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SOLVED PAPERS

[PREVIOUS YEARS SOLVED QUESTION PAPERS]

[B.Tech]

FIRST SEMESTER

Applied Mathematics-I

(ETMA-101)

Rs.63.00/-

AKASH BOOKS
NEW DELHI

FIRST TERM EXAMINATION [SEPT. 2017]
FIRST SEMESTER [B.TECH]
APPLIED MATHEMATICS–I [ETMA–101]

Time : 1.5 hrs. M.M : 30

Note: *Question No. 1 is compulsory. Attempt any two questions from rest.*

Q.1. (a) Find the nth differential coefficient of $\frac{1}{x^2+5x+6}+\sin x \sin 2x$. **(2.5)**

Ans. Let $y=\frac{1}{x^2+5x+6}+\sin x\sin 2x$

$$\Rightarrow y=\frac{1}{(x+2)(x+3)}+\frac{1}{2}(2\sin x\sin 2x)$$

$$\Rightarrow y=\frac{1}{x+2}-\frac{1}{x+3}+\frac{1}{2}(\cos x-\cos 3x)$$

Taking n times derivative

$$\Rightarrow y_n=\frac{d^n}{dx^n}\left(\frac{1}{x+2}\right)-\frac{d^n}{dx^n}\left(\frac{1}{x+3}\right)+\frac{1}{2}\frac{d^n}{dx^n}(\cos x-\cos 3x)$$

$$\Rightarrow y_n=\frac{(-1)^n n!}{(x+2)^{n+1}}-\frac{(-1)^n n!}{(x+3)^{n+1}}+\frac{1}{2}\left[\cos\left(x+\frac{n\pi}{2}\right)-3^n\cos\left(3x+\frac{n\pi}{2}\right)\right]$$

Q.1. (b) State Maclaurin's theorem with Lagrange's form of remainder for $f(x)=\cos x$. **(2.5)**

Ans.

$$f^n(x)=\frac{d^n}{dx^n}(\cos x)$$

$$=\cos\left(\frac{n\pi}{2}+x\right) \text{ so that}$$

$$f^n(0)=\cos\left(\frac{n\pi}{2}\right)$$

Thus $f(0)=1$

$$f^{2n}(0)=\cos\left(\frac{2n\pi}{2}\right)=(-1)n$$

$$f^{2n+1}(0)=\cos\left[\frac{(2n+1)\pi}{2}\right]=0$$

Substituting these values in Maclaurin's theorem with Lagrange's form of Remainder

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots\frac{x^{2n}}{(2n)!}f^{2n}(0)+\frac{x^{2n+1}}{(2n+1)!}f^{2n+1}(\theta x)$$

We get

$$\cos x=1+0+\frac{x^2}{2!}(-1)+0+\ldots+\frac{x^{2n}}{(2n)!}(-1)^n+\frac{x^{2n+1}}{(2n+1)!}(-1)^n(-1)\cos(\theta x)$$

i.e.,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} \cos(\theta x)$$

Q.1. (c) Test the convergence of the series $\sum \frac{(n - \log n)^n}{2^n \times n^n}$ (2.5)

Ans. Let $\sum U_n = \frac{(n - \log n)^n}{2^n \cdot n^n}$

$$\text{Let } U_n = \frac{(n - \log n)^n}{n^n \cdot 2^n}$$

$$\text{Consider } (U_n)^{1/n} = \frac{n - \log n}{n.2}$$

$$\Rightarrow \quad (U_n)^{1/n} = \frac{n\left(1 - \frac{\log n}{n}\right)}{n \cdot 2}$$

$$\Rightarrow \quad (U_n)^{1/n} = \frac{1 - \frac{\log n}{n}}{2}$$

$$\lim_{n \to \infty} (U_n)^{1/n} = \lim_{n \to \infty} \frac{\left(1 - \frac{\log n}{n}\right)}{2} = \frac{1}{2} < 1$$

$\therefore$ By Cauchy's nth root test ΣU_n is convergent.

Q.1. (d) Find the rank of the following matrices $\begin{bmatrix} 1 & 2 & 32 \\ 2 & 3 & 51 \\ 1 & 3 & 45 \end{bmatrix}$ (2.5)

Ans. Let

$$A = \begin{bmatrix} 1 & 2 & 32 \\ 2 & 3 & 51 \\ 1 & 3 & 45 \end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 32 \\ 0 & -1 & -13 \\ 0 & 1 & 13 \end{bmatrix}$$

Operating $R_3 \to R_3 + R_2 \sim \begin{bmatrix} 1 & 2 & 32 \\ 0 & -1 & -13 \\ 0 & 0 & 0 \end{bmatrix}$

2×2 minor is $\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$

$\therefore r(A) = 2.$

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Q.2. (a) If $y^{1/m} + y^{-1/m} = 2x$, show that $(x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0.$

Ans. Let $w = y^{1/m}$

$$\Rightarrow \quad w + \frac{1}{w} = 2x \Rightarrow w^2 - 2xw + 1 = 0$$

$$\therefore \quad w = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

Case (i) $w = x + \sqrt{x^2-1}$

$\Rightarrow y^{1/m} = x + \sqrt{x^2-1}$

$\Rightarrow y = (x+\sqrt{x^2-1})^m$ ...(i)

Differentiating once w.r.t. 'x'.

$$\Rightarrow \quad y_1 = m\left[x+\sqrt{x^2-1}\right]^{m-1}\left(1+\frac{2x}{2\sqrt{x^2-1}}\right)$$

$$\Rightarrow \quad y_1 = m\left[x+\sqrt{x^2-1}\right]^{m-1}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)$$

$$\Rightarrow \quad y_1 = \frac{my}{\sqrt{x^2-1}}$$

$$\Rightarrow \quad y_1^2(x^2-1) = m^2y^2$$

Again differentiating w.r.t 'x'

$$\Rightarrow \quad 2y_1y_2(x^2-1) + 2xy_1^2 = 2m^2yy_1$$

$$\Rightarrow \quad y_2(x^2-1) + xy_1 - m^2y = 0$$

Applying Leibnitz theorem, we get

$y_{n+2}(x^2-1) + {}^nc_1 y_{n+1}\cdot 2x + {}^nc_2 y_n.2 + y_{n+1}.x + nc_1y_n - m^2y_n = 0$

$\Rightarrow y_{n+2}(x^2-1) + 2nx\,y_{n+1} + n\,(n-1)\,y_n + x\,y_{n+1} + ny_n - m^2yn = 0$

$\Rightarrow y_{n+2}(x^2-1) + (2n+1)x\,y_{n+1} + (n^2-m^2)\,y_n = 0$

Case (ii) $\omega = x - \sqrt{x^2-1}$

$$\Rightarrow \quad y = \left(x-\sqrt{x^2-1}\right)^m$$

Similarly
$$y_1 = m\left[x-\sqrt{x^2-1}\right]^{m-1}\left(1-\frac{2x}{2\sqrt{x^2-1}}\right)$$

$$\Rightarrow \quad y_1 = \frac{-my}{\sqrt{x^2-1}}$$

$$\Rightarrow \quad y_1^2(x^2-1) = m^2y^2$$

$$\Rightarrow \quad 2y_1y_2(x^2-1) + 2xy_1^2 = 2m^2yy_1$$

$$y_2(x^2-1) + xy_1 = m^2y.$$

Differentiating 'n' times using Leibnitz theorem

$y_{n+2}(x^2-1) + n\,y_{n+1}\,2x + n(n-1)\,y_n + xy_{n+1} + n\cdot y_n = m^2y_n$

$\Rightarrow (x^2-1)\,y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-m^2)\,y_n = 0.$

Q.2. (b) Discuss the convergence of the following series (5)

$$1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \frac{3.6.9.12}{7.10.13.16}x^4 + \ldots..$$

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Ans. Neglecting first term

$$\text{Let } U_n = \frac{3\cdot 6\cdot 9.....3n}{7\cdot 10\cdot 13.....(3n+4)}x^n$$

$$\text{and } U_{n+1} = \frac{3\cdot 6\cdot 9.....(3n+3)}{7\cdot 10\cdot 13.....(3n+7)}x^{n+1}$$

$$\text{Consider } \frac{U_n}{U_{n+1}} = \frac{3\cdot 6\cdot 9.....(3n)}{7\cdot 10\cdot 13.....(3n+4)} \times \frac{7.10.13.....(3n+7)}{3.6.9.....(3n+3)}\frac{x^n}{x^{n+1}}$$

$$= \frac{3n+7}{3n+3}\cdot\frac{1}{x}$$

$$= \frac{n(3+7/n)}{n(3+3/n)}\cdot\frac{1}{x} = \left(\frac{3+7/n}{3+3/n}\right)\cdot\frac{1}{x}$$

$$\Rightarrow \lim_{n\to\infty} = \lim_{n\to\infty}\frac{3+7/n}{3+3/n}\cdot\frac{1}{x} = \frac{1}{x}$$

$\therefore$ By Ratio test ΣU_n is convergent if $\frac{1}{x} > 1$ *i.e.*, if $x < 1$.

and divergent if $\frac{1}{x} < 1$ *i.e.*, if $x > 1$

Test fails for $x = 1$

for $x = 1$

$$\frac{U_n}{U_{n+1}} - 1 = \frac{3n+7}{3n+3} - 1$$

$$= \frac{3n+7-3n-3}{3n+3}$$

$$\Rightarrow n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{4n}{n\left(3+\frac{3}{n}\right)}$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{4}{3} > 1$$

$\therefore$ By Raabe's test ΣU_n is Convergent.

Thus the series ΣU_n is convergent if $x \le 1$ and divergent if $x > 1$.

Q.3. (a) If $f(x) = x^3 + 2x^2 - 5x + 11$, calculate the value of $f\left(\frac{9}{10}\right)$ by using Taylor's series expansion.

Ans. $f(x) = x^3 + 2x^2 - 5x + 11$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ...$$

Put $x = 1, h = \frac{-1}{10}$

$$\Rightarrow f\left(\frac{9}{10}\right) = f(1) - \frac{1}{10}f'(1) + \frac{1}{2!}\left(\frac{1}{10}\right)^2 f''(1) - \frac{1}{3!}\left(\frac{1}{10}\right)^3 f'''(1) + ...$$

Let $f(x) = x^3 + 2x^2 - 5x + 11$

$f(1) = 1 + 2 - 5 + 11 = 9$

$f'(x) = 3x^2 + 4x - 5,$ $\qquad f'(1) = 3 + 4 - 5 = 2$

$f''(x) = 6x + 4,$ $\qquad f''(1) = 6 + 4 = 10$

$f'''(x) = 6,$ $\qquad f'''(1) = 6$

By equation (*i*), we get

$$f\left(\frac{9}{10}\right) = 9 - \frac{1}{10}\times 2 + \frac{1}{2}\left(\frac{1}{10}\right)^2 \times 10 - \frac{1}{6}\times\left(\frac{1}{6}\right)^3 \times 6$$

$$= 9 - \frac{2}{10} + \frac{1}{20} - \frac{1}{1000}$$

$$= 9 - 0.2 + 0.5 - 0.001$$

$$= 8.849.$$

Q.3. (*b*) For what value of *x* is the following series convergent

$x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \frac{x^5}{\sqrt{5}} \ldots\ldots$ **(5)**

Ans. Given series is $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{\sqrt{n}}$

For Absolute Convergence

Consider $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{x^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{|x|^n}{\sqrt{n}}$

Let $U_n = \frac{|x|^n}{\sqrt{n}}$ and $U_{n+1} = \frac{|x|^{n+1}}{\sqrt{n+1}}$

Consider $\left| \frac{U_n}{U_{n+1}} \right| = \frac{|x|^n}{\sqrt{n}} \times \frac{\sqrt{n+1}}{|x|^{n+1}}$

$$= \frac{\sqrt{n}\sqrt{1+\frac{1}{n}}}{\sqrt{n}\,|x|} = \frac{\sqrt{1+\frac{1}{n}}}{|x|}$$

$$\lim_{n\to\infty}\left|\frac{U_n}{U_{n+1}}\right| = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}}{|x|} = \frac{1}{|x|}$$

$\therefore$ By Ratio test series is convergent if $\frac{1}{|x|} > 1$ *i.e.*, $|x| < 1$ and divergent if $\frac{1}{|x|} < 1$ *i.e.*, $|x| > 1$

Thus series is absolutely convergent if $|x| < 1$.

Test fails for $|x| = 1$ or $x = 1$ or -1.

When $x = 1$. Series becomes $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \ldots\ldots$

Here $U_n = \frac{1}{\sqrt{n}}, U_{n+1} = \frac{1}{\sqrt{n+1}}$

(i) As $n < n + 1 \Rightarrow \sqrt{n} < \sqrt{n+1}$

$\Rightarrow \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$

$\Rightarrow U_n > U_{n+11}$

$\Rightarrow \langle U_n \rangle$ is monotonically decreasing.

(ii) $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\therefore$ By Leibnitz test $\Sigma(-1)^{n-1} U_n$ is Convergent.

When $x = -1$ Series becomes $-1 + \frac{1}{\sqrt{2}} \frac{-1}{\sqrt{3}} + \ldots..$

i.e., $\sum_{n=1}^{\infty} (-1)^n U_n$

Here $U_n = \frac{1}{\sqrt{n}}$

Similarly $U_{n+1} = \frac{1}{\sqrt{n+1}}$

It is monotonically decreasing and $\lim_{n\to 0} U_n = 0$

This is also convergent by Leibnitz test.

This series is convergent if $|x| \le 1$ or $-1 \le x \le 1$.

Q.4. (a) Use Guass–Jordan method to find the inverse of the following matrice.

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Ans. Let $A = IA$

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

operate $R_2 \to R_2 + 2R_1, R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 3 & 6 \\ 0 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

operate $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} A$$

operate $R_2 \to \frac{R_2}{3}, R_3 \to \frac{R_3}{-9}$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1/3 & 0 \\ 2/9 & 2/9 & -1/9 \end{bmatrix} A$$

Operate $R_1 \to R_1 - 2R_2$

$$\sim\begin{bmatrix}-1 & 0 & -2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-1/3 & -2/3 & 0\\ 2/3 & 1/3 & 0\\ 2/9 & 2/9 & -1/9\end{bmatrix}A$$

Operate $R_1 \to R_1 + 2R_3, R_2 \to R_2 - 2R_3$

$$\sim\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1/9 & -2/9 & -2/9\\ 2/9 & -1/9 & 2/9\\ 2/9 & 2/9 & -1/9\end{bmatrix}A$$

Operate $R_1 \to -R_1$

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-1/9 & 2/9 & 2/9\\ 2/9 & -1/9 & 2/9\\ 2/9 & 2/9 & -1/9\end{bmatrix}A$$

$$\Rightarrow \qquad I = A^{-1}A$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix}-1/9 & 2/9 & 2/9\\ 2/9 & -1/9 & 2/9\\ 2/9 & 2/9 & -1/9\end{bmatrix}$$

Q.4. (*b*) For what value of *K* does the following system of equations $x + y + z = 1, x + 2y + 4z = K, x + 4y + 10z = K^2$ have a solution and solve them completely in each case. **(5)**

Ans. Given $x + y + z = 1$

$x + 2y + 4z = K$

$x + 4y + 10z = K^2$

In matrix form $AX = B$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 4 & 10\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ K\\ K^2\end{bmatrix}$$

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 \to R_1$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 0 & 3 & 9\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ K-1\\ K^2-1\end{bmatrix}$$

Operating $R_3 \to R_3 - 3R_2$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ K-1\\ K^2-3K+2\end{bmatrix} \qquad ...(i)$$

$\Rightarrow r(A) = 2$

and

$$r(A:B) = \begin{bmatrix}1 & 1 & 1 & : & 1\\ 0 & 1 & 3 & : & K-1\\ 0 & 0 & 0 & : & K^2-3K+2\end{bmatrix}$$

System is consistent iff $r(A) = r(A : B)$

$\Rightarrow K^2 - 3K + 2 = 0$

$\Rightarrow K = 1, 2$

for $K = 1, 2, r(A) = r(A : B) <$ no. of unknowns

$\therefore$ system has infinite many solutions.

For $K = 1$, equation (i) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + y + z &= 1 \\ y + 3z &= 0 \end{aligned}$$

$$\Rightarrow \quad y = -3z$$

$$\Rightarrow \quad x = 1 + 2z$$

Let $z = t$

$\therefore x = 1 + 2t, y = -3t$ and $z = t$

For $K = 2$, equation (i) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow x + y + z = 1$

$y + 3z = 1$

Let $z = t \Rightarrow y = 1 - 3t$

$\Rightarrow x + 1 - 3t + t = 1$

$\Rightarrow x = 2t$

Thus $x = 2t, y = 1 - 3t, z = t$.

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END TERM EXAMINATION [DEC. 2017]
FIRST SEMESTER [B.TECH]
APPLIED MATHEMATICS–I [ETMA–101]

Time : 3 hrs. **M.M : 75**

Note: Attempt any five question including Q.No.1 which is compulsory. Select one question from each unit.

Q.1. (a) Assuming the possibility of expansion prove that $\sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{\sqrt{2}}\left[1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\ldots\right]$ **(2.5)**

Ans. Let $f(x+\theta) = \sin(x+\theta)$

Putting $\theta = 0$, $f(x) = \sin x$

$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$

$\therefore \sin(x+\theta) = f(x+\theta)$

$$= f(x)+\theta f'(x)+\frac{\theta^2}{2!}f''(x)+\frac{\theta^3}{3!}f'''(x)\ldots$$

$$= \sin x+\theta\cos x-\frac{\theta^2}{2!}\sin x-\frac{\theta^3}{3!}\cos x+\ldots$$

Putting $x = \frac{\pi}{4}$, we get

$$f\left(\frac{\pi}{4}+\theta\right) = \sin\left(\frac{\pi}{4}+\theta\right) = \sin\frac{\pi}{4}+\theta\cos\frac{\pi}{4}-\frac{\theta^2}{2!}\sin\frac{\pi}{4}-\frac{\theta^3}{3!}\cos\frac{\pi}{4}+\ldots..$$

$$= \frac{1}{\sqrt{2}}\left(1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\ldots\right)$$

$$\left[\because \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

Q.1. (b) Test for convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$ **(4)**

Ans.

$$U_n = \frac{n^2}{2^n}+\frac{1}{n^2} = (W_n+T_n)\text{ (say)}$$

$$\text{Let } W_n = \frac{n^2}{2^n},\ W_{n+1} = \frac{(n+1)^2}{2^{n+1}}$$

$$\text{Consider } \frac{W_n}{W_{n+1}} = \frac{n^2}{2^n}\cdot\frac{2^{n+1}}{(n+1)^2} = \frac{n^2}{2^n}\cdot\frac{2^n\cdot 2}{n^2\left(1+\frac{1}{n}\right)^2}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{W_n}{W_{n+1}} = \lim_{n\to\infty}\frac{2}{\left(1+\frac{1}{n}\right)^2} = 2 > 1$$

$\therefore$ By Ratio Test ΣW_n Converges

Let $T_n = \frac{1}{n^2}$ $\qquad (p = 2 > 1)$

$\therefore \Sigma T_n$ Converges by p-test

$\Rightarrow \Sigma U_n = \Sigma(W_n + T_n)$ is convergent.

Q.1. (C) Find the asymptotes of the curve $y^3 - 2xy^2 - x^2y + 2x^3 + 2x^2 - 3xy + x - 2y + 1 = 0$.

Ans. $y^3 - 2xy^2 - x^2y + 2x^3 + 2x^2 - 3xy + x - 2y + 1 = 0$

Since Coefficient of x^3 and y^3 are constants

$\therefore$ no parallel asymptotes to x-axis or y-axis

Oblique Asymptote

$\phi_3(x,y) = y^3 - 2xy^2 - x^2y + 2x^3$

$\phi_2(x,y) = 2x^2 - 3xy$

$\phi_1(x,y) = x - 2y, \qquad \phi_0(x,y) = 1$

Put $x = 1, y = m$

$\phi_3(m) = m^3 - 2m^2 - m + 2, \phi_2(m) = 2 - 3m$

Value of m $\phi_3(m) = 0$

$\Rightarrow m^3 - 2m^2 - m + 2 = 0$

$\Rightarrow (m-1)(m^2 - m - 2) = 0$

$\Rightarrow (m-1)(m+1)(m-2) = 0$

$\Rightarrow m = 1, -1, 2$

$$C = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-(2-3m)}{3m^2-4m-1}$$

For $m = 1$

$$C = \frac{-(2-3)}{3-4-1} = \frac{1}{-2} = \frac{-1}{2}$$

For $m = -1$

$$C = \frac{-(2-3)}{3+4-1} = \frac{-5}{6}$$

For $m = 2$

$$C = \frac{-(2-6)}{12-8-1} = \frac{4}{3}$$

$\therefore$ Asymptotes are $y = mx + c$

$y = x - \frac{1}{2}, y = -x - \frac{5}{6}, y = 2x + \frac{4}{3}$

$\Rightarrow$ $2y - 2x + 1 = 0, 6y + 6x + 5 = 0, 3y - 6x - 4 = 0$

Q.1. (d) Show that $\int_0^1 (x \log x)^3 dx = \frac{-3}{128}$ **(2.5)**

Ans.

$$\text{Let } I_{m,n} = \int_0^1 x^m (\log x)^n dx$$

$$I_{m,n} = \left[\frac{x^{m+1}}{m+1}(\log x)^n\right]_0^1 - \int_0^1 n\frac{(\log x)^{n-1}}{x}\cdot\frac{x^{m+1}}{m+1}dx$$

$\Rightarrow$
$$I_{m,n} = -\frac{n}{m+1}\int_0^1 x^m (\log x)^{n-1} dx$$

$\Rightarrow$
$$I_{m,n} = -\frac{n}{m+1} I_{m,n-1} \quad ...(i)$$

Let
$$n = 3, m = 3$$

$\Rightarrow$
$$I_{3,3} = -\frac{3}{4} I_{3,2}$$

$$I_{3,2} = -\frac{2}{4} I_{3,1}$$

$$I_{3,1} = -\frac{1}{4} I_{3,0}$$

$$\text{Consider } I_{3,0} = \int_0^1 x^3 dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$\Rightarrow$
$$I_{3,1} = \frac{-1}{4}\times\frac{1}{4} = \frac{-1}{16}$$

$\Rightarrow$
$$I_{3,2} = \frac{-2}{4}\times\left(\frac{-1}{16}\right) = \frac{1}{32}$$

$\Rightarrow$
$$I_{3,3} = -\frac{3}{4}\times\frac{1}{32} = \frac{-3}{128}$$

$\therefore$
$$\int_0^1 x^3 (\log x)^3 dx = -\frac{3}{128}$$

Q.1. (e) Evaluate $\int_0^1 (1 - x^{1/n})^{m-1} dx$ **(2.5)**

Ans.
$$\int_0^1 (1 - x^{1/n})^{m-1} dx$$

Let $x^{1/n} = t$

$\Rightarrow$ $x = t^n$

$\Rightarrow$ $dx = nt^{n-1}\,dt$

Consider $\int_0^1 (1-x^{1/n})^{m-1}dx = \int_0^1 (1-t)^{m-1}.nt^{n-1}dt$

$$= n\int_0^1 (1-t)^{m-1}\cdot t^{n-1}dt$$

$$= n\,\beta(n,m)$$

$$= n\,\beta(m,n)$$

Q.1. (f) Show that the matrix $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

Ans.
$$A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

To show $AA^\theta = A^\theta A = I$

$$A^\theta = (\overline{A})'$$

$$\text{Now } \overline{A} = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$$

$$A^\theta = (\overline{A})' = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

Now,
$$AA^\theta = \frac{1}{4}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (1+i)(1-i)+(-1+i)(-1-i) & (1+i)(1-i)+(-1+i)(1+i) \\ (1+i)(1-i)+(1-i)(-1-i) & (1+i)(1-i)+(1-i)(1+i) \end{bmatrix}$$

$$AA^\theta = \frac{1}{4}\begin{bmatrix} (1+1)+(1+1) & (1+1)+(-1-1) \\ (1+1)+(-1-1) & (1+1)+(1+1) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^\theta A = \frac{1}{4}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$A^\theta A = \frac{1}{4}\begin{bmatrix} (1-i)(1+i)+(1-i)(1+i) & (1-i)(-1+i)+(1-i)^2 \\ (-1-i)(1+i)+(1+i)^2 & (-1-i)(-1+i)+(1+i)(1-i) \end{bmatrix}$$

$$A^\theta A = \frac{1}{4}\begin{bmatrix} 2+2 & 2i-2i \\ -2i+2i & 2+2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore A^{\theta} A = I = AA^{\theta}$

Thus A is unitary matrix.

Q. 1. (g) Test whether the vectors (1, 1, 1, 3), (1, 2, 3, 4) and (2, 3, 4, 9) are linearly dependent or not. If depentent, find the relations between them. (2.5)

Ans. Consider

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = (0, 0, 0, 0) \quad ...(i)$$

$$\Rightarrow \lambda_1(1, 1, 1, 3) + \lambda_2(1, 2, 3, 4) + \lambda_3(2, 3, 4, 9) = (0, 0, 0, 0)$$

$$\Rightarrow \left.\begin{aligned} \lambda_1 + \lambda_2 + 2\lambda_3 &= 0 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 &= 0 \\ \lambda_1 + 3\lambda_2 + 4\lambda_3 &= 0 \\ 3\lambda_1 + 4\lambda_2 + 9\lambda_3 &= 0 \end{aligned}\right\} \quad ...(ii)$$

This is a homogeneous system of equation

Matrix form is AX = B

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 9 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2, R_3 \to \frac{1}{2}R_3$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $r(A) = 3$ = no. of unknowns

$\therefore$ System has zero or trivial solution.

Thus all scalars $\lambda_1, \lambda_2, \lambda_3$ are zero

$\therefore x_1, x_2, x_3$ are linearly independent

Also $\therefore \lambda_1 + \lambda_2 + 2\lambda_3 = 0$

$\lambda_2 + \lambda_3 = 0$

$2\lambda_3 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$

Q.1. (h) Solve $(x^4 e^x - 2mxy^2)\, dx + 2mx^2 y\, dy = 0$ (2.5)

Ans.

$$\text{Here } M = x^4 e^x - 2m\, xy^2$$

$$N = 2mx^2 y$$

$$\frac{\partial M}{\partial y} = -4mx\,y,\ \frac{\partial N}{\partial x} = 4m\,xy$$

Eqution is not exact.

Now
$$\frac{\dfrac{\partial M}{\partial x} - \dfrac{\partial N}{\partial x}}{N} = \frac{-4m\,xy - 4m\,xy}{2mx^2 y}$$

$$= \frac{-8\,m\,xy}{2m\,x^2 y} = \frac{-4}{x} = f(x).$$

$$\text{I.F} = e^{\int -\frac{4}{x} dx} = e^{-4\log x} = x^{-4} = \frac{1}{x^4}$$

Multiply given equation by $\dfrac{1}{x^4}$

$$\Rightarrow \quad \left(e^x - \frac{2m\,y^2}{x^3}\right)dx + \left(\frac{2m\,y}{x^2}\right)dy = 0$$

$$\text{New } M' = e^x - \frac{2my^2}{x^3}, N' = \frac{2my}{x^2}$$

$$\frac{\partial M'}{\partial y} = \frac{-4my}{x^3},\ \frac{\partial N'}{\partial x} = \frac{-4m\,y}{x^3}$$

Now equation is exact.

$$\text{Consider } \int_{y\ \text{constt}} M'\,dx = \int_{y\ \text{constt}} \left(e^x - \frac{2my^2}{x^3}\right)dx$$

$$\Rightarrow \qquad e^x + \frac{2my^2}{2x^2} = e^x + \frac{my^2}{x^2}$$

and
$$\int (N')^* dy = \int 0\, dy = 0$$

Required. solution

$$e^x + \frac{my^2}{x^2} = C$$

Q.1. (i) Show that $\frac{d}{dx}\{J_n^2(x)\} = \frac{x}{2n}\{J_{n-1}^2(x) - J_{n+1}^2(x)\}$ **(2.5)**

Ans. As we know

$$J_n(x) = \frac{x}{2n}\,[J_{n-1}(x) + J_{n+1}(x)] \qquad ...(i)$$

$$Jn'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \qquad ...(ii)$$

$$\text{Consider } \frac{d}{dx}\left[J_n^2(x)\right] = 2\,J_n(x)\,J_n'(x)$$

$$= 2\cdot\frac{x}{2n}\,[J_{n-1}(x) + J_{n+1}(x)]\cdot\frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

[Using (i) and (ii)]

$$= \frac{x}{2^n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

Q.1. (j) Find the rank of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ by reducing it in its normal form. **(2.5)**

Ans. Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Operating $R_2 \to R_2 + 2R_1, R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

Operating $R_3 \to R_3 - \frac{1}{2} R_2$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Operating $C_2 \to C_2 + \frac{1}{3} C_1, C_3 \to C_3 - \frac{2}{3} C_1$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Operating $C_1 \to \frac{1}{3} C_1$, $C_3 \to \frac{1}{8} C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Operating $C_2 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & - & - \\ - & - & 0 \\ 0 & - & - \end{bmatrix}$$

Reduced to a normal form of order 2

$\therefore r(A) = 2.$

UNIT-II

Q.2. (a) If $y = x^n \log x$, prove that

(i) $y_{n+1} = \dfrac{n!}{x}$ (ii) $y_n = ny_{n-1} + (n-1)!$

Ans. (i) Let $y = x^n \log x$

$$\therefore \quad y_1 = x^n \cdot \frac{1}{x} + nx^{n-1} \cdot \log x$$

$$\Rightarrow \quad xy_1 = x^n + nx^n \log x$$

$$xy_1 = x^n + ny$$

Differentiating n times by Leibnitz theorem

$$y_{n+1}x + {}^nC_1 y_n = n! + ny_n$$

$$\Rightarrow \quad y_{n+1}x + ny_n = n! + ny_n$$

$$xy_{n+1} = n!$$

or

$$y_{n+1} = \frac{n!}{x}$$

(ii) Let $y = x^n \log x$

Let $u = \log x$,

$$U_n = \frac{(-1)^{n-1}(n-1)!}{x^n} \qquad V = x^n$$
$$\qquad V_1 = nx^{n-1}$$

$$U_{n-1} = \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} \qquad V_2 = n(n-1)x^{n-2}$$

$$U_{n-2} = \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} \qquad V_n = n!$$

By Leibnitz theorem

$$\frac{d^n}{dx^n}(x^n \log x) = \frac{d^n}{dx^n}(\log \cdot x^n)$$

$$= \frac{{}^nC_0(-1)^{n-1}(n-1)!}{x^n}\cdot x^n + \frac{{}^nC_1(-1)^{n-2}(n-2)!}{x^{n-1}}n\,x^{n-1}$$

$$+\ {}^nC_2(-1)^{n-3}\frac{(n-3)!}{x^{n-2}}n(n-1)x^{n-2} + \ldots + {}^nC_n n!$$

$$= (-1)^{n-1}n!\left[(n-1) - n(n-2) + \frac{n(n-1)(n-3)}{2} + \ldots\right]$$

Q.2. (b) Test the convergence of the Series **(6.5)**

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)$$

Ans. Let $U_n = 1+\dfrac{1}{\sqrt{n}}$

Let $f(x) = 1+\dfrac{1}{\sqrt{x}}, x \geq 1$

then $f(x)$ is non-negative

consider $x_1 < x_2$

$$\Rightarrow \qquad \sqrt{x_1} < \sqrt{x_2}$$

$$\Rightarrow \qquad \frac{1}{\sqrt{x_1}} > \frac{1}{\sqrt{x_2}}$$

$$\Rightarrow \qquad 1+\frac{1}{\sqrt{x_1}} > 1+\frac{1}{\sqrt{x_2}}$$

$$\Rightarrow \qquad f(x_1) > f(x_2)$$

$\Rightarrow f(x)$ is monotonic decreasing

Thus $f(x)$ is positive, monotonic decreasing and integrable function of x such that

$$f(n) = 1+\frac{1}{\sqrt{n}}$$

As
$$U_n = 1+\frac{1}{\sqrt{n}}$$

$\Rightarrow f(n) = U_n$ for all +*ve* integral value of n.

$\therefore$ By Cauchy's integral test

$$\text{Consider } \int_1^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_1^t f(x)dx \quad ...(i)$$

Now
$$\int_1^t f(x)dx = \int_1^t\left(1+\frac{1}{x^{1/2}}\right)dx$$
$$= \left[x+2x^{1/2}\right]_1^t$$
$$= t+2t^{1/2}-1-2 = t+2t^{1/2}-3$$

By (*i*)
$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty}(t+2t^{1/2}-3)$$
$$= \lim_{t\to\infty} t+2\lim_{t\to\infty} t^{1/2}-3$$
$$= \infty+2\infty-3 = \infty\text{(in finite)}$$

$\therefore$
$$\int_1^{\infty} f(x)\,dx \text{ is divergent.}$$

$\Rightarrow$ ΣU_n is divergent.

Q.3. (*a*) Use Maclaurin's theorem to show that $\sqrt{1+x+2x^2} = 1+\frac{x}{2}+\frac{7}{8}x^2-\frac{7}{16}x^3+.....$ **(6)**

Ans.
$$\text{Let } f(x) = \sqrt{1+x+2x^2}$$

By Maclaurin's theorem

$$f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+..... \quad ...(i)$$

Consider $f(x) = \sqrt{1+x+2x^2}$, $f(0) = 1$

$$f'(x) = \frac{1}{2}(1+x+2x^2)^{-1/2}(1+4x), f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{-(1+4x)}{4}(1+x+2x^2)^{-3/2}(1+4x)+2(1+x+2x^2)^{-1/2}$$

$$f''(0) = -\frac{1}{4}+2 = \frac{7}{4}$$

$$f'' = \frac{-3}{8}(1+4x)^3(1+x+2x^2)^{-5/2}-2(1+4x)(1+x+2x^2)^{-3/2}-(1+x+2x^2)^{-3/2}(1+4x)$$

$f''(0) = \frac{-3}{8} - 2 - 1 = \frac{-21}{8}$

Substituting all values in (*i*), we get

$$\sqrt{1+x+2x^2} = 1 + \frac{1}{2}x + \frac{x^2}{2!} \times \frac{7}{4} + \frac{x^3}{3!} \times \left(\frac{-21}{8}\right) + \ldots$$

$$= 1 + \frac{x}{2} + \frac{7x^2}{8} - \frac{7}{16}x^3 + \ldots\ldots$$

Q.3. (b) Test the following series for Convergence and absolute convergence

$-\frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \frac{9}{16} \ldots\ldots$ **(6.5)**

Ans. By neglecting first term

Let the series be

$$\sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)}{2^n}$$

Absolute Convergence

Consider $\sum_{n=1}^{\infty} \left| (-1)^n \frac{(2n+1)}{2^n} \right| = \sum_{n=1}^{\infty} \frac{2n+1}{2^n}$

Let $U_n = \frac{2n+1}{2^n}, U_{n+1} = \frac{2n+3}{2^{n+1}}$

Consider
$$\frac{U_n}{U_{n+1}} = \frac{2n+1}{2^n} \cdot \frac{2^{n+1}}{2n+3}$$

$$= \frac{n\left(2+\frac{1}{n}\right)}{2^n} \cdot \frac{2^n \cdot 2}{n\left(2+\frac{3}{n}\right)}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{2\left(2+\frac{1}{n}\right)}{2+\frac{3}{n}} = 2 . > 1$$

$\therefore$ By Ratio test ΣU_n is Convergent.

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)}{2^n}$ is absolutely convergent and hence covergent.

UNIT-II

Q.4. (a) If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, show that $(I_n + I_{n-2}) = \frac{1}{n-1}$. Deduce value of I_5. **(6)**

Ans.
$$I_n = \int_0^{\pi/4} \tan^n x \cdot dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \cdot \tan^2 x\,dx$$

$$= \int_0^{\pi/4} \tan_x^{n-2}(\sec^2 x - 1)dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \cdot \sec^2 x\,dx - \int_0^{\pi/4} \tan^{n-2} x\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore \quad I_n = \int_0^{\pi/4} t^{n-2}dt - \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$= \left.\frac{t^{n-1}}{n-1}\right|_0^{\pi/4} \int_0^{\pi/4} \tan^{n-2} x dx$$

$$\Rightarrow \quad I_n = \frac{1}{n-1}\left|\tan^{n-1} x\right|_0^{\pi/4} - I_{n-2}$$

$$\Rightarrow \quad I_n = \frac{1}{n-1} - I_{n-2}$$

$$\Rightarrow \quad I_n + I_{n-2} = \frac{1}{n-1} \quad \text{...(i)}$$

Putting $n = 5$ and 3 in (i) successively, we get

$I_5 + I_3 = \frac{1}{4}$ and $I_3 + I_1 = \frac{1}{2}$

Now
$$I_1 = \int_0^{\pi/4} \tan\theta\,d\theta = \left|\log\sec\theta\right|_0^{\pi/4}$$

$$= \log\sqrt{2} = \frac{1}{2}\log 2$$

$$\therefore \quad I_3 = \frac{1}{2} - \frac{1}{2}\log 2$$

$$I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\log 2$$

$$\Rightarrow \quad I_5 = \frac{1}{2}\log 2 - \frac{1}{4}.$$

Q.4. (b) Find the radius of curvature at any point on the curve $x = a\cos^3 t, y = a\sin^3 t$. **(6.5)**

Ans. Let (x, y) be any point. $x = a\cos^3 t, y = a\sin^3 t$

$$x' = -3a\cos^2 t\ \sin t, y' = 3a\sin^2 t\cos t$$

$$x'' = -3a[\cos^2 t\cos t - 2\cos t\sin^2 t]$$

$$= -3a\,[\cos^3 t - 2\cos t \sin^2 t]$$
$$= -3a \cos t\,(\cos^2 t - 2\sin^2 t)$$
$$y'' = 3a\,[2\sin t\cos^2 t - \sin^3 t]$$
$$= 3a\sin t\,(2\cos^2 t - \sin^2 t)$$

Now $(x')^2 + (y')^2 = (-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2$

$$= 9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t$$
$$= 9a^2\cos^2 t\sin^2 t\,(\cos^2 t + \sin^2 t)$$
$$= 9a^2\cos^2 t\sin^2 t$$

and $x'y'' - y'x''$

$$= (-3a\cos^2 t\sin t)\,[3a\sin t\,(2\cos^2 t - \sin^2 t)]$$
$$-(3a\sin^2 t\cos t)\,[-3a\cos t\,(\cos^2 t - 2\sin^2 t)]$$

$$x'y'' - y'x'' = -9a^2\cos^2 t\sin^2 t\,(2\cos^2 t - \sin^2 t) + 9a^2\sin^2 t\cos^2 t\,(\cos^2 t - 2\sin^2 t)$$
$$= 9a^2\cos^2 t\sin^2 t\,[\cos^2 t - 2\sin^2 t - 2\cos^2 t + \sin^2 t]$$
$$= -9a^2\cos^2 t\sin^2 t\,(\cos^2 t + \sin^2 t)$$
$$= -9a^2\cos^2 t\sin^2 t$$

$$\text{Consider } \rho = \frac{\left[x'^2 + y'^2\right]^{3/2}}{x'y'' - y'x''}$$

$$= \frac{(9a^2\sin^2 t\cos^2 t)^{3/2}}{-9a^2\cos^2 t\sin^2 t} = -(9a^2\sin^2 t \times \cos^2 t)^{1/2}$$

$\Rightarrow \rho = 3a\sin t\cos t$ (in magnitude)

Q.5. (a) Trace the Curve $y^2(2a - x) = x^3$

Ans. Equation of Curve is $y^2(2a - x) = x^3$

(i) **Symmetry:** It contains even powers of y, the curve is symmetrical about x-axis.

(ii) **Origin:** Curve passes through origin.

(iii) **Axes intersection:** Curve meets x-axis and y-axis at origin only.

(iv) **Tangents: At (0, 0)** $y^2 = 0$ *i.e.*, $y = 0, 0$.

Since two tangents are real and coincident.

$\therefore$ origin is a cusp.

(v) **Asymptotes:** Equating to zero, the highest degree term in y, the asymptote parallel to y-axis is $x - 2a = 0$.

$\Rightarrow x = 2a$.

Oblique. $\phi_3 = x^3 + y^2 x$

$$\phi_2 = 2ay^2$$
$$\text{Put } x = 1, y = m$$
$$\Rightarrow \quad \phi_3(m) = 1 + m^2$$
$$\text{Now } \phi_3(m) = 0$$
$$\Rightarrow \quad m^2 + 1 = 0$$

Imaginary values. No ablique asymptotes.

(vi) **Region of existence** $y = x\sqrt{\dfrac{x}{2a - x}}$

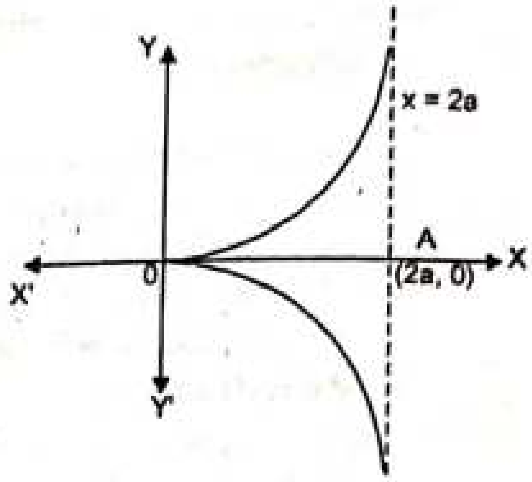
When $x < 0$, y is imaginary.

$\therefore$ No portion of curve lies to left of line $x = 0$.

When $0 < x < 2a$, y is real

When $x > 2a$, y is imaginary.

$\therefore$ No portion of curve lies to right of the line $x = 2a$.



