Discrete Fourier Transform :- freq. domain Sampling.

To perform freq. Analysis on a discrete-time signal $x(n)$, we convert the time domain sequence to an equivalent freq. domain representation. Such a representation is given by fourier transform $X(\omega)$ of sequence $x(n)$. However, $X(\omega)$ is a continous function of freq. and therefore, it is not a computationally convenient representation of sequence $x(n)$.

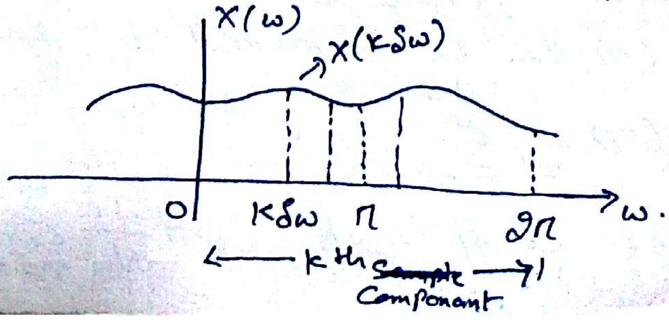
Discrete fourier transformation is the representation of a sequence $x(n)$ by samples of its spectrum $X(\omega)$. DFT is a powerful ~~tool~~ computational tool for performing frequency analysis of discrete time signal.

Fourier series — Periodic signal frequency Analysis
Fourier Transform → freq. Analysis of aperiodic signals by assuming them periodic signal.

Discrete fourier Transform :- freq. Analysis using samples of spectrum $X(\omega)$.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}.$$ → fourier transform of $x(n)$.

Now suppose we sample $X(\omega)$ periodically in freq. at a spacing of $\delta\omega$ radians between two successive samples. Since $X(\omega)$ is periodic with period 2π, only samples in the fundamental freq. range are neccessary.



fig① freq domain sampling of fourier transform.

For Convenience, we take N equidistant samples in the $0 \le \omega \le 2\pi$. with spacing $\delta\omega = \frac{2\pi}{N}$.

First we consider the selection of N, the number of samples in freq. domain.

now put $\omega = \left(\frac{2\pi}{N}\right)k$ → for k-th Component

$$X(\omega) = X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi k}{N}n}. \qquad k = 0, 1, \dots N-1.$$

This summation can be subdivided into infinite summations, where each term contains N terms. Thus.

$$X\left(\frac{2\pi k}{N}\right) = \dots\dots + \sum_{n=-N}^{-1} x(n)\, e^{-j\frac{2\pi k}{N}.n} + \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k}{N}n}$$

$$+ \sum_{n=N}^{2N-1} x(n)\, e^{-j\frac{2\pi k}{N}.n} + \dots\dots\dots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j\frac{2\pi k}{N}.n}$$

if we change the index in the inner summation from n to $(n-lN)$ and interchange the order of summation we obtain the result

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j2\pi kn/N}. \quad \text{—(1)}$$

for $k = 0, 1, \dots N-1$.

The signal $x_p(n) = \sum_{l=-\infty}^{\infty}(x-lN)$ obtained by the periodic repetion of $x(n)$ every N samples, is clearly periodic with fundamental period N.

$x_p(n)$ can be expanded in fourier series as.

$$x_p(n) = \sum_{k=0}^{N-1} C_k\, e^{j2\pi kn/N} \qquad n = 0, 1, 2 \dots N-1.$$

with fourier coefficient $C_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N}$ —(2)

$k = 0, 1, 2 \dots N-1.$

Comparing ① & ②, we conclude that.

$$C_K = \frac{1}{N} X\left(\frac{2\pi}{N}K\right) \qquad K = 0, 1, 2 .. N-1.$$

Therefor $$x_p(n) = \frac{1}{N}\sum_{K=0}^{N-1} X\left(\frac{2\pi K}{N}\right) e^{j2\pi K n/N} \quad -③$$

$n = 0, 1, 2 .... N-1.$

The releation ship provides the reconstruction of periodic signal $x_p(n)$ from the samples of spectrum $X(\omega)$. Since $X_p(n)$ is the periodic extention of $x(n)$. It is clear that $x(n)$ can be ~~cleared~~ recovered from $x_p(n)$ if there is no aliasing in the time domain

Q.1 Compute 4-Point DFT of causal three sample sequence given by.

$$x(n) = 1/3 \qquad 0 \leq n \leq 2$$
$$= 0 \qquad \text{otherwise}$$

Sol^n:-

$$X[K] = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi K n}{N}} \qquad \text{Here } N = 4.$$

therefore 4-point DFT $$X[K] = \sum_{n=0}^{3} x(n) e^{\frac{-j2\pi K}{4}n} = \sum_{n=0}^{3} x(n) e^{\frac{-j\pi K}{2}n}$$

$$= x(0)e^{0} + x(1)e^{-j\pi K/2} + x(2)e^{-j\pi K} + x(3)e^{-3\pi K/2}$$

$$= 1/3 + \frac{1}{3}e^{-j\pi K/2} + 1/3\, e^{-j\pi K} + 0.$$

$$= 1/3\left[1 + \cos\frac{\pi K}{2} - j\sin\frac{\pi K}{2} + \cos\pi K - j\sin\pi K\right]$$

Now for $K=0$.

$$X[0] = 1/3[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0] = 1/3[1+1+1] = 1\angle 0.$$

$$X[1] = 1/3[1 + \cos\pi/2 - j\sin\pi/2 + \cos\pi - j\sin\pi] = 1/3\angle -\pi/2.$$

$$X[2] = 1/3[1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi] = 1/3\angle 0$$

$$X[3] = 1/3[1 + \cos 3\pi/2 - j\sin 3\pi/2 + \cos 3\pi - j\sin 3\pi] = 1/3\angle \pi/2.$$

Hence 4 point DFT sequence.

$$X[k] = \{1\angle 0,\ \tfrac{1}{3}\angle -\pi/2,\ \tfrac{1}{3}\angle 0,\ \tfrac{1}{3}\angle \pi/2\}$$

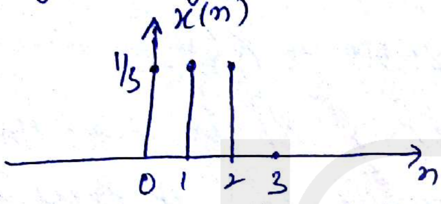
Magnitude plot

$$|X(k)| = \{1, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\}$$

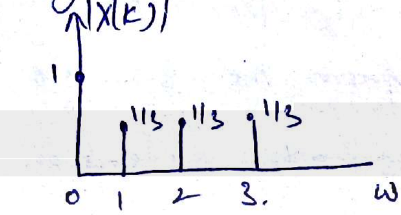
Phase plot

$$\angle X(k) = \{0, -\pi/2, 0, \pi/2\}$$

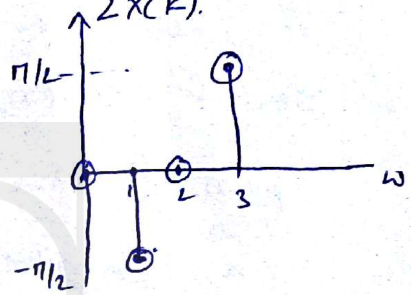
given Signal plot.



Magnitude plot.



Phase plot



Q.2 Compute the DFT of sequence $x(n) = \{0, 1, 2, 3\}$. ⇒ sketch the magnitude & phase plot.

Sol^n ⇒

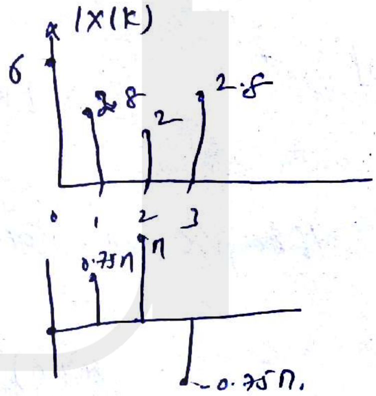
~~$X[k] = \{6\angle 0, 2.8\angle 0, 0.75\angle\pi$~~

multi. $\pi/180$

$$X[k] = \{6\angle 0,\ 2.8\angle 0.75\pi,\ 2\angle\pi,\ 2.8\angle -0.75\pi\}.$$

$$|X(k)| = \{6, 2.8, 2, 2.8\}$$

$$\angle X(k) = \{0, 0.75\pi, \pi, -0.75\pi\}$$



Q.3 Compute a 5-point DFT of the following finite length sequence
$x(n) = e^{-n}$; $0 \le n \le 4$.

Sol^n Given $x(n) = e^{-n}$; $0 \le n \le 4$ Hence $X[k] = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$

$\boxed{\text{where } N = 5}$

or $X[k] = \sum_{n=0}^{4} e^{-n} e^{-j\frac{2\pi}{5}kn} = \sum_{n=0}^{4} \left[e^{-1} e^{-j\frac{2\pi}{5}k}\right]^n$

$$= \frac{1 - \left[e^{-1} e^{-j\frac{2\pi}{5}k}\right]^5}{1 - e^{-1} e^{-j\frac{2\pi}{5}k}} = \boxed{\frac{1 - e^{-5}}{1 - e^{-1} e^{-j2\pi/5\, k}}}$$

Ans

$\boxed{\text{Sol}^n = \frac{1 - A^n}{1 - A}}$

as $\boxed{e^{-j2\pi k} = 1}$

(1) Lect-2

Properties of DFT

→ The DFT has a number of mathematical properties which can be used to simplify problems or which leads to useful applications.

$$DFT \Rightarrow X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad K = 0, 1, \ldots N-1.$$

$$IDFT \Rightarrow x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, \ldots N-1.$$

where $W_N = e^{-j2\pi/N}$. → twiddle factor

Hence N-point DFT pair $x(n)$ and $X(K)$ is

$$x(n) \underset{\substack{IDFT \\ N\text{-point}}}{\overset{DFT}{\longleftrightarrow}} X(K)$$

(1) **Periodicity** :- if $x(n)$ and $X(K)$ are ~~an~~ N-point DFT pair, then.

$$x(n+N) = x(n) \quad \text{for all } n.$$

$$X(K+N) = X(K) \quad \text{for all } K.$$

(2) **Linearity** :- if

$$x_1(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(K)$$

and $$x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_2(K)$$

then for any real valued or complex valued constants a_1 and a_2

$$a_1 x_1(n) + a_2 x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} a_1 X_1(K) + a_2 X_2(K).$$

(3) **Symmetry** :-

$$Re[X(N-K)] = Re[X(K)]$$ Symmetry of amplitude (even) fxn.

$$Im[X(N-K)] = -Im[X(K)]$$ antisymmetry property of phase. odd. fxn.

④ **Even functions** :- if $x(n)$ is even function x_e.
that is $x_e(n) = x_e(-n)$ then.

$$F_D[x_e(n)] = X_e(k) = \sum_{n=0}^{N-1} x_e(n) \cos\left(\frac{2\pi kn}{N}\right)$$

⑤ **odd function** :- if $x(n)$ is odd function $x_o(n)$
that is $x_o(n) = -x_o(-n)$ then.

$$F_D[x_o(n)] = X_o(k) = \sum_{n=0}^{N-1} x_o(n) \sin\left(\frac{2\pi kn}{N}\right)$$

⑥ **Parseval's theorem** :- The normalized energy in the signal is given by either of expression.

$$\sum_{n=0}^{N-1} x^2(n) = \frac{1}{N} \sum_{n=0}^{N-1} |X(k)|^2$$

⑦ **Circular Convolution** :- DFTs may also be used in the computation of circular convolution. The time domain convolution theorem states that

$$x_3(n) = x_1(n) \circledast x_2(n) = \sum X_1(k) X_2(k)$$

finite periodic sequence. → circular convolution.

$$x_3(n) = \sum_{m=0}^{N-1} x_1(n)\, x_2(n-m).$$

Example :-

$$x_1(n) = [\underset{\uparrow}{2}, 1, 2, 1] \qquad x_2 = [\underset{\uparrow}{1}, 2, 3, 4]$$

By circular convolution

$$x_3(n) = [14, 16, 14, 16].$$

find $x_3(n)$ using DFT.

