15. COMPLEX INTEGRATION

5

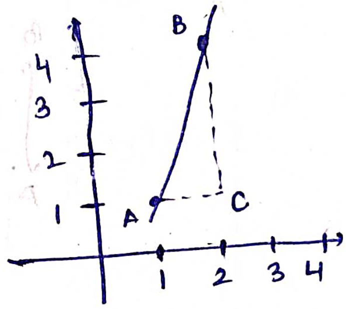
Let $f(z)$ be a continuous function of two independent variables x and y.

If $f(z) = u(x,y) + i\,v(x,y)$ where $z = x+iy$, then we have $\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$, which is the evaluation of a line integral of a complex function.

<u>Ex</u>: Evaluate $\int_A^B z^2\,dz$ where $A=(1,1)$, $B=(2,4)$

along

(a) the line segment AC parallel to x-axis and then along CB parallel to y-axis.



<u>Solution</u>. For any point <u>on AC</u>

we have $y=1$

$\therefore\ z = x+iy = x+i$

and $dz = dx$, $dy = 0$.

<u>On BC</u>

we have

$x = 2$

$dx = 0$.

$z = (2+iy)$.

$dz = i\,dy$

$$\therefore \int_C f(z)\,dz = \int_A^C z^2\,dz + \int_C^B z^2\,dz.$$

$$= \int_1^2 (x+i)^2\,dx + \int_1^4 (2+iy)^2\, i\,dy$$

$$= \left[\frac{(x+i)^3}{3}\right]_1^2 + \left[\frac{i\,(2+iy)^3}{3i}\right]_1^4$$

$$= \frac{1}{3}\left[(2+i)^3 - (1+i)^3\right] + \frac{1}{3}\left[(2+4i)^3 - (2+i)^3\right] \quad (2)$$

$$= \frac{-1}{3}\left[1+i^3+3i-3\right] + \frac{1}{3}\left[8 - 64i + 48i - 96\right]$$

$$= \frac{7}{3} + \frac{i}{3} - i + 1 - \frac{64i}{3} + 16i - 32$$

(b) the straight line AB joining the two pts A and B.

Solution The equation of line $y-1 = \frac{4-1}{2-1}(x-1)$

$$\Rightarrow y = 3x-2$$

$$\therefore \int_A^B z^2 dz = \int_A^B (x+iy)^2 (dx+idy)$$

$$= \int_A^B (x+i(3x-2))^2 (dx+i\,3dx) = \int_1^2 ((1+3i)x-2i)^2 (1+3i)\,dx$$

$$= \frac{-86}{3} - 6i.$$

(c) The curve C : $y = x^2$.

Solution $\int_C z^2 dz = \int_C (x+iy)^2(dx+idy)$ let $x=t$, $y=t^2$.

$$= \int_1^2 (t+it^2)(dt+2it\,dt) = -\frac{86}{3} - 6i.$$

X

Ex Evaluate $\int_C (z-a)^n dz$ where C is the circle with centre 'a' and radius r.

Also, discuss the case when $n=-1$, that is evaluate $\int_C \frac{dz}{(z-a)}$.

Solution The equation of the circle C is $|z-a| = r$

$$\Rightarrow \quad z-a = re^{i\theta}, \quad 0 \le \theta \le 2\pi$$

$$\Rightarrow \quad dz = ire^{i\theta} d\theta$$

$$\therefore \int_C (z-a)^n dz = \int_0^{2\pi} r^n e^{ni\theta} \cdot ire^{i\theta} d\theta = r^{n+1} \int_0^{2\pi} e^{(n+1)i\theta} i \, d\theta$$

$$= \frac{i r^{n+1}}{i(n+1)} \left[e^{(n+1)i\theta} \right]_0^{2\pi}$$

$$= 0.$$

when $n=-1$

$$\int_C \frac{dz}{(z-a)} = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

X

Ex Evaluate the integral $\int_c \bar{z}\,dz$ along the right half of the circle $c : |z| = 2$.

Solution $c: |z| = 2 \Rightarrow z = 2e^{i\theta}$

$$\bar{z} = 2e^{-i\theta}$$

$$dz = 2ie^{i\theta}\,d\theta.$$

$$\therefore \int_c \bar{z}\,dz = \int_{-\pi/2}^{\pi/2} 2e^{-i\theta} \cdot 2ie^{i\theta}\,d\theta = 4i\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 4\pi i$$

Ex Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$ along the line $y = x/2$.

