Legendre Equation & Polynomials

The Legendre equation given by

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0} \quad \text{— (A)}$$

where $n \geq 0$ is a real parameter. This eq^n arises in a variety of applications, particular in connection with physical problems with spherical symmetry.

The coefficients of the Legendre eq^n are analytic at the origin and the leading co-eff $(1-x^2)$ only vanishes at $x = \pm 1$, so a power series solution of the Legendre eq^n exists about the point $x=0$ in the interval $-1 < x < 1$. Solutions of eq^n (A) are called Legendre functions & examples of special functions.

Let $y = a_0 y_1 + a_1 y_2$ be the solution of eq^n (A)

$$y = a_0 y_1 + a_1 y_2$$

where, $y_1(x) = 1 - n(n+1)\dfrac{x^2}{2!} + (n-2)n(n+1)(n+3)\dfrac{x^4}{4!} - \cdots$

& $y_2(x) = x - (n-1)(n+2)\dfrac{x^3}{3!} + (n-3)(n-1)(n+2)(n+4)\dfrac{x^5}{5!} - \cdots$

The series for $y_1(x)$ & $y_2(x)$ converge for $-1 < x < 1$ and further since $y_1(x)$ contains even powers of x only and $y_2(x)$ contains odd powers of x only, hence these are the two linearly independent solutions of eq^n (A).

Legendre Polynomials

If n is even, the series $y_1(x)$ will reduce to a polynomial of degree n in even powers of x, whereas if n is odd, the series $y_2(x)$ will reduce to a polynomial of degree n in odd powers of x. For example

For $n = 0$, $y_1(x) = 1$

For $n = 2$, $y_1(x) = 1 - 3x^2$

For $n = 4$, $y_1(x) = 1 - 10x^2 + \frac{35}{3}x^4$

For $n = 1$, $y_2(x) = x$

For $n = 3$, $y_2(x) = x - \frac{5}{3}x^3$

For $n = 5$, $y_2(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$

These polynomials multiplied with suitable constants are called Legendre polynomials and denoted by $P_n(x)$, where n denotes the degree of the polynomial.

Rodrigue's Formula

$$\boxed{P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]}$$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x^2)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

Generating Function for Legendre Polynomials

The generating function of $P_n(x)$ can be given by

$$\boxed{L(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n}$$

that is, if $L(x,t)$ is expanded as a power series in t, then the co-efficient of t^n is $P_n(x)$, the Legendre polynomial of degree n.

Recurrence Relations for $P_n(x)$

I $(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$

Proof:- We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \text{—} (1)$$

diff wrt t (partially), we get

$$\left(-\frac{1}{2}\right)(-2x+2t)(1-2x+t^2)^{-3/2} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

multiplying by $(1-2x+t^2)$, gives

$$(x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

or $$(x-t)\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

Now equating the co-eff of t^n on both sides, we get

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2nxP_n(x) + (n-1)P_{n-1}(x)$$

or $$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

II $P_n = P'_{n+1} - 2xP'_n + P'_{n-1}$

Proof:- diff (1) partially wrt x, we get

$$\left(-\frac{1}{2}\right)(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} t^n P_n'(x)$$

$$t(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} t^n P_n'(x)$$

$$t\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)\sum_{n=0}^{\infty} t^n P_n'(x)$$

Now, equating on both sides, the co-eff of t^{n+1}, we get

$$P_n(x) = P_{n+1}'(x) - 2x P_n'(x) + P_{n-1}'(x)$$

(III) $$nP_n = xP_n' - P_{n-1}'$$

<u>Proof</u>:- Here we differentiate the recurrence relation (I) partially w.r.t. x, to get

$$(n+1)P_{n+1}' = (2n+1)xP_n' + (2n+1)P_n - nP_{n-1}'$$

Substituting here for P_{n+1}' from recurrence relation (II), we get

$$(n+1)P_n = -2(n+1)xP_n' + (n+1)P_{n-1}' + (2n+1)xP_n' + (2n+1)P_n - nP_{n-1}'$$

$$nP_n = xP_n' - P_{n-1}'$$

(IV) $$(n+1)P_n = P_{n+1}' - xP_n'$$

Adding the relations (II) & (III), gives

$$(n+1)P_n = P_{n+1}' - xP_n'$$

(V) $$(2n+1)P_n = P_{n+1}' - P_{n-1}'$$

Adding the relations (III) & (IV), gives

$$(2n+1)P_n = P_{n+1}' - P_{n-1}'$$

(VI) $$(1-x^2)P_n' = n\left[P_{n-1} - xP_n\right]$$

Replacing n by $(n-1)$ in relation (IV), gives we get $n P_{n-1} = P_n' - x P_{n-1}'$ ——— (i)

