# Unit 4 Theory of Relativity.

**Relativity:-** The term relativity is used to explain primary quantities of Mechanics for example space, time & mass.

Theory of Relativity can be divided into two parts.

I) **Special theory of Relativity:-**

It deals with object & system which are moving with constant speed.

II) **General theory of Relativity:-**

It deals with object & system which are accelerating or deaccelerating with respect to each other.

**Refrence frame**

A system of coordinate axis which define the position of a particle or event in two or three dimension space is called frame of Refrence.

Refrence frame with four coordinates x, y, z, t is referred to as space time frame.

**Two type of Refrence frame:-**

I) **Inertial Frame of Refrence:-**

A frame of refrence that is not undergoing acceleration. The inertial frame of refrence are non-accelerating frames. In inertial frame else a body is at rest or in uniform motion.

Eg. A car stand still or a bus moving with constant speed are considered to be inertial frame of Refrence

Inertial Frame of Refrence Hold Newton law of motion.

## Non-Inertial Frame of Refrence:-

A frame of Refrence that undergoes acceleration wrt an inertial frame.

~~Eg.~~ Non-inertial frame don't hold Newton law of Motion.

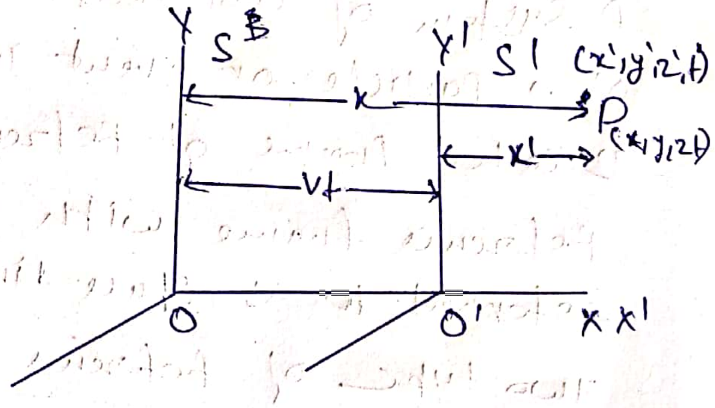
E.g. A car just started Moving from a standstill.

## Galilean Transformation:-

Galilean Transformations are those equation which relates the coordinates of a Particle/object in two Different inertial frames

OR.

The Gallilean Transformations are used to transform the coordinates of Position and time from one inertial frame to another.




Consider

S-frame of Refrence is Stationary (Rest)

S' - Non-Stationary (In Motion v velocity along X-Axis)

V - Constant velocity

O, O' → Two observer

P - Point P at any Particular time.

(x,y,z,t) → coordinates wrt S

(x',y',z',t') → coordinates wrt S'

Now the Relationship between coordinates of both the frame S & S' will be

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

These ~~four~~ four coordinates are called Galilean Transformation

## Lorentz Transformation:- (Used for high Speed moving objects)

Lorentz Transformation is the transformation between two inertial frame of reference when one is moving with constant velocity with respect to other.

According to Lorentz Equation, measurement in x-direction made in frame S must be linearly proportional to that made in S' frame. Hence a constant K should be there.

Consider

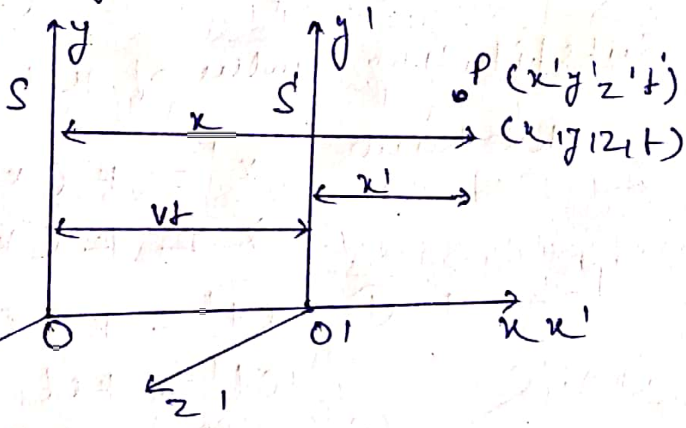
S - Rest frame

S' - Moving with velocity v along X-Axis

O, O' - Two observer of frame S & S'. ~~watchi~~

Now pulse of light reaches point P whose coordinates of position & time are..

(x, y, z, t) & (x', y', z', t')




Transform Equation of x & x' can be given as-

$$x' = k(x - vt) \quad \text{--- (1)}$$

where k is proportionality constant.

