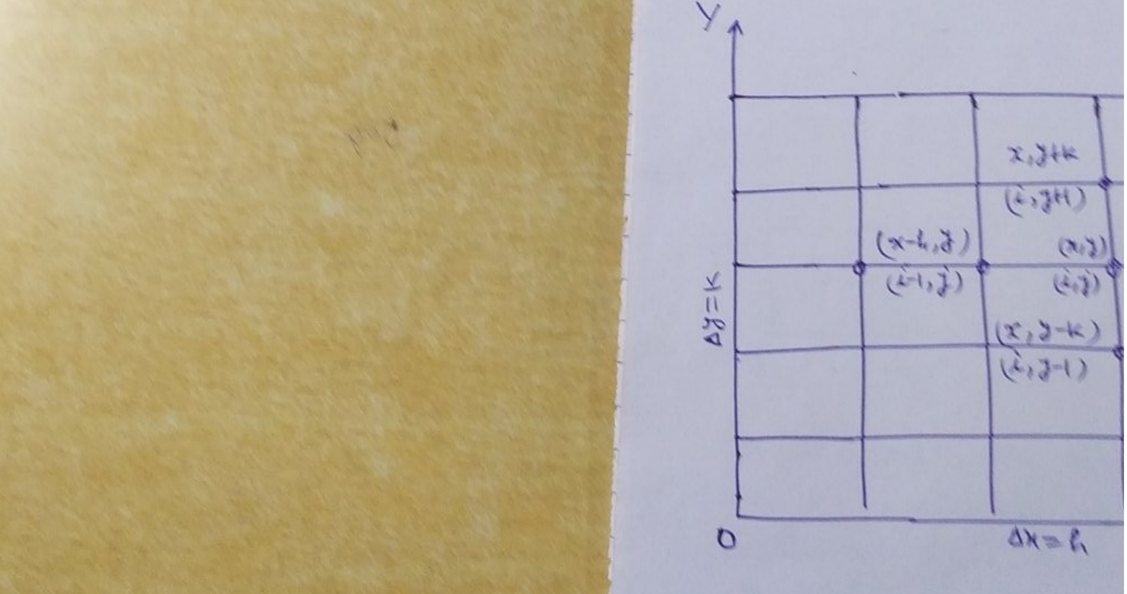
Numerical Solution

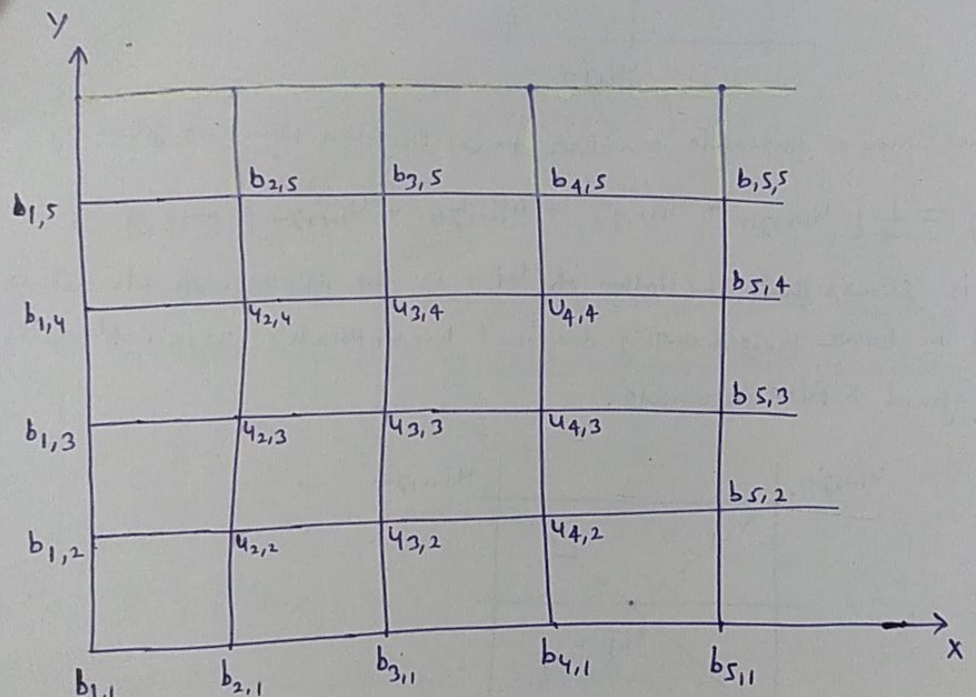
Finite difference appr

Consider a rectangul
region into a rectangular
shown in fig. The points o
called mesh points, Nodal



Solution of Elliptic Equation $\nabla^2 u = 0$ or

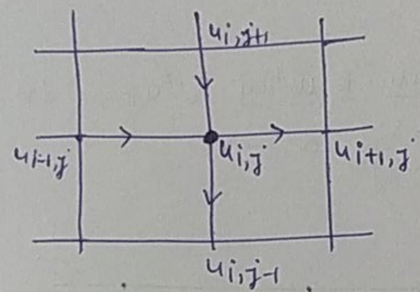
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow (1)$$