Solution $x = 2y \Rightarrow z = x + iy$

$$\Rightarrow z = 2y + iy$$

$$\Rightarrow dz = (2+i)\,dy.$$

$$\therefore \int_0^{2+i} (\bar{z})^2\,dz = \int_0^1 (\underset{2y}{\underset{\parallel}{x}} - iy)^2 (2+i)\,dy = \int_0^1 (2-i)^2 y^2 (2+i)\,dy$$

$$= 5\left[\frac{y^3}{3}\right]_0^1 (2-i) = \frac{5(2-i)}{3}.$$

FUNDAMENTAL THEOREM OF CONTOUR INTEGRATION

Statement :- let a continuous function $f(z)$ be the derivative of an analytic function $F(z)$. let $f(z)$ and $F(z)$ both have domain D and C be a contour in D from $z = z_1$ to $z = z_2$ then

$$\int_C f(z)\,dz = F(z_2) - F(z_1).$$

Properties of Contour Integrals

$\boxed{(1)}$ If the contour C is divided into two parts C_1 and C_2, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

$\boxed{(2)}$ If the sense of integration is reversed, the sign of the integral also changes.

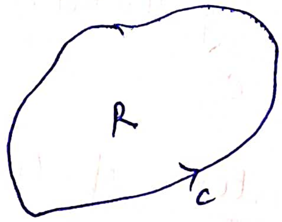
$$\int_{z_0}^{z_1} f(z)\,dz = -\int_{z_1}^{z_0} f(z)\,dz.$$

$\boxed{(3)}$. $$\int_C (k_1 f_1(z) + k_2 f_2(z))\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz$$

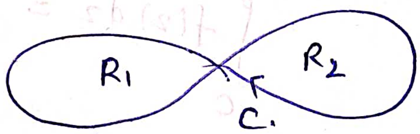
$\boxed{(4.)}$ If L is the length of a contour C and M the upper bound of $f(z)$. i.e $|f(z)| \leq M$ on the track of C then $\int_C f(z)\,dz \leq ML$.

SOME DEFINITIONS

1. **SIMPLE CURVE** :- A simple curve is one which does not cross itself.



2. **MULTIPLE CURVE** :- A multiple curve is one which crosses itself.



3. **SIMPLY CONNECTED REGION** :-

CAUCHY'S INTEGRAL THEOREM.

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a closed contour C, then $\int_C f(z)\,dz = 0$.

COROLLARY (Independence of Path). If $f(z)$ is analytic in R and if two points A and B in R are joined by two different curves C_1 and C_2 lying wholly in R, then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

Ex Evaluate $\int_C z^2\,dz$ using Cauchy's integral theorem, where $C: |z| = 1$.

Solution $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$

$\therefore$ $u = x^2 - y^2$ and $v = 2xy$

Hence, $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial u}{\partial y} = -2y$, $\dfrac{\partial v}{\partial x} = 2y$, $\dfrac{\partial v}{\partial y} = 2x$

$\Rightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Since z^2 is analytic, CR-equations holds good.

$\therefore$ By Cauchy's integral theorem, the given integral $= 0$.

Ex Evaluate $\int_C \frac{dz}{z^2}$ where $C: |z| = 1$.

Solution :- $C: |z| = 1$

$\Rightarrow z = 1e^{i\theta}$.

$dz = ie^{i\theta}\, d\theta$. $\quad , 0 < \theta < 2\pi$

$$\therefore \int_0^{2\pi} \frac{ie^{i\theta}\, d\theta}{e^{2i\theta}} = \int_0^{2\pi} ie^{-i\theta}\, d\theta = i\left[\frac{e^{-i\theta}}{-i}\right]_0^{2\pi}$$

$$= -\left[e^{-i\theta}\right]_0^{2\pi}$$

$$= -\left[e^{-2\pi i} - 1\right]$$

$$= 0.$$

x

CAUCHY'S INTEGRAL FORMULA.

If $f(z)$ is analytic within and on a closed curve C and if 'a' is any point within C, then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{z-a}.$$

COROLLARY

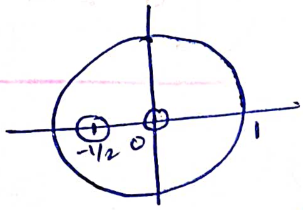
$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a)^{n+1}}, \quad n = 0, 1, 2 \ldots$$

Ex Use Cauchy's integral formula to calculate $\int_c \frac{z^2+1}{z(2z+1)}\,dz$ where $c : |z| = 1$.

