PAGE :

Unit - III

Matrices and Determinants

(Topic → Rank of a Matrix)

Lec → 1

Rank of matrix is denoted by

RANK(A), or

$\rho(A)$, or

$r(A)$

Rank of Matrix

Minor

↳ Part

↳ square order

4×4

3×3

2×2

1×1

Flow Chart

Matrix (A)

⇓

Largest Possible Minor(s)

⇓

Determinant of all

⇓

if all Det = 0

Yes → Lowest order minor(s)

⇓ No

Rank = ?

Rank of matrix

$\Downarrow$

Matrix(A)

Two ways

Minor(s) $\Downarrow$ Det

Elementary Transformation → optional

$\Downarrow$

Minor(s)

$\Downarrow$

Det

But more preferred as max elements are 0, then easier to find rank of matrix.

$A \xrightarrow[\text{Transformation}]{\text{Elementary}} B$

equivalent

Note: Rank of two equivalents matrices is always same. ~~is a~~ (property)

Definition of Rank

The rank of a matrix A is said to be r

- → At least one minor of order r is non-zero.
- → All the minor(s) of order $r+1$ & higher are zero.

Q. Find Rank of Matrix.

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

Sol→ Let the given matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

By applying $R_1 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 3 & 1 & 4 \end{bmatrix}$$

By applying ; $R_3 \rightarrow R_3 + 3R_1$
$R_4 \rightarrow R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 8 \\ 0 & 10 & 16 \\ 0 & -5 & -8 \end{bmatrix}$$

By applying ; $C_2 \rightarrow C_2 - 2C_1$ & $C_3 \rightarrow C_3 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 8 \\ 0 & 10 & 16 \\ 0 & -5 & -8 \end{bmatrix}$$

By applying

$C_2 \to \frac{1}{5} C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 8 \\ 0 & 2 & 16 \\ 0 & -1 & -8 \end{bmatrix}$$

By applying $R_3 \to R_3 - 2R_2$ & $R_4 \to R_4 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By applying $C_3 \to C_3 - 8C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times 3} \begin{array}{l} \\ \\ \to \text{Zero Row} \\ \to \text{zero Row} \end{array}$$

Since all the minor(s) of order 3×3 will contain at least one zero row.
So their determinants will be zero.

Now one of the minor of order 2×2

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$$

$\boxed{\text{Rank}(A) = 2}$ Ans

(Ek sir ka tareeke upper triangular
matrix formation & find rank of matrix)
by no. of diagonal elements.

DATE: ___/___/___
PAGE: ______

(Topic → Consistency of Linear System of Equation)

Lec - 1

$\boxed{\text{Steps to be followed}}$

Set of linear Equations

$\Downarrow$

Coefficent Matrix $A = ?$, $X = ?$ → Variable Matrix, $B = ?$ → R.H.S. matrix

$\Downarrow$

Augmented Matrix

$[A : B]$

$\Downarrow$ By Applying Elementary Row transformation

Row Rank $(A) = ?$; Row Rank $(A : B) = ?$

$\gamma(A) = \gamma(A : B)$

Then consistent soln

$\gamma(A) \neq \gamma(A : B)$

Then inconsistent soln

↓

No soln

$\gamma(A) = \gamma(A : B) = n$ ← no of variables

Then unique soln

$\gamma(A) = \gamma(A : B) < n$

Then infinite soln

Solution

$AX = B$

$\boxed{\text{Row Rank = No of non-zero rows in a matrix}}$

Q. Check the consistency of following system of linear equations & if possible solve it too.

$$x + 2y + z = 3$$
$$2x + 3y + 2z = 5$$
$$3x - 5y + 5z = 2$$
$$3x + 9y - z = 4$$

Sol → $A =$ Coefficient Matrix $= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \\ 3 & 9 & -1 \end{bmatrix}$

$X =$ variable Matrix $= \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$B =$ RHS Matrix $= \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$

Now augmented Matrix is written as

$[A : B]$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{array}\right]$$

By applying

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 3R_1$

Good Write

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{array}\right]$$

By applying $R_2 \rightarrow -(R_1)$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{array}\right]$$

By applying $R_3 \rightarrow R_3 + 11R_2$ & $R_4 \rightarrow R_4 - 3R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{array}\right]$$

By applying $R_4 \rightarrow R_4 + 2R_3$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now Row rank $(A) = 3 = \gamma(A)$

Row rank $(A:B) = 3 = \gamma(A:B)$

Now $\gamma(A) = \gamma(A:B)$

So, consistent system of linear equation

Now; $\gamma(A) = \gamma(A:B) = n = 3$

So, system has unique solution

DATE: ___/___/___
PAGE: ______

Now solution can be written as $Ax = B$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \end{bmatrix}$$

