

Fourier - Transformation

L-1

Fourier Transform or Infinite F.T :- Let $f(t)$ be a function defined for $t \in (-\infty, \infty)$. Then infinite F.T of $f(t)$ denoted by $F(f(t))$.

$$\therefore F[f(t)] = \int_{-\infty}^{\infty} e^{-ist} \cdot f(t) dt$$

or $\therefore \bar{f}(s) = F[f(t)] = \int_{-\infty}^{\infty} e^{-ist} \cdot f(t) dt \quad \therefore F[G(t)] = \bar{f}(s)$

Inverse formula for F.T :-

$$f(t) = \bar{F}[\bar{f}(s)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s) e^{ist} ds$$

Fourier Sine Transformation :-

The infinite Fourier

Sine transformation for function $f(t)$ when $t < \infty$

denoted by $F_s(f(t))$

$$\bar{f}_s(s) = F_s(f(t)) = \int_0^{\infty} f(t) \cdot \sin st dt$$

Inverse Formula

$$f(t) = \bar{F}^{-1}(\bar{f}_s(s)) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_s(s) \sin st ds$$

Fourier Cosine Transformation :-

$$F_c(f(t)) = \bar{f}_c(s) = \int_0^{\infty} f(t) \cdot \cos st dt$$

Inverse formula

$$f(t) = \bar{F}^{-1}(\bar{f}_c(s)) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_c(s) \cos st ds$$

Linearity Proof :-

⑤ If $f(x) + g(x)$ are any two function. When F, f, g exist c_1, c_2 arbit. const. Then

① $F\{c_1 f(x) + c_2 g(x)\} = c_1 F(f(x)) + c_2 F(g(x))$

$$\begin{aligned}
 \text{R.H.S.} \quad & F\{c_1 f(x) + c_2 g(x)\} = \int_{-\infty}^{\infty} e^{-isx} \{c_1 f(x) + c_2 g(x)\} dx \\
 &= \int_{-\infty}^{\infty} c_1 e^{-isx} \cdot f(x) dx + \int_{-\infty}^{\infty} c_2 e^{-isx} \cdot g(x) dx \\
 &= c_1 F(f(x)) + c_2 F(g(x))
 \end{aligned}$$

\longleftrightarrow

l.h.s.

② $F_s\{c_1 f(x) + c_2 g(x)\} = c_1 F_s(f(x)) + c_2 F_s(g(x))$

$$\begin{aligned}
 \text{R.H.S.} \quad & F_s\{c_1 f(x) + c_2 g(x)\} = \int_0^{\infty} (c_1 f(x) + c_2 g(x)) \sin sx \cdot dx \\
 &= c_1 \int_0^{\infty} f(x) \sin sx \cdot dx + c_2 \int_0^{\infty} g(x) \cdot \sin sx \cdot dx \\
 &= c_1 F(f(x)) + c_2 F(g(x))
 \end{aligned}$$

l.h.s. $F_c\{c_1 f(x) + c_2 g(x)\} = c_1 F_c(f(x)) + c_2 F_c(g(x))$

$F_c\{c_1 f(x) + c_2 g(x)\} = \int_0^{\infty} (c_1 f(x) + c_2 g(x)) \cos sx \cdot dx.$

$= c_1 \int_0^{\infty} f(x) \cdot \cos sx \cdot dx + c_2 \int_0^{\infty} g(x) \cdot \cos sx \cdot dx = c_1 F_c(f(x)) + c_2 F_c(g(x))$

W.L.O.G.

large scale Prop:-

\Rightarrow The F.T. of $f(ax)$ is $\frac{1}{a} \bar{f}(s/a)$. Then F.T. of $f(ax)$ is

$$\text{Pl :- } \bar{f}(s) = \int_{-\infty}^{\infty} e^{-isx} \cdot f(x) dx.$$

$$\begin{aligned} \text{Now } F(f(ax)) &= \int_{-\infty}^{\infty} e^{-isx} \cdot f(ax) dx && \text{Put } ax=t \\ &= \int_{-\infty}^{\infty} e^{-ist/a} \cdot f(t) \frac{dt}{a} && x = t/a \\ &= \frac{1}{a} \int_{-\infty}^{\infty} e^{-ist/a} f(t) dt = \frac{1}{a} \bar{f}(s/a) && dtx = dt/a \end{aligned}$$

\longleftrightarrow

Isly
Now

$$\begin{aligned} F_g f(ax) &= \int_0^{\infty} f(ax) \cdot \sin sx \cdot dx \\ F_c f(ax) &= \int_0^{\infty} f(ax) \cdot \cos sx \cdot dx \end{aligned} \quad \left. \begin{array}{l} \text{D.S. Complete} \\ \text{done.} \end{array} \right\}$$

Shifting Prop:- If $\bar{f}(s)$ is the Fourier T.S. of $f(x)$

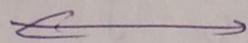
To show that $e^{iax} \bar{f}(s)$ is F.T. of $f(x-a)$

$$\begin{aligned} \text{Proof :- } F\{f(x-a)\} &= \int_{-\infty}^{\infty} e^{-isx} \cdot f(x-a) dx \\ &= \int_{-\infty}^{\infty} e^{-is(x-a)} \cdot f(x-a) dx && \text{Put } x-a=t \\ &= \int_{-\infty}^{\infty} e^{-is(x-a)} \cdot f(x-a) dx && x = a+t \\ &= \int_{-\infty}^{\infty} e^{-is(a+t)} \cdot f(t) \cdot a dt && dx = a dt \\ &= \int_{-\infty}^{\infty} e^{-isa} \cdot e^{-ist} \cdot f(t) dt && \text{H.R.} \end{aligned}$$