Solution. $I = \int_c \frac{z^2+1}{z(2z+1)}\,dz = \int_c (z^2+1)\left[\frac{1}{z} - \frac{2}{2z+1}\right] dz$

$$= \int \frac{(z^2+1)}{z}\,dz - \int \frac{z^2+1}{z+1/2}\,dz$$

Now $c : |z| = 1$



By Cauchy's integral formula

since $z=0$ and $z=-1/2$ both lie inside C

$$\therefore\ I = \oint 2\pi i\,(z^2+1)_{z=0} - 2\pi i\,(z^2+1)_{z=-1/2}$$

$$= 2\pi i - 2\pi i\left(\frac{5}{4}\right) = 2\pi i\left(\frac{-1}{4}\right) = -\frac{\pi i}{2}.$$

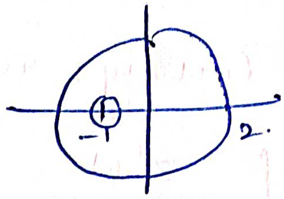
x

Ex. Evaluate (1) $\int_C \frac{e^{-z}}{z+1}\,dz$ where $c : |z| = 2$

(2) $\int_C \frac{e^{z}\,dz}{z-2}$ $\cancel{dz}$ where $c : |z| = 3$.

(3) $\int_c \frac{e^{z}\,dz}{z^2-9}$ where $c : |z| = 1$.

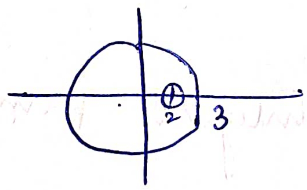
Solution 1(1) $C: |z| = 2$



Since the given integral e^{-z} is analytic inside and on the circle $C: |z| = 2$ and $z = -1$ lies inside C. ∴ By cauchy's integral formal

$$I = \int \frac{e^{-z}}{z+1}\,dz = 2\pi i\,(e^{-z})_{z=-1} = 2\pi i e.$$

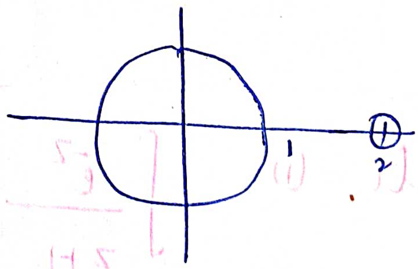
1(2) C:



Same as above,

$$I = \int \frac{e^{z}}{z-2}\,dz = 2\pi i\,(e^{z})_{z=2} = 2\pi i e^{2}$$

3. $C: |z| = 1$



Since $z = 2$ lies outside the circle C and e^{z} is analytic inside and on C

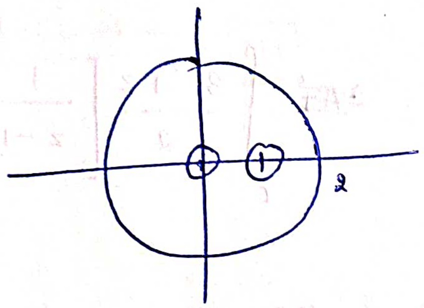
$$\therefore \int \frac{e^{z}}{z-2}\,dz = 0$$ by Cauchy integral theorem.

Ex Evaluate :- (1) $\int \frac{5z-2}{z^2-z}\,dz$ $c : |z| = 2$

(2) $\int \frac{3z^2+z}{z^2-1}\,dz$ where (a) $c : |z-1| = 1$

(b) $c : |z| = 2$

Solution **(1)** $\int_c \frac{5z-2}{z^2-z}\,dz = \int_c (5z-2)\left[\frac{1}{z-1} - \frac{1}{z}\right] dz$