& multiplying III by x, gives

$$n x P_n = x^2 P_n' - x P_{n-1}' \quad \text{——— (ii)}$$

(i) − (ii), gives

$$(1-x^2) P_n' = n [P_{n-1} - x P_n]$$

VII $\quad (1-x^2) P_n' = (n+1) [x P_n - P_{n+1}]$

Replacing n by $(n+1)$ in recurrence relation III, gives

$$(n+1) P_{n+1} = x P_{n+1}' - P_n' \quad \text{——— (iii)}$$

and the multiply IV by x, we get

$$(n+1) x P_n = x P_{n+1}' - x^2 P_n' \quad \text{——— (iv)}$$

(iv) − (iii)

$$(1-x^2) P_n' = (n+1) [x P_n - P_{n+1}]$$

Question 1:- Show that $P_n'(1) = \frac{n(n+1)}{2}$

Solution: We know that

$$\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)^{-1/2}$$

diff (1) partially w.r.t. x, gives

$$\sum_{n=0}^{\infty} t^n P_n'(x) = -\frac{1}{2}(1-2xt+t^2)^{-3/2} = t(1-t)^{-3}$$

$$= t\left[1 + (-3)(-t) + \frac{(-3)(-4)}{2!}(-t)^2 + \frac{(-3)(-4)(-5)(-t)^3}{3!} + \dots + \frac{(-3)(-4)(-5)\dots(-n-2)(-t)^{n-1}}{(n-1)!} + \dots\right]$$

Comparing the co-eff of t^n on bothsides, we get $P_n'(1) = \frac{(n+1)n}{2}$

H.P.

Question 2:- Prove that

$$(1-x^2)P_n'(x) = \frac{n(n+1)}{2n+1}[P_{n-1}(x) - P_{n+1}(x)]$$

Solution:- Multiplying the recurrence relation (I) by n & (VI) by $(2n+1)$ and adding them, we get

$$n(n+1)P_{n+1} + (2n+1)(1-x^2)P_n' = -n^2 P_{n-1} + n(2n+1)P_{n-1}$$

$$\therefore (1-x^2)P_n' = \frac{n(n+1)}{2n+1}[P_{n-1} - P_{n+1}]$$

H.P.

Question 3:- Find the values of m & n, if

$$3x^2 = mP_2(x) + nP_0(x)$$

Solution:- Since $P_0(x) = 1$, $P_1(x) = x$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

hence $3x^2 - 1 = 2P_2(x)$

or $3x^2 = 2P_2(x) + 1$

$= 2P_2(x) + P_0(x)$

But $3x^2 = mP_2(x) + nP_0(x)$

$\Rightarrow m = 2, \; n = 1$ Answer

Question 4:- Show that $P_n(1) = 1$ & $P_n(-1) = (-1)^n$

Solution:- We know that

$$\sum P_n(x)t^n = (1-2xt+t^2)^{-1/2} \quad \text{—(1)}$$

(i) Putting $x=1$ in (1), we get

$$\sum_{n=0}^{\infty} P_n(1)t^n = (1-2t+t^2)^{-1/2} = (1-t)^{-1}$$

$$= \sum_{n=0}^{\infty} t^n$$

$\Rightarrow P_n(1) = 1.$

(ii) Next, putting $x = -1$ in (1), we get

$$\sum_{n=0}^{\infty} P_n(-1) t^n = (1+2t+t^2)^{-1/2} = (1+t)^{-1} = \sum_{n=0}^{\infty}(-1)^n t^n$$

$$\Rightarrow P_n(-1) = (-1)^n$$

H.P.