DFT of $x_1(n) \Rightarrow X_1[k] = \sum_{n=0}^{3} x_1(n) e^{-j2\pi nk/4}$ $\quad k = 0,1,2,3.$

$= 2 + e^{-j\pi k/2} + 2e^{-j\pi k} + e^{-3\pi k/2}.$

Thus $X_1[0]=6$, $X_1[1]=0$, $X_2[2]=2$ $\quad X_1[3]=0$

DFT of $x_2(n)$

$$X_2[k] = \sum_{n=0}^{3} x_2(n) e^{-j2\pi nk/4} \qquad k = 0,1,2,3.$$

$$= 1 + 2e^{-j\pi k/2} + 3e^{-j\pi k} + 4^{-j3\pi k/2}.$$

Thus $X_2[0]=10$, $X_2[1]=-2+j2$ $\quad X_2[2]=-2$, $X_2[3]=-2-2j$

when we multiply the two DFTs, we obtain the product.

$$X_3[k] = X_1[k]\, X_2[k].$$

$X_3[0]=60$, $X_3[1]=0$, $X_3[2]=-4$, $X_3[3]=0.$

$$X_3[k] = [\underset{\uparrow}{60}, 0, -4, 0]$$

Hence

$$x_3(n) = \frac{1}{N} \sum_{k=0}^{3} X(k) e^{j2\pi nk/4}.$$

$$= \frac{1}{4}(60 - 4e^{j\pi n}]$$

Thus $x_3(0)=14$ $\quad x_3(1)=16$ $\quad x_3(2)=14$ $\quad x_3(3)=16$

Same as obtained by Circular Convolution.

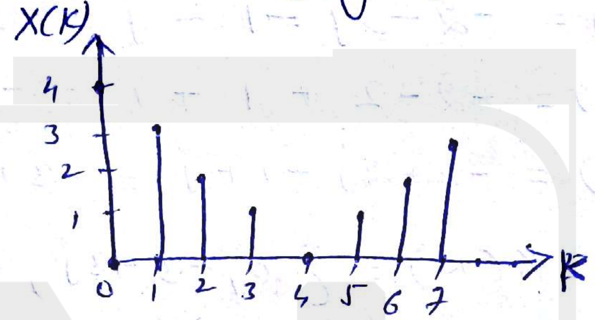
$x_3(n) = [\underset{\uparrow}{14}, 16, 14, 16]$ Ans

DFT :- Exercise

1. A finite duration sequence of $x(n)$ of length 8 has 8-point DFT $X(K)$ as shown in fig(1). A new sequence $y(n)$ of length 16 is defined by.

$$y(n) = \begin{cases} x(n/2) & ; n \text{ even} \\ 0 & ; n \text{ odd.} \end{cases}$$

find the 16-point DFT of $y(n)$.



Ans we have a finite duration sequence $x(n)$ of length (8). Suppose we interpolate it by a factor 2. That is we wish to double the size of $x(n)$ by inserting zeros at all values of n for $0 \le n \le 15$.

$$\text{i.e} \quad y(n) = \begin{cases} x(n/2) & n \text{ even} \quad 0 \le n \le 15 \\ 0 & n \text{ odd.} \end{cases}$$

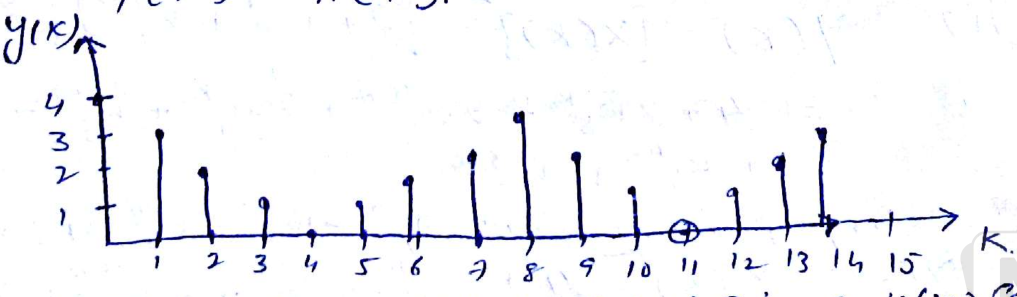
The 16-point DFT of $y(n)$ is

$$Y(K) = \sum_{n=0}^{N-1} y(n) e^{-\frac{j2\pi k}{N} n} \qquad 0 \le k \le 15$$

$$Y(K) = \sum_{n=0}^{N/2-1} x(n) e^{-\frac{j2\pi k n}{N/2}} \qquad 0 \le k \le 15$$

$$= \sum_{n=0}^{7} x(n) e^{-j\frac{2\pi k n}{8}}$$

$$Y(K) = X(K). \qquad 0 \le k \le 15.$$



Therefor 16 point DFT of 16 Interpolated signal $y(n)$ contains two copies of 8-point DFT of $x(n)$. Since $Y(k)$, $N=8$, $Y(k)$ is periodic with

Q.2 Compute the DFT of sequence

$$x(n) = [-2, 2, 1, -1]$$

Sol^n :- $X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn} = \sum_{n=0}^{3} x(n) W_4^{kn}$ $\quad K = 0, 1, 2, 3.$

$X(K) = -2 + 2W_4^K + W_4^{2K} - W_4^{3K}.$

$X(0) = -2 + 2 + 1 - 1 = 0.$ $\quad W_4 = e^{-j2\pi/4}$

$X(1) = -2 - 2j - 1 - j = -3 - 3j$ $\quad W_4 = -j$

$X(2) = -2 - 2 + 1 + 1 = -2$

$X(3) = -2 + 2j - 1 + j = -3 + 3j$

Hence $X(K) = \{0, -3-3j, -2, -3+3j\}$

Q.3 Consider a finite length sequence

$$x(n) = 2\delta(n) + \delta(n-1) + \delta(n-3).$$

we perform the following op^{n}s. on this sequence.

(i) first compute 5-point DFT $X(K)$.

(2) Compute IDFT of $Y(K) = \{X(K)\}^2$ to obtain a seq^n $y(n)$.

Determine the sequence $y(n)$ for $n = 0, 1, 2, 3, 4.$

Sol^n :- (i) $X(K) = \sum_{n=0}^{4} x(n) W_5^{kn}$ $\quad K = 0, 1, 2, 3, 4.$

$X(K) = x(0)W_5^0 + x(1)W_5^K + x(2)W_5^{2K} + x(3)W_5^{3K} + x(4)W_5^{4K}.$

$X(K) = 2 + W_5^K + 0 + W_5^{3K} + 0 = 2 + W_5^K + W_5^{3K}.$

(ii) $Y(K) = [X(K)]^2 = [2 + W_5^K + W_5^{3K}][2 + W_5^K + W_5^{3K}]$

$= 4 + 2W_5^K + 2W_5^{3K} + 2W_5^K + W_5^{2K} + W_5^{4K} + 2W_5^{3K} + W_5^{4K} + W_5^{6K}$

Using Periodic property. $W_5^5 = W_5^0 = 1$, $W_5^{6K} = W_5^K.$

$Y(K) = X^2(K) = 4 + 5W_5^K + W_5^{2K} + 4W_5^{3K} + 2W_5^{4K}.$ $\boxed{0 \le n \le 4}$

So IDFT of $Y(K) = y(n) = 4\delta(n) + 5\delta(n-1) + \delta(n-2) + 4\delta(n-3) + 2\delta(n-4)$

Discrete fourier transform

We can recover the $x(n)$ from freq. samples.

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi k}{N} n} \qquad n = 0, 1 \dots, N-1.$$

and is called Inverse DFT (IDFT)

DFT :-

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi k}{N} n} \qquad k = 0, 1, 2 \dots N-1.$$

IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi k}{N} n}. \qquad n = 0, 1, 2 \dots N-1$$

The $X(k\omega)$ have real & Imaginary Components in general so that

$$X(K) = R(k) + jI(k)$$

and $$|X(k)| = \left[R^2(k) + I^2(k)\right]^{1/2} \text{ — magnitude.}$$

and $X(k)$ has the associated phase angle.

$$\phi(k) = \tan^{-1}\left(I(k)/R(k)\right).$$

Example 1 :- $x(n) = [\underset{\uparrow}{1}, 0, 0, 1]$

means
$$\left.\begin{aligned} x(0) &= 1 \\ x(T) &= 0 \\ x(2T) &= 0 \\ x(3T) &= 1 \end{aligned}\right]$$
Recorded at time Intervals T.

Thus $N = 4$ from 0, to 3

It is required to find Complex values $X(k)$ for $k = 0$, $k = 1$, $k = 2$ and $k = 3$. (Since $N-1 = 3$).

$$X(0) = \sum_{n=0}^{3} x(nT)\, e^{-j0} = \sum_{n=0}^{3} x(nT)$$

$$= x(0) + x(T) + x(2T) + x(3T)$$

$$= 1 + 0 + 0 + 1 = 2$$

So $X(0)$ is entirely real of magnitude 2 and phase angle $\phi(0) = 0$.

$$X(1) = \sum_{n=0}^{3} x(nT)\, e^{-j2\pi n/N}$$

$$= 1 + 0 + 0 + 1 \cdot e^{-j2\pi 3/4} = 1 + e^{-j3\pi/2}$$

$$= 1 + \cos[3\pi/2] - j\sin[3\pi/2] =$$

$$= 1 + 0 - j(-1) = 1 + j$$

Thus $X(1) = 1 + j \rightarrow$ magnitude $= \sqrt{1+1} = \sqrt{2}$

$\phi(k) = \tan^{-1}\frac{1}{1} = \tan^{-1} 1 = 45^\circ$

$$X(2) = \sum_{n=0}^{3} x(nT)\, e^{-j\omega nT} = \sum_{n=0}^{3} x(nT)\, e^{-j2\pi n \cdot 2\pi/N}.$$

$$= 1 + 0 + 0 + 1 \cdot e^{-j4\pi 3/4} = 1 + e^{-j3\pi}$$

$$= 1 + \cos[3\pi] - j\sin[3\pi] = 1 - 1 = 0$$

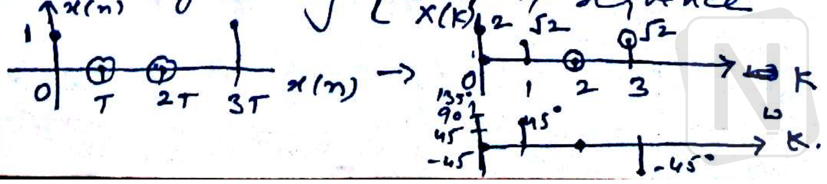
Hence magnitude 0 & phase Interminate.

$$X(3) = \sum_{n=0}^{3} x(nT)\, e^{-j\omega nT} = \sum_{n=0}^{3} X(nT) = e^{-j2\pi 3n/N}$$

$$= 1 + 0 + 0 + 1 \cdot e^{-j\pi 9/2} = 1 - j$$

Magnitude $= \sqrt{2}$. $\phi(3) = -45^\circ$.

Time Series $[1, 0, 0, 1]$ has DFT given by complex sequence $[2, 1+j, 0, 1-j]$



(1)

Linear filtering Technique based on DFT:- Linear Convolution using DFT

As a matter of fact, linear filtering is same as linear convolution.

we know that the linear convolution of $x(n)$ and $h(n)$ is given by.

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$\text{or } y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \qquad -(1)$$

if we obtain fourier transform of $x(n)$ and $h(n)$, then we shall get $X(\omega)$ and $H(\omega)$. we know that convolution is equivalent to multiplication in freq. domain. Therefore

$$Y(\omega) = X(\omega)\, H(\omega) \qquad -(2)$$

if we take Inverse ~~Con~~ fourier transform we get $y(n)$

(*) However we cannot use fourier transform to obtain linear Convolution due to 2 reasons:-

(1) In fourier transform ω is continuous fxn of freq. Hence computation can not be done on digital computers.

Because of digital signal processor discrete signals are required in place of continuous signals.

(2) if we use DFT, then the computation will be more efficient b'cos of availability of fast fourier transform (FFT) algorithms. Therefor we must use DFT to obtain linear filtering opns.

(*) But in case of DFT, we know that multiplication of two DFT in freq. domain is equivalent to circular convolution.

$$X(k) \cdot H(\omega) = x(n) \circledast h(n).$$

However we want linear convolution and not the circular convolution.

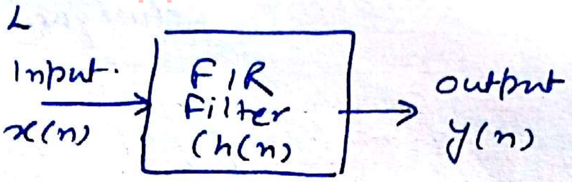
*(Earlier we have studied that if we adjust the length of two sequences $x(n)$ and $h(n)$ then the same result can be obtained using linear convolution and circular convolution).

Let $x(n)$ = Input seq having length L

$x(n) = \{0, 1, 2 \ldots L-1\}$

$h(n)$ = Impulse Response length M.

$h(n) = \{0, 1, 2 \ldots M-1\}$.



Therefor linear convolution of $x(n)$ and $h(n)$ produces output sequence of length $(L+M-1)$.

Hence $y(n) \Rightarrow$ length $(L+M-1)$.