On expanding

$$x + 2y + z = 3 \quad -(1)$$
$$y = 1 \quad -(2)$$
$$2z = 4 \quad -(3)$$

Now, from eq (3); $2z = 4$
$\boxed{z = 2}$

from eq (2); $\boxed{y = 1}$

As value of y & z are 1 & 2 respectively
So, from eq (1)

$$x + 2y + z = 3$$
$$x + 2(1) + 2 = 3$$
$$\boxed{x = -1}$$

So, solution of system:

$\boxed{x = -1 \; ; y = 1 \; ; z = 2}$ Ans

Q. Test the consistency of following equations & solve them if consistent.

$$x + 2y + 3t = 2$$
$$2x + y + z + t = -4$$
$$4x - 3y + z + 7t = 8$$

Sol→ from the given equations

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \end{bmatrix}$$

Coefficient Matrix

$$X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$$

Now Augmented Matrix can be written as $[A : B]$

$$\left[\begin{array}{cccc:c} 1 & 2 & 0 & 3 & 2 \\ 2 & 1 & 1 & 1 & -4 \\ 4 & -3 & 1 & 7 & 8 \end{array}\right]$$

By applying $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 4R_1$, we get

$$\sim \left[\begin{array}{cccc:c} 1 & 2 & 0 & 3 & 2 \\ 0 & -3 & 1 & -5 & -8 \\ 0 & -11 & 1 & -5 & 0 \end{array}\right]$$

By multiplying ; $R_2 \to \left(\frac{1}{-3}\right) R_2$

$$\sim \left[\begin{array}{cccc:c} 1 & 2 & 0 & 3 & 2 \\ 0 & 1 & -1/3 & 5/3 & 8/3 \\ 0 & -11 & 1 & -5 & 0 \end{array}\right]$$

By applying

$R_3 \to R_3 + 11 R_2$; we get

$$\sim \left[\begin{array}{cccc:c} 1 & 2 & 0 & 3 & 2 \\ 0 & 1 & -1/3 & 5/3 & 8/3 \\ 0 & 0 & 1 - 11/3 & -5 + \frac{55}{3} & 88/3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc:c} 1 & 2 & 0 & 3 & 2 \\ 0 & 1 & -1/3 & 5/3 & 8/3 \\ 0 & 0 & -8/3 & 40/3 & 88/3 \end{array}\right]$$

Now Row rank $(A) = 3$
Row rank $(A:B) = 3$

So as $\underline{r(A) = r(A:B)}$
$\hookrightarrow$ consistent system

Here; $n = 4$
So, $\underline{r(A) = r(A:B) < n}$
So, infinite soln

Now soln can be written as
$AX = B$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & -8/3 & 40/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 8/3 \\ 88/3 \end{bmatrix}$$

On expanding;

$x + 2y + 3t = 2$ — (1)

$y - \frac{z}{3} + \frac{5}{3}t = \frac{8}{3}$ — (2)

$-\frac{8}{3}z + \frac{40}{3}t = \frac{88}{3}$ — (3)

Let $t = k$

Put the value of t in eqn (3), we get

$$-\frac{8}{3} z + \frac{40}{3} k = \frac{88}{3}$$

$$-\frac{8}{3} z = \frac{88}{3} - \frac{40}{3} k$$

$$\boxed{z = 4k - 10}$$

Now put values of z & t in eq (2), we get

$$y - \frac{(4k-10)}{3} + \frac{5}{3} k = \frac{8}{3}$$

$$y - \frac{4k}{3} \cancel{+\frac{10}{3}} + \frac{10}{3} + \frac{5}{3} k = \frac{8}{3}$$

$$y + \frac{k}{3} + \frac{10}{3} = \frac{8}{3}$$

$$y = \frac{8}{3} - \frac{10}{3} - \frac{k}{3}$$

$$\boxed{y = -\frac{k}{3} - \frac{2}{3}}$$

Now put values of y, z & t in eq (1)

$$x + 2\left(-\frac{k}{3} - \frac{2}{3}\right) + 0.z + 3k = 2$$

$$x - \frac{2k}{3} - \frac{4}{3} \cancel{-\frac{4}{3}} + 3k = 2$$

$$x + \frac{7k}{3} - \frac{4}{3} = 2$$

$$x = 2 + \frac{4}{3} - \frac{7k}{3}$$

$$\boxed{x = -\frac{7k}{3} + \frac{10}{3}}$$

Ans

So, soln ~~are~~ is

$$x = -\frac{7k}{3} + \frac{10}{3}$$

$$y = -\frac{k}{3} - \frac{2}{3}$$

$$z = 4k - 10$$

$$t = k$$

Good Write

DATE: ___/___/___
PAGE: ___

[Topic: Determinants: Cramer's Rule]

Note: We use determinants to solve systems of "linear" equations.

Cramer's Rule

Let

$a_1 x + a_2 y + a_3 z = d_1$

$b_1 x + b_2 y + b_3 z = d_2$

$c_1 x + c_2 y + c_3 z = d_3$

This system can be Homogeneous or non-Homogeneous eqn.

