

# UNIT - II

## Quantum Statistics

## Basic Idea About the statistics:-

System: — Microscopic (Small no. of atom) Macroscopic (Large no. of atoms).

In thermodynamics, we have studied the pressure, volume, temperature for a matter in a bulk on macroscopic level. As we know that every matter made with atom-nucleus on the basis of thermodynamics, we can not study the behaviour of atoms, nucleus and their mutual interaction.

In statistics mechanics, we study the behaviour of atoms, nucleus of matter in a bulk. In classical mechanics to know the behaviour of atoms, nucleus for material, we have to find the position and momentum for the particle at different time. Then by using classical mechanics we can find out set of equation to describe the macroscopic behaviour of atom & nucleus but the number of atoms in small material  $10^{23}$  eqn. which is not possible.

So, In static method we find the describe the macroscopic behaviour of atoms and nucleus by finding the most probable behaviour of every atom or nucleus.

In this method we decide that how then different atoms or nucleus for any ensemble are divided in different accessible states and what is their most probable behaviour.

The exact meaning of statistics is the study of behaviour of any dynamical system on the basis of statistics.

The statistics mechanics and used only where there are large no. of atoms or nucleus (particle) are present.

There are some basic definition of statistical form.

Probability :-

¶ The probability of any event, the total no. of outcomes can occur divided by the all possible no. of outcomes.

$$\therefore \text{the probability} = \frac{\text{No. of Outcomes can occur}}{\text{Total no. of outcomes}}$$

Let the no. of the possible outcomes, that event can occur is 'n' and the no. of possible outcomes, that event not occur, then —

The probability of event occurring —

$$P = \frac{n}{n+y}$$

and probability of event, not to occur =

$$P' = \frac{y}{n+y}$$

i.e.  $P+P'=1$ ; the total probability is always ONE.

System :-

A system which is composed by small particles (atoms, nucleus). for these particle we will study the macroscopic properties.

## Ensembles: —

- A system is defined as a collection of particles.
- An ensemble is defined as a collection of a large no. macroscopically identical but essentially independent systems.

## Microscopically Identical: —

Each of the systems constituting an ensemble have same macroscopic condition i.e.— Volume, energy, pressure, total no. of particles etc.

## Independent System: —

Each of the systems constituting an ensemble are mutually non-interacting.

## Types of ensembles: —

### ① Microcanonical Ensemble: —

It is the collection of large no. of essentially independent systems having the same energy, volume and number of particles ( $N$ ). The individual systems of micro canonical ensembles are separated by rigid impermeable well isolated wall such that values of  $E$ ,  $N$  &  $V$  remain constant.

EVN	EVN	EVN
EVN	EVN	EVN
EVN	EVN	EVN

### ② Canonical Ensemble: —

Conducting wall.

TVN	TVN	TVN	TVN
TVN	TVN	TVN	TVN

It is a large number of essentially independent system having the same temp. ( $T$ ), Volume ( $V$ ) and same no. of particles ( $N$ ). The individual system is separated by rigid impermeable but conducting walls. At the separating walls are conducting, heat can be exchange between the system as result all the system active at common temperature or of the thermal equilibrium state.

### (3) Ground Canonical Ensembles:-

Permissible	$TV_u$	$TV_{\bar{u}}$	$TV_{\bar{\bar{u}}}$
$\frac{S}{\dots}$	$TV_u$	$TV_u$	$TV_u$
Conducting wall	$TV_u$	$TV_{\bar{u}}$	$TV_{\bar{\bar{u}}}$

$u \rightarrow$  chemical potential.

It is the collection of a large no. of essentially independent system having the same temperature ( $T$ ), Volume ( $V$ ) & chemical potential ( $u$ ). The individual systems of a grand canonical ensemble are separated by rigid but permeable conducting walls.

As a separating wall is conducting & permeable the exchange of heat energy as well as that of particle between the system takes place in such a way that all system arrives at common temperature ( $T$ ) & chemical potential ( $u$ ).

System:

Static System: — The system in which its composing particle are rest.

Dynamic System: —

The system in which its composing particles are in constant state of motion.

Microstate and Macrostate of System: —

Let us consider a system consisting of large number of gas molecules in the phase space which is divided into cells. Each cell represents a small region of position and momentum. Each molecules may be specified by a point (phase point) lying somewhere inside one of the cell.

The microstate of the system at a particular instant be defined when we specify as to which particle in cell each molecule of the system belongs at that instant.

A macrostate of the system, on the other hand be defined by just giving the number of molecule in each cell such as ' $n_1$ ' molecules are in cell '1'; ' $n_2$ ' are in cell '2' and so on.

There may be large number of microstate corresponding to same macrostate. An example; Let us consider a system of three molecules only  $\alpha$ ,  $\beta$ ,  $\gamma$  which are distributed in two halves of box, the left half 'L' and Right half 'R'

There are four possible distribution —

Left Half (L)

3  
2  
1  
0

Right Half (R)

0  
1  
2  
3

Let, in cell there distribution as  $(3,0), (2,1), (1,2)$   
 $(1,3)$  respectively.

Now, there are possible of microstates —

Distribution	L	R	No. of Microstates ( $\Omega_L$ )	Total microstates $\sum \Omega_L$	Probability of Macrostate $P_i = \frac{\Omega_L}{\sum \Omega_L}$
$(3,0)$	$\alpha\beta\gamma$	0	1	$1 + 3 + 3$ $+1 = 8$	$\frac{1}{8}$
$(2,1)$	$\alpha\beta$ $\alpha\gamma$ $\gamma\beta$	$\gamma$ $\beta$ $\alpha$	3		$\frac{3}{8}$
$(1,2)$	$\alpha$ $\beta$ $\gamma$	$\beta\gamma$ $\alpha\gamma$ $\beta\alpha$	3		$\frac{3}{8}$
$(0,3)$	0	$\alpha\beta\gamma$	1		$\frac{1}{8}$

## Phase Space:-

In classical mechanics, an instantaneous dynamical state of a particle is completely specified by its three position co-ordinates ( $x, y, z$ ) and corresponding momentum component ( $P_x, P_y, P_z$ ). This 'P' is co-ordinates are needed to specify a one particle system completely. Gibbs suggested that any instantaneous state (position & momentum) of that an the particle may be conveniently represented by some point in an imaginary six-dimensional space in which the six co-ordinates ( $x, y, z, P_x, P_y, P_z$ ) are marked along six mutually perpendicular axes in space. The six dimensional space is known as "Phase Space" or u-space.

The point in the phase space representing instantaneous state of the particle is called the Phase Point."

As the time progress, the phase point moves in the phase space. The path of the phase point represents a "trajectory of the particle.

If the system contains a large number of particles, then every particle is completely specified by a point in phase space. Thus the instantaneous state of a system of particles is represented by the corresponding distribution of phase points in the phase space.

## Division of Phase Space into Cells:-

The momentum & position of the particle in dynamic system continuously change in phase space, and there a continuously change in values of momentum & position can be taken. To represent the state of any particle there infinite points are present possible and in definite value element of phase space there are infinite microstates.

In classical mechanics, the phase space divided into the small-small phase cell.

Since these phase cell are 6-in-dimensional, whose sides are —

$$\delta x, \delta y, \delta z, \delta p_x, \delta p_y, \delta p_z.$$

The volume of each of these cells is —

$$\delta V = \delta x \delta y \delta z \delta p_x \delta p_y \delta p_z$$

Classically there is no restriction on the volume of phase cell. It may be reduced to any extent tending to zero also, without the classical result.

However according to uncertainty principle —

$$\delta x \delta p_y \geq \hbar.$$

$$\delta x \delta p_y \geq \hbar$$

$$\delta x \delta p_z \geq \hbar$$

Hence,

$$\delta V \geq (\hbar)^3$$

Thus the point in the phase space actually a cell, whose minimum volume is of the order of  $\hbar^3$ .

So, the number of phase cell (or micro states) in volume element of phase space are —

$$n = dx dy dz dp_x dp_y dp_z$$

$$\hbar^3$$



Volume of element of

Phase space

Now the state of system can be described by specifying the distribution of the particles of the system among the phase cells. We can determine the probabilities of occurrence of all possible distributions that are permitted by the nature of system. Out of these we can select the most probable one.

### Constraints, Accessible and Non-Accessible microstates:-

#### ① Constraints:-

When we think about the distribution of particles in the system. There are some restriction imposed by physical laws. For ex -

① The total number of particles remain constant

$$N = \sum n_x = \text{Constant}$$

Let, there are ' $n_1$ ' molecules in cell '1'; ' $n_2$ ' molecules in cell '2'; —  $n_x$  in cell 'x'.

② The total energy of system remain constant  $E = \text{constant}$

If ' $n_1$ ', ' $n_2$ ', ' $n_3$ ' are distributed for energy ' $E_1$ ', ' $E_2$ ', ' $E_3$ ' —

then total energy -

$$E = n_1 E_1 + n_2 E_2 + n_3 E_3 + \dots + n_x E_x$$

$$= \text{Constant}$$

$$\sum_{x=1}^X$$

By following these two constraints the particles will be distributed in the system or you can say among the phase cells in phase space.

(2)

## Accessible and Non- Accessible Microstates :-

The microstates which are permitted under the constraints imposed upon the system are called Accessible microstates and remaining those are not permitted known as Non- Accessible microstates.

Example:-

Let we have a, b, c three particle distributed in two box X and Y.

In box 'X' the energy of the particle will be 7 ev and in box 'Y' it will be 8 ev,

The constraints <sup>for</sup> the distribution are -

(i)

The no. particle remain same

(ii)

The total energy of the system will be between 21 ev and 23 ev.

If is clear that according constraints - (I)

$(abc, 0), (ab, c), (bc, a), (ac, b), (a, bc),$

$(b, ca), (c, ab)$  and  $(0, bc)$  are accessible microstates.

But  $(0, ab), (ab, 0), (bc, 0)$  -- are non-accessible states.

These micro-states should follow constraint (II).

Now, for (3,0) macro-states the total energy remain confor

$$E(3,0) = 3 \times 7 + 0 \times 8 \\ = 21 \text{ eV}$$

for (2,1)

$$E(2,1) = 2 \times 7 + 1 \times 8 = 22 \text{ eV}$$

for (1,2)

$$E(1,2) = 1 \times 7 + 2 \times 8 = 23 \text{ eV}$$

for (0,3)

$$E(0,3) = 0 \times 7 + 3 \times 8 = 24 \text{ eV}$$

So, according to constraints (ii) (abc,0), (ab,c) (bc,0),  
(ac,b) (b,ac) and (c,ab) are  $\rightarrow$  accessible states but  
(0, abc) is non-accessible state.

finally there are 7 accessible microstates  
are possible for above example.

Principle of equal a Priori Probabilities:

The probability of any macro-states + micro-states is equal to the ratio of the num of accessible microstate and total no. of microstates —  
i.e. the probability of any microstate 'i' is equal to —

$$P_i = \frac{n_i}{\sum n_i} = \frac{\text{No. of accessible micro-state}}{\text{total no. of micro-states}}$$

$$P_i = n_i \times P$$

Hence,  $\frac{1}{\sum n_i} = P$ .  $\rightarrow$  The probability of any microstate.

$$P_i = \alpha n_i$$

i.e. the probability of macro-state & microstates is proportional to no. of accessible microstates there for that.

