Gamma dist

Gamma function is defined as

$$\boxed{\Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx}$$

$$\boxed{\frac{\Gamma n}{a^n} = \int_0^\infty x^{n-1} e^{-ax}\, dx}$$

$$\boxed{\begin{aligned} \Gamma(n+1) &= n\Gamma n \\ \Gamma n &= \lfloor (n-1) \end{aligned}}$$

Gamma dist :

A cont. R.V X is said to follow Gamma dist. if its prob. density function is defined as

1st kind → standard form

2nd kind

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} & ; x > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

where $\alpha > 0$

$$\int_{-\infty}^{\infty} \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)}\, dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x}$$

$$= \frac{1}{\Gamma(\alpha)} \Gamma\alpha = 1$$

1. Impact fun
2. First decision
3. Format / prints

Exponential dist.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{elsewhere,} \end{cases}$$

$$\int_{-\infty}^{\infty} \lambda e^{-\lambda x}\, dx = 1$$

$$\boxed{\int x^{n-1} e^{-ax}\, dx = \frac{\Gamma n}{a^n}}$$

* Mean $= \displaystyle\int_0^{\infty} \lambda x e^{-\lambda x}\, dx$

$= \lambda \displaystyle\int_0^{\infty} x e^{-\lambda x}\, dx$

$= \lambda \left[\left. x \times \frac{e^{-\lambda x}}{-\lambda} \right|_0^{\infty} + \displaystyle\int_0^{\infty} \frac{e^{-\lambda x}}{-\lambda}\, dx \right]$

$= \lambda \left[\frac{1}{\lambda} \left. \frac{e^{-\lambda x}}{-\lambda} \right|_0^{\infty} \right]$

$= -\frac{1}{\lambda}(0-1) = \frac{1}{\lambda}$

* Var =

$^{2}\quad E(x^2) - E(x)^2$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx = \lambda \int_0^{\infty} x^2 e^{-\lambda x}\,dx = \lambda \frac{\Gamma 3}{\lambda^3} = \frac{2}{\lambda^2}$$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\,dx = \lambda \int_0^{\infty} x\, e^{-\lambda x}\,dx = \cancel{\lambda} \times \frac{1}{\lambda^{\cancel{2}}}$$

$$= \frac{1}{\lambda}$$

$$\boxed{var = \frac{1}{\lambda^2}}$$

<u>MGF</u>

$$E(e^{tx}) = \int_{-\infty}^{\infty} e^{xt} f(x)\,dx$$

$$= \lambda \int_0^{\infty} e^{xt} e^{-\lambda x}\,dx$$

$$= \lambda \left[\int_0^{\infty} e^{-x(\lambda - t)}\,dx\right]$$

$$= \lambda \left| -\frac{e^{-x(\lambda - t)}}{(\lambda - t)} \right|_0^{\infty} = \frac{\lambda}{\lambda - t}$$

Name of student:
Enrolment numb

of exponential dist.

$$P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{0}^{x} \lambda e^{-\lambda x}\,dx$$

$$= -\lambda \left[\frac{e^{-\lambda x}}{\lambda} \right]_0^x$$

$$= -(e^{-\lambda x} - 1)$$

$$\boxed{F_X(x) = 1 - e^{-\lambda x}}$$

Ex) The time (in hours) required to repair a machine is exponential distributed with parameter $1/3$. What is the prob. that the repair time exceeds 3 hours?

Sol^n^ Required prob $= P(X > 3)$

$$= \int_{3}^{\infty} \frac{1}{3} e^{-x/3}\,dx$$

$$= 1/e.$$

(5)

$1 - F(3)$

$= 1 - (1 - e^{-\frac{1}{3} \times 3})$

$= e^{-1}$

(Memoryless property)

exponential dist, then $P(X > m \mid X > n) = P(X > m-n)$

for $m, n > 0$.

Question: A Bank finds that the Avg time customers have to wait for service is 45 seconds. If the waiting time can be treated as an exponential R.V. What is the prob that a customer will have to wait more than 5 minutes given that already he waited for 2 minutes?

