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Unit-1

Discrete Mathematics is a study of discrete structures which are abstract mathematical models dealing with discrete object and their relationship between them.

Objects → Sets, Permutation & Combination, functions, Graphs, Trees, Finite state automata (FSA), Boolean Algebra, Relation, Logic

Logic → Ability of Problem solving

Propositional logic: A Proposition is an assertion which can either be true or false but can never be both.

a) This is a table → True It is a Proposition
b) Why are you laughing? → It is not a Proposition
c) Buy four Books of discrete maths → It is not a proposition
d) $x = 3$ → It is not a Proposition

Proposition

- Compound
  - Combining Propositions
  - Eg: If there is rain tomorrow then there is no class
- Primitive
  - Proposition which cannot be broken into different Propositions
  - Eg: There is a chair

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Basic logic operations

Conjuction (AND) ($P \wedge q$) | Disjunction/Inclusive or (OR) ($P \vee q$) | Negation (NOT) ($\neg P, \sim P, \bar{P}$) | Exclusive or (EXOR) ($P \oplus q$)

1 → True
0 → false

multiply

P	q	$P \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

P	q	$P \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

P	$\neg P$
0	1
1	0

P	q	$P \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

Implication/Conditional Statement

① Statement $P \rightarrow q$ is true when
↳ implies

a) both P & q are true result true
b) both P & q are false the result is true
c) When P is false no matter the truth value q has

Eg: If moon is made of cheese then the Earth is not round.

* If, then — 0 → false, 1 → true

Conjuction – but, and
Disjunction – or
Exclusive – both
Implication – If, then
Equivalence – If and only if

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$\sim P$ (Negation) = 1
$\wedge$ (Conjuction) = 2
$\vee$ (Disjunction) = 3
$\rightarrow$ (Implication) = 4
$\leftrightarrow$ Equivalence = 5

Implication ($\rightarrow$) ($\Rightarrow$) | Equivalence

P	q	$P \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

P	q	$P \Leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Identify P & q

① If P, then q
② P only if q
③ P is sufficient condition for q
④ q is necessary condition for P
⑤ q if P
⑥ q follows from P
⑦ q provided P
⑧ q is logical sequence of P
⑨ q whenever P

Q "The hometeam wins whenever it is raining" What is the implication, Conjuction & Disjunction of statement?

Sol q → The hometeam wins
P → It is raining

Implication: It is raining and hometeam wins
~~Conjuction: But~~

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Implication

$p \longrightarrow q$

p → Hypothesis, Antecedent, Premise

q → Conclusion, consequence

$(p \rightarrow q)$ Implication / Conditional statement

- Converse $(q \rightarrow p)$
- Contrapositive $(\neg q \rightarrow \neg p)$
- Inverse $(\neg p \rightarrow \neg q)$

Q The home team wins whenever it is raining

Sol $p \rightarrow$ It is raining
$q \rightarrow$ The home team wins

Implication: If it rains then home team wins

Contrapositive: If the home team does not win then it is not raining

Inverse: If it is not raining then the home team wins

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p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	1	1	0	0	1	1

Different ways to Equivalence

⇒ p is Equivalent to q
⇒ p if and only if q
⇒ p is necessary and sufficient condition for q

Q $[(p \wedge q) \vee \neg r] \Longleftrightarrow p$ Example of Contingency

p	q	r	$p \wedge q$	$\neg r$	$(p \wedge q) \vee \neg r$	$[(p \wedge q) \vee \neg r] \Longleftrightarrow p$
0	0	0	0	1	1	0
0	0	1	0	0	0	1
0	1	0	0	1	1	0
0	1	1	0	0	0	1
1	0	0	0	1	1	1
1	0	1	0	0	0	0
1	1	0	1	1	1	1
1	1	1	1	0	1	1

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Propositional Equivalence

Tautology $(P \vee \neg P)$
Propositional form whose truth value is true for all possible values

P	$\neg P$	$P \vee \neg P$
0	1	1
1	0	1

Absurdity/Contradiction $(P \wedge \neg P)$
Propositional form whose truth value is always false

P	$\neg P$	$P \wedge \neg P$
0	1	0
1	0	0

Contingency $(q \wedge \neg P) \Rightarrow P$
Propositional form which is neither tautology nor contradiction

Q The home team wins whenever it is raining

Sol P:- It is raining
q:- The home team wins

Implication $(P \rightarrow q)$:- If it is raining then home team wins

Inverse $(\neg P \rightarrow \neg q)$:- If it is not raining then home team does not win

Converse $(q \rightarrow P)$:- If the home team wins then it is raining

Contrapositive $(\neg q \rightarrow \neg P)$:- If the home team does not win, then it is not raining

Q Check whether the statement is tautology, contradiction or contingency
$(P \rightarrow q) \rightarrow r$

Sol This is contingency

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$(P \rightarrow q) \rightarrow r$

P	q	r	$P \rightarrow q$	$(P \rightarrow q) \rightarrow r$
0	0	0	1	0
0	0	1	1	1
0	1	0	1	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

Q Determine the Converse, Inverse, Contrapositive for the statement
" If John is a Poet then he is Poor"

Sol $P \Rightarrow$ John is a Poet
$q \Rightarrow$ he is Poor

Converse: $(q \rightarrow P)$
If John is Poor then he is a Poet

Inverse: $(\neg P \rightarrow \neg q)$
If John is not a Poet then he is not Poor.

Contrapositive: $(\neg q \rightarrow \neg P)$
If John is not a Poor the he is not a Poet

Q let P be "Erik reads Newsweek"
q be "Erik reads the New Yorker"
r be "Erik reads time"
Write the following statements given below in symbolic form.

i) Erik reads Newsweek & The New Yorker or he does not read Newsweek & Time.

ii) It is not true and Erik reads Time or the New Yorker but not Newsweek.

Sol i) $(P \wedge q) \vee (\neg P \wedge r)$

2) $\sim[(p \vee q) \wedge \sim p]$

or

$\neg[(r \vee q) \wedge \neg p]$

* $\Leftrightarrow$ or $\equiv$ Equivalence

* $\rightarrow$ or $\Rightarrow$ Implication

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Logical Identities

① $P \Leftrightarrow P \vee P$ or $P \equiv P \vee P$:- Idempotence of $\vee$ } Identity laws
(Equivalence, Equivalence)

② $P \Leftrightarrow P \wedge P$ or $P \equiv P \wedge P$:- Idempotence of $\wedge$

③ $(P \vee q) \Leftrightarrow (q \vee P)$:- Commutativity of $\vee$ } Commutative laws

④ $(P \wedge q) \Leftrightarrow (q \wedge P)$:- Commutativity of $\wedge$

⑤ $[(P \vee q) \vee r] \Leftrightarrow [P \vee (q \vee r)]$ } Associative laws

⑥ $[(P \wedge q) \wedge r] \Leftrightarrow [P \wedge (q \wedge r)]$

⑦ $\neg(P \vee q) \Leftrightarrow (\neg P \wedge \neg q)$ } De Morgans Law

⑧ $\neg(P \wedge q) \Leftrightarrow (\neg P \vee \neg q)$

⑨ $[P \wedge (q \vee r)] \Leftrightarrow [(P \wedge q) \vee (P \wedge r)]$ } Distributive Law

⑩ $[P \vee (q \wedge r)] \Leftrightarrow [(P \vee q) \wedge (P \vee r)]$

$(P \vee 1) \Leftrightarrow 1$ $(q \vee q) \Leftrightarrow q$

$(P \wedge 1) \Leftrightarrow P$

$(P \wedge 0) \Leftrightarrow 0$

$(P \vee \neg P) \Leftrightarrow 1$

$(P \wedge \neg P) \Leftrightarrow 0$

$P \Leftrightarrow \neg(\neg P)$:- Double Negation

$(P \Leftrightarrow q) \Leftrightarrow (\neg P \vee q)$:- Implication

$(P \Leftrightarrow q) \Leftrightarrow [(P \Leftrightarrow q) \wedge (q \Leftrightarrow P)]$:- Equivalence

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$(P \Rightarrow q) \Leftrightarrow (\neg P \vee q)$: Implication

$(P \Leftrightarrow q) \Leftrightarrow [(P \Rightarrow q) \wedge (q \Rightarrow P)]$: Equivalence

$[(P \wedge q) \Rightarrow r] \Leftrightarrow [P \Rightarrow (q \Rightarrow r)]$: Exportation

$[(P \Rightarrow q) \wedge (P \Rightarrow \neg q)] \Leftrightarrow \neg P$: Contradiction

$(P \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg P)$: Contrapositive

① Check whether statements are Tautology or not?

$[(P \vee q) \wedge (P \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

P	q	r	$(P \vee q)$	$(P \rightarrow r)$	$(q \rightarrow r)$	$(P \vee q) \wedge (P \rightarrow r)$	$(P \vee q) \wedge (P \rightarrow r) \wedge (q \rightarrow r)$	$[(P \vee q) \wedge (P \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$
0	0	0	0	1	1	0	0	1
0	0	1	0	1	1	0	0	1
0	1	0	1	1	0	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

The above truth table the statement is Tautology

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Propositional functions / Predicates

⇒ Propositional function (often statement);

⇒ Statement involving one or more variables;

⇒ e.g. $x - 3 = 5$; $y = x$ → predicate

⇒ Let us call this Propositional function $P(x)$ often where P is the Predicate and x is the variable.

What is the truth value of $P(2)$? : False

$2 - 3 = 5$

What is the truth value of $P(8)$? : True

$8 - 3 = 5$

$5 = 5$

What is the truth value of $P(9)$? False

$9 - 3 = 5$

$6 = 5$

Quantification

⇒ It expresses the extent to which a Predicate is true for over range of elements.

⇒ English words like all, some, many, none, few are used in Quantification.

⇒ Two types of Quantification:

- Universal Quantification: In this type a Predicate is true for every element under consideration.
- Existential Quantification: In this there is one or more element under consideration.

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Set Theory

Set: Any well defined Collection, Class or ~~Lists~~ list of distinct object is called a Set. The objects in a set are called members or elements of set

$A = \{1, 2, 3, 4\}$

Representation of Sets:

⇒ Roster form or Tabular form: In this form all the member of the set are listed. These members are ~~listed~~ separated via comma and enclosed within { }

eg: $\{a, e, i, o, u\}$

⇒ Set Builder form or Rule form of a set:

eg:

$A = \{x : x$ is a Positive integer$\}$

$A = \{x \mid x$ is a ~~Posit~~ Positive integer$\}$

In this form a set is defined by stating the Properties which are satisfied by the elements of the set. Here the rule governing the members of the set is stated.

Symbol for set builder $\{x : x'$ or $x \mid x$

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Cardinality / Cardinal Number / Dimension / order of set

The number of distinct element in set A is also called number or cardinal number of the set and is denoted

$n(A)$ or $O(A)$ or $|A|$

Finite and Infinite no of set

Finite Set: A set is called Finite if it has n distinct elements where n belongs to N $n \in N$ such a set consist of a specific no of different element the. In a set process of counting different no's comes to an end.

Infinite Set: otherwise set is Finite.

Equality of a set

If every member of set A is a member of set B and every member of set B is a member of set A then two set A and B are said to be equal.

Eg: $A = \{1, 2, 3\}$

$B = \{3, 1, 2\}$

$A = B$

Null Set

A Set which contain no element is called a null set. It is denote by ϕ, $\{\ \}$, $\{\phi\}$

Eg: The set of person greater than age above 400

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Singleton Set
A Set Containing Only one element is Called a singleton set.
Eg: $A = \{3\}$

Subset
If Every member in set A is also a member of set B then A is said to be subset of B.
Eg:
$B = \{1, 2, 3, 4, 5\}$
$A = \{1, 2, 3\}$
$B \subseteq A$ $A \subseteq B$
$A \subseteq A$ = This condition is also true

Superset
If A is Subset of B then B is said to be the superset of A
Eg: $B \supset A$

Universal Set $\boxed{U}$
The set of which all sets Under investigation subsets is called Universal Set.
It is denoted by U

Power Set
The set Containing all the subsets of a set set A as its elements is called a power set of A and is denoted by
$P(A)$
Eg: $A = \{1, 2\} \rightarrow 2^m$
$P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

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Universal Set
$A = \{1, 2, 3, 4\}$
$B = \{5, 6, 7, 8\}$
$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Disjoint Set
If no element of A is a member of B and no element of B is a member of A then A and B are said to be disjoint set.
$A \not\subset B$, $B \not\subset A$, $A \neq B$ These are known as disjoint set.
$A = \{1, 2, 3\}$
$B = \{4, 5, 6\}$

Set of Sets / Family of Sets
It is a set of elements which are sets themselves
Eg: $A = \{\{1, 3, 5\}, \{2, 4, 6, 8\}, \{9\}\}$

Proper Subset
If B is subset of A and B is not equal to A there exist at least an element in a which does not belong to B then B is said to be Proper Subset of a
Eg: $A = \{1, 2, 3, 4\}$
$B = \{1, 2\}$

Condition of Proper Subset
- B is subset of A
- B is not equal to A
- Element in A which does not belong to B

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Set Operations

⇒ **Union (U):** The Union of two sets A and B is the set of all elements which belongs to A or to B or A to B

$A = \{1, 2, 3, 4\}$
$B = \{2, 5, 6, 7\}$

$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

⇒ **Intersection of Sets (∩):** The Intersection of two sets A and B is the set of elements which belongs to A as well as to B.

$A = \{1, 2, 3, 4\}$
$B = \{2, 5, 6, 7\}$

$A \cap B = \{2\}$

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

⇒ **Difference of Sets:** The difference of two sets A and B is the set of element which belong to set A but do not belong to set B. $A - B$, $A \sim B$, $A \setminus B$

$A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$
$A - B = \{1, 2\}$, $B - A = \{4, 5\}$

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⇒ **Complement of Set:** The complement of set A is the set of element which do not belong to A. It is therefore the difference of Universal set and set A. A' or A^c

$U = \{1, 2, 3, 4\}$
$A = \{1\}$
$A' = U - A$
$= \{2, 3, 4\}$

Ex: $A = \{x \mid x \text{ is odd natural number}\}$
$U = \{x \mid x \text{ is natural number}\}$
$A' = \{x \mid x \text{ is even number}\}$

Ex: $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$A = \{2, 4, 6, 8, 10\}$
$A' = U - A$
$= \{1, 3, 5, 7, 9\}$

Ex: $U = \{x \mid x \text{ is letter of English language}\}$
$A = \{u, v, w, x, y, z\}$
$A' = \{a, b, c, \ldots, s, t\}$

Ex: $A = \{x \mid x \text{ is integer and } x < 7\}$
$U = Z$ (set of integer)

$A' = \{x : x \text{ is integer and } x \geq 7\}$

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Symmetric difference of two sets

$A \oplus B$ or $A \Delta B = (A-B) \cup (B-A)$

Min Set, $U \rightarrow$ subsets P & Q

$P \cap Q$ $\quad P \cap Q'$
$P' \cap Q$ $\quad P' \cap Q'$

Ex: $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$P = \{1, 3, 5, 7\}$
$Q = \{4, 6, 8, 10\}$
Min set

$P \cap Q = \{\phi\}$
$P \cap Q' = P \cap (U - Q)$
$= P \cap \{1, 2, 3, 5, 7, 9\}$
$= \{1, 3, 5, 7\} \cap \{1, 2, 3, 5, 7, 9\}$
$= \{1, 3, 5, 7\}$

$P' \cap Q = (U - P) \cap Q$
$= \{2, 4, 6, 8, 9, 10\} \cap \{4, 6, 8, 10\}$
$= \{4, 6, 8, 10\}$

$P' \cap Q' = (U - P) \cap (U - Q)$
$= \{2, 4, 6, 8, 9, 10\} \cap \{1, 2, 3, 5, 7, 9\}$
$= \{2, 9\}$

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Max Set : $U \rightarrow$

$P \cup Q$ $\quad P \cup Q'$
$P' \cup Q$ $\quad P' \cup Q'$

Min Set: let P & Q be two subset of a Universal set U then a ~~universal~~ Possible set that are $\cap$ of P, P', Q, Q' are Min Set i.e $P \cap Q$, $P \cap Q'$, $P' \cap Q$, $P' \cap Q'$

Max Set: let P & Q be two subset of a Universal set U then all Possible ~~set~~ that are Union of P, P', Q, Q' are called Max set with respect to P & Q. These are $P \cup Q$, $P \cup Q'$, $P' \cup Q$, $P' \cup Q$

Algebra of Set

(1) Idempotent law

a) $A \cup A = A$
b) $A \cap A = A$

(2) Associative law

a) $(A \cup B) \cup C = A \cup (B \cup C)$
b) $(A \cap B) \cap C = A \cap (B \cap C)$

(3) Commutative law

a) $A \cup B = B \cup A$
b) $A \cap B = B \cap A$

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④ Distributive law

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

⑤ Identity law

a) $A \cup \phi = A$

b) $A \cap U = A$

⑥ Domination law

a) $A \cup U = U$

b) $A \cap \phi = \phi$

⑦ Involution Law

$(A')' = A$

⑧ Complement Laws

$A \cup A' = U$

$A \cap A' = \phi$

⑨ $U' = \phi$

$\phi' = U$

⑩ De Morgan's law

a) $(A \cup B)' = A' \cap B'$

b) $(A \cap B)' = A' \cup B'$

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⑪ $A - (B \cup C) = (A - B) \cap (A - C)$

$A - (B \cap C) = (A - B) \cup (A - C)$

Imp Results

① no of elements belonging to A or B or Both $= n(A \cup B)$

② no of elements not belonging to either of A or B $= n(A \cup B)'$

③ no of elements belonging to both A & B $= n(A \cap B)$

④ $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

⑤ $n(A - B) = n(A) - n(A \cap B)$

⑥ $n(B - A) = n(B) - n(B \cap A)$

⑦ $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$

Q $U = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$
$X = \{1, 3, 6, 8\}$
$Y = \{4, 5, 7, 9\}$
$Z = \{3, 5, 7\}$

a) X'
$= U - X$
$= \{4, 5, 7, 9, 10\}$

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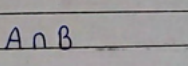
b) $(X \cup Y)'$
$\{2,3,4,5,6,7,8,9\}$
$U - \{2,3,4,5,6,7,8,9\}$
$\{2,3,4,5,6,7,8,9,10\} - \{2,3,4,5,6,7,8,9\}$
$\{10\}$

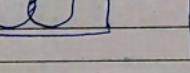
c) $(Y - Z)'$
$\{4,9\}$
$U - \{4,9\}$
$\{2,3,5,6,7,8,10\}$

Venna Diagram
Geomtrical Figures or Diagrams that are Use to represent set are called ~~seem~~ Venndiagram.

