Date 14/4/23

$\left[\begin{matrix}\text{APPLIED} \\ \text{MATHEMATICS - 2}\end{matrix}\right]$

Unit - I
Complex Analysis - I

Complex Numbers

As ; $\sqrt{9} = \pm 3$

As ; $\sqrt{16} = \pm 4$

Now $\Rightarrow \sqrt{-9} = \sqrt{-1 \times 9}$

$= \sqrt{-1} \times \sqrt{9}$

$\Rightarrow \sqrt{-1} \times (\pm 3)$

where $\sqrt{-1} = i =$ imaginary number

$\sqrt{-9} = i(\pm 3) = \pm 3i$

Means ; $\sqrt{-1} = i$

Square both side

$\boxed{i^2 = -1}$

$\Rightarrow$ Complex Numbers

Complex Number $(z) =$ Real + Imaginary Part

$\therefore \boxed{z = \underbrace{x}_{\text{Real Part}} + \underbrace{iy}_{\text{Imaginary Part}}}$

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⇒ Representation / Forms of Complex No.

1. Cartesian form $(z = x + iy)$
2. Polar form $(z = re^{i\theta} = r(\cos\theta + i\sin\theta))$
3. Exponential form $(z = re^{i\theta})$

⇒ Operations of Complex Numbers

1. Addition : If $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

then, $(z_1 + z_2) = (x_1 + iy_1) + (x_2 + iy_2)$

$\Rightarrow x_1 + iy_1 + x_2 + iy_2$

$\Rightarrow \underbrace{(x_1 + x_2)}_{\text{Real}} + i\underbrace{(y_1 + y_2)}_{\text{Imaginary}}$

2. Subtraction : If $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

then $(z_1 - z_2) = (x_1 + iy_1) - (x_2 + iy_2)$

$\Rightarrow x_1 + iy_1 - x_2 - iy_2$

$\Rightarrow \underbrace{(x_1 - x_2)}_{\text{Real}} + i\underbrace{(y_1 - y_2)}_{\text{Imaginary}}$

3. Multiplication: $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

then $z_1 . z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$

$\Rightarrow x_1x_2 + ix_1y_2 + ix_2y_1 + i^2 y_1y_2$

where $i^2 = -1$

$\Rightarrow x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2$

$\Rightarrow \underbrace{(x_1x_2 - y_1y_2)}_{\text{Real}} + i\underbrace{(x_1y_2 + x_2y_1)}_{\text{Imaginary}}$

④ Powers of i :

$i = \sqrt{-1} \quad \therefore \quad i^2 = -1$

$i^3 = i^{2+1} \Rightarrow i^2 . i^1 \quad \Rightarrow -1 \times i \Rightarrow -i$

$i^4 = i^{2+2} = i^2 . i^2 = (-1) \times (-1) \quad = 1$

$i^5 = i^{3+2} = i^3 . i^2 = -1 \times i^3 = -1 \times -i = i$

$i^6 = i^{3+3} \quad = i^3 . i^3 = (-i)(-i) \quad = i^2 = -1$

$i^2 = -1$
$i^3 = -1$
$i^4 = 1$
$i^5 = i$
$i^6 = -1$

⑤ Conjugate of Complex Number
(change only sign of imaginary no.)

If $z = x + iy$;
then conjugate of z is $\bar{z}$ i.e.,

$$\boxed{\bar{z} = x - iy}$$

⑥ Division of Complex Numbers

If $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

$$\therefore \frac{z_1}{z_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)}$$

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Apply Rationalization;

$\therefore \frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2}$

$\Rightarrow \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2)^2-(iy_2)^2}$

$\Rightarrow \frac{(x_1x_2 - ix_1y_2 + ix_2y_1 - i^2y_1y_2)}{x_2^2+y_2^2}$

$\Rightarrow \frac{x_1x_2 - ix_1y_2 + ix_2y_1 + y_1y_2}{x_2^2+y_2^2}$

$\Rightarrow \frac{(x_1x_2+y_1y_2)}{(x_2^2+y_2^2)} + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2+y_2^2}\right)$

Q. Express in form of $a+ib$ if $\frac{3+2i}{2-3i}$ / Separate Real & Imaginary.

Sol→ $z = \frac{3+2i}{2-3i}$

(Rationalize)

$z = \frac{3+2i}{2-3i} \times \left(\frac{2+3i}{2+3i}\right) \Rightarrow \frac{6+9i+4i-6}{2^2+9}$

$\Rightarrow \frac{13i}{13} \Rightarrow 0+1i$

$\boxed{a+ib = \underbrace{\frac{0}{13}}_{\text{Real}} + i\underbrace{1}_{\text{Imaginary}}}$ Ans

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Q. Express in form of $(a+ib)$ if $z = \frac{2-\sqrt{3}i}{1+i}$.

Sol→ $z = \frac{2-\sqrt{3}\,i}{1+i}$

(Rationalize)

$$z = \frac{2-\sqrt{3}i}{1+i} \times \frac{1-i}{1-i} \Rightarrow \frac{2-2i-\sqrt{3}i-\sqrt{3}}{1+1}$$

$$\Rightarrow z = \frac{(2-\sqrt{3})+i(-2-\sqrt{3})}{2}$$

$$\Rightarrow z = \frac{(2-\sqrt{3})}{2} + i\left(\frac{-2-\sqrt{3}}{2}\right)$$

Ans→ $$\boxed{a+ib = \left(\frac{2-\sqrt{3}}{2}\right) + i\left(\frac{-2-\sqrt{3}}{2}\right)}$$

Q. If $x+iy = \frac{1}{a+ib}$; Prove that $(x^2+y^2)(a^2+b^2) = 1$.

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Sol→ Consider;

$$x^2+y^2 = x^2-(iy)^2 = (x+iy)(x-iy)$$

$\because$ $x+iy = \frac{1}{a+ib}$ then $x-iy = \frac{1}{a-ib}$

$$\therefore (x+iy)(x-iy) = \frac{1}{(a+ib)} \cdot \frac{1}{(a-ib)}$$

$$\boxed{(x^2+y^2)(a^2+b^2) = 1}$$

Hence, proved

Q. Separate Real & Imaginary of $z = \frac{5+i}{(1-i)^2}$

Sol→ $z = \frac{5+i}{(1-i)^2} = \frac{5+i}{1-2i+i^2} = \frac{5+i}{1-2i-1}$

$$z = \frac{5+i}{0-2i} \times \frac{0+2i}{0+2i}$$

$$z = \frac{10i-2}{2^2}$$

$$z = \frac{-2+10i}{4}$$

$$\boxed{z = \underbrace{-\frac{1}{2}}_{\text{Real}} + i\underbrace{\left(\frac{5}{2}\right)}_{\text{Imaginary}}}$$

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$\Rightarrow$ Square root of complex number

(1) Find : $\sqrt{3-4i}$

Sol $\rightarrow$ consider ; $z = x+iy = \sqrt{3-4i}$

$\therefore$ Square both sides

$\Rightarrow z^2 = 3-4i$

$\Rightarrow (x+iy)^2 = 3-4i$

$\Rightarrow x^2-y^2+2ixy = 3-4i$

$\Rightarrow x^2+2ixy-y^2 = 3-4i$

$\Rightarrow (x^2-y^2)+2ixy = 3-4i$ — (i)

$\therefore$ Comparing Real & Imaginary Part

$\therefore (x^2-y^2) = 3$ & $2xy = -4$

$xy = -2$

$y = -2/x$

$\Rightarrow$ substitution of y in $x^2-y^2 = 3$;