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k).$$

<u>Example</u>:- Determine the response of FIR filter using DFT if $x(n) = \{\underset{\uparrow}{1}, 2\}$ and $h(n) = \{\underset{\uparrow}{2}, 2\}$

<u>Soln</u>

(i) Here length L of $x(n) = 2$
length M of $h(n) = 2$

Hence length of $y(n) = L+M-1 = 2+2-1 = 3 = N$.

Hence $N = 0, 1, 2, 3$. $N = 4$ 4 point DFT is required.

(ii) make $x(n)$ & $h(n)$ of length 4. by adding zeros at the end.

$x(n) = \{\underset{\uparrow}{1}, 2, 0, 0\}$ $h(n) = \{\underset{\uparrow}{2}, 2, 0, 0\}$

(iii) Calculation of $X(k)$ using Matrix method.

$$X(k) = W_N \cdot x_N.$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad x_N = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Hence

$$X[k] \Rightarrow \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1-2j \\ -1 \\ 1+2j \end{bmatrix}$$

④ Calculate of $H(K) = W_N \cdot h_N$

$$\begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+2 \\ 2-2j \\ 2-2 \\ 2+2j \end{bmatrix} = \begin{bmatrix} 4 \\ 2-2j \\ 0 \\ 2+2j \end{bmatrix}$$

⑤ Now calculate $Y[K] = X[K]\, H[K]$

$$= [3, 1-2j, -1, 1+2j]\,[4, 2-2j, 0, 2+2j]$$

$$= \{12, -2-6j, 0, -2+6j\}.$$

⑥ obtain $y(n)$ by Inverse IDFT of $Y(K)$

$$y(n) = \frac{1}{N} W_N^{*}\, Y(K).$$

where W_N^{*} = Complex Conjugate of W_N, by changing j by $-j$

Hence

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 12 \\ -2-6j \\ 0 \\ -2+6j \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ 24 \\ 16 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 0 \end{bmatrix}$$

Therefor $y(n) = \{2, 6, 4, 0\}$. Ans.

Fast Fourier Transform (FFT) Algorithms.

FFT usually refers to a class of algorithms for efficiently computing the DFT. The basic idea behind all ~~for~~ fast algorithms for computing ~~DFT~~ FFT is to decompose successively the N-point DFT computation into computation of smaller size DFTs and to take the advantage of periodicity & symmetry of complex number W_N^{kn}. This basic approach leads to a family of an efficient computational algorithms known collectively as FFT algorithms.

Consider DFT of a finite length of sequence.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad -(1)$$

Similarly the IDFT becomes.

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, 2 .. N-1.$$

The sequence $x(n)$ is also assumed to be a complex value.

We observed from eqn (1)

(1) N complex multiplications for each value of k.
(2) $N-1$ complex additions for each value of k.
(3) N^2 complex multiplications for N values of k
(4) $N(N-1)$ additions for N values of k.

for a complex seqn.

$$X_R[k] = \sum_{n=0}^{N-1} \left[x_R(n) \cos\frac{2\pi kn}{N} + x_I(n) \sin\frac{2\pi kn}{N} \right]$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin\frac{2\pi kn}{N} - x_I(n) \cos\frac{2\pi kn}{N} \right]$$

$$\therefore X[k] = X_R(k) + jX_I(k) \quad -(3)$$

The direct computation of eqⁿ (3) requires.

① 4N real multiplications for each value of K.

② (4N-2) real Additions for each value of K.

③ $4N^2$ real multiplications of N value of K.

④ N(4N-2) real Additions for N value of K.

we have seen that amount of computations & thus the computation time is approximately proportional to N^2. It is evident that number of arithmetic opⁿs required to compute the DFT by direct method becomes very large values of N.

Hence two properties

$$W_N^{(K+N/2)} = -W_N^K \quad \text{Symmetry property}$$

$$W_N^{K+N} = W_N^K \quad \text{periodic property.}$$

Exploit of these two basic properties results in computionally efficient algorithms which are collectively known as FFT. Algorithms.

Two Basic classes of FFT algorithms

① Decimation in Time (DIT) :- The sequence $x(n)$ [time sequence] is decomposed into successively smaller subsequences, Hence it is called "Decimation in Time (DIT) FFT"

② Decimation in frequency :- The sequence $X(K)$ is decomposed into smaller subsequences, hence it is called as "Decimation in frequency FFT."

Radix of FFT :- In N-Point sequence, if N can be expressed as $N = r^m$, then the sequence can be decimated into r point sequence. for each r point sequence, r-point sequence can be computed. From the result of r-point DFT, the r^2-point DFTs are computed, then from r^2 point DFT to r^3-point DFT and so on till r^m point DFT.

r is called radix of FFT algorithm.

Radix-2 Algorithm :- In a radix 2 FFT the N point sequence is decimated into 2-point seq^n & 2 point DFT for each sequence is computed. then 4-point, then 8-point until N point DFT.

$$N = 2^m.$$

Here decimation can be performed m times when

$$m = \log_2 N.$$

In Radix 2 FFT, the total no. of complex additions are reduced to $N \log_2 N$ and total number of complex multiplications are reduced to $(N/2) \log_2 N$.

Compare $N(N-1)$ to $N \log_2 N$.
N^2 to $(N/2) \log_2 N$

if $N = 16. (2^4)$.

Direct method.

$16(16-1) = 240$ Additions.

$(16)^2 = 256$ multiplications

Radix 2-FFT

$16 \log_2 16 = 16 \log_2 2^4 = 16 \times 4 \log_2 2$
$= 64$ Addn.

$N/2 \log_2 (2^4) = \frac{16}{2} \times 4(1) = 32$ multi.

Hence 8 point DFT using Radix-2 DIT FFT.

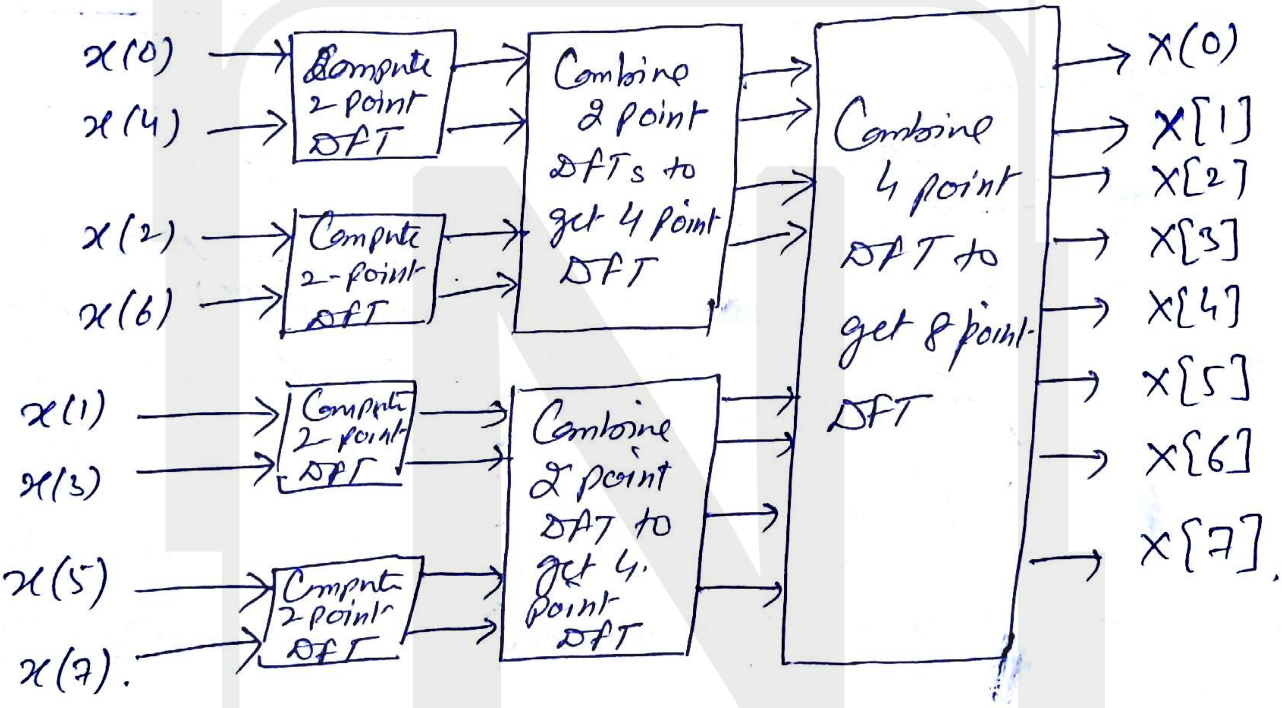
$N = 8 = 2^3$ Hence $r = 2$, $m = 3$.

$x(n)$ is decimated in time into 4 numbers of 2 point sequence

(1) $x(0)$ & $x(4)$ (2) $x(2)$ & $x(6)$.

(3) $x(1)$ & $x(5)$ (3) $x(5)$ & $x(7)$.

8 point DFT is computed as.



An 8-point sequence is given by: $x(n) = \{2,2,2,2,1,1,1,1\}$. Compute 8-point DFT of $x(n)$ by radix-2 DIT FFT. Also sketch the magnitude & phase spectrum.

Ans The given seq^n is first Rearranged in bit reversed order.

normal seq^n		Bit Reversed seq^n.
$x(0) = 2$	$= x(000) \rightarrow$	$x(000) = x(0) = 2$
$x(1) = 2$	$= x(001) \rightarrow$	$x(100) = x(4) = 1$
$x(2) = 2$	$= x(010) \rightarrow$	$x(010) = x(2) = 2$
$x(3) = 2$	$= x(011) \rightarrow$	$x(110) = x(6) = 1$
$x(4) = 1$	$= x(100) \rightarrow$	$x(001) = x(1) = 2$.
$x(5) = 1$	$= x(101) \rightarrow$	$x(101) = x(5) = 1$
$x(6) = 1$	$= x(110) \rightarrow$	$x(011) = x(3) = 2$.
$x(7) = 1$.	$= x(111) \rightarrow$	$x(111) = x(7) = 1$

Now we need to W_8^{nk}. as we know we have to compute only N/2 computations. Compute W_8^{nk} only for $n = 0,1,2,3$.

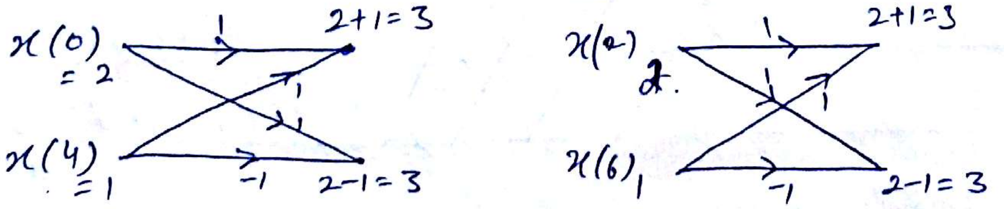
$$W_8^0 = 1$$

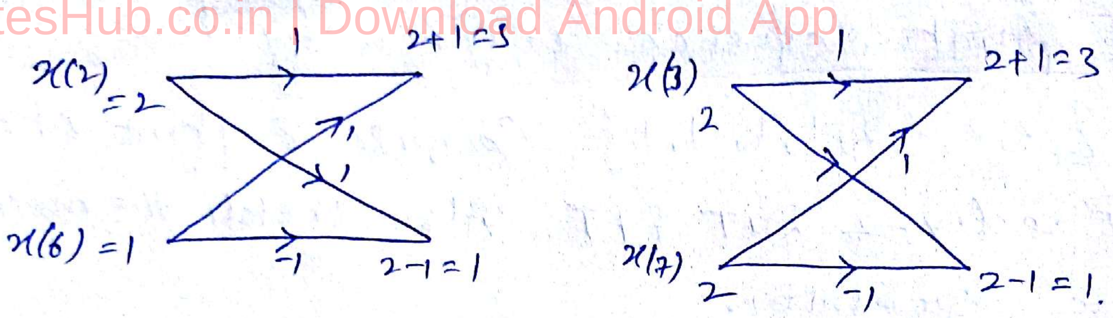
$$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = -j$$

$$W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

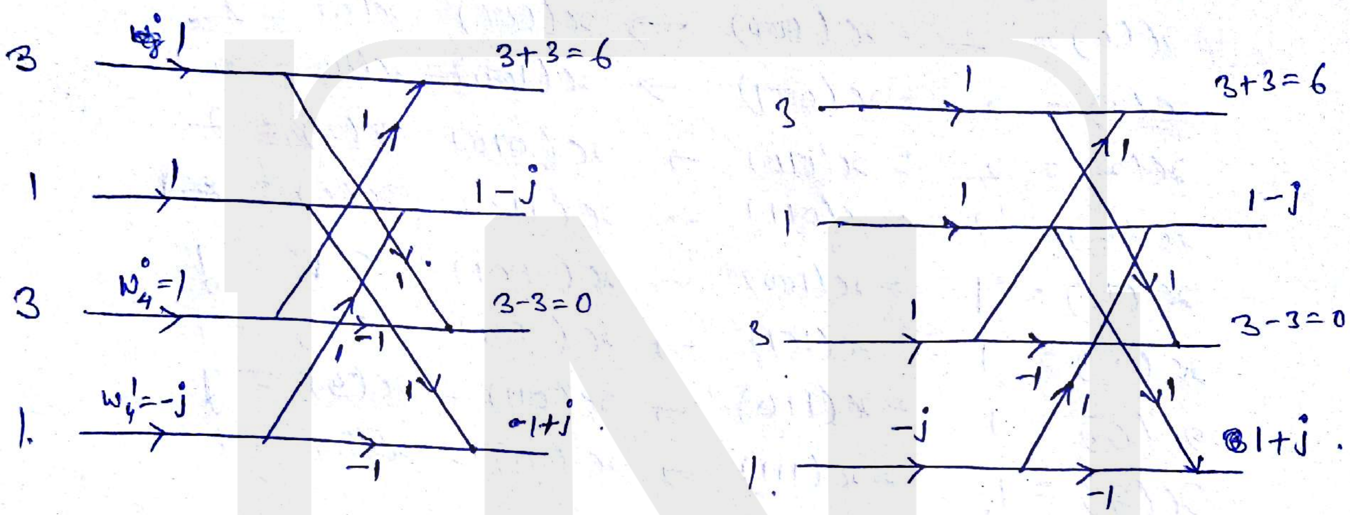
First stage Computation $\overset{N/4\text{ DIT FFT}}{:-}$ I/P sequence $\{2,1,2,1,2,1,2,1\}$