[if $d_1 = d_2 = d_3 = 0$] [if d_1, d_2 & d_3 koi bhi ek minimum non zero]

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$D_1 = \begin{vmatrix} d_1 & a_2 & a_3 \\ d_2 & b_2 & b_3 \\ d_3 & c_2 & c_3 \end{vmatrix} \qquad D_2 = \begin{vmatrix} a_1 & d_1 & a_3 \\ b_1 & d_2 & b_3 \\ c_1 & d_3 & c_3 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & a_2 & d_1 \\ b_1 & b_2 & d_2 \\ c_1 & c_2 & d_3 \end{vmatrix}$$

$$\boxed{x = \frac{D_1}{D}\ ;\ y = \frac{D_2}{D}\ ;\ z = \frac{D_3}{D}}$$

(but D should be non zero if we want to find unique solution)

Q// Solve the equation by using determinants

$$x + y + z = 4$$
$$x - y + z = 0$$
$$2x + y + z = 5$$

Sol→ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ $\quad X = x, y$

So, $D = \det(A) = 1(-1-1) - 1(1-2) + 1(1+2)$

$= -2 + 1 + 3$

$\Rightarrow 2$

By Cramer's Rule

$$D_1 = \begin{vmatrix} 4 & 1 & 1 \\ 0 & -1 & 1 \\ 5 & 1 & 1 \end{vmatrix} = 4(-1-1) - 1(0-5) + 1(0+5)$$

$$D_1 = -8 + 5 + 5 = 2$$

DATE: __/__/__
PAGE:

$$D_2 = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 0 & 1 \\ 2 & 5 & 1 \end{vmatrix} = 1(0-5) - 4(1-2) + 1(5)$$

$$D_2 = -5 + 4 + 5 = 4$$

$$D_3 = \begin{vmatrix} 1 & 1 & 4 \\ 1 & -1 & 0 \\ 2 & 1 & 5 \end{vmatrix} = 1(-5) - 1(5) + 4(1+2)$$

$$D_3 = -5 - 5 + 12$$

$$D_3 = 2$$

$$x = \frac{D_1}{D} = \frac{2}{2} = 1$$

$$y = \frac{D_2}{D} = \frac{4}{2} = 2$$

$$z = \frac{D_3}{D} = \frac{2}{2} = 1$$

So, $\boxed{x = 1;\ y = 2;\ z = 1}$ Ans

(Topic → Gauss Elimination Method)

Q Solve the following equations by Gauss Elimination method.

$$x - y + 2z = 3$$
$$x + 2y + 3z = 5$$
$$3x - 4y - 5z = -13$$

Sol →
$$\underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix}}_{AX = B}$$

Augmented matrix → $[A : B] \Rightarrow \begin{bmatrix} 1 & -1 & 2 & : & 3 \\ 1 & 2 & 3 & : & 5 \\ 3 & -4 & -5 & : & -13 \end{bmatrix}$

⇒ By applying; $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 2 & : & 3 \\ 0 & 3 & 1 & : & 2 \\ 0 & -1 & -11 & : & -22 \end{bmatrix}$$

⇒ By applying; $R_3 \to 3R_3 + R_2$

$$\begin{bmatrix} 1 & -1 & 2 & : & 3 \\ 0 & 3 & 1 & : & 2 \\ 0 & 0 & -32 & : & -64 \end{bmatrix}$$

⇒ On expanding

$$x - y + 2z = 3 \quad -(I)$$
$$3y + z = 2 \quad -(II)$$
$$-32z = -64 \quad -(III)$$

DATE : ___/___/___
PAGE : ______

from eqn (III)

$z = -64/-32 = 2 \Rightarrow \boxed{z = 2}$

Put value of z in eqn (II)

$3y + 2 = 2 \Rightarrow 3y = 2 - 2$

$y = 0/3 \Rightarrow \boxed{y = 0}$

Put value of y & z in eqn (I)

$x - y + 2z = 3 \Rightarrow x - 0 + 2(2) = 3 \Rightarrow x = 3 - 4 \Rightarrow \boxed{x = -1}$

Solution is $\boxed{x = -1, y = 0 \ \& \ z = 2}$

(Topic → Gauss Jordan Method)

Q. Apply Gauss Jordan Method to solve the equation

$$x + y + z = 9$$
$$2x - 3y + 4z = 13$$
$$3x + 4y + 5z = 40$$

Sol → $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

Augmented Matrix = $[A : B]$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

By applying; $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

By applying; $R_1 \to 5R_1 + R_2$ & $R_3 \to 5R_3 + R_2$

$$\left[\begin{array}{ccc:c} 5 & 0 & 7 & 40 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

By applying; $R_2 \to 6R_2 - R_3$ & $R_1 \to 12R_1 - 7R_3$

$$\left[\begin{array}{ccc:c} 60 & 0 & 0 & 60 \\ 0 & -30 & 0 & -90 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

DATE: __/__/__
PAGE: ____

On expanding;

$60x = 60 \Rightarrow \boxed{x = 1}$

$-30y = -90 \Rightarrow y = \frac{-\cancel{90}^{3}}{\cancel{30}_{1}} = 3 \Rightarrow \boxed{y = 3}$

$12z = 60 \Rightarrow z = \frac{\cancel{60}^{5}}{\cancel{12}_{1}} \Rightarrow \boxed{z = 5}$

So, solution is $\boxed{x = 1;\ y = 3\ \&\ z = 5}$

M.M. Imp. (10 marks) (Topic → Eigen Values & Eigen vectors) (Lec - 1)

Application Areas
- Physics
- Electrical Engineering
- Computer Graphics, etc.

