Equations (... from space and time.

Example 7.4 *Show that* $(x^2 + y^2 + z^2 - c^2t^2)$ *is invariant under Lorentz transformation.*

[GGSIPU, Dec. 2013 (6.5 marks)]

Solution. Let (x, y, z, t) and (x', y', z', t') be the space-time coordinates of an event observed by two observers in reference frames S and S' respectively, the frame S' moving constant velocity v relative to S. Using Lorentz transformation equations, we have to simply prove that

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

The Lorentz equations are

$$x' = \frac{x - vt}{\sqrt{(1 - v^2/c^2)}}, \quad y' = y, \quad z' = z \quad \text{and} \quad t' = \frac{t - (xv/c^2)}{\sqrt{(1 - v^2/c^2)}}$$

Then, we have : $x'^2 + y'^2 + z'^2 - c^2t'^2$

$$= \frac{(x-vt)^2}{1-(v^2/c^2)} + y^2 + z^2 - \frac{c^2\left\{t - \dfrac{xv}{c^2}\right\}^2}{1-(v^2/c^2)}$$

$$= \frac{1}{1-\dfrac{v^2}{c^2}}\left[x^2 + v^2t^2 - 2xv\,t - c^2\left[t^2 + \frac{x^2v^2}{c^4} - \frac{2txv}{c^2}\right]\right] + y^2 + z^2$$

$$= \frac{1}{1-\dfrac{v^2}{c^2}}\left[x^2 + v^2t^2 - 2xv\,t - c^2t^2 - \frac{x^2v^2}{c^2} + 2txv\right] + y^2 + z^2$$

$$= \frac{1}{1-\dfrac{v^2}{c^2}}\left[x^2\left(1 - \frac{v^2}{c^2}\right) - c^2t^2\left(1-\frac{v^2}{c^2}\right)\right] + y^2 + z^2$$

$= (x^2 - c^2t^2) + y^2 + z^2 = x^2 + y^2 + z^2 - c^2t^2$ hence proved.

Example 7.6 *Calculate the length contraction and orientation of a rod of length 5 m in a frame of reference which is moving with a velocity 0.6 c in a direction making an angle 30° with the rod.*

[GGSIPU, Nov. 2013 (3 marks) ; Dec. 2013 (6 marks)]

Solution. The length contracts only in the direction of motion. The component of length in the direction of motion

$$l_x = 5\cos 30° = \frac{5\sqrt{3}}{2}, \qquad l_y = 5\sin 30° = \frac{5}{2}$$

when seen from the stationary observer,

$$l'_x = l_x\sqrt{\left(1-\frac{v^2}{c^2}\right)} = \frac{5\sqrt{3}}{2}\sqrt{1-(0.6)^2} = 2\sqrt{3} \qquad \Rightarrow \quad 1\,l'_y = ly = \frac{5}{2}$$

Hence $$l = \sqrt{l_x'^2 + l_y'^2} = \sqrt{(2\sqrt{3})^2 + \left(\frac{5}{2}\right)^2} = 4.2 \text{ m}$$

and $$\tan\theta = \frac{l'_y}{l'_x} = \frac{5/2}{2\sqrt{3}} = 0.72 \qquad \text{or} \qquad \theta = \tan^{-1}(0.72) = 35.8°$$

Example 7.7 *Calculate the velocity of rod when its length appears three-fourth of its proper length.*

[GGSIPU, Dec. 2015 (3 marks)]

Solution. A rod of proper length l appears to get contracted to a length l' for an observer with respect to whom it is moving with velocity v, such that

$$l' = l\sqrt{1-\frac{v^2}{c^2}}$$

Here $$l' = \frac{3}{4}l \qquad \therefore \qquad \frac{3}{4}l = l\left(1-\frac{v^2}{c^2}\right)^{1/2}$$

This gives $$1-\frac{v^2}{c^2} = \frac{9}{16}$$

$$\frac{v^2}{c^2} = \frac{7}{16} \qquad \Rightarrow \qquad \frac{v}{c} = \sqrt{\frac{7}{16}} = 0.66 \qquad \Rightarrow \qquad v = 0.66\,c$$

$\therefore$ The rod moves with a velocity 0.66 c.