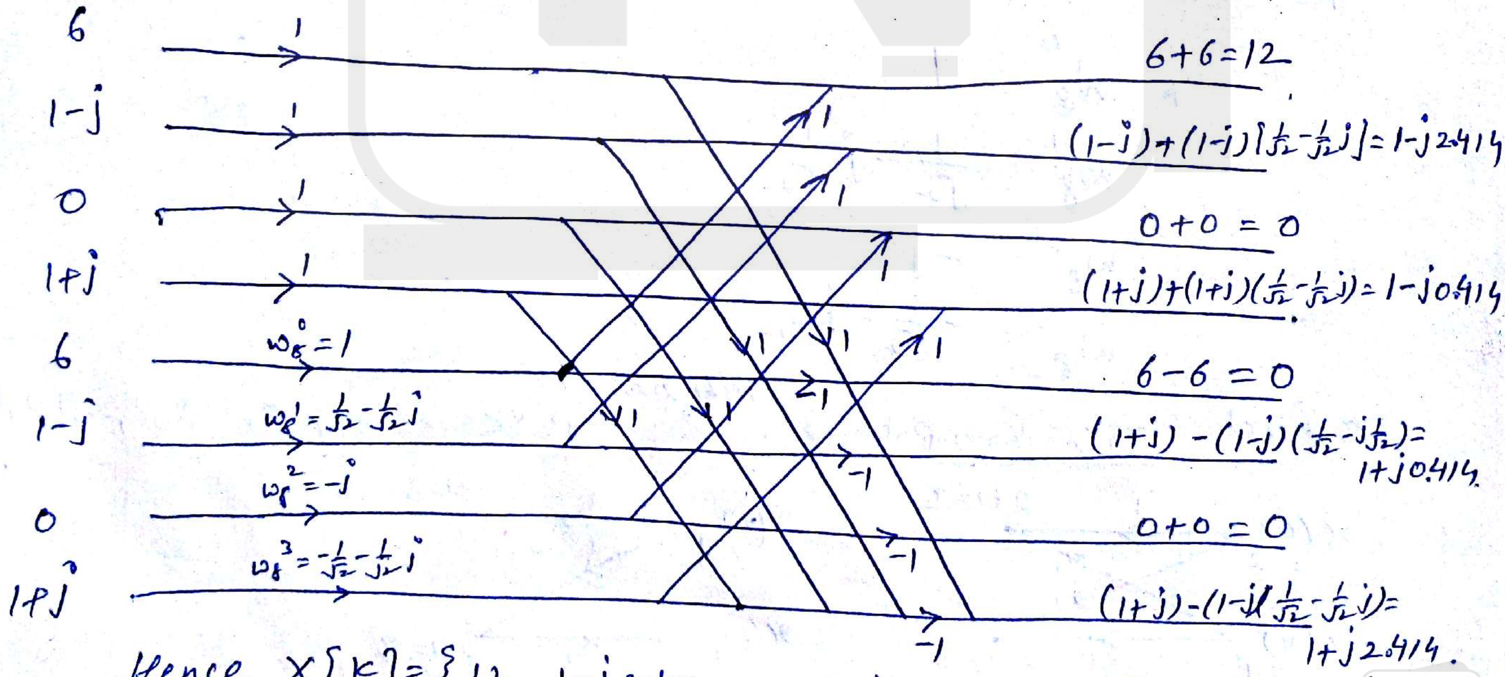


Second stage Computation N/2 DIT FFT → called Butterfly Computation

now sequence is $\{3, 1, 3, 1, 3, 1, 3, 1\}$. $W_4^0 = 1$, $W_4^1 = -j$.



Third stage $W_8^0 = 1$, $W_8^1 = \frac{1}{\sqrt2} - j\frac{1}{\sqrt2}$, $W_8^2 = -j$, $W_8^3 = -\frac{1}{\sqrt2} - j\frac{1}{\sqrt2}$.

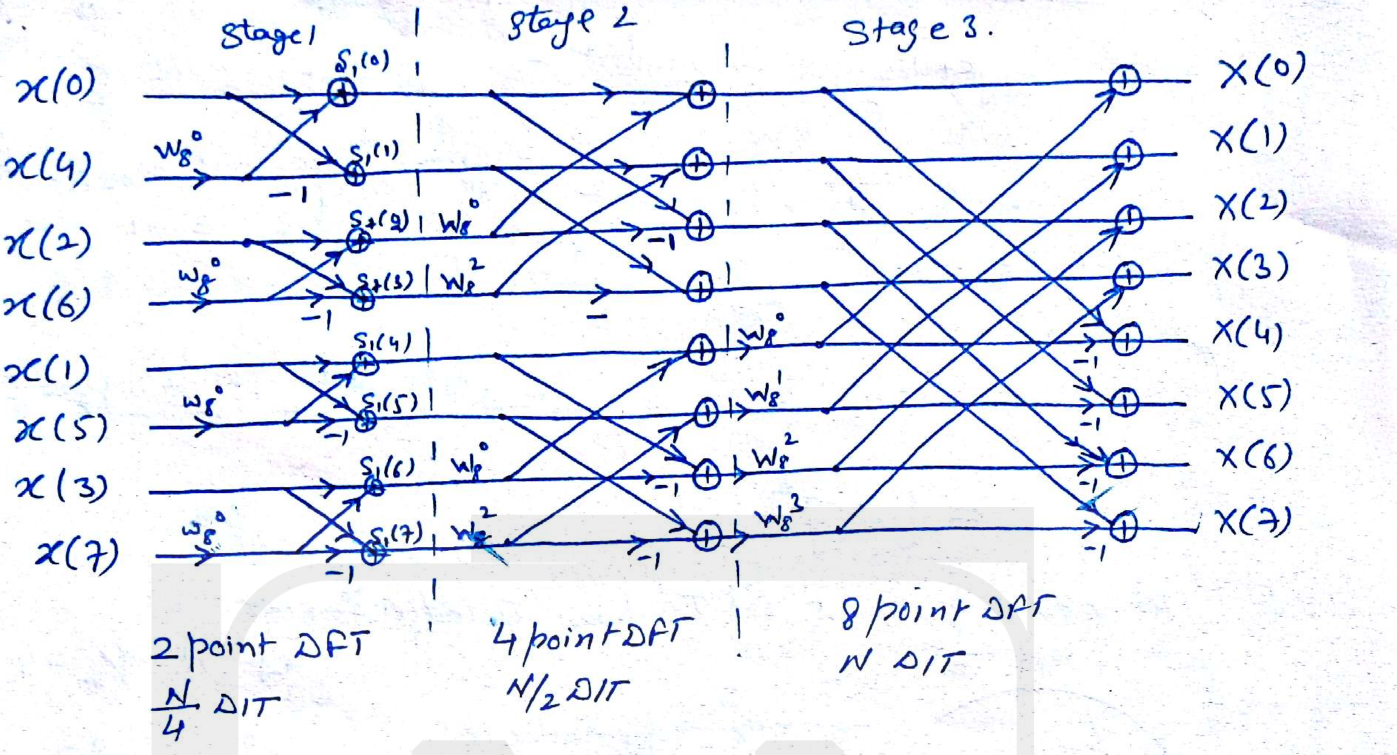


Hence $X[K] = \{12, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$

$|X[K]| = [12, 2.61, 0, 1.08, 0, 1.08, 0, 2.61]$ → Draw on

$\angle X[K] = \{0, -0.37\pi, 0, -0.12\pi, 0, 0.12\pi, 0, 0.37\pi\}$.

DIT FFT :- Total Signal flow Graph of 8 point DIT DFFT



① Radix-2 FFT Algorithm

N point DFT Can be factorized as $N = r_1 \cdot r_2 \cdot r_3 \cdots r_v$

where $[r_j]$ are prime.

There is important case when $r_1 = r_2 = r_3 = r_v = r$.

Then we can write $N = r^v$

where r is radix and v is no. of stages in FFT.

Thus for Computing $8 = N$ point DFT, we have

$8 = 2^3 \Rightarrow r = 2$ and $v = 3$, Therefore, for 8-point DFT, there are three stages of FFT algorithm.

② Types :- ⓐ Radix 2 - Decimation in Time (DIT) algorithm

ⓑ Radix 2 - Decimation in frequency (DIF) algorithm

③ Twiddle factor :- $W_N = e^{-j\frac{2\pi}{N}}$.

ⓐ Twiddle factor is periodic $\rightarrow$ $\boxed{W_N^{k+N} = W_N^{k}}$

Proof :- $W_N^{k+N} = \left[e^{-j\frac{2\pi}{N}}\right]^{k+N} = e^{-j\frac{2\pi}{N}k} \cdot e^{-j2\pi}$

Hence $e^{-j\frac{2\pi}{N}k} = W_N^{k}$ proved.

$\left[e^{-j2\pi} = \cos 2\pi - j\sin 2\pi = 1\right]$

(b) Twiddle Factor is Symmetric :- $W_N^{k+N/2} = -W_N^k$.

$$W_N^{k+N/2} = \left[e^{-j\frac{2\pi}{N}}\right]^{(k+N/2)} = e^{-j\frac{2\pi}{N}k} \cdot e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}}$$

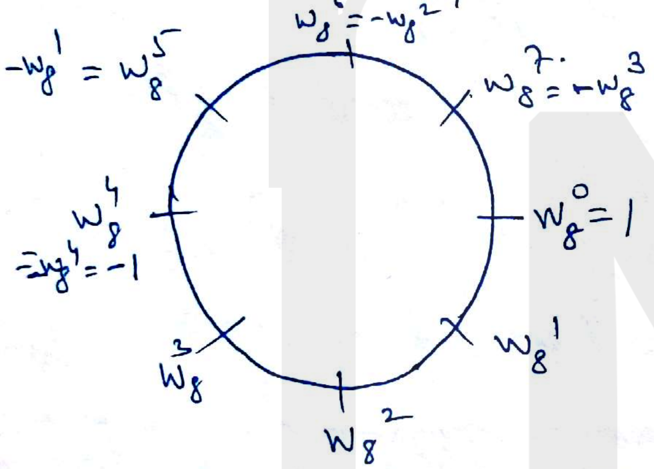
$$= e^{-j\frac{2\pi k}{N}} \cdot e^{-j\pi} = -e^{-j\frac{2\pi k}{N}} \quad \Big| \; e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

$$= -W_N^k$$

(c) $W_N^2 = W_{N/2} \Rightarrow W_{N/2} = e^{-j\frac{2\pi}{N/2}} = e^{-j\frac{2\pi \cdot 2}{N}} = \left[e^{-j\frac{2\pi}{N}}\right]^2$

$$= [W_N]^2$$

Hence 8-point DFT has Twiddle factor



Periodicity means. ~~Symmetric means~~

$\Rightarrow$

if $N = 8/2 = 4$, then

$W_8^4 = -W_8^0$

First Stage of Decimation :- let $x(n)$ be the given sequence of N Samples. Divide $x(n)$ into even and odd seq^n.

$$x(n) = f_1(m) + f_2(m)$$

$\boxed{\text{Put } \frac{n}{2} = m. \text{ Hence } n = 2m}$

where

Even seq^n $\quad f_1(m) = x(2m), \quad m = 0, 1, \ldots N/2 - 1.$

odd seq^n $\quad f_2(m) = x(2m+1) \quad m = 0, 1, \ldots N/2 - 1.$

Hence both sequences contains $N/2$ Samples.