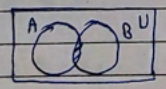
1) Universal Set

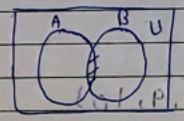


2) Disjoint Set $A \cap B = \phi$



3) ~~A~~ $A \cap B$





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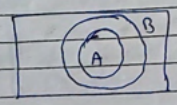
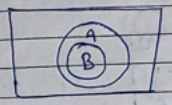
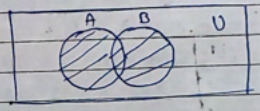
⇒ A is a Subset of B

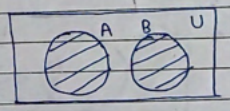


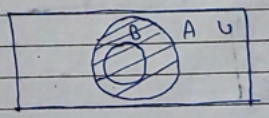
⇒ B is a Subset of A



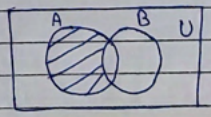
4) $A \cup B$







5) $A - B$



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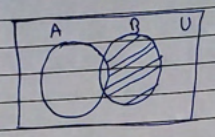
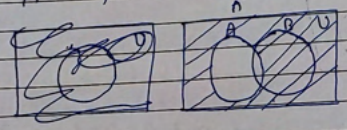
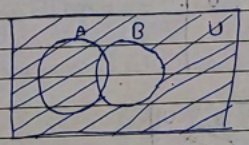
⇒ $B - A$



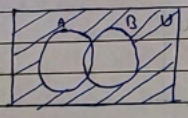
(6) A' , B'



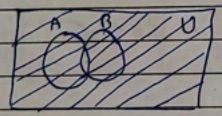
(7) B'



(8) $(A \cup B)'$



(9) $(A \cap B)'$



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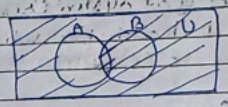
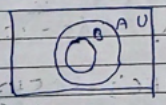
(10) $(A \cup B)'$



(11) $B \subset A$



Q Given Universal set $U = \{a, b, c, d, e, f, g\}$

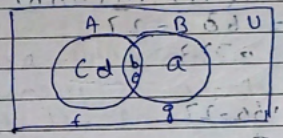
$A = \{b, c, d, e\}$

$B = \{a, b, e\}$

i) $A' = U - A = \{a, f, g\}$

ii) $A - B = \{c, d\}$

iii) $B - A = \{a\}$



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Q In a class of certain college 42 students got distinction 60 got distinction in FMCC 27 student both the subject. Use Venn diagram to find

(1) In accounts only $42 - 27 = 15$

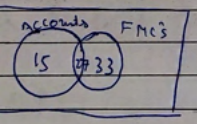
(2) In MFC's only $60 - 27 = 33$

(3) Accounts or in MFC's or in both

i) $n(A - A \cap B) = n(A) - n(A \cap B)$
$= 42 - 27$
$= 15$

ii) $n(B - A \cap B) = n(B) - n(A \cap B)$
$= 60 - 27$
$= 33$

iii) $n(A \cup B) = 42 + 60 - 27$
$= 75$



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Q If $n(U) = 60$, $n(A) = 27$, $n(B) = 19$, $n(A \cup B)' = 31$

Sol i) $n(A \cap B)$

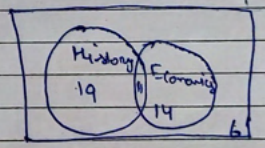
$n(A \cup B) = n(U) - n(A \cup B)'$
$= 60 - 31$
$= 29$

$n(A \cap B) = n(A) + n(B) - n(A \cup B)$
$= 27 + 19 - 29$
$= 46 - 29$
$= 17$

i) $n(A - B)$
$n(A) - n(A \cap B)$
$= 27 - 17$
$= 10$

Q In a class of 50 students 30 study history, 25 study economics, 11 study both the language. Draw Venn Diagram to represent this.

Sol



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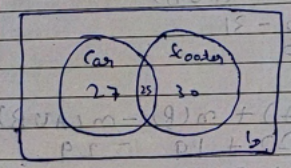
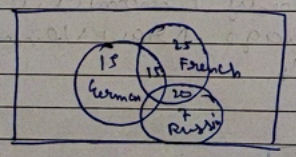
Q It is found that out of 100 students, 18 can drive neither a scooter nor a car, while 25 can drive both they and 55 of them a scooter. How many can drive a car.

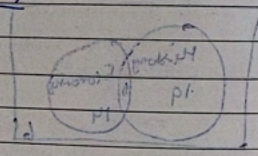
Sol



Q Suppose that 100 of the mathematics students at a college atleast one of the languages French, German & Russian. Also suppose 60 study French.
40 study Russian,
15 study French & German
20 study French & Russian
10 study German & Russian

Sol





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Predicate

Quantifiers

Universal Quantifier

Existential Quantifier

It is denoted by $\boxed{\forall x P(x)}$

(All), Every

$(\exists x P(x))$

(Some)

Few, Many

Several, Each

Q What is the truth value of $\forall x P(x)$ where $P(x)$ is the statement "$x^2 < 10$" & the domain consist of positive integers not exceeding 4.

Sol $P(x)$: "$x^2 < 10$"

0 True

1 True

2 True

3 True

4 False

False → Existential Quantifier

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Rules of inference

By an argument, we mean a sequence of statement that end with a conclusion. By valid we mean that the conclusion or final statement of the argument must follow from the truth of the preceding statement or the premises of the statement.

Premises { Statement 1, 2, 3, 4 } Argument

5 → conclusion

Ex: "If you have a current password, then you can log on to the network"
Therefore
"You can log on to the network"

p = You have a current password
q = You can log on to the network

$p \to q$
p
$\therefore q$ — Tautology

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Rules of inference

Rule of inference	Tautology	Name
(1) p, $p \to q$, $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus Ponens (law of detachment)
(2) $\neg q$, $p \to q$, $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
(3) $p \to q$, $q \to r$, $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical Syllogism
(4) $p \vee q$, $\neg p$, $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive Syllogism
(5) p, $\therefore p \vee q$	$p \to (p \vee q)$	Addition
(6) $p \wedge q$, $\therefore p$	$(p \wedge q) \to p$	Simplification
(7) p, q, $\therefore (p \wedge q)$	$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
(8) $p \vee q$, $\neg p \vee r$, $q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

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Q State which rule of inference is the basis of the following argument:
"It is below freezing now. Therefore it is either below freezing or raining now."

Sol P → It is below
Q → Raining now

$\frac{P}{P \vee Q}$ — Addition

Q State which rule of inference is the basis of the following argument
"It is below freezing and raining therefore it is below freezing now"

Sol $\frac{P \wedge Q}{\therefore P}$ ← Simplification

Q Show that the following argument is valid
If Mohan is a lawyer, then he is ambitious
If Mohan is an early riser, then he does not like idlies.
If Mohan is ambitious then he is an early riser.
Then if Mohan is a lawyer, then he does not like idlies.

Sol ① $L \to A$ ④ $L \to E$ (from ① and ③)
② $E \to \neg I$ ⑤ $L \to \neg I$ Hypothetical
③ $A \to E$ ← $(P \vee \neg r) \wedge (p \vee q)$ Syllogism

$\frac{p \vee q \quad r \vee \neg p}{q \vee r}$

$\frac{q \vee r}{r \vee p}$

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Method of Proof

Theorem: The statement that is true to be true it is called a Theorem. Special type of theorem are referred to as lemma's and corallries.

~~lemma~~
lemma - It is a theorem that is usually not to interesting in it's own write but it is useful in proving another Theorem.

Corallary - It is a theorem that follows quickly from another theorem.

Conjecture: It is a statement whose truth value is unknown. When a proof a conjecture is found it becomes a theorem.

Proof - An argument that establishes the truth of the theorem is called proof.

Logic - Tool for analysis of the proof is called logic.

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① Direct Proof

② Indirect Method → Proof by Contrapositive
→ Proof by Contradiction.

⇒ Direct Proof (Modus Ponens (Law of Affirmity)). A direct Proof $P \Rightarrow q$ is a logical valid argument in which we start with a assumption that hypothesis is true that P is true.

* A direct proof assumes that $P(x_1, x_2, \dots x_n)$ is true then using $P(x_1, x_2, \dots x_n)$ as well as other axioms, definition and Previously ~~already~~ derived theorem $P(x_1, x_2, \dots x_n)$

Ex: Give a direct Proof of the statement "The Product of two odd integers is odd"

$x = 2m+1, \; y = 2n+1$

$xy = (2m+1)(2n+1)$

$= 4mn + 2n + 2m + 1$

$= 2(2mn + n + m) + 1$

$= 2P + 1$

$P \in Z$ is an odd integer ∴ the result follows.

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⇒ "The Square of an even integer is an even integer" Give a direct Proof of the Statement

Sol $x = 2n$

Squaring both sides

$x^2 = 4n^2$

$= 2(2n^2)$

$= 2P$

$P \in Z$

P is an even integer

⇒ Let $x \in R$ (set of real numbers) such that $x^2 = K$, then either $x = \sqrt{K}$ or $x = -\sqrt{K}$

Sol $P, x^2 = K$

$q \; x = \sqrt{K}$ or $x = -\sqrt{K}$

Let $x^2 = K$

$x = \pm\sqrt{K}$

is an even integer

* Proof by Contrapositive

$P \Rightarrow q$

$\sim q \Rightarrow \sim P$

P	q	$P \to q$	$\sim P$	$\sim q$	$\sim q \to \sim P$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

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Ex If x is an integer and x^2 is even, then x is also even by proving its contrapositive

Sol p - x is an integer and x^2 is even
q - x is also even

$\sim q$ = x is odd integer

& $\sim q \Rightarrow \sim p$

$x = 2m+1$
$x^2 = (2m+1)^2$
$= 4m^2+1+4m$
$= 2(2m^2+2m)+1$
$= 2k+1$
$k = 2m^2+2m \in \mathbb{Z}$

x^2 is an odd integer
$\therefore \sim p$ holds, so the result is true by method of contrapositive.

Q Give an indirect proof (contrapositive) of the theorem: If "$3m+2$" is odd, then m is odd
(p → q)

Sol p = $3m+2$ is odd
q = m is odd
We need to prove $p \Rightarrow q$
Consider the contrapositive
$\sim q \Rightarrow \sim p$
$\sim q$: m is even
$\Rightarrow m = 2k$, $k \in \mathbb{Z}$

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Consider $3m+2$
$= 3(2k)+2$
$= 6k+2$
$= 2(3k+1)$ = even
$\therefore 3m+2$ is even $= \sim p$ holds
$\therefore$ By method of contrapositive result holds

<u>Proof by Contradiction</u>
This method is also called reductio and deductum because it believes on reducing a given assumption to an absurdity.
For the argument $p \to q$ we need to prove the If p is true then q is true. In this method we start by assuming that q is false.

Ex Use the method of proof by contradiction show that for $x \in \mathbb{R}$ if $x^3+4x=0$, then $x=0$
assume that $x^3+4x=0$ & $x \neq 0$

$x^3+4x=0$
$x(x^2+4)=0$
$x \neq 0$ or $x^2+4=0$
$x^2=-4$
$x^2=-4$ not possible
$\therefore$ our assumption is false
Hence the given statement is true

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Normal form

The Problem of finding whether a given statement is a Tautology or Contradiction or satisfiable a finite no of steps is called a decision Problem. For a Decision Problem the Contradiction of Truth table may not be one is practical. We consider alternate Procedure called as reduction to Normal form.

Well formed formula consist of
Propositional Variable, Connector & Parenthesis

$p \wedge q$

$(q \vee r)$

$(p \vee q) \wedge (\neg r \vee s)$

=) Elementary Product - A Product of Variables and their negation in a formula is called Elementary Product

$\neg p \wedge q$

=) Elementary Sum - An Elementary Sum is a disjunction of literals

$p \vee \neg q$

Unnessary and sufficient Condition For an Elementary to be identically false (contradiction) it contains atleast one Pair literals in which one is negation of other

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If $p \wedge \neg p \wedge \ldots$ is formula then reduce $F \wedge \ldots$ is equal to F

Condition of Elementary Sum is to be Identity (True)

Tautology it contains atleast one Pair of literals in which one is negation of other

i.e. $p \vee \ldots \vee \neg p \vee \ldots$ True

=) DNF (Disjunctive Normal form) (SOP form)

$(p \wedge q \wedge r) \vee (p \wedge r) \vee (q \wedge r)$

A Conjunction of statement variables and their negation is called as Fundamental Conjunction. A statement form which consist consist of Fundamental Disjunction

Q obtain disjunctive Normal form

1) $(p \rightarrow q) \wedge \neg q$

$(\neg p \vee q) \wedge \neg q$

$(\neg p \wedge \neg q) \vee (q \wedge \neg q)$

2) $\neg (p \wedge q) \longleftrightarrow (p \vee q)$

=) CNF (Conjunctive Normal form)

$(p \vee q) \wedge (r \vee s)$

A disjunction of statement variables and their negation is called a Fundamental Disjunction or max terms. A statement which consist of Fundamental Disjunction is called Fundamental normal form

Q $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

$= p \vee (\sim p \wedge q \wedge r) \wedge (q \vee (\sim p \wedge q \wedge r))$

$= (p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r)$

$= (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge q \wedge (q \vee r)$

Principal Conjunctive Normal form (PCNF)

↓

Maxterms

Disjunction (sum) of variable & its negation appear exactly one

Maxterms of a no of variable is a formula which consist of Disjunction in which each variable or its negation but not the both appear exactly once. Maxterms are the dual of minterms. Each of the maxterms has the truth table value for exactly one combination of the truth value of the variable.

Variable		Maxterms and their truth values			
p	q	$p \vee q$	$p \vee \neg q$	$\neg p \vee q$	$\neg p \vee \neg q$
T	T	T	T	T	F
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	T	T	T

PCNF Principal Conjunctive Normal form

Or

Product of Sum Conical form

A formula equivalent to a given formula consisting of conjunctions of the maxterms only is called PCNF.

