CHAPTER

12

Application of Network Theorems in AC Circuits

12.1 INTRODUCTION

When two or more sources are present in a network and they are connected in such a way that they are not mutually in series or in parallel, special network solution methods like *mesh analysis* or *nodal analysis* must be employed. The application of these two techniques being described earlier for d.c. networks, they are well applied in a.c. network analysis too. The difference is that in a.c. application the *complex numbers* (*i.e.*, impedances) are used as the *coefficients of the equations* (instead of real numbers *i.e.*, resistances as done in d.c. applications) and the *variables* (like currents and voltages) have phasors instead of scalar voltage or currents.

In addition to application of mesh or nodal analysis in a.c. networks, application of network theorems like *Thevenin's theorem, Norton theorem, superposition theorem, Millman's theorem, Reciprocity theorem* and *Maximum Power Transfer theorems* are also shown in this chapter in steady state conditions. Since the discussion of theoretical aspects have been well covered in the chapter showing their d.c. applications, hence in theoretical discussions regarding these theorems here, only the variations required in a.c. application are considered.

12.2 MESH ANALYSIS

Mesh current analysis for a.c. networks can be done exactly in the same manner like d.c. network, the only difference is that the coefficients in the mesh current equations of the a.c. network have complex numbers (*i.e.*, impedances) instead of real numbers (resistances) and having phasors instead of real currents or voltages as the unknowns.

The mesh current analysis technique is applied for all networks with independent sources or networks with dependent sources where the controlling variables are not a part of the network under investigation. If the controlling variable is part of the network being examined, additional care must be taken when applying the above technique.

EXAMPLE 12.1 *Find the transfer function (V_2 / V_1) for the network shown in Fig. E12.1 using mesh analysis.*

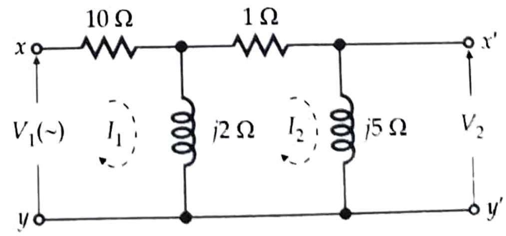


Fig. E12.1

Solution. The mesh currents being designated by dotted arrows, *mesh analysis* yields :

$$(10+j\,2)\,I_1 - j2\,(I_2) = V_1 \qquad ...(1)$$

and $$(-j\,2)\,I_1 + (1+j\,7)\,I_2 = 0 \qquad ...(2)$$

In matrix form, we can rearrange as

$$\begin{bmatrix} (10+j\,2) & -(j\,2) \\ (-j\,2) & (1+j\,7) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}$$

However, at the output,

$$V_2 = (j\,5)\,I_2 \qquad ...(3)$$

Solving for I_2,

$$I_2 = \frac{\begin{vmatrix} 10+j\,2 & V_1 \\ -j\,2 & 0 \end{vmatrix}}{\begin{bmatrix} 10+j\,2 & -j\,2 \\ -j\,2 & 1+j\,7 \end{bmatrix}}$$

$$= \frac{(j\,2)\,V_1}{10+j\,70+j\,2-14+4} = \frac{(j\,2)\,V_1}{+j\,72}$$

$$\therefore \quad V_2 = (j.5)\frac{(j\,2)\,V_1}{j\,72} = -\frac{10\,V_1}{j\,72}$$

$$= -0.138\,\angle 90°\,V_1$$

$\therefore$ $\mathbf{(V_2 / V_1) = 0.138\,\angle 90°}$.

(*i.e.*, the required transfer function.)

Example 12.2 *Find the current through Z_L using mesh analysis for the network shown in Fig. E12.2.*

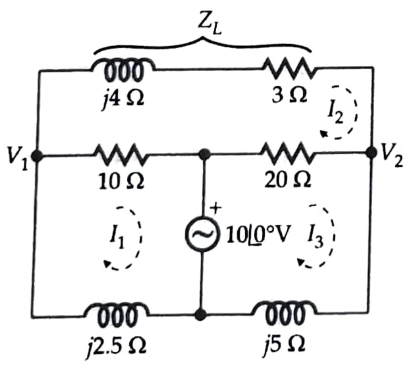


Fig. E12.2

Solution. The mesh currents being designated by dotted arrowhead lines, *mesh analysis* gives

$$(10+j\,2.5)\,I_1 - 10\,I_2 = -10\,\angle 0° \qquad ...(1)$$

$$(20+j\,5)\,I_3 - 20\,I_2 = 10\,\angle 0° \qquad ...(2)$$

$$-10\,I_1 - 20\,I_3 + I_2\,(33+j\,4) = 0 \qquad ...(3)$$

Rearranging in matrix form,

$$\begin{bmatrix} (10+j2.5) & -10 & 0 \\ 0 & -20 & (20+j\,5) \\ -10 & (33+j4) & -20 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -10\angle 0° \\ 10\angle 0° \\ 0 \end{bmatrix}$$

$$\therefore\ I_2 = \frac{\begin{vmatrix} 10+j\,2.5 & -10\,\angle 0° & 0 \\ 0 & 10\angle 0° & 20+j\,5 \\ -10 & 0 & -20 \end{vmatrix}}{\begin{vmatrix} (10+j\,2.5) & -10 & 0 \\ 0 & -20 & (20+j\,5) \\ -10 & (33+j\,4) & -20 \end{vmatrix}}$$

$$= \frac{(10+j2.5)\,10\angle 0° \times (-20) - (-10\angle 0°)\ \{10\,(20+j\,5)\}}{\Delta Z},$$

where ΔZ = determinant of the matrix in the denominator;

$$= \frac{-2000 - j\,500 + 2000 + j\,500}{\Delta Z} = \frac{0}{\Delta Z} = \mathbf{0.}$$

Thus, it is seen that no current would flow through Z_L, since I_2 is found to be zero and at the starting of the solution, I_2 had been the current flowing through Z_L.

Example 12.3 *Using mesh current analysis, find the drop in the capacitor for the network shown in Fig. E12.3.*

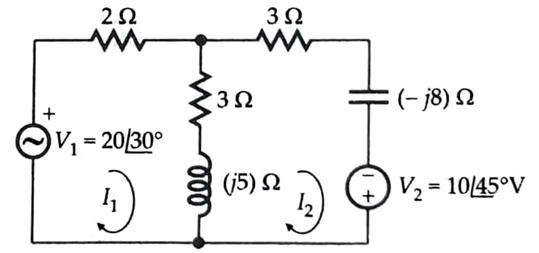


Fig. E12.3

Solution. Let us first designate the mesh currents as I_1 and I_2. Application of *loop analysis* yields

$$(2+3+j5)\,I_1 - (3+j\,5)\,I_2 = V_1$$

or $$(5+j\,5)\,I_1 - (3+j\,5)\,I_2 = V_1 = 20\,\angle 30°\text{ V} \qquad ...(1)$$

and $$(3-j\,8+3+j\,5)\,I_2 - (3+j\,5)\,I_1 = 10\,\angle 45°$$

or $$(6-j\,3)\,I_2 - (3+j\,5)\,I_1 = 10\,\angle 45° \qquad ...(2)$$

In matrix form equations (1) and (2) are rearranged as

$$\begin{bmatrix} (5+j\,5) & -(3+j\,5) \\ -(3+j\,5) & (6-j\,3) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 20\,\angle 30° \\ 10\,\angle 45° \end{bmatrix}$$

$$\therefore I_2 = \frac{\begin{vmatrix} 5+j5 & 20\angle 30^\circ \\ -(3+j5) & 10\angle 45^\circ \end{vmatrix}}{\begin{vmatrix} 5+j5 & -(3+j5) \\ -(3+j5) & (6-j3) \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 7.07\angle 45^\circ & 20\angle 30^\circ \\ -5.83\angle 59^\circ & 10\angle 45^\circ \end{vmatrix}}{\begin{vmatrix} 7.07\angle 45^\circ & -5.83\angle 59^\circ \\ -5.83\angle 59^\circ & 6.71\angle -26.56^\circ \end{vmatrix}}$$

$$= \frac{70.7\angle 90^\circ + 116.6\angle 89^\circ}{47.44\angle 18.44 - 33.99\angle 118^\circ}$$

$$= \frac{j70.7 + 2.03 + j116.58}{45 + j15 + 15.96 - j30} = \frac{2.03 + j187.28}{60.96 - j15}$$

$$= \frac{187.29\angle 89.38^\circ}{62.78\angle -13.81} = 2.98\angle 103.29^\circ \text{ A}$$

$\therefore$ Drop in the capacitor

$$= I_2(-j8) = \mathbf{23.84\angle 13.19^\circ\ V.}$$

EXAMPLE 12.4 *Find the drop across 4 Ω and 10 Ω resistors using mesh current analysis. (Fig. E12.4).*

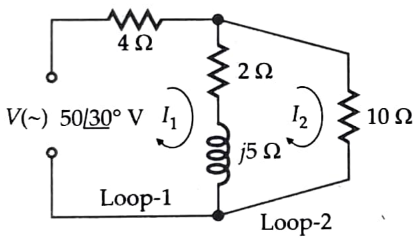


Fig. E12.4

SOLUTION. Let us first designate the mesh currents by arrowhead lines indicating the loops.

In *loop-1*, mesh current analysis provides

$$(4 + 2 + j5)\,I_1 - (2 + j5)\,I_2 = 50\angle 30^\circ$$

or $$(6 + j5)\,I_1 - (2 + j5)\,I_2 = 50\angle 30^\circ \quad \text{...(1)}$$

In *loop-2*, mesh analysis provides

$$-(2 + j5)\,I_1 + (2 + 10 + j5)\,I_2 = 0$$

or $$-(2 + j5)\,I_1 + (12 + j5)\,I_2 = 0 \quad \text{...(2)}$$

Rearranging (1) and (2) in matrix form,

$$\begin{bmatrix} (6+j5) & -(2+j5) \\ -(2+j5) & (12+j5) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 50\angle 30^\circ \\ 0 \end{bmatrix}$$

$$\therefore I_1 = \frac{\begin{vmatrix} 50\angle 30^\circ & -(2+j5) \\ 0 & (12+j5) \end{vmatrix}}{\begin{vmatrix} (6+j5) & -(2+j5) \\ -(2+j5) & (12+j5) \end{vmatrix}}$$

$$= \frac{50\angle 30(12 + j5)}{(6 + j5)(12 + j5) - (2 + j5)(2 + j5)}$$

$$= \frac{650\angle 52.62^\circ}{72 + j30 + j60 - 25 - 4 - j10 - j10 + 25}$$

$$= \frac{650\angle 52.62^\circ}{68 + j70}$$

$$= \frac{650\angle 52.62^\circ}{97.60\angle 45.83^\circ} = 6.66\angle 6.79^\circ \text{ A}$$

$\therefore$ Drop across 4 Ω resistor is

$$4 \times I_1 = 4 \times 6.66\angle 6.79^\circ \text{ } i.e., 26.64\angle 6.79^\circ \text{ V.}$$

Similarly,

$$I_2 = \frac{\begin{vmatrix} 6+j5 & 50\angle 30^\circ \\ -(2+j5) & 0 \end{vmatrix}}{\begin{vmatrix} (6+j5) & -(2+j5) \\ -(2+j5) & (12+j5) \end{vmatrix}}$$

$$= \frac{50\angle 30^\circ(2 + j5)}{68 + j70}$$

$$= \frac{5.385\angle 68.19^\circ \times 50\angle 30^\circ}{97.60\angle 45.83^\circ} = 2.76\angle 52.36^\circ \text{ A}$$

Thus, drop across 10 Ω resistor is

$$10\,I_2 = 10 \times 2.76\angle 52.36^\circ$$
$$= \mathbf{27.6\angle 52.36^\circ\ V.}$$

EXAMPLE 12.5 *What is the value of V_2 such that the current in $(2 + j3)\Omega$ impedance is zero (Fig. E12.5) ?*

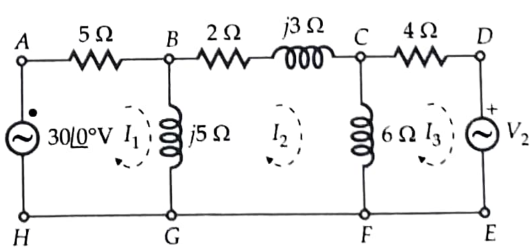


Fig. E12.5

SOLUTION. Let us first designate the loops alongwith the loop currents in Fig. E12.5.

In loop ABGH, mesh analysis gives

$$5I_1 + j\,5I_1 - j\,5I_2 = 30\angle 0°$$

or $$I_1\,(5 + j\,5) - j\,5I_2 = 30\,\angle 0° \quad ...(1)$$

In loop BCFG, mesh analysis gives

$$(2 + j3 + 6 + j\,5)\,I_2 - j\,5I_1 - 6I_3 = 0$$

or $$(8 + j\,8)\,I_2 - j\,5\,I_1 - 6\,I_3 = 0 \quad ...(2)$$

In loop CDEF, mesh analysis gives

$$(4 + 6)\,I_3 - 6I_2 = -V_2$$

or $$10\,I_3 - 6\,I_2 = -V_2 \quad ...(3)$$

Rearranging the three equations in matrix form,

$$\begin{bmatrix} (5 + j\,5) & -j\,5 & 0 \\ -j\,5 & 8 + j\,8 & -6 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 30\,\angle 0° \\ 0 \\ -V_2 \end{bmatrix} \quad ...(4)$$

$$\therefore \quad I_2 = \frac{\begin{vmatrix} (5 + j\,5) & 30\,\angle 0° & 0 \\ (-j\,5) & 0 & -6 \\ 0 & -V_2 & 10 \end{vmatrix}}{\Delta Z}$$

[where ΔZ is the determinant of the matrix equation (4)]

$$= \frac{(5 + j\,5)(-6V_2) - 30\,\angle 0°(-j\,5 \times 10)}{\Delta Z}$$

But I_2 being the current through $(2 + j\,3)\,\Omega$, as per the question, $I_2 = 0$.

i.e., $$(5 + j5)(-6\,V_2) - 30\,\angle 0° \times (-j\,50) = 0$$

$$-V_2\,(5 + j5) = 5\,\angle 0° \times (-j\,50) = -j\,250.$$

$$\therefore \quad V_2 = \frac{j\,250}{5 + j\,5} = 35.36\angle 45°\text{ V}.$$

Thus, $\mathbf{V_2 = 35.36\,\angle\,45°\text{ V}}$.

EXAMPLE 12.6 *Find the mesh transformation matrix of the given circuit (Fig. E12.6).*

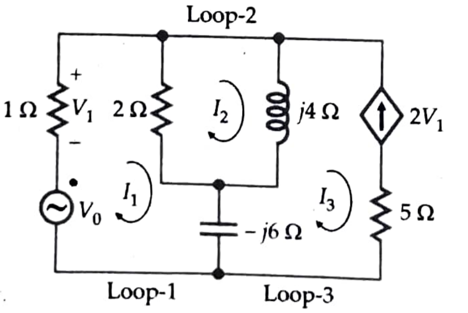


Fig. E12.6

SOLUTION. Let us first designate the loops as shown in the given figure.