UNIT-III

Q.6. (a) Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ **by reducting it to echelon form.** **(5)**

Ans.
$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to \frac{1}{3}R_2, R_3 \to -\frac{1}{2}R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1/3 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & -4/3 \end{bmatrix}$$

$R_3 \to -3R_3,\ R_4 \to \frac{-3}{4}R_4$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{3}R_3$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix is reduced to echelon form

$\therefore r(A) = 3$

Q.6. (b) Find the modal matrix of $A\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ **and diagonalize it.**

Ans. The characterstic equation of A is

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)[(3-\lambda)(1-\lambda)-8]-2[5\lambda-5+4]-2[-20+6-2\lambda]=0$

$\Rightarrow (4-\lambda)(3-4\lambda+\lambda^2-8)-2(5\lambda-1)-2(-14-2\lambda)=0$

$(4-\lambda)(\lambda^2-4\lambda-5)-10\lambda+2+28+4\lambda=0$

$\Rightarrow 4\lambda^2-16\lambda-20-\lambda^3+4\lambda^2+5\lambda-6\lambda+30=0$

$\Rightarrow -\lambda^3+8\lambda^2-17\lambda+10=0$

$\Rightarrow \lambda^3-8\lambda^2+17\lambda-10=0$

$\Rightarrow (\lambda-1)(\lambda^2-7\lambda+10)=0$

$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-5)=0$

$\Rightarrow \lambda = 1, 2, 5$ (eigen values)

For $\lambda = 1$

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 + \frac{5}{3}R_1,\ R_3 \to R_3 + \frac{2}{3}R_1$

$$\sim\begin{bmatrix} 3 & 2 & -2 \\ 0 & 16/3 & -4/3 \\ 0 & 16/3 & -4/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$\sim\begin{bmatrix} 3 & 2 & -2 \\ 0 & 16/3 & -4/3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 2x_2 - 2x_3 = 0$$

$$\frac{16}{3}x_2 - \frac{4}{3}x_3 = 0$$

Since $r(A) = 2 < 3$

$\Rightarrow n - r = 3 - 2 = 1$ Variable can be given arbitrary value.

$$\Rightarrow \quad 16x_2 = 4x_3$$

$$4x_2 = x_3$$

$$\Rightarrow \quad \frac{x_2}{1} = \frac{x_3}{4}$$

Also. $$3x_1 + 2 - 8 = 0$$

$\Rightarrow$ $$3x_1 = 6$$

$$x = 2$$

$\therefore$ eigen vector is $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$.

For $\lambda = 2$

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + \frac{5}{2} R_1, R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 2 & 2 & -2 \\ 0 & 6 & -3 \\ 0 & 6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 2 & 2 & -2 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 2x_2 - 2x_3 = 0$$

$\Rightarrow$ $$x_1 + x_2 - x_3 = 0$$

and $$6x_2 - 3x_3 = 0$$

$\Rightarrow$ $$2x_2 - x_3 = 0$$

$\Rightarrow$ $$\frac{x_2}{1} = \frac{x_3}{2}$$

Now $2x_1 + 2 - 4 = 0$ $\Rightarrow 2x_1 = 2$

$\Rightarrow x_1 = 1$

$\therefore$ eigen vector is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

For $\lambda = 5$.

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1, R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} -1 & 2 & -2 \\ 0 & -12 & 12 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now
$$-x_1 + 2x_2 - 2x_3 = 0$$
$$-12x_2 + 12x_3 = 0$$

$\Rightarrow$
$$\frac{x_2}{1} = \frac{x_3}{1}$$

Also,
$$-x_1 + 2 - 2 = 0$$

$\Rightarrow$
$$x_1 = 0$$

$\therefore$ Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$
$$\text{Modal matrix } M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$\Rightarrow |M| = 1$

$$M^{-1} = \frac{1}{1}\begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\text{Now } D = M^{-1}AM = \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 5 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Q.7. (a) Use Caley – Hamilton theorem for the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & -1 \end{bmatrix}$, to find A^{-1}. (6)

Ans. $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & -1 \end{bmatrix}$

The characteristic equation of A is
$$|A - \lambda I| = 0$$

i.e.,
$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)[(-3-\lambda)(-1-\lambda) - 8] + 8[-4 - 4\lambda + 6] - 2[-16 + 9 + 3\lambda] = 0$$

$\Rightarrow (8-\lambda)(3+4\lambda+\lambda^2-8)+8(-4\lambda+2)-2(-3\lambda-7)=0$

$\Rightarrow (8-\lambda)(\lambda^2+4\lambda-5)-32\lambda+16-6\lambda+14=0$

$\Rightarrow 8\lambda^2+32\lambda-40-\lambda^3-4\lambda^2+5\lambda-38\lambda+30=0$

$\Rightarrow -\lambda^3+4\lambda^2+-\lambda-10=0$

$\Rightarrow \lambda^3-4\lambda^2+\lambda+10=0$

To verify cayley hamilton theorem

$$A^3-4A^2+A+10I = 0$$

This verifies cayley Hamilton.

Multiply both sides by A^{-1}, we get

$$A^2-4A+I+10A^{-1} = 0$$

$$\Rightarrow \quad 10^{A-1} = 4A-A^2-I$$

$$\Rightarrow A^{-1} = \frac{1}{10}\left\{4\begin{bmatrix}8 & -8 & -2\\4 & -3 & -2\\3 & -4 & -1\end{bmatrix}-\begin{bmatrix}26 & -32 & 2\\14 & -15 & 0\\5 & -8 & 3\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \frac{1}{10}\left\{\begin{bmatrix}32 & -32 & -8\\16 & -12 & -8\\12 & -16 & -4\end{bmatrix}-\begin{bmatrix}26 & -32 & 2\\14 & -15 & 0\\5 & -8 & 3\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right\}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{10}\begin{bmatrix}5 & 0 & -10\\2 & 2 & -8\\7 & -8 & -8\end{bmatrix}$$

Q.7. (b) Invertigate whether the set of equations $2x-y-z=2, x+2y+z=2, 4x-7y-5z=2$ is consistent or not, if consistent, solve it. (6.5)

Ans. Given system of equations, can be written as

$$AX = B$$

$$\begin{bmatrix}2 & -1 & -1\\1 & 2 & 1\\4 & -7 & -5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$$

operating $R_2 \leftrightarrow R_1$

$$\begin{bmatrix}1 & 2 & 1\\2 & -1 & -1\\4 & -7 & -5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$$

operating $R_2 \to R_2-2R_1$, $R_3 \to R_3-4R_1$

$$\sim\begin{bmatrix}1 & 2 & 1\\0 & -5 & -3\\0 & -15 & -9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\-2\\-6\end{bmatrix}$$

$$R_3 \to \frac{-1}{3}R_3$$

$$\sim\begin{bmatrix}1 & 2 & 1\\0 & -5 & -3\\0 & 5 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\-2\\2\end{bmatrix}$$

operating $R_3 \to R_3 + R_2$

$$\sim\begin{bmatrix}1 & 2 & 1\\0 & -5 & -3\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\-2\\0\end{bmatrix}$$

$$\text{Since } r(A) = 2, r(A:B) = 2.$$
$$i.e., r(A) = r(A:B)$$

$\therefore$ System is consistent

As $r(A) = r(A:B) = 2 < 3$ (no. of unknowns)

Thus system has inifnite many solutions.

We give $n - r = 3 - 2 = 1$ variable arbitrary value

$$\text{Let } z = k$$
$$x + 2y + z = 2$$
$$-5y - 3z = -2$$
$$\Rightarrow \quad -5y - 3k = -2$$
$$\Rightarrow \quad y = \frac{2-3k}{5}$$
$$\text{Also} \quad x = 2 - 2y - z$$
$$\Rightarrow \quad x = 2 - \frac{(4-6k)}{5} - k$$
$$\Rightarrow \quad x = \frac{10-4+6k-5K}{5} = \frac{6+K}{5}$$
$$\therefore \quad x = \frac{6+K}{5}, y = \frac{2-3K}{5}, Z = K.$$

UNIT-IV

Q.8. (a) Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x\sin x$ (6.5)

Ans. Given equation is $(D^2 + D + 1)y = x\sin x$

A.E. $D^2 + D + 1 = 0$

$$\Rightarrow \quad D = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore CF = e^{-1/2x}\left[C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^2+D+1}x\sin x$$

$$= \text{Img part of}\left[\frac{1}{D^2+D+1}(xe^{ix})\right] \quad ...(i)$$

$$\text{Now } \frac{1}{D^2+D+1}xe^{ix} = e^{ix}\frac{1}{(D+i)^2+D+i+1}x$$

$$= e^{ix}\frac{1}{D^2-1+2iD+D+i+1}x$$

$$= e^{ix}\frac{1}{D^2+(2i+1)D+i}x$$

$$= \frac{e^{ix}}{i}\left[1+\frac{1}{\frac{D^2+(2i+1)D}{i}}\right]x$$

$$= \frac{e^{ix}}{i}\left[1+\left(\frac{D^2}{i}+2D+\frac{D}{i}\right)\right]^{-1}x$$

$$= \frac{e^{ix}}{i}\left[1-\left(\frac{D^2}{i}+2D+\frac{D}{i}\right)+...\right]x$$

$$= \frac{e^{ix}}{i}\left(1-2D-\frac{-D}{i}\right)x$$

(Neglecting higher order terms)

$$= \frac{e^{ix}}{i}\left(x-2-\frac{1}{i}\right)$$

$$= -i(\cos x + i\sin x)(x-2+i)$$

$$= (-i\cos x + \sin x)(x-2+i)$$

By (i) $P.I. =$ Img. part of $[(-i\cos x+\sin x)(x-2+i)]$

$$= -x\cos x + 2\cos x + \sin x$$

$\therefore$ Complete soln is

$$y = e^{-1/2x}\left(C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right) - x\cos x + 2\cos x + \sin x$$

Q.8. (b) Solve $\frac{d^2y}{dx^2} + y = -\cot x$ by method of variation of parameters. (6.5)

Ans. Given $(D^2 + 1)y = -\cot x$

A.E. $D^2 + 1 = 0$

$D^2 = -1 \Rightarrow D = \pm i$

$\Rightarrow$ $CF = C_1 \cos x + C_2 \sin x$

$\therefore$

Here $y_1 = \cos x, y_2 = \sin x, X = -\cot x$

Now $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$W = \cos^2 x + \sin^2 x + 1$$

$P.I = uy_1 + vy_2$

where $$u = -\int \frac{y_2 X}{W} dx = -\int \frac{-\sin x \cot x}{1} dx$$

$$= \int \sin x \cdot \frac{\cos x}{\sin x} dx = \int \cos x \, dx$$

$$= \sin x$$

$$V = \int \frac{y_1 x}{W} dx = -\int \frac{\cos x \cot x}{1} dx$$

$$= -\int \frac{\cos^2 x}{\sin x} dx = -\int \frac{1 - \sin^2 x}{\sin x} dx_2 - \int (\operatorname{cosec} x - \sin x) dx$$

$$= -\log(\operatorname{cosec} x - \cot x) - \cos x$$

$$PI = \sin x \cos x - [\log(\operatorname{cosec} x - \cot x) + \cos x] \sin x$$

$$= -\sin x [\log(\operatorname{cosec} x - \cot x]$$

$\therefore$ Complete soln is

$$y = C_1 \cos x + C_2 \sin x - \sin x [\log(\operatorname{cosec} x - \cot x)]$$

Q.9. (a) Solve $(x^2 D^2 + 3x + D + 1)y = \frac{1}{(1-x)^2}$

Ans. $$(x^2 D^2 + 3x D + 1)y = \frac{1}{(1-x)^2}$$

Putting $x = e^z$, we get

$$[D(D-1) + 3D + 1]y = \frac{1}{(1-x)^2}$$

$\Rightarrow$ $(D+1)^2 y = \frac{1}{(1-x)^2}$, where $D = \frac{d}{dz} = \frac{xd}{dx}$

$$A.E = (D+1)^2 = 0$$

$\Rightarrow$ $$D = -1, -1$$

$$C.F. = (C_1 + C_2 z)e^{-z} = (C_1 + C_2 \log x)\frac{1}{x}$$

$$P.I. = \frac{1}{(\theta+1)^2}\frac{1}{(1-x)^2}, \theta = x\frac{d}{dx} \quad ...(i)$$

Let $\frac{1}{\theta+1}\left[\frac{1}{(1-x)^2}\right] = v$ So that $(\theta + 1)u = \frac{1}{(1-x)^2}$

$$= x\frac{du}{dx} + u = \frac{1}{(1-x)^2} \text{ or}$$

$$\frac{du}{dx} + \frac{u}{x} = \frac{1}{x(1-x)^2} \quad ...(ii)$$

I.F. of (2) $$= e^{\int dx/x} = e^{\log x} = x$$

Sol of (2) $$u \cdot x = \int \frac{x\,dx}{x(1-x)^2} = \frac{1}{1-x}$$

$$\text{or } u = \frac{1}{x(1-x)}$$

By (*i*), we get

$$P.I = \frac{1}{\theta+1} \cdot u = V(\text{say}), \text{ Then } U = (\theta + 1)V$$

or $$\frac{1}{x(1-x)} = x\frac{dV}{dx} + V \text{ or } \frac{dV}{dx} + \frac{V}{x} = \frac{1}{x^2(1-x)} \quad ...(iii)$$

I.F. of (*iii*) is x and solution of (*iii*) is

$$V.x = \int \frac{x\,dx}{x^2(1-x)}$$

$$= \int \frac{dx}{x(1-x)} = \int\left(\frac{1}{1-x} + \frac{1}{x}\right)dx$$

or $$Vx = \log\frac{x}{1-x}$$

$\Rightarrow$ $$V = \frac{1}{x}\log\frac{x}{1-x} = P.I$$

$$\text{Hence } y = x^{-1}(C^1 + C^2 \log x) + \frac{1}{x}\log\frac{x}{1-x}$$

Q.9. (b) Prove that $\int_{-1}^{1}(x^2-1)P_{n+1}P_n' dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$.

Ans. By recurrence relation

$$(1-x^2)Pn'(x) = n[P_{n-1}(x)-xPn(x)] \quad ...(i)$$

and $(1-x_2)Pn'(x) = (n+1)(x\,Pn(x)-P_{n+1}(x)$ ...(ii)

Multiplying (i) by $n+1$ and (ii) by n and adding, we get

$$(n+1)(1-x^2)Pn'(x)+n(1-x^2)Pn'(x) = n(n+1)P_{n-1}(x)-n(n+1)P_{n+1}(x)$$

$$\Rightarrow (2n+1)(1-x^2)Pn' = n(n+1)[P_{n-1}(x)-P_{n+1}(x)]$$

$$\Rightarrow (2n+1)(x^2-1)Pn' = n(n+1)[P_{n+1}(x)-P_{n-1}(x)]$$

$$\Rightarrow (x^2-1)P_n' = \frac{n(n+1)}{2n+1}[P_{n+1}(x)-P_{n-1}(x)]$$

Multiply both sides by P_{n+1} and integrate w.r.t. 'x' between limits –1 to 1, we get

$$\int_{-1}^{1}(x^2-1)P_{n+1}P_n' dx = \frac{n(n+1)}{2n+1}\left[\int_{-1}^{1}P_{n+1}^2 dx - P_{n+1}P_{n-1} dx\right]$$

By Orthogonality

$$\int_{-1}^{1}P_m(x)P_n(x)\,dx = \begin{cases} 0, \text{ if } m \neq n \\ \frac{2}{2n+1}, \text{ if } m = n \end{cases}$$

$$\Rightarrow \int_{-1}^{1}(x^2-1)P_{n+1}P_n' dx = \frac{n(n+1)}{2n+1}\left(\frac{2}{2(n+1)+1}\right) = \frac{2n(n+1)}{(2n+1)(2n+3)}$$

FIRST TERM EXAMINATION [SEPT. 2018]
FIRST SEMESTER [B.TECH]
APPLIED MATHEMATICS-I [ETMA-101]

Time : 1.5 hrs. **M.M. : 30**

Note: *Q. 1. is compulsory. Attempt any two questions.*

Q.1. (a) For the matrix $A = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & -i \\ -2-3i & -i & 0 \end{bmatrix}$

Show that $\bar{A}$ is skew - hermitian matrix. **(2.5)**

Ans. Let
$$\bar{A} = B = \begin{bmatrix} -i & 1-i & 2+3i \\ -1-i & -2i & i \\ -2+3i & i & 0 \end{bmatrix}$$

Now to show
$$\bar{B}' = -B.$$

$$\bar{B} = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & -i \\ -2-3i & -i & 0 \end{bmatrix}$$

$$\bar{B}' = \begin{bmatrix} i & -1+i & -2-3i \\ 1+i & 2i & -i \\ 2-3i & -i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} -i & 1-i & 2+3i \\ -1-i & -2i & i \\ -2+3i & i & 0 \end{bmatrix} = -B$$

$\therefore$ $\bar{A}$ is skew hermitian matrix.

Q. 1. (b) Reduce the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to the canonical form. Also write the nature of the quadratic form. **(2.5)**

Ans. $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$

Here
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

As
$$A = I_3 A I_3$$

i.e.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_2 + 3R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 \to C_2 + 3C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 24 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$\Rightarrow$ $$\text{Diag}(1, 3, 24) = P'AP$$

Canonical form is given by

$$Y'(P'AP)Y = [y_1\, y_2\, y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 24 \end{bmatrix} A \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [y_1 \quad y_2 \quad y_3] \begin{bmatrix} y_1 \\ 3y_2 \\ 24y_3 \end{bmatrix}$$

$$= y_1^2 + 3y_2^2 + 24y_3^2$$

$$r(A) = 3, \text{index} = 3, \text{signature} = 3$$

It is positive definite.

Q. 1. (c) Find the nth derivative of $e^x \cos x \cos 2x$. (2.5)

Ans. Let $$y = e^x \cos x \cos 2x$$

$$= \frac{1}{2} e^x (2\cos x \cos 2x)$$

$$= \frac{1}{2} e^x (\cos x + \cos 3x)$$

$$= \frac{1}{2} (e^x \cos x + e^x \cos 3x)$$

$\Rightarrow$ $$y_n = \frac{1}{2}\Big[\left\{(1^2 + 1^2)^{n/2} e^x \cos(x + n\tan^{-1} 1)\right\}$$

$$+ \left\{(1^2 + 3^2)^{n/2} e^x \cos(3x + n\tan^{-1} 3)\right\}\Big]$$

$$= \frac{1}{2} e^x \left[2^{n/2} \cos\left(x + \frac{n\pi}{4}\right) + 10^{n/2} \cos(3x + n\tan^{-1} 3)\right]$$

Q. 1. (d) Prove that the matrix $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ is orthogonal. **(2.5)**

Ans. Let
$$A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$A' = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

Now
$$AA' = \frac{1}{81}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AA' = I$$

A is an orthogonal matrix.

Q. 2. (a) Find the characteristic equation of matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Hence find A^{-1}

Also use it to find the matrix represented by $A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$. **(5)**

Ans. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda - 5 = 0$$

By Cayley Hamilton theorem

$$A^2 - 4A - 5I = 0 \qquad \text{...(1)}$$

$\Rightarrow$ Multiply by A^{-1}

$$\Rightarrow \quad A - 4AA^{-1} - 5IA^{-1} = 0$$

$$A - 4I - 5A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}[A - 4I]$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}\left\{\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix} - \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}\right\}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}\begin{bmatrix}-3 & 4\\ 2 & -1\end{bmatrix}$$

Now $\quad A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$

$$= A^3(A^2 - 4A - 5I) + 4A^4 + 5A^3 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$$

$$= 9A^4 - A^3 + 2A^2 - 4A + 7I \qquad \text{[Using (1)]}$$

$$= 9A^2(A^2 - 4A - 5I) + 36A^3 + 45A^2 - A^3 + 2A^2 - 4A + 7I$$

$$= 35A^3 + 47A^2 - 4A + 7I$$

$$= 35A(A^2 - 4A - 5I) + 140A^2 + 175A + 47A^2 - 4A + 7I$$

$$= 187A^2 + 171A + 7I$$

$$= 187(A^2 - 4A - 5I) + 748A + 935I + 171A + 7I$$

$$= 919A + 942I$$

$$= 919\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix} + 942\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}919 & 3676\\ 1838 & 2757\end{bmatrix} + \begin{bmatrix}942 & 0\\ 0 & 942\end{bmatrix}$$

$$= \begin{bmatrix}1861 & 3676\\ 1838 & 3699\end{bmatrix}$$

Q. 2. (b) Find values of k for which the following system of equation may have non-trivial solution. $3x_1 + x_2 - kx_3 = 0$, $4x_1 - 2x_2 - 3x_3 = 0$, $2kx_1 + 4x_2 + kx_3 = 0$. For each permissible value of k, find the general solution. (5)

Ans. Given system

$$3x_1 + x_2 - kx_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2kx_1 + 4x_2 + kx_3 = 0$$

In matrix notation, $AX = 0$

where $\quad A = \begin{bmatrix}3 & 1 & -k\\ 4 & -2 & -3\\ 2k & 4 & k\end{bmatrix}, \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = X$

For non-trivial solution $|A| = 0$

$$\begin{vmatrix}3 & 1 & -k\\ 4 & -2 & -3\\ 2k & 4 & k\end{vmatrix} = 0$$

$$\Rightarrow 3(-2k+12)-1(4k+6k)-k(16+4k) = 0$$

$$-6k+36-10k-16k-4k^2 = 0$$

$$\Rightarrow \quad 36-32K-4k^2 = 0$$

$$4k^2+32k-36 = 0$$

$$\Rightarrow \quad k^2+8k-9 = 0$$

$$\Rightarrow \quad (k-1)(k+9) = 0$$

$$\Rightarrow \quad k = 1, -9$$

Now,
$$\begin{bmatrix} 3 & 1 & -k \\ 4 & -2 & -3 \\ 2k & 4 & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

for k = 1

$$\begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \to 2R_3 - R_2$$

$$-\begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 0 & 10 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \to R_3 + 5R_2$$

$$-\begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 0 & 0 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$3x_1 + x_2 - x_3 = 0$$

$$4x_1 - 2x_2 - 3x_3 = 0$$

$$-10x_3 = 0$$

from first 2 equations

$$\frac{x_1}{-3-2} = \frac{x_2}{-4+9} = \frac{x_3}{-6-4} = t \text{ (say)}$$

$$\Rightarrow \quad \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-10} = t$$

$$\Rightarrow \quad x_1 = t,\ x_2 = -t,\ x_3 = 2t$$

for $k = -9$

$$\begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & 3 \\ -18 & 4 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$3x_1 + x_2 + 9x_3 = 0$$
$$4x_1 - 2x_2 + 3x_3 = 0$$
$$-18x_1 + 4x_2 - 9x_3 = 0$$

from first and third equation

$$\frac{x_1}{45} = \frac{x_2}{-135} = \frac{x_3}{30} = t \text{ (any)}$$

$$x_1 = 15t,\ x_2 = -45t, x_3 = 10t$$

$\Rightarrow$ $$x_1 = 3t,\ x_2 = -9t, x_3 = 2t$$

Q. 3. (a) Find $y_n(0)$, if $y_r = e^{m \sin^{-1} x}$

Ans. $$y = e^{m \sin^{-1} x} \quad ...(1)$$

$\Rightarrow$ $$y_1 = e^{m \sin^{-1} x} \frac{m}{\sqrt{1-x^2}}$$

$\Rightarrow$ $$y_1 = \frac{m}{\sqrt{1-x^2}} y \quad ...(2)$$

$\Rightarrow$ $$(1-x^2)y_1^2 = m^2y^2$$

Differentiating again w.r.t (x)

$\Rightarrow$ $$(1-x^2)\,2y_1y_2 + y_1^2(-2x) = 2yy_1.m^2$$

$\Rightarrow$ $$(1-x^2)y_2 - xy_1 = m^2y \quad ...(3)$$

Differentiating n times using Leibnitz Theorem.

$$\{y_{n+2}(1-x^2) + ny_{n+1}(-2x) + n\frac{(n-1)}{2!}y_n(-2)] - [y_{n+1}.x + ny_n] = m^2y_n$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n. - xy_{n+1} - ny_n - m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \quad ...(4)$$

Putting x = 0 in (1), (2), (3) and (4), we get

(1) $\Rightarrow$ $$y(0) = e^{m \sin^{-1}0} = e^0 = 1$$

(2) $\Rightarrow$ $$y_1(0) = m\,y(0) = m$$

(3) $\Rightarrow$ $$y_2(0) = m^2y(0) = m^2$$

(4) $\Rightarrow$ $$y_{n+2}(0) = (m^2+n^2)y_n(0) \quad ...(5)$$

Putting $n = 1, 2, 3, 4 \ldots$ in (5)

$$y_3(0) = (m^2+1)y_1(0) = (m^2+1)m$$
$$y_4(0) = (m^2+2^2)y_2(0) = (m^2+2^2).m^2$$
$$y_5(0) = (m^2+3^2)y_3(0) = (m^2+3^2)(m^2+1^2)m$$
$$y_6(0) = (m^2+4^2)y_4(0)$$
$$= (m^2+4^2)(m^2+2^2).m^2$$

Thus when n is odd.

$$y_n(0) = m[m^2+1^2][m^2+3^2]\ldots..[m^2+(n-2)^2]$$

when n is even

$$y_n(0) = m^2[m^2+2^2][m^2+4^2] \ldots [m^2+(n-2)^2]$$

Q. 3. (b) A transformation from variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by Y = AX and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by z = BY where

$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}, B \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 3 & 5 \end{bmatrix}$. Obtain transformation from x_1, x_2, x_3 to z_1, z_2, z_3. (5)

Ans. Given

$$Y = AX$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and

$$Z = BY$$

where

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 3 & 5 \end{bmatrix}$$

$\Rightarrow$ $$Z = B(AX)$$

$\Rightarrow$ $$Z = (BA)X$$

$$BA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 3 \\ 5 & -3 & 1 \\ -3 & 14 & 11 \end{bmatrix}$$

$\therefore$ $$Z = (BA)X$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ 5 & -3 & 1 \\ -3 & 14 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow$

$$z_1 = x_1 + 4x_2 + 3x_3$$

$$z_2 = 5x_1 - 3x_2 + x_3$$

$$z_3 = -3x_1 + 14x_2 + 11x_3$$

Q. 4. (a) Apply Taylor's theorem to calculate value of $f\left(\frac{11}{10}\right)$, where $f(x) = x^3 + 3x^2 + 5x - 10$. (5)

Ans. By Taylor's theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots\ldots$$

Put $$x = 1, h = \frac{1}{10}$$

$$f\left(\frac{11}{10}\right) = f(1) + \frac{1}{10}f'(1) + \frac{1}{2!}\left(\frac{1}{10}\right)^2 f''(1) + \ldots (1)$$

Here $f(x) = x^3 + 3x^2 + 5x - 10$, $\quad f(1) = -1$

$$f'(x) = 3x^2 + 6x + 5, \quad f'(1) = 14$$

$$f''(x) = 6x + 6, \quad f''(1) = 12$$

$$f'''(x) = 6, \quad f'''(1) = 6$$

By (1), we have $$f\left(\frac{11}{10}\right) = -1 + \frac{1}{10}\times 14 + \frac{1}{2}\cdot\frac{1}{100}\times 12 + \frac{1}{6}\cdot\frac{1}{1000}\times 6$$

$$= -1 + 1.4 + 0.06 + 0.001$$

$$f\left(\frac{11}{10}\right) = 0.461$$

Q. 4. (b) Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ (5)

Ans. Characterstic equation of given matrix is $|A - \lambda I| = 0$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$$

$\Rightarrow$ $$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$\Rightarrow$ $$\lambda = 2, 3, 5 \text{ (eigen values).}$$

For $$\lambda = 2, \text{ eigen vector is given by}$$

$$(A + 2I)X = 0$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{2}R_2$$

$$-\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $$r(A) = 2 < 3$$

$$x_3 = 0$$

Let $x_1 + x_2 + 4x_3 = 0$

$\Rightarrow$ $x_1 + x_2 = 0$

$\Rightarrow$ $x_1 = -x_2$

$\therefore$ eigen vector is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

For $\lambda = 5$ eigen vector is given by $(A + 3I)X = 0$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{5}R_2$$

$$\sim \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As $r(A) = 2 < 3$

$\Rightarrow$ $x_2 + 4x_3 = 0,\ 10x_3 = 0$

$\Rightarrow$ $x_3 = 0,\ x_2 = 0$

Let $x_1 = 1$

$\therefore$ eigenvector is $X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

for $\lambda = 5$, eigenvector is given by

$$(A + 5I)X = 0$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 + 4x_3 = 0$$
$$-3x_2 + 6x_3 = 0$$
$$x_2 = 2x_3$$
$$\frac{x_2}{2} = \frac{x_3}{1}$$

and
$$-2x_1 + 2 + 4 = 0$$
$$x_1 = 3$$

$\Rightarrow$

$\therefore$ eigenvector is
$$X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

END TERM EXAMINATION [NOV.-DEC. 2018]
FIRST SEMESTER [B.TECH]
APPLIED MATHEMATICS-I [ETMA-101]

Time : 3 hrs. **M.M. : 75**

Note: *Attempt five questions in all including Q. No. 1 which is compulsory. Select one question from each unit. Assume suitable missing data, if any.*

Q.1. (a) Find the n^{th} derivative of sin 4x. cos 2x. **(3)**

Ans. Let
$$y = \sin 4x \cos 2x$$
$$\Rightarrow \quad y = \frac{1}{2}(2\sin 4x \cos 2x)$$
$$= \frac{1}{2}(\sin 6x + \sin 2x)$$
$$\Rightarrow \quad y_n = \frac{1}{2}\left[6^n \sin\left(6x + \frac{n\pi}{2}\right) + 2^n \sin\left(2x + \frac{n\pi}{2}\right)\right]$$

Q. 1. (b) Test the convergence of the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\infty$. **(2)**

Ans. It is an alternating series $\Sigma(-1)^{n-1} U_n$

where
$$U_n = \frac{1}{n^2}$$
$$U_{n+1} = \frac{1}{(n+1)^2}$$

(i) As
$$n^2 < (n+1)^2$$
$$\Rightarrow \quad \frac{1}{n^2} > \frac{1}{(n+1)^2} \ \forall n$$
$$\therefore \quad U_n > U_{n+1} \ \forall n$$

(ii)
$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$\therefore$ By Leibnitz test series is convergent

$\Rightarrow \Sigma(-1)^n U_n$ is convergent.

Q. 1. (c) Find the asymptotes of the curve $y^3 + x^2y + 2xy^2 - y + 1 = 0$ **(3)**

Ans.
$$y^3 + x^2y + 2xy^2 - y + 1 = 0$$

Since coefficient of y^3 is constant, so there is no asymptote || to y – axis.

coefficient of x^2 is $y = 0$

$\therefore$ $y = 0$ is asymptote || to x–axis.

Oblique Asymtotes

$$\phi_3(x, y) = y^3 + x^2y + 2xy^2$$
$$\phi_2(x, y) = 0$$
$$\phi_1(x, y) = -y$$
$$\phi_0 = 1$$

Put $x = 1, y = m$

$$\phi_3(m) = m^3 + m + 2m^2$$
$$\phi_2(m) = 0$$
$$\phi_1(m) = -m$$
$$\phi_0 = 1$$

Slopes of asymptotes are roots of $\phi_3(m) = 0$

$$m^3 + 2m^2 + m = 0$$

$\Rightarrow$ $$m(m^2 + 2m + 1) = 0$$

$\Rightarrow$ $$m = 0,\ m^2 + 2m + 1 = 0$$

$\Rightarrow$ $$m = 0,\ (m+1)^2 = 0$$

$\Rightarrow$ $$m = 0, -1, -1$$

$\Rightarrow$

For non-repeated value $m = 0$

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = 0$$

For repeated values $$m = -1, -1$$

$$\frac{c^2}{2!}\theta_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

$$\phi_3'(m) = 3m^2 + 1 + 4m$$
$$\phi_3''(m) = 6m + 4$$
$$\phi_2'(m) = 0$$

$\Rightarrow$ $$\frac{c^2}{2!}(6m+4) + (-m) = 0$$

for $$m = -1$$

$\Rightarrow$ $$-2.\frac{c^2}{2} + 1 = 0$$

$\Rightarrow$ $$c^2 - 1 = 0 \Rightarrow c^2 = 1$$

$\Rightarrow$ $$c = \pm 1$$

asymptotes are $$y = mx + c$$

i.e. $$y = 0,\ y = -x + 1, y = -x - 1$$

$\Rightarrow$ $$y = 0,\ y + x - 1 = 0, y + x + 1 = 0$$

Q. 1. (d) If $I_n = \int_0^{\pi/4} \tan^n\theta d\theta$, , show that $I_n + I_{n-2} = \dfrac{1}{n-1}$ (3)

Ans. Let $$I_n = \int_0^{\pi/4} \tan^n\theta d\theta$$

$\Rightarrow$ $$I_n = \int_0^{\pi/4} \tan^{n-2}\theta . \tan^2\theta d\theta$$

$$= \int_0^{\pi/4} \tan^{n-2}\theta\,(\sec^2\theta - 1)\,d\theta$$

$$= \int_0^{\pi/4} \tan^{n-2}\theta.\sec^2\theta\,d\theta - \int_0^{\pi/4} \tan^{n-2}\theta.d\theta$$

Let $\tan\theta = t$

$\sec^2\theta\,d\theta = dt$

$\Rightarrow \quad I_n = \int_0^1 t^{n-2}\,dt - I_{n-2}$

$\Rightarrow \quad I_n + I_{n-2} = \left.\dfrac{t^{n-1}}{n-1}\right|_0^1$

$\Rightarrow \quad I_n + I_{n-2} = \dfrac{1}{n-1}$

Q. 1. (e) Show that the matrix A = $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ is a Hermitian matrix. **(2)**

Ans. To show $\bar{A}' = A$.

Given
$$A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$$

$$\bar{A}' = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix} = A$$

$\therefore$ A is a hermitian matrix.

Q. 1. (f) If 1 λ be an eigen value of a non-singular matrix A, then show that λ^{-1} is an eigen value of A^{-1}. **(3)**

Ans. Since λ is eigenvalue of A then

$$(A - \lambda I)X = 0 \quad ...(1)$$

Premultiply (1) by A^{-1}

$\Rightarrow \quad A^{-1}(A - \lambda I)X = 0$

$$(I - \lambda A^{-1} I)X = 0$$

$\Rightarrow$

Multiply (2) throughout by λ^{-1}.

$$(\lambda^{-1} I - A^{-1})X = 0$$

$$-(A^{-1} - \lambda^{-1} I)X = 0$$

$\Rightarrow$

$$(A^{-1} - \lambda^{-1} I)X = 0$$

$\therefore$ λ^{-1} is an eigenvalue of A^{-1}.

Q. 1. (g) Solve

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3, y(0) = \frac{\pi}{4}.$$

(3)

Ans. $\sec^2 y \dfrac{dy}{dx} + 2x \tan y = x^3$

Let

$$Z = \tan y$$

$\Rightarrow$

$$\frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$$

$\therefore$

$$\frac{dz}{dx} + 2xz = x^3 \text{ (linear in z).}$$

$$I.F = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ Required solution is

$$Ze^{x^2} = \int x^3 e^{x^2} dx$$

$$Ze^{x^2} = \frac{x^3 e^{x^2}}{2x} - \int 3x^2 \frac{e^{x^2}}{2x} dx \quad [x^2 = t]$$

$$= \frac{x^2 e^{x^2}}{2} - \frac{3}{2}\int x e^{x^2} dx$$

$$Ze^{x^2} = \frac{x^2 e^{x^2}}{2} - \frac{3}{4}\int e^t dt$$

$$= \frac{x^2 e^{x^2}}{2} - \frac{3}{4} e^{x^2} + c$$

$$Z = \frac{x^2}{2} - \frac{3}{4} + ce^{-x^2}$$

$\Rightarrow$

$$\tan y = \frac{x^2}{2} - \frac{3}{4} + ce^{-x^2}$$

At

$$x = 0$$

$$\tan \frac{\pi}{4} = \frac{-3}{4} + c$$

$$\Rightarrow \quad c = 1+\frac{3}{4}=\frac{7}{4}$$

$$\therefore \quad \tan y = \frac{x^2}{2}-\frac{3}{4}+\frac{7}{4}e^{-x^2}$$

Q. 1. (h) Find the particular solution of the differential equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=10x$. **(3)**

Ans. Given $\quad (D^2-3D+2)y = 10x$

$$\text{P.I.} = \frac{1}{D^2-3D+2}10x$$

$$\Rightarrow \quad 10\left(\frac{1}{D^2-3D+2}\right)x$$

$$\Rightarrow \quad \frac{10}{2}\left[\frac{1}{\frac{D^2}{2}-\frac{3D}{2}+1}\right]x$$

$$\Rightarrow \quad 5\left\{1-\left(\frac{-D^2}{2}+\frac{3D}{2}\right)\right\}^{-1}x$$

$$\Rightarrow \quad 5\left[1+\left(\frac{3D}{2}-\frac{D^2}{2}\right)+.....\right]x$$

$$= 5\left[1+\frac{3D}{2}\right]x$$

$$= 5\left[x+\frac{3}{2}Dx\right]$$

$$= 5\left[x+\frac{3}{2}\right]=5x+\frac{15}{2}$$

Q. 1. (i) Show that $(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$. **(3)**

Ans. Consider $(1-2xz+z+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x)$ ...(1)

Differentiating (1) both sides partially w.r.t 'z', we get

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z)=\sum_{n=1}^{\infty} nz^{n-1}P_n(x)$$

$$\Rightarrow \quad (x-z)(1-2xz+z^2)^{-1/2}=(1-2xz+z^2)\sum_{n=1}^{\infty} nz^{n-1}P_n(x)$$

$$\Rightarrow (x-z)\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

Equating Co-efficients of z^n on both sides

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2x\,nP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow \quad (2n+1)x.P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

$$\Rightarrow \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

UNIT-I

Q. 2. (a) If $y = (\sin^{-1} x)^2$, then prove that $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - n^2y_n = 0$ (6)

Ans. Let $\quad y = (\sin^{-1} x)^2$

$$\Rightarrow \quad y_1 = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad y_1^2(1-x^2) = 4(\sin^{-1} x)^2$$

$$\Rightarrow \quad y_1^2(1-x^2) = 4y \quad ...(1)$$

Differentiating again w.r.t 'x'

$$(1-x^2)\,2y_1y_2 - 2x\,y_1^2 = 4y_1$$

$$\Rightarrow \quad (1-x^2)y_2 - xy_1 = 2 \quad ...(2)$$

Using Leibnitz theorem n times, we get

$$[(1-x^2)\,y_{n+2} - {}^nC_1(-2x)\,y_{n+1} + {}^nC_2(-2)\,y_n] - [xy_{n+1} - {}^nC_1 y_n] = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - 2nx\,y_{n+1} - \frac{n(n-1)}{2}.2y_n - xy_{n+1} - ny_n = 0$$

$$\Rightarrow \; (1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - n^2y_n = 0$$

Q. 2. (b) Test the convergence of the series $\frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} + ...x > 0.$ (6.5)

Ans. Given $\frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} + ...$

Let $\quad \Sigma u_n = \Sigma v_n + \Sigma w_n$

where $\quad \Sigma v_n = \sum \frac{1}{x-n}$

and $\quad \Sigma w_n = \sum \frac{1}{x+n}$

for Σv_n

Here $\quad v_n = \frac{1}{x-n}$

Let $\quad a_n = \frac{1}{n}$

$$\Rightarrow \quad \frac{v_n}{a_n} = \frac{\frac{1}{x-n}}{1/n}$$

$$\Rightarrow \quad \frac{v_n}{a_n} = \frac{n}{x-n} = \frac{n}{n\left(\frac{x}{n}-1\right)}$$

$$\lim_{n\to\infty}\frac{v_n}{a_n} = \lim_{n\to\infty}\frac{1}{\left(\frac{x}{n}-1\right)} = -1 \text{ (finite and non zero)}$$

$\therefore$ By L.C. T Σv_n and Σa_n have same behaviour.

Consider $\Sigma a_n = \sum \frac{1}{n}$ (p = 1)

$\therefore \Sigma a_n$ is divergent by p - series

$\Rightarrow \Sigma v_n$ is divergent.

Now, for Σw_n

Here $w_n = \frac{1}{x+n}$

Let $b_n = \frac{1}{n}$

$$\Rightarrow \quad \frac{w_n}{b_n} = \frac{1/x+n}{1/n} = \frac{n}{x+n}$$

$$\Rightarrow \quad \frac{w_n}{b_n} = \frac{n}{n(1+x/n)}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{w_n}{b_n} = \lim_{n\to\infty}\frac{1}{1+x/n} = 1 \text{ (finite and non zero)}$$

$\therefore$ By L.C.T Σw_n and Σb_n both have same behaviour consider

$$b_n = \frac{1}{n} \quad (p = 1)$$

$\therefore \Sigma b_n$ is divergent by p – series

$\Rightarrow \Sigma w_n$ is also divergent

Thus $\Sigma u_n = \Sigma v_n + \Sigma w_n$ is also divergent.