For the method of obtaining PCNF in each disjunction missing variable is provided in the negated and non negated form

$\Rightarrow \neg(p \vee q)$

$\neg p \wedge \neg q$

$\Rightarrow (\neg p \vee F) \wedge (\neg q \vee F)$

$(\neg p \vee (q \wedge \sim q)) \wedge (\neg q \vee (p \wedge \neg p))$

$(\neg p \vee q) \wedge (\neg p \vee \sim q) \wedge (\neg q \vee p) \wedge (\neg q \vee p)$

$(\neg p \vee q) \wedge (\neg p \vee \sim q) \wedge (\neg q \vee p)$

Q $\neg(p \rightarrow q)$ obtain PCNF

$= \neg(\neg p \vee q)$

$= (p \wedge \neg q)$

$= (p \vee F) \wedge (\neg q \vee F)$

$= (p \vee (q \wedge \neg q)) \wedge (\neg q \vee (p \wedge \neg p))$

$= (p \vee q) \wedge (p \vee \neg q) \wedge (\neg q \vee p) \wedge (\neg q \vee \neg p)$

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Principal Disjunctive Normal form (PC(N)F)

Definition of Miniterm

Miniterms of the given no of variable is a formula which consist of in which each variable or negation take only once. Each of the miniterm has truth value T for exactly one combination of truth values of the variables.

p	q	$p \wedge q$	$p \wedge \neg q$	$\neg p \wedge q$	$\neg p \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
T	F	F	F	F	T

PDNF - If we are given a statement formula than an equivalent formula consisting of disjunctions of miniterms only is known as PDNF or sum of products or canonical form of the given formula

$\Rightarrow$ $p \vee q$

$(p \wedge T) \vee (q \wedge T)$ (By identity law)

$(p \wedge (q \vee \neg q)) \vee (q \wedge (p \vee \neg p))$ (By Negation law)

$(p \wedge q) \vee (p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg p)$ (By distributive law)

$(p \wedge q) \vee (p \wedge \neg q) \vee (q \wedge \neg p)$

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Q $p \to q$ Ans: $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

Q $\neg p \vee q$ Ans: $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

Q $\neg(p \wedge q)$ Ans: $(\neg p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

Some Important laws & Identities

1. $\neg(p \wedge q) \equiv \neg p \vee \neg q$ } De Morgan's
2. $\neg(p \vee q) \equiv \neg p \wedge \neg q$ } law
3. $(p \vee q) \vee r \equiv p \vee (q \vee r)$ } Associative
4. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ } Law
5. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ } Distributive
6. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ } Law

Logical equivalences involving conditional statements

$\Rightarrow$ $(q \to \neg r) \equiv (\neg q \vee \neg r)$

1. $p \to q \equiv \neg p \vee q$
2. $p \to q \equiv \neg q \to \neg p$
3. $p \vee q \equiv \neg p \to q$
4. $\neg(p \to q) \equiv p \wedge \neg q$
5. $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
6. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
7. $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
8. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Q

$\Rightarrow$ $p \overset{\text{if and only if}}{\longleftrightarrow} q \equiv (p \to q) \wedge (q \to p)$

$\Rightarrow$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$\Rightarrow$ $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\Rightarrow$ $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

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Q Obtain PDNF

1) $P \to q$
$= \neg P \vee q$
$= (\neg P \wedge T) \vee (q \wedge T)$
$= (\neg P \wedge (q \vee \neg q)) \vee (q \wedge (P \vee \neg P))$
$= (\neg P \wedge q) \vee (\neg P \wedge \neg q) \vee (q \wedge P) \vee (q \wedge \neg P)$
$= (P \wedge q) \vee (\neg P \wedge q) \vee (\neg P \wedge \neg q)$

2) $\neg P \vee q$
$= (\neg P \wedge T) \vee (q \wedge T)$
$= (\neg P \wedge (q \vee \neg q)) \vee (q \wedge (P \vee \neg P))$
$= (\neg P \wedge q) \vee (\neg P \wedge \neg q) \vee (q \wedge P) \vee (q \wedge \neg P)$
$= (\neg P \wedge q) \vee (\neg P \wedge \neg q) \vee (q \wedge P)$

3) $\neg (P \wedge q)$
$= \neg P \vee \neg q$
$= (\neg P \wedge T) \vee (\neg q \wedge T)$
$= (\neg P \wedge (q \vee \neg q)) \vee [\neg q \wedge (P \vee \neg P)]$
$= [(\neg P \wedge q) \vee (\neg P \wedge \neg q)] \vee (\neg q \wedge P) \vee (\neg q \wedge \neg P)]$
$= (\neg P \wedge q) \vee (\neg P \wedge \neg q) \vee (P \wedge \neg q) \vee (\neg P \wedge \neg q)$

Normal forms for first order logic / Prenex Normal form

$= (Q_i x_i)\ (M)$
Prefix — formula containing no quantifiers

- We will consider Normal forms for formulas in the first order logic called Prenex Normal form.
- A Formula F in the first order logic is said to be in Prenex Normal form if and only if the formula F is in the form of

$(Q_1 x_1) \text{ ---------- } (Q_m x_m)(M)$ where every $(Q_i x_i)$, $i = 1, \ldots, m$ is either $(\forall x_i)$ or $(\exists x_i)$ and M is formula containing no quantifiers.

$(Q_1 x_1) \text{ ---- } (Q_m x_m)$ is called the Prefix
$M \to$ Matrix of formula F

Steps carried out to bring a formula of first order logic to ~~Prenex~~ Prenex Normal form

Step ① Replace $\leftrightarrow$ and $\to$ using $\wedge, \vee, \neg$
Step ② Use double negation and De Morgan's law repeatedly and the laws (1) and (2)
Step ③ Remove variable if necessary
Step ④ Use rules ③ and ④ to bring the quantifiers to the left

Q $\forall (x) (Q(x) \to (\exists x) R(x,y))$

Sol $\forall (x) (\neg Q(x) \vee (\exists x) R(x,y))$
We check the dependancy replace x with z
$\forall (x) (\neg Q(x) \vee (\exists z) R(z,y))$
$\forall (x) \exists (z) (\neg Q(x) \vee R(z,y))$

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Permutation

$P(n, r) = \frac{n!}{(n-r)!}$

Q $^nP_2 = 72$, find value of n

$^nP_2 = \frac{n!}{(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 72$

$n^2 - n = 72$

$n = 9$

$n^2 - n - 72 = 0$

$n^2 - 9n + 8n - 72 = 0$

$n(n-9) + 8(n-9) = 0$

$(n-9)(n+8) = 0$

$n = 9$

Q In how many ways can we select three students from a group of five students to stand in a line for a picture? In how many ways can we arrange all five of these students in a line for the picture?

Sol $^5P_3 = \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2!}$

$= \frac{5 \times 4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}}$

$= 60$

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Q How many different numbers lie b/w 100 and 1000 can we formed with a digits 1, 2, 3, 4 & 5 no digit being repeated

Sol $^5P_3 = \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}}$

$= 60$ ways

Q In how many of the permutation of 10 things taken 4 at a time will

a) Two things always occur.

b) Never occur

Sol a) $^8P_2 \times {}^4P_2$

$= 672$

b) $^8P_4 = 1680$

Q In how many way can 7 boys and 5 girls be seated in a row so that no two girls may sit together

Sol $7! \times {}^8P_5$

Q Find the no of permutation that can be had from the letters of the word DAUGHTER?

i) Taken all the letters together.

ii) Begining with D

iii) Begining with D and Ending with R

iv) Vowels being always together.

v) Not all vowels together

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Sol i) 8!

ii) D _____ 7! = 7!

iii) D _____ R = 6!

iv) 8 − 3 + 1 = 6! × 3!
= 720 × 6
= 4320

v) 7! − 4320
= 36000

Q How many no of 3-Digits can be formed with the digits 2, 4, 6 & 8 where digit may be repeated any no of times.

Sol $4 \times 4 \times 4 = 4^3$
= 64 ways

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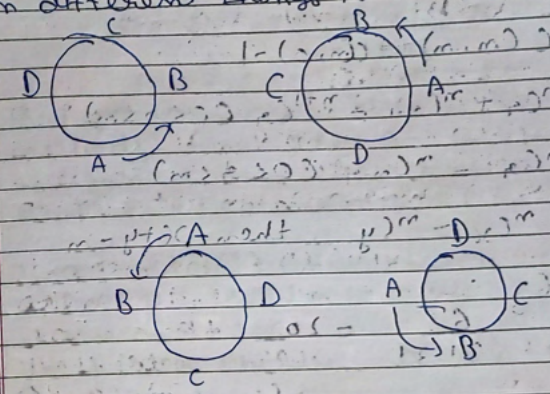
Circular Permutation
The no of circular arrangements of n different things is (n−1)!



Q In how many ways 5 Boys and 4 girls can be seated at a round table if
① There is no restriction
② all four girls sit together.
③ all four girls do not sit together
④ no two girls sit together

Sol a) (9−1)! = 8!
b) (9−4+1) = 6! ⇒ (6−1)! = 5! × 4!
c) 8! − (5! × 4!)
d) $^5P_4 \times (5-1)!$

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Combination

$C(n,r) = \frac{n!}{r!(n-r)!}$

$\Rightarrow C(n,n) = C(n,0) = 1$

$\Rightarrow {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \quad (0 \le r \le n)$

$\Rightarrow {}^nC_r = {}^nC_{n-r} \quad (0 \le r \le n)$

$\Rightarrow$ If ${}^nC_x = {}^nC_y$, then $x + y = n$

Q ${}^6C_3$

$\frac{6!}{3!\,3!} = 20$

Q In how many ways can a committee of 5 teachers and 4 students be chosen from 9 teachers and 15 students?

Sol ${}^9C_5 \times {}^{15}C_4$

Q Out of 5 men and 2 women a committee of 3 is to be formed. In how many ways can this be done so as to include

$\Rightarrow$ Exactly one woman

$\Rightarrow$ at least one woman

Sol $\Rightarrow {}^5C_2 \times {}^2C_1$

$\Rightarrow {}^2C_1 \times {}^5C_2 + {}^2C_2 \times {}^5C_1$

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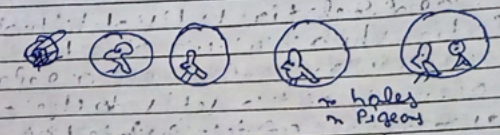
Pigeonhole Principle / Dirichlet Principle



The Pigeonhole Principle it is useful in counting methods if n pigeons are assigned to m pigeonholes then atleast one pigeon hole contains two or more pigeons ($m < n$)

Q In a group of 13 children, there must be atleast two children who were born in the same month?

Sol The thirteen children can be thought of as pigeons 12

13 children $\rightarrow$ pigeon

12 months $\rightarrow$ pigeonhole

$12 < 13$

So acc to pigeonhole principle we can conclude that out of 13 children atleast 2 were born in the same month.

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Relations

The word relation is used to indicate relationship b/w two object.
The direct way to express a relationship b/w elements of two set is to use ordered pairs made up of two related elements.
let A and B be two set as follows:

A = {Calcutta, Patna, Lucknow, Chennai}
B = {West Bengal, Bihar, Uttar Pradesh, Tamil Nadu}

R = {(Calcutta, West Bengal), (Patna, Bihar), (Lucknow, Uttar Pradesh), ...}

Domain:
The set {a ∈ A : (a, b) ∈ R for some b ∈ B} is called domain of R and is denoted by Dom(R)

Range:
The set {b ∈ B : (a, b) ∈ R for some a ∈ A} is called the range of R and denoted by Ran(R)

Q let A = {2, 3, 5}, B = {2, 4, 6, 10} A relation R from A to B is given as follows:
2R2, 2R4, 2R6, 2R10, 3R6, 5R10. Write R as set of ordered pairs

Sol R = {(2, 2), (2, 4), (2, 6), (2, 10), (3, 6), (5, 10)}

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Q let A = {2, 3, 4} and B = {3, 4, 5}
list the elements of each relation R defined below and the domain and range

a) a ∈ A is related to b ∈ B that is aRb if and only if a < b

(b) a ∈ A is related to b ∈ B that is aRb if a and b are both odd no

a) R = {(2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}
Range = {2, 3, 4}
Domain = {3, 4, 5}

b) R = {(3, 3), (3, 5)}
Range = {3}
Domain = {3, 5}

~~Q If A = {x, y, z}, B = {X, Y, Z}, C = {x, y} and D = {Y, Z}. R is a relation from A to B defined by R = {(x, X), (y, Z)}. Find R', R∪S, R∩S and R − S.~~

~~Sol~~

Q If A = {x, y, z} B = {X, Y, Z} C = {x, y} and D = {Y, Z}. R is a relation from A to B defined by R = {(x, X), (x, Y), (y, Z)} and S is relation from C to D defined by S = {(x, Y), (y, Z)}. Find R', R∪S, R∩S and R − S

Sol A × B = {(x, X), (x, Y), (x, Z), (y, X), (y, Y),

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$(y, z), (z, x), (z, y), (z, z)\}$

$R = \{(x, x), (x, y), (y, z)\}$

~~$R' = \{(x,x), (x,y), (y,z)\}$~~

$R' = \{(x, z), (y, x), (y, y), (z, x), (z, y), (z, z)\}$

$R \cup S = \{(x, x), (x, y), (y, z)\}$

$R \cap S = \{(x, y), (y, z)\}$

$R - S = (x, x)$

Type of Relations

① Inverse Relation $(a, b) \rightarrow (b, a)$

Let R be any relation from a set A to a set B. The inverse of R denoted by R^{-1} is the relation from B to A which consist of those ordered pairs which, when reversed belong to R

$R^{-1} = \{(b, a) : (a, b) \in R\}$

$xRy = yR^{-1}x$

② Identity Relation :

$I_A = \{(x, x) : x \in A\}$

A relation R in a set A is said to be identity

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relation generally denoted by I_A if

$I_A = \{(x, x) : x \in A\}$

Ex: $A = \{1, 2, 3\}$

$I_A = \{(1,1), (2,2), (3,3)\}$

Properties of Relation

① Reflexive Relation:

A relation R on a set A is reflexive if a Ra for every $a \in A$ if $(a, a) \in R$ for every $a \in A$. This simply means that each element of A is related to itself

Ex: If $R\{(1,1), (1,2), (3,3), (2,3)\}$ be a relation on $A = \{1, 2, 3\}$

② Irreflexive Relation:

A relation R on a set A is irreflexive if, for every $a \in A$, $(a, a) \notin R$, ie there is no $a \in A$ such that a Ra.

Ex: Relation $R = \{(1, 2), (1, 3), (2, 3)$

$A = \{1, 2, 3\}$

③ Non-Reflexive Relation: A relation ~~R on a~~ R on a set A is non-reflexive if R is neither reflexive nor irreflexive

Ex $R = \{(1, 2) (1, 1) (2, 3) (2, 2)\}$

$A = \{(1, 2, 3)\}$

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Symmetric Relation

A relation R on a set A is symmetric if whenever $(a,b) \in R$, then $(b,a) \in R$ i.e if $aRb \Rightarrow bRa$. This means if any one element is related to any other element, then the second element is related to the first

Ex: $A = \{1,2,3\}$
$R_1 = \{(1,1),(1,2),(1,3),(2,1),(2,1),(3,1)\}$

Properties of Relation

⇒ Asymmetric Relation: A relation R on a set A is asymmetric if whenever $(a,b) \in R$, then $(b,a) \notin R$ for $a \neq b$ ie if $aRb \Rightarrow b \not R a$. This means that the presence of (a,b) in R exclude the possibility of presence of (b,a) in R

Ex: $R = \{(1,1),(1,2),(2,3),(3,1)\}$ on $A = \{1,2,3\}$ is an asymmetric Relation

⇒ Antisymmetric Relation

A relation R on a set A is antisymmetric if for all $a,b \in A$ $(aRb$ and $bRa) \Rightarrow a = b$

For eg:
$R_1 = \{(1,1),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

$R = \{(x,y) \in N \mid x$ is a divisor of $y\}$ is (an antisymmetric relation since x divided y & y divides x.

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⇒ Transitive Relation

A relation R on a set A is transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
ie aRb and $bRc \Rightarrow aRc$

Ex $A = \{1,2,3\}$
$R_1 = \{(1,2),(2,3),(1,3)\}$

⇒ Equivalence Relation

A relation on a set A is called an equivalence relation if it is reflexive, symmetric & transitive.
ie R is an equivalence relation if it has following property

1) $(a,a) \in R$ for all $a \in A$ (reflexive)
2) $(a,b) \in R$ implies $(b,a) \in R$ (Symmetric)
3) (a,b) and $(b,c) \in R$ implies $(a,c) \in R$ (Transitive)

Closure of Relation :-

⇒ Reflexive Closure

$S = \{1,2,3,4\}$
$R = \{(1,2),(2,1),(1,1),(1,2)\}$

$R^{(r)} = R \cup \{(3,3),(4,4)\}$
$R = \{(1,2),(2,1),(1,1),(2,2),(3,3),(4,4)\}$

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⇒ Symmetric Closure

$R = \{(1,2), (4,3), (2,2), (2,1), (3,1)\}$ be a relation on $S = \{1,2,3,4\}$ find Symmetric Closure

$R^{-1} = \{(2,1), (3,4), (2,2), (1,2), (1,3)\}$

$R^{(s)} = R \cup R^{-1} = \{(1,2), (2,1), (4,3), (3,4), (3,1), (1,3)\}$

* PMI (Principle of Mathematical Induction)

Induction means the methods of ~~Infer~~ inferring a general statement from the validity of Particular Cases.
Mathematical Induction is the technique by which one can Proof mathematical statements involving Positive Integers.

