Linear Differential Equation

A differential equation is said to be linear if the dependent variable and its derivative occur only in first degree and are not multiplied together. Its general form is

$$\boxed{\frac{dy}{dx} + Py = Q} \quad \text{—— (1)}$$

where P and Q are functions of x only.

It was introduced by Leibneitz and is therefore better known as Leibneitz's linear differential equation.

Solution of (1) can be written as

$$\boxed{y(I.F.) = \int Q(I.F.)\,dx + C}$$

where, $I.F. = e^{\int P dx}$ (Integrating factor)

Similarly, the equation $\boxed{\frac{dx}{dy} + Px = Q}$ —(2)

where P and Q are functions of y only, is linear in x. Here the I.F. is $e^{\int P dy}$ and the general solution will be written as

$$\boxed{x(I.F.) = \int Q(I.F.)\,dy + C}$$

where, $I.F. = e^{\int P dy}$

Question 1: Solve $\frac{dy}{dx} + y\sec x = \tan x$

Solution: We have $P = \sec x$, $Q = \tan x$

$$I.F. = e^{\int P dx} = e^{\int \sec x\, dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

The solution is given by

$$y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + C$$

$$= \int \tan x \sec x\,dx + \int \tan^2 x\,dx + C$$

$$= \sec x + \int (\sec^2 x - 1)\,dx + C$$

$$y(\sec x + \tan x) = \sec x + \tan x - x + C$$

Question 2: Solve $\frac{dy}{dx} = y\tan x - 2\sin x$

Solution: We have $P = -\tan x$, $Q = -2\sin x$

$$I.F. = e^{\int P dx} = e^{\int -\tan x dx} = e^{-\log \sec x} = e^{\log \cos x} = \cos x$$

Hence the solution is

$$y\cos x = \int -2\sin x \cos x\,dx + C$$

$$= -\int \sin 2x\,dx + C = \frac{\cos 2x}{2} + C$$

Question 3: Solve $x\log x \frac{dy}{dx} + y = 2\log x$

Solution: We have $P = \frac{1}{x\log x}$, $Q = \frac{2}{x}$

$$I.F. = e^{\int \frac{dx}{x\log x}} = e^{\log(\log x)} = \log x$$

Hence the solution is

$$y\cdot\log x = \int \frac{2\log x}{x}\,dx + C$$

$$y\cdot\log x = (\log x)^2 + C$$

Question 4: $\frac{dy}{dx} + \frac{1-2x}{x^2}y = 1$

Solution: We have $P = \frac{1-2x}{x^2}$, $Q = 1$

$$I.F. = e^{\int\left(\frac{1-2x}{x^2}\right)dx} = e^{-\frac{1}{x} - 2\log x} = e^{-\frac{1}{x}}\cdot e^{\log x^{-2}} = \frac{e^{-1/x}}{x^2}$$

The required solution is

$$y(x^{-2}e^{-1/x}) = \int x^{-2}e^{-1/x}\,dx + C$$

$$= \int e^{t}\,dt + C \qquad \text{where } t = -\frac{1}{x},$$

$$= e^{t} + C$$

$$= e^{-1/x} + C$$

$y = x^2(1 + Ce^{1/x})$ is the required solution.

Question 5: Solve $(x+1)\frac{dy}{dx} - ny = e^{x}(x+1)^{n+1}$

Solution: We have $P = \frac{-n}{x+1}$, $Q = e^{x}(x+1)^{n}$

$$I.F. = e^{\int \frac{-n}{(x+1)}dx} = e^{-n\log(x+1)} = (x+1)^{-n}$$

The required solution is

$$y(x+1)^{-n} = \int e^{x}(x+1)^{n}\cdot(x+1)^{-n}\,dx + C$$

$$\frac{y}{(x+1)^{n}} = \int e^{x}\,dx + C$$

$$\frac{y}{(x+1)^{n}} = e^{x} + C$$

$$y = (e^{x} + C)(x+1)^{n}$$

Question 6: $\frac{dy}{dx} + \frac{y}{(1-x^2)^{3/2}} = \frac{x + \sqrt{1-x^2}}{(1-x^2)^2}$