Modulation Prop :- If $\tilde{f}(s)$ is infinite F.T of $f(x) \cos ax$ is $\frac{1}{2} \tilde{f}(s-a) + \frac{1}{2} \tilde{f}(s+a)$

Proof :-

$$\begin{aligned}
 F\{f(x) \cos ax\} &= \int_{-\infty}^{\infty} e^{-isx} f(x) \cos ax dx \\
 &= \int_{-\infty}^{\infty} e^{-isx} \left(\frac{e^{iax} + e^{-iax}}{2} \right) f(x) dx \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-isx} e^{iax} f(x) dx + \frac{1}{2} \int_{-\infty}^{\infty} e^{-isx} e^{-iax} f(x) dx \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-i(s-a)x} f(x) dx + \frac{1}{2} \int_{-\infty}^{\infty} e^{-i(s+a)x} f(x) dx \\
 &= \frac{1}{2} \tilde{f}(s-a) + \frac{1}{2} \tilde{f}(s+a).
 \end{aligned}$$



Eg Find the F.T of $f(x) = e^{-ax|x|}$ when $a > 0$ and

$$\begin{aligned}
 \tilde{f}(s) &= \int_{-\infty}^{\infty} f(x) e^{-isx} dx = \int_{-\infty}^{\infty} e^{-ax|x|} e^{-isx} dx \quad x \in (-\infty, \infty) \\
 &= \int_{-\infty}^0 e^{-ax} e^{isx} dx + \int_0^{\infty} e^{-ax} e^{-isx} dx. \quad \left\{ \begin{array}{l} |x| = -x \text{ if } x \leq 0 \\ |x| = x \text{ if } x \geq 0 \end{array} \right. \\
 &= \int_{-\infty}^0 e^{(a-is)x} dx + \int_0^{\infty} e^{-(a+is)x} dx \\
 &\left[\frac{e^{(a-is)x}}{a-is} \right]_{-\infty}^0 + \left[\frac{e^{-(a+is)x}}{-a-is} \right]_0^{\infty} = \frac{1}{a-is} + \left(0 + \frac{1}{a+is} \right)
 \end{aligned}$$

$$\frac{1}{a-is} + \frac{1}{a+is} = \frac{a+is}{(a^2+s^2)} = \frac{a+is+a-is}{a^2+s^2} \quad P-3$$

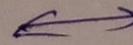
$$= \frac{2a}{a^2+s^2}$$

Ex 1.2 Find the F.T of the function:-

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{otherwise.} \end{cases}$$

Sol.

$$\begin{aligned} \therefore F[f(x)] &= \int_{-\infty}^{\infty} f(x) e^{-isx} dx = \int_{-\infty}^{0} 0 dx + \int_{0}^{a} k e^{-isx} dx + \int_{a}^{\infty} 0 dx \\ &= \int_{-\infty}^{0} f(x) e^{-isx} dx + \int_{0}^{a} f(x) e^{-isx} dx + \int_{a}^{\infty} e^{-isx} \cdot f(x) dx \\ &= \int_{-\infty}^{0} 0 dx + \int_{0}^{a} k e^{-isx} dx + \int_{a}^{\infty} 0 dx \\ &= k \int_{0}^{a} e^{-isx} dx = -\frac{k}{is} [e^{-isx}]_{x=0}^{x=a} \\ &= \frac{-k}{is} \left[e^{-isa} - e^{0} \right] = \frac{-k}{is} \left(e^{-isa} - 1 \right) \\ &= k \left(\frac{1 - e^{-isa}}{is} \right) = -ik \left(\frac{1 - e^{-isa}}{s} \right) \\ \text{or } &\frac{ik}{s} (e^{-isa} - 1) \end{aligned}$$



Ex 1.3 Find the F.T

$$f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2\ell} & -\ell < x < \ell \\ 0 & |x| > \ell \end{cases}$$

Sol. $F[f(x)] = \int_{-\ell}^{-\ell} e^{-isx} \cdot f(x) dx + \int_{-\ell}^{\ell} e^{-isx} \cdot f(x) dx + \int_{\ell}^{\infty} e^{-isx} \cdot f(x) dx$

Do complete

EPR

Find the F.T.

$$f(x) = \begin{cases} -1 & -\alpha < x < 0 \\ 1 & 0 < x < \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Sol

$$\tilde{f}(s) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) \cdot e^{-isx} \cdot dx$$

$$\begin{aligned} &= \int_{-\infty}^{-a} f(x) e^{-isx} dx + \int_{-a}^0 f(x) e^{-isx} dx + \int_0^a f(x) e^{-isx} dx + \int_a^{\infty} f(x) e^{-isx} dx \\ &= 0 + (-1) \left[\frac{e^{-isx}}{-is} \right]_{-a}^0 + \left[\frac{e^{-isx}}{-is} \right]_0^a + 0 \\ &\quad \text{Do complete} \end{aligned}$$

EPR 5

Find the F.T.