Inverse of above relation can be given as -

$$x = k(x' + vt') \quad \text{--- (2)}$$

Using first postulate of special theory of Relativity

All the laws of physics are valid and same for all inertial frame of reference (also called Principle of Equivalency)

∴ x should satisfy x'

Substitute value of x' from (1) in (2)

$$x = k(k(x - vt) + vt')$$

$$\frac{x}{k} = (kx - kvt + vt')$$

$$\frac{x}{k} - kx + kvt = vt'$$

$$t' = \frac{x}{kv} - \frac{kx}{v} + kt$$

$$= kt - \frac{kx}{v} + \frac{x}{kv}$$

$$= kt - \frac{kx}{v}\left(1 - \frac{1}{k^2}\right) \quad -(3)$$

According to second postulate of special Theory of Relativity speed of light remains constant in both the frame of Refrence, therefore velocity of light at O & O' should be same.

$$\therefore \quad x = ct \quad \& \quad x' = ct' \quad -(4)$$

Substituting value of $x$ & $x'$ from (4) in (1) & (2)

we have
$$x' = k(x - vt) \quad -(1)$$
$$x = k(x' + vt') \quad -(2)$$

In (1)
$$ct' = k(\cancel{ct} - vt)$$
$$ct' = kct - kvt$$
$$ct' = kt(c - v) \quad -(5)$$

Similarly $\quad ct = kt'(c + v) \quad -(6)$

Multiplying (5) & (6)

$$c^2 tt' = k^2 tt'(c^2 - v^2)$$

After solving we get $\quad k = \frac{1}{\sqrt{1 - v^2/c^2}} \quad -(7)$

Now substitute (7) in (1) then Lorentz transformation in position will be

$$\boxed{x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z}$$

Now Calculation of time:- Substitute (7) in (3)

$$t' = kt - \frac{kx}{v}\left(1 - \frac{1}{k^2}\right) \quad (3)$$

$$k = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \frac{1}{k} = \sqrt{1-\frac{v^2}{c^2}}$$

$$1-\frac{v^2}{c^2} = \frac{1}{k^2}$$

$$\frac{v^2}{c^2} = 1-\frac{1}{k^2}$$

Using in (3)

From (3) $$t' = kt - \frac{kx}{v}\left(1-\frac{1}{k^2}\right)$$

$$t' = kt - \frac{kx}{v}\left(\frac{v^2}{c^2}\right)$$

$$t' = kt - \frac{kxv}{c^2}$$

$$= k\left(t - \frac{xv}{c^2}\right)$$

Substitute $\left(k = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \text{ from } (7)\right)$

$$\Rightarrow \boxed{t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}}$$

Hence Lorentz transformation Equations becomes -

$$x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}},\ y'=y,\ z'=z \ \& \ t' = \frac{t-\frac{xv}{c^2}}{\sqrt{1-v^2/c^2}}$$

Inverse of Lorentz

$$x = \frac{x'+vt'}{\sqrt{1-v^2/c^2}},\ y=y',\ z=z' \ \& \ t = t'-\frac{x'v}{c^2} \Big/ \sqrt{1-v^2/c^2}$$

Special case:- when velocity of moving frame S' is smaller than the velocity of light $v \ll c$ that is $\sqrt{1-\frac{v^2}{c^2}} = 1$ & $\left(t-\frac{x'v}{c^2}\right) = t$

Then Lorentz transformation Reduce to Galilean Transformation

ie $x' = x - vt,\ y' = y,\ z' = z,\ t' = t$

## Michelson Morley Experiment:-

According to Michelson Morley theory, light should travel at different speed through ether.

(Note:- Ether is a theoretical universal substance to act as a medium for transmission of Electromagnetic waves.

Michelson Morley designed an interferometer to spot minute difference in the arrival of light beams.

Experiment compared the relative speed of light to the relative motion of Earth through ether.

