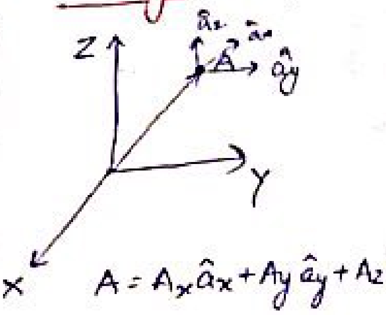
EMFT (unit-1)
Coordinate System

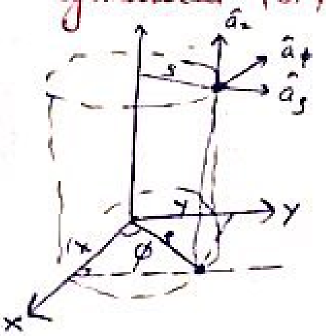
Rectangular (x, y, z)



$$A = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$$

$$\boxed{\begin{matrix} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{matrix}}$$

Cylindrical (ρ, ϕ, z)



Que - Range of 3 coordinates in cylindrical coordinate system (2.5 M) (2018)

$$\boxed{\begin{matrix} 0 \le \rho < \infty \\ 0 \le \phi < 2\pi \\ -\infty < z < \infty \end{matrix}}$$

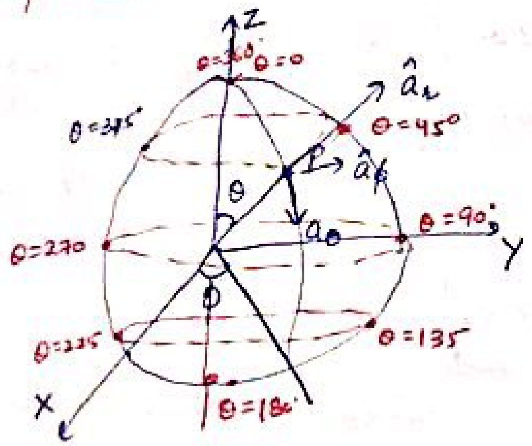
$$X = \rho\cos\phi \qquad \rho = \sqrt{x^2 + y^2}$$
$$Y = \rho\sin\phi \quad \text{or} \quad \phi = \tan^{-1}\frac{y}{x}$$
$$Z = Z \qquad Z = Z$$

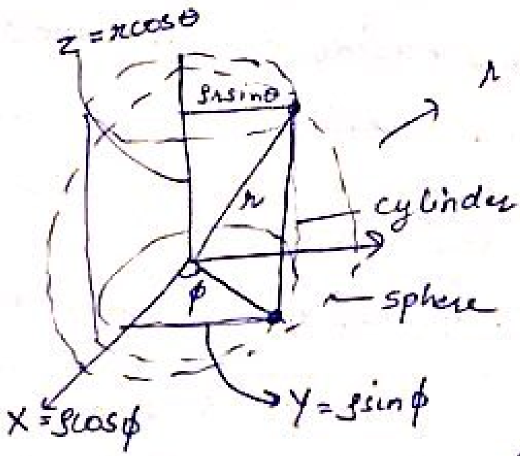
$\partial/\partial\rho$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

for remembering
X comes before Y and C comes before S so $X = \rho\cos\phi$, $Y = \rho\sin\phi$

Spherical coordinate (r, θ, ϕ)





$$r = \sqrt{z^2 + \rho^2} = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\frac{\rho}{z}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$\rho = r\sin\theta$$

$$\boxed{\begin{matrix} 0 \le r < \infty \\ 0 \le \theta < \pi \\ 0 \le \phi < 2\pi \end{matrix}}$$

$$X = \rho\cos\phi = r\sin\theta\cos\phi$$
$$Y = \rho\sin\phi = r\sin\theta\sin\phi$$
$$Z = r\cos\theta$$

As angle 45° and 315° form the same cone in sphere, $\theta = 90°$ and 270° form the same hemisphere & $\theta = 135°$ & 225° form same cone. Hence $\boxed{0 < \theta < \pi}$

$\partial/\partial\theta$ $\qquad$ $\partial/\partial\phi$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\theta \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Que: Transform the vector $A = 2\hat{a}_r + 3\hat{a}_\phi + 4\hat{a}_z$ into cartesian coordinates (March 2020) ②

Solⁿ: Since the vector $A = 2\hat{a}_s + 3\hat{a}_\phi + 4\hat{a}_z$ is a (r,ϕ,z) i.e. cylindrical coordinate vector.

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow \begin{array}{l} A_x = 2\cos\phi - 3\sin\phi\ \hat{a}_x \\ A_y = 2\sin\phi + 3\cos\phi\ \hat{a}_y \\ A_z = 4\hat{a}_z \end{array}$$

We know $x = \rho\cos\phi$ and $\rho = \sqrt{x^2+y^2}$
$y = \rho\sin\phi$

So, $\cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$ and $\sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}$

$$A_x = \frac{2\cdot x}{\sqrt{x^2+y^2}} - \frac{3y}{\sqrt{x^2+y^2}}\hat{a}_x,\quad A_y = \frac{2y}{\sqrt{x^2+y^2}} + \frac{3x}{\sqrt{x^2+y^2}}\hat{a}_y,\quad A_z = 4\hat{a}_z$$

$$\boxed{A = \frac{2x-3y}{\sqrt{x^2+y^2}}\hat{a}_x + \frac{2y+3x}{\sqrt{x^2+y^2}}\hat{a}_y + 4\hat{a}_z}$$

Que: Transform the vector $A = y\hat{a}_x + x\hat{a}_y + \dfrac{x^2}{\sqrt{x^2+y^2}}\hat{a}_z$ into cylindrical coordinates (2019 sessional)

Solⁿ:
$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \Rightarrow \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x \\ \frac{x^2}{\sqrt{x^2+y^2}} \end{bmatrix}$$

$A_\rho = y\cos\phi + x\sin\phi = \rho\sin\phi\cos\phi + \rho\cos\phi\sin\phi = 2\rho\sin\phi\cos\phi$
$= \rho\sin 2\phi$

$A_\phi = -y\sin\phi + x\cos\phi = -\rho\sin\phi\sin\phi + \rho\cos\phi\cos\phi = \rho(\cos^2\phi - \sin^2\phi)$
$= \rho\cos 2\phi$

$A_z = \dfrac{x^2}{\sqrt{x^2+y^2}} = \dfrac{\rho^2\cos^2\phi}{\sqrt{\rho^2\cos^2\phi + \rho^2\sin^2\phi}} = \rho\cos^2\phi$

$$\boxed{A(\rho,\phi,z) = \rho\sin 2\phi\ \hat{a}_\rho + \rho\cos 2\phi\ \hat{a}_\phi + \rho\cos^2\phi\ \hat{a}_z}$$

$$\boxed{\begin{array}{l} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \\ \cos\phi = \frac{x}{\sqrt{x^2+y^2}} \\ \sin\phi = \frac{y}{\sqrt{x^2+y^2}} \end{array}}$$

Q: Convert the point $P(1,3,5)$ from Cartesian to cylindrical & spherical coordinate geometry (3 Marks, 2019)(end term)

Solⁿ: In cartesian $P(1,3,5) \Rightarrow x = 1, y = 3$ & $z = 5$

For cylindrical coordinate $\rho = \sqrt{x^2+y^2} = \sqrt{1+9} = \sqrt{10} = 3.16$
$\phi = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\dfrac{3}{1} = 71.56$

For spherical geometry $r = \sqrt{\rho^2+z^2} = \sqrt{1+9+25} = \sqrt{35} = 5.91$
$\theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\dfrac{\sqrt{10}}{5} = 32.29°$

$P(1,3,5) = P(3.16, 71.56°, 5) = P(5.91, 32.29°, 71.56°)$

①

Q:- Transform the vector $A = y\hat{a}_x - x\hat{a}_y + z\hat{a}_z$ into cylindrical coordinate
(End Term, 2019, 6 Marks)

Sol^n:
$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ -x \\ z \end{bmatrix}$$

$y = \rho\sin\phi$
$x = \rho\cos\phi$

or $A_\rho = (\cos\phi\, y - x\sin\phi)\hat{a}_\rho = (\rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi)\hat{a}_\rho = 0$

$A_\phi = [-y\sin\phi + \cos\phi(-x)]\hat{a}_\phi = (-\rho\sin^2\phi - \rho\cos^2\phi)\hat{a}_\phi = -\rho\hat{a}_\phi$

$A_z = z\hat{a}_z$

~~A~~ $\boxed{A(\rho,\phi,z) = -\rho\hat{a}_\phi + z\hat{a}_z}$

(end term 2018)

Q:- Transform the vector $A = \hat{i}_\theta + \hat{i}_\phi$ to cartesian coordinate system (6.5).