$$f(x) = \begin{cases} x \cdot e^{-x} & x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Sol

$$\tilde{f}(s) = \int_{-\infty}^{\infty} f(x) \cdot e^{-sx} \cdot dx = \int_{-\infty}^0 0 + \int_0^{\infty} x \cdot e^{-x} \cdot e^{-sx} \cdot dx$$

$$= \int_0^{\infty} x e^{-sx} e^{-x} \cdot dx$$

$$= \int_0^{\infty} x e^{-(s+1)x} \cdot dx$$

~~complete~~

$$\left[\frac{x \cdot e^{-(s+1)x}}{-(s+1)} \right]_0^{\infty} - \left[\frac{e^{-(s+1)x}}{-(s+1)} \right]_0^{\infty}$$

Do complete

$$\lim_{x \rightarrow \infty} \left(\frac{-x}{1+s} e^{-(s+1)x} - 0 \right)$$

Find the sine and cosine Fourier Transformation.

P-4

$$of \quad 2e^{-5x} + 5e^{-2x}$$

$$\text{Note:- } \int_0^\infty e^{-ax} \cdot \sin bx dx = \frac{b}{a^2+b^2}$$

$$\int_0^\infty e^{-ax} \cos bx dx = \frac{a}{a^2+b^2}$$

Sol

$$f(x) = 2e^{-5x} + 5e^{-2x}$$

$$\therefore \bar{f}_S(s) = \int_0^\infty f(x) \cdot \sin sx dx$$

$$= \int_0^\infty (2e^{-5x} + 5e^{-2x}) \cdot \sin sx dx$$

$$= 2 \int_0^\infty e^{-5x} \cdot \sin sx dx + 5 \int_0^\infty e^{-2x} \cdot \sin sx dx$$

$$= 2 \cdot \frac{s}{s^2+25} + 5 \cdot \frac{s}{s^2+4}$$

$$= \frac{2s}{s^2+25} + \frac{5s}{s^2+4}$$

\longleftrightarrow

$$(ii) \quad \bar{f}_C(s) = \int_0^\infty (2e^{-5x} + 5e^{-2x}) \cos sx dx$$

$$= 2 \int_0^\infty e^{-5x} \cos sx dx + 5 \int_0^\infty e^{-2x} \cos sx dx$$

$$= \frac{2}{s^2+25} + 5 \cdot \frac{(2)}{s^2+4}$$

Note:-

$$\begin{aligned} & \int e^{ax} \cdot \sin(bx+c) dx \\ &= \frac{e^{ax}}{a^2+b^2} [a \sin(bx+c) - b \cos(bx+c)] \end{aligned}$$

$$\int e^{ax} \cos(bx+c) dx =$$

$$= \frac{e^{ax}}{a^2+b^2} [a \cos(bx+c) + b \sin(bx+c)]$$

Expt. 2 Find the Fourier Sine transformation.

Sol

$$f(x) = \frac{e^{-ax}}{x}$$

$$\therefore \bar{f}_S(s) = F_S(f(x))$$

$$\bar{f}_S(s) = \int_0^\infty \frac{e^{-ax}}{x} \cdot \sin sx dx$$

This problem is Diff under

The integral sign.. using Leibnitz rule.

Dif w.r.t 's'

$$\frac{d}{ds} \bar{f}_S(s) = \int_0^\infty \frac{e^{-ax}}{x} \cdot \cos sx \cdot x dx$$

$$\begin{aligned} &= \int_0^\infty e^{-ax} \cdot \cos sx dx \\ &= \frac{a}{a^2+s^2} \end{aligned}$$

$$\therefore \frac{d}{ds} \overline{f_s(s)} = \frac{a}{a^2 + s^2}$$

I.B.S. w.t + 8

$$\overline{f_s}(s) = a \int \frac{ds}{a^2 + s^2} = \frac{a}{a} \tan^{-1} \frac{s}{a} + c \quad \text{--- (i)}$$

Put $s=0$ in Eq(i)

$$\overline{f_s}(0) = 0. \quad \text{Putting (i)} = 0 = \tan^{-1} \frac{0}{a} + c.$$

$$\therefore \overline{f_s}(s) = \tan^{-1} \frac{s}{a} \quad 0 = 0 + c \\ c = 0$$

hence

$$\overline{f_s}(s) = F_s(f(x)) = \tan^{-1} \frac{s}{a}. \quad \longleftrightarrow$$

E8 Find the Fourier cosine Transformation of e^{-x^2}

$$\underline{\text{Sol}} \quad \therefore f(x) = e^{-x^2}$$

$$\overline{f_s}(s) = F_c(f(x)) = \underline{\text{To find}}$$

$$\text{Let } I = \int_0^\infty f(x) \cos sx dx \quad \text{--- (i)}$$

$$\overline{f_s}(s) = \int_0^\infty e^{-x^2} \cos sx dx$$

s is the parameter. Using

DH under the integral sign.

