JACOBIANS

Let $u = u(x, y)$ and $v = v(x, y)$ be two continuous functions of the independent variables x and y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in x and y.

Then the following determinant is called the Jacobian of u and v with respect to x and y, and is denoted by

$$\boxed{J\left(\frac{u, v}{x, y}\right) \text{ or } \frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.}$$

Similarly, if u, v, w are functions of x, y, z then Jacobian of u, v, w with respect to x, y, z is defined as

$$\boxed{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = J\left(\frac{u, v, w}{x, y, z}\right) = \frac{\partial (u, v, w)}{\partial (x, y, z)}.}$$

Chain Rule for Jacobians:

If u, v are functions of x, y and x, y are themselves functions of r, s, then

$$\boxed{\frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (r, s)} = \frac{\partial (u, v)}{\partial (r, s)}}$$ [GGSIPU 2011]

Proof: $$\frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (r, s)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} \end{vmatrix} \quad ...(1)$$

On the other hand, since $u = u(x, y), v = v(x, y)$ and $x = x(r, s), y = y(r, s)$, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s}.$$

Therefore (1) becomes $$\frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (r, s)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} = \frac{\partial (u, v)}{\partial (r, s)}.$$

In the above chain rule, if we replace r, s by u, v this corollary immediately follows.

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=1$$

If $J=\dfrac{\partial(u,v)}{\partial(x,y)}$ then the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ is denoted by J'. and we have $\boxed{JJ'=1.}$

Also, in general, $\boxed{\dfrac{\partial(x_1,x_2,\ldots,x_n)}{\partial(u_1,u_2,\ldots,u_n)}\cdot\dfrac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)}=1.}$

JACOBIAN OF IMPLICIT FUNCTIONS:

Let u and v are implicit functions of the variables x and y, connected by the relations

$$f_1(u,v,x,y)=0 \quad \text{and} \quad f_2(u,v,x,y)=0.$$

Differentiating the above relations partially *w.r.t.* x and y separately, we get

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x}=0,\quad \frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial y}=0$$

and $$\frac{\partial f_2}{\partial x}+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x}=0,\quad \frac{\partial f_2}{\partial y}+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y}=0$$

Now, $$\frac{\partial(f_1,f_2)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix}\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial y}\\ \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}-\frac{\partial f_1}{\partial x} & -\frac{\partial f_1}{\partial y}\\ -\frac{\partial f_2}{\partial x} & -\frac{\partial f_2}{\partial y}\end{vmatrix}=(-1)^2\frac{\partial(f_1,f_2)}{\partial(x,y)}$$

Thus, we have $$\boxed{\frac{\partial(u,v)}{\partial(x,y)}=(-1)^2\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}}$$

Next, in the case when u, v, w are implicit functions of x, y, z, given by the relations

$$f_1(u,v,w,x,y,z)=0$$
$$f_2(u,v,w,x,y,z)=0$$
$$f_3(u,v,w,x,y,z)=0$$

then we have $$\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}}.$$

EXAMPLE 1.34. If $x = r\cos\theta,\ y = r\sin\theta,\ z = z$, find (i) $\dfrac{\partial(r,\theta)}{\partial(x,y)}$ and (ii) $\dfrac{\partial(r,\theta,z)}{\partial(x,y,z)}$

[AKTU 2019]

SOLUTION: (i) Since $r^2 = x^2 + y^2$ we have $\dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta$ and $\dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta$

and as $\theta = \tan^{-1}(y/x)$, we have $\dfrac{\partial\theta}{\partial x} = \dfrac{-y}{x^2+y^2} = \dfrac{-\sin\theta}{r}$ and $\dfrac{\partial\theta}{\partial y} = \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}$.

Therefore, $$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial\theta}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \dfrac{-\sin\theta}{r} \\ \sin\theta & \dfrac{\cos\theta}{r} \end{vmatrix} = \frac{1}{r}[\cos^2\theta + \sin^2\theta] = \frac{1}{r} \quad \textbf{Ans.}$$

(ii) $$\frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial\theta}{\partial x} & \dfrac{\partial z}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial\theta}{\partial y} & \dfrac{\partial z}{\partial y} \\ \dfrac{\partial r}{\partial z} & \dfrac{\partial\theta}{\partial z} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{r} & -\dfrac{y}{x^2+y^2} & 0 \\ \dfrac{y}{r} & \dfrac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\theta & -\dfrac{\sin\theta}{r} & 0 \\ \sin\theta & \dfrac{\cos\theta}{r} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{r} \quad \textbf{Ans.}$$

EXAMPLE 1.35. (*a*) If $x = u(1+v),\ y = v(1+u)$ find $\dfrac{\partial(u,v)}{\partial(x,y)}$ [AKTU 2018]

(*b*) Find the value of the Jacobian $\dfrac{\partial(u,v)}{\partial(r,\theta)}$ where $u = x^2 - y^2,\ v = 2xy$ and $x = r\cos\theta,\ y = r\sin\theta$. [GGSIPU 2009]

SOLUTION: (*a*) $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+u+v \quad \therefore \quad \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{1+u+v}. \quad \textbf{Ans.}$$

(*b*) We know that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = \dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,\theta)}$.

Here $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) = 4r^2$$

and $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Therefore, $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$. **Ans.**

EXAMPLE 1.36. (*a*) For the transformation $x = a(u+v)$, $y = b(u-v)$ and $u = r^2 \cos 2\theta$, $v = r^2 \sin 2\theta$, find $\dfrac{\partial(x, y)}{\partial(r, \theta)}$.

(*b*) If $y_1 = \dfrac{x_2 x_3}{x_1}$, $y_2 = \dfrac{x_3 x_1}{x_2}$, $y_3 = \dfrac{x_1 x_2}{x_3}$, find the value of $\dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$.

[*GGSIPU 2003; AKTU 2016, 2018*]

SOLUTION: (a) From the relations $x = a(u+v)$, $y = b(u-v)$, we have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & a \\ b & -b \end{vmatrix} = -2ab$$

And from the relations $u = r^2 \cos 2\theta$, $v = r^2 \sin 2\theta$, we have

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial u}{\partial r} & \dfrac{\partial u}{\partial \theta} \\ \dfrac{\partial v}{\partial r} & \dfrac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & -2r^2 \sin 2\theta \\ 2r \sin 2\theta & 2r^2 \cos 2\theta \end{vmatrix} = 4r^3$$

By chain rule, $$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(r,\theta)} = -2ab \,.\, 4r^3 = -8\,abr^3.$$ **Ans.**

(*b*)
$$J = \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \dfrac{\partial y_2}{\partial x_3} \\ \dfrac{\partial y_3}{\partial x_1} & \dfrac{\partial y_3}{\partial x_2} & \dfrac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\dfrac{x_2 x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\ \dfrac{x_3}{x_2} & -\dfrac{x_3 x_1}{x_2^2} & \dfrac{x_1}{x_2} \\ \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & -\dfrac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= -\frac{x_2 x_3}{x_1^2}\left(\frac{x_1^2}{x_2 x_3} - \frac{x_1^2}{x_2 x_3}\right) + \frac{x_3}{x_1}\left(\frac{x_1 x_2}{x_2 x_3} + \frac{x_1 x_2 x_3}{x_2 x_3^2}\right) + \frac{x_2}{x_1}\left(\frac{x_3 x_1}{x_2 x_3} + \frac{x_1 x_2 x_3}{x_3 x_2^2}\right)$$

$$= 0 + \frac{x_3}{x_1}\left(\frac{x_1}{x_3} + \frac{x_1}{x_3}\right) + \frac{x_2}{x_1}\left(\frac{x_1}{x_2} + \frac{x_1}{x_2}\right) = 2 + 2 = 4.$$ **Ans.**

EXAMPLE 1.37. (*a*) If $x + y + z = u$, $y + z = uv$, $z = uvw$, find $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$. [*GGSIPU 2017*]

(*b*) If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$ find $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$. [*GGSIPU 2010; 2014*]

SOLUTION: (*a*) The given relations can be written as

$z = uvw$, $y = uv - z = uv(1 - w)$ and $x = u - (y + z) = u(1 - v)$.