Core formula used

$$\boxed{A\,x = \lambda x}$$

Eigen vector

Matrix

Eigen value

• Flow chart to find Eigen values & Eigen vectors

Square Matrix (A)

$\Downarrow$

$[A - \lambda I] = ?$ → Eigen value; I → unit matrix

$\Downarrow$

Put $|A - \lambda I| = 0$
to get characteristic equation

$\Big\}$ Most mistakeprone area

$\Downarrow$

On solving

$\Downarrow$

Values of $\lambda = ?$ } Error prone Area

Eigen values

$\Downarrow$

Eigen vectors

by Putting $(A - \lambda I)x = 0$ → Zero matrix

→ Eigen vector

(M.M Imp)

Q. Find the Eigen values & Eigen vectors for the Matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol→ The given matrix is

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Now;

$$[A - \lambda I] = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

⇒ Now for characteristic equation;

Put $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

On expanding;

$$\Rightarrow (1-\lambda)\left((5-\lambda)(1-\lambda)-1\right) - 1(1-\lambda-3) + 3\left(1-3(5-\lambda)\right) = 0$$

$$\Rightarrow (1-\lambda)(-6\lambda + 4 + \lambda^2) - 1(-\lambda-2) + 3(3\lambda - 14) = 0$$

$$\Rightarrow \boxed{\lambda^3 - 7\lambda^2 + 36 = 0} \quad -①$$

DATE: __/__/__
PAGE ____

Now by apply hit & trial method

Hit $\lambda = 1$; $(1)^3 - 7(1)^2 + 36 \Rightarrow 30 \neq 0$

Hit $\lambda = -1$; $(-1)^3 - 7(-1)^2 + 36 \Rightarrow 28 \neq 0$

Hit $\lambda = 2$; $(2)^3 - 7(2)^2 + 36 \Rightarrow 16 \neq 0$.

Hit $\boxed{\lambda = -2}$; $(-2)^3 - 7(-2)^2 + 36 \Rightarrow \boxed{0 = 0}$.

Since $\lambda = -2$ is satisfying equation (1); so it is one of the roots of eqn (1)

Now by synthetic division method

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

Now the rest two roots will be given by the equation

$$1(\lambda^2) - 9\lambda + 18 = 0$$
$$\lambda^2 - 3\lambda - 6\lambda + 18 = 0$$
$$\lambda(\lambda - 3) - 6(\lambda - 3) = 0$$
$$(\lambda - 6)(\lambda - 3) = 0$$
$$\boxed{\lambda = 6} \quad \boxed{\lambda = 3}$$

$\boxed{\text{So, Eigen values are } -2, 3, 6}$ Ans

⇒ Now Eigen vectors are given as

$$(A - \lambda I)\, X = 0$$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

⇒ Now for $\lambda = -2$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On multiplying, we get

$$3x + y + 3z = 0$$
$$x + 7y + z = 0$$
$$3x + y + 3z = 0$$

Note: It is fixed nature of these are infinite soln for all ques.

On solving by cross multiplication method, we get

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$\frac{x}{1-21} = \frac{y}{3-3} = \frac{z}{21-1}$$

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20}$$

$$\frac{x}{+1} = \frac{y}{0} = \frac{z}{-1}$$

So, Eigen vector when $\lambda = -2$

$$\langle 1, 0, -1 \rangle$$

Now for $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + y + 3z = 0$$
$$x + 2y + z = 0$$
$$3x + y + (-2z) = 0$$

By cross multiplication method

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x}{-5} = \frac{y}{5} = \frac{z}{-5}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

for Eigen vector when $\lambda = 3$

$\langle 1, -1, 1 \rangle$

$\Rightarrow$ Now for $\lambda = 6$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-5x + y + 3z = 0$$
$$x - y + z = 0$$
$$3x + y - 5z = 0$$

By cross multiplication method

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & -5 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\Rightarrow \frac{x}{4} = \frac{y}{8} = \frac{z}{4} \Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

$\Rightarrow$ So, Eigen vectors when $\lambda = 6$

$\langle 1, 2, 1 \rangle$

Eigen vectors

when

$\lambda = -2$; $\langle 1, 0, -1 \rangle$

$\lambda = 3$; $\langle 1, -1, 1 \rangle$

$\lambda = 6$; $\langle 1, 2, 1 \rangle$

Ans

Eigen bases

[Topic → Diagonalisation]

Two types

Eigen values distinct

Eigen values same (AM & GM ka locha)

□ Case : Eigen values distinct

Q.1 → Show that the following matrix A is diagonalizable.

