Gauss - Jordan Method

The elementary row transformations which reduce a square matrix A to the unit matrix when applied to the unit matrix, give the inverse matrix A^{-1}

Let A be a non-singular square matrix.

Then $A = IA$

Apply suitable elementary row operations to A on the L.H.S. so that A is reduced to I.

Simultaneously, apply the same elementary row operations to the pre-factor I on R.H.S.

Let I reduces to B, so that $\boxed{I = BA}$

Post multiplying by A^{-1}, we get

$$IA^{-1} = BAA^{-1}$$

$$\Rightarrow A^{-1} = B(AA^{-1})$$

$$= BI$$

$$= B$$

$$\therefore B = A^{-1}$$

Note 1 :- In practice, to find the inverse of A by E-row operations, we write A and I side by side in the form [A : I] and the same E-row operations are performed on both. As soon as A is reduced to I, I will reduce to A^{-1}!

Note 2 :- This method fails when $|A| = 0$

Note 3 :- [A : I] is known as augmented matrix.

Problem :- Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by elementary row operations.

Solution :- Writing the given matrix side by side with unit matrix I_3, we get

$$[A : I_3] = \left[\begin{array}{ccc:ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

operating $R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

operating $R_3 \longrightarrow R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right]$$

operating $R_1 \rightarrow R_1 - 2R_2$, $R_3 \longrightarrow R_3 + 5R_2$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right]$$

operating $R_3 \longrightarrow \frac{1}{2} R_3$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

operating $R_1 \rightarrow R_1 + R_3$, $R_2 \longrightarrow R_2 - 2R_3$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$= [I_3 : A^{-1}]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Exercise

Using Gauss-Jordan method, find the inverse of the following matrices

1) $A = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$

Solⁿ $A^{-1} = \begin{bmatrix} -23 & 29 & \frac{-64}{5} & \frac{-18}{5} \\ 10 & -12 & \frac{26}{5} & \frac{7}{5} \\ 1 & -2 & \frac{6}{5} & \frac{2}{5} \\ 2 & -2 & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Solⁿ $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

3) $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 2 \end{bmatrix}$

Solⁿ $A^{-1} = \frac{1}{40}\begin{bmatrix} -12 & 14 & 2 \\ 14 & -23 & 11 \\ 2 & 11 & -7 \end{bmatrix}$

4) $A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$

Solⁿ $A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$

5) $A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix}$

Solⁿ $A^{-1} = \begin{bmatrix} 1 & 4 & 8 \\ 1 & 3 & 7 \\ 0 & -1 & -2 \end{bmatrix}$