<u>Question 5:-</u> $P_{2m}(0) = (-1)^m \dfrac{1.3.5\ldots(2m-1)}{2.4.6\ldots 2m}$ &

$P_{2m+1}(0) = 0.$

<u>Solution:-</u> We know that

$$\sum_{n=0}^{\infty} P_n(x) t^n = (1-2xt+t^2)^{-1/2} \quad \text{--- (1)}$$

In eq^n (1), let us put $x=0$ to get

$$\sum_{n=0}^{\infty} P_n(0) t^n = (1+t^2)^{-1/2}$$

$$= 1 - \frac{1}{2}t^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}t^4 + \cdots + \frac{(-1)^r . 1.3.5 \ldots (2r+1)}{2.4.6\ldots 2r} t^{2r} + \cdots$$

Since on the R.H.S. there are only even power terms in t, on comparing the co-efficients of t^n on both sides, we have

$P_{2m+1}(0) = 0$ when n is odd

$P_{2m}(0) = \dfrac{(-1)^m 1.3.5\ldots(2m-1)}{2.4.6\cdots(2m)}$ when n is even

H.P.

<u>Question 6:-</u> Prove that $P_n(-x) = (-1)^n P_n(x)$.

<u>Solution:-</u> Replacing x by $-x$ in the generating function for $P_n(x)$, gives

$$\sum_{n=0}^{\infty} P_n(-x) t^n = (1+2xt+t^2)^{-1/2} \quad \text{--- (1)}$$

Again replacing t by $-t$ in (1), we have

$$\sum_{n=0}^{\infty} P_n(x)(-t)^n = (1+2xt+t^2)^{-1/2}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)(-t)^n = \sum_{n=0}^{\infty} P_n(-x)t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n P_n(x) t^n = \sum_{n=0}^{\infty} P_n(-x) t^n$$

$$\Rightarrow P_n(-x) = (-1)^n P_n(x)$$

H.P.

Question 7:- Express $x^3 - 5x^2 + x + 2$ in terms of Legendre polynomials

Solution:- We know that

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\Rightarrow x = P_1$$

$$x^2 = \frac{1}{3}(2P_2 + P_0)$$

$$x^3 = \frac{1}{5}(2P_3 + 3P_1)$$

$$\therefore x^3 - 5x^2 + x + 2 = \left(\frac{2P_3 + 3P_1}{5}\right) - 5\left(\frac{2P_2 + P_0}{3}\right) + P_1 + 2P_0$$

$$= \frac{2}{5}P_3 - \frac{10}{3}P_2 + \frac{8}{5}P_1 + \frac{1}{3}P_0$$

Question 8:- Express the following polynomials in terms of Legendre polynomials

(i) $1 - 2x + x^2 + 5x^3$

(ii) $4x^3 + 6x^2 + 7x + 2$

Answer:- (i) $2P_3(x) + \frac{2}{3}P_2(x) + P_1(x) + \frac{4}{3}P_0(x)$

(ii) $\frac{8}{5}P_3 + 4P_2 + \frac{47}{5}P_1 + 4P_0$

Question 9:- Show that $\frac{1+t}{t(1-2xt+t^2)^{1/2}} - \frac{1}{t} = \sum_{n=0}^{\infty}(P_n+P_{n+1})t^n$

Solution:- L.H.S. $\left(1+\frac{1}{t}\right)(1-2xt+t^2)^{-1/2} - \frac{1}{t}$

$$= \left(1+\frac{1}{t}\right)\sum_{n=0}^{\infty} t^n P_n(x) - \frac{1}{t}$$

$$= \sum_{n=0}^{\infty} t^n P_n(x) + \sum_{n=0}^{\infty} t^{n-1} P_n(x) - \frac{1}{t}$$

$$= \sum_{n=0}^{\infty} t^n P_n(x) + \frac{1}{t}P_0(t) + \sum_{n=0}^{\infty} t^n P_{n+1}(x)$$

$$- \frac{1}{t}P_0(t) \qquad \text{since } P_0(t)=1$$

$$= \sum_{n=0}^{\infty} t^n [P_n(x) + P_{n+1}(x)]$$

$$= \text{R.H.S.}$$

H.P.