Application of KVL around *loop-1* yields

$$-V_0 + 1.\,I_1 + 2\,(I_1 - I_2) + (-j\,6)(I_1 - I_3) = 0$$

or $$I_1\,(3 - j\,6) - 2\,I_2 + j\,6.\,I_3 = V_0 \quad ...(1)$$

Application of KVL at the *loop-2* yields

$$2\,(I_2 - I_1) + j\,4(I_2 - I_3) = 0$$

or $$-2\,I_1 + I_2\,(2 + j\,4) - (j\,4)\,I_3 = 0 \quad ...(2)$$

Presence of current source in the *third loop* makes the use of KVL redundant and using KCL,

$$I_3 = -2\,V_1 = -2\,I_1$$

or $$I_3 + 2\,I_1 = 0 \quad ...(3)$$

Thus, arranging (1), (2) and (3) in matrix form, we get

$$\begin{bmatrix} (3 - j6) & -2 & +j6 \\ -2 & (2 + j4) & (-j4) \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_0 \\ 0 \\ 0 \end{bmatrix}$$

Left hand matrix in the above eqn. gives *mesh transformation matrix*.

EXAMPLE 12.7 *Develop mesh equations for the network shown in Fig. E12.7 and find the power absorbed by the 3 Ω resistor.*

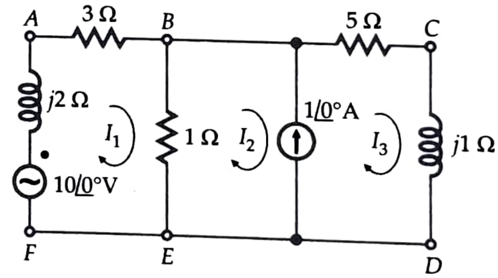


Fig. E12.7

SOLUTION. Application of mesh analysis in loop *ABEF* yields,

$$I_1\,(4 + j\,2) - 1 \times I_2 = 10\,\angle 0°$$

or $$(4 + j\,2)\,I_1 - I_2 = 10\,\angle 0° \quad ...(1)$$

Similarly, in *loop BCDE*, mesh analysis yields

$$(5 + j\,1)\,I_3 + I_2 - I_1 = 0$$

or $$-I_1 + I_2 + (5 + j\,1)\,I_3 = 0 \quad ...(2)$$

But it is evident from the given figure that

$$I_3 - I_2 = 1\angle 0° \quad ...(3)$$

Next, subtracting equation (3) from (1),

$$(4+j\,2)\,I_1 - I_3 = 10\,\angle 0^\circ - 1\,\angle 0^\circ = 9$$

or $$I_3 = [(4+j\,2)\,I_1 - 9] \qquad \ldots(4)$$

Also, adding equation (1) to (2), we get

$$(4+j\,2)\,I_1 - I_1 + (5+j\,1)\,I_3 = 10\,\angle 0^\circ$$

or $$(3+j\,2)\,I_1 + (5+j\,1)[(4+j\,2)\,I_1 - 9] = 10$$

or $$3I_1 + j2\,I_1 + (5+j1)(4+j2)\,I_1 - 9(5+j1) = 10$$

or $$3\,I_1 + j\,2\,I_1 + 20\,I_1 + j\,10\,I_1 + j\,4\,I_1 - 2\,I_1 - 45 - j\,9 = 10$$

or $$21\,I_1 + j\,16\,I_1 = 10 + 45 + j\,9 = 55 + j\,9$$

or $$I_1 = \frac{55+j\,9}{21+j\,16} = \frac{55.73\,\angle 9.29^\circ}{26.4\,\angle 37.3^\circ}$$

$$= 2.11\,\angle -28.01^\circ \text{ A.}$$

∴ Power absorbed by the 3 Ω resistor is

$$(2.11)^2 \times 3 = \mathbf{13.36\ W.}$$

EXAMPLE 12.8 *Find the drop across 2Ω resistor in the network of Fig. E12.8 using mesh analysis.*

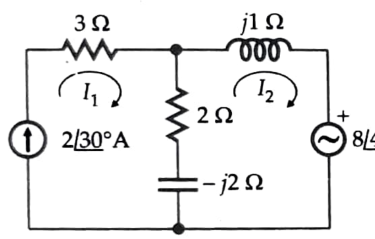


Fig. E12.8

SOLUTION. Due to presence of the current source, the loop current of the left loop is $I_1 = 2\angle 30^\circ$ A while that of the right loop is assumed to be I_2 A.

Obviously, the current through 2 Ω resistor is $(I_1 - I_2)$, the vector difference.

Application of KVL at the *right hand loop* yields

$$(j1)\,I_2 + (2-j\,2)\,I_2 - 2\,\angle 30\,(2-j2) = -8\,\angle 45^\circ$$

or $$I_2 = \frac{2\,\angle 30\,(2-j\,2) - 8\,\angle 45^\circ}{(2-j\,1)}$$

$$= 3.18\,\angle -65^\circ \text{ A.}$$

However, since the drop across the 2 Ω resistor is $2\,(I_1 - I_2)$ hence we find $V_{drop(2\Omega)} = 2\,(I_1 - I_2)$

$$= 2\,(2\,\angle 30^\circ - 3.18\,\angle -65^\circ)$$

$$= 2\,(0.38 + j\,3.88) = (0.76 + j\,7.76)\text{ V}$$

$$= \mathbf{7.80\,\angle 84.41^\circ\ V.}$$

12.3 SOURCE CONVERSIONS

When applying the network theorems in the a.c. networks, it may be necessary to convert a current source to voltage source and vice versa. This can be accomplished in the same manner as we do for d.c. networks, except now we shall be using phasors and impedances instead of real numbers and resistors.

12.4 NODAL ANALYSIS

Node voltage analysis of a.c. networks is identical to that of d.c. networks. In the frequency domain network having *n*-principal nodes, one of them is designated as the reference node and we require $(n-1)$ node voltage equations to solve for the desired result. Regarding *sign convention* of nodal currents, we take the currents entering the nodes as –ve while the currents leaving the nodes are +ve.

12.5 INDEPENDENT AND DEPENDENT SOURCES

The term *independent* indicates that the magnitude of the source is independent of the network to which it is applied and that it exhibits its terminal characteristics even if completely isolated.

A *dependent* or *controlled* source is that whose magnitude is governed by a current or voltage of the system in which it is situated.

The diagrammatic representation of the dependent sources is similar to that we have shown in analysing the d.c. network.

EXAMPLE 12.9 *Convert the given current source (Fig. E12.9) to voltage source.*

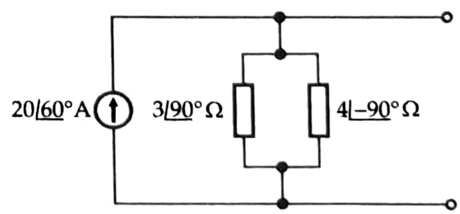


Fig. E12.9

SOLUTION. $Z_{\text{equivalent}}$ (Fig. E12.9)

$$= \frac{3\,\angle 90^\circ \times 4\,\angle -90^\circ}{3\,\angle 90^\circ + 4\,\angle -90^\circ}$$

$$= \frac{j\,3 \times (-j\,4)}{j\,3 - j\,4} = \frac{12}{-j} = 12\,\angle 90^\circ\,\Omega$$

$\therefore$ V (equivalent voltage source)

$$= I \times Z_{\text{equivalent}}$$

$$= 20 \angle 60° \times 12 \angle 90° = 240 \angle 150° \text{ V}$$

Thus, the equivalent voltage source of the given current source becomes **240 $\angle$150° V** [Fig. E12.9(a)].

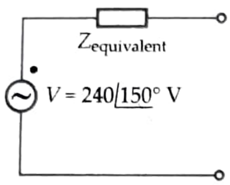


Fig. E12.9 (a)

EXAMPLE 12.10 *Convert the current source circuit (Fig. E12.10) to a single voltage source circuit.*

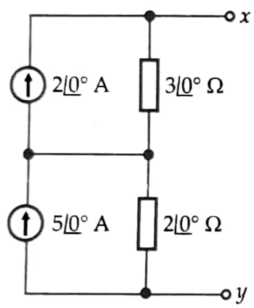


Fig. E12.10

SOLUTION. The current sources are converted into equivalent voltage source as shown in Fig. E12.10(a) and then to single equivalent voltage source as shown in Fig. E12.10(b).

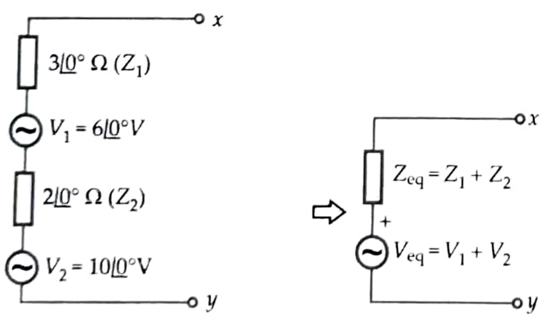


Fig. E12.10 (a) (b)

In Fig. E12.10(b),

$$V_{eq.} = V_1 + V_2 = 2 \angle 0° \times 3 \angle 0° + 5 \angle 0° \times 2 \angle 0°$$

$$= 6 \angle 0° + 10 \angle 0° = \mathbf{16 \angle 0° \text{ V}}$$

$$Z_{eq.} = 3 \angle 0° + 2 \angle 0° = \mathbf{5 \angle 0° \Omega}.$$

EXAMPLE 12.11 *Convert the given voltage source (Fig. E12.11) to a current source.*

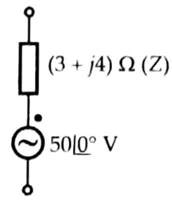


Fig. E12.11

SOLUTION. Figure E12.11(a) represents the conversion of voltage source to equivalent current source when,

$$I_{eq} = \frac{V}{Z} = \frac{50 \angle 0°}{(3 + j\,4)} = \frac{50 \angle 0°}{5 \angle 53.13°} = \mathbf{10 \angle -53.13° \text{ A}}.$$

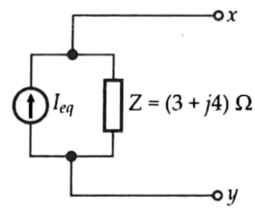


Fig. E12.11 (a)

EXAMPLE 12.12 *Transform the given circuit (Fig. E12.12) to the circuit having current source only.*

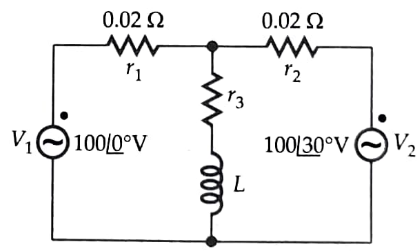


Fig. E12.12

SOLUTION. Figure E12.12(a) represents the conversion of the given circuit from voltage source fed circuit to current source fed circuit.

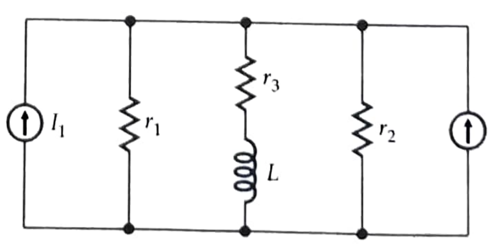


Fig. E12.12 (a)

Here, $\quad I_1 = V_1 / r_1 = \dfrac{100 \angle 0°}{0.02} = \mathbf{5000 \angle 0° \text{ A}}$

and $\quad I_2 = V_2 / r_2 = \dfrac{100 \angle 30°}{0.02} = \mathbf{5000 \angle 30° \text{ A}}.$

EXAMPLE 12.13 *Write the node transformation matrix of the network shown in Fig. E12.13.*

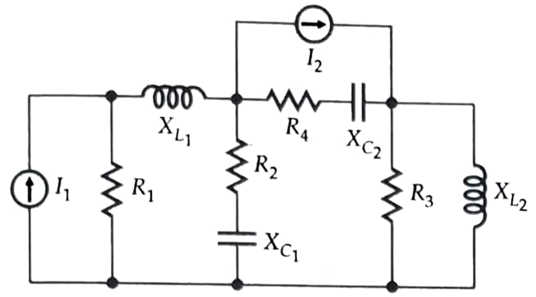


Fig. E12.13

SOLUTION. Let the given network be formatised as shown in Fig. E12.13(*a*) indicating the respective node voltages.

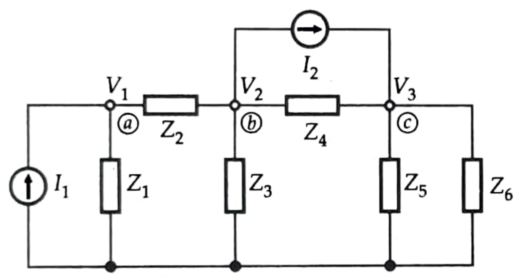


Fig. E12.13 (*a*)

The nodal equations are as follows :

At node (a),

$$\frac{V_1}{Z_1} + \frac{V_1 - V_2}{Z_2} = I_1$$

or $$V_1\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) - \frac{V_2}{Z_2} = I_1$$

or $$V_1\,(Y_1 + Y_2) - V_2\,Y_2 = I_1 \qquad \text{...(1)}$$

where $Y_1 = \dfrac{1}{Z_1}$; $Y_2 = \dfrac{1}{Z_2}$

At node (b),

$$\frac{V_2 - V_1}{Z_2} + \frac{V_2}{Z_3} + \frac{V_2 - V_3}{Z_4} + I_2 = 0$$

or $$-V_1 \cdot \frac{1}{Z_2} + V_2\left(\frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4}\right) - V_3 \cdot \frac{1}{Z_4} = -I_2$$

or $$-V_1\,Y_2 + V_2\,(Y_2 + Y_3 + Y_4) - V_3\,Y_4 = -I_2 \qquad \text{...(2)}$$

where $Y_3 = \dfrac{1}{Z_3}$; $Y_4 = \dfrac{1}{Z_4}$

At node (c),

$$\frac{V_3}{Z_5} + \frac{V_3}{Z_6} + \frac{V_3 - V_2}{Z_4} = I_2$$

or $$-V_2 \cdot \frac{1}{Z_4} + V_3\left(\frac{1}{Z_4} + \frac{1}{Z_5} + \frac{1}{Z_6}\right) = I_2$$

or $$-V_2\,Y_4 + V_3\,(Y_3 + Y_5 + Y_6) = I_2 \qquad \text{...(3)}$$

where $Y_5 = \dfrac{1}{Z_5}$; $Y_6 = \dfrac{1}{Z_6}$.