Soln Mean = 45 seconds $\quad \frac{1}{\lambda} = 45$

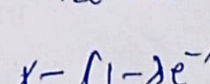
$\lambda = \frac{1}{45}$

$P(X > 5 \text{ min} \mid X > 2 \text{ min})$

$P(X > 300 \mid X > 120)$

$$P(X > 300 \mid X > 120) = \frac{P(X > 300 \cap X > 120)}{P(X > 120)}$$

$$= \frac{P(X > 300)}{P(X > 120)}$$



$$= \frac{1 - P(X \le 300)}{1 - P(X \le 120)} = \frac{1 - F(300)}{1 - F(200)} = \frac{1 - (1 - \lambda e^{-\lambda x})}{1 - (1 - \lambda e^{-\lambda x})}$$

$$= \frac{e^{-\frac{1}{45} \times 300}}{e^{-\frac{1}{45} \times 120}} = e^{-4}$$

(in months)
length of an electric component follows an exponential dist. with parameter 1/2. What is the prob. that the component survives at least 10 months given that already it had survived for more than 9 months?

Sol^n:

$\lambda = 1/2$

$$P(X>10 \mid X>9) = \frac{P(X>10 \cap X>9)}{P(X>9)}$$

$$= \frac{P(X>10)}{P(X>9)} =$$

$$= \cancel{P(X|1)} \quad P(X>1)$$

$$= 1 - P(X \le 1)$$

$$= \cancel{1} - (\cancel{1} - e^{-\lambda x})$$

$$= e^{-\lambda x} = e^{-\frac{1}{2}} \text{ (Ans)}$$

Negative B.D (No. of trials needed to get rth success)

Suppose we have a succession of n bernoulli trials. We assume that (i) trials are independent.

(ii) The prob. of success 'p' in a trial remains constant from trial to trial.

Let $P(x, r, p)$ denote the prob. that there are x failures preceding the rth success in $(x+r)$ trials

Now the last trial must be a success, whose prob is p. In the remaining $(x+r-1)$ trials we must have $(r-1)$ successes whose prob is

by B.D. $\binom{x+r-1}{r-1} p^{r-1} q^{x+r-1-r+1}$

$= \binom{x+r-1}{r-1} p^{r-1} q^{x}$ $\quad x+r=n$

$\therefore$ By probability then $f(x, r, P) = \binom{x+r-1}{r-1} p^{r-1} \cdot q^{x} \cdot p$

$$\boxed{P(x, r, p) = \binom{x+r-1}{r-1} p^{r} q^{x}}$$

Mean =

$$\boxed{P(x) = \binom{n-1}{r-1} p^{r} q^{n-r}.}$$

$\binom{5-1}{3-1} p^{3} q^{5-3} = {}^{4}C_{3}\, p^{3} q^{2}.$

In ~~negative~~ B.D we have a fixed no. of trials. but in negative B.D we have a fixed no. of success.

$$\boxed{\text{mean} = \sum x P(x) = \frac{rq}{p} \qquad \text{var} = \frac{rq}{p^2}}$$

__Ex:__ If we flip a coin for a fixed no. of times and count the no. of times the coin turns out heads is a B.D. If we continue flipping the coin until it has turned a particular no. of heads say 4th head on flipping 10 times ~~get~~ ~~head~~ is an example of negative B.D.

Q-1 Ram ~~Sita~~ is writing an exam with multi-Choice questions, and it's prob. of attemping the question with the right answer is 60%. What is the prob. that Ram gives the third correct answer for the fifth attempted question?

Sol^n $P(x) = \binom{n-1}{r-1} p^r q^{n-r}$.

$x+r=n$

$x = n-r$.

$n=5$, $r=3$

$p = 0.6$

$q = 0.4$

$P(X=3) = {}^4C_2 (0.6)^3 (0.4)^2$

$= 0.020736$. 0.20736

Unit-2

Probability

[redacted]

Probability is a concept which numerically measure the degree of uncertainty of occurence of events.

If A is an event that can happen in 'm' ways & can fail in 'n' ways, then probability of happening of A is $\frac{m}{m+n}$ and that of not happening of A is $\frac{n}{m+n}$.