$\Rightarrow x^2 - \left(4/x^2\right) = 3$

$\Rightarrow x^2 - 4/x^2 = 3$

$\Rightarrow x^4 - 4 = 3x^2$

$\rightarrow \boxed{x^4-3x^2-4=0}$

Put $x^2 = t$

$t^2-3t-4=0$

$t^2+\cancel{t}\,\cancel{4t}\,t - 4 = 0$

$t(t+1)-4(t+1)=0$

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$(t-4)\ (t+1)=0$

$(x^2-4)=0 \ ; (x^2+1)=0$

$x^2=4$ & $x^2=-1$

not possible

$$\boxed{x = \pm 2}$$

$\therefore$ Substituting in $y = -2/x$

when $x=2 \ \therefore y=-1$
when $x=-2 \ \therefore y=1$

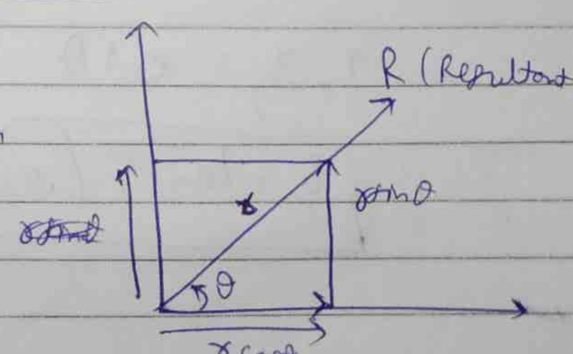
$\therefore$ Square roots of $\sqrt{3-4i}$ are;

$x+iy = 2-i$ &
$x+iy = -2+i$

H/w
Q. find : $\sqrt{5+6i}$

$\Rightarrow$ Modulus & Argument of Complex No

As complex number is;
$z = x+iy$ — Cartesian form
$z = r\cos\theta + ir\sin\theta$
$$\boxed{z = r(\cos\theta + i\sin\theta)}$$
Polar form / Parametric form



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Compare Real & Imaginary part of cartesian & Polar form.

$\therefore x = r\cos\theta$; $y = r\sin\theta$

$\therefore$ Square both side

$x^2 = r^2\cos^2\theta$; $y^2 = r^2\sin^2\theta$

Adding these equation;

$$\Rightarrow x^2 + y^2 = r^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{\to 1}$$

$$\Rightarrow x^2 + y^2 = r^2$$

$$\boxed{r = \pm\sqrt{x^2+y^2}}$$

is force

$$\boxed{r = \sqrt{x^2+y^2}}$$

As ; $x = r\cos\theta$, $y = r\sin\theta$

Divide these two equation

$$\Rightarrow x/y = \frac{r\cos\theta}{r\sin\theta}$$

$$\Rightarrow x/y = \cot\theta \Rightarrow \tan\theta = y/x$$

$$\Rightarrow \boxed{\theta = \tan^{-1}(y/x)}$$

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$\therefore$ Modulus $\Rightarrow r = \sqrt{x^2+y^2}$ &

Argument $\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Q. Express $\frac{1+2i}{1-3i}$ in the form of $r(\cos\theta + i\sin\theta)$.

Sol → $z = \frac{1+2i}{1-3i}$

$\Rightarrow \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$

$\Rightarrow \frac{1+3i+2i+6i^2}{1-9i^2} \Rightarrow \frac{1-6+5i}{1+9}$

$\Rightarrow \frac{-5+5i}{10}$

$\Rightarrow z = \frac{-1+i}{2}$

(Seperating Real & Imaginary Part)

$z = -\frac{1}{2} + i\left(\frac{1}{2}\right)$

$\therefore x = \frac{1}{2}$ & $y = \frac{1}{2}$

Modulus $r = \sqrt{x^2+y^2} = \sqrt{\left(\left(\frac{1}{2}\right)\right)^2 + \left(\frac{1}{2}\right)^2} \Rightarrow \sqrt{\frac{1}{4}+\frac{1}{4}}$

$\boxed{r = \sqrt{\frac{1}{2}}}$

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And Argument $\theta = \tan^{-1}(y/x)$

$$\theta = \tan^{-1}\left(\frac{1/2}{-1/2}\right)$$

$$\theta = \tan^{-1}(-1)$$

$[\because \tan^{-1}(-\theta) = -\tan^{-1}(\theta)$

$$\theta = -\tan^{-1}(1)$$

$$\theta = -\pi/4$$

$$\boxed{\theta = 3\pi/4}$$

Ans $\therefore$ $r(\cos\theta + i\sin\theta) = \sqrt{\frac{1}{2}}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

H/w Q. Express in polar form where $z = (1 - \sqrt{2} + i)$.

Q. Find the modulus & Argument of $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$.

Sol → Let consider $z = r(\cos\theta + i\sin\theta) = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

comparing Real & Imaginary Part

$\therefore$ $r\cos\theta = -1/2$ & $r\sin\theta = \sqrt{3}/2$

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(Squaring & adding)

$$r^2 \cos^2\theta + r^2 \sin^2\theta = \left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\therefore r^2(\cos^2\theta + \sin^2\theta) = \frac{1}{4} + \frac{3}{4}$$

$$r^2 = \frac{4}{4}$$

$$\boxed{r = 1}$$

→ Reason: r is magnitude

As; $r\cos\theta = -\frac{1}{2}$; $r\sin\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{1}{2}$ & $\sin\theta = \frac{\sqrt{3}}{2}$

~~(as r = 1)~~ $(r = 1)$

Since; $\cos\theta$ is $-ve$ & $\sin\theta$ is positive.
θ lies in second quadrant.

$$\boxed{\theta = \frac{2\pi}{3}}$$

$\therefore$ Modulus of Given complex no. is

$\boxed{r = 1}$ &

Argument is $\boxed{\theta = \frac{2\pi}{3}}$

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Q. find modulus & Argument of $\frac{(3 - i\sqrt{2})^2}{1+2i}$

Sol: Let us consider ;

$$z = r(\cos\theta + i\sin\theta) = \frac{(3 - i\sqrt{2})^2}{1+2i}$$

$$\Rightarrow \frac{3^2 - 2\times3\times i\sqrt{2} + (i\sqrt{2})^2}{1+2i}$$

$$\Rightarrow \frac{9 - 2 - 6\sqrt{2}i}{1+2i}$$

$$\Rightarrow \frac{7 - 6\sqrt{2}i}{1+2i}$$

(Rationalization)

$$\Rightarrow \frac{7 - 6\sqrt{2}i}{1+2i} \times \frac{1-2i}{1-2i}$$

$$\Rightarrow \frac{7 - 14i - 6\sqrt{2}i + 12\sqrt{2}i^2}{1 - i^2 4}$$

$$\Rightarrow \frac{7 - 12\sqrt{2} - 14i - 6\sqrt{2}i}{1+4}$$

$$\Rightarrow \frac{(7 - 12\sqrt{2})}{5} + i\left(\frac{-14 - 6\sqrt{2}}{5}\right)$$

Further find modulus & argument : H/w

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⇒ Exponential form of Complex Number

According to series expansion; i.e.; $e^{i\theta}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$$

Assuming equation;

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \ldots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\right) + \left(i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!}\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

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$\Rightarrow$ Exponential form:

$z = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta$ — polar form

$z = re^{i\theta}$

comparing these forms

$r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\therefore \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ —(i)

$e^{-i\theta} = \cos(\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

$\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$ —(ii)

Add above (i) & (ii) eqns:

$e^{i\theta} + e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\therefore \boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$

Subtracting (ii) from (i)

$e^{i\theta} - e^{-i\theta} = \cancel{\cos\theta} + i\sin\theta - \cancel{\cos\theta} + i\sin\theta$

$\therefore \boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$

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Q(1) → find the complex number z if ; $\arg(z+1) = \frac{\pi}{6}$ & $\arg(z-1) = \frac{2\pi}{3}$.