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$$

Sol → $[A - \lambda I] = \begin{bmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{bmatrix}$

Put $|A - \lambda I| = 0$

$\lambda^2 - 7\lambda + 6 = 0$

$\boxed{\lambda = 1, 6}$

$\lambda_1 = 1$ & $\lambda_2 = 6$

Eigen vectors

for $\lambda_1 = 1$

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x_1 + 2x_2 = 0$ —(i)

$3x_1 + 2x_2 = 0$ —(ii)

Take eqn (i); $3x_1 + 2x_2 = 0$

Let $x_2 = 3k$

$3x_1 = -6k$

$\boxed{x_1 = -2k}$

$\langle x_1, x_2 \rangle$

$\langle -2k, 3k \rangle$

$\cancel{\langle -2, 3 \rangle}$

$\langle 2, -3 \rangle$

DATE: / /
PAGE

for $\lambda_2 = 6$

$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 + 2x_2 = 0$ — (iii)

$3x_1 - 3x_2 = 0$ — (iv)

Take eqn (iii); $-2x_1 + 2x_2 = 0$

Let $x_2 = k$

$-2x_1 = -2k$

$x_1 = k$

So; $\langle x_1, x_2 \rangle \Rightarrow \langle k, k \rangle \Rightarrow \underline{\langle 1, 1 \rangle}$

$\Rightarrow$ Now; matrix formed from both Eigen vectors is P.

$$P = \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ -3 & 6 \end{bmatrix}$$

$$\Rightarrow \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 30 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

So, $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Hence, A is a diagonalizable matrix

DATE: / /
PAGE

Q.2 → Show that the following matrix $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ diagonalizable.

Sol →
$$[A - \lambda I] = \begin{bmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{bmatrix}$$

Put;
$$|A - \lambda I| = 0$$
$$(\lambda-1)(\lambda+2)(\lambda-3) = 0$$
$$\boxed{\lambda = 1, -2, 3}$$

Eigen vectors

for $\lambda_1 = 1$
$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$\langle x_1, x_2, x_3 \rangle \Rightarrow \langle -1, 4, 1 \rangle$

for $\lambda_2 = -2$
$$\begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$\langle x_1, x_2, x_3 \rangle \Rightarrow \langle 1, -1, -1 \rangle$

for $\lambda_3 = 3$
$$\begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$\langle x_1, x_2, x_3 \rangle \Rightarrow \langle 1, 2, 1 \rangle$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 4 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{-1}{6} \begin{bmatrix} 1 & -2 & 3 \\ -2 & -2 & 6 \\ -3 & 0 & -3 \end{bmatrix}$$

$$P^{-1}AP = \frac{-1}{6} \begin{bmatrix} 1 & -2 & 3 \\ -2 & -2 & 6 \\ -3 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 4 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow \text{So, } P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Hence; A is diagonalizable matrix.

- Case: Eigen values are repeated

Concept of AM & GM

AM = Algebraic Multiplicity
GM = Geometric Multiplicity

[for A ~~to~~ to be diagonalizable ; AM = GM.
in this case especially.]

Eg. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 7 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

Eigen values:
$\lambda_1 = 1$; $\lambda_2 = 7$; $\lambda_3 = 7$

So, $(\lambda - 1)^1 (\lambda - 7)^{2}$

so this is AM

AM is 2 of matrix A

GM = $n(A - \lambda I)$

nullity ; 0 rows in matrix
(by Elementary Row transformation)

Note:
$\lambda = 1$ is distinct so,
AM = GM = 1

if $\lambda = 7$

$$[A - 7I] = \begin{bmatrix} -6 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

1

So, $n(A - 7I) = 1$, & $(A.M)_7 = 1$

GM = 1

So, as AM = GM in all Eigen values case so ; A is diagonalizable.

Q. Show that the Matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ is diagonalizable.

Soln $[A - \lambda I] = \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}$

Put;

$|A - \lambda I| = 0$

$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$

$$\boxed{\lambda = \underbrace{-1, -1}_{\text{Repeated}}, 8}$$

Eigen vectors (On checking; ~~AM = GM for all Eigen values~~)

for $\lambda = 8$

By putting;

$[A - 8I]X = 0$

$\langle x_1, x_2, x_3 \rangle = \langle 2, 1, 2 \rangle$

for $\lambda_2 = \lambda_3 = -1$

By putting;

$(A + I)X = 0$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

DATE: __/__/__
PAGE ____

$$4x_1 + 2x_2 + 4x_3 = 0$$
$$2x_1 + x_2 + 2x_3 = 0$$
$$4x_1 + 2x_2 + 4x_3 = 0$$

$x_2 = 0; x_3 = k$ | $x_2 = 2k; x_3 = 0$

$4x_1 + 4k = 0$	$4x_1 + 4k = 0$
$x_1 = -k$	$x_1 = -k$
$\langle x_1, x_2, x_3 \rangle \Rightarrow \langle -1, 0, 1 \rangle$	$\langle x_1, x_2, x_3 \rangle \Rightarrow \langle -1, 1, 0 \rangle$

$$P = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = -\frac{1}{5} \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & -3 \\ 1 & -4 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Hence, A is diagonalizable

(Topic → Cayley Hamilton Theorem)

• **Statement** (Definition)

Every square Matrix satisfies its own characteristic equation.