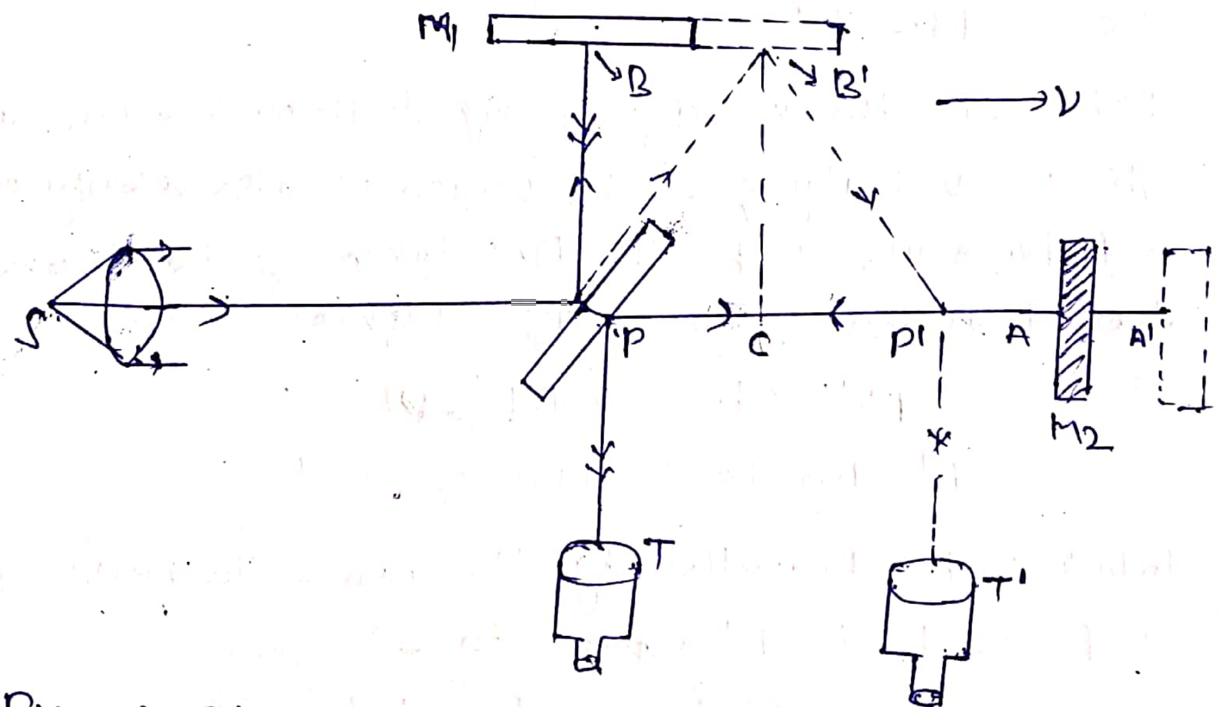



Figure shows that a parallel beam of light from a monochromatic source falls on a semi silvered glass plate at an angle of 45° to the beam. Now reflected beam moves towards $M_1$ & rest transmitted to $M_2$. These two beams moves perpendicular to each other and reaches to mirror $M_1$ & $M_2$. As the mirrors are highly silvered two beams thus reflected to point P and finally reached towards telescope T.

In this ~~Experiment~~ apparatus is at rest in ether & distance of $M_1$ & $M_2$ from P are equal, then two beam would take same time to return to P.

But in Actual Experiment Mirror on Apparatus is moving in the Direction of motion of Earth round the Sun and speed is $3 \times 10^4$ m/s.

Therefore, due to the motion of Earth or apparatus, the path of two beams and their reflection from the mirror are shown by dotted lines

Now optical path from P to Mirror $M_1$ & $M_2$ be $PA = PB = l$

Let c be the velocity of light through ether and v be the velocity of earth which is also velocity of Apparatus, t be the time taken by the beam to travel distance from P to mirror then.

$$PB' = ct \quad , \quad BB' = vt$$

( B' due to motion of Earth)

Total path travelled by the beam to move from P & reach to P' equal to -

$$PB'P' = PB' + B'P'$$
$$= 2PB'$$

Also $B'B = PC$

$B'C$ is $\perp$ ~~from~~ ~~B'~~ to PC then,

$$(PB')^2 = (PC)^2 + (CB')^2$$
$$(ct)^2 = (vt)^2 + l^2$$

$$\boxed{t = \frac{l}{\sqrt{c^2 - v^2}}}$$

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∴ Now time taken by the ray to move distance $PB'P'$ is

$$t_1 = 2t = \frac{2l}{\sqrt{c^2 - v^2}}$$

$$= \frac{2l}{c\sqrt{1-\frac{v^2}{c^2}}} = \frac{2l}{c}\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

$$\boxed{t_1 = \frac{2l}{c}\left(1+\frac{v^2}{2c^2}\right)}$$

The transmitting ray through P will move towards $M_2$ with velocity $(c-v)$ relative to apparatus and with velocity $(c+v)$ it will move back from $M_2$. Therefore total time from P to A' & A' to P' will be

$$t_2 = \frac{l}{c-v} + \frac{l}{c+v}$$

$$= \frac{2lc}{c^2 - v^2}$$

$$= \frac{2lc}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-1}$$

$$\boxed{t_2 = \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right)}$$

Hence the difference between time of travel of two rays is

$$t_2 - t_1 = \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right) - \frac{2l}{c}\left(1+\frac{v^2}{2c^2}\right)$$

$$= \frac{lv^2}{c^3}$$

Thus path difference, $\delta$ between two rays is -

$$\boxed{\delta = c(t_2 - t_1) = \frac{lv^2}{c^2}}$$

In Michelson Morley Experiment

$l = 11$ meter $\quad v = 3 \times 10^4$ m/s

$\lambda = 5890$ Å $\quad c = 3 \times 10^8$ m/s

$$\delta n = \frac{l v^2}{\lambda c^2}$$

$$= \frac{2 \times 11 \times (3 \times 10^4)^2}{5890 \times 10^{-10} \times (3 \times 10^8)^2}$$

$$= \frac{2.2}{5.89} = \underline{0.37}$$

In this Experiment design of Apparatus was such that it could detect a shift ~~but and~~ or Path difference but no fringe shift was observed in this Experiment. The Experiment was repeated several time along gap of about six month and at several place, but the Result was again Negative.

But this Experiment Results in the Special theory of Relativity.

<u>Special theory of Relativity</u> :→ (Postulates)

1) All the laws of physics have same form in all frame of Refrence moving with uniform velocity with respect to each other

11) Speed of light is constant in free space or in vaccum in all inertial frame of Refrence moving with uniform velocity with Each other.