$\therefore$ series is divergent

Q. 3. (a) Assuming the possibility of expansion expand $\tan^{-1}x$ as for as containing x^5 by Maclaurin's series. **(6)**

Ans. Let

$$y = f(x) = \tan^{-1} x, \; y(0) = 0$$

$$y_1 = \frac{1}{1+x^2}, \quad y_1(0) = 1 \quad ...(1)$$

$$y_1(1+x^2) = 1 \quad ...(2)$$

$$\Rightarrow \quad (1+x^2)y_2 + 2xy_1 = 0$$

$$y_2(0) = 0 \quad ...(3)$$

Using Leibnitz theorem n times, we get

$$\left[(1+x^2)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2}.2y_n\right] + 2xy_{n+1} + 2ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2nxy_{n+1} + n^2y_n - ny_n + 2ny_n + 2xy_{n+1} = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (n+1)2xy_{n+1} + (n^2+n)y_n = 0$$

$$y_{n+2}(0) = -n(n+1)\,y_n(0) \quad ...(4)$$

Putting n = 1, 2, 3..... in (4), we get

$$y_3(0) = -2\,y_1(0) = -2$$

$$y_4(0) = -2\times 3\,y_2(0) = 0 \quad \text{[by (2)]}$$

$$y_5(0) = -3\times 4\,y_3(0) = 24 \quad \text{[by (3)]}$$

By Maclauring expansion

$$y = \tan^{-1} x = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + ...$$

$$\Rightarrow \quad \tan^{-1}x = x + \frac{x^3}{3!}(-2) + \frac{x^5}{5!}\times 24 + ...$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + ...$$

Q. 3. (b) Test the convergence of the series $\frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \frac{4!}{5^4}x^4 + ...$ **(6.5)**

Ans. Here

$$U_n = \frac{n!}{(n+1)^n}x^n$$

$$U_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}}x^{n+1}$$

$$\frac{U_n}{U_{n+1}} = \frac{n!}{(n+1)^n}.x^n.\frac{(n+2)^{n+1}}{(n+1)!}\frac{1}{x^{n+1}}$$

$$= \frac{n!}{n^n\left(1+\frac{1}{n}\right)^n}.\frac{n^n.n\left(1+\frac{2}{n}\right)^{n+1}}{(n+1)n!}\frac{1}{x}$$

$$= \frac{n}{n\left(1+\frac{1}{n}\right)} \frac{(1+2/n)^n.(1+2/n)}{\left(1+\frac{1}{n}\right)^n} \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{(1+2/n)^n(1+2/n)}{\left(1+1/n\right)\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x}$$

$$= \frac{e^2}{e} \cdot \frac{1}{x} = \frac{e}{x}$$

By Ratio test

series is convergent if $\frac{e}{x} > 1 \Rightarrow$ if $x < e$

series is divergent if $\frac{e}{x} < 1 \Rightarrow$ if $x > e$

Test fails if $\frac{e}{x} = 1 \Rightarrow x = e$

By log test

$$\log \frac{U_n}{U_{n+1}} = \log \left[\frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1} e} \right]$$

$$\Rightarrow \quad \log \frac{U_n}{U_{n+1}} = \log\left(1+\frac{2}{n}\right)^{n+1} - \log e - \log\left(\frac{1+1}{n}\right)^{n+1}$$

$$= (n+1)\log\left(1+\frac{2}{n}\right) - 1 - (n+1)\log\left(1+\frac{1}{n}\right)$$

$$= (n+1)\left[\frac{2}{n} - \frac{4}{n^2.2} + \frac{8}{n^3.3} + \ldots\right] - 1 - (n+1)\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right]$$

$$= (n+1)\left[\left(\frac{2}{n} - \frac{1}{2}\frac{4}{n^2} + \frac{1}{3}\cdot\frac{8}{n^3} \ldots\right) - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\right)\right] - 1$$

$$= (n+1)\left[\frac{1}{n} - \frac{3}{2n^2} + \frac{7}{3n^3} \ldots\right] - 1$$

$$= 1 - \frac{3}{2n} + \frac{7}{3n^2} + \frac{1}{n} - \frac{3}{2n^2} + \frac{7}{3n^2} + \ldots - 1$$

$$\log \frac{U_n}{U_{n+1}} = -\frac{1}{2n} + \frac{5}{6n^2} + \frac{7}{3n^3} + \ldots\ldots$$

$$n \log \frac{U_n}{U_{n+1}} = n\left[-\frac{1}{2n} + \frac{5}{6n^2} + \frac{7}{3n^3} + \ldots\ldots\right]$$

$$= \frac{-1}{2} + \frac{5}{6n} + \frac{7}{3n^2} \ldots\ldots\ldots$$

$$\lim_{n\to\infty} n \log \frac{U_n}{U_{n+1}} = \frac{-1}{2} < 1$$

$\therefore$ By log test, series is divergent. Hence, given series ΣU_n converges if $x < e$ and diverges if $x \geq e$.

UNIT–II

Q. 4. (a) If ρ_1, ρ_2 be the radii of curvature at extremities of two conjugate diameters of an ellipse $x = a \cos \theta$ and $y = b \sin \theta$, then prove that $\left(\rho_1^{2/3} + \rho_2^{2/3}\right)(ab)^{2/3} = a^2 + b^2$. **(6)**

Ans. Given

$$x = a\cos\theta \quad , \quad y = b\sin\theta$$
$$x' = -a\sin\theta \ , \quad y' = b\cos\theta$$
$$x'' = -a\cos\theta \ , \quad y'' = -b\sin\theta.$$

$$\rho = \frac{\left[(x')^2 + (y')^2\right]^{3/2}}{x'y'' - y'x''}$$

$$\Rightarrow \qquad \rho = \frac{\left[a^2\sin^2\theta + b^2\cos^2\theta\right]^{3/2}}{ab\sin^2\theta + ab\cos^2\theta}$$

$$\Rightarrow \qquad \rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

At point D,

$$x = -a\sin\theta, y = b\cos\theta.$$
$$x' = -a\cos\theta, y' = -b\sin\theta.$$
$$x'' = a\sin\theta, \ y'' = -b\cos\theta.$$

$$\rho_2 = \frac{\left[a^2\cos^2\theta + b^2\sin^2\theta\right]^{3/2}}{ab\cos^2\theta + ab\sin^2\theta}$$

$$= \frac{\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{3/2}}{ab}$$

At point C

$$x = a\cos\theta, \quad y = b\cos\theta$$

$$\rho_1 = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

Consider
$$\rho_1^{2/3} + \rho_2^{2/3} = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{(ab)^{2/3}} + \frac{a^2\cos^2\theta + b^2\sin^2\theta}{(ab)^{2/3}}$$

$$\Rightarrow \quad \left[\rho_1^{2/3} + \rho_2^{2/3}\right](ab)^{2/3} = a^2 + b^2.$$

Q. 4. (b) If $\overline{|n}\ \overline{|1-n} = \dfrac{\pi}{\sin n\pi}$, then show that $\displaystyle\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}$. (6.5)

Ans. As
$$B(n, m) = \frac{\overline{|n}\ \overline{|m}}{\overline{|n+m}}$$

Put
$$m = 1-n$$

$$\Rightarrow \quad B(n, 1-n) = \frac{\overline{|n}\ \overline{|1-n}}{\overline{|1}}$$

$$\Rightarrow \quad \overline{|n}\ \overline{|1-n} = B(n, 1-n) = B(1-n, n) \text{ (By symmetry)} \qquad ...(1)$$

Consider
$$\beta(1-n, n) = \int_0^1 y^{-n}(1-y)^{n-1}dy$$

Let
$$y = \frac{1}{1+x} \Rightarrow dy = \frac{-1}{(1+x)^2}dx$$

Now
$$\beta(1-n, n) = \int_\infty^0 \left(\frac{1}{1+x}\right)^{-n}\left(\frac{x}{1+x}\right)^{n-1}\left(\frac{-1}{(1+x)^2}\right)dx$$

$$= -\int_\infty^0 \frac{x^{n-1}}{(1+x)^{-n+n-1+2}}dx$$

$$\Rightarrow \quad \beta(1-n, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)}dx$$

By (1)
$$\overline{|n}\ \overline{|1-n} = \int_0^\infty \frac{x^{n-1}}{(1+x)}dx = \frac{\pi}{\sin n\pi}$$

Q. 5. (a) Trace the curve $x^3 + y^3 = 3axy$. (6)

Ans. Given
$$x^3 + y^3 = 3axy \qquad ....(1)$$

(i) Symmetry– The curve is symmetrical about the line $y = x$.

(ii) Origin– Curve passes through origin and tangents at origin are $x = 0, y = 0$. These tangents being real and distinct, origin is a node.

(iii) Putting $x = 0$ we get $y = 0$

Putting $y = 0$, we get $x = 0$

$\therefore$ Curve meets the axes only at origin (0, 0)

Curve meets $y = x$ at point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

(iv) Tangent at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Let $$x = X + \frac{3a}{2},\ y = Y + \frac{3a}{2}$$

$$\Rightarrow \quad \left(X + \frac{3a}{2}\right)^3 + \left(Y + \frac{3a}{2}\right)^3 = 3a\left(X + \frac{3a}{2}\right)\left(Y + \frac{3a}{2}\right)$$

$$\Rightarrow X^3 + Y^3 - 3a\,XY + \frac{9a}{2}(X^2 + Y^2) + \frac{9a^2}{4}(X + Y) = 0$$

Lowest degree term $$X + Y = 0$$

Tangent at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is

$$x - \frac{3a}{2} + y - \frac{3a}{2} = 0 \Rightarrow x + y = 3a$$

(v) Asymptotes: No parallel asymtotes.

Oblique– Put $x = 1$ and $y = m$ in the highest degree term and equating to zero

$$\phi_3 = 1 + m^3 = 0$$
$$\Rightarrow m = -1$$
$$\phi_2 = -3a\,xy$$
$$\phi_2(m) = -3a\,m.$$
$$\phi_3' = 3m^2.$$

Now for $$m = -1$$

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{3am}{3m^2}$$

$$\Rightarrow \quad c = \frac{a}{m} = -a$$

$\therefore$ asymptote is $$y = -x - a$$

(vi) Region– x and y cannot both be – ve.

If x and y are both – ve, then LHS of (1) is –ve and RHS of (1) is +ve which is impossible.

$\therefore$ No portion of curve lies in 3rd quadrant.

Transforming (1) to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, we get

$$r^3(\cos^3\theta + \sin^3\theta) = 3ar^2 \sin\theta\cos\theta$$

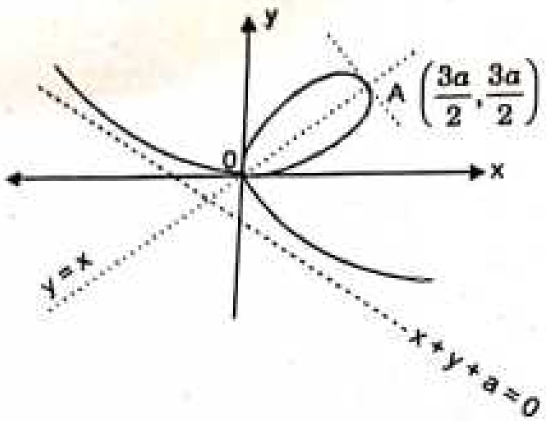
$$\Rightarrow \qquad r = \frac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}$$

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	0	$\frac{3a\sqrt{2}}{2}$	$\frac{6a\sqrt{3}}{1+3\sqrt{3}}$	0	$\frac{-6a\sqrt{3}}{-1+3\sqrt{3}}$	0

r must increase from 0 and then decrease to 0.

Thus there is a loop between $\theta = 0$ and $\theta = \frac{\pi}{2}$

r. is –ve in range $\frac{2\pi}{3}$. Thus, we have portion of curve in fourth quadrant.

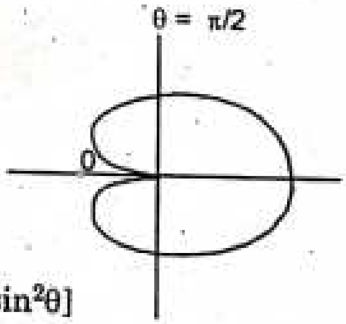


Q. 5. (b) Find the entire length of the cardiod r = a (1 + cos θ). (6.5)

Ans.

$$r = a(1+\cos\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta.$$



$$S = 2\int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$r^2+\left(\frac{dr}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2[1+\cos^2\theta + 2\cos\theta + \sin^2\theta]$$

$$= a^2[2+2\cos\theta]$$

$$= 2a^2(1+\cos\theta)$$

$$= 4a^2\cos^2\theta/2$$

$$S = 2\int_0^{\pi}\sqrt{4a^2\cos^2\frac{\theta}{2}}\, d\theta$$

$$\Rightarrow \qquad S = 4a\int_0^{\pi}\cos\frac{\theta}{2}d\theta$$

$$\Rightarrow \qquad S = 4a\times 2\left|\sin\frac{\theta}{2}\right|_0^{\pi}.$$

$$\Rightarrow \qquad S = 8a.$$

UNIT-III

Q. 6. (a) Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ **by elementary row operations.** (6)

Ans. Let consider

$$A = IA$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_3 \leftrightarrow R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$$R_3 \rightarrow R_3 + 5R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$$R_3 \rightarrow \frac{1}{2} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 + R_3 \;,\; R_2 \rightarrow R_2 - 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$$\Rightarrow \quad I = A^{-1}A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Q. 6. (b) Using matrix method, show that the equations $3x + 4y + 2z = 1$, $x + 2y = 4$, $10y + 3z = -2$, $2x - 3y - z = 5$ are consistent and hence obtain solutions for x, y and z. **(6.5)**

Ans.

$$3x + 4y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

In matrix notation $\quad AX = B$

$$\begin{bmatrix} 3 & 4 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 2 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 5 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1, R_4 \to R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 10 & 3 \\ 0 & -7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -2 \\ -3 \end{bmatrix}$$

$$R_2 \to -2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 10 & 3 \\ 0 & -7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11/2 \\ -2 \\ -3 \end{bmatrix}$$

$$R_3 \to R_3 - 10R_2, \quad R_4 \to R_4 + 7R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 13 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11/2 \\ -57 \\ 71/2 \end{bmatrix}$$

$$R_3 \to \frac{1}{13}R_3,\ R_4 \to \frac{-1}{8}R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11/2 \\ -57/13 \\ -71/16 \end{bmatrix}$$

$$R_4 \to R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11/2 \\ -57/13 \\ -11/208 \end{bmatrix}$$

$$r(A) = 3,\ r(A:B) = 4$$

$\therefore$ System has inconsistent solution.

Q. 7. (a) Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. **(6)**

Ans. Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(4-\lambda) - 10 = 0$$

$$\Rightarrow \quad \lambda^2 - 5\lambda - 6 = 0$$

$$\Rightarrow \quad \lambda = 6, -1$$

For $\lambda = 6$

$$[A - 6I]X = 0$$

$$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$r(A) = 1$$

$$-5x_1 - 2x_2 = 0$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{-5}$$

$$X_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

For $\lambda = -1$ $\quad [A + I]X = 0$

$$\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 = -2x_2$$

$$x_1 = x_2$$

$$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Q. 7. (b) Use Cayley - Hamilton theorem to express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A where $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. **(6.5)**

Ans. Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda - 5 = 0$$

By cayley Hamilton theorem

$$A^2 - 4A - 5I = 0 \quad \text{....(1)}$$

Consider

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\,I$$

$$\Rightarrow \quad A^3(A^2 - 4A - 5I) + 5A^3 - 7A^3 + 11A^2 - A - 10I$$

$$\Rightarrow \quad -2A^3 + 11A^2 - A - 10I \quad \text{[By (1)]}$$

$$\Rightarrow \quad -2A(A^2 - 4A - 5I) - 8A^2 - 10A + 11A^2 - A - 10I$$

$$\Rightarrow \quad 3A^2 - 11A - 10I$$

$$\Rightarrow \quad 3(A^2 - 4A - 5I) - 11\,A - 10I + 12A + 15I$$

$$\Rightarrow \quad A + 5I.$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$$

UNIT-IV

Q. 8. (a) Solve $(D^3 - 3D^2 + 4D - 2)\,y = e^x + \cos x$. (6)

Ans. $$(D^3-3D^2+4D-2)\,y = e^x+\cos x$$

A.E. $$D^3-3D^2+4D-2 = 0$$

$$(D-1)(D^2-2D+2) = 0$$

$\Rightarrow$ $$D = 1,\ 1\pm i$$

$\Rightarrow$ $$C.F = C_1e^x + e^x(C_1\cos x + C_2\sin x)$$

$$PI = \frac{1}{D^3-3D^2+4D-2}e^x + \frac{1}{D^3-3D^2+4D-2}\cos x$$

Consider $$\frac{1}{D^3-3D^2+4D-2}e^x$$

Replace $$D \to 1$$

$$= \frac{1}{0}\,e^x \quad \text{(Case of failure)}$$

$$= x\frac{1}{3D^2-6D+4}e^x$$

$$= x\,\frac{1}{3-6+4}e^x = xe^x$$

Consider $$\frac{1}{D^3-3D^2+4D-2}\cos x$$

Replace $$D^2 = -1$$

$$= \frac{1}{-D+3+4D-2}\cos x$$

$$= \frac{1}{3D+1}\cos x$$

$$= \frac{(3D-1)}{(3D+1)(3D-1)}\cos x$$

$$= \frac{(3D-1)}{9D^2-1}\cos x$$

$$= \frac{3D\cos x-\cos x}{-9-1}$$

$$= \frac{-3\sin x-\cos x}{-10}$$

$$= \frac{3\sin x+\cos x}{10}$$

$$\therefore \qquad PI = xe^x + \frac{3\sin x + \cos x}{10}$$

Complete soln is

$$y = c_1e^x + e^x(c_1\cos x + c_2\sin x) + xe^x + \frac{3\sin x + \cos x}{10}$$

Q. 8. (b) Solve $(D^2 + 1)y = \tan x$ by using variation of parameter method. **(6.5)**

Ans. $(D^2 + 1)y = \tan x$

A.E. $D^2 + 1 = 0$

$\Rightarrow$ $D = \pm i$

$$y = c_1\cos x + c_2\sin x$$

Let $y_1 = \cos x,\ y_2 = \sin x,\ X = \tan x$

$$y_1' = -\sin x,\ y_2' = \cos x$$

Now
$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1$$

$$PI = uy_1 + vy_2$$

where
$$u = -\int \frac{y_2 X}{w}\,dx$$

$$= -\int \frac{\sin x.\tan x}{1.}\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx$$

$$= -\int \frac{(1-\cos^2 x)}{\cos x}\,dx$$

$$= -\int (\sec x - \cos x)\,dx$$

$$u = -[\log(\sec x + \tan x) - \sin x]$$

$$v = -\int \frac{y_1 X}{w}\,dx = \int \cos x \tan x\,dx$$

$$= \int \sin x\,dx = -\cos x$$

$$PI = \cos x[\sin x - \log(\sec x + \tan x)] - \sin x\cos x$$

$$PI = -\cos x\log(\sec x + \tan x)$$

complete solution is

$$y = c_1\cos x + c_2\sin x - \cos x[\log(\sec x + \tan x)]$$

30–2018

Q. 9. (a) Solve $(x^2 D^2 - 3xD + 4)y = x^3$. (6)

Ans.

$$(x^2 D^2 - 3xD + 4)y = x^3$$

Let

$$x = e^z \quad , z = \log x, \quad D = \frac{d}{dz}$$

$$xDy = Dy$$

$$x^2D^2y = D(D-1)y$$

$$(D(D-1) - 3D + 4)y = e^{3z}$$

$$(D^2 - 4D + 4)y = e^{3z}$$

$\Rightarrow$

A.E

$$D^2 - 4D + 4 = 0$$

$$D = 2, 2$$

$$CF = (C_1 + C_2 z)e^{2z}$$

$$PI = \frac{1}{(D-2)^2} e^{3z} = e^{3z}$$

$$y = (C_1 + C_2 z)e^{2z} + e^{3z}$$

$\therefore$

$$= (C_1 + C_2 \log x)x^2 + x^3$$

Q. 9. (b) Show that $\frac{d}{dx}\left\{x\, J_n(x)\, J_{n+1}(x)\right\} = x\left\{J_n^2(x) - J_{(n+1)}^2(x)\right\}$ (6.5)

Ans. $\frac{d}{dx}\{x\, J_n(x)\, J_{n+1}(x)\}$

$$= J_n(x)\, J_{n+1}(x) + x[xJ_n'(x)J_{n+1}(x) + J_{n+1}'(x)J_n(x)]$$

$$\Rightarrow \quad \frac{d}{dx}\{x\, J_n J_{n+1}\} = J_n J_{n+1} + J_{n+1}\{x J'_n\} + J_n\{x J'_{n+1}\} \quad ...(1)$$

As we know by relation

$$xJ'_n(x) = nJ_n(x) - x J_{n+1}(x) \quad ...(2)$$

Also

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x) \quad ...(3)$$

Changing n to $n+1$ in (3), we get

$$J'_{n+1}(x) + \frac{n+1}{x}J_{n+1}(x) = J_n(x)$$

$$\Rightarrow \quad xJ'_{n+1}(x) = x J_n(x) - (n+1)J_{n+1}(x) \quad ...(4)$$

Putting (2) and (4) in (1), we have

$$\frac{d}{dx}\{xJ_nJ_{n+1}\} = J_nJ_{n+1} + J_{n+1}\{n J_n - x J_{n+1}\} + J_n\{x J_n - (n+1)J_{n+1}\}$$

$$= J_n J_{n+1} + nJ_n J_{n+1} - x J^2_{n+1} + x J_n^2 - (n+1)J_n J_{n+1}$$

$$= x(J_n^2 - J^2_{n+1})$$

$$\Rightarrow \quad \frac{d}{dx}\{x\, J_n(x)J_{n+1}(x)\} = x\{J_n^2(x) - J_{n+1}^2(x)\}$$

SYLLABUS

APPLIED MATHEMATICS-I—[ETMA-101]

UNIT-I

Successive differentiation: Leibnitz theorem for nth derivative (without proof). Infinite series: Convergence and divergence of infinite series, positive terms infinite series, necessary condition, comparison test (Limit test), D'Alembert ratio test, Integral Test, Cauchy's root test, Raabe's test and Logarithmic test(without proof). Alternating series, Leibnitz test, conditional and absolutely convergence. Taylor's and Maclaurin's expansion (without proof) of function (ex, log(1+x), cos x , sin x) with remainder terms, Taylor's and Maclaurin's series, Error and approximation. **[T1], [T2] [No. of hrs. 12]**

UNIT- II

Asymptotes to Cartesian curves. Radius of curvature and curve tracing for Cartesian, parametric and polar curves. Integration: integration using reduction formula for , . Application of integration : Area under the curve, length of the curve, volumes and surface area of solids of revolution about axis only. Gamma and Beta functions. **[T1], [T2][No. of hrs. 12]**

UNIT- III

Matrices: Orthogonal matrix, Hermitian matrix, Skew-Hermitian matrix and Unitary matrix. Inverse of matrix by Gauss-Jordan Method (without proof). Rank of matrix by echelon and Normal (canonical) form. Linear dependence and linear independence of vectors. Consistency and inconsistency of linear system of homogeneous and non homogeneous equations . Eigen values and Eigen vectors. Properties of Eigen values (without proof). Cayley-Hamilton theorem (without proof). Diagonlization of matrix. Quadratic form, reduction of quadratic form to canonical form. **[T1], [T2][No. of hrs. 12]**

UNIT-IV

Ordinary differential equations: First order linear differential equations, Leibnitz and Bernaulli's equation. Exact differential equations, Equations reducible to exact differential equations. Linear differential equation of higher order with constant coefficients, Homogeneous and non homogeneous differential equations reducible to linear differential equations with constant coefficients. Method of variation of parameters. Bessel's and Legendre's equations (without series solutions), Bessel's and Legendre's functions and their properties.

FIRST TERM EXAMINATION
FIRST SEMESTER (B. TECH) ETMA-101
APPLIED MATHEMATICS-2013

Time : 1 1/2 hrs. **M.M. : 30**

Note: ***Attempt question no.1 and any two more questions.***

Q.1. (*a*) Discuss the convergence of series $\sum_{n=1}^{\infty} \frac{1}{\log(n+7)}$ **(3)**

Sol. Let
$$U_n = \frac{1}{\log(n+7)}$$

As $\log n < n$

$$\Rightarrow \quad \log(n+7) < n+7$$

$$\Rightarrow \quad \frac{1}{\log(n+7)} > \frac{1}{n+7}$$

$$\Rightarrow \quad u_n \geq kv_n$$

Let
$$v_n = \frac{1}{n+7}$$

$$v_n = \frac{1}{n(1+7/n)}$$

Let
$$\omega_n = \frac{1}{n}$$

Now
$$\frac{v_n}{\omega_n} = \frac{1}{1+7/n}$$

$$\lim_{n\to\infty} \frac{v_n}{\omega_n} = \lim_{n\to\infty} \frac{1}{1+7/n} = 1 \text{ (non zero and finite)}$$

Now
$$\Sigma\omega_n = \Sigma\frac{1}{n} \quad (p = 1)$$

$\Sigma\omega_n$ is divergent.

$\Rightarrow \Sigma v_n$ divergent by limit comparison Test

$\therefore \Sigma u_n$ also diverges by comparison test.

Q.1. (*b*) Find the Maclaurin's series expansion of function $e^x \log(1+x)$ (3)

Ans. As
$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

$$= 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\ldots$$

and $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\ldots$

$$\therefore \quad e^x \log(1+x) = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\ldots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\ldots\right)$$

$$= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\ldots x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\ldots$$

$$+\frac{x^3}{2}-\frac{x^4}{4}+\frac{x^5}{6}+\ldots+\frac{x^4}{6}-\frac{x^5}{12}+\ldots+\frac{x^5}{24}+\ldots$$

$$= x+\frac{x^2}{2}+\frac{x^3}{3}+x^5\left(\frac{1}{5}-\frac{1}{4}+\frac{1}{6}-\frac{1}{12}+\frac{1}{24}+\ldots\right)$$

$$= x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{3x^5}{40}+\ldots$$

Q.1. (c) If $T = 2\pi\sqrt{\frac{l}{g}}$, find the error in T corresponding to an error of 2% in l where g is constant. **(2)**

Ans. $$T = 2\pi\sqrt{\frac{l}{g}}$$

Given $$\frac{\delta l}{l} = 2$$

$$\frac{dT}{dl} = \frac{2\pi}{\sqrt{g}}.\frac{1}{2}\frac{1}{\sqrt{l}} = \frac{\pi}{\sqrt{lg}}$$

error in T is given by

$$\delta T = \frac{dT}{dl}\times\delta l$$

$$\Rightarrow \quad \frac{\delta T}{T}\times 100 = \frac{dT}{dl}\times\frac{\delta l}{T}\times 100$$

$$= \frac{\pi}{\sqrt{lg}}\times\delta l.\times\frac{100}{T} = \frac{\pi\sqrt{l}}{\sqrt{g}}\times\frac{\delta l}{l}\times\frac{100}{T}$$

$$= \frac{2}{2}\times 100\times\frac{T}{T} = 1$$

$\therefore$ error in T = 1%

Q.1. (d) Evaluate $\int_0^{\pi/2} \sin^7 x\, dx$ ...(2)

Sol. By reduction formula.

$$n = 7$$

$$I_n = \frac{(n-1)(n-3)\ldots 2}{n(n-2)\ldots 3}$$

$$= \frac{6.4.2}{7.5.3} = \frac{16}{35}$$

Q.2. (a) If $y = \sin(m \sin^{-1} x)$ then show that
$(1 - x^2)y_{n+2} - (2n + 1)\, x\, y_{n+1} - (n^2 - m^2)y_n = 0$ **(5)**

Sol. Let $\quad y = \sin(m \sin^{-1} x)$

On differentiating w.r.t 'x'

$$y_1 = \cos(m \sin^{-1} x).\frac{m}{\sqrt{1-x^2}}$$

Squaring and cross multiplying

$$(1-x^2)y_1^2 = m^2 \cos^2(m \sin^{-1} x)$$
$$= m^2[1 - \sin^2(m \sin^{-1} x]$$
$$(1-x^2)y_1^2 = m^2(1-y^2)$$

Again differentiating wr.t. 'x'.

$$-2xy_1^2 + 2y_1 y_2(1-x^2) = m^2(-2yy_1)$$
$$\Rightarrow \quad (1-x^2)y_2 - xy_1 + m^2 y = 0$$

Using Leibnitz theorem, differentiating 'n' times, we get

$$(1-x^2)y_{n+2} + n\, y_{n+1}(-2x) + \frac{n(n-1)}{2} y_n(-2) - [x\, y_{n+1} + n\, y_n] + m^2 y_n = 0$$
$$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} + (m^2 - n^2)y_n = 0$$
$$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2 - m^2)\, y_n = 0$$

Q.2. (b) Test the convergence of $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n(n+3)}$, $x \in R$ **(5)**

Sol. $\sum_{n=1}^{\infty}(-1)^n u_n$ is an alternating series.

Here $\quad u_n = \frac{x^n}{n(n+3)}$

Consider $\sum_{n=1}^{\infty}\left|(-1)^n \frac{x^n}{n(n+3)}\right|$

$$= \sum_{n=1}^{\infty}|(-1)^n u_n| = \sum_{n=1}^{\infty}|u_n|$$

$$u_n = \frac{|x|^n}{n(n+3)}$$

$$u_{n+1} = \frac{|x|^{n+1}}{(n+1)(n+4)}$$

Now $\quad \frac{u_n}{u_{n+1}} = \frac{|x|^n}{n(n+3)} \frac{(n+1)(n+4)}{|x|^{n+1}}$

$$= \frac{n^2(1+1/n)(1+4/n)}{n^2(1+3/n)|x|}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{|x|}$$

$\therefore$ By Ratio Test $\Sigma|u_n|$ is convergent if $\frac{1}{|x|} > 1$ or $|x| < 1$.

and divergent if $\frac{1}{|x|} < 1$ or $|x| > 1$

Thus series is absolutely convergent if $|x| < 1$ and hence convergent if $-1 < x < 1$.

Test fails for $|x| = 1$

or $$x = -1, 1$$

When $x = 1$ series becomes.

$$-\frac{1}{1.4} + \frac{1}{2.5} - \frac{1}{3.7} + \ldots$$

Here $$u_n = \frac{1}{n(n+3)}$$

$$u_{n+1} = \frac{1}{(n+1)(n+4)}$$

(*i*) As $n < n+1$

and $$n+3 < n+4$$

$\Rightarrow$ $$n(n+3) < (n+1)(n+4)$$

$\Rightarrow$ $$\frac{1}{n(n+3)} > \frac{1}{(n+1)(n+4)}$$

$\Rightarrow$ $$u_n > u_{n+1}$$

$\therefore$ $<u_n>$ is monotonic decreasing.

(*ii*) $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n(n+3)} = 0$$

$\therefore$ By leibnitz test Σu_n is convergent

$\Rightarrow \Sigma(-1)^n u_n$ is convergent.

When $$x = -1 \text{ series becomes.}$$

$$\frac{1}{1.4} \frac{-1}{2.5} + \frac{1}{3.7} \ldots$$

i.e., $\sum(-1)^{n+1} \frac{1}{n(n+3)}$

$$u_n = \frac{1}{n(n+3)}$$

$$u_{n+1} = \frac{1}{(n+1)(n+4)}$$

It is convergent by Leibnitz test *i.e.*, $\Sigma(-1)^{n+1} u_n$ is convergent.

Thus given series is convergent if $-1 \leq x \leq 1$ or $|x| \leq 1$.

Q.3. (a) Find the asymptotes of the curve **(4)**

$x^3 + x^2y + xy^2 + y^3 + 2x^2 + 3xy - 4y^2 + 7x + 2y = 0$

Sol. Since coefficient of x^3 and y^3 are constants.

$\therefore$ there are no asymptotes parallel to x–axis or y–axis.

Oblique Asymptotes

$$\phi_3(x, y) = x^3 + x^2y + xy^2 + y^3$$
$$\phi_2(x, y) = 2x^2 + 3xy - 4y^2$$
$$\phi_1(x, y) = 7x + 2y$$
$$\phi_0 = 0$$

Put $x = 1, y = m$, We have

$$\phi_3(m) = 1 + m + m^2 + m^3.$$
$$\phi_2(m) = 2 + 3m - 4m^2$$
$$\phi_1(m) = 7 + 2m$$

Put $\phi_3(m) = 0$

$$1 + m + m^2 + m^3 = 0$$
$$m^2(m + 1) + 1(m + 1) = 0$$
$$m + 1 = 0, m^2 + 1 = 0$$
$$m = -1, m = \pm i \text{ (imaginary)}$$

Only $m = -1$ will be considered.

$$c = -\frac{\phi_2}{\phi_3'}$$
$$= \frac{-(2+3m-4m^2)}{3m^2+2m+1}$$

For $m = -1$

$$C = \frac{-(-5)}{2} = \frac{5}{2}$$

$\therefore$ asymptotes is $y = mx + c$

$$y = -x + \frac{5}{2}$$

$\Rightarrow$ $2y + 2x = 5$

Q. (b) Find the radius of curvature if $y = c \log \sec\left(\frac{x}{c}\right)$ at (x, y) where C is constant. **(4)**

Sol. $y = c \log \sec\left(\frac{x}{c}\right)$

$$\frac{dy}{dx} = c\frac{1}{\sec(x/c)}\sec\left(\frac{x}{c}\right)\tan\left(\frac{x}{c}\right)\frac{1}{c}$$
$$= \tan\frac{x}{c}$$
$$\frac{d^2y}{dx^2} = \left(\sec^2\frac{x}{c}\right).\frac{1}{c}$$

Now $$\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}$$

$$= \frac{[1+(\tan x/c)^2]^{3/2}}{\frac{1}{c}.\sec^2\frac{x}{c}} = \frac{c\left[1+\tan^2\frac{x}{c}\right]^{3/2}}{\sec^2 x/c}$$

$$= \frac{c\left(\sec^2\frac{x}{c}\right)^{3/2}}{\sec^2 x/c} = c\sec\frac{x}{c}$$

Q. (c) State Cauchy Integral Test. (2)

Sol. Let $u(x)$ be positive, monotonic decreasing and integrable on $[1, \infty)$. For each $n \varepsilon N$, let $u_n = u(n)$. Then the series $\sum_{n=1}^{\infty} u_n$ and the integral

$\int_1^{\infty} u(x)dx$ either both converge or both diverge.

Q.4. (a) Trace the curve $9ay^2 = x(x-3a)^2$ (5)

Sol. (1) It is symmetric about x–axis as there are even powers of y only.

2. for $x = 0, y = 0$

It passes through origin.

Tangents at origin

$$9\,ay^2 = x(x^2+9a^2-6ax)$$

$\Rightarrow$ $$9\,ay^2 = x^3+9\,a^2x-6ax^2$$

Lowest degree term

$$9\,a^2x = 0 \Rightarrow x = 0.$$

3. For $x = 3a., y = 0$

Point of intersection $(3a, 0)$ and $(0, 0)$

Tangent at $(3a, 0)$

Put

$$X = x-3a, Y = y.$$

Equation becomes.

$$9\,aY^2 = (X+3a)X^2$$

$\Rightarrow$ $$9\,aY^2 = X^3+3\,aX^2$$

Lowest degree terms.

$$9\,aY^2-3\,aX^2 = 0$$

$\Rightarrow$ $$Y^2 = \frac{1}{3}X^2 \Rightarrow Y = \pm\frac{1}{\sqrt{3}}X$$

$\Rightarrow$ $$y = \pm\frac{1}{\sqrt{3}}(x-3a)$$

i.e., Inclined at an angle of 30° to the x-axis.

4. Aymptotes: No asymptote parallel to any axes.

Oblique:
$$\phi_3 = x^3$$
$$\Rightarrow \quad \phi_3(m) = 1$$
$$\therefore \quad 1 = 0$$

Not possible

$\Rightarrow$ no asymptotes.

5. Region of existences.

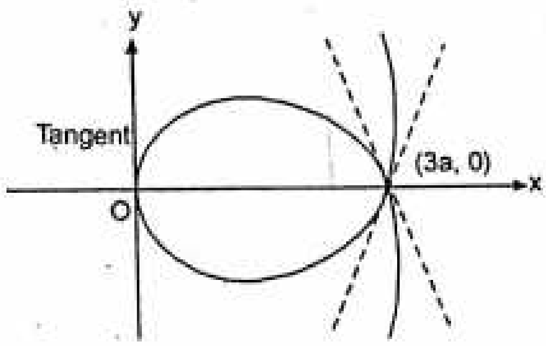
$$y^2 = \frac{x}{9a}(x-3a)^2$$

Region	Points	Sign	Existence
$x < 0$	$x = -a$	$-ve$	No.
$0 < x < 3a$	$x = 2a$	+ ve	Yes
$x > 3a$	$x = 4a$	+ ve	Yes.

$\therefore$ Curve lies in second and the fourth quadrants.

As x increases from 0 to a, y also increases, when x increases from a to $3a$, y decreases to zero.

When x increase beyond $3a$, y goes on increasing and as $x \to \infty$, $y \to \infty$.



Q.(b) Evaluate $\int_0^{\pi/2} \sin^5 x \cos 2x \, dx$ **(3)**

Sol. Let
$$I = \int_0^{\pi/2} \sin^5 x(1-2\sin^2 x)dx$$
$$\Rightarrow \quad I = \int_0^{\pi/2} \sin^5 x - 2\sin^7 x\, dx$$
$$I = \int_0^{\pi/2} \sin^5 x\, dx - 2\int_0^{\pi/2} \sin^7 x\, dx$$
$$I = I_1 + I_2$$
$$I_1 = \int_0^{\pi/2} \sin^5 x\, dx$$
$$= \frac{4.2}{5.3} = \frac{8}{15}$$

$$I_2 = -2\int_0^{\pi/2} \sin^{-3} x \, dx$$

$$= -2\left(\frac{6.4.2.}{7.5.3}\right) = \frac{-32}{35}$$

Now
$$I = \frac{8}{15} - \frac{32}{35}$$

$$= \frac{56-96}{105} = \frac{-40}{105} = \frac{-8}{21}$$

Q. (c) Find Taylor series expansion of $y = \sin x$ about a point $x = \frac{\pi}{4}$ **(2)**

Sol.
$$f(x) = y = \sin x$$

$$f(\pi/4) = \frac{1}{\sqrt{2}}$$

$$f'(x) = \cos x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f''(x) = -\sin x$$

$$f''\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

$$f'''(x) = -\cos x$$

$$f'''\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

$$f^{iv}(x) = \sin x$$

$$f^{iv}(\pi/4) = \frac{1}{\sqrt{2}}$$

By Taylor's series

$$f(x) = f\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \left(\frac{x-\pi/4}{2}\right)^2 f''\left(\frac{\pi}{4}\right) + \frac{(x-\pi/4)^3}{3!} f'''\left(\frac{\pi}{4}\right) + \ldots$$

$$f(x) = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - \frac{x-(\pi/4)^2}{2!}\frac{1}{\sqrt{2}} - \frac{(x-\pi/4)^3}{3!}\frac{1}{\sqrt{2}} + \frac{(x-\pi/4)^4}{4!}\frac{1}{\sqrt{2}} + \ldots$$

SECOND TERM EXAMINATION
FIRST SEMESTER (B. TECH) ETMA-101
APPLIED MATHEMATICS-2013

Time : 1 1/2 hrs. **M.M. : 30**

Note: ***Attempt question no.1 and any two more questions.***

Q.1. (*a*) Show that the system of three vector (1, 3, 2), (1, – 7, –8) and (2, 1, –1) is linearly dependent. **(2)**

Sol. Consider the matrix equation

$$\lambda_1(1, 3, 2) + \lambda_2(1, -7, -8) + \lambda_3(2, 1, -1) = 0 \quad ...(1)$$

$$\Rightarrow \quad \left.\begin{aligned} \lambda_1 + \lambda_2 + 2\lambda_3 &= 0 \\ 3\lambda_1 - 7\lambda_2 + \lambda_3 &= 0 \\ 2\lambda_1 - 8\lambda_2 - \lambda_3 &= 0 \end{aligned}\right\} \quad ...(2)$$

In matrix form $AX = B$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -10 & -5 \\ 0 & -10 & -5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$.

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As $r(A) = 2 < 3$ (no. of unknowns) $\therefore$ It has infinite number of non-trivial solution *i.e.*, $\exists$ scalars $\lambda_1, \lambda_2, \lambda_3$ not all zero such that.

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$$

Thus they are linearly dependent.

(*b*) For the matrix $A = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & -i \\ -2-3i & -i & 0 \end{bmatrix}$ ...(2)

Show that $\overline{A}$ is skew hermitian matrix.

Sol. Let $\overline{A} = B = \begin{bmatrix} -i & 1-i & 2+3i \\ -1-i & -2i & i \\ -2+3i & i & 0 \end{bmatrix}$

To show $$B = -(\bar{B})'$$

Now $$\bar{B} = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & -i \\ -2-3i & -i & 0 \end{bmatrix}$$

$$(\bar{B})' = \begin{bmatrix} i & -1+i & -2-3i \\ 1+i & 2i & -i \\ 2-3i & -i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} -i & 1-i & 2+3i \\ -1-i & -2i & i \\ -2+3i & i & 0 \end{bmatrix} = -B$$

$\therefore B = \bar{A}$ is skew-hermitian.

(c) Solve $\frac{dy}{dx} + \frac{y}{x} = y^2$. **(3)**

Sol. $$y^{-2}\frac{dy}{dx} + \frac{1}{yx} = 1$$

Let $$z = \frac{1}{y}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{-1}{y^2}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{-dz}{dx} + \frac{z}{x} = 1$$

$$\Rightarrow \quad \frac{dz}{dx} - \frac{z}{x} = -1$$

which is linear in z

Let $$P = \frac{-1}{x}$$

$$\therefore \quad \text{I.F.} = e^{\int -1/x\,dx}$$

$$= e^{\log x^{-1}} = x^{-1}$$

Complete solution is

$$zx^{-1} = \int (-1)x^{-1}dx + c$$

$$\Rightarrow \quad \frac{z}{x} = -\log x + c$$

$$\Rightarrow \quad \frac{1}{xy} = -\log x + c.$$

(d) By using Gamma function evaluate $\int_0^2 (4-x^2)^{3/2} dx$ **(3)**

Sol. Put
$$x = 2\sin\theta$$
$$\Rightarrow \quad dx = 2\cos\theta \, d\theta$$
$$\therefore \quad \int_0^2 (4-x^2)^{3/2} dx = \int_0^{\pi/2} (4-4\sin^2\theta)^{3/2} 2\cos\theta d\theta$$
$$= 16\int_0^{\pi/2} \cos^4\theta \, d\theta$$
Here
$$n = 4$$
$$= 16\frac{\sqrt{5/2}}{\sqrt{6/2}} \cdot \frac{\sqrt{\pi}}{2}$$
$$= 8\sqrt{\pi}\frac{\sqrt{5/2}}{\sqrt{3}}$$
$$= \frac{8}{2}\sqrt{\pi}\frac{3}{2}\times\frac{1}{2}\frac{\sqrt{\pi}}{2} = \frac{3\pi}{2}$$

Q.2. (a) For what values of λ and μ, the system of equations $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ has **(4)**

(i) Unique solution

(ii) Infinite number of solution

(iii) No solution.