Now according to DFT

$$X[k] = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n\,even} x(n) W_N^{kn} + \sum_{n\,odd} x(n) W_N^{kn}$$

$$= \sum_{m=0}^{N/2-1} x(2m) W_N^{2km} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{k(2m+1)}$$

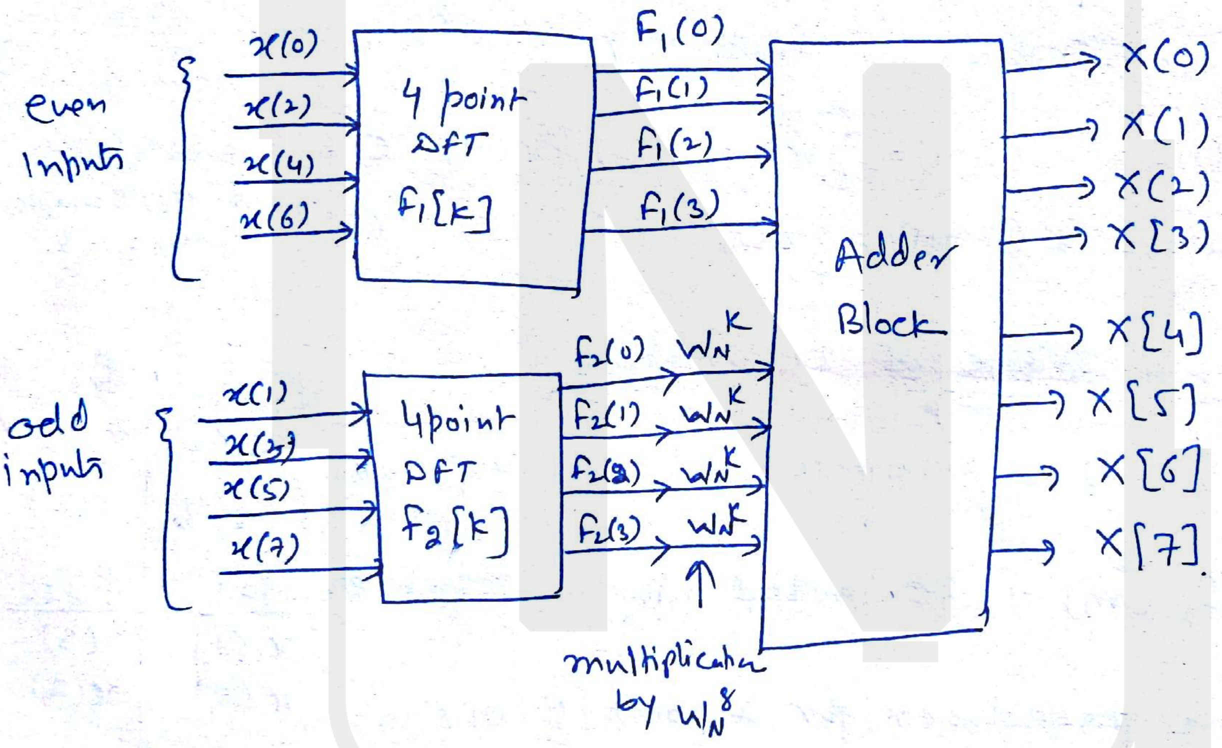
$$X[k] = \sum_{m=0}^{N/2-1} f_1(m)(W_N^2)^{km} + \sum_{m=0}^{N/2-1} f_2(m)(W_N^{km^2}) \cdot W_N^k$$

$$X[k] = \sum_{m=0}^{N/2-1} f_1(m)(W_N^2)^{km} + W_N^k \sum_{m=0}^{N/2-1} f_2(m)(W_N^2)^{km}$$

But we know that $\boxed{W_N^2 = W_{N/2}}$.

$$X[k] = \sum_{m=0}^{N/2-1} f_1(m)(W_{N/2})^{km} + W_N^k \sum_{m=0}^{N/2-1} f_2(m) W_{N/2}^{km}$$

$$= F_1[k] + W_N^k F_2[k] \quad —— (1)$$



Now $F_1[k]$ & $F_2[k]$ are 4 point $[N/2]$ DFT. They are periodic with period $N/2$.

Using periodicity property.

$$F_1[k+N/2] = F_1[k] \quad \text{and} \quad F_2[k+N/2] = F_2[k]$$

By Replacing k by $k+N/2$ in eqn (1).

$$X[k+N/2] = F_1[k+N/2] + W_N^{k+N/2} \cdot F_2[k+N/2]$$

$$= F_1[k+N/2] - W_N^k F_2[k+N/2] \qquad \because W_N^{k+N/2} = -W_N^k$$

$$X[k+N/2] = F_1[k] - W_N^k F_2[k]$$

So we can write:

$$X[0] = F_1(0) + W_N^0 F_2(0)$$
$$X[1] = F_1[1] + W_N^1 F_2[1]$$
$$X[2] = F_1[2] + W_N^2 F_2[2]$$
$$X[3] = F_1[3] + W_N^3 F_2[3].$$

and $X[0+4] = X[4] = F_1(0) - W_N^0 F_2(0)$

$$X[1+4) = X(5) = F_1(1) - W_N^1 F_2(1)$$
$$X[2+4] = X[6] = F_1[2] - W_N^2 F_2(2)$$
$$X[3+4] = X[7] = F_1[3] - W_N^2 F_2[3].$$

Hence $\ominus$ sign in Butterfly structure.

$f_1(m) = x(2m) = \{x(0), x(2), x(4), x(6)\}$
$f_2(m) = x(2m+1) = \{x(1), x(3), x(5), x(7)\}.$] both contain $N/2$ samples $n = 0, 1, 2, 3$.

(b) Second stage of Decimation

$f_1(m) \rightarrow$ Decimated into two sequences.

$f_2(m) \rightarrow$ Decimated into two sequences.

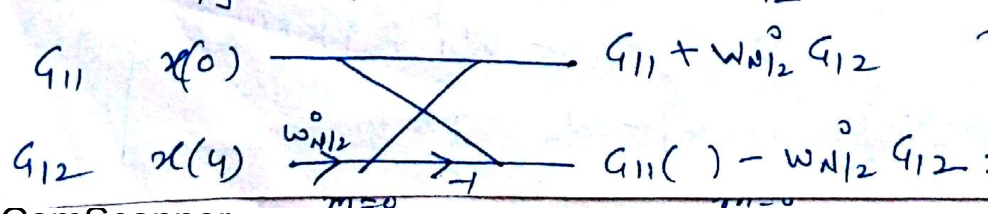
f_{11}	f_{12}
$x(0)$	$x(2)$
$x(4)$	$x(6)$

f_{21}	f_{22}
$x(1)$	$x(3)$
$x(5)$	$x(7)$.

Same proceducer for 2-point $[\frac{N}{4}]$ DFT.

~~$F_1[0 + 8/4] = f_1(2) =$~~

$$F_1(0) = G_{11}(0) + W_{N/2}^0 G_{12}(0)$$
$$F_1(1) = G_{11}(1) + W_{N/2}^1 G_{12}(1)$$
$$F_1[0 + 8/4] = F_1[2] = G_{11}(0) - W_{N/2}^0 G_{12}(0)$$
$$F_1[1 + 8/4] = F_1(3) = G_{11}(1) - W_{N/2}^1 G_{12}(1)$$



] this structure look like Butterfly. Hence called Butterfly FFT

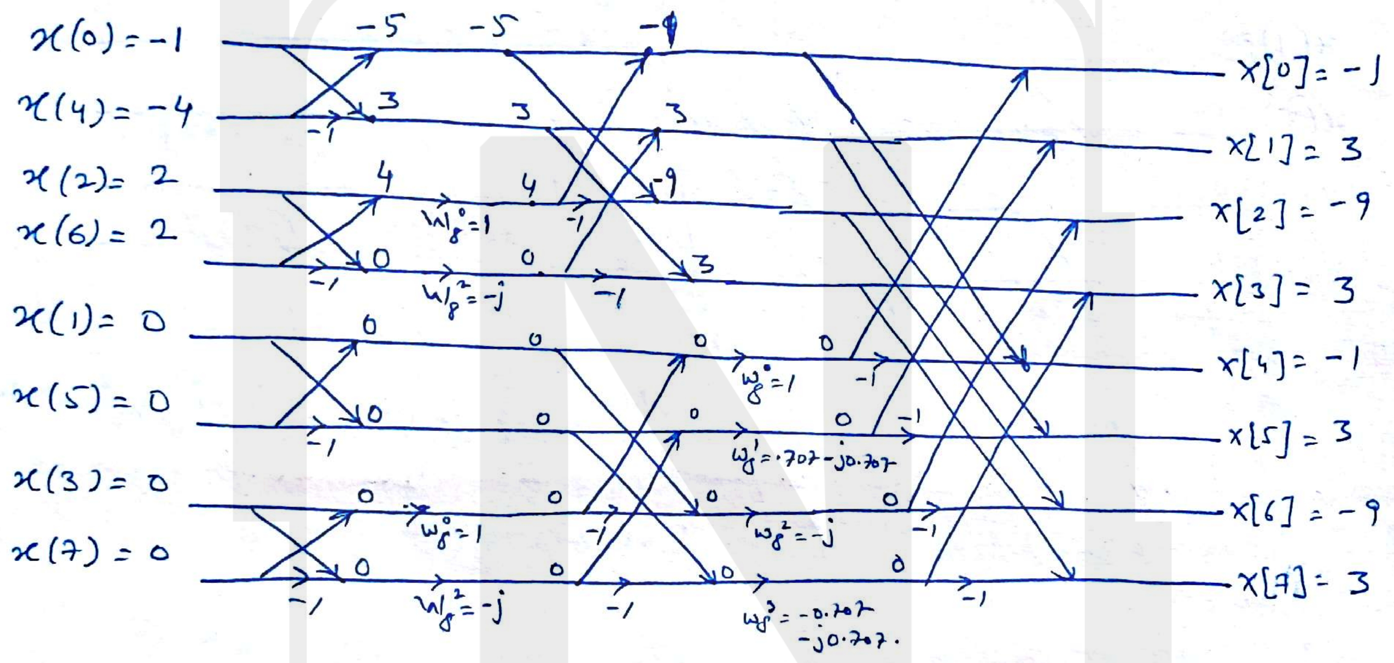
Q.1 Let $x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$
$n = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

find $X[K]$ using DIT FFT flow graph.

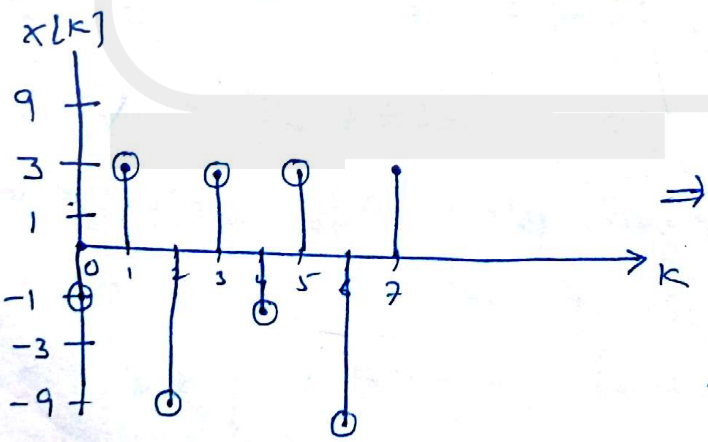
Ans Since it is a 8 point DIT FFT. First find Twiddle factors. as.

$W_8^0 = 1$, $W_8^1 = e^{-j\pi/4} = 0.707 - j0.707$

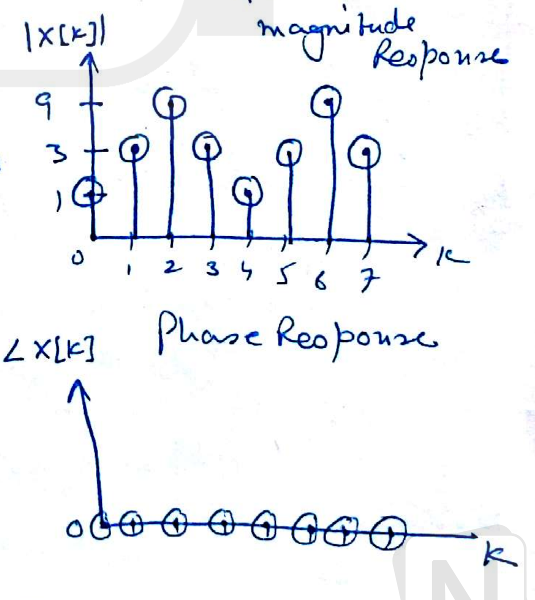
$W_8^2 = e^{-j\pi/2} = -j$ $W_8^3 = e^{-j\pi 3/4} = -0.707 - j0.707$.



