Rank of a Matrix

Let A be any $m \times n$ matrix. It has square sub-matrices of different orders. The determinants of these square sub-matrices are called minors of A. If all minors of order $(r+1)$ are zero but there is atleast one non-zero minor of order r, then r is called the rank of A, is denoted by $\boxed{\rho(A)}$.

Note (i) If A is a null matrix, then $\rho(A) = 0$

(ii) If A is not a null matrix, then $\rho(A) \geq 1$.

(iii) If A is a non-singular $n \times n$ matrix, then $\rho(A) = n$

(iv) If A is an $m \times n$ matrix, then $\rho(A) =$ minimum of m and n

(v) If all minors of order r equal to zero, then $\rho(A) < r$.

Normal forms

If A is an $m \times n$ matrix & by a series of elementary (row or column or both) operations, it can be put into one of the following forms (normal forms)

$$\left[\begin{array}{c:c} I_r & 0 \\ \hdashline 0 & 0 \end{array}\right], \quad \left[\begin{array}{c} I_r \\ \hdashline 0 \end{array}\right], \quad [I_r : 0], \quad [I_r]$$, where I_r is the unit matrix of order r.

Since the rank of a matrix is not changed as a result of elementary transformations, it follows that $\boxed{\rho(A) = r}$

For an $m \times n$ matrix A of rank r, to find square matrices P & Q of order m & n respectively, such that PAQ is in the normal form.

Method

① Write $A = IAI$

② Reduce the matrix on LHS to normal form by affecting elementary row or column transformations.

③ Every elementary row transformation on A must be accompanied by the same transformation on the pre-factor on RHS.

④ Every elementary column transformation on A must be accompanied by the same transformation on the post-factor on RHS.

The Echelon Form of a matrix

A matrix of order $m \times n$ is said to be in row (column) echelon form if

(i) the entries in a row (column) appear to the right (below) of the first non-zero entry.

(ii) the number of zeros preceding the first non-zero element in the i^{th} row (column) is less than that of the number of such zeros in the $(i+1)^{th}$ row (column) and

(iii) all rows (columns) that consists entirely of zeros lie at the bottom (right) of the matrix. For example, the matrices

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ are in their row-echelon form.

while the matrices $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \end{bmatrix}$$

are in their column echelon form.

Note:- In case of a square matrix the row-echelon form is an upper triangular matrix and the column echelon form is a lower triangular matrix.

Problem 1:- Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$$

Solution: Here $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ is a 2×4 matrix.

$$\therefore \rho(A) \leq 2$$

As $\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4 \neq 0$

$$\therefore \rho(A) = 2.$$

Problem 2:- Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Solution:- Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 1 & 6 & 5 \end{bmatrix}$ is a 3×3 matrix.

$$\therefore \rho(A) \leq 3$$

operating $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

operating $R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

As $\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 \neq 0$

$$\therefore \rho(A) = 2$$

Problem 3: – Reduce the following matrices to normal form and hence, find their ranks:

(i) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

Solution (i) Let $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

operating $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & 7 \end{bmatrix}$$

operating $R_4 \rightarrow R_4 - (R_1 + R_2 + R_3)$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_2 \rightarrow C_2 + C_1$, $C_3 \rightarrow C_3 + 2C_1$, $C_4 \rightarrow C_4 + 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $R_2 \rightarrow R_2 - 2R_1$, $R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $R_2 \rightarrow R_2 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $R_3 \rightarrow R_3 - 4R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_3 \to C_3 + 6C_2$, $C_4 \to C_4 + 3C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_3 \to \frac{1}{33} C_3$, $C_4 \to \frac{1}{22} C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_4 \to C_4 - C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} I_3 & 0 \\ \hline 0 & 0 \end{array}\right]$$ which is the required normal form.

$\Rightarrow \rho(A) = 3$

(ii) Let $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

operating $C_1 \to \frac{1}{8} C_1$

operating $R_3 \to R_3 + R_1$

operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 - 6C_1$

operating $C_2 \to \frac{1}{3} C_2$

operating $C_3 \to C_3 - 2C_2$, $C_4 \to C_4 - 2C_2$

operating $C_4 \to \frac{1}{10} C_4$

operating $C_3 \leftrightarrow C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [I_3 \ 0]$$ which is the required normal form.

$\rho(A) = 3$

Problem 4: For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$, find non-singular matrices P & Q such that PAQ is in the normal form. Hence find the rank of A.

Solution:- We write $A = IAI$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - 2C_1$ (Subjecting the post-factor on R.H.S. to same operations)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$ (Subjecting the pre-factor on R.H.S. to same operation)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 \to C_3 - C_2$ (Subjecting the post-factor on R.H.S. to same operation)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + R_2$ (Subjecting the pre-factor on R.H.S. to same operation)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ \text{ where}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \;\&\; Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise

1) Find the ranks of the following matrices:

(i) $\begin{bmatrix} 3 & -4 & -1 & 2 \\ 1 & 7 & 3 & 1 \\ 5 & -2 & 5 & 4 \\ 9 & -3 & 7 & 7 \end{bmatrix}$ Sol^n $\rho(A) = 3$

(ii) $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$ Sol^n $\rho(A) = 2$

(iii) $\begin{bmatrix} 2 & 4 & 3 & -2 \\ -3 & -2 & -1 & 4 \\ 6 & -1 & 7 & 2 \end{bmatrix}$ Sol^n $\rho(A) = 3$

(iv) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ Sol^n $\rho(A) = 2$

2) Reduce each of the following matrices to normal form and hence, find their ranks:

(i) $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$ Sol^n $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, 2

(ii) $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$ Sol^n $[I_3 \; 0]$, 3

(iii) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{bmatrix}$ Solⁿ $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$, 3

(iv) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ Solⁿ $[I_3 \; 0]$, 3

3) For the matrix $A = \begin{bmatrix} 2 & 1 & -3 & 6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ find non-singular matrices P & Q such that PAQ is in the normal form. Hence find the rank of A. Solⁿ: $PAQ = [I_3 \; 0]$; $\rho(A) = 3$