Sol. In matrix form it can written as
$$AX = B$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$
$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & \lambda-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu-6 \end{bmatrix}$$
$$R_3 \to R_3 - R_2$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu-10 \end{bmatrix}$$

Case 1. If $\lambda \neq 3$, μ may have any value. Then $r(A) = r[A:B] = n = 3$.

$\therefore$ system has unique solution.

Case 2. If $\lambda = 3, \mu = 10$. Then $r[A] = r[A:B] = 2 <$ number of unknowns

$\therefore$ system has infinite many solution.

Case 3. If $\lambda = 3, \mu \neq 10$

then $r[A]=2$, $r[A:B]=3 \Rightarrow r[A] \neq r(A:B)$

$\therefore$ system has no solution.

Q. (*b*) FInd normal form of matrix.

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & -4 \\ -3 & 1 & -1 \end{bmatrix} \quad ...(4)$$

Sol.

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & -4 \\ -3 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_3$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 + \frac{1}{3}C_1$

$$\sim \begin{bmatrix} 3 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_1 \to \frac{1}{3}C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{c:c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$$

(c) State Cayley Hamilton theorem. (1)

Sol. Cayley Hamilton states every square matrix satisfies its characterstic equation.

Q.3. (a) Find diagonal form of matrix. (5)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ if possible.}$$

Sol.
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(1-\lambda)(1-\lambda)^2 - 1(0-0) = 0$

$\Rightarrow$ $(1-\lambda)^3 = 0$

$\Rightarrow$ $\lambda = 1, 1, 1$ (eigen values)

For $\lambda = 1$

$$[A - I]X = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x_2 = 0, x_3 = 0$

As $r(A) = 2 <$ no of unknowns.

$\therefore$ 1 variable be given arbitrary value.

Let $x_1 = 1$

$\therefore$ eignvector $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Not possible as other two eignvectors are not feasible.

Q. (b) Using method of variation of parameter. Find solution of differential equation $\frac{d^2y}{dx^2} + 4y = \tan 2x$. (5)

Ans. $(D^2 + 4)y = \tan 2x$

Auxiliary equation is

$$D^2 + 4 = 0$$

$\Rightarrow$ $D = \pm 2i$

$$\text{C.F.} = C_1 \cos 2x + C_2 \sin 2x$$

Here $y_1 = \cos 2x, y_2 = \sin 2x$

and $X = \tan 2x$

$$y'_1 = -2 \sin 2x, y'_2 = 2 \cos 2x$$

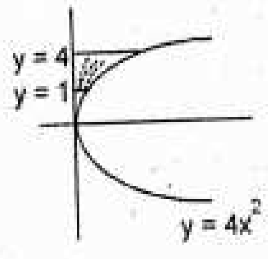
Now $$W = y_1 y_2' - y_1' y_2$$
$$= 2\cos^2 2x + 2\sin^2 2x$$
$$W = 2$$
$\Rightarrow$ $$P.I = y_1 u + y_2 V$$
$$u = -\int \frac{y_2 X}{W} dx$$
$$= -\int \frac{\sin 2x \tan 2x}{2} dx = \frac{-1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$$
$$= \frac{-1}{2}\int \frac{1-\cos^2 2x}{\cos 2x} dx = \frac{-1}{2}\int (\sec 2x - \cos 2x) dx$$
$$= \frac{-1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$
$\Rightarrow$ $$u = \frac{1}{4}[\sin 2x - \log(\sec 2x + \tan 2x)]$$
and $$v = \int \frac{y_1 X}{W} dx$$
$$= \int \frac{\cos 2x \tan 2x}{2} dx$$
$$= \frac{1}{2}\int \sin 2x\, dx = -\frac{\cos 2x}{4}$$
Now $$P.I. = \frac{\cos 2x}{4}[\sin 2x - \log(\sec 2x + \tan 2x)] - \frac{\sin 2x \cos 2x}{4}$$
$$= \frac{-\cos 2x}{4}\log(\sec 2x + \tan 2x)$$

$\therefore$ complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x \; \frac{-\cos 2x}{4}\log(\sec 2x + \tan 2x)$$

Q.4. (a) Find the area of the region bounded by the parabola $y = 4x^2$, the axis of y and the two absciaae $y = 1$ and $y = 4$ lying in the first quadrant. **(4)**

Sol.



Point of intersection (1, 1/2), (4, 1), (0, 0).

$$\text{Area} = \int_{x=a}^{b} y\,dx$$

$$= \int_0^{1/2} 4x^2dx + \int_{1/2}^{1} 4x^2dx = \frac{4}{3}|x^3|_0^{1/2} + \frac{4}{3}|x^3|_{1/2}^{1}$$

$$= \frac{4}{3}\left[\frac{1}{3}\right] + \frac{4}{3}\left[1-\frac{1}{8}\right]$$

$$= \frac{4}{3}\left[\frac{1}{8}+\frac{7}{8}\right] = \frac{4}{3}$$

Q. (*b*) Find inverse of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$ **by Guass Jordan method. (4)**

Sol.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$A = I_3A$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

$$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}A$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}A$$

$$R_1 \to R_1 + 2R_2$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 2 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}A$$

$$R_1 \to R_1 - 3R_3, R_2 \to R_2 - 3R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}A$$

$$R_2 \to -R_2, R_3 \to -R_3$$

$$= \frac{mn}{m+n}\beta(m,n)$$

Consider $m\beta(m, n+1)$ $$= \frac{m\lceil m \lceil n+1}{\lceil m+n+1} = \frac{m\lceil m \cdot n\lceil n}{(m+n)\lceil m+n}$$

$$= \frac{mn}{m+n}\beta(m,n)$$

$\therefore$ L.H.S. = R.H.S.

(d) Prove that $u = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ **is a Unitary matrix.**

Sol.
$$U = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

To show. $(\bar{U})^T = U^{-1}$ or $U\bar{U}^* = \bar{U}^*U = I$

$$\bar{U} = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$$

$$\bar{U} = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

Consider $$U\bar{U}^* = \frac{1}{4}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$\bar{U}^*U = \frac{1}{4}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ U is Unitary matrix.

Q. (e) Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2xz - 2xy$ to th canonical form. Also write the nature of the quadratic form.

Sol. Given quadratic form can be written as $X'AX$

where $$X' = [x\ y\ z] \text{ and}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

As $$A = I_3AI_3.$$

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + \frac{1}{3}R_1$, $R_3 \to R_3 - \frac{1}{3}R_1$ (same on prefactor)

$$\sim\begin{bmatrix}3 & -1 & 1\\0 & 14/3 & -2/3\\0 & -2/3 & 8/3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1/3 & 1 & 0\\-1/3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\sim\begin{bmatrix}3 & -1 & 1\\0 & 14/3 & -2/3\\0 & -2/3 & 8/3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1/3 & 1 & 0\\-1/3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$C_2 \to C_2 + \frac{1}{3}C_1, C_3 \to C_3 - \frac{1}{3}C_1$ [same on post factor]

$$\sim\begin{bmatrix}3 & 0 & 0\\0 & 14/3 & -2/3\\0 & -2/3 & 8/3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1/3 & 1 & 0\\-1/3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 1/3 & -1/3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$R_3 \to R_3 + \frac{1}{7}R_2$$

$$\sim\begin{bmatrix}3 & 0 & 0\\0 & 14/3 & -2/3\\0 & 0 & 54/21\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1/3 & 1 & 0\\-6/21 & 1/7 & 1\end{bmatrix} A \begin{bmatrix}1 & 1/3 & -1/3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$C_3 \to C_3 + \frac{1}{7}C_2$$

$$\sim\begin{bmatrix}3 & 0 & 0\\0 & 14/3 & 0\\0 & 0 & 54/21\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1/3 & 1 & 0\\-6/21 & 1/7 & 1\end{bmatrix} A \begin{bmatrix}1 & 1/3 & -6/21\\0 & 1 & 1/7\\0 & 0 & 1\end{bmatrix}$$

$$diag\left(3, \frac{14}{3}, \frac{18}{7}\right) = P'AP$$

Canonical form is given by

$$Y'(P'AP)Y = \begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix}\begin{bmatrix}3 & 0 & 0\\0 & 14/3 & 0\\0 & 0 & 18/7\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$$

$$= \begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix}\begin{bmatrix}3y_1\\14/3y_2\\18/7\,y_3\end{bmatrix}$$

$$= 3y_1^2 + \frac{14}{3}y_2^2 + \frac{18}{7}y_3^2$$

Here $r(A) = 3$, index $(p) = 3$

signature $= 6 - 3 = 3$

As rank = no. of variables and $p = r$

$\therefore$ Quadratic form is positive definite.

(f) Solve the differential equation $(xy^2+x)dx+(y^2x^2+y)dy=0$

Sol. Here
$$Mdx+Ndy = 0$$
$$M = xy^2+x, N = yx^2+y$$
$$\frac{\partial M}{\partial y} = 2xy$$
$$\frac{\partial N}{\partial x} = 2xy$$

$\therefore$ equation is exact $\therefore$ Solution is

$$\int_{y\text{ constant}} (xy^2+x)dx+\int y\,dy=c$$

or
$$x^2y^2+x^2+y^2 = c$$
$\Rightarrow$
$$x^2(1+y^2)+y^2 = c$$

Q. (g) Prove that $\frac{d}{dx}[x^nJ_n(x)]=x^nJ_{n-1}(x)$.

Sol. As
$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\lceil n+r+1}\left(\frac{x}{2}\right)^{n+2r}$$

$\Rightarrow$
$$x^nJ_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^\mu x^{2n+2r}}{r!\lceil n+r+1 2^{n+2r}}$$

Consider $\frac{d}{dx}\left[x^nJn(x)\right]$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r(2n+2r)x^{2n+2r-1}}{r!\lceil n+r+1\,2^{n+2r}}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r 2(n+r).x^n x^{n+2r-1}}{r!(n+r)\lceil n+r\,2^{n+2r}}$$

$$= x^n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\lceil n+r}\left(\frac{x}{2}\right)^{n+2r-1} = x^nJ_{n-1}(x)$$

Q.(h) Express the following in terms of Legendere's polynomial x^3+2x^2-x-3.

Sol.
$$f(x) = x^3+2x^2-x-3$$

As
$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$\Rightarrow$
$$x^3 = \frac{3}{5}x+\frac{2}{5}p_3(x)$$

Now
$$f(x) = \frac{3}{5}x+\frac{2}{5}P_3(x)+2x^2-x-3$$

$\Rightarrow$
$$f(x) = \frac{2}{5}P_3(x)+2x^2-\frac{2}{5}x-3$$

Now
$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$\Rightarrow$
$$x^2 = \frac{2}{3}P_2(x)+\frac{1}{3}$$

$$\Rightarrow \quad f(x) = \frac{2}{5}P_3(x)+2\left[\frac{2}{3}p_2(x)+\frac{1}{3}\right]-\frac{2}{5}x-3$$

$$= \frac{2}{5}P_3(x)+\frac{4}{3}P_2(x)+\frac{2}{3}-\frac{2}{5}x-3$$

$$f(x) = \frac{2}{5}P_3(x)+\frac{4}{3}P_2(x)-\frac{2}{5}P_1(x)-\frac{7}{3}P_0(x)$$

Q. (i) Prove that $\lim\limits_{n\to\infty} U_n = 0$ is a necessary but not sufficient condition for the convergence of positive term series Σu_n.

Sol. Consider the series $\Sigma u_n = \Sigma \frac{1}{\sqrt{n}}$

$$\Sigma\frac{1}{\sqrt{n}} = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$$

Here
$$S_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\ldots+\frac{1}{\sqrt{n}} \qquad \left[\because m<n \Rightarrow \frac{1}{\sqrt{m}} > \frac{1}{\sqrt{m}}\right]$$

$$= \frac{n}{\sqrt{n}} = \sqrt{n}$$

$$\therefore \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty}\sqrt{n} = \infty$$

Thus series is divergent.

$\therefore$ It is not sufficient condition for convergence of Σu_n.

Q. (j) The pressure p and volume V of a gas are connected by the relation $pv^{1.4} = c$ where c is a constant. Find the percentage increase in the pressure corresponding to a decrease of $\frac{1}{2}$% in the volume.

Sol.
$$p = cv^{-1.4}$$

Let change in pressure is given by $\frac{dp}{dv}.\delta v$.

As
$$p = cv^{-1.4}$$

$$\Rightarrow \quad \frac{dp}{dv} = -1.4\, v^{-2.4}\, c.$$

Now
$$\delta p = \frac{dp}{dv}\times\delta v$$

Given
$$\frac{\delta v}{v} = \frac{-1}{2}$$

Percentage increase is given by

$$\frac{\delta p}{p}\times 100 = \frac{dp}{dv}\times\frac{\delta v}{p}\times 100$$

$$= \frac{-1.4}{v^{2.4}}c\times\delta v\times 100\times\frac{1}{p}$$

$$= \frac{-1.4c}{v^{1.4}} \times \frac{\delta v}{v} \times 100 \times \frac{1}{p}$$

$$= -1.4p \times \left(\frac{-1}{2}\right) \times \frac{100}{p} = 0.7\%.$$

UNIT–I

Q.2. (a) If $y = \sin(m \sin^{-1} x)$, prove that $(1 - x^2)\, y_2 - xy_1 + m^2 y = 0$ and $(1-x^2)\, y_{n+2} - 2(n+1)\, x\, y_{n+1} + (m^2 - n^2)\, y_n = 0$. Also find $y_n(0)$.

Sol. Let $y = \sin(m \sin^{-1} x), y(0) = 0$

$$y_1 = \cos(m \sin^{-1} x).\frac{m}{\sqrt{1-x^2}}$$

Now $\quad y_1(0) = m \quad$ ...(1)

$\Rightarrow \quad (1-x^2)y_1^2 = m^2 \cos^2(m \sin^{-1} x)$

$= m^2 [1 - \sin^2(m \sin^{-1} x)]$

$\Rightarrow \quad (1-x^2)y_1^2 = m^2(1-y^2)$

Again differentiating w.r.t 'x'

$\Rightarrow \quad 2y_1y_2(1-x^2) - 2y_1^2 x = m^2(-2yy_1)$

$\Rightarrow \quad (1-x^2)y_2 - xy_1 = -m^2 y.$

Where $\quad y_2(0) = 0 \quad$ ...(2)

Differentiating 'n' times by Leibnitz theorem.

$$(1-x^2)y_{n+2} - 2nx\, y_{n+1} - \frac{2n(n-1)}{2!} y_n - x\, y_{n+1} \;\; -ny_n + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$$

for $x = 0$

$y_{n+2}(0) = (n^2 - m^2)\, y_n(0) \quad$ ...(3)

Putting $\quad n = 1, 2, 3\ldots$ in (3), we get

$y_3(0) = (1^2 - m^2)\, y_1(0) = m(1^2 - m^2)$

$y_4(0) = (2^2 - m^2)\, y_2(0) = 0$

$y_5(0) = (5^2 - m^2)\, y_3(0) = m(1^2 - m^2)(5^2 - m^2)$

$y_6(0) = (4^2 - m^2)\, y_4(0) = 0$

In general $\quad y_n(0) = 0$ if n is even

$y_{2n+1}(0) = 0$

$m(1^2 - m^2)(3^2 - m^2)\ldots[(2n-1)^2 - m^2]$

if n is odd.

(b) Discuss the convergence of the series $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \ldots$ **(6)**

Sol. Let $\quad u_n = \frac{x^n n^n}{n!}$

$$u_{n+1} = \frac{x^{n+1}(n+1)^{n+1}}{(n+1)!}$$

Consider
$$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1} x^{n+1}}$$

$$= \frac{n^n (n+1) n!}{n!(n+1)(n+1)^n} \cdot \frac{1}{x}$$

$$= \frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n}\frac{1}{x} = \frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}$$

$$\Rightarrow \qquad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{ex}$$

$\therefore$ By ratio test Σu_n is convergent if $\frac{1}{ex} > 1$ *i.e.,* $x < \frac{1}{e}$

and divergent if $\frac{1}{ex} < 1$ *i.e.,* $x > \frac{1}{e}$

$$\text{Test fails if } \frac{1}{ex} = 1$$

By lograthmic test,

Consider
$$\log\frac{u_n}{u_{n+1}} = \log\frac{e}{\left(1+\frac{1}{n}\right)^n}$$

$$= \log e - n\log\left(1+\frac{1}{n}\right)$$

$$= 1-n\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\right] = 1-\left[1-\frac{1}{2n}+\frac{1}{3n^2}+\ldots\right]$$

$$= \frac{1}{2n}-\frac{1}{3n^2}+\ldots$$

$$n\log\frac{u_n}{u_{n+1}} = \frac{1}{2}-\frac{1}{3n}+\ldots$$

$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \frac{1}{2} < 1.$$

Thus Σu_n is divergent Hence the series Σu_n is convergent if $x < 1/e$ and divergent if $x \geq 1/e$.

Q.3. (*a*) Discuss the absolute convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n+1}$. **(6)**

Sol. For absolute convergence

Given series $\sum_{n=1}^{\infty}(-1)^n u_n$ where $u_n = \frac{x^n}{n+1}$

$$\left|\sum_{n=1}^{\infty}(-1)^n u_n\right| = \sum_{n=1}^{\infty}|u_n|$$

$$|u_n| = \frac{|x|^n}{n+1}, |u_{n+1}| = \frac{|x|^{n+1}}{n+2}$$

Consider
$$\frac{|u_n|}{|u_{n+1}|} = \frac{|x|^n}{n+1}\frac{n+2}{|x|^{n+1}}$$

$$= \frac{n(1+2/n)}{n(1+1/n)}\cdot\frac{1}{|x|}$$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \frac{1}{|x|}$$

Thus by ratio test Σu_n is convergent if $\frac{1}{|x|} > 1$ or $|x| < 1$ and divergent if $\frac{1}{|x|} < 1$ or $|x| > 1$

$\therefore$ series is absolute convergent for $|x| < 1$ and hence convergent.

Test fails if $\quad |x| = 1$

i.e., $\quad x = \pm 1$

for $x = 1$ series becomes.

$$\frac{-1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

This is an alternating series where

$$u_n = \frac{1}{n+1}, \quad u_{n+1} = \frac{1}{n+2}$$

(*i*) $\quad n+1 < n+2$

$\Rightarrow \quad \frac{1}{n+1} > \frac{1}{n+2}$

$\Rightarrow \quad u_n > u_{n+1}$

$\therefore$ Σu is monotonically decreasing

(*ii*) Consider $\quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\therefore$ By Leibnitz test $\Sigma (-1)^n u_n$ is convergent.

For $x = -1$, series becomes.

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1}$$

$$u_n = \frac{1}{n+1} = \frac{1}{n(1+1/n)}$$

Let $\quad v_n = \frac{1}{n}$

$\Rightarrow \quad \frac{u_n}{v_n} = \frac{1}{1+1/n}$

$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$ (finite and non-zero)

$\therefore$ By L.C.T. ΣV_n and Σu_n both have same behaviour.

As $\quad \Sigma v_n = \Sigma \frac{1}{n}$ $(p = 1)$

$\Rightarrow \Sigma v_n$ is divergent

$\Rightarrow \Sigma u_n$ is also divergent

$\therefore$ Given series is convergent if $-1 < x \leq 1$.

Q. (*b*) Expand $\sin x$ in power of $\left(x + \frac{\pi}{2}\right)$. Hence find the value of 91° correct to 4 decimal places. (6.5)

Sol. Let $\quad f(x) = \sin x$

$\Rightarrow \quad f(x) = f\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right]$

$$= f(x+h)$$

$$x = \frac{\pi}{2}, h = x - \frac{\pi}{2}$$

By Taylor's expansion

$$f(x) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + ...$$

$$\Rightarrow \quad f(x) = f\left(\frac{\pi}{2}\right) + hf'\left(\frac{\pi}{2}\right) + \frac{h^2}{2!}f''\left(\frac{\pi}{2}\right) + ... \quad (1)$$

As $f(x) = \sin x, \quad f\left(\frac{\pi}{2}\right) = 1$

$$f'(x) = \cos x, \quad f'\left(\frac{\pi}{2}\right) = 0$$

$$f''(x) = -\sin x, \quad f''\left(\frac{\pi}{2}\right) = -1$$

$$f'''(x) = -\cos x, \quad f'''\left(\frac{\pi}{2}\right) = 0$$

$$f^{iv}(x) = \sin x, \quad f^{iv}\left(\frac{\pi}{2}\right) = 1$$

$$f^{v}(x) = \cos x, \quad f^{v}(\pi/2) = 0$$

By (1), we have

$$f(x) = \sin x = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + ...$$

Now $\sin 91° = \sin(90° + 1°) = f(x+h)$

here $x = 90° = \frac{\pi}{2}$

$$h = 1° = \frac{\pi}{180} = 0.0174$$

$$\sin x = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + ...$$

$$\sin 91° = \sin(90° + 1°)$$

$$= \sin x + h\cos x - \frac{h^2}{2!}\sin x \frac{-h^3}{3!}\cos x + ...$$

$$= \sin 90° + 0.0174 \times \cos 90° - \frac{(0.0174)^2}{2}\sin 90° - \frac{(0.0174)^3}{3!}\cos 90° + ...$$

$$= 1 - 0.00015138 + ...$$

$$= 0.9998 \text{ (approx)}.$$

2. Put

$$x = 0,$$
$$y = 0$$

$\Rightarrow$
Curve passes through the origin, Tangent at origin are

$$y^2 = x^2 \Rightarrow y = \pm x$$

Thus origin is a node.

3. Intersection with axes:

for

$$y = 0$$
$$x = 0$$
$$x = \pm a$$

Tangent at $(a, 0)$
Shift origin to point

$$X = x - a, Y = y$$

$\therefore$ equation becomes.

$$Y^2[a^2 + (X+a)^2] = (X+a)^2[a^2 - (X+a)^2]$$

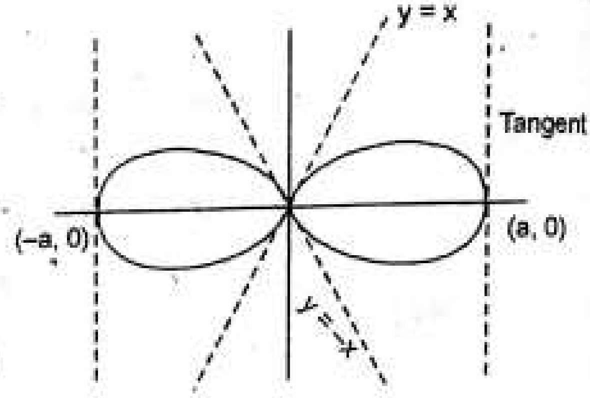
$$\Rightarrow \quad Y^2(2a^2 + 2aX + X^2) = -(X+a)^2(2aX + X^2)$$

equating lowest degree term to zero.

$$-2a^3 X = 0$$

$$\Rightarrow \quad X = 0$$

$$\therefore \quad x = a$$



Tangent at $(a, 0)$ is a line parallel to y–axis at distance 'a' from it. Tangent at $(-a, 0)$ is also parallel to y–axis.

4. Curve has no asymptotes.

5. $$y = \pm x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$$

If $x^2 > a^2$ *i.e.*, $|x| > a$, then y is imaginary
No curve lies beyond the lines

$$x = \pm a$$

Q. (b) Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$ is $4a\cos\dfrac{\theta}{2}$. (6.5)

Sol.

$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$$
$$x' = a(1 + \cos\theta), y' = a\sin\theta$$
$$x'' = -a\sin\theta, y'' = a\cos\theta$$

$$\rho = \frac{\left[(x')^2 + (y')^2\right]^{3/2}}{x'y'' - y'x''}$$

$$= \frac{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2\cos\theta(1+\cos\theta) + a^2\sin^2\theta}$$

$$= \frac{a^3\left[1 + \cos^2\theta + 2\cos\theta + \sin^2\theta\right]^{3/2}}{a^2[\cos\theta + \cos^2\theta + \sin^2\theta]}$$

$$= \frac{a[2(1+\cos\theta)]^{3/2}}{1+\cos\theta}$$

$$= 2^{3/2}\, a\,(1 + \cos\theta)^{1/2}$$

$$= 2^{3/2} a (2\cos^2\theta/2)^{1/2}$$

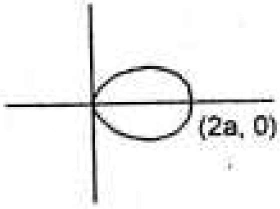
$$= 4a\cos\frac{\theta}{2}$$

Q.5. (a) Show that the whole area of the curve $a^2 y^2 = x^3 (2a - x)$ is πa^2 **(6)**

Sol.
$$a^2y^2 = x^3(2a-x)$$

1. Curve is symmetrical about x–axis.
2. for $y = 0, x = 0, 2a$.
3. It has no. asymptotes parallel to any axes.

$$y = \frac{x}{a}\sqrt{x(2a-x)}$$



If $x > 2a$, then y is imaginary. If $x < 0$, then y is imaginary. As it is symmetrical about x-axis.

Area of whole curve

$$= 2\int_0^{2a} y\,dx$$

$$= 2\int_0^{2a} \frac{x}{a}\sqrt{x(2a-x)dx}$$

Put $\quad x = 2a\sin^2\theta.$

$$dx = 4a\sin\theta\cos\theta\,d\theta$$

for $\quad x = 0, \theta = 0$

$$x = 2a, \theta = \pi/2$$

$$\text{Area} = 2\int_0^{\pi/2} \frac{2a\sin^2\theta}{a}\sqrt{4a^2\sin^2\theta(1-\sin^2\theta)} \times 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a\int_0^{\pi/2} 2a.\sin^3\theta\sin\theta\cos^2\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2} \sin^4\theta\cos^2\theta d\theta$$

$$= 32a^2.\frac{3.1}{6.4.2.}\times\frac{\pi}{2}$$

$$= \pi a^2$$

Q. (b) Express $\int_0^1 x^m(1-x^n)^p dx$ in terms of gamma function and hence evaluate $\int_0^1 x^5(1-x^3)^{10} dx$ **...(6.5)**

Sol. $\int_0^1 x^m(1-x^n)^p dx$

Let $\quad x^n = y \Rightarrow x = y^{1/n}$

$$\Rightarrow \quad dx = \frac{1}{n}y^{1/n-1}dy$$

Consider $\int_0^1 y^{m/n}(1-y)^p \frac{1}{n}y^{1/n-1}dy$

$$= \frac{1}{n}\int_0^1 y^{\frac{m+1}{n}-1}(1-y)^p\,dy$$

$$= \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$$

$$= \frac{1}{n}\frac{\overline{\left|\frac{m+1}{n}\right.}\overline{|p+1}}{\overline{\left|\frac{m+1}{n}+p+1\right.}} \quad \text{...(1)}$$

Consider $\int_0^1 x^5(1-x^3)^{10}dx$

Here $m = 5, n = 3, p = 10$

$$= \frac{1}{3}\frac{\overline{|2}\,\overline{|11}}{\overline{|13}} \quad \text{[by (1)]}$$

$$= \frac{1\overline{|1}\times 10!}{3\times 12!} = \frac{10!}{3\times 12\times 11\times 10!}$$

$$= \frac{1}{396}$$

UNIT-III

Q.6. (*a*) Find the values of a and b for which the equations. $x + ay + z = 3$, $x + 2y + 2z = b$, $x + 5y + 3z = 9$ are consistent. When will these equations have a unique solution? **(6)**

Sol. In matrix notation it can be written as $AX = B$

$$\begin{bmatrix}1 & a & 1\\1 & 2 & 2\\1 & 5 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\b\\9\end{bmatrix}$$

for consistent solution

(*i*) $r(A) = r(A:B)$

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\sim\begin{bmatrix}1 & a & 1\\0 & 2-a & 1\\0 & 5-a & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\b-3\\6\end{bmatrix}$$

Operating $R_3 \to R_3 - 2R_2$

$$\sim\begin{bmatrix}1 & a & 1\\0 & 2-a & 1\\0 & 1+a & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\b-3\\12-2b\end{bmatrix}$$

for consistent system $r(A) = r(A:B) = 3$

(*i*) $r(A) = r(A:B) = 3 =$ no. of unknown

system is consistent and have unique solution.

Let $a \neq -1$, b can have any value.

(*ii*) If $r(A) = r(A:B) <$ no. of unknown, system is consistent and have infinite solution.

Let $a = -1, b = 6$.

Q. (*b*) Diagonalize the matrix $A = \begin{bmatrix}3 & 1 & -1\\-2 & 1 & 2\\0 & 1 & 2\end{bmatrix}$ **(6.5)**

Sol. Characterstic equation is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (3-\lambda)[(1-\lambda)(2-\lambda)-2]+2[2-\lambda+1] = 0$$

$$\Rightarrow \quad (3-\lambda)(\lambda^2-3\lambda)+2(3-\lambda) = 0$$

$$\Rightarrow \quad (3-\lambda)[\lambda^2-3\lambda+2] = 0$$

$$\Rightarrow \quad (3-\lambda)(\lambda-1)(\lambda-2) = 0$$

$$\lambda = 1, 2, 3$$

For $\lambda = 1$ Consider $[A-I]X = 0$.

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_1 \to R_1 + R_2$

$$\sim \begin{bmatrix} 0 & 1 & 1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 0 & 1 & 1 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r(A) = 2 < 3$ (no. of unknowns)

$\therefore$ one variable be given arbitrary value.

$$x_2 + x_3 = 0$$

$$-2x_1 + 2x_3 = 0$$

Let $\quad x_1 = 1$

$$x_3 = x_1$$

$$x_2 = -1$$

$\therefore$ Eigen vector is $\quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

for $\lambda = 2$ Consider $[A-2I]X = 0$

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

operating $R_2 \to R_2 + 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$-\begin{bmatrix} 1 & & \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r(A) = 2 < 3$ (no. of unknowns).

$\therefore$ one variable be given artibtrary value.

$$x_1 + x_2 - x_3 = 0$$

$$x_2 = 0$$

Let

$$x_1 = 1$$

$\Rightarrow$

$$x_1 = x_3 = 1$$

$$\text{eigenvector } X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 3$ Consider $[A - 3I]X = 0$

$$\begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(A) = 2 < 3 \text{ (no. of unknowns)}$$

$$x_2 - x_3 = 0$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$x_2 = x_3$$

Let

$$x_1 = 0$$

$\therefore$

$$\text{eigen vector } X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\textbf{Modal matrix is } B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|B| = 1 \neq 0$$

Now

$$B^{-1} = \frac{adjB}{|B|}$$

$$B^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & -1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$r(A) = 2 < 3$ (no. of unknowns).

$\therefore$ one variable be given artibtrary value.

$$x_1 + x_2 - x_3 = 0$$
$$x_2 = 0$$

Let $x_1 = 1$

$\Rightarrow$ $x_1 = x_3 = 1$

$\therefore$ eigenvector $X_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$

For $\lambda = 3$ Consider $[A - 3I]X = 0$

$$\begin{bmatrix}0 & 1 & -1\\-2 & -2 & 2\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$R_3 \to R_3 - R_1$$

$$\begin{bmatrix}0 & 1 & -1\\-2 & -2 & 2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$r(A) = 2 < 3 \text{ (no. of unknowns)}$$
$$x_2 - x_3 = 0$$
$$-2x_1 - 2x_2 + 2x_3 = 0$$
$$x_2 = x_3$$

Let $x_1 = 0$

$\therefore$ eigen vector $X_3 = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Modal matrix is $B = \begin{bmatrix}1 & 1 & 0\\-1 & 0 & 1\\1 & 1 & 1\end{bmatrix}$

$$|B| = 1 \neq 0$$

Now $B^{-1} = \dfrac{adjB}{|B|}$

$$B^{-1} = \begin{bmatrix}-1 & -1 & 1\\2 & 1 & -1\\-1 & 0 & 1\end{bmatrix}$$

$$D = B^{-1}AB$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Q.7. (*a*) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(6)}$$

Sol. Characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)[(2-\lambda)(1-\lambda)] - 1(1-\lambda) = 0$$

$$\Rightarrow \quad (1-\lambda)[(2-\lambda)^2 - 1] = 0$$

$$\Rightarrow \quad (1-\lambda)(4 + \lambda^2 - 4\lambda - 1) = 0$$

$$\lambda = 1, \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 1, \lambda = 1, \lambda = 3 \text{ (eigenvalues)}$$

For $\lambda = 1$ Consider $[A - I]X = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(A) = 1 < 3 \text{ (no. of unknowns)}$$

$\therefore$ 2 variables be given arbitrary value.

$$x_1 + x_2 + x_3 = 0$$

Let $\quad x_2 = 0$

$\Rightarrow \quad x_1 = -x_3$

$\Rightarrow \quad \dfrac{x_1}{-1} = \dfrac{x_3}{1}$

$\therefore \quad$ eigevector $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Let $\quad x_3 = 0 \Rightarrow x_1 = -x_2$

$\Rightarrow \quad \dfrac{x_1}{-1} = \dfrac{x_2}{1}$

$\therefore \quad$ eigenvector $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Let $\quad x_3 = 0 \Rightarrow x_1 = -x_2$

$$\Rightarrow \qquad \frac{x_1}{-1} = \frac{x_2}{1}$$

$$\therefore \qquad \text{eigenvector } x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 3$. Consider $[A - 3I]X = 0$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(A) = 2 < 3 \text{ (no of unknowns)}$$

$\therefore$ one variable be given abitrary value.

$$-x_1 + x_2 + x_3 = 0$$
$$2x_3 = 0$$
$$\Rightarrow \qquad x_3 = 0$$
$$x_1 = x_2$$

$$\therefore \qquad \text{eignvector is } X_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(b) Verify cayley Hamilton theorem for the given matrix and find its inverse.

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad (6.5)$$

Sol. Characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(1-\lambda)^2 - 4] - 2[\lambda - 1 - 2] = 0$$
$$\Rightarrow \quad (1-\lambda)(1 + \lambda^2 - 2\lambda - 4) - 2(\lambda - 3) = 0$$
$$\Rightarrow \quad \lambda^2 - 2\lambda - 3 - \lambda^3 + 2\lambda^2 + 3\lambda - 2\lambda + 6 = 0$$
$$\Rightarrow \quad -\lambda^3 + 3\lambda^2 - \lambda + 3 = 0$$
$$\Rightarrow \quad \lambda^3 - 3\lambda^2 + \lambda - 3 = 0$$

To show,

$$A^3 - 3A^2 + A - 3I = 0$$

$$A^2 = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$$

Consider

$$\begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix} - 3\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix} - \begin{bmatrix} -3 & 12 & 12 \\ 0 & 9 & 12 \\ 0 & 18 & 15 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ Hence proved.}$$

Consider $A^3 - 3A^2 + A - 3I = 0$

$$A^{-1}(A^3 - 3A^2 + A - 3I) = 0$$

$$\Rightarrow \quad A^2 - 3A + I - 3A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{3}(A^2 - 3A + I)$$

$$A^{-1} = \frac{1}{3}\left(\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 0 \\ -3 & 3 & 6 \\ 3 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$$

UNIT-IV

Q.8. (a) Solve by the method of variation of parameters $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$ **(6)**

Sol. $$y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$$

$$\text{Aux. equation is } D^2 - 6D + 9 = 0$$
$$(D-3)^2 = 0$$
$$D = 3, 3$$
$$\text{C.F} (C_1 + C_2 x)e^{3x}$$
$$\Rightarrow \quad C_1 e^{3x} + C_2 x e^{3x}$$

Let
$$y_1 = e^{3x}, y_2 = xe^{3x}, X = \frac{e^{3x}}{x^2}$$
$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix}$$
$$= e^{6x}$$
$$\text{P.I.} = y_1 u + y_2 v$$
$$u = -\int \frac{y_2 X}{W} dx$$
$$= -\int \frac{xe^{3x} e^{3x}}{e^{6x} . x^2} . dx$$
$$= -\int \frac{1}{x} dx = -\log x$$
$$V = \int \frac{y_1 X}{w} . dx = \int \frac{e^{3x} e^{3x}}{e^{6x} . x^2} dx$$
$$= \int \frac{1}{x^2} dx = \frac{-1}{x}$$
$$P.I. = -\log x e^{3x} - \frac{1}{x} x e^{3x}$$
$$= -e^{3x}(\log x + 1)$$

Complete solution is
$$y = (c_1 + c_2 x)e^{3x} - e^{3x}(1 + \log x)$$

(b) Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x$ **(6.5)**

Sol. Aux. equation is
$$D^3 + 2D^2 + D = 0$$
$$\Rightarrow \quad D(D^2 + 2D + 1) = 0$$
$$\Rightarrow \quad D(D+1)^2 = 0$$
$$\Rightarrow \quad D = 0, -1, -1$$
$$C.F = C_1 + (C_2 + C_3 x)e^{-x}$$

$$P.I \frac{1}{D^3 + 2D^2 + D}(e^{2x} + \sin 2x)$$
$$= \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x$$
$$= \frac{1}{8+8+2} e^{2x} + \frac{1}{-4D - 8 + D} \sin 2x$$
$$= \frac{e^{2x}}{18} - \frac{1}{8+3D} \sin 2x = \frac{e^{2x}}{18} - \frac{(8-3D)}{64 - 9D^2} \sin 2x$$

$$= \frac{e^{2x}}{18} - \frac{(8-3D)}{100}\sin 2x = \frac{e^{2x}}{18}\frac{-1}{100}(8\sin 2x - 6\cos 2x)$$

Complete solution is

$$y = C_1(C_2 + C_3x)e^{-x} + \frac{e^{2x}}{18}\frac{-1}{100}(8\sin 2x - 6\cos 2x)$$

Q.9. (a) Prove that

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x) \qquad \textbf{(6)}$$

Sol. By using relation.

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x) \qquad ...(1)$$

Replace n by 3 in (1), we get

$$\Rightarrow \qquad J_4(x) = \frac{6}{x}J_3(x) - J_2(x) \qquad ...(2)$$

Put n as 2 in (1), we get

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x) \qquad ...(3)$$

Using (3) in (2), we get

$$J_4(x) = \frac{6}{x}\left[\frac{4}{x}J_2(x) - J_1(x)\right] - J_2(x)$$

$$= \frac{24}{x^2}J_2(x) - \frac{6}{x}J_1(x) - J_2(x) \qquad ...(4)$$

Put n as 1 in (1), we get

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$$

from (4), we have

$$J_4(x) = \frac{24}{x^2}\left[\frac{2}{x}J_1(x) - J_0(x)\right] - \frac{6}{x}J_1(x) - \frac{2}{x}J_1(x) + J_0(x)$$

$$J_4(x) = \left(1 - \frac{24}{x^2}\right)J_0(x) + \left(\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right)J_1(x)$$

$$\Rightarrow \qquad J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$$

(b) Prove that $\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\ m = n$ **(6.5)**

Sol. When $m = n$

Consider $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$

On squaring both sides, we get

$$(1 - 2xz + z^2)^{-1} = \sum_{n=0}^{\infty}[z^n P_n(x)]^2$$

$$\Rightarrow \quad (1-2xz+z^2)^{-1} = \sum_{n=0}^{\infty} z^{2n}[P_n(x)]^2 + \sum_{\substack{m=0\\n=0\\m\neq n}}^{\infty} z^{m+n}P_m(x)P_n(x)$$

Integrating both sides w.r.t 'x' between limits -1 to 1, we have

$$\Rightarrow \quad \int_{-1}^{1}(1-2xz+z^2)^{-1}dx = \sum_{n=0}^{\infty}\int_{-1}^{1} z^{2n}[P_n(x)]^2 dx + \sum_{\substack{m=0\\n=0\\m\neq n}}^{\infty}\int_{-1}^{1} z^{m+n}P_m(x)P_n(x)dx$$

By orthogonality of $P_m(x)$, if $m \neq n$

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty}\int_{-1}^{1} z^{2n}[P_n(x)]^2 dx = \int_{-1}^{1}\frac{dx}{1-2xz+z^2} \quad \text{...(1)}$$

$$= \frac{-1}{2z}[\log(1-2z+z^2)-\log(1+2z+z^2)$$

$$= \frac{-1}{2z}[\log(1-z)^2-\log(1+z)^2] = \frac{1}{z}[\log(1+z)-\log(1-z)]$$

$$= \frac{1}{z}[\log(1+z)-\log(1-z)]$$

$$= \frac{1}{z}\left[\left(z-\frac{z^2}{2}+\frac{z^3}{3}\frac{-z^4}{4}+...\right)\right] + \left(z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+...\right)$$

$$= \frac{1}{z}\left[2z+\frac{2z^3}{3}+...\right] = \frac{2}{z}\left[z+\frac{z^3}{3}+\frac{z^5}{5}+...\right]$$

$$\int_{-1}^{1}\frac{dx}{1-2xz+z^2} = 2\left[1+\frac{z^2}{3}+\frac{z^4}{5}+...+\frac{z^{2n}}{2n+1}+...\right]$$

By (1), we have

$$\sum_{n=0}^{\infty} z^{2n}\int_{-1}^{1}[P_n(x)]^2 dx = 2\left(1+\frac{z^2}{3}+\frac{z^4}{5}+...\frac{z^{2n}}{2n+1}+...\right)$$

Equating coefficient of z^{2n} both sides we get

$$\int_{-1}^{1}[P_n(x)]^2 dx = \frac{2}{2n+1}$$

$$\Rightarrow \quad \int_{-1}^{1}P_m(x)P_n(x)dx = \frac{2}{2n+1}, m=n$$

FIRST SEMESTER (B.TECH)
FIRST TERM EXAMINATION [2014]
APPLIED MATHEMATICS-I [ETMA-101]

Time : 1.30 hrs. M.M. : 30

Note: *Attempt three questions. Q. No. 1 is compulsory.*

Q.1. (a) Find the nth derivative of $\frac{x^3}{x^2-1}$. (2)

Sol.
$$\frac{x^3}{x^2-1} = x+\frac{x}{x^2-1}$$
$$y = \frac{x^3}{x^2-1} = x+\frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)$$

nth derivative is

$$y_n = \frac{1}{2}(-1)^n\, n!\left[\frac{1}{(x+1)^{n+1}}+\frac{1}{(x-1)^{n+1}}\right]$$

Q.1. (b) Find all the asymptotes of the curve $x^2y+xy^2+xy+y^2+3x=0$ (2)

Sol. Coefficient of x^2 is $y = 0$.