Solution: We have $P = \frac{1}{(1-x^2)^{3/2}}$, $Q = \frac{x+\sqrt{1-x^2}}{(1-x^2)^2}$

$$\int P\,dx = \int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\,d\theta}{\cos^3\theta} \qquad \text{, Let } x = \sin\theta,\ dx = \cos\theta\,d\theta$$

$$= \int \sec^2\theta\,d\theta = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$$I.F. = e^{\int P\,dx} = e^{x/\sqrt{1-x^2}}$$

The required solution is

$$y e^{x/\sqrt{1-x^2}} = \int e^{\frac{x}{\sqrt{1-x^2}}} \cdot \frac{x+\sqrt{1-x^2}}{(1-x^2)^2}\, dx + C$$

$$= \int e^{\tan\theta} \frac{(\sin\theta + \cos\theta)\cdot\cos\theta}{\cos^4\theta}\, d\theta + C \qquad , x = \sin\theta$$

$$= \int e^{\tan\theta}\left(\frac{\sin\theta}{\cos^3\theta} + \frac{1}{\cos^2\theta}\right) d\theta + C$$

$$= \int e^{\tan\theta} (\sec^2\theta + \tan\theta\sec^2\theta)\, d\theta + C$$

$$= \int e^y (y+1)\, dy + C \qquad y = \tan\theta$$

$$= (y+1)e^y - \int e^y dy + C \qquad \text{Integrating by parts}$$

$$= (y+1)e^y - e^y + C$$

$$= y e^y + C$$

$$= \tan\theta\, e^{\tan\theta} + C$$

$$= \frac{x}{\sqrt{1-x^2}} e^{x/\sqrt{1-x^2}} + C$$

$$y = \frac{x}{\sqrt{1-x^2}} + C e^{-x/\sqrt{1-x^2}}$$

Question 7: Solve $(1+x+xy^2)\,dy + (y+y^3)\,dx = 0$

Solution: Given equation can be written as

$$\frac{dx}{dy} + \frac{1+x(1+y^2)}{y(1+y^2)} = 0$$

$$\frac{dx}{dy} + \frac{1}{y}\cdot x = -\frac{1}{y(1+y^2)}$$

where $P = \frac{1}{y}$, $Q = \frac{-1}{y(1+y^2)}$

$$I.F. = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

The required solution is

$$xy = -\int \frac{1}{y(1+y^2)}\cdot y\, dy + C = -\int \frac{dy}{1+y^2} + C = -\tan^{-1} y + C$$

$xy + \tan^{-1} y = c$

Question 8: Solve $dr + (2r\cot\theta + \sin 2\theta)\,d\theta = 0$

Solution: The equation is $\frac{dr}{d\theta} + 2r\cot\theta = -\sin 2\theta$ ___ (1)

Here $P = 2\cot\theta$ & $Q = -\sin 2\theta$

$I.F. = e^{\int 2\cot\theta\, d\theta} = e^{2\log\sin\theta} = \sin^2\theta$

The required solution (1)

$r\,(I.F.) = \int Q\,(I.F.)\,d\theta + C$

$r\sin^2\theta = -2\int \sin^3\theta\cos\theta\, d\theta = -\frac{2}{4}\sin^4\theta + C$

$r = -\frac{1}{2}\sin^2\theta + C\,\mathrm{cosec}^2\theta$

Exercise

Question 1:- Solve $\cos^3 x\,\frac{dy}{dx} + y\cos x = \sin x$

Question 2:- Solve $(1+x^2)\,\frac{dy}{dx} + y = \tan^{-1} x$

Question 3:- $(x+y+1)\,\frac{dy}{dx} = 1$

Question 4:- $(x+2y^3)\,\frac{dy}{dx} = y$

Question 5:- $(1+y^2)\,dx + (x - e^{-\tan^{-1} y})\,dy = 0$

Question 6:- Solve $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$

Question 7:- Solve $2(1-xy)\,\frac{dy}{dx} = y^2$

Question 8:- Solve $ye^y = (y^3 + 2xe^y)\,\frac{dy}{dx}$

Question 9:- $(2xy\cos x^2 - 2xy + 1)\,dx + (\sin x^2 - x^2)\,dy = 0$

Question 10:- Solve $x(1-x^2)\,\frac{dy}{dx} + (2x^2-1)\,y = x^3$.