Question 10:- Using Rodrigues formula, prove that $P_n'(x) = x P_{n-1}'(x) + n P_{n-1}(x)$.

Solution:- By Rodrigues formula, we get

$$P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n}(x^2-1)^n$$

$$\therefore \quad P_n'(x) = \frac{1}{2^n . n!} \frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n$$

$$= \frac{1}{2^n . n!} \frac{d^n}{dx^n}[n(x^2-1)^{n-1} . 2x]$$

$$= \frac{1}{2^n . n!}\left[2nx \frac{d^n}{dx^n}(x^2-1)^{n-1} + n\frac{d}{dx}(2nx) . \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right]$$

$$= \frac{2nx}{2^n . n!}\frac{d}{dx}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1} + \frac{n . 2n}{2^n . n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}$$

$$= x \cdot \frac{d}{dx}\left[\frac{1}{2^{n-1}(n-1)!} \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right] +$$

$$n \cdot \frac{1}{2^{n-1}(n-1)!} \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}$$

or $P_n'(x) = n P_{n-1}'(x) + n P_{n-1}(x)$

H.P.

Question 11:- Prove that

$$P_n'(x) - P_{n-2}'(x) = (2n-1) \cdot P_{n-1}(x)$$

Solution:- We have the recurrence relⁿ

$$(n+1) P_{n+1} = (2n+1) x P_n - n P_{n-1}$$

$$n P_n = (2n-1) x P_{n-1} - (n-1) P_{n-2}$$

$$\Rightarrow n P_n' = (2n-1) x P_{n-1}' + (2n-1) P_{n-1} - (n-1) P_{n-2}' \quad (1)$$

Also we have another recurrence relation as

$$P_n = P_{n+1}' - 2x P_n' + P_{n-1}'$$

$$P_{n-1} = P_n' - 2x P_{n-1}' + P_{n-2}' \quad \text{——} (2)$$

Eliminating $x P_{n-1}'$ in (1) & (2), we get

$$2n P_n' + (2n-1) P_{n-1} = 2(2n-1) P_{n-1} - 2(n-1) P_{n-2}'$$
$$+ 2(n-1) P_n' - 2x(2n-1) P_{n-1}' + (2n-1) P_{n-2}'$$

which on simplification, gives

$$P_n' - P_{n-2}' = (2n-1) P_{n-1}$$

H.P.

Question 12 (i):- Prove that $\int_{-1}^{1} P_n(x)\,dx = 0 \quad , n \neq 0$

(ii) Show that $\int_{0}^{1} P_n(x)\,dx = \frac{1}{n+1} P_{n-1}(0)$.

Solution:- (i) The Legendre's eq^n can be written as

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0$$

Integrating both sides in $(-1, 1)$, we get

$$\int_{-1}^{1} n(n+1)P_n(x)\,dx = -\left[(1-x^2)P_n'(x)\right]_{-1}^{1}$$

$$\Rightarrow \int_{-1}^{1} P_n(x)\,dx = \frac{-1}{n(n+1)}\left[(1-x^2)P_n'(x)\right]_{-1}^{1} = 0$$

for $n \neq 0$.

(ii) Integrating on both sides of the above Legendre's eq^n, in $(0, 1)$, we get

$$n(n+1)\int_0^1 P_n(x)\,dx = -\left[(1-x^2)P_n'(x)\right]_0^1$$

$$= P_n'(0)$$

$$\int_0^1 P_n(x)\,dx = \frac{1}{n(n+1)}P_n'(0)$$

Now the recurrence relation

$(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$ at $x = 0$,

gives $P_n'(0) = nP_{n-1}(0)$.

$$\therefore \int_0^1 P_n(x)\,dx = \frac{1}{n(n+1)}nP_{n-1}(0)$$

$$= \frac{1}{n+1}P_{n-1}(0).$$

M.P.