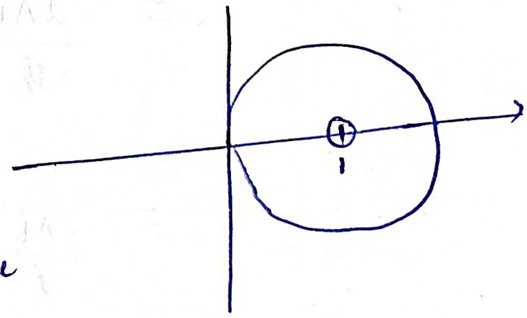
$$= \int \frac{5z-2}{z-1}\,dz - \int \frac{5z-2}{z}\,dz$$

$$= 2\pi i\,(5z-2)_{z=1} - 2\pi i\,(5z-2)_{z=0}$$

$$= 6\pi i + 4\pi i = 10\pi i$$

---

(2) (a) $\int \frac{3z^2+z}{z^2-1}\,dz$



Since only $z=1$ is lying inside the circle c

$\therefore$ we take $f(z) = \frac{3z^2+z}{z+1}$

That is, $\int \frac{3z^2+z}{z^2-1}\,dz = \int \frac{3z^2+z}{(z-1)(z+1)}\,dz$

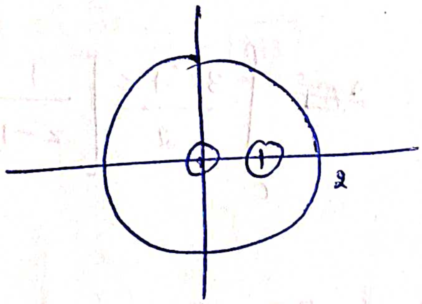
$$= \int \frac{\frac{3z^2+z}{z+1}}{z-1}\,dz$$

Ex Evaluate :- (1) $\int \frac{5z-2}{z^2-z}\,dz$ $\quad c: |z| = 2$ (11)

(2) $\int \frac{3z^2+z}{z^2-1}\,dz$ where (a) $c: |z-1| = 1$

(b) $c: |z| = 2$

Solution $\boxed{(1)}$ $\int_c \frac{5z-2}{z^2-z}\,dz = \int_c (5z-2)\left[\frac{1}{z-1} - \frac{1}{z}\right] dz$

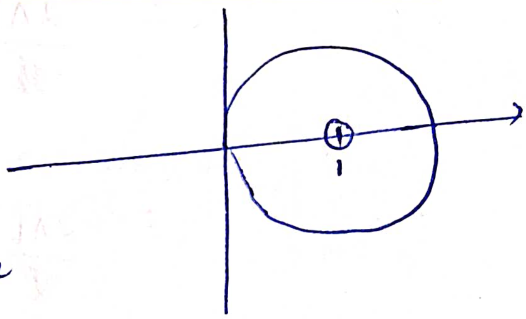


$= \int \frac{5z-2}{z-1}\,dz - \int \frac{5z-2}{z}\,dz$

$= 2\pi i (5z-2)_{z=1} - 2\pi i (5z-2)_{z=0}$

$= 6\pi i + 4\pi i = 10\pi i$

$\boxed{(2)\ (a)}$ $\int \frac{3z^2+z}{z^2-1}\,dz$



Since only $z=1$ is lying inside the circle c

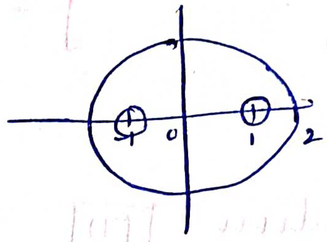
$\therefore$ we taken $f(z) = \frac{3z^2+z}{z+1}$

That is, $\int \frac{3z^2+z}{z^2-1}\,dz = \int \frac{3z^2+z}{(z-1)(z+1)}\,dz$

$= \int \frac{\frac{3z^2+z}{z+1}}{z-1}\,dz$

$$= 2\pi i \left[\frac{3z^2+z}{z+1} \right]_{z=1} = 4\pi i$$

$\boxed{(2)\ (b)}$ $\quad C : |z| = 2$



In this case both $z=1$ & $z=-1$ are lying inside the circle C

$$\therefore \quad I = \int_C \frac{3z^2+z}{z^2-1}\,dz = \cancel{2\pi i} \int_C \frac{3z^2+z}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$$

$$= \int_C \frac{3z^2+z}{2(z-1)}\,dz - \int \frac{3z^2+z}{2(z+1)}\,dz$$

$$= \frac{2\pi i}{\cancel{2}}\left[\frac{3z^2+z}{2}\right]_{z=1} - 2\pi i\left[\frac{3z^2+z}{-2}\right]_{z=-1}$$