Sol → Let $z = x + iy$ & $z+1 = x + iy + 1$

$$(z+1) = (x+1) + iy$$

∴ Taking argument on bothsides;

$$\arg(z+1) = \frac{\pi}{6} \text{ (given)} \quad -(i)$$

$$\arg(z+1) = \tan^{-1}\left(\frac{y}{x+1}\right) \quad -(ii)$$

⇒ Comparing ~~one~~ (i) & (ii)

$$\tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{6} \Rightarrow \therefore \frac{y}{x+1} = \tan\frac{\pi}{6}$$

$$\therefore \frac{y}{x+1} = \frac{1}{\sqrt{3}}$$

$$\therefore \sqrt{3}y = x+1$$

$$x - \sqrt{3}y + 1 = 0 \quad -(iii)$$

Again ; $z = x + iy$

$(z-1) = (x-1) + iy$

∴ Taking argument on both side

$$\cancel{\tan^{-1}} \arg(z-1) = \frac{2\pi}{3} = \tan^{-1}\left(\frac{y}{x-1}\right)$$

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$\Rightarrow \tan^{-1}\left(\frac{y}{x-1}\right) = \frac{2\pi}{3}$

$\Rightarrow \frac{y}{x-1} = \tan\left(\frac{2\pi}{3}\right)$

$\Rightarrow \frac{y}{x-1} = -\sqrt{3}$

$\Rightarrow y = -\sqrt{3}x + \sqrt{3}$

$\Rightarrow \sqrt{3}x + y - \sqrt{3} = 0 \quad -(iv)$

~~from eqn (iii) & (iv)~~

Multiply eqn (iv) by $\sqrt{3}$

$3x + \sqrt{3}y - 3 = 0 \quad -(v)$

from eqn (iii) & (v)

$$\begin{array}{r} x - \sqrt{3}y + 1 = 0 \\ + \quad 3x + \sqrt{3}y - 3 = 0 \\ \hline 4x - 2 = 0 \end{array}$$

$x = 2/4 = 1/2$

$\boxed{x = 1/2}$

$\boxed{y = \frac{\sqrt{3}}{2}}$

$\therefore$ Complex No.,

$z = x + iy$

$\boxed{z = \frac{1}{2} + i\frac{\sqrt{3}}{2}}$ Ans

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Q If $z_1 = 1+2i$; $z_2 = 2+3i$. find $z_1 \cdot \bar{z}_1$, $|z_1 \cdot z_2|$, $\left|\frac{z_1}{z_2}\right|$.

Sol→ $z_1 = 1+2i$

$\bar{z}_1 = 1-2i$

$\Rightarrow z_1 \cdot \bar{z} = (1+2i)(1-2i) \Rightarrow 1 - i^2 4$

$\Rightarrow 1+4 \Rightarrow \underline{5}$ Ans

$\Rightarrow |z_1 \cdot z_2| \Rightarrow$.

$(z_1 \cdot z_2) = (1+2i) \cdot (2+3i)$

$\Rightarrow 2+3i+4i+6i^2$

$\Rightarrow 2-6+7i \Rightarrow -4+7i$

$\Rightarrow |z_1 \cdot z_2| = |-4+7i| \Rightarrow \sqrt{(-4)^2+7^2}$

$\Rightarrow \sqrt{16+49} \Rightarrow \sqrt{65}$ Ans

$\Rightarrow \frac{z_1}{z_2} = \frac{1+2i}{2+3i}$

$\Rightarrow \frac{1+2i}{2+3i} \times \frac{2-3i}{2-3i}$ (Rationalize)

$\Rightarrow \frac{2-3i+4i-6i^2}{4-9i^2} \Rightarrow \frac{2+6+i}{4+9} \Rightarrow \frac{8+i}{13}$

$\Rightarrow \frac{8}{13} + \frac{i}{13}$

$\Rightarrow \left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{8}{13}\right)^2+\left(\frac{1}{13}\right)^2} \Rightarrow \sqrt{\frac{64+1}{169}} \Rightarrow \frac{\sqrt{65}}{13}$ Ans

Complex Analysis

• Analytic function

Complex Number :

$$\boxed{z = x + iy}$$

→ iota = $\sqrt{-1}$

Imaginary Part

Complex Numbers | Real Part

Complex variable function:

$$\boxed{w = f(z) = u + iv = \phi + i\psi}$$

$u(x, y)$, $v(x, y)$, $\phi(x, y)$, $\psi(x, y)$

Eg:

$$f(z) = z^2$$

$$= (x+iy)^2 = x^2 + (iy)^2 + 2x(iy)$$

$$\Rightarrow w = f(z) = \underbrace{(x^2 - y^2)}_{\substack{u \\ \phi}} + \underbrace{2xy\,i}_{\substack{v \\ \psi}}$$

$$f(z) = \bar{z} = \overline{(x + iy)}$$

$$= (\underbrace{x}_{u} + i\underbrace{(-y)}_{v})$$

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Modulus $= |z| = \sqrt{(Re)^2 + (Img)^2}$

$$|z| = \sqrt{x^2 + y^2}$$

Argument $= \theta = \tan^{-1}\left(\frac{Img}{Re}\right)$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Conjugate $= \bar{z} = (\overline{x+iy})$

$$\Rightarrow x - iy$$

⇒ Analytic / Regular / Holomorphic / Monogenic function

V.Imp

A complex variable function $f(z)$ or w, is said to be Analytic if f ~~st~~ satisfies following conditions:

(1) $f(z)$ should be single valued

Eg:→ $f(z) = z^2$	$f(z) = \sqrt{z}$
$f(4) = 4^2$	$f(4) = \sqrt{4}$
$= 16$	$= \pm 2$
Single value	Multiple values
✓	X

(2) $f(z)$ should have unique derivatives w.r.t. z at each point of it's domain D.

(3) $f(z)$ should satisfy Cauchy - Riemann (C-R) Equations:-

i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ or $u_x = v_y$

&

ii) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ or $u_y = -v_x$

if u & v are functions of x & y.

Cartesian form / Rectangular form

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Note:

C-R equations alone are necessary but not sufficient to check any function analytic.

Note: Some standard functions such as :-

① Polynomials eg → $x^2 - 2x + 7$; $x^{10} - 7x^5 + 20x$ etc.

② Rationals with non-zero denominators eg: $\frac{xy}{x^2+y^2}$, etc.

③ Sine or cosine functions . eg → $\sin 3x$, $\cos 7x$, $x \sin x$, etc.

④ Exponential functions eg: e^{2x}, $e^{-x} \sin x$, $x^2 e^{10x}$, etc.

(always have unique derivatives & single valued)

Conclusion: To check any function $f(z)$ as Analytic

Case 1: In case of, $f(z)$ belongs to any of above standard functions then only C-R equation must be checked. (Only last condition)

Case 2: In other case; $f(z)$ must be checked for single valued, differentiability along with C-R equations. (All 3 conditions)

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Entire function:

A function $f(z)$ which is Analytic everywhere is called as Entire function.

Eg → $f(z) = z^2$

$$f(z) = (x+iy)^2$$
$$= x^2 + (iy)^2 + 2(x)(iy)$$

$\boxed{f(z) = x^2 - y^2 + 2xyi}$ Here; $u = x^2 - y^2$; $v = 2xy$

Since, u & v are polynomials ; so, single valued & ~~desirable~~ derivable everywhere as well as satisfies C-R equations ; so analytic.

So, $f(z) = z^2$ is an entire function as well.

→ Cauchy – Riemann Equations in Polar form

If u & v are the functions of r & θ

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

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Numericals

Q.1→ Examine the Analyticity of the function : $2xy + i(x^2-y^2)$

Sol→ Let

$f(z) = 2xy + i(x^2-y^2)$

$u + iv = 2xy + i(x^2-y^2)$

Now by comparing both sides, we get :-

$\boxed{u = 2xy}$ $\boxed{v = x^2 - y^2}$

Since both u & v are Polynomials, so they must be always single valued and having unique derivative.

