Labor Economics Homework 2

Due:

1. Robinson Crusoe's Coconuts Revisited

Recall our settings from homework 1. This time, Robinson Crusoe's utility is a CRRA function

$$u(c;\gamma) = \lim_{\gamma' \to \gamma} \frac{c^{1-\gamma'}}{1-\gamma'}$$

with $\gamma \geq 0$. At the beginning of each period t, Crusoe owns y_t coconuts. He can either consume them or plant them. If he plants k_{t+1} coconuts, he will obtain $y_{t+1} = z_{t+1} f(k_{t+1})$ coconuts in the next period, where $z_{t+1} \overset{iid}{\sim} Lognormal(\mu, \sigma^2)$. The coconuts rot after each period, thus k_t is fully depreciated. We assume that $f(k) = k^\alpha$ with $\alpha \in (0,1)$. With discount factor $\beta \in (0,1)$, Crusoe's problem is

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \mathbb{E}_0 \left[\beta^t u(c_t) \right] \quad \text{s.t.} \quad c_t + k_{t+1} = y_t \quad \text{and} \quad y_{t+1} = z_{t+1} f(k_{t+1})$$

1. Write down the Bellman equation and define the Bellman operator.
2. Show that the Bellman operator is a contraction mapping.
   (Hint: Verify the Blackwell's sufficient conditions.)
3. Given $\alpha = 0.8$, $\beta = 0.96$, $\gamma = 1$, $\mu = 0.0$, $\sigma = 0.3$. Solve the model by value function iteration for $y \in (0, 10)$.
   (Hint: You may use the linear interpolation and monte carlo simulation to approximate the expectation of v.)
4. Given the same parameters, solve the model by policy function iteration.
5. Given the same parameters, solve the model by envelope condition methods. Use both the exogenous grids and the endogenous grids. Compare the running time of the four methods using macro `@benchmark` in Julia.

6. The model has a famous closed-form solution with $\gamma = 1$:
$$c^*(y) = (1 - \alpha\beta)y.$$
Plot the consumption policies for the analytical and numerical solutions. Do your numerical solutions fit the analytical one well?

7. Solve the model for $\sigma = 0.1, 0.15, 0.2, 0.25$ and 0.3 with $\gamma = 1.5$. Plot the consumption policies for different σ. Explain your findings.

Solution. 1.1.
The Bellman equation is
$$v(y) = \max_c u(c) + \beta \int v(zf(y-c))F(dz),$$
where F stands for the CDF of z. The Bellman operator is
$$(Tv)(y) = \max_c u(c) + \beta \int v(zf(y-c))F(dz).$$
■

Solution. 1.2.
We are going to check the conditions in Blackwell's theorem are satisfied. First, if $v \leq w$, $v, w \in B(X)$, let c_v and c_w be the optimal in Bellman equations for v and w, respectively. Then
$$\begin{aligned} Tv(y) &= u(c_v) + \beta \int v(zf(y-c_v))\phi(dz) \\ &\leq u(c_v) + \beta \int w(zf(y-c_v))\phi(dz) \\ &\leq u(c_w) + \beta \int w(zf(y-c_w))\phi(dz) = Tw(y). \end{aligned}$$
Thus the monotonicity is satisfied. Second, for any $a \in \mathbb{R}_+$,
$$\begin{aligned} T(v+a)(y) &= \max_c u(c) + \beta \int (v+a)(zf(y-c))\phi(dz) \\ &= \max_c u(c) + \beta \int v(zf(y-c)) + a\phi(dz) = Tv(y) + \beta a. \end{aligned}$$
Note that here by $(v+a)(y)$ we mean $v(y) + a$. This completes the proof. ■

Solution. 1.3, 1.4, 1.5, 1.6, 1.7.
See the Julia code file.