Therefore, $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$ (Applying $R_2 \to R_2 + R_3$)

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} = uv\,[u(1-v) + uv] = u^2 v.$$ **Ans.**

(*b*) Let us first calculate the value of

$$J=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix} yz & zx & xy\\ 2x & 2y & 2z\\ 1 & 1 & 1\end{vmatrix}=\begin{vmatrix} yz & z(x-y) & y(x-z)\\ 2x & 2(y-x) & 2(z-x)\\ 1 & 0 & 0\end{vmatrix}$$

$$=2z(x-y)(z-x)-2y(y-x)(x-z)=-2(x-y)(z-x)(y-z)$$

Using the fact that $JJ'=1$, we have $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=J'=\dfrac{1}{J}=\dfrac{-1}{2(x-y)(y-z)(z-x)}$. **Ans.**

EXAMPLE 1.38. If $u^3+v+w=x+y^2+z^2$, $u+v^3+w=x^2+y+z^2$ and $u+v+w^3=x^2+y^2+z$, find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$. [GGSIPU 2006]

SOLUTION: The given relations can be written as implicit functions, as

$$f_1(u,v,w,x,y,z)=u^3+v+w-x-y^2-z^2=0$$
$$f_2(u,v,w,x,y,z)=u+v^3+w-x^2-y-z^2=0$$
$$f_3(u,v,w,x,y,z)=u+v+w^3-x^2-y^2-z=0$$

Now $$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}=\begin{vmatrix}3u^2 & 1 & 1\\ 1 & 3v^2 & 1\\ 1 & 1 & 3w^2\end{vmatrix}=3u^2(9v^2w^2-1)-1(3w^2-1)+1(1-3v^2)$$
$$=27u^2v^2w^2-3u^2-3v^2-3w^2+2$$

and $$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}=\begin{vmatrix}-1 & -2y & -2z\\ -2x & -1 & -2z\\ -2x & -2y & -1\end{vmatrix}=-1(1-4yz)+2y(2x-4xz)-2z(4xy-2x)$$
$$=-1+4(xy+yz+zx)-16xyz$$

Therefore $$\frac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\Big/\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$$
$$=\frac{1-4(xy+yz+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}.$$ **Ans.**

PARTIAL DERIVATIVES FROM IMPLICIT FUNCTIONS:

Suppose u, v are implicit functions of the independent variables x, y connected by the functional relations

$$f_1(u,v,x,y)=0 \quad ...(1)$$

and $$f_2(u,v,x,y)=0. \quad ...(2)$$

To obtain partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ we partially differentiate (1) and (2) w.r.t. x and y separately, and get

$$\frac{\partial f_1}{\partial x}.1+\frac{\partial f_1}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}.\frac{\partial v}{\partial x}=0,\quad \frac{\partial f_2}{\partial x}.1+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x}=0 \quad ...(3)$$

and $$\frac{\partial f_1}{\partial y}.1+\frac{\partial f_1}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial f_1}{\partial v}.\frac{\partial v}{\partial y}=0,\quad \frac{\partial f_2}{\partial y}.1+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y}=0 \quad ...(4)$$

Solving (3) for $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$, we get

$$\frac{\partial u}{\partial x} = -\begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} \Bigg/ \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = -\frac{\dfrac{\partial (f_1, f_2)}{\partial (x, v)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}} \qquad \text{[GGSIPU 2005]}$$

and
$$\frac{\partial v}{\partial x} = -\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial x} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial x} \end{vmatrix} \Bigg/ \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = -\frac{\dfrac{\partial (f_1, f_2)}{\partial (u, x)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}}$$

Similarly, solving (4) for $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$, we get

$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial (f_1, f_2)}{\partial (y, v)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\dfrac{\partial (f_1, f_2)}{\partial (u, y)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}}.$$

EXAMPLE 1.39. (*a*) If $u^2 + xv^2 = x + y,\ v^2 + yu^2 = x - y$, find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial y}$, using Jacobians.

(*b*) Using Jacobians find $\dfrac{\partial u}{\partial x}$ if $u^2 + xy^2 - xy = 0$ and $u^2 + u\,vx + v^2 = 0$.

[GGSIPU 2011]

SOLUTION: (*a*) The functional relations are

$f_1(u, v, x, y) = u^2 + xv^2 - x - y = 0$ and $f_2(u, v, x, y) = v^2 + yu^2 - x + y = 0$

Now
$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial (f_1, f_2)}{\partial (x, v)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}} = -\begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} \Bigg/ \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= -\begin{vmatrix} v^2 - 1 & 2xv \\ -1 & 2v \end{vmatrix} \Bigg/ \begin{vmatrix} 2u & 2xv \\ 2uy & 2v \end{vmatrix} = -\frac{1}{2}\frac{(v^3 - v + xv)}{uv - uv\,xy} = \frac{-1\,(v^2 - 1 + x)}{2u(1 - xy)}.$$

and
$$\frac{\partial v}{\partial y} = \frac{-\dfrac{\partial (f_1, f_2)}{\partial (u, y)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}} = -\begin{vmatrix} 2u & -1 \\ 2yu & u^2-1 \end{vmatrix} \Bigg/ \begin{vmatrix} 2u & 2xv \\ 2uy & 2v \end{vmatrix} = \frac{-1}{2}\frac{\overset{2u(u^2+1+y)}{(u + yu)}}{(uv - xy\,uv)} = \frac{-(1 + y + u^2)}{2v(1 - xy)}. \quad \textbf{Ans.}$$

(b) Here, we have $\quad f_1 \equiv u^2 + xy^2 - xy = 0, \quad f_2 \equiv u^2 + uvx + v^2 = 0$

$$\therefore \quad \frac{\partial (f_1, f_2)}{\partial (x, v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} y^2 - y & 0 \\ uv & ux + 2v \end{vmatrix} = -\frac{u^2}{x}(ux + 2v)$$

and
$$\frac{\partial (f_1, f_2)}{\partial (u, v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 2u + vx & ux + 2v \end{vmatrix} = 2u\,(ux + 2v)$$

$\therefore$ By the method of Jacobians
$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial (f_1, f_2)}{\partial (x, v)}}{\dfrac{\partial (f_1, f_2)}{\partial (u, v)}} = \frac{\dfrac{u^2}{x}(ux + 2v)}{2u\,(ux + 2v)} = \frac{u}{2x}. \quad \textbf{Ans.}$$

EXAMPLE 1.40. If $u = x + y^2,\ v = y + z^2,\ w = z + x^2,$ prove that $\dfrac{\partial x}{\partial u} = -(1 + 8xyz)^{-1}.$

[GGSIPU 2005]

SOLUTION: We are given three functional relations in $u, v, w, x, y, z,$ as

$$f_1 = u - x - y^2 = 0,$$
$$f_2 = v - y - z^2 = 0$$
$$f_3 = w - z - x^2 = 0.$$

then
$$\frac{\partial x}{\partial u} = \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)}}{\dfrac{\partial(f_1, f_2 f_3)}{\partial(x, y, z)}} = \frac{\begin{vmatrix} 1 & -2y & 0 \\ 0 & -1 & -2z \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & -2y & 0 \\ 0 & -1 & -2z \\ -2x & 0 & -1 \end{vmatrix}} = \frac{-1}{1 + 8xyz}.$$

Hence Proved.

JACOBIANS TO DETERMINE FUNCTIONAL DEPENDENCE:

Jacobian is also used in determining whether or not two functions are functionally dependent. Two functions $f(x, y)$ and $\phi(x, y)$ are called functionally dependent if they are functions of each other.

Assume that $f(x, y)$ and $\phi(x, y)$ are functionally dependent then there exists a relation $F(f, \phi) = 0$.

Differentiating it partially *w.r.t.* x and y, we get

$$\frac{\partial F}{\partial f}\frac{\partial f}{\partial x} + \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial f}\frac{\partial f}{\partial y} + \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial y} = 0$$

These are homogeneous equations. Their non-trivial solution would exist only when

$$\begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial f}{\partial y} & \dfrac{\partial \phi}{\partial y} \end{vmatrix} = 0, \quad \text{that is,} \quad \frac{\partial(f, \phi)}{\partial(x, y)} = 0$$

Hence **$f(x, y)$ and $\phi(x, y)$ are functionally dependent if their Jacobian vanishes identically.**

The idea can be easily extended to three functions and, in general, to n functions as well.

EXAMPLE 1.41. (*a*) Examine the functional dependence of $u = \dfrac{x - y}{1 + xy}$ and $v = \tan^{-1}x - \tan^{-1}y.$
If dependent, find the relation.

[GGSIPU 2011; 2012]

(*b*) Prove that the function $u = x + y - z,\ v = x - y + z,\ w = x^2 + y^2 + z^2 - 2yz$ are not functionally independent. If so find the relation between then.