Q.1 → Verify Cayley-Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

Sol: $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$[A - \lambda I] = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[A - \lambda I] = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$$

Put;

$$|A \lambda I| = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\boxed{\lambda^2 - 4\lambda - 5 = 0} \quad - (1)$$

~~Characteristic~~ eqn

Now by Cayley Hamilton Theorem; Matrix A will satisfy eqn (1)

$$A^2 - 4A - 5I = 0$$

Now, L.H.S $= A^2 - 4A - 5I$

$$A^2 = A \cdot A \Rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

Now; $A^2 - 4A - 5I$

$$\Rightarrow \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow 0 = R.H.S.$$

So, $A^2 - 4A - 5I = 0$

Hence proved.

So, Cayley Hamilton Theorem verified.

Q.2→ Now find A^{-1} further; after previous question.

Sol→ $A^2 - 4A - 5I = 0$ (from previous Ques. soln)

$$A^{-1}(A^2 - 4A - 5I) = 0$$

$$A - 4AA^{-1} - 5A^{-1}I = 0$$

$$[AA^{-1} = I]$$

$$A - 4I - 5A^{-1} = 0$$

$$5A^{-1} = A - 4I$$

$$A^{-1} = \frac{1}{5}(A - 4I) \Rightarrow \frac{1}{5}\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$$ Ans

Q.3→ If matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then verify

Cayley Hamilton Theorem; hence ~~you~~ find A^{-1}.

Sol→ Put $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 5\lambda + 6) - 4(2-\lambda) = 0$$

$$\boxed{\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0} \quad -(1)$$

Now by Cayley Hamilton Theorem A will satisfy eq (1)

So, $L.H.S. = A^3 - 6A^2 + 7A + 2I$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$\Rightarrow$ L.H.S.

$\Rightarrow A^3 - 6A^2 + 7A + 2I$

$$\Rightarrow \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 5 \\ 2 & 4 & 8 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow 0 = R.H.S.$$

So, Cayley Hamilton Theorem verified on matrix A.

Now find $A^{-1} = ?$

$$(A^3 - 6A^2 + 7A + 2I = 0) \text{ (Proved above)}$$

$$A^{-1}(A^3 - 6A^2 + 7A + 2I) = 0$$
$$A^2 - 6A + 7I + 2A^{-1} = 0$$
$$2A^{-1} = -A^2 + 6A - 7I$$
$$A^{-1} = \frac{1}{2}(-A^2 + 6A - 7I)$$

$$A^{-1} = \frac{1}{2}\left(-\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$A^{-1} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & 1/2 & 1/2 \\ 2 & 0 & -1 \end{bmatrix} \text{ Ans}$$

DATE: ___ PAGE ___

(Topic → Linear Independence)

• Define →

A set of vectors are linearly dependent if atleast one of them can be expressed as linear combination of others & if there doesn't exist any Linear Combination b/w them then they are called Linear Independent.

· Methods to find Linear Independent & Linear Dependent vectors

1. General Method

i) [Rank = No of vectors] then Linear Independent

ii) [Rank ≠ No. of vectors] then Linear Dependent

2. Tricky Method

i) $\text{Det}(A) \neq 0 \Rightarrow$ Linear Independent (L.I.)

ii) $\text{Det}(A) = 0 \Rightarrow$ Linear Dependent (L.D.)

[Note: Dimensions → No. of L.I. vectors.
Basis → Set of L.I. vectors.]

Q.1 → Find whether the set of vectors $v_1 = (1,2,1)$, $v_2 = (3,1,5)$, $v_3 = (3,-4,7)$ is linearly independent or dependent.

