

Labor Economics Homework 1

Due:

1. Julia and VSCode

1. Go to <https://julialang.org/downloads/> and download the latest stable version of Julia.
2. Go to <https://code.visualstudio.com/download> and download Visual Studio Code.
3. Go to the extensions tab in VSCode and install the Julia extension.

Now you should be able to run Julia code in VSCode. To install packages, you can open the terminal and type `julia`; then type `!add PackageName` to install the package `PackageName`. For more details, see <https://julia.quantecon.org/intro.html>.

2. Methodology

Illustrate the procedure of investigating the economic questions via the structural model approach.

Solution. 2.

In general, the structural model approach consists of four steps:

1. **Modeling:** Formulate the economic problem and the environment.
2. **Solving:** Solve the model to obtain the policy functions or the equilibrium.
3. **Estimation:** Estimate the model parameters using the obtained policy functions or the equilibrium and the data. Find the parameters that produce the best fits between the model and the data.
4. **Counterfactual Analysis:** Conduct the policy experiments. For example, one may ask what if the minimum wage increases by 10%?

Typically the most difficult parts are the solving and the estimation steps. However, one should always keep in mind that the counterfactual analysis is the ultimate goal of the entire research. ■

3. Robinson Crusoe's Coconuts

Robinson Crusoe finds z_0 units of coconut on the beach in the morning of day 0. Each day, he can either consume the coconuts or leave them for the next day. If he consumes c_t coconuts on day t , he will have $z_{t+1} = z_t - c_t$. In each day, the utility of consuming c_t coconuts is $u(c_t) = \sqrt{c_t}$. Furthermore, the coconuts will rot at the end of day T . Given the time discount factor $\beta \in (0, 1)$, Crusoe's problem is thus formulated as follows:

$$\max_{\{c_t\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t) \quad \text{s.t.} \quad z_{t+1} = z_t - c_t.$$

1. Write down the value function $v(z, t)$ and the Bellman equation.
2. Write down the first-order condition of the problem.
3. Solve the optimal consumption c_0^* corresponding to z_0 analytically.
4. Now instead of the coconuts, Crusoe finds z_0 units of canned food. Since the canned

food will not rot, the problem is formulated as

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad z_{t+1} = z_t - c_t.$$

Write down the value function and the Bellman equation. Solve the optimal consumption c_0^* corresponding to z_0 analytically.

Solution. 3.1.

The value function is

$$v(z, t) = \max_{\{c_s\}_{s=t}^T} \sum_{s=t}^T \beta^s u(c_s)$$

with $z = \sum_{s=t}^T c_s$. The Bellman equation is

$$v(z, t) = \max_{c_t} \{u(c_t) + \beta v(z - c_t, t + 1)\}.$$

■

Solution. 3.2.

The F.O.C. is

$$u'(c_t) = \beta u'(c_{t+1}) \quad \forall t = 0, 1, \dots, T - 1.$$

Since $u(c_t) = \sqrt{c_t}$, the F.O.C. becomes

$$\frac{1}{2\sqrt{c_t}} = \beta \frac{1}{2\sqrt{c_{t+1}}}.$$

■

Solution. 3.3.

Rearranging the F.O.C. yields

$$c_{t+1} = \beta^2 c_t.$$

Combining this with the budget constraint $z_0 = \sum_{s=0}^T c_s$, we have

$$z_0 = c_0 + \beta^2 c_0 + \dots + \beta^{2T} c_0 = c_0 \sum_{s=0}^T \beta^{2s} = c_0 \frac{1 - \beta^{2(T+1)}}{1 - \beta^2}.$$

Thus

$$c_0^* = \frac{z_0(1 - \beta^2)}{1 - \beta^{2(T+1)}}.$$

■

Solution. 3.4.

Taking the limit as $T \rightarrow \infty$, the value function is

$$v(z) = \max_{\{c_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^s u(c_s).$$

Observe that the value function does not depend on t since today's problem is the same as tomorrow's.

The Bellman equation becomes

$$v(z) = \max_c \{u(c) + \beta v(z - c)\}.$$

The F.O.C. is the same as 3.3. Thus the optimal consumption is

$$c_0^* = z_0(1 - \beta^2)$$

by taking $T \rightarrow \infty$.