Coefficient of y^2 is $(x+1)=0$

$\Rightarrow$
$$x = -1$$
$$y = 0, x = -1 \text{ are parallel asymptotes}$$

Oblique Asymptotes

$$\phi_3 = x^2y+xy^2$$
$$\phi_2 = xy+y^2$$
$$\phi_1 = 3x, \phi_0 = 0$$
$$\text{Replace } x = 1, y = m.$$
$$\phi_3(m) = m+m^2$$
$$\phi_2(m) = m+m^2$$
$$\phi_1 = 3$$
$$\phi_3(m) = 0$$

Roots are
$$m+m^2 = 0$$
$\Rightarrow$
$$m(m+1) = 0$$
$\Rightarrow$
$$m = 0, m = -1$$
$\Rightarrow$

Non-repeated values of c, m is

$$c = \frac{-\phi_2(m)}{\phi_3'(m)}$$
$$= \frac{-(m+m^2)}{2m+1}$$

$$m = 0$$

For

$$C = 0$$

$$m = -1,$$

For

$$C = \frac{-(1-1)}{-1} = 0.$$

Asymptotes are.

$$y = 0, y = -x.$$

Thus asymptotes are

$$y = 0, x = -1, x + y = 0$$

Q.1. (c) Find the value of $\int_0^{\pi/2} \sin^3\theta \cos^5\theta \, d\theta$ using reduction formula.

Ans. Using

$$I_{m,n} = \frac{n-1}{m+n} I\, m, n-2$$

$$I_{3,5} = \frac{4}{8} I_{3,3}$$

$$I_{3,3} = \frac{2}{6} I_{3,1}$$

$$I_{3,1} = \int_0^{\pi/2} \sin^3\theta \cos\theta \, d\theta$$

Let

$$\cos\theta = t$$

$$-\sin\theta \, d\theta = dt$$

$$= -\int_1^0 (1-t^2) t \, dt$$

$$= \int_0^1 (t - t^3) dt = \left| \frac{t^2}{2} - \frac{t^4}{4} \right|_0^1$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$I_{3,3} = \frac{2}{6} \times \frac{1}{4} = \frac{1}{12}$$

$$I_{3,5} = \frac{4}{8} \times \frac{1}{12} = \frac{1}{24}$$

Q. (d) What is the necessary condition for convergence of a positive term series? Give an example to show that the converse of this result is not true.

Sol. A necessary condition for convergence of Σu_n is $\lim_{n\to\infty} u_n = 0$, i.e. If Σu_n converges then $\lim_{n\to\infty} u_n = 0$

But converse is not true For example, consider the series

$$\sum \frac{1}{\sqrt{n}} = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$$

Here
$$S_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\ldots+\frac{1}{\sqrt{n}}$$

$$= \frac{n}{\sqrt{n}} = \sqrt{n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} = \infty$$

$\therefore$ Series $\sum \frac{1}{\sqrt{n}}$ is divergent.

Q.2. (a) If $y = \cos(m \sin^{-1} x)$, prove that $(1-x^2)\, y_{n+2} - (2n+1)_x\, y_{n+1} - (n^2 - m^2)\, y_n = 0$, Also find $(y_n)_0$. **(5)**

Sol.
$$y = \cos(m \sin^{-1} x) \quad \ldots(1)$$

$$y_1 = -\sin(m \sin^{-1} x)\frac{m}{\sqrt{1-x^2}}$$

$$y_1 = \frac{-m\sqrt{1-y^2}}{\sqrt{1-x^2}} \quad \ldots(2)$$

$$\Rightarrow \quad y_1^{\,2}(1-x^2) = m^2(1-y^2)$$

Again differentiating w.r.t. 'x'.

$$2y_1 y_2 (1-x^2) - 2xy_1^{\,2} = -2m^2 y y_1$$

$$\Rightarrow \quad y_2(1-x^2) - xy_1 = -m^2 y$$

$$\Rightarrow \quad y_2(1-x^2) - xy_1 + m^2 y = 0 \quad \ldots(3)$$

By Leibnitz theorem

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - [y_{n+1}x + {}^nC_1\, y_n] + m^2 y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - 2nx\, y_{n+1} - n(n-1)y_n - x\, y_{n+1} - ny + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} + (m^2 - n^2)\, y_n = 0 \quad \ldots(4)$$

Putting $x = 0$ in (1), (2), (3), (4).

$$y(0) = 1, y_1(0) = 0$$

$$y_2(0) = -m^2$$

$$y_{n+2}(0) + (m^2 - n^2)\, y_n(0) = 0$$

$$\Rightarrow \quad y_{n+2}(0) = (n^2 - m^2)\, y_n(0)$$

Putting $\quad n = 1, 2, 3, 4, \ldots$

$$y_3(0) = (1^2 - m^2)\,y_1(0) = 0$$
$$y_4(0) = (2^2 - m^2)\,y_2 = (0) = -m^2(2^2 - m^2)$$
$$y_5(0) = (3^2 - m^2)\,y_3(0) = 0$$
$$y_6(0) = (4^2 - m^2)\,y_4(0)$$
$$= -m^2(2^2 - m^2)(4^2 - m^2)$$

In general
$$y_n(0) = \begin{cases} 0, & \text{if n is odd} \\ -m^2[((n-2)^2 - m^2)\ldots(z^2 - m^2)], & \text{if n is even} \end{cases}$$

Q.2. (b) Discuss the convergence of the series

$$\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \ldots\infty (x > 0)$$ (5)

Sol. Let
$$u_n = \frac{x^n}{(2n-1)\,.\,2n}$$
$$u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$$

consider

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{(2n-1).2n}\frac{(2n+1)(2n+2)}{x^{n+1}}$$
$$= \frac{2n(1+1/2n)\,.\,2n(1+1/n)}{2n\left(1-\frac{1}{2n}\right).2n\,.\,x} = \frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1}{x} = \frac{1}{x}$$

By Ratio Test

Σu_n converges if $\frac{1}{x} > 1$ i.e. $x < 1$

Σu_n diverges if $\frac{1}{x} < 1$ i.e. $x > 1$

Test tails if $x = 1$

By Raabe's test.

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+1)(2n+2)}{2n(2n-1)} - 1$$
$$= \frac{(2n+1)(2n+2) - 2n(2n-1)}{2n(2n-1)}$$
$$= \frac{4n^2 + 6n + 2 - 4n^2 + 2n}{4n^2 - 2n} = \frac{8n+2}{2n(2n-1)}$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{2n^2(4+1/n)}{2n^2(2-1/n)}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = 2 > 1$$

$\therefore$ Σu_n converges.

Thus series Σu_n converges if $x \le 1$ and diverges if $x > 1$.

Q.3. (a) Show that $\tan\left(\frac{\pi}{4}+x\right) = 1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\dots$ and hence evaluate the value of 46° correct to 4 places of decimal. **(5)**

Sol. Let

$$f(x) = \tan(x+\pi/4), f(0) = 1$$

$$f'(x) = \sec^2\left(x+\frac{\pi}{4}\right), f'(0) = 1+1 = 2$$

$$= 1+\tan^2\left(x+\frac{\pi}{4}\right)$$

$$f''(x) = 2\tan\left(x+\frac{\pi}{4}\right)\sec^2\left(x+\frac{\pi}{4}\right)$$

$$= 2f(x)f'(x)$$

$$f''(0) = 2f(0)f'(0) = 4$$

$$f'''(x) = 2f(x)f''(x)+2[f'(x)]^2$$

$$f'''(0) = 2f(0)f''(0)+2[f'(0)]^2$$

$$= 8+(2\times 4) = 16$$

$$f^{iv}(x) = 2f(x)f'''(x)+2f'(x)f''(x)+4f'(x)f''(x)$$

$$f^{iv}(0) = 2f(0)f'''(0)+2f'(0)f''(0)f''(0)+4f'(0)f''(0)$$

$$= 80$$

$$\therefore \quad \tan\left(\frac{\pi}{4}+x\right) = f(0)+xf'(0)+\frac{x^2}{2'}f''(0)+\dots$$

$$= 1+2x+\frac{4x^2}{2!}+\frac{16}{3!}x^3+\frac{80}{4!}x^4+\dots$$

$$= 1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\dots$$

Take

$$x = 1° = \frac{\pi}{180} = 0.0174$$

$$\tan 46° = 1+2(0.0174)+2(0.0174)^2+\frac{8}{3}(0.0174)^3+\frac{10}{3}(0.0174)^4+\dots$$

$$= 1.035 \text{ (approx)}$$

Q.3.(b) Discuss the absolute convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)}$

Sol. Let $$|0_n| = \left|\frac{x^n}{n+1}\right| = \frac{|x|^n}{n+1}$$

$$u_{n+1} = \frac{|x|^{n+1}}{n+2}$$

Consider $$\frac{u_n}{u_{n+1}} = \frac{|x|^n}{n+1} \times \frac{n+2}{|x|^{n+1}}$$

$$= \frac{n(1+2/n)}{n(1+1/n)|x|}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{|x|}$$

By ratio test

$\Sigma|u_n|$ converges if $\frac{1}{|x|} > 1$ or $|x| < 1$

or $-1 < x < 1$

and diverges if $\frac{1}{|x|} < 1$ i.e. $|x| > 1$

Test fails if $|x| = 1$

or $x = 1, -1$

for $x = 1 \Rightarrow$ $$\Sigma u_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

$$u_n = \frac{1}{n+1}, u_{n+2} = \frac{1}{n+2}$$

As $n+1 < n+2$

$\Rightarrow$ $$\frac{1}{n+1} > \frac{1}{n+2}$$

$\Rightarrow$ $u_n > u_{n+1}$

$\Rightarrow < u_n >$ is mon. decreasing

Now, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$\therefore$ Σu_n is convergent by Leibnitz test.

For $x = -1$ $$\Sigma u_n = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{n+1}$$

$$u_n = \frac{1}{n+1}$$

$\therefore$ Covnvergent by leibnitz test.

Thus series is abs. Convergent if $|x| \leq 1$

Q.4. (a) Show that the radius of curvature at $\left(\frac{a}{4}, \frac{a}{4}\right)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $a/\sqrt{2}$ **(5)**

Sol. $$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

Different w.r.t 'x' we get

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x^{-1/2}}{y^{-1/2}} = \frac{-\sqrt{y}}{\sqrt{x}}$$

Again differentiating, we get

$$\frac{d^2y}{dx^2} = \frac{\left(x^{1/2}\frac{1}{2}y^{-1/2}\frac{dy}{dx} - y^{1/2}\frac{x^{-1/2}}{2}\right)}{x}$$

$$= \frac{\frac{-\sqrt{x}}{\sqrt{y}}\left(\frac{-\sqrt{y}}{\sqrt{x}}\right) + \frac{\sqrt{y}}{\sqrt{x}}}{2x}$$

$$= \frac{1 + \sqrt{y}/\sqrt{x}}{2x} = \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}}$$

$$= \frac{\sqrt{a}}{2x\sqrt{x}}$$

At $$\left(\frac{a}{4}, \frac{a}{4}\right)$$

$$\frac{dy}{dx} = -1, \frac{d^2y}{dx^2} = \frac{\sqrt{a}}{\frac{2a}{4}\frac{\sqrt{a}}{2}} = \frac{4}{a}$$

$\therefore$ ρ at $\left(\frac{a}{4}, \frac{a}{4}\right)$

$$= \frac{[1 + (dy)/dx)^2]^{3/2}}{d^2y/dx^2}$$

$$= \frac{[1+(-1)^2]^{3/2}}{4/a}$$

$$= \frac{2^{3/2}}{4/a} = \frac{a.2\sqrt{2}}{4}$$

$$= \frac{a}{\sqrt{2}}$$

Q.4. (*b*) Trace the Curve

$$y^2(a^2+x^2) = x^2(a^2-x^2)$$

Sol. Equation of curve.

$$y^2(a^2+x^2) = x^2(a^2-x^2)$$

1. Symmetry: Powers of both x and y are even, the curve is symmetrical about both axes.

2. Curve passes through origin

3. Axes intersection:

Put $x = 0$

$\Rightarrow$ $y = 0$

Put $y = 0$

$$x^2(a-x^2) = 0$$

$\Rightarrow$ $x = 0, a, -a$

$\therefore$ Points of intersection are

$(0, 0), (a, 0), (-a, 0)$

4. Tangents

At (0, 0).

equating lowest degree terms

$$y^2 = x^2$$

$\Rightarrow$ $y = \pm x$ (real & distinct)

Origin is a node.

At (a, 0)

Transferring to origin

Let $X = x-a, y = y-0.$

$\Rightarrow$ $x = X+a, y = Y$

equation of curve becomes

$$Y^2[a^2+(X+a)^2] = (X+a)^2[a^2-(X+a)^2]$$

$$\Rightarrow \quad Y^2[2a^2+X^2+2aX] = (X^2+a^2+2aX)(a^2-X^2-a^2-2aX)$$

$$\Rightarrow \quad Y^2[2a^2+X^2+2aX] = -X(X+2a)(X^2+a^2+2aX)$$

equating lowest degree term to zero

$\Rightarrow$ $-a^2X = 0$

$\Rightarrow$ $X = 0$

$\Rightarrow$ $x-a = 0$

$\Rightarrow$ $x = a$ is tangent || to y-axis

By symmetry, tangent at $(-a, 0)$ is $x = -a$ || to y–axis.

5. Asymptotes: No parallel asymptotes

oblique: $y^2 a^2 + y^2 x^2 - x^2 a^2 + x^4 = 0$

$$\phi_4 = y^2 x^2 + x^4$$

Put $x = 1, y = m$

$$\Rightarrow \quad m^2 + 1 = 0$$

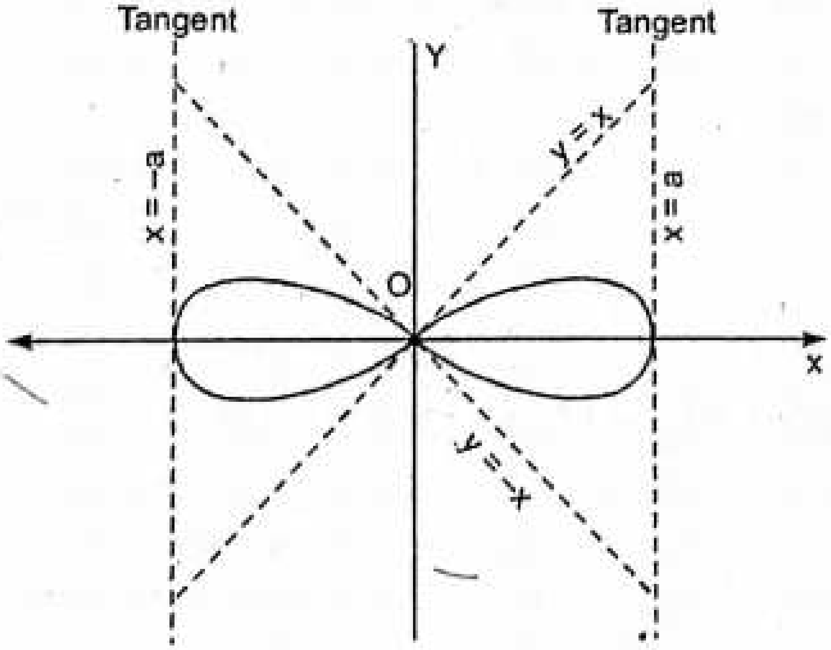
$$\Rightarrow \quad m^2 = -1$$

$\therefore$ No asymptotes.

6. Region: $$y = \pm x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$$

Region	Points	Sign on RHS	Nature of Curve
$x < -a$	$x = -2a$	– ve	No
$-a < x < 0$	$x = -a/2$	+ ve	Yes
$0 < x < a$	$x = a/2$	+ ve	Yes
$x\ a$	$x = 2a$	– ve	No

$\therefore$ Curve lies between $-a$ and a.



FIRST SEMESTER (B.TECH)
SECOND TERM EXAMINATION [2014]
APPLIED MATHEMATICS-I [ETMA-101]

M.M. : 30

Time : 1.30 hrs.

Note: *Attempt three questions. Q. No. 1 is compulsory.*

Q.1. (a) Prove that every Hermition matrix can be written as $A + iB$, where A is real and symmetric and B is real and skew symmetric matrix. **(2.5)**

Sol. Let
$$C = A + iB$$
Let C be hermitian matrix.

then
$$C^\theta = C.$$

Let us take
$$A = \frac{1}{2}(C + \bar{C})$$

and
$$B = \frac{1}{2i}(C - \bar{C})$$

then A and B are real matrices [as if $z = x + iy$ is a complex number, then $\frac{1}{2}(z + \bar{z})$ and $\frac{1}{2i}(z + \bar{z})$ are real.]

Now,
$$C = \frac{1}{2}(C + \bar{C}) + i\left[\frac{1}{2i}(C - \bar{C})\right]$$
$$= A + iB$$

To show A is symmetric and B is skew-symmetric.

$$A' = \left[\frac{1}{2}(C + \bar{C})\right]'$$
$$= \frac{1}{2}(C + \bar{C})' = \frac{1}{2}[C' + C^\theta]$$
$$= \frac{1}{2}[(C^\theta)' + C] \qquad [\because C^\theta = C]$$
$$= \frac{1}{2}\left[\{(\bar{C})'\}' + C\right] = \frac{1}{2}(\bar{C} + C) = A$$

$\therefore$ A is symmetric.

Consider
$$B' = \left[\frac{1}{2i}(C - \bar{C})\right]'$$
$$= \frac{1}{2i}(C - \bar{C})' = \frac{1}{2i}\left[C' - (\bar{C})'\right]$$

$$= \frac{1}{2i}\left[C' - C^{\theta}\right] = \frac{1}{2i}\left[(C^{\theta})' - C\right]$$

$$= \frac{1}{2i}\left[\left((\bar{C})'\right)' - C\right]$$

$$= \frac{1}{2i}\left[\bar{C} - C\right]$$

$$= \frac{-1}{2i}(C - \bar{C}) = -B$$

$\therefore B$ is skew-symmetric Hence Proved.

Q.1.(b) Reduce the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to cononical form. Also write the nature of the quadratic form. (2.5)

Sol. Here

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

As

$$A = I_3 A I_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + \frac{1}{3}R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 8/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 \to C_3 + \frac{1}{3}C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

Diag $(1, 3, 8/3) = P'AP$

Cononical form is given by

$$Y'(P'AP)Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8/3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 3y_2^2 + \frac{8}{3}y_3^2$$

$$\text{rank}(r) = 3, \text{ index }(p) = 3$$

$$\text{signature} = 3'$$

It is positive definite.

Q.1. (c) Solve the differential equation $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

Sol. Here

$$M = x^2y - 2xy^2$$

$$N = 3x^2y - x^3$$

Now,

$$\frac{\partial M}{\partial y} = x^2 - 4xy$$

$$\frac{\partial N}{\partial y} = 6xy - 3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ equation is not exact.

Consider

$$Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$$

$$= x^2y^2 \neq 0$$

$$\text{I.F.} = \frac{1}{x^2y^2}$$

Equation changes to

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

Again,

$$M = \frac{1}{y} - \frac{2}{x} \quad N = \frac{3}{y} - \frac{x}{y^2}$$

Now

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2}, \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

$\therefore$ equation is exact

Consider

$$\int_{y\,\text{constant}} M dx = \int\left(\frac{1}{y} - \frac{2}{x}\right)dx$$

$$= \frac{x}{y} - 2\log x$$

$$\int N\,dy = \int \frac{3}{y}\,dy = 3\log y$$

Thus, required solution is

$$\frac{x}{y} - 2\log x + 3\log y = c.$$

Q.1. (d) Prove that $\beta(m+1, n) + \beta(m, n+1) = \beta(m, n)$

Sol. Consider $\beta(m+1, n) + \beta(m, n+1)$

$$= \int_0^1 x^m (1-x)^{n-1} dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x + (1-x)]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1} dx = \beta(m, n).$$

Q.2. (a) Show that, if $\lambda \neq -5$, the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$, have a unique solution. If $\lambda = -5$, show that the equations are consistent. Also determine the solution when $\lambda = -5$. **(5)**

Sol. In matrix form system is

$$\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

Operating $R_2 \to 3R_2 - R_1, R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 7 & \lambda - 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -9 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & \lambda + 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix}$$

Now

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & \lambda + 5 \end{bmatrix}$$

$$[A:B] = \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & \lambda + 5 & 0 \end{bmatrix}$$

If $\lambda \neq -5$

Then $\rho(A) = \rho(A:B) = 3 = $ no. of unknowns

$\therefore$ system has unique solution given by

$$3x - y + 4z = 3$$
$$7y - 13z = -9$$
$$(\lambda + 5)z = 0$$

$$\Rightarrow \quad z = 0, y = \frac{-9}{7}$$

and $$3x = 3-\frac{9}{7}$$

$\Rightarrow$ $$x = \frac{4}{7}$$

If $$\lambda = -5$$

Then $$\rho(A) = \rho(A:B) \neq 2 < \text{no. of unknowns.}$$

$\Rightarrow$ system is consistent and has many solution.

Now
$$3x - y + 4z = 3$$
$$7y - 13z = -9$$

Let $$z = k$$

$\Rightarrow$ $$y = \frac{13k-9}{7}$$

and $$3x - \frac{(13k-9)}{7} = 3-4k$$

$\Rightarrow$ $$3x = 3-4k+\frac{(13k-9)}{7}$$

$\Rightarrow$ $$3x = \frac{12-15k}{7}$$

$\Rightarrow$ $$x = \frac{4-5k}{7}$$

$\therefore$ $$x = \frac{4-5k}{7}, y = \frac{13k-9}{7}, z = k.$$

Q.2.(b) Find the eigen values and corresponding eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \quad (5)$$

Sol. Characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$\Rightarrow$ $$\lambda = 1, 2, 3 \text{ (given values)}$$

For $\lambda = 1$

$$[A - I][X] = 0$$

$\Rightarrow$ $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 + x_3 = 0$

$$r(A) = 2 < 3 \text{ (no. of unknowns)}$$

$$x_2 + x_3 = 0$$

$$2x_1 + 2x_3 = 0$$

$$x_1 + x_3 = 0$$

$\Rightarrow$ $$x_1 = -x_3$$

and $$x_2 = -x_3$$

$\therefore$ $$\text{eigen vector } x_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

For $\lambda = 2$

$$[A - 2I]X = 0$$

$\Rightarrow$ $$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(A) = 2 < 3 \text{ (no. of unknowns)}$$

As $$x_1 = 0, x_3 = 0$$

Let $$x_2 = 1$$

$\therefore$ $$\text{eigenvector } X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$

$$[A - 3I]X = 0$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(A) = 2 < 3 \text{ (no. of unknown)}$$

$$-2x_1 = 0 \Rightarrow x_1 = 0$$

$$-x_2 + x_3 = 0$$

$$\Rightarrow \quad \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore \quad \text{eigenvector } X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Q.3. (a) Solve $\dfrac{dy}{dx} + \left(\dfrac{y}{x}\right)\log y = \dfrac{y}{x}(\log y)^2$ (5)

Sol. Multiplying equation by $\dfrac{1}{y(\log y)^2}$

$$\Rightarrow \quad \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x\log y} = \frac{1}{x}$$

Let $\quad \dfrac{1}{\log y} = z$

$$\Rightarrow \quad \frac{-1}{(\log y)^2}\frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \quad -\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{dz}{dx} - \frac{z}{x} = \frac{-1}{x} \qquad \text{(linear in } z)$$

Now, $\quad \text{I.F.} = e^{\int -1/x\, dx} = e^{-\log x}$

$$= e^{\log x^{-1}} = \frac{1}{x}$$

$\therefore$ required solution is

$$z\frac{1}{x} = \int \frac{-1}{x}\cdot\frac{1}{x}\,dx + c$$

$$\Rightarrow \quad \frac{z}{x} = -\frac{(-1)}{x} + c$$

$$\Rightarrow \quad \frac{1}{\log y} = 1 + cx$$

Q.3. (b) Using the M.O.V.P solve $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=\frac{e^{3x}}{x^2}$

Sol. Consider

$$(D^2-6D+9)y = 0\ e^{3x}/x^2$$

A.E.
$$D^2-6D+9 = 0$$

$\Rightarrow$
$$(D-3)^2 = 0$$
$$D = 3, 3$$

C.F.
$$(C_1+C_2x)e^{3x}$$

$\Rightarrow$
$$C_1e^{3x}+C_2xe^{3x}$$

$$y_1 = e^{3x}, y_2 = xe^{3x}, X = \frac{e^{3x}}{x^2}$$

$$y'_1 = 3e^{3x}, y'_2 = 3xe^{3x}+e^{3x}$$

$$W = y_1y'_2 - y'_1y_2$$
$$= e^{3x}.e^{3x}(3x+1) - xe^{3x}.3e^{3x}$$
$$= 3xe^{6x}+e^{6x}-3xe^{6x}$$
$$= e^{6x}.$$

$$\text{P.I.} = uy_1+vy_2$$

$$u = -\int\frac{y_2X}{W}dx$$
$$= -\int\frac{xe^{3x}.e^{3x}}{e^{6x}.x^2}dx$$
$$= -\int\frac{dx}{x} = -\log x$$

$$V = \int\frac{y_1x}{W}dx$$
$$= \int\frac{e^{3x}.e^{3x}}{e^{6x}.x^2}dx$$
$$= \int\frac{1}{x^2}dx = -\frac{1}{x}$$

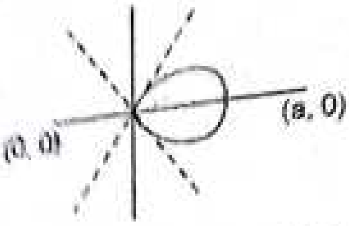
$$\text{P.I.} = -\log x\, e^{3x} - \frac{1}{x}.xe^{3x}$$
$$= -e^{3x}(\log x+1)$$

Complete solution is

$$y = (C_1+C_2x)e^{3x} - e^{3x}(\log x+1)$$

Q.4. (a) Find the perimeter of the loop of the curve $ay^2 = x^2(a-x)$.

Sol. It is symmetric about x-axis. Curve passes through (0, 0) and $(a, 0)$.



$$S = 2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

As
$$y^2 = \frac{x^2(a-x)}{a}$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = \frac{1}{a}\left[(a-x).2x - x^2\right]$$

$$2y\frac{dy}{dx} = \frac{1}{a}[2ax - 3x^2]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x(2a-3x)}{a.2y}$$

$$= \frac{x(2a-3x).\sqrt{a}}{2a\,x\sqrt{a-x}}$$

$$= \frac{(2a-3x)}{2\sqrt{a}\,\sqrt{a-x}}$$

Now
$$S = 2\int_0^a \sqrt{1+\frac{(2a-3x)^2}{4a(a-x)}}\,dx$$

$$= 2\int_0^a \frac{\sqrt{4a^2-4ax+4a^2+9x^2-12ax}\,dx}{2\sqrt{a}\,\sqrt{a-x}}$$

$$= \frac{1}{\sqrt{a}}\int_0^a \sqrt{\frac{8a^2+9x^2-16ax}{a-x}}\,dx$$

Q.4. (b) Prove that $\int_0^\infty x^{n-1} e^{-ax}\cos bx\,dx$

$$= \frac{\sqrt{n}}{(a^2+b^2)^{n/2}}\cos n\left(\tan^{-1}\frac{b}{a}\right)$$

Sol. Consider $\int_0^\infty e^{-ax}x^{n-1}dx$

Put
$$ax = z$$
$$\Rightarrow \quad dx = \frac{dz}{a}$$

$\therefore$
$$\int_0^\infty e^{-ax}\, x^{n-1}\, dx = \int_0^\infty e^{-z}\left(\frac{z}{a}\right)^{n-1}\frac{dz}{a} = \frac{\overline{|n}}{a^n}$$

Replace a by $a + ib$.

$$\int_0^\infty e^{-(a+ib)x} x^{n-1} dx = \frac{\sqrt{n}}{(a+ib)^n} \quad \text{...(1)}$$

Now
$$e^{-(a+ib)x} = e^{-ax}\, e^{-ibx}$$
$$= e^{-ax}(\cos bx - i\sin bx)$$

Putting
$$a = r\cos\theta, b = r\sin\theta$$

$\Rightarrow$
$$r^2 = a^2 + b^2, \quad \theta = \tan^{-1}\frac{b}{a}$$
$$(a+ib)^n = r^n(\cos\theta + i\sin\theta)^n$$
$$= r^n\,(\cos n\theta + i\sin n\theta)$$

By (1), we have

$$\int_0^\infty e^{-ax}.\left(\cos bx - i\sin\frac{bx}{1}\right)x^{n-1}dx = \frac{\overline{|n}}{r^n(\cos n\theta + i\sin n\theta)}$$
$$= \frac{\overline{|n}}{r^n}(\cos n\theta - i\sin n\theta)$$

Equating real and imaginary part.

$$\int_0^\infty e^{-ax}\, x^{n-1}\cos bx\, dx = \frac{\overline{|n}}{r^n}\cos n\theta$$
$$= \frac{\overline{|n}}{(a^2+b^2)\,n/2}\cos n\left(\tan^{-1}\frac{b}{a}\right)$$

FIRST SEMESTER (B.TECH)
END TERM EXAMINATION [2014]
APPLIED MATHEMATICS-I [ETMA-101]

Time : 3 hrs. **M.M. : 7**

Note: *Attempt any five questions including Ques No. 1 which is compulsory. Select one question from each unit.*

Q.1. (a) Find the nth differential coefficient of $e^x \sin^2 x$. **(10 × 2.5 = 25)**

Sol. Let
$$y = e^x \sin^2 x.$$
$$= e^x\left(\frac{1-\cos 2x}{2}\right)$$
$$= \frac{1}{2}[e^x - e^x \cos 2x]$$
$$\therefore \quad y_n = \frac{1}{2}\left[e^x - (1^2+2^2)^{n/2} e^x \cos\left(2x + n\tan^{-1}\frac{2}{1}\right)\right]$$
$$= \frac{e^x}{2}\left[1 - 5^{n/2}\cos(2x + n\tan^{-1} 2\right]$$

Q.1. (b) Given $\sin 30° = \frac{1}{2}$, use Taylor's theorem to evaluate $\sin 31°$ correct to 4 decimal places ($\cos 30° = 0.8660$)

Sol. Let
$$f(x+h) = \sin(x+h)$$
Put
$$h = 0$$
$$\Rightarrow \quad f(x) = \sin x$$
$$f'(x) = \cos x, f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$\therefore \quad \sin(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$
$$= \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \ldots$$
Putting
$$x = 30°, h = 1° = \frac{\pi}{180} = 0.0174$$
$$\sin 31° = \sin 30° + 0.0174\cos 30° - \frac{(0.0174)^2}{2}\sin 30° - \frac{(0.0174)^3}{6}\cos 30° + \ldots$$
$$= 0.5 + (0.174)(0.866) - \frac{(0.0174)^2}{2}(0.5) - \frac{(0.0174)^3}{6}(0.866) + \ldots$$

$$= 0.5150684 - 0.00007569 - \frac{(0.0174)^3}{6} 0.866 + \ldots = 0.5149 \text{ (Aprrox)}$$

Q.1. (c) Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

Sol. Let

$$u_n = \frac{1}{(\log n)^n}$$

$$\Rightarrow \quad u_n^{1/n} = \frac{1}{\log n}$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$$

$\therefore$ By Cauchy root test Σu_n is convergent

Q.1. (d) Find the asymptotes of the curve $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 0$

Sol. It has no parallel asymptotes.

Oblique asymptotes:

$$\phi_3 = y^3 - xy^2 - x^2y + x^3$$
$$\phi_2 = x^2 - y^2$$
$$\phi_1 = 0$$

Put

$$x = 1 \text{ and } y = m$$
$$\phi_3(m) = m^3 - m^2 - m + 1$$
$$\phi_2(m) = 1 - m^2$$

Put

$$\phi_3(m) = 0$$

$$\Rightarrow \quad m^3 - m^2 - m + 1 = 0$$
$$(m-1)(m^2-1) = 0$$
$$m = 1, \pm 1$$

$\Rightarrow$

For non repeated $m = -1$

$$c = \frac{-\phi_2(m)}{\phi_3^1(m)}$$

$$= -\frac{(1-m^2)}{3m^2 - 2m - 1}$$

$$C = \frac{-(1-1)}{3+2-1} = 0$$

For repeated $m = 1, 1$

$$\frac{c^2}{2!} \phi_3''(m) + c\, \phi_2'(m) + \phi_1(m) = 0$$

$$\Rightarrow \quad \frac{c^2}{2}(6m-2) + c(-2m) + 0 = 0$$

When $m = 1$, $\quad \frac{c^2}{2}(6-2)+c(-2) = 0$

$$\Rightarrow \quad 2c^2 - 2c = 0$$

$$2c(c-1) = 0$$

$$\Rightarrow \quad c = 0, 1$$

Asymptotes are

$$y = -x, y = x, y = x+1$$

i.e. $$y = \pm x, y - x - 1 = 0$$

Q.1. (e) Show that $\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}dx = 2^{p+q-1}\beta(p,q)$

Sol. Put $$1+x = 2y$$

$$\Rightarrow \quad x = 2y-1$$

$$\Rightarrow \quad dx = 2\,dy$$

and $$1-x = 1-2y+1 = 2-2y$$

Now $$\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}dx$$

$$= \int_0^1 (2y)^{p-1}(2-2y)^{q-1}2\,dy$$

$$= 2^{p-1+q-1+1}\int_0^1 y^{p-1}(1-y)^{q-1}dy$$

$$= 2^{p+q-1}\beta(p,q)$$

Q.1. (f) Show that $\int_0^1 x^{3/2}(1-x)^{3/2}dx = \frac{3\pi}{128}$

Sol. Consider $\int_0^1 x^{3/2}(1-x)^{3/2}dx$

$$= \beta\left(\frac{5}{2},\frac{5}{2}\right)$$

$$= \frac{\Gamma(5/2)\Gamma(5/2)}{\Gamma(5/2+5/2)}$$

$$= \frac{\Gamma(5/2)\Gamma(5/2)}{\Gamma(5)}$$

$$= \frac{3/2\,\Gamma(3/2) - 3/2\,\Gamma(3/2)}{4!}$$

$$= \frac{9}{4}\times\frac{1}{2}\times\frac{1}{2}\,\frac{\Gamma(1/2)\Gamma(1/2)}{4\times 3\times 2}$$

$$= \frac{9\sqrt{\pi}\times\sqrt{\pi}}{4\times 4\times 4\times 3\times 2} = \frac{3\pi}{128}$$

Q.1. (g) If A and B are unitary matrices show that AB is a unitary matrix.

Sol. As A and B are unitary

$\therefore$
$$AA^{\theta} = A^{\theta}A = I \text{ and } BB^{\theta} = B^{\theta}B = I$$

Let
$$C = AB$$

To show
$$CC^{\theta} = C^{\theta}C = I$$

Consider

$$\begin{aligned} CC^{\theta} &= (AB)(AB)^{\theta} \\ &= AB\,B^{\theta}A^{\theta} \\ &= A\,(BB^{\theta})\,A^{\theta} \\ &= A\,I\,A^{\theta} && [\because B \text{ is unitary}] \\ &= AA^{\theta} \\ &= I && [\because A \text{ is unitary}] \end{aligned}$$

Also
$$\begin{aligned} C^{\theta}C &= (AB)^{\theta}(AB) \\ &= B^{\theta}A^{\theta}A\,B \\ &= B^{\theta}(A^{\theta}A)B \\ &= B^{\theta}I^{B} \\ &= B^{\theta}B = I \end{aligned}$$

Hence proved.

Q.1. (h) Reduce the matrix $\begin{bmatrix} 8 & 1 & 36 \\ 0 & 3 & 22 \\ -8 & -1 & -34 \end{bmatrix}$ to normal form and hence, find its rank.

Sol.
$$\text{Let } A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

Operating
$$c_1 \to \frac{c_1}{8}$$

$$\sim A = \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -1 & -1 & -3 & 4 \end{bmatrix}$$

Operating $R_3 \to R_3 + R_1$

$$\sim A = \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Operating $c_2 \to c_2 - c_1, c_3 \to c_3 - 3c_1, c_4 \to c_4 - 6c_1$

$$\sim A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Operating $c_2 \to \frac{1}{3}c_2, c_3 \to \frac{1}{2}c_2, c_4 \to \frac{1}{2}c_4$

$$\sim A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Operating $c_3 \to c_3 - c_2, c_4 \to c_4 - c_2$

$$\sim A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$c_4 \to \frac{1}{5}c_4$$

$$\sim A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $c_3 \leftrightarrow c_4$

$$\sim A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A = [I_3 : 0]$$

$$\text{Rank of } A = 3.$$

Q.1. (i) Solve $(y^3 - 3xy^2)\,dx + (2x^2y - xy^2)\,dy = 0$

Sol. Here

$$M = y^3 - 3xy^2$$

$$N = 2x^2y - xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 - 6xy$$

$$\frac{\partial N}{\partial x} = 4xy - y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ (Not exact)}$$

Consider

$$Mx + Ny = xy^3 - 3x^2y^2 + 2x^2y^2 - xy^3$$

$$= -x^2y^2 = 0$$

$$\text{I.F.} = \frac{-1}{x^2y^2}$$

Equation changes to

$$\left(\frac{-y}{x^2} + \frac{3}{x}\right)dx + \left(\frac{-2}{y} + \frac{1}{x}\right)dy = 0$$

Now

$$M = \frac{-y}{x^2}+\frac{3}{x}, N = \frac{1}{x}\frac{-2}{y}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{x^2}, \frac{\partial N}{\partial x} = \frac{-1}{x^2}$$

$\therefore$ equation is exact

$$\text{consider} \int_{y\text{ constt}} M dx = \int\left(-\frac{y}{x^2}+\frac{3}{x}\right)dx$$

$$= \frac{y}{x}+3\log x.$$

$$\int N^* dy = \int\left(-\frac{2}{y}\right)dy$$

$$= -2\log y$$

Complete solution is

$$\frac{y}{x}+3\log x-2\log y = c$$

Q.1. (j) Show that $J_{3/2} = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x}-\cos x\right]$

Sol. As we know

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \quad ...(1)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad ...(2)$$

By recurrence relation

$$J_{n+1}(x)+J_{n-1}(x) = \frac{2n}{x}J_n(x)$$

Replacing n by $\frac{1}{2}$

$$J_{3/2}(x)+J_{-1/2}(x) = \frac{1}{x}J_{1/2}(x)$$

$$\Rightarrow \quad J_{3/2}(x) = \frac{1}{x}J_{1/2}(x)-J_{-1/2}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x-\sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x}-\cos x\right]$$

UNIT–I

Q.2. (a) If $y = (\sin^{-1} x)^2$, show that $(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - n^2 y_n = 0.$ **(6.5)**

Sol. Let
$$y = (\sin^{-1} x)^2$$
$$\Rightarrow \quad y_1 = 2\sin^{-1} x \frac{1}{\sqrt{1-x^2}}$$
$$\Rightarrow \quad \sqrt{1-x^2}\, y_1 = 2\sin^{-1} x$$
$$\Rightarrow \quad (1-x^2)\, y_1^2 = 4(\sin^{-1} x)^2$$
$$\Rightarrow \quad (1-x^2)\, y_1^2 = 4y$$

Again differentiating w.r.t 'x'
$$\Rightarrow \quad (1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = 4y_1$$
$$\Rightarrow \quad (1-x^2)\, y_2 - x y_1 = 2$$

Using Leibnitz theorem n times
$$y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - y_{n+1} x - {}^nC_1 y_n = 0$$
$$\Rightarrow (1-x^2)\, y_{n+2} - 2n\, x y_{n+1} - \frac{n(n-1)}{x} y_n \,.\, 2 - x\, y_{n+1} - n\, y_n = 0$$
$$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - n^2 y_n = 0$$

Q.2. (b) Test for convergence of the series $\frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 \ldots \infty$ **(6)**

Sol. Let
$$u_n = \frac{3.6.9\ldots.3_n}{7.10.13\ldots(3n+4)} x^n$$
$$u_{n+1} = \frac{3.6.9\ldots(3n+3)}{7.10.13\ldots(3n+7)} x^{n+1}$$
$$\frac{u_n}{u_{n+1}} = \frac{3.6.9\ldots 3n}{7.10.13\ldots(3n+4)} \frac{7.10.13\ldots(3n+7)x^n}{3.6.9\ldots(3n+3)x^{n+1}}$$
$$= \frac{3n+7}{3n+3} \cdot \frac{1}{x}$$
$$= \frac{n(3+7/n)}{n(3+3/n)} \frac{1}{x}$$
$$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By Ratio test

Σu_n is convergent if $\frac{1}{x} > 1$

i.e. if $\quad x < 1$

Σu_n is divergent if $\frac{1}{x} < 1$ i.e. if $x > 1$

Test fails for $x = 1$

for $x = 1$
$$\frac{u_n}{u_{n+1}} - 1 = \frac{3n+7}{3n+3} - 1$$
$$= \frac{3n+7-3n-3}{3n+3}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{4n}{3n+3} = \frac{4n}{n(3+3/n)}$$
$$= \frac{4}{3(1+1/n)}$$
$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{4}{3} > 1$$

$\therefore$ By Raabe's Test Σu_n is convergent

Thus the series Σu_n is convergent if $x \leq 1$ and divergent if $x > 1$

Q.3. (a) Find if the series $\frac{1}{2} - \frac{1}{2.2^2} + \frac{1}{3.2^3} \cdots$ converges absolutely or conditionally. **(6.5)**

Sol. Here
$$u_n = \frac{(-1)^{n+1}}{n.2^n}$$

Absolute Convergence

Consider
$$|u_n| = \left|\frac{(-1)^{n+1}}{n.2n}\right| = \frac{1}{n.2n}$$

$$\Rightarrow \quad u_n = \frac{1}{n.2^n}$$
$$u_{n+1} = \frac{1}{(n+1).2^{n+1}}$$

Consider
$$\frac{u_n}{u_{n+1}} = \frac{1}{n.2n} \cdot \frac{(n+1)2^{n+1}}{1}$$
$$= \frac{n(1+1/n).2}{n}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 2 > 1$$

$\therefore \Sigma|u_n|$ is convergent By ratio test $\Sigma(-1)^{n+1}u_n$.