"for any macro-states, there are equal probability for all micro-states!"

or

The fundamental postulate of statistical physic is that for a system in equilibrium "all accessible microstates corresponding to a given macro-states are equally probable". The postulate is known as the postulate of equal priori probability".

### Fundamental Postulates of Statistical Mechanica:-

To Analyse the properties of any system in micro-states—

(i) Volume , Temperature      (ii) Pressure

(iii) E, there are some postulates are statistical

(i) If the degree of freedom of system is 'f' then we will take '2f' Phase space . where 'f' is for position and another P is for momentum

(ii) Phase space devided in to the phase cell . The volume of phase cell is equal. Each phase cell represents the micro-state of the system . If the degree of freedom of system is 'f' then the volume of phase cell will be ' $\hbar^f$ ', where ' $\hbar$ ' is plank constant.

following the constraints, we calculate the accessible micro-state. The all micro-state all together makes statistical ensemble.

Now we group the accessible microstates according to constraint energy. The no. of accessible microstates in unit volume of phase space between energy  $E$  &  $E+dE$  is represent by  $\Omega(E)$ .

Now by the postulate of equal a priori probability since all accessible microstates corresponding to a given macrostate are equally probable. Then for a system the probability of finding the system in any microstate —

— micro-state between energy  $E$  at and  $E+dE$  —

$$P(E) = \Omega(E)$$

where,  $E$  is proportionality constant.

Now, find out the state of equilibrium for a system because the equilibrium position for a system is a state for which the probability is maximum.

In last the properties of system can be find out, by average (avg) value of particular properties in therm equilibrium.

$$\langle E \rangle = \sum_i E_i P(E_i)$$

## Classification of Statistics:-

or

Every solid, liquid, gas is an assembly of an enormous number of microscopic particles. Likewise, radiation is an assembly of photons. Obviously, the actual motion or interaction of individual particles can not be investigated. However, the macroscopic properties of such assemblies can be explained in terms of the statistical distribution of the individuals among different possible states and their most probable behaviour.

For example:-

From Maxwell distribution of speeds among the molecules of a gas, we can calculate mean speed (which is related to the momentum carried by the molecule) and mean square speed (which is related to the energy of the molecule) and so on. From the average quantities we calculate observable properties like pressure and temperature of the gas.

There are Three Kinds of Identical Particles:-

i

Identical particles of any spin which are so much separated in the assembly that they can be distinguished from one another. The molecules of gas are particles of this kind.

iii

Identical particles of zero or integral ( $S=0, 1, 2, 3 \dots$ ) which can not be distinguished from one another. These are called Bare particles (or bosons) and do not obey Pauli's exclusion principle. Photons and  $\alpha$ -particle are the particles of this kind. -

(iii)

Identical particles of Half integral spin which can not be distinguished from one to another. There are Fermi particles (or fermions) and do obey Pauli's exclusion principle. Electron, proton, neutron are the particles of this kind.

15

- The first kind of the particles are the classical and obey Maxwell-Boltzmann Energy distribution law.

- The second and third kinds of particles are quantum particles and energy distribution laws from them can be derived by method of quantum statistics (Bose-Einstein and Fermi-Dirac statistics) only.

→ The essential difference between these three statistics may be illustrated in the following manner;

Let, there are three states and only two particles of a collection and only two cells to be occupied.

The essential difference between these statistics is in following manner —

Classical statistics —

P	Q	Q	P	PP
		PQ		

Bose-Einstein Statistics —

.	.	..		
		..		

Fermi-Dirac Statistics —

.	.

# MAXWELL BOLTZMAN

(Classical Statistics):-

Let we have a system composed of 'n' - number of identical & distinguishable particles. The particles 1, 2, 3 - - -  $\gamma$  are distributed in 'r' cells. The energy of particles are limited to  $E_1, E_2, \dots, E_r$  (These are represent the energy of these particles or mean energy b/w the different integrals). The no. of particle and energy will be remain constant (Constraint). Let the particles are distributed, with respect to their energy such that the cell '1' contain ' $n_1$ ' particles, with each particles have energy  $E_1$ , in cell '2', there are ' $n_2$ ' particles with energy  $E_2$  - - - i.e.  $(n_1, n_2, n_3, \dots, n_r)$  is the macrostate of the system we want to find out macrostate at the system we equilibrium position for the system. According to statistics, the distribution of equilibrium will be that has maximum probability.

If  $g_1, g_2, g_3, \dots, g_r$  are the degeneracy or no. of microstates corresponds to the energy  $E_1, E_2, \dots, E_r$  in the cell 1, 2, 3 - - -  $\gamma$ .

We can write -

The total no. of involved microstate for the macro-state  $(n_1, n_2, \dots, n_r)$

$$-2 = \ln \frac{g_1^{n_1} g_2^{n_2} g_3^{n_3} \dots g_r^{n_r}}{n_1! n_2! n_3! \dots n_r!}$$

→ I

By eq<sup>n</sup> —  $\delta \log_e W = 0$

or

$$-\sum_{\gamma} S_{n\gamma} \log_e n_{\gamma} + \sum_{\gamma} n_{\gamma} \frac{S_{n\gamma}}{(n_{\gamma})} + \sum_{\gamma} S_{n\gamma}$$

$$\log_e g_{\gamma} = 0$$

$\because 'n'$  is constant in system

$$-\sum_{\gamma} \left( -\log_e g_{\gamma} + \log_e n_{\gamma} + 1 \right) S_{n\gamma} = 0$$

$$-\sum_{\gamma} \left\{ \log_e \left( \frac{n_{\gamma}}{g_{\gamma}} \right) + 1 \right\} S_{n\gamma} = 0$$

$$\therefore \frac{n_{\gamma}}{g_{\gamma}} > 1, \text{ so,}$$

$$\log_e \left( \frac{n_{\gamma}}{g_{\gamma}} \right) + 1 \approx \log \frac{n_{\gamma}}{g_{\gamma}}$$

So, the most probable macro-state:-

$$\boxed{\sum_{\gamma} \log_e \left( \frac{n_{\gamma}}{g_{\gamma}} \right) S_{n\gamma} = 0} \quad (4)$$

There should be two restriction will be imposed —

- ① The total no. of particle should be constant that is —

$$n = \sum n_{\gamma} = \text{Constant or } S_n = 0$$

$$\text{i.e. } \sum_{\gamma} S_{n\gamma} = 0 \quad (5)$$

(ii) The total energy should be constant at constant temperature.

$$E = n_1 E_1 + n_2 E_2 + n_3 E_3 + \dots + n_r E_r = \text{Constant}$$

$$E = \sum_{\gamma} n_{\gamma} E_{\gamma} = \text{Constant}$$

$$\delta E = 0$$

$$\text{i.e. } \sum_{\gamma} E_{\gamma} S_{n\gamma} \quad \text{--- (6)}$$

the equation (4), (5) & (6) they are independent to each other. but they should be imposed together. for this we have to use Langrange's method of undetermined multipliers. By which we will multiply eqn (4) to  $\alpha$  and eqn (6) to  $\beta$ . and then add them, in eqn (4) -

$$\sum_{\gamma} \left[ \log_e \left( \frac{n_{\gamma}}{g_{\gamma}} \right) + \alpha + \beta E_{\gamma} \right] S_{n\gamma} = 0 \quad \text{--- (7)}$$

$\therefore$  The total number of particle are constant so to follow this condition eqn (7) the each term of addition should be ZERO.

$$\log_e \left( \frac{n_{\gamma}}{g_{\gamma}} \right) + \alpha + \beta E_{\gamma} = 0$$

$$\log_e \left( \frac{n_{\gamma}}{g_{\gamma}} \right) = -\alpha + \beta E_{\gamma}$$

$$n_{\gamma} = g_{\gamma} e^{-(\alpha + \beta E_{\gamma})} = g_{\gamma} e^{(\beta E_{\gamma} - \alpha)}$$

$$n_{\gamma} = g_{\gamma} e^{-\alpha} e^{-\beta E_{\gamma}} = A g_{\gamma} e^{-\beta E_{\gamma}}$$

--- (8)

where  $A = e^{-\alpha} = \text{constant}$  because  $\alpha$  is the factor of number of particle, it's not depend on temp. ( $T$ ).

If system is under thermal equilibrium at temp ( $T$ ). then constant, ' $\beta$ ' has value  $\frac{1}{KT}$ , where  $K$  is Boltzmann constant ( $1.38 \times 10^{-23}$  Joule/K), then eqn. (8) —

$$n_r = A g_r e^{-E/KT} \quad (9)$$

The above eqn. is known as Classical or Maxwell Boltzmann distribution function, Here particle are identical but distinguishable i.e. we change the phase point of the particle, then we get new microstate.

In classical statistics the position & momentum of the particle can be find exactly.  
So, the total volume of phase cell is not-fixed,  
it can be small or large.

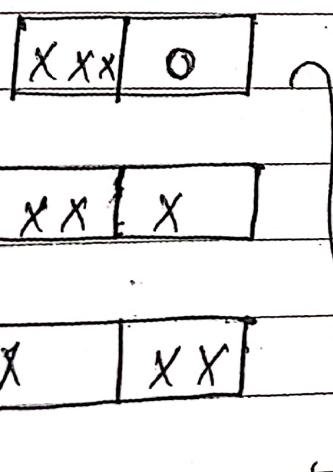
# BOSE-EINSTEIN STATISTICS:

(B.E)

B.E.

Statistic is applied to the distinguishable & identical particles having integral spin. These particle like photons, mesons etc. are called bosons. Any number of particles can occupy single cell i.e. Bosons don't obey Pauli exclusion principle. Let us consider an isolated system volume 'V' of 'n' bosons obeying B.E statistics with volume 'V'. Let the system be divided into N compartments such that  $n_1, n_2, n_3, \dots, n_N$  are the no. particles with mean energy  $E_1, E_2, E_3, \dots, E_N$  in compartment 1, 2, 3, ..., N respectively. Let,  $g_1, g_2, g_3, \dots, g_N$  be no. of elementary cell or degenerate energy level in compartment 1, 2, ..., N respectively.