Consider a rectangular region R for which $u(x,y)$ is known at the boundary. Divide this region into a network of square mesh of side h as shown in figure. Replacing the derivatives in (1) by their difference approximations, we have

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}\right] \longrightarrow (2)$$

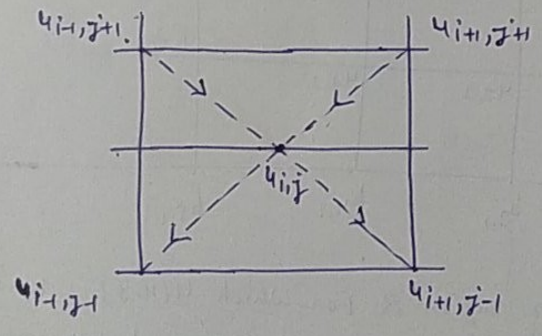
This shows that the value of u at any interior mesh point is the average of its values at four neighbouring points to the left, right, above and below. (2) is called the standard 5 points formula.



Sometimes a formula similar to (2) is used which is given by

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right] \longrightarrow (3)$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighbouring diagonal mesh points. (3) is called the diagonal 5 point formula.



<u>Note</u>. Diagonal 5 point formula is less accurate than standard 5-point formula.

Now we find ~~out~~ the Initial values of u at the ~~integ~~ interior mesh points, we first use the diagonal five point formula (3) and compute $u_{3,3}$, $u_{2,4}$, $u_{4,4}$, $u_{4,2}$ and $u_{2,2}$. In ~~order~~ this order, we get

$$u_{3,3} = \frac{1}{4}\left[b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1} \right]$$

$$u_{2,4} = \frac{1}{4}\left[b_{1,5} + u_{3,3} + b_{3,5} + b_{1,3} \right]$$

$$u_{4,4} = \frac{1}{4}\left[b_{3,5} + b_{5,3} + b_{5,5} + u_{3,3} \right]$$

$$u_{4,2} = \frac{1}{4}\left[u_{3,3} + b_{5,1} + b_{3,1} + b_{5,3} \right]$$

$$u_{2,2} = \frac{1}{4}\left[b_{1,3} + b_{3,1} + u_{3,3} + b_{1,1} \right]$$

The values at the remaining interior points i.e $u_{2,3}, u_{3,4}$, $u_{4,3}$ and $u_{3,2}$ are computed by standard five point formula (2).

$$u_{2,3} = \frac{1}{4}\left[b_{1,3} + b_{3,3} + u_{2,4} + u_{2,2} \right]$$

$$u_{3,4} = \frac{1}{4}\left[u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3} \right]$$

$$u_{4,3} = \frac{1}{4}\left[u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2} \right]$$

$$u_{3,2} = \frac{1}{4}\left[u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1} \right]$$

Having found all nine values of $u_{i,j}$ once, their accuracy is improved by Jacobi Method or Gauss-Seidal method or any method.

Jacobi's Method

Denoting the nth iterative value of $u_{i,j}$ by $u_{i,j}^{(n)}$.

The iterative formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)} \right]$$

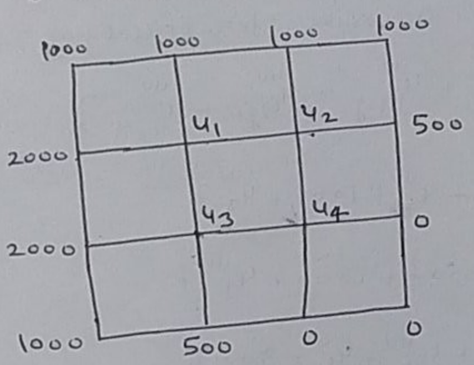
It gives improved values of $u_{i,j}$ at the interior mesh points and is called the Point Jacobi's Formula.

Gauss-Seidal Method

In this method, the iteration formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)} \right]$$

1. Given the values of $u(x,y)$ on the boundary of the square in the figure. Evaluate the function $u(x,y)$ satisfying the Laplace Equation $\nabla^2 u = 0$ at the pivotal points of this figure by Jacobi's method.



Solution:

To get Initial Values of u_1, u_2, u_3, u_4 we assume that $u_4 = 0$, Then

We Find u_1 by Diagonal Five Point Formula

$u_1 = \frac{1}{4}[1000 + u_4 + 2000 + 1000] \quad \because u_4 = 0$

$= \frac{1}{4}[1000 + 0 + 2000 + 1000] = 1000$

Now we find u_2 by Five Point Standard Formula

$u_2 = \frac{1}{4}[u_1 + 500 + 1000 + u_4] \quad \because u_1 = 1000, u_4 = 0$

$u_2 = \frac{1}{4}[1000 + 500 + 1000 + 0] = 625$

Now we find out u_3 by Five Point Standard Formula

$u_3 = \frac{1}{4}[u_1 + 500 + 2000 + u_4]$

$= \frac{1}{4}[1000 + 500 + 2000 + 0] = \cancel{875}\ 875$

Now we find out u_4 by Standard Five Point Formula

$$u_4 = \frac{1}{4}\left[u_2 + 0 + u_3 + 0\right]$$

$$u_4 = \frac{1}{4}\left[625 + 875\right] = 375$$

We carry out the successive Iterations using Jacobi's Formulae.