[AKTU 2018]

SOLUTION: (*a*) $\dfrac{\partial u}{\partial x} = \dfrac{1 + y^2}{(1 + xy)^2},\ \dfrac{\partial u}{\partial y} = \dfrac{-(1 + x^2)}{(1 + xy)^2},\ \dfrac{\partial v}{\partial x} = \dfrac{1}{1 + x^2}$ and $\dfrac{\partial v}{\partial y} = \dfrac{-1}{1 + y^2}.$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{1 + y^2}{(1 + xy)^2} & \dfrac{-(1 + x^2)}{(1 + xy)^2} \\ \dfrac{1}{1 + x^2} & \dfrac{-1}{1 + y^2} \end{vmatrix} = \frac{1}{(1 + xy)^2}\begin{vmatrix} 1 + y^2 & -(1 + x^2) \\ \dfrac{1}{1 + x^2} & \dfrac{-1}{1 + y^2} \end{vmatrix} = 0$$

$\Rightarrow$ u and v are functionally dependent. Clearly the relation between them is $u = \tan v.$ **Ans.**

(b) $u = x+y-z,\ v = x-y+z,\ w = x^2+y^2+z^2-2yz$. Hence

$$\therefore \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 2x & 2(y-z) & 2(z-y) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2(y-z) & 0 \end{vmatrix} = 0$$

Hence functionally dependent. The relation between them is $2w = u^2 + v^2$. **Ans.**

EXAMPLE 1.42. Show that the functions $u = x+y+z$, $v = x^3+y^3+z^3-3xyz$, and $w = x^2+y^2+z^2-xy-yz-zx$ are functionally dependent.
Find the relation between them. [GGSIPU 2005]

SOLUTION: The functions u, v, w are functionally dependent if $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \\ 2x-y-z & 2y-z-x & 2z-x-y \end{vmatrix} \quad \text{(Applying } C_2-C_1, C_3-C_1)$$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ x^2-yz & y^2-x^2+z(y-x) & z^2-x^2+y(z-x) \\ 2x-y-z & 3(y-x) & 3(z-x) \end{vmatrix}$$

$$= 6\begin{vmatrix} (y-x)(x+y+z) & (z-x)(x+y+z) \\ y-x & z-x \end{vmatrix} = 6(y-x)(z-x)(x+y+z)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

Therefore u, v, w are functionally dependent.

Next, $\quad v = x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) = uw$

Hence $v = uw$ is the desired relation. **Ans.**

EXAMPLE 1.43. If u, v, w are the roots of the equation $(x-a)^3+(x-b)^3+(x-c)^3 = 0$, find $\dfrac{\partial(u,v,w)}{\partial(a,b,c)}$.
[AKTU 2019]

SOLUTION: Given equation is $(x-a)^3+(x-b)^3+(x-c)^3 = 0$

or $$3x^3 - 3(a+b+c)x^2 + 3(a^2+b^2+c^2)x - (a^3+b^3+c^3) = 0 \quad \text{...(1)}$$

Roots of (1) are u, v, w hence

$$u+v+w = a+b+c,\quad uv+vw+wu = a^2+b^2+c^2,\quad uvw = \frac{1}{3}(a^2+b^3+c^3)$$

or

$$f_1 \equiv u+v+w-(a+b+c) = 0,\quad f_2 \equiv uv+vw+wu-(a^2+b^2+c^2) = 0$$

and

$$f_3 \equiv uvw - \frac{1}{3}(a^3+b^3+c^3) = 0.$$

$$\frac{\partial(u,v,w)}{\partial(a,b,c)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(a,b,c)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = -\frac{\begin{vmatrix} -1 & -1 & -1 \\ -2a & -2b & -2c \\ -a^2 & -b^2 & -c^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & uw & uv \end{vmatrix}} = \frac{2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & uw & uv \end{vmatrix}}$$

$$= \frac{2\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix}} = \frac{2(b-a)(c-a)}{(u-v)(u-w)} \frac{\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & a+b & c+a \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ v+w & 1 & 1 \\ vw & w & v \end{vmatrix}}$$

$$= \frac{(-2)(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}.\quad \textbf{Ans.}$$

EXERCISE 1D

1. (a) For the transformation $x = e^u \cos v,\ y = e^u \sin v$ show that $\dfrac{\partial(u, v)}{\partial(x, y)} = e^{-2u}$. [AKTU 2019]

   (b) If $u = \dfrac{y^2}{2x},\ v = \dfrac{x^2 + y^2}{2x}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$.

2. Find the Jacobian of the following transformations $u = \cos x,\ v = \sin x \cos y,\ w = \sin x \sin y \cos z$.

3. If (x, y, z) and (r, θ, ϕ) are respectively the cartesian and spherical polar coordinates of a point then show that $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$.

4. Verify $JJ' = 1$ when $x = v^2 + w^2,\ y = w^2 + u^2,\ z = u^2 + v^2$. where $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$. [AKTU 2017]

5. If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$ show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{x^2 - y^2}{u^2 + v^2}$.

6. If $u^3 + v^3 + w^3 = x + y + z,\ \ u^2 + v^2 + w^2 = x^3 + y^3 + z^3,\ \ u + v + w = x^2 + y^2 + z^2$, find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$.

7. If $u = xyz,\ v = x^2 + y^2 + z^2,\ w = x + y + z$, find $\dfrac{\partial x}{\partial u}$.

8. If $x + y + z + u + v = a$ and $x^2 + y^2 + z^2 + u^2 + v^2 = b$ where a and b are constants, find $\left(\dfrac{\partial v}{\partial y}\right)_{(x,u)}$ and $\left(\dfrac{\partial y}{\partial v}\right)_{(x,z)}$.

9. If $u = \sin^{-1} x + \sin^{-1} y,\ v = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$ find out if they are functionally dependent and if so, find the relation between them.

10. If $u = x + y + z,\ v = x^2 + y^2 + z^2,\ w = xy + yz + zx$ investigate whether u, v, w are functionally dependent on x, y, z, or not. If yes, find the relation between them. [GGSIPU 2019]

11. Investigate if $u = \dfrac{x - y}{x + z},\ v = \dfrac{x + z}{y + z}$ are functionally dependent and if so, find the functional relation between them.

12. Given that $x = u + v + w,\ y = u^2 + v^2 + w^2,\ z = u^3 + v^3 + w^3$, find $\dfrac{\partial u}{\partial x}$.

13. Show that $u = \dfrac{x + y}{z},\ v = \dfrac{y + z}{x},\ w = \dfrac{y(x + y + z)}{xz}$ are not independent, find the relation between them. [AKTU 2019]

14. If $u^3 + v^3 = x + y,\ u^2 + v^2 = x^3 + y^3$, show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{y^2 - x^2}{2uv(u - v)}$.

[GGSIPU 2013; 2016]

MAXIMA AND MINIMA OF A FUNCTION OF TWO OR MORE VARIABLES:

We shall simply extend the definition of maxima and minima of functions of one variable to functions of two variables. The function $f(x, y)$ has a maximum value for a certain pair of values of x and y, if $f(x, y) > f(x+h, y+k)$ where h and k are small. Similary, a minimum value of $f(x, y)$ is defined when $f(x, y) < f(x+h, y+k)$

Thus, the quantity $f(x, y) - f(x+h, y+k)$ must retain a constant sign for small variations in h and k. When this constant sign is positive, there exists a maximum value of $f(x, y)$ and when it is negative there exists a minimum value of $f(x, y)$. Clearly, if the sign does not remain constant, there will be *neither a* ***maximum*** *nor a* ***minimum*** and then such a point is called ***saddle point***.

To obtain the criterion for maxima and minima, recall the Taylor's expansion of $f(x+h, y+k)$ hence

$$f(x+h, y+k) - f(x, y) = (hf_x + kf_y) + \frac{1}{2\,!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) + \ldots \quad \ldots(1)$$

When h and k are small the first term on the R.H.S. of (1) governs the sign of the R.H.S. and hence, for constant sign on the R.H.S., we must have $\boldsymbol{f_x = p = 0}$ **and** $\boldsymbol{f_y = q = 0}$ ...(2)
as essential condition for the existence of maxima and minima. A point satisfying the conditions in (2), is called a **stationary point or critical point**. Then, in that case, eqn. (1) becomes

$$f(x+h, y+k) - f(x, y) = \frac{1}{2\,!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) + \ldots \quad \ldots(3)$$

Again since h and k are small, the sign of R.H.S. of (3) will be governed by the sign of its first term.

$$h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy} = \frac{1}{f_{xx}}\left[h^2 f_{xx}^2 + 2hk\, f_{xy}\, f_{xx} + k^2\, f_{xx}\, f_{yy}\right]$$

$$= \frac{1}{f_{xx}}\left[(h\, f_{xx} + k\, f_{xy})^2 + k^2(f_{xx}\, f_{yy} - f_{xy}^2)\right] = \frac{1}{r}[(hr+ks)^2 + k^2(rt - s^2)]$$

If $\boldsymbol{rt - s^2 \geq 0}$, the above expression in the square brackets is always positive and there will be maxima if $f_{xx}(=r)$ is negative and minima if $f_{xx}(=r)$ is positive. Whereas, if $\boldsymbol{rt - s^2 < 0}$, the expression in the square brackets can change in sign and therefore there will be neither maxima nor minima. Thus, the criteria for the existence of maxima and minima runs as follows:

(*i*) $f_x = f_y = 0$, or $p = q = 0$ the solution of which gives pairs of values of x and y, called *critical points or stationary points.*

(*ii*) Compute $f_{xx}(=r)$, $f_{yy}(=t)$ and $f_{xy}(=s)$ for each pair (x, y). If $f_{xx}\, f_{yy} - f_{xy}^2 > 0$, *i.e.*, $rt - s^2 > 0$, there will be a maxima if f_{xx} (or f_{yy}) < 0 and minimum if f_{xx} (or f_{yy}) > 0.