$$= \frac{2\pi i}{\cancel{2}}\left[2\right] - 2\pi i\left[1\right]$$

$$= \frac{2\pi i}{\cancel{2}} \cancel{4\pi i}$$

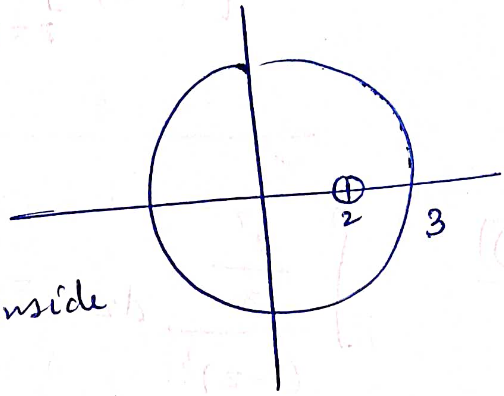
Ex Evaluate (1) $\int_C \frac{2z}{(z-2)^2} + \frac{3(z-1)}{(z-2)^3}\, dz$

where $C : |z| = 3$

(2) $\int \frac{e^z\, dz}{z(1-z)^3}$ where (a) $C : |z| = \frac{1}{2}$

(b) $C : |z-1| = \frac{1}{2}$

Solution $\boxed{(1)}$ $C : |z| = 3$



The singularity $z = 2$ is lying inside the circle C

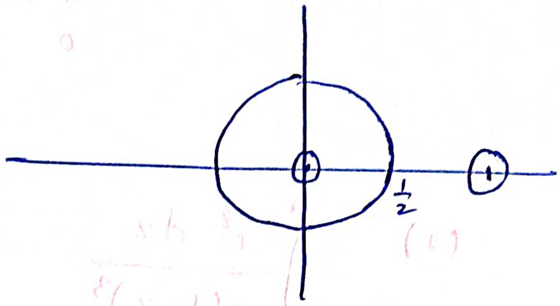
$$\therefore \quad I = \int_C \frac{2z\, dz}{(z-2)^2} + \int \frac{3(z-1)}{(z-2)^3}\, dz$$

$$= 2\pi i \left[\frac{d}{dz} 2z\right]_{z=2} + 2\pi i \left[\frac{d^2}{dz^2}(3z-1)\right]_{z=2}$$

$$= 2\pi i\,[2] + \frac{2\pi i}{2!}[0]$$

$$= 4\pi i$$

2. (a) $\int \frac{e^z\,dz}{z(1-z)^3}$ $\quad C: |z| = \frac{1}{2}$

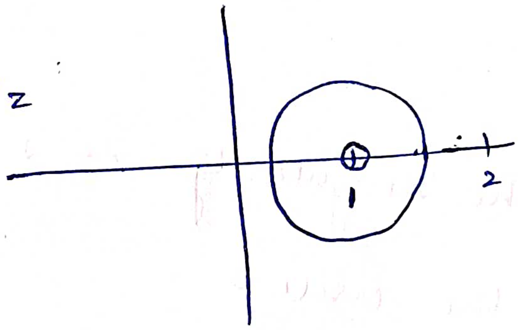


$$= \int_C \frac{\frac{e^z}{(1-z)^3}}{z}\,dz$$

$$= 2\pi i\left[\frac{e^z}{(1-z)^3}\right]_{z=0} = 2\pi i$$

(b)

$$\int \frac{\frac{e^z}{z}}{(1-z)^3}\,dz = \int \frac{-\frac{e^z}{z}}{(z-1)^3}\,dz$$



$$= \frac{-2\pi i}{2!}\left[\frac{d^2}{dz^2}\left(\frac{e^z}{z}\right)\right]_{z=1}$$

$$= \frac{-2\pi i}{2!}\left[\frac{d}{dz}\left(\frac{ze^z - e^z(1)}{z^2}\right)\right]_{z=1}$$

$$= -\pi i\left[\frac{z^2(ze^z+e^z-e^z) - (z(e^z)-e^z)\,2z}{z^4}\right]_{z=1}$$

$$= -\pi i\,[e] = -\pi i e$$

Ex. Evaluate (1) $\displaystyle\int_C \frac{e^{2z}\,dz}{z^2-3z+2}$ where $C: |z| = 3$

(2) $\displaystyle\int_C \frac{e^{z}\,dz}{(z-1)(z-4)}$ where $C: |z| = 2$

(3) $\displaystyle\int_C \frac{z^4\,dz}{(z+1)(z-i)^2}$ where $C: 9x^2+4y^2 = 36$

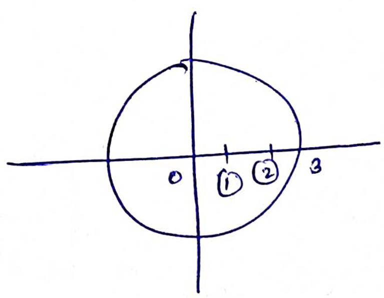
(4) $\displaystyle\int_C \frac{(z-3)\,dz}{z^2+2z+5}$ $C: |z| = 1$