In matrix form, the three equations can be shown as

$$\begin{bmatrix} (Y_1 + Y_2) & (-Y_2) & 0 \\ (-Y_2) & (Y_2 + Y_3 + Y_4) & (-Y_4) \\ 0 & (-Y_4) & (Y_3 + Y_5 + Y_6) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_1 \\ -I_2 \\ I_2 \end{bmatrix}$$

The matrix in the left hand side is called *node transformation matrix.*

EXAMPLE 12.14 *Form node-transformation matrix for the given network using a.c nodal analysis (Fig. E12.14).*

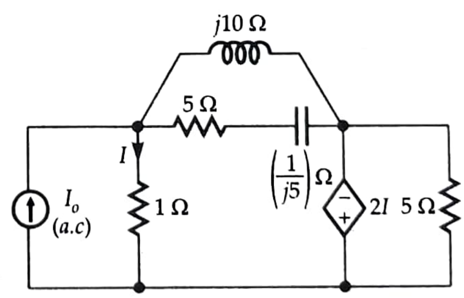


Fig. E12.14

SOLUTION. Let us first designate nodes as shown in Fig. E12.14(*a*) with assumed nodal voltages.

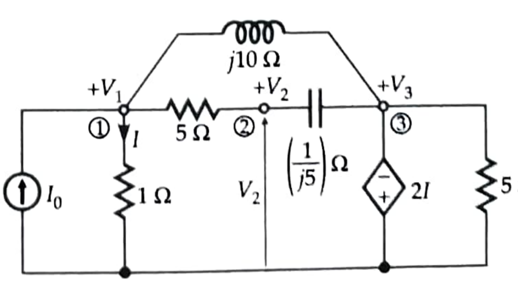


Fig. E12.14 (*a*)

At *node-1*, nodal equation can be written as

$$-I_0+\frac{V_1}{1}+\frac{V_1-V_2}{5}+\frac{V_1-V_3}{j\,10}=0$$

or $V_1\,(1+0.2-0.1\,j)+V_2\,(-0.2)+V_3\,(+0.1\,j)=I_0$

or $$V_1\,(1.2-j\,0.1)-0.2\,V_2+0.1\,j\,V_3=I_0 \quad ...(1)$$

At *node-2*, nodal equation can be written as

$$\frac{V_2-V_1}{5}+\frac{V_2-V_3}{\frac{1}{j\,5}}=0$$

or $-0.2\,V_1+0.2\,V_2+j\,5\,V_2-j\,5V_3=0$

or $-0.2\,V_1+V_2\,(0.2+j\,5)-j\,5\,V_3=0 \quad ...(2)$

At *node-3*, the nodal equation is

$$V_3=-2I=-2\left(\frac{V_1}{1}\right)=-2\,V_1$$

or $2V_1+0.V_2+V_3=0 \quad ...(3)$

Thus, in matrix form, equations (1), (2) and (3) are

$$\begin{bmatrix} (1.2-j0.1) & -0.2 & j0.1 \\ -0.2 & (0.2+j5) & -j5 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}=\begin{bmatrix} I_0 \\ 0 \\ 0 \end{bmatrix}$$

The left hand matrix is the required *node transformation matrix.*

EXAMPLE 12.15 *In the equivalent circuit of an op-amp (Fig. E12.15) obtain an expression for the output voltage V_L using nodal analysis.*

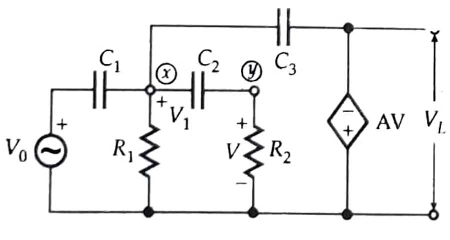


Fig. E12. 15

SOLUTION. Let the *nodes (x) and (y)* be marked in Fig. E12.15 assuming the node voltages to be V_1 and V (which has already been given).

Nodal analysis at *node* (x) yields

$$\frac{V_1-V_0}{1/j\,\omega C_1}+\frac{V_1}{R_1}+\frac{V_1-V}{1/j\omega C_2}+\frac{V_1+AV}{1/j\omega C_3}=0$$

or $$(V_1-V_0)\,j\omega C_1+V_1\cdot\frac{1}{R_1}+(V_1-V)\,j\,\omega C_2+(V_1+AV)\,j\omega C_3=0$$

or $$V_1\left(j\,\omega C_1+\frac{1}{R_1}+j\,\omega C_2+j\,\omega C_3\right)-V\,(j\,\omega C_2-Aj\omega C_3)=j\,\omega C_1\,V_0 \quad ...(1)$$

Nodal analysis at *node* (y) yields

$$\frac{V-V_1}{1/j\omega C_2}+\frac{V}{R_2}=0 \text{ or } (V-V_1)\,j\,\omega C_2+\frac{V}{R_2}=0$$

or $$-V_1\,(j\,\omega C_2)+V\left(j\,\omega C_2+\frac{1}{R_2}\right)=0 \quad ...(2)$$

In matrix form, (1) and (2) can be arranged as

$$\left[\begin{array}{c:c} \left(j\omega C_1+\dfrac{1}{R_1}+j\omega C_2+j\omega C_3\right) & -(j\omega C_2-Aj\omega C_3) \\ \hdashline (-j\,\omega C_2) & \left(j\omega C_2+\dfrac{1}{R_2}\right) \end{array}\right]$$

$$\begin{bmatrix} V_1 \\ V \end{bmatrix}=\begin{bmatrix} j\,\omega C_1\,V_0 \\ 0 \end{bmatrix}$$

$$\therefore \quad V=\frac{\left|\begin{array}{c:c} \dfrac{1}{R_1}+j\,\omega(C_1+C_2+C_3) & j\,\omega C_1\,V_0 \\ \hdashline -j\,\omega C_2 & 0 \end{array}\right|}{\left|\begin{array}{c:c} \dfrac{1}{R_1}+j\,\omega(C_1+C_2+C_3) & -(j\,\omega C_2-j\,\omega C_3\,A) \\ \hdashline -j\,\omega C_2 & j\,\omega C_2+\dfrac{1}{R_2} \end{array}\right|}$$

$$=\frac{-\omega^2\,C_1\,C_2\,V_0}{\left[\dfrac{1}{R_1}+j\omega\,(C_1+C_2+C_3)\right]\left(j\,\omega C_2+\dfrac{1}{R_2}\right)-j\,\omega C_2\,(j\,\omega C_2-j\,\omega C_3\,A)}$$

$$=\frac{-\omega^2\,C_1\,C_2\,R_1\,R_2\,V_0}{1+j\,\omega[R_2\,C_2+R_1\,(C_1+C_2+C_3)]-\omega^2\,R_1\,R_2\,C_2\,(C_1+C_3+C_3\,A)}\text{ V.}$$

By observation, it is revealed that

$$V_L\text{ (output voltage)}=-AV$$

$$\therefore \quad V_L=\frac{(-A)\,(-\omega^2\,C_1\,C_2\,R_1\,R_2\,V_0)}{\{1+j\,\omega[R_2\,C_2+R_1\,(C_1+C_2+C_3)]-\omega^2\,R_1\,R_2\,C_2\,(C_1+C_3+C_3\,A)\}}\text{ V.}$$

EXAMPLE 12.16 *Assuming the circuit variables to be a.c. phasor quantities, find V as well as V_0 using nodal analysis in the circuit of Fig. E12.16.*

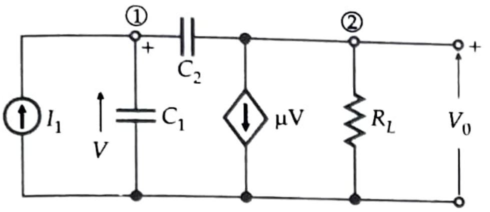


Fig. E12.16

SOLUTION. Application of nodal analysis at the input *node (1)*, nodal analysis yields

$$\frac{V}{1/j\,\omega C_1} + \frac{V - V_0}{1/j\,\omega C_2} - I_1 = 0$$

[Output node being (2) having nodal voltage V_0]

$$V_1\,(j\,\omega C_1) + (V - V_0)\,j\,\omega C_2 = I_1$$

or $$(j\,\omega C_1 + j\,\omega C_2)\,V - j\,\omega C_2\,V_0 = I_1 \qquad ...(1)$$

Similarly, at *node (2)*, nodal analysis reveals

$$\frac{(V_0 - V)}{1/j\,\omega C_2} + \mu\,V + \frac{V_0}{R_L} = 0$$

$$(V_0 - V)\,j\,\omega C_2 + \mu\,V + \frac{V_0}{R_L} = 0$$

or $$(\mu - j\,\omega C_2)\,V + \left(j\,\omega C_2 + \frac{1}{R_L}\right)V_0 = 0 \qquad ...(2)$$

In matrix form, (1) and (2) can be arranged as follows :

$$\begin{bmatrix} (j\,\omega C_1 + j\,\omega C_2) & (-j\,\omega C_2) \\ (\mu - j\,\omega C_2) & \left(j\,\omega C_2 + \dfrac{1}{R_L}\right) \end{bmatrix} \begin{bmatrix} V \\ V_0 \end{bmatrix} = \begin{bmatrix} I_1 \\ 0 \end{bmatrix}$$

Solving for V,

$$V = \frac{\begin{vmatrix} I_1 & -j\,\omega C_2 \\ 0 & j\,\omega C_2 + \dfrac{1}{R_L} \end{vmatrix}}{\begin{vmatrix} j\,\omega(C_1 + C_2) & (-j\,\omega C_2) \\ (\mu - j\,\omega C_2) & \left(j\,\omega C_2 + \dfrac{1}{R_L}\right) \end{vmatrix}}$$

$$= \frac{I_1\left(j\,\omega C_2 + \dfrac{1}{R_L}\right)}{j\omega(C_1+C_2)\left(j\omega C_2 + \dfrac{1}{R_L}\right) + j\omega C_2\,(\mu - j\omega C_2)}$$

and $$V_0 = \frac{\begin{vmatrix} j\,\omega(C_1 + C_2) & I_1 \\ (\mu - j\,\omega C_2) & 0 \end{vmatrix}}{\begin{vmatrix} j\,\omega(C_1 + C_2) & (-j\,\omega C_2) \\ (\mu - j\,\omega C_2) & \left(j\,\omega C_2 + \dfrac{1}{R_L}\right) \end{vmatrix}}$$

$$= \frac{-I_1\,(\mu - j\,\omega C_2)}{j\omega(C_1+C_2)\left(j\omega C_2 + \dfrac{1}{R_L}\right) + j\omega C_2\,(\mu - j\omega C_2)}.$$

EXAMPLE 12.17 *If $Z_L = -j\,2\,\Omega$, find V_L in the circuit of Fig. E12.17 using nodal analysis.*

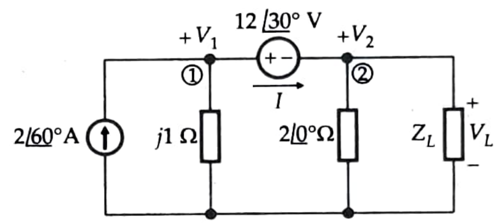


Fig. E12.17

SOLUTION. Let us mark nodes alongwith the node voltages in the given figure.

Applying nodal equations at respective nodes,

At node (1) :

$$\frac{V_1}{j\,1} - 2\,\angle 60° = -\,I_1 \qquad ...(1)$$

At node (2) :

$$\frac{V_2}{2} + \frac{V_2}{Z_L} + I_2 = 0$$

or $$\frac{V_1 - 12\,\angle 30°}{2} + \frac{V_1 - 12\,\angle 30°}{-j\,2} = -\,I_2 \qquad ...(2)$$

[$\because$ using KVL, $-V_1 + 12\,\angle 30° + V_2 = 0$
or, $V_2 = (V_1 - 12\,\angle 30°)$ V]

But actually $I_2 = -\,I_1$,

$\therefore$ (2) becomes $$\frac{V_1 - 12\,\angle 30°}{2} + \frac{V_1 - 12\,\angle 30°}{-j\,2} = I_1 \qquad ...(3)$$

Comparing (1) and (3),

$$\frac{V_1}{j\,1} - 2\,\angle 60° = -\left[\frac{V_1 - 12\,\angle 30°}{2} + \frac{V_1 - 12\,\angle 30°}{-j\,2}\right]$$

or $$\frac{V_1}{j\,1} + \frac{V_1 - 12\,\angle 30°}{2} + \frac{V_1 - 12\,\angle 30°}{-j\,2} = 2\,\angle 60°$$

CHAPTER 12

or $V_1\left(\frac{1}{j1}+\frac{1}{2}+\frac{1}{-j2}\right)=2\angle 60°+\frac{12\angle 30°}{2}-\frac{12\angle 30°}{j2}$

or $(0.71\angle -45°)\,V_1=3.2+j\,9.93$

$\therefore \quad V_1=14.7\angle 117°$ V.

But we have seen earlier,

$$V_2=(V_1-12\angle 30°)\text{ V}$$

$\therefore$
$$V_2=14.7\angle 117°-12\angle 30°$$
$$=-6.7+j\,13.07-10.4-j\,6$$
$$=-17.1+j\,7.07=18.5\angle 157.55°\text{ V.}$$

By inspection,

$$V_2=V_L=\mathbf{18.5\angle 157.55°\ V.}$$

EXAMPLE 12.18 *Using nodal analysis, find V_L in the circuit of Fig. E12.18.*

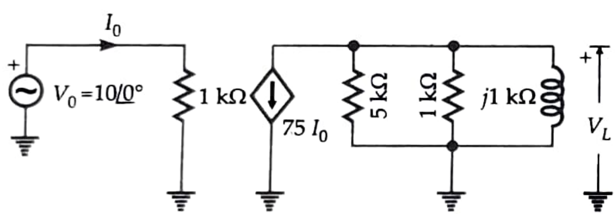


Fig. E12.18

SOLUTION. Let us simplify the figure as shown in Fig. E12.18 (*a*) where

$$Y_1=\frac{1}{5\times 10^3}=0.2\times 10^{-3}\angle 0°\text{ mho.}$$

$$Y_2=\frac{1}{1\times 10^3}=1\times 10^{-3}\angle 0°\text{ mho.}$$

$$Y_3=\frac{1}{j\,1\times 10^3}=1\times 10^{-3}\angle -90°\text{ mho,}$$
$$=-j\,(1\times 10^{-3})\text{ mho.}$$

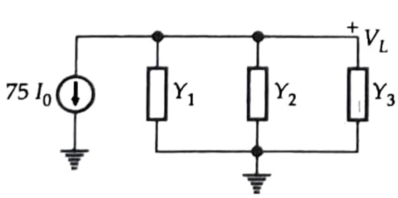


Fig. E12.18(*a*)

Application of nodal analysis at the load node (voltage being V_L),

$$-75\,I_0=V_L\,(Y_1+Y_2+Y_3)$$

[$\because$ current in $Y_1=V_1\,Y_1$, in $Y_2=V_2\,Y_2$ and in $Y_3=V_3\,Y_3$]

or
$$V_L=-\frac{75\,I_0}{Y_1+Y_2+Y_3}$$
$$=-\frac{75\,I_0}{0.2\times 10^{-3}+10^{-3}+(-j)\times 10^{-3}}$$
$$=-\frac{75\,I_0}{(1.2-j)10^{-3}}\text{ V.}$$

But it is evident from the given circuit that

$$I_0=\frac{V_0}{1\text{ k}\Omega}=\frac{10\angle 0°}{1000}=0.01\angle 0°\text{ A;}$$

$\therefore$
$$V_L=-\frac{75\times 0.01\times 10^3}{1.2-j}=-\frac{750\,(1.2+j)}{2.44}$$
$$=-307.38\,(1.2+j)$$
$$=(-368.86-j\,307.38)\text{ V}$$
$$=\mathbf{480\angle -140.2°\ V.}$$

12.6 THEVENIN'S AND NORTON'S THEOREM

The *Thevenin's theorem* is changed only in the a.c. application to include the impedance instead of resistance. Since the reactances of a circuit are frequency dependent, the Thevenin's circuit found for a particular network is applicable only at the specified frequency.

