Fourier - Transformation

Fourier Transform of Infinite F.T :- Let $f(t)$ be a function defined for $t \in (-\infty, \infty)$ Then Infinite F.T of $f(t)$ denoted by $F(f(t))$.

$$\therefore F[f(t)] = \int_{-\infty}^{\infty} e^{-ist} \cdot f(t) \cdot dt$$

or $$\therefore \bar{f}(s) = F[f(t)] = \int_{-\infty}^{\infty} e^{-ist} \cdot f(t) dt \quad \Big| \therefore F[f(t)] = \bar{f}(s)$$

Inverse formula for F.T :- $$f(t) = F^{-1}[\bar{f}(s)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s) e^{+ist} ds$$

Fourier Sine Transformation :- The infinite Fourier sine transformation for function $f(t)$ where $0 < t < \infty$ denoted by $F_s(f(t))$

$$\bar{f}_s(s) = F_s(f(t)) = \int_0^{\infty} f(t) \cdot \sin st \, dt$$

Inverse Formula $$f(t) = F^{-1}(\bar{f}(s)) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_s(s) \sin st \, ds$$

Fourier Cosine Transformation :-

$$F_c(f(t)) = \bar{f}_c(s) = \int_0^{\infty} f(t) \cdot \cos st \cdot dt$$

Inverse formula $$f(t) = F^{-1}(\bar{f}_c(s)) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_c(s) \cdot \cos st \, ds$$

Linearity Prop :-

(I) If $f(x)$ & $g(x)$ are any two function. whose F.T exist c_1, c_2. arbit const. Then

(1) $F\{c_1 f(x) + c_2 g(x)\} = c_1 F(f(x)) + c_2 F(g(x))$

Pf:- $F[c_1 f(x) + c_2 (g(x))] = \int_{-\infty}^{\infty} e^{-isx} \{c_1 f(x) + c_2 g(x)\} dx$

$= \int_{-\infty}^{\infty} c_1 . e^{-isx} . f(x) dx + \int_{-\infty}^{\infty} c_2 e^{-isx} . g(x) dx$

$= c_1 F(f(x)) + c_2 F(g(x))$

IIsly. (2) $F_s\{c_1 f(x) + c_2 g(x)\} = c_1 F_s\{f(x)\} + c_2 F_s(g(x))$

Pf:

$F_s[c_1 f(x) + c_2 g(x)] = \int_0^{\infty} (c_1 f(x) + c_2 g(x)) \sin sx . dx$

$= c_1 \int_0^{\infty} f(x) \sin sx \, dx + c_2 \int_0^{\infty} g(x) . \sin sx \, dx$

$= c_1 F\{f(x)\} + c_2 F(g(x))$

IIsly $F_c\{c_1 f(x) + c_2 g(x)\} = c_1 F_c\{f(x)\} + c_2 F_c(g(x))$

$F_c\{c_1 f(x) + c_2 g(x)\} = \int_0^{\infty} (c_1 f(x) + c_2 g(x)) \cos sx \, dx.$

$= c_1 \int_0^{\infty} f(x) . \cos sx \, dx + c_2 \int_0^{\infty} g(x) . \cos sx \, dx = c_1 F_c(f(x)) + c_2 F_c(g(x))$

Hence

Whose [illegible] ...nge Scale Prop :-

Ex:① The F.T of $f(x)$ is $\bar{f}(s)$. Then F.T of $f(ax)$ is $\frac{1}{a}\bar{f}(s/a)$

Pf:- $\bar{f}(s) = \int_{-\infty}^{\infty} e^{-isx} \cdot f(x)\,dx.$

Now $F(f(ax)) = \int_{-\infty}^{\infty} e^{-isx} \cdot f(ax)\,dx$ Put $ax = t$, $x = t/a$, $dx = dt/a$

$$= \int_{-\infty}^{\infty} e^{-\frac{ist}{a}} \cdot f(t)\frac{dt}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} e^{-\frac{ist}{a}} f(t)\,dt = \frac{1}{a}\bar{f}(s/a)$$

llsly
||ly

$F_s f(ax) = \int_0^{\infty} f(ax)\cdot \sin sx \cdot dx$

$F_c f(ax) = \int_0^{\infty} f(ax)\cdot \cos sx$] - Do. Complete Same.

Shifting Prop :- If $\bar{f}(s)$ is the Fourier ~~[illegible]~~ Tras $f(x)$

To show that $e^{-iax}\bar{f}(s)$ is F.T of $f(x-a)$

Proof:- $F[f(x-a)] = \int_{-\infty}^{\infty} e^{-isx} \cdot f(x-a)\,dx$

$=$ Put $x-a = (dt)$, $x = a+t$, $dx = dt$

$$= \int_{-\infty}^{\infty} e^{-is(a+t)} \cdot f(t) \cdot dt$$

$$= \int_{-\infty}^{\infty} e^{-isa} \cdot e^{-ist} \cdot f(t)\,dt$$

$$= e^{-isa}\int_{-\infty}^{\infty} e^{-ist} \cdot f(t)\,dt$$

$$= e^{-isa}\bar{f}(s)$$

H.P

Modulation Prop:- If $\bar{f}(s)$ is infinite F.T of $f(x)$. Then F.T of $f(x)\cos ax$ is $\frac{1}{2}\bar{f}(s-a)+\frac{1}{2}\bar{f}$