Now Partially differentiate ~~eq~~ u & v wrt x & y, we get :-

$\frac{\partial u}{\partial x} = 2y$ | $\frac{\partial v}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = 2x$ | $\frac{\partial v}{\partial y} = -2y$

So now for Cauchy - Reimann Equations:

$\frac{du}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{du}{\partial y} = -\frac{dv}{\partial x}$

(Not equal) (Doesn't matter)

So, C - R equations are not satisfied.

So, given function $2xy + i(x^2-y^2)$ is not Analytic.

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Q.2→ Using the C-R equations show that $f(z) = z^3$ is analytic in entire z-plane.

Sol→ Now according to the question:

$$f(z) = z^3$$
$$= (x+iy)^3$$
$$= x^3 + (iy)^3 + 3x(iy)(x+iy)$$

$$(\because (a+b)^3 = a^3 + b^3 + 3ab(a+b))$$

$$= x^3 + i^3y^3 + 3xyi(x+iy)$$
$$= x^3 + i^3y^3 + 3x^2yi + 3xy^2i^2$$
$$= x^3 - iy^3 + 3x^2yi - 3xy^2$$

$$u + iv = (x^3 - 3xy^2) + (3x^2y - y^3)i$$

Now by comparing both sides; we get:

$\boxed{u = x^3 - 3xy^2}$ $\boxed{v = 3x^2y - y^3}$

Since, u & v are polynomials. So they must be single valued & having unique derivatives.

Now partially differentiate u & v wrt x & y, we get

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \; ; \quad \frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = -6xy \; ; \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

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Now by C-R equations:
as we know that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So, both C-R equations are true in this question.

So; given function i.e.

$f(z) = z^3$ is Analytic.

• Important Formulae

1. $e^{i\theta} = \cos\theta + i\sin\theta$
2. $e^{-i\theta} = \cos\theta - i\sin\theta$
3. $\cos(i\theta) = \cosh\theta$
4. $\sin(i\theta) = i\sinh\theta$
5. $\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$
6. $\cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2}$
7. $\sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2}$

Date..............

Q. 3→ Examine $\sin z$ is Analytic or not?

Sol→ Let

$$f(z) = \sin z$$

$$\Rightarrow \sin(x+iy)$$

$$\Rightarrow \sin x \cos(iy) + (\cos x)\sin(iy)$$

$$[\because \sin(A+B) = \sin A \cos B + \cos A \sin B)$$

$$f(z) = \sin x \cosh y + \cos x(i \sinh y)$$

$$u + iv = \sin x \cosh y + i \cos x \sinh y$$

Now by comparing both sides; we get

$$u = \sin x \cosh y$$
$$v = \cos x \sinh y$$

Since ~~u~~ & v contain only sine, cosine & their hyperbola. So, they must be single valued & having unique derivatives.

Now partially differentiate u & v wrt x & y, we get:

$$\frac{\partial u}{\partial x} = \cos x \cosh y \quad \Bigg| \quad \frac{dv}{dx} = -\sin x \sinh y$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y \quad \Bigg| \quad \frac{\partial v}{\partial y} = \cos x \cosh y$$

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Now as per C-R equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So, C-R equations are also satisfied.

~~Since~~ So, the function $\sin z$ is an Analytic function.

Q.4→ Verify the C-R Equations for the function:

$$f(z) = e^{-\bar{z}}$$

Sol→ The given function is

$$f(z) = e^{-\bar{z}}$$
$$= e^{-(\overline{x+iy})}$$
$$= e^{-x+iy}$$
$$= e^{-x} . e^{iy}$$
$$= e^{-x}(\cos y + i \sin y)$$

$$u + iv = e^{-x}\cos y + i e^{-x}\sin y$$

Now by comparing both sides; we get:

$$u = e^{-x}\cos y \; ; \; v = e^{-x}\sin y$$

Now, partially differentiate u & v wrt x & y ; we get:

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$\frac{\partial u}{\partial x} = -e^{-x}\cos y$ | $\frac{\partial v}{\partial x} = -e^{-x}\sin y$

$\frac{\partial u}{\partial y} = -e^{-x}\sin y$ | $\frac{\partial v}{\partial y} = e^{-x}\cos y$

Now by C-R equations:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

× (Not satisfied) × (Not satisfied)

→ So, C-R equations are not verified.

Q.5→ Show that $f(z) = \log z$ is Analytic. $(z \neq 0)$

Sol→ According to the question;

$$f(z) = \log z$$
$$= \log(x+iy)$$

$$u + iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

Now by comparing both sides we get:

$$\boxed{u = \frac{1}{2}\log(x^2+y^2)} \quad \boxed{v = \tan^{-1}(y/x)}$$

So, u & v are both single valued and have unique derivative.

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Now :

- $\frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{1}{x^2+y^2}\right) \times (2x) \Rightarrow \frac{x}{x^2+y^2}$

- $\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{1}{x^2+y^2}\right) 2y \Rightarrow \frac{y}{x^2+y^2}$

- $\frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2} \times \left(\frac{-y}{x^2}\right) \Rightarrow \frac{-y}{x^2+y^2}$

- $\frac{\partial v}{\partial y} = \frac{1}{1+(y/x)^2} \times \frac{1}{x} \Rightarrow \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} \Rightarrow \frac{x}{x^2+y^2}$

⇒ Now ; by C-R equations ; we know

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

✓ ✓

So, C-R equations are satisfied.

So, Analytic ✓

↳

Date..............

Q.6 → Check whether $\left(\frac{1}{z^3}\right)$ is Analytic or not?

Sol → Let

$$f(z) = \frac{1}{z^3}$$

$$= \frac{1}{(x+iy)^3} \quad \text{—(i)}$$

So; it will be difficult to handle in cartesian form

⇒ Now by changing equation (i) in Polar form.

⇒ Put $x = r\cos\theta$; $y = r\sin\theta$

Now from eq(i), we get

$$f(z) = \frac{1}{(r\cos\theta + ir\sin\theta)^3}$$

$$= \frac{1}{r^3(\cos\theta + i\sin\theta)^3} = \frac{1}{r^3}(\cos\theta + i\sin\theta)^{-3}$$

By Demoiver's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$= \frac{1}{r^3}(\cos(-3\theta) + i\sin(-3\theta))$$

$$= \frac{1}{r^3}(\cos 3\theta - i\sin 3\theta)$$

$(\because \cos(-\theta) = \cos\theta \ ; \ \sin(-\theta) = -\sin\theta)$

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$$u + iv = \frac{1}{r^3}\cos 3\theta - i\frac{1}{r^3}\sin 3\theta$$

Now by comparing both sides we get:

$$\boxed{u = \frac{1}{r^3}\cos 3\theta} \qquad \boxed{v = \frac{-1}{r^3}\sin 3\theta}$$

Since, u, v contain r and terms of sine & cosine only, so they must be single valued and have unique derivative.

Now partially ~~derivat~~ differentiate u & v w.r.t. r & θ; we get:

$$\frac{\partial u}{\partial r} = \frac{-3}{r^4}\cos 3\theta \qquad ; \qquad \frac{\partial v}{\partial r} = \frac{+3\sin 3\theta}{r^4}$$

$$\frac{\partial u}{\partial \theta} = \frac{1}{r^3}(-3\sin 3\theta) \qquad ; \qquad \frac{\partial v}{\partial \theta} = -\frac{1}{r^3}(3\cos 3\theta) = \frac{-3\cos 3\theta}{r^3}$$

Now C-R equations in polar form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

So, C-R equations are satisfied

So, given function f(z) = $\frac{1}{z^3}$ is an analytic function.

Hence, proved.

Date..............

Q.7→ Prove that the function $\sinh z$ is Analytic or not.