For independent and dependent sources (with a control variable *not* in the network under study) the application of the Thevenin's theorem in the a.c. circuits needs modification only for inclusion of impedances instead of resistances, as stated above, and using current and or voltage phasors. However, for dependent sources of the special type where the control variable is governed by the network variable, any of the following methods is to be applied to find the Thevenin's impedance.

Method-1

Removing the load impedance, $V_{o.c.}$ is obtained. Next, these terminals being shorted, the short circuit current ($I_{s.c.}$) is determined though the shorted link. Established network solution techniques like node voltage or loop current methods can be used for these purposes.

Then, $\quad Z_{int}=\frac{V_{o.c}}{I_{s.c}}$

Method-2

Open circuiting the load impedance, a source voltage (V) is applied. The source current (I), entering through the load terminals is determined, the original source voltage of the network being set to zero.

The Thevenin impedance is then given by

$$Z_{\text{int}} = \frac{V}{I}$$

where $V \equiv V_{o.c}$

In the application of *Norton's theorem* in a.c. circuits, the resistances are replaced by impedances, the circuit variables being current and or voltage phasors. For independent sources and dependent sources (where the control variable does not contain any network variable) the internal impedance of the network (being same to that for Thevenin's theorem application) can be determined by the usual procedure as described for d.c. networks. On the other hand, where the network does have the dependent sources, whose control variables depend on network parameters, the way of finding the internal impedance of the network is same as the two methods described for Thevenin's network.

Also, the Norton's equivalent circuit, like the Thevenin's equivalent circuit, is applicable only on one frequency since the reactances are frequency dependent.

The Norton's and Thevenin's equivalent circuits can also be found from each other by using the source transformation as illustrated in Fig. 12.1.

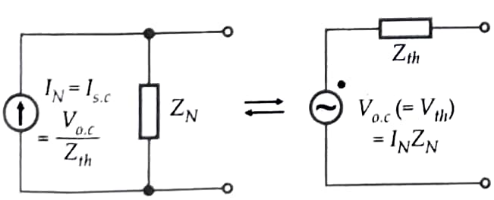


Fig. 12.1 Conversion of Norton to Thevenin Equivalent circuit and vice versa

$[Z_N \equiv Z_{th} \equiv Z_{\text{int}} : V_{Th} \equiv V_{o.c} ; I_N \equiv I_{s.c}]$

EXAMPLE 12.19 *If $I = 33\angle -13°$ A, find the Thevenin's equivalent circuit to the left of terminals x-y in the network of Fig. E12.19.*

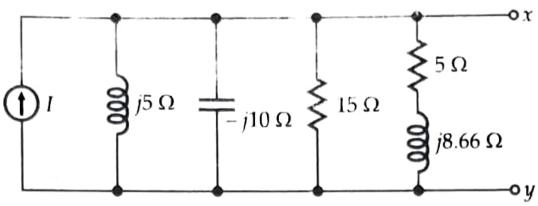


Fig. E12.19

SOLUTION. Let us first find the equivalent impedance across the current source. However, assuming the equivalent admittance to be Y_{eq}, we find that

$$Y_{eq} = Y_1 + Y_2 + Y_3 + Y_4$$

where Y_1 .... Y_4 is the branch admittance of each branch.

Obviously,

$$Y_1 = \frac{1}{j5} = -j\,0.2 \text{ mho};$$

$$Y_2 = \frac{1}{-j10} = j\,0.1 \text{ mho}$$

$$Y_3 = \frac{1}{15} = 0.067 \text{ mho}$$

$$Y_4 = \frac{1}{5 + j\,8.66} = \frac{1}{10\angle 60°}$$

$$= (0.05 - j\,0.0866) \text{ mho},$$

$$\therefore \quad Y_{eq.} = (0.117 - j\,0.1866) \text{ mho}$$

Then,

$$V_{x-y}\ (= V_{o.c}) = I / Y_{eq}$$

$$= 33\angle -13 / (0.117 - j\,0.1866)$$

$$= 33\angle -13 / 0.22\angle -58°$$

$$= 150\angle 45° \text{ V}.$$

To find Z_{in} (= Z_{Th}), the current source is deactivated and by inspection it is observed that

$$Z_{in} = \frac{1}{Y_{eq}} = \frac{1}{0.117 - j\,0.1866}$$

$$= \frac{1}{0.22\angle -58°} = 4.545\angle 58° \text{ ohm}.$$

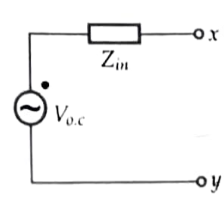


Fig. E12.19(a)

Figure E12.19 (*a*) represents the Thevenin's equivalent circuit where

$$Z_{in} = \mathbf{4.545 \angle 58° \Omega}$$

and $$V_{o.c} = \mathbf{150 \angle 45° \, V.}$$

EXAMPLE 12.20 *Using Thevenin's theorem, find the Thevenin's equivalent circuit of the network across x-y in Fig. E12.20.*

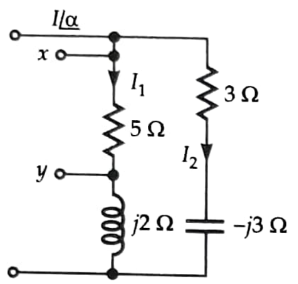


Fig. E12.20

SOLUTION. Given : $I_2 = 15 \angle 0°$ A

$\therefore$ Drop across $(3 - j\,3)\,\Omega$ branch is

$$V_{drop} = (3 - j\,3)\,15 \angle 0° = 63.6 \angle -45° \text{ V}$$

This gives

$$I_1 = \frac{V_{drop}}{5 + j\,2} = \frac{63.6 \angle -45°}{5 + j\,2}$$

or $$I_1 = 11.8 \angle -66.8° \text{ A} = (4.64 - j\,10.85) \text{ A}$$

$\therefore$ $V_{o.c}$(*i.e.*, V_{x-y}, the drop across *xy* terminal) is given by

$$V_{o.c} = 5\,(4.64 - j\,10.85)$$

$$[\because V_{o.c} = (I_1 \times 5\,\Omega) \text{ V}]$$

$$= 59 \angle -66.8 \text{ V}$$

In order to find Z_{int} $(= Z_{Th})$ we deactivate the current source and find the internal impedance (Z_{int}) across *x-y* looking into the circuit through *x-y* [Fig. E12.20(*a*)].

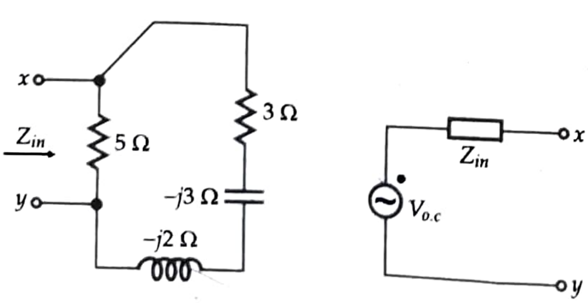


Fig. E12.20 (*a*) (*b*)

Here $$Z_{in} = (3 - j\,3 + j\,2) \,||\, 5$$

$$= \frac{(3 - j\,1)\,5}{3 - j\,1 + 5} = \frac{15 - j\,5}{8 - j\,1}$$

$$= \frac{15.81 \angle -18.42°}{8.06 \angle -7.125}\,\Omega$$

$$= 1.96 \angle -11.295° \Omega$$

Thus, Thevenin's equivalent circuit is obtained as shown in E12.20(*b*) where

$$Z_{in} = \mathbf{1.96\,\Omega \angle -11.295°}$$

$$V_{o.c.} = \mathbf{59 \angle -66.8° \, V.}$$

EXAMPLE 12.21 *Find the current through 10 Ω resistor using Thevenin's Theorem. (Fig. E12.21).*

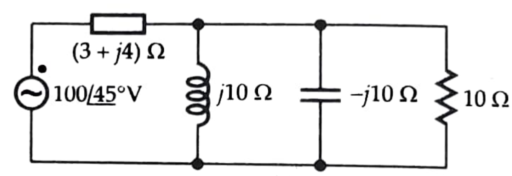


Fig. E12.21

SOLUTION. Let us first remove the 10 Ω resistor and designate the open circuit voltage across it by $V_{o.c}$ [Fig. E12.21 (*a*)]. Let the mesh currents be I_1 and I_2.

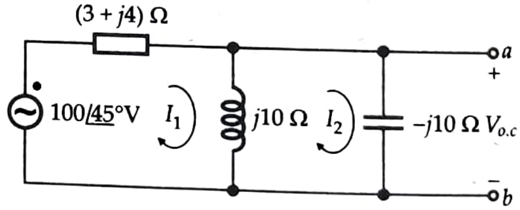


Fig. E12.21 (*a*)

The mesh equations are as follows :

$$(3 + j\,4)\,I_1 + j\,10 \,.\, I_1 - j\,10\,I_2 = 100 \angle 45°$$

or $$(3 + j\,14)\,I_1 - j\,10\,I_2 = 100 \angle 45° \qquad ...(1)$$

also, $$-j\,10\,I_1 + I_2\,(j\,10 - j\,10) = 0 \qquad ...(2)$$

From (2), $I_1 = 0$

Thus from (1),

$$I_2 = \frac{100 \angle 45°}{-j\,10} = 10 \angle 135° \text{ A}$$

and V_{a-b} $(= V_{o.c}) = -j\,10 \times 10 \angle 135° = 100 \angle 45°$ V.

It may be noted here since in the parallel branches, $X_L = X_C$, resonance would appear making $I_1 = 0$. Since $I_1 = 0$, $V_{o.c} \equiv V_{supply}$.

To find the internal impedance of the circuit through a-b, let us deactivate the voltage source [Fig. E12.21(b)].

$$Z_{\text{int}} = \frac{1}{Y_{\text{int}}} = \frac{1}{\dfrac{1}{-j\,10} + \dfrac{1}{j\,10} + \dfrac{1}{3 + j\,4}} = (3 + j\,4)\text{ ohm.}$$

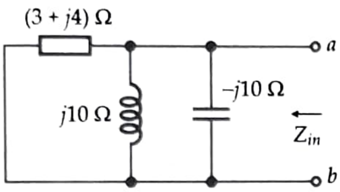


Fig. E12.21 (b)

In Fig. E12.21(c), the Thevenin's equivalent circuit is drawn where $Z_{in} = (3 + j\,4)\,\Omega$ and $V_{o.c} = 100\,\angle 45°$ V.

$$I_L = \frac{V_{o.c}}{Z_{in} + Z_L} = \frac{100\,\angle 45°}{(3 + j\,4) + 10} = 7.35\,\angle 28.3°\text{ A}$$

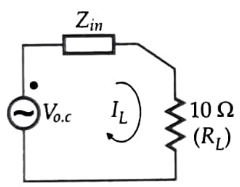


Fig. E12.21 (c)

Hence, **the current through the 10 Ω resistor is 7.35 ∠28.3° A**.

EXAMPLE 12.22 *Find Thevenin's equivalent circuit of the network shown in Fig. E12.22.*

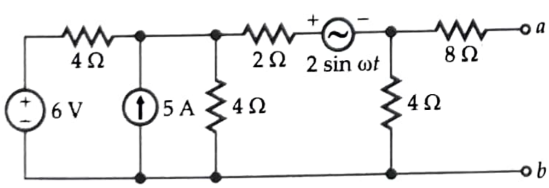


Fig. E12.22

SOLUTION. Let us redraw the given network with mesh currents indicated [Fig. E12.22(a)]. Let the open circuit voltage across a-b be $V_{o.c.}$ It may be noted here that the 5 A current source has been converted to necessary voltage source as shown in the figure.

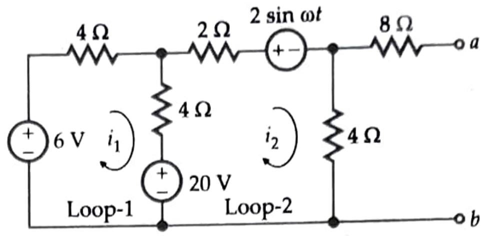


Fig. E12.22 (a)

In *loop 1*, mesh analysis gives

$$(4 + 4)\,i_1 - 4i_2 = 6 - 20 = -14$$

or $\quad 8i_1 - 4i_2 = -14 \qquad \text{...(1)}$

In *loop-2*, mesh analysis gives

$$i_2\,(2 + 4 + 4) - 4i_1 = 20 - 2\sin\omega t$$

or $\quad -4i_1 + 10\,i_2 = 20 - 2\sin\omega t$

or $\quad -2i_1 + 5i_2 = 10 - \sin\omega t \qquad \text{...(2)}$

From (1),

$$8\,i_1 = 4\,i_2 - 14$$

or $\quad i_1 = 0.5\,i_2 - 1.75 \qquad \text{...(3)}$

Using the value of i_1 from (3) in (2),

$$-2\,(0.5\,i_2 - 1.75) + 5i_2 = 10 - \sin\omega t$$

or $\quad -i_2 + 3.5 + 5\,i_2 = 10 - \sin\omega t$

or $\quad 4\,i_2 = 6.5 - \sin\omega t$

$\therefore \quad i_2 = 1.625 - 0.25\sin\omega t \qquad \text{...(4)}$

$\therefore \quad V_{o.c} = i_2 \times 4$

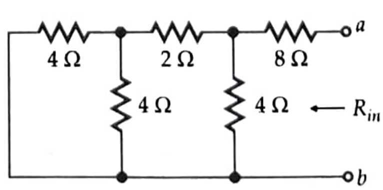
$$= V_{a-b}$$

$$= (1.625 - 0.25\sin\omega t)\,4$$

or $\quad V_{o.c} = 6.5 - \sin\omega t$

To find the Thevenin's internal resistance, the sources are replaced by their internal resistances and the network is then shown in Fig. E12.22(b).

$$R_{in} = [\{(4\,||\,4) + 2\}\,||\,4] + 8 = 10\,\Omega$$



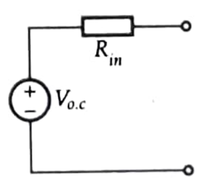


Fig. E12.22 (b) (c)

The Thevenin's equivalent circuit is formed where

$$\mathbf{V_{o.c.} = (6.5 - \sin\omega t)\ V}$$

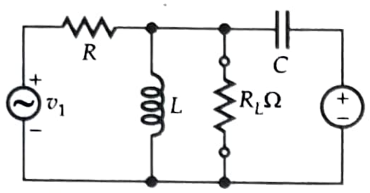
and $\quad \mathbf{R_{in}\ (= R_{th}) = 10\,\Omega.}$

[Refer to Fig. E12.22(c)].