Thus $\Sigma(-1)^n + 1\, u_n$ is absolutely convergent Hence convergent.

Q.3. (b) Use maclaurin's theorem to show that
$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5 + \ldots}{5!} \qquad (6)$$

Sol. Let
$$y = e^x \cos x,\ y(0) = 1 = f(x)$$
$$f'(x) = e^x(\cos x - \sin x), f'(0) = 1$$
$$f''(x) = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x)$$
$$f''(0) = 1 - 1 = 0$$
$$f'''(x) = e^x(\cos x - \sin x) - e^x(\sin x + \cos x)$$
$$-e^x(\sin x + \cos x) - e^x(\cos x - \sin x)$$
$$f'''(x) = -2e^x(\sin x + \cos x)$$
$$f'''(0) = -2$$
$$f^{iv}(x) = -2e^x(\sin x + \cos x) - 2e^x(\cos x - \sin x)$$
$$f^{iv}(0) = -2 - 2 = -4$$
$$f^{iv}(x) = -2e^x(2\cos x) = -4e^x \cos x$$
$$f^{v}(x) = -4e^x \cos x + 4e^x \sin x$$
$$= 4e^x(\sin x - \cos x)$$
$$f^{v}(0) = -4.$$

By Maclaurin's series
$$f(x) = f'(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$
$$e^x \cos x = 1 + x - \frac{2}{3!}x^3 - \frac{4x^4}{4!} - \frac{4x^5}{5!} + \ldots$$
$$= 1 + x - \frac{2}{3!}x^3 - \frac{2^2}{4!}x^4 - \frac{2^2}{5!}x^5 + \ldots$$

UNIT–II

Q.4. (a) Trace the curve
$$x^2y^2 = a^2(y^2 - x^2) \tag{6.5}$$

Sol. Symmetry: Even powers of x and y, curve is symmetrical about both axes.

2. Curve passes through origin.

3. **Axes Intersection.**

Put
$$x = 0, y = 0$$
It does not cut any axes.

4. **Tangents:** At origin (0, 0)

equating lowest degree terms
$$y^2 = x^2$$
$$\Rightarrow \quad y = \pm x \text{ (real and distinct)}$$
origin is a node.

5. Asymptotes:
$$x^2y^2 - a^2y^2 + a^2x^2 = 0$$
$$x = \pm a \text{ are parallel asymptotes}$$

Obliques
$$\phi_4 = x^2y^2$$

$$\phi_2 = x^2 a^2 - a^2 y^2 i\ \phi_2(m) = a^2 - m^2 a^2$$

Put $\quad x = 1, y = m$

$\Rightarrow \quad m^2 = 0 \Rightarrow m = 0, 0.$

For Repeated

$$\frac{c^2}{2!}\phi_4''(m) + c\phi_3'(m)\phi_2(m) = 0$$

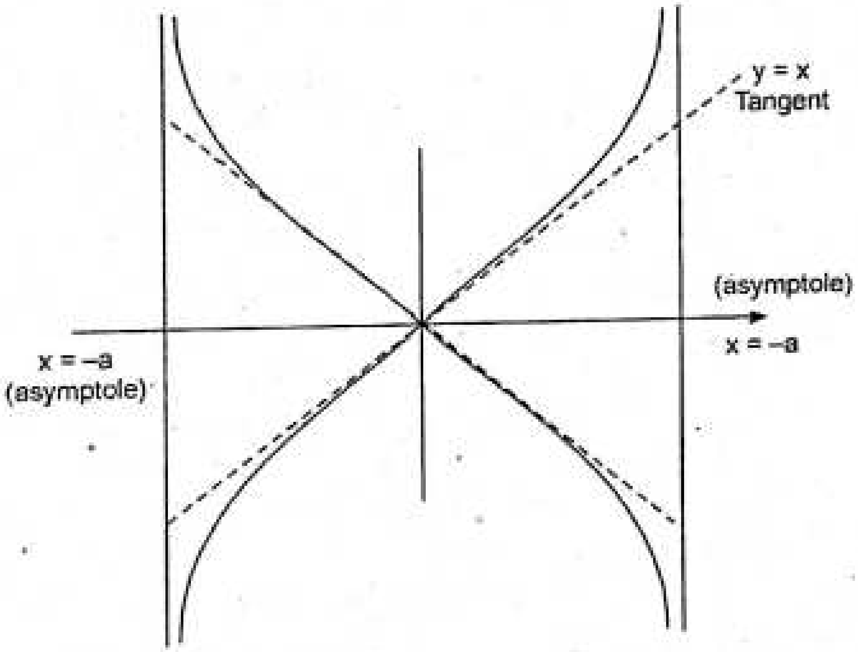
$$\Rightarrow \quad \frac{c^2}{2!}.2 + a^2(1-m^2) = 0$$

$$\Rightarrow \quad c^2 + a^2 = 0$$

$$\Rightarrow \quad c^2 = -a^2$$

$$\Rightarrow \quad c = \pm ia$$

only $x = \pm a$ are asymptotes.



Region of existence

$$y^2 = \frac{a^2x^2}{a^2 - x^2}$$

Region	example	sign	Region of existence
$x < -a$	$x = -2a$	– ve	No
$-a < x < 0$	$x = -a/2$	+ve	Yes
$0 < x < a$	$x = a/2$	+ve	Yes
$x > a$	$x = 2a$	–ve	No

Curves lies bet $-a$ and a

Q.4. (b) Find the radius of curvature at any point t on the curve

$$x = a\cos t + \log\left(\tan\frac{t}{2}\right) y = a\sin t$$

Sol.

$$\rho = \frac{[x'^2 + y'^2]^{3/2}}{x'y'' - y'x''}$$

$$x' = a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}\sec^2\frac{t}{2}.\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{\cos t/2}{\sin t/2}\frac{1}{\cos^2 t/2}.\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin t/2\cos t/2}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= \frac{a(1-\sin 2t)}{\sin t}$$

$$= \frac{a\cos^2 t}{\sin t}$$

$$x'' = a\left[\frac{-2\cos t\sin^2 t - \cos^2 t.\cos t}{\sin^2 t}\right]$$

$$x'' = -a\left[\frac{\cos t\,(2\sin^2 t + \cos^2 t)}{\sin^2 t}\right]$$

$$y = a\sin t$$
$$y' = a\cos t$$
$$y'' = -a\sin t.$$

$$\rho = \frac{\left[a^2\frac{\cos^4 t}{\sin^4 t} + a^2\cos^2 t\right]^{3/2}}{-a^2\cos^2 t + \frac{a^2\cos^2 t(2\sin^2 t + \cos^2 t)}{\sin^2 t}}$$

$$= \frac{a^3\left[\frac{\cos^4 t}{\sin^4 t} + \cos^2 t\right]^{3/2}}{a^2\left[-\frac{\sin^2 t\cos^2 t + 2\cos^2 t\sin^2 t + \cos^4 t}{\sin^2 t}\right]}$$

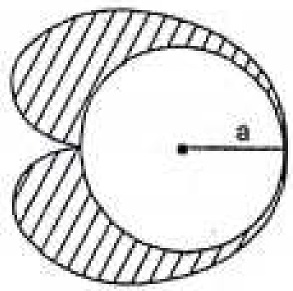
$$= \frac{a\cos^3 t\left[1 + \frac{\cos^2 t}{\sin^4 t}\right]^{3/2}}{\frac{\cos^2 t\sin^2 t + \cos^4 t}{\sin^2 t}}$$

$$= \frac{\dfrac{a\cos t\,(\sin^4 t+\cos^2 t)^{3/2}}{\sin^5 t}}{\dfrac{1}{\sin^2 t}}$$

$$= \frac{a\cos t}{\sin^3 t}(\sin^4 t+\cos^2 t)^{3/2}$$

Q.5. (*a*) Find the area lying inside the cardiode $r = a(1 + \cos\theta)$ and outside the circler = $2\,a\cos\theta$. **(6.5)**

Sol.



Point of intersection are

$$2\,a\cos\theta = a(1+\cos\theta)$$

$$\cos\theta = 1 \Rightarrow \theta = 0, 2\pi$$

since curve is symmetrical about initial line.

$$\therefore \quad \text{Area} = 2\times\frac{1}{2}\,[\text{Area of cardiode} - \text{Area of circle}]$$

$$= \int_0^{\pi} r_1^2 d\theta - \int_0^{\pi} r_2^2 d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2 d\theta - \int_0^{\pi} 4a^2\cos^{2\theta} d\theta$$

$$= a^2\int_0^{\pi}(1+\cos^2\theta+2\cos\theta)d_0 - 4a^2.2\int_0^{\pi.2}\cos^2\theta d\theta$$

$$= a^2\left[\theta\Big|_0^{\pi} + 2\int_0^{\pi/2}\cos^2\theta\, d_0 + 2\sin\Big|_0^{\pi}\right] - 8a^2.\frac{1}{2}.\frac{\pi}{2}$$

$$= a^2\left[\pi + 2.\frac{1}{2}.\frac{\pi}{2}\right] - 2a^2\pi$$

$$= \frac{3\pi a^2}{2} - 2\pi a^2$$

$$= \frac{-\pi a^2}{2}$$

$$\text{Area} = \frac{\pi a^2}{2}$$

Q.5. (b) Evaluate $\int_0^{\pi} \sin^2\theta(1+\cos\theta)^4 d\theta$

Sol. Since it is an even function

$$= \int_0^{\pi} \sin^2\theta(1+\cos^4\theta+6\cos^2\theta+4\cos\theta+4\cos^3\theta)d\theta$$

$$= \int_0^{\pi}\sin^2\theta\, d\theta + \int_0^{\pi}\sin^2\theta\cos^4\theta\, d\theta + 6\int_0^{\pi}\sin^2\theta\cos^2\theta d\theta$$

$$+4\int_0^{\pi}\sin^2\theta\cos\theta d\theta + 4\int_0^{\pi}\sin^2\theta\cos^3\theta d\theta$$

All are even functions

$$\Rightarrow\ 2\int_0^{\pi/2}\sin^2\theta\, d\theta + 2\int_0^{\pi}\sin^2\theta\cos^4\theta\, d\theta$$

$$+12\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta + 8\int_0^{\pi/2}\sin^2\theta\cos\theta d\theta\ +8\int_0^{\pi/2}\sin^2\theta\cos^3\theta d\theta$$

$$= 2\times\frac{1}{2}\times\frac{\pi}{2}+\frac{2\sqrt{3/2}\,\sqrt{5/2}}{2\sqrt{4}}+\frac{12\sqrt{3/2}\,\sqrt{3/2}}{2\sqrt{3}}$$

$$+\frac{8\sqrt{3/2}\,\sqrt{1}}{2\sqrt{5/2}}+\frac{8\sqrt{3/2}\,\sqrt{2}}{2\sqrt{7/2}}$$

$$= \frac{\pi}{2}+\frac{1}{3!}\frac{1}{2}\sqrt{\pi}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}+\frac{6}{2}\frac{1}{4}.\pi$$

$$+\frac{4\sqrt{3/2}}{\frac{3}{2}\sqrt{3/2}}+\frac{4\sqrt{3/2}}{\frac{5}{2}.\frac{3}{2}\sqrt{\frac{3}{2}}}$$

$$= \frac{\pi}{2}+\frac{\pi}{16}+\frac{3\pi}{4}+\frac{8}{3}+\frac{16}{15}$$

$$= \frac{15\pi}{16}+\frac{56}{15}$$

UNIT–III

Q.6. (a) Find the values of a and b such that the system of equations

$$3x-2y+z = 6,\ 5x-8y+9z=3,$$
$$2x+y+az = -b \text{ may have a unique solution.}$$

Sol. Consider $AX = B$

$$\begin{bmatrix}3 & -2 & 1\\ 5 & -8 & 9\\ 2 & 1 & a\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}6\\ 3\\ -b\end{bmatrix}$$

Operating $R_2 \to R_2 - \frac{5}{3}R_1,\ R_3 \to R_3 - \frac{2}{3}R_1$

$$\sim\begin{bmatrix}3 & -2 & 1\\ 0 & -14/3 & 22/3\\ 0 & 7/3 & a-2/3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}6\\ -7\\ -b-4\end{bmatrix}$$

Operating $R_3 \to R_3 + 1/2\, R_2$

$$\sim \begin{bmatrix} 3 & -2 & 1 \\ 0 & -14/3 & 22/3 \\ 0 & 0 & a+3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ b-15/2 \end{bmatrix}$$

For unique solution

$$\rho(A) = \rho(A:B) = 3$$

$$a = -3, b \text{ can have any value.}$$

Q.6. (*b*) The eigen vectors of a 3 × 3 matrix A corresponding to the eigen values 1, 2, 3 are $[1, 2, 1]^T$, $[2, 3, 4]^T$, $[1, 4, 9]^T$ respectively Find the matrix *A*.

Sol. Modal matrix of *A* is $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 9 \end{bmatrix} = B$

and diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Now as we know

$$D = B^{-1}AB$$

$\Rightarrow$ $$BDB^{-1} = A$$

$\Rightarrow$ $$A = BDB^{-1}$$

$$\text{As} \to B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 9 \end{bmatrix}$$

$$|B| = 1(27-16) - 2(18-4) + 1(8-3) = -12$$

$$B^{-1} = \frac{-1}{12}\begin{bmatrix} 11 & -14 & 5 \\ -14 & 8 & -2 \\ 5 & -2 & -1 \end{bmatrix}$$

$$A = -\frac{1}{12}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 11 & -14 & 5 \\ -14 & 8 & -2 \\ 5 & -2 & -1 \end{bmatrix}$$

$$A = -\frac{1}{12}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} 11 & -14 & 5 \\ -28 & 16 & -4 \\ 15 & -6 & -3 \end{bmatrix}$$

$$= \frac{-1}{12}\begin{bmatrix} -30 & 12 & -6 \\ -2 & -4 & -14 \\ 34 & -4 & -38 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 15 & -6 & 3 \\ 1 & 2 & 7 \\ 17 & 2 & 19 \end{bmatrix}$$

Q.7. (a) Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$. Hence find A^{-1}. **(6.5)**

Sol. Characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 4 & 1-\lambda & 0 \\ 8 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda)^2 + 1[4-4\lambda] + 1[4-8+8\lambda] = 0$$

$$\Rightarrow \quad (1-\lambda)^3 + 4 - 4\lambda - 4 + 8\lambda = 0$$

$$\Rightarrow \quad 1 - \lambda^3 - 3\lambda + 3\lambda^2 + 4\lambda = 0$$

$$\Rightarrow \quad -\lambda^3 + 3\lambda^2 + \lambda + 1 = 0$$

$$\Rightarrow \quad \lambda^3 - 3\lambda^2 - \lambda - 1 = 0$$

Its characteristic equation is

$$A^3 - 3A^2 - A - I = 0$$

Multiply by A^{-1}

$$A^2 - 3A - AA^{-1} - A^{-1}I = 0$$

$$\Rightarrow \quad A^2 - 3A - I - A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = A^2 - 3A - I$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 8 & -3 & 4 \\ 20 & -6 & 9 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5 & -1 & 2 \\ 8 & -3 & 4 \\ 20 & -6 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$$

Q.7. (b) Find the eigen values and eigen-vectors of the matrix.

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \quad ...(6)$$

Sol. Characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda)-16]+6[-18+6\lambda+8]+2[24-14+2\lambda]=0$

$\Rightarrow (8-\lambda)(21-10\lambda+\lambda^2-16)+6(6\lambda-10)+2(10+2\lambda)=0$

$\Rightarrow (8-\lambda)(5-10\lambda+\lambda^2)+36\lambda-60+20+4\lambda=0$

$\Rightarrow 40-80\lambda+8\lambda^2-5\lambda+10\lambda^2-\lambda^3-40+40\lambda=0$

$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0$

$\Rightarrow -\lambda(\lambda^2-18\lambda+45)=0$

$\Rightarrow \lambda=0, (\lambda^2-15\lambda-3\lambda+45)=0$

$(\lambda-15)(\lambda-3)=0$

$\Rightarrow \lambda = 0, 3, 15$ (eigen values)

For $\lambda = 0$, eigen vector

$$[A - OI][X] = 0$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operating $R_3 \to R_3 - \dfrac{R_1}{4}, R_2 \to R_2 + \dfrac{3}{4}R_1$

$$\sim \begin{bmatrix} 8 & -6 & 2 \\ 0 & 5/2 & -5/2 \\ 0 & -5/2 & 5/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 8 & -6 & 2 \\ 0 & 5/2 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(A) = 2 < 3$$

One variable be given arbitrary value.

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$\frac{5}{2}x_2 - \frac{5}{2}x_3 = 0$$

$\Rightarrow$
$$4x_1 - 3x_2 + x_3 = 0$$
$$x_2 - x_3 = 0$$

$\Rightarrow$
$$x_2 = x_3$$

Now
$$4x_1 - 3x_2 + x_2 = 0$$

$\Rightarrow$
$$4x_1 - 2x_2 = 0$$

$\Rightarrow$
$$2x_1 = x_2$$

$\Rightarrow$
$$\frac{x_1}{1} = \frac{x_2}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

For $\lambda = 3$, eigenuector is

$$[A - 3I]X = 0$$

$\Rightarrow$
$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operation
$$R \to R_2 + \frac{6}{5}R_1, R_3 \to R_3 - \frac{2}{5}R_1$$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16/5 & -8/5 \\ 0 & -8/5 & -4/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_3 \to R_3 - \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16/5 & -8/5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As
$$r(A) = 2 < 3 \text{ (infinite many solution)}$$
$$5x_1 - 6x_2 + 2x_3 = 0$$

and
$$\frac{-16}{5}x_2 - \frac{8}{5}x_3 = 0$$

$\Rightarrow$

$\Rightarrow$
$$2x_2 + x_3 = 0$$
$$2x_2 = -x_3$$

$\Rightarrow$
$$\frac{x_2}{-1} = \frac{x_3}{2}$$
$$x_1 = -2$$

$$X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

For $\lambda = 15$ eigenvector is

$$[A - 15I][X] = 0$$

$$\Rightarrow \quad \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operating $R_2 \to \frac{1}{2}R_2, R_3 \to \frac{1}{2}R_3$

$$\sim \begin{bmatrix} -7 & -6 & 2 \\ -3 & -4 & -2 \\ 1 & -2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-3x_1 - 4x_2 - 2x_3 = 0$$

$$x_1 - 2x_2 - 6x_3 = 0$$

on solving last two equations.

$$\frac{x_1}{24-4} = \frac{x_2}{-2-18} = \frac{x_3}{6+4}$$

$$\Rightarrow \quad \frac{x_1}{20} = \frac{x_2}{-20} = \frac{x_3}{10}$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

UNIT–IV

Q.8. (a) Solve $\dfrac{d^2y}{dx^2} - \dfrac{2dy}{dx} + y = xe^x \cos x$ **(6.5)**

Sol. Given equation is

$$(D^2 - 2D + 1)y = x e^x \cos x.$$

A.E.

$$D^2 = -2D + 1 = 0$$

$$\Rightarrow \quad D = 1, 1$$

$$\text{C.F} \Rightarrow (c_1 + c_2 x)e^x$$

P.I.

$$= \frac{1}{(D^2 - 2D + 1)}(x e^x \cos x)$$

$$= \frac{1}{(D-1)^2}(xe^x \cos x)$$

$$\Rightarrow \quad e^x \frac{1}{D^2} x \cos x$$

$$\text{P.I.} = e^x \frac{1}{D} \int x \cos x \, dx$$

$$= e^x \frac{1}{D}[x \sin x + \cos x]$$

$$= e^x \int (x \sin x + \cos x) dx$$

$$= e^x [x(-\cos x) + \sin x + \sin x]$$

$$= e^x[-x \cos x + 2 \sin x].$$

Complete solution is

$$y = (c_1 + c_2 x)e^x + e^x(2\sin x - x\cos x)$$

Q.8. (*b*) Solve by M.O.V.P. (6

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = e^x \log x$$

Sol. Given equation is

$$(D^2 - 2D + 1)y = e^x \log x$$

A.E. $D^2 - 2D + 1 = 0$

$$D = 1, 1$$

C.F. $(c_1 + c_2 x) e^x$

$$= c_1 e^x + c_2 x e^x$$

Let $y_1 = e^x, y_2 = xe^x$

$$X = e^x \log x$$

$$y'_1 = e^x, y'_2 = xe^x + e^x$$

$$W = y_1 y'_2 - y'_1 y_2$$

$$W = e^x (x e^x + e^x) - e^x . xe^x$$

$$W = e^{2x}$$

$$\text{P.I.} = uy_1 + Vy_2$$

$$u = -\int \frac{y_2 x}{W} dx$$

$$= -\int \frac{xe^x . e^x \log x}{e^{2x}} dx$$

$$= -\int x \log x \, dx$$

$$= -\left[\log x \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2} dx\right]$$

$$= -\left[\frac{x^2 \log x}{2} - \int \frac{x}{2} dx\right]$$

$$= -\left[\frac{x^2 \log x}{2} - \frac{x^2}{4}\right]$$

$$= \frac{x^2}{4} - \frac{x^2 \log x}{2}$$

$$V = \int \frac{y_1 x}{W} dx$$

$$= \int \frac{e^x . e^x \log x}{e^{2x}} dx$$

$$= \int \log x \, dx$$

$$= \log x . x - \int \frac{1}{x} . x \, dx$$

$$= x \log x - x$$

$$\text{P.I.} = \frac{x^2 e^x}{4} - \frac{x^2 e^x \log x}{2} + x^2 e^x \log x \; - x^2 e^x .$$

$$= \frac{-3x^2 e^x}{4} + \frac{x^2 e^x \log x}{2}$$

$$= \frac{x^2 e^x}{2} \left(\log x - \frac{3}{2} \right)$$

Complete solution is

$$y = (c_1 + c_2 x) e^x + \frac{x^2 e^x}{2} \left(\log x - \frac{3}{2} \right)$$

Q.9. (a) Show that

$$\frac{d}{dx}[x J_n J_{n+1}] = x[J_n{}^2 - J_{n+1}{}^2]$$

Sol. $$\frac{d}{dx}[x J_n J_{n+1}] = J_n J_{n+1} + x\{J'_n J_{n+1} + J'_{n+1} J_n\}$$

$$\Rightarrow \quad \frac{d}{dx}[x J_n J_{n+1}] = J_n J_{n+1} + J_{n+1}\{x J'_n\} + J_n\{x J'_{n+1}\}$$

As we know by relation

$$x J'_n(x) = n J_n(x) - x J_{n+1}(x) \quad \text{...(2)}$$

also, $$J'_n(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \quad \text{...(3)}$$

Changing n to $(n + 1)$ in (3), we get

$$J'_{n+1} + \frac{(n+1)}{x} J_{n+1} = J_n$$

$$\Rightarrow \quad x J'_{n+1} = x J_n - (n+1) J_{n+1} \quad \text{...(4)}$$

Putting (2) and (4) in (1), we get

$$\frac{d}{dx}\{x J_n J_{n+1}\} = J_n J_{n+1} + J_{n+1}\{n J_n - x J_{n+1}\}$$

$$+ J_n\{x J_n - (n+1) J_{n+1}\}$$

$$= J_n J_{n+1} + n J_n J_{n+1} - x J_{n+1}^2$$

$$+ x J_n^2 - (n+1) J_n J_{n+1}$$

$$= x\{J_n{}^2 - J_{n+1}^2\}$$

Q.9. (b) Prove that $\int_{-1}^{1} x^2\, p_{n+1}(x)\, p_{n-1}(x)\, dx$ ...(6

$$= \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

Sol. By recurrence relation

$$(2n+1)\,x\,P_n(x) = (n+1)\,P_{n+1} + n\,P_{n-1} \quad ...(1)$$

Replace n by $n-1$

$$(2n-1)x\,P_{n-1} = n\,P_n + (n-1)\,P_{n-2}$$

$$\Rightarrow \qquad x\,P_{n-1} = \frac{n}{2n-1}P_n + \frac{n-1}{2n-1}P_{n-2} \quad ...(2)$$

Replace n by $(n+2)$ is (2)

$$\Rightarrow \qquad x\,p_{n+1} = \frac{n+2}{2n+3}P_{n+2} + \frac{(n+1)}{2n+3}P_n \quad ...(3)$$

Multiply (2) and (3), we get

$$x^2 P_{n-1} P_{n+1} = \left[\frac{n}{2n-1}P_n + \frac{n-1}{2n-1}P_{n-2}\right] \times \left[\frac{n+2}{2n+3}P_{n+2} + \frac{n+1}{2n+3}P_n\right]$$

$$= \frac{1}{(2n-1)(2n+3)}\,[n(n+1)P_n^2 + n(n+2)P_nP_{n+2} + (n-1)(n+2)P_{n-2}P_{n+2} + (n^2-1)P_{n-2}P_n$$

As by orthogonality

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{2}{2n+1} & m = n \end{cases} \quad ...(5)$$

Integrating (4) w.r.t. x, from $a-1$ to 1 and using (5), we get

$$\int_{-1}^{1} x^2 P_{n-1}(x)\,P_{n+1}(x)dx = \frac{n(n+1)}{(2n-1)(2n+3)}\int_{-1}^{1} P_n^2(x)dx$$

$$= \frac{(n+1)2n}{(2n-1)(2n+3)(2n+1)}$$

FIRST TERM EXAMINATION [SEPT.-2015]
FIRST SEMESTER [B. TECH]
APPLIED MATHEMATICS-I [ETMA-101]

Time: 1.30 Hrs.

Note: *Q. No. 1 is compulsory. Attempt any 2 more questions.* **MM : 30**

Q.1. Find the value of a, b, c if matrix.

$$A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$$

is orthogonal.

Ans. For matrix A to be orthogonal $AA^T = I$ **(2)**

$$\text{Consider } AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$$

$$= \begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix}$$

$$\text{Since } AA^T = I$$

$$\text{On solving } 2b^2-c^2 = 0$$

and $$a^2-b^2-c^2 = 0$$

$\Rightarrow$ $$c = \pm\sqrt{2}b$$

and $$a^2 = b^2+c^2$$

$\Rightarrow$ $$a^2 = 3b^2$$

$\Rightarrow$ $$a = \pm\sqrt{3}b$$

Comparing $$4b^2+c^2 = 1$$

$\Rightarrow$ $$6b^2 = 1$$

$\Rightarrow$ $$b = \pm\sqrt{\frac{1}{6}}$$

$\Rightarrow$ $$c = \pm\sqrt{2}\cdot\frac{1}{\sqrt{6}} = \pm\frac{1}{\sqrt{3}}$$

$$a = \pm\sqrt{3}\cdot\frac{1}{\sqrt{6}} = \pm\frac{1}{\sqrt{2}}, b = \pm\frac{1}{\sqrt{6}}$$

Q.1. (b) Test the convergence of the series

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\dots$$

Ans. Neglecting first term

Let
$$U_n = \frac{n^n}{(n+1)^{n+1}}$$

$$(U_n)^{1/n} = \frac{n}{(n+1)^{1+1/n}} = \frac{n}{n.n^{1/n}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{1/n}}$$

$$= \frac{1}{n^{1/n}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty}(U_n)^{1/n} = \frac{1}{e}<1$$

$\therefore \Sigma U_n$ Converges by Cauchy nth root test.

Q.1. (c) If $y=(x^2-1)^n$, prove that
$(x^2-1)\,y_{n+2}+2x\,y_{n+1}-n(n+1)y_n=0$ (2.5)

Ans.
$$y = (x^2-1)^n$$
$$y_1 = n(x^2-1)^{n-1}.2x$$
$$= \frac{2\,nx}{x^2-1}.y$$
$$(x^2-1)y_1 = 2\,nxy$$

Again differentiating w.r.t 'x'.

$$(x^2-1)y_2+2\,xy_1 = 2\,n(xy_1+y).$$

By Leibnitz theorem 'n' times

$$(x^2-1)y_{n+2}+{}^nC_1\,.\,2x\,y_{n+1}+{}^nC_2\,.\,2\,y_n+2\,[xy_{n+1}+{}^nC_1y_n]$$
$$= 2\,nxy_{n+1}+2\,ny_n+2n^2\,y_n$$

$$\Rightarrow (x^2-1)y_{n+2}+2nx\;y_{n+1}+\frac{n(n-1)}{2}y_n.2+2\;xy_{n+1}+2\;ny_n\;-2n^2y_n-2\;nxy_{n+1}-2\;ny_n=0$$

$y_{n+2}+2\,xy_{n+1}-n(n+1)y_n=0$

Q.1. (d) Reduce the quadratic form

$12\,x_1^2+4x_2^2+5x_3^2-4x_2\,x_3+6x_1x_3-6x_1x_2$ to Canonical form. Also find the nature of quadratic form. (2.5)

Ans. It can be written as $X'AX$ where

$$X' = [x_1\;\; x_2\;\; x_3]\text{ and }A=\begin{bmatrix}12 & -3 & 3\\ -3 & 4 & -2\\ 3 & -2 & 5\end{bmatrix}$$

As
$$A = I_3AI_3$$

$$\begin{bmatrix} 12 & -3 & 3 \\ -3 & 4 & -2 \\ 3 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 \to R_2 + \frac{1}{4}R_1, R_3 \to R_3 - \frac{1}{4}R_1$

$$\sim \begin{bmatrix} 12 & -3 & 3 \\ 0 & 13/4 & -5/4 \\ 0 & -5/4 & 17/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \to C_2 + \frac{1}{4}C_1, C_3 \to C_3 - \frac{1}{4}C_1$

$$\sim \begin{bmatrix} 12 & 0 & 0 \\ 0 & 13/4 & -5/4 \\ 0 & -5/4 & 17/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + \frac{5}{13}R_2$

$$\begin{bmatrix} 12 & 0 & 0 \\ 0 & 13/4 & -5/4 \\ 0 & 0 & 49/13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -2/13 & 5/13 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 \to C_3 + \frac{5}{13}C_2$

$$\sim \begin{bmatrix} 12 & 0 & 0 \\ 0 & 13/4 & 0 \\ 0 & 0 & 49/13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -2/13 & 5/13 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/4 & -2/13 \\ 0 & 1 & 5/13 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Diag}\left(12, \frac{13}{4}, \frac{49}{13}\right) = P' \, A \, P$$

Canonical form is

$$Y'(P'' AP)Y = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 12 & 0 & 0 \\ 0 & 13/4 & 0 \\ 0 & 0 & 49/13 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 12y_1^2 + \frac{13}{4}y_2^2 + \frac{49}{13}y_3^2$$

rank $(r) = 3$, index $(p) = 3$, Dignature $= 3$, $n = 3$. Nature is positive definite.

Q.1. (e) If .... are eigenvalues of a matrix A, then A^4 has eigenvalues. (1)

Ans. $\lambda_1^4, \lambda_2^4 \dots \lambda_n^4$.

Q.2. (a) Expand $e^{a\sin^{-1}x}$ by Mclaurin's theorem upto x^4. (5)

Ans. Let
$$y = e^{a\sin^{-1}x} \quad ...(1)$$
Differentiating w.r.t 'x'.
$$y_1 = e^{a\sin^{-1}x}\frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$
$$\Rightarrow \quad (1-x^2)y_1^2 = a^2y^2 \quad ...(2)$$
Again differentiating w.r.t 'x'.
$$(1-x^2)\,2y_1y_2 - 2xy_1^2 = 2a^2yy_1$$
$$\Rightarrow \quad (1-x^2)y_2 - xy_1 = a^2y \quad ...(3)$$
Differentiating 'n' times by Leibnitz theorem, we get
$$y_{n+2}(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2) - [x.y_{n+1} + ny_n] = a^2y_n.$$
$$= a^2y_n$$
$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0 \quad ...(4)$$
Putting
$$x = 0 \text{ in } (1), (2), (3), (4)$$
$$y(0) = e^0 = 1$$
$$y_1(0) = a$$
$$y_2(0) = a^2$$
and
$$y_{n+2}(0) = (n^2+a^2)y_n(0) \quad ...(A)$$
Putting
$$n = 1, 2, 3 \dots \text{ in } (A), \text{ we get.}$$
$$y_3(0) = (1^2+a^2)y_1(0) = a(1^2+a^2)$$
$$y_4(0) = (2^2+a^2)y_2(0) = a^2(2^2+a^2)$$
$$y_5(0) = (3^2+a^2)y_3(0) = a(1^2+a^2)(3^2+a^2)$$
$$\therefore e^a\sin^{-1}x = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \dots$$
$$e^{a\sin^{-1}x} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a(1^2+a^2)}{3!}x^3 + \frac{a^2(2^2+a^2)}{4!}x^4 + \dots$$

Q.2. (b) Determine the values of λ for which the following system of equation have non-trivial solution $3x_1 + x_2 - \lambda x_3 = 0$, $4x_1 - 2x_2 - 3x_3 = 0$, $2\lambda x_1 + 4x_2 + \lambda x_3 = 0$. For each permissible value of λ, find the solution. (5)

Ans. In matrix notation, it can be written as
$$AX = 0$$

$$\begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

For non trivial solution $|A| = 0$

$$\Rightarrow \quad \begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow 3(-2\lambda + 12) - 1(4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0$$

$$\Rightarrow \quad -6\lambda + 36 - 10\lambda - 16\lambda - 4x^2 = 0$$

$$\Rightarrow \quad -4\lambda^2 - 32\lambda + 36 = 0$$

$$\Rightarrow \quad \lambda^2 + 8\lambda - 9 = 0$$

$$\Rightarrow \quad (\lambda + 9)(\lambda - 1) = 0$$

$$\lambda = 1, -9$$

for $\lambda = 1$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_2 \to R_2 - \frac{4}{3}R_1, R_3, \to R_3 \frac{-2}{3}R_1$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & -10/3 & -5/3 \\ 0 & 10/3 & 5/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 3 & 1 & -1 \\ 0 & -10/3 & -5/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$3x_1 + x_2 - x_3 = 0$$

$$-\frac{10}{3}x_2 - \frac{5}{3}x_3 = 0$$

Let $\quad x_2 = k \Rightarrow 5x_3 = -10k \Rightarrow x_3 = -2k.$

$$3x_1 + 2k + k = 0$$

$$x_1 = k.$$

$\Rightarrow \quad x_1 = k, x_2 = k, x_3 = -2k.$

For $\lambda = -9$

$$\begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & -3 \\ -18 & 4 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + 6R_1, R_2 \to R_2 - \frac{4}{3}R_1$

$$\sim \begin{bmatrix} 3 & 1 & 9 \\ 0 & 10/3 & -5 \\ 0 & 10 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_2 \to 3R_2$$

$$\sim \begin{bmatrix} 3 & 1 & 9 \\ 0 & -10 & -45 \\ 0 & 10 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 3 & 1 & 9 \\ 0 & -10 & -45 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$3x_1 + x_2 + 9x_3 = 0$$

$$-10x_2 - 45x_3 = 0$$

Let $$x_2 = k$$

$\Rightarrow$ $$45x_3 = -10k.$$

$$x_3 = \frac{-2}{9}k$$

$\Rightarrow$ $$3x_1 + k - \frac{9 \times 2}{9}k = 0.$$

$\Rightarrow$ $$3x_1 = k \Rightarrow x_1 = \frac{k}{3}$$

$\Rightarrow$ $$x_1 = \frac{k}{3}, x_2 = k, x_3 = -\frac{2}{9}k.$$

Q.3. (*a*) For what values of *x* the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n(n+3)}$ is convergent. **(5)**

Ans. Let $$u_n = \frac{x^n}{n(n+3)}$$

Consider $$\Sigma|(-1)^n u_n| = \Sigma|u_n|$$

$$u_n = \frac{|x|^n}{n(n+3)}, u_{n+1} = \frac{|x|^{n+1}}{(n+1)(n+4)}$$

Now $$\frac{U_n}{U_{n+1}} = \frac{|x|^n}{n(n+3)} \frac{(n+1)(n+4)}{|x|^{n+1}}$$

$$= \frac{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{4}{n}\right)}{n^2\left(1+\frac{3}{n}\right)|x|}$$

$$\lim_{n\to\infty}\frac{U_n}{U_{n+1}} = \frac{1}{|x|}$$

$\therefore$ By Ratio test $\Sigma|U_n|$ is convergent if $\frac{1}{|x|} > 1$ or $|x| < 1$ and divergent if $\frac{1}{|x|} < 1$ or $|x| > 1$

Test fails if $|x| = 1 \Rightarrow x = -1, 1$.

When $x = 1$, series becomes.

$$\frac{-1}{1.4}+\frac{1}{2.5}-\frac{1}{3.7}+\ldots$$

Here $$u_n = \frac{1}{n(n+3)},\quad u_{n+1} = \frac{1}{(n+1)(n+4)}$$

$n < n+1$ and $n+3 < n+4$

$\Rightarrow n(n+3) < (n+1)(n+4)$

$$\Rightarrow \quad \frac{1}{n(n+3)} > \frac{1}{(n+1)(n+4)}$$

$$\Rightarrow \quad U_n > U_{n+1}.$$

$\therefore$ $<U_n>$ is monotonic decreasing

Also, $$\lim_{n\to\infty} U_n = \lim_{n\to\infty}\frac{1}{n(n+3)} = 0$$

$\therefore$ By Leibnitz test ΣU_n is convergent.

When $x = -1$, series is

$$\frac{1}{1.4}-\frac{1}{2.5}+\frac{1}{3.7}\ldots$$

$$U_n = \frac{1}{n(n+3)}$$

Convergent by Leibnitz test.

$\therefore$ series is convergent if $|x| \le 1$.

Q.3. (b) Discuss the convergence of the following series $1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+\ldots$ **(5)**

Ans. Let $$U_n = \frac{x^n . n!}{(n+1)^n},\quad u_{n+1} = \frac{x^{n+1}(n+1)!}{(n+2)^{n+1}}$$

$$\frac{U_n}{U_{n+1}} = \frac{x^n \cdot n!}{(n+1)^n} \cdot \frac{(n+2)^{n+1}}{x^{n+1}(n+1)!}$$

$$= \frac{(n+2)^n \cdot (n+2)}{(n+1)^n \cdot (n+1) \cdot x} = \frac{n^n \cdot n\left(1+\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)}{n^n . n\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right).x}$$

$$\lim_{m \to \infty} \frac{U_n}{U_{n+1}} = \frac{e^2}{ex} = \frac{e}{x}.$$

By ratio test ΣU_n converges if $\frac{e}{x} > 1$ or $x < e$ and diverges if $\frac{e}{x} < 1$ or $x > e$.

Test fails if $x = e$

$$\log \frac{U_n}{U_{n+1}} = (n+1)\log\left(1+\frac{2}{n}\right) - (n+1)\log\left(1+\frac{1}{n}\right) - \log e$$

$$= (n+1)\left[\left(\frac{2}{n} - \frac{1}{2}.\frac{4}{n^2} + \frac{1}{3}.\frac{8}{n^3}....\right) - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\right)\right] - 1$$

$$= (n+1)\left[\frac{1}{n} - \frac{3}{2n^2}...\right] - 1$$

$$= 1 - \frac{3}{2n} + \frac{1}{n} - \frac{3}{2n^2} + ... - 1$$

$$= -\frac{1}{2n} - \frac{3}{2n^2}$$

$$n \log \frac{U_n}{U_{n+1}} = -\frac{1}{2} - \frac{3}{2n}$$

$$\lim_{n \to \infty} n \log \frac{U_n}{U_{n+1}} = -\frac{1}{2} < 1$$

∴ By log test, series is divergent.

ΣU_n converges if $x < e$ and diverges if $x \geq e$.

Q.4. (a) State Cayley Hamilton Theorem and hence use it to find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$, where **(5)**

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Ans. Cayley Hamilton Theorem – Every square matrix satisfies its characterstic equation

Characteristic equation of A is

$$(A-\lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)[(1-\lambda)(2-\lambda)] - 1[0] + 1[\lambda - 1] = 0$

$\Rightarrow (4+\lambda^2-4\lambda)(1-\lambda) + \lambda - 1 = 0$

$\Rightarrow 4 - 4\lambda + \lambda^2 - \lambda^3 - 4\lambda + 4\lambda^2 + \lambda - 1 = 0$

$\Rightarrow -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley Hamilton theorem

$$A^3 - 5A^2 + 7A - 3I = 0 \qquad \ldots(1)$$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$\Rightarrow A^5(A^3 - 5A^2 + 7A - 3I) + A^4 - 5A^3 + 8A^2 - 2A + I$

$\Rightarrow A(A^3 - 5A^2 + 7A - 3I) - 7A^2 + 3A + 8A^2 - 2A + I$

$\Rightarrow A^2 + A + I.$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Q.4. (b) Find eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$ **(5)**

Ans. Characterstic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 2 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda) - 2] - 1[2-\lambda+2] + 1[1+3-\lambda] = 0$

$\Rightarrow (2-\lambda)[6 - 5\lambda + \lambda^2 - 2] - 4 + \lambda - 4 - \lambda = 0$

$\Rightarrow \qquad (2-\lambda)(\lambda^2 - 5\lambda + 4) = 0$

$$\Rightarrow \quad \lambda = 2, \lambda^2 - \lambda - 4\lambda + 4 = 0$$
$$\Rightarrow \quad (\lambda - 1)(\lambda - 4) = 0$$
$$\Rightarrow \quad \lambda = 1, 2, 4 \text{ (eigenvalues)}$$

For $\lambda = 1$, eigenvector $[A - I]x = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + R_1, R_2 \to R_2 - R_1$

$$\sim\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 - 2R_2$

$$\sim\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 + x_2 + x_3 = 0$$
$$x_2 + x_3 = 0$$
$$\rho(A) = 2 < 3$$

Let $\quad x_2 = 1 \Rightarrow x_3 = -1$

$$x_1 = 0$$

for $\lambda = 1$, $\quad X_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\lambda = 2$, eigenvector is $[A - 2I]X = 0$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + R_2$

$$\sim\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 - 2R_1$

$$\sim\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$r(A) = 2 < 3.$$

$$x_2 + x_3 = 0$$

and $$x_1 + x_2 + 2x_3 = 0$$

Let $$x_2 = 1 \quad \Rightarrow x_3 = -1$$

$$x_1 = 1$$

$\therefore$ $$X_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\lambda = 4$, eigenvector $[A - 4I]X = 0$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + R_1$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ -1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$r(A) = 2 < 3$$

$$-2x_1 + x_2 + x_3 = 0$$

$$-x_1 + 3x_3 = 0$$

$$x_1 = 3x_3$$

$\Rightarrow$ $$\frac{x_1}{3} = \frac{x_3}{1}$$

Also, $$-6x_3 + x_3 + x_2 = 0$$

$$x_2 = 5x_3 \Rightarrow \frac{x_2}{5} = \frac{x_3}{1}$$

$\therefore$ $$X_3 = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

END TERM EXAMINATION [DEC.-2015]
FIRST SEMESTER [B. TECH]
APPLIED MATHEMATICS-I [ETMA-101]

MM : 75

Time: 3 Hrs.