Sol→ Let $f(z) = \sinh z$

$\Rightarrow \sinh(x+iy)$

$(\because \sin(i\theta) = i\sinh\theta)$

$= \frac{1}{i}\sin(i(x+iy))$

$\Rightarrow \frac{i}{i^2}\sin(ix + i^2 y)$

$\Rightarrow -i\sin(ix - y)$

$\Rightarrow -i(\sin(ix)\cos(y) - \cos(ix)\sin(y))$

$(\because \sin(A-B) = \sin A\cos B - \cos A\sin B)$

$\Rightarrow -i(i\sinh x\cos y - \cosh x\sin y)$

$\Rightarrow -i^2\sinh x\cos y + i\cosh x\sin y$

$u + iv = \sinh x\cos y + i\cosh x\sin y$

Now by comparing both sides, we get

$\boxed{u = \sinh x\cos y}$ $\boxed{v = \cosh x\sin y}$

Since; u & v contain only sine, cosine & their hyperbolics, so they are always single valued & have unique derivatives.

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Now partially differentiate u & v wrt x & y, we get:

$$\frac{\partial u}{\partial x} = \cosh x \cos y \quad ; \quad \frac{\partial v}{\partial x} = \sinh x \sin y$$

$$\frac{\partial u}{\partial y} = -\sinh x \sin y \quad ; \quad \frac{\partial v}{\partial y} = \cosh x \cos y$$

Now for C - R equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So, C - R equations are also satisfied

So, the given function $\sinh z$ is an Analytic function.

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Q.8 → Find the values of a, b, c and d so that the function $f(z)$ is Analytic where

$$f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$$

Sol. Now, According to the question

$$f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$$

$$\Rightarrow u + iv = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$$

Now by comparing both sides we get:

$$u = x^2 + axy + by^2$$
$$v = cx^2 + dxy + y^2$$

Now, since the given function $f(z)$ is Analytic.
So, it must satisfy C-R equations.

So ⇒

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Now partially differentiate u & v w.r.t. x & y, we get:

$$\frac{\partial u}{\partial x} = 2x + ay + 0$$

$$\frac{\partial u}{\partial y} = ax + 2by$$

$$\frac{\partial v}{\partial x} = 2cx + dy.$$

$$\frac{\partial v}{\partial y} = dx + 2y$$

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$\Rightarrow$ Now ~~B~~; as

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow 2x + ay = dx + 2y$$

Now by comparing both sides, we get:

$$\boxed{2 = d}\ ;\ \boxed{a = 2}$$

Now from equation 2:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow ax + 2by = -(2cx + dy)$$

$$\Rightarrow ax + 2by = -2cx - dy$$

Now by comparing both sides, we get

$$a = -2c \qquad 2b = -d$$

$$2 = -2c \qquad 2b = -2$$

$$\boxed{c = -1} \qquad \boxed{b = -1}$$

So, $\boxed{a = 2,\ b = -1,\ c = -1,\ d = 2}$ Ans

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Q.9→ Determine p such that $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$ is an analytic function

Sol→ Now, as per the given question

$$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$$

$$\Rightarrow u+iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$$

Now, by comparing both sides, we get:

$$u = \frac{1}{2}\log(x^2+y^2$$

$$v = \tan^{-1}\left(\frac{px}{y}\right)$$

⇒ Now since the given function $f(z)$ is an analytic function. So, it must satisfy C-R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -(1) \qquad ; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad -(2)$$

⇒ Now partially differentiate u and v w.r.t. x and y, we get:

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{1}{x^2+y^2} \times 2x\right)$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}}$$

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$$\frac{dv}{dy} = \frac{1}{1+\left(\frac{px}{y}\right)^2} \times \left(-\frac{px}{y^2}\right) = \frac{-px}{p^2x^2+y^2}$$

$\Rightarrow$ So, now from eqn (1), we get

$$\Rightarrow \frac{du}{dx} = \frac{dv}{dy}$$

$$\Rightarrow \frac{x}{x^2+y^2} = \frac{-px}{p^2x^2+y^2}$$

(by putting value $-p=1$ by comparing numerator)

$\Rightarrow$ $-p = 1$ $\Rightarrow$ $\boxed{p=-1}$ Ans

Date..............

Polar form of C-R Eqn

- Forms of Complex No.

① $z = x + iy$ → cartesian form
② $z = r(\cos\theta + i\sin\theta)$ → Polar form
③ $z = r e^{i\theta}$ → Exponential form

∴ Comparing

$$x = r\cos\theta ; y = r\sin\theta$$

$$\therefore z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\therefore f(z) = u + iv = f(re^{i\theta})$$

Differentiate $f(z)$ w.r.t. r;

$$\therefore \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\, e^{i\theta}$$

Differentiate $f(z)$ wrt θ;

$$\therefore \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(r e^{i\theta}) \cdot r \cdot e^{i\theta} \cdot i$$

$$(f'(re^{i\theta})\, e^{i\theta})\, ri$$

⇒

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) ri$$

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$$\therefore \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = r i\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Comparing RP & IP

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

& $\boxed{\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}}$

Q. Find p, if $f(z) = r^2\cos 2\theta + i r^2 \sin p\theta$ is analytic.

Sol: $f(z) = r^2\cos 2\theta + i r^2 \sin p\theta$

$\therefore u = r^2\cos 2\theta$; $v = r^2 \sin p\theta$

$u_r = 2r\cos 2\theta$; $v_\theta = r^2(\cos p\theta.)\, p$

As the given function is analytic
$\therefore$ C-R equation in polar form is

$$u_r = \frac{1}{r} v_\theta$$

$$2r\cos 2\theta = \frac{1}{r}.r^2 p\cos p\theta$$

$$2r\cos 2\theta = r p \cos p\theta$$

$$\boxed{p = 2}$$

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Harmonic function

Any function $f(x, y)$ is said to be harmonic if f satisfy Laplace equation ie.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$f(z) = u + iv$

$f'(z) = u_x + iv_x$ & $f'(z) = u_y + iv_y$

According to C-R equation

$u_x = v_y$ & $u_y = -v_x$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Differentiate w.r.t. x and second part w.r.t y.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \& \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

Comparing

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

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& $\boxed{\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0}$

Q. ① → If $u(x, y)$ is harmonic function then prove that, $f(z) = u_x - i u_y$ is an analytic function.

Sol: Since $u(x, y)$ is harmonic function.

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore f(z) = u_x - i u_y$$

Let $u_x = U$; $-u_y = V$

$$\therefore f(z) = U + iV$$

$$\therefore U = u_x$$

$$U_x = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$U_y = \frac{\partial^2 u}{\partial x \partial y}$$

$$V = -u_y$$

$$V_x = -\frac{\partial^2 u}{\partial x \partial y}$$

$$V_y = -\frac{\partial^2 u}{\partial y^2}$$

$\therefore U_x = V_y$ & $U_y = -V_x$

$\therefore f(z) = U + iV$ is analytic

Date..............

Euler's Identities

$e^{i\theta} = \cos\theta + i\sin\theta$ $\qquad e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ $\qquad \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cosh\theta = \dfrac{e^{\theta} + e^{\theta}}{2}$ $\qquad \sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

Given: complex no. $\to z = x \pm iy$

Trigonometric: $\sin iy = i\sinh y$; $\cos iy = \cosh y$

$\sin(x+iy) = \sin x \cosh y + i\cos x \sinh y$

$\sin(x-iy) = \sin x \cosh y - i\cos x \sinh y$

$\cos(x+iy) = \cos x \cosh y - i\sin x \sinh y$

$\cos(x-iy) = \cos x \cosh y + i\sin x \sinh y$

$\tan z = \dfrac{\sin z}{\cos z}$

$\cot z = \dfrac{\cos z}{\sin z}$

$\sec z = \dfrac{1}{\cos z}$

$\operatorname{cosec} z = \dfrac{1}{\sin z}$

Date...............