Example 7.11 *A beam of particle of half life 2.0×10^{-8} s, travels in the laboratory with speed 0.96 c. How much distance does the beam travel before the number of particle is reduced to half times of the initial value.*

[GGSIPU, Nov. 2011 (3 marks)]

Solution. 2.0×10^{-8} s is the proper half life of the particles *i.e.*, the time interval in the particles own frame of reference in which its flux reduces to half of its initial flux. The ordinary half in the laboratory frame Δt and proper half life $\Delta t'$ are related by the relation

$$\Delta t = \Delta t' \sqrt{1 - \frac{v^2}{c^2}}$$

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}}$$

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.0 \times 10^{-8}}{\sqrt{1 - (0.96)^2}} = 7.1 \times 10^{-8} \text{ s.}$$

$\therefore$ The distance travelled by the beam in this time in the laboratory frame

$$= 0.96\, c \times 7.1 \times 10^{-8} = 20.45 \text{ m.}$$

Then the relativistic velocity is given by

$$v = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{(0.5 + 0.9)c}{1 + \frac{(0.5 \times 0.9)c^2}{c^2}} = 0.9655c$$

Classically $\quad u = u' + v = 0.5\,c + 0.9\,c = 1.4\,c$ which is not possible.

Problem 7.3 *Calculate the percentage contraction of a rod moving with a velocity 0.8 c in a direction inclined at 60° to its own length.* **[GGSIPU, Dec. 2016 reappear (3.5 marks)]**

Solution. The component of the length of the rod to its direction of motion.

$$= l\cos 60° = \frac{1}{2}l$$

and the component of its length, perpendicular to its direction of motion

$$= l\sin 60° = \frac{\sqrt{3}}{2}l$$

where l = length of the rod, placed along the x-axis in S frame (at rest).

Hence, $\quad l'_x = \frac{1}{2}l\sqrt{1 - \left(\frac{0.8\,c}{c}\right)^2} = \frac{1}{2}l \times 0.6 = 0.3\,l$

other component $\; l'_y = \frac{\sqrt{3}}{2}l$

Total length of the rod in frame S' (moving) frame is

$$l' = (l'^2_x + l'^2_y)^{1/2}$$

$$= \sqrt{(0.3\,l)^2 + \left(\frac{\sqrt{3}}{2}l\right)^2} = \sqrt{0.09\,l^2 + \frac{3}{4}l^2} = \sqrt{0.84}\,l = 0.92\,l$$

∴ Percentage contracted produced in the length of the rod

$$= \frac{l - 0.9165\,l}{l} \times 100 = 8.3\%.$$

Problem 7.4 *A beam of particle of half life 2×10^{-8} s travels in a laboratory with speed 0.96 times the speed of light. How much distance does the beam travel before the flux falls to 1/2 times the initial flux ?*

Solution. Given : $\; v = 0.96\,c,$

$$\Delta t' = \text{proper half life} = 2 \times 10^{-8}\,\text{sec}$$

$$\Delta t = ? = \text{ordinary half life in the lab frame}$$

We know that

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2 \times 10^{-8}}{\sqrt{1 - (0.96)^2}} = 7.1 \times 10^{-8}\,\text{s}$$

Then distance travelled by the beam

$$= \text{Velocity} \times \text{time} = v \times \Delta t = 0.96c \times 7.1 \times 10^{-8}\,\text{s} = 20.45\ \text{m}.$$

or $v^2 = \frac{24}{25} c^2$ or $v = 0.98c$

Problem 7.6 *Calculate the speed of the particle if its total energy is exactly twice of its rest mass energy.*

Or

At what speed a body must move so as to have its mass double ? **[GGSIPU, Dec. 2009 (3.5 marks)]**

Or

Find the speed of an electron having mass double its rest mass. **[GGSIPU, Dec. 2012 (2.5 marks)]**

Or

What is the speed of an electron having mass double its rest mass ? **[GGSIPU, Dec. 2011 (2 marks)]**

Or

At what speed the mass of an electron shall be double its rest mass ? **[GGSIPU, Dec. 2004 (3 marks)]**

Or

The total energy of particle is exactly twice its rest energy. Calculate the velocity of particle. **[GGSIPU, Dec. 2015 (3 marks)]**

Solution. The total energy E of a moving particle is mc^2 while its rest energy E_0 is m_0c^2.