$g_1$	1	2	3	4
	5	6	7	8
	$i+1$	-	-	$ng_N$



$$n = 3, g = 2$$

Total 4 Arrangement

In terms of 'n' & 'g' we can write -

$$3+2-1=4$$

$$(n_1 + g_1 - 1) = 4$$

Now,

The no. of ways or microstates to distribute  $n$  particle is-

$$\Omega_r = \frac{(n_r + g_r - 1)!}{n_r! (g_r - 1)!}$$

The total no. of microstates to distribute 'n' particles-  
(no. of ways)

$$\Omega = \frac{(n_1 + g_1 - 1)!}{n_1! (g_1 - 1)!} \times \frac{(n_2 + g_2 - 1)!}{n_2! (g_2 - 1)!} \times \dots \times \frac{(n_r + g_r - 1)!}{n_r! (g_r - 1)!}$$

The principle of priori we can write -

$$W \propto \Omega$$

$$W = \frac{(n_1 + g_1 - 1)!}{n_1! (g_1 - 1)!} \times \frac{(n_2 + g_2 - 1)!}{n_2! (g_2 - 1)!} \times \dots \times \frac{(n_r + g_r - 1)!}{n_r! (g_r - 1)!}$$

\* constant  $\rightarrow$  (I)

By taking log -

$$\log_e W = \sum_r [\log_e (n_r + g_r - 1)! - \log n_r! -$$

$$-\log_e [(g_r - 1)!] + \text{constant}$$

$$\therefore (n_r + g_r) \gg 1 \text{ and } (n_r + g_r - 1) \approx n_r + g_r$$

So -

$$\log_e W = \sum_r [\log_e (n_r + g_r)! - \log n_r! -$$

$$-\log_e (g_r - 1)! - \log_e n_r! -$$

$$-\log_e (g_r - 1)! + \text{constant}$$

$n_r$  and  $g_r$  are too large, by Stirling formula

$$[\log n! = n \log n - n]$$

$$\begin{aligned}
 &= \log_e W = \sum_r \left[ \{ (n_r + g_r) \log_e (n_r + g_r) - (n_r + g_r) \} \right. \\
 &\quad \left. - \{ n_r \log_e n_r - n_r \} - \{ (g_r - 1) \log_e (g_r - 1) - (g_r - 1) \} \right] \\
 &\quad + \text{Constant} \\
 &= \sum_r \left[ (n_r + g_r) \log_e (n_r + g_r) - n_r \log_e n_r - (g_r - 1) \right. \\
 &\quad \left. - \log_e (g_r - 1) - 1 \right] + \text{constant} \quad \text{II}
 \end{aligned}$$

for most probable distribution the value of 'W' should be maximum —

By eqn II —

$$\begin{aligned}
 &\sum_r \left[ S_{n_r} \log_e (n_r + g_r) + S_{n_r} \frac{(n_r + g_r)}{(n_r + g_r)} \right. \\
 &\quad \left. - S_{n_r} \log_e n_r - \frac{n_r}{n_r} S_{n_r} \right] = 0
 \end{aligned}$$

$\therefore$  The no. of energy levels at any energy states are constant, then  $S_{g_r} = 0$  —

or

$$\begin{aligned}
 &\sum_r \left[ \{ \log_e (n_r + g_r) - \log_e n_r \} S_{n_r} \right] = 0 \\
 &\quad \boxed{\sum_r \log_e \left( \frac{n_r}{n_r + g_r} \right) S_{n_r} = 0} \quad \text{III}
 \end{aligned}$$

There are two constraints will be imposed -

$$\textcircled{I} \quad n_1 + n_2 + \dots + n_r = \text{constant}$$

$$n = \sum_y n_y = \text{constant}$$

$$S_n = 0$$

So -

$$\boxed{\sum_y S_n y = 0} \quad \textcircled{IV}$$

- \textcircled{II} The energy will be constant at constant temperature.  
i.e. -

$$E = n_1 E_1 + n_2 E_2 + \dots + n_r E_r = \text{constant}$$

$$E = \sum_y n_y E_y = \text{constant}$$

or

$$S_E = 0$$

or

$$\boxed{\sum_y E_y S_n y = 0} \quad \textcircled{V}$$

eqn \textcircled{III}, \textcircled{IV} & \textcircled{V} are independent to each other, but they should be imposed on the system together. for this we use Langrieng's method of undetermined multipliers: we will multiply \textcircled{IV} to ' $\alpha$ ' and \textcircled{V} to ' $\beta$ ', then add this multiplication in \textcircled{III} eqn -

$$\boxed{\left[ \sum_y \left\{ \log_e \left( \frac{n_y}{n_1 + n_2 + \dots + n_r} \right) + \alpha + \beta E_y \right\} S_n y \right] = 0} \quad \textcircled{VI}$$

$\therefore$  The change in number of particles between the energy level will not depend on the particles, so, they eqn = (VI), the terms of addition should be equal to ZERO individually-

$$\log_e \left( \frac{n_r}{n_r + g_r} \right) + \alpha + \beta E_r = 0$$

or

$$\log_e \left( \frac{n_r}{n_r + g_r} \right) = -(\alpha + \beta E_r)$$

$$= \frac{n_r}{n_r + g_r} = e^{-(\alpha + \beta E_r)}$$

$$\text{or } \frac{n_r + g_r}{n_r} = e^{(\alpha + \beta E_r)}$$

$$\frac{g_r}{n_r} = e^{(\alpha + \beta E_r) - 1}$$

$$n_r = \frac{g_r}{e^{(\alpha + \beta E_r) - 1}}$$

If the system is at thermal equilibrium at absolute temperature, then constant  $\alpha$  or  $\beta$  has value,  $\beta = \frac{1}{KT}$ , then —

$$n_r = \frac{g_r}{e^{\alpha-1} \left( \frac{E_r}{KT} - 1 \right)}$$

$$n_r = \frac{g_r}{A e^{\left( \frac{E_r}{KT} - 1 \right)}}$$

$$A e^{\left( \frac{E_r}{KT} - 1 \right)}$$

→ When  $e^\alpha = A$  (constant),  $\alpha$  is the multiple factor of no. particles, which is not depend on T.

# FERMI - DIRAC Statistics (F.D.S):-

A F-D Statistic is applied to identical & distinguishable particles which obey the Pauli Exclusion Principle. These particles have half integral spin like electrons, protons, neutrons etc. Only one particle occupy a single cell. Let us consider an isolated system of 'n' fermions obeying Fermi - Dirac statistics with volume 'V'.

Let the system be obeying F-D stats divided into 'x' compartments such that  $n_1, n_2, n_3, \dots, n_x$  are the no. of particles with mean energy values  $E_1, E_2, \dots, E_x$  in compartments '1', '2', '3' ... 'x' respectively.

Let,  $g_1, g_2, \dots, g_x$  be the no. of elementary cells in compartments 1, 2, 3 ... x.

As such cells can contain either zero or one particle according to cell should be more or equal to no. of particle in each compartment.

			$n_1$		No. of cells
1	2	3			$g_1$
4	5	6			
7	- -	$n_x$			

1	2	3	
1	2	3	
4	5	6	
7	8	9	
$i+1$	- -	$x$	

$$\therefore n_c = \frac{n}{L^x (n-x)}$$

To arrange the ' $n_i$ ' particle in a cell, the thermodynamical probability for  $n_i$  -

$$\log_e W = \sum g_r$$

$$-2_1 = g_1 c_{n_1}$$

Similarly -

$$-2_r = g_r c_{n_r} = \frac{g_r!}{n_r! (g_r - n_r)!}$$

(I) ←

Total no. of ways for  $n_r$  particles -

$$-2 = \frac{g_1!}{n_1! (g_1 - n_1)!} \times \frac{g_2!}{n_2! (g_2 - n_2)!} \times \dots \times \frac{g_r!}{n_r! (g_r - n_r)!}$$

By the principle of a priori probability -

$$W \propto -2$$

$$W = -2 \times \text{constant}$$

$$W = \frac{g_1!}{n_1! (g_1 - n_1)!} \times \frac{g_2!}{n_2! (g_2 - n_2)!} \times \dots \times \frac{g_r!}{n_r! (g_r - n_r)!} \times \text{constant}$$

By taking log both sides -

$$\log_e W = \sum_r [\log g_r! - \log n_r! - \log (g_r - n_r)!] + \text{constant}$$

$\therefore n_r$  and  $g_r$  are too large so by using Stirling formula -

$$\log_e n! = n \log_e n - n \approx 0$$

$$\log_e W = \sum_{\gamma} \left[ \left\{ g_{\gamma} \log_e g_{\gamma} - g_{\gamma} \right\} - \left\{ n_{\gamma} \log_e n_{\gamma} - n_{\gamma} \right\} - \left\{ (g_{\gamma} - n_{\gamma}) \log_e (g_{\gamma} - n_{\gamma}) - (g_{\gamma} - n_{\gamma}) \right\} \right] + \text{constant}$$

(II) ←

for the most probable state, the value of thermodynamic probability must be maximum at thermal equilibrium —  
for this —

$$S \log_{e} k = 0$$

So, by eqn (II) —

$$-\sum \left[ \frac{n_{\gamma}}{n_{\gamma}} + S_{\gamma} \log_e n_{\gamma} + \frac{g_{\gamma} - n_{\gamma}}{g_{\gamma} - n_{\gamma}} (-S_{n_{\gamma}}) + (-S_{n_{\gamma}}) \log_e (g_{\gamma} - n_{\gamma}) \right] = 0$$

## APPLICATIONS of STATISTICS

### APPLICATION OF M.B. STATISTICS:-

Molecular Speed of ideal gas (M.B. Speed wise distribution law) :-

As we see the molecule of gas moves in random manner. The speed of particle (molecule) can be zero to infinite anything. we have to find out the no. of molecule b/w the range  $C \leq v \leq C + dv$  and most probable speed.

Let the vessel is filled with ideal gas at absolute temp. and the no. of molecule in unit volume are ' $N$ '. Let the vessel is filled with ideal gas at absolute temperature and the no. of molecules in unit volume are  $N$ . Let molecule which has velocity  $v$  component  $V_x, V_y \& V_z$  along the axis  $X, Y, Z$  at instant and speed ' $C$ ' then -

$$C^2 = V_x^2 + V_y^2 + V_z^2 \quad \text{--- (I)}$$

According to velocity distribution law of Maxwell and Boltzmann the no. of molecule b/w the velocity component  $V_x \& V_x + dv_x, V_y \& V_y + dv_y$  and  $V_z \& V_z + dv_z$  -

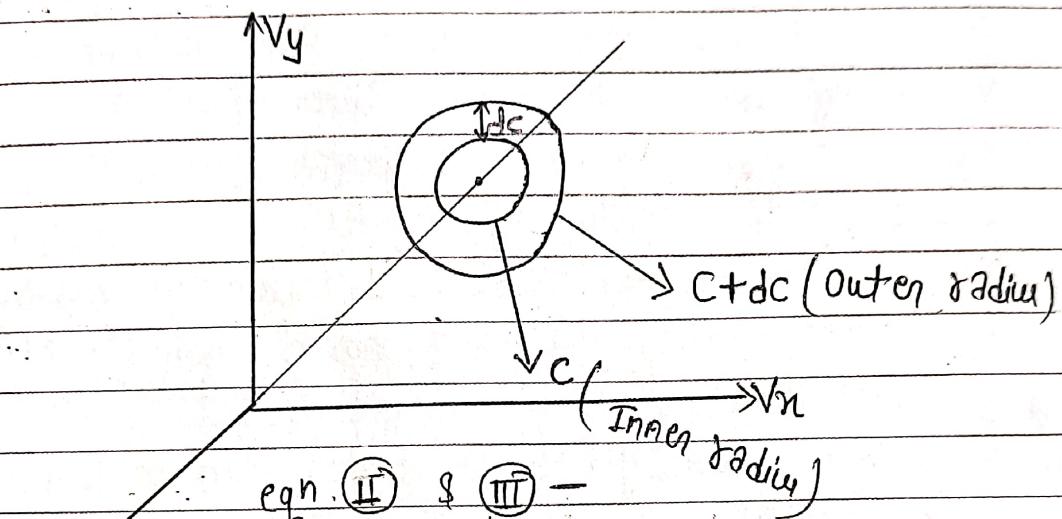
$$dN = N \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{m(V_x^2 + V_y^2 + V_z^2)}{2kT}} \cdot dV_x dV_y dV_z \quad \text{--- (II)}$$

where  $dV_x, dV_y, dV_z$  shows the volume component of velocity space. Since eqn (I)  $V_x^2 + V_y^2 + V_z^2 = C^2$  show the eqn of Sphere the eqn which has radius ' $C$ '.