Sol → Let a_1, a_2, a_3 be three scalars such that $a_1v_1 + a_2v_2 + a_3v_3 = 0$

$a_1(1,2,1) + a_2(3,1,5) + a_3(3,-4,7) = 0$

$(a_1 + 3a_2 + 3a_3,\ 2a_1 + a_2 - 4a_3,\ a_1 + 5a_2 + 7a_3) = 0$

DATE: / /
PAGE

$\Rightarrow a_1 + 3a_2 + 3a_3 = 0$
$2a_1 + a_2 - 4a_3 = 0$
$a_1 + 5a_2 + 7a_3 = 0$

Coefficient matrix of these eqn is A

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}$$

$|A| = 27 - 54 + 27 = 0$

~~$\& \Delta \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$~~

Now, $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}$

$R_2 \rightarrow R_2 - 2R_1 \;\&\; R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & 2 & 4 \end{bmatrix}$$

$R_3 \rightarrow 5R_3 + 2R_2$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

So, $\det(A) = 0$

So, one of minor of order $2\times2 \Rightarrow M = \begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix}$

Now $|M| = -5 \neq 0$

So, $\text{Rank}(A) = 2$

i.e. the rank of matrix $A <$ no. of vectors

So, the set of vectors are linearly dependent.

Are the vectors $\overbrace{(2,2,2,4)}^{\alpha_1}$, $\overbrace{(2,-2,-4,0)}^{\alpha_2}$, $\underbrace{(4,-2,-5,2)}_{\alpha_3}$, $\underbrace{(4,2,1,6)}_{\alpha_4}$ linearly independent.?

Let a_1, a_2, a_3 & a_4 be four scalars such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$

Here; $\alpha_1 = (2,2,2,4)$, $\alpha_2 = (2,-2,-4,0)$, $\alpha_3 = (4,-2,-5,2)$ and $\alpha_4 = (4,2,1,6)$

$a_1(2,2,2,4) + a_2(2,-2,-4,0) + a_3(4,-2,-5,2) +$
$a_4(4,2,1,6) = 0$

$\Rightarrow (2a_1 + 2a_2 + 4a_3 + 4a_4, 2a_1 - 2a_2 - 2a_3 + 2a_4,$
$2a_1 - 4a_2 - 5a_3 + a_4, 4a_1 + 2a_3 + 6a_4) = (0, 0, 0, 0)$

Coefficient matrix of these equation is

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 2 & -2 & -2 & 2 \\ 2 & -4 & -5 & 1 \\ 4 & 0 & 2 & 6 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ and $R_4 \to R_4 - 2R_1$;

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & -6 & -9 & -3 \\ 0 & -4 & -6 & -2 \end{bmatrix}$$

Applying $R_3 \to 2R_3 - 3R_2$ and $R_4 \to R_4 - R_2$;

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = r(A) = \rho(A) = 2$

i.e.; the rank of matrix A < no. of vectors
The system of eqn will have $4 - 2 = 2$; nonzero soln
& hence the set of vectors are linearly dependent.
Hence given vectors are not linearly independent.

DATE: / /
PAGE

(Topic → Vector space)

V ↓ vector

F → field

$V(F)$ = vector space

↔ $F(+,\cdot)$ is field; (has $a, b, c, \ldots$)
↳ scalar elements

field ki ~~khaas~~ baat

i) $(F,+)$ is a abelian group
Or
commutative group (have 5 condt)

(a) $a, b \in F \Rightarrow a+b \in F$ — Closure Property

(b) $(a+b)+c = a+(b+c)$ — Associative Property

(c) $a+0 = a = 0+a$ — Identity Property
0 is identity element; $0 \in F$

(d) $a+(-a) = 0 = (-a)+a$ — Inverse Property
$-a \in F$

(e) $a+b = b+a$ — Commutative Property

ii) $(F,\cdot)$ is abelian group

(a) $a, b \in F \Rightarrow a \cdot b \in F$ — Closure Property

(b) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ — Associative "

(c) $a \cdot 1 = a = 1 \cdot a$ — Identity Property
1 → identity element
$1 \in F$

(d) $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a$ — Inverse "
$a^{-1} = 1/a \in F$

(e) $a \cdot b = b \cdot a$ — Commutative "

DATE: ___/___/___
PAGE ______

iii) $a \cdot (b+c) = a \cdot b + a \cdot c$
$\cdot$ is distributive over addition
(Distributive property)

→ V ke pass do operation $+$, $\cdot$ ~~addition~~ ~~scalar multiplication~~

$+$ (vector addition) internal Binary Composition $u, v \in V$

$\cdot$ (scalar multiplication) External Binary Composition $a \in F; u \in V$

i) $(V, +)$ — should have abelian group
a) $u, v \in V$; $u + v \in V$ (Closure)
b) $u + (v + w) = (u + v) + w$ (Associativity)
c) $u + 0 = u = 0 + u$ (Identity)
d) $u + (-u) = 0 = (-u) + u$ (Inverse)
e) $u + v = v + u$ (Commutative)

ii) $a \in F$; $u \in V$
$\boxed{a \cdot u \in V}$

iii) $a \cdot (u + v) = a \cdot u + a \cdot v$
iv) $(a + b) \cdot u = a \cdot u + b \cdot u$
v) $(a \cdot b) \cdot u = a \cdot (b \cdot u)$
vi) $1 \in F$; $u \in V \Rightarrow 1 \cdot u = u$; $u \in V$

bcoz it satisfies all 9 properties $3 + 6$

Note: R → field C → field (complex no.)
$C(R)$ → vector space
vector bana wrt. R always.
But $R(C)$ → is not a vector space

Q1→ Check whether vector $V = \{ x, y, z \in R^3 :$
$2x + 3y^3 - 4z^2 = 0\}$ is vector space or not?