(*iii*) If $f_{xx}\, f_{yy} - f_{xy}^2 < 0$, that is, $rt - s^2 < 0$ there will be neither a maxima nor minima and points are called ***saddle points.***

EXAMPLE 1.44. (*a*) Find the maximum and minimum values of the function
$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2.$ [GGSIPU 2009, 2013]

(b) Find the dimensions of a rectangular box (without top) with a given volume, so that the material used is minimum. [GGSIPU 2007, 2017]

SOLUTION: (a) $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$

$\therefore\ p=\dfrac{\partial f}{\partial x}=4x^3-4x+4y$ and $q=\dfrac{\partial f}{\partial y}=4y^3+4x-4y.$

For stationary points $p=0,\ q=0$, that is $x^3-x+y=0$ ...(1) and $y^3-y+x=0$ ...(2)

Adding (1) and (2), gives $x^3+y^3=0 \quad \therefore \quad y=-x$

and substracting (2) from (1), gives $x^3-y^3-2(x-y)=0$ or $(x-y)(x^2+y^2+xy-2)=0$

$\Rightarrow x=y$ or $x^2+y^2+xy=2.$

Thus, the stationary points are $(0,0),\ (\sqrt{2},\sqrt{2}),\ (\sqrt{2},-\sqrt{2}),\ (-\sqrt{2},\sqrt{2}),\ (-\sqrt{2},-\sqrt{2})$.

Now, $r=12x^2-4,\ t=12y^2-4,\ s=4.\ \therefore\ rt-s^2=16(3x^2-1)(3y^2-1)-16.$

At $(0,0)$, $r=-4,\ s=4,\ t=-4$ so $rt-s^2=0.$

and at $(\pm\sqrt{2},\pm\sqrt{2})$, $r=44=t$ and $rt-s^2>0.$

Hence $\operatorname{Max} f(x,y)=8$ and $\operatorname{Min} f(x,y)=-8.$ **Ans.**

(b) Let the length, breadth and height of the box be x, y and z respectively and V be the given volume, then $V=xyz=$ constant and the surface area $S=xy+2yz+2zx.$ ...(1)

Substituting $z=\dfrac{V}{xy}$ in (1), we get $S=xy+2(x+y)\dfrac{V}{xy}=xy+\dfrac{2V}{y}+\dfrac{2V}{x}.$ ...(2)

Therefore, $p=\dfrac{\partial S}{\partial x}=y-\dfrac{2V}{x^2}$ and $q=\dfrac{\partial S}{\partial y}=x-\dfrac{2V}{y^2}.$

At stationary points $\dfrac{\partial S}{\partial x}=0,\ \dfrac{\partial S}{\partial y}=0$, *i.e.*, $x^2y=2V$ and $xy^2=2V.$

$\Rightarrow$ $xy(x-y)=0$ which gives $x=0,\ y=0,$ and when $x=y,\ x^3=y^3=2V.$

$\therefore$ Stationary points are $(0,0)$ and $\left((2V)^{1/3},(2V)^{1/3}\right).$

Next, $r=\dfrac{\partial^2 S}{\partial x^2}=\dfrac{4V}{x^3},\ q=\dfrac{\partial^2 S}{\partial x\,\partial y}=1,\ t=\dfrac{\partial^2 S}{\partial y^2}=\dfrac{4V}{y^3}.$

At $(0,0)$ $\dfrac{\partial^2 S}{\partial x^2}$ and $\dfrac{\partial^2 S}{\partial y^2}$ are not defined, therefore, neither maxima nor minima at $(0,0)$.

At $\left((2V)^{1/3},(2V)^{1/3}\right)$, $r=2,\ t=2$ and $s=1$ hence $rt-s^2=3>0$ and $r=t>0$

Thus S has minimum value at $x=(2V)^{1/3},\ y=(2V)^{1/3}.$

At this point $z=\dfrac{V}{xy}=\dfrac{V}{(2V)^{1/3}.(2V)^{1/3}}=\dfrac{V^{1/3}}{2^{2/3}}=\dfrac{x}{2}.$

Therefore, the box should have a square base and the height should be half the length of the base for material being used to be least, in making the box. **Ans.**

EXAMPLE 1.45. (*a*) Examine the function $f(x,y)=\sin x+\sin y+\sin(x+y)$ for maximum and minimum values. [*GGSIPU 2005*]

(*b*) In a plane triangle *ABC* find the maximum value of $\cos A\cos B\cos C$. [*GGSIPU 2010; 2012*]

SOLUTION: (a) $f(x,y)=\sin x+\sin y+\sin(x+y)$

$\therefore$ $\dfrac{\partial f}{\partial x}=p=\cos x+\cos(x+y)=2\cos\left(x+\dfrac{y}{2}\right)\cos\dfrac{y}{2}$

and $\dfrac{\partial f}{\partial y}=q=\cos y+\cos(x+y)=2\cos\left(y+\dfrac{x}{2}\right)\cos\dfrac{x}{2}.$

For f to be maximum or minimum, $\frac{\partial f}{\partial x}=0=\frac{\partial f}{\partial y}$, that is,

$\cos\left(x+\frac{y}{2}\right)\cos\frac{y}{2}=0$ and $\cos\left(y+\frac{x}{2}\right)\cos\frac{x}{2}=0$ which gives $x=\pi$, $y=\pi$; $x=\pi/3$, $y=\pi/3$

$\therefore$ Critical points are $(\pi/3, \pi/3)$, and (π, π) if x and $y \in [0, \pi]$.

Next, $\frac{\partial^2 f}{\partial x^2}=r=-\sin x-\sin(x+y)$, $\frac{\partial^2 f}{\partial y^2}=t=-\sin y-\sin(x+y)$ and $\frac{\partial^2 f}{\partial x\partial y}=s=-\sin(x+y)$.

At (π, π), $r=s=t=0$, hence it is a saddle point.

At $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$, $r=-\sqrt{3}$, $t=-\sqrt{3}$, $s=-\sqrt{3}/2$. $\therefore$ $rt-s^2=\frac{9}{4}>0$ and $r<0$.

Hence $f(x, y)$ is maximum at $(\pi/3, \pi/3)$ and (π, π) is saddle point. **Ans.**

(*b*) In triangle ABC, $A+B+C=\pi$, hence

$$\cos A\cos B\cos C = \cos A\cos B\cos(\pi-A-B) = -\frac{1}{2}[\cos(A+B)+\cos(A-B)]\cos(A+B)$$

$$= -\frac{1}{4}[1+\cos(2A+2B)+\cos 2A+\cos 2B] = f(A, B), \text{ say.}$$

then $\frac{\partial f}{\partial A}=\frac{1}{2}\sin(2A+2B)+\frac{1}{2}\sin 2A$ and $\frac{\partial f}{\partial B}=\frac{1}{2}\sin(2A+2B)+\frac{1}{2}\sin 2B$.

For f to be maximum $\frac{\partial f}{\partial A}=0$, $\frac{\partial f}{\partial B}=0$, i.e., $\sin(2A+B)\cos B=0$ and $\sin(2B+A)\cos A=0$.

Now if $\cos B=0$ then $B=\frac{\pi}{2}$ which gives $\sin A\cos A=0$ or $A=\frac{\pi}{2}$ which is not possible. Therefore, $2A+B=\pi$ and $2B+A=\pi$ $\Rightarrow$ $A=\pi/3$, $B=\pi/3$ which is critical point.

Next, $\frac{\partial^2 f}{\partial A^2}=\cos(2A+2B)+\cos 2A$, $\frac{\partial^2 f}{\partial A\partial B}=\cos(2A+2B)$, $\frac{\partial^2 f}{\partial B^2}=\cos(2A+2B)+\cos 2B$.

At $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$, $\frac{\partial^2 f}{\partial A^2}=-1$, $\frac{\partial^2 f}{\partial A\partial B}=-1$, $\frac{\partial^2 f}{\partial B^2}=-1$.