Proof:- $F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{-isx} f(x)\cos ax . dx$

$$= \int_{-\infty}^{\infty} e^{-isx}\left(\frac{e^{iax}+e^{-iax}}{2}\right) f(x)\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-isx}.e^{iax} f(x)dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-isx}.e^{-iax} f(x)dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(s-a)x}. f(x)dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(s+a)x}. f(x)dx$$

$$= \frac{1}{2}\bar{f}(s-a) + \frac{1}{2}\bar{f}(s+a)$$

Exp ① Find the F.T of $f(x) = e^{-a|x|}$ when $a>0$ and $x\in(-\infty,\infty)$

Sol $\bar{f}(s) = \int_{-\infty}^{\infty} f(x)\,e^{-isx}.dx = \int_{-\infty}^{\infty} e^{-a|x|}.e^{-isx}.dx$

$$\int_{-\infty}^{0} e^{-a(-x)} e^{-isx}.dx + \int_{0}^{\infty} e^{-a(x)}.e^{isx}dx. \quad \left[\because |x| = \begin{cases} -x \text{ if } x\le 0 \\ x \text{ if } x>0. \end{cases}\right.$$

$$= \int_{-\infty}^{0} e^{ax}.e^{-isx}dx + \int_{0}^{\infty} e^{-ax}.e^{-isx}.dx$$

$$\int_{-\infty}^{0} e^{(a-is)x} dx + \int_{0}^{\infty} e^{-(a+is).x}.dx$$

$$\left[\frac{e^{(a-is)x}}{a-is}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+is)x}}{-(a+is)}\right]_{0}^{\infty} = \frac{1}{a-is} + \left(0+\frac{1}{a+is}\right)$$

$$\Rightarrow \frac{1}{a-is}+\frac{1}{a+is} = \frac{a+is+a-is}{(a^2-i^2s^2)} = \frac{a+is+a-is}{a^2+s^2}$$

$$= \frac{2a}{a^2+s^2}$$

Ex 1.2 Find the F.T of the function:-

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{otherwise.} \end{cases}$$

Sol.

$$\therefore F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx = \int_{-\infty}^{0}(\,) + \int_{0}^{a}(\,) + \int_{a}^{\infty}(\,)$$

$$= \int_{-\infty}^{0} f(x)\cdot e^{-isx}\, dx + \int_{0}^{a} f(x)\, e^{-isx}\, dx + \int_{a}^{\infty} e^{-isx}\cdot f(x)\, dx$$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{a} k\cdot e^{-isx}\cdot dx + \int_{a}^{\infty} 0\cdot dx$$

$$= k\cdot\int_{0}^{a} e^{-isx}\cdot dx = -\frac{k}{is}\left[e^{-isx}\right]_{x=0}^{x=a}$$

$$= \frac{-k}{is}\left(e^{-isa} - e^{-0}\right) = \frac{-k}{is}\left(e^{-isa}-1\right)$$

$$= k\left(\frac{1 - e^{-isa}}{is}\right) = -ik\left(\frac{1 - e^{-isa}}{s}\right)$$

$$\text{or } \frac{ik}{s}\left(e^{-isa}-1\right)$$

$\longleftrightarrow$

Ex Find the F.T

$$f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2l} & -l < x < l \\ 0 & |x| > l \end{cases}$$

Sol

$$F\{f(x)\} = \int_{-\infty}^{-l} e^{-isx}\cdot \underbrace{f(x)}_{0}\, dx + \int_{-l}^{l} e^{-isx}\, \underbrace{f(x)}_{\frac{\sqrt{2\pi}}{2l}}\, dx + \int_{l}^{\infty} e^{-isx}\cdot \underbrace{f(x)}_{0}\, dx$$

Do complete

Ex-4 Find the F.T $f(x) = \begin{cases} -1 & -a < x < 0 \\ 1 & 0 < x < a \\ 0 & \text{otherwise.} \end{cases}$

Sol

$$\bar{f}(s) = F(f(x)) = \int_{-\infty}^{\infty} f(x) \cdot e^{-isx} \cdot dx$$

$$\int_{-\infty}^{-a} \underbrace{f(x)}_{0} e^{-isx} dx + \int_{-a}^{0} \underbrace{f(x)}_{(-1)} \cdot e^{-isx} dx + \int_{0}^{a} \underbrace{f(x)}_{1} e^{-isx} dx + \int_{a}^{\infty} \underbrace{f(x)}_{0} e^{-isx} dx$$

$$= 0 + (-1)\left[\frac{e^{-isx}}{-is}\right]_{-a}^{0} + \left[\frac{e^{-isx}}{-is}\right]_{0}^{a} + 0$$

Do complete

Ex-5 Find the F.T $f(x) = \begin{cases} x \cdot e^{-x} & x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

Sol

$$F(f(x)) = \int_{-\infty}^{\infty} f(x) \cdot e^{-isx} \cdot dx = \int_{-\infty}^{0} 0 + \int_{0}^{\infty} x \cdot e^{-x} \cdot e^{-isx} \cdot dx.$$