EXAMPLE 12.23 *In the network of Fig. E12.23,*

$$v_1 = 4 \sin \omega t\ ;\quad v_2 = \cos \omega t.$$

Find the current through R_L using Thevenin's theorem at $t = 0.015$ sec.



[f = 50 Hz, R = 1 Ω, L = 0.1 H, C = 10 μF, R_L = 10 Ω]

Fig. E12.23

SOLUTION. Let R_L be removed and the circuit is drawn as shown in Fig. E12.23(a). Let the open circuit voltage be $v_{o.c}$

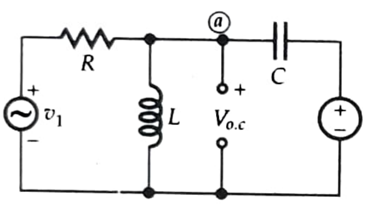


Fig. E12.23 (a)

Application of nodal analysis at node "a" gives :

$$\frac{v_{o.c} - v_2}{X_C} + \frac{v_{o.c}}{X_L} + \frac{V_{o.c} - v_1}{R} = 0$$

or $$v_{o.c}\left[\frac{1}{R} + \frac{1}{j\,\omega L} + j\,\omega C\right] - \frac{v_1}{R} - j\,\omega C\, v_2 = 0$$

$$\left[\because X_C = \frac{1}{j\omega C}\,;\, X_L = j\,\omega L\right]$$

or $$v_{o.c} = -\left(\frac{v_1}{R} + j\,\omega C\, v_2\right)\Big/\left(\frac{1}{R} + \frac{1}{j\,\omega L} + j\,\omega C\right)$$

However,

$$R = 1\Omega,\ v_1 = 4 \sin \omega t = 4 \sin 314\, t$$

$$v_2 = \cos 314\, t\ (= \cos \omega t)$$

$$\omega = 2\pi f = 2 \times \pi \times 50 = 314 \text{ rad/sec}$$

$$L = 0.1\text{ H}; C = 10\,\mu\text{ F}$$

$$\therefore\ v_{ac} = \frac{\left(\frac{4}{1}\sin 314\, t + j \times 314 \times 10^{-6} \times 10 \cos 314\, t\right)}{\left(\frac{1}{1} + \frac{1}{j\,314 \times 0.1} + j\,314 \times 10 \times 10^{-6}\right)}$$

$$= \frac{(4 \sin 314\, t + j\,3.14 \times 10^{-3} \cos 314\, t)}{1 + \frac{1}{j\,31.4} + j\,3.14 \times 10^{-3}}\text{ V}$$

$$= \frac{-0.099 \cos 314\, t + j\,125.6 \sin 314\, t}{0.9 + j\,31.4}$$

$$= [(-0.0891 + j\,3.1086) \cos 314\, t + (3943.84 + j\,113.04) \sin 314\, t] \times (1/986.77)$$

$$= [(-9.03 \times 10^{-5} + j\,3.15 \times 10^{-3}) \cos 314\, t + (4 + j\,0.115) \sin 314\, t]\text{ V}$$

Internal impedances can be obtained from Fig. E12.23(b).

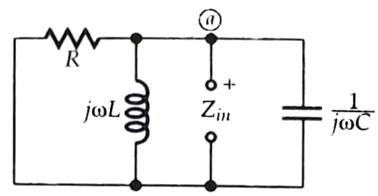


Fig. E12.23 (b)

Here the admittance

$$Y = Y_1 + Y_2 + Y_3 = \frac{1}{R} + \frac{1}{j\,\omega L} + \frac{1}{(1/j\,\omega C)}$$

or $$Y = \frac{1}{R} + \frac{1}{j\,\omega L} + j\,\omega C = \frac{1}{R} - \frac{j}{\omega L} + j\,\omega C$$

$$= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

$$= 1 + j\left(314 \times 10 \times 10^{-6} - \frac{1}{314 \times 0.1}\right)$$

$$= (1 - j\,0.0287)\text{ mho}$$

$$\therefore Z_{in} = \frac{1}{Y} = \frac{1}{1 - j\,0.00287}$$

$$= \frac{1 + j\,0.0287}{1.000824} = (0.99918 + j\,0.0287)\,\Omega$$

At $t = 0.015$ sec, the open circuit voltage (Thevenin's voltage) becomes

$$v_{o.c} = -9.03 \times 10^{-5} \cos 314 \times 0.015 + j\,0.115 \sin 314 \times 0.015 + j\,3.15 \times 10^{-3} \cos 314 \times 0.015 + 4.0 \sin 314 \times 0.015$$

$$= 2.16 \times 10^{-7} - 0.115\, j - j\,7.53 \times 10^{-6} - 4$$

$$= (-4 - j\,0.115)\text{ V}.$$

Thevenin's equivalent circuit being drawn in Fig. E12.23(c), at $t = 0.015$ sec,

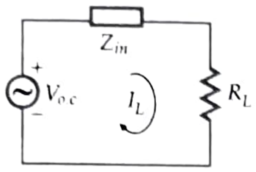


Fig. E12.23 (c)

$$I_L \text{ (current through } R_L) = \frac{v_{o.c.}}{Z_{in} + R_L}$$

$$= \frac{(-4 - j\,0.115)}{0.99918 + j\,0.0287 + 10}$$

$$= \frac{4\angle -178.35^\circ}{10.992\angle 0.15^\circ}$$

$$= \mathbf{0.364\angle -178.49^\circ\ A.}$$

Example 12.24 *Using Thevenin's theorem, find the current through the coil $(5 + j\,4)\,\Omega$ in the bridge circuit shown in Fig. E12.24.*

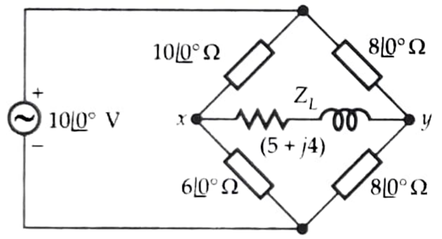


Fig. E12.24

Solution. Let us first remove Z_L [Fig. E12.24(a)].

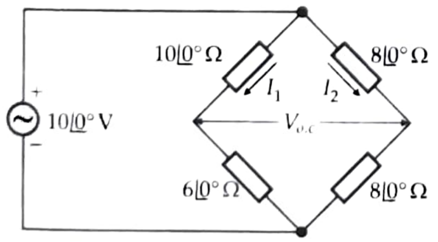


Fig. E12.24 (a)

$$I_1 = \frac{10\angle 0^\circ}{10 + 6} = 0.625\angle 0^\circ \text{ A};$$

$$I_2 = \frac{10\angle 0^\circ}{8 + 8} = 0.625\angle 0^\circ \text{ A.}$$

$$\therefore \quad V_{o.c} = V_{xy} \text{ (the Thevenin's voltage)}$$

$$= -6I_1 + 8I_2$$

$$= -6 \times 0.625\angle 0^\circ + 8 \times 0.625\angle 0^\circ$$

$$= -3.75\angle 0^\circ + 5.0\angle 0^\circ = 1.25\angle 0^\circ \text{ V}$$

To calculate Z_{int} (Thevenin's impedance), the voltage source is deactivated [Fig. E12.24(b)].

$$Z_{int} = \frac{10 \times 6}{10 + 6} + \frac{8 \times 8}{8 + 8} = 7.75\,\Omega$$

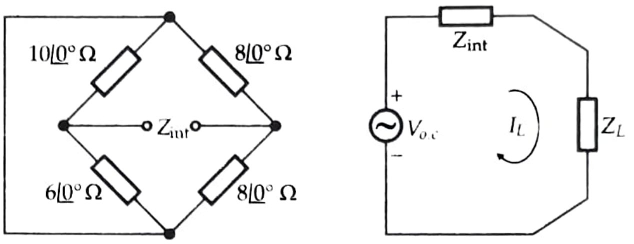


Fig. E12.24 (b) (c)

Thevenin's equivalent circuit can now be formed where,

$$V_{o.c} = 1.25\angle 0^\circ \text{ V} \quad \text{and} \quad Z_{int} = 7.75\angle 0^\circ\,\Omega$$

[Fig. E12.24(c)]

Here,

$$I_L = \frac{V_{o.c}}{Z_{int} + Z_L} = \frac{1.25\angle 0^\circ}{7.75 + 5 + j\,4} = \frac{1.25\angle 0^\circ}{13.36\angle 17.4^\circ}$$

$$= 0.094\angle -17.4^\circ \text{ A.}$$

The load current though Z_L would be **0.094 ∠ – 17.4° A.**

Example 12.25 *Find Thevenin's equivalent network at the left of terminals x-y in the network of Fig. E12.25.*

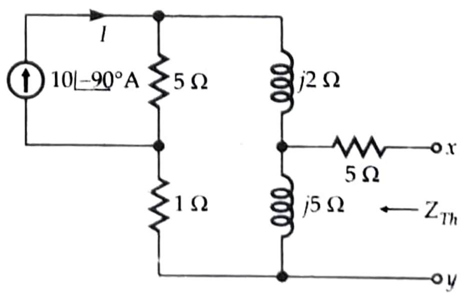


Fig. E12.25

Solution. Let us redraw the circuit as shown in Fig. E12.25(a). Voltage across x-y being $V_{o.c.}$, no current would flow through r_2. The current division has been shown in Fig. E12.25(a).

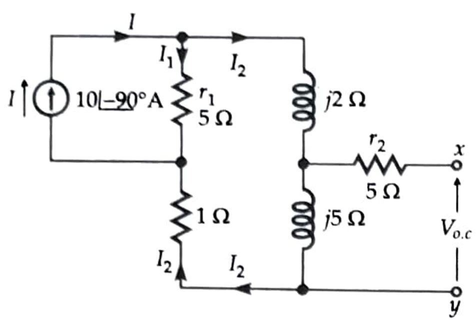


Fig. E12.25 (*a*)

Here $I_2 = I \dfrac{r_1}{r_1 + 1 + j\,7}$ A

$$= 10\ \angle -90^\circ \frac{5}{6 + j\,7} = \frac{50\ \angle -90^\circ}{9.22\ \angle 49.40^\circ}$$

$$= 5.423\ \angle -139.4^\circ \text{ A.}$$

$$\therefore \quad V_{o.c} = I_2 \times j\,5 = 5.423\ \angle -139.4^\circ \times 5\ \angle 90^\circ$$

$$= 27.115\ \angle -49.4^\circ \text{ V.}$$

The internal impedance across x-y can be obtained as shown in Fig. E12.25(*b*).

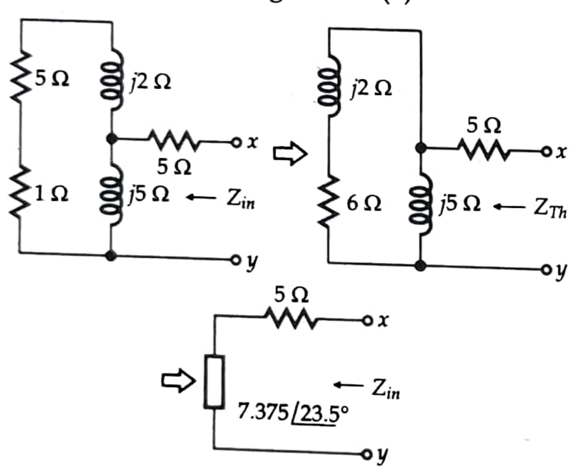


Fig. E12.25 (*b*)

Here, $Z_{in} = 5 + \dfrac{j\,5\,(6 + j\,2)}{6 + j\,7} = 5 + \dfrac{j\,30 - 10}{6 + j\,7}$

$$= \frac{30 + j\,35 + j\,30 - 10}{6 + j\,7} = \frac{20 + j\,65}{6 + j\,7}$$

$$= \frac{68\ \angle 72.9^\circ}{9.22\ \angle 49.4^\circ} = 7.375\ \angle 23.5^\circ\ \Omega.$$

The Thevenin's equivalent circuit is shown in Fig. E12.25(c) where

$\mathbf{V_{o.c}} = \mathbf{27.115\ \angle -49.4^\circ\ V}$

$\mathbf{Z_{int}} = \mathbf{7.375\ \angle 23.5^\circ\ \Omega.}$

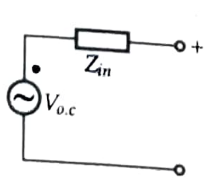


Fig. E12.25 (c)

EXAMPLE 12.26 *Find Norton's equivalent circuit for the network shown in Fig. E12.26 at the left of terminals 1-2. Assume $I = 5\ \angle 0^\circ$ A.*

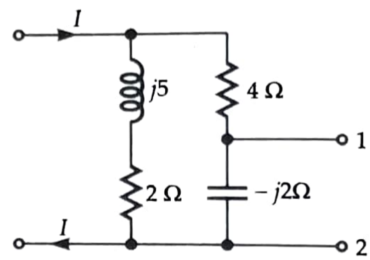


Fig. E12.26

SOLUTION. Terminals 1-2 are shorted [Fig. E12.26(*a*)]. No current would flow through $(-j\,2\ \Omega)$ capacitive reactance due to shorting of terminal 1-2.

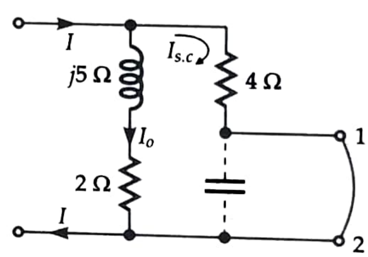


Fig. E12.26 (*a*)

Obviously,

$$I_{s.c} = \frac{(2 + j\,5)}{(2 + j\,5 + 4)} \times I = \frac{5\ \angle 0^\circ\,(2 + j\,5)}{6 + j\,5}$$

$$= \frac{10 + j\,25}{6 + j\,5} = \frac{26.926\ \angle 68.198^\circ}{7.81\ \angle 39.81^\circ}$$

$$= 3.45\ \angle 28.39^\circ \text{ A.}$$

Internal impedance can be obtained from Fig. E12.26(*b*) deactivating the current source.