Here, $\frac{\partial^2 f}{\partial A^2}\cdot\frac{\partial^2 f}{\partial B^2}-\left(\frac{\partial^2 f}{\partial A\partial B}\right)^2=1-\frac{1}{4}=\frac{3}{4}>0$ and $\frac{\partial^2 f}{\partial A^2}<0$, hence Maxima at $(\pi/3, \pi/3)$.

and Max. value of $(\cos A\cos B\cos C)=\left(\frac{1}{2}\right)^3=\frac{1}{8}$. **Ans.**

EXAMPLE 1.46. (*a*) Examine the function $f(x,y)=x^3-3x^2-4y^2+1$ for maximum and minimum. [*GGSIPU 2005; 2011*]

(*b*) Discuss the maxima and minima of $x^3y^2(1-x-y)$.

SOLUTION: (*a*) $f(x,y)=x^3-3x^2-4y^2+1$. Here $\frac{\partial f}{\partial x}=3x^2-6x$, $\frac{\partial f}{\delta y}=-8y$.

For $f(x,y)$ to have maximum or minimum, $\frac{\partial f}{\partial x}=0=\frac{\partial f}{\delta y}$. Thus, the critical points are $(0, 0)$ and $(2, 0)$.

Next, $r=\frac{\partial^2 f}{\partial x^2}=6x-6$, $s=\frac{\partial^2 f}{\partial x\partial y}=0$, $t=\frac{\partial^2 f}{\partial y^2}=-8$.

At $(0, 0)$, $r<0$, $t<0$ and $rt-s^2=48>0$ hence maxima.

At $(2, 0)$, $r=6$, $s=0$, $t=-8$ hence $rt-s^2<0$ $\therefore$ saddle point at $(2, 0)$. **Ans.**

(b) Given function is $f=x^3y^2(1-x-y)$.

$\therefore$ $p=-3x^2y^2(x+y-1)-x^3y^2=-x^2y^2[4x+3y-3]$ and $q=-x^3y[2x+3y-2]$.

For f to be maximum or minimum $p=q=0$. Thus, the critical points are $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{3}\right)$.

Next, $r=-2xy^2(4x+3y-3)-4x^2y^2$, $t=-x^3(2x+3y-2)-3x^3y$

and $s=-2x^2y(4x+3y-3)-3x^2y^2$.

At $(0, 0)$ $r=s=t=0$ hence neither nor maxima

and at $\left(\frac{1}{2}, \frac{1}{3}\right)$, $r=-\frac{1}{9}$, $s=\frac{-1}{12}$, $t=\frac{-1}{8}$ hence $rt-s^2=\frac{1}{72}-\frac{1}{144}$.

Since $rt-s^2>0$ and r and t are negative, we have a maxima at $\left(\frac{1}{2}, \frac{1}{3}\right)$. **Ans.**

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

At occasions it would be necessary to find the maxima or minima of a function, subject to one or two conditions (or constraints) being satisfied. Suppose **we are to maximise or minimise the function $u=f(x,y)$ subject to the condition $g(x,y)=0$**. We take recourse to Lagrange's method of undetermined multipliers as explained below:

Since $u=f(x,y)$ we have $du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$ hence $\frac{du}{dx}=\frac{\partial f}{\partial x}.1+\frac{\partial f}{\partial y}\frac{dy}{dx}$ ...(1)

and from the relation $g(x,y)=0$ we have $\frac{\partial g}{\partial x}.1+\frac{\partial g}{\partial y}\frac{dy}{dx}=0$ ...(2)

From (2), we have $\frac{dy}{dx}=\frac{-\partial g/\partial x}{\partial g/\partial y}=-\frac{g_x}{g_y}$ on the curve $g(x,y)=0$.

But on $g(x,y)=0$ the function $u=f(x,y)$ becomes a function of x only, as mentioned above, hence the stationary points are given by $\frac{du}{dx}=0$, that is,

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0 \quad \text{or} \quad \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\left(\frac{-\partial g/\partial x}{\partial g/\partial y}\right)=0 \quad \text{or} \quad f_x g_y-f_y g_x=0 \qquad ...(3)$$

which can be solved subject to the condition $g(x,y)=0$. ...(4)

Algebraically, this is equivalent to finding the stationary points of a new function F given by

$\boxed{F(x,y)=f(x,y)+\lambda g(x,y)}$ where λ is a parameter to be determined. ..(5)

The stationary points of F (x,y) are given by the equations $\boxed{f_x+\lambda g_x=0}$ and $\boxed{f_y+\lambda g_y=0}$ which will have a solution if the condition (3) is satisfied. The maxima or minima can then be obtained by examining the condition in the neighbourhood of the stationary points.

Further, suppose it is required to find the stationary values of a function $u=f(x,y,z)$ ...(6)

subject to conditions $\phi(x,y,z)=0$ and $\psi(x,y,z)=0$. ...(7)

In this case we construct the function F as $\boxed{F=f+\lambda_1\phi+\lambda_2\psi}$ ...(8)

where λ_1, λ_2 are called non-zero Lagrange's undetermined multipliers.

Next, form the equations $\dfrac{\partial F}{\partial x}=0, \quad \dfrac{\partial F}{\partial y}=0, \quad \dfrac{\partial F}{\partial z}=0$ ...(9)

Eliminating $x, y, z, \lambda_1, \lambda_2$ between (6), (7) and (9) we shall get an equation in u, the roots of which give the stationary values of $u=f(x,y,z)$.

The method would be more clear through the following examples.

Example 1.47. (a) Use Lagrange's method of multipliers to find the smallest and largest value of $x+2y$ on the circle $x^2+y^2=1$. [GGSIPU 2011]

(b) Find the shortest distance between the line $2xy+y-10=0$ and the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$. [GGSIPU 2015]

Solution: (a) Here $f(x,y)=x+2y, \ \phi(x,y)=x^2+y^2-1=0$.

Let $\quad F=x+2y+\lambda(x^2+y^2-1)$ where λ is undetermined multiplier.

$\therefore \quad \dfrac{\partial F}{\partial x}=1+2\lambda x, \quad \dfrac{\partial F}{\partial y}=2+2\lambda y.$ For F to be Max. and Min. $\dfrac{\partial F}{\partial x}=0, \quad \dfrac{\partial F}{\partial y}=0.$

or $\quad 1+2\lambda x=0 \quad$ and $\quad 2+2\lambda y=0 \quad$ or $\quad x=\dfrac{-1}{2\lambda}, \quad y=\dfrac{-1}{\lambda}.$

Since $\quad x^2+y^2=1 \quad$ we have $\quad \dfrac{1}{4\lambda^2}+\dfrac{1}{\lambda^2}=1, \quad \therefore \quad \lambda=\dfrac{+\sqrt{5}}{2}$

Thus, $f(x,y)=x+2y=-\dfrac{1}{2\lambda}-\dfrac{2}{\lambda}=-\dfrac{5}{2\lambda} \quad \therefore$ Max $(x+2y)=\sqrt{5}$ and Min $(x+2y)=-\sqrt{5}$. **Ans.**

(b) Let D be the distance between the line $2x+y-10=0$ and the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ then

$D=\dfrac{2x+y-10}{\sqrt{4+1}}$ where $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ and D is to be least.

Let $$F=\frac{1}{\sqrt{5}}(2x+y-10)+\lambda\left(\frac{x^2}{4}+\frac{y^2}{9}-1\right) \quad ...(1)$$

then $\quad \dfrac{\partial F}{\partial x}=\dfrac{2}{\sqrt{5}}+\dfrac{2\lambda x}{4} \quad$ and $\quad \dfrac{\partial F}{\partial y}=\dfrac{1}{\sqrt{5}}+\dfrac{2\lambda y}{9}$

For D to be minimum, $\dfrac{\partial F}{\partial x}=0$ and $\dfrac{\partial F}{\partial y}=0.$ hence $x=\dfrac{-4}{\lambda\sqrt{5}}$ and $y=\dfrac{-9}{2\lambda\sqrt{5}}$. ...(2)

Since $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ we have $\dfrac{16}{4(5\lambda^2)}+\dfrac{9}{4(5\lambda^2)}=1$ hence $\lambda=\pm\dfrac{\sqrt{5}}{2}$

Thus $\quad x=\dfrac{4}{\sqrt{5}}\cdot\dfrac{2}{\sqrt{5}}=\dfrac{8}{5}$ and $y=\dfrac{9}{2\sqrt{5}}\left(\dfrac{2}{\sqrt{5}}\right)=\dfrac{9}{5}$

Then least $\quad D=\dfrac{1}{\sqrt{5}}2\left(\dfrac{8}{5}\right)+\dfrac{9}{5}-10=\sqrt{5}$ **Ans.**

MPLE 1.48. (*a*) Determine the maxima of the function $u = (x+1)(y+1)(z+1)$ subject to the condition $a^x b^y c^z = k$ where a, b, c, k are constants.

(*b*) The temperature T at any point (x, y, z) of space is given by $T = 400\, xyz^2$ find the highest temperature at the surface of the sphere $x^2+y^2+z^2=1$.