I.w.s + 's'

$$\begin{aligned} \frac{d}{ds} \overline{f_s}(s) &= \int_0^\infty e^{-x^2} (-\sin sx) \cdot x dx \\ &= \frac{1}{2} \int_0^\infty -2x \cdot \frac{e^{-x^2}}{2} \cdot \underline{\sin sx dx} \quad \text{I.B.S.} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left[\sin sx \cdot e^{-x^2} \right]_0^\infty - \int_0^\infty \cos sx \cdot s e^{-x^2} dx \\ &\stackrel{dx}{=} 0 - \frac{1}{2} \cdot s \int_0^\infty e^{-x^2} \cdot \cos sx dx \\ &\quad \text{I (Bj(i))} \end{aligned}$$

$$\frac{dI}{ds} = -\frac{s}{2} I$$

$$\frac{\frac{dI}{ds}}{s} = -\frac{1}{2} I$$

I.B.S. w.t
s.

$$\log I = -\frac{s^2}{4} + \log C.$$

$$\log I - \log C = -\frac{s^2}{4}$$

$$\log \frac{I}{C} = -\frac{s^2}{4} \quad \text{Taking e on both sides}$$

$$e^{F_D(\frac{x}{c})} = e^{-\frac{8x^2}{a}}$$

$$\frac{x}{c} = e^{-\frac{8x^2}{a}}$$

Put $x=0$ in (i)

$$I = c \cdot e^{-\frac{8x^2}{a}}$$

$$= \frac{1}{2} \int_0^a [\cos((1-s)x) - \cos((1+s)x)] dx \quad P-5$$

$$I = \int_0^\infty e^{-x^2} \cos 8x dx = \int_0^\infty e^{-x^2} dx$$

$$I = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad (\text{Beta + Gamma function results.})$$

$$I = \frac{\sqrt{\pi}}{2}$$

B) (+) $s=0$

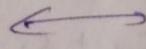
$$I = c \cdot e^0 = c \dots$$

$$I = c$$

$$c = \frac{\sqrt{\pi}}{2}$$

$$B) (+) = I = \frac{\sqrt{\pi}}{2} e^{-\frac{8x^2}{a}}$$

$$f_S(s) = F_C(f(x)) = I = \frac{\sqrt{\pi}}{2} e^{-\frac{8s^2}{a}}$$



Ans

~~V.G.M.P.~~

~~E.H.~~

Find the Fourier

coseine transformation. $\frac{1}{1+x^2}$

and deduce the sine transformation of $\frac{x}{1+x^2}$

$$F_C\left(\frac{1}{1+x^2}\right) = \int_0^\infty \frac{1}{1+x^2} \cdot \cos 8x dx$$

$$I = \int \frac{1}{1+x^2} \cdot \cos 8x dx = I(1)$$

Or H.C. $w \cdot z + y$

$$\frac{dI}{ds} = \int \frac{-1}{(1+x^2)^2} \cdot \sin 8x dx \dots x$$

$$= \int_0^\infty \left(\frac{-x}{(1+x^2)^2} \right) \cdot \sin 8x dx$$

$$= F_S\left(\frac{-x}{1+x^2}\right)$$

(2)

Sol

$$F_S(f(x)) = \int_0^\infty f(x) \cdot \sin 8x dx$$

$$= \int_0^a \sin x \cdot \sin 8x dx + \int_a^\infty 0 \cdot \sin 8x dx$$

$$= \frac{1}{2} \int 2 \sin x \cdot \sin 8x dx + 0$$

$$= \int_0^\infty -\frac{x^2 \cdot \sin 8x}{x(1+x^2)} dx$$

$$= - \int_0^\infty \frac{(1+x^2-1) \cdot \sin 8x dx}{x(1+x^2)}$$

$$= - \int_0^\infty \left(\frac{\sin 8x}{x} - \frac{\sin 8x}{x(1+x^2)} \right) dx$$

$$= \int_0^\infty \frac{\sin 8x}{x} dx + \int_0^\infty \frac{\sin 8x}{x(1+x^2)} dx$$

$\downarrow \quad \downarrow$

$$I_1 + I_2$$

$$I_1 = \int_0^\infty \frac{\sin 8x}{x} dx \quad (\text{we solve by Laplace Transform})$$

$$\because \text{Laplace Transform} = \int_0^\infty e^{-sx} f(x) dx$$

$$\boxed{\text{E.P. 2} \quad (L.T.)}$$

$$\int_0^\infty \frac{\sin 8x}{x} dx = \frac{\pi}{2} \quad \text{if } s > 0$$

$$= -\frac{\pi}{2} \quad \text{if } s < 0$$

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin 8x}{x(1+x^2)} dx \quad \boxed{3}$$

Again diff. $\Rightarrow s + s^-$

$$\begin{aligned} \frac{d^2I}{ds^2} &= 0 + \int_0^\infty \frac{1}{x(1+x^2)} \cdot \cos 8x \cdot x dx \\ &= \int_0^\infty \frac{\cos 8x}{1+x^2} dx \\ &= I \end{aligned} \quad \boxed{B.J.(1)}$$

$$\frac{d^2I}{ds^2} = I$$

~~$$s \\ s \\ \partial \\ \partial^2 - 1 = 0 \\ \partial^1 = 1 \\ \partial = \pm 1$$~~

$$I = c_1 e^s + c_2 e^{s^-} \quad \boxed{4}$$

Put $s=0$ in (4)

$$I = \int_0^\infty \frac{\cos 0}{1+x^2} dx = \int_0^\infty \frac{1}{1+x^2}$$

$$= \left(\tan^{-1} x \right)_0^\infty = \tan^{-1} \infty - \tan^{-1} 0$$

$$= \underline{\tan^{-1} \frac{\pi}{2}} - 0$$

$$= \frac{\pi}{2}$$

$$\boxed{I = \frac{\pi}{2}}$$

Putting. B.J. (4)