So, the volume component  $dV_x dV_y dV_z$  will be equal to the volume of Hollow sphere, which has Inner radius 'C' and outer radius  $C+dc$  in velocity. i.e. —

$$dV_x dV_y dV_z = \frac{4}{3} \pi (C+dc)^3 - \frac{4}{3} \pi C^3$$

$$= 4\pi C^2 dc \quad \left. \begin{array}{l} \text{The higher powers of} \\ \text{dc can be neglected} \end{array} \right\}$$



The no. of molecules b/w speed range  $C$  and  $C+dc$  —

$$N(C) dc = N \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mc^2}{2kT}} \cdot (4\pi C^2 dc)$$

or

$$N(C) dc = 4\pi N \left( \frac{m}{2\pi kT} \right)^{3/2} C^2 e^{-\frac{mc^2}{2kT}} dc \quad \boxed{\text{IV}}$$

Known as maxwell speed distribution law.

The no. of molecules b/w speed 0 &  $C$  —

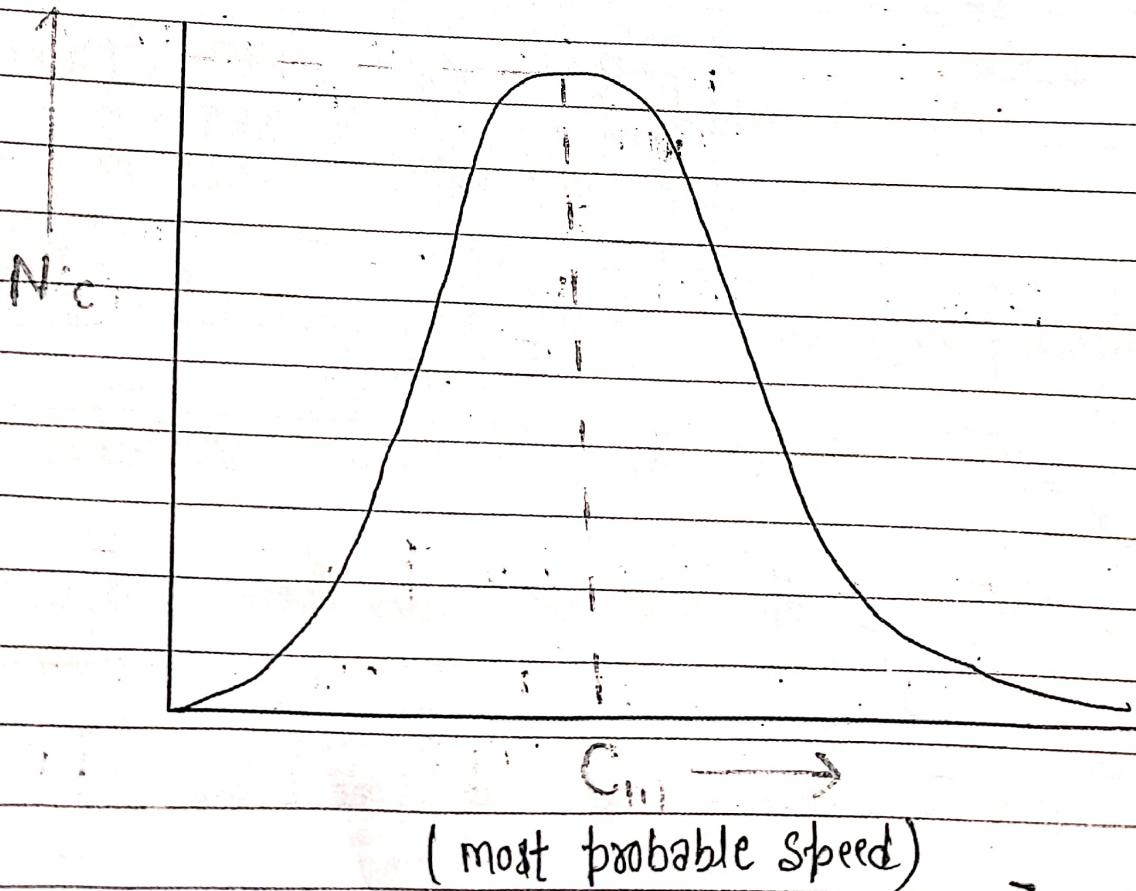
$$N(C) = 4\pi N \left( \frac{m}{2\pi kT} \right)^{3/2} C^2 e^{-\frac{mc^2}{2kT}} \rightarrow \text{Temperature}$$

Boltzmann Constant

## Analysis of Speed Distribution :-

- (i) When  $C = 0$ ,  $N(c) = 0$  i.e. in translational motion, no molecule has zero speed.
- (ii) For small value of  $C$ , the term  $C^2$  is more effective than the term of  $e^{-\frac{mc^2}{2KT}}$  in eqn (iv). So as the value of  $C$  increase, the value of no. of molecule also increases and for particular value of  $C$ . At the value of there is maximum value of no. of molecules. After this the term of  $e^{-\frac{mc^2}{2KT}}$  is more effective than  $C^2$  for large value of  $C$ . As the value of  $C$  increases the no. of molecule decrease exponentially. At  $C = \infty$  the value of no. of molecule  $N(c)$  become zero.

we get a fat peak, when we plot graph b/w  $N(c)$  &  $C$ .



(iii)

### Most probable speed -

speed of molecules can possess. The speed corresponds to peak of curve.

at  $C = C_m$ .

$|N(c)| = \text{maxm.}$

So -

$$\left| \frac{d}{dc} N(c) \right|_{c=c_m} = 0$$

$$\frac{d}{dc} \left[ 4\pi N \left( \frac{m}{2\pi kT} \right)^{3/2} c^2 e^{-\frac{mc^2}{2kT}} \right]_{c=c_m} = 0$$

$$4\pi N \left( \frac{m}{2\pi kT} \right)^{3/2} \left[ 2ce^{-\frac{mc^2}{2kT}} + c^2 e^{-\frac{mc^2}{2kT}} \left( -\frac{2mc}{2kT} \right) \right]_{c=c_m} = 0$$

$$\left( 1 - \frac{mc^2}{2kT} \right)_{c=c_m} = 0$$

or

$$C_m = \sqrt{\frac{2kT}{m}}$$

i.e. - most probable speed of molecules (a) proportional to tem of gas (b) inversely proportional to temperature of gas  
root of mass of  $m$  of the molecules.

The no. of molecules with most probable speed:

$$\text{at } C = C_m = \sqrt{\frac{2kT}{m}}$$

$$|N(c)|_{\max} = 4\pi N \left( \frac{m}{2\pi kT} \right)^{3/2} \times$$

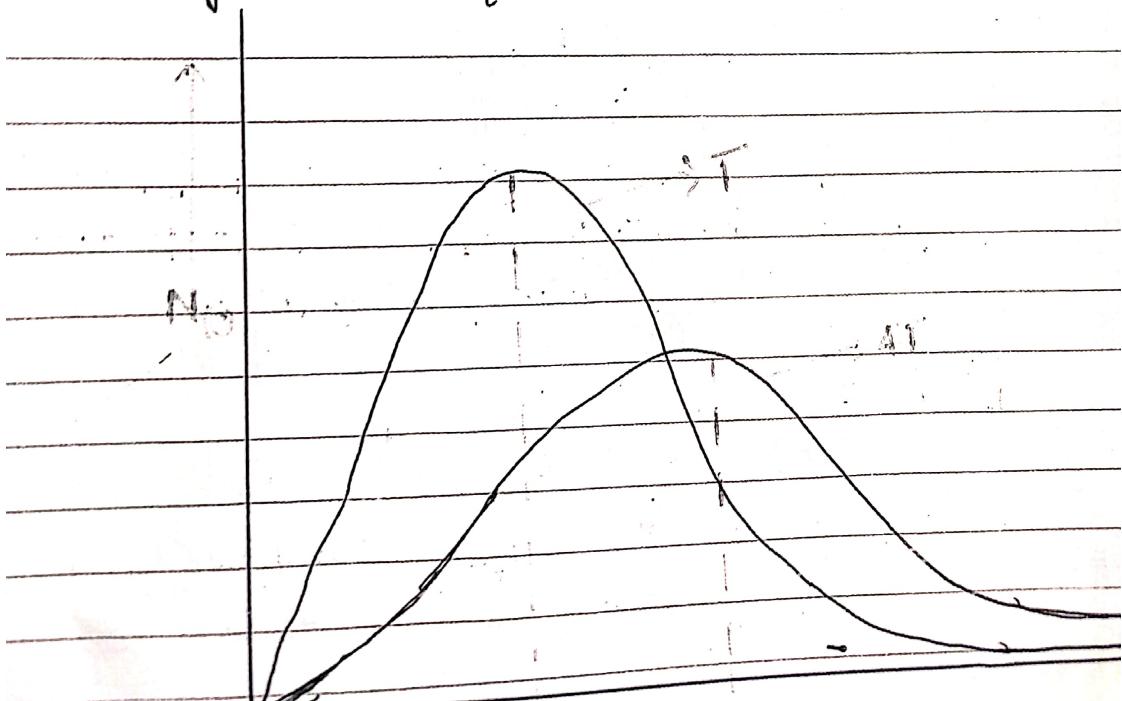
$$\left( \frac{2kT}{m} \right) e^{-\frac{mc^2}{2kT}}$$

$$|N(c)|_{\max} = 4N \left( \frac{m}{2\pi kT} \right)^{1/2} e^{-\frac{c^2}{2kT}}$$

So, the no. of molecules with most-probable speed

- (a) inversely proportional to the root of temperature of gas and (b) proportional to the root mean of molecules.

for gas at the temp. increases the, most probable speed increases, but the value of no. of molecules decreases and  $N(c)$  decreases exponentially with the value of  $C$  but the area of curve, covered by the curve of  $N$  and  $C$ , with (axis) remain equal i.e. curve is going flatten. The area shows the no. of molecules from  $C=0$  to  $C=\infty$ .



(vi) At particular temperature the speed distribution for different gases will be different. As the mass of molecule decreases, the no. of molecules corresponds to most probable speed also inverse and the most probable speed decreases and it decreases fast exponentially. i.e. at equal temperature the most probable speed of molecule of heavy gases is low but corresponding no. of molecules will be large and if most probable speed increases, the no. of molecules decrease.