~~complete.~~

$$= \int_{0}^{\infty} x e^{-isx} \cdot e^{-x} \cdot dx$$

$$= \int_{0}^{\infty} \underset{\mathrm{I}}{x}\, \underset{\mathrm{II}}{e^{-(1+is)x}} \cdot dx$$

$$\left[x \cdot \frac{e^{-(1+is) \cdot x}}{-(1+is)}\right]_{0}^{\infty} - \int_{0}^{\infty} 1 \cdot \frac{e^{-(1+is)x}}{-(1+is)}$$

$$\lim_{x \to \infty} \left[\frac{-x}{(1+is)\, e^{(1+is)x}} - 0\right]$$

Do comply

Find the Sine and Cosine Fourier Transformation of $2e^{-5x} + 5\cdot e^{-2x}$

Note:-

$$\int_0^\infty e^{-ax}\cdot \sin bx\,dx = \frac{b}{a^2+b^2}$$

$$\int_0^\infty e^{-ax} \cos bx\,dx = \frac{a}{a^2+b^2}$$

Note:-

$$\int e^{ax}\cdot \sin(bx+c)\cdot dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$

$$\int e^{ax} \cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right]$$

Sol

$$f(x) = 2e^{-5x} + 5\cdot e^{-2x}$$

$$\therefore \bar{f}_s(s) = \int_0^\infty f(x)\cdot \sin sx\,dx$$

$$= \int_0^\infty (2e^{-5x} + 5e^{-2x})\cdot \sin sx\,dx$$

$$= 2\int_0^\infty e^{-5x}\cdot \sin sx\,dx + 5\int_0^\infty e^{-2x}\cdot \sin sx\,dx$$

$$= 2\cdot\frac{s}{5^2+s^2} + 5\cdot\frac{s}{2^2+s^2}$$

$$= \frac{2s}{25+s^2} + \frac{5s}{4+s^2}$$

(ii) $$\bar{f}_c(s) = \int_0^\infty (2e^{-5x}+5e^{-2x})\cos sx\,dx$$

$$= 2\int_0^\infty e^{-5x}\cos sx\,dx + 5\int_0^\infty e^{-2x}\cdot\cos sx\,dx$$

$$2\cdot\frac{5}{25+s^2} + 5\,\frac{(2)}{4+s^2}$$

Imp

Exp. 2 Find the Fourier Sine transformation. $\dfrac{e^{-ax}}{x}$

Sol

$$f(x) = \frac{e^{-ax}}{x}$$

$$\therefore \bar{f}_s(s) = F_s(f(x))$$

$$\bar{f}_s(s) = \int_0^\infty \frac{e^{-ax}}{x}\cdot \sin sx\,dx \quad \text{——(1)}$$

This problem is Diff under the Integral sign.. using Leibnitz Rule.

Diff (1) w.r.t 's'

$$\frac{d}{ds}\bar{f}_s(s) = \int_0^\infty \frac{e^{-ax}}{\cancel{x}}\cdot \cos sx\cdot \cancel{x}$$

$$= \int_0^\infty e^{-ax}\cdot\cos sx\,dx$$

$$= \frac{a}{a^2+s^2}$$

$\therefore \frac{d}{ds}\overline{f_s(s)} = \frac{a}{a^2+s^2}$

I.B.S. w.r.t s

$\overline{f_s}(s) = a\int \frac{ds}{a^2+s^2} = \frac{a}{a}\tan^{-1}\frac{s}{a} + c$ —— (*)

Put $s=0$ in Eq (i)

$\overline{f_s}(0) = 0$. Putting (*) = $0 = \tan^{-1}\frac{0}{a} + c$.

$0 = 0 + c$

$c = 0$

$\therefore \overline{f_s}(s) = \tan^{-1}\frac{s}{a}$

Hence $\overline{f_s}(s) = F_s(f(x)) = \tan^{-1}s/a$.

Ex 8 Find the Fourier cosine Transformation of e^{-x^2}

Sol $\therefore f(x) = e^{-x^2}$

$\overline{f_c}(s) = \underbrace{F_c(f(x))}_{} =$

Let $I = \int_0^\infty f(x)\cos sx$ —— (1)

$\overline{f_s}(s) = \int_0^\infty e^{-x^2}\cdot\cos sx\,dx$

's' is the Parameter. Using Diff under the integral sign.

D w.r.t 's'

$\frac{d}{ds}\overline{f_s}(s) = \int_0^\infty e^{-x^2}\cdot(-\sin sx).x$

$= \frac{1}{2}\int_0^\infty \underset{\text{II}}{-2x\cdot e^{-x^2}}\cdot\underset{\text{I}}{\sin sx}\,dx$ I.B.S.

$= \frac{1}{2}\left[\sin sx\cdot e^{-x^2}\right]_0^\infty - \int_0^\infty \cos s\cdot x\cdot s\,e^{-x^2}dx$

$\frac{dI}{ds} = 0 - \frac{1}{2}\cdot s\underbrace{\int_0^\infty e^{-x^2}\cdot\cos sx}_{I \text{ (By (i))}}$

$\frac{dI}{ds} = -\frac{s}{2}I$

$\frac{\frac{dI}{ds}}{I} = -s/2$

I.B.S. w.r.t s.