Random experiment: Experiments which are performed essentially under some conditions and whose results can not be predicted are called random experiments. for example: rolling a die, tossing a coin.

Sample space: The set of all possible outcomes of a random experiment is called sample space, & denoted by S. For example $S = \{H, T\}$ in a toss of a coin.

Event: The outcome of a random experiment is called an event. Thus every subset of a sample space S is an event.

Mutually exclusive events: Two events are said to mutually exclusive, when the occurrence of one of the excludes the occurrence of the other. For example, on tossing of a coin, either we get head or tail, but not both.

Addition law of Probability: If $p_1, p_2, \ldots, p_n$ are separate probabilities of mutually exclusive events, then the probability P, that any these events will happen is given by

$$P = p_1 + p_2 + \cdots + p_n.$$

Q1 An urn contains 10 black and 10 white balls. Find the probability of drawing two balls of the same colour.

soln:

$$P(\text{Black}) = \frac{{}^{10}C_2}{{}^{20}C_2}$$

$$P(\text{Red}) = \frac{{}^{10}C_2}{{}^{20}C_2}$$

$\therefore$ required probility $= \dfrac{{}^{10}C_2}{{}^{20}C_2} + \dfrac{{}^{10}C_2}{{}^{20}C_2}$ (by addition law of Probability)

$$= \boxed{\frac{9}{19}}$$

b) **Independent events** : Two events may be independent, when the actual happening of one does not influence in any way the probability of the happening of the other. For example, getting a head after tossing a coin and rolling a 5 on a die.

Multiplication Law of Probability : If there are two independent events, the respective probabilities of which are known, then the probability that both will happen is the product of their respective probabilities

$$P(A \text{ and } B) = P(A) \times P(B)$$

Ques A bag contains 10 white and 15 black balls. Two balls are drawn in succession. What is the probability the first is white & second is black?

soln:

$$P(\text{White}) = \frac{10}{25}$$

$$P(\text{Black ball without replacement}) = \frac{15}{24}$$

$$\therefore \text{Required probability} = \frac{10}{25} \times \frac{15}{24}$$

$$= \boxed{\frac{1}{4}}$$

Ques A six-faced die is so biased that, wh... it is twice as likely to show an even ... an odd number. If it is thrown twice, ... is the probability that the sum of two num... thrown is odd?

Soln: A biased die, when thrown, shows even no twice than an odd no.

$$\therefore P(\text{even no.}) = \frac{2}{2+1} = \frac{2}{3}$$

$$P(\text{odd no.}) = \frac{1}{2+1} = \frac{1}{3}$$

Sum of two numbers is odd if the first is even & the second is odd or vice versa.

$$\therefore P(\text{odd sum}) = P(\text{even}) \times P(\text{odd}) + P(\text{odd}) \times P(\text{even})$$

$$= \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3}$$

$$= \boxed{\frac{4}{9}}$$

Conditional Probability : Let A and B be two events of a sample space S and let $P(B) \neq 0$. Then conditional probability of the event A, given B, denoted by $P(A/B)$, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If the events A and B defied on a sample space S of a random experiment are independe then

$$P(A/B) = P(A) \text{ and } P(B/A) = P(B).$$

Baye's Theorem

If $B_1, B_2, ---, B_n$ are mutually exclusive events with $P(B_i) \neq 0$ $(i=1,2,---,n)$ of a random experiment then for any arbitrary even A of the sample space of the above experimen with $P(A)>0$, we have.

$$P(B_i|A) = \frac{P(B_i)\, P(A|B_i)}{\sum_{i=1}^{n} P(B_i)\, P(A|B_i)}$$

Ques 1: In a bulb factory, machines A, B and C manufacture respectively 25%, 35% and 40 of the total. If their output 5, 4 and 2 percen are defective bulbs. A bulb is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine B?

Sol

Soln A: bulb is manufactured by machine A

B: bulb is manufactured by machine B

C: bulb is manufactured by machine C

D: bulb is defective.