Sol^n: Since the vector has θ and ϕ component, it is in spherical coordinate geometry.

and in end term (2017) (6.5)

So.
$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\theta \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r = 0 \\ A_\theta = 1 \\ A_\phi = 1 \end{bmatrix}$$

$A_x = [\cos\theta\cos\phi - \sin\phi]\hat{a}_x =$

$A_y = [\cos\theta\sin\phi + \cos\theta]\hat{a}_y$

$A_z = [-\sin\theta]\hat{A}_z$

We know $z = r\cos\theta \Rightarrow \cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

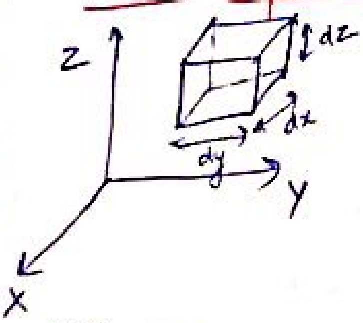
$\cos\phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}$, $\rho = r\sin\theta$

$\sin\phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}$, $\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$

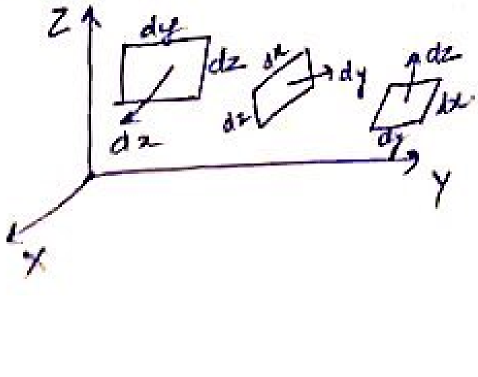
$$\boxed{A_x = \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\cdot\frac{x}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}\right)\hat{a}_x + \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\cdot\frac{y}{\sqrt{x^2+y^2}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\right)\hat{a}_y - \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\hat{a}_z}$$

~~Q: Determine the unit vector normal to the surface $S(x,y,z) = x^2+y^2+z$ at $(1,3,0)$ (End term 2018)(6.4)~~

Vector Representation of Surfaces



Differential displacement $\rightarrow dl = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$

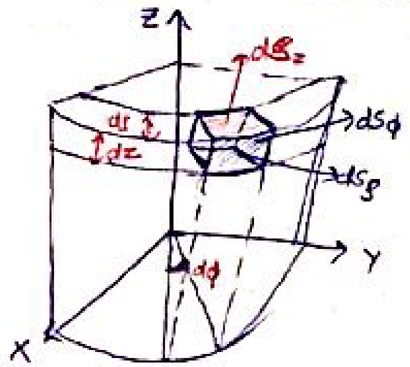


differential surface $ds = dydz\,\hat{a}_x$ or $dxdy\,\hat{a}_z$ or $dxdz\,\hat{a}_y$

Differential volume is given by $dV = dxdydz$

Scanned with CamScanner

In Cylindrical Coordinate Geometry



Differential displacement

$$dl = d\rho\, \hat{a}_\rho + \rho\, d\phi\, \hat{a}_\phi + dz\, \hat{a}_z$$

Differential normal surface

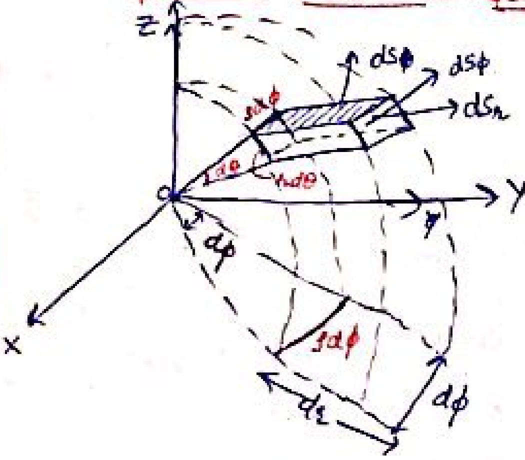
$$ds = \rho\, d\phi\, dz\, \hat{a}_\rho \quad \text{or}$$
$$d\rho\, dz\, \hat{a}_\phi \quad \text{or}$$
$$\rho\, d\rho\, d\phi\, \hat{a}_z$$

$dl = \rho\, d\phi$

Differential volume

$$dV = \rho\, d\rho\, d\phi\, dz$$

In Spherical Coordinate Geometry



$\rho = r\sin\theta$

Differential displacement

$$dl = dr\, \hat{a}_r + r\, d\theta\, \hat{a}_\theta + r\sin\theta\, d\phi\, \hat{a}_\phi$$

Differential Surface Area

$$ds = r^2 \sin\theta\, d\theta\, d\phi\, \hat{a}_r \quad \text{or}$$
$$r\sin\theta\, dr\, d\phi\, \hat{a}_\theta \quad \text{or}$$
$$r\, dr\, d\theta\, \hat{a}_\phi$$

Differential volume $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Vector Analysis

Vector differential operator del (∇) is defined as $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

∇ has direction but no magnitude but applies it to some vector or scalar.

∇ operator corresponding to each coordinate

Cartesian: $\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$

Cylindrical $\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$

Spherical $\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi$ ← only for remembering

∇ → scalar → Vector (Gradient)
∇ → (dot) vector → scalar (Divergence)
∇ → (cross product) vector → vector (Curl)

$dl = d\rho\, \hat{a}_\rho + \rho\, d\phi\, \hat{a}_\phi + dz\, \hat{a}_z$
$dl = dr\, \hat{a}_r + r\, d\theta\, \hat{a}_\theta + r\sin\theta\, d\phi\, \hat{a}_\phi$

Gradient:- It represents the rate of change.

Cartesian: $\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$

Cylindrical: $\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$

Spherical $\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$

(Q in 2018 (2.5M))

→ ∇V points in the direction of maximum rate of change in V
→ ∇V is normal to constant value surface at a given point.

Divergence: It gives the rate per unit volume at which the physical entity is issuing from that point.

→ It is a measure of how much a field diverges at a given point.

Cartesian: $\nabla\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

Cylindrical: $\nabla\cdot\vec{V} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$

Spherical: $\nabla\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$

Divergence Theorem:– Total outward flux through any closed surface of a vector is equal to volume integral of divergence of that vector.

$$\boxed{\oint_S \vec{A}\cdot d\vec{s} = \int_V \nabla\cdot\vec{A}\, dV}$$

* A vector is said to be divergence less or solenoid if $\nabla\cdot\vec{A} = 0$

Curl: It is a measure of how much a field curl or rotate around a point.

→ It is a measure of angular velocity of each point of the vector field.

Cartesian:

$$\nabla\times\vec{V} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Cylindrical

$$\nabla\times\vec{V} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\phi & V_z \end{vmatrix}$$

Spherical

$$\nabla\times\vec{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & rV_\theta & r\sin\theta\, V_\phi \end{vmatrix}$$

Strokes Theorem: Circulation of a vector in closed path is equal to surface integral of curl of that vector bounded by that closed loop.

$$\boxed{\oint A\cdot dl = \iint(\nabla\cdot\vec{A})\,ds}$$

* A vector is said to be conservative or irrotational if $\nabla\times\vec{A} = 0$

Second Derivative

Gradient (vector)
- Divergence ↓ Scalar Laplacian operator ($\nabla^2 V$)
- Curl ↓ DC = 0

Divergence (scalar)
↓ Gradient ↓ X

Curl (vector)
- Divergence (DC = 0)
- Curl ↓ vector Laplacian operator.

