CHAPTER 22

Numerical Solution to Ordinary Differential Equations and Partial Differential Equations

Taylor Series Method, Picard's Method, Euler's Method, Modified Euler's Method, Runge-Kutta Method of Fourth Order, Milne's Predictor-Corrector Method, Partial Differential Equations, Liebmann Iteration, Crank-Nicholson Iteration.

NUMERICAL SOLUTION TO ORDINARY DIFFERENTIAL EQUATIONS:

In many practical problems we come across differential equations having initial conditions and boundary conditions. A few types of these differential equations can be solved by analytical methods whereas others have to be solved numerically. That is why numerical methods play an important role in solving the differential equations which cannot be solved by analytical methods. Here we shall first discuss the methods for obtaining approximate solutions to first order and first degree ordinary differential equations

of the form $\dfrac{dy}{dx} = f(x, y)$ with the initial conditions $y = y_0$ when $x = x_0$. ...(1)

Actually the numerical solution to a differential equation means finding the discrete values of y at pivotal points $x_0 + ih$ $(i = 0, 1, 2, \ldots n)$, h being small. Later, the concept will be extended to cover the second order differential equations also. The numerical methods developed for solving equations of the form (1) will give the solution in one of the following forms:

(*i*) A series for y in terms of powers of x from which the values of y can be obtained by directly substituting the values of x. The methods of Taylor and Picard belong to this category.

(*ii*) A set of tabulated values of x and y. The methods of Euler, Runge-Kutta, Milne belong to this category.

TAYLOR SERIES METHOD:

This is purely a numerical step by step method. Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \qquad \ldots(1)$$

The Taylor's expansion of $y(x)$ about the point (x_0, y_0), is

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \ldots \text{ where } y'(x_0) = \left(\frac{dy}{dx}\right)_{x = x_0}, y''(x_0) = \left(\frac{d^2y}{dx^2}\right)_{x = x_0}, \text{etc.}$$

We first obtain $y_1 = y(x_1) = y(x_0 + h)$, h being small, using the Taylor's expansion

$$\boxed{y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots} \quad \text{where } h = x_1 - x_0. \qquad \ldots(2)$$

The coefficients $y'_0, y''_0, y'''_0, \ldots$ are computed by successively differentiating (1). Once y_1 is obtained we easily calculate $y'_1, y''_1, y'''_1, \ldots$ using $y' = f(x, y)$.

Expanding $y(x)$ now in a Taylor series about the point (x_1, y_1), gives

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$$

Repeating the process, we get, in general,

$$\boxed{y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \ldots} \quad \text{where} \quad y_n^{(r)} = \left(\frac{d^r y}{dx^r}\right)_{(x_n, y_n)} \quad \ldots(3)$$

In (3) there is an infinite series which has to be truncated at some term to get the numerical value calculated. Actually at this stage we retain the terms upto h^n and neglect the terms involving h^{n+1} and higher order of h. The Taylor algorithm used here is said to be of order n. Of course, the error can be reduced further by including more number of terms for calculations.

EXAMPLE 22.1. **(a)** Solve $\frac{dy}{dx} = x - y^2$, given that $y(0) = 1$, for $y(0.1)$ correct to four places of decimal using Taylor series method.

(b) Apply Taylor's method to find $y(0.2)$ from $y' - 4y = 0$, given that $y(0) = 1$. [GGSIPU 2014]

SOLUTION: (a) The Taylor series for $y(x)$ about $x = x_0 = 0$, is given by

$$y(x) = y_0 + x y_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{4!} y_0^{iv} + \ldots \quad \ldots(1)$$

Here $y'(x) = x - y^2$ hence $y'_0 = x_0 - y_0^2 = 0 - 1^2 = -1,$

$y''(x) = 1 - 2yy'$ hence $y''_0 = 1 - 2y_0 y_0' = 1 - 2(1)(-1) = 3,$

$y'''(x) = -2yy'' - 2y'^2$ hence $y'''_0 = -2y_0 y_0'' - 2y_0'^2 = -2(1)(3) - 2(-1)^2 = -8$

$y^{iv}(x) = -2yy''' - 2y'y'' - 4y'y'' = -2y_0 y_0''' - 6y_0' y_0'' = -2(1)(-8) - 6(-1)(3) = 34$

and so on. Using these values in (1), we get

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 + \ldots$$

To get $y(0.1)$ correct to four places of decimal it is found that the terms upto x^4 are to be taken and others neglected. Thus, we get $y(0.1) = 0.9138$ approx. **Ans.**

(b) By Taylor's method (here $x_0 = 0$, $y_0 = 1$)

$$y = y_0 + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \ldots$$

$$= 1 + (x - 0)y'(x_0) + \frac{(x - 0)^2}{2!} y''(x_0) + \frac{(x - 0)^3}{3!} y'''(x_0) + \ldots$$

Further, since $y' = 4y$ we have $y'(x_0) = 4y_0 = 4$ and $y'' = 4y'$ $\therefore$ $y''(x_0) = 4y'(x_0) = 16.$

Also $y''' = 4y''$ $\therefore$ $y'''(x_0) = 4y''(x_0) = 4(16) = 64$

Thus, $y = 1 + 4x + \frac{16x^2}{2!} + \frac{64}{3!}x^3 + \ldots = 1 + 4x + 8x^2 + (32/3)x^3 + \ldots$

$\therefore$ $y(0.2) = 1 + 4(0.2) + 8(0.2)^2 + (32/3)(0.2)^3 = 3.2$ approx. **Ans.**

EXAMPLE 22.2. Using Taylor series scheme find the numerical solution to

$$\frac{dy}{dx} = 2y + 3e^x \quad \text{at} \quad x = 0.2, \quad \text{given that} \quad y(0) = 1.$$

Compare the result with the analytical solution.

SOLUTION: Here we shall find $y(0.2)$ in two steps taking $h = 0.1$

Step I To find y at $x = 0.1$, take $h = 0.1$, $x_0 = 0$, $y_0 = 1$.

then $y' = 2y + 3e^x$, $y'' = 2y' + 3e^x$,

$y''' = 2y'' + 3e^x$, $y^{iv} = 2y''' + 3e^x$, ...

$\therefore$ $y_0' = 2y_0 + 3e^{x_0} = 5$, $y_0'' = 2y_0' + 3e^{x_0} = 13$,

$y_0''' = 2y_0'' + 3e^{x_0} = 29$, $y_0^{iv} = 2y_0''' + 3e^{x_0} = 61.$, etc.

Therefore, the Taylor series solution for the given differential equation, is

$$y = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \ldots$$

Thus, $y(0.1) = 1 + 0.1\,(5) + \frac{(0.1)^2}{2!}(13) + \frac{(0.1)^3}{3!}(29) + \frac{(0.1)^4}{4!}(61) = 1.5701.$

Step II. For Obtaining $y(0.2)$, we take $y_0 = 1.5701$ at $x_0 = 0.1$ and obtain

$y_0' = 2y_0 + 3e^{0.1} = 2\,(1.5701) + 3e^{0.1} = 6.4557$,

$y_0'' = 2y_0' + 3e^{0.1} = 2\,(6.4557) + 3e^{0.1} = 16.2270$,

$y_0''' = 2y_0'' + 3e^{0.1} = 2\,(16.2270) + 3e^{0.1} = 35