Then $P(A) = 0.25$

$P(B) = 0.35$

$P(C) = 0.40$

Probability of drawing defective bulb manufactured by machine A is $P(D/A) = 0.05$

Similarly, $P(D/B) = 0.04$

$P(D/C) = 0.02$

To find : $P(B/D)$

By Baye theorem,

$$P(B/D) = \frac{P(B)\ P(D/B)}{P(A)\ P(D/A) + P(B).P(D/B) + P(C)\ P(D/C)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02}$$

$$= \boxed{0.41}$$

Practice Problems :

1 An Urn I contains 3 white and 4 red balls and an Urn II contains 5 white and 6 red balls. One ball is drawn at random from one of the urns and is found to be white. Find the probability that it was drawn from urn I.

[Ans: 33/68]

Q2 Three machines I, II and III manufacture respectively 0.4, 0.5 and 0.1 of the total production. The percentage of defective items produced by I, II and III is 2, 4 and 1 %, respectively. For an item chosen at random, what is the probability that it is defective?

[Ans: 0.029].

Q3 A can hit a target 3 times in 5 shots, B 2 times in 5 shots and C three times in 4 shots. All of them fire one shot each simultaneously at the target. What is the probability that

(i) 2 shots hit

(ii) At least two shots hit.

$\left[\frac{9}{20}, \frac{63}{100}\right]$

Q4 There are 6 positive and 8 negative numbers. Four numbers are chosen at random, without replacement and multiplied. What is the prob. that the product is a positive number.

[Ans: $\frac{505}{1001}$]

Random variable :

If a real variable X is associated with the outcome of a random experiment then the values which X takes depend on chance and is called a random variable.

Example: Random experiment: Tossing a pair of dice

$X \rightarrow$ sum of two numbers

Then X is a random variable which can take values $2, 3, 4, ---, 12$.

If a random variable take a finite set of values, it is called discrete variate.

If it takes values in an interval, then it is called continuous variate.

Discrete probability distribution :

Let X be a discrete variate and it takes values x_i, $i = 1, 2, \ldots$

If $P(X = x_i) = p_i$ or $p(x_i)$, $i = 1, 2, \ldots$

where (i) $p(x_i) \geq 0$ for all i.

(ii) $\sum_i p(x_i) = 1$.

The set of values x_i with their probabilities p_i constitute a discrete probability distribution of the discrete variate X.

Distribution function :

The distribution function $F(x)$ of the discrete variate X is defined by

$$F(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$$, where x is an integer.

It is also called cumulative distribution function.

Ques 1 : The probability density function of a variate X is

X	0	1	2	3	4	5	6
$p(x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(i) Find $P(X<4)$, $P(X \geq 5)$, $P(3<X\leq 6)$

(ii) What is the minimum value of k so that

$$P(X \leq 2) \geq 0.3.$$

Soln: (i) First find k

Since $\sum_{i=0}^{6} p(x_i) = 1$

$\Rightarrow k + 3k + \cdots + 13k = 1$

$\Rightarrow \boxed{k = 1/49}$

$\therefore P(X<4) = k + 3k + 5k + 7k$

$= 16k = 16/49$

$P(X \geq 5) = 11k + 13k$

$= 24k = 24/49$

$P(3<X\leq 6) = 9k + 11k + 13k$

$= 33k = 33/49$

(ii) $P(X \leq 2) \geq 0.3$

$\Rightarrow k + 3k + 5k \geq 0.3$

$\Rightarrow 9k \geq 0.3$

$\Rightarrow \boxed{k \geq \frac{1}{30}}$

$\therefore$ minimum value of $k = \boxed{\frac{1}{30}}$

6.)

that

Continuous probability distribution

If X takes values in an interval then it gives rise to continuous distribution of X.

The probability distribution of a continuous variate x is defined by a function $f(x)$ such that the probability of the variate x falling in the small interval $(x - \frac{1}{2}dx, x + \frac{1}{2}dx)$ is $f(x)dx$.

Symbolically

$$P\left(x - \frac{1}{2}dx \le x \le x + \frac{1}{2}dx\right) = f(x)\,dx$$

$f(x)$ is called probability density function and the continuous curve $y = f(x)$ is called probability curve.

(i) $f(x) \ge 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(that is, total area under the probability curve is 1.).