Scalar Laplacian

Cartesian: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

Cylindrical: $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

Spherical: $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r^2}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\theta^2}$

Vector Laplacian

$\nabla^2\vec{V} = \nabla(\nabla\cdot\vec{V}) - \nabla\times\nabla\times\vec{V}$

Que:- Determine the unit vector normal to the surface $S(x,y,z) = x^2+y^2-z$ at $(1,3,0)$ (ET, 2018) (6M)

Soln: Unit vector normal to the surface is given by Gradient

$$\nabla S = \frac{\partial S}{\partial x}\hat{a}_x + \frac{\partial S}{\partial y}\hat{a}_y + \frac{\partial S}{\partial z}\hat{a}_z$$

$$= \frac{\partial}{\partial x}(x^2+y^2-z)\hat{a}_x + \frac{\partial}{\partial y}(x^2+y^2-z)\hat{a}_y + \frac{\partial}{\partial z}(x^2+y^2-z)\hat{a}_z$$

$$= 2x\hat{a}_x + 2y\hat{a}_y + 2z\hat{a}_z$$

Que: Write Divergence & Stokes theorem in mathematical form (ET 2018, 2.5M)

∇S at $(1,3,0)$

$$= 2\hat{a}_x + 6\hat{a}_y \qquad \boxed{\nabla S = 2\hat{a}_x + 6\hat{a}_y}$$

Que:- Explain the condition when a field is solenoidal and when irrational (ET, 2017, 2.5M)

Soln: Define divergence & curl here, then

if divergence $= 0$ → Field is solenoidal

if curl $= 0$ → Field is irrotational

Que:- Give a vector function $A = (x+c_1z)\hat{a}_x + (c_2x+3z)\hat{a}_y + (x+c_3y+c_4z)\hat{a}_z$ (ET, 2017, 6M)

a) Calculate c_1, c_2 and c_3 if A is irrotational

b) Determine the constant c_4 if A is solenoidal also

Solution: To determine if its irrotational $\nabla\times A = 0$

$$\nabla\times A = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ x+c_1z & c_2x+3z & x+c_3y+c_4z \end{vmatrix} = 0$$

$$= \left[\frac{\partial}{\partial y}(x+c_3y+c_4z) - \frac{\partial}{\partial z}(c_2x+3z)\right]\hat{a}_x - \left[\frac{\partial}{\partial x}(x+c_3y)+c_4z) - \frac{\partial}{\partial z}(x+c_1z)\right]\hat{a}_y + \left[\frac{\partial}{\partial x}(c_2x+3z) - \frac{\partial}{\partial y}(x+c_1z)\right]\hat{a}_z$$

$$= (c_3-3)\hat{a}_x + (c_1-1)\hat{a}_y + (c_2-0) = 0 \Rightarrow \boxed{\begin{matrix} c_3 = 0 \\ c_2 = 0 \\ c_1 = 1 \end{matrix}}$$

For solenoidal $\nabla\cdot A = 0$

$$\nabla\cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial}{\partial x}(x+c_1z) + \frac{\partial}{\partial y}(c_2x+3z) + \frac{\partial}{\partial z}(x+c_3y+c_4z) = 0$$

$$1+0+c_4 = 0 \Rightarrow \boxed{c_4 = -1}$$

Que:- If $F = \text{grad}\,[(x^3+y^3+z^3) - 3xyz]$ Find curl F? (ET, 2017, 2.5M)

$$F = \text{grad}\,[x^3+y^3+z^3] - 3xyz$$

$$\nabla F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\hat{a}_y + \frac{\partial F}{\partial z}\hat{a}_z$$

$$= \frac{\partial}{\partial x}[x^3+y^3+z^3]-3xyz) + \frac{\partial}{\partial y}[x^3+y^3+z^3-3xyz]\hat{a}_y + \frac{\partial}{\partial z}[x^3+y^3+z^3-3xyz]\hat{a}_z$$

$$= (3x^2-3yz)\hat{a}_x + (3y^2-3xz)\hat{a}_y + [3z^2-3xy)\hat{a}_z$$

curl of $F = \nabla\times F$

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix} = \hat{a}_x\left[\frac{\partial}{\partial y}[3z^2-3xy] - \frac{\partial}{\partial z}[3y^2-3xz]\right] + \left[\frac{\partial}{\partial z}(3x^2-3yz) - \frac{\partial}{\partial x}(3z^2-3xy)\right]\hat{a}_y + \hat{a}_z\left[\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$$= (-3x+3x)\hat{a}_x + [-3y+3y]\hat{a}_y + [-3z+3z]\hat{a}_z$$

$$= 0$$

Que:- Find the laplacian of the scalar vector field $V = \rho z \sin\phi + z^2\cos^2\phi + \rho^2$ [ET, 2017, 6M] (7)

Sol^n. Laplacian → Divergence of gradient

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\left(\frac{\partial}{\partial\rho}\rho z\sin\phi + z^2\cos^2\phi + \rho^2\right)\right) + \frac{1}{\rho^2}\left(\frac{\partial^2}{\partial\phi^2}(\rho z\sin\phi + z^2\cos^2\phi + \rho^2)\right) + \frac{\partial^2}{\partial z^2}(\rho z\sin\phi + z^2\cos^2\phi + \rho^2)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho(z\sin\phi + 2\rho)\right) + \frac{1}{\rho^2}\left(\frac{\partial}{\partial\phi}(\rho z\cos\phi - z^2\cdot 2\cdot\sin\phi)\right) + \frac{\partial}{\partial z}(\rho\sin\phi + 2z\cos^2\phi)$$

$$= \frac{1}{\rho}[z\sin\phi + 4\rho] + \frac{1}{\rho^2}[-\rho z\sin\phi - 2z^2\cos\phi) + 2\cos^2\phi$$

$$\boxed{\nabla^2 V = \frac{1}{\rho}[z\sin\phi + 4\rho] + \frac{1}{\rho^2}[-\rho z\sin\phi - 2z^2\cos\phi) + 2\cos^2\phi}$$

Electrostatics

Coulomb's Law → $F = \frac{kQ_1Q_2}{R^2}\hat{a}_r$ or $F = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q_1Q_2}{R^2}\hat{a}_r$

where ϵ_0 = permittivity of free space

$\epsilon_0 = 8.854\times10^{-12}$ or $\frac{10^{-9}}{36\pi}$ F/m or $k = 9\times10^9$ m/F

Electric Field: Force per unit charge when placed in an electric field.

Thus $E = \lim_{Q\to0}\frac{F}{Q}$ or $E = \frac{Q_1}{4\pi\epsilon_0 R^2}\hat{a}_r$

Continuous Charge Distribution:

Line charge density ρ_L $Q = \int_L \rho_L dl$

Surface charge density ρ_s $Q = \int_S \rho_s ds$

Volume charge density ρ_v $Q = \int_v \rho_v dv$

Electric Field Intensity

1. Due to line charge, $\vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\hat{a}_\rho$
2. Due to surface charge $\vec{E} = \frac{\rho_s}{2\epsilon_0}\hat{a}_n$
3. Due to volume charge $\vec{E} = \frac{Q}{4\pi\epsilon_0 z^2}\hat{a}_z$

Electric Flux: It is a measure of electric field lines passing through any surface $\psi = \int_S \vec{E}\cdot d\vec{s}$

and electric flux density is given by $\vec{D} = \epsilon_0\vec{E}$

↳ also called electric displacement.

Gauss Law: The total electric flux passing through any closed surface is equal to charge enclosed by that surface. (ET2018, 2.5M)

$$\psi = \oint \vec{E}\cdot d\vec{s} = \frac{Q_{enclosed}}{\epsilon_0}$$

$$\oint \epsilon_0\vec{E}\cdot d\vec{s} = Q_{enclosed}$$

gaussian surface → $\oint \vec{D}\cdot d\vec{s} = Q_{enclosed}$

Gaussian surface: $d\vec{s}$ is ⊥ to the surface and moving outward from the surface.
2. $\vec{E}$ is equal at all points on the surface.