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)}\right]$$

$$\therefore\ u_1^{(n+1)} = \frac{1}{4}\left[2000 + u_2^{(n)} + 1000 + u_3^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n)} + 500 + 1000 + u_4^{(n)}\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[2000 + u_4^{(n)} + u_1^{(n)} + 500\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[u_3^{(n)} + 0 + u_2^{(n)} + 0\right]$$

For First Iteration Put $n=0$

$$\therefore\ u_1^{(1)} = \frac{1}{4}\left[2000 + 625 + 1000 + 875\right] = 1125$$

$$u_2^{(1)} = \frac{1}{4}\left[1000 + 500 + 1000 + 375\right] = 719$$

$$u_3^{(1)} = \frac{1}{4}\left[2000 + 375 + 1000 + 500\right] = 969$$

$$u_4^{(1)} = \frac{1}{4}\left[875 + 0 + 625 + 0\right] = 375$$

For Second Iteration Put $n=1$

$$u_1^{(2)} = \frac{1}{4}\left[2000 + 719 + 1000 + 969\right] = 1172$$

$$u_2^{(2)} = \frac{1}{4}\left[1125 + 500 + 1000 + 375\right] = 750$$

$$u_3^{(2)} = \frac{1}{4}\left[2000 + 375 + 1125 + 500\right] = 1000$$

$$u_4^{(2)} = \frac{1}{4}[969 + 0 + 719 + 0] = 422$$

For Third Iteration put $n = 2$

$$u_1^{(3)} = \frac{1}{4}[2000 + u_2^{(2)} + 1000 + u_3^{(2)}]$$
$$= \frac{1}{4}[2000 + 750 + 1000 + 1000] = 1187.5 \approx 1188$$

$$u_2^{(3)} = \frac{1}{4}[u_1^{(2)} + 500 + 1000 + u_4^{(2)}]$$
$$= \frac{1}{4}[1172 + 500 + 1000 + 422] = 773.5 \approx 774$$

$$u_3^{(3)} = \frac{1}{4}[2000 + u_4^{(2)} + u_1^{(2)} + 500]$$
$$= \frac{1}{4}[2000 + 422 + 1172 + 500] = 1023.5 \approx 1024$$

$$u_4^{(3)} = \frac{1}{4}[u_3^{(2)} + 0 + u_2^{(2)} + 0]$$
$$= \frac{1}{4}[1000 + 750] = 437.5 \approx 438$$

For Fourth Iteration put $n = 3$

$$u_1^{(4)} = \frac{1}{4}[2000 + u_2^{(3)} + 1000 + u_3^{(3)}] =$$
$$= \frac{1}{4}[2000 + 774 + 1000 + 1024] = 1199.5 \approx 1200$$

$$u_2^{(4)} = \frac{1}{4}[u_1^{(3)} + 500 + 1000 + u_4^{(3)}]$$
$$= \frac{1}{4}[1188 + 500 + 1000 + 438] = 781.5 \approx 782$$

$$u_3^{(4)} = \frac{1}{4}[2000 + u_4^{(3)} + u_1^{(3)} + 500]$$
$$= \frac{1}{4}[2000 + 438 + 1188 + 500] = 1031.5 \approx 1032$$

$$u_4^{(4)} = \frac{1}{4}[u_3^{(3)} + 0 + u_2^{(3)} + 0] =$$
$$= \frac{1}{4}[1024 + 774] = 449.5 \approx 450$$

Similarly

$$u_1^{(5)} = 1204, \quad u_2^{(5)} = 788, \quad u_3^{(5)} = 1038, \quad u_4^{(5)} = 454$$

$$u_1^{(6)} = 1206.5, \quad u_2^{(6)} = 790, \quad u_3^{(6)} = 1040, \quad u_4^{(6)} = 456.5$$

$$u_1^{(7)} = 1208, \quad u_2^{(7)} = 791, \quad u_3^{(7)} = 1041, \quad u_4^{(7)} = 458$$

$$u_1^{(8)} = 1208, \quad u_2^{(8)} = 791.5, \quad u_3^{(8)} = 1041.5, \quad u_4^{(8)} = 458$$

Thus there is no significant difference between the seventh and Eighth iteration values

Hence $u_1 = 1208$

$u_2 = 792$

$u_3 = 1042$

$u_4 = 458$ Ans

(b) We Carry out the successive Iterations, using Gauss-Seidal Formula

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)} \right]$$

$$\left.\begin{aligned}
\therefore u_1^{n+1} &= \frac{1}{4}\left[2000 + u_2^{(n)} + 1000 + u_3^{(n)} \right] \\
u_2^{(n+1)} &= \frac{1}{4}\left[u_1^{n+1} + 500 + 1000 + u_4^{(n)} \right] \\
u_3^{(n+1)} &= \frac{1}{4}\left[2000 + u_4^{(n)} + u_1^{(n+1)} + 500 \right] \\
u_4^{(n+1)} &= \frac{1}{4}\left[u_3^{(n+1)} + 0 + u_2^{(n+1)} + 0 \right]
\end{aligned}\right] \longrightarrow (A)$$