### vii) Mean Speed:-

The mean speed b/w the range 0 to infinite can be find out by adding the multiplication of speed of molecule b/w zero to infinite and corresponding no. of molecules and by deviding the total number of molecules in this addition.

$$\bar{C} \text{ or } \langle C \rangle = \frac{N_1 C_1 + N_2 C_2 + \dots}{N_1 + N_2 + N_3}$$

$$= \frac{N_1 C_1 + N_2 C_2 + \dots}{N}$$

$$\boxed{\langle C \rangle = \frac{\int_0^{\infty} CN(C) dC}{N}}$$

$$\boxed{\langle C \rangle = \sqrt{\frac{8KT}{m\pi}}}$$

VIII

## Root mean Square Speed :-

By the dividing the addition of multiplications of square of speed and corresponding no. of molecules b/w the range zero to infinite

$$C^{-2} = \langle C^2 \rangle = \frac{N_1 C_1^2 + N_2 C_2^2 + \dots}{N_1 + N_2 + N_3 + \dots}$$

$$= \frac{N_1 C_1^2 + N_2 C_2^2 + \dots}{N}$$

$$\langle C^2 \rangle = \int_0^\infty C^2 N(C) dC$$

$$C_{rms} = \sqrt{\langle C^2 \rangle} = \sqrt{\frac{3kT}{m}}$$

$$\langle C^2 \rangle = \frac{3kT}{m}$$

Finally

$$C_m = \sqrt{\frac{2kT}{m}}$$

$$\langle C \rangle \text{ or } C_{av.} = \sqrt{\frac{8kT}{m\pi}}$$

$$C_{rms} = \sqrt{\frac{3kT}{m}}$$

## classmate Date 35

### Application of M.B Statistics :-

Energy of molecules of Ideal gas ( Maxwell - Boltzmann's Energy wise ) :

We know that the gas is the composition of molecules since, there is no time b/w the molecules of ideal gas.

The molecules in gas are identical but distinguishable. The molecules have all possible value and energy from zero to infinite. we want to calculate the no. of molecules b/w the range  $E$  &  $E + dE$ .

In monoatomic gas, any the translational motion is free. Let one mole of atomic gas, in thermal equilibrium at absolute zero temp. in the box of volume ( $V$ ).

The no. of molecules according to maxwell - Boltzma distribution law -

$$dN = C g(E) e^{-\frac{E}{kT}} dE \quad \text{--- (1)}$$

Since monoatomic gases, molecules only do translational motion to represent their state have to consider a phase space, which every phase cell has volume of  $\hbar^3$ .

So, the no. of microstate b/w  $E$  and  $E + dE$  -

$$g(dE) = \frac{1}{\hbar^3} \iiint dxdydz dP_x dP_y dP_z \quad \text{--- (II)}$$

where,  $\iiint dxdydz$  is the volume of position space.

$\iiint dP_x dP_y dP_z$  is the volume of momentum space.

If molecule has component of momentum in all three direction -

$P_x, P_y, P_z$   
then -

$$P^2 = P_x^2 + P_y^2 + P_z^2$$

$\therefore$  It is the equation of SPHER in phase space which radius is 'P'.

The total volume of phase space b/w the momentum 'P' and ' $P+dp$ ' will be equal to the volume of a hollow SPHER inner radius is 'P' and outer radius is

So the volume of phase space =  $\frac{4}{3}\pi(P+dp)^3$

$$= \frac{4}{3}\pi P^3$$

$$= 4\pi P^2 dp$$

( $\because$  the higher power of  $dp$  can be neglected)

So, we can write -

$$g(dE) = \frac{1}{h^3} V \times 4\pi P^2 dp \quad (3)$$

$$\therefore P = \sqrt{2mE}$$

$$dP = \sqrt{2m} \frac{1}{2E} dE = \sqrt{\frac{m}{2E}} dE$$

By eqn (3)-

$$g(dE) = \frac{1}{h^3} V \times 4\pi (2mE) \int \frac{m}{2E} dE$$

$$g(dE) = \frac{4\sqrt{2}\pi V m^{3/2} E^{1/2}}{h^3} \quad (4)$$

Using eqn ④ - put the value of  $g(dE)$  in eqn ④ -

$$dN = C \times \frac{4\sqrt{2}\pi Vm^{3/2} E^{1/2}}{h^3} \int_0^E E^{1/2} e^{-\frac{E}{KT}} dE$$

But;

$$E = \frac{P^2}{2m}, \quad dE = \frac{P dP}{dm}$$

$$\text{So, } N = \int dN = C \times \frac{4\sqrt{2}\pi Vm^{3/2}}{h^3} \times \frac{1}{\sqrt{2m}^{3/2}} \int_0^\infty P^2 e^{-\frac{P^2}{2mKT}} dP \quad \boxed{⑤}$$

But by the authentic integration -

$$\int_0^\infty n^2 e^{-\alpha n^2} dn = \frac{1}{4\alpha} \sqrt{\frac{\pi}{\alpha}}$$

$$\therefore \int_0^\infty P^2 e^{-\frac{P^2}{2mKT}} dP = \frac{1}{4\left(\frac{1}{2}mKT\right)} \sqrt{\frac{\pi}{\frac{1}{2}mKT}}$$

So,

$$N = C \times \frac{4\pi V}{h^3} \frac{1}{4(2mKT)} \sqrt{\frac{\pi}{\frac{1}{2}mKT}}$$

$$C = \frac{Nh^3}{V(2\pi mKT)^{3/2}} \quad \boxed{⑥}$$

put the value of 'C' in eqn ⑤ -

$$dN = \frac{Nh^3}{V(2\pi mKT)^{3/2}} \times \frac{4\sqrt{2}\pi m^{3/2} E^{1/2}}{h^3} e^{-\frac{E}{KT}} \times dE$$

$$dN = 2\pi N \left( \frac{1}{\pi KT} \right)^{3/2} e^{-\frac{E}{KT}} dE$$

If shows the no. of particles, molecules b/w  $E$  &  $E+dE$ .

The probability of energy of molecule b/w  $E$  &  $E + dE$

$$P(E) dE = \frac{dN}{N} = 2\pi \left( \frac{1}{\pi kT} \right)^{3/2} E^{1/2} e^{-\frac{E}{kT}}$$

This eqn is known as distribution function to energy of maxwell - Boltzman.

There are some conclusion firo this eqn -

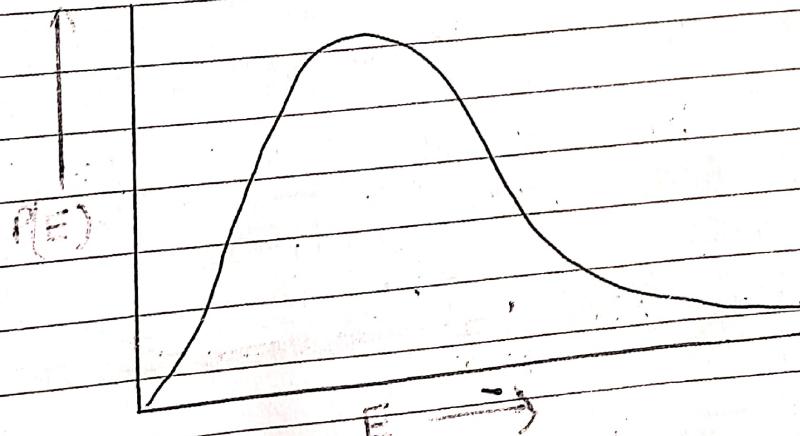
(i) When  $E=0$  then  $P(E)=0$ , i.e. the probability of kinetic energy of molecule to be ZERO is always ZERO, that means the K.E. of molecule can never be ZERO.

(ii) For small values of  $E$  (when  $\frac{E}{kT} \ll 1$ )

the term  $E^{1/2}$  is more effective than  $e^{-\frac{E}{kT}}$  and for some particular value of  $E$ ,  $P(E)$  become maximum.

Let, that value is  $E_m$ . After that when further we increase the value of  $E$  ( $\frac{E}{kT} \gg 1$ ), the

term  $e^{-\frac{E}{kT}}$  become more effective. The result is as the value of ' $E$ ' increases; the probability decreases exponentially. Such that we get a peice when we plot a graph b/w  $P(E)$  &  $E$ .



### III Most Probable Energy:-

$\therefore E = E_m$  then  $P(E) = \text{maximum}$

So,

$$\left[ \frac{d}{dE} P(E) \right]_{E=E_m} = 0$$

$$2\pi \left( \frac{1}{\pi KT} \right)^{3/2} \frac{d}{dE} \left[ \left( E^{1/2} e^{-\frac{E}{KT}} \right) \right]_{E=E_m} = 0$$

$$\left[ \frac{1}{2\sqrt{E}} e^{-\frac{E}{KT}} - \frac{1}{KT} E^{1/2} e^{-\frac{E}{KT}} \right]_{E=E_m} = 0$$

$$\frac{1}{2E_m} - \frac{1}{KT} = 0$$

$$E_m = \frac{KT}{2}$$

The value of most probable energy ' $E_m$ ' is proportional to  $KT$  and does not depend on the nature of the gas or mass of molecules.

### IV Maximum Probability:-

$$\text{at } E = E_m = \frac{KT}{2}$$

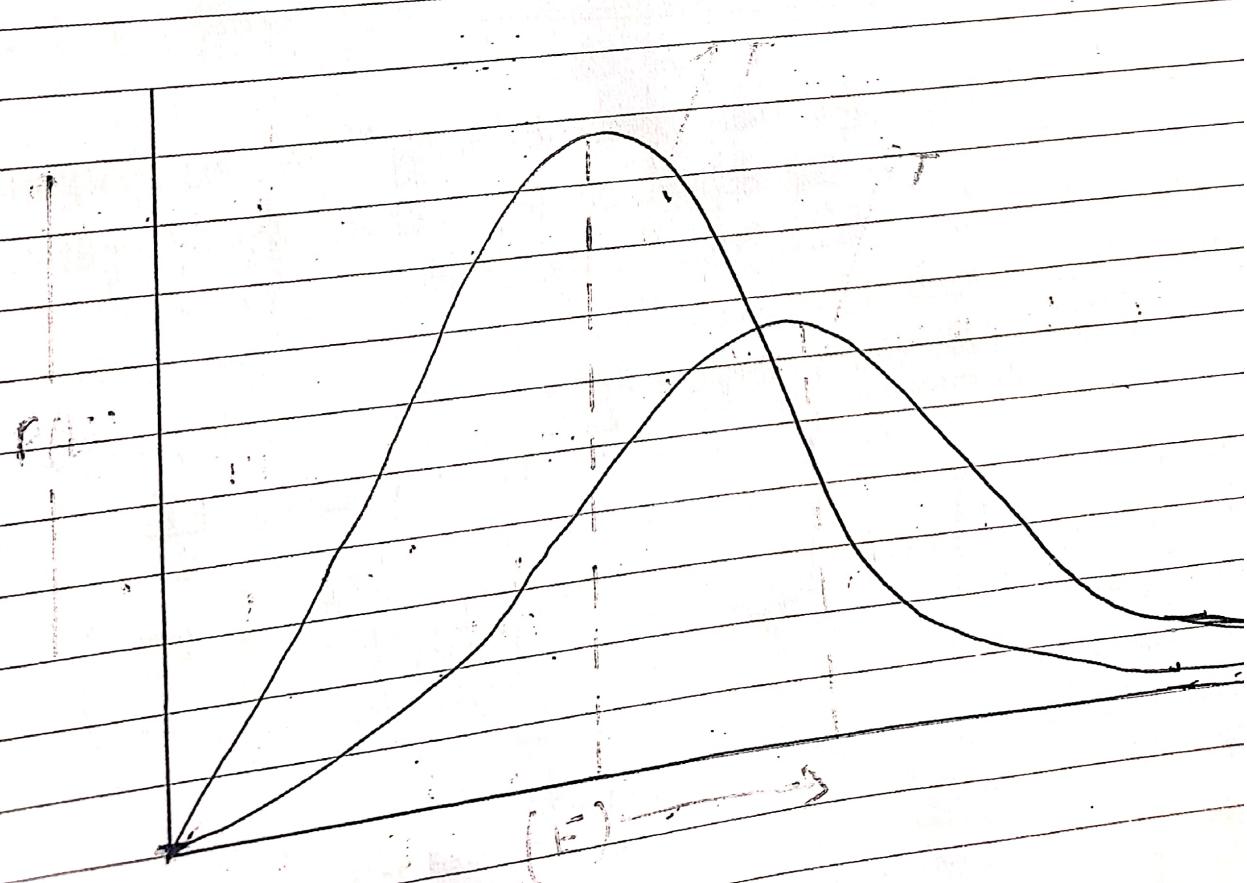
$$P(E)_{\max} = 2\pi \left( \frac{1}{\pi KT} \right)^{3/2} \left( \frac{KT}{2} \right)^{2/2} e^{-\frac{KT}{2KT}}$$

$$= \sqrt{\frac{2}{\pi e}} \times \frac{1}{KT}$$

i.e. the value of maximum probability is inversely proportional to the absolute temperature. It means that as we increase the temp., the value of maximum probability decreases and the value of most probable energy 'Em' increases. But the no. of molecules of most probable energy will decrease.