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Length contraction:→ (Applications of Lorentz transformation)

According to Special theory of Relativity, length of a moving object decrease than the length of an object at rest. This process is called length contraction.

Let a Rod be placed parallel to x-axis in the system S' then the coordinates of the rod be $x_1'$ & $x_2'$ & length $l'$

$$l' = x_2' - x_1'$$

Also length $l$ of Rod in system S is given by

$$l = x_2 - x_1$$

According to Lorentz transformation Equation.

$$x_2' = \frac{(x_2 - vt)}{\sqrt{1 - v^2/c^2}}$$

$$x_1' = \frac{(x_1 - vt)}{\sqrt{1 - v^2/c^2}}$$

using $l' = x_2' - x_1'$

$$l' = \frac{(x_2 - vt)}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} - \frac{(x_1 - vt)}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

$$= \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} = \frac{l}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \boxed{l = l' \sqrt{1 - \frac{v^2}{c^2}}}$$

Special case:→ when $v \lll c$ then. $\sqrt{1 - \frac{v^2}{c^2}} \simeq 1$

therefore ~~$l = l_0$~~ $\boxed{l = l'}$

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Numerical (Length contraction).

Q. A rod has a length of 100 cm, when the Rod in a satellite moving with velocity 0.9c relative to laboratory. What is the length of the Rod determined by an observer in the laboratory.

Sol.

$$l' = 100\,cm. \quad v = 0.9c$$

then length of the Rod determined by observer is given by

$$l = l'\sqrt{1-\frac{v^2}{c^2}}$$

$$= 100\sqrt{1-\frac{(0.9c)^2}{c^2}}$$

$$= 100\sqrt{1-(0.9)^2} = 43.58\,cm.$$

Time Dilation: Time slow down at very fast speed. This Phenomenon is known as time Dilation.

Let the clock be situated ~~in the~~ in the frame S giving signals at the interval

$$\Delta t = t_2 - t_1$$

If this interval is recorded ~~by an~~ in the frame S'

the $$\Delta t' = t_2' - t_1'$$

From Lorentz transformation we have-

$$t_1' = \frac{t_1 - \frac{vx}{c^2}}{\sqrt{1-v^2/c^2}}$$

$$t_2' = \frac{t_2 - \frac{vx}{c^2}}{\sqrt{1-v^2/c^2}}$$

$$t_2' - t_1' = \frac{t_2 - t_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \boxed{\Delta t' = \frac{\Delta t}{\sqrt{1-v^2/c^2}}}$$

Special case :- I) $V < C$ then $\frac{V^2}{C^2}$ will be Negligible therefore $\Delta t' = \Delta t$

II) $V = C$ then $\frac{V^2}{C^2}$ will be less than Unity. then interval recorded by a moving clock greater than the clock when it is at Rest.

III

Numerical (Time Dilation)

Q A Certain Process require $10^{-6}s$ to occur in an atom at rest in Laboratory. How Much time this Process require to an observer in Laboratory, when the atom is Moving with a speed $5\times10^{7}\,ms^{-1}$

Sol:
$$\Delta t = 10^{-6}s$$
$$V = 5\times10^{7}\,ms^{-1} = 5\times10^{9}\,cms^{-1}$$
$$\Delta t' = \frac{10^{-6}}{\sqrt{1-\left(\frac{5\times10^{9}}{3\times10^{10}}\right)^2}} = \frac{10^{-6}}{\sqrt{1-\frac{25}{900}}}$$
$$= 1.013\times10^{-6}\,s.$$

Q. Prove that length is Invariant Under galilean Transformation.

Sol: Consider $S$ & $S'$ be two ~~inertial~~ Inertial frame. $S$ is in rest $S'$ moves with uniform velocity $V$ relative to $S$ along x-axis. Suppose a Rod AB is placed along x-axis in $S$.

Then
length of Rod in frame $S = x_2 - x_1$
$$L_0 = x_2 - x_1$$
length of Rod in frame $S'$ · $L = x_2' - x_1'$

According to Galilean transformation -
$$x_1' = x_1 - vt$$
$$x_2' = x_2 - vt$$
$$(x_2' - x_1') = (x_2 - x_1) - vt + vt$$
$$\Rightarrow \boxed{L = L_0}$$

In Galilean length is same in $S$ & $S'$ hence it is invariant.

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Relativistic Addition of velocities:→

Consider an inertial frame S moving with uniform velocity v relative to stationary observer S. If particle moves through a distance dx in time dt the velocity is -

$$u = \frac{dx}{dt} \quad \text{--- (1)}$$

To an observer in S' frame let velocity be

$$u' = \frac{dx'}{dt'} \quad \text{--- (2)}$$

We have Lorentz Equation:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \& \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{--- (3)}$$

Taking differential of above Equation, we get -

$$\cancel{dx' = \frac{dx - vdt}{dt - \frac{vdx}{c^2}}} \quad \text{and} \quad dt' = \frac{dt - \frac{vdx}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}} \quad \text{--- (4)}$$

Using (4) in (2)

$$u' = \frac{dx - vdt}{dt - vdx/c^2} = \frac{\frac{dx}{dt} - v}{\left(1 - \frac{vdx}{c^2 dt}\right)}$$

$$\boxed{u' = \frac{u - v}{1 - uv/c^2}}$$

This is the relativistic velocity addition formula.