Hyperbolic :

$\sinh iy = i\sin y$ $\cosh iy = \cos y$

$\sinh(x+iy) = \sinh x \cos y + i\cosh x \sin y$

$\sinh(x-iy) = \sinh x \cos y - i\cosh x \sin y$

$\cosh(x+iy) = \cosh x \cos y + i\sinh x \sin y$

$\cosh(x-iy) = \cosh x \cos y - i\sinh x \sin y$

$\tanh z = \dfrac{\sinh z}{\cosh z}$ $\operatorname{sech} z = \dfrac{1}{\cosh z}$

$\coth z = \dfrac{\cosh z}{\sinh z}$ $\operatorname{cosech} z = \dfrac{1}{\sinh z}$

Date..............

Euler's formula

$z = a + ib \Rightarrow z = r(\cos\theta + i\sin\theta)$

$r \rightarrow$ modulus : $r = \sqrt{a^2+b^2}$

$\theta \rightarrow$ argument : $\theta = \tan^{-1}(b/a)$

$\boxed{z = re^{i\theta}}$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x + \sin x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$x = i\theta$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

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$$\Rightarrow e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

$$\cancel{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$z \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \ldots$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z = r(\cos\theta + i\sin\theta)$$

$\boxed{z = re^{i\theta}}$ → Euler's formula.

If $\theta = \pi$: $e^{i\theta} = e^{i\pi} = \cos\pi + i\sin\pi$

$$\Rightarrow e^{i\pi} = -1 + i(0)$$

$$\Rightarrow e^{i\pi} = -1$$

$$\Rightarrow \boxed{e^{i\pi} + 1 = 0}$$

Date...............

Q. Express the equation $z = \dfrac{(1+\sqrt{3}i)^7 (1-\sqrt{3}i)^4}{(1-i)^{16}}$ in the form of $a + ib$

Sol: $z = re^{i\theta}$

$1+\sqrt{3}i \Rightarrow r, \theta$

$r = \sqrt{1^2+3} = 2$

$\theta = \alpha = \tan^{-1}\left|\dfrac{\sqrt{3}}{1}\right|$

$\theta = \pi/3$

$$\boxed{1+\sqrt{3}\,i = 2\cdot e^{i\pi/3}}$$

$1-\sqrt{3}i \Rightarrow r = 2$

$\theta = -\alpha = -\tan^{-1}(b/a)$

$\Rightarrow -\tan^{-1}\left|-\dfrac{\sqrt{3}}{1}\right|$

$\theta = -\pi/3$

$$\boxed{1-\sqrt{3}i = 2e^{-i\pi/3}}$$

$1-i \Rightarrow r, \theta$

$r = \sqrt{1+(-1)^2} = \sqrt{2}$

$\alpha = \tan^{-1}\left|-\dfrac{1}{1}\right| = \pi/4$

$\theta = -\alpha = -\pi/4$

$1-i = \sqrt{2}\cdot e^{-i\pi/4}$

$\Rightarrow$ Equation

$$z = \frac{(2e^{i\pi/3})^7 (2e^{-i\pi/3})^4}{(\sqrt{2}e^{-i\pi/4})^{16}} = 8e^{i\pi + i4\pi}$$

$z = 8\cdot e^{i(5\pi)}$ Ans

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$z = 8(\cos 5\pi + i \sin 5\pi)$

$z = 8(-1 + i(0))$

$\boxed{z = -8 + i \cdot 0}$

From the desk of

DATE 17/4/23

MAL - II

Euler's Formula

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$$\boxed{\frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta}$$

$$\boxed{\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}}$$

Q1 → PT

$\sin(\alpha+n\theta) - e^{i\alpha}\sin n\theta = e^{-in\theta}\sin\alpha$

Q2 →

$\frac{z^2-1}{z^2+1} = i\tan\theta$

where $z = e^{i\theta}$

Neelgagan

1.

$\sin\alpha\cos n\theta + \cos\alpha\sin n\theta - (\cos\alpha + i\sin\alpha)\sin n\theta$

$\Rightarrow \cos\alpha\sin n\theta - i\sin\alpha\sin n\theta$

$\sin\alpha\cos n\theta - i\sin\alpha\sin n\theta$

$\sin\alpha(\cos n\theta - i\sin n\theta)$

$\boxed{\sin\alpha \; e^{-in\theta}}$

$e^{-in\theta}\sin\alpha = RHS$

Neelgagan

From the desk of

DATE $\overset{\circ}{e}$

$$\frac{z^2-1}{z^2+1} = i\tan\theta \qquad z = e^{i\theta}$$

Sol → LHS.

$$\frac{z^2-1}{z^2+1} \Rightarrow \frac{e^{2i\theta}-1}{e^{2i\theta}+1}$$

$$\frac{\cancel{e^{i\theta}}\left(e^{i\theta}-e^{-i\theta}\right)}{\cancel{e^{i\theta}}\left(e^{i\theta}+e^{-i\theta}\right)}$$

~~$\cos\theta \mp i\sin\theta$ — (.~~

$$\frac{\cancel{2}\, i\sin\theta}{\cancel{2}\cos\theta}$$

$i\tan\theta$

Neelgagan

DATE

Exam Ques

Q: If α & β are imaginary cube roots of unity.

P.T.

$$\alpha e^{\alpha x} + \beta e^{\beta x} = -e^{-x/2}\left(\cos \frac{\sqrt{3}}{2}x + \sqrt{3}\sin\frac{\sqrt{3}}{2}x\right)$$

Sol → $\omega^3 = 1$

$$\omega^3 - 1 = 0$$

$$(x-1)(x^2 + x + 1) = 0$$

$x = 1$ | $x^2 + 1 + 1 = 0$

$$x = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$1 , \quad \frac{-1+\sqrt{3}\, i}{2} , \quad \frac{-1-\sqrt{3}\, i}{2}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$1 \qquad \alpha \qquad \beta$$

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$\alpha \quad e^{\alpha x} + \beta e^{\beta x}$

$\frac{-1+\sqrt{3}i}{2} \qquad \frac{-1-\sqrt{3}i}{\cancel{\sqrt{2}}\,2}$

$\frac{-1+\sqrt{3}i}{2}\left(e^{-\frac{1+\sqrt{3}i}{2}x}\right) + \frac{-1-\sqrt{3}i}{2}\, e^{\left(-\frac{1-\sqrt{3}i}{2}\right)x}$

$\frac{-1+\sqrt{3}i}{2}\left(e^{-x/2} \cdot e^{\frac{\sqrt{3}}{2}ix}\right) + \frac{-1-\sqrt{3}i}{2} \cdot e^{-x/2} \cdot e^{-\frac{\sqrt{3}}{2}ix}$

$e^{-x/2}\left(\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)e^{\frac{\sqrt{3}i}{2}x} + \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)e^{-\frac{\sqrt{3}}{2}ix}\right)$

$\frac{\pi}{2}+\frac{\pi}{6}$

$\cos \frac{2}{3}\pi$

$\frac{3\pi+\pi}{6} \quad \frac{4\pi}{6}$

$e^{-x/2}\left(-\frac{1}{2}e^{\sqrt{3}i/2\,x} + \frac{\sqrt{3}i}{2}e^{\frac{\sqrt{3}i}{2}x} - \frac{1}{2}e^{-\sqrt{3}i/2\,x} - \frac{\sqrt{3}i}{2}e^{-\sqrt{3}/2\,ix}\right)$