Note: *Attempt any 5 questions All questions carry equal marks.*

Q.1. (a) Solve $(D-1)^2 (D+1)^2 y = \sin^2 \frac{x}{2} + e^x + x$. (2)

Ans.
$$(D-1)^2 (D+1)^2 y = \sin^2 \frac{x}{2} + e^x + x$$

A.E.
$$(D-1)^2 (D+1)^2 = 0$$

$\Rightarrow$
$$D = 1, 1, -1, -1$$

C.F is $(C_1 + C_2 x) e^x + (C_3 + C_4 x) e^{-x}$

P.I.
$$\frac{1}{(D-1)^2 (D+1)^2} = \sin^2 \frac{x}{2} + e^x + x$$

$$\Rightarrow \frac{1}{(D-1)^2 (D+1)^2} \sin^2 \frac{x}{2} + \frac{1}{(D-1)^2 (D+1)^2} e^x + \frac{1}{(D-1)^2 (D+1)^2} x$$

$$\Rightarrow \frac{1}{[(D-1)(D+1)]^2} \left[\frac{1}{2} - \frac{\cos x}{2} + e^x + x\right]$$

$$\Rightarrow \frac{1}{(D^2-1)^2} \left(e^x - \frac{\cos x}{2} + x + \frac{1}{2}\right)$$

$$\Rightarrow \frac{1}{(D^4 - 2D^2 + 1)} e^x - \frac{1}{D^4 - 2D^2 + 1} \frac{\cos x}{2} + \frac{1}{(D^4 - 2D^2 + 1)} \left(x + \frac{1}{2}\right)$$

Consider $\frac{1}{D^4 - 2D^2 + 1} e^x = \frac{1}{(1-2+1)} e^x [D=1]$ Case of failure

$$= x \frac{1}{4D^3 - 4D} e^x = x \frac{1}{4-4} e^x \ [D=4]$$

gain case of failure

$$= x^2 \frac{1}{12 D^2 - 4} e^x = x^2 \frac{1}{12-4} e^x = \frac{x^2}{8} e^x$$

Consider $\frac{1}{D^4 - 2D^2 + 1} \left(\frac{\cos x}{2}\right) = \frac{1}{2[(-1)^2 - 2(-1) + 1]} \cos x$ $[D^2 = 1]$

$$= \frac{\cos x}{2[4]} = \frac{\cos x}{8}$$

$$\frac{1}{D^4-2D^2+1}\left(x+\frac{1}{2}\right) = \frac{1}{1-2D^2+D^4}\left(x+\frac{1}{2}\right)$$

$$= [1-(2D^2-D^4)]^{-1}\left(x+\frac{1}{2}\right)$$

$$= [1+2D^2]\left[x+\frac{1}{2}\right]$$

[Neglecting higher power terms)

$$= x+\frac{1}{2}+2D^2\left(x+\frac{1}{2}\right)$$

$$= x+\frac{1}{2}$$

$$\therefore \qquad \text{P.I.} = \frac{x^2e^x}{8}-\frac{\cos x}{8}+x+\frac{1}{2}$$

Complete solution is.

$$y = (c_1+c_2x)e^x+(c_3+c_4x)e^{-x}+\frac{x^2e^x}{8}-\frac{\cos x}{8}+\left(x+\frac{1}{2}\right)$$

Q.1. (b) Express the matrix $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ as sum of a symmetric and a skew symmetric matrix.

Ans. Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

i.e., A can be written as

$$A = \frac{1}{2}(A+A')+\frac{1}{2}(A-A')$$

$$A' = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 2 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A+A' = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 1 \\ 0 & 2 & -2 \end{bmatrix}+\begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 2 \\ 3 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 3 \\ 0 & 8 & 3 \\ 3 & 3 & -4 \end{bmatrix}$$

$$A - A^1 = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 1 \\ 0 & 2 & -2 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad A = \frac{1}{2}\begin{bmatrix} 4 & 0 & 3 \\ 0 & 8 & 3 \\ 3 & 3 & -4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

Q.2. (a) Show that the values of $(1+i\sqrt{3})^{1/3}$ are given by $2^{1/3}\left(\cos\frac{r\pi}{9}+i\sin\frac{r\pi}{9}\right)$, $r = 1, 7, 13$ .... [Not in new syllabus].

Ans.

$$1+i\sqrt{3} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$= 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$(1+i\sqrt{3})^{1/3} = 2^{1/3}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{1/3}$$

$$= 2^{1/3}\left[\cos\left(\frac{\pi}{3}+2k\pi\right)+i\sin\left(\frac{\pi}{3}+2k\pi\right)\right]^{1/3}$$

$$= 2^{1/3}\left[\cos\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right)\right]$$

for $k = 0, 1, 2$

$$= 2^{1/3}\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right), 2^{1/3}\left(\cos\frac{7\pi}{9}+i\sin\frac{7\pi}{9}\right)$$

$$\text{and } 2^{1/3}\left(\cos\frac{13\pi}{9}+i\sin\frac{13\pi}{9}\right)$$

$$\therefore \quad (1+i\sqrt{3})^{1/3} = 2^{1/3}\left(\cos\frac{r\pi}{9}+i\sin\frac{r\pi}{9}\right), r = 1,7,13.$$

Q.2. (b) If $f(x) = x^2 e^{-x/a}$ show that $fn\ (0) = \dfrac{n(n-1)(-1)^n}{a^{n-2}}$ where $fn = \dfrac{d^n f}{dx^n}$.

Ans. $f(x) = x^2 e^{-x/a}$

Let $u = e^{-x/a}, \quad v = x^2.$

$$u_1 = -\frac{1}{a}e^{-x/a}, \quad v_1 = 2x$$

$$u_2 = \frac{1}{a^2} e^{-x/a}, \quad v_2 = 2.$$

$$u_3 = -\frac{1}{a^3} e^{-x/a}$$

$$u_n = \frac{(-1)^n}{a^n} e^{-x/a}.$$

By Leibnitz theorem.

$$f_n = {}^nc_0 u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \ldots$$

$$f_n = \frac{(-1)^n}{a^n} e^{-x/a} x^2 + n \frac{(-1)^{n-1}}{a^{n-1}} 2xe^{x/a} + \frac{n(n-1)}{2} \cdot \frac{(-1)^{n-2}}{a^{n-2}} . 2e^{-x/a}$$

$$= \frac{(-1)^n}{a^n} x^2 e^{-x/a} + \frac{2nx(-1)^{n-1}}{a^{n-1}} e^{-x/a} + \frac{n(n-1)(-1)^n.(-1)^2 e^{-x/a}}{a^{n-2}}$$

for $\quad x = 0$

$$f_n(0) = \frac{(-1)^n n(n-1)}{a^{n-2}}$$

Q.3. (a) The tangents at two points P and Q on the cycloid $x = a\,(\theta - \sin\theta), y = a(1 - \cos\theta)$ are at right angles. If $\rho_1 = \rho_2$ are the radii of curvature at these points than show that $\rho_1^2 + \rho_2^2 = 16\,a^2$.

Ans.

$$x = a\,(\theta - \sin\theta), \quad y = a\,(1 - \cos\theta)$$
$$x' = a(1 - \cos\theta), \quad y' = a\sin\theta$$
$$x'' = a\sin\theta, \quad y'' = a\cos\theta$$

As

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'\,y'' - y'\,x''}$$

$$x'^2 + y'^2 = a^2(1 - \cos\theta)^2 + a^2\sin^2\theta$$
$$= a^2(1 + \cos^2\theta - 2\cos\theta + a^2\sin^2\theta)$$
$$= a^2(2 - 2\cos\theta) = 2a^2(1 - \cos\theta)$$
$$= -4a^2\sin^2\theta/2$$

$$x'y'' - y'x'' = a^2\cos\theta\,(1 - \cos\theta) - a^2\sin^2\theta$$
$$= a^2(\cos\theta - \cos^2\theta - \sin^2\theta)$$
$$= a^2(\cos\theta - 1) = -a^2(1 - \cos\theta)$$
$$= -2\,a^2\sin^2\theta/2$$

$$\therefore \quad \rho = \frac{(4a^2\sin^2\theta/2)^{3/2}}{-2a^2\sin^2\theta/2} = \frac{8\,a^3\sin^3\theta/2}{-2\,a^2\sin^2\theta/2}$$

$$= 4\,a\sin\theta/2 \text{ (in magnitude)}$$

Let the points P and Q correspond to $\theta = \theta_1$ and $\theta = \theta_2$.

$$\rho_1 = 4a\sin\frac{\theta_1}{2}, \quad \rho_2 = 4a\sin\frac{\theta_2}{2}$$

Now,
$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$
$$= \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$$

$$\text{Slope of tangent is } = \cot\frac{\theta}{2}$$

$\therefore$ Slope of tangent at P and Q are $\cot\frac{\theta_1}{2}$ and $\cot\frac{\theta_2}{2}$

Since tangent at P and Q are perpendicular.

$$\therefore \quad \cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2} = -1$$
$$\Rightarrow \quad \cot\frac{\theta_2}{2} = -\tan\frac{\theta_1}{2} = \cot\left(\frac{\pi}{2}+\frac{\theta_1}{2}\right)$$
$$\Rightarrow \quad \frac{\theta_2}{2} = \frac{\pi}{2}+\frac{\theta_1}{2} \Rightarrow \theta_2 = \pi+\theta_1$$
$$\Rightarrow \quad \rho_2 = 4a\sin\left(\frac{\pi}{2}+\frac{\theta_1}{2}\right) = 4a\cos\frac{\theta_1}{2}$$
$$\text{Thus} \quad \rho_1^2+\rho_2^2 = 16a^2.$$

Q.3. (b) Solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^4\sin x$.

Ans. It is homogeneous form of equation.

Put
$$Z = \log x \Rightarrow x = e^z.$$
$$x\frac{dy}{dx} = Dy \quad \text{where} \quad D = \frac{d}{dz}$$
$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Now
$$D(D-1)y - 4Dy + 6y = e^{4z}\sin e^z.$$
$$(D^2-5D+6)y = e^{4z}\sin e^z.$$

A.E.
$$D^2-5D+6 = 0$$
$$\Rightarrow \quad (D-2)(D-3) = 0$$
$$\Rightarrow \quad D = 2,3 \Rightarrow \text{CF}: C_1e^{2z}+C_2e^{3z}$$
$$PI = \frac{1}{D^2-5D+6}e^{4z}\sin e^z.$$
$$= e^{4z}\frac{1}{(D+4)^2-5(D+4)+6}\sin e^z.$$

$$= e^{4z}\frac{1}{D^2+16+8D-5D-20+6}\sin e^z.$$

$$= e^{4z}\frac{1}{D^2+3D+2}\sin e^z.$$

$$= e^{4z}\frac{1}{(D+1)(D+2)}\sin e^z = e^{4z}\left[\frac{1}{D+1}-\frac{1}{D+2}\right]\sin e^z$$

$$= e^{4z}\left[\frac{1}{D+1}\sin e^z - \frac{1}{D+2}\sin e^z\right]$$

Using $\quad \frac{1}{D-a}X = e^{az}\int e^{-az}X\,dz$

$$\frac{1}{D+1}\sin e^z = e^{-z}\int e^z \sin e^z\,dz \qquad [\text{Put } e^z = t]$$

$$= e^{-z}\int \sin t\,dt$$

$$= -e^{-z}\cos e^z$$

$$\frac{1}{D+2}\sin e^z = e^{-2z}\int e^{2z}\sin e^z\,dz$$

$$= e^{-2z}\int t\sin t\,dt$$

$$= e^{-2z}[t(-\cos t)-(-\sin t)]$$

$$= e^{-2z}[-e^z\cos e^z+\sin e^z]$$

$$\text{PI} = e^{4z}[-e^{-z}\cos e^z + e^{-z}\cos e^z - e^{-2z}\sin e^z]$$

$$= -e^{2z}\sin e^z \text{ Complete solution.}$$

$\therefore$

$$y = c_1e^{2z}+c_2e^{3z}-e^{2z}\sin e^z$$

$$y = c_1x^2+c_2x^3-x^2\sin x$$

Q.4. (a) Show that $\int_0^a \frac{x^n}{\sqrt{ax-x^2}}dx = \frac{1.3.5\ldots(2n-1)}{2.4\ldots2n}\pi a^n$

Ans. Consider $\int_0^a \frac{x^n}{\sqrt{ax-x^2}} = \int_0^a \frac{x^n}{x^{1/2}\sqrt{a-x}}dx$

Put
$$x = a\sin^2\theta$$
$$dx = 2a\sin\theta\cos\theta\,d\theta.$$

$$\int_0^a \frac{x^n}{\sqrt{ax-x^2}}dx = \int_0^{\pi/2}\frac{a^n\sin^{2n}\theta}{a^{1/2}\sin\sqrt{a-a\sin^2\theta}}.2a\sin\theta\cos\theta\,d\theta$$

$$= 2a^n\int_0^{\pi/2}\frac{\sin^{2n}\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$$= 2a^n \int_0^{\pi/2} \sin^{2n}\theta \, d\theta$$

Using reduction formula

$$= 2a^n \frac{(2n-1)(2n-3)....3.1)}{2n(2n-2)...4.2} \frac{\pi}{2}$$

$$= \frac{\pi a^2 (1.3 \ldots (2n-1)}{2.4 \ldots 2n}$$

Q.4. (*b*) Find the area outside the circle $r = 2a\cos\theta$ and inside the cardioid $r = a(1+\cos\theta)$

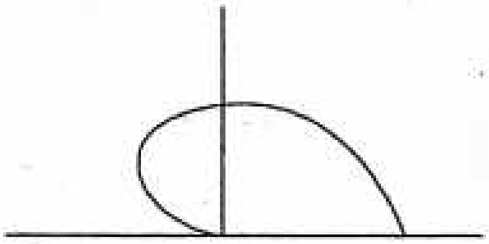
Ans. Eliminating r from two equations.

$$2a\cos\theta = a(1+\cos\theta)$$

$$2\cos\theta = 1+\cos\theta \quad \Rightarrow \cos\theta = 1.$$

$$\Rightarrow \qquad \theta = 0, \pi$$

r values from. $2a\cos\theta$ to $a(1+\cos\theta)$.



$$\therefore \quad \text{Requd. area} = 2\int_0^{\pi}\int_{2a\cos\theta}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_{2a\cos\theta}^{a(1+\cos\theta)} d\theta$$

$$= \frac{2}{2}\int_0^{\pi}(a^2(1+\cos\theta)^2 - 4a^2\cos^2\theta)d\theta$$

$$= \int_0^{\pi} a^2(1+\cos^2\theta + 2\cos\theta - 4\cos^2\theta)d\theta$$

$$= a^2\int_0^{\pi}(1-3\cos^2\theta+2\cos\theta)\,d\theta$$

$$= a^2\left[|\theta|_0^{\pi} - 3.2\int_0^{\pi/2}\cos^2\theta\,d\theta. + 2\int_0^{\pi}\cos\theta\,d\theta\right]$$

$$= a^2\left[\pi - 6.\frac{1}{2}.\frac{\pi}{2} + 2|\sin\theta|_0^{\pi}\right] = a^2\left[\pi - \frac{3\pi}{2}\right]$$

$$= a^2\left(-\frac{\pi}{2}\right) = \frac{\pi a^2}{2} \text{ (in magnitude).}$$

Q.5. (*a*) Reduce the following matrix into its normal form and hence find its rank.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Ans. Let $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Operating $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$C_2 \to C_2 + C_1, C_3 \to C_3 + 2C_1, C_4 \to C_4 + 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 \to R_2 - R_3, R_4 \to R_4 - 2R_3.$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

$R_4 \to R_4 - R_2.$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 6C_2, C_4 \to C_4 + 3C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to \frac{1}{33} C_3$, $\quad C_4 \to \frac{1}{22} C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_4 \to C_4 - C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{(Normal form)}$$

$$r(A) = 3.$$

Q. 5. (b) Trace the curve $y^2 (2a - x) = x^3$.

Ans. $$y^2 = \frac{x^3}{2a - x}$$

1. It has even power of y, symmetrical about x-axis.
2. Curve passes through the origin.
3. It cuts x axis and y axis at origin only.

Tangents at origin are $y^2 = 0$.

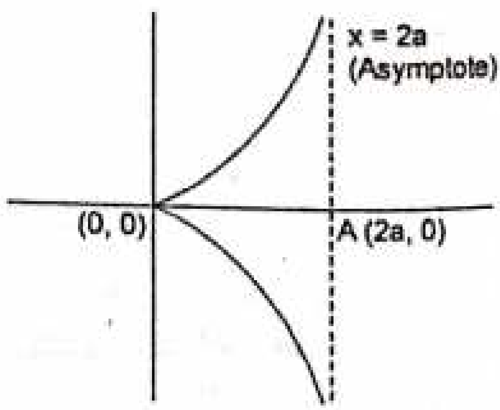
$\therefore$ tangents are real and coincident

$\therefore$ origin is a cusp.

4. **Asymptotes** $\parallel$ to y axis $x - 2a = 0$

$\Rightarrow$ $$x = 2a$$

No oblique asymptotes.



5. **Region** $$y = \sqrt{\frac{x}{2a - x}}$$

when $x < 0$, y is imaginary

$x > 2a$, y is imaginary.

$x < 2a$, y is + ve.

Curve does not lie to the left of y-axis and to the right of line $x = 2a$.

When $x = 0$, $y = 0$. As x increases from 0 to $2a$, y also increases from 0 to ∞.

Q.6. (a) Show that the asymptotes of the cubic curve $x^3 + x^2y - xy^2 - y^3 - 2x^2 + 5x + y + 1 = 0$ cut the curve in three points which lie on the straight line $2x + y + 1 = 0$ **(Out of syllabus)**

Ans. $x^3 + x^2 y - xy^2 - y - 2x^2 + 2y^2 - 5x + y + 1 = 0$

Highest degree terms has constant coefficients.

$\therefore$ No parallel asymptotes.

Oblique. Put $x = 1$ and $y = m$

$$\phi_3 = x^3 - y^3 + x^2 y - xy^2$$

$$\phi_2 = -2x^2 + 2y^2 = -2 + 2m^2$$

$$\phi_1 = -5x + y = -5 + m$$

$$\phi_0 = 1$$

$$\phi_3 = 1 - m^3 + m - m^2 = 0$$

$$\Rightarrow \quad m^3 - 1 - m + m^2 = 0$$

$$\Rightarrow (m-1)(m^2 - 1 + m) + m(m-1) = 0$$

$$\Rightarrow \quad (m-1)(m^2 + 1 + 2m) = 0$$

$$\Rightarrow \quad m = 1, \quad (m+1)^2 = 0$$

$$\Rightarrow \quad m = 1, -1, -1$$

For $m = 1$,
$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-2(m^2 - 1)}{-3m^2 + 1 - 2m}$$

$$= \frac{-2(1-1)}{-3+1-2} = 0$$

For $m = 1$, $\frac{c^2}{2!}\phi_2''(m) + c\phi_1'(m) + \phi_0(m) = 0$

$$\frac{c^2}{2!}.4 + c + 1 = 0$$

$$\Rightarrow \quad 2c^2 + c + 1 = 0 \quad \Rightarrow c^2 + \frac{c}{2} + \frac{1}{2} = 0$$

$$c = -1, 3$$

Asymptotes are

$$y = x, \quad y = -x - 1, y = -x + 3$$

Joint equation of asymptotes is

$$F_3 = (x - y)(x + y + 1)(y + x - 3) = 0 \quad \ldots(1)$$

Q.6. (b) Test for convergence, the series $\frac{1}{3} + \frac{3}{5} + \frac{7}{9} + \frac{15}{17} + \ldots$

Ans. Let
$$u_n = \frac{2^n - 1}{2^n + 1}, \quad U_{n+1} = \frac{2^{n+1} - 1}{2^{n+1} + 1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^n - 1}{2^n + 1} \cdot \frac{2^{n+1} + 1}{2^{n+1} - 1}$$

$$= \frac{2^n\left(1-\frac{1}{2^n}\right).2^n\left(2+\frac{1}{2^n}\right)}{2^n\left(1+\frac{1}{2^n}\right).2^n\left(2-\frac{1}{2^n}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1.2}{1.2}=1$$

$\therefore$ Ratio test fails.

Consider
$$u_n = \frac{2^n-1}{2^n+1}=\frac{2^n\left(1-\frac{1}{2^n}\right)}{2^n(1+1/2^n)}=\frac{1-1/2^n}{1+1/2^n}$$

$$\lim_{n\to\infty} U_n = 1 \neq 0$$

$\therefore$ Σu_n does not converge.

Q.7. (a) Show that the system of equations
$x+y+3z=6$, $2x+3y+z=8$, $x+5y+7z=20$, $x+z=10$ has no solution.

Ans. It can be written as $AX=B$

$$\begin{bmatrix}1&1&3\\2&3&1\\1&5&7\\1&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\8\\20\\10\end{bmatrix}$$

$$R_2\to R_2-2R_1, R_3\to R_3-R_1, R_4\to R_4-R_1$$

$$\sim\begin{bmatrix}1&1&3\\0&1&-5\\0&4&4\\0&-1&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\-4\\14\\4\end{bmatrix}$$

$$R_4\to R_4+R_2, R_3\to R_3/4$$

$$\sim\begin{bmatrix}1&1&3\\0&1&-5\\0&1&1\\0&0&-7\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-4\\7/2\\0\end{bmatrix}$$

$$R_3\to R_3-R_2$$

$$\sim\begin{bmatrix}1&1&3\\0&1&-5\\0&0&6\\0&0&-7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\-4\\15/2\\0\end{bmatrix}$$

$$R_3\to\frac{1}{6}R_3 \qquad R_4\to\frac{1}{7}R_4$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 5/4 \\ 0 \end{bmatrix}$$

$$R_4 \to \quad R_4 + R_3$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 5/4 \\ 5/4 \end{bmatrix}$$

$$r(A) = 3, \quad r(A:B) = 4$$

since $\quad r(A) \neq r(A:B)$

Thus it has no solution.

Q.7. (b) If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$, show that

$$2\theta = n\pi + \frac{\pi}{2} + \alpha, e^{2\phi} = \pm\cot\left(\frac{\alpha}{2}\right) \text{ (Not in new syllabus)}$$

Ans.

$$\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$$

$$\tan(\theta - i\phi) = \tan\alpha - i\sec\alpha$$

$$\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$= \frac{(\tan\alpha + i\sec\alpha) + (\tan\alpha - i\sec\alpha)}{1 - (\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)}$$

$$= \frac{2\tan\alpha}{1 - (\tan^2\alpha + \sec^2\alpha)} = \frac{2\tan\alpha}{-2\tan^2\alpha}$$

$$\Rightarrow \quad \tan 2\theta = -\cot\alpha$$

$$= \tan\left(\frac{\pi}{2} + \alpha\right)$$

$$2\theta = n\pi + \frac{\pi}{2} + \alpha$$

Now

$$\tan 2i\phi = \tan[(\phi + i\phi) - (\theta - i\phi)]$$

$$= \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$= \frac{(\tan\alpha + i\sec\alpha) - (\tan\alpha - i\sec\alpha)}{1 + (\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)}$$

$$= \frac{2i \sec \alpha}{1 + \left(\tan^2 \alpha + \sec^2 \alpha\right)} = \frac{2i \sec \alpha}{2 \sec^2 \alpha}$$

$$\tan 2i\phi = i \cos \alpha$$

$$\Rightarrow \quad i \tanh 2\phi = i \cos \alpha$$

$$\tanh 2\phi = \cos \alpha$$

$$\frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\cos \alpha}{1}$$

By componendo and dividendo

$$\frac{e^{2\phi}}{e^{-2\phi}} = \frac{1 + \cos \alpha}{1 - \cos \alpha}$$

$$= \frac{2\cos^2 \alpha / 2}{2\sin^2 \alpha / 2} = \cot^2 \alpha / 2$$

$$e^{2\phi} = \pm \cot \alpha/2$$

FIRST SEMESTER [B.TECH]
FIRST TERM EXAMINATION [SEPT. 2016]
APPLIED MATHS–I [ETMA–101]

Time : 1½ hrs. M.M. : 30

Note: Attempt any three questions including Q. No. 1. is complsory

Q. 1. (a) Find all the asymptotes of the curve (4)

$$x^2y + xy^2 + xy + y^2 + 3x = 0$$

Ans.

$$x^2y + xy^2 + xy + y^2 + 3x = 0$$

Coefficient of highest power of x and y are

$$y = 0, x + 1 = 0$$

$$\Rightarrow y = 0, x = -1$$

These are parallel asmptotes.

Oblique Asymptotes.

$$\phi_3(x,y) = x^2y + xy^2$$

$$\phi_2(x,y) = xy + y^2$$

$$\phi_1(x,y) = 3x$$

put $x = 1, y = m$

$$\phi_3(m) = m + m^2$$

$$\phi_2(m) = m + m^2, \phi_3'(m) = 1 + 2m$$

$$\phi_1(m) = 3$$

Roots are $\phi_3(m) = 0$

$$\Rightarrow \quad m(m+1) = 0$$

$$\Rightarrow \quad m = 0, -1 \text{ (Distinct)}$$

Value of C for distinct values of m are

$$C = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{-(m+m^2)}{1+2m}$$

$$\frac{\text{for } m = 0}{\text{for } m = -1} \quad \begin{matrix} C = 0 \\ C = -\dfrac{(-1+1)}{1-2} = 0 \end{matrix}$$

$\therefore$ Asymptotes are $y = mx + c$

$$y = 0, y = -x$$

Required asymptotes are

$$y = 0, y = -x, x = -1$$

Q.1. (b) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(n+1)^n x^n}{n^{n+1}}$

Ans. Let

$$U_n = \frac{(n+1)^n . x^n}{n^{n+1}}$$

$$(U_n)^{1/n} = \left[\frac{(n+1)^n \cdot x^n}{n^n \cdot n}\right]^{1/n} = \frac{(n+1).x}{n - n^{1/n}}$$

$$= \frac{n\left(1+\frac{1}{n}\right).x}{n.\,n^{1/n}} = \frac{\left(1+\frac{1}{n}\right).x}{n^{1/n}}$$

$$\lim_{n\to\infty}(U_n)^{1/n} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)x}{n^{1/n}} = x$$

$\therefore$ By cauchy's root test ΣU_n is covergent if $x < 1$ and divergent if $x > 1$.

Test fails if $x = 1$

When $x = 1$,

$$U_n = \frac{(n+1)^n}{n^{n+1}} = \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n.n} = \frac{\left(1+\frac{1}{n}\right)^n}{n}$$

Let

$$V_n = \frac{1}{n}$$

$\therefore$

$$\frac{U_n}{V_n} = \left(1+\frac{1}{n}\right)^n$$

$\lim\limits_{n\to\infty}\frac{U_n}{V_n} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ =e (which is finite and non-zero)

$\therefore$ By comparison test ΣU_n and ΣV_n will have same behaviour.

Since

$$\Sigma V_n = \sum\frac{1}{n} \quad (p = 1)$$

$\Rightarrow$ ΣV_n is divergent by p-series

Thus, ΣU_n is divergent

Hence, ΣU_n is convergent if $x < 1$ and divergent if $x \geq 1$.

Q.1. (c) Show that the radius of curvature of the curve $y^2 = x^2\dfrac{(a+x)}{a-x}$ at the origin is $a\sqrt{2}$. (3)

Ans. Equation of curve is $y^2(a-x) = x^2(a+x)$ ...(1)

It passes through origin.

Equating to zero the lowest degree terms, we get

$$a(y^2 - x^2) = 0 \quad \therefore y = \pm x \text{ are tangents at origin.}$$

$\therefore$ Newton's method is not applicable.

Let
$$y = px + \frac{qx^2}{2!} + ...$$

Putting this value of y in (1), we get

$$\left(px + \frac{qx^2}{2} + ...\right)^2 (a - x) = (a + x)x^2$$

Equating coefficients of x^2 both sides

$$ap^2 = a \quad \therefore p = \pm 1$$

Equating coefficient of x^3 both sides

$$-p^2 + apq = 1 \qquad ...(2)$$

When $p = 1$, $(2) \Rightarrow -1 + aq = 1$

$\Rightarrow$
$$q = 2/a$$

$\therefore$
$$\rho \text{ (at origin)} = \frac{(1+p^2)^{3/2}}{q}$$

$$= \frac{(1+1)}{2/a} = 2\sqrt{2}.\frac{a}{2}$$

$$= a\sqrt{2}$$

When $p = -1$, $(2) \Rightarrow -1 - aq = 1 \Rightarrow q = -2/a$

$$\therefore \rho \text{ (at origin)} = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+1)^{3/2}}{-2/a} = \sqrt{2}a$$

Q.2. (a) Test for what values of x series convergent $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n . n^2}$. **(5)**

Ans. Here
$$U_n = \frac{(-1)^n (x+1)^n}{2^n . n^2}$$

$$|U_n| = \frac{|x+1|^n}{2^n . n^2},$$

$$|U_{n+1}| = \frac{|n+1|^{n+1}}{2^{n+1}(n+1)^2}$$

$\Rightarrow$
$$\frac{|U_n|}{|U_{n+1}|} = \frac{|x+1|^n}{2^n . n^2} . \frac{2^{n+1}(n+1)^2}{|x+1|^{n+1}}$$

$$= \frac{n^2\left(1+\frac{1}{n}\right)^2 .2}{n^2 |x+1|} = \frac{2\left(1+\frac{1}{n}\right)^2}{|x+1|}$$

$$\lim_{n\to\infty}\frac{U_n}{U_{(n+1)}} = \frac{2}{|x+1|}$$

$\therefore$ By ratio test $\sum |U_n|$ is convergent if $\frac{2}{|x+1|} > 1$

i.e. if $|x+1| < 2$ or $-2 < x+1 < 2 \Rightarrow -3 < x < 1$

and divergent if $\frac{2}{|x+1|} < 1$ i.e. if $|x+1| > 2$

i.e. if $x+1 > 2$ or $x+1 < -2$

$\Rightarrow$ if $x > 1$ or $x < -3$.

Test fails for $x = 1$ and $x = -3$.

when $x = 1$ series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n.2^n}{2^n.n^2}$

$$\Rightarrow \quad \sum U_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$= \sum_{n=1}^{\infty}(-1)^n V_n$$ which is an alternating series,

where

$$V_n = \frac{1}{n^2}$$

$$V_{n+1} = \frac{1}{(n+1)^2}$$

As $n < n+1$

$\Rightarrow$ $n^2 < (n+1)^2$

$\Rightarrow$ $\frac{1}{n^2} > \frac{1}{(n+1)^2}$

$\Rightarrow V_n > V_{n+1}$ for all n.

Also $\lim_{n\to\infty} V_n = \lim_{n\to\infty}\frac{1}{n^2} = 0$

$\therefore$ By leibnitz test ΣV_n is convergent

$\Rightarrow \Sigma U_n$ is convergent

When $x = -3$.

$$\Sigma U_n = \sum_{n=1}^{\infty}\frac{(-1)^n(-2)^n}{2^n.n^2}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{2n}2^n}{2^n.n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Here

$$U_n = \frac{1}{n^2}(p = 2 > 1)$$

$\therefore \sum \frac{1}{n^2}$ is convergent by p-series test

Thus, $\sum |U_n|$ is absolutely convergent when $-3 \le x \le 1$.

since, absolute convergence $\Rightarrow$ convergence

Thus ΣU_n is convergent if $-3 \le x \le 1$.

If $|x+1| > 2$, $\lim_{n\to\infty} V_n \ne 0$

$\Rightarrow \Sigma U_n$ is not convergent.

ΣU_n is convergent if $|x+1| \le 2$

$\Rightarrow -3 \le x \le 1$.

Q.2. (b) Test for convergence the series

$$\frac{x}{1} + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots$$

Ans. Neglecting first term

$$U_n = \frac{1.3.5..(2n-1)}{2.4.6..(2n)}\frac{x^{2n+1}}{2n+1}$$

$$U_{n+1} = \frac{1.3.5...(2n+1)}{2.4.6...(2n+2)}\frac{x^{2n+3}}{2n+3}$$

Consider $$\frac{U_n}{U_{n+1}} = \frac{1.3.5...(2n-1)}{2.4.6...2n}\frac{x^{2n+1}}{2n+1} \times \frac{2.4.6...(2n+2)}{1.3.5...(2n+1)}\frac{2n+3}{x^{2n+3}}$$

$$\Rightarrow \quad \frac{U_n}{U_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}.\frac{1}{x^2}$$

$$= \frac{n^2\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}{n^2\left(2+\frac{1}{n}\right)^2}.\frac{1}{x^2} = \frac{\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}{x^2\left(2+\frac{1}{n}\right)^2}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{U_n}{U_{n+1}} = \lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}{x^2\left(2+\frac{1}{n}\right)^2}$$

$$= \frac{2.2}{x^2.4} = \frac{1}{x^2}$$

$\therefore$ By ratio test ΣU_n is convergent if $\frac{1}{x^2} > 1$ i.e. if $x < 1$ and divergent if $\frac{1}{x^2} < 1$ i.e if $x > 1$

Test fails for $x = 1$

for $x = 1$,
$$\frac{U_n}{U_{n+1}} - 1 = \frac{(2n+2)(2n+3)}{(2n+1)^2} - 1$$
$$= \frac{4n^2+10n+6}{4n^2+4n+1} - 1$$
$$= \frac{4n^2+10n+6-4n^2-4n-1}{\left(4n^2+4n+1\right)}$$

$\Rightarrow$
$$n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{n(6n+5)}{4n^2+4n+1} = \frac{n^2(6+5/n)}{n^2\left(4+\frac{4}{n}+\frac{1}{n^2}\right)}$$
$$\lim_{(n\to\infty)}\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{6}{4} = \frac{3}{2} > 1$$

$\therefore$ By Raabe's test ΣU_n is convergent.

Hence, the series ΣU_n is convergent if $x \le 1$ and divergent if $x > 1$.

Q.3. (a) The tangents at two points P, Q on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ are at right angles. show that if ρ_1, ρ_2 be the radii of curvature at these points then $\rho_1^2+\rho_2^2 = 16a^2$

Ans. Here $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

$\Rightarrow$
$$x' = a(1-\cos\theta),\ y^1 = a\sin\theta$$
$$x'' = a\sin\theta,\ y^{11} = a\cos\theta$$

$\therefore$
$$x'^1 + y'^2 = a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)$$
$$= 2a^2(1-\cos\theta) = 4a^2\sin^2\theta/2$$
$$x'y''-y'x'' = a^2\cos\theta(1-\cos\theta) - a^2\sin^2\theta$$
$$= a^2(\cos\theta-\cos^2\theta-\sin^2\theta) = a^2(\cos\theta-1)$$
$$= -a^2(1-\cos\theta)$$
$$= -2a^2\sin^2\theta/2$$
$$\rho \text{ at } '\theta' = \frac{\left(x'^2+y'^2\right)^{3/2}}{x'y''-y'x''} = \frac{(4a^2\sin^2\theta/2)^{3/2}}{-2a^2\sin^2\theta/2}$$
$$= \frac{8a^3\sin^3\theta/2}{-2a^2\sin^2\theta/2} = 4a\sin\theta/2$$

Let the points P and Q correspond to
$$\theta = \theta_1 \text{ and } \theta = \theta_2 \text{ them}$$
$$\rho_1 = 4a\sin\frac{\theta_1}{2},\quad \rho_2 = 4a\sin\frac{\theta_2}{2}$$

Now,
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{y'}{x'} = \frac{\sin\theta}{1-\cos\theta}$$
$$= \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = \cot\frac{\theta}{2}$$

$\Rightarrow$ Slope of tangent at 'θ' $= \cot\dfrac{\theta}{2}$.

$\therefore$ Slopes of tangents at P and Q are $\cot\dfrac{\theta_1}{2}$ and $\cot\dfrac{\theta_2}{2}$.

Since tangents at P and Q are perpendicular.

$\therefore$
$$\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2} = -1$$

$\Rightarrow$
$$\cot\frac{\theta_2}{2} = -\tan\frac{\theta_1}{2} = \cot\left(\frac{\pi}{2}+\frac{\theta_1}{2}\right)$$

$\Rightarrow$
$$\frac{\theta_2}{2} = \frac{\pi}{2}+\frac{\theta_1}{2} \Rightarrow \theta_2 = \pi+\theta_1$$

$\therefore$
$$\rho_2 = 4a\sin\left(\frac{\pi}{2}+\frac{\theta_1}{2}\right) = 4a\cos\frac{\theta_1}{2}$$

Hence
$$\rho_1^2+\rho_2^2 = 16a^2\left(\sin^2\frac{\theta_1}{2}+\cos^2\frac{\theta_1}{2}\right) = 16a^2$$

Q.3. (*b*) Show that $\int_0^\infty x^{m-1}\cos(ax)d_x = \dfrac{\Gamma m}{a^m}\cos\dfrac{m\pi}{2}$. **(5)**

Ans. Consider $\int_0^\infty x^{m-1}e^{-iax}dx$

let
$$iax = t$$

$\Rightarrow$
$$ia\,dx = dt$$

$\Rightarrow$
$$\int_0^\infty \frac{t^{m-1}}{i^{m-1}a^{m-1}}e^{-t}\frac{dt}{ia}$$

$$\frac{1}{i^m a^m}\int_0^\infty e^{-t}t^{m-1}dt$$

$\Rightarrow$
$$\frac{\Gamma m}{i^m a^m} \quad (\text{by def}^n)$$

$\Rightarrow$
$$\int_0^\infty x^{m-1}e^{-iax}dx = \frac{\Gamma m}{i^m a^m}$$

$$\int_0^\infty x^{m-1}(\cos ax - i\sin ax)dx = \frac{\Gamma m}{i^m a^m} \qquad \text{(A)}$$

Now

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$\Rightarrow$

$$i^m = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^m$$

$$= \cos\frac{m\pi}{2} + i\sin\frac{m\pi}{2}$$

By (A), we get

$$\int_0^\infty x^{m-1}(\cos ax - i\sin ax)\,dx = \frac{\overline{|m}}{a^m} i^{-m}$$

$$= \frac{\overline{|m}}{a^m}\left(\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right)$$

Comparing imaginary parts both sides, we get

$$\int_0^\infty x^{m-1}\cos ax\,dx = \frac{\overline{|m}}{a^m}\cos\frac{m\pi}{2}$$

Q.4. (a) if $y^{1/m} + y^{-1/m} = 2x$, prove that $(x^2-1)\,y_{n+2} + (2n+1)\,y_{n+1}\,x + (n^2-m^2)\,y_n = 0$ (5)

Ans.

$$Y^{1/m} + y^{-1/m} = 2x$$

Put

$$y^{1/m} = z, \text{ then}$$

$$z + \frac{1}{z} = 2x \text{ or } z^2 - 2xz + 1 = 0$$

$$z = \frac{2x \pm \sqrt{4x^2-4}}{2} \Rightarrow z = x \pm \sqrt{x^2-1}$$

$\therefore$

$$y^{1/m} = x \pm \sqrt{x^2-1}$$

i.e.

$$y = \left(x \pm \sqrt{x^2-1}\right)^m$$

$\Rightarrow$

$$y = \left(x + \sqrt{x^2-1}\right)^m, \text{ them}$$

if

$$y_1 = m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(1 + \frac{1}{2}\frac{1}{\sqrt{x^2-1}}.2x\right)$$

$$= m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$

$$= \frac{m\left(x+\sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$$

$$y = \left(x - \sqrt{x^2-1}\right)^m \text{, then}$$

$$y_1 = m\left(x-\sqrt{x^2-1}\right)^{m-1}\left(1-\frac{1}{2}\cdot\frac{1}{\sqrt{x^2-1}}.2x\right)$$

$$= m\left(x-\sqrt{x^2-1}\right)^{m-1}\left(\frac{\sqrt{x^2-1}-x}{\sqrt{x^2-1}}\right)$$

$$= \frac{-m\left(x-\sqrt{x^2-1}\right)^{m-1}\left[x-\sqrt{x^2-1}\right]}{\sqrt{x^2-1}}$$

$$= \frac{-m\left(x-\sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}} = \frac{-my}{\sqrt{x^2-1}}$$

On squaring, we have, in either case

$$y_1^2 = \frac{m^2y^2}{x^2-1} \text{ or } (x^2-1)y_1^2 = m^2y^2.$$

Differentiating again

$$(x^2-1)2y_1y_2 + y_1^2(2x) = m^2.2y\,y_1.$$

Dividing both sides by $2y_1$.

$$\Rightarrow \quad (x^2-1)y_2 + xy_1 - m^2y = 0$$

Differentiating every term n times by leibnity theorem.

$$\left[y_{n+2}(x^2-1)+n.y_{n+1}.2x+\frac{n(n-1)}{2!}y_n.2\right]+[x.y_{n+1}+ny_n]-[m^2y_n]=0$$

$$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-n+n-m^2)y_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Q.4. (*b*) Calculate the approximate value of $\sqrt{10}$ to four decimal places by taking the first four terms of an appropriate Taylor's series. (5)

Ans. Let $\quad f(x+h) = \sqrt{x+h}$

put $\quad h = 0$ then $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2}x^{-1/2} = -\frac{1}{2\sqrt{x}}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} = \frac{-1}{4x\sqrt{x}}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8x^2\sqrt{x}}$$

$\therefore \qquad \sqrt{x+h} = f(x+h)$

$$= f(x) + hf'(x) + \frac{h^2}{2i} f''(x) + (x) + \frac{h^3}{3!} f'''(x) + \ldots.$$

$$= \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8x\sqrt{x}} + \frac{h^3}{48x^2\sqrt{x}} + \ldots.$$

put $\qquad x = 9 \text{ and } h = 1$

$$\sqrt{10} = 3 + \frac{1}{2(3)} - \frac{1^2}{8(9)(3)} + \frac{1^3}{48(9^2)\times 3} + \ldots$$

$$= 3 + 0.16666 - 0.00463 + 0.00009 + \ldots$$

$$= 3.1621 \text{ (approx)}$$

END TERM EXAMINATION [DEC. 2016]
FIRST SEMESTER [B.TECH]
APPLIED MATHEMATICS–I [ETMA–101]

Time : 3 Hrs. M.M. : 75

Note: *Attempt any five questions including Q. No. 1 which is compulsory. Select one question from each unit.*

Q.1. (a) Find the nth derivative of sin 2x sin 3x. (3)

Ans. Let

$$y = \sin 2x \sin 3x$$
$$= \cos x - \cos 5x$$
$$y_n = \frac{d^n}{dx^n}\cos x - \frac{d^n}{dx^n}\cos 5x$$
$$= 1^n \cos\left(\frac{n\pi}{2} + x\right) - 5^n \cos\left(\frac{n\pi}{2} + 5x\right)$$
$$= \cos\left(\frac{n\pi}{2} + x\right) - 5^n \cos\left(\frac{n\pi}{2} + 5x\right).$$

Q.1. (b) Find all the asymptotes of the curve $y^2(x-2a) = x^3 - a^3$. (3)

Ans. $y^2x - 2ay^2 - x^3 + a^3 = 0$

Coefficient of highest degree term in x is constant.