Therefore $X[K] = [-1, 3, -9, 3, -1, 3, -9, 3]$ **Ans**



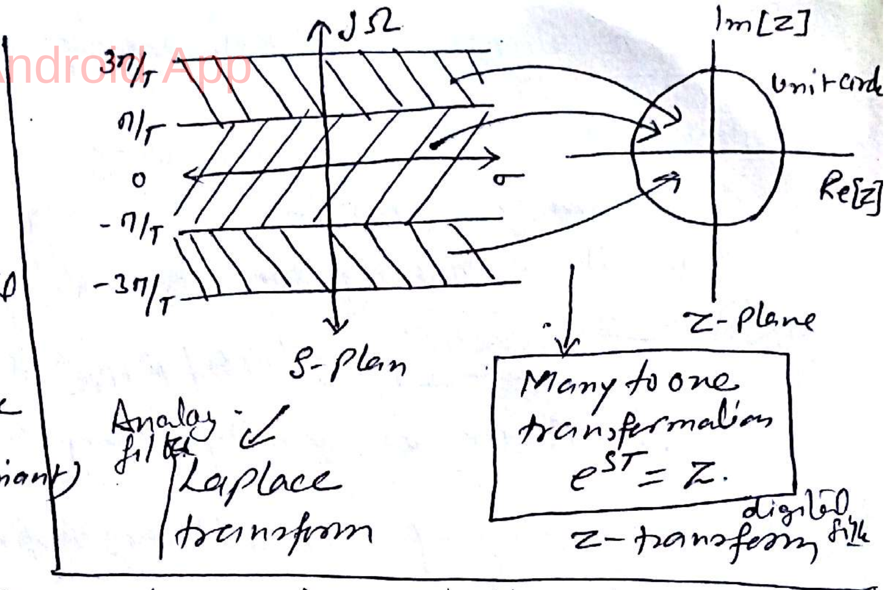
$\Rightarrow$



(1)

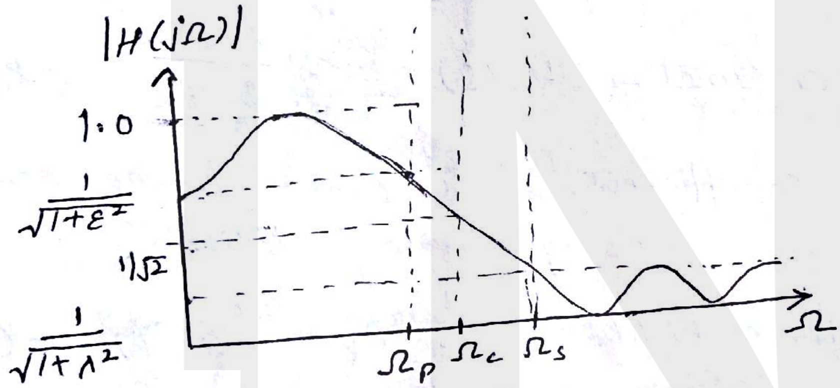
Impulse Invariant Method

The Impulse Invariant method converts analog filter transfer fxns to digital filter transfer functions in such a way that the Impulse response is the same (Invariant) at the sampling Instant.

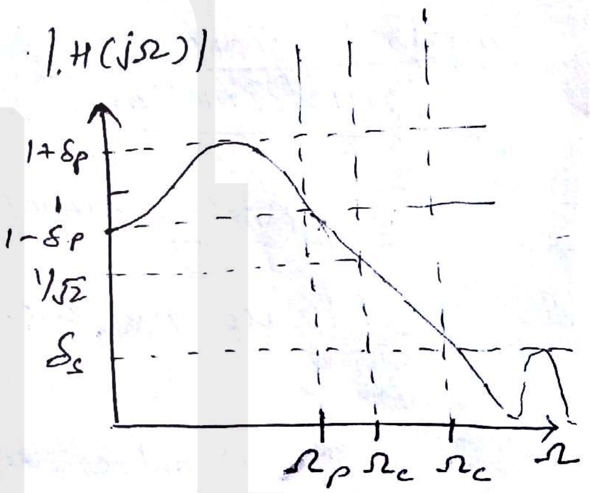


However the digital filter response is an aliased version of analog filter's frequency response.

Analog filter specifications:-



fig(1) Analog low pass filter



Alternate specifications of low pass filter.

Ω_p = Passband freq. rad/sec.

Ω_c = 3dB cut off freq. rad/sec.

Ω_s = stopband freq. rad/sec.

ϵ = parameter specifying allowable passband

λ = parameter specifying allowable stopband

$\delta_p \rightarrow$ passband error toler

$\delta_s \rightarrow$ ~~stopband~~ Maximum allowable magnitude

Analog lowpass filter design

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} a_i s^i}{1 + \sum_{i=1}^{N} b_i s^i}$$

where $H(s)$ is the laplace transform of the Impulse response $h(t)$. and $N \geq M$.

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

for stable filter the poles of $H(s)$ lies in the left of S-plane

Digital filters :- selection of filters — ① IIR → Infi...
② FIR → finite ...

In most cases, the order (N_{FIR}) of an FIR filter Considerably higher than the order (N_{IIR}) of an equivalen IIR filter meeting the same magnitude specifications.

* Generally $N_{FIR}/N_{IIR} \geq$ order of ten or more. Hence IIR filter is usually computationally more efficient.

Designing of IIR filters from Analog filters.

① Impulse Invariance. ③ Matched Z- transform
② Bilinear transformation ④ Approximations of derivatives

Impulse Invariance Method
Transformation

Analog transfer function $H_a(S) = \sum_{k=1}^{N} \frac{A_K}{S - P_K}$ → ①

Where A_K are co-efficients in the partial fraction expansion. P_K are the poles of analog filter.

The Impulse response $h_a(t) = \sum_{K=1}^{N} A_K e^{P_K t} u_a(t)$. – ②

if we sample $h_a(t)$ periodically at $t = nT$ we have

$$h(n) = h_a(nT) = \sum_{K=1}^{N} A_K e^{P_K nT} u_a(nT) \quad -③$$

Z- transform of $h(n)$ is

$$H(Z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} \sum_{K=1}^{N} A_K e^{P_K nT} z^{-n}$$

$$= \sum_{K=1}^{N} \sum_{n=0}^{\infty} A_K . (e^{P_K T} z^{-1})^n = \sum_{K=1}^{N} A_K \frac{1}{1 - e^{P_K T} z^{-1}} \quad -④$$

provided that $|e^{P_K T}| < 1$, which is always satisfied if $P_K < (0)$ indicating that $H_a(S)$ is stable transfer function.

Hence $Z_K = e^{P_K T}$ $K = 1, 2, \ldots N$.

Comparing ~~two transformations~~ ① & ④, we see the Impulse Invariance transformation is accomplished by mapping $\frac{1}{S - P_K} \rightarrow \frac{1}{1 - e^{P_K T} z^{-1}}$ | $\frac{1}{S + P_K} \rightarrow \frac{1}{1 - e^{-P_K T} z^{-1}}$

Q: For the Analog filter transfer function $Ha(s) = \frac{2}{(s+1)(s+2)}$ determine the $H(z)$ using Impulse Invariant method.

Sol^n → $$Ha(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{(s+1)} - \frac{2}{s+2}$$

$\frac{2s+4-2s-2}{}$ ②.

- Using the Impulse Invariance transformation. The digital filter transfer function

$$H[z] = \frac{Y[z]}{X[z]} = \frac{2}{1-e^{-T}z^{-1}} - \frac{2}{1-e^{-2T}z^{-1}}$$

$$= \frac{2[1-e^{-2T}z^{-1}] - 2[1-e^{-T}z^{-1}]}{[1-e^{-T}z^{-1}][1-e^{-2T}z^{-1}]} = \frac{2e^{-T}z^{-1}[1-e^{-T}]}{[1-e^{-T}z^{-1}][1-e^{-2T}e^{-1}]}$$

$\alpha_p = -20 \log(1-\delta_p)$ | $\alpha_s = -20\log(\delta_s)$ | $Ha(j\Omega)$ in dB.

Peak passband ripple — ~~Peak~~ minimum stopband ripple. — $-20 \log|Ha(j\Omega)|$

Q:1 Let the desired peak pass band ripple of a low pass filter be 0.01 dB and the minimum stopband attenuation in the stopband be 70 dB. Determine δp & δs.

Ans Given data $\alpha p = 0.01$ dB

$\alpha s = 70$ dB

(a) To find δp

$$\alpha p = -20 \log_{10}(1-\delta p).$$

$$\delta p = 1 - 10^{-\alpha p/20} = 0.00115.$$

(b) To find δs

$$\alpha_s = -20 \log_{10}(\delta_s)$$

Hence $\delta s = 10^{-\alpha s/20} = 0.0003162.$

Q.3 Convert the analog filter with system fxn.

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}.$$

into digital IIR filter by means of Impulse Invariance method.

Ans :- system has zero at $s = -0.1$, and a pair of complex poles at

$$p_K = -0.1 \pm j3$$

By partial fraction expansion of $H_a(s)$, we have

$$H(s) = \frac{1/2}{s+0.1-j3} + \frac{1/2}{s+0.1+j3}$$

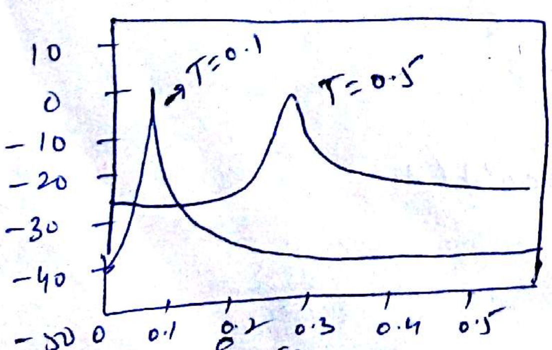
Then

$$H[z] = \frac{1/2}{1-e^{-0.1T}\cdot e^{j3T}z^{-1}} + \frac{1/2}{1-e^{-0.1T}e^{-j3T}z^{-1}}$$

Combine to form a single filter (single two pole filter)

$$\boxed{H[z] = \frac{1-(e^{-0.1T}\cos 3T)z^{-1}}{1-(2e^{-0.1T}\cos 3T)z^{-1}+e^{-0.2T}z^{-2}}}$$

Ans



fig(1)

~~Bi~~linear Transform Method

1. ~~Inverse~~ Impulse Invariance technique is appropriate only for band limited signals.
2. In Impulse Invariance technique
   A strip in S plane between $\Omega=(2k-1)\pi/T$ and $\Omega=(2k+1)\pi/T$ is mapped completely into z-plane. Therefore aliasing is caused by mapping of all such strips on top of each other in z-plane (called many to one mapping).

To overcome aliasing problem we need a one-to-one mapping technique. This type of mapping is called Bilinear transformation.

Bilinear transformation is a conformal mapping that transforms $j\omega$ axis into the unit circle in the z-plane only once. also points points in left half of s-plane are mapped inside the z-plane and all points in right half of s-plane are mapped outside the z-plane.

(1) Development of transformation:- use trapezoidal rule of integration.

Let us consider Analog filter transfer fxn.

$$H_a(s) = \frac{Y_a(s)}{X_a(s)} = 1/s. \quad -(1)$$

also characterized by differential eqn as.

$$\frac{dy_a(t)}{dt} = x_a(t).$$

by integrating both sides :- $\int_{nT-T}^{nT} \frac{dy_a(t)}{dt} = \int_{nT-T}^{nT} x_a(t).$

$$\Rightarrow \int_{nT-T}^{nT} dy_a(t) = \int_{nT-T}^{nT} x_a(t)\,dt$$

This integration can be approximated by trapezoidal rule of integration

$$y_a(nT) - y(nT-T) = T/2\left[x_a(nT) + x_a(nT-T)\right].$$

If we assume $y(n) = y_a(nT)$ & $x(n) = x_a(nT)$ then.

$$y(n) - y(n-1) = T/2\,[x(n) + x(n-1)]$$

Taking Z-transform

$$Y[z] - z^{-1}Y[z] = T/2\,[X[z] + z^{-1}X[z]]$$

$$H[z] = \frac{Y[z]}{X[z]} = \frac{T}{2}\left[\frac{1+z^{-1}}{1-z^{-1}}\right] = \frac{1}{\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} \quad \text{—(2)}$$

Comparing eqn (1) & (2) we have

$$\boxed{s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$ desired mapping.