Power of mathematical Induction

$$n = \frac{n(n+1)}{2}$$

From this statement we cannot conclude that the statement is true for all Positive n as one can never be sure that the statement does not fail for some Untryed Values of n

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Formal Statement of Principle Of Mathematical Induction

Let $S(n)$ be a statement that involves +ve integers $n = 1, 2, 3, \ldots$ Then $S(n)$ is true for all Positive integer n Provided that

① $S(1)$ is true ② $S(K+1)$ is true whenever $S(K)$ is true.

So, there are 3 steps of Proof Using PMI

Step1 Inductive Base: Verify that $S(1)$ is true

Step2 Inductive Hypothesis: Assume that $S(K)$ is true for an arbitrary value of K

Step3 Inductive Step: Verify that $S(K+1)$ is true on basic of the inductive hypothesis.

Q Show that $1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6}$ $(n \geq 1)$ by mathematical induction

Sol Let $S(n)$ be the given statement

① Inductive base:- For $n = 1$ we have

$$1^2 = \frac{1(1+1)(2+1)}{6}$$

$$= 1$$

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② Inductive Hypothesis: Assume that $S(k)$ is true

$$1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$$

③ Inductive Step:

$$1^2+2^2+3^2+\cdots+(k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$$= \frac{(k+1)(k+2)(2k+1)}{6}$$

$$\underbrace{1^2+2^2+3^2+\cdots+k^2}+(k+1)^2$$

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$\frac{(k+1)(k+1)(2k+3)}{6}$$

Q. Proof that the sum of cubes of n natural numbers is equal to $\left(\frac{n(n+1)}{2}\right)^2$

Sol. let $S(n)$ be the given statement

$$1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

① Inductive base: For $n=1$

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2$$

$$1 = 1$$

② Inductive hypothesis: Assume that $S(k)$ is true.

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$$1^3+2^3+3^3+\cdots+k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

③ Inductive Step:

$$1^3+2^3+3^3+\cdots+(k+1)^3 = \left(\frac{k+1(k+2)}{2}\right)^2$$

$$\underbrace{1^3+2^3+3^3+\cdots+k^3}+(k+1)^3$$

$$\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

$$\cancel{\frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4}}$$

$$\cancel{(k+1)^2\left[\frac{k^2+4k+4}{4}\right]}$$

$$\cancel{\left(\frac{k(k+1)}{2}\right)^2 = \left(\frac{(k+1)(k+2)}{2}\right)^2}$$

$$(k+1)^2\left(\frac{k^2+4(k+1)}{4}\right)$$

$$(k+1)^2\left(\frac{k^2+4k+1}{4}\right)$$

$$\frac{(k+1)^2(k+4)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2$$

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Q. Show that
$1+3+5+\cdots+(2n-1) = n^2$

Sol. $S(1)$

$1 = 1$

$S(k)$

$1+3+5+\cdots+(2k-1) = k^2$

$S(k+1)$

$1+3+5+\cdots+(2k+1) = (k+1)^2$

$1+3+5+\cdots+(2k-1)+(2k+1)$

$= k^2+(2k+1)$

$= k^2+2k+1$

$= (k+1)^2$

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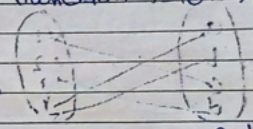
Unit - 2

FUNCTIONS

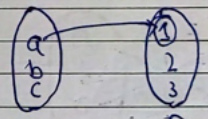
Let f denote a function from A to B, then denoted as

$f: A \to B$



"f is a function from A to B"
"f takes or maps A into B"

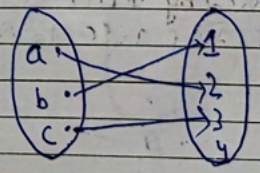
If $a \in A$ then $f(a)$ (read as "f of a") denotes unique element of B which f assign to a. This called the $\boxed{\text{Image of a Under f}}$ or $\boxed{\text{the value of f at a}}$



Domain, Co-Domain, Range

Type of functions:

(1) One to One Mapping / Injective Mapping



$A \to B$ is injective if all minterms of A are related to different members of B

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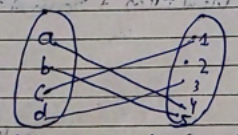
Q Determine whether function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, $f(d) = 3$ is one to one.

Sol



This function is one to one.

Q Determine whether function $f(x) = x^2$ of function is one to one.

Sol

$f(1) = (1)^2 = 1$
$f(-1) = (-1)^2 = 1$

② Onto Mapping / Surjective Mapping

A function f from A to B is called onto or surjective if and only if for every element $b \in B$, there is an element $a \in A$ with $f(a) = b$

Q let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. Is f an onto function?

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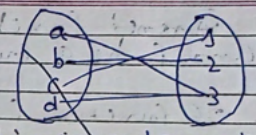
Sol



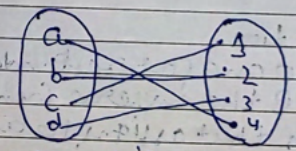
This is onto mapping

③ Bijective Mapping / One to one Correspondence

The function f is one to one correspondance or Bijection

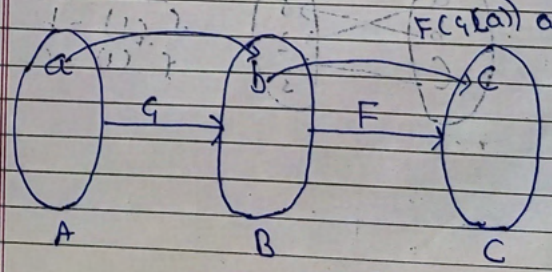
Q Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. If it a Bijection?

Sol



Yes it is a bijection

Composition of Functions



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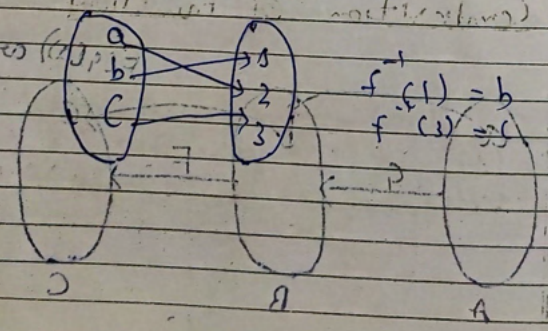
Suppose g is a ~~function~~ function from set A to B and f be a function from set B to C
The composition of function f and g is

$$f \circ (g(a)) = f(g(a))$$

Q Let g is a function from set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, $g(c) = a$ & if be a function from set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, $f(c) = 1$. Find composition of F & G

Sol $g(a) = b$ $(F \circ G)a = F(G(a)) = f(b) = 2$
$g(b) = c$ $F \circ G(b) = F(G(b)) = F(c) = 1$
$g(c) = a$ $F \circ (c) = F(G(c)) = F(a) = 3$

Inverse function
If f be one to one correspondance from set A to set B the inverse function of f is a function that assigns to an element $b \in B$ the unique element in a in A such that $f(a) = b$, $f^{-1}(b) = a$ when $f(a) = b$



$f^{-1}(1) = b$
$f^{-1}(3) = c$

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Partial order Relation

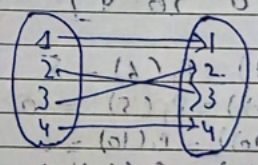
POSET
① Reflexive: $\{1, 2, 3\} \to \{(1,1), (2,2), (3,3)\}$
② Transitive: $\{a, b, c\} \to \{(a,b), (b,c), (a,c)\}$
③ Antisymmetric: $\{(1,1), (2,2), (3,3), (1,3), (2,3)\}$

$(a,b) \to (b,a)$ reversing chain
$a = b$

Q $A = \{1, 2, 3\}$
$R_1 = \{\ \}$ ✗
$R_2 = \{(1,1), (2,2), (3,3)\}$ ✗ Not antisymmetric
$R_3 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ ✗ (Not antisymmetric)
$R_5 = \{(1,1), (1,2), (1,3), (1,3)\}$ ✗

Permutation function

① Function must be Bijective (one to one, onto)



$F: \{(1,1), (2,3), (3,2), (4,4)\}$
$F: \{(1,1), (2,2), (3,3), (4,4)\}$
$4! = 4 \times 3 \times 2 \times 1$
$= 24$ ways

Representation of Permutation function

$A = \{x_1, x_2, x_3, \ldots, x_n\}$
$P = \{P(x_1)\ P(x_2)\ P(x_3) \cdots P(x_n)\}$ → Image of

$$P = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ P(x_1) & P(x_2) & P(x_3) & \cdots & P(x_n) \end{bmatrix} \to \text{Image of}$$

$$\begin{bmatrix} a & b & c & d & e \\ c & e & d & a & b \end{bmatrix}\begin{bmatrix} a & b & c & d & e \\ r & y & z & w & v \end{bmatrix} = \begin{bmatrix} a & b & c & d & e \\ c & e & d & a & b \end{bmatrix}$$

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Inverse of Permutation Function PF (P^{-1})

Q $P = \begin{bmatrix} a & b & c & d & e \\ c & e & d & a & b \end{bmatrix}$ Find inverse

i) Image row on top

ii) Rearrangement of Top Row

Sol i) $P^{-1} = \begin{bmatrix} c & e & d & a & b \\ a & b & c & d & e \end{bmatrix}$

ii) $\begin{bmatrix} a & b & c & d & e \\ d & e & a & c & b \end{bmatrix}$

Composition of PF or Product of PF

$$P_1 = \begin{bmatrix} 5 & 10 & 20 & 6 & 4 \\ 10 & 6 & 4 & 5 & 20 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 5 & 10 & 20 & 6 & 4 \\ 6 & 5 & 10 & 20 & 4 \end{bmatrix}$$

$P_1 \circ P_2 = P_1(P_2(5)) = P_1(6) = 5$

$P_1(P_2(10)) = P_1(5) = 10$

$P_1(P_2(20)) = P_1(10) = 6$

$P_1(P_2(6)) = P_1(20) = 4$

$P_1(P_2(4)) = P_1(4) = 20$

$$P_1 \circ P_2 = \begin{bmatrix} 5 & 10 & 20 & 6 & 4 \\ 5 & 10 & 6 & 4 & 20 \end{bmatrix}$$

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$P_2 \circ P_1 = P_2(P_1(5)) = 5$

$= P_2(P_1(10)) = 20$

$= P_2(P_1(20)) = 4$

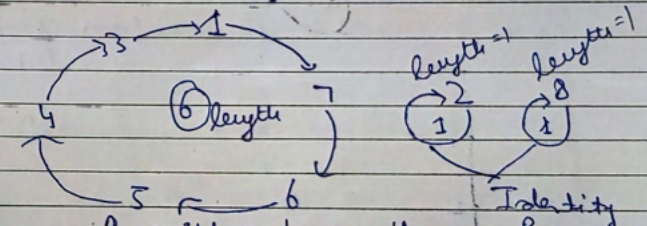
$= P_2(P_1(6)) = 6$ 5 10 20 6 4

$= P_2(P_1(4)) = 10$ 5 10 4 6 10

$$P_2 \circ P_1 \begin{bmatrix} 5 & 10 & 20 & 6 & 4 \\ 5 & 20 & 4 & 6 & 10 \end{bmatrix}$$

Cyclic Permutation

$$P = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 2 & 1 & 3 & 4 & 5 & 6 & 8 \end{bmatrix}$$



3 possible cycles are there

Identity Permutation

Hasse diagram

1. Create Vertex for element [U-Pward direction]
2. If $a R b$ then draw edge from a to b
3. Remove self loop if any.

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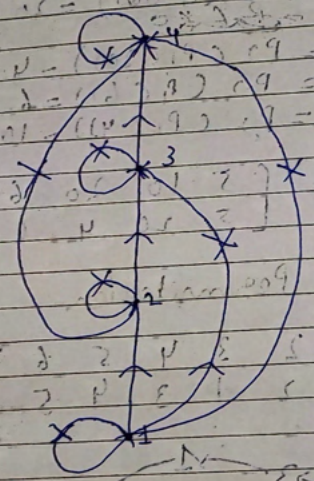
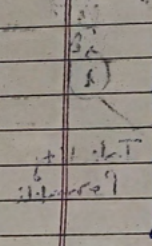
Q Construct Hasse diagram for $(\{1,2,3,4\} \leq)$





Q Draw Hasse diagram representing partial ordering $\{(a,b) / a \text{ divided } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$

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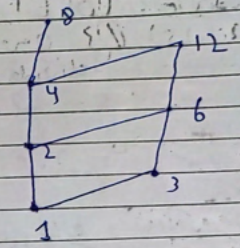
Sol

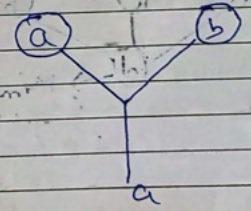


Eg

Maximal & Minimal elements

Maximal: It is an element that is not related to any other elements. An element of an POSET is called maximal if it is not less than any element of POSET

Top elements of Hasse diagram



Minimal: An element of POSET is called minimal if it is not greater than any element of POSET. Minimal is equal bottom element of Hasse diagram

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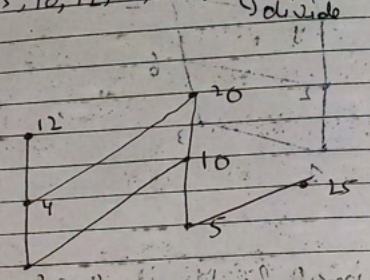
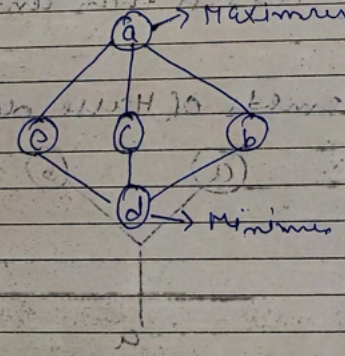
Q Which elements of POSET $\{2, 4, 5, 10, 12, 20, 25\}$, divide are maximal

Sol



Maximal - 12, 20, 25

Minimal - 4, 5



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Recurrence Relation

$S = \{3, 3^2, 3^3 \ldots 3^n\}$

$S_n = 3^n$

$S_n = 3 \cdot S_{n-1}$, $n \geq 2$, $S = 3$

Q From the recurrence relation
$S = \{5, 8, 11, 14, 17, \ldots\}$

Sol $S_1 = 5$

$S_n = (S_{n-1} + 3)$, $n \geq 2$

Q Fibonacci Series $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$

Sol $S = (S_{n-1}) + (S_{n-2})$

Q Find first 4 terms of each of the follow

1) $a_k = 2a_{k-1} + k$ for all integers $k \geq 2$, $a_1 = 1$

Sol For $k = 2$

$a_2 = 2a_{2-1} + 2$

$a_2 = 2(a_1) + 2$

$= 2 + 2$

$a_2 = 4$

$a_3 = 2(4) + 3 = 11$

$a_4 = 2(11) + 4 = 26$

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Q $a_K = a_{K-1} + 3a_{K-2}$ for all integers $K \geq 2$, $a_0 = 1$, $a_1 = 2$

Sol $K = 2$

$a_2 = a_{(2-1)} + 3a_{(2-2)}$

$= a_1 + 3a_0$

$= 2 + 3$

$a_2 = 5$

$a_3 = 11$

Order Of Recurrence

The order of recurrence relation is the difference b/w greatest suffix and last suffix.

eg: $a_n = a_{n-1} + a_{n-2}$

$n - (n-2) = 2$

Linear Recurrence Relation

A recurrence relation having only the first power of the sequence $\{\langle a_n \rangle\}$ and not having any product term is said to be linear.

Ex: $5a_n - 3a_{n-1} + a_{n-2} = 0$

Linear Recurrence Relation with Constant Coefficient

A recurrence relation of the form

$C_0 a_n + C_1 a_{n-1} + \cdots C_n (a) = f(n)$

is called linear recurrence relation with constant coefficient.

