma)^{-1}\pi_\sigma$, where Q_σ is the transition matrix given the policy and π_σ is the profit flow given the policy.)

5. The above two methods are fundamental. For this problem, with some additional assumptions, it is possible to solve the model faster. Assume that the shock ϵ_{t+1} and the demand D_t are conditionally independent, i.e.,

   $$\Pr(I_{t+1} \mid \epsilon_t, I_t, s_t) = \Pr(I_{t+1} \mid I_t, s_t).$$

   Derive the function equation

   $$\mathbb{E}_\epsilon[v(I)] = \log\left\{\sum_{s\in\{0,1\}} \exp\left[\mathbb{E}_D[\pi(s, I, D)] + \rho \sum_{I'} (\mathbb{E}_\epsilon[v(I')])Q(I', I, s)\right]\right\},$$

   where γ is the Euler-Mascheroni constant. Note that iterating on this equation forms the inner loop of the NFXP algorithm.

6. Simulate the data $\{(\min\{D_t, I_t\}, I_t, s_t)\}_{t=1}^{500} = \{(d_t, I_t, s_t)\}_{t=1}^{500}$. Estimate the transition matrix parameters α and β for the inventory level I_t using the simulated data.

7. Estimate the parameters $(p, c, \theta_0, \theta_1, \rho)$ using NFXP algorithm.
8. Estimate the parameters $(p, c, \theta_0, \theta_1, \rho)$ using CCP.
9. Simulate and plot the inventory level I_t against t for 100 periods for different price levels $p \in \{1.1, 1.3, 1.5\}$ and storage cost $c \in \{0.8, 1.0, 1.2\}$. Explain your findings.

Solution. 1.1, 1.3, 1.4, 1.9.
See the Julia code file. ∎

Solution. 1.2.
The Bellman equation is

$$v(I_t, \epsilon_t) = \max_{s_t} \left\{ \mathbb{E}_D[\pi(s_t, I_t, D_t)] + \epsilon_t(s) + \rho \sum_{I_{t+1}, \epsilon_{t+1}} v(I_{t+1}, \epsilon_{t+1}) \Pr(I_{t+1}, \epsilon_{t+1} \mid I_t, s_t) \right\}.$$

∎

Solution. 1.5.
By the conditional independence assumption, we have

$$\begin{aligned}
\mathbb{E}_{(\epsilon_{t+1}, I_{t+1})}[v(I_{t+1}, \epsilon_{t+1})] &= \sum_{I_{t+1}, \epsilon_{t+1}} v(I_{t+1}, \epsilon_{t+1}) \Pr(I_{t+1}, \epsilon_{t+1} \mid I_t, s_t) \\
&= \sum_{I_{t+1}} \sum_{\epsilon_{t+1}} v(I_{t+1}, \epsilon_{t+1}) \Pr(I_{t+1} \mid \epsilon_{t+1}, I_t, s_t) \Pr(\epsilon_{t+1} \mid I_t, s_t) \\
&= \sum_{I_{t+1}} \sum_{\epsilon_{t+1}} v(I_{t+1}, \epsilon_{t+1}) \Pr(I_{t+1} \mid I_t, s_t) \Pr(\epsilon_{t+1} \mid I_t, s_t) \\
&= \sum_{I_{t+1}} \mathbb{E}_{\epsilon_{t+1}}[v(I_{t+1}, \epsilon_{t+1})] \Pr(I_{t+1} \mid I_t, s_t).
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}_\epsilon[v(I, \epsilon)] &= \int_\epsilon \max_s \left\{ \mathbb{E}_D[\pi(s, I, D)] + \epsilon(s) + \rho \sum_{I'} \mathbb{E}_\epsilon \left[v(I', \epsilon') \right] P(I' \mid I, s) \right\} dF(\epsilon) \\
&= \log \left\{ \sum_{s \in \{0,1\}} \exp \left[\mathbb{E}_D[\pi(s, I, D)] + \rho \sum_{I'} \mathbb{E}_\epsilon \left[v(I', \epsilon') \right] Q(I', I, s) \right] \right\}
\end{aligned}$$

by the *T1EV* assumption. ∎

Solution. 1.6.

We derived the likelihood function here.

$$\mathcal{L} = \prod_{t=1}^{500} \Pr(\min\{D_t, I_t\} = d_t \mid I_t) = \prod_{t=1}^{500} f(d_t)^{\mathbb{1}\{d_t < I_t\}} (1 - F(I_t))^{\mathbb{1}\{d_t = I_t\}},$$

where f is the probability mass function of the Beta-Binomial distribution and F is the cumulative distribution function. ■

Solution. 1.7.

Once we obtain $\mathbb{E}_\epsilon[v(I, \epsilon)]$ for the given parameters, we can calculate the associated likelihood via

$$\begin{aligned}
\mathcal{L} = \prod_{t=1}^{500} &\Pr(\mathbb{E}_D[\pi(0, I_t, D)] + \epsilon_t(0) + \rho \sum_{I_{t+1}} \mathbb{E}_\epsilon[v(I_{t+1}, \epsilon_{t+1})] Q(I_{t+1}, I_t, 0) \\
&\quad > \mathbb{E}_D[\pi(1, I_t, D)] + \epsilon_t(1) + \rho \sum_{I_{t+1}} \mathbb{E}_\epsilon[v(I_{t+1})] Q(I_{t+1}, I_t, 1))^{\mathbb{1}\{s_t = 0\}} \\
&\times \Pr(\mathbb{E}_D[\pi(1, I_t, D)] + \epsilon_t(1) + \rho \sum_{I_{t+1}} \mathbb{E}_\epsilon[v(I_{t+1}, \epsilon_{t+1})] Q(I_{t+1}, I_t, 1) \\
&\quad > \mathbb{E}_D[\pi(0, I_t, D)] + \epsilon_t(0) + \rho \sum_{I_{t+1}} \mathbb{E}_\epsilon[v(I_{t+1})] Q(I_{t+1}, I_t, 0))^{\mathbb{1}\{s_t = 1\}} \\
= \prod_{t=1}^{500} \prod_s &\left\{ \frac{\mathbb{E}_D[\pi(s, I_t, D)] + \rho \sum_{I_{t+1}} \mathbb{E}_\epsilon[v(I_{t+1}, \epsilon_{t+1})] Q(I_{t+1}, I_t, s)}{\sum_{s'} \exp\left[\mathbb{E}_D[\pi(s', I_t, D)] + \rho \sum_{I_{t+1}} \mathbb{E}_\epsilon[v(I_{t+1}, \epsilon_{t+1})] Q(I_{t+1}, I_t, s')\right]} \right\}^{\mathbb{1}\{s_t = s\}}.
\end{aligned}$$

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