If charge is distributed over volume V which contains an surface ds

$$Q = \int_V \rho_v \, dv$$

$$\oint_S \vec{D} \cdot ds = Q_{enclosed} = \int \rho_v \, dv \qquad \boxed{\oint_S \vec{D} \cdot \vec{ds} = \int_V \rho_v \, dv}$$ Maxwell's first equation.

Applying divergence theorem $\rightarrow \oint_S \vec{D} \cdot ds = \int_V \nabla \cdot D \, dv$

$$\Rightarrow \boxed{\nabla \cdot \vec{D} = \rho_v} \rightarrow$$ Maxwell's first equation in differential form.

$\vec{E}$ due to point charge

$$\oint \vec{D} \cdot \vec{ds} = Q_{enclosed}$$

$$\vec{D} \oint_S \vec{ds} = Q_{enclosed}$$

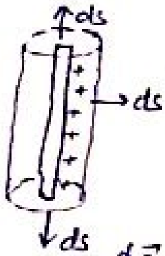
$$\vec{D} \cdot 4\pi R^2 = Q$$

$$\vec{D} = \frac{Q}{4\pi R^2} \cdot \hat{a}_r$$

$$\epsilon_0 \vec{E} = \frac{Q}{4\pi R^2} \hat{a}_r$$

$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 R} \hat{a}_r}$$

- $\vec{E}$ due to line charge



$$Q = \int \rho_L \, dl$$

$\oint_S \vec{D} \cdot \vec{ds} = US + LS +$ Round surface

For US & LS angle b/w $\vec{D}$ & $\vec{ds}$ is $90°$ so $\vec{D} \cdot \vec{ds} = 0$

$$\oint \vec{D} \cdot \vec{ds} = \oint_{round} \vec{D} \cdot \vec{ds} = \rho_L \cdot L$$

$$\vec{D} \oint ds = \rho_L L$$

$$\epsilon_0 \vec{E} \cdot 2\pi R \cancel{L} = \rho_L \cancel{L}$$

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 R} \hat{a}_s}$$

Electric Potential $\rightarrow$ It is a scalar quantity and is used to defined electric field.

Work done in moving a charge is

$$dW = -F \cdot dl$$ -ve sign indicates that the work is done

$$= -E \cdot q \cdot dl$$ by an external agent.

or $\boxed{W = -Q \int_A^B E \cdot dl}$ $\qquad \boxed{V = -\int E \cdot dl \quad \text{or } E = -\nabla V}$

$$V_{AB} = \frac{W}{Q} = -\int_A^B E \cdot dl = -\int_A^B \frac{Q}{4\pi\epsilon_0 r^2} \hat{a}_r \, dr \, \hat{a}_r = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

potential energy per unit charge

$$\boxed{V_{AB} = V_B - V_A}$$

$$V_{AB} = -V_{BA}$$

$$V_{AB} = -\int_A^B E \cdot dl$$

$$-V_{AB} = \int_A^B E \cdot dl = -\int_B^A \vec{E} \cdot dl = V_{BA}$$

$$-V_{AB} = V_{BA}$$

$$V_{AB} + V_{BA} = 0 \Rightarrow -\int_A^B \vec{E} \cdot dl - \int_B^A \vec{E} \cdot dl = 0 \Rightarrow \boxed{\oint \vec{E} \cdot \vec{dl} = 0}$$ Maxwell's 2nd equation

It states that the net work done in moving a point charge in a closed loop in presence of $\vec{E}$ is zero.

Apply stoke's theorem $\oint \vec{E} \cdot \vec{dl} = \int_S (\nabla \times \vec{E}) ds$

$$\because \int_S (\nabla \times \vec{E}) \, ds = 0$$

$$\boxed{\nabla \times \vec{E} = 0}$$

Q: Given that the field $E = \frac{100}{r}\hat{a}_r$ Volt/meter in cylindrical coordinate. Show that the work needed to move a point charge Q from any radial distance r to a point twice the radial distance is independent of r.

Sol^n: $W = -Q\int_r^{2r} \vec{E}\cdot d\vec{l}$ $\qquad dl = dr\,\hat{a}_r$

$= -Q\int_r^{2r} \frac{100}{r}\hat{a}_r \cdot dr\,\hat{a}_r$

$= -Q\int_r^{2r} \frac{100}{r}dr = -100Q\left[\ln[r]\right]_r^{2r} = -100Q\left[\ln\left[\frac{2r}{r}\right]\right]$

$= -100q\ln[2]$ $\qquad \ln(2) = 0.6931$

$\boxed{W = -69.31q}$

Q: Given that the field $V = \frac{10}{r^2}\sin\theta\cos\phi$. Find the electric flux density D at $(10,0,0)$ and work done in moving a charge of $100\mu C$ from point $A(2,60°,60°)$ to $B(4,30°,30°)$.

Sol^n: $D = \epsilon_0 E$

and $E = -\nabla V$

Since V is given in spherical coordinates

$E = -\nabla = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right]$

$= -\left[-\frac{20}{r^3}\sin\theta\cos\phi\hat{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\hat{a}_\phi + \frac{10}{r^3}\sin\phi\,\hat{a}_\phi\right]$

At $(10,0,0) \rightarrow D = \epsilon_0 E$

$D = -\epsilon_0\left[\frac{-20}{(10)^3}\sin 0°\cos 0°\hat{a}_r - \frac{10}{(10)^3}\cos 0°\cos 0° + \frac{10}{(10)^3}\cos 0°\,\hat{a}_\phi\right]$

$= -\epsilon_0\left[0 - 0 + \frac{1}{10^2}\hat{a}_\phi\right) = -\frac{8.854\times 10^{-12}}{10^2} = -8.854\times 10^{-14}\hat{a}_\phi$

Work done in moving a charge of $100\mu C$ from $A(2,60°,60°)$ to $B(4,30°,30°)$

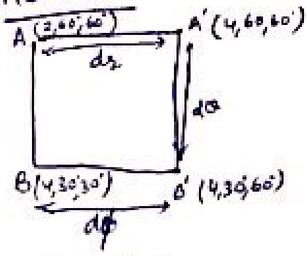
Method I

$V_A = \frac{10}{r^2}\sin\theta\cos\phi = \frac{10}{4}\sin 60°\cos 60° = \frac{10}{4}\times\frac{\sqrt{3}}{2}\times\frac{1}{2} = \frac{10\sqrt{3}}{16}$

$V_B = \frac{10}{r^2}\sin\theta\cos\phi = \frac{10}{16}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{10\sqrt{3}}{64}$

$W = q(V_B - V_A) = 100\times\left(\frac{10\sqrt{3}}{64} - \frac{10\sqrt{3}}{16}\right) = 100\times\frac{10\sqrt{3}}{64}(1-4)$ $\boxed{W = -8.10\mu J}$

Method II



$AA' = dl = dr\,\hat{a}_r$

$A'B' = dl = r\,d\theta\,\hat{a}_\theta$

$B'B = dl = r\sin\theta\, d\phi\,\hat{a}_\phi$

$W = -Q\int E\cdot dl = -Q\left[\int_{AA'} E\cdot dl + \int_{A'B'} E\cdot dl + \int_{B'B} E\cdot dl\right]$

$-\frac{W}{Q} = \int_2^4 \frac{20\sin\theta\cos\phi\,dr}{r^3} + \int_{\theta=60}^{30} -\frac{10\cos\theta\cos\phi}{r^3}\cdot r\,d\theta + \int_{\phi=60°}^{30°}\frac{10\sin\phi}{r^3}r\sin\theta\,d\phi$

$= 20\sin 60°\cos 60°\left[-\frac{1}{2r^2}\right]_2^4 - \frac{10}{r^2}\left[\sin\theta\right]_{60}^{30}\cos\phi + \frac{10}{r^2}\sin\theta\left[-\cos\phi\right]_{60}^{30}$

$= 20\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\left[\frac{-1}{16-4}\right] - \frac{10}{r^2}\left[\sin 30° - \sin 60°\right]\cos 60° + \frac{10}{16}\cdot\frac{1}{2}\left[\cos 60° - \cos 30°\right]$

$= \frac{10\sqrt{3}}{2}\cdot\frac{1}{2}\left[-\frac{3}{16}\right] - \frac{10}{16}\left[\frac{1}{2} - \frac{\sqrt{3}}{2}\right]\cdot\frac{1}{2} + \frac{10}{16}\cdot\frac{1}{2}\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\right]$

