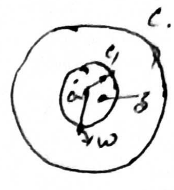
(1). (unit 4)

I **Taylor's theorem** → if a function $f(z)$ is analytic at all points inside a circle C_1 with its centre at the point 'a' and radius R, then at each point z inside C.

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{\lfloor 2}(z-a)^2 + \dots + \frac{f^n(a)}{\lfloor n}(z-a)^n + \dots$$

proof → Take any point z inside C. Draw a circle C_1 with centre a, enclosing the point z. Let w be a point on circle C_1



$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{w-a-(z-a)}$$

$$= \frac{1}{w-a}\left(\frac{1}{1-\frac{(z-a)}{(w-a)}}\right) = \frac{1}{(w-a)}\left[1-\frac{(z-a)}{(w-a)}\right]^{-1}$$

Applying Binomial.

$$\frac{1}{w-z} = \frac{1}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2 + \dots + \left(\frac{z-a}{w-a}\right)^n + \dots\right]$$

$$= \left[\frac{1}{(w-a)} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \dots + \frac{(z-a)^n}{(w-a)^{n+1}} + \dots\right] \to (1)$$

As $|z-a| < |w-a| \Rightarrow \frac{|z-a|}{|w-a|} < 1$

so that series converges uniformly. Hence the series is integrable.

multiply (1) by $f(w)$

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-a} + (z-a)\frac{f(w)}{(w-a)^2} + (z-a)^2\frac{f(w)}{(w-a)^3} + \dots + (z-a)^n\frac{f(w)}{(w-a)^{n+1}} + \dots$$

On integrating w.r.to w for C_1

$$\int_{C_1}\frac{f(w)}{w-z}\,dw = \int_{C_1}\frac{f(w)}{(w-a)}\,dw + (z-a)\int_{C_1}\frac{f(w)}{(w-a)^2}\,dw + \dots + (z-a)^n\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw$$

We know that

$$\int \frac{f(w)}{w-z}\, dz = 2\pi i\, f(z), \quad \int_{c_1} \frac{f(w)}{(w-a)}\, dw = 2\pi i\, f(a)$$

$$\int_c \frac{f(w)}{(w-a)^2} = \frac{f'(a)}{\lfloor 1}(z-a)' 2\pi i$$ and so on. [by nth derivative formula]

substituting in ⑤:

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{\lfloor 2}(z-a)^2 + \cdots + \frac{f^n(a)\,(z-a)^n}{\lfloor n} + \cdots$$

Q. find the first four term of the Talor's series expansion of the complex variable function.

$$f(z) = \frac{z+1}{(z-3)(z-4)}$$ about $z = 2$ find the Region of convergence.

Q find the Taylor's Series Expansion of a function of complex variable.

$$f(z) = \frac{1}{(z-1)(z-3)} \quad \text{about } z = 4$$

Sol^n $$f(z) = \frac{1}{(z-1)(z-3)} = \frac{1}{2}\left[\frac{1}{(z-3)} - \frac{1}{(z-1)}\right] \text{ by partial fraction.}$$

$$= \frac{1}{2}\left[\frac{1}{(z-4+1)} - \frac{1}{(z-4+3)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\{1+(z-4)\}} - \frac{1}{3\{1+\left(\frac{z-4}{3}\right)\}}\right]$$

$$= \frac{1}{2}\left[\{1+(z-4)\}^{-1} - \frac{1}{3}\left\{1+\frac{(z-4)}{3}\right\}^{-1}\right]$$

$$= \frac{1}{2}\left[1-(z-4)+(z-4)^2-(z-4)^3+\cdots\right] - \frac{1}{6}\left[1-\frac{(z-4)}{3}+\frac{(z-4)^2}{9}-\cdots\right]$$

Alternative method — Taylor's theorem is given by:

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots \quad (1)$$

$$f(z) = \frac{1}{2}\left[\frac{1}{(z-3)} - \frac{1}{(z-1)}\right] \quad \text{Here } a = 4$$

$$z=4 \quad f(4) = \frac{1}{2}\left[\frac{1}{(4-3)} - \frac{1}{(4-1)}\right] = 1/3$$

$$f'(z) = \frac{1}{2}\left[-\frac{1}{(z-3)^2} + \frac{1}{(z-1)^2}\right] \Rightarrow f'(4) = \frac{1}{2}\left[\frac{-1}{(4-3)^2} + \frac{1}{(4-1)^2}\right] = \frac{4}{9}$$

$$= -4/9$$

$$f''(z) = \frac{1}{2}\left[+\frac{2}{(z-3)^3} - \frac{2}{(z-1)^3}\right] \Rightarrow f''(4) = 26/27$$

$$f'''(z) = \frac{1}{2}\left[-\frac{6}{(z-1)^4} + \frac{6}{(z-1)^4}\right] \Rightarrow f'''(4) = -80/27$$

from (1)

$$f(z) = 1/3 + (z-4)(-4/9) + \frac{(z-4)^2}{2!}\left(\frac{26}{27}\right) + \frac{(z-4)^3}{3!}\left(\frac{-80}{27}\right) - \cdots$$

Q Expand the function by Taylor Series and Maclaurin Series.

$$f(z) = \frac{(z+1)}{(z-3)(z-4)} \quad \text{about } z=2$$

Soln $f(z) = \frac{-4}{z-3} + \frac{5}{(z-4)}$ by partial fraction.

$$= \frac{-4}{[(z-2)-1]} + \frac{5}{[(z-2)-2]} = 4[1-(z-2)]^{-1} - \frac{5}{2}\left[1-\left(\frac{z-2}{2}\right)\right]$$

$$= 4\left[1+(z-2)+(z-2)^2+\cdots\right] - \frac{5}{2}\left[1+\frac{(z-2)}{2}+\frac{(z-2)^2}{4}+\frac{(z-2)^3}{8}+\cdots\right]$$

Maclaurin Series: A Taylor series with centre $a=0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{\lfloor n} z^n$$ is reffered to as

Maclaurin series

$$f(z) = f(0) + \frac{f'(0)}{\lfloor 1} \times z + \frac{f''(0)}{\lfloor 2} z^2 + \cdots + 0$$

Soln $f(z) = \frac{-4}{z-3} + \frac{5}{z-4} \Rightarrow f(0)^{z=0} = \frac{-4}{-3} + \frac{5}{-4} = \frac{1}{12} = \frac{1}{12}$

$$f'(z) = \frac{4}{(z-3)^2} - \frac{5}{(z-4)^2} \Rightarrow f'(0) = \frac{4}{9} - \frac{5}{16} = \frac{19}{144}$$

$$f''(z) = \frac{-8}{(z-3)^3} + \frac{10}{(z-4)^3} \Rightarrow f'(0) = \frac{-8}{27} + \frac{10}{64} = -\frac{242}{1728}$$

$$= \frac{121}{864}$$

$$f(z) = \frac{1}{12} + \frac{19z}{144} + \frac{121}{864 \times 2} z^2 + \cdots$$ [Maclaurin Series]

Q Show that

$a = 1$ by comp. $(z-1)$

$$\log z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \dots$$

when $|z-1| < 1$

$$\log z = \log[1+(z-1)]$$

TAYLOR'S series

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + -$$

$f(z) = \log z$ $\qquad f(1) = \log 1 = 0$

$$f'(z) = \frac{1}{z}$$

$$f'(1) = 1$$

$$f''(z) = -\frac{1}{z^2} \qquad f''(1) = -1$$

$$f'''(z) = \frac{2}{z^3} \qquad f''' = \frac{2}{1} = 2$$

$$f(z) = 0 + \frac{(z-1)\,1}{1!} + \frac{(z-a)^2 \cdot 1}{2} + \frac{(z-a)^3\, 2}{6} - - -$$

$$\boxed{f(z) \to 0 + (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - - - - -}$$