[*GGSIPU 2005*]

TION: (*a*) From the given function and the given condition, we can write

$$\log u = \log(x+1) + \log(y+1) + \log(z+1) \qquad \text{...(1)}$$

and $$x \log a + y \log b + z \log c = \log k \qquad \text{...(2)}$$

Now, when u is maximum $\log u$ will also be maximum. For finding critical points, we differetiate (1) and (2) and get $$\frac{dx}{x+1} + \frac{dy}{y+1} + \frac{dz}{z+1} = 0 \qquad \text{...(3)}$$

and $$(\log a)\, dx + (\log b)\, dy + (\log c)\, dz = 0. \qquad \text{...(4)}$$

Multiplying (4) by λ and then adding to (3) and equating to zero the co-efficients of dx, dy, dz, we get $\dfrac{1}{x+1} + \lambda \log a = 0$, $\dfrac{1}{y+1} + \lambda \log b = 0$, $\dfrac{1}{z+1} + \lambda \log c = 0$

or $$\left.\begin{aligned} &1 + \lambda x \log a + \lambda \log a = 0 \\ &1 + \lambda y \log b + \lambda \log b = 0 \\ &1 + \lambda z \log c + \lambda \log c = 0 \end{aligned}\right\} \qquad \text{...(5)}$$

Adding these, we get $3 + \lambda(x \log a + y \log b + z \log c) + \lambda(\log a + \log b + \log c) = 0$

or $$3 + \lambda \log k + \lambda \log(abc) = 0 \quad \therefore \quad \lambda = \frac{-3}{\log(k\, abc)}.$$

Substituting this value of λ in (5), we get

$$x+1 = \frac{-1}{\lambda \log a} = \frac{\log(k\, abc)}{3 \log a}, \quad y+1 = \frac{\log(k\, abc)}{3 \log b}, \quad z+1 = \frac{\log(k\, abc)}{3 \log c}$$

Thus, the maxima is given by $$x = \frac{\log\left(k \dfrac{bc}{a^2}\right)}{3 \log a}, \quad y = \frac{\log\left(k \dfrac{ca}{b^2}\right)}{3 \log b}, \quad z = \frac{\log\left(k \dfrac{ab}{c^2}\right)}{3 \log c}. \qquad \textbf{Ans.}$$

(*b*) $T = 400\, xyz^2$ where $x^2+y^2+z^2 = 1$. Let $T' = xyz^2 = xy(1-x^2-y^2)$

hence $$\frac{\partial T'}{\partial x} = y(1-x^2-y^2) + xy(-2x) = -y(3x^2+y^2-1)$$

and $$\frac{\partial T'}{\partial y} = x(1-x^2-y^2) + xy(-2y) = -x(3y^2+x^2-1).$$

For T' to be maximum or minimum $\dfrac{\partial T'}{\partial x} = \dfrac{\partial T'}{\partial y} = 0$, that is, $y(3x^2+y^2-1) = 0$ and $x(3y^2+x^2-1) = 0$.

The above relations when solved for x and y, give $x = 0$, $y = 0$ and $x^2 = \dfrac{1}{4}$, $y^2 = \dfrac{1}{4}$.

Thus the critical points are $(0, 0)$, $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$, $\left(\dfrac{1}{2}, \dfrac{-1}{2}\right)$, $\left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$, $\left(-\dfrac{1}{2}, \dfrac{-1}{2}\right)$.

Next, $$r = \frac{\partial^2 T'}{\partial x^2} = -6xy, \quad t = \frac{\partial^2 T'}{\partial y^2} = -6xy \text{ and } s = \frac{\partial^2 T'}{\partial x \partial y} = -3x^2 - 3y^2 + 1.$$

At $(0, 0)$, $r = t = 0, s = 1$ so $rt < s^2$. Hence $(0, 0)$ is a saddle point.

At $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{-1}{2}, \frac{-1}{2}\right)$, $r = -\frac{3}{4},\ t = \frac{-3}{4},\ s = \frac{-1}{2}$ $\therefore\ rt - s^2 = \frac{5}{16} > 0$ and $r < 0$

and at $\left(\frac{1}{2}, \frac{-1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $r = \frac{3}{4} = t$ and $s = \frac{-1}{2}$ $\therefore\ rt - s^2 = \frac{5}{16} > 0$ and $r > 0$.

$\therefore$ $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{-1}{2}, \frac{-1}{2}\right)$ are the point of maxima and $\left(\frac{1}{2}, \frac{-1}{2}\right)$ and $\left(\frac{-1}{2}, \frac{1}{2}\right)$ are the point of minima.

Thus maximum $T = 400.\frac{1}{2}.\frac{1}{2}.\left(1 - \frac{1}{4} - \frac{1}{4}\right) = 50$. **Ans.**

EXAMPLE 1.49. (*a*) Find the stationary value of $a^3 x^2 + b^3 y^2 + c^3 z^2$ subject to the fulfilment of the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

(*b*) If $\frac{3}{x} + \frac{4}{y} + \frac{5}{z} = 6$, find the values of x, y and z which make $x + y + z$ maximum.

[GGSIPU 2013]

SOLUTION: (*a*) Let $f(x, y, z) = a^3 x^2 + b^3 y^2 + c^3 z^2$ and $\phi(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$.

Now, consider the function $F = f + \lambda\phi$ where λ is an unknown non-zero quantity

or $$F = a^3 x^2 + b^3 y^2 + c^3 z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right).$$

For stationary points, we have $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$.

or $$2a^3 x - \frac{\lambda}{x^2} = 0,\quad 2b^3 y - \frac{\lambda}{y^2} = 0,\quad 2c^3 z - \frac{\lambda}{z^2} = 0 \quad \text{or} \quad 2a^3x^3 = 2b^3y^3 = 2c^3y^3 = \lambda.$$

or $$ax = by = cz = K, \text{ say.} \Rightarrow x = \frac{K}{a},\ y = \frac{K}{b},\ z = \frac{K}{c}.$$

Substituting these in $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, we get $K = a + b + c$.

Hence the stationary point is given by $x = \frac{a+b+c}{a}, y = \frac{a+b+c}{b}, z = \frac{a+b+c}{c}$

$\therefore$ Stationary value of $f = \frac{a^3 (a+b+c)^2}{a^2} + \frac{b^3 (a+b+c)^2}{b^2} + \frac{c^3 (a+b+c)^2}{c^2} = (a+b+c)^3$. **Ans.**

(*b*) Let $f(x, y, z) = x + y + z$. Here $\phi(x, y, z) = \frac{3}{x} + \frac{4}{y} + \frac{5}{z} - 6 = 0$.

Consider $F = f + \lambda\phi = x + y + z + \lambda\left(\frac{3}{x} + \frac{4}{y} + \frac{5}{z} - 6\right)$. $f(x, y, z)$ will be maximum when $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$.

that is, when $1 - \frac{3\lambda}{x^2} = 0,\ 1 - \frac{4\lambda}{y^2} = 0,\ 1 - \frac{5\lambda}{z^2} = 0$ or $x = \sqrt{3\lambda},\ y = \sqrt{4\lambda},\ z = \sqrt{5\lambda}$.

Putting these in $\frac{3}{x} + \frac{4}{y} + \frac{z}{z} = 6$ we get $\frac{3}{\sqrt{3\lambda}} + \frac{4}{\sqrt{4\lambda}} + \frac{5}{\sqrt{5\lambda}} = 6$ or $\sqrt{\lambda} = \frac{1}{6}\left[\sqrt{3} + \sqrt{4} + \sqrt{5}\right]$

$\therefore$ $x + y + z = \sqrt{\lambda}\left(\sqrt{3} + \sqrt{4} + \sqrt{5}\right) = \frac{1}{6}(\sqrt{3} + \sqrt{4} + \sqrt{5})^2$ **Ans.**

EXAMPLE 1.50. (*a*) Find the volume of the greatest rectangular parallelopiped that can be inscribed inside the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$ [*GGSIPU 2012; AKTU 2019*]

(*b*) Find the largest and the smallest distances from the origin to the ellipsoid $\dfrac{x^2}{4}+\dfrac{y^2}{5}+\dfrac{z^2}{25}=1$ and the plane $z=x+y.$

SOLUTION: (*a*) Let the edges of the parallelopiped be $2x$, $2y$ and $2z$ parallel to the co-ordinate axes. The volume V of the parallelopiped, is given by $V=8\,xyz$ which is to be maximised

subject to the condition $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

Consider the function $F=8\,xyz+\lambda\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)$ where λ is an unknown multiplier.