$\log I = -\frac{s^2}{4} + \log c$.

$\log I - \log c = -s^2/4$

$\log I/c = -s^2/4$ Taking e on both s

$c \cdot \log\left(\frac{I}{c}\right) = e^{-s^2/4}$

$I/c = e^{-s^2/4}$

$I = c \cdot e^{-s^2/4}$ — (*)

Put $s=0$ in (i)

$$I = \int_0^\infty e^{-x^2} \cdot \cos 0 \cdot dx = \int_0^\infty e^{-x^2} \cdot dx.$$

$$I = \int_0^\infty e^{-x^2} \cdot dx = \frac{\sqrt{\pi}}{2}$$ ($\because$ Beta & Gamma function Results.)

$$I = \frac{\sqrt{\pi}}{2}$$

By (*) $s=0$ $I = c \cdot e^0 = c \cdot 1$

$I = c$

$c = \frac{\sqrt{\pi}}{2}$

By (*) = $I = \frac{\sqrt{\pi}}{2} e^{-s^2/4}$

$$\bar{f}_s(s) = F_c(f(x)) = I = \frac{\sqrt{\pi}}{2} e^{-s^2/4}$$

$\longleftrightarrow$ H.W

Ex Find Sine Transformation.

$$f(x) = \begin{cases} \sin x & 0 < x < a \\ 0 & x > a \end{cases}$$

Sol

$$F_s(f(x)) = \int_0^\infty f(x) \cdot \sin sx \, dx$$

$$= \int_0^a \sin x \sin sx \, dx + \int_a^\infty 0 \cdot \sin sx \, dx$$

$$= \frac{1}{2} \int 2 \sin x \cdot \sin sx \cdot dx + 0$$

P-5

$$= \frac{1}{2} \int_0^a \cos(1-s)x - \cos(1+s)x$$

$$= \frac{1}{2} \left[\frac{\sin(1-s)x}{1-s} - \frac{\sin(1+s)x}{1+s} \right]_0^a$$

$$= \frac{1}{2} \left[\frac{\sin(1-s)a}{1-s} - \frac{\sin(1+s)a}{1+s} - 0 - 0 \right]$$

$$= \frac{1}{2} \left[\frac{\sin(1-s)a}{1-s} - \frac{\sin(1+s)a}{1+s} \right]$$

$\longleftrightarrow$

V. Imp

Ex Find the Fourier cosine Transformation $\frac{1}{1+x^2}$ and deduce the Sine transformation of $\frac{x}{1+x^2}$

Sol

$$F_c\left(\frac{1}{1+x^2}\right) = \int_0^\infty \frac{1}{1+x^2} \cdot \cos sx \, dx = I(s)$$

$$I = \int_0^\infty \frac{1}{1+x^2} \cdot \cos sx \, dx \quad \text{—— (1)}$$

Diff w.r.t s

$$\frac{dI}{ds} = \int_0^\infty \frac{-1}{1+x^2} \cdot \sin sx \cdot x$$

$$= \int_0^\infty \left(\frac{-x}{1+x^2}\right) \cdot \sin sx$$

$$= F_s\left(\frac{-x}{1+x^2}\right) \quad \text{—— (2)}$$

$$= \int_0^\infty \frac{-x^2 \cdot \sin sx}{x(1+x^2)}\,dx$$

$$= -\int_0^\infty \frac{(1+x^2-1)\sin sx\,dx}{x(1+x^2)}$$

$$= -\int_0^\infty \left(\frac{\sin sx}{x} - \frac{\sin sx}{x(1+x^2)}\right)dx$$

$$-\int_0^\infty \frac{\sin sx}{x}\,dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx$$

$$I_1 \quad + \quad I_2$$

$I_1 = \int_0^\infty \frac{\sin sx}{x}\,dx$ (we solve by laplace Transfor

$(\because$ ~~L[sin sx]~~ $] =$ ~~$\frac{s}{s^2+s^2}$~~

$\frac{\text{Exp. 2}}{\text{p. 5.39}}$ (L.T)

$$\int_0^\infty \frac{\sin sx}{x} = \frac{\pi}{2} \text{ If } s>0$$
$$-\frac{\pi}{2} \text{ if } s<0$$

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx \quad \text{—— (3)}$$

Again Diff w.r.t s

$$\frac{d^2I}{ds^2} = 0 + \int_0^\infty \frac{1}{\cancel{x}(1+x^2)}\cdot\cos sx \cdot \cancel{x}$$

$$= \int_0^\infty \frac{\cos sx}{1+x^2}\,dx \quad \text{— By (1)}$$

$$= I$$

$$\frac{d^2I}{ds^2} = I$$

~~I = ...~~

~~$\frac{d^2I}{ds^2}$~~ $=$ $\quad D^2 - 1 = 0$

$$D^2 = 1 \qquad D = \pm 1$$

$$I = c_1 e^{-s} + c_2 e^{s} \quad \text{—— (4)}$$

Put $s=0$ in (1)

$$I = \int_0^\infty \frac{\cos 0}{1+x^2}\,dx = \int_0^\infty \frac{1}{1+x^2}$$

$$= (\tan^{-1}x)_0^\infty = \tan^{-1}\infty - \tan^{-1}0$$

$$= \tan^{-1}\tan\frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

$$I = \frac{\pi}{2}$$

Putting. By (4)

$$I = c_1 \cdot e^0 + c_2 e^0 \quad (\because s=0$$

$$\frac{\pi}{2} = c_1 + c_2 \quad \text{—— (5)}$$

By (3) $\quad I = c_1 e^{-s} + c_2 e^{s}$

D.w.r.t s.