(V) The energy distribution does not depend on the mass of molecules i.e. the energy distribution at particular temp. for different type of gases will remain same.

(VI) As temp. increases, the most probable energy will be increased but the value of maximum probability decreases that mean in the graph of  $P(E) - (E)$  the peak state to shift near to origin, as temp. on the other hand,  $P(E)$  decreases slowly and exponentially with 'E' but the area covered along the 'E' axis by  $P(E) - E$  curve remain same.



## VII Mean Energy of Molecules :-

$$\langle E \rangle = \int_0^{\infty} EP(E) dE$$

$$= \int_0^{\infty} EX \cdot 2\pi \left(\frac{1}{\pi KT}\right)^{3/2} E^{1/2} e^{-\frac{E}{KT}} dE$$

Let,  $E^{1/2} = n$  or  $E = n^2 \Rightarrow dE = 2ndn$

$$- \frac{n^2}{KT} (2ndn)$$

$$\langle E \rangle = 2\pi \left(\frac{1}{\pi KT}\right)^{3/2} \int_0^{\infty} n^4 e^{-\frac{n^2}{KT}} dn$$

By the authentic integration

$$\int_0^{\infty} n^4 e^{-\frac{n^2}{KT}} dn = \frac{3}{8\alpha^2} \sqrt{\frac{\pi}{\alpha}}$$

$$\text{So } \int_0^{\infty} n^4 e^{-\frac{n^2}{KT}} dn = \frac{3}{8\left(\frac{1}{KT}\right)^2} \sqrt{\frac{\pi}{1/KT}}$$

$$\langle E \rangle = 4\pi \left(\frac{1}{\pi KT}\right)^{3/2} \times \frac{3}{8\left(\frac{1}{KT}\right)^2} \left(\frac{1}{\sqrt{1/KT}}\right)$$

$$\langle E \rangle = \frac{3}{2} KT$$

So, that the mean translational energy of molecule is  $\frac{3}{2} KT$ , which is proportional to temp. The mean energy of one mole gas

$$\langle E \rangle = N \times \frac{3}{2} KT$$

$$= \frac{3}{2} RT$$

where  $\rightarrow N$  is Avogadro no.

# Black Body Radiation:-

A black body radiation is an identical object that absorbs completely all electromagnetic radiation, of whatever wavelength that falls on it. It appears black, when it is a black body not only completely absorb all the radiation falling on it, but also conversely, behave as a perfect radiator and emits all the radiations falling on it, but also conversely, when heated to suitable high temp.

→ The radiation from such body depends only on the temperature to which it is raised and not at all on the nature of the body.

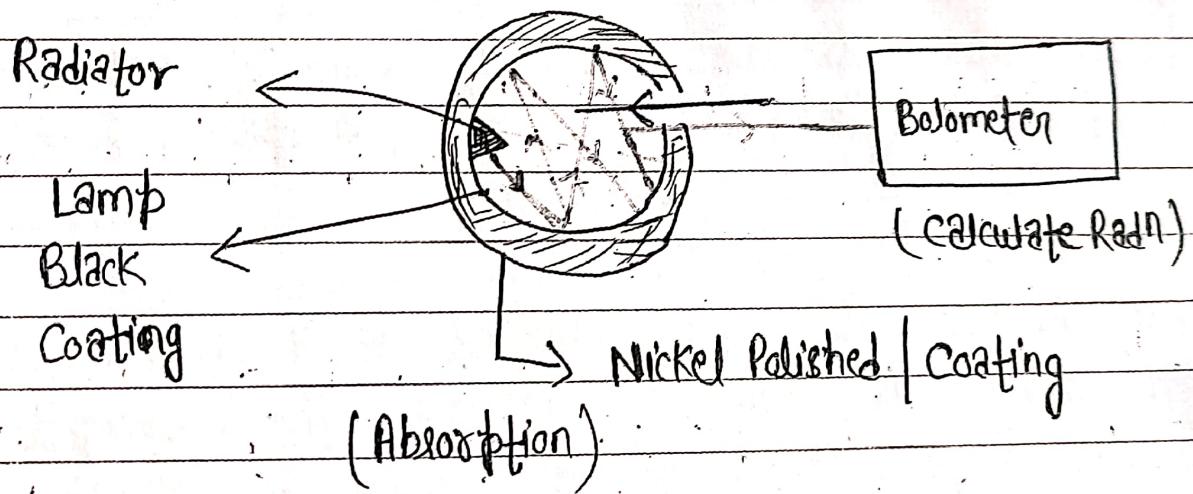
→ Fowny has devised a black body which approximates very closely to the properties of a perfect black body.

→ It consists of a metallic cavity in the form of double-walled hollow copper sphere with lamp black coating from inside and nickel polished from outside. The spherical enclosure has a narrow conical hole (o) space.

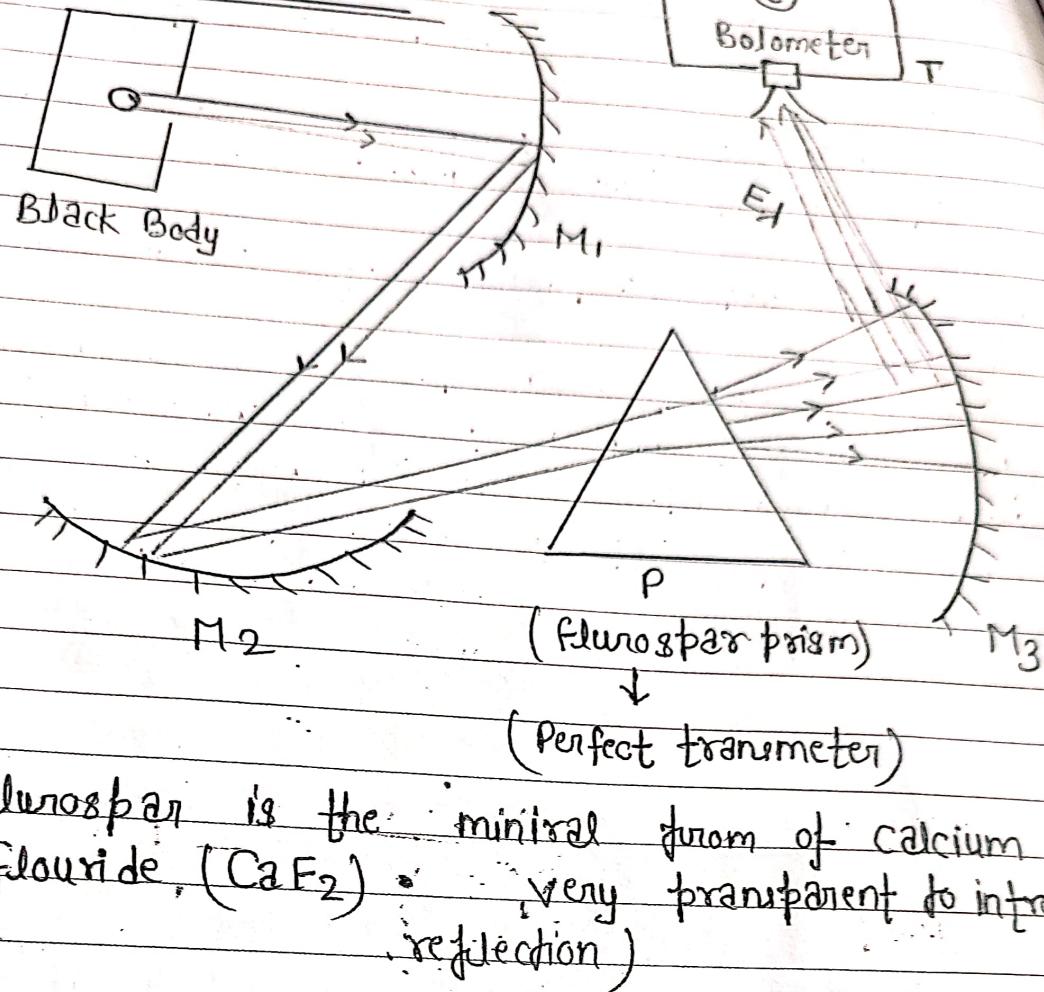
→ The inner sphere below the walls of the spherical enclosure is evaluated to prevent loss of heat energy by conducting and convection.

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Page No. 43
- To avoid any chance of radiation getting cut by reflection falling on the hole(s) and involved conical projection P is made.
  - Any Radiation falling on the hole enters the enclosure multiple reflection at the interior surfaced before is escaped.
  - At each reflection a part of radiation is absorbed.

In this way the spherical enclosure behaves like a perfect absorber which is the main characteristic of a black body. When the spherical enclosure is heated at constant temp. the radiation coming out of the enclosure for any temperature of the wall. called Black body radiation and black body behaves very much like a perfect radiator.