(5) $\displaystyle\int_C \frac{(z-3)\,dz}{z^2+2z+5}$ $C: |z+1+i| = 2$

Solution (1) Since $\dfrac{1}{z^2-3z+2} = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{(z-2)} - \dfrac{1}{(z-1)}$

we have

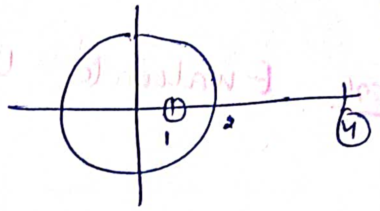
$$I = \int_C \frac{e^{2z}\,dz}{z^2-3z+2} = \int \frac{e^{2z}\,dz}{(z-2)} - \int \frac{e^{2z}\,dz}{(z-1)}$$



$$= 2\pi i\left[e^{2z}\right]_{z=2} - 2\pi i\left[e^{2z}\right]_{z=1}$$

$$= 2\pi i\left[e^4 - e^2\right]$$

(2) $\int \frac{e^z\,dz}{(z-1)(z-4)}$

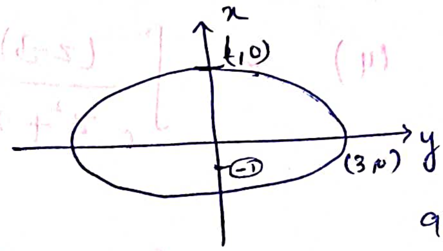


$$= \int \frac{\frac{e^z}{(z-4)}}{(z-1)}\,dz = 2\pi i\left[\frac{e^z}{z-4}\right]_{z=1}$$

$$= \frac{2\pi i e}{-3} = -\frac{2\pi i e}{3}.$$

---

(3). $\int \frac{z^4\,dz}{(z+1)(z-i)^2}$



$9x^2+4y^2=36$

$\frac{x^2}{4}+\frac{y^2}{9}=1$

$$\frac{1}{(z+1)(z-i)^2} = \frac{-i}{2(z+1)} + \frac{1-i}{2(z-i)^2} + \frac{i}{2(z-i)}$$ (using partial fractions)

$$\therefore \int \frac{z^4\,dz}{(z+1)(z-i)^2} = \int \frac{-z^4 i\,dz}{2(z+1)} + \int \frac{z^4(1-i)}{2(z-i)^2}\,dz + \int \frac{z^4 i\,dz}{2(z-i)}$$

$$= 2\pi i\left[-\frac{z^4 i}{2}\right]_{z=-1} + \frac{2\pi i}{1!}\frac{d}{dz}\left[\frac{z^4(1-i)}{2}\right]_{z=i}$$

$$+ 2\pi i\left[\frac{z^4 i}{2}\right]_{z=i}$$

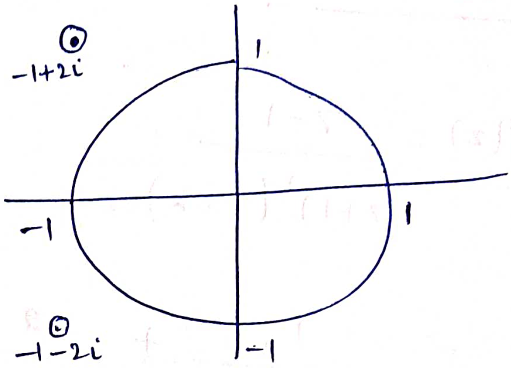
$$= +\pi i(-i) + 2\pi i\left[\frac{4z^3(1-i)}{2}\right]_{z=i} + \frac{2\pi i \times i}{2}$$