$$\frac{dI}{ds} = -c_1 e^{-s} + c_2 e^{s}$$

Put $s=0$ (By (3))

At $s=0$

$$\therefore \frac{dI}{ds} = -\pi/2 + 0$$

$$-c_1 + c_2 = -\pi/2$$

$$c_1 - c_2 = \pi/2 \quad \text{—— (6)}$$

From (5) + (6)

$c_1 + c_2 = \pi/2$

$c_1 - c_2 = +\pi/2$

$c_1 = \pi/2$ $\boxed{c_1 = \pi/2}$

$c_2 = 0$

$I = c_1 e^{-s} + c_2 e^{s}$

$I = \frac{\pi}{2} e^{-s}$

$I = \frac{\pi}{2} e^{-s}$

$F_c\left(\frac{1}{1+x^2}\right) = \frac{\pi}{2} e^{-s}$

By (2) $F_s\left(\frac{-x}{1+x^2}\right) = \frac{dI}{ds}$

$= \frac{d}{ds}\left(\frac{\pi}{2} e^{-s}\right)$

$= -\frac{\pi}{2} e^{-s}$

$F_s\left(\frac{-x}{1+x^2}\right) = -\frac{\pi}{2} e^{-s}$

$$\boxed{F_s\left(\frac{x}{1+x^2}\right) = \frac{\pi}{2} e^{-s}}$$

Hence proved

Ex P 7 Find the Fourier Sine and cosine Transformation of x^{m-1}.

Sol we know that P-6

$F_c\{f(x)\} = \int_0^\infty f(x) \cos sx \, dx$

$F_s\{f(x)\} = \int_0^\infty f(x) \cdot \sin sx \, dx$

$\therefore f(x) = x^{m-1}$

$F_c(f(x)) = \int_0^\infty x^{m-1} \cdot \cos sx \cdot dx$

$F_s f(x) = \int_0^\infty x^{m-1} \cdot \sin sx \, dx$

$\therefore$ As we Know that Gamma function

$\Gamma m = \int_0^\infty x^{m-1} \cdot e^{-x} \cdot dx$

Put $x = isy$

D.w.r.t 'y' $dx = i \cdot s \cdot dy$

$= \int_0^\infty (isy)^{m-1} \cdot e^{-isy} \cdot i \cdot s \cdot dy$

$= \int_0^\infty i^{m-1+1} \cdot s^{m-1+1} \cdot y^{m-1} \cdot e^{-isy} dy$

$= \int_0^\infty i^m \cdot s^m \cdot y^{m-1} \cdot e^{-isy} \cdot dy$

$\cancel{\Gamma m} = s^m \cancel{\int_0^\infty \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)}$

$\therefore i^m = \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^m = \left(e^{i\frac{\pi}{2}}\right)^m$

$\Gamma m = s^m \int_0^\infty \left(e^{i\pi/2}\right)^m y^{m-1} \cdot e^{-isy} \cdot dy$

P.T.O

$$\Gamma m = s^m e^{\frac{im\pi}{2}} \int_0^\infty e^{-isy} y^{m-1} \cdot dy$$

$$\frac{\Gamma m}{s^m} e^{-\frac{im\pi}{2}} = \int_0^\infty e^{-isy} \cdot y^{m-1} \cdot dy$$

Now change $y \to x$

$$\frac{\Gamma m}{s^m} e^{-\frac{im\pi}{2}} = \int_0^\infty e^{-isx} \cdot x^{m-1} \cdot dx$$

By Euler's thm

$$\frac{\Gamma m}{s^m} \left[\cos \frac{m\pi}{2} - i \sin \frac{m\pi}{2}\right] =$$

$$= \int_0^\infty (\cos sx - i \sin sx)\, x^{m-1} \cdot dx$$

Equating R.P + I.P

$$\int_0^\infty \cos sx\, x^{m-1} \cdot dx = \frac{\Gamma m}{s^m} \cos m\pi/2$$

$$\int_0^\infty \sin sx\, x^{m-1} dx = \frac{\Gamma m}{s^m} \sin m\pi/2$$

Type-III Inverse Fourier Transforms.

$$f(t) = F^{-1}(f(s)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s) \cdot e^{ist} ds$$