$$I = c_1 e^0 + c_2 e^0 \quad (\because s=0)$$

$$\frac{I}{2} = c_1 + c_2 \quad \boxed{5}$$

$$\boxed{B.J.(3)} \quad I = c_1 e^s + c_2 e^{s^-}$$

D.W. $s+1$

$$\frac{dI}{ds} = -c_1 e^s + c_2 e^{s^-}$$

Put $s=0$

$$-c_1 + c_2 = -\frac{\pi}{2}$$

$$c_1 - c_2 = \frac{\pi}{2}$$

From (5) + (6)

$$\begin{cases} \therefore \frac{dI}{ds} = -\frac{\pi}{2} + 0 \\ \therefore \frac{dI}{ds} = 0 \end{cases}$$

$$\begin{aligned}
 c_1 + c_2 &= \pi/2 \\
 c_1 - c_2 &= +\pi/2 \\
 \hline
 2c_1 &= \pi/2 \\
 c_1 &= \pi/4 \\
 c_2 &= \pi/2
 \end{aligned}$$

$$I = c_1 e^{-s} + c_2 e^s$$

$$I = \frac{\pi}{2} e^{-s}$$

$$I = \frac{\pi}{2} e^{-s}$$

$$F_C\left(\frac{1}{1+x^2}\right) = \frac{\pi}{2} e^{-s}$$

$$\begin{aligned}
 \text{Q3 (g)} \quad F_S\left(\frac{-x}{1+x^2}\right) &= \frac{dI}{dx} \\
 &= \frac{d}{dx} \left(\frac{\pi}{2} e^{-s} \right) \\
 &= -\frac{\pi}{2} e^{-s}
 \end{aligned}$$

$$F_S\left(\frac{-x}{1+x^2}\right) = -\frac{\pi}{2} e^{-s}$$

$$F_S\left(\frac{x}{1+x^2}\right) = \frac{\pi}{2} e^{-s}$$

Hence proved

E.P 7 Find the Fourier sine
and cosine Transformation
of x^{m-1}

Sol we know that P-6

$$F_C\{f(x)\} = \int_0^\infty f(x) \cos \omega x dx$$

$$F_S\{f(x)\} = \int_0^\infty f(x) \sin \omega x dx$$

$$\therefore f(x) = x^{m-1}$$

$$F_C(f(x)) = \int_0^\infty x^{m-1} \cdot \cos \omega x \cdot dx$$

$$F_S(f(x)) = \int_0^\infty x^{m-1} \cdot \sin \omega x \cdot dx$$

\therefore As we know that Grammer function

$$\Gamma_m = \int_0^\infty x^{m-1} \cdot e^{-x} \cdot dx$$

Put $x = iy$
D.W.S.T. $dx = i \cdot dy$

$$= \int_0^\infty (iy)^{m-1} \cdot e^{-iy} \cdot i \cdot dy$$

$$= \int_0^\infty i^{m-1} \cdot y^{m-1} \cdot y^{m-1} \cdot e^{-iy} dy$$

$$= \int_0^\infty i^m \cdot y^m \cdot y^{m-1} \cdot e^{-iy} dy$$

$$\Gamma_m = s^m \int_0^\infty (\cos \frac{\omega y}{2} + i \sin \frac{\omega y}{2}) dy$$

$$\therefore i^m = \left(\cos \frac{\omega}{2} + i \sin \frac{\omega}{2}\right)^m = \left(e^{i\frac{\omega}{2}}\right)^m$$

$$\Gamma_m = s^m \int_0^\infty \left(e^{i\frac{\omega}{2}}\right)^m y^{m-1} \cdot e^{-iy} dy$$

f.i.d

$$\Gamma_m = s^m e^{im\frac{\pi}{2}} \int_0^\infty e^{-sy} y^{m-1} dy$$

$$\frac{\Gamma_m}{s^m} e^{-im\frac{\pi}{2}} = \int_0^\infty e^{-sy} \cdot y^{m-1} dy$$

Now change $y \rightarrow x$

$$\frac{\Gamma_m}{s^m} e^{-im\frac{\pi}{2}} = \int_0^\infty e^{-isx} \cdot x^{m-1} dx$$

By Euler's thm

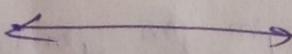
$$\frac{\Gamma_m}{s^m} \left[\cos \frac{m\pi}{2} - i \sin \frac{m\pi}{2} \right] =$$

$$= \int_0^\infty (\cos sx - i \sin sx) x^{m-1} dx$$

Equating R.P + I.P

$$\int_0^\infty \cos sx x^{m-1} dx = \frac{\Gamma_m}{s^m} \cos \frac{m\pi}{2}$$

$$\int_0^\infty \sin sx x^{m-1} dx = \frac{\Gamma_m}{s^m} \sin \frac{m\pi}{2}$$



Type-II

Inverse Fourier

Transforms.

$$\tilde{f}(t) = \tilde{F}(f(s)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \cdot e^{ist} ds$$

EPR Find $f(x)$ if $\tilde{f}(x) =$
Fourier Sine Transform
i1 $\frac{s}{1+s^2}$
or