Sol→ Given: $v = \{ x, y, z \in R^3 : 2x + 3y^3 - 4z^3 = 0\}$

so, take $(2, 0, 1), (8, 0, 2) \in V$

V.S. 1:- Now, if $(2, 0, 1) + (8, 0, 2) = (10, 0, 3) = -36 \neq 0$

i.e. additive vector $(10, 0, 3)$ doesnot belong to V.

Therefore set V is not a vector space.

Q.2→ Check whether set $V = \{ (a, b, c) \in R^3 : (a-b)c = 0\}$
is vector space or not?

Sol→ Given: $V = \{(a, b, c) \in R^3 : (a-b)c = 0\}$

So, take, $(1, 1, 2), (2, 5, 0) \in V$

V.S.1 ⇒ $(1, 1, 2) + (2, 5, 0) = (3, 6, 2)$

So, $(3-6)2 = -6 \neq 0$

Therefore; the closure property is not satisfied.

So, set V is not a vector space.

DATE: ___/___/___
PAGE ______

(Topic: Quadratic form)

A quadratic form in variables $(x_1, x_2, \dots x_n)$ is a polynomial such that every term has degree '2'.

① $Q(x) = X^T A X$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $X^T = [x_1 \; x_2 \; \dots \; x_n]$

② The condt ;

if you want to make quadratic from matrix A then $\boxed{A = A^T}$

Means A should be a symmetric matrix.

Q. find the quadratic form of

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

Sol → (rough: Checked; $A^T = A$, so quadratic form can be made)

$Q(x) = X^T A X$

$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} 4x_1 & 3x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Q(x) = 4x_1^2 + 3x_2^2$ Ans

DATE: PAGE

Q. Find the quadratic form of

$$A = \begin{bmatrix} 4 & -5 & 7 \\ -5 & -6 & 8 \\ 7 & 8 & -9 \end{bmatrix}$$

Sol → $Q(x) = X^T A X$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & -5 & 7 \\ -5 & -6 & 8 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow 4x_1^2 - 6x_2^2 - 9x_3^2 - 10x_1x_2 + 14x_1x_3 + 16x_2x_3$$ Ans

Q. Find the matrix of quadratic form

(i) $x_1^2 - 6x_1x_2 - 3x_2^2$

Sol → $A = \begin{bmatrix} \text{coefficient of } x_1^2 & \tfrac{1}{2} \text{ of coefficient of } x_1x_2 \\ \tfrac{1}{2} \text{ of coefficient of } x_1x_2 & \text{coefficient of } x_2^2 \end{bmatrix}_{2\times 2}$

$$A = \begin{bmatrix} 1 & -3 \\ -3 & -3 \end{bmatrix}$$ Ans

(ii) $3x^2 - 8xy + y^2 - 16xz + 16yz + 5z^2$

Sol → $3x^2 + y^2 + 5z^2 - 8xy + 16yz - 16xz$

$$A = \begin{bmatrix} \text{coeff of } x^2 & \tfrac{1}{2} \text{ of coeff of } xy & \tfrac{1}{2}\text{coeff of } xz \\ \tfrac{1}{2}\text{coeff of } xy & \text{coeff of } y^2 & \tfrac{1}{2} \text{ coeff of } yz \\ \tfrac{1}{2} \text{ coeff of } xz & \tfrac{1}{2} \text{ coeff of } yz & \cancel{\text{coefficient of } z^2}\ \text{coeff of } z^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -4 & -8 \\ -4 & 1 & 8 \\ -8 & 8 & 5 \end{bmatrix}$$ Ans

(Topic ⇒ Orthogonal Matrix)

Square matrix is said to be orthogonal matrix if

$A A^T = A A^{-1} = I$

means $\boxed{A^{-1} = A^T}$

Q. Prove that if A is orthogonal matrix then $|A| = \pm 1$.

Sol → $A A^{-1} = A^{-1} A = I$ (B'coz A is orthogonal Matrix)

$A A^T = A^T A = I$

$A A^T = I$

$|A A^T| = |I|$

$|A| |A^T| = 1$

(We know that $|A| = |A^T|$)

$|A| \, |A| = 1$

$|A|^2 = 1$

$\boxed{|A| = \pm 1}$

Hence, proved

Q. Verify that Matrix is orthogonal :

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

Sol → $A^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

Now, $A A^T$

$\Rightarrow \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I$

So, as $\underline{A A^T = I}$

So, A is ~~a~~ orthogonal Matrix.

Hence, verified.

Note: Extra

Orthogonal Matrix → Proper → if det(Matrix) = +1

Orthogonal Matrix → Improper → if det(Matrix) = −1