Hence $$H[z] = H_a[s]\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

(2) Characteristics of Bilinear transformation.

as we know

$s = \sigma + j\Omega$ Ω = Analog freq.

$z = re^{j\omega}$ ω = Discrete freq.

$$S = \frac{2}{T}\left[\frac{z-1}{z+1}\right] = \frac{2re^{j\omega}-1}{2re^{j\omega}+1} = \frac{2}{T}\left[\frac{r(\cos\omega + j\sin\omega) - 1}{r(\cos\omega - j\sin\omega) + 1}\right]$$

$$= \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\,\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right] = \sigma + j\omega.$$

Hence $$\sigma = \frac{r^2-1}{1+r^2+2r\cos\omega}\ ; \quad \omega = \frac{2r\sin\omega}{1+r^2+2r\cos\omega}.$$

Case I when $\Omega < 1$ then $\sigma < 0$ LHP inside z plane

Case II when $\Omega > 1$ then $\sigma > 0$ RHP outside z-plane

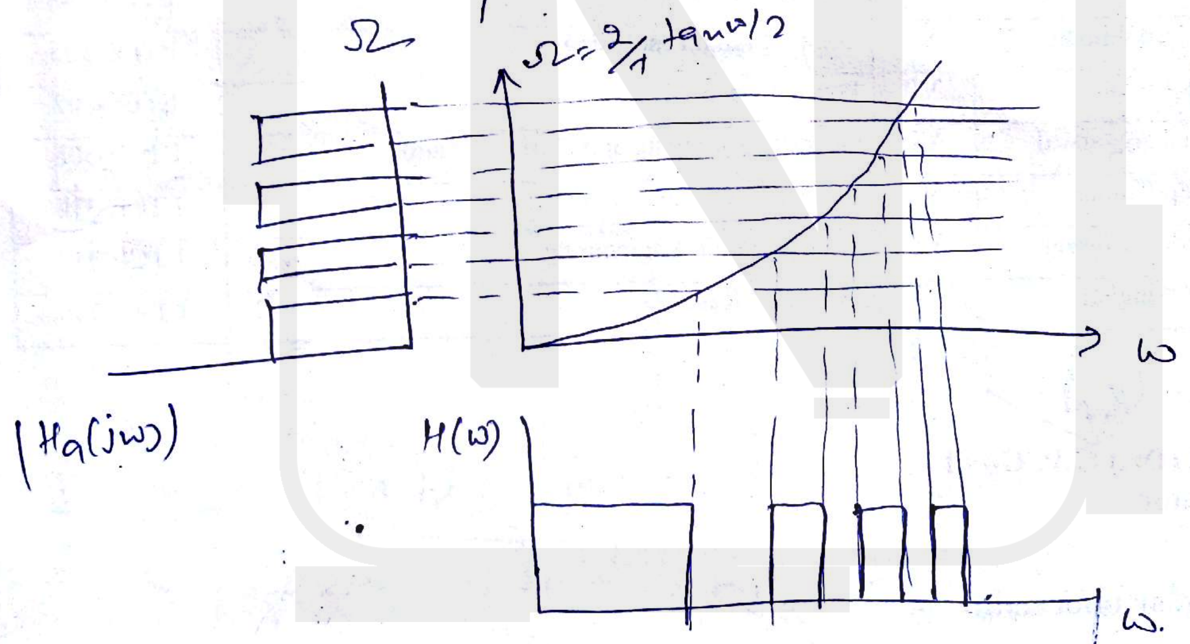
Case III when $\Omega = 1$ then $\sigma = 0$ $j\omega$ axis onto unit circle of z plane

for Case three. we have $r = 1$, $\sigma = 0$ then.

$$\boxed{\Omega = \frac{2}{T}\left[\frac{\sin\omega}{1+2\cos\omega}\right] = \frac{2}{T}\left[\frac{2\sin\omega/2\cos\omega/2}{2\cos^2\omega/2}\right] = \frac{2}{T}\tan\omega/2}$$

$$\& \quad \omega = 2\tan^{-1}\frac{\Omega T}{2}$$

(3) **warping effect**:- The mapping of freq. from Ω to ω is approximately linear for small values of ω & Ω. for higher values it is highly non linear. This distortion in freq. scale is known as warping effect.



influence of warping effect.

(4) **Prewarping** :- introduce T in Ω as

$$\Omega^* = \frac{2}{T}\tan\frac{\Omega T}{2}$$

Hence $\omega = 2\tan^{-1}\left[\frac{\Omega^* T}{2}\right] = 2\tan^{-1}\left[\frac{T}{2}\times\frac{2}{T}\tan\frac{\Omega T}{2}\right] = \Omega T$

Q. Convert the analog filter with system fxn.

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$$ into a digital IIR filter by means of BLT. Resonant freq. of a digital filter is given as $\omega_r = \pi/2$.

Ans

$$\Omega_r = \sqrt{16} = 4$$

(ii) $\Omega = \frac{2}{T} \tan\frac{\omega}{2}$

$$4 = \frac{2}{T} \tan \pi/4$$

$$T = 1/2$$

(iii) $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] = 4\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$

(iv) $H[z] = H_a(s) = \dfrac{4\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+0.1}{\left[4\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\}+0.1\right]^2+16}$

$$= \frac{(z+1)(z-0.95)}{(z-0.987e^{-j\pi/2})(z-0.987e^{j\pi/2})}$$

zeros $= -1, 0.95$

Poles $= 0.982e^{\pm j\pi/2}$

Frequency Transformation

Until now we have studied only design of low pass filters. If we have to design other filters like high pass, band pass etc, then we have to do frequency transformation.

If the cut off freq ~~Ω_c~~ of LPF is equal to 1. i.e $\Omega_c = 1$. then it is called normalized filter. To design other filters, first normalized LPF is designed & then by using freq. transformation, we can get the required fxn. for other filters.

① Low pass to low pass :- New passband edge freq. Ω_{LP}

then $s \to \frac{\Omega_c}{\Omega_{LP}} s$. means replace s by new s.

② Low pass to high pass. :- Ω_{HP} cutoff freq.

$$\text{Replace } s \to \frac{\Omega_c \Omega_{HP}}{s}.$$

③ Lowpass to band ~~reject~~ pass $\Omega_u \to$ upper cutoff Ω_L = lower cutoff.

$$\text{Replace } s \to \frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)} \Omega_c$$

---

* Example Lowpass to Bandstop or Notch or Band reject.

Transformation is $s = \cancel{\frac{s^2 + \Omega_u \Omega_L}{s(\Omega_u - \Omega_L)}}$ $\frac{s(\Omega_u - \Omega_L)}{s^2 + \Omega_u \cdot \Omega_L} \cdot \Omega_c$.

Q.1 Design a second order band pass digital Butterworth filter with passband of 300 Hz to 500 Hz. and sampling freq of 1500 Hz. using Bilinear transformation. Given LPF with Tx fxn $H_a(s) = \frac{\Omega_c}{s + \Omega_c}$.

Ans $f_u = 500\ Hz$ Sampled at 1500 Hz $\to f_u = \frac{500}{1500} = 0.33$ ~~rad/sample~~

~~$\Omega_u = 2\pi \times 500 = 1000\pi$ rad/sec~~

$f_L = 300\ Hz$ ~~$\Omega_L = 2\pi \times 300 = 600\pi$ rad/sec~~ $= \frac{300}{1500} = 0.2$

$w_u = 2\pi \times 0.33 = 2.073$ rad./sample

$w_L = 2\pi \times 0.2 = 1.257$ rad/sample.

(1) Higher digital cut-off $\omega_u = 2.073$ rad./sample
(2) lower digital cut-off $\omega_L = 1.257$ rad./sample.
(3) order of filter $N = 2$.

(1) Calculate Analog ~~cut~~ frequencies.

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

(a) $\Omega_u = (2 \times 1500) \tan\left(\frac{2.073}{2}\right) = 5070.11$

$\Omega_L = (2 \times 1500) \tan\left(\frac{1.257}{2}\right) = 2180.46$.

(2) Given system Transfer fxn.

$$H(_a s) = \frac{\Omega_c}{s + \Omega_c} \quad \text{for normalised LPF } \omega_c = 1$$

$$= \frac{1}{s+1}.$$

Convert to Band pass filter using

$$s \rightarrow \Omega_c \frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)}.$$

$$= 1 \cdot \frac{s^2 + (5070.11)(2180.46)}{s(5070.11 - 2180.46)}$$

$$\boxed{s \rightarrow \frac{s^2 + 11.05 \times 10^6}{2889.65 s}}$$

$$\rightarrow H_a(s) = \frac{1}{\frac{s^2 + 11.05 \times 10^6}{2889.65 s} + 1}$$

$$= \frac{2889.6 s}{s^2 + 2889.65 s + 11.05 \times 10^6}$$

This is Transfer fxn for desired BPF. Put Butterworth transformation $s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$.

$$H[z] = \frac{2889.6 [2 \times 1500]\left[\frac{z-1}{z+1}\right]}{(2 \times 1500)^2 \left[\frac{z-1}{z+1}\right]^2 + 2889.6 [2 \times 1500]\left(\frac{z-1}{z+1}\right) + 11.05 \times 10^6}$$

$$\boxed{H[z] = \frac{8.67(z-1)(z+1)}{9(z-1)^2 + 8.67(z^2-1) + 11.05(z+1)^2}}$$

Digital Filter Structures

The linear, time invariant system is described by the eq^n.

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + \dots = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \dots$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k). \quad -(1)$$

a_k & b_k are constants with $a_0 = 1$.

For Recursive → eq^n (1) is recursive past outputs are available

Structures of IIR System

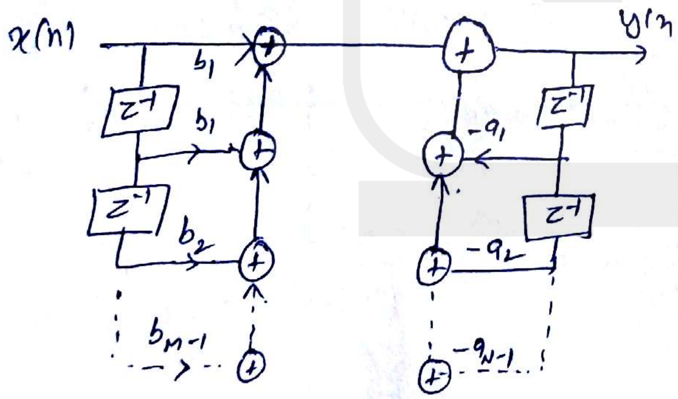
(1) Direct from Structures

$$H(z) = \frac{Y[z]}{X[z]} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}.$$

Hence $y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$

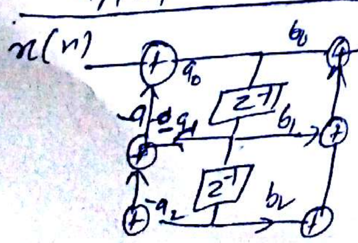
Poles   Zeros

Structure



All zero System $H_1(z)$ All pole System $H_2(z)$

Direct I

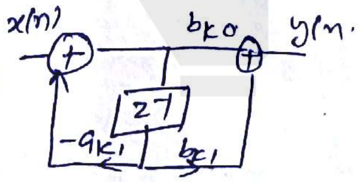


Direct (2)

(2) Cascade from Structures

$$H(z) = A\, H_1(z).H_2(z)\dots H_3(z) = A \prod_{k=1}^{K} H_k(z).$$

Divide into 1st order structures and add in series to find final structure.