[Note We Write only Boundary Value only]

For First Iteration : put $n = 0$

$$\therefore u_1^{(1)} = \frac{1}{4}\left[2000 + u_2^{(0)} + 1000 + u_3^{(0)} \right]$$

$$= \frac{1}{4}\left[2000 + 625 + 1000 + 875 \right] = 1125$$

$$u_2^{(1)} = \frac{1}{4}\left[u_1^{(1)} + 500 + 1000 + u_4^{(0)} \right]$$

$\therefore u_2^{(1)} = \frac{1}{4}[1125 + 500 + 1000 + 375] = 750$

$u_3^{(1)} = \frac{1}{4}[2000 + u_4^{(0)} + u_1^{(1)} + 500]$

$u_3^{(1)} = \frac{1}{4}[2000 + 375 + 1125 + 500] = 1000$

$u_4^{(1)} = \frac{1}{4}[u_3^{(1)} + 0 + u_2^{(1)} + 0]$

$u_4^{(1)} = \frac{1}{4}[1000 + 750] = 438$

Now For Second Iteration : put $n = 1$ in Ⓐ

$\therefore u_1^{(2)} = \frac{1}{4}[2000 + u_2^{(1)} + 1000 + u_3^{(1)}]$

$u_1^{(2)} = \frac{1}{4}[2000 + 750 + 1000 + 1000] = 1188$

$u_2^{(2)} = \frac{1}{4}[u_1^{(2)} + 500 + 1000 + u_4^{(1)}]$

$u_2^{(2)} = \frac{1}{4}[1188 + 500 + 1000 + 438] = 782$

$u_3^{(2)} = \frac{1}{4}[2000 + u_4^{(1)} + u_1^{(2)} + 500]$

$u_3^{(2)} = \frac{1}{4}[2000 + 438 + 1188 + 500] = 1032$

$u_4^{(2)} = \frac{1}{4}[u_3^{(2)} + 0 + u_2^{(2)} + 0]$

$u_4^{(2)} = \frac{1}{4}[1032 + 782] = 454$

For Third Iteration : put $n = 2$ in Ⓐ

$u_1^{(3)} = \frac{1}{4}[2000 + u_2^{(2)} + 1000 + u_3^{(2)}] = 1204$

$u_2^{(3)} = \frac{1}{4}[u_1^{(3)} + 500 + 1000 + u_4^{(2)}] = 789$

$u_3^{(3)} = \frac{1}{4}[2000 + u_4^{(2)} + u_1^{(3)} + 500] = 1040$

$u_4^{(3)} = \frac{1}{4}[u_3^{(3)} + 0 + u_2^{(3)} + 0] = 458$

Fourth Iteration : Put $n=3$ in Ⓐ, we get

$u_1^{(4)} = \frac{1}{4}\left[2000 + u_2^{(3)} + 1000 + u_3^{(3)}\right] = 1207$

$u_2^{(4)} = \frac{1}{4}\left[u_1^{(4)} + 500 + 1000 + u_4^{(3)}\right] = 791$

$u_3^{(4)} = \frac{1}{4}\left[2000 + u_4^{(3)} + u_1^{(4)} + 500\right] = 1041$

$u_4^{(4)} = \frac{1}{4}\left[u_3^{(4)} + 0 + u_2^{(4)} + 0\right] = 458$

For fifth Iteration : Put $n=4$ in Ⓐ,

$u_1^{(5)} = \frac{1}{4}\left[2000 + u_2^{(4)} + 1000 + u_3^{(4)}\right] = 1208$

$u_2^{(5)} = \frac{1}{4}\left[u_1^{(5)} + 500 + 1000 + u_4^{(4)}\right] = 791.5$

$u_3^{(5)} = \frac{1}{4}\left[2000 + u_4^{(4)} + u_1^{(5)} + 500\right] = 1041.5$

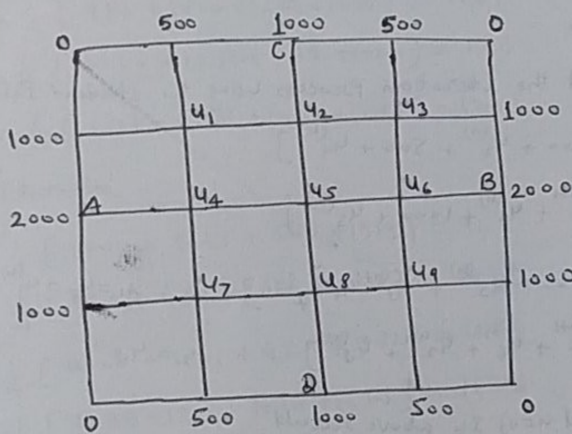
$u_4^{(5)} = \frac{1}{4}\left[u_3^{(5)} + 0 + u_3^{(5)} + 0\right] = 458.25$

Thus there is no significant difference between the fourth and fifth iteration values.