$\frac{2}{3}\pi$

$\frac{2}{3} \times \cancel{180}\ 60$

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$$e^{-x/2}$$

$$\left(-\frac{1}{2}\left(e^{\frac{\sqrt{3}i}{2}x}+e^{-\frac{\sqrt{3}i}{2}x}\right)\right.$$

$$\left.+\frac{\sqrt{3}}{2}i\left(e^{\frac{\sqrt{3}i}{2}x}-e^{-\sqrt{3}/2\,ix}\right)\right)$$

$$e^{-x/2}\left(-\frac{1}{2}\,2\cos\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{2}2\sin\frac{\sqrt{3}}{2}x\right)$$

$$-e^{-x/2}\left(\cos\frac{\sqrt{3}}{2}x+\sqrt{3}\sin\frac{\sqrt{3}}{2}x\right)=R.H.S.$$

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De-Moivre's Theorem

$$\boxed{(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)} = z^n \quad (\because r = 1)$$

$$\boxed{(\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - i\sin n\theta)} = z^{-n} \quad (\because r = 1)$$

$$\boxed{\cos n\theta = \frac{z^n + z^{-n}}{2}}$$

$$\boxed{\sin n\theta = \frac{z^n - z^{-n}}{2i}}$$

① Evaluate: $(1+i)^{100} + (1-i)^{100}$

Sol → For $(1+i)$

$\therefore x = 1, y = 1$

$\therefore r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}(y/x) = \tan^{-1}(1) = \pi/4$

for $(1-i)$;

$\therefore x = 1, y = -1$

$\therefore r = \sqrt{x^2+y^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = -\pi/4$

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$\therefore$ To convert cartesian to polar;

$$x + iy = r(\cos\theta + i\sin\theta)$$

$$\therefore (1+i) = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$\therefore (1-i) = \sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$$

$\therefore$ Taking 100^{th} power & adding

$$(1+i)^{100} + (1-i)^{100} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{100} + \left[\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^{100}$$

- According to De-Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

$$\therefore (1+i)^{100} + (1-i)^{100} = (\sqrt{2})^{100}\left[\cancel{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}}\right]^{100}$$

$$(\sqrt{2})^{100}\left[\left(\cos\frac{100\pi}{4} + \cancel{i\sin\frac{100\pi}{4}} + \cos\frac{100\pi}{4} - \cancel{i\sin\frac{100\pi}{4}}\right)\right]$$

$$= 2^{50}\left[2\cos 25\pi\right] = 2^{50}\left[2\cos(24\pi + \pi)\right]$$

$$= 2^{50}(-2\cos 24\pi) = 2^{50}(-2(1)) = \underline{\underline{-2^{51}}}$$

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Q. 2 → Prove that:

$$(1+\sqrt{3}i)^8 + (1-\sqrt{3}i)^8 = -2^8$$

Sol → Consider $z = 1+\sqrt{3}i = x+iy$

$\therefore x = 1, y = \sqrt{3}$

$\therefore r = \sqrt{x^2+y^2} = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$ = magnitude

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$\therefore (1+\sqrt{3}i) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ — ①

Also, $z = (1-\sqrt{3}i) = x+iy$

$\therefore x = 1, y = -\sqrt{3}$

$\therefore r = \sqrt{1+3} = 2$; $\theta = \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) = -\frac{\pi}{3}$

$\therefore (1-\sqrt{3}i) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$ — ②

Taking 8^{th} power of eqn ① & ②

$(1+i\sqrt{3})^8 = \left(2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right)^8$

$= 2^8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

(Apply De-Moivre's Theorem)

$\Rightarrow 2^8\left(\cos 8\frac{\pi}{3} + i\sin 8\frac{\pi}{3}\right)$

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$$(1 - i\sqrt{3})^8 = 2^8\left(\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right)$$

$\therefore$ Adding above equation

$$LHS = (1+i\sqrt{3})^8 + (1-i\sqrt{3})^8 = 2^8\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3} + \cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right)$$

$$= 2^8\left(2\cos\frac{8\pi}{3}\right)$$

$$= 2^8\left(2\left(-\frac{1}{2}\right)\right) \Rightarrow -2^8 = R.H.S.$$

Hence, Proved.

Q.3 → If α, β are roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$
Prove that $\alpha^3 + \beta^3 = 0$.

Sol: As the equation is

$$x^2 - 2\sqrt{3}x + 4 = 0$$

This is quadratic equation.

$$\text{Root} = x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Comparing with $ax^2 + bx + c = 0$

$a = 1$, $b = -2\sqrt{3}$, $c = 4$

$\therefore x = \dfrac{2\sqrt{3} \pm \sqrt{12-16}}{2}$

$x = \dfrac{2\sqrt{3} \pm \sqrt{4}\, i}{2}$

$x = 2\left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)$

$x = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

Let $\alpha = 2\left[\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right]$ &

$\beta = 2\left[\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right]$

$\therefore$ (Taking cube & adding & applying Demoivre)

$L.H.S = \alpha^3 + \beta^3 = 2^3\left[\cos 3\dfrac{\pi}{6} + i\sin 3\dfrac{\pi}{6} + \cos 3\dfrac{\pi}{6} - i\sin 3\dfrac{\pi}{6}\right]$

$= 2^3\left(2\cos\dfrac{3\pi}{6}\right)$

$= 2^3\left(2\cos\dfrac{\pi}{2}\right) = 0 = R.H.S$

Hence, proved

Q.4→ If α, β are roots of $x^2 - 2x\cos\theta + 1 = 0$, find the equation whose roots are α^n, β^n.

Sol→ As the equation is

$x^2 - 2x\cos\theta + 1 = 0$

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Comparing with $ax^2+bx+c=0$

$\therefore a=1, b=-2\cos\theta, c=1$

Roots $= x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta-4}}{2}$

$\Rightarrow \dfrac{2\cos\theta \pm 2\sqrt{\sin^2\theta}}{2}$

$\Rightarrow \boxed{x = \cos \pm i\sin\theta}$

$\therefore \alpha$ & β are roots

$\alpha = \cos\theta + i\sin\theta$

$\beta = \cos\theta - i\sin\theta$

$\therefore$ Taking n^{th} power

$\alpha^n = (\cos\theta + i\sin\theta)^n$

$$\boxed{\begin{aligned}\alpha^n &= \cos n\theta + i\sin n\theta \\ \beta^n &= \cos n\theta - i\sin n\theta\end{aligned}}$$

$\therefore$ Quadratic equation can be written as

$x^2 - (\text{Sum of Roots})x + (\text{Product of Roots}) = 0$

Sum of Roots $= \alpha^n + \beta^n = 2\cos n\theta$

Product of Roots $= \alpha^n \cdot \beta^n = (\cos n\theta + i\sin n\theta)(\cos n\theta - i\sin n\theta)$

$= (\cos n\theta)^2 - (i\sin n\theta)^2$

$= (\cos n\theta)^2 + (\sin n\theta)^2 = 1$

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$$\Rightarrow \boxed{x^2 - (2\cos n\theta)x + 1 = 0}$$

$\Rightarrow$ Expansion of $\cos n\theta$ & $\sin n\theta$

			1				
		1	2	1			
		1	3	3	1		
	1	4	6	4	1		
	1	5	10	10	5	1	
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Binomial Theorem

$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b^1 + {}^nC_2 a^{n-2}b^2 + \dots + b^n\,{}^nC_n$

$(\cos\theta + i\sin\theta)^n = \cos^n\theta + {}^nC_1 \cos^{n-1}\theta(i\sin\theta) + {}^nC_2 \cos^{n-2}\theta(i\sin\theta)^2 + \dots + (i\sin\theta)^n$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$
$= a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

~~$(a+b)^n$~~

Q. Express $\sin 6\theta$ in terms of powers of $\sin\theta$ & $\cos\theta$.

Sol. According to De-Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

$\therefore$ Expanding L.H.S.