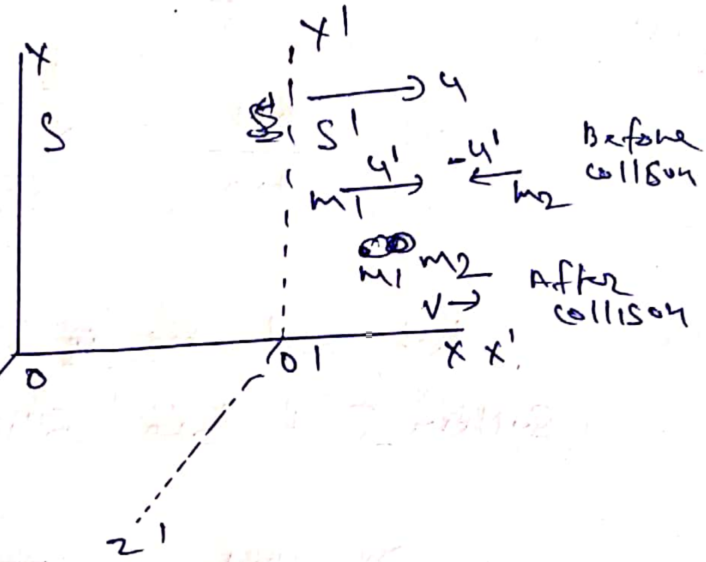
Inverse ⇒ $u = \frac{u' + v}{1 + u'v/c^2}$

Special Case:- If u & v are small compared to speed of light ($\frac{uv}{c^2} = 0$) then above formula reduce to Newtonian velocity addition formula.

$$\boxed{u' = u - v}$$

## Variation of Mass with velocity:-

Using the law of Addition of velocities. The velocity $u_1$ & $u_2$ in System S corresponding to $u'$ & $-u'$ are given by -




$$u_1 = \frac{u'+v}{1+\frac{u'v}{c^2}} \qquad u_2 = \frac{-u'+v}{1-\frac{u'v}{c^2}} \quad \text{—(1)}$$

According to principle of conservation of momentum

$$m_1u_1 + m_2u_2 = (m_1+m_2)V$$

Put the value of $u_1$ & $u_2$ from (1)

$$m_1\frac{u'+v}{1+\frac{u'v}{c^2}} + m_2\frac{-u'+v}{1-\frac{u'v}{c^2}} = (m_1+m_2)v$$

$$m_1\left(\frac{u'+v}{1+\frac{u'v}{c^2}} - v\right) + m_2\left(v - \frac{-u'+v}{1-\frac{u'v}{c^2}}\right)$$

$$m_1\left(\frac{u'+v-v-u'v^2/c^2}{1+\frac{u'v}{c^2}}\right) = m_2\left(\frac{v-\frac{u'v^2}{c^2}+u'-v}{1-\frac{u'v}{c^2}}\right)$$

$$\Rightarrow \frac{m_1}{m_2} = \frac{1+\frac{u'v}{c^2}}{1-\frac{u'v}{c^2}} \quad \text{—(2)}$$

From Eqn (1) It can be proved that

$$\frac{u'v}{c^2} = \frac{2c^2-u_1^2-u_2^2-2\sqrt{(c^2-u_1^2)(c^2-u_2^2)}}{u_1^2-u_2^2} \quad \text{—(3)}$$

Substituting value of $\frac{u'v}{c^2}$ from (3) to (2)

$$\Rightarrow \quad \frac{m_1}{m_2} = \frac{\sqrt{1+\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

Let body $m_2$ is moving with zero velocity in System S before collison then $u_2 = 0$

$$\Rightarrow \quad \frac{m_1}{m_2} = \frac{1}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

In commonly used notation we use

$m_1 = m$, $m_2 = m_0$ and $u_1 = v$.

$$\Rightarrow \quad \frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \boxed{m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}}$$

Case I $\quad v \ll c \quad \Rightarrow m = m_0$

~~$\Rightarrow m = m_0$~~

Case II $\quad v \simeq c \quad \Rightarrow m > m_0$

Case III $\quad v = c \quad \Rightarrow m = \infty$ or imaginary.

(This is not possible)

# Mass energy Equivalence:-

We have amount of Energy E related to Mass m as

$$E = mc^2$$

Rate of change of Momentum as force

$$F = \frac{d(P)}{dt} = \frac{d(mv)}{dt}$$

According to Relativity mass as well as velocity are variables

$$F = m\frac{dv}{dt} + v\frac{dm}{dt}$$

When particle is displaced through a distance dx by force F, then increase in kinetic Energy dk is.

$$dk = Fdx$$

$$dk = m\frac{dv}{dt}dx + v\frac{dm}{dt}dx$$

$$\text{or } \boxed{dk = mvdv + v^2dm} \quad -(1)$$

We have variation of Mass with velocity as.