If $\tilde{f}(s) = \frac{s}{1+s^2}$, Then find

$f(x)$

or

Find $\tilde{F}_s^{-1} \left(\frac{s}{1+s^2} \right)$

Sol

$$\therefore \tilde{f}(s) = \frac{s}{1+s^2}$$

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \tilde{f}(s) \cdot \sin sx ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \cdot \sin sx ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s^2}{s(1+s^2)} \cdot \sin sx ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{(1+s^2-1)}{s(1+s^2)} \sin sx ds$$

$$= \frac{2}{\pi} \int_0^\infty \left(\frac{1}{s} \sin sx - \frac{\sin sx}{s(1+s^2)} \right) ds$$

$$= \frac{2}{\pi} \left[\int_0^\infty \frac{\sin sx}{s} ds - \int_0^\infty \frac{\sin sx}{s(1+s^2)} ds \right]$$

$$= \frac{2}{\pi} \left[\frac{\pi}{2} - \int_0^\infty \frac{\sin sx}{s(1+s^2)} ds \right]$$

$$f(x) = \frac{2}{\pi} \cdot \frac{\pi}{x} - \frac{2}{\pi} \int_0^{\infty} \frac{\ln sx}{s(1+s^2)} \cdot ds$$

$$\therefore f(x) = 1 - \frac{2}{\pi} \int_0^{\infty} \frac{\ln sx}{s(1+s^2)} \cdot ds \quad \text{--- (1)}$$

Dif. w.r.t. x . By Leibnitz Rule

$$\therefore \frac{df}{dx} = 0 - \frac{2}{\pi} \int_0^{\infty} \frac{\cos sx \cdot s}{s(1+s^2)} \cdot ds$$

Again Dif. w.r.t. x .

$$\frac{d^2}{dx^2} f(x) = + \frac{2}{\pi} \int_0^{\infty} \frac{\sin sx \cdot s}{1+s^2} \cdot ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{s}{1+s^2} \cdot \sin sx \cdot ds \quad \text{By (*)}$$

$$\frac{d^2 f}{dx^2} = f(x)$$

$$\therefore f(x) = f$$

$$\textcircled{1} \quad \frac{d^2 f}{dx^2} - f = 0$$

$$(D^2 - 1) f = 0$$

$$D^2 - 1 = 0$$

$$\textcircled{2} \quad D = 1 \quad | \quad \textcircled{3} \quad D = -1$$

Sol.

$$f(x) = A \cdot e^x + B \cdot e^{-x} \quad \text{--- (2)}$$

$$\frac{df}{dx} = A \cdot e^x - B \cdot e^{-x} \quad \text{--- (3)}$$

$$f(0) = 1 \quad \text{C. : By (1)}$$

$$\textcircled{4} \quad \text{Put } x = \infty \text{ in (2).}$$

$$f(0) = A \cdot e^0 + B \cdot e^0$$

$$1 = A + B. \quad \text{--- (4)}$$

$$\text{Put } x = \infty \text{ in Eq (3)}$$

$$\frac{df}{dx} = - \frac{2}{\pi} \int_0^{\infty} \frac{\cos s \cdot ds}{1+s^2}.$$

$$= - \frac{2}{\pi} \int_0^{\infty} \frac{ds}{1+s^2} \dots \text{P.M}$$

$$\frac{-2}{\pi} \left[\tan^{-1}s \right]^\infty = \frac{-2}{\pi} \left[\tan^{-1}\infty - \tan^{-1}0 \right] = \frac{-2}{\pi} \left(\frac{\pi}{2} - 0 \right)$$

$$\frac{df}{dx} = -1 \quad \text{At } x=\infty$$

Now

Eq (3)

$$\frac{df}{dx} = A e^0 - B \cdot e^0$$

$$-1 = A - B$$

$$A - B = -1$$

→ (1)

From Eq (4) + (5)

$$A + B = 1$$

$$A - B = -1$$

$$2A = 0$$

$$\boxed{A=0}$$

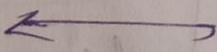
$$\boxed{B=1}$$

∴ By (2)

$$f(x) = 0 + 1 \cdot e^{-x}$$

$$f(x) = e^{-x}$$

By



Q Find $f(x)$ given that $\tilde{f}_s(s) = \frac{e^{-as}}{s}$, $a > 0$, $s > 0$

and hence deduce $\tilde{F}_s(Y_s)$

Sol

$$\tilde{f}_s(s) = \frac{e^{-as}}{s}$$

By Inv. Inversion formula.

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{e^{-as}}{s} \cdot \sin sx ds \quad \text{--- (1)}$$

DH w.r.t α

P-8

$$\frac{df}{dx} = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{e^{-ax}}{x} \cdot 8 \cos 8x dx$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} e^{-ax} \cdot \cos 8x dx = \frac{2}{\pi} \left[\frac{e^{-ax}}{a^2 + x^2} \left[-a \cos 8x + x \sin 8x \right] \right]_{-\infty}^{\infty}$$

$$= \frac{2}{\pi} \left[0 - \frac{1}{a^2 + x^2} (-a \cos 0 + 0) \right] = \frac{2}{\pi} \left(\frac{a}{a^2 + 0^2} \right)$$

$$\frac{df}{dx} = \frac{2a}{\pi(a^2 + x^2)}$$

S.B.S

$$f = \frac{2a}{\pi} \int \frac{dx}{a^2 + x^2} = \frac{2a}{\pi} \left[\frac{1}{a} \tan^{-1} \frac{x}{a} \right] + C$$

$$f(x) = \frac{2}{\pi} \tan^{-1} \frac{x}{a} + C \quad \text{--- (1)}$$

At $x = \infty$ in Eq (1) $f(\infty) = 0$

$$f(\infty) = \frac{2}{\pi} \tan^{-1} \infty + C \quad \therefore C = 0$$

$$f(x) = \frac{2}{\pi} \tan^{-1} \frac{x}{a}$$

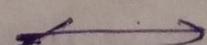
Deduction

At $a = 0$

$$f(x) = \frac{2}{\pi} \tan^{-1} \frac{x}{a}$$

$$= \frac{2}{\pi} (\pi/2)$$

$$f(x) = F^{-1}(Y_s) = \frac{\pi}{2} + \frac{x}{a} \pi \quad \underline{\underline{=}}$$