$\therefore (\cos\theta + i\sin\theta)^6 = \cos^6\theta + 6\cos^5\theta(i\sin\theta) + 15\cos^4\theta(i\sin\theta)^2 + 20\cos^3\theta(i\sin\theta)^3 + 15\cos^2\theta(i\sin\theta)^4 + 6\cos\theta(i\sin\theta)^5 + 1(i\sin\theta)^6$

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(Apply De-Moivre's Theorem)

$\Rightarrow (\cos 6\theta + i\sin 6\theta) = \cos^6\theta + i6\cos^5\theta \sin\theta - 15\cos^4\theta \sin^2\theta - i20 \sin^3\theta \cos^3\theta$
$+ 15\cos^2\theta \sin^4\theta + 6i\cos\theta(\sin^5\theta) - \sin^6\theta$

$\Rightarrow (\cos 6\theta + i\sin 6\theta) = (\cos^6\theta - 15\cos^4\theta \sin^2\theta + 15\cos^2\theta \sin^4\theta - \sin^6\theta) +$
$i(6\cos^5\theta \sin\theta - 20\sin^3\theta \cos^3\theta + 6\cos\theta \sin^5\theta)$

$\therefore$ Comparing Real & Imaginary Part

$\therefore \cos 6\theta = \cos^6\theta - 15\cos^4\theta \sin^2\theta + 15\cos^2\theta \sin^4\theta - \sin^6\theta$

Ans $\rightarrow$ $\sin 6\theta = 6\cos^5\theta \sin\theta - 20\sin^3\theta \cos^3\theta + 6\cos\theta \sin^5\theta$

Q. Use De-Moivre's Theorem to show that;

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

Sol $\rightarrow$ As $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$ — (1)

According to De-Moivre's Theorem
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$\Rightarrow (\cos\theta + i\sin\theta)^5 = [\cos^5\theta (i\sin\theta)^0] + [5\cos^4\theta (i\sin\theta)^1] + [10\cos^3\theta (i\sin\theta)^2]$
$+ [10\cos^2\theta (i\sin\theta)^3] + [5(\cos\theta)(i\sin\theta)^4] + [(i\sin\theta)^5]$

(Apply De-Moivre Theorem)

$\Rightarrow (\cos 5\theta + i\sin 5\theta) = \cos^5\theta + 5i\cos^4\theta \sin\theta - 10\cos^3\theta \sin^2\theta - 10i\cos^2\theta \sin^3\theta$
$+ 5\cos\theta \sin^4\theta + i\sin^5\theta$
$= (\cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta) + i[5\cos^4\theta \sin\theta -$
$10\cos^2\theta \sin^3\theta + \sin^5\theta]$

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Compare Real & Imaginary Part

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta \cdot \sin^2\theta + 5\cos\theta \sin^4\theta$

$\sin 5\theta = 5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta$

$\Rightarrow \tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$

$\Rightarrow \dfrac{5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta}$

(Dividing Numerator & Denominator by $\cos^5\theta$.)

$$\tan 5\theta = \dfrac{\dfrac{5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta}{\cos^5\theta}}{\dfrac{\cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta}{\cos^5\theta}}$$

$$\boxed{\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}}$$

Hence, proved.

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⇒ Expansion of 'n' th power of $\sin\theta$ / $\cos\theta$

Note: By Demoivre's Theorem

$x = \cos\theta + i\sin\theta$ $\qquad \frac{1}{x} = x^{-1} = \cos\theta - i\sin\theta$

$\boxed{x + \frac{1}{x} = 2\cos\theta}$ $\qquad \therefore \cos\theta = \frac{1}{2}\left(x + \frac{1}{x}\right)$

$\boxed{x - \frac{1}{x} = 2i\sin\theta}$ $\qquad \therefore \sin\theta = \frac{1}{2i}\left(x - \frac{1}{x}\right)$

Q. Show that : $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Sol → suppose $x = \cos\theta + i\sin\theta$

$\frac{1}{x} = x^{-1} = \cos\theta - i\sin\theta$ ($\because$ Demoivre)

$\therefore$ subtracting ;

$\left(x - \frac{1}{x}\right) = 2i\sin\theta$

Take 5^{th} power

$(2i\sin\theta)^5 = \left(x - \frac{1}{x}\right)^5$

$\therefore 32\, i^5 \sin^5\theta = x^5 - 5x^4\left(\frac{1}{x}\right) + 10\frac{x^3}{x^2} - 10x^2\left(\frac{1}{x}\right)^3 + \frac{5x}{x^4} - \frac{1}{x^5}$

$32 i\sin^5\theta = x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$

$\Rightarrow 32 i\sin^5\theta = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$

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$$\left(x^n - \frac{1}{x^n} = 2i\sin n\theta\right)$$ (Apply Demoivre)

→ So;

$32 i\sin^5\theta = (2i\sin 5\theta) - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$32 i\sin^5\theta = 2i[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$$\boxed{\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)}$$

Hence, Proved.

<u>Note</u>: $x = \cos\theta + i\sin\theta$ $\quad$ $1/x = \cos\theta - i\sin\theta$

Taking x^n

$(\cos\theta + i\sin\theta)^n$

$x^n = (\cos n\theta + i\sin n\theta)$

Taking x^{-n}

$(\cos\theta + i\sin\theta)^{-n}$

$x^{-n} = (\cos n\theta) - i\sin n\theta$

$$\boxed{\left(x^n + \frac{1}{x^n}\right) = 2\cos n\theta}$$

$$\boxed{\left(x^n - \frac{1}{x^n}\right) = 2i\sin n\theta}$$

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Q Expand $\sin^5\theta \cdot \cos^3\theta$ in a series of sines of multiples of θ.

Sol → ~~sta~~

Suppose;

$x = \cos\theta + i\sin\theta$ & $1/x = \cos\theta - i\sin\theta$

$x^n + \frac{1}{x^n} = 2\cos n\theta$ & $x^n - \frac{1}{x^n} = 2i\sin n\theta$

$$\therefore (2i\sin\theta)^5 (2\cos\theta)^3 = \left(x - \frac{1}{x}\right)^5 \left(x + \frac{1}{x}\right)^3$$

$$= \left(x - \frac{1}{x}\right)^3 \left(x - \frac{1}{x}\right)^2 \left(x + \frac{1}{x}\right)^3$$

$$= \left(x^2 - \frac{1}{x^2}\right)^3 \left(x - \frac{1}{x}\right)^2$$

$$= \left[(x^2)^3 - 3(x^2)^2\left(\frac{1}{x^2}\right) + 3(x^2)\left(\frac{1}{x^2}\right)^2 - \left(\frac{1}{x^2}\right)^3\right]\left[x^2 - 2x\cdot\frac{1}{x} + \frac{1}{x^2}\right]$$

$$= \left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right]\left[x^2 - 2 + \frac{1}{x^2}\right]$$

$$= x^8 - 2x^6 + x^4 - 3x^4 + 6x^2 - 3 + 3 - \frac{6}{x^2} + \frac{3}{x^4}$$

$$- \frac{1}{x^4} + \frac{2}{x^6} - \frac{1}{x^8}$$

$$= \left(x^8 - \frac{1}{x^8}\right) - 2\left(x^6 - \frac{1}{x^6}\right) - 2\left(x^4 - \frac{1}{x^4}\right) + 6\left(x^2 - \frac{1}{x^2}\right)$$

$$\left(x^n - \frac{1}{x^n} = 2i\sin n\theta\right)$$

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$$= 2i\sin 8\theta - 2(2i\sin 6\theta) - 2(2i\sin 4\theta) + 6(2i\sin 2\theta)$$

$$\therefore (2i\sin\theta)^5(2\cos\theta)^3 = 2i[\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta]$$

$$32i\sin^5\theta\, 8\cos^3\theta = 2i(\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta)$$

Ans → $\boxed{\sin^5\theta\cos^3\theta = \frac{1}{128}(\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta)}$