Hence $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$, $u_4 = 458$

Ans.

Solve the elliptic equation $u_{xx}+u_{yy}=0$ For the following square mesh with boundary values as shown in fig.



Sol^n: Let $u_1, u_2, u_3, \dots u_9$ be the values of u at the interior mesh-points. since the boundary values of u are symmetrical about AB so $u_1 = u_7$, $u_2 = u_8$, $u_3 = u_9$

Also the values of u being symmetrical about CD.

so $u_3 = u_1$, $u_4 = u_6$, $u_7 = u_9$

Thus It is sufficient to find the value u_1, u_2, u_4 and u_5

Now we Find their Initial values in following order:

$$u_5 = \frac{1}{4}\left[2000 + 2000 + 1000 + 1000\right] = 1500 \quad \text{(Std. Formula)}$$

$$u_1 = \frac{1}{4}\left[0 + u_5 + 2000 + 1000\right]$$

$$= \frac{1}{4}\left[0 + 1500 + 1000 + 2000\right] = 1125$$

$$u_2 = \frac{1}{4}\left[u_1 + u_3 + 1000 + u_5\right] \quad \because u_1 = u_3$$

$$= \frac{1}{4}\left[1125 + 1125 + 1000 + 1500\right] = 1188 \quad \text{(Std. Formula)}$$

$$u_4 = \frac{1}{4}\left[2000 + u_5 + u_1 + u_7\right]$$

$$= \frac{1}{4}\left[2000 + 1500 + 1125 + 1125\right] \quad \because u_1 = u_7 \quad \text{(Std. Formula)}$$

$$u_4 = 1438$$

Now we carry out the iteration process using the standard formulae.

$$u_1^{n+1} = \frac{1}{4}\left[1000 + u_2^{(n)} + 500 + u_4^{(n)}\right]$$

$$u_2^{n+1} = \frac{1}{4}\left[u_1^{n+1} + u_3^{(n)} + 1000 + u_5^{(n)}\right]$$

$$u_4^{n+1} = \frac{1}{4}\left[2000 + u_5^{(n)} + u_1^{(n+1)} + u_7^{(n)}\right] \quad \because u_1 = u_7 = u_1^{(n)}$$

$$u_5^{n+1} = \frac{1}{4}\left[u_4^{n+1} + u_6^{n} + u_2^{n+1} + u_8^{(n)}\right] \quad \because u_2 = u_8$$

First Iteration (Put $n=0$) In above result

$$u_1^{(1)} = \frac{1}{4}\left[1000 + u_2^{(0)} + 500 + u_4^{(0)}\right]$$

$$= \frac{1}{4}\left[1000 + 1188 + 500 + 1438\right] = 1032$$

$$u_2^{(1)} = \frac{1}{4}\left[u_1^{(1)} + u_3^{(0)} + 1000 + u_5^{(0)}\right]$$

$$= \frac{1}{4}\left[1032 + 1125 + 1000 + 1500\right] = 1164$$

$$u_4^{(1)} = \frac{1}{4}\left[2000 + 1500 + 1032 + 1125\right] = 1414$$

$$u_5^{(1)} = \frac{1}{4}\left[1414 + 1438 + 1164 + 1138\right] = 1301$$

For second Iteration (put $n=1$)

$$u_1^{(2)} = \frac{1}{4}\left[1000 + u_2^{(1)} + 500 + u_4^{(1)}\right] =$$

$$= \frac{1}{4}\left[1000 + 1164 + 500 + 1414\right] = 1020$$

$$u_2^{(2)} = \frac{1}{4}\left[1220 + 1032 + 1000 + 1301\right] = 1088$$

$$u_4^{(2)} = \frac{1}{4}\left[2000 + 1301 + 1020 + 1032\right] = 1338$$

$$u_5^{(2)} = \frac{1}{4}\left[1338 + 1414 + 1088 + 1164\right] = 1251$$

ird Iteration:

$u_1^{(3)} = \frac{1}{4}[1000 + 1088 + 500 + 1338] = 982$

$u_2^{(3)} = \frac{1}{4}[982 + 1020 + 1000 + 1251] = 1063$

$u_4^{(3)} = \frac{1}{4}[2000 + 1251 + 982 + 1020] = 1313$

$u_5^{(3)} = \frac{1}{4}[1313 + 1338 + 1063 + 1088] = 1201$

Fourth Iteration

$u_1^{(4)} = \frac{1}{4}[1000 + 1063 + 500 + 1313] = 969$

$u_2^{(4)} = \frac{1}{4}[969 + 982 + 1000 + 1201] = 1038$

$u_4^{(4)} = \frac{1}{4}[2000 + 1201 + 969 + 982] = 1288$

$u_5^{(4)} = \frac{1}{4}[1288 + 1313 + 1038 + 1063] = 1176$

Fifth Iteration

$u_1^{(5)} = \frac{1}{4}[1000 + 1038 + 500 + 1288) = 957$

$u_2^{(5)} = \frac{1}{4}(957 + 969 + 1000 + 1176) = 1026$

$u_4^{(5)} = \frac{1}{4}[2000 + 1176 + 957 + 969] = 1276$

$u_5^{(5)} = \frac{1}{4}[1276 + 1288 + 1026 + 1038] = 1157$

Similarly

$u_1^{(6)} = 951, \quad u_2^{(6)} = 1016, \quad u_4^{(6)} = 1266, \quad u_5^{(6)} = 1146$