For stationary values $\dfrac{\partial F}{\partial x}=\dfrac{\partial F}{\partial y}=\dfrac{\partial F}{\partial z}=0,$ *i.e.*, $8yz+\dfrac{2\lambda x}{a^2}=0,\ 8zx+\dfrac{2\lambda y}{b^2}=0,\ 8xy+\dfrac{2\lambda z}{c^2}=0.$

Eliminating λ between the above equations, taken in pairs, we get $\dfrac{x^2}{a^2}=\dfrac{y^2}{b^2}=\dfrac{z^2}{c^2}.$

Substituting these in the condition $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1,$ we get $x=+\dfrac{a}{\sqrt{3}},\ y=+\dfrac{b}{\sqrt{3}},\ z=+\dfrac{c}{\sqrt{3}}.$

Note that as x increases, volume also increases. Hence V is maximum when

$x=\dfrac{a}{\sqrt{3}},\ y=\dfrac{b}{\sqrt{3}},\ z=\dfrac{c}{\sqrt{3}}.$ $\therefore$ Maximum volume $=8xyz=\dfrac{8abc}{3\sqrt{3}}.$ **Ans.**

(*b*) Let $u=x^2+y^2+z^2$ be the square of the distance of any point (x,y,z) on the ellipsoid from the origin. We are to maximise and minimise u subject to the constraints

$$\phi_1=\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1=0 \quad ...(1) \qquad \text{and} \qquad \phi_2=x+y-z=0. \quad ...(2)$$

Now consider the function $F=u+\lambda_1\,\phi_1+\lambda_2\,\phi_2=x^2+y^2+z^2+\lambda_1\left(\dfrac{x^2}{4}+\dfrac{y^2}{5}+\dfrac{z^2}{25}-1\right)+\lambda_2(x+y-z).$

where λ_1 and λ_2 are non-zero indetermined multipliers.

For critical points we have $\dfrac{\partial F}{\partial x}=0,\ \dfrac{\partial F}{\partial y}=0,\ \dfrac{\partial F}{\partial z}=0.$

Hence $2x+\dfrac{\lambda_1 x}{2}+\lambda_2=0,\ 2y+\dfrac{2\lambda_1 y}{5}+\lambda_2=0$ and $2z+\dfrac{2\lambda_1 z}{25}-\lambda_2=0.$

Above equations when solved for $x, y, z,$ give $x=\dfrac{-2\lambda_2}{\lambda_1+4},\ y=\dfrac{-5\lambda_2}{2\lambda_1+10},\ z=\dfrac{25\lambda_2}{2\lambda_1+50}.$ ... (3)

Substituting these in the constraint $x+y-z=0$ and on dividing by $\lambda_2\,(\neq 0)$, we get

$$\frac{2}{\lambda_1+4}+\frac{5}{2\lambda_1+10}+\frac{25}{2\lambda_1+50}=0$$

or
$$2\,(2\,\lambda_1+10)\,(2\,\lambda_1+50)+5\,(\lambda_1+4)\,(2\,\lambda_1+50)+25\,(\lambda_1+4)\,(2\,\lambda_1+10)=0$$

or
$(17\lambda_1+75)\,(\lambda_1+10)=0$ which gives $\lambda_1=-10,\ -\dfrac{75}{17}.$

When $\lambda_1 = -10$ we have, from (3) $x = \frac{\lambda_2}{3}$, $y = \frac{\lambda_2}{2}$, $z = \frac{5}{6}\lambda_2$.

These values should satisfy (1), hence $\lambda_2^2\left(\frac{1}{36}+\frac{1}{20}+\frac{1}{36}\right)=1$ or $\lambda_2 = \pm 6\sqrt{\frac{5}{19}}$.

This gives two critical points $\left(2\sqrt{\frac{5}{19}}, 3\sqrt{\frac{5}{19}}, 5\sqrt{\frac{5}{19}}\right), \left(-2\sqrt{\frac{5}{19}}, -3\sqrt{\frac{5}{19}}, -5\sqrt{\frac{5}{19}}\right)$.

$\therefore$ The stationary value of $x^2+y^2+z^2$ corresponding to these critical points, is $= 10$.

Next, when $\lambda_1 = -\frac{75}{17}$ we have from (3). $x = \frac{34}{7}\lambda_2$, $y = -\frac{17}{4}\lambda_2$, $z = \frac{17}{28}\lambda_2$.

Substituting these in (1), gives $\lambda_2 = \pm\frac{140}{17\sqrt{646}}$ which, in turn, gives the critical points as

$$\left(\frac{40}{\sqrt{646}}, \frac{-35}{\sqrt{646}}, \frac{5}{\sqrt{646}}\right) \text{ and } \left(\frac{-40}{\sqrt{646}}, \frac{35}{\sqrt{646}}, \frac{-5}{\sqrt{646}}\right)$$

The stationary value of $x^2+y^2+z^2$ corresponding to these critical points, is $\frac{75}{17}$.

Thus, the required maximum value is 10 and the minimum value is $\frac{75}{17}$. **Ans.**

EXAMPLE 1.51. (*a*) If $u = ax^2+by^2+cz^2$ where $x^2+y^2+z^2=1$ and $lx+my+nz=0$ prove that the stationary value of u satisfy the equation $\frac{l^2}{a-u}+\frac{m^2}{b-u}+\frac{x^2}{c-u}=0$.

[GGSIPU 2014]

(*b*) Show that the stationary values of $u = \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$ where $lx+my+nz=0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, are roots of equation $\frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0$.

SOLUTION: (a) We are to maximize or minimize $u = ax^2+by^2+cz^2$ subject to the conditions $\phi_1 = x^2+y^2+z^2-1=0$ and $\phi_2 = lx+my+nz=0$.

Let $F = u+\lambda\phi_1+\mu\phi_2$. For u to be maximum and minimum, we have

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}+\lambda\frac{\partial \phi_1}{\partial x}+\mu\frac{\partial \phi_2}{\partial x} = 2ax+\lambda(2x)+\mu(l)=0, \quad ...(1)$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y}+\lambda\frac{\partial \phi_1}{\partial y}+\mu\frac{\partial \phi_2}{\partial y} = 2by+\lambda(2y)+\mu(m)=0, \quad ...(2)$$

$$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z}+\lambda\frac{\partial \phi_1}{\partial z}+\mu\frac{\partial \phi_2}{\partial z} = 2cz+\lambda(2z)+\mu(n)=0 \quad ...(3)$$

Multiplying (1) by x, (2) by y, (3) by z and adding, we get $2u+2\lambda(1)+\mu(0)=0$ hence $\lambda=-u$.

Putting $\lambda=-\mu$ in (1), (2), (3), we get $x = \frac{-\mu l}{2a-2u}$, $y = \frac{-\mu m}{2b-2u}$, $z = \frac{-\mu n}{2c-2u}$

Substituting these in $lx+my+nz=0$, gives

$$-\frac{\mu l^2}{2a-2u}-\frac{\mu m^2}{2b-2u}-\frac{\mu n^2}{2c-2u}=0 \quad \text{or} \quad \frac{l^2}{a-u}+\frac{m^2}{b-u}+\frac{n^2}{c-u}=0. \quad \textbf{Hence Proved.}$$

(b) Let us write $u = f(x, y, z) = \dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}$,

subject to $\phi(x, y, z) = lx + my + nz = 0$ ... (1) and $\psi(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1 = 0$. ... (2)

Now construct the function $F = f + \lambda_1 \phi + \lambda_2 \psi$, where λ_1, λ_2 are non-zero Lagrangian multipliers. ... (3)

For stationary values of u, we have $\dfrac{\partial F}{\partial x} = 0$, $\dfrac{\partial F}{\partial y} = 0$, $\dfrac{\partial F}{\partial z} = 0$, that is,

$$\frac{2x}{a^4} + \lambda_1 l + \lambda_2 \frac{2x}{a^2} = 0 \quad ...(4) \qquad \frac{2y}{b^4} + \lambda_1 m + \lambda_2 \frac{2y}{b^2} = 0 \quad ...(5) \qquad \frac{2z}{c^4} + \lambda_1 n + \lambda_2 \frac{2z}{c^2} = 0. \quad ...(6)$$

Multiplying (4), (5), (6) by x, y, z respectively and adding, gives

$$2u + 0 \cdot \lambda_1 + 2\lambda_2 = 0 \quad \text{hence} \quad \lambda_2 = -u$$

Substituting this value of λ_2 in (4), (5) and (6), we get

$$x = \frac{-a^4 \lambda_1 l}{2(1 - a^2 u)} \quad \text{and} \quad y = \frac{-b^4 \lambda_1 m}{2(1 - b^2 u)}, \quad z = \frac{-c^4 \lambda_1 n}{2(1 - c^2 u)}.$$

These values must satisfy (1), hence $-\dfrac{\lambda_1 a^4 l^2}{2(1 - a^2 u)} - \dfrac{\lambda_1 b^4 m^2}{2(1 - b^2 u)} - \dfrac{\lambda_1 c^4 n^2}{2(1 - c^2 u)} = 0$

Since $\lambda_1 \neq 0$ we have $\dfrac{a^4 l^2}{1 - a^2 u} + \dfrac{b^4 m^2}{1 - b^2 u} + \dfrac{c^4 n^2}{1 - c^2 u} = 0$.

which gives the stationary values of u. **Hence Proved.**

EXAMPLE 1.52. (*a*) Find a point on the plane $ax + by + cz = p$ at which the function $f = x^2 + y^2 + z^2$ has a minimum value and find the minimum f. [*GGSIPU 2012, 2019*]

(*b*) Find the max. and min. distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$. [*GGSIPU 2016*]

SOLUTION: **(a)** We are to minimise $f = x^2 + y^2 + z^2$ such that $ax + by + cz = p$.