Ex. Find $f(x)$ if its $f(x) =$ Fourier sine Transformation is $\frac{s}{1+s^2}$

or

If $\bar{f}(s) = \frac{s}{1+s^2}$. Then find $f(x)$

or

Find $F_s^{-1}\left(\frac{s}{1+s^2}\right)$

Sol

$$\because \bar{f}_s(s) = \frac{s}{1+s^2}$$

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \bar{f}(s) \cdot \sin sx\, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \cdot \sin sx\, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s^2}{s(1+s^2)} \cdot \sin sx\, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{(1+s^2-1) \sin sx\, ds}{s(1+s^2)}$$

$$= \frac{2}{\pi} \int_0^\infty \left(\frac{1}{s} \sin sx - \frac{\sin sx}{s(1+s^2)}\right) ds$$

$$= \frac{2}{\pi} \left[\int_0^\infty \frac{\sin sx}{s} ds - \int_0^\infty \frac{\sin sx\, ds}{s(1+s^2)}\right.$$

$$\frac{2}{\pi} \left[\frac{\pi}{2} - \int_0^\infty \frac{\sin sx\, ds}{s(1+s^2)}\right.$$

$$f(x) = \frac{2}{\pi}\cdot\frac{\pi}{2} - \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\,ds$$

$$\therefore f(x) = 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\,ds \quad \text{--- (1)}$$

Diff w.r.t. x. By Leibnitz Rule

$$\therefore \frac{d f(x)}{dx} = 0 - \frac{2}{\pi}\int_0^\infty \frac{\cos s.x . \cancel{s}}{\cancel{s}(1+s^2)}\,ds \quad \text{--- (**)}$$

Again Diff. w.r.t x.

$$\frac{d^2}{dx^2} f(x) = +\frac{2}{\pi}\int_0^\infty \frac{\sin sx . s}{1+s^2}\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\cdot\sin sx\,ds$$

By (*)

$$\frac{d^2 f}{dx^2} = f(x)$$

$$\frac{d^2 f}{dx^2} - f = 0$$

$$(D^2 - 1) f = 0$$

$$D^2 - 1 = 0$$

$$D^2 = 1 \quad | \quad D = \pm 1$$

Sol

$$f(x) = A.e^{x} + B.e^{-x} \quad \text{--- (2)}$$

$$\frac{df}{dx} = A.e^{x} - B.e^{-x} \quad \text{--- (3)}$$

$\therefore f(x) = f$

$f(0) = 1$ $\quad$ ($\because$ By (1)) Put $x=0$

Put $x=0$ in (2)

$$f(0) = A.e^0 + B.e^0$$

$$1 = A + B \quad \text{--- (4)}$$

Put $x=0$ in Eq (**)

$$\frac{df}{dx} = -\frac{2}{\pi}\int_0^\infty \frac{\cos 0 . ds}{1+s^2}$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{ds}{1+s^2}$$

P.T.O

$= \frac{-2}{\pi}\left[\tan^{-1} s\right]_0^{\infty} = \frac{-2}{\pi}\left[\tan^{-1}\infty - \tan^{-1} 0\right) = \cancel{\frac{-2}{\pi}\left(\frac{\pi}{2}\right)}$ $\frac{df}{dx}$ "

$\frac{df}{dx} = -1$ At $x = \infty$

Now Eq (3) $\frac{df}{dx} = A e^0 - B \cdot e^0$

$-1 = A - B$

$A - B = -1$ — (5)

From Eq (4) & (5)

$$\begin{array}{r} A + B = 1 \\ A - B = -1 \\ \hline 2A = 0 \end{array}$$

$\boxed{A = 0}$

$\boxed{B = 1}$

$\therefore$ By (2)

$f(x) = 0 + 1 \cdot e^{-x}$

$f(x) = e^{-x}$ Ans

Q Find $f(x)$ given that $\bar{f}_s(s) = \frac{e^{-as}}{s}$, $a > 0, s > 0$

and hence deduce $F_s^{-1}(1/s)$

Sol $\bar{f}_s(s) = \frac{e^{-as}}{s}$.

By sine. Inversion formula.

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{e^{-as}}{s} \cdot \sin sx \, ds \quad — (1)$$

Diff w.r.t 'x'

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{\cancel{s}}\cdot \cancel{s}\cos sx\,dx$$

$$= \frac{2}{\pi}\int_0^\infty e^{-as}\cdot\cos sx\,dx = \frac{2}{\pi}\left[\frac{e^{-as}}{a^2+x^2}\left[-a\cos sx + x\sin sx\right]\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{1}{a^2+x^2}(-a\cdot 1+0)\right] = \frac{2}{\pi}\left(\frac{a}{a^2+x^2}\right)$$

$$\frac{df}{dx} = \frac{2a}{\pi(a^2+x^2)}$$

I.B.S

$$f = \frac{2a}{\pi}\int\frac{dx}{a^2+x^2} = \frac{2\cancel{a}}{\pi}\left[\frac{1}{\cancel{a}}\tan^{-1}\frac{x}{a}\right]+c$$

$$f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a}+c \quad \text{—— (2)}$$

At $x=0$ in Eq (1) $\quad f(0)=0$

$$\underset{0}{f(0)} = \frac{2}{\pi}\underset{0}{\tan^{-1}0}+c \qquad \therefore c = 0$$

$$f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a}$$

$$f(x) = F_s^{-1}(1/s) = \frac{\pi}{2}\cdot\frac{2}{\pi} = 1$$

$$=\underline{1}$$

Deduction At $a \to 0$

$$f(x) = \frac{2}{\pi}\underset{1}{\tan^{-1}\infty}$$

$$= \frac{2}{\pi}\left(\frac{\pi}{2}\right)$$

Q Find the Fourier Transform of $f(x)$.