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A recurrence relation

$C_0 a_n + C_1 a_{n-1} + \cdots + C_K a_{n-K} = f(n)$

is known as K^{th} order recurrence relation provided $C_0 \neq 0$, $C_K \neq 0$

Homogenous recurrence relation with Constant Coefficients

Suppose second order Homogenous linear Recurrence relation is $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$

The Characteristic eq (Auxillary eq) is

$C_0 m^2 + C_1 m + C_2 = 0$

Case 1 If the Auxillary Equation are real and unequal $m_1 \neq m_2$

General Sol: $a_n = C_1 m_1^n + C_2 m_2^n$

Case 2 If the roots of A.E are real and equal

$m_1 = m_2 = m$

General sol is $a_n = (C_1 + n C_2) m^n$

Case 3 If the roots of A.E are in complex no

$m = \alpha \pm i\beta$

General sol is $a_n = C_1 \cos n\theta + C_2 \sin \theta) R^n$

where $R = \sqrt{\alpha^2 + \beta^2}$ & $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

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Q Solve the recurrence relation $a_r + 5a_{r-1} + 6a_{r-2} = 0$

Sol $a_r + 5a_{r-1} + 6a_{r-2} = 0$ Order = 2

$m^2 + 5m + 6 = 0$

~~$m^2 + 6m - m + 6 = 0$~~

~~$m(m+6) - 1(m+6) = 0$~~

$m^2 + 2m + 3m + 6$

$m(m+2) + 3(m+2) = 0$

$(m+2)(m+3) = 0$

$m = -2, -3$ Both are distinct

General Sol

$a_r = c_1 m_1^r + c_2 m_2^r$

$a_r = c_1 (-2)^r + c_2 (-3)^r$

Q Solve the recurrence relation

$a_r - 7a_{r-1} + 10a_{r-2} = 0$

Given that $a_0 = 0$, $a_1 = 3$

Sol $m^2 - 7m + 10 = 0$ Order = 2

$m^2 - 5m - 2m + 10 = 0$

$m(m-5) - 2(m-5) = 0$

$(m-5)(m-2) = 0$

$\boxed{m = 5, 2}$

When $a_0 = 0$

$a_r = c_1 m_1^r + c_2 m_2^r$

$a_0 = c_1 (5)^0 + c_2 (2)^0$

$a_0 = c_1 + c_2$

$0 = c_1 + c_2$ — ①

~~$a_1 = c_1 (5)^1 + c_2 (2)^1$~~

~~$a_1 = 5c_1 + 2c_2$~~

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~~$a_1 = 3$~~

~~$3 = c_1 (2)^3 + c_2 (5)^3$~~

~~$3 = 8c_1 + 125c_2$~~

~~$8c_1 + 125c_2 = 3$~~

~~$c_1 + c_2 = 0$~~

~~$c_1 = -c_2$~~

~~$-8c_2 + 125c_2 = 3$~~

$a_1 = 3$

$3 = c_1 (2)^1 + c_2 (5)^1$

$3 = 2c_1 + 5c_2$

$$\boxed{\begin{aligned} 2c_1 + 5c_2 &= 3 \\ c_1 + c_2 &= 0 \end{aligned}}$$

$c_1 = -c_2$

$2c_1 - 5c_1 = 3$

$-3c_1 = 3$

$$\boxed{\begin{aligned} c_1 &= -1 \\ c_2 &= 1 \end{aligned}}$$

$a_r = -(2)^r + (5)^r$

$= 5^r - 2^r$

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Generating function

$\{A\} \to a_0, a_1, a_2$ — the Generating Function $G(A, z)$ of a sequence A is infinite series

$$G(A, z) = \sum_{n=0}^{\infty} a_n \cdot z^n = a_0 + a_1 z + a_2 z^2 + \cdots \infty$$

Generating function:

① $a_n = C, n \geq 0$

$$G(s, z) = \sum_{n=0}^{\infty} a_n \cdot z^n$$

$$= \sum_{n=0}^{\infty} C \cdot z^n$$

$$= C \sum_{n=0}^{\infty} z^n$$

$$= C \cdot (1 + z + z^2 + \cdots \infty) \text{ With } a = 1, r = z, S_\infty = \frac{a}{1-r}$$

$$= C \cdot \left(\frac{1}{1-z}\right)$$

$$= \frac{C}{1-z}$$

② $a_n = b^n, n \geq 0$

$$G(s, z) = \sum_{n=0}^{\infty} b^n \cdot z^n = \sum_{n=0}^{\infty} (bz)^n$$

$$= 1 + bz + (bz)^2 + \cdots \infty \text{ Plaitoo With } a = 1, r = bz$$

$$GF = \frac{1}{1-bz}$$

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③ $a_n = n, n \geq 0$

$$G(s, z) = \sum_{n=0}^{\infty} n \cdot z^n$$

$$= 0 + z + 2z^2 + 3z^3$$

$$= z(1 + 2z + 3z^2 + \cdots)$$

$$= z(1-z)^{-2} \text{ (By binomial theory)}$$

④ $a_n = C \cdot b^n$

$$GF = \frac{C}{1-bz}$$

General Sequence	GF	
C	$\frac{C}{1-x}$	$(-1)^k = \frac{1}{1+x}$
k+1	$\frac{1}{(1-x)^2}$	$k = \frac{x}{(1-x)^2}$
k(k+1)	$\frac{2x}{(1-x)^3}$	$(-a)^k = \frac{1}{1+ax}$
(k+1)(k+2)	$\frac{2}{(1-x)^3}$	$k^2 = \frac{x(1+x)}{(1-x)^3}$
a^k	$\frac{1}{1-ax}$	$ka^k = \frac{ax}{(1-ax)^2}$

Sol of Recurrence Relation by Generating f(x)

① In given equation, multiply by x^r (If variable is x)

② Take Summation from $r = 1$ to ∞ if one initial condition is given take summation from $r = 2$ to ∞ if two initial conditions are given.

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③ Write each summation in terms of $G(x)$ or closed form

④ Put values of each summation & find $G(x)$

⑤ Find Partial fraction of $G(x)$

$$\boxed{\sum_{r=0}^{\infty} a_r \cdot x^r}$$

⑥ Write as

$$G(x) = a_0 + a_1(x) + a_2 x^2 + a_3 x^3 + \cdots$$

Q Solve

$$a_r - 2a_{r-1} - 3a_{r-2} = 0$$

$a_0 = 3$, $a_1 = -1$ by Using Generating Functions

Sol ① $a_r x^r - 2a_{r-1} \cdot x^r - 3a_{r-2} \cdot x^r = 0$

② $\sum_{r=2}^{\infty} a_r x^r - \sum_{r=2}^{\infty} 2a_{r-1} x^r - \sum_{r=2}^{\infty} 3a_{r-2} x^r = 0$ — ①

③ $\sum_{r=2}^{\infty} a_r \cdot x^r$

$$GF = a_0 + a_1 x + a_2 x^2 + \cdots \infty$$

$$GF = a_0 - a_1 x - a_2 x^2 + \cdots \infty$$

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$$\sum_{r=2}^{\infty} a_{r-1} x^r = a_1 x^2 + a_2 x^3 + a_3 x^4$$
$$= x[a_1 x + a_2 x^2 + a_3 x^3 + \cdots]$$
$$= x[GF - a_0]$$

$$\sum_{r=2}^{\infty} a_{r-2} x^r = a_0 x^2 + a_1 x^3 + a_2 x^4$$
$$= x^2[a_0 + a_1 x + a_2 x^2 + \cdots]$$
$$= GF \cdot x^2$$

Put value in eq ①

$$G(x) = (G(x) - a_0 - a_1 x) - 2x(G(x) - a_0) - 3x^2(G(x))$$

$$G(x)[1 - 2x - 3x^2] = a_0 + a_1 x - 2x a_0$$
$$= 3 + x - 6x$$
$$= 3 - 5x$$

$$G(x) = \frac{3 - 5x}{1 - 2x - 3x^2}$$

$$= \frac{5x - 3}{-3x^2 - 2x + 1} = \frac{5x - 3}{3x^2 + 2x - 1}$$

$$= \frac{5x - 3}{(3x - 1)(x + 1)} = \frac{A}{3x - 1} + B$$

$3x^2 + 2x - 1$
$3x^2 + 3x - x - 1$
$3x(x+1) - 1(x+1)$
$x = \frac{1}{3}$

$$= \frac{A}{x+1} + \frac{B}{3x-1}$$
$$= \frac{3x - 1A + x + 1B}{(x+1)(3x-1)}$$

$5 \times \frac{1}{3} - 3$
$\frac{5}{3} - 3$
$\frac{5-9}{3} = -\frac{4}{3}$

$$5x - 3 = (3x - 1)A + (x + 1)B$$

Put $x = \frac{1}{3}$

$$5x - 3 = \left(3 \times \frac{1}{3} - 1\right)A + \left(\frac{1}{3} + 1\right)B$$

$$\frac{5}{3} - 3 = \frac{4}{3} B$$

$B = -1$

When $x = -1$
$5(-1) - 3$
$-5 - 3$
-8
$3(-1) - 1A + B$

$-A + B = -\frac{4}{3}$

$-4A + B = -1$

$-5A = -\frac{7}{3}$

$A = \frac{7}{5}$

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Put $x = -1$

$5(-1) - 3 = (3(-1) - 1)A + (-1+1)B$

$-5 - 3 = (-3-1)A$

$\cancel{-}8 = \cancel{-}4A$

$A = 2$

$\frac{2}{x+1} - \frac{1}{3x+1}$

$\frac{2}{1+x} - \frac{1}{1+3x}$

$2(-1)^k - (-3)^k$

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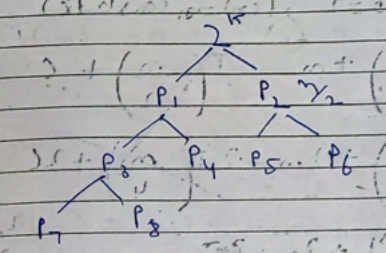
Divide Conquer and Combine



Height of tree = k

$T(n) = T(n/2) + T(n/2) + f(n)$

$T(n) = 2T(n/2) + f(n)$

In general

$$\boxed{T(n) = aT(n/b) + f(n)}$$

Recurrence relation for divide and conquer algorithm

① $T(n) = 2T(n/2) + n$

$T(n) = (4T(n/4) + n) + n$

$8T(n/8) + 3n$

2^k

$nT\left(\frac{n}{n}\right) + kn$

$T(n) = n(T(1)) + k(n)$

Const

$T(n) = n\log n$

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Q $T(n) = a\,T(n/b) + f(n)$

$a = 1, b = 2$, $f(n) =$ constant (c)

Sol $\cancel{2T\left(\frac{n}{2}\right) + n}$ $\cancel{T\left(\frac{n}{2}\right) + c}$

$= \cancel{4T\left(\frac{n}{4}\right) + 2n}$ $\cancel{2T\left(\frac{n}{4}\right) + 2c}$

$= \cancel{8T\left(\frac{n}{8}\right) + 3n}$

$T(n) = T(n/2) + c$

$= T(n/4) + 2c$

$= T(n/8) + 3c$

$+ kc$

$T\left(\frac{n}{2^k}\right) + kc$

$T(1) + (\log n)$

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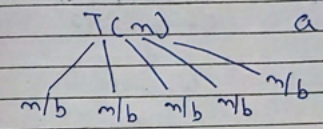
Master's Theorem

$T(n) = a\,T(n/b) + f(n)$



$a = 5$

Size $= b$

$f(n)$ = Cost of dividing the subproblem and combining the result of all sub problem.

$a \geq 1$, $a \rightarrow$ Natural no

$k > 0$

$\Theta(n) = \Theta(n^k \log^p n)$

Case-1 $a > b^k$, $T(n) = \Theta(n^{\log_b a})$

Case-2 $a = b^k$

a) $P > -1$ $T(n) = \Theta(n^{\log_b a} \log_n^{P+1})$

b) $P = -1$ $T(n) = \Theta(n^{\log_b a} \log(\log n))$

c) $P < -1$ $T(n) = \Theta(n^{\log_b a})$

Case-3 If $a < b^k$

① $P \geq 0$ $T(n) = \Theta(n^k \log^p n)$

② $P < 0$ $T(n) = \Theta(n^k)$

Q $T(n) = 2T(n/2) + \frac{n}{\log n}$

Sol $f(n) = \frac{n}{\log n} = n^k \log^p n$ $\quad k=1, p=-1$

$a = 2$
$b = 2$
$k = 1$
$p = -1$

$b^k = 2^1 = 2$
$a = b^k$

~~Case (2) iii) $T(n) = \Theta(n \log^a b$~~

Case 2 $T(n) = \Theta(n^{\log_b a} \log(\log n))$

$= \Theta(n^{\log_2 2} \log(\log n))$

$= \Theta(n \log(\log n))$

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Unit-3

Algebric Structure
↓
Semi group
↓
Monoid
↓
group
↓
Albelian group

$N = \{1, 2, 3, 4 \ldots \infty\}$ Decimal, -ve no, 0 are not N
$Z = \{$set of all integers$\}$ -1.43, 3.14 are not integers
$Q = \{$set of all rational no$\}$
$R = \{$set of all Real no$\}$
$C = \{$set of all complex no$\}$

$\boxed{Q^* = Q - 0}$

Algebric Structure

A non empty set S is called Algebraic structure w.r.t binary operation if $(a*b) \in S$ where $(a, b) \in S$
Binary operation
$*$ is closure operation on S

① $(N, +)$ $5 + 7 = 12$
Natural no
$+ \rightarrow$ closure on N
Natural no = 0 not a natural no

② $(N, \cdot) \rightarrow 2 \cdot 3 = 6$

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③ $(N, -)$
$1-1=0$ No

④ $(N, \div)$
$3 \div 2 = 1.5$ No

⑤ $(Z, +, \cdot, -, \div)$
$4+2=6$, $4 \cdot 2 = 8$, $4-2=2$, $3 \div 2 =$ No

⑥ $(R, +, \cdot, -, \div)$
Yes Yes Yes Yes

⑦ $(Q \div 0)$
$3/0 = \infty$ No

⑧ $(Q^*, \div)$
Yes

Semigroup: An algebric structure $(S, *)$ is called semigroup if it follows associative property

$(a*b)*c = a*(b*c) \ \forall a, b, c \in S$

$\Rightarrow (N, +) =$ Yes

$(1+2)+3 = 1+(2+3)$
$3+3 = 1+5$
$6 = 6$

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$\Rightarrow (Z, *) =$ Yes

$\Rightarrow (N, -) =$ No

Monoid: A Semigroup $(S, *)$ is called monoid if there exists an element $e \in S$ such that

$(a*e) = (e*a) = a \ \forall a \in S$

element e is called identity element of S w.r.t $*$

$\Rightarrow (N, \cdot)$
$3 \cdot e = 3$
$\boxed{e = 1}$

$\Rightarrow (Z, +)$
$3 + e = 4$
$e = 4-3$
$\boxed{e = 1}$

$3 + e = 3$
$\boxed{e = 0}$

Group: A monoid $(S, *)$ with identity element 'e' is called a group if to each element $b \in S$ such that

$(a*b) = (b*a) = e$

then 'b' is called inverse of a denoted as a^{-1}

$a^{-1} = b$ and $b^{-1} = a$

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$5 + __ = e$

$e \rightarrow 5 + e = 5$
$5 + 0 = 5$
$e = 0$

$5 + (-5) = 0$

AS → Closure
SG → Associativity
Monoid → Identity

Group → Inverse

$5 \cdot __ = e$
$5 \cdot e = 5$
$e = 1$

$5 \cdot e = 1$
$e = \frac{1}{5}$

Homomorphism of Groups: If (G, O) and (G', O') are two groups then a mapping function of such that $G \rightarrow G$ is called homomorphism if

$f(a \, O \, b) = f(a) \, O' \, f(b) \;\; \forall \; a, b \in G$

$(G, O) \quad (G', O')$

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Eg: $(G, +)$ $G \rightarrow$ set of real no
$(G', *)$ $G' \rightarrow$ set of non-zero numbers
$f(x) = 2^x, x \in G$, check if function is homomorphism

Sol let $(a, b) \in G$
$f(a) = 2^a, f(b) = 2^b$

$f(a+b) = 2^{a+b} = 2^a \cdot 2^b$

$f(a+b) = f(a) \cdot f(b)$

Isomorphism in Groups

If (G, O) & (G', O') are two groups, then a one-one onto bijective mapping such that $f: G \rightarrow G$ is called isomorphism from G to G' is f is homomorphism
also
$G \xrightarrow{f} G'$ is isomorphism

G' elements mapped to only one element if G & each G element

$f(a \cdot b) = f(a) \, O' \, f(b) \;\; \forall \; a, b \in G$

Ex $(G, +) \rightarrow G$ is a set of real no
$(G', +) \rightarrow G'$ is set of non-zero
$f(x) = 2^x \;\; \forall \; x \in G$
show that f is isomorphism
Prove that f is one-one, onto, homomorphism

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Col i) for one-one

Let $x_1, x_2 \in G$ i.e $f(x_1) = 2^{x_1}$

$f(x_2) = 2^{x_2}$

$f(x_1) = f(x_2)$

$2^{x_1} = 2^{x_2}$

$x_1 = x_2$

$f: G \to G'$ is one-one

ii) for onto

$f(x) = 2^x$

Let $f(x) = y$

$y = 2^x$

$\log_2 y = \log_2 2^x$

$\log_2 y = x$

For onto - y do not having value whose G' value is not available

An y non zero +ve value will always give real no.

iii) f is homomorphism

Let $a, b \in G$

i.e $f(a) = 2^a$ and $f(b) = 2^b$

and $f(a+b) = 2^{a+b}$

$= 2^a \cdot 2^b$

$f(a+b) = f(a) \times f(b)$

$\therefore$ f is isomorphism.