$\therefore$ No asymptotes parallel to x-axis.

Coefficient of highest degree term in y is

$$x - 2a = 0 \Rightarrow x = 2a$$

$\therefore$ $x = 2a$ is asymptote parallel to y-axis.

Oblique asymptotes

$$\phi_3 = y^2x - x^3$$
$$\phi_2 = -2ay^2$$
$$\phi_1 = 0$$
$$\phi_0 = a^3$$

Put $x = 1, y = m$

$$\phi_3(m) = m^2 - 1, \quad \phi_2(m) = -2am^2$$

The slopes of the asymptotes are the roots of $\phi_3(m) = 0$.

$$\Rightarrow \quad m^2 = 1 \Rightarrow m = \pm 1$$

For non-repeated root $m = 1$, c is given by

$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-(-2am^2)}{2m}$$
$$= am = a$$

For non-repeated root $m = -1$, c is given by

$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = -\frac{(-2am^2)}{2m}$$
$$= am = -a$$

Corresponding asmptote is $y = mx + c$

i.e., $$y = x + a \text{ and } y = -x - a$$

All asymptotes are

$$x = 2a, x + y + a = 0, x - y + a = 0$$

Q.1. (c) Find the value of $\int_0^{\pi/2} \sin^3\theta \cos^{5/2}\theta \, d\theta$.

Ans. $\int_0^{\pi/2} \sin^3\theta \cos^{5/2}\theta \, d\theta$

Here $$p = 3, q = \frac{5}{2}$$

$$\Rightarrow \int_0^{\pi/2} \sin^3\theta \cos^{5/2}\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(2, \frac{7}{4}\right)$$

$$= \frac{\overline{|2}\ \overline{|7/4}}{2\ \overline{|15/4}}$$

$$= \frac{1\overline{|1}\ \overline{|7/4}}{2 \times \frac{11}{4}\overline{|11/4}} = \frac{\overline{|7/4}}{\frac{11}{2} \times \frac{7}{4}\overline{|7/4}} = \frac{8}{77}$$

Q.1. (d) Prove that every Hermitian matrix can be written as $A + iB$, where A is real and symmetric and B is real and skew Symmetric matrix. (3)

Ans. Let $C = A + iB$

Let C be hermitian *i.e.* $C^\theta = C$

Let $$A = \frac{1}{2}(C + \bar{C}), \; B = \frac{1}{2i}(C - \bar{C})$$

$\Rightarrow A$ and B are real matrices

Now $$C = \frac{1}{2}(C + \bar{C}) + i\left[\frac{1}{2i}(C - \bar{C})\right]$$

$$= A + iB$$

To show A is symmetric and A is skew-symmetric matrix.

$$A' = \frac{1}{2}[C + \bar{C}]'$$

$$= \frac{1}{2}(C + \bar{C})' = \frac{1}{2}[C' + C^\theta]$$

$$= \frac{1}{2}[(C^\theta)' + C] \qquad [\because C^\theta = C]$$

$$= \frac{1}{2}\left[((\bar{C})')' + C\right] = \frac{1}{2}(\bar{C} + C) = A$$

$\therefore$ A is symmetric

$$\text{Consider } B' = \left[\frac{1}{2i}(C - \bar{C})\right]'$$

$$= \frac{1}{2i}(C - \bar{C})' = \frac{1}{2i}[C' - (\bar{C})']$$

$$= \frac{1}{2i}\left[C' - C^{\theta}\right] = \frac{1}{2i}[(C^{\theta})' - C]$$

$$= \frac{1}{2i}[((\bar{C})')' - C]$$

$$= \frac{1}{2i}[\bar{C} - C]$$

$$= \frac{-1}{2i}(C - \bar{C}) = -B$$

$\therefore$ B is skew Symmetric.

Q.1. (e) Reduce the qudratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to the canonical form. Also write the nature of the quadratic form. **(3)**

Ans. The given quadratic form can be written as $X'AX$, where $X' = [x_1, x_2, x_3]$ and symmetric matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

As
$$A = I_3 A I_3$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + \frac{1}{3}R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 \to C_3 + \frac{1}{3}C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Diag.}\left(1, 3, \frac{8}{3}\right) = P'AP.$$

$\therefore$ The canonical form of given quadratic form is

$$Y'(P'AP)Y = [y_1\ y_2\ y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [y_1\ \ y_2\ \ y_3] \begin{bmatrix} y_1 \\ 3y_2 \\ \frac{8}{3}y_3 \end{bmatrix} = y_1^2 + 3y_2^2 + \frac{8}{3}y_3^2$$

Here index, $p = 3$, rank $= 3$, signature

$$= 2p - r = 2 \times 3 - 3 = 3$$

It is positive definite.

Q.1. (f) Solve the differential equation $(x^2y - 2xy^2)\, dx - (x^3 - 3x^2y)\, dy = 0$. (3)

Ans. Here $M = x^2y - 2xy^2$, $N = 3x^2y - x^3$

$$\frac{\partial M}{\partial y} = x^2 - 4xy, \quad \frac{\partial N}{\partial x} = 6xy - 3x^2$$

This is not exact.

Consider
$$Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$$
$$= x^2y^2 \neq 0$$

$$\therefore \qquad \text{I.F.} = \frac{1}{x^2y^2}$$

Multiply throughout by $\frac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

Again, here
$$M' = \frac{1}{y} - \frac{2}{x}, \quad N' = \frac{3}{y} - \frac{x}{y^2}$$

Now
$$\frac{\partial M'}{\partial y} = \frac{-1}{y^2}, \quad \frac{\partial N'}{\partial x} = -\frac{1}{y^2}$$

This is exact.

Consider
$$\int_{y\text{ constant}} M'dx = \int\left(\frac{1}{y}-\frac{2}{x}\right)dx$$
$$= \frac{x}{y} - 2\log x$$

and
$$\int N'' dy = \int\frac{3}{y}dy = 3\log y$$

$\therefore$ required solution is
$$\frac{x}{y} - 2\log x + 3\log y = C.$$

Q.1. (g) Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$. **(3)**

Ans. By definition of $J_n(x)$, we have
$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

Taking $n = \frac{1}{2}$, we get
$$J_{1/2}(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma\left(r+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{2r+1/2}$$
$$= \frac{1}{\Gamma(3/2)}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{1!\,\Gamma(5/2)}\left(\frac{x}{2}\right)^{5/2} + \frac{1}{2!\,\Gamma(7/2)}\left(\frac{x}{2}\right)^{9/2} + \ldots$$
$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\Gamma(3/2)} - \frac{1}{\Gamma(5/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(7/2)}\left(\frac{x}{2}\right)^4 + \ldots\right]$$
$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 + \ldots\right]$$
$$= \frac{\sqrt{x}}{\Gamma(2)\,\Gamma(1/2)}\left[\frac{2}{1!} - \frac{2.2}{3}\frac{x^2}{2^2} + \frac{8}{5.3.2.1}\frac{x^4}{2^4} + \ldots\right]$$
$$= \frac{\sqrt{x}}{\Gamma(2)\sqrt{\pi}}\left[2 - \frac{2x^2}{3!} + \frac{2\cdot x^4}{5!} + \ldots\right]$$
$$= \frac{2\sqrt{x}}{\sqrt{2\pi}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right]$$

$$= \sqrt{\frac{2x}{\pi}}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}\cdots\right]$$

$$= \frac{\sqrt{2x}\times\sqrt{x}}{\sqrt{\pi}\,x\times\sqrt{x}}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right]$$

$$\Rightarrow \qquad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x\,.$$

Q.1. (*h*) Prove that $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, m \neq n.$ (2)

Ans. Consider the differential equation

$$(1-x^2)u''-2xu'+m(m+1)\,u = 0 \qquad ...(1)$$

$$(1-x^2)\,v''-2xv'+n(n+1)\,v = 0 \qquad ...(2)$$

$P_m(x)$ and $P_n(x)$ are required solution of equation (1).

(1) and (2) i.e. $u = P_m(x)$ and $v = P_n(x)$.

Multiply (1) by v and (2) by u and subtracting, we get

$$(1-x^2)\,(u''v-v''u)-2x(u'v-v'u)+[m(m+1)-n(n+1)]\,uv = 0$$

$$\Rightarrow \frac{d}{dx}[(1-x^2)\,(u'v-v'u)]+(m^2+m-n^2-n)\,uv = 0$$

$$\Rightarrow \frac{d}{dx}[(1-x^2)\,(u'v-v'u)]+(m-n)\,(m+n+1)\,uv = 0$$

$$\Rightarrow (n-m)\,(m+n+1)\,uv = \frac{d}{dx}\,[(1-x^2)\,(u'v-v'u)]$$

Integrating w.r.t. 'x' from –1 to 1, we get

$$(n-m)\,(m+n+1)\int_{-1}^{1} uv\,dx = \left|(1-x^2)\,(u'v-v'u)\right|_{-1}^{1}$$

$$= 0$$

$$\Rightarrow \qquad \int_{-1}^{1} uv\,dx = 0 \qquad (\text{as } m \neq n)$$

$$\Rightarrow \qquad \int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0.$$

Q.1. (*i*) Discuss the convergence of the series, $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$ using leibnitz rule. (2)

Ans. Given series is an alternating series $\Sigma(-1)^{n-1}$ with

$$u_n = \frac{1}{\sqrt{n}} \qquad \forall\, n \geq 1$$

(*i*) $$u_n = \frac{1}{\sqrt{n}}, \qquad u_{n+1} = \frac{1}{\sqrt{n+1}}$$

$$n < n+1 \quad \Rightarrow \quad \sqrt{n} < \sqrt{n+1}$$

$$\Rightarrow \quad \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} \quad \forall\, n$$

(*ii*) $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Thus by leibnitz test, series is convergent.

UNIT-I

Q.2. (a) If $y = e^{m(\cos^{-1} x)}$, prove that $(1-x^2)y_{n+2} - (2n+1)_x y_{n+1} - (n^2+m^2)y_n = 0$. Also find $(y_n)_0$.

Ans.

$$y = e^{m(\cos^{-1} x)} \quad ...(1)$$

$$y_1 = e^{m(\cos^{-1} x)} \cdot \frac{(-m)}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad y_1 = \frac{-my}{\sqrt{1-x^2}} \quad ...(2)$$

$$\Rightarrow \quad y_1^2(1-x^2) = m^2y^2$$

Again differentiating w.r.t. 'x', we get.

$$2y_1y_2(1-x^2) - y_1^2 \cdot 2x = 2m^2yy_1$$

$$\Rightarrow \quad y_2(1-x^2) - y_1x = m^2y \quad ...(3)$$

Differentiating w.r.t. 'n'.

$$\Rightarrow y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2} y_n(-2) - xy_{n+1} - ny_n - m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n - m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)\,xy_{n+1} - (n^2+m^2)\,y_n = 0 \quad ...(4)$$

Putting $x = 0$. Now

1. $\Rightarrow$ $$y(0) = e^{m\pi/2}$$

2. $\Rightarrow$ $$y_1(0) = -my(0) = -me^{m\pi/2}$$

3. $\Rightarrow$ $$y_2(0) = m^2y(0) = m^2e^{m\pi/2}$$

4. $\Rightarrow$ $$y_{n+2}(0) = (n^2+m^2)\,y_n(0) \quad ...(5)$$

Putting $n = 1, 2, 3, 4, 5, ...$ in (5),

$$y_3(0) = (1^2+m^2)\,y_1(0) = -(1^2+m^2)\,me^{m\pi/2}$$

$$y_4(0) = (2^2+m^2)\,y_2(0) = (2^2+m^2)\,m^2e^{m\pi/2}$$

$$y_5(0) = (3^2+m^2)\,y_3(0) = -(1^2+m^2)(3^2+m^2)\,me^{m\pi/2}$$

Now $$y_{2n+1}(0) = -me^{m\pi/2}(1^2+m^2)(3^2+m^2)\,...\,[(2n-1)^2+m^2]$$

$$y_{2n}(0) = m^2e^{m\pi/2}(2^2+m^2)(4^2+m^2)\,...\,[(2n-2)^2+m^2].$$

Q.2. (b) Discuss the convergence of the series $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+...\infty$.

Ans. Here
$$U_n = \frac{x^n}{n(n+1)},\; U_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$$

Consider
$$\frac{U_n}{U_{n+1}} = \frac{x^n}{n(n+1)}\cdot\frac{(n+1)(n+2)}{x^{n+1}}$$
$$= \frac{n+2}{n\cdot x} = \frac{n(1+2/n)}{n\cdot x}$$
$$\frac{U_n}{U_{n+1}} = \frac{(1+2/n)}{x}$$
$$\lim_{n\to\infty}\frac{U_n}{U_{n+1}} = \lim_{n\to\infty}\frac{(1+2/n)}{x} = \frac{1}{x}$$

By Ratio Test

ΣU_n is convergent if $\frac{1}{x} > 1$ *i.e.* $x < 1$.

ΣU_n is divergent if $\frac{1}{x} < 1$ *i.e.* $x > 1$.

Test fails if $n = 1$.

Consider
$$U_n = \frac{1}{n(n+1)} = \frac{1}{n^2\left(1+\frac{1}{n}\right)}$$

Let
$$V_n = \frac{1}{n^2}$$
$$\frac{U_n}{V_n} = \frac{1}{\left(1+\frac{1}{n}\right)}$$

$\therefore$
$$\lim_{n\to\infty}\frac{U_n}{V_n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1 \text{ (Finite and non-zero)}$$

$\therefore$ By comparison test ΣU_n and ΣV_n have same behaviour.

Since
$$\Sigma V_n = \Sigma\frac{1}{n^2} \qquad (p = 2 > 1)$$

ΣV_n converges by p-test.

ΣU_n also converges.

Thus ΣU_n converges if $x \le 1$ and diverges if $x > 1$.

Q.3. (a) Show that $\tan\left(\frac{\pi}{4}+x\right) = 1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\ldots$ and hence evaluate the value of 46° correct up to 4 places of decimal. (6.5)

Ans. Let $f(x+h) = \tan(x+h)$

Putting $h=0$,

$$\begin{aligned}
f(x) &= \tan x \\
f'(x) &= \sec^2 x \\
f''(x) &= 2\sec^2 x \tan x \\
f'''(x) &= 2(\sec^4 x + 2\sec^2 x\tan^2 x) \\
&= 2\sec^2 x(\sec^2 x + 2\tan^2 x) \\
&= 2\sec^2 x(1+3\tan^2 x) = 2\sec^2 x + 6\sec^2 x\tan^2 x \\
f^{iv}(x) &= 16\tan x + 40\tan^3 x + 24\tan^5 x
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \tan(x+h) &= f(x+h) \\
&= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots \\
&= \tan x + h\sec^2 x + h^2\sec^2 x\tan x + \frac{h^3}{3}\sec^2 x(1+3\tan^2 x) \\
&\quad + \frac{h^4}{4!}(16\tan x + 40\tan^3 x + 24\tan^5 x) + \ldots
\end{aligned}$$

Putting $x = \frac{\pi}{4}$

$$\tan\left(\frac{\pi}{4}+h\right) = 1 + h\cdot 2 + h^2\cdot 2 + \frac{h^3}{3}\cdot 2\ (1+3) + \frac{h^4}{24}\times 80 + \ldots$$

$$\tan\left(\frac{\pi}{4}+h\right) = 1 + 2h + 2h^2 + \frac{8}{3}h^3 + \frac{10}{3}h^4 + \ldots$$

Putting $h = x$

$$\tan\left(\frac{\pi}{4}+x\right) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \ldots$$

Put $\quad x = 1° = \frac{\pi}{180} = 0.0174, \quad \frac{\pi}{4} = 45°$

$$\begin{aligned}
\tan 46° &= \tan(46°) = 1 + 2\times 0.0174 + 2\times(0.0174)^2 \\
&\quad + \frac{8}{3}\times(0.0174)^3 + \frac{10}{3}(0.0174)^4 + \ldots \\
\tan 46° &= 1 + 0.0348 + 0.00060552 + \ldots \\
&= 1.03540552 \text{ (approx)}
\end{aligned}$$

Q.3. (b) Discuss the absolute convergence of $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n+1}$. (6)

Ans. The given series is

$$\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n+1} = \sum_{n=1}^{\infty}(-1)^n U_n$$

$$U_n = \frac{x^n}{n+1}, U_{n+1} = \frac{x^{n+1}}{n+2}$$

$$\frac{|U_n|}{|U_{n+1}|} = \frac{|x|^n}{n+1} \cdot \frac{n+2}{|x|^{n+1}} = \frac{n(1+2/n)}{n(1+1/n)} \cdot \frac{1}{|x|}$$

$$\lim_{n\to\infty} \frac{|U_n|}{|U_{n+1}|} = \lim_{n\to\infty} \left(\frac{1+2/n}{1+1/n}\right) \frac{1}{|x|} = \frac{1}{|x|}$$

$\therefore$ By ratio test $\sum_{n=1}^{\infty} |U_n|$ converges if $\frac{1}{|x|} > 1$

i.e. $|x| < 1$ i.e. $-1 < x < 1$

Divergent if $\frac{1}{|x|} < 1$ *i.e.* $|x| > 1$

Test fails if $|x| = 1 \Rightarrow x = 1$

When $x = 1$, the series becomes

$$\Sigma(-1)^n U_n = \Sigma(-1)^n \frac{1}{n+1}$$

It is an alternating series

Here $$U_n = \frac{1}{n+1}, \quad U_{n+1} = \frac{1}{n+2}$$

Since $n + 1 < n + 2$

$$\Rightarrow \quad \frac{1}{n+1} > \frac{1}{n+2} \quad \forall n$$

Also $$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$\therefore$ By Lebenitz test series is convergent

Thus series is absolutely convergent if $|x| < 1$ and convergent if $-1 \le x \le 1$.

UNIT–II

Q.4. (a) Show that the radius of curvature at $\left(\frac{a}{4}, \frac{a}{4}\right)$ on the cur[ve] $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $a/\sqrt{12}$.

Ans. Equation of curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

Differentiating w.r.t 'x', we get

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\frac{y}{x}}$$

Differentiating again w.r.t 'x', we have

$$\frac{d^2y}{dx^2} = \frac{-x^{1/2}\cdot\frac{1}{2}y^{-1/2}\frac{dy}{dx} - y^{1/2}\cdot\frac{1}{2}x^{-1/2}}{x}$$

$$= \frac{-\sqrt{\frac{x}{y}}\left(-\sqrt{\frac{y}{x}}\right)-\sqrt{\frac{y}{x}}}{2x} = \frac{1+\sqrt{\frac{y}{x}}}{2x}$$

$$= \frac{\sqrt{x}+\sqrt{y}}{2x\sqrt{x}} = \frac{\sqrt{a}}{2x\sqrt{x}}$$ [By (1)]

At $\left(\frac{a}{4},\frac{a}{4}\right)$, $\frac{dy}{dx} = -1$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{a}}{2\frac{a}{4}\cdot\frac{\sqrt{a}}{2}} = \frac{4}{a}$$

$$\Rightarrow \quad \rho_{\left(\frac{a}{4},\frac{a}{4}\right)} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = \frac{(1+1)^{3/2}}{4/a} = \frac{a}{\sqrt{2}}.$$

Q.4. (b) Trace the curve $r = a\cos 3\theta$. **(6.5)**

Ans. $r = a\cos 3\theta$ ...(1)

1. Symmetry: On changing θ to $-\theta$, it remains unchanged.

$\therefore$ Symmetric about initial line.

2. Pole: Putting $r = 0$, we get $\cos 3\theta = 0$, which gives

$$3\theta = 0$$

$$\Rightarrow \quad 3\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \ldots$$

$$\Rightarrow \quad \theta = \pm\frac{\pi}{6} \pm\frac{\pi}{2} \ldots$$

By putting $\theta = 0$, we get $r = a$.

$\therefore$ Curve cuts initial line at $r = a$.

3. Asymptotes: For no value of θ, $r\to\infty$ thus curve has no asymptotes.

4. Region

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
r	a	0	$-a$	0	a	0	$-a$

5. Value of ϕ: By (1),

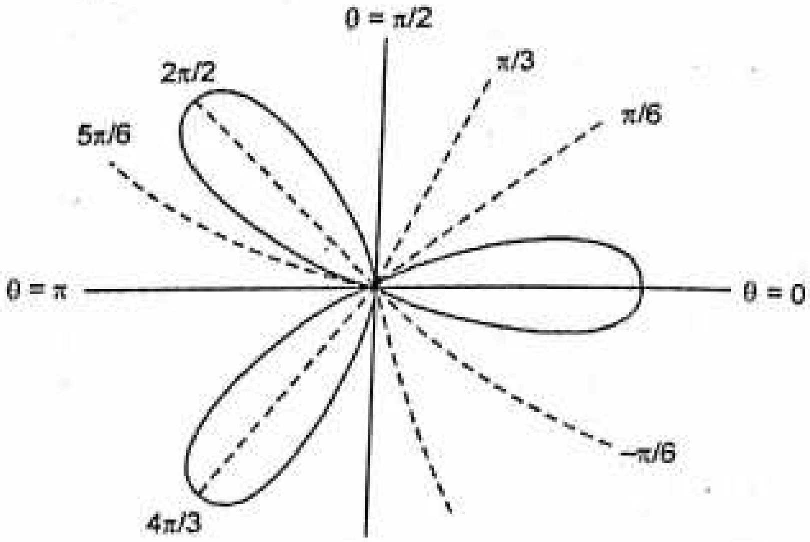
$$\frac{dr}{dQ} = -3a\sin 3\theta$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{a\cos 3\theta}{-3a\sin 3\theta}$$

$$\Rightarrow \qquad \tan\phi = \frac{-1}{3}\cot 30 = \frac{1}{3}\tan\left(\frac{\pi}{2}+30\right)$$

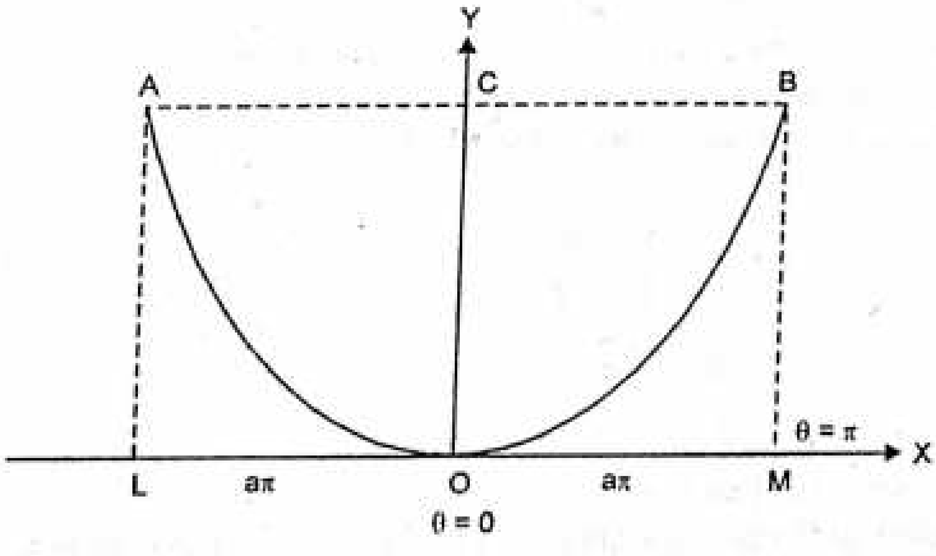
$$\Rightarrow \qquad \phi = \frac{\pi}{2}+30$$

When $\theta = 0, \phi = \dfrac{\pi}{2}$. Thus the points $(a, 0)$ are tangents $\perp$ to initial line.



Q.5. (a) Find the area include between the cycloid $x = a\,(\theta + \sin\theta), y = a\,(1 - \cos\theta)$ and its base. (6.5)

Ans. Since θ lies between $-\pi$ and π



Line joining A and B is base of cycloid.

$$\text{Area } AOB = 2 \times \text{Area of } OBC \qquad \text{(As it is symmetric about } y\text{-axis)}$$

$$\text{Area of } OBC = \int_0^{\pi} x\cdot\frac{dy}{d\theta}\cdot d\theta$$

$$= \int_0^{\pi} a(\theta + \sin\theta)\,a\sin\theta\,d\theta$$

$$= a^2\left[\int_0^{\pi}\theta\cdot\sin\theta\,d\theta + \int_0^{\pi}\sin^2\theta\,d\theta\right]$$

$$= a^2\left[(-\theta\cos\theta)_0^\pi + \int_0^\pi \cos\theta\, d\theta\right] + 2a^2\int_0^{\pi/2}\sin^2\theta\, d\theta$$

$$= a^2\left[\pi + (\sin\theta)_0^\pi\right] + 2a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= a^2\pi + \frac{a^2\pi}{2}$$

Required area $= 2\left(a^2\pi + \dfrac{a^2\pi}{2}\right)$

$$= 2a^2\left(\pi + \frac{\pi}{2}\right) = 3\pi a^2.$$

Q.5. (b) Evaluate $\int_0^{\pi/2} \sin^6 x\, dx$ using reduction formula. (6)

Ans. Let $\quad I_6 = \int_0^{\pi/2} \sin^6 x\, dx$

As $\quad I_n = \dfrac{(n-1)}{n} I_{n-2}$

$\Rightarrow \quad I_6 = \dfrac{5}{6} I_4 \quad$ ...(1)

$I_4 = \dfrac{3}{4} I_2 \quad$ ...(2)

$I_2 = \dfrac{1}{2} I_0 \quad$ ...(3)

Now, $\quad I_0 = \int_0^{\pi/2} \sin^0 x\, dx = \int_0^{\pi/2} dx = \dfrac{\pi}{2}$

$\Rightarrow \quad I_2 = \dfrac{1}{2}\cdot\dfrac{\pi}{2}$

$I_4 = \dfrac{3}{4}\cdot\dfrac{1}{2}\times\dfrac{\pi}{2}$

$I_6 = \dfrac{5}{6}\times\dfrac{3}{4}\times\dfrac{1}{2}\times\dfrac{\pi}{2} = \dfrac{5\pi}{32}.$

UNIT-III

Q.6. (a) Show that, if $\lambda \neq -5$, the system of the equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$, have a unique solution. If $\lambda = -5$, show that the equations are consistent. Also determine the solution when $\lambda = -5$. (6.5)

Ans. In matrix form system is

$$\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & 3 \\ 6 & 5 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1, R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 7 & \lambda - 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -9 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & \lambda + 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix} \quad ...(1)$$

Now, $A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & \lambda + 5 \end{bmatrix}$, $[A : B] = \left[\begin{array}{ccc:c} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & \lambda + 5 & 0 \end{array}\right]$

If $\lambda \neq -5$

Then $\rho(A) = \rho(A : B) = 3 =$ No. of unknowns.

$\therefore$ System has a unique solution given by:

1. $\Rightarrow$

$$3x - y + 4z = 3$$
$$7y - 13z = -9$$
$$(\lambda + 5)z = 0$$

$\Rightarrow$

$$z = 0, \; y = \frac{-9}{7}$$

Now,

$$3x + \frac{9}{7} = 3 \Rightarrow x = \frac{4}{7}$$

$\therefore$

$$x = \frac{4}{7}, \quad y = \frac{-9}{7}, \; z = 0$$

If $\lambda = -5$

Then $\rho(A) = \rho(A : B) = 2 <$ No. of unknowns.

$\Rightarrow$ System is consistent and has many solution.

(1) $\Rightarrow$

$$3x - y + 4z = 3$$
$$7y - 13z = -9$$

Let

$$z = k$$

$\Rightarrow$

$$y = \frac{13k - 9}{7}$$

and

$$3x - \frac{(13k - 9)}{7} + 4k = 3$$

$$3x = 3 + \frac{13k - 9}{7} - 4k = \frac{21 + 13k - 9 - 28k}{7}$$

$$= \frac{12-15k}{7}$$

$$x = \frac{4-5k}{7}$$

$\Rightarrow$

$$x = \frac{4-5k}{7},\ y = \frac{13k-9}{7},\ z = k.$$

$\therefore$

Q.6. (b) Use Guass-Jordan method to find the inverse of the matrix

$$= \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}.$$ **(6)**

Ans. It can be written as

$$A = I_3 A$$

$$\Rightarrow \quad \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operating $R_2 \to R_2 - \frac{2}{3}R_1$

$$\sim \begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operating $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 4/3 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A$$

Operating $R_1 \to R_1 - 3R_2$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 4/3 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A$$

Operating $R_2 \to R_2 + 4R_3$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ 2 & -3 & 4/3 \\ 2/3 & -1 & 1 \end{bmatrix} A$$

Operating $R_2 \to -R_2, R_3 \to -3R_3, R_1 \to \frac{1}{3}R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4/3 \\ -2 & 3 & -3 \end{bmatrix} A$$

$$\Rightarrow \quad I = A^{-1}A$$

Thus, inverse of
$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4/3 \\ -2 & 3 & -3 \end{bmatrix}$$

Q.7. (a) Find the eigenvalues and the corresponding Eigen vectors of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$. (6)

Ans. The characteristic equation of given matrix is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\lambda = 1, 2, 3 \text{ (eigen values)}$$

Eigen vectors: For $\lambda = 1$

$$[A - I]X = 0$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Since $\rho(A) = 2, < 3$ (no. of unknowns)

$$x_2 + x_3 = 0$$
$$2x_1 + 2x_3 = 0$$

Let $\quad x_2 = 1$

$\Rightarrow \quad x_3 = -1$ and $x_1 = 1$

Eigen vector for $\lambda = 1$ is $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

For $\lambda = 2$: $[A - 2I]X = 0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operating $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operating $R_3 \to R_3 + R_1$

$$\sim\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Since $\rho(A) = 2 < 3$ (no. of unknowns)

Let
$$x_1 = 0,\ x_3 = 0$$
$$x_2 = 1$$

$\therefore$ Eigen vector for $\lambda = 2$ is $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 3$:
$$[A - 3I]X = 0$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operating $R_3 \to R_3 + R_1$

$$\sim\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Since
$$\rho(A) = 2 < 3$$

$\therefore$ One variable can be given arbitrary value

$$-2x_1 = 0 \Rightarrow x_1 = 0$$
$$-x_2 + x_3 = 0 \Rightarrow x_2 = x_3$$

$\therefore$ Eigen vector for $\lambda = 3$ is $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Q.7. (b) Find the characteristics equation of the matrix, $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ show that the equation is satisfied by A and hence find the inverse of the given matrix. (6.5)

Ans. The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(1-\lambda) - 6] - 3[4 - 4\lambda - 3] + 7[8 - 2 + \lambda] = 0$$
$$\Rightarrow (1-\lambda)(2 - 3\lambda + \lambda^2 - 6) - 3(1 - 4\lambda) + 7(6 + \lambda) = 0$$
$$\Rightarrow (1-\lambda)(\lambda^2 - 3\lambda - 4) - 3 + 12\lambda + 42 + 7\lambda = 0$$
$$\Rightarrow \lambda^2 - 3\lambda - 4 - \lambda^3 + 3\lambda^2 + 4\lambda + 19\lambda + 39 = 0$$

$$-\lambda^3+4\lambda^2+20\lambda+35 = 0$$

$$\Rightarrow \quad \lambda^3-4\lambda^2-20\lambda-35 = 0$$

To show that

$$A^3-4A^2-20A-35I = 0$$

$$A^2 = \begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}20 & 23 & 23\\15 & 22 & 37\\10 & 9 & 14\end{bmatrix}$$

$$A^3 = \begin{bmatrix}20 & 23 & 23\\15 & 22 & 37\\10 & 9 & 14\end{bmatrix}\begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}135 & 152 & 232\\140 & 163 & 208\\60 & 76 & 111\end{bmatrix}$$

Now, $\begin{bmatrix}135 & 152 & 232\\140 & 163 & 208\\60 & 76 & 111\end{bmatrix} - 4\begin{bmatrix}20 & 23 & 23\\15 & 22 & 37\\10 & 9 & 14\end{bmatrix} - 20\begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix} - \begin{bmatrix}35 & 0 & 0\\0 & 35 & 0\\0 & 0 & 35\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}135 & 152 & 232\\140 & 163 & 208\\60 & 76 & 111\end{bmatrix} - \begin{bmatrix}80 & 92 & 92\\60 & 88 & 148\\40 & 36 & 56\end{bmatrix} - \begin{bmatrix}20 & 60 & 140\\80 & 40 & 60\\20 & 40 & 20\end{bmatrix} - \begin{bmatrix}35 & 0 & 0\\0 & 35 & 0\\0 & 0 & 35\end{bmatrix}$$

$$= 0$$

$\therefore$ A satisfies this equation

Multiply both sides of (1) by A^{-1}, we have

$$A^2-4A-20I-35A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{35}(A^2-4A-20I)$$

$$\Rightarrow A^{-1} = \frac{1}{35}\left\{\begin{bmatrix}20 & 23 & 23\\15 & 22 & 37\\10 & 9 & 14\end{bmatrix} - 4\begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix} - \begin{bmatrix}20 & 0 & 0\\0 & 20 & 0\\0 & 0 & 20\end{bmatrix}\right\}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{35}\begin{bmatrix}-4 & 11 & -5\\-1 & -6 & 25\\6 & 1 & -10\end{bmatrix}$$

UNIT-IV

Q.8. (a) Solve $\dfrac{dy}{dx}+\left(\dfrac{y}{x}\right)\log y=\dfrac{y}{x^2}(\log y)^2$. (6)

Ans. Consider $\dfrac{dy}{dx}+\dfrac{y\log y}{x} = \dfrac{y(\log y)^2}{x^2}$

Multiply equation by $\dfrac{1}{y(\log y)^2}$

$$\Rightarrow \quad \frac{1}{y(\log y)^2}\frac{dy}{dx}+\frac{1}{x\log y} = \frac{1}{x^2}$$

Let $$\frac{1}{\log y} = z$$

$$\Rightarrow \quad \frac{-1}{(\log y)^2}\cdot\frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$$

∴ Equation becomes

$$\frac{-dz}{dx}+\frac{z}{x} = \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dz}{dx}-\frac{z}{x} = \frac{-1}{x^2}, \text{ which is linear}$$

Now, $$\text{I.F.} = e^{-\int 1/x\,dx} = e^{-\log x} = e^{\log x^{-1}}$$

$$\text{I.F.} = \frac{1}{x}$$

∴ Required solution is

$$z\cdot\frac{1}{x} = \int -\frac{1}{x^2}\cdot\frac{1}{x}dx + C$$

$$\Rightarrow \quad z\cdot\frac{1}{x} = \frac{-x^{-2}}{-2}+C$$

$$\Rightarrow \quad \frac{1}{x\log y} = \frac{1}{2x^2}+C$$

$$\Rightarrow \quad \frac{1}{\log y} = \frac{1}{2x}+Cx$$

Q.8. (b) Solve $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=4\cos[\log(1+x)]$. (6.5)

Ans. Let $1+x=e^z \quad \Rightarrow \quad z = \log(1+x)$

$$\Rightarrow \quad (1+x)\frac{dy}{dx} = Dy$$

$$(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y \quad \text{where } D = \frac{d}{dz}$$

Given equation becomes

$$D(D-1)y + Dy + y = 4\cos z$$

$$(D^2+1)y = 4\cos z$$

A.E. $D^2+1=0 \Rightarrow D=\pm i$

C.F. $C_1\cos z + C_2\sin z$

P.I. $\frac{1}{D^2+1}4\cos z$

$$\Rightarrow \quad 4\frac{1}{D^2+1}\cos z$$

For

$$D^2 = -1,\ f(D)=0$$

Case of failure arises

$$\Rightarrow \quad \text{P.I.} = 4\frac{z}{2D}\cos z$$

$$= 2z\int\cos z\,dz = 2z\sin z$$

$\therefore$ Complete solution is

$$y = C_1\cos z + C_2\sin z + 2z\sin z$$

$$= C_1\cos\{\log(1+x)\} + C_2\sin\{\log(1+x)\} + 2\log(1+x)\sin\{\log(1+x)\}$$

Q.9. (a) Prove that $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$. (6)

Ans. Consider recurrence relation

$$\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$$

$$\Rightarrow \quad x^nJ_n'(x) + nx^{n-1}J_n(x) = x^nJ_{n-1}(x)$$

$$\Rightarrow \quad J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x) \quad ...(A)$$

And from $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$$\Rightarrow \quad x^{-n}J_n'(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow \quad -J_n'(x) + \frac{n}{x}J_n(x) = J_{n+1}(x) \quad ...(B)$$

Subtracting (B) from (A), we get

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \quad ...(1)$$

Differentiating (1) w.r.t. 'x', we get

$$J''_n(x) = \frac{1}{2}[J'_{n-1}(x) - J'_{n+1}(x)]$$

$$J''_n(x) = \frac{1}{2}J'_{n-1}(x) - \frac{1}{2}J'_{n+1}(x).$$

Again applying (1), we get

$$J''_n(x) = \frac{1}{2}\left[\frac{1}{2}\{J_{n-2}(x) - J_n(x)\}\right] - \frac{1}{2}\left[\frac{1}{2}\{J_n(x) - J_{n+2}(x)\}\right]$$

$$= \frac{1}{4}[J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)]$$

$$= \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$$

$$\Rightarrow \quad 4J''_n(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x).$$

Q.9. (b) Prove that $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$. **(6.5)**

Ans. Let us consider $V = (x^2-1)^n$

$$\Rightarrow \quad V_1 = \frac{dV}{dx} = n(x^2-1)^{n-1}\cdot 2x$$

Multiplying both sides by (x^2-1), we get

$$(x^2-1)V_1 = 2nx(x^2-1)^n$$

$$\Rightarrow \quad (x^2-1)V_1 = 2nxV$$

$$\Rightarrow \quad (1-x^2)V_1 + 2nxV = 0$$

Differentiating $(n+1)$ times by leibnitz theorem, we get

$$[(1-x^2)V_{n+2} + (n+1)(-2x)V_{n+1} + \frac{n(n+1)}{2!}(-2)V_n] + 2n[xV_{n+1} + (n+1)V_n] = 0$$

$$\Rightarrow (1-x^2)V_{n+2} - 2xV_{n+1} + n(n+1)V_n = 0$$

$$\Rightarrow (1-x^2)\frac{d^2V_n}{dx^2} - 2x\frac{d(V_n)}{dx} + n(n+1)V_n = 0$$

Which is Legendre's equation and V_n is its solution. As solution of Legendre's equation are $P_n(x)$ and $Q_n(x)$ and $V_n = \frac{d^n}{dx^n}(x^2-1)^n$ contains only +ve powers of x, it must be constant multiple of $P_n(x)$.

$$\Rightarrow \quad V_n = CP_n(x) \quad ...(1)$$

$$\Rightarrow \quad CP_n(x) = \frac{d^n}{dx^n}(x^2-1)^n$$

$$= \frac{d^n}{dx^n}[(x-1)^n (x+1)^n]$$

$$= (x-1)^n \frac{d^n}{dx^n}(x+1)^n + {}^nC_1 n(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x+1)^n$$

$$+ \ldots + (x+1)^n \frac{d^n}{dx^n}(x-1)^n$$

$$= (x-1)^n n! + {}^nC_1\, n(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \ldots + (x+1)^n n!$$

$$= n!(x+1)^n + \text{terms containing powers of } (x-1).$$

Putting $x = 1$ on both sides, we get

$$CP_n(1) = n!\, 2^n$$

$$\Rightarrow \qquad C = n!\, 2^n \qquad [\because P_n(1) = 1]$$

By (1), we get

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n.$$