Orthogonality of Legendre Polynomials

$$\boxed{\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0 & , m \neq n \\ \frac{2}{2n+1} & , m = n \end{cases}}$$

Question 13:- Show that $\int_{-1}^{1}(1-x^2)\,P_m'(x)\,P_n'(x)\,dx =$

$$\begin{cases} 0 & , \text{ if } m \neq n \\ \frac{2n(n+1)}{2n+1} & , \text{ if } m = n \end{cases}$$

Solution:- $\int_{-1}^{1}(1-x^2)\,P_m'(x)\,P_n'(x)\,dx$

$$= \int_{-1}^{1}\left[(1-x^2)\,P_m'(x)\right] P_n'(x)\,dx$$

Integrate by parts

$$= \left[(1-x^2)\,P_m'\,P_n\right]_{-1}^{1} - \int_{-1}^{1} P_n \frac{d}{dx}\left[(1-x^2)\,P_m'(x)\right] dx$$

$$= 0 - \int_{-1}^{1} P_n\left[(1-x^2)\,P_m''(x) - 2x\,P_m'(x)\right] dx.$$

But $P_m(x)$ is the solution of the equation,

$$(1-x^2)\,P_m'' - 2x\,P_m' + m(m+1)\,P_m = 0$$

Hence $\int_{-1}^{1}(1-x^2)\,P_m'\,P_n'\,dx = (-1)^2 m(m+1)\int_{-1}^{1} P_n\,P_m\,dx$

(by using the orthogonality property)

$$\therefore \int_{-1}^{1}(1-x^2)\,P_m'\,P_n'\,dx = \begin{cases} 0 & \text{ if } m \neq n \\ \frac{2n(n+1)}{2n+1} & , \text{ if } m = n \end{cases}$$

H.P.

Question 14:- (i) Evaluate $\int_{-1}^{1} P_5^2(x)\,dx$

(ii) Prove that $\int_{-1}^{1} x^2 P_{n+1}(x) P_{n-1}(x)\,dx = \dfrac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

Solution :- (i) We know by orthogonal property of Legendre polynomials, that

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$$

$$\therefore \int_{-1}^{1} P_5(x)\,dx = \frac{2}{2(5)+1} = \frac{2}{11} \quad \text{Answer}$$

(ii) Let us use the recurrence relation

$$x P_{n-1} = \frac{nP_n + (n-1)P_{n-2}}{(2n-1)}$$

and (on replacing n by $n+2$), we have

$$x P_{n+1} = \frac{(n+2)P_{n+2} + (n+1)P_n}{(2n+3)}$$

Then $\displaystyle\int_{-1}^{1} x P_{n-1} \times x P_{n+1}\,dx = \int_{-1}^{1} \frac{[nP_n + (n-1)P_{n-2}][(n+2)P_{n+2} + (n+1)P_n]}{(2n-1)(2n+3)}\,dx$

$$= \frac{1}{(2n-1)(2n+3)} \int_{-1}^{1} [n(n+2)P_n P_{n+2} + n(n+1)P_n^2 + (n-1)(n+2)P_{n-2}P_{n+2} + (n^2-1)P_{n-2}P_n]\,dx$$

$$= \frac{n(n+1)}{(2n-1)(2n+3)} \int_{-1}^{1} P_n^2\,dx + 0 \quad \text{(by orthogonal property)}$$

$$= \frac{n(n+1)2}{(2n-1)(2n+3)(2n+1)}$$

Thus $\displaystyle\int_{-1}^{1} x^2 P_{n-1} P_{n+1}\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

H.P.

Question 15 :- Let $P_n(x)$ be the Legendre polynomial of degree n. Show that, for any function $f(x)$ for which the nth derivative is continuous,

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{(-1)^n}{2^n . n!} \int_{-1}^{1} (x^2-1)^n f^{(n)}(x)\, dx$$

Solution :- By Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n, \text{ we have}$$

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \int_{-1}^{1} f(x) \frac{1}{2^n . n!} \frac{d^n}{dx^n} (x^2-1)^n dx$$

Now integrating by parts

$$= \left[f(x) \frac{1}{2^n . n!} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right]_{-1}^{1} - \int_{-1}^{1} f'(x) \frac{1}{2^n . n!} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx$$

$$= 0 - \frac{1}{2^n n!} \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx \quad \left[\text{Since } \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \text{ has a factor } (x^2-1) \right]$$

$$= \frac{(-1)^n}{2^n . n!} \int_{-1}^{1} f^n(x) . (x^2-1)^n dx$$ (on integrating by parts repeatedly)

H. P.