$\therefore f(x) = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1 \end{cases}$

Hence deduce $\int_0^{\infty} \frac{\sin x}{x}\, dx$.

Sol

$F(f(x)) = \int_{-\infty}^{\infty} f(x) \cdot e^{-isx} \cdot dx$

$= \int_{-\infty}^{-1} f(x)\, e^{-isx} dx + \int_{-1}^{1} f(x)\, e^{-isx} dx + \int_{1}^{\infty} f(x) e^{-isx} dx$

$0 + \int_{-1}^{1} f(x)\, e^{-isx} \cdot dx + 0$

$= \int_{-1}^{1} 1 \cdot e^{-isx} \cdot dx$ $\quad (\because f(x) = 1 \text{ for } |x|<1)$

$= \frac{-1}{is}\left[e^{-isx}\right]_{-1}^{1}$

$\frac{-1}{is}\left[e^{-is} - e^{+is}\right] = \frac{+2}{s}\left[\frac{e^{is} - e^{-is}}{2i}\right]$

$= \frac{2}{s} \cdot \sin s$

$F(f(x)) = \begin{cases} \frac{2 \sin s}{s} & \text{if } s \neq 0 \\ 2 & \text{if } s = 0. \end{cases}$

$\because$ Note: $s = 0$

$\int_{-1}^{1} 1 \cdot e^{0} \cdot ds = \int_{-1}^{1} 1 \cdot ds$

$= \left[(s)\right]_{-1}^{1} = 1 + 1 = 2$

$\therefore$ By Inversion formula

$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx} \cdot ds$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cdot 2\frac{\sin s}{s}\, e^{isx} \cdot ds \quad (s \neq 0$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin s}{s}\, e^{isx}\, ds =$$

$$\therefore \int_{-\infty}^{\infty} \frac{2\sin s}{s}\, e^{isx}\, ds = 2\pi\, f(x) = 2\pi \begin{cases} 1 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

$$\int_{-\infty}^{\infty} \frac{2\sin s}{s}\, e^{isx}\, ds = \begin{cases} 2\pi & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

At $x=0$

$$\int_{-\infty}^{\infty} \frac{2\sin s}{s} \cdot e^{0}\, ds = 2\pi$$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s} \cdot ds = \pi$$

$\because f(s) = \frac{\sin s}{s}$

$f(-s) = \frac{-\sin s}{-s} = f(s)$

Even function

$$= 2\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \pi$$

$$\boxed{\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \pi/2}$$

Def:- Convolution :- The convolution of two function $f(x)$ & $g(x)$ over the int $(-\infty, \infty)$, denoted by. $f*g$

$$\therefore f*g = \int_{-\infty}^{\infty} f(u) \cdot g(x-u)\, du. \qquad (\because f*g = g*f$$

Convolution thm of Fourier Transformation:-

Statment:- If $F(f(x))$ and $F(g(x))$ F.T of two function $f(x)$ + $g(x)$. Then the F.T of Convolution of $f(x)+g(x)$ is the product of their F.T.

$$F[f(x) * g(x)] = F[f(x) * F(g(x))$$

Proof:-

$$F[f(x) * g(x)] = \int_{-\infty}^{\infty} e^{-isx} (f(x) * g(x))$$ (∵ By def of F.T)

$$= \int_{-\infty}^{\infty} e^{-isx} \left(\int_{-\infty}^{\infty} f(u) \cdot g(x-u)\, du\, dx \right.$$ (∵ By def of Convolution)

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} e^{-isx} g(x-u)\, dx \right] du$$

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} e^{-is(x-u)} \cdot g(x-u)\, e^{-isu}\, dx \right] du$$

(∵ By change of order of the integration)

$$= \int_{-\infty}^{\infty} f(u)\, e^{-isu} \left[\int_{-\infty}^{\infty} e^{-is(x-u)} \cdot g(x-u)\, dx \right) du$$

Put $x-u = t$.

$dx = dt$

$$= \left(\int_{-\infty}^{\infty} f(u) \cdot e^{-isu} \right) \left[\int_{-\infty}^{\infty} e^{-ist} \cdot g(t). \right] du$$

Replace $u \to x$, $t \to x$

$$\left(\int_{-\infty}^{\infty} f(u) \cdot e^{-isu} \right) F(g(t) \cdot du = \left(\int_{-\infty}^{\infty} f(x) \cdot e^{-isx} \right) F(g(x)).$$

$$= \cdot F(f(x) * F(g(x))$$ H.S

T of the derivative :-

Thm :- If the F.T of $f(x)$ is $\bar{f}(s)$. Then the F.T of $f'(x)$ is given by $is\bar{f}(s)$

or

To prove $F(f'(x)) = is\bar{f}(s)$

Proof :- $F(f'(x)) = \int_{-\infty}^{\infty} e^{-isx} \cdot f'(x)$ By Part

$$\left[e^{-isx} \cdot f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -is \cdot e^{-isx} \cdot f(x)\,dx$$

$$= 0 + is \int_{-\infty}^{\infty} e^{-isx} \cdot f(x)\,dx$$

$$= is\,\bar{f}(s)$$

Ilsy. $F\left[\frac{d^n}{dx^n} f\right] = (is)^n \cdot \bar{f}(s)$

Thm :- If the Fourier sine and cosine Transformations of $f(x)$ and $f'(x)$ exist and $f(x) \to 0$ as $x \to \infty$