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4. McCall's Job Search Model

A worker is finding a job. In each period, the worker receives a job offer with wage w_t . The wage is i.i.d. across periods in $Beta(\alpha, \beta)$. The worker can either accept the offer ($s_t = 1$) and work at the wage w_t forever or reject it ($s_t = 0$) and receives \underline{c} as the

unemployment benefit. Once the worker accepts an offer, he cannot search for another job or quit. Assume that the worker cannot borrow or save; thus $u(c_t) = w_t$. With the discount factor $\rho \in (0, 1)$, the worker's problem is

$$\max_{\{s_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \rho^t (w_t s_t + (1 - s_t) \underline{c}) \right] \quad \text{s.t. } \{s_t \mid s_t \in \{0, 1\}\} \text{ is a non-decreasing sequence.}$$

1. Write down the Bellman equation.
2. Define the Bellman operator and verify that it is a contraction mapping on the space of bounded continuous functions defined on $[0, 1]$ with the supremum norm.
(Hint: $|\max\{a, b\} - \max\{a, c\}| \leq |b - c|$ for real a, b, c .)
3. Given $\rho = 0.95$, $\underline{c} = 0.1$, $\alpha = 2$, and $\beta = 5$, solve the model numerically.
4. Plot the value function and the optimal policy function as functions of w . What is the reservation wage?
5. Simulate the model for 1000 agents. Plot the distribution of the time to find a job.

Solution. 4.1.

The Bellman equation is

$$v(w) = \max \left\{ \frac{w}{1 - \rho}, \underline{c} + \rho \int v(w') F(dw') \right\},$$

where F is the CDF of $\text{Beta}(\alpha, \beta)$. ■

Solution. 4.2.

The Bellman operator is

$$(Tv)(w) = \max \left\{ \frac{w}{1-\rho}, \underline{c} + \rho \int v(w') F(dw') \right\}.$$

Now let v_1 and v_2 be two bounded continuous functions on $[0, 1]$ and $\|\cdot\|$ denote the supremum norm. Then

$$\begin{aligned} \|Tv_1 - Tv_2\| &= \sup_{w \in [0,1]} \left| \max \left\{ \frac{w}{1-\rho}, \underline{c} + \rho \int v_1(w') F(dw') \right\} \right. \\ &\quad \left. - \max \left\{ \frac{w}{1-\rho}, \underline{c} + \rho \int v_2(w') F(dw') \right\} \right| \\ &\leq \sup_{w \in [0,1]} \left| \rho \int v_1(w') F(dw') - \rho \int v_2(w') F(dw') \right| \\ &\leq \rho \int \sup_{w \in [0,1]} |v_1(w') - v_2(w')| F(dw') \\ &= \rho \int \|v_1 - v_2\| F(dw') = \rho \|v_1 - v_2\|. \end{aligned}$$

Thus T is a contraction mapping. ■

An Alternative Approach:

One may observe that there exists a “reservation wage”, \bar{w} , where the worker is indifferent between accepting and rejecting the offer. That is,

$$\frac{\bar{w}}{1-\rho} = \underline{c} + \rho \int v(w') F(dw').$$

By the Bellman equation, substituting $v(w')$ yields

$$\frac{\bar{w}}{1-\rho} = \underline{c} + \rho \int \max \left\{ \frac{w'}{1-\rho}, \frac{\bar{w}}{1-\rho} \right\} F(dw'),$$

which is equivalent to

$$\bar{w} = (1-\rho)\underline{c} + \rho \int \max \{w', \bar{w}\} F(dw').$$

■

Solution. 4.2. Conti.

Define the new operator U defined on $[0, 1]$ as

$$U\bar{w} = (1 - \rho)\underline{c} + \rho \int \max \{w', \bar{w}\} F(dw').$$

We claim that U is a contraction mapping.

Let $\bar{w}_1, \bar{w}_2 \in [0, 1]$. Then

$$\begin{aligned} |U\bar{w}_1 - U\bar{w}_2| &= \left| \rho \int \max \{w', \bar{w}_1\} F(dw') - \rho \int \max \{w', \bar{w}_2\} F(dw') \right| \\ &= \rho \left| \int \max \{w', \bar{w}_1\} - \max \{w', \bar{w}_2\} F(dw') \right| \\ &\leq \rho \int |\max \{w', \bar{w}_1\} - \max \{w', \bar{w}_2\}| F(dw') \\ &\leq \rho \int |\bar{w}_1 - \bar{w}_2| F(dw') = \rho |\bar{w}_1 - \bar{w}_2| \int F(dw') = \rho |\bar{w}_1 - \bar{w}_2|. \end{aligned}$$

■

Solution. 4.3, 4.4, 4.5.

See the Julia code file.

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