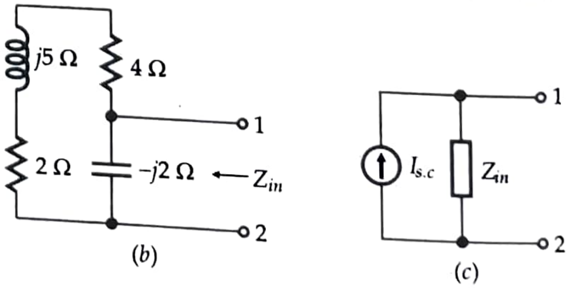


Fig. E12.26

$$Z_{in} = \frac{-j\,2\,(4 + 2 + j\,5)}{-j\,2 + 4 + j\,5 + 2} = \frac{-j\,8 - j\,4 + 10}{6 + j\,3}$$

$$= \frac{10 - j\,12}{6 + j\,3} = \frac{15.62\ \angle -50.19^\circ}{6.71\ \angle 26.56^\circ}$$

$$= 2.33\ \angle -76.75^\circ\ \Omega.$$

Norton's equivalent circuit is shown in Fig. E12.26(c) where

$$I_{s.c} = \mathbf{3.45 \angle 28.39° \ A;}$$
$$Z_{in} = \mathbf{2.33 \angle -76.75° \ \Omega.}$$

EXAMPLE 12.27 *Find Thevenin's equivalent network across x-y (Fig. E12.27).*

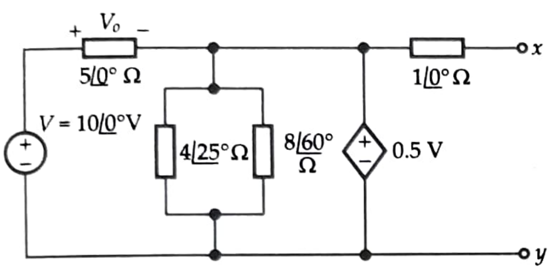


Fig. E12.27

SOLUTION. The circuit is simplified as shown in Fig. E12.27(a)

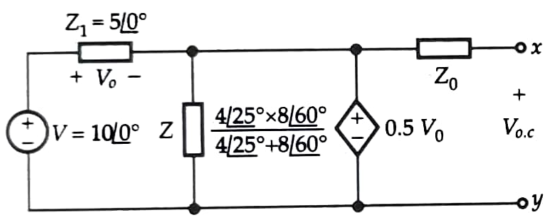


Fig. E12.27 (a)

Here
$$Z = \frac{4 \angle 25° \times 8 \angle 60°}{4 \angle 25° + 8 \angle 60°}$$

$$= \frac{32 \angle 85°}{7.625 + j\,8.62} = \frac{32 \angle 85°}{11.51 \angle 48.5°}$$

$$= 2.78 \angle 36.5° \Omega$$

Let the current through Z be I.

By inspection, $IZ = 0.5\ V_0 = V_{o.c}$

[$\because$ no current flows through Z_0]

Also, in the leftmost loop,

$$-V + V_0 + 0.5\ V_0 = 0 \qquad [\because IZ = 0.5\ V_0]$$

or
$$V_0 = \frac{V}{1.5} = \frac{10 \angle 0°}{1.5} = 6.67 \angle 0° \text{ V.}$$

$$\therefore \quad V_{o.c} = 0.5\ V_0 = 0.5 \times 6.67 \angle 0°$$

$$= \mathbf{3.335 \angle 0° \ V.}$$

To find Z_{int}, a current source of $1 \angle 0°$ A is applied at x-y deactivating the voltage source [Fig. E12.27(b)].

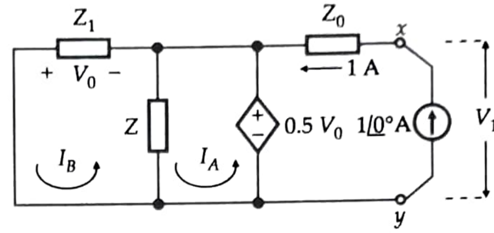


Fig. E12.27 (b)

Here, $-V_1 + 1 \angle 0 \times 1 \angle 0° + 0.5\ V_0 = 0$

or $\qquad V_1 = 0.5\ V_0 + 1 \qquad$ ...(1)

Also, $\qquad -0.5\ V_0 + (I_A - I_B)\,Z = 0$

or $\qquad 0.5\,(I_B\ Z_1) + (I_A - I_B)\,Z = 0$

$[\because -I_B\ Z_1 = V_0\ ]$

or $\qquad I_B\,(0.5\ Z_1 - Z) + I_A\ Z = 0 \qquad$ ...(2)

Again in the leftmost loop,

$$I_B\ Z - I_A\ Z + I_B\ Z_1 = 0$$

or $\qquad -I_A\ Z + I_B\,(Z + Z_1) = 0 \qquad$ ...(3)

Using (2) in (3),

$$I_B\,(0.5\ Z_1 - Z) + I_B\,(Z + Z_1) = 0$$

or $\qquad 1.5\ I_B\ Z_1 = 0 \qquad \therefore\ I_B = 0$

Thus from (1),

$$V_1 = 1 \text{ or } \frac{V_1}{1} = 1\Omega \quad \textit{i.e.,}\ \mathbf{Z_{int} = 1\Omega.}$$

Figure E12.27(c) represents the Thevenin's equivalent source across x-y where $V_{o.c} = 3.335 \angle 0°$ and $Z_{int} = 1\Omega$.

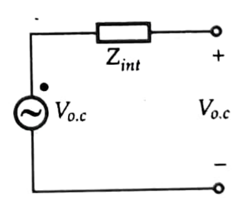


Fig. E12.27 (c)

EXAMPLE 12.28 *The Thevenin's equivalent voltage and Thevenin's equivalent impedance of a circuit are 40 ∠30° volts and 10 ∠30° Ω respectively (Fig. E12.28). Determine the Norton equivalent circuit.*

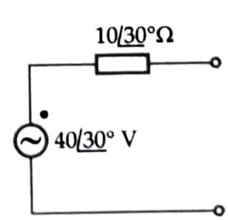


Fig. E12.28

CHAPTER 12

SOLUTION. V_{Th} (Thevenin's voltage)

$$= 40\ \angle 30°\ \text{V};$$

Z_{Th} (Thevenin's impedance) $= 10\ \angle 30°\ \Omega$

$$\therefore \quad I_{Nor} = \frac{V_{Th}}{Z_{Th}} = \frac{40\ \angle 30°}{10\ \angle 30°} = \mathbf{4\ \angle 0°\ A.}$$

Obviously, Thevenin's impedance being equal to Norton's impedance,

$$\mathbf{Z_{Nor} = Z_{Th} = 10\ \angle 30°\ \Omega.}$$

EXAMPLE 12.29 *Find Thevenin's equivalent circuit of the network shown in Fig. E12.29 across the load.*

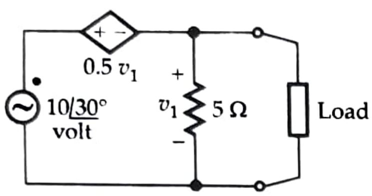


Fig. E12.29

SOLUTION. Removing the load, assuming the open circuit voltage to be $V_{o.c}$, it is seen that $V_{o.c} = v_1$. Applying KVL in the left loop of Fig. E12.29(*a*),

$$-10\ \angle 30° + 0.5\ v_1 + v_1 = 0$$

or $$1.5\ v_1 = 10\angle 30°$$

$$\therefore \quad v_1 = 6.67\ \angle 30°\ \text{V} \quad (= V_{o.c}).$$

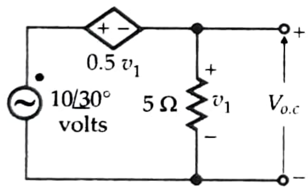


Fig. E12.29 (*a*)

Applying short circuit across the output, following Fig. E12.29(*b*), it is evident that $v_1 = 0$ (since the output is short-circuited). This makes the dependent source also to be zero.

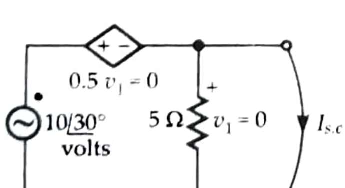


Fig. E12.29 (*b*)

$$\therefore \quad I_{s.c} = \frac{10\angle 30°\ \text{V}}{0} \quad \text{[Since there is no other impedance in the loop]}$$

i.e., $$I_{s.c} = \infty.$$

$$\therefore \quad Z_{int} = \frac{V_{o.c}}{I_{s.c}} = \frac{V_{o.c}}{\infty} = 0\ \Omega.$$

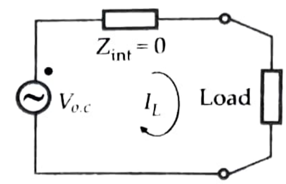


Fig. E12.29 (*c*)

Thevenin's equivalent circuit is then shown in Fig. E12.29(*c*).

$$\left[I_L = \frac{V_{o.c.}}{Z_{int} + Z_{Load}} \right]$$

EXAMPLE 12.30 *Find the Thevenin's equivalent network for the network shown in Fig. E12.30 of the previous worked out example if the 10 ∠30° voltage source has internal resistance of 1 Ω.*

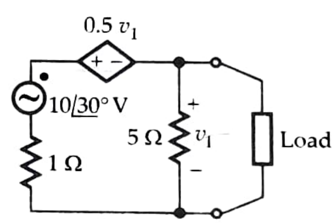


Fig. E12.30

SOLUTION. Let us redraw the network with 1 Ω internal resistance of the 10 ∠30° V source [Fig. E12.30(*a*)].

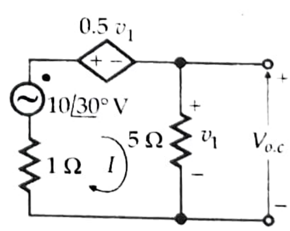


Fig. E12.30 (*a*)

Next, the load being removed, the loop equation in the left loop gives [Fig. E12.30(*a*)]

$$-10\ \angle\ 30° + 0.5\ v_1 = 5\ I + 1\ I$$

However, $I = \dfrac{V_{o.c}}{5}$;

$$\therefore \; -10(\cos 30^\circ + j\sin 30^\circ) + 0.5\,v_1$$

$$= 5.\frac{V_{o.c.}}{5} + 1.\frac{V_{o.c.}}{5}$$

or $\quad -8.66 - j\,5 = V_{o.c} - 0.5\,V_{o.c} + 0.2\,V_{o.c.}$

$$[\because V_{o.c} \equiv v_1]$$

or $\quad V_{o.c} = (-8.66 - j\,5)/0.7$

$$= -12.37 - j\,7.14 \text{ V}$$

To find Z_{Th}, let the output be shorted [Fig. E12.30(*b*)]. No current passes through $5\,\Omega$ resistor and $v_1 = 0$. This leads to $0.5\,v_1$ to be zero also.

$$\therefore \quad I_{s.c.} = \frac{10\angle 30^\circ}{1\Omega} = 10\angle 30^\circ \text{ A}$$

Thus, $Z_{int} = \dfrac{V_{o.c.}}{I_{s.c.}} = \dfrac{-12.37 - j\,7.14}{8.66 + j\,5}$

$$\therefore \quad Z_{int} = \frac{14.28\angle -150^\circ}{9.999\angle 30^\circ} = 1.43\angle -180^\circ.$$

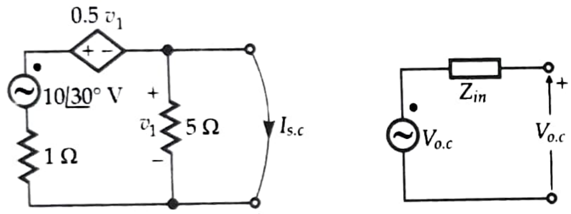


Fig. E12.30 (*b*) (*c*)

Thevenin's circuit is drawn in Fig. E12.30(*c*) so that

$$\mathbf{V_{o.c}} = \mathbf{(-12.37 - j7.14)\ V}$$
$$\mathbf{Z_{int}} = \mathbf{1.43\angle -180^\circ\,\Omega.}$$

EXAMPLE 12.31 *Find Thevenin's equivalent network of the circuit shown across x-y and find the current through the $1\,\Omega$ resistor. Verify the result using Norton's theorem (Fig. E12.31).*

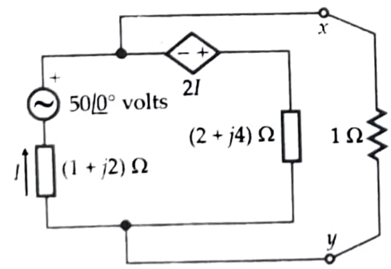


Fig. E12.31

SOLUTION. Let us remove $1\,\Omega$ resistor from x-y and keep it open circuited [Fig. E12.31(*a*)].

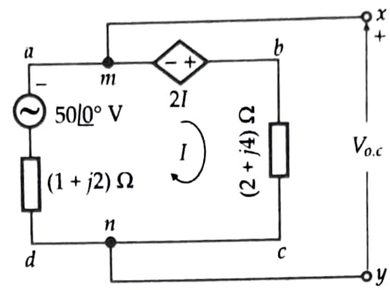


Fig. E12.31 (*a*)

Application of loop equation in loop *abcd* yields

$$-50\angle 0^\circ - 2I + I(2 + j\,4 + 1 + j\,2) = 0$$

or $\quad -50 = 2I - I(3 + j\,6)$

or $\quad I(-1 - j\,6) = -50$

$$\therefore \quad I = \frac{50}{1 + j\,6} = (1.35 - j\,8.11) \text{ A}$$

Application of loop equation in *loop amnd* yields

$$(1 + j\,2)\,I - 50 + V_{o.c} = 0$$

or $\quad V_{o.c} = 50 - I(1 + j\,2)$

$$\therefore \quad V_{o.c} = 50 - (1.35 - j\,8.11)(1 + j\,2)$$
$$= 50 - [(17.57 - j\,5.41)]$$
$$= (32.43 + j\,5.41) \text{ V} = 32.88\angle 9.47^\circ \text{ V}.$$

Next, short circuit is applied across x-y [Fig. E12.31(*b*)]

$$I_1 = \frac{50\angle 0^\circ\, V}{(1 + j\,2)\Omega} = 22.36\angle -63.43^\circ \text{ A},$$

$$I_2 = \frac{2I_1}{(2 + j\,4)} = \frac{2 \times 22.36\angle -63.434^\circ}{4.472\angle 63.434^\circ}$$

$$= 10\angle -126.87^\circ \text{ A}$$

$$\therefore \quad I_{s.c} = I_1 - I_2$$
$$= 22.36\angle -63.43^\circ - 10\angle -126.87^\circ$$
$$= 10 - j\,20 + 6 + j\,8 = (16 - j\,12) \text{ A}$$
$$= 20\angle -36.87^\circ \text{ A}$$

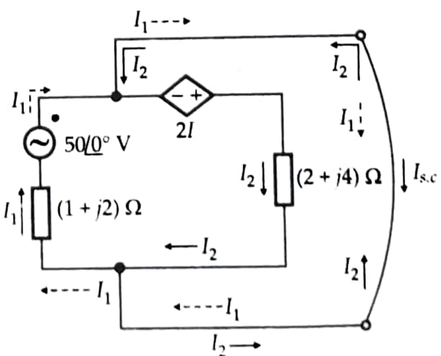


Fig. E12.31 (*b*)

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Thus, in Fig. E12.45(b),

$$I = \frac{10\angle 0°}{Z_2 + Z_{1(eq)} + Z_{1(eq)}}$$

$$= \frac{10\angle 0°}{8 + j3 + j3} = \frac{10\angle 0°}{(8 + j6)}\text{ A}$$

Observation reveals that

$$V_{o.c} = \text{drop across } (Z_{1(eq)} + Z_2)$$

$$= I(Z_{1(eq.)} + Z_2)$$

$$= \frac{10\angle 0°}{(8 + j6)} \times (8 + j3)$$

$$= \frac{8.54\angle 20.56° \times 10\angle 0°}{10\angle 36.87°}$$

$$= 8.54\angle -16.31°\text{ V.}$$

In order to find Z_{int} (Thevenin's impedance) looking across the open circuited terminals, we deactivate the voltage source of Fig. E12.45(b) as redrawn in Fig. E12.45(c).