Then. (i) $F_s[f'(x)] = -s\,F_c[f(x)]$

(ii) $F_c[f'(x)] = s\,F_c(f(x)) - f(0)$

Proof :- $\therefore F_s[f'(x)] = \int_0^{\infty} f'(x) \cdot \sin sx \cdot dx$. (By def of sine Transform) By Part

$$= \left[\sin sx\, f(x)\right]_{x=0}^{\infty} - \int_0^{\infty} \cos s\cdot x \cdot s \cdot f(x)\,dx$$

$$0 - 0 - s \int_0^\infty f(x) \cdot \cos sx \, dx = - s \, F_c[f(x)]$$

$$F_s(f'(x)) = - s \, F_c(f(x))$$

(ii) $$F_c[f'(x)] = \int_0^\infty \underset{\text{II}}{f'(x)} \cdot \underset{\text{I}}{\cos sx} \, dx$$

By Part

$$= \left[\cos sx \cdot f(x)\right]_0^\infty + \int_0^\infty \sin sx \cdot s \, f(x) \cdot dx$$

$$\left[0 - f(0) \cdot \cos 0\right] + s \int_0^\infty \underbrace{f(x) \cdot \sin sx \, dx}$$

$$- f(0) \cdot 1 + s \, F_s(f(x))$$

$$F_c(f'(x)] = s \, F_s(f(x)) - f(0)$$

IIsly we can prove

(i) $$F_s[f''(x)] = - s \, F_c(f'(x)) = - s \left[s \, F_s(f(x)) - f(0) \right]$$

$$= - s^2 F_s(f(x)) + s \, f(0)$$

(ii)

$$F_c(f''(x)] = s \, \underline{F_s} (f'(x)) - f'(0)$$

$$= s \left[- s \, F_c(f(x)) \right] - f'(0)$$

$$= - s^2 F_c(f(x)) - f'(0)$$

Relation between Fourier and Laplace Transformation.

Let us define a function $f(t)$. s.t

$$f(t) = \begin{cases} e^{-xt}\,\phi(t), & t > 0 \\ 0 & t < 0 \end{cases}$$

$$F[f(t)] = \int_{-\infty}^{\infty} e^{-iyt} f(t)\,dt \qquad (\because \text{By def of F.T})$$

$$= \int_{-\infty}^{0} e^{-iyt}\cdot f(t) + \int_{0}^{\infty} e^{-iyt}\cdot f(t)\,dt$$

$$= \int_{-\infty}^{0} e^{-iyt}(0) + \int_{0}^{\infty} e^{-iyt}\cdot e^{-xt}\,\phi(t)\,dt$$

$$0 + \int_{0}^{\infty} e^{-(x+iy)t}\,\phi(t)\,dt$$

$$= \int_{0}^{\infty} e^{-st}\,\phi(t)\,dt \qquad \text{Put } x+iy = s$$

$$= L[\phi(t)] \qquad \text{By def of L.T}$$

$$F[f(t)] = L[\phi(t)]$$

Parseval's Identity for F.T :-

If $F(s)$ and $G(s)$ denoted the F.T of $f(x)$ and $g(x)$. Then

P.T.O

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\cdot\overline{G(s)}\,ds = \int_{-\infty}^{\infty} f(x)\cdot\overline{g(x)}\,dx$

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} [F(s)]^2\,ds = \int_{-\infty}^{\infty}[f(x)]^2\,dx$ (∵ Bar showing that complex conjugate.)

Proof:-

(i) $\int_{-\infty}^{\infty} f(x)\cdot\overline{g(x)}\,dx = \int_{-\infty}^{\infty} f(x)\cdot\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G(s)}\cdot e^{isx}\cdot ds\right]$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G(s)}\left[\underline{\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\cdot dx}\right]ds$ (∵ By inverse Transformation of $\overline{g}(x)$)

$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G(s)}\cdot F(s)\,ds$ (∵ By def of Trans

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\cdot\overline{G(s)}\cdot ds$

H.P

(ii) Put $g(x) = f(x)$ in (i) Part

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cdot\overline{f}(x) = \int_{-\infty}^{\infty}|f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)\cdot\overline{F(s)}\,ds$

$\Rightarrow \frac{1}{2\pi}\int|F(s)|^2\,ds = \int_{-\infty}^{\infty}|f(x)|^2\,dx$

Parseval's Identity for Fourier Sine and cosine Transformation:-

Thm:- If $F_c(s)$ and $G_c(s)$ denoted the Fourier cosine Transformation of $f(x)$ & $g(x)$. r.t.