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Abelian Group: A group $(G, *)$ is said to be abelian if $(a*b) = (b*a)$ $\forall a, b \in G$

① $(Z, +)$

$10+7 = 7+10$

$-10+7 = 7+(-10)$

$-3 = -3$

② $(N, \cdot)$

It is not a abelian group because it is not a group

③ $(R^*, \cdot)$

It is an abelian group

Q Show that the set $\{1, 2, 3, 4, 5\}$ is not a group under addition & multiplication Modulo 6. Let $G = \{1, 2, 3, 4, 5\}$

$+_6$	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

6 is having

$\therefore$ G: Group is $(G, +_6)$ Not closed

$6 \overline{)7} \; 1$

$*_6$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

$6 \overline{)2} \; 6$

0

2

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Subgroup

A subset H of G is called a subgroup of G if H also forms a group under the operation *. Denoted by $H \subseteq G$ read as "H is a subgroup of G".

Basic properties of Subgroup:

① A subset H of the group G if and only if it is non empty and closed under products and inverses

⇒ $a \in H$ and $b \in H$

⇒ $a \in H$, then $a^{-1} \in H$

⇒ $a \in H$, $b \in H$ and $ab^{-1} \in H$

② The identity of subgroup is the identity of the group.

③ The inverse of an element in a subgroup is the inverse of an element in the group

④ The intersection of subgroups A and B is again a subgroup.

⑤ The union of subgroups A and B may be or be a subgroup.

Q let $G = \{1, -1, i, -i\}$ is a multiplicative group and $H = \{1, -1\}$ where $H \subseteq G$ then show that H is a subgroup.

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Sol

X	1	-1
1	1	-1
-1	-1	1

① Closure
② Associative
③ Identity element
④ Inverse

⇒ Closure property

In which all elements which follows closure property $\{1, -1\}$

⇒ Associative property

⇒ Identity element

1 It is an identity element

$1 \times 1 = 1$

⇒ Inverse element

$-1^{-1} = -1$

So inverse element is present

Cosets

let H be a subgroup of a group G where composition have been denoted multiplicative

let $a \in G$

$Ha = \{ha : h \in G\} \rightarrow$ Right coset

$aH = \{ah : h \in G\} \rightarrow$ left coset

Q① Multiplicative group $G = \{1, -1, i, -i\}$

$H = \{1, -1\}$

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Right Coset	Left Coset
$H\cdot 1 = \{1,-1\}$	$1\cdot H = \{1,-1\}$
$H\cdot -1 = \{-1,1\}$	$-1\cdot H = \{-1,1\}$
$H\cdot i = \{i,-i\}$	$i\cdot H = \{i,-i\}$
$H\cdot -i = \{-i,i\}$	$-i\cdot H = \{-i,i\}$

Properties of Coset

① $H\cdot e = e\cdot H = H$
if e is the identity element

② $Hh = H \iff h \in H$

$H = \{1,-1\}$
$H\cdot 1 = \{1,-1\}$

③ If a and b are element of Group G

$Ha \cap Hb = \phi$ —— or $Ha = Hb$

$H\cdot 1 = \{1,-1\}$ $\quad$ $H\cdot 1 = H = \{1,-1\}$
$H\cdot i = \{i,-i\}$ $\quad$ $H\cdot -1 = H = \{-1,1\}$
$H\cdot 1 \cap H\cdot i = \phi$

④ $a \in Ha$ {any element lies in the right coset}

$H\cdot 1 = \{1,-1\} = H$
$H\cdot -i = \{-i,i\} = H$

⑤ Order of the right coset is equal to order of subgroup H Equal to order of left coset.

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$H = \{1,-1\}$
$H\cdot 1 = \{1,-1\} = H$
$H\cdot i = \{i,-i\} = H$

⑥ In general $Ha \neq aH$

⑦ $Ha = aH$, G is abelian group

⊛ If the combination in the Group G has been denoted additively, then

Right Coset $= H + a = \{h + a : h \in H\}$
left Coset $= a + H = \{a + h : h \in H\}$

Ex $(Z,+)$: addition group of integers

$H = (3Z, +)$

$Z = \{-3,-2,-1,0,1,2,3,\ldots\}$
$H = \{-9,-6,-3,0,3,6,9,\ldots\}$

$H + 0 = \{-9,-6,-3,0,3,6,9,\ldots\}$
$H + 1 = \{-8,-5,-2,1,4,7,10,\ldots\}$
$H + 2 = \{-7,-4,-1,2,5,8,11,\ldots\}$
$H + 3 = \{-6,-3,0,3,6,9,\ldots\}$
$H + 5 = \{-4,-1,2,5,8,11,15\ldots\}$

$H = H + 3$
$H + 2 = H + 5$
$H + 1$
$(H) \cup (H+2) \cup (H+1) = Z$

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Lagrange's Theorem

Statement: The order of each subgroup of a finite group is divisor of the order of group.

$O(H) \mid O(G)$

$O(G) \rightarrow$ order of group
$O(H) \rightarrow$ order of subgroup

let $O(G) = n \rightarrow$ elements of group
let $O(H) = m \rightarrow$ elements of subgroup

$\boxed{m/n}$ ie $n = Km$
↳ it means m divides n

$2/4 \Rightarrow 4 = 2$
↳ K

let $H = \{h_1, h_2, h_3, \ldots h_m\}$

Coset $a_1 \in G$
$Ha_1 = \{h_1a_1, h_2a_1, h_3a_1 \ldots h_ma_1\}$
$a_2 \in G$
$Ha_2 = \{h_1a_2, \ldots h_ma_2\}$
n coset can be formed
let K = distinct coset

$Ha_3 = \{h_1a_3 \ldots h_ma_3\}$
⋮
$Ha_K = \{h_1a_K \ldots h_m(a_K)\}$

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$G = H \cup Ha_1 \cup Ha_2 \ldots \cup Ha_K$
$= $ no of elements in H + ------ + no of elements in Ha_K

$m + m + m + \ldots$ K times

$O(G) = K \cdot m$
$n = Km$
$m/n \rightarrow$ m divides n
Hence Proved

Permutation and Permutation group

let P be a finite set having n distinct element then a one-one, onto mapping from P to itself is called a Permutation of degree n

Ex: $P = \{1, 2, 3\}$

$f: P \rightarrow P \Rightarrow$ Permutation

$\sigma = (P_1) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$3! = 6 \rightarrow$ Total no of Permutation

$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

It is easy to verify that the collection of all permutation of a known finite set a forms a group under the operation composition. This group is called a

Permutation group. It is denoted by S_n

$f : A \longrightarrow A, \quad A \neq \phi$

$S_n = \{ f_1 : f_1 : A \to A$ is a Permutation$\}$

(S_n, o) → Operation.
↳ Permutation group

Identity Permutation

If each element of a Permutation is replaced by itself: Then it is called the identity Permutation and it is denoted by symbol I. For eg

$$I = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$$

Equality of Permutation

let f and g be two Permutation on a set X. Then f = g if and only if $f(x) = g(x)$ for all x in X

Eg: let f and g be given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad g = \begin{pmatrix} 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$f(1) = 2 = g(1) \qquad f(3) = 4 = g(3)$

$f(2) = 3 = g(2) \qquad f(4) = 1 = g(4)$

Product of Permutation

$$f = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{pmatrix} \quad g = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ c_1 & c_2 & c_3 & \cdots & c_n \end{pmatrix}$$

$$fog = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ c_1 & c_2 & c_3 & \cdots & c_n \end{pmatrix}$$

Q Find the Product of two Permutation and Show that it is not Commutative

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

Sol $fg = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$gf = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

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Q $U = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 7 & 5 & 8 & 6 \end{pmatrix}$

$V = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 5 & 8 & 7 & 6 \end{pmatrix}$

find: UV, VU

$UV = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 7 & 5 & 8 & 6 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 & 8 \\ 5 & 8 & 7 & 6 \end{pmatrix}$

$= \begin{pmatrix} 5 & 6 & 7 & 8 \\ 7 & 5 & 8 & 6 \end{pmatrix} \begin{pmatrix} 7 & 5 & 8 & 6 \\ 7 & 5 & 6 & 8 \end{pmatrix}$

$= \begin{pmatrix} 5 & 6 & 7 & 8 \\ 7 & 5 & 6 & 8 \end{pmatrix}$

$VU = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 5 & 8 & 7 & 6 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 & 8 \\ 7 & 5 & 8 & 6 \end{pmatrix}$

$= \begin{pmatrix} 5 & 6 & 7 & 8 \\ 5 & 8 & 7 & 6 \end{pmatrix} \begin{pmatrix} 5 & 8 & 7 & 6 \\ 7 & 6 & 8 & 5 \end{pmatrix}$

$= \begin{pmatrix} 5 & 6 & 7 & 8 \\ 7 & 6 & 8 & 5 \end{pmatrix}$

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Normal Subgroup

Let $G(x)$ be a group and H is a subgroup under the same binary operation $*$, then for any $a \in G$, if

$aH = \{a * h \mid h \in H\}$ $H \Delta G$ $\in$ Representation

$Ha = \{h * a \mid h \in H\}$

then $aH = Ha$ {Left Coset = Right Coset}

that subgroup H is called as Normal Subgroup of G

For abelian group $aH = Ha$

H = Normal subgroup $\forall a \in G$.

Improper / Trivial Normal subgroup -

$\{e\}$ $ea = ae \ \forall a \in G$

$\{G\}$ $Ga = aG \ \forall a \in G$

Each group has two normal subgroups - $\{e\}$ and $\{G\}$

Other than $\{e\}$ and $\{G\}$ normal subgroup are called Proper Normal subgroup.

Q Let H be a subgroup of G, then H is a normal subgroup if $aha^{-1} \in H \ \forall a \in G, h \in H$

Sol Let H is a normal subgroup ie $aH = Ha$

We have to prove that $aha^{-1} \in H$

Let $x \in aH$, $x \in Ha$

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$x = a * h$ — ① $x = h_1 * a$ — ②

from ① and ②

$a * h = h_1 * a$

Multiply a^{-1} both on LHS & RHS

$(a*h)\,a^{-1} = (h_1 * a) * a^{-1}$

$= h_1 * (a * a^{-1})$

$= h_1 * e$

$(a*h) * a^{-1} = h_1$ & $h_1 \in H$

$(a*h) * a^{-1} \in H$

or

$aha^{-1} \in H \;\forall\; h \in H,\; a \in G$

let $aha^{-1} \in H$ and we have to prove that H is a normal subgroup $aH = Ha$

let $x = aH$ — ①

$x = a * h = ah$

Multiply a^{-1} on both sides

$xa^{-1} = aha^{-1}$, here $aha^{-1} \in H$

So $xa^{-1} \in H$

Multiply a on both sides

$(xa^{-1})\,a \in Ha$

$x(a^{-1}a) \in Ha$

$x e \in Ha$ — ②

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$x \in Ha$ — ②

From ① & ② $aH \subseteq Ha$

Hence $aH = Ha$

Factor / Quotient group

A group whose element are the coset of a normal group of a given group is called as Quotient or a factor group.

If G is a group and N is a Normal subgroup of G, then the set of all cosets of N in G forms a group. This group is called as quotient group or factor group of G relative to N.

It is denoted by G/N.

$G/N = \{Na \mid a \in G\}$ or

$\{aN \mid a \in G\}$

as $aN = Na$

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Cayley's theorem

Every group is isomorphic to a group of Permutations.

V	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\lambda_e = \begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}$ $\theta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)$

$\lambda_a = \begin{pmatrix} e & a & b & c \\ e & e & c & b \end{pmatrix}$ $\theta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34)$

$\lambda_b = \begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix}$ $\theta_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24)$

$\lambda_c = \begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix}$ $\theta_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$

Hence G is isomorphic to the Subgroup of S_4

$\{1, (12)(34), (13)(24), (14)(23)\} =$

$\{(12)(34), (13)(24)$

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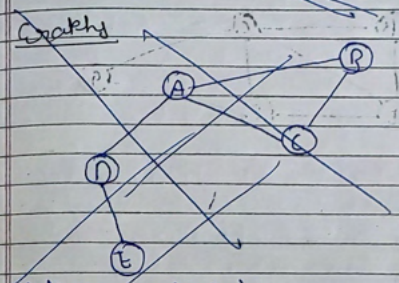
Unit 4

Graphs



1. Adjacency Matrix
2. Adjacency list

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Algorithm of BFS

Step 1 ⇒ Define a queue of size of total no of vertices in the graph.

Step 2 ⇒ Select any vertex as starting point for traversal visit that vertex and insert it into the queue.

Step 3 ⇒ Visit all the known visited adjacent vertices of the vertex which is at the front of the queue and insert them into the queue.

Step 4 ⇒ When there is no more vertex to be visited from the vertex which is at the front of the queue then delete that vertex.

Step 5 ⇒ Repeat step 3 & 4 until queue becomes empty.

Step 6 ⇒ When queue becomes empty then produce final spanning tree by removing unused edges from the graph.

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Unit-4

Graphs

A graph is a data structure that consist of the following two components:

① A finite set of ~~variables~~ vertices is also called as nodes.

② A finite set of ordered pair of the form (U, V) called as edge. The pair is ordered because (U, V) is not the same as (V, U) in case of a directed graph (di-graph).

(U, V) → indicates that there is an edge from vertex U to vertex V.

The edge may contain weight/value/cost.

Use of Graphs:

⇒ Represent networks

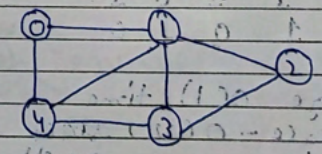
⇒ Include path in a city or telephone network.

⇒ Used in social networks like linkedIn, facebook.

FB → Person as node

→ Contain info like person id, name, gender etc

Ex:



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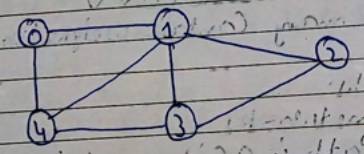
Representation of graph:

⇒ Adjacency Matrix
⇒ Adjacency list

★ Adjacency Matrix: Adjacency Matrix is a 2D array of size V×V where V is the no of vertices in a graph. Let the 2D array be adj[][], a slot adj[i][j]=1 indicates that there is an edge from vertex i to vertex j.

- Adjacency matrix for undirected graph is always symmetric
- If adj[i][j] = w, then there is an edge from vertex i to vertex j with weight w.



	0	1	2	3	4
0	0	1	0	0	1
1	1	0	1	1	1
2	0	1	0	1	0
3	0	1	1	0	1
4	1	1	0	1	0

Removing edge O(1) time
Space - $O(V^2)$ space
adding a vertex - $O(V^2)$ time

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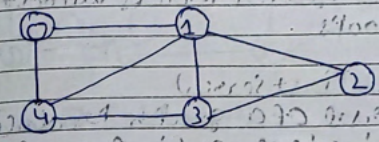
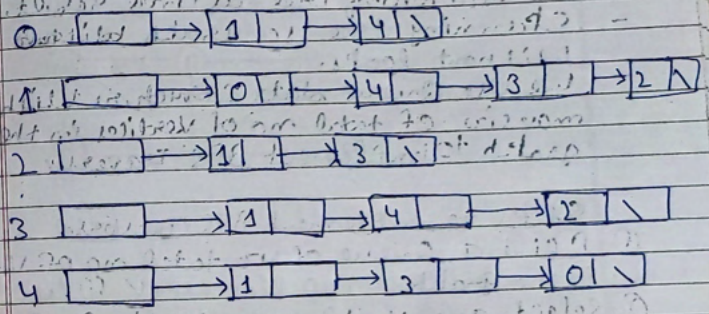
★ Adjacency list: An array of list is used. The size of array is equal to the no of vertices. This representation can also be used to represent a weighted graph.





space = $O(|V| + |E|)$
Worst case - $O(V^2)$

Queries like where there is an edge (u,v) etc

Graph Traversal
↙ BFS ↘ DFS

Graph Traversal is a technique used for searching a vertex in a graph.

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- Graph Traversal is also used to decide the order of vertices is visited in the search process.
- Graph Traversal find the edges to be used in the search process without creating loops.

⇒ BFS (Breath First Search)
- BFS traversal of a graph produces a spanning tree as final result.
- Spanning tree is a graph without without loops.
- We use Queue data structure with max size of total no of vertices in the graph to implement BFS Traversal.