Unit - 2

Laurent's theorem → If $f(z)$ is analytic on C_1 and C_2, and the annular region R bounded by two concentric centre C_1 and C_2 of radii r_1 and r_2 ; ($r_1 > r_2$) with the centre a, then for all z in R ;

$$f(z) = \left[a_0 + a_1 (z-a) + a_2 (z-a)^2 + \cdots \right] + \left[\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots \right]$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw$, $b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)\, dw}{(w-a)^{-n+1}}$

Cut AB

1| Q expand $\frac{1}{z^2-3z+2}$ in the Region. [Question based on Taylor's and Laurent's series]

(a) $|z|<1$ (b) $1<|z|<2$ (c) $|z|>2$

(d) $0<|z-1|<1$

Soln.
$$f(z) = \frac{1}{(z^2-3z+2)} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{(z-1)}$$

for $|z|<1$ ∵ $|z|<1$, $|z/2|<1$,

(a)
$$\frac{1}{(1-z)} + \frac{1}{(z-2)} = (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= (1-z)^{-1} - \frac{1}{2}\left(1-z/2\right)^{-1}$$

$$= (1+z+z^2+z^3+\cdots) - \frac{1}{2}\left(1+\frac{z}{2}+\frac{z}{2}^2+\cdots\right)$$

$$f(z) = \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \cdots$$

(b) $1<|z|<2$, $\frac{1}{|z|}<1$, $\frac{|z|}{2}<1$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{-2}\left(1-z/2\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left(1+z/2+z^2/2^2+\cdots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right)$$

(c) $|z|>2$ $\quad 1>\left|\frac{2}{|z|}\right|$ ∵ $\frac{2}{z}<1$ ∴ $\left|\frac{1}{z}\right|<1$

by ∵ inequality property $\frac{1}{|z|}<\frac{1}{2} \Rightarrow \frac{2}{|z|}<1$

$$f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\right) - \frac{1}{z}\left(1+\left(\frac{1}{z}\right)^1+\left(\frac{1}{z}\right)^2+\cdots\right]$$

$0 < |z-1| < 1$ (2)

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)} = \frac{1}{(z-1)-1} - \frac{1}{(z-1)}.$$

$$= -(z-1)^{-1} - [1-(z-1)]^{-1}$$

$$= -(z-1)^{-1} - [1+(z-1)+(z-1)^2+\cdots].$$

Q prove that $\cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n+\frac{1}{z^n}\right)$.

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cosh(2\cos\theta)\, d\theta$

Sol suppose $f(z) = \cosh(z+z^{-1})$ then the function $f(z)$ is analytic in every finite part of z plane except at $z=0$ Hence function is analytic in the annulus $r \le |z| \le R$, where r is small and R is large therefore $f(z)$ can be expanded in Laurent's series

$$\cosh(z+z^{-1}) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

where $a_n = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z^{n+1}}$, $b_n = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z^{-n+1}}\,dz$

where C is a unit circle $|z|=1$ so that $z = e^{i\theta}$

$$a_n = \frac{1}{2\pi i}\int_c \frac{\cosh(z+z^{-1})}{z^{n+1}}\,dz = \int_0^{2\pi} \frac{\cosh(e^{i\theta}+e^{-i\theta})}{e^{i(n+1)\theta}}\, ie^{i\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\, e^{-in\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)(\cos n\theta - i\sin n\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta - \frac{i}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\sin n\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta - 0$$

by definite Integral property VI

$f(2\pi-\theta) = -f(\theta)$ then $\int_0^{2\pi} f(\theta)\, d\theta = 0$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta$$

$$b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \cos(2\cos\theta)\cos(-n\theta)\, d\theta = a_n$$

$$\cosh(z+z^{-1}) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n z^{-n}$$

$$\cosh(z+z^{-1}) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n z^{-n}$$

$$\cosh(z+z^{-1}) = a_0 + \sum_{n=1}^{\infty} a_n (z^n + z^{-n}).$$

this complete the proof.