$u_1^{(7)} = 946, \quad u_2^{(7)} = 1011, \quad u_4^{(7)} = 1260, \quad u_5^{(7)} = 1138$

$u_1^{(8)} = 943, \quad u_2^{(8)} = 1007, \quad u_4^{(8)} = 1257, \quad u_5^{(8)} = 1134$

$u_1^{(9)} = 941, \quad u_2^{(9)} = 1005, \quad u_4^{(9)} = 1255, \quad u_5^{(9)} = 1131$

$u_1^{(10)} = 940, \quad u_2^{(10)} = 1003, \quad u_4^{(10)} = 1253, \quad u_5^{(10)} = 1129$

$u_1^{(11)} = 939, \quad u_2^{(11)} = 1002, \quad u_4^{(11)} = 1252, \quad u_5^{11} = 1128$

$u_1^{12} = 939, \quad u_2^{12} = 1001, \quad u_4^{12} = 1251, \quad u_5^{(12)} = 1126$

Hence $u_1 = 939, \; u_2 = 1001, \; u_4 = 1251, \; u_5 = 1126$ Ans

Numerical Solution of Parabolic Equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

is one dimensional heat Equation

where $\alpha^2 = \frac{K}{\rho C}$ = Total Diffusivity of the substance ($cm^2/sec.$)

Here K is Thermal conductivity

ρ is density

The finite difference equation for (1) is

$$u_{i,j+1} = \lambda \left[u_{i-1,j} + u_{i+1,j} \right] + (1-2\lambda) u_{i,j} \longrightarrow (2)$$

Where $\lambda = \frac{k\alpha^2}{h^2}$ (Is also called mesh Ratio Parameter)

Here k represent step size of t (along y-axis)

h " " " " x axis (along x axis)

Equation (2) is called Bendre - Schmidt recurrence relation (formulae). This equation is Explicit form of (1)

When $\lambda = \frac{1}{2}$, then

$$u_{i,j+1} = \frac{1}{2} \left[u_{i-1,j} + u_{i+1,j} \right] + 0$$

$$\boxed{u_{i,j+1} = \frac{1}{2} \left[u_{i-1,j} + u_{i+1,j} \right]}$$ (Implicit Form)

Note If $\lambda = \frac{1}{2}$,

$u_{i,j+1}$

$\frac{1}{2} u_{i-1,j}$ $\quad$ $\frac{1}{2} u_{i+1,j}$

If $\lambda \neq \frac{1}{2}$

$u_{i,j+1}$

$\lambda (u_{i-1,j})$ $\quad$ $u_{i,j}$ $(1-2\lambda)$ $\quad$ $(u_{i+1,j})\lambda$

2. Find the value of $u(x,t)$ satisfying the Parabolic equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0,t)=0$, $u(8,t)=0$ and $u(x,0) = 4x - \frac{1}{2}x^2$ at the Points $x = i : i = 0,1,2,3,4$ and $t = \frac{1}{8}j : j = 0,1,2,3,4,5$.

Solution

Given $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$

Given step size of $x = h = 1$

step size of $y = t = \frac{1}{8}$

We know that standard Parabolic eq^n is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$ i.e compair with (1)

so $\alpha^2 = 4$

We Also know that

$$\lambda = \frac{K\alpha^2}{h^2} = \frac{(1/8)\cdot 4}{1} = \frac{1}{2}$$

$$\boxed{\lambda = \frac{1}{2}}$$

Here $\lambda = 1/2$, so

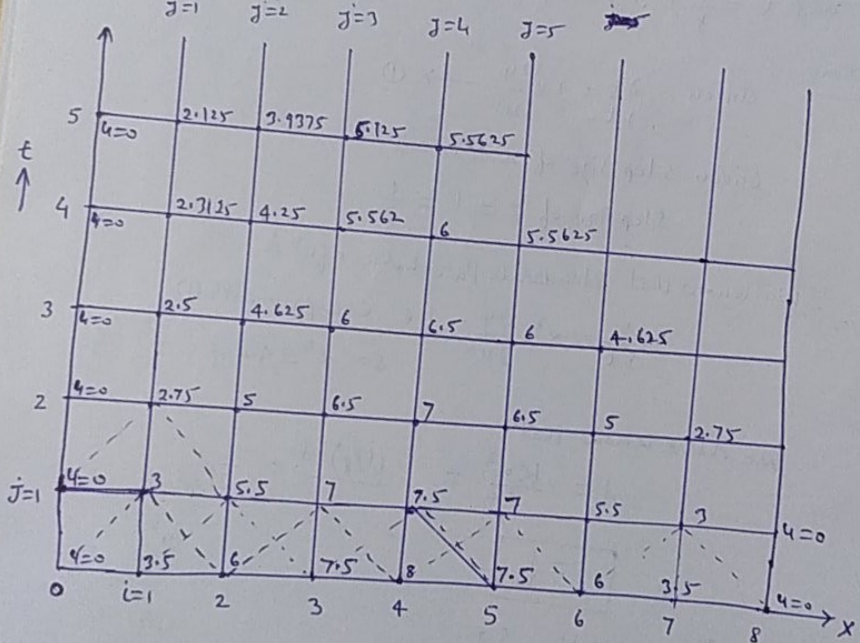
$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