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$m^2 = \frac{m_0^2}{1-\frac{v^2}{c^2}} = \frac{m_0^2c^2}{c^2-v^2}$$

$$\Rightarrow m^2c^2 - m^2v^2 = m_0^2c^2$$

on Differentiating

$$c^2 2m dm - v^2 2m dm - m^2 2v dv = 0$$

$$\Rightarrow c^2dm - v^2dm - mvdv = 0$$

$$\Rightarrow c^2dm = v^2dm + mvdv \quad -(2)$$

Comparing ①, ②

$$\Rightarrow \boxed{dK = c^2 dm}$$

If mass of body increase from $m_0$ to $m$

$$\int dK = \int_{m_0}^{m} c^2 dm$$

$$K = c^2(m - m_0)$$

This is increase in Kinetic Energy due to increase in Mass

Total Energy = Kinetic Energy + Rest Energy

$$= K + m_0c^2$$

$$= c^2(m - m_0) + m_0c^2$$

$$\boxed{E = mc^2}$$

This gives Universal Equivalence between Mass & Energy.

**Example 7.2** *What will be the expected fringe shift on the basis of stationary ether hypothesis in Michelson-Morley experiment if the effective length of each path is 8 m and wavelength of light used is 8000 Å. Given $v = 3\times10^4\, m/s$.* **[GGSIPU, Dec. 2013 (2.5 marks)]**

***Solution.*** Given $d = 8\,\text{m}$, $\lambda = 8000\,\text{Å} = 8.00\times10^{-7}\,\text{m}$, $c = 3\times10^8\,\text{m/s}$, $v = 3\times10^4\,\text{m/s}$

The fringe shift in Michelson-interferometer experiment is

$$= \frac{2dv^2}{c^2\lambda} = \frac{2\times8\times(3\times10^4)^2}{(3\times10^8)^2\times(8\times10^{-7})}$$

$$= 2\times10^8\times10^{-16}\times10^7 = 2.0\times10^{-1} = 0.2.$$

**Example 7.3** *Calculate the expected fringe shift in a Michelson-Morley experiment if the distance from each path is 2 metres and light has wavelength 6000 Å. Given $v = 3\times10^4\, m/s$, $c = 3\times10^8\, m/s$.*

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***Solution.*** Given $d = 2$ m, $v = 3 \times 10^4$ m/s, $\lambda = 6000$ Å $= 6.0 \times 10^{-7}$ m

The fringe shift in Michelson-Morley experiment is

$$= \frac{2dv^2}{c^2\lambda} = \frac{2 \times 2 \times (3 \times 10^4)^2}{(3 \times 10^8)^2 \times 6.0 \times 10^{-7}} = 0.07$$

***Example 7.4*** *Show that* $(x^2 + y^2 + z^2 - c^2t^2)$ *is invariant under Lorentz transformation.*

**[GGSIPU, Dec. 2013 (6.5 marks)]**

***Solution.*** Let $(x, y, z, t)$ and $(x', y', z', t')$ be the space-time coordinates of an event observed by two observers in reference frames $S$ and $S'$ respectively, the frame $S'$ moving constant velocity $v$ relative to $S$. Using Lorentz transformation equations, we have to simply prove that

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

The Lorentz equations are

$$x' = \frac{x - vt}{\sqrt{(1 - v^2/c^2)}}, \quad y' = y, \quad z' = z \quad \text{and} \quad t' = \frac{t - (xv/c^2)}{\sqrt{(1 - v^2/c^2)}}$$

Then, we have : $x'^2 + y'^2 + z'^2 - c^2t'^2$

$$= \frac{(x-vt)^2}{1-(v^2/c^2)} + y^2 + z^2 - \frac{c^2\left\{t - \frac{xv}{c^2}\right\}^2}{1-(v^2/c^2)}$$

$$= \frac{1}{1-\frac{v^2}{c^2}}\left[x^2 + v^2t^2 - 2xv - c^2\left[t^2 + \frac{x^2v^2}{c^4} - \frac{2txv}{c^2}\right]\right] + y^2 + z^2$$

$$= \frac{1}{1-\frac{v^2}{c^2}}\left[x^2 + v^2t^2 - 2xv - c^2t^2 - \frac{x^2v^2}{c^2} + 2txv\right] + y^2 + z^2$$

$$= \frac{1}{1-\frac{v^2}{c^2}}\left[x^2 + \left(1 - \frac{v^2}{c^2}\right) - c^2t^2\left(1 - \frac{v^2}{c^2}\right)\right] + y^2 + z^2$$

$$= (x^2 - c^2t^2) + y^2 + z^2 = x^2 + y^2 + z^2 - c^2t^2 \text{ hence proved.}$$

***Example 7.5** As measured by O, a flash bulb goes off at x = 100 km, y = 10 km, z = 1 km at t = 5×10⁻⁴ s. What are the coordinates x', y', z' and t' of this event as determined by a second observer, O', moving relative to O at – 0.8 c along the common x–x' axis ?*