When $E = 2E_0$, then $mc^2 = 2m_0c^2$ or $m = 2m_0$

The mass of the moving particle is related to its rest mass m_0 by

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Here $m = 2m_0$

$$2m_0 = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

or

$$1 - \frac{v^2}{c^2} = \frac{1}{4} \quad \text{or} \quad \frac{v^2}{c^2} = \frac{3}{4}$$

$$v = \frac{\sqrt{3}}{2} c = 0.886\, c$$

This result does not depend on the rest mass of the particle.

1.6×10^{-19}

Problem 7.10 *Calculate the mass and speed of an electron accelerated to a kinetic energy of 2 MeV.*

Solution. Given : $K.E. = 2 \text{ MeV} = 2 \times 10^{6} \times 1.6 \times 10^{-19}$

We know that the kinetic energy

$$E_k = mc^2 - m_0c^2 = (m - m_0)c^2$$

Or

A rod 1 m long is moving along its length with velocity 0.6 c. Calculate the length as it appears to an observer on the surface of earth. **[GGSIPU, Nov. 2011 reappear (2 marks) ; Dec. 2017 (2.5 marks)]**

Hint : (*a*) Since the rod is at rest in satellite its length as measured by an observer in the satellite, will be its proper length *i.e.*, $l' = l = 1$ m.

$$l' = l\sqrt{1-\frac{v^2}{c^2}}$$

Here l = length of rod in the system in which it is at rest (*i.e.*, satellite) = 100 cm

$$v = 0.6\,c \qquad \therefore \qquad l' = 100\sqrt{1-(0.6)^2} = 80 \text{ cm}.$$

7.3 Length of a moving spaceship is found to be 80% of its actual length to an observer at rest. Calculate the speed of the spaceship. **[GGSIPU, Dec. 2016 (2 marks)]**

Hint : $l' = l\sqrt{\left(1-\frac{v^2}{c^2}\right)} \Rightarrow v = 0.6c$

7.4 At what speed will an object of length 100 cm be measured as 50 cm an observer at rest.

Hint : $l' = l\sqrt{\left(1-\frac{v^2}{c^2}\right)} \Rightarrow \frac{1}{2} = \sqrt{\left(1-\frac{v^2}{c^2}\right)} \Rightarrow \frac{1}{4} = 1-\frac{v^2}{c^2}$

$$\Rightarrow \frac{v^2}{c^2} = \frac{3}{4} \Rightarrow \frac{v}{c} = \sqrt{\frac{3}{4}} \Rightarrow v = \frac{\sqrt{3}}{2}c = 0.866\ c.$$

7.5 The ratio of the proper life to the mean life of a moving fundamental particle is 1/5. Calculate the speed of the fundamental particle. **[GGSIPU, Dec. 2016 (2 marks)]**

Hint : $\Delta t' = \frac{\Delta t}{\sqrt{\left(1-\frac{v_2}{c^2}\right)}} \Rightarrow v = 0.9c$

7.6 A space craft is moving relative to earth. An observer on the earth finds that, between 1 PM and 2 PM according to her clock, 3601 s elapse on space-craft's clock. What is the space craft's speed relative to earth. **[GGSIPU, Dec. 2015 (3 marks) – Main Campus]**

Hint : $$\Delta t = \frac{\Delta t'}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \Rightarrow \frac{v^2}{c^2} = 1-\left(\frac{3600}{3601}\right)^2$$

$$\Rightarrow \quad v = 3\times10^8\left[1-\left(\frac{3600}{3601}\right)^2\right]^{1/2} = 7.1\times10^6 \text{ m/s.}$$

Today's space craft are much slower than this.

7.7 A muon decays with a mean life time of 2.2×10^{-6} seconds measured in a frame of reference in which it is at rest. If the muon velocity is $0.99\,c$ with respect to the laboratory, what is its mean life as observed from laboratory frame ? **[GGSIPU, Dec. 2008 (4 marks)]**

Hint : $$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.2\times10^{-6}}{\sqrt{1-(0.99)^2}} = 1.57\times10^{-5}\text{ s.}$$

7.8 Two twin brothers are aged 40. One of them stays on earth and the other goes into space in a rocket. If the latter returns to earth after 5 years, who will be younger and why ? **[GGSIPU, Nov. 2013 (2 marks)]**

Hint : As we know that 'moving clocks go slow' and as per time-dilation, the brother goes to space, will be younger.