Q
P

Find the Fourier Transform. of $f(x)$.

$$\therefore f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Hence deduce

Sol

$$\begin{aligned} F(f(x)) &= \int_{-\infty}^{\infty} f(x) \cdot e^{-isx} dx \\ &= \int_{-\infty}^{-1} f(x) \cdot e^{-isx} dx + \int_{-1}^1 f(x) \cdot e^{-isx} dx + \int_1^{\infty} f(x) \cdot e^{-isx} dx \\ &\quad 0 + \int_{-1}^1 f(x) \cdot e^{-isx} dx + 0 \\ &= \int_{-1}^1 1 \cdot e^{-isx} dx \quad (\because f(x) = 1 \text{ for } |x| < 1) \\ &= \frac{-1}{is} \left[e^{isx} \right]_{-1}^1 \\ -\frac{1}{is} \left[e^{-is} - e^{+is} \right] &= \frac{+1}{is} \left[\frac{e^{is} - e^{-is}}{2i} \right] \\ &= \frac{2}{s} \cdot \sin s \end{aligned}$$

$$\left\{ \begin{array}{l} \therefore \text{not } s = 0 \\ \int_{-1}^1 1 \cdot e^s ds = \int_{-1}^1 1 ds \end{array} \right.$$

$$F(f(x)) = \begin{cases} \frac{2 \sin s}{s} & \text{if } s \neq 0 \\ 2 & \text{if } s = 0. \end{cases}$$

$$= [s]_1^{\infty} = 1 + 1 = 2$$

∴ By Inversion formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{isx} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin s}{s} e^{isx} ds \quad (\text{if } 0)$$

P-Q

$$\pi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin s}{s} e^{isx} ds = \boxed{0}$$

$$\therefore \int_0^\infty \frac{2\sin s}{s} e^{isx} ds = 2\pi f(x) = 2\pi \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

$$\int_{-\infty}^{\infty} \frac{2\sin s}{s} e^{isx} ds = \begin{cases} 2\pi & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

$$\text{At } x=0 \quad \int_{-\infty}^{\infty} \frac{2\sin s}{s} \cdot e^0 ds = 2\pi$$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi \quad \left| \begin{array}{l} \text{if } f(s) = \frac{\sin s}{s} \\ \text{if } f(-s) = -\frac{\sin s}{s} = f(s) \end{array} \right.$$

$$= 2 + \int_0^{\infty} \frac{\sin s}{s} ds = \pi$$

$$f(-s) = -\frac{\sin s}{s} = f(s)$$

Even function

$$\boxed{\int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}}$$

Def:- Convolution :- The convolution of two functions

$f(x) * g(x)$ over the int $(-\infty, \infty)$, denoted by $f * g$

$$\therefore f * g = \int_{-\infty}^{\infty} f(u) \cdot g(x-u) du \quad (\because f * g = g * f)$$

Convolution theorem of Fourier Transformation..

Statement:- If $F(f(x))$ and $F(g(x))$ F.T of two function $f(x) + g(x)$. Then the F.T of convolution of $f(x) + g(x)$ is the product of their F.T.

$$F\{f(x) * g(x)\} = F[f(x) + F(g(x))]$$

Proof:-

$$\begin{aligned} F[f(x) + g(x)] &= \int_{-\infty}^{\infty} e^{-isx} (f(x) * g(x)) \quad (\because \text{By def of F.T.}) \\ &= \int_{-\infty}^{\infty} e^{-isx} \left(\int_{-\infty}^{\infty} f(u) \cdot g(x-u) du \right) dx \quad (\because \text{By def of Convolution}) \\ &= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} e^{-isx} g(x-u) dx \right] du \\ &= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} e^{-is(x-u)} + g(x-u) e^{-isu} dx \right] du \quad (\because \text{By change of order of the integration}) \\ &= \int_{-\infty}^{\infty} f(u) e^{-isu} \left[\int_{-\infty}^{\infty} e^{-is(x-u)} + g(x-u) dx \right] du \end{aligned}$$

$$\text{Put } x-u=t.$$

$$dt = dx$$

$$\begin{aligned} &\left(\int_{-\infty}^{\infty} f(u) \cdot e^{-isu} \right) \left(\int_{-\infty}^{\infty} e^{-ist} \cdot g(t) dt \right) du \\ &\left(\underbrace{\int_{-\infty}^{\infty} f(u) \cdot e^{-isu} du}_{\bullet} \right) F(g(t)) \cdot dt = \left(\int_{-\infty}^{\infty} f(x) \cdot e^{-isx} \right) F(g(x)). \\ &\bullet = F(f(x) * g(x)) \quad \text{H.S.} \end{aligned}$$

of the derivative:

P-10

Thm.: If the F.T. of $f(x)$ is $\bar{f}(s)$. Then the F.T. of $f'(x)$ is given by $i\pi \bar{f}(s)$

prove
$$F(f'(x)) = i\pi \bar{f}(s)$$

Proof:
$$F(f'(x)) = \int_{-\infty}^{\infty} e^{isx} \cdot f'(x) dx$$
 By Part

$$\begin{aligned} & \left[e^{isx} \cdot f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} i\pi \cdot e^{isx} \cdot f(x) dx \\ &= 0 + i\pi \int_{-\infty}^{\infty} e^{-isx} \cdot f(x) dx \\ &= i\pi \bar{f}(s) \end{aligned}$$

Ex. $F\left[\frac{d^n}{dx^n} f\right] = (is)^n \cdot \bar{f}(s).$

Thm.: If the Fourier Sine and Cosine Transformations of $f(x)$ and $f'(x)$ exist and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$

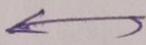
Then. (i) $F_S[f'(x)] = -s F_C[f(x)]$

(ii) $F_C[f'(x)] = s F_C[f(x)] - f(0).$

Proof:
$$\begin{aligned} F_S[f'(x)] &= \int_0^{\infty} f'(x) \cdot \sin sx \cdot dx \quad (\text{By def of SFT}) \\ &= \left[\sin sx \cdot f(x) \right]_{x=0}^{\infty} - \int_0^{\infty} \cos sx \cdot s \cdot f(x) dx \end{aligned}$$

$$0 - 0 - \delta \int_0^\infty f(x) \cdot \cos 8x dx = -\delta F_c[f(x)]$$

$$F_S[f'(x)] = -\delta F_c[f(x)]$$

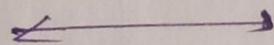


(ii) $F_c[f'(x)] = \int_0^\infty f'(x) \cdot \cos 8x dx$ By Part

$$= [\cos 8x \cdot f(x)]_0^\infty + \int_0^\infty \sin 8x \cdot -\delta f(x) dx$$

$$\left[0 - f(0) \cdot \cos 0 \right] + \delta \underbrace{\int_0^\infty f(x) \cdot \sin 8x dx}_{-f(0) \cdot 1 + \delta F_S[f(x)]}$$

$$F_c[f'(x)] = \delta F_S[f(x)] - f(0)$$



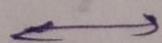
11. we can prove $\delta F_S[f''(x)] = -\delta F_c[f'(x)] = -\delta \{ \delta F_S[f(x)] - f(0) \}$

$$= -\delta^2 F_S[f(x)] + \delta f(0)$$

(ii) $F_c[f''(x)] = \underline{\delta F_S[f'(x)] - f'(0)}$

$$= \delta \left[-\delta F_c[f(x)] \right] - f'(0)$$

$$= -\delta^2 F_c[f(x)] - f'(0)$$



Relation between Fourier and Laplace Transformation.

P-11

Let us define a function $f(t)$. s.t

$$f(t) = \begin{cases} e^{-xt} \phi(t), & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$F[f(t)] = \int_{-\infty}^{\infty} e^{-iyt} f(t) dt \quad (\because \text{By def of F.T})$$

$$= \int_0^{\infty} e^{iyt} \cdot f(t) dt + \int_0^{\infty} e^{-iyt} \cdot f(t) dt$$

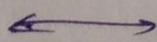
$$= \int_0^{\infty} e^{iyt} \phi(t) dt + \int_0^{\infty} e^{-iyt} \cdot e^{-xt} \phi(t) dt$$

$$0 + \int_0^{\infty} e^{-(x+iy)t} \phi(t) dt$$

$$= \int_0^{\infty} e^{-st} \phi(t) dt \quad \text{Put } x+iy=s$$

$$\Rightarrow L[\phi(t)] \quad \text{By def of L.T}$$

$$F[f(t)] = L[\phi(t)]$$



Parseval's Identity for F.T.

If $F(s)$ and $G(s)$ denoted the F.T of $f(x)$ and $g(x)$. Then

p.t.o

$$(i) \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cdot \overline{G_c(s)} ds = \int_{-\infty}^{\infty} f(x) \cdot \overline{g(x)} dx$$

$$(ii) \frac{1}{2\pi} \int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$

\therefore Both sides
have complex
conjugate.

Proof:-

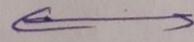
$$\begin{aligned} \textcircled{1} \quad \int_{-\infty}^{\infty} f(x) \cdot \overline{g(x)} dx &= \int_{-\infty}^{\infty} f(x) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G_c(s)} \cdot e^{isx} ds \right] dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G_c(s)} \left\{ \underbrace{\int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx}_{ds} \right\} ds \quad (\because \text{By inverse transformation of } g(x)) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G_c(s)} \cdot F(s) ds \quad (\because \text{By def of F}) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cdot \overline{G_c(s)} ds \end{aligned}$$

H.P

(ii) Put $g(x) = f(x)$ in (i) Part

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \cdot \overline{f(x)} = \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\Rightarrow \frac{1}{2\pi} \int (F(s))^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$



Parseval's Identity For Fourier Sine and cosine

Transformations

Def^- If $F_c(s)$ and $G_c(s)$ denoted the Fourier cosine transformation of $f(x) + g(x)$.