Distribution function

If $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$, then

$F(x)$ is defined as the cumulative distribution function or just distribution function of continuous variate X.

<u>Properties</u>

(i) $F'(x) = f(x) \geq 0$

$\therefore F(x)$ is a non-decreasing function

(ii) $F(-\infty) = 0$

(iii) $F(\infty) = 1$

(iv) $$P(a \leq x \leq b) = \int_a^b f(x)\,dx$$

$$= \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= F(b) - F(a)$$

<u>Ques 1</u> (i) Is the function defined as follows a density function?

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

(ii) If so, determine the probability that the variate having this density will fall in the interval $(1, 2)$?

(iii) Also find the cumulative probability function $F(2)$.

continuous

Sol

Soln (i) $f(x) \geq 0$

$(\because e^{-x} \geq 0$ for all x in $(1,2))$

Also,

$$\int_{-\infty}^{\infty} f(x).dx = \int_{-\infty}^{0} 0.dx + \int_{0}^{\infty} e^{-x} dx$$

$$= -e^{-x}\Big|_0^{\infty}$$

$$= \boxed{1}$$

$\therefore f(x)$ satisfies the requirements for a density f^n.

(ii) $P(1 \leq x \leq 2) = \int_1^2 e^{-x} dx$

$$= -e^{-x}\Big|_1^2$$

$$= \boxed{0.233}$$

(iii) cummulative probability f^n.

$$F(2) = \int_{-\infty}^{2} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0.dx + \int_0^2 e^{-x} dx$$

$$= 1 - e^{-2}.$$

$$= \boxed{0.865}$$

Expectation :

The mean value (μ) of the probability distribution of a variate X is known as expectation and denoted by E(X).

If $f(x)$ is a probability density function of the variate X, then

$$E(X) = \sum_i x_i f(x_i) \quad \text{(discrete distribution)}$$

and

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \quad \text{(continuous distribution)}$$

→ Expectation of any function $\phi(x)$ is given by

$$E[\phi(x)] = \sum_i \phi(x_i) f(x_i)$$

$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x) f(x)\, dx.$$

→ Variance : variance of a distribution is given by

$$\sigma^2 = \sum_i (x_i - \mu)^2 f(x_i) \quad \text{(discrete)}$$

or

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx \quad \text{(continuous)}$$

$\sigma \rightarrow$ standard deviation.

rth moment about mean

It is denoted by μ_r and is defined by

$$\mu_r = \sum_i (x_i-\mu)^r f(x_i) \quad \text{(discrete)}$$

or

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx \quad \text{(continuous)}$$

Mean deviation about mean

$$E(|X-\mu|) = \sum_i |x_i-\mu| f(x_i)$$

or

$$E(|X-\mu|) = \int_{-\infty}^{\infty} |x-\mu| f(x)\, dx$$

Ques 1. A distribution function is defined as follows

$$F(x) = \begin{cases} 0 & , \quad x \le 1 \\ \frac{1}{16}(x-1)^4 & , \quad 1 \le x \le 3 \\ 1 & , \quad x > 3. \end{cases}$$

Find the density function $f(x)$. Find the mean of x.

soln: Density function is given by

$$f(x) = F'(x) = \begin{cases} \frac{1}{4}(x-1)^3 & , \quad 1 \le x \le 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$\text{Mean} = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \frac{1}{4}\int_{1}^{3} x(x-1)^3\, dx$$

$$= \boxed{2.6}$$

Ques 2: A person draws 2 balls from a bag containing 3 white and 4 red balls. If he is to receive 10 P for every white ball which he draws and 20 P for each red ball. Find his expectation.

Soln: Three possibilities are:

(i) Both white In this case he receives 20 P

and $P(X=20) = \frac{^3C_2}{^7C_2} = \frac{3}{21}$

(ii) 1 white and 1 red In this case he receives 30 P and

$P(X=30) = \frac{^3C_1 \times {^4C_1}}{^7C_2} = \frac{12}{21}$.

(iii) 2 red. In this case he receives 40 P

and

$P(X=40) = \frac{^4C_2}{^7C_2} = \frac{6}{21}$