i.e $u_{i,j+1}$

$\frac{1}{2}(u_{i-1,j})$ $\frac{1}{2}(u_{i+1,j})$

Boundary condition

$$u(0,t) = 0 , \quad 0 \leq t \leq 5$$
$$u(8,t) = 0 .$$
$$u(x,0) = 4x - \frac{1}{2}x^2$$



$$u(x,0) = 4x - \frac{1}{2}x^2$$
$$u(1,0) = 4 - \frac{1}{2}(1) = 3.5$$
$$u(2,0) = 8 - \frac{1}{2}(4) = 6$$
$$u(3,0) = 12 - \frac{1}{2}(9) = 7.5$$
$$u(4,0) = 16 - \frac{1}{2} \times 16 = 8$$
$$u(5,0) = 20 - \frac{1}{2}(25) = 7.5$$
$$u(6,0) = 24 - \frac{1}{2}(36) = 6$$

9

$u(7,0) = 28 - \frac{1}{2}\times 49 = 3.5$

First grid Point $\frac{0+6}{2} = 3$

$\frac{3.5+7.5}{2} = \frac{11}{2}$

$\frac{6+8}{2} = 7$

2. Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$, given $u(0,t)=0$, $u(4,t)=0$

$u(x,0) = x(4-x)$, Assume $h=1$. Find the value of u upto $t=5$.

Solⁿ We know that

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial u}{\partial x^2}$$

Given $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$

$$\because \lambda = \frac{k\alpha^2}{h^2} = \frac{1\cdot(1/2)}{1} = \frac{1}{2} \qquad \text{Let } \boxed{k=1}$$

$\lambda = 1/2$ i.e at $k=1$, $h=1$

We use Bendre-Schmidt formula

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

$u_{i,j+1}$

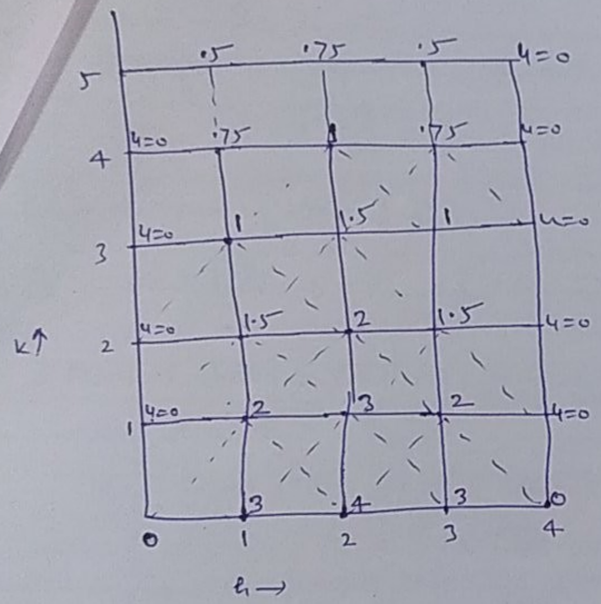
$\frac{1}{2}(u_{i-1,j})$ $\qquad$ $\frac{1}{2}(u_{i+1,j})$

Here Boundary condition

$u(0,t) = 0$

$u(4,t) = 0$

$u(x,0) = x(4-x)$.



$\because$ $u(x,0) = x(4-x)$

$u(1,0) = 1(4-1) = 3$

$u(2,0) = 2(4-2) = 4$

$u(3,0) = 3(4-3) = 3$

$u(4,0) = 4(4-0) = 0$ Ans.

$\therefore$ $u(1$

Date
DELTA Pg No.

e Dimensional Wave equation (Hyperbolic Equation)

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow ① \quad \text{Where } a^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass}}$$

Finite difference equation of ① is

$$y_{i,j+1} = \lambda\left[y_{i-1,j} + y_{i+1,j}\right] + (2-2\lambda)\, y_{i,j} - y_{i,j-1} \longrightarrow ②$$

y represent displacement of string corresponding to point x ~~axis~~ and time t

Where $\lambda = \frac{a^2 k^2}{h^2}$

Here $h \rightarrow$ stepsize of 'x' [along x axis]

$k \rightarrow$ stepsize of 't' [along y-axis]