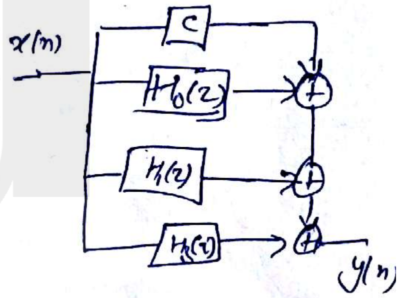


(3) Parallel form Structures

$$H(z) = \frac{Y(z)}{X[z]} = C + \sum_{k=1}^{N} H_k(z)$$

where $H_k(z) = \sum_{k=1}^{N} \frac{A_k}{1+P_k z^{-1}}$

$C = \frac{b_N}{a_N}$, P_k are poles. a_k are coefficients



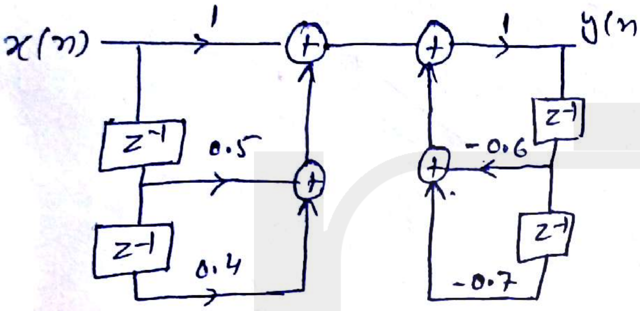
Direct & transposed form

$y(n) = x(n) + 0.5x(n-1) + 0.4x(n-2) - 0.6y(n-1) - 0.7y(n-2)$

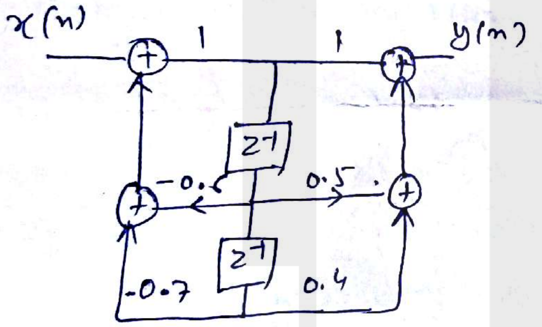
Z transform.

$Y[Z] = X[Z] + 0.5z^{-1}X[Z] + 0.4z^{-2}X[Z] - 0.6z^{-1}Y[Z] - 0.7z^{-2}Y[Z]$

Direct I form.



Direct form II transposed form.

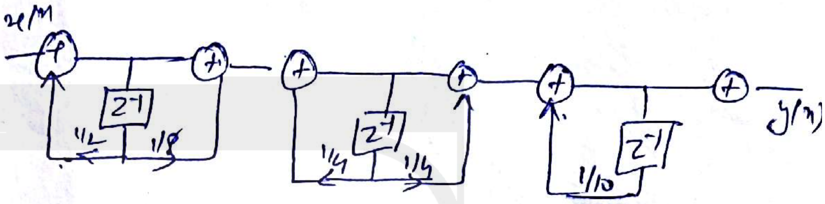


Q.2 Using first order section, obtain Cascade realization form

$$H(z) = \frac{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{10}z^{-1}\right)}$$

$$H_1(z) = \frac{\left(1+\frac{1}{8}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)}, \quad H_2(z) = \frac{1+\frac{1}{4}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)}$$

$$H_3(z) = \frac{1}{1-\frac{1}{10}z^{-1}}$$



Q.3 Parallel form.

$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2)$

taking z-transform

$Y[Z] = -\frac{3}{8}z^{-1}Y[Z] + \frac{3}{32}z^{-2}Y[Z] + \frac{1}{64}z^{-3}Y(Z) + X[Z] + 3z^{-1}X[Z] + 2z^{-2}X[Z]$

or. $$\frac{Y[Z]}{X[Z]} = \frac{1+3z^{-1}+2z^{-2}}{1+\frac{3}{8}z^{-1}-\frac{3}{32}z^{-2}-\frac{1}{64}z^{-3}} = \frac{(1+z^{-1})(1+2z^{-1})}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

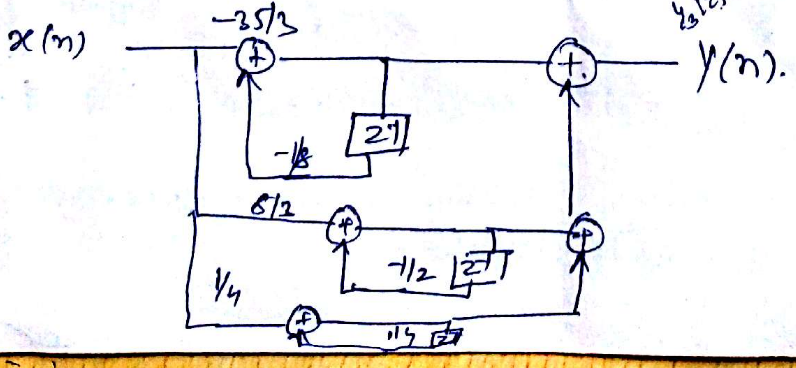
By Partial fraction

$$H(Z) = \frac{A}{1+\frac{1}{8}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}} + \frac{C}{1-\frac{1}{4}z^{-1}} = \frac{-35/3}{1+1/8\,z^{-1}} + \frac{8/3}{1+1/2\,z^{-1}} + \frac{10}{1-1/4\,z^{-1}}.$$

$H(Z) = \frac{Y[Z]}{X[Z]}$ => Hence $Y_1[Z] = -\frac{1}{8}z^{-1}Y_1[Z] - \frac{35}{3}X[Z]$

$Y_2[Z] = -\frac{1}{2}z^{-1}Y_2[Z] + 8/3\,X[Z]$

$Y_3[Z] = \frac{1}{4}z^{-1}Y_3[Z] + 10X[Z].$



Parallel Realization

Design technique for linear phase filters

(1) By Fourier series method (2) freq. Sampling method (3) optimal filter design

following two Concepts leads to design of FIR filter by fourier series method,

(a) The freq. response of a digital filter is periodic with period equal to sampling freq.

(b) Any periodic function can be expressed as a linear combination of complex exponentials.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

z transform of sequence.

$$H(z) = \sum_{n=-\infty}^{\infty} h_d(n)\, z^{-n}$$ → Non Caused digital filter of infinite duration

Therefore. $h(n) = h_d(n)$ for $|n| \le \frac{N-1}{2}$

$= 0$ otherwise.

Then $$H[z] = \sum_{n=-(\frac{N-1}{2})}^{\frac{N-1}{2}} h(n)\, z^{-n}.$$

$$= h\left[\tfrac{N-1}{2}\right] z^{-[\frac{N-1}{2}]} + \dots + h(1) z^{-1} + h(0) z^{0} + h(-1) z^{1} + h\left[-\left(\tfrac{N-1}{2}\right)\right] z^{\left[\frac{N-1}{2}\right]}$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n) z^{-n} + h(-n) z^{n}\right]$$

for a Symmetrical Impulse response. $h(-n) = h(n)$.

Hence $$\boxed{H[z] = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^{n} + z^{-n}\right]}$$ → This response is physically not realizable

Hence Realizable Response

$$H'[z] = z^{-(N-1/2)}\, H[z]$$

$h(n) = \{1/2,\ 0.3183,\ 0,\ -0.106,\ 0,\ 0.063\}$

~~Ho~~ Transfer function [z-transform]

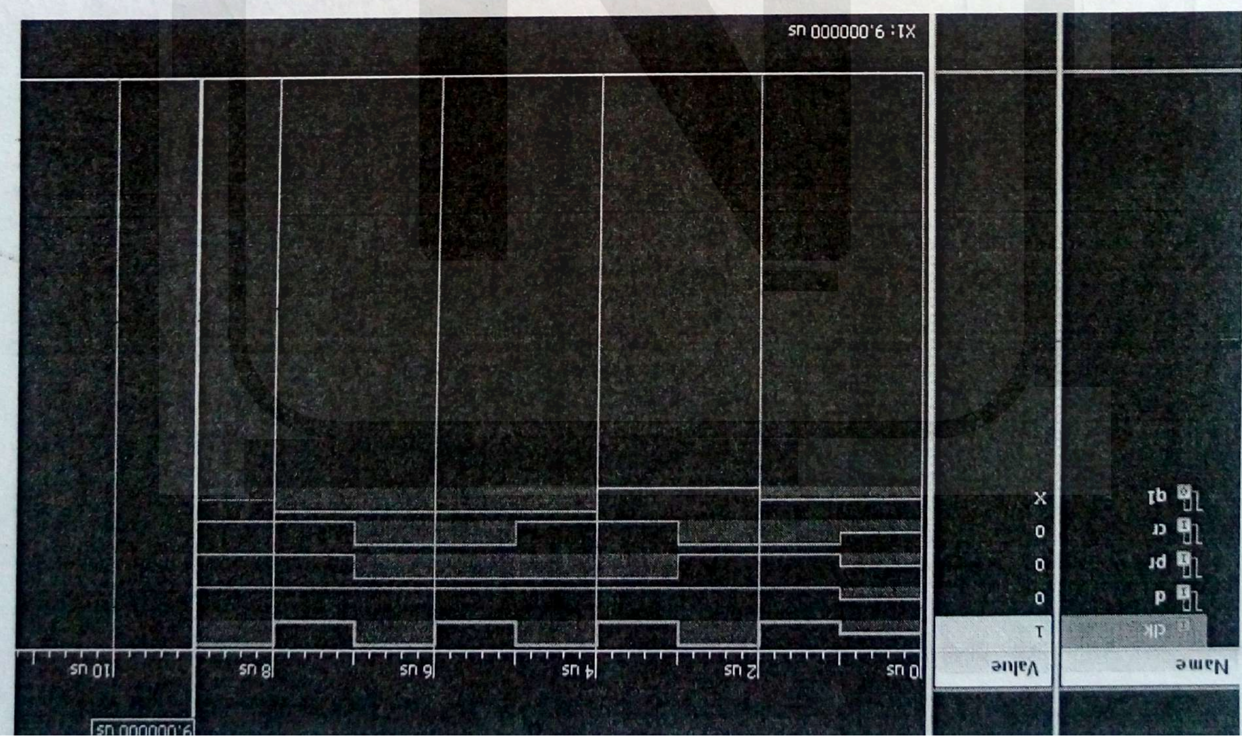
$$H[z] = h(0) + \sum_{n=1}^{N-1/2} h(n)\,[z^{n} + z^{-n}]$$

$$= \frac{1}{2} + 0.3183\,[z + z^{-1}] - 0.106\,[z^{3} + z^{-3}] + 0.06\,[z^{5} + z^{-5}]$$

To find Realizable filter multiply by $z^{-(N-1/2)} = z^{-5}$.

$$H'[z] = 0.063 - 0.106\,z^{-2} + 0.3183\,z^{-4} + 0.5\,z^{-5} + 0.3183\,z^{-6} - 0.106\,z^{-8} + 0.063\,z^{-10}$$

$h(n) = \{0.063, 0, -0.106, 0, 0.3183, 0.5, -0.106, 0, 0.065$

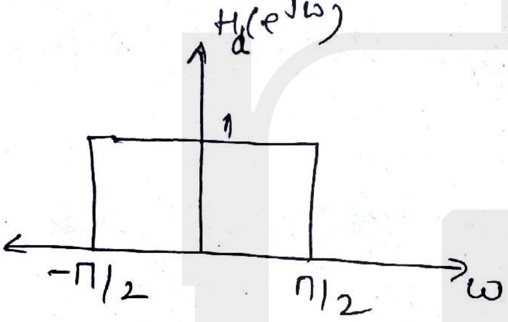


Q.1 Design an Ideal low pass filter with a frequency response.

$$H_d(e^{j\omega}) = 1 \quad \text{for } -\pi/2 \le \omega \le \pi/2$$
$$= 0 \quad \text{for } \pi/2 \le |\omega| \le \pi$$

Find the values of $h(n)$ for $N = 11$.

Ans Draw the desired freq. response



and $N=11$. Hence $\frac{N-1}{2} = 5$

$N =$ odd symmetry function

① Find $h_d(n)$. in general $h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}).e^{j\omega n} d\omega.$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1.\, e^{j\omega n} d\omega = \frac{1}{2\pi j n}\left[e^{j\omega n}\right]_{-\pi/2}^{\pi/2}$$

$$= h_d(n) = \frac{\sin\frac{\pi n}{2}}{\pi n} \qquad -\alpha \le n \le \alpha.$$

② find $h(n) = h_d(n)$ for $|n| \le \frac{N-1}{2} = 5$

$$= \frac{\sin \pi n/2}{\pi n} \quad \text{for } |n| \le 5$$
$$= 0 \quad \text{otherwise}$$

③ $h(0) = \lim_{n\to 0} \frac{\sin \pi n/2}{\pi n} = \frac{1}{2}\lim_{n\to 0}\frac{\sin \pi/2 \cdot n}{\pi/2 \cdot n}.$ $\qquad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$

$$\boxed{h(0) = \frac{1}{2}.}$$

for $n=1 \Rightarrow h(1) = h(-1) = \frac{\sin \pi/2}{\pi} = \frac{1}{\pi} = 0.3183$

$n=2 \Rightarrow h(2) = h(-2) = \frac{\sin \pi}{2\pi} = 0$

$n=3 \Rightarrow h(3) = h(-3) = \frac{\sin 3\pi/2}{3\pi} = \frac{1}{3\pi} = -0.106$

$n=4 \Rightarrow h(4) = h(-4) = 0$

$n=5 \Rightarrow h(5) = h(-5) = 0.06366.$