**Solution.** From the Lorentz transformations

$$x' = \frac{x - vt}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \frac{100\text{ km} - (-0.8\times 3\times 10^5\text{ km/s})(5\times 10^{-4}\text{ s})}{\sqrt{(1-(0.8)^2)}} = 367\text{ km}$$

$$y' = y = 10\text{ km}\ ; \quad z' = z = 1\text{ km}$$

and

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \frac{5\times 10^{-4}\text{ s} - \frac{(0.8)(100\text{ km})}{3\times 10^5\text{ km/s}}}{\sqrt{[1-(0.8)^2]}} = 12.8\times 10^{-4}\text{ s}$$

*Example 7.6 Calculate the length contraction and orientation of a rod of length 5 m in a frame of reference which is moving with a velocity 0.6 c in a direction making an angle 30° with the rod.*

**[GGSIPU, Nov. 2013 (3 marks) ; Dec. 2013 (6 marks)]**

***Solution.*** The length contracts only in the direction of motion. The component of length in the direction of motion

$$l_x = 5\cos 30° = \frac{5\sqrt{3}}{2}, \qquad l_y = 5\sin 30° = \frac{5}{2}$$

when seen from the stationary observer,

$$l'_x = l_x\sqrt{\left(1-\frac{v^2}{c^2}\right)} = \frac{5\sqrt{3}}{2}\sqrt{1-(0.6)^2} = 2\sqrt{3} \qquad \Rightarrow \quad 1l'_y = ly = \frac{5}{2}$$

Hence $$l = \sqrt{l'^2_x + l'^2_y} = \sqrt{(2\sqrt{3})^2 + \left(\frac{5}{2}\right)^2} = 4.2 \text{ m}$$

and $$\tan\theta = \frac{l'_y}{l'_x} = \frac{5/2}{2\sqrt{3}} = 0.72 \quad \text{or} \quad \theta = \tan^{-1}(0.72) = 35.8°$$

*Example 7.7 Calculate the velocity of rod when its length appears three-fourth of its proper length.*

**[GGSIPU, Dec. 2015 (3 marks)]**

***Solution.*** A rod of proper length $l$ appears to get contracted to a length $l'$ for an observer with respect to whom it is moving with velocity $v$, such that

$$l' = l\sqrt{1-\frac{v^2}{c^2}}$$

Here $$l' = \frac{3}{4}l \qquad \therefore \quad \frac{3}{4}l = l\left(1-\frac{v^2}{c^2}\right)^{1/2}$$

This gives $$1-\frac{v^2}{c^2} = \frac{9}{16}$$

$$\frac{v^2}{c^2} = \frac{7}{16} \qquad \Rightarrow \quad \frac{v}{c} = \sqrt{\frac{7}{16}} = 0.66 \qquad \Rightarrow \quad v = 0.66\,c$$

∴ The rod moves with a velocity 0.66 c.

*Example 7.10 A clock is moving with a speed* $0.95c$ *relative to an observer stationed on the earth. If the speed is increased by 5% by what % does time dilation increases ?*

**Solution.** Given $v = 0.95c$ and $v' = 1.05 \times v$

We know that time dilation formula

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{or} \quad \frac{\Delta t'}{\Delta t} = \frac{1}{\sqrt{1-(0.95)^2}}$$

or
$$\frac{\Delta t'}{\Delta t} = 3.2$$

Now $\quad v' = 1.05\, v = 1.05 \times 0.95c = 0.9975c \quad$ (on increment of 5% velocity)

Then
$$\left(\frac{\Delta t'}{\Delta t}\right)^{\text{New}} = \frac{1}{\sqrt{1-(0.9975)^2}} = 14.14$$

$$\% \text{ time dilation increases} = \frac{14.14 - 3.2}{14.14} \times 100 = 77.3\%$$

*Example 7.11 A beam of particle of half life* $2.0 \times 10^{-8}$ *s, travels in the laboratory with speed* $0.96\,c$. *How much distance does the beam travel before the number of particle is reduced to half times of the initial value.*

**[GGSIPU, Nov. 2011 (3 marks)]**

**Solution.** $2.0 \times 10^{-8}$ s is the proper half life of the particles *i.e.,* the time interval in the particles own frame of reference in which its flux reduces to half of its initial flux. The ordinary half in the laboratory frame $\Delta t$ and proper half life $\Delta t'$ are related by the relation

$$\Delta t = \Delta t' \sqrt{1 - \frac{v^2}{c^2}}$$

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.0 \times 10^{-8}}{\sqrt{1-(0.96)^2}} = 7.1 \times 10^{-8} \text{ s.}$$

∴ The distance travelled by the beam in this time in the laboratory frame

$$= 0.96\, c \times 7.1 \times 10^{-8} = 20.45 \text{ m.}$$

***Example 7.19*** *Find the speed that a proton must be given if its mass is to be twice its rest mass of $1.67 \times 10^{-27}$ kg. What energy must be given the proton to achieve this speed ?*