$= \frac{15\sqrt{3}}{32} - \frac{5}{32} + \frac{5\sqrt{3}}{32} - \frac{5\sqrt{3}}{32} + \frac{5}{32}$

$-\frac{W}{Q} = 0.810$

$\boxed{W = -0.810\times q = -8.10\mu J}$

Que:- Define the terms Electric field intensity (E) and Electric flux density (D). Also give the relationship b/w E & D. (ET 2019, 2.5 M)

Que Potential is given by $V = 2(x+1)^2(y+2)^2(z+3)^2$ Volt in free space. At a point $P(2,-1,4)$. calculate (a) Potential at point P (b) Electric field intensity $\vec{E}$ at point P. (c) Electric flux density D at point P (d) Volumetric charge density ρ_v at P (ET 2019, M-8)

Solⁿ. $V = 2(x+1)^2(y+2)^2(z+3)^2$

$P(2,-1,4)$

1) $V = 2(2+1)^2(-1+2)^2(4+3)^2 = 2(3)^2(1)^2(7)^2 = 2\times9\times1\times49 = 882V$

2) $E = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)$

$= -4(x+1)(y+2)^2(z+3)^2\,\hat{a}_x - 4(x+1)^2(y+2)(z+3)^2\,\hat{a}_y - 4(x+1)^2(y+2)^2(z+3)\,\hat{a}_z$

At $P(2,-1,4$

$\vec{E} = 4(3)(1)^2(7)^2\hat{a}_x - 4(3)^2(1)(7)^2\hat{a}_y - 4(3)^2(1)^2(7)\hat{a}_z$

$\boxed{\vec{E} = -588\,\hat{a}_x - 1764\,\hat{a}_y - 252\,\hat{a}_z}$

(iii) $D = \epsilon_0\vec{E} = 8.854\times10^{-12}[-588\,\hat{a}_x - 1764\,\hat{a}_y - 252\,\hat{a}_z]$

$= -5.21\,\hat{a}_x - 15.62\,\hat{a}_y - 2.23\,\hat{a}_z\ nC/m^2$

(iv) $\rho_v = \nabla\cdot D = \epsilon_0\left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right]$

$= \epsilon_0\left[-4(y+2)^2(z+3)^2 - 4(x+1)^2(z+3)^2 - 4(x+1)^2(y+2)^2\right]$

$\rho_v = -17.67\ nC/m^3$

Electric field is usually determined using coulomb's law or Gauss law when the charge distribution is known or using $E = -\nabla V$ when the potential distribution is known.

However in most practical situations, when neither the charge distribution nor the potential distribution is known but only electrostatic conditions (charge & potential) at some boundaries are known & it is desired to find $\vec{E}$ and $\vec{V}$ through the region. Such problems are tackled using Poisson, Laplace or method of images.

Poisson and Laplace Equation

(May 2017) M-6.5.

(May 2018) M-4.5

From Gauss law $\nabla\cdot D = \nabla\cdot\epsilon\vec{E} = \rho_v$ —(1)

$E = -\nabla V$ —(2)

Put (2) in (1) $\nabla\cdot(\epsilon(-\nabla V)) = \rho_v$

For Homogeneous medium ie. when ϵ is constant

$\epsilon\nabla^2 V = -\rho_v$ or $\boxed{\nabla^2 V = \frac{-\rho_v}{\epsilon}}$ → Poisson equation

For a charge free region

$\boxed{\nabla^2 V = 0}$ → Laplace Equation → Laplace eqⁿ is all different geometries is given an 2nd derivatives page.

Uniqueness Theorem states that if a solution to a laplace equation can be found that satisfies the boundary conditions, then the solution is unique i.e. it is the only solution regardless of the method used. (10)

Que: In cylindrical coordinate, two ϕ = constant planes are insulated along Z-axis. Find the expression for $\vec{E}$ b/w the planes assuming a potential of 100 V for $\phi = \alpha$ and zero at $\phi = 0$. (Sessional 2018)

Soln: Since the potential is constant with r & z. Laplace eqn is given by

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \Rightarrow \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Integrating $\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0$

$$\frac{\partial^2 V}{\partial\phi^2} = 0 \quad \frac{\partial V}{\partial\phi} = A = \text{constant}$$

$$\boxed{V = A\phi + B}$$

Apply Boundary conditions

At $\phi = 0 \rightarrow A(0) + B = 0 \Rightarrow B = 0$

At $\phi = \alpha \quad A(\alpha) + B = 100 \Rightarrow A = \frac{100}{\alpha} \longrightarrow \boxed{V = \frac{100}{\alpha}\phi}$

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial\rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial\phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z\right] = -\left[\frac{1}{\rho}\frac{\partial V}{\partial\phi}\hat{a}_\phi + 0 + 0\right]$$

$$= -\frac{1}{\rho}\frac{\partial V}{\partial\phi}\hat{a}_\phi = -\frac{1}{\rho}\frac{\partial}{\partial\phi}\left[\frac{100\phi}{\alpha}\right]\hat{a}_\phi = -\frac{100}{\rho\alpha}\hat{a}_\phi \text{ V/m}$$

$$\boxed{|E| = \left|\frac{100}{\alpha\rho}\right|}$$

Que: Two conducting cones ($\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{10}$) of infinite extent are seperated by a very small gap at $r = 0$. If V at $\theta = \frac{\pi}{10} = 0$ and v at $\theta = \frac{\pi}{6}$ is 50 V. Find $\vec{E}$. (Sessional 2020)

Soln: The Laplacian eqn in spherical coordinates is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 V}{\partial\theta^2} \text{ reduces to}$$

$$\nabla^2 V = 0 = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \Rightarrow \frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right] = 0$$

Integration $\sin\theta\frac{\partial V}{\partial\theta} = A$ or $\frac{\partial V}{\partial\theta} = \frac{A}{\sin\theta}$ —(1)

Integrating $V = A\int\frac{1}{\sin\theta}d\theta = A\int\frac{1}{2\sin\theta/2\cos\theta/2}d\theta = A\int\frac{\frac{1}{2}\sec^2\theta/2}{\tan\theta/2}d\theta$

$$= A\int\frac{d(\tan\theta/2)}{\tan\theta/2} = A\ln(\tan\theta/2) + B$$

Apply Boundary conditions to determine A & B

$V(\theta = \theta_1) = 0 = A\ln(\tan\theta_1/2) + B \Rightarrow B = -A\ln(\tan\theta_1/2)$

So, $V = A\ln\tan\theta/2 - A\ln(\tan\theta_1/2)$

$$V = A\ln\left[\frac{\tan\theta/2}{\tan\theta_1/2}\right] \quad \text{—(2)}$$

At $V(\theta=\theta_2) = V_0 = A \ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]$ or $A = \frac{V_0}{\ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]}$

Put A & B value in ②

$$V = \frac{V_0 \ln\left[\frac{\tan\theta/2}{\tan\theta_1/2}\right]}{\ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]}$$

from eq ①

$$E = -\nabla V = -\frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta = \frac{-A}{r\sin\theta}\hat{a}_\theta = \frac{-V_0}{r\sin\theta \ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]}\hat{a}_\theta$$

Putting value of $\theta_1 = \frac{\pi}{10}$, $\theta_2 = \frac{\pi}{6}$ and $V_0 = 50$

$$V = \frac{50 \ln\left[\frac{\tan\theta/2}{\tan\pi/20}\right]}{\ln\left[\frac{\tan\pi/12}{\tan\pi/20}\right]} = 95.1 \ln\left[\frac{\tan\theta/2}{0.1584}\right] V \quad -③$$

Comparing ② & ③ $A = 95.1$

and $\boxed{E = -\frac{95.1}{r\sin\theta}\hat{a}_\theta \ V/m}$

Que: Derive the poisson eq? State the conditions when the equation becomes the laplace equation. In spherical coordinate geometery $V=0$ at $r=0.2$m and $V=200$ at $r=4$m then calculate $\vec{E}$ and $\vec{D}$ assuming free space b/w the shells. (ET May 2018, 6.5M)