7.9 In a laboratory two particles are observed to travel in opposite directions with speed 2.8×10^{10} cm/s. Deduce the relative speed of the particles. **[GGSIPU, Nov. 2004, Oct. 2013 (4 marks)]**

Hint : $$u_x' = \frac{u_x - v}{1-\left(\frac{u_x v}{c^2}\right)}$$

Putting the value, we get :

$$u_x' = \frac{2.8\times10^8-(-2.8\times10^8)}{1-\frac{(-2.8\times10^8)(2.8\times10^8)}{(3\times10^8)^2}} = 2.99\times10^8 \text{ m/sec.}$$

7.10 Calculate the velocity of a particle at which its mass will become 8 times of its rest mass. **[GGSIPU, Jan 2015 (3 marks)]**

Hint : Using relation,

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow 8m_0 = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \Rightarrow v = 0.99\,c$$

∴ The velocity of the first particle relation to the second is 2.99×10^8 m / s.

7.11 An electron has an initial speed of 1.4×10^8 m/s. How much additional energy must be imported to it for its speed to double ? **[GGSIPU, Dec. 2010 (4 marks)]**

Hint : (*i*) $m_0 = 9.1 \times 10^{-31}$ kg, $v = 1.4 \times 10^8$ m/s then $m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$

$\Rightarrow$ $m = 1.0289 \times 10^{-30}$ kg $\Rightarrow$ $E = mc^2 = 9.26 \times 10^{-14}$ J ;

and (*ii*) $v = 2 \times 1.4 \times 10^8$ m/s then $m = 2.5347 \times 10^{-31}$ kg

and $E' = mc^2 = 2.28 \times 10^{-31}$ J

Additional Energy $\Delta E = E' - E = 1.26 \times 10^{-13}$ J .

7.12 Find the velocity of a particle, when its mass increases by three times.

[GGSIPU, Dec. 2019 (2.5 marks)]

Hint : $m = \dfrac{m_0}{\sqrt{(1-v^2/c^2)}}$ $\Rightarrow$ $3m_0 = \dfrac{m_0}{\sqrt{(1-v^2/c^2)}}$

$\left(1-\dfrac{v^2}{c^2}\right) = \dfrac{1}{9}$ or $\dfrac{v^2}{c^2} = 1 - \dfrac{1}{9} = \dfrac{8}{9}$ $\Rightarrow$ $v = \sqrt{\dfrac{8}{9}}c = 0.94\,c.$

7.13 What is the length of 1 metre stick moving parallel to its length when its mass is 1.5 times of its rest mass ?

[GGSIPU, Dec. 2018 (2 marks)]

Hint : $m = \dfrac{m_o}{\sqrt{\left(1-\dfrac{v^2}{c^2}\right)}}$ $\Rightarrow 1.5\, m_o = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$ so $\left(1-\dfrac{v^2}{c^2}\right) = \dfrac{2}{3}$

$\therefore$ The contracted length $l' = l\sqrt{\left(1-\dfrac{v^2}{c^2}\right)} = \dfrac{2}{3} \times$ length in moving stick frame.

7.14 How much energy must be given to an electron to accelerate it 0.95 c ?

Hint : Use the mass-increase formula

$$m = \frac{m_o}{\sqrt{\left(1-\dfrac{v^2}{c^2}\right)}} = \frac{9.1\times10^{-31}\,\text{kg}}{\sqrt{1-\left(\dfrac{0.95c}{c}\right)^2}} = 2.91 \times 10^{-31} \text{ kg}$$

$$E = (m - m_0)c^2 = [(29.1-9.1)\times10^{-31}]\times(9\times10^6) = 1.8 \times 10^{-13}\text{J} = 1.125 \text{ MeV}.$$

7.15 A relativistic electron ($m_0 = 0.511\text{MeV}/c^2$) and a photon ($m_0 = 0$) both have momenta of 2.00 MeV/c. Find the total energy of each.

[GGSIPU, Dec. 2007 (5 marks)]

Hint : (*a*) For electron, $E^2 = p^2c^2 + m_0^2c^4 = 2.064$ MeV

(*b*) For photon $m_0 = 0$

$E = pc = 2.00\left(\dfrac{\text{MeV}}{\text{c}}\right)\text{c} = 2.00$ MeV.

7.16 Calculate the velocity that one atomic mass unit will have if its kinetic energy is equal to twice the rest mass energy.