Equations Reducible to Linear Form

Bernoulli's Equation

The differential equation $\boxed{\frac{dy}{dx} + Py = Qy^n}$

where P & Q are functions of x and n is a real number. It is known as Bernoulli's equation and can be reduced to Leibneitz's linear form by dividing throughout by y^n, and get

$$\boxed{y^{-n}\frac{dy}{dx} + P.y^{1-n} = Q}$$

Putting here $y^{1-n} = t$ so $(1-n)y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$

we get

$$\boxed{\frac{dt}{dx} + (1-n)P.t = (1-n)Q}$$

which is linear in t and can be solved easily.

Another type of equation, which can be reduced to the linear form, is

$$\boxed{f'(y)\frac{dy}{dx} + Pf(y) = Q} \quad \text{— (A)}$$

where P & Q are functions of x alone.

The substitution $t = f(y)$, so $\frac{dt}{dx} = f'(y)\frac{dy}{dx}$

transforms the equation (A) into linear in ~~A~~ t, as $\frac{dt}{dx} + Pt = Q$

Question 1: Solve $x\dfrac{dy}{dx} + y^2 x = y$

Solution:- $\dfrac{dy}{dx} - \dfrac{y}{x} = -y^2$

dividing by $-y^2$, we get

$$-\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = 1 \qquad -(1)$$

put $\dfrac{1}{y} = z \Rightarrow -\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$, (1) can be written as

$$\frac{dz}{dx} + \frac{1}{x}\cdot z = 1 \qquad -(2)$$

$$I.F. = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

The required solution is

$$z\cdot x = \int 1\cdot x\,dx + c = \frac{x^2}{2} + c$$

Hence $\dfrac{x}{y} = \dfrac{x^2}{2} + c$

Question 2: Solve $\sec^2 y \dfrac{dy}{dx} + 2x \tan y = x^3$, $y(0) = \dfrac{\pi}{4}$

Solution:- Putting $\tan y = t$ in the given eq^n becomes $\dfrac{dt}{dx} + 2x\,t = x^3$, which is linear in t.

$$I.F. = e^{\int 2x\,dx} = e^{x^2}$$

The solution is

$$t e^{x^2} = \int x^3 e^{x^2} dx + c$$

$$\tan y\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2 - 1) + c$$

Since $y(0) = \dfrac{\pi}{4}$, we get $1 = \dfrac{1}{2}(0-1) + c$

$\therefore c = \frac{3}{2}$

$\therefore$ Solution is $\tan y \, e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + \frac{3}{2}$

$$\tan y = \frac{1}{2}(x^2-1) + \frac{3}{2} e^{-x^2}$$

Exercise

Question 1:- Solve $x \frac{dy}{dx} + y = y^2 x^3 \cos x$.

Question 2:- Solve $x^3 \frac{dy}{dx} - x^2 y + y^4 \cos x = 0$

Question 3:- Solve $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$

Question 4:- Solve $\frac{1}{y}\frac{dy}{dx} + \frac{x}{1-x^2} = x y^{-1/2}$

Question 5:- Solve $\frac{dy}{dx} + xy = y^2 e^{\frac{x^2}{2}} \log x$

Question 6:- Solve $(xy^2 - e^{1/x^3})\,dx - x^2 y\,dy = 0$

Question 7:- Solve $(x^3 y^2 + xy)\,dx = dy$

Question 8:- Solve $(1-x^2)\frac{dy}{dx} + xy = xy^2$

Question 9:- Solve $y(2xy + e^x)\,dx - e^x\,dy = 0$

Question 10:- Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$