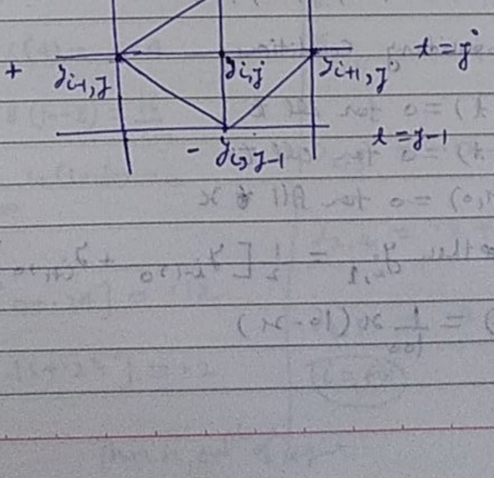
If $\lambda = 1$ then Equation (2) becomes.

$$y_{i,j+1} = \left[y_{i-1,j} + y_{i+1,j}\right] + 0 - y_{i,j-1}$$

$$y_{i,j+1} = y_{i-1,j} + y_{i+1,j} - y_{i,j-1}$$

Graphically we present as.

I am taking grid point



① Determine the displacement $y(x,t)$ of string for $x = 0, 2, 4, 6,$ and upto one ~~half~~ half period of vibration. If $y(x,t)$ satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}$$ and boundary condition.

$y(0,t) = 0$, $y(10,t) = 0$, $\frac{\partial y}{\partial t} = 0$ and

$y(x,0) = \frac{1}{100}x(10-x)$

Sol^n

Periodic Length $= \frac{2l}{a}$

Given $h = 2$, Step size of x : 2

$k = ?$ Range x : $0 \le x \le 10$

But given Half period of vibration $= \frac{10}{2} = 5$

using $\lambda = \frac{a^2k^2}{h^2}$ $\because \lambda = 1$

$1 = \frac{4 \times k^2}{4} \Rightarrow k^2 = \frac{4}{4} = 1$

$\boxed{k=1}$ Step size of t is 1

Now $x = 0, 2, 4, 6, 8, 10$

$t = 0, 1, 2, 3, 4, 5$

Using boundary condition,

$y(0,t) = 0$ for all t

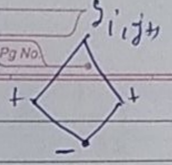
$y(10,t) = 0$ for all t

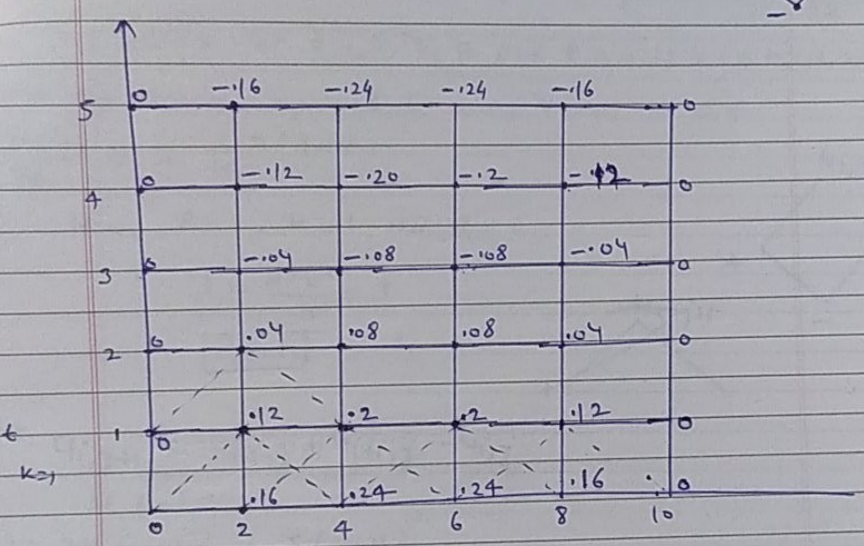
$\frac{\partial y}{\partial t}(x,0) = 0$ for All x

then $y_{i,1} = \frac{1}{2}\left[y_{i-1,0} + y_{i+1,0}\right]$

$y(x,0) = \frac{1}{100}x(10-x)$

Grid point





$h = 2$

$\because \; y(x,0) = \frac{1}{100} x(10-x)$

$y(2,0) = \frac{1}{100}(2)(8) = \frac{16}{100} = .16$

$y(4,0) = \frac{1}{100} 4(6) = \frac{24}{100} = .24$

$y(6,0) = \frac{1}{100} 6(4) = .24$

$y(8,0) = \frac{1}{100} 8(10-8) = \frac{16}{100}$

$y(10,0) = \frac{1}{100} 10(10-10) = 0$

$y_{2,1} = \frac{1}{2}[0 + .24] = .12$

$y_{3,1} = \frac{1}{2}[.16 + .24] = .2$

$y_{4,1} = \frac{1}{2}[.24 + .16] = .2$

$y_{5,1} = \frac{1}{2}[.24 + 0] = .12$

$\cancel{y \quad 0 + .2 - .16}$

$y_{2,2} = 0 + .2 - .16 = .04$

$y_{2,3} = .12 + .2 - .24 = .08$

$y_{2,4} = .2 + .12 - .24 = .08$

$y_{2,5} = .2 + 0 - .16 = .04$

$i = \text{fix}$

This is sol^n of eqn