Sol^n:- Laplace eq^n spherical coordinate geometery

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \quad \text{(since variation is only in } r)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \rightarrow r^2\frac{\partial V}{\partial r} = A \qquad dV = \frac{A}{r^2}dr \qquad V = -\frac{A}{r} + B$$

Applying Boundary conditions

$V=0$ at $r=0.2$ $\quad 0 = \frac{-A}{0.2} + B \quad -A + 0.2B = 0 \quad \boxed{A = 0.2B}$

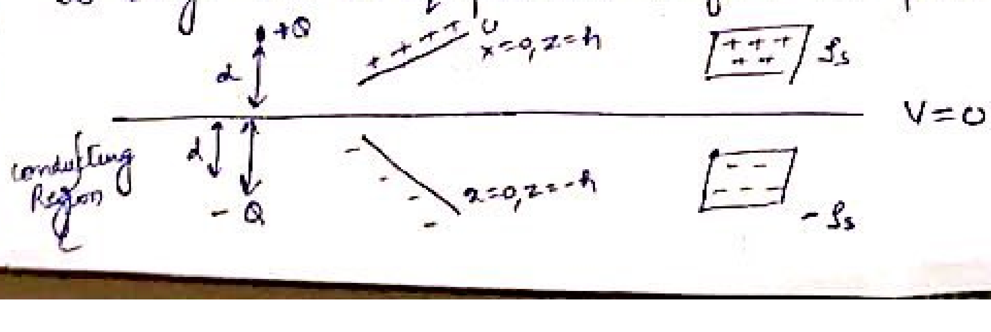
$V=200$ at $r=4$ $\quad 200 = -\frac{A}{4} + B \quad 200 = -\frac{0.2B}{4} + B \Rightarrow \boxed{B = \frac{4000}{19} \text{ and } A = \frac{800}{19}}$

$$V = \frac{-800}{19r} + \frac{4000}{19}$$

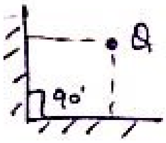
$$E = -\nabla V = -\left(+\frac{800}{19r^2}\hat{a}_r\right) = -\frac{800}{19r^2}\hat{a}_r \qquad \vec{D} = -\frac{\epsilon_0 \cdot 800}{19r^2}\hat{a}_r$$

Method of Images

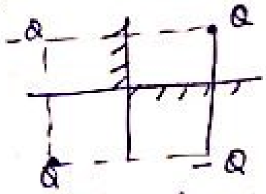
If any charge configuration placed above infinite grounded perfect conducting plane than the charge is replaced by the charge configuration itself and its image and ~~its~~ equipotential surface in place of grounded plane.



The number of images formed is $N = \frac{360^\circ}{\phi} - 1$
where ϕ = angle b/w conducting planes.

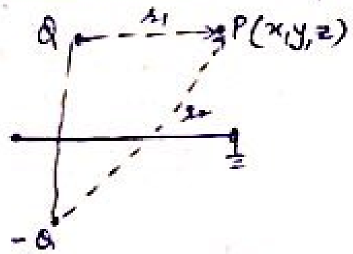


$\phi = 90^\circ \qquad N = \frac{360^\circ}{90^\circ} - 1 = 3$



For applying method of images, two conditions must always be satisfied
1) The image charge(s) must be located in the conducting region → to satisfy Poisson Equation
2) The image charge(s) must be located such that on the conducting surface(s) the potential is zero or constant. → this condition ensures that boundary conditions are satisfied.

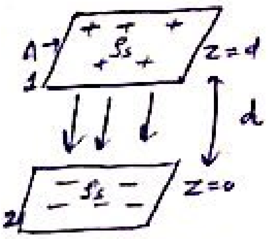
$\vec{E}$ and $\vec{V}$ due to point charge placed above grounded conducting plane



$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{a}_{r_1}}{r_1^2} + \frac{-Q}{4\pi\epsilon_0}\frac{\hat{a}_{r_2}}{r_2^2}$$
$$= \frac{Q}{4\pi\epsilon_0 r_1^3}\vec{r_1} - \frac{Q}{4\pi\epsilon_0 r_2^3}\vec{r_2}$$
$$r_1 = x\hat{a}_x + y\hat{a}_y + (z-h)\hat{a}_z$$
$$|r_1| = \sqrt{x^2+y^2+(z-h)^2} \qquad |r_2| = \sqrt{x^2+y^2+(z+h)^2}$$
$$V = \frac{Q}{4\pi\epsilon_0 r_1} - \frac{Q}{4\pi\epsilon_0 r_2} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

Capacitance: When two conductors of equal and opposite charges are placed at a distance they exhibit capacitance.

Parallel Plate Capacitor

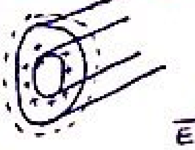


$$C = \frac{Q}{V_{21}}$$
$$Q = \int \rho_s ds = \rho_s \cdot A$$

$V_{21} = -\int_2^1 \vec{E}\cdot\vec{dl} \qquad \vec{E} = \frac{\rho_s}{2\epsilon_0}(\hat{a}_n)$

$$\vec{E} = \underbrace{\frac{\rho_s}{2\epsilon_0}(-\hat{a}_z)}_{\text{due to +ve sheet}} - \underbrace{\frac{\rho_s}{2\epsilon_0}(\hat{a}_z)}_{\text{due to -ve sheet}} = \frac{-\rho_s}{\epsilon_0}\hat{a}_z = -\frac{Q}{A\epsilon_0}\hat{a}_z$$
$$V_{21} = -\int_2^1 \vec{E}\cdot\vec{dl} = -\int_2^1 -\frac{Q}{A\epsilon_0}\hat{a}_z\cdot dz\,\hat{a}_z$$
$$= -\int_0^d -\frac{Q}{A\epsilon_0}dz = \frac{Q}{A\epsilon_0}[d-0] = \frac{Qd}{A\epsilon_0}$$
$$C = \frac{Q}{V_{21}} = \frac{Q}{\frac{Qd}{A\epsilon_0}} \qquad \boxed{C = \frac{A\epsilon_0}{d}}$$

Coaxial / Cylindrical Capacitor



$V_{21} = -\int_2^1 \vec{E}\cdot\vec{dl} \qquad \oint \vec{E}\cdot\vec{ds} = \frac{Q_{enclosed}}{\epsilon_0}$ ← gauss law

$$E\oint ds = \frac{Q}{\epsilon_0}$$
$$\vec{E}\cdot 2\pi r L = \frac{Q}{\epsilon_0} \qquad \vec{E} = \frac{Q}{2\pi\epsilon_0 r L}\hat{a}_r$$
$$V_{21} = -\int_2^1 \frac{Q}{2\pi\epsilon_0 r L}\hat{a}_r \cdot dr\,\hat{a}_r$$
$$= -\int_b^a \frac{Q}{2\pi\epsilon_0 r L}dr = \frac{-Q}{2\pi\epsilon_0 L}\ln[r]_a^b$$
$$= \frac{-Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$$
$$C = \frac{Q}{\frac{-Q}{2\pi\epsilon_0 L}\ln[b/a]} \qquad C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$$

Spherical capacitance

$\oint \vec{E}\cdot\vec{ds} = \frac{Q_{enclosed}}{\epsilon_0}$ $\qquad \oint E \cdot ds \cos 0^\circ = \frac{Q}{\epsilon_0}$ $\qquad E = \frac{Q}{4\pi\epsilon_0 r^2}$

$$V_{21} = -\int_2^1 \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r \cdot dr\,\hat{a}_r = -\int_b^a \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{-Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_b^a$$

$$V_{21} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right]$$

$$C = \frac{Q}{V_{21}} = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right]} \qquad \boxed{C = \frac{4\pi\epsilon_0}{\frac{1}{a}-\frac{1}{b}}}$$

If $B = \infty$ $\quad C = 4\pi\epsilon_0 a$.