We can write $f = x^2 + y^2 + \dfrac{1}{c^2}(p - ax - by)^2$

$$= \left(1 + \frac{a^2}{c^2}\right)x^2 + \left(1 + \frac{b^2}{c^2}\right)y^2 - \frac{2ap}{c^2}x - \frac{2bp}{c^2}y + \frac{2abxy}{c^2} + \frac{p^2}{c^2}$$

Hence $\dfrac{\partial f}{\partial x} = 2\left(1 + \dfrac{a^2}{c^2}\right)x - \dfrac{2ap}{c^2} + \dfrac{2aby}{c^2}$ and $\dfrac{\partial f}{\partial y} = 2\left(1 + \dfrac{b^2}{c^2}\right)y - \dfrac{2bp}{c^2} + \dfrac{2abx}{c^2}$.

For maxima and minima of f, we have $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$.

Thus, $(a^2 + c^2)x + aby = ap$ and $(b^2 + c^2)y + abx = bp$

which, on solving for x and y, gives $x = \dfrac{ap}{a^2 + b^2 + c^2}$ and $y = \dfrac{bp}{a^2 + b^2 + c^2}$.

and defines a critical point $\left(\dfrac{ap}{a^2 + b^2 + c^2}, \dfrac{bp}{a^2 + b^2 + c^2}\right)$.

Next, $r = \dfrac{\partial^2 f}{\partial x^2} = 2\left(1 + \dfrac{a^2}{c^2}\right)$, $s = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{2ab}{c^2}$ and $t = \dfrac{\partial^2 f}{\partial y^2} = 2\left(1 + \dfrac{b^2}{c^2}\right)$.

$$\therefore \quad rt-s^2=\frac{4}{c^4}(a^2+c^2)(b^2+c^2)-\frac{4a^2b^2}{c^4}=\frac{4}{c^4}[c^4+a^2c^2+b^2c^2]=\frac{4}{c^2}[a^2+b^2+c^2)>0.$$

Here $r>0$ hence at the critical point, function f has minima.

When $x=\frac{ap}{a^2+b^2+c^2}$, $y=\frac{bp}{a^2+b^2+c^2}$, $z=\frac{1}{c}[p-ax-by]=\frac{pc}{a^2+b^2+c^2}$

$$\therefore \quad \text{Min.} f = x^2+y^2+z^2=\frac{p^2}{(a^2+b^2+c^2)^2}[a^2+b^2+c^2]=\frac{p^2}{a^2+b^2+c^2}.$$ **Ans.**

(*b*) Distance of the point (x, y) on the curve from the origin $=\sqrt{x^2+y^2}$.

Let $f(x,y)=x^2+y^2$ subject to the condition $\phi(x,y)=5x^2+6xy+5y^2-8=0$ ...(1)

Consider the function $F(x,y)=x^2+y^2+\lambda(5x^2+6xy+5y^2-8)$

For f to be max. or min. $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$.

Thus, $2x+10\lambda x+6\lambda y=0$ and $2y+6\lambda x+10\lambda y=0$

or $(1+5\lambda)x+3\lambda y=0$ and $(1+5\lambda)y+3\lambda x=0$ ...(2)

or $\frac{y}{x}=-\frac{(1+5\lambda)}{3\lambda}$ and $\frac{y}{x}=-\frac{3\lambda}{1+5\lambda}$

$\Rightarrow \frac{1+5\lambda}{3\lambda}=\frac{3\lambda}{1+5\lambda}$ or $1+5\lambda=\pm 3\lambda$ $\therefore$ $\lambda=-1/2$ and $\lambda=-1/8$.

Next, when $\lambda=-1/2$ we have $x+y=0$ or $y=-x$. Then (1) gives $5x^2-6x^2+5x^2=8$

or $x=\pm\sqrt{2}$ and $y=\pm\sqrt{2}$. $\therefore$ The critical points are $(-\sqrt{2},\sqrt{2})$ and $(\sqrt{2},-\sqrt{2})$.

Hence, the critical distance $=\sqrt{x^2+y^2}=2$.

And when $\lambda=-1/8$ we have $x-y=0$ or $y=x$. Then (1) gives $5x^2+6x^2+5x^2-8=0$

or $x=\pm\frac{1}{\sqrt{2}}$, $y=\pm\frac{1}{\sqrt{2}}$. $\therefore$ The critical points are $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2},-1/\sqrt{2})$.

Hence, the critical distance $=\sqrt{x^2+y^2}=1$.

Thus, the max. and min. distances are 2 and 1 respectively. **Ans.**

EXERCISE 1E

1. Find the maximum and minimum value of the function $f(x, y)=x^3+y^3-3axy$. [AKTU 2019]

2. Show that $u=x^2 y^2 (x^2+y^2-1)$ has a maximum value of $-\frac{1}{27}$ for four sets of values of x and y and a minimum value at $x=0, y=0$.

3. (a) Find the maximum and minimum values of $\sin x \sin y \sin (x+y)$.

   (b) Find the maximum and minimum values of $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$. [GGSIPU 2003]

4. A tent on a square base of side $2a$ consists of four vertical sides of height b surmounted by a regular pyramid of height h. If the volume enclosed is V show that the area of the canvas in the tent is $\frac{2V}{a}-\frac{8}{3}ah+4a\sqrt{h^2+a^2}$. Also show that if a and h can both vary, the least area of the canvas corresponding to a given volume V is given by $a=\frac{\sqrt{5}}{2}h,\ b=\frac{h}{4}$.

5. Find the extreme values of $x^2+y^2+z^2$ subject to the constraints

$$x^2+\frac{y^2}{2}+\frac{z^2}{3}=2 \quad \text{and} \quad 3x+2y+z=0.$$

6. Find the minimum value of x^2+y^2 subject to the condition $ax+by=c$.

7. Find the points on the ellipse $x^2 + 4y^2 = 1$ whose distances from the straight line $2x+3y=6$ are the greatest and the least.

8. (a) Find the minimum value of $f(x,y,z)=x^2+y^2+z^2$ where $x+y+z=1$.

   (b) Find the maximum value of the fucntion $f=z-2x^2-2y^2$ where $3xy-z+7=0$. [AKTU 2019]

9. Determine the values of x, y, z that maximise the function $3x+5y+z-x^2-y^2-z^2$ subject to the constraint $x+y+z=6$.

10. Divide 120 into three parts in such a way that the sum of their products taken two at a time, is maximum.

11. Find a point within a given triangle so that the sum of the squares of its distances from the sides of the triangle is minimum.

12. If r is the distance of a point on the conic $ax^2+by^2+cz^2=1,\ lx+my+nz=0$ from the origin, then show that the stationary values of r are given by $\frac{l^2}{1-ar^2}+\frac{m^2}{1-br^2}+\frac{n^2}{1-cr^2}=0$ [AKTU 2016]

13. Find the extreme value of $x^l \cdot y^m \cdot z^n$ subject to the conditions $ax+by+cz=l+m+n$.

14. Divide $2u$ into three parts such that the continued product of the first, square of the second and cube of the third may be maximum. [AKTU 2017]

15. Find the points on the surface $z^2 = xy + 1$ nearest to origin.

16. Find the maximum and minimum distances of the point $(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$. [GGSIPU 2011]

17. Find the maximum distance from the point $(1, 2, 0)$ to the cone $z^2 = x^2 + y^2$.

18. A rectangular box open at the top has a capacity of 256 cubic feet. Applying the Lagrange's method of undetermined multipliers determine the dimensions of the box such that least material is required for the construction of the box. [GGSIPU 2017]

19. Find the maximum and minimum distance of the point $(1, 2, -1)$ from the sphere $x^2 + y^2 + z^2 = 24$. [AKTU 2018, 2019]

20. Find the relation between u, v, w for the values $u = x + 2y + 3z$, $v = x - 2y + 3z$, $w = 2xy - zx + 9yz - 2z^2$. [AKTU 2017]

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