$$= \pi(+1) + 4\pi i\left[-i(1-i)\right] + \pi(-1)$$

$$= \cdots 0 + 4\pi[(1-i)] = 4\pi(1-i)$$

---

(4). $\displaystyle\int \frac{(z-3)\,dz}{(z^2+2z+5)}$ $\quad c: |z|=1$



Now, $z^2+2z+5=0$

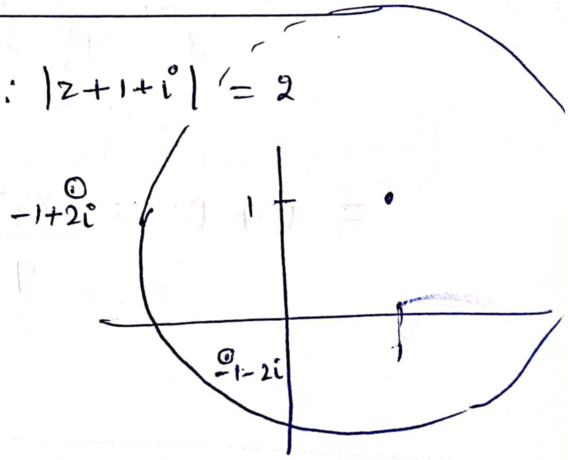
$\Rightarrow z = -1 \pm 2i$

Both the singularities are lying outside the circle

$$\therefore \int \frac{(z-3)\,dz}{(z^2+2z+5)} = 0.$$

---

(5). $\displaystyle\int \frac{(z-3)\,dz}{z^2+2z+5}$ $\quad c: |z+1+i| = 2$



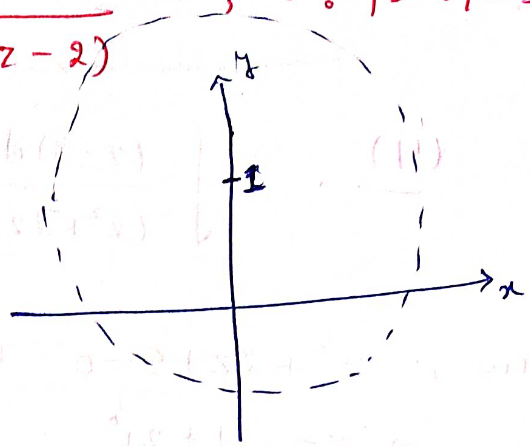
$z = -1+2i$ is lying outside the circle, but $z=-1-2i$ are lying inside.

$$\int \frac{\frac{(z-3)}{(z+1-2i)}}{(z+1+2i)}\,dz = 2\pi i\left[\frac{z-3}{z+1-2i}\right]_{z=-1-2i}$$

$$= 2\pi i\left[\frac{-1-2i-3}{\cancel{-1}-2i+\cancel{1}-2i}\right]$$

$$= 2\pi i\left[\frac{4+2i}{4i}\right] = \cancel{2}\pi \cancel{i}\left[\frac{2+i}{\cancel{2}\cancel{i}}\right] = \pi(2+i).$$

Ex Evaluate $\oint_C \frac{(z-1)\,dz}{(z+1)^2(z-2)}$, $C : |z-i| = 2$



Solution.

$$f(z) = \frac{z-1}{(z+1)^2(z-2)}$$

$$= \underbrace{\frac{1}{9(z-2)}}_{\text{outside}} + \underbrace{\frac{2}{3(z+1)^2}}_{\text{inside}} - \underbrace{\frac{1}{9(z+1)}}_{\text{inside}}.$$

$$= 0 + \frac{2}{3} \cdot \frac{2\pi i}{1!} \frac{d}{dz}\left[1\right]_{z=-1} - \frac{2\pi i}{9} \cdot 1 .$$

$$= 0 + 0 - \frac{2\pi i}{9}.$$

Ex Use cauchy's integral formula to evaluate

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz,$$ where $c: |z| = 3$.

Solution: $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{(z-2)} - \dfrac{1}{(z-1)}$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz - \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}$$

$$= 2\pi i \left[\sin \pi z^2 + \cos \pi z^2 \right]_{z=2} - 2\pi i \left[\sin \pi z^2 + \cos \pi z^2 \right]_{z=1}$$

$$= 2\pi i \left[\sin 4\pi + \cos 4\pi \right] - 2\pi i \left[\sin \pi + \cos \pi \right]$$

$$= 2\pi i [0+1] - 2\pi i [0-1]$$

$$= 4\pi i$$

Ex Using cauchy's integral formula,

evaluate $$\oint_C \frac{e^z dz}{(z^2+\pi^2)^2}$$ where $c: |z| = 4$.

Solution $\dfrac{1}{(z^2+\pi^2)^2} = \dfrac{1}{(z+\pi i)^2 (z-\pi i)^2}$