Steps to implement BFS Traversal

① Define a Queue of size total no of vertices in the graph.
② Select any vertex as starting point for traversal. Visit that vertex and insert it into the Queue.
③ Visit all the non-visited adjacent vertices of the vertex which is at front of the Queue and insert them into the Queue.
④ When there is no new vertex to be visited from the vertex which is at front of the Queue then delete that vertex.
⑤ Repeat steps 3 and 4 until queue becomes empty.
⑥ When queue becomes empty, then produce

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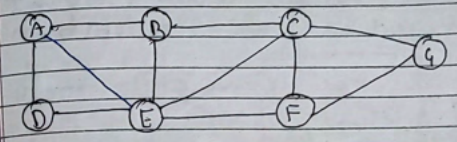
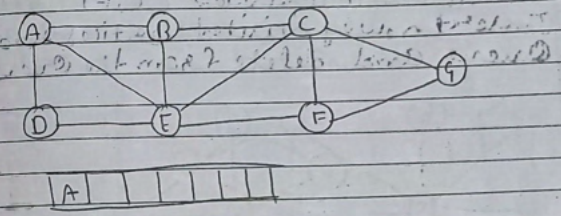
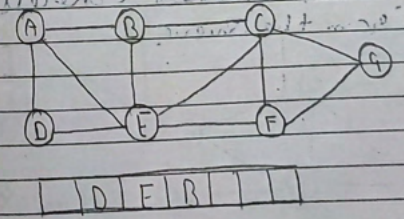
final spanning tree by removing unused edges from the graph.

①


Sol Step ① Select the vertex A as start point (visit A) Insert A into the Queue.



Step ② Visit all adjacent vertices of A which are not visited (D, E, B)
- Insert newly visited vertices into the Queue & delete A from the queue.



Step ③ Visit all adjacent vertices of D which are not visited (there is no vertex)

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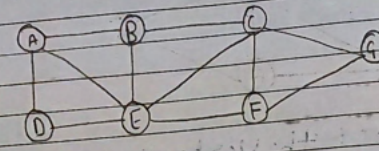
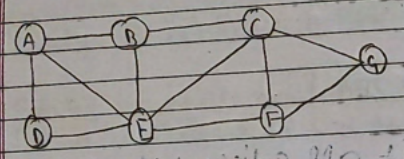
- Delete D from the Queue

		E	B			

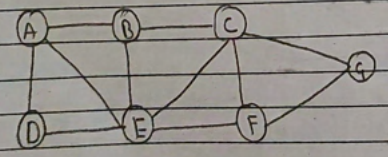


Step ④ Visit all adjacent vertices of E which are not visited (C, F)
- Insert newly visited vertices into the Queue and delete from the Queue.



		B	C	F		

Step ⑤ Visit all adjacent vertices of B which are not visited (there is no vertex)
- Delete B from the Queue.



			C	F		

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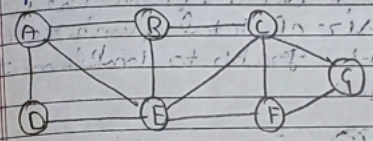
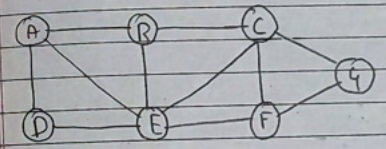
Step-6 Visit all adjacent vertices of C which are not visited (G)
- Insert newly visited vertex into the Queue and delete C from the Queue.



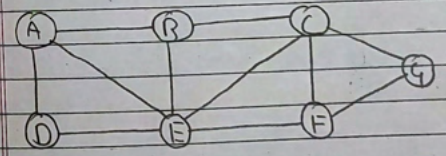
					F	G

Step-7 Visit all adjacent vertices of F which are not visited (there is no vertex)
- Delete F from the Queue.

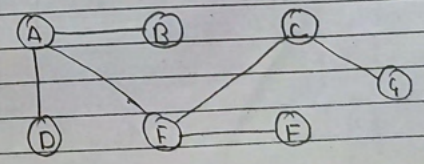


						G

Step-8 Visit all adjacent vertices of G which are not visited (there is no vertex)
- Delete G from the Queue.



Result



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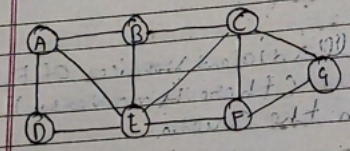
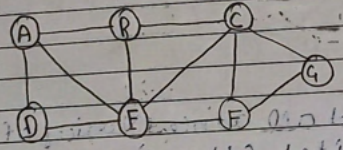
DFS (Depth First Search)

DFS traversal of a graph produces a spanning tree as final result.

- Spanning tree is a graph without loops
- We use stack data structure with maximum size of total number of vertices in the graph to implement DFS traversal.

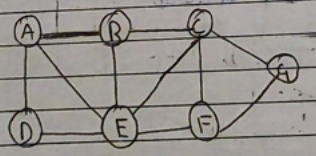


Step ①



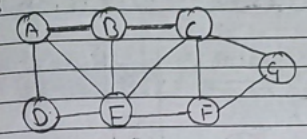
A

Step-2



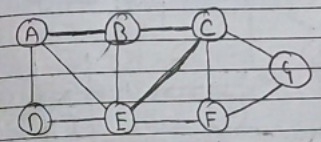
B
A

Step-3



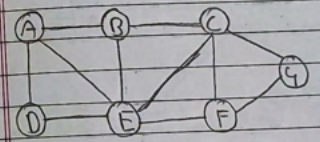
C
B
A

Step-4



E
C
B
A

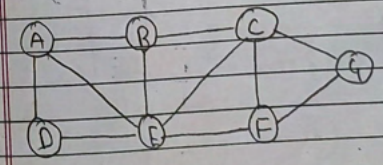
Step-5



D
E
C
B
A

Step-6 There is no new vertex to be visited from D. So use backtrack

- Pop D from the stack



E
C
B
A

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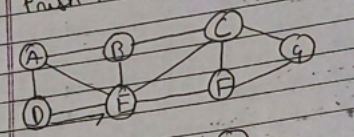
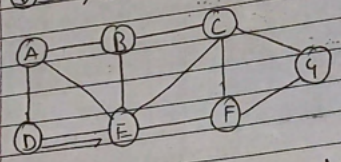
Step-7 Visit adjacent vertex of E which is not visited (F)

- Push F into the stack



F
E
C
B
A

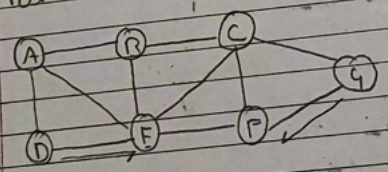
Step-8



G
F
E
C
B
A

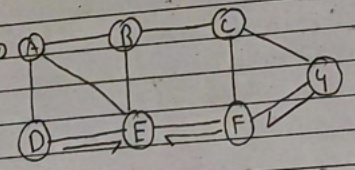
Step-9 There is no new vertex to be visited from G, so use backtrack

- Pop G from the stack



F
E
C
B
A

Step-10



E
C
B
A

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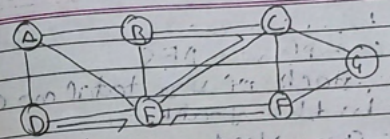
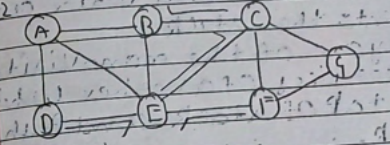
Step-11



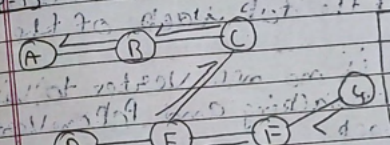
C
B
A

Step-12



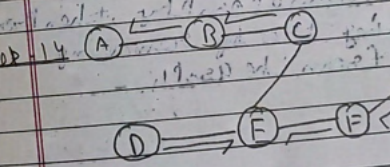
B
A

Step-13



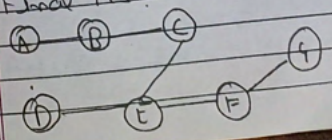
A

Step-14



Stack becomes empty. So stop DFS traversal.

Final MST of DFS



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Steps to implement DFS

Step① Define a stack of size total no of Vertices in the graph.

Step② Select any vertex as starting point for traversal. Visit that vertex and Push it onto the stack.

Step③ Visit anyone of the non-visited adjacent vertices of a vertex which is at the top of stack and Push it onto stack.

Step④ Repeat step ③ Until there is no new vertex to be visited from the vertex which is at the top ~~stack~~ of the stack.

Step⑤ When there is no new vertex to visit then use backtracking and Pop one vertex from the stack.

Step⑥ Repeat step 3, 4 and 5 Until stack becomes empty.

Step⑦ When stack becomes empty, then produce final spanning tree by removing unused edges from the graph.

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Graph Theory

$G = (V, E)$

V = Vertex Set $\{V_1, V_2, V_3, V_4, \ldots V_n\}$

E = Edge Set $\{E_1, E_2, E_3 \ldots E_n\}$

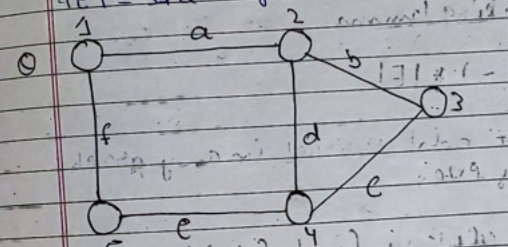
In which $l_k \in E$ is $l_k = \{V_i, V_j\}$ is called Undirected graph.

$V_i \xrightarrow{E_1} V_j$ is called directed graph

$V_i \overset{E_1}{—} V_j$ is called Undirected graph

$|V|$ = order of graph (no of Vertex)

$|E|$ = Size of graph (no of Edge's)



$G = (V, E)$

$V = \{1, 2, 3, 4, 5\}$

$E = \{a, b, c, d, e, f\}$

Adjacent Vertex = Common Edge b/w two vertex

$a = \{1, 2\}$

Adjacent Edge: Used in two commonly vertex

a is in common edge

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Self loop - Edge have start and end point are same.

Multi edge - Multiple edges b/w two vertices

Pseudograph - Graph contain self loop & multiedge

Multigraph - Graph have multiple edges b/w two vertices known as multigraph

Simple Graph - Not contain self loop, multi edges.

* Handshaking lemma

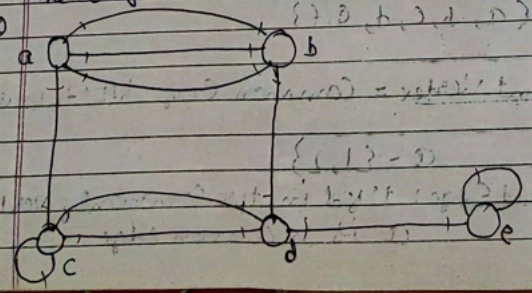
$\Sigma d(v) = 2 * |E|$

The no of odd vertices in any graph is always even.

Degree is always find for vertex

The degree of self loop is 2.

Q



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Sol Degree of vertex a = 4
Degree of vertex b = 4
Degree of vertex c = 5
Degree of vertex d = 4
Degree of vertex e = 3

Sum of degree of vertex = 5+4+4+4+3
= 20

Total no of edge = 10 - self loop count one edge

$\Sigma d(v) = 2 * |E|$
$20 = 2 \times 10$
$20 = 20$

* Havel-Hakimi Theorem
For a given degree sequence, whether a simple graph exist or not.

Q (2, 2, 2, 2)
Arrange in Descending order

(2, 2, 2, 2) Remove First element and subtract -1 from the elements is remove

(1, 1, 2)

(2, 1, 1)

(0, 0) graph exist

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Q (), 3, 0, 1, 1)

Sol Arrange in descending order

(~~3~~, 2, 1, 1, 0)
-1

(~~1~~, 0, 0, 0)
-1

(-1, 0, 0) Graph not exist

Q (6, 5, 4, 3, 3, 1)

(~~6~~, 5, 4, 3, 3, 1)
-1

Not exist graph because we delete 6
Subtract -1 in 6 elements but element is 5

Q (7, 6, 5, 4, 4, 3, 2, 1)

(~~7~~, 6, 5, 4, 4, 3, 2, 1)
-1

(~~6~~, 4, 3, 3, 2, 1, 0)
-1

(~~3~~, 2, 2, 1, 0, 0)
-1

(~~1~~, 1, 0, 0, 0)
-1

(0, 0, 0, 0) Graph exist

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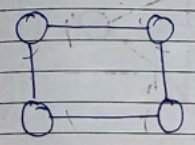
Regular Graph

A Vertex has same degree in graph is called regular graph.

No of Edges $E = \frac{m \times d}{2}$

m = no of Vertex
d = degree of each vertex



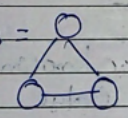
$E = \frac{4 \times 2}{2} = \frac{8}{2} = 4$

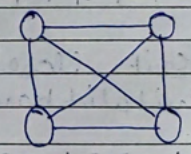
Complete graph

A simple graph is Complete graph (K_n) in which every pair of vertices are adjacent.

K_n = where n is no of vertices

$K_1 =$ O , $K_2 =$ O——O

$K_3 =$ 

$K_4 =$ 

No of Edges $= {}^nC_2 = \frac{n \times (n-1)}{2} = 6$

${}^5C_2 = \frac{5 \times (4)}{2} = 10$

Degree of Every vertex = n-1

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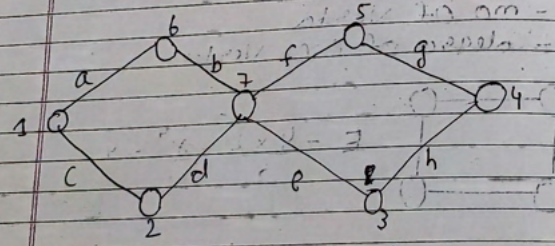
The no of vertex are 'n' how many unlike graph Possible $= 2^{E} = 2^{\frac{n(n-1)}{2}}$

* Every Regular graph is not Complete graph

* Every Complete graph is regular graph



Walk - Repetion are allowed. Opened Walk and Closed Walk. Open means start Kahe se kahe end
1a6b7d Close mean Jaha se start kar uski end

Trail - Edges repetition not allowed
Vertex repetion allowed.

7 5 4 3 7

Circuit - Closed Trail also known as Circuit without repeat (Edge)

1 6 7 5 4 3 7 2 1

Path - Vertex repetion not allowed
1 6 7 2 1

Vertices start and end vertices are same
Closed Path

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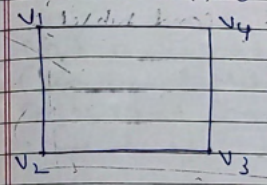
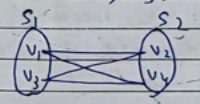
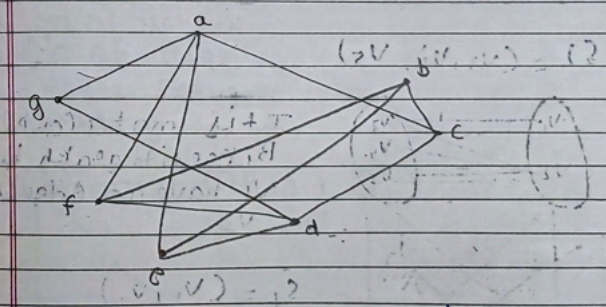
Bipartite Graph
'G' is bipartite if its vertex set 'V' can be partitioned into two disjoint non empty set V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 so that no edge in G connects either two vertices in V_1 or two vertices in V_2.



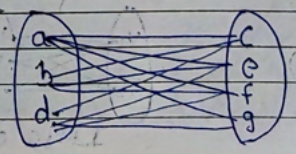
$S_1 = \{V_1, V_3\}$ $S_2 = \{V_2, V_4\}$



* Take the set in which not common edge in b/w two vertex in same set



$V(G) = (a, b, c, d, e, f, g)$



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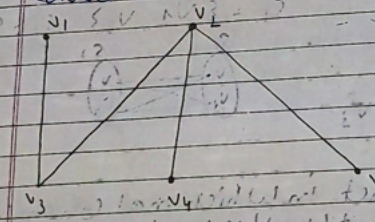
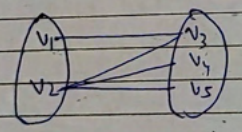
Complete Bipartite Graph: If it has its vertex set partitioned into two subsets of 'm' and 'n' vertices respectively. There is an edge b/w two vertices if and only if one vertex is in first subset and other in second subset.

* It means vertex of first subset is completely connected vertices of second subset.

