{} Application of Analytic function to flow Problems: →

Let the two dimensional irrotational motion of an incompressible fluid, in plane ∥ to xy plane.

Let $\vec{v}$ be the velocity of a fluid particle as

$$\vec{V} = v_x \hat{i} + v_y \hat{j} \longrightarrow (1)$$

Since motion is irrotational, ∃ a scalar function $\phi(x,y)$.

s.t

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$$

$$\vec{V} = -\nabla\phi$$

$$\vec{V} = -\frac{\partial\phi}{\partial x}\hat{i} - \frac{\partial\phi}{\partial y}\hat{j} \quad \text{—— (2)}$$

By Comparing eq1 and eq2.

$$\left.\begin{aligned} v_x &= \frac{-\partial\phi}{\partial x} \\ v_y &= \frac{-\partial\phi}{\partial y} \end{aligned}\right\} \quad (3)$$

ie Scalar function $\phi(x,y)$, which gives the velocity Components, is Called Velocity potential function or simply velocity Potential.

Since fluid is incompressible ⇒ $\text{div}\,\vec{v} = 0$

$$\nabla\cdot\vec{V} = 0$$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\cdot\left(V_x\hat{i} + V_y\hat{j}\right) = 0$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial y}\right) = 0$$

$$\Rightarrow -\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

⇒ ϕ is Harmonic and Can be treated as real part of an Analytic function and $\psi(x,y)$ is Imaginary

Let $w = f(z) = \phi(x,y) + i\psi(x,y)$

By CR's eq $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ and $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

Pa diff w.r.t y

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \phi}{\partial y \partial x}$$

Pa diff w.r.t x

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial^2 \phi}{\partial y \partial x}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{\partial^2 \psi}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$\Rightarrow \psi$ will harmonic function

We can say $\phi(x,y)$ and $\psi(x,y)$ both are conjugate Harmonic function to each other.

Where. $\phi(x,y) = c_1$ is known as Equipotential line curve

$\psi(x,y) = c_2$ is known as stream line curve

And $\psi(x,y)$ is known as stream function.

$w = f(z) = \phi(x,y) + i\psi(x,y)$

w is known as complex potential

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = -V_x - i(-V_y).$$

$$\frac{dw}{dz} = -V_x + iV_y$$

$\left|\frac{dw}{dz}\right| = \sqrt{V_x^2 + V_y^2}$, which is Resultant Velocity of fluid Flow.

Note: In the study of Electrostatics and Gravitational ~~flo~~ field, the curve $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$ are called Equipotential line and line of force respectively.

In Heat flow Problems the curves $\phi(x,y)=c_1$ and $\psi(x,y)=c_2$ are known as Isothermals and Heat flow lines respectively.