Boundary Conditions

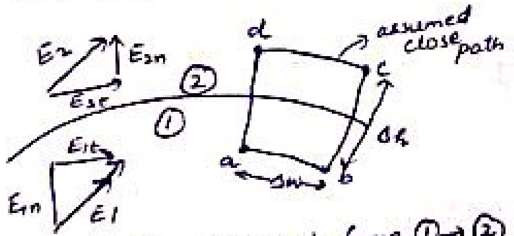
When $\vec{E}$ exists in a region consisting of two different media, the condition that the field must satisfy at the interface seperating the medium is called boundary condition.

To determine boundary conditions Maxwell's equation are used

1. $\oint \vec{E}\cdot\vec{dl} = 0$
2. $\oint_s \vec{D}\cdot\vec{ds} = 0$

Decompose $\vec{E}$ and $\vec{D}$ into normal and tangential components i.e. $\vec{E_n}$ and $\vec{E_t}$, $\vec{D_n}$ & $\vec{D_T}$.

Dielectric – Dielectric Boundary Condition



When $\vec{E}$ is directed from ① → ②

Apply $\oint \vec{E}\cdot\vec{dl} = 0$

$$= \int_a^b \vec{E}\cdot\vec{dl} + \int_b^c \vec{E}\cdot\vec{dl} + \int_c^d \vec{E}\cdot\vec{dl} + \int_d^a \vec{E}\cdot\vec{dl}$$

$$E_{1t}\Delta w + \cancel{E_{1n}\frac{\Delta h}{2}} + \cancel{E_{2n}\frac{\Delta h}{2}} - E_{2t}\Delta w - \cancel{E_{2n}\frac{\Delta h}{2}} - \cancel{E_{1n}\frac{\Delta h}{2}} = 0$$

$$(E_{1t} - E_{2t})\Delta w = 0$$

or $\boxed{E_{1t} = E_{2t}}$

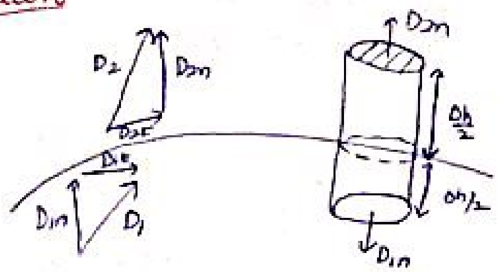
i.e tangential component of $\vec{E}$ i.e. $\vec{E_t}$ is said to be continuous (constant) across the boundary.

$D = \epsilon_0 E \quad \therefore D_{1t} = \epsilon E_{1t}$

$E_{1t} = \frac{D_{1t}}{\epsilon_1} \qquad E_{2t} = \frac{D_{2t}}{\epsilon_2}$

$$\boxed{\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}}$$

Tangential comp



$$\int_{top} \vec{D}\cdot\vec{ds} + \cancel{\int_{curved} \vec{D}\cdot\vec{ds}} + \int_{bottom} \vec{D}\cdot\vec{ds} = Q_{enclosed}$$

cancelled due to symmetry

$D_{2n}\Delta S - D_{1n}\Delta S = \rho_s \Delta S$

$$\boxed{D_{2n} - D_{1n} = \rho_s}$$

If the surface has no charge i.e. $\rho_s = 0$

$$\boxed{D_{1n} = D_{2n}}$$

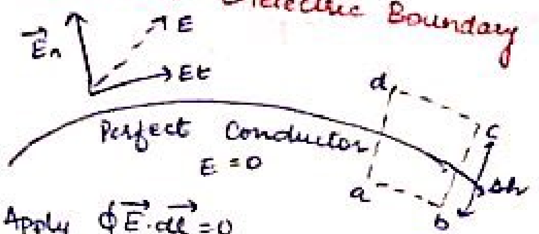
normal component of $\vec{D}$ is continuous across interface.

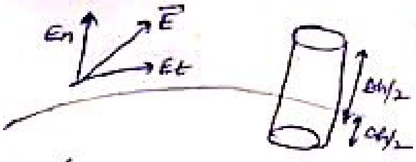
Since $D = \epsilon E$

$$\boxed{\epsilon \vec{E_{1n}} = \epsilon_2 \vec{E_{2n}}}$$

$\vec{E_n}$ is not continuous across interface.

Conductor - Dielectric Boundary condition





Apply $\oint \vec{E}\cdot\vec{dl} = 0$

$$\underbrace{0\cdot\Delta w}_{ab} + \underbrace{0\cdot\frac{\Delta h}{2} + E_n\cdot\frac{\Delta h}{2}}_{bc\ path} - \underbrace{E_t\,\Delta w}_{cd} - \underbrace{E_n\frac{\Delta h}{2} - 0\cdot\frac{\Delta h}{2}}_{da} = 0$$

$E_t\cdot\Delta w = 0$ or $\boxed{E_t = 0}$

$$\int_{top}\vec{D}\cdot ds + \int_{bottom}\vec{D}\cdot ds + \underbrace{\int_{surface}\vec{D}\cdot ds}_{\text{zero due to symmetry}} = Q_{enclosed}$$

$D_n\cdot\Delta s - 0\cdot\Delta s = \rho_s\cdot\Delta s$

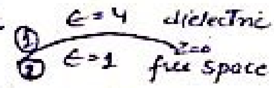
$\boxed{D_n = \rho_s}$

Que:- The region 1 ($z<0$) is free space while region 2 ($z>0$) is dielectric for which relative permittivity is equal to 4. Given that $D_1 = 2\hat{a}_x - 3\hat{a}_y + 4\hat{a}_z$ C/m². Find E_2 and angles θ_1 and θ_2. (Sessional 2020)

Solⁿ: $D_1 = 2\hat{a}_x - 3\hat{a}_y + 4\hat{a}_z$

Boundary condition for dielectric-dielectric medium are

$E_{1t} = E_{2t}$ and $D_{1n} = D_{2n}$



$D_{1n} = 4\hat{a}_z = D_{2n}$ (since $z=0$ seperates both the spaces. So $\hat{a}_n$ is in z-direction)

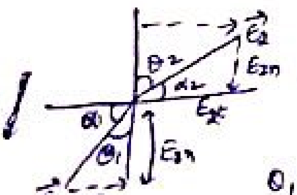
$D_{1t} = 2\hat{a}_x - 3\hat{a}_y$

$E_{1t} = E_{2t}$

$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$ $\qquad D_{2t} = \frac{\epsilon_2}{\epsilon_1}D_{1t} = 4(2\hat{a}_x - 3\hat{a}_y) = 8\hat{a}_x - 12\hat{a}_y$

$\vec{D}_2 = \vec{D}_{2t} + \vec{D}_{2n} = 8\hat{a}_x - 12\hat{a}_y + 4\hat{a}_z$

$\vec{E}_2 = \frac{\vec{D}_2}{\epsilon_0} = \frac{1}{\frac{10^{-9}}{36\pi}\times 4}(8,-12,4) \Rightarrow \boxed{\vec{E}_2 = 226.08\,\hat{a}_x - 339.12\,\hat{a}_y + 113.76\,\hat{a}_z}$



$\alpha_1 = 90 - \theta_1$
$\alpha_2 = 90 - \theta_2$

$\tan\alpha_1 = \frac{D_{1n}}{D_{1t}}$ or $\frac{E_{1n}}{E_{1t}} = \frac{4}{\sqrt{2^2+3^2}} = 1.11 \rightarrow \alpha = 47.98°$

$\tan\alpha_2 = \left|\frac{D_{2n}}{D_{2t}}\right| = \frac{4}{\sqrt{8^2+12^2}} = 0.27 \rightarrow \alpha = 15.10°$

$\theta_1 = 90° - 47.98° = 42.02°$
$\theta_2 = 90° - 15.10° = 74.9°$

$$\boxed{\frac{\tan\theta_2}{\tan\theta_1} = \frac{\tan 74.9°}{\tan 42.02°} = \frac{3.706}{0.9010} = 4.11 \approx \frac{\epsilon_{r1}}{\epsilon_{r1}}}$$

For verification only.

Que: The region ($x<0$) is free space while region ($x>0$) is a dielectric for which the relative permittivity is 2.5. Find E_2 and angle θ_1 and θ_2. Given that $D_1 = 3\hat{a}_x - 4\hat{a}_y + 6\hat{a}_z$.