Q prove that the function $\sin\left[c\left(z+\frac{1}{z}\right)\right]$ can be expanded in a series of type

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

in which the coefficient of both of z^n

z^{-n} are $\frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta\, d\theta$

Q $e^{c/2\,(z-z^{-1})} = \sum_{-\infty}^{\infty} a_n z^n$ where.

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\, d\theta.$$

$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - -$ when $|z| <$

Obtain the Taylor's and Laurent's series which represent the function

$$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

in the region

(i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$

Solⁿ:- We're

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{(z^2+5z+6)} = 1 - \frac{5z+7}{(z^2+5z+6)}$$

Now we're resolve $\frac{5z+7}{z^2+5z+6}$ into Partial fractions

$$\frac{5z+7}{(z+2)(z+3)} = \frac{A}{(z+2)} + \frac{B}{(z+3)}$$

$$5z+7 = A(z+3) + B(z+2)$$

Putting $z = -3$

$$5(-3)+7 = B(-3+2)$$

$$B = 8$$

Putting $z = -2$

$$5(-2)+7 = A(-2+3)$$

$$A = -3$$

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 - \left\{\frac{-3}{(z+2)} + \frac{8}{z+3}\right\}$$

$$= 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)}$$

(i) for $|z| < 2$ $\therefore \frac{|z|}{2} < 1$

$$f(z) = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)} = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$= 1 + \frac{3}{2}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right)$ ②

$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$

(ii) for $2 < |z| < 3$

$\frac{|z|}{3} < 1 \quad , \quad \frac{2}{|z|} < 1$

$f(z) = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)} = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$

$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right)$

$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$

(iii) for $|z| > 3$

$\frac{3}{|z|} < 1$

$f(z) = 1 - \frac{3}{(z+2)} - \frac{8}{(z+3)} = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$

$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$

$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right) - \frac{8}{z}\left(1 - \frac{3}{z} + \frac{3^2}{z^2} - \frac{3^3}{z^3} + \cdots\right)$

$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$

Q Obtain the expansion for $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ which are valid (i) $|z| < 1$ (ii) $1 < |z| < 4$ (iii) $|z| > 4$.

Find the Laurent expression for

$$f(z) = \frac{7z-2}{z^3 - z^2 - 2z}$$

in the regions given by

i) $0 < |z+1| < 1$ (ii) $1 < |z+1| < 3$ (iii) $|z+1| > 3$

Solⁿ: We've

$$f(z) = \frac{7z-2}{z^3 - z^2 - 2z} = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$$

$$= \frac{1}{(z+1)-1} - \frac{3}{z+1} + \frac{2}{(z+1)-3}$$

(i) $0 < |z+1| < 1$

$$f(z) = -1[1-(z+1)]^{-1} - \frac{3}{(z+1)} - \frac{2}{3}\left\{1 - \frac{(z+1)}{3}\right\}^{-1}$$

$$= \frac{-3}{(z+1)} - \sum_{n=0}^{\infty} (z+1)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

This is a series in negative and positive powers of $(z+1)$ hence it is an expension of $f(z)$ in Laurent's Series within the annulus $0 < |z+1| < 1$

(ii) $1 < |z+1| < 3$

$$f(z) = \frac{1}{z+1}\left(1 - \frac{1}{z+1}\right)^{-1} - \frac{3}{z+1} - \frac{2}{3}\left\{1 - \left(\frac{z+1}{3}\right)\right\}^{-1}$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{(z+1)} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

This is also a series in negative and positive powers of $(z+1)$ hence it is an expression of $f(z)$ in Lausent's series within the annulus $1 < |z+1| < 3$

(iii) $|z+1| > 3$

$$f(z) = \frac{1}{z+1}\left(1 - \frac{1}{z+1}\right)^{-1} - \frac{3}{(z+1)} + \frac{2}{(z+1)}\left(1 - \frac{3}{z+1}\right)^{-1}$$

$$= \frac{1}{(z+1)}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{(z+1)} + \frac{2}{(z+1)}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n$$

This is a series in negative powers of $(z+1)$ hence it is an expansion of $f(z)$ in Laurent's Series within the annulus $3 < |z+1| < R$ where R is large.