***Solution.*** Given $m = 2m_0$, $m_0 = 1.67 \times 10^{-27}$ kg

We know that the variation of mass with velocity

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \qquad 2m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \Rightarrow \qquad v = 2.6 \times 10^8 \text{ m/s}$$

To achieve this speed, the energy given to proton $= mc^2$

$$= \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1.67\times 10^{-27}\times(3\times 10^8)^2}{\sqrt{1-\left(\frac{2.6\times 10^8}{3\times 10^8}\right)^2}} = 3.01\times 10^{-10}\ \text{J} = 1.88\ \text{GeV}$$

**Problem 7.1** *Show that if* $(x_1, y_1, z_1, t_1)$ *and* $(x_2, y_2, z_2, t_2)$ *be the coordinates of one event in* $S_1$ *frame and the corresponding event in* $S_2$ *frame respectively, then the expression*

$$dS_1^2 = dx_1^2 + dy_1^2 + dz_1^2 - c^2 dt_1^2$$

*is invariant under a Lorentz transformation of coordinates.*

*Or*

*Show that space-time interval in Lorentz transformation is invariant.* **[GGSIPU, Dec. 2011 (4.5 marks)]**

***Solution.*** The inverse Lorentz transformation equations are :

$$x_1 = \frac{x_2 + vt_2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}, \quad y_1 = y_2; \quad z_1 = z_2$$

and

$$t_1 = \frac{t_2 + \frac{vx_2}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \quad ...(i)$$

Differentiating,

$$dx_1 = \frac{dx_2 + vdt_2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}; \quad dy_1 = dy_2; \ dz_1 = dz_2$$

and

$$dt_1 = \frac{dt_2 + \frac{v}{c^2} dx_2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \quad ....(ii)$$

$$dS_1^2 = \left[\frac{dx_2 + vdt_2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}\right]^2 + dy_2^2 + dz_2^2 - c^2 \left[\frac{dt_2 + \frac{v}{c^2} dx_2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}\right]^2$$

On simplification,

$$dS_1^2 = dx_2^2 + dy_2^2 + dz_2^2 - c^2 dt_2^2 = dS_2^2.$$ Hence proved.

**Problem 7.3** *Calculate the percentage contraction of a rod moving with a velocity* 0.8 *c in a direction inclined at* 60° *to its own length.* **[GGSIPU, Dec. 2016 reappear (3.5 marks)]**

***Solution.*** The component of the length of the rod to its direction of motion.

$$= l\cos 60° = \frac{1}{2}l$$

and the component of its length, perpendicular to its direction of motion

$$= l\sin 60° = \frac{\sqrt{3}}{2}l$$

where $l$ = length of the rod, placed along the $x$-axis in $S$ frame (at rest).

Hence, $$l'_x = \frac{1}{2}l\sqrt{1-\left(\frac{0.8\,c}{c}\right)^2} = \frac{1}{2}l\times 0.6 = 0.3\,l$$

other component $l'_y = \frac{\sqrt{3}}{2}l$

Total length of the rod in frame $S'$ (moving) frame is

$$l' = (l'^2_x + l'^2_y)^{1/2}$$

$$= \sqrt{(0.3\,l)^2 + \left(\frac{\sqrt{3}}{2}l\right)^2} = \sqrt{0.09\,l^2 + \frac{3}{4}l^2} = \sqrt{0.84}\,l = 0.92\,l$$

∴ Percentage contracted produced in the length of the rod

$$= \frac{l - 0.9165\,l}{l}\times 100 = 8.3\%.$$

**Problem 7.6** *Calculate the speed of the particle if its total energy is exactly twice of its rest mass energy.*

*Or*

*At what speed a body must move so as to have its mass double ?* **[GGSIPU, Dec. 2009 (3.5 marks)]**

*Or*

*Find the speed of an electron having mass double its rest mass.* **[GGSIPU, Dec. 2012 (2.5 marks)]**

*Or*

*What is the speed of an electron having mass double its rest mass ?* **[GGSIPU, Dec. 2011 (2 marks)]**

*Or*

*At what speed the mass of an electron shall be double its rest mass ?* **[GGSIPU, Dec. 2004 (3 marks)]**

*Or*

*The total energy of particle is exactly twice its rest energy. Calculate the velocity of particle.*
**[GGSIPU, Dec. 2015 (3 marks)]**

***Solution.*** The total energy $E$ of a moving particle is $mc^2$ while its rest energy $E_0$ is $m_0c^2$.

When $E = 2E_0$, then $\quad mc^2 = 2m_0c^2 \quad$ or $\quad m = 2m_0$

The mass of the moving particle is related to its rest mass $m_0$ by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Here $m = 2m_0$

$$2m_0 = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

or

$$1-\frac{v^2}{c^2} = \frac{1}{4} \quad \text{or} \quad \frac{v^2}{c^2} = \frac{3}{4}$$

$$v = \frac{\sqrt{3}}{2}c = 0.886\,c$$

This result does not depend on the rest mass of the particle.