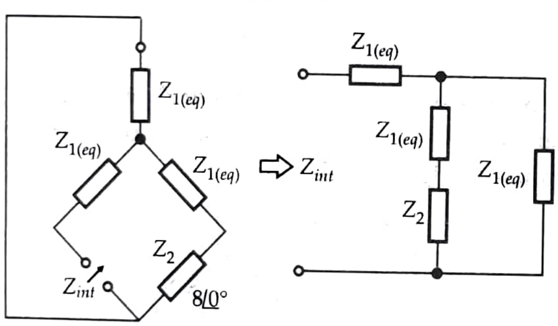

Fig. E12.45 (c)

Here

$$Z_{\text{int}} = Z_{1(eq)} + \frac{Z_{1(eq)}(Z_{1(eq)} + Z_2)}{Z_{1(eq)} + Z_{1(eq)} + Z_2}$$

$$= j3 + \frac{j3(j3 + 8)}{j3 + j3 + 8} = j3 + \frac{-9 + j24}{8 + j6}$$

$$= j3 + \frac{(-9 + j24)(8 - j6)}{64 + 36}$$

$$= j3 + 0.72 + j2.46$$

$$= (0.72 + j5.46)\,\Omega.$$

Thus, to have maximum power transfer, Z_L must be the complex conjugate of Z_{int}

i.e., $\quad Z_L = \mathbf{(0.72 - j5.46)\,\Omega}$.

Also, $P_{\max} = \dfrac{V_{o.c}^2}{4R_L} = \dfrac{(8.54)^2}{4 \times 0.72} = \mathbf{25.32\ W}$.

12.11 TELLEGEN'S THEOREM

The basic concept of the theorem being identical in d.c. or a.c. systems, for the application in a.c. systems, it can be stated as follows :

In any linear, non-linear, passive, active time variant network, excited by alternating sources, the summation of instantaneous or complex power of the sources is zero.

For a network excited by sinusoidal sources, if the number of branches be "b",

$$\sum_{b=1}^{b} v_b\, i_b = 0$$

where v_b and i_b represent the instantaneous voltage and current of source at each branch.

When considering the complex power, if V_b and I_b be the voltage and current of each branch, as per this theorem,

$$\sum_{b=1}^{b} V_b\, I_b^* = 0$$

where I_b^* is the complex conjugate of I_b.

12.11.1 Proof of Tellegen's Theorem

With reference to the network shown in Fig. 12.9, let the node voltages be V_1, V_2 and V_3 at nodes 1, 2 and 3 respectively. The current directions are shown arbitrarily.

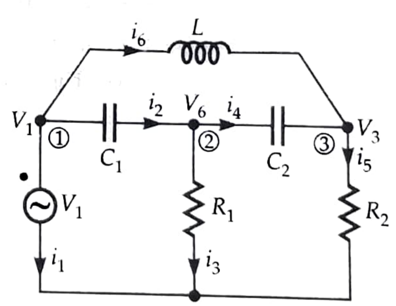

Fig. 12.9 Application of Tellegen's theorem.

The summation of instantaneous power in the network is given by

$$\sum_{b=1}^{b} v_b\, i_b = V_1 i_1 + V_{c_1} i_2 + V_{R_1} i_3 + V_{c_2} i_4 + V_{R_2} i_5 + V_L i_6$$

[where, V_{C_1}, V_{C_2}, V_L, V_{R_1} etc. indicate the respective voltages across the elements C_1, C_2, L, R_1 etc.]

or $$\sum_{b=1}^{b} v_b i_b = V_1 i_1 + (V_1 - V_2) i_2 + V_2 i_3 + (V_2 - V_3) i_4 + V_3 i_5 + (V_1 - V_3) i_6$$
$$= V_1 (i_1 + i_2 + i_6) + V_2 (i_3 - i_2 + i_4) + V_3 (i_5 - i_4 - i_6)$$

Application of KCL at *node (1)* reveals that $i_1 + i_2 + i_6 = 0$, at *node (2)*, $i_3 + i_4 - i_2 = 0$ and at *node (3)*, $i_5 - i_4 - i_6 = 0$.

Thus finally,

$$\sum_{b-1}^{b} v_b i_b = V_1 \times 0 + V_2 \times 0 + V_3 \times 0 = 0$$

Hence the theorem is proved.

EXAMPLE 12.46 *In the network shown in Fig. E12.46, $V_1 = 10$ V, $V_2 = 4$ V, $V_4 = 6$ V. Also it is given that $I_1 = 2$ A $= I_2$ while $I_3 = 4$ A. Check the validity of Tellegen's theorem.*

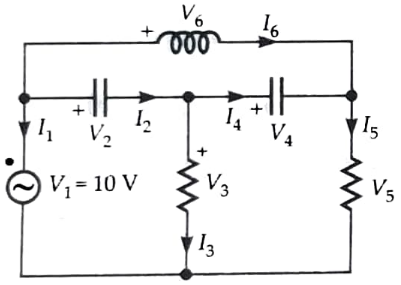

Fig. E12.46

SOLUTION. Let us redraw the given circuit with node representation [Fig. E12.46(*a*)] and loop formations.

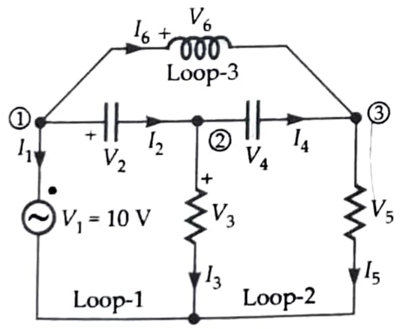

Fig. E12.46 (*a*)

In *loop-1*, KVL yields

$$-V_1 + V_2 + V_3 = 0$$

or $$V_3 = V_1 - V_2 = 10 - 4 = 6 \text{ V.}$$

In *loop-2*, KVL yields

$$-V_3 + V_4 + V_5 = 0$$

or $$V_5 = V_3 - V_4 = 6 - 6 = 0 \text{ V.}$$

Similarly, for *loop-3*, KVL yields

$$V_6 - V_4 - V_2 = 0$$

or $$V_6 = V_4 + V_2 = 6 + 4 = 10 \text{ V.}$$

Next, application of KCL at *node (1)* yields :

$$I_2 + I_1 + I_6 = 0$$

$\therefore$ $$I_6 = -I_2 - I_1 = -4 \text{ A}$$

Also, application of KCL at *node (2)* yields

$$I_2 = I_3 + I_4$$

or $$I_4 = I_2 - I_3 = 2 - 4 = -2 \text{ A.}$$

Similarly, for node (3),

$$I_5 = I_4 + I_6 \text{ or } I_5 = -2 + (-4) = -6 \text{ A}$$

$\therefore$ Summation of branch voltages and currents are

$$\sum_{b=1}^{b} v_b i_b = V_1 I_1 + V_2 I_2 + V_3 I_3 + V_4 I_4 + V_5 I_5 + V_6 I_6$$
$$= 10 \times 2 + 4 \times 2 + 6 \times 4 + 6 \times (-2) + (0)(-6) + 10 \times (-4)$$
$$= 20 + 8 + 24 - 12 + 0 - 40 = 0.$$

Hence the **Tellegen's theorem is Verified.**

It may be noted that for the application of Tellegen's theorem, it is also true that when voltages and currents in the network branches are measured at different instants t_1 and t_2, the set of voltage corresponding to time t_1 can be paired to the set of currents corresponding to time t_2 and vice versa.

i.e., $$\sum_{b=1}^{b} v_b(t_1)\, i_b(t_2) = \sum_{b=1}^{b} v_b(t_2)\, i_b(t_1) = 0.$$

ADDITIONAL EXAMPLES

EXAMPLE 12.47 *Find the current coming out from $150\angle 120°$ volts source in Fig. E12.47 by mesh current analysis.*

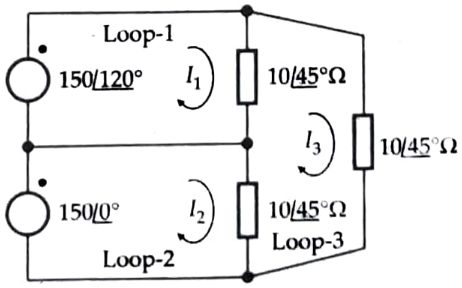

Fig. E12.47

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SOLUTION. Applying mesh analysis in loop 1,

$$10\angle 45^\circ (I_1) - 10\angle 45^\circ (I_3) = 150\angle 120^\circ \quad \ldots(1)$$

then for loop 2 mesh analysis gives

$$(I_2)10\angle -45^\circ - (I_3)10\angle -45^\circ = 150\angle 0^\circ \quad \ldots(2)$$

Loop analysis for loop 3 gives

$$(-I_1)10\angle 45^\circ + (-I_2)10\angle -45^\circ + (I_3)30\angle 45^\circ = 0 \quad \ldots(3)$$

Rearranging the three equations in matrix form,

$$\begin{vmatrix} 10\angle 45^\circ & 0 & -10\angle 45^\circ \\ 0 & 10\angle -45^\circ & -10\angle -45^\circ \\ -10\angle 45^\circ & -10\angle -45^\circ & 30\angle 45^\circ \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \\ I_3 \end{vmatrix} = \begin{vmatrix} 150\angle 120^\circ \\ 150\angle 0^\circ \\ 0 \end{vmatrix}$$

The determinant of Z of the above equation is given by

$$\Delta Z = 10\angle 45^\circ [10\angle -45^\circ \times 30\angle 45^\circ - 10\angle -45^\circ \times 10\angle -45^\circ]$$
$$- 10\angle 45^\circ [10\angle -45^\circ \times 10\angle 45^\circ]$$

$$= 10\angle 45^\circ [300\angle 0^\circ - 100\angle -90^\circ]$$
$$- 10\angle 45^\circ \times 100\angle 0^\circ$$

$$= 3000\angle 45^\circ - 1000\angle -45^\circ - 1000\angle 45^\circ$$
$$= 1000\angle 135^\circ$$

$$\therefore I_1 = I_A = \frac{\begin{vmatrix} 150\angle 120^\circ & 0 & -10\angle 45^\circ \\ 150\angle 0^\circ & 10\angle -45^\circ & -10\angle -45^\circ \\ 0 & -10\angle -45^\circ & 30\angle 45^\circ \end{vmatrix}}{\Delta Z}$$

[I_A is the current coming out of $150\angle 120^\circ$ V source]

$$= \frac{150\angle 120^\circ [10\angle -45^\circ \times 30\angle 45^\circ - 10\angle -45^\circ \times 10\angle -45^\circ] + (-10\angle 45^\circ)[-10\angle -45^\circ \times 150\angle 0^\circ]}{\Delta Z}$$

$$= \frac{45000\angle 210^\circ - 15000\angle 210^\circ + 15000\angle 90^\circ}{1000\angle 135^\circ}$$

$$= 45\angle 75^\circ - 15\angle 75^\circ + 15\angle -45^\circ$$

$$= 11.65 + j\,43.47 - 3.88 - j\,14.49 + 10.6 - j\,10.6$$

$$= 18.37 + j\,18.38 = 25.99\angle 45^\circ \text{ A}$$

$\therefore$ The current coming out of $150\angle 120^\circ$ V source is $25.99\angle 45^\circ$ A.

EXAMPLE 12.48 *Determine the values of I_1 and I_2 in Fig. E12.48 using mesh analysis.*

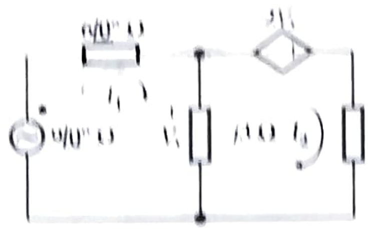

Fig. E12.48

SOLUTION. Using mesh analysis at the *left loop*,

$$6\angle 0^\circ \times I_1 + (j3)[I_1 - I_2] = 9\angle 0^\circ \quad \ldots(1)$$

Mesh analysis at the *right side loop* yields

$$-V_1 - 2V_1 + 3\angle 0^\circ \cdot I_2 = 0$$

or $$-3V_1 + 3I_2 = 0$$

or $$-V_1 + I_2 = 0 \quad \ldots(2)$$

But $$V_1 = (-j3)(I_1 - I_2) \quad \ldots(3)$$

Using (3) in (2),

$$j3(I_1 - I_2) + I_2 = 0$$

or $$j3\,I_1 + (1 - j3)\,I_2 = 0 \quad \ldots(4)$$

Simplification of (1) yields

$$(2 + j)\,I_1 - j\,I_2 = 3. \quad \ldots(5)$$

Equations (4) and (5) can be arranged in matrix form as

$$\begin{bmatrix} j3 & 1 - j3 \\ 2 + j & -j \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\therefore I_1 = \frac{\begin{vmatrix} 0 & 1 - j3 \\ 3 & -j \end{vmatrix}}{\begin{vmatrix} j3 & 1 - j3 \\ 2 + j & -j \end{vmatrix}} = \frac{-3(1 - j3)}{3 - (2 + j)(1 - j3)}$$

$$= \frac{-3 + j9}{3 - 2 + j6 - j - 3}$$

$$= \frac{-3 + j9}{-2 + j7} = \frac{9.487\angle 108.43^\circ}{7.28\angle 106^\circ}$$

$$= 1.30\angle 2.43^\circ \text{ A.}$$

and $$I_2 = \frac{\begin{vmatrix} j3 & 0 \\ 2 + j & 3 \end{vmatrix}}{-2 + j7} = \frac{j9}{7.28\angle 106^\circ}$$

$$= 1.24\angle -16^\circ \text{ A.}$$

EXAMPLE 12.49 *Obtain the output voltage V_0 across x-y using mesh analysis in the network of Fig. E12.49.*

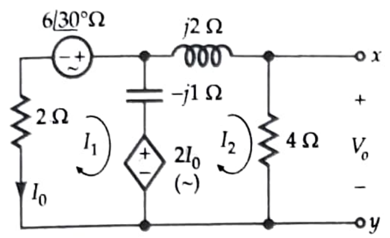

Fig. E12.49

SOLUTION. Let us first designate the mesh currents I_1 and I_2. It may be noted here that $I_0 \equiv I_1$.