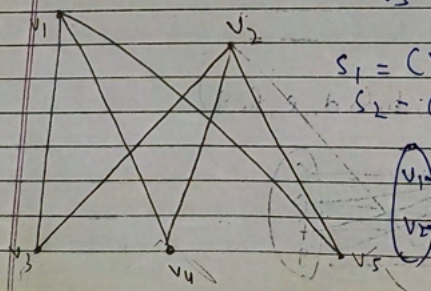


$S_1 = (v_1, v_2)$

$S_2 = (v_3, v_4, v_5)$

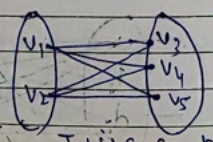


It is not a complete Bipartite graph because v_1 have no edge b/w v_4 & v_5



$S_1 = (v_1, v_2)$

$S_2 = (v_3, v_4, v_5)$



It is a complete bipartite graph

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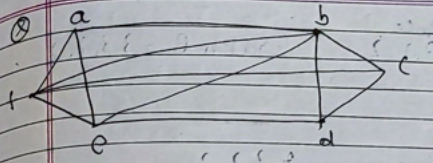


$S_1 = (a, d, c)$

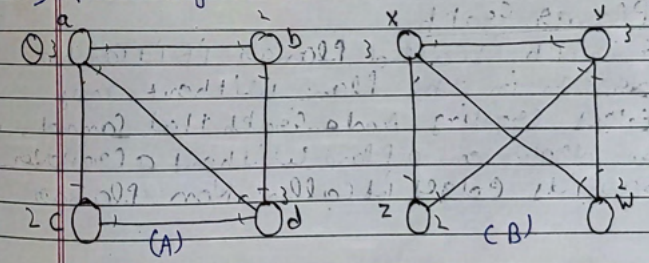
but d & c have common edge

Not a Bipartite graph: it's vertex can't be partitioned in two sets such that edges can't connect two vertices from same subset.

* Isomorphic Graph

Two graph are different to each other but they are same known as isomorphic. Check the properties the graph is isomorphic:

⇒ No of vertex
⇒ No of edge
⇒ Degree sequence
⇒ Mapping



Sol No of vertex Graph A = 4, Graph B = 4
No of edge Graph A = 5, Graph B = 5

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Degree Sequence
Graph A = 3, 3, 2, 2 Graph B = 3, 3, 2, 2

Mapping

a has degree sequence 3, 2, 2
b has degree sequence 3, 3
c has degree 2 Connected sequence 3, 3
d has degree 3 Connected sequence 3, 2, 2

x has degree 3 Connected sequence 3, 2, 2
y has degree 3 Connected sequence 3, 2, 2
w has degree 2 Connected sequence 3, 3
z has degree 2 Connected sequence 3, 3

Edge wise
a = x, y, x & z have edge
b = z, y & w have not edge
b & c have not edge
c = w, w & y have edge
c & d have edge
d = y

Thus the graph is isomorphic

Planar Graph
A Graph is called Planar if it can be drawn in the plane without any edges crossing. And a Graph that cannot be drawn on a plane without a crossover b/w its edges it called Non Planar

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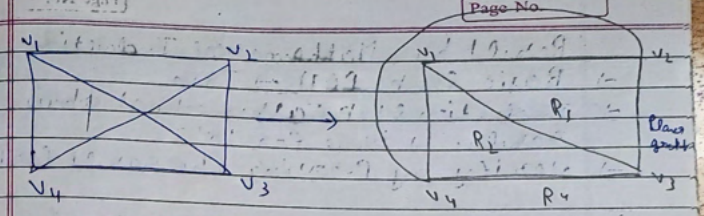
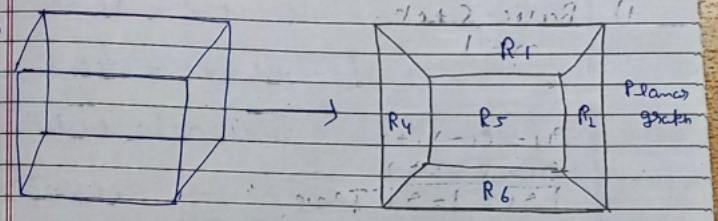
i)

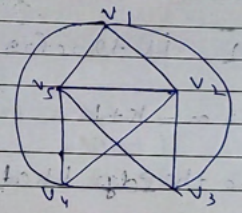


ii)



$R_6 \rightarrow$ outer region

Theorem: A Complete graph of five vertices is Non-Planar



Non-Planar graph

VV Imp
Euler's formula Proof
Let G be a Connected Planar simple Graph with 'e' edges and 'v' vertices. Let x be no of regions in a Planar representation of G.

$|x = e - v + 2|$

Proof:

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Result by Mathematical Induction:

⇒ Basic Step P(1) → True

⇒ Induction Step [assume that above result is true for 'k']

⇒ verify by Proving the result for k+1

i) Basic Step:

$m = 1$

$r_1 = e_1 - v_1 + 2$

$r_1 = 1 - 2 + 2$

$\boxed{r_1 = 1 = r}$ True

ii) Induction Step

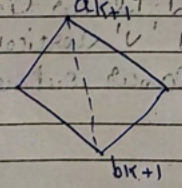
Assume the equation is true for $m = k$

$r_k = e_k - v_k + 2$ is true for G_k

iii) Verification for $m = k+1$

let (a_{k+1}, b_{k+1}) be the edge that is added to G_k

Case 1:



Both the vertices are in G_k

$r_{k+1} = r_k + 1$

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$e_{k+1} = e_k + 1$

$v_{k+1} = v_k$ both vertices are in G_k

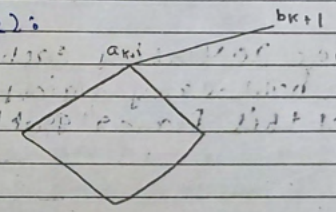
$r_{k+1} = e_{k+1} - v_{k+1} + 2$

$r_k + \cancel{1} = e_k + \cancel{1} - v_k + 2$

$\boxed{r_k = e_k - v_k + 2}$

It is also True

Case 2:



$r_{k+1} = r_k$

$v_{k+1} = v_k + 1$

$e_{k+1} = e_k + 1$

$r_{k+1} = e_{k+1} - v_{k+1} + 2$

$r_k = e_k + \cancel{1} - [v_k + \cancel{1}] + 2$

$\boxed{r_k = e_k - v_k + 2}$

It is also true

By induction method Euler Formula is Proved

Hence Prove

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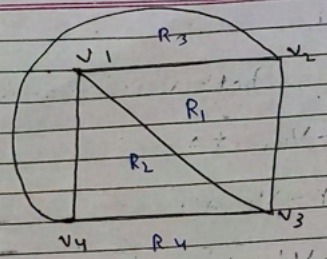
Q



$e = 6$, $v = 4$

$r = 6 - 4 + 2$

$r = 4$

Q If there are 20 vertices each of degree 3 then into how many region does a represent of this planar graph split the plane?

Sol $r = ?$

$v = 20$

$\boxed{\sum \deg(v) = 2e}$

degree $= 3$

$\sum \deg(v) = 20 \times 3 = 60$

$60 = 2e$

$e = 30$

$r = e - v + 2$

$r = 30 - 20 + 2$

$r = 12$

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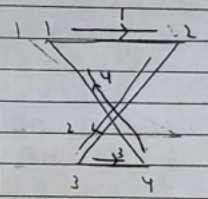
Euler Graph

Euler Path - It is a path that traverse each edge exactly one and only once. (vertex can be repeated but edge can't).

A graph that contains an Euler path is called as Euler graph.



$1 \to 2 \to 3 \to 4$ it is Euler graph (Each edge visited only once)

$1 \to 2 \to 3 \to 4 \to 1$ It is Euler circuit

Euler graph is always connected because Euler path contain all the edges of the graph.

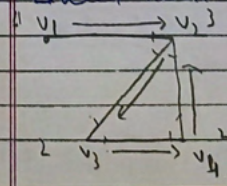
Euler circuit: First and last vertex are same. It is a circuit that traverse each edge once & only once.

Imp:-

⇒ A connected Euler graph iff it has atmost 2 odd degree vertices.

⇒ Euler circuit → Each vertex is of even degree.

⇒ Euler path → Max two vertices odd degree



$V_1 \to V_2 \to V_3 \to V_4$ It is a Euler path.

But not Euler circuit because start and end point are not same.

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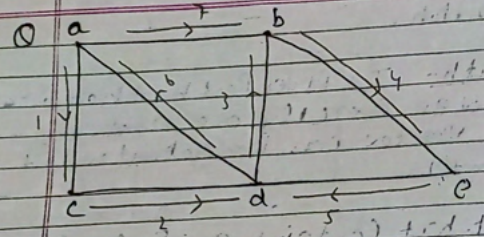
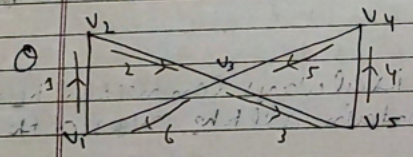


$a \to c \to d \to b \to e \to d \to a \to b$

Degree
$a \to 3$
$b \to 3$ } Two vertex with odd degree
$c \to 2$
$d \to 4$
$e \to 2$

Q



It is an Euler circuit
$V_1 \to V_2 \to V_3 \to V_5 \to V_4 \to V_3 \to V_1$

$V_1 - 2$
$V_2 - 2$
$V_3 - 4$
$V_4 - 2$
$V_5 - 2$
} Even degree

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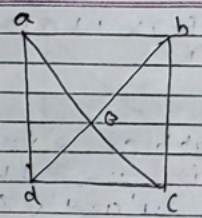
Q



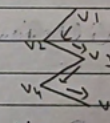
$a = 3$
$b = 3$
$c = 3$
$d = 3$
$e = 4$
} 4 vertices with odd degree

It is not a Euler graph

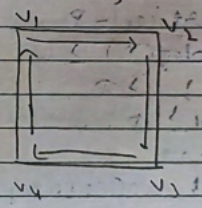
☆ Hamiltonian Path

Hamiltonian Path Contain each vertex exactly once

Hamiltonian circuit: First and last vertex are same



$V_1 \to V_2 \to V_3 \to V_4 \to V_5$
It is an Hamilton Path



$V_1 \to V_2 \to V_3 \to V_4 \to V_1$
It is Hamiltonian circuit

On removing any one edge from a Hamiltonian circuit, then we are left with Hamiltonian Path.

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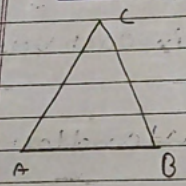
⇒ length of Hamiltonian Path in a connected graph of 'n' vertices is

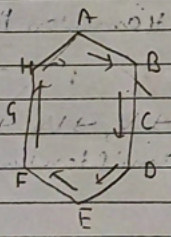
$\boxed{(n-1) \text{ edges}}$

Theorem: let G be a graph of 'n' vertices then 'G' has a hamiltonian Path if for only two vertices 'u' and 'v' of G,

$\boxed{\deg(u) + \deg(v) \geq n}$



no of vertices (n) = 3
Consider two vertex (A, B)
deg(A) = 2 , deg(B) = 2
deg(A) + deg(B) = 4 ≥ n
4 ≥ n
It is Hamiltonian Path.



no of vertices = 8
deg(A) + deg(D) ≥ n
2 + 2 ≥ 8
4 ≥ 8
Not True

This theorem is not neccesary true

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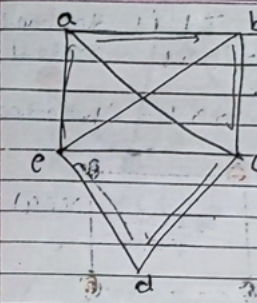
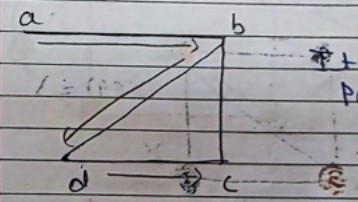
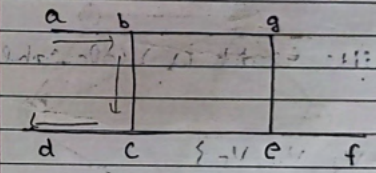
Q 

It is an Hamiltonian circuit
(a, b, c, d, e, a)

Q 

It is an Hamiltonian Path
(a, b, d, c)

Q 

It is an Hamiltonian Path/circuit

☆ Chromatic number of a graph

Graph Coloring
Coloring of Graph Constitutes coloring vertices / edges of the graph
Coloring all the vertices of a graph is the property that no two adjacent vertices have same colour.

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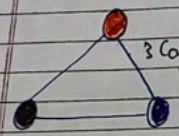
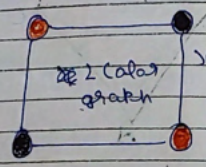
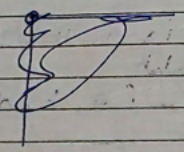
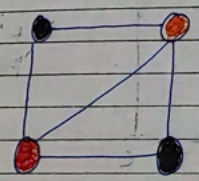
Chromatic Number: It is defined as the least no of Colors needed for Coloring the Graph. denoted by $\chi(G)$ or K- Chromatic Graph.





$\chi(G) = 2$



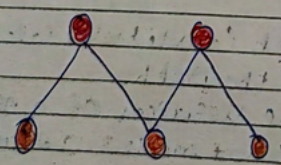


$\chi(G) = 3$

⇒ Every Bipartite Graph is 2 Colourable
$\chi(G) = 2$

$V = \{v_1, v_2, v_3, v_4, v_5\}$

To vertices same set to belong karti ha
Unke beach ma edge nhi hai



We can Color the Vertices of 1 one set with one color.

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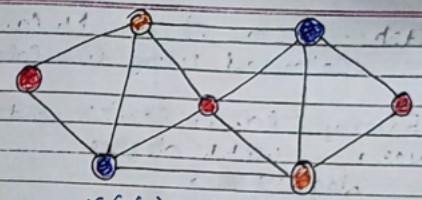
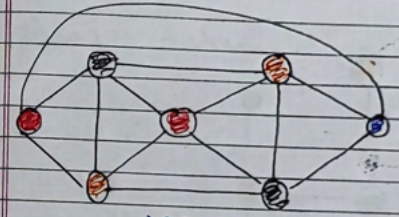


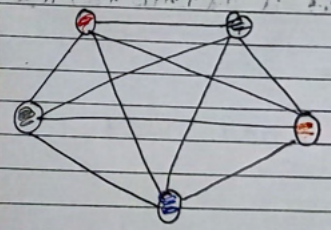
$\chi(G) = 3$

Q



$\chi(G) = 4$

Q



$\chi(G) = 5$

☆ 5 Color Theorem Proof

Theorem: Every Planar Graph with 'n' Vertices Can be Colored Using atmost 5 Colors.

Proof:
Theorem by Mathematical Induction Method

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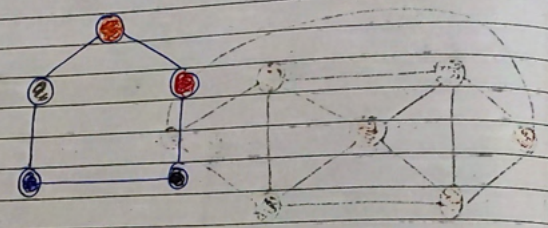
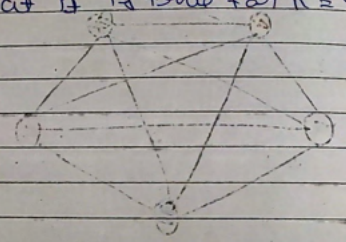
i) Basic Step:- $P(n < 5) \rightarrow$ Graph can be coloured using five colours.

lemma: Every Planar Graph contains a vertex with degree $\deg(V) < 5$

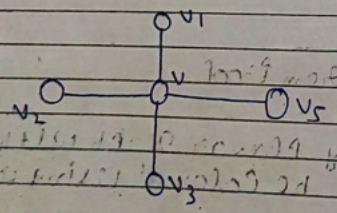


ii) Induction Step:
let assume that it it true for $K = 4$

$\deg(V) \leq 4$



iii) Verify for $K+1$
$\deg(V) < 5$



Case 1: No direct edge

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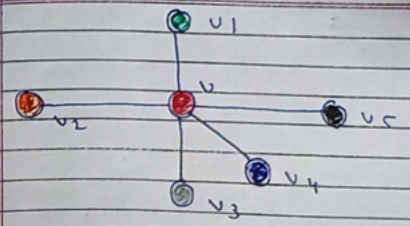
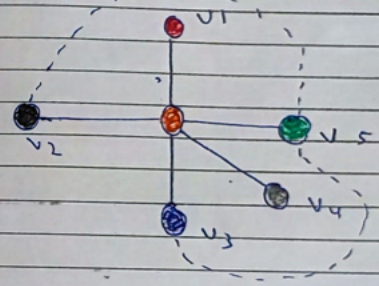


Case 2: Assume direct edge



Hence Prove.