Solⁿ $D_1 = 3\hat{a}_x - 4\hat{a}_y + 6\hat{a}_z$

The boundary is along $\hat{a}_x$ and boundary conditions are

$E_{1t} = E_{2t}$ and $D_{2n} = D_{1n}$

$D_{1n} = 3\hat{a}_x = D_{2n}$

$D_{1t} = -4\hat{a}_y + 6\hat{a}_z$

$E_{1t} = E_{2t}$ or $\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$ or $D_{2t} = \frac{\epsilon_2}{\epsilon_1} D_{1t}$

$D_{2t} = 2.5(-4\hat{a}_y + 6\hat{a}_z) = |-10\hat{a}_y + 15\hat{a}_z|$

$D_2 = 3\hat{a}_x - 10\hat{a}_y + 15\hat{a}_z$

$\vec{E_2} = \frac{\vec{D_2}}{\epsilon_0} = \frac{1}{\frac{10^{-9}}{36\pi} \times 2.5}(3, -10, 15) = 45.216 \times 10^{-9}(3, -10, 15)$

$\boxed{\vec{E_2} = 135.648\,\hat{a}_x - 452.16\,\hat{a}_y + 678.24\,\hat{a}_z\ \text{c/m}^2}$

$\tan\alpha_1 = \frac{3}{\sqrt{4^2+6^2}} = \frac{3}{\sqrt{36+16}} = \frac{3}{7.21} = 0.416$ $\boxed{\alpha_1 = 22.58}$ $\theta_1 = 90° - 22.58° = 67.42°$

$\tan\alpha_2 = \frac{3}{\sqrt{10^2+15^2}} = \frac{3}{\sqrt{325}} = \frac{3}{18.02} = 0.166$ $\boxed{\alpha_2 = 9.42}$ $\theta_2 = 90° - 9.42° = 80.58°$

$\boxed{\frac{\tan\theta_2}{\tan\theta_1} = \frac{\tan 80.58°}{\tan 67.42°} = \frac{6.027}{2.4047} = 2.50 \rightarrow \text{Ratio of } \frac{\epsilon_2}{\epsilon_1}}$ → for verification purpose only.

Que:- There exists a boundary b/w two dielectric medium at $z=0$, $\epsilon_{r_1} = 2.5$ in region 1, $z<0$ and $\epsilon_{r_2} = 4$ in region 2, $z>0$. If the field in the region 1 is $E_1 = -30\,\hat{i}_x + 50\,\hat{i}_y + 70\,\hat{i}_z$ V/m. Find (ET 2018, 6M)

(i) Normal and tangential components of E_1

(ii) Normal and tangential components of D_2

(iii) Angle of incidence & angle of refraction. $(\theta_1 \& \theta_2)$

Sol^n: $E_1 = -30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z$

The boundary is defined at $z=0$ so $\hat{a}_n = \hat{a}_z$

Acc. to Boundary conditions

$E_{1t} = E_{2t}$

and $D_{1n} = D_{2n}$

$E_{1n} = 70\hat{a}_z$

$E_{1t} = -30\hat{a}_x + 50\hat{a}_y = E_{2t}$

$D_{1n} = D_{2n}$

$\epsilon_1 E_{1n} = E_{2n}\epsilon_2$

$E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{2.5}{4}(70\hat{a}_z)$ $\vec{E_{2n}} = 43.75\hat{a}_z$

$\boxed{\vec{E_2} = -30\hat{a}_x + 50\hat{a}_y + 43.75\hat{a}_z}$

$\vec{D_2} = \epsilon_2\vec{E_2} = \epsilon_0\epsilon_{r_2}\vec{E_2} = \frac{10^{-9}}{36\pi} \times 4 \times (-30, 50, 43.75)$

$\boxed{\vec{D_2} = -1.061\hat{a}_x + 1.768\hat{a}_y + 1.547\hat{a}_z\ \text{nC/m}^2}$

$\tan\alpha_1 = \frac{E_{1n}}{E_{1t}} = \frac{70}{\sqrt{(30)^2+50^2}} = \frac{70}{\sqrt{900+2500}} = \frac{70}{65.57} = 1.067 =$ $\boxed{\alpha_1 = 46.85°} \rightarrow \theta_1 = 43.15°$

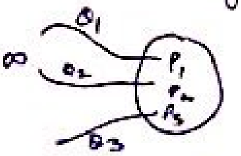
$\tan\alpha_2 = \frac{E_{2n}}{E_{2t}} = \frac{43.75}{\sqrt{30^2+50^2}} = \frac{43.75}{65.57} = 0.667$ $\boxed{\alpha_2 = 33.70} \rightarrow \theta_2 = 56.3$

$\boxed{\frac{\tan\theta_2}{\tan\theta_1} = \frac{\tan 56.3}{\tan 43.15} = \frac{1.499}{0.9374} = 1.59 \rightarrow \text{Ratio of } \frac{\epsilon_2}{\epsilon_1}}$

$\frac{\epsilon_2}{\epsilon_1} = \frac{4}{2.5} = 1.6$

Electrostatic Energy / Energy density in Electrostatics [ET 2019, 6.5 M] (16)

To determine the energy present in an assembly of charges, determine the amount of work done to assemble them.



No work is done to transfer Q_1 from ∞ to P_1 because the space is initially charge free & there is no electric field.

Work done in moving Q_2 from ∞ to P_2 is equal to product of Q_2 and potential V_{21} due to Q_1

Illy $W_3 = Q_3(V_{32} + V_{31})$

$$W_E = W_1 + W_2 + W_3 = 0 + Q_2 V_{21} + Q_3(V_{31} + V_{32}) \quad -①$$

If the charges are positioned in reverse order i.e. $Q_3 \to Q_2 \to Q_1$

so $W_E = W_3 + W_2 + W_1 = 0 + Q_2 V_{23} + Q_1(V_{12} + V_{13}) \quad -②$

By ①+② $\quad 2W_E = Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32})$

$= Q_1 V_1 + Q_2 V_2 + Q_3 V_3$ where V_1, V_2 and V_3 are total potential at P_1, P_2 and P_3 respectively.

$W_E = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3)$.

for n-charges $\quad W_E = \frac{1}{2}\sum_{K=1}^{n} Q_K V_K$ in Joules $\qquad Q = \int_v \rho_v dv$

or $W_E = \frac{1}{2}\int_v V \cdot \rho_v dV$ for continuous charge distribution

Acc. to Gauss law $\nabla \cdot D = \rho_v$

$$\boxed{W_E = \frac{1}{2}\int_v V(\nabla \cdot \vec{D})dV}$$

Using vector identity $\quad \nabla \cdot (VD) = V\underbrace{(\nabla \cdot \vec{D})}_{\text{div of } D} - \vec{D}\underbrace{(\nabla V)}_{\text{grad of } V}$

or $V \cdot (\nabla D) = \nabla \cdot (V\vec{D}) - \vec{D}(\nabla V)$

$$W_E = \frac{1}{2}\int_v \{\nabla \cdot (V\vec{D}) - \vec{D}(\nabla V\} dV$$

$$= \frac{1}{2}\int_v \nabla \cdot (V\vec{D})dV - \frac{1}{2}\int \vec{D} \cdot (\nabla V)dV$$

divergence theorem

$$\frac{1}{2}\oint_s \underbrace{V\vec{D}}_{0} \cdot ds - \frac{1}{2}\int_v \vec{D} \cdot (\nabla V)dV$$

$\left.\begin{array}{l} V \propto \frac{1}{R} \\ D \propto \frac{1}{R^2} \end{array}\right\} VD \propto \frac{1}{R^3}$ and $dS \propto R^2$

$\oint V\vec{D} ds \propto \frac{R^2}{R^3} = \frac{1}{R} = \frac{1}{\infty} = 0$

$$W_E = -\frac{1}{2}\int \vec{D}(\nabla V)dV$$

$E = -\nabla V$

$$W_E = \frac{1}{2}\int \vec{D}\vec{E}dV$$

$$\boxed{\begin{array}{l} W_E = \frac{1}{2}\int \epsilon_0 \vec{E}^2 dV \\ \text{or } W_E = \frac{1}{2}\int \frac{D^2}{\epsilon_0} dV \end{array}}$$