Loop equation at the *left loop* is :

$$2\,I_1 + (I_1 - I_2)(-j\,1) = 6\,\angle 30° - 2\,I_0$$

or $$4I_1 - j\,1.I_1 + j\,.1.\,I_2 = 6\,\angle 30° \qquad ...(1)$$

Similarly, in the *right sided loop*,

$$-I_1(-j\,1) + I_2(4 + j\,2) - 2\,I_0 + I_2(-j\,1) = 0$$

or $$I_1(-2 + j\,1) + I_2(4 + j\,1) = 0$$

$$[\because I_1 = I_0] \qquad ...(2)$$

These two equations can be configured in matrix form as

$$\begin{bmatrix} 4 - j\,1 & j\,1 \\ -2 + j\,1 & 4 + j\,1 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 6\,\angle 30° \\ 0 \end{bmatrix}$$

Here $$I_2 = \frac{\begin{vmatrix} 4 - j\,1 & 6\,\angle 30° \\ -2 + j\,1 & 0 \end{vmatrix}}{\begin{vmatrix} 4 - j\,1 & j\,1 \\ -2 + j\,1 & 4 + j\,1 \end{vmatrix}}$$

$$= \frac{6\,\angle 30\,(2 - j\,1)}{(4 - j\,1)(4 + j\,1) - (j\,1)(-2 + j\,1)}$$

$$= \frac{6\,\angle 30° \times 2.24\,\angle -26.565°}{16 + 1 + j\,2 + 1}$$

$$= \frac{13.44\,\angle 3.435°}{18 + j\,2}$$

$$= \frac{13.44\,\angle 3.435°}{18.11\,\angle 6.34°} = 0.742\,\angle -2.91$$

$\therefore$ V_0 (output voltage) $= 4I_2 =$ **2.96 $\angle$ − 2.91° V.**

EXAMPLE 12.50 *Using mesh current analysis, find V_2 such that the source current of V_2 is zero.*

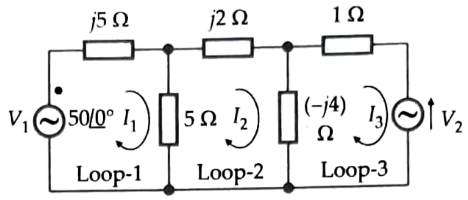

Fig. E12.50

SOLUTION. In *loop-1*, mesh analysis yields

$$(5 + j\,5)\,I_1 - 5I_2 = 50\,\angle 0° \qquad ...(1)$$

Mesh analysis in *loop-2* yields

$$-5\,I_1 + [(j\,2 + \{-j\,4\}) + 5]\,I_2 - (-j\,4)\,I_3 = 0$$

or $$-5\,I_1 + (5 - j\,2).\,I_2 + j\,4\,I_3 = 0 \qquad ...(2)$$

On the other hand, mesh analysis in loop-3 yields

$$-(-j\,4)\,I_2 + [(-j\,4) + 1]\,I_3 = -V_2$$

or $$j\,4I_2 + (1 - j\,4)\,I_3 = -V_2 \qquad ...(3)$$

In matrix form, the three equations can be rearranged as

$$\begin{bmatrix} (5 + j\,5) & -5 & 0 \\ -5 & (5 - j\,2) & j\,4 \\ 0 & j\,4 & (1 - j\,4) \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 50\,\angle 0° \\ 0 \\ -V_2 \end{bmatrix}$$

$$\therefore I_3 = \frac{\begin{vmatrix} 5 + j\,5 & -5 & 50\,\angle 0 \\ -5 & (5 - j\,2) & 0 \\ 0 & j\,4 & -V_2 \end{vmatrix}}{\begin{vmatrix} 5 + j\,5 & -5 & 0 \\ -5 & (5 - j\,2) & j\,4 \\ 0 & j\,4 & 1 - j\,4 \end{vmatrix}}$$

$$= \frac{(5 + j5)\,[\{(5 - j2) \times (-V_2)\}] + 5\,[(-5)(-V_2)] + 50\,\angle 0°(-j\,20)}{(5 + j\,5)\,[(5 - j2)(1 - j4) - j^2 16] + 5\,[(-5)(1 - j4)]}$$

$$= \frac{-(5 + j\,5)(5 - j2)\,V_2 + 25\,V_2 - j\,1000}{[(5 + j\,5)(-3 - j\,22 + 16) + 5(-5 + j\,20)]}$$

$$= \frac{V_2(-10 - j\,15) - j\,1000}{175 - j\,45 + j\,100 - 25}$$

$$= \frac{-V_2\,(10 + j\,15) - j\,1000}{150 + j\,55}$$

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But as per the question, the current through V_2 or coming out from V_2 must be zero.

$\therefore \quad I_3 = 0.$

This gives $$\frac{-V_2(10+j\,15)-1000\,j}{150+j\,55}=0$$

or $$V_2(10+j\,15)=-1000\,j$$

$$\therefore \quad V_2=\frac{-j\,1000}{(10+j\,15)}=\frac{-1000\angle 90°}{18.03\angle 56.31}$$

$$=-55.46\angle 33.69° \text{ V}.$$

Thus, the value of V_2 is $\mathbf{-55.46\angle 33.69°\ V}$.

EXAMPLE 12.51 *Transform the circuit given in Fig. E12.51 to s-domain and apply mesh analysis to find the current through the dependent voltage source (in s-domain).*

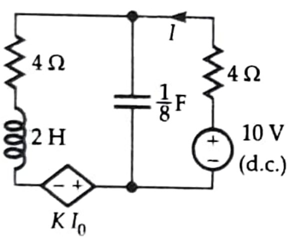

Fig. E12.51

SOLUTION. Figure 12.51(*a*) represents the s-domain form of the given circuit.

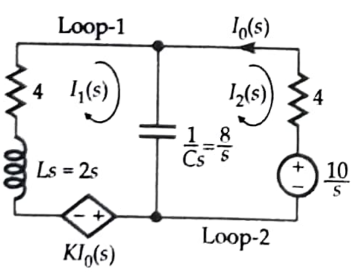

Fig. E12.51 (*a*)

In *loop-1* :

$$\left(4+2s+\frac{8}{s}\right)I_1(s)-\frac{8}{s}I_2(s)+K\,I_0(s)=0$$

or $$\left(4+2s+\frac{8}{s}\right)I_1(s)-\frac{8}{s}I_2(s)-KI_2(s)=0$$

$[\because\ I_0(s)=-I_2(s)]$

or $$\left(4+2s+\frac{8}{s}\right)I_1(s)-I_2(s)\left(\frac{8}{s}+K\right)=0 \quad ...(1)$$

In *loop-2* :

$$4I_2(s)+\frac{10}{s}+I_2(s).\frac{8}{s}-I_1(s)\frac{8}{s}=0$$

or $$-\frac{8}{s}I_1(s)+I_2(s)\left(4+\frac{8}{s}\right)=-\frac{10}{s} \quad ...(2)$$

In matrix form (1) and (2) are arranged as

$$\begin{bmatrix}\left(4+2s+\frac{8}{s}\right) & -\left(\frac{8}{s}+K\right)\\ -\frac{8}{s} & \left(4+\frac{8}{s}\right)\end{bmatrix}\begin{bmatrix}I_1(s)\\ I_2(s)\end{bmatrix}=\begin{bmatrix}0\\ -\frac{10}{s}\end{bmatrix}$$

$\therefore$ $I_1(s)$, the current through the dependent source is given by

$$I_1(s)=\frac{\begin{vmatrix}0 & -\frac{8}{s}-K\\ -\frac{10}{s} & 4+\frac{8}{s}\end{vmatrix}}{\begin{vmatrix}2s+4+\frac{8}{s} & -\frac{8}{s}-K\\ -\frac{8}{s} & 4+\frac{8}{s}\end{vmatrix}}=\frac{\mathbf{-10-s\left(\frac{10}{8}K\right)}}{\mathbf{s\,(s^2+4s+8-K)}}.$$

EXAMPLE 12.52 *Using nodal analysis, find the node voltage V_1 and V_2 (Fig. E12.52).*

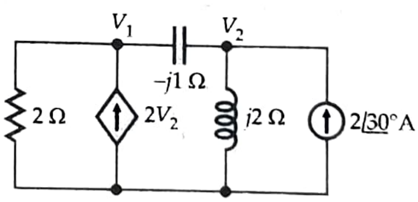

Fig. E12.52

SOLUTION. The respective nodal equations are :

$$\frac{V_1}{2}+\frac{V_1-V_2}{-j1}-2\,V_2=0$$

or $$\frac{V_1}{2}+\frac{V_1-V_2}{-j1}=2\,V_2 \quad ...(1)$$

and $$\frac{V_2-V_1}{-j1}+\frac{V_2}{j2}-2\angle 30°=0$$

or $$\frac{V_2-V_1}{-j1}+\frac{V_2}{j2}=2\angle 30° \quad ...(2)$$

Simplification of equations (1) and (2) leads

$$\left(\frac{1}{2}+j.1\right)V_1-(2+j\,1)\,V_2=0 \quad ...(3)$$

$$-(j.1)V_1 + \left(j\frac{1}{2}\right)V_2 = 2\angle 30° \qquad ...(4)$$

or
$$\begin{bmatrix} \left(\frac{1}{2}+j1\right) & -(2+j1) \\ (-j.1) & \left(j\frac{1}{2}\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\angle 30° \end{bmatrix}$$

$$\therefore \quad V_1 = \frac{\begin{vmatrix} 0 & -(2+j1) \\ 2\angle 30° & j.\frac{1}{2} \end{vmatrix}}{\begin{vmatrix} \left(\frac{1}{2}+j1\right) & -(2+j1) \\ (-j1) & \left(j.\frac{1}{2}\right) \end{vmatrix}}$$

$$= \frac{(2\angle 30°)(2+j1)}{\frac{1}{2} - j1.75} = \frac{4.48\angle 56.57°}{1.82\angle -74.05°}$$

$$= 2.46\angle 130.62° \text{ V}$$

and
$$V_2 = \frac{\begin{vmatrix} \left(\frac{1}{2}+j1\right) & 0 \\ (-j1) & 2\angle 30° \end{vmatrix}}{\begin{vmatrix} \left(\frac{1}{2}+j1\right) & -(2+j1) \\ (-j1) & \left(j.\frac{1}{2}\right) \end{vmatrix}}$$

$$= \frac{2\angle 30° \times 1.12\angle 63.43°}{1.82\angle -74.05°}$$

$$= 1.23\angle 167.48° \text{ V}$$

Thus, $V_1 = \mathbf{2.46\angle 130.62° \text{ V}}$

and $V_2 = \mathbf{1.23\angle 167.48° \text{ V}}$.

EXAMPLE 12.53 *Assuming $\alpha = \beta = 90$, $V_0 = 1\angle 0°$ V, find the voltage V_L at load node in the circuit of Fig. E12.53 using nodal analysis.*

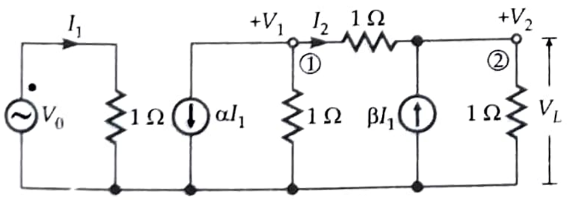

Fig. E12.53

SOLUTION. In the left hand loop of the given circuit,

$$I_1 = \frac{V_0}{1} = 1\angle 0° \text{ A} \qquad ...(1)$$

Let us designate the nodal voltages at the right hand part of the given network designating the nodes as shown in Fig. E12.53.

By inspection, at *node (1)*,

$$\frac{V_1}{1} + \frac{V_1 - V_2}{1} + \alpha I_1 = 0$$

or
$$V_1(1+1) - V_2 = -\alpha I_1 \qquad ...(2)$$

Similarly, at *node (2)*,

$$\frac{V_2}{1} + \frac{V_2 - V_1}{1} = \beta I_2$$

or
$$V_2(1+1) - V_1 = \beta I_2$$

or
$$2V_2 - V_1 = \beta(V_1 - V_2) \qquad ...(3)$$

$$\left[\because I_2 = \frac{V_1 - V_2}{1} = (V_1 - V_2)\right]$$

Reorienting (2) and (3),

$$2V_1 - V_2 = -\alpha I_1 \qquad ...(4)$$

and
$$-(1+\beta)V_1 + (2+\beta)V_2 = 0 \qquad ...(5)$$

From (4) and (5),

$$V_2 = V_L$$

$$= \frac{\begin{vmatrix} 2 & -\alpha I_1 \\ -(1+\beta) & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ -(1+\beta) & (2+\beta) \end{vmatrix}} = \frac{\begin{vmatrix} 2 & -\alpha \\ -(1+\beta) & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ -(1+\beta) & (2+\beta) \end{vmatrix}}$$

[Since $I_1 = 1\angle 0°$ from (1)]

$$\therefore \quad V_L = \frac{-\alpha(1+\beta)}{2(2+\beta) - 1(1+\beta)} = \left[\frac{-\alpha(1+\beta)}{3+\beta}\right]$$

$$= \frac{-90 \times 91}{3+90} = -88.065 \text{ V}$$

Thus $V_L = \mathbf{-88.065 \text{ V}}$.

EXAMPLE 12.54 *Taking $I(s) = \dfrac{2\omega}{s^2 + \omega^2}$, find $V_L(s)$ in Fig. E12.54 using Nodal analysis.*

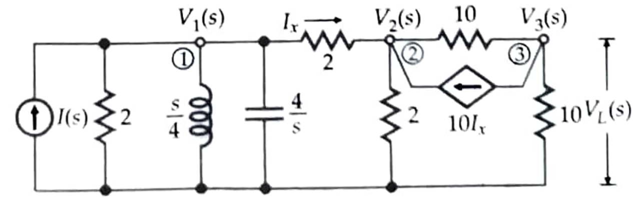

Fig. E12.54