## Black Body Radiation :-



(Fluor spar is the mineral form of calcium fluoride,  $\text{CaF}_2$ ) • very transparent to infrared reflection)

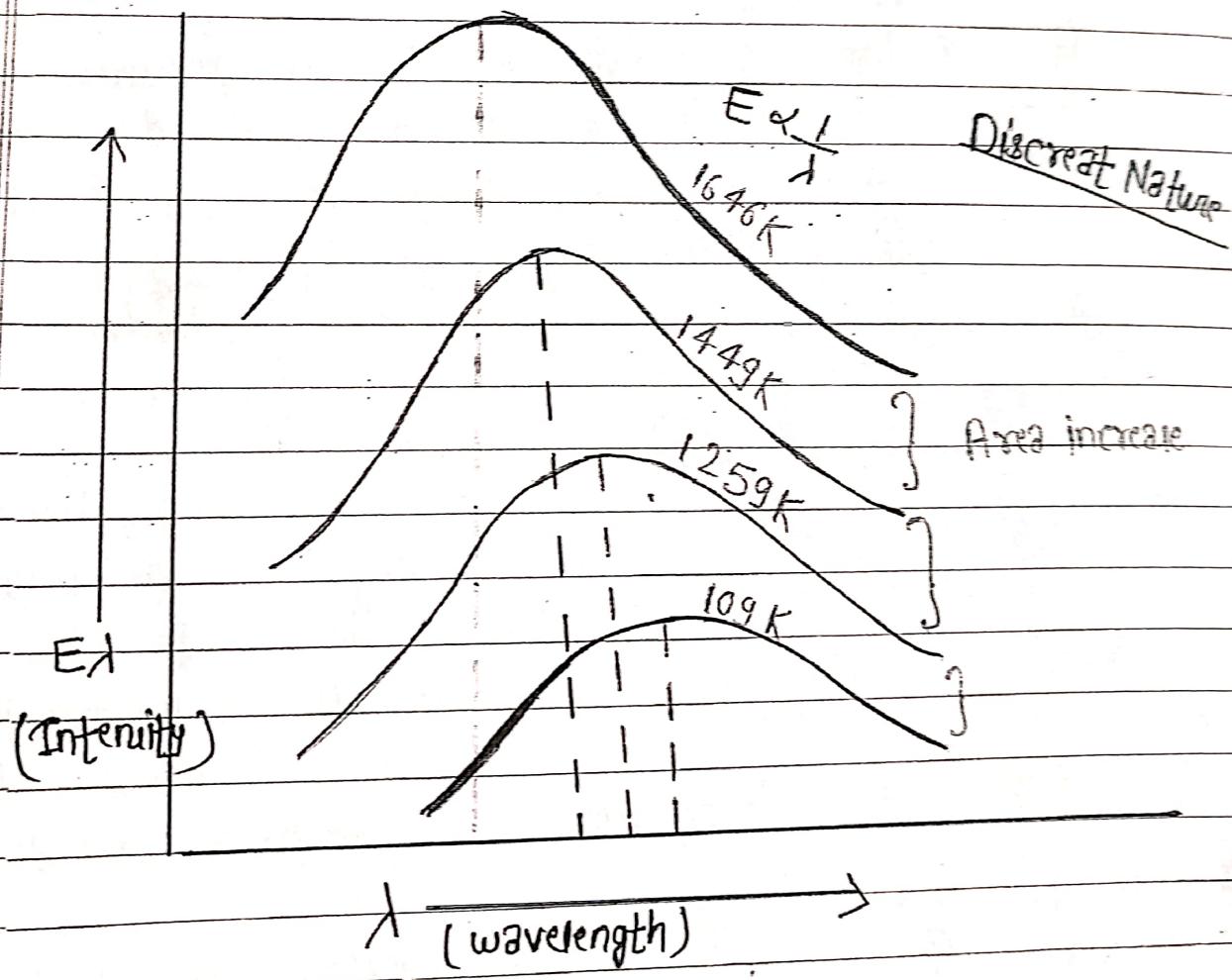
→ The amount of energy emitted is a narrow wavelength range is plotted against the wavelength of each range. The different curves are obtained for the different temperatures of the black body.

① The energy distribution are not uniform. As the temp. increases, the intensity of radiation for each wavelength increases.

② At given temp., the intensity of radiation increases with increase in wavelength and become maximum at a

particular wave-length with further increase in wavelength the intensity of radiation decreases.

- (3) The point of maximum energy shifts towards the shorter wavelength as the temperature increases.
- (4) For a given temp. the total energy of radiation is represented by the area between the curve and the horizontal axis and the area increases with increase of temp. being directly proportional to the fourth power of absolute temperature.



## Stefan Boltzmann Law:-

The total amount of heat  $E$  radiated by perfectly black body per second per unit area is directly proportional to the fourth power of its absolute temp. ( $T$ ).

$$\text{i.e. } E \propto T^4$$

$$E = \sigma T^4$$

where,  $\sigma$  is Stefan constant.

→ Value of  $\sigma = 5.67 \times 10^{-5} \text{ Cm}^{-2} \text{ sec K}^4$ .

If black body at absolute temp. ' $T$ ' is surrounded by another black body at absolute to then the net amount of heat ( $E$ ) lost by the former per second  $\text{cm}^2$  is given by -

$$E = \sigma (T^4 - T_0^4)$$

## Wein's Law of energy Distribution:-

The distribution of energy among the different wavelength at various temp. of the black body showed that the peak in the black body spectrum shifted to progressively shorter wavelength (higher-frequency) as the temperature increased and enunciated this empirical fact in the form of law as -

\* In the black body spectrum the wave-length ( $\lambda_m$ ) for which emissive power is max. is inversely proportional to the absolute temp. ( $T$ ) of the body, that is -

$$\lambda_m \propto \frac{1}{T} \text{ or } \lambda_m \times T = \text{Constant}$$

→ Wein also showed that the energy emitted at the wavelength of maximum emission, is proportional to the fifth power of absolute temp. of the black body.

That is -

$$E_{dm} \propto T^5$$

$$E_{dm} \times T^{-5} = \text{constant}$$

we can combine -

$$\frac{E_\lambda}{\lambda^5} = C_1 f(\Delta T)$$

$$E_\lambda = C_1 \lambda^{-5} f(\Delta T)$$

•  $C_1$  constant and  $f(\Delta T)$  is the undetermined function of product ( $\Delta T$ ).

→ To find  $f(\Delta T)$ , wein assumed that the radiation inside ha enclosure may be supposed to be emitted by a resonator of molecular dimension having Max-well-ellian distribution and the frequency of emitted radiation is proportional to the kinetic energy of the corresponding resonator.

$$E_\lambda = \frac{C_1}{\lambda^5 e^{\frac{C_2}{\lambda T}}}$$

where,  $C_1$  &  $C_2$  are constant.

→ Wein's energy distribution formula agreed with experimental obser curves for shorter wave-length but not for long wa

## Rayleigh-Jeans Law :-

The total amount of energy emitted by black body per unit-volume at an absolute temperature  $T$  in the wavelength range ' $\lambda$ ' and  $d\lambda$  is given as →

$$E_\lambda d\lambda = \frac{8\pi K T}{\lambda^4} d\lambda$$

'K' is boltzmann constant and its value is  $1.381 \times 10^{-31} \text{ J/K}$ .

The energy radiated in a given wavelength range ' $\lambda$ ' and  $\lambda + d\lambda$  increases rapidly as  $\lambda$  decreases and approaches infinite for very short wavelength which however can't be true. This law explains the energy distribution at longer wavelength at all temperatures and fails totally for shorter wavelength.

## Plank Radiation Law:-

Wien's law and Rayleigh-Jeans law failed to explain experimental results over the entire range of black body spectrum, although they were free from error in the application of classical principles.

The failure of classical theory to explain black body radiation spectrum was due to the assumption that the energy changes of radiations take place continuously i.e. it can take any arbitrary close consecutive values.

To explain the energy distribution in the spectrum of black body Max- $\epsilon$ . Plank proposed a new revolutionary hypothesis which laid the foundation of Modern Physics.

According to plank's hypothesis the exchange of energy by radiation with matter do not take continuously but discontinuously and discrete represented by the relation  $E = h\nu$ , where  $\nu$  is the frequency of vibrations and 'h' is plank's constant.

Thus, the can oscillate only with integral every values -

$h\nu, 2h\nu, 3h\nu \dots nh\nu$  or

or in general -

$$E_n = nh\nu$$

$$\{ n = 1, 2, 3, \dots \}$$

and hence emission and absorption of energy by the particles of a radioactive body radiating body interchanging energy with radiation. Oscillation occur discretely not in a continuous sequence.

In relation  $E_n = nh\nu$ , 'n' is called a quantum number and the energies of the ~~radiations~~ are said to be quantized and allowed energy states are quantum states.

On the basis of this assumption blank derived a relation energy density ( $U\nu$ ) of resonator emitting radiation of frequency lying between  $\nu$  and  $\nu + d\nu$  which is given by -

$$\left\{ U\nu d\nu = \frac{8\pi h^3}{c^3} \frac{d\nu}{e^{\frac{h\nu}{kT}} - 1} \right\} \quad \textcircled{I}$$

$$\left\{ U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{\frac{hc}{\lambda kT}} - 1} \right\} \quad \textcircled{II}$$

These eqn are known as plank's radiation law. The energy distribution in the spectrum of a black body at various temperatures was successfully explained by the plank's hypothesis and plank's formula agrees well with experimental results.

①

Wein's Law from Planck's Radiation Law :-

For shorter wavelength  $\lambda T$  will be small and hence —

$$e^{\frac{hc}{\lambda T}} \gg 1$$

So, Planck's formula —

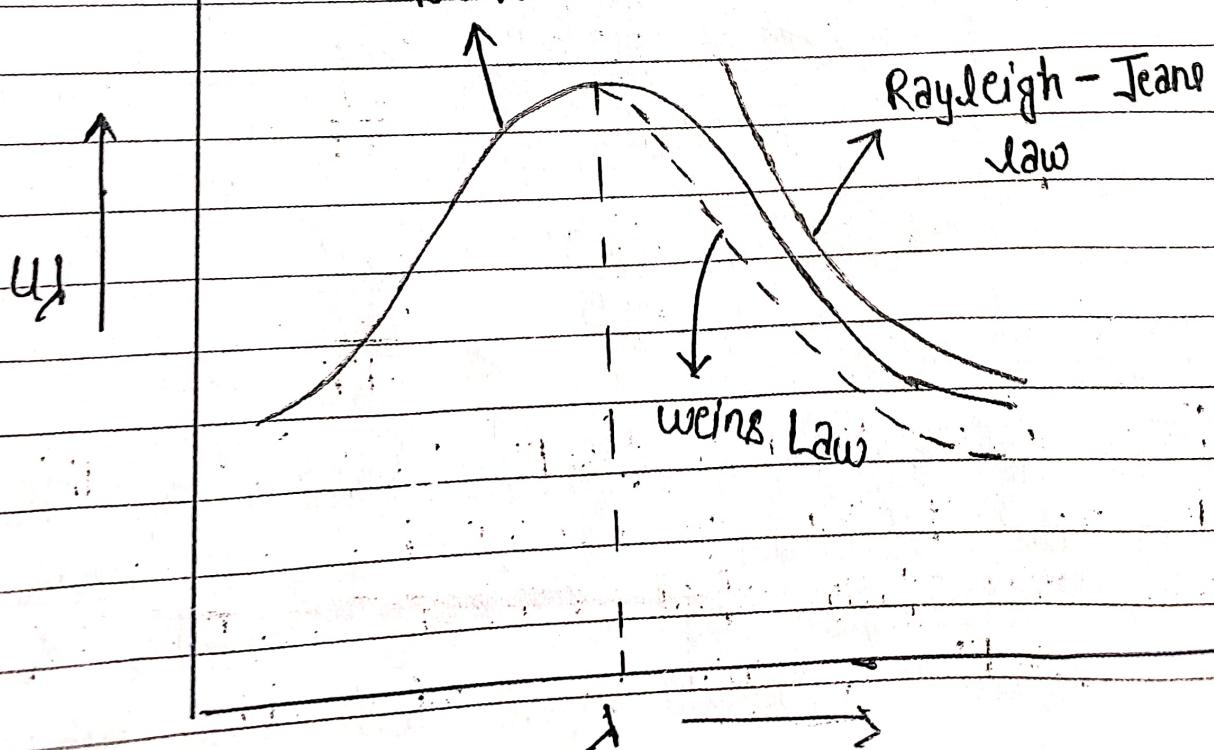
$$u_{\lambda} d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{\frac{hc}{\lambda T}}}$$

$$= 8\pi hc \lambda^{-5} e^{-\frac{hc}{\lambda T}} \cdot d\lambda$$

$$\boxed{u_{\lambda} d\lambda = A e^{-\frac{c}{\lambda T}} d\lambda} \quad \text{III}$$

$A = 8\pi hc$  and  $a = hc/k$  are constant. This is eqn of Wein's Law. This result shows that at shorter wavelength Planck's Law approaches Wien's Law and hence at shorter wavelength Planck's Law and Wien's Law agree.