[GGSIPU, Nov. 2004 (5 marks)]

Hint : Since we know that K.E. $= mc^2 - m_0c^2$ = Total energy – Rest mass energy

According to question, K.E. $= 2m_0c^2$

$2m_0c^2 = mc^2 - m_0c^2$ or $3m_0c^2 = mc^2$ or $\dfrac{m}{m_0} = 3$

1.6×10

7.18 What is the total energy of a 2.5 MeV electron ?

[GGSIPU, Nov. 2008, Nov. 2013 (2 marks)]

Hint : $E = \text{K.E.} + m_0c^2 = 2.5\ \text{MeV} + \dfrac{9.1 \times 10^{31} \times (3 \times 10^8)^2}{16 \times 10^{-19}}$

$$= 2.5\ \text{MeV} + 0.511\ \text{MeV} = 3.011\ \text{MeV}.$$

7.19 Find the velocity that an electron which must be given so that its momentum is 11 times its rest mass time the speed of light. What is the energy at this speed ? **[GGSIPU, Nov. 2008 (4 marks)]**

Hint :

$$p = mv = \frac{m_0 v}{\sqrt{1 - \dfrac{v^2}{c^2}}}, \quad \text{But } p = 11\, m_0 c$$

$$11 m_0 c = \frac{m_0 v}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

$$\Rightarrow \qquad v = 2.99 \times 10^8\ \text{m/s}$$

and $$m = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \frac{9.1\times10^{-31}}{1-(0.996)^2} = \frac{9.1\times10^{-31}}{\sqrt{1-.992}} = 10.17\times10^{-30}\text{ kg}$$

$$E = mc^2 = 10.17\times10^{-30}\times(2.99\times10^8)^2 = 9.09\times10^{-13}\text{ J}$$

Energy may also be calculated using formula :

$$E^2 = p^2c^2 + m_0^2c^4.$$

7.20 A particle of rest mass m_0 moves with a speed $c/\sqrt{2}$. What are mass, momentum, total energy and kinetic energy. **[GGSIPU, Dec. 2011 (4 marks)]**

Hint : $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ as per question $v = c/\sqrt{2}$; then mass $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\Rightarrow \quad m = \sqrt{2}\,m_0\text{ ; momentum } p = mv = \sqrt{2}\,m_0\times\frac{c}{\sqrt{2}} = m_0c\text{ ;}$$

Energy $E = mc^2 = \sqrt{2}\,m_0c^2$ and

Kinetic energy K.E. $=(m-m_0)c^2 = m_0(\sqrt{2}-1)c^2 = 0.472\,m_0c^2$.

7.21 A particle of rest mass m_0 moves with a speed of 0.9 c. What are its mass, momentum, total energy and kinetic energy ? **[GGSIPU, Dec. 2012 (4.5 marks)]**

Hint : Given $v = 0.9$ c

(*i*) mass $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1-\left(\frac{0.9c}{c}\right)^2}}$

$= 2.29\,m_0 = 2.29\times9.1\times10^{-31}\text{ kg} = 2.08\times10^{-30}\text{ kg}$

(*ii*) momentum $p = mv = 2.08\times10^{-30}\times0.9c = 2.08\times10^{-30}\times0.9\times3\times10^8 = 5.62\times10^{-22}$ kg m/s

(*iii*) Total energy $E = mc^2 = 2.08\times10^{-30}\times9\times10^{16} = 1.872\times10^{-13}$ J

(*iv*) Kinetic energy $(KE) = E - \text{rest energy} = mc^2 - m_0c^2$

$= (m-m_0)c^2 = 1.17\times10^{-30}\times9\times10^{16}$

$= 1.053\times10^{-13}$ J.

7.22 Having the same momentum, which will move faster an electron or a photon ? **[GGSIPU, Dec. 2008 ; Nov. 2012 Reappear (2.5 marks)]**

Hint : $p_e = p_p$

$\Rightarrow \quad m_ev_e = m_pv_p \qquad m_p << m_e$

$v_e = \frac{m_p}{m_e}v_p \qquad \frac{m_p}{m_e} << 1$ so $v_p > v_e$.

7.23 Calculate the ratio of the mass of electron moving with K.E. 20 MeV to its rest mass which is 0.51 MeV. **[GGSIPU, Dec. 2013 (2 marks)]**