Planck's Law



② Rayleigh - Jean's Law :-

small and  $e^{\frac{hc}{\lambda T}} = 1 + \frac{hc}{\lambda T} \approx \frac{hc}{\lambda T}$  for longer wave-length  $e^{\frac{hc}{\lambda T}}$  is

So,

$$4\pi dd = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda T}}} d\lambda$$

$$4\pi dd = \frac{8\pi KT}{\lambda^4} d\lambda \quad (IV)$$

Eqn (IV) shows that for longer wave length Planck's law approaches to Rayleigh - Jean's law and thus longer wave length Planck's law and Rayleigh - Jean's law agree.

Thus it is concluded that the Planck's radiation law successfully explained. The entire shape of the curve giving the energy distribution in black body.

## FAILURE OF CLASSICAL THEORY:-

Wein's formula and Rayleigh-Jeans formula of energy distribution derived on the basis of classical theory lead to wrong conclusions for example, in Wein's formula for  $T = \infty$ ,  $E$  is still finite which is in contradiction with experimentally verified Stefan's Law, similarly in Rayleigh-Jeans formula, the energy radiated in a given wave-length ( $\lambda$ ) increases rapidly as ' $\lambda$ ' decreases and approaches infinite for very for very short wavelength.

None of these theoretical formulae could therefore account for the shape of the radiation curve over its entire wavelength range.

To fit the entire experimental facts, Planck in 1901 proposed entirely new Hypothesis to explain the distribution of energy among the various wavelength of the cavity radiation for the entire wave-length range.

He pointed out that the classical theory had led to inconsistent results due to the wrong assumption that the energy changes could take place only discontinuously and discretely instead of continuously.

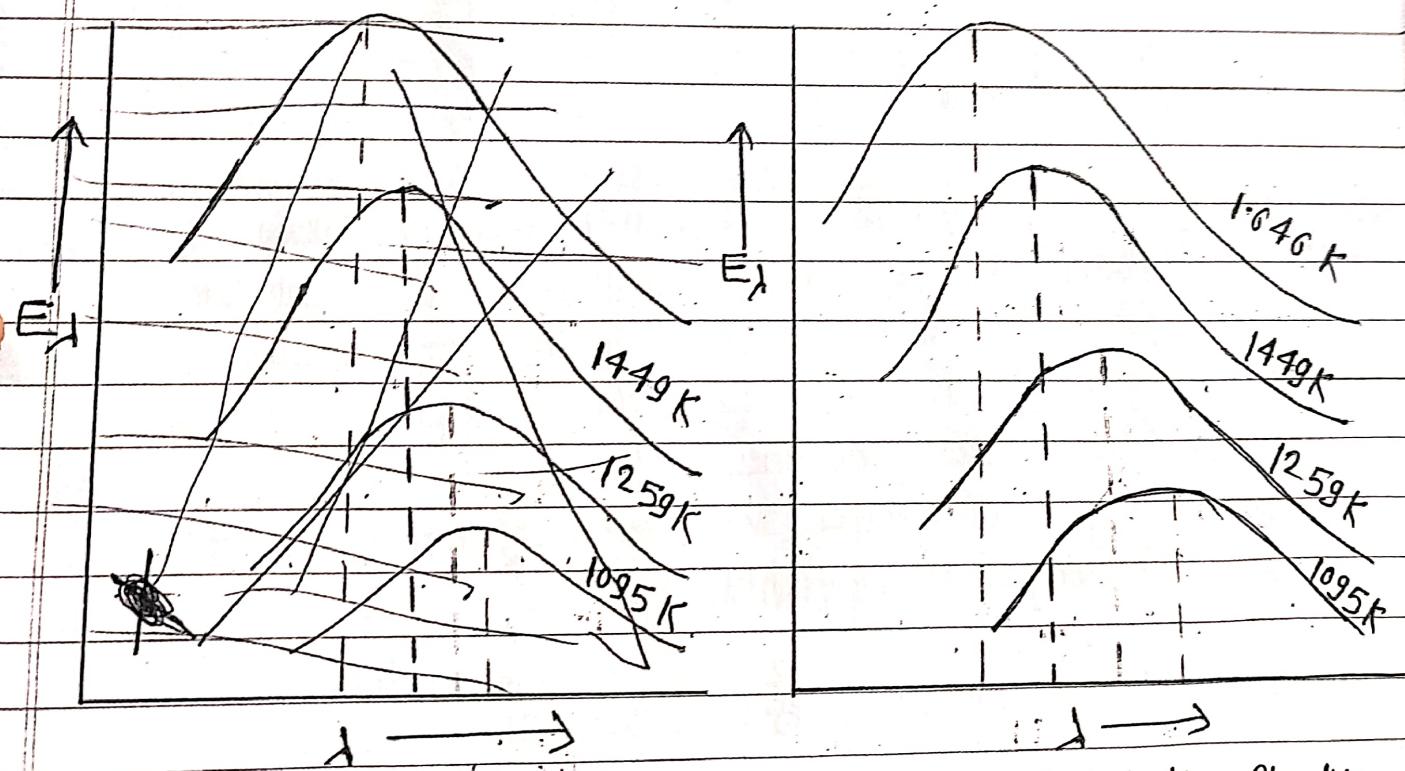
On the basis of such new revolution quantum hypothesis, he devised a radiation formula that was in complete agreement with experimental result for the entire wavelength range at all temperature.

The proposed of quantization is known as theory of quantum.

## Application of Bose-Einstein (B-E) Statistic:

### Derivation of Planck's formula from Bose-Einstein Statistics for Black Body Energy Distribution:

As we studied the energy distribution for black body radiation. In various experiment, it is found that the black body radiation curve is not continuous. At any temp., the energy distribution in the emitted radiation of black-body along the various wave-length is not continuous but it is distributed discretely. At particular temp., the energy is maximum corresponds to maximum wave-length. If we further increase in temp., the maximum value of energy decreasing.



To describe the black-body radiation energy distribution Planck assumed that the energy absorption and emission is in form of quantum energy ( $h\nu$ ). That means the absorption and emission of energy is not continuous but it is discrete in term of bundles of energy.

These bundles are known as Photon. Now, we can let; the black body is a cavity of volume 'V', filled with quantum of energy ' $h\nu$ ' and momentum  $\frac{h\nu}{c}$ . That means a black body is a system of  $c$  the potential and distinguishable particle. This system is just like gas, filled in vessel. but the different i. difference is —

- (1) Photon has zero mass zero.
- (2) There is continuous process of construction & destruction process of Photons, still the energy in the cavity remain constant.

$$\sum n_{\gamma} \neq \text{constant}$$

$$\sum \delta n_{\gamma} \neq 0.$$

- (3) Photon has spin one  $S=1$ , that mean they are Boson. Every photon has two possible spin. i.e. — there are two possible direction of free spin. In the perpendicular direction of transmission of magnetic field lines and photon can possess any one of them. It means, the no. of microstates are double of which we get from applying B.E. statistics in the

Now by using B.E. statistics —

$$n_{\gamma} = \frac{g_{\gamma}}{e^{\frac{E_{\gamma}}{kT}} + \left(\frac{E_{\gamma}}{kT}\right)^{-1}}$$
(I)

But in above eqn, we had considered that the total number of particle remain constant, but here in cavity of black-body the no. of photon are not constant i.e. —

$\sum_{\gamma} S_{\gamma} \neq 0$ , i.e- the value of factor ' $\alpha$ ' will be zero, then —

$$n_{\gamma} = \frac{g_{\gamma}}{e^{\left(\frac{E_{\gamma}}{kT}\right)} - 1} \quad \text{--- (II)}$$

$\therefore$  The energy of Photon is  $E = h\nu$ , then the number of photon b/w the range  $\nu$  and  $\nu + d\nu$ .

$$n_{\nu} d\nu = \frac{g(\nu) d\nu}{e^{\left(\frac{h\nu}{kT}\right)} - 1} \quad \text{--- (III)}$$

where  $g(\nu) d\nu$  are the possible quantum states in the range  $\nu$  and  $(\nu + d\nu)$ . According to quantum statistics the volume of phase cell in phase space corresponds to quantum state in so the possible no. of quantum states b/w the momentum range ' $P$ ' and  $(P + dP)$  in the unit volume of phase space.

$$g'(P) dP = \frac{4\pi P^2 dP}{h^2}$$

$$\text{where unit Volume } dV = 4\pi P^2 dP$$

Since there are two possible spin for photon. then —

$$g'(P) dP = 2 \times \frac{4\pi P^2 dP}{h^3} \quad \text{--- (IV)}$$

But for photon -  $P = \frac{h\nu}{c}$ ,  $dP = \frac{h\nu d\nu}{c}$

So the possible quantum (micro) states in the range  $\nu$  and  $(\nu + d\nu)$  are -

$$g(\nu) d\nu = \frac{2 \times 1\pi \left(\frac{h\nu}{c}\right)^2 \left(\frac{h\nu d\nu}{c}\right)^2}{h^3} \\ = \frac{8\pi \nu^2 d\nu}{h^3}$$

By eqn (III) -

Put the value of  $g(\nu) d\nu$  in eqn (III) we get -

$$h\nu(d\nu) = \frac{8\pi \nu^2}{c^3} \left( e^{\frac{h\nu}{kT}} - 1 \right)$$

(V)

This is the eqn for no. of photon, in the unit volume of phase space, b/w the range  $\nu$  and  $(\nu + d\nu)$ .

So the radiation energy (or spectrum energy density per unit volume) in the range  $\nu$  and  $(\nu + d\nu)$  for black body -

$$U(\nu) d\nu = h(\nu) d\nu + h\nu$$

$$U(\nu) d\nu = \frac{8\pi h\nu^3 d\nu}{c^3 \left( e^{\frac{h\nu}{kT}} - 1 \right)} = \frac{8\pi h\nu^3 d\nu}{c^3 \left( e^{\frac{h\nu}{kT}} - 1 \right)}$$

(VI)

→ This is the plank radiation law in term of frequency.

$$\therefore C = V\lambda, \quad V = \frac{C}{\lambda}, \quad dV = -\frac{C}{\lambda^2} d\lambda$$

Now, we can write (vi) eqn in terms of 'λ' -

$$V_\lambda(d\lambda) = \frac{8\pi h \left(\frac{C}{\lambda}\right)^3 \left(\frac{C}{\lambda^2}\right) d\lambda}{C^3 \left(e^{\frac{hc}{\lambda KT}} - 1\right)}$$

$$V_\lambda(d\lambda) = \frac{8\pi hc d\lambda}{\lambda^5 \left(e^{\frac{hc}{\lambda KT}} - 1\right)}$$

→ VII

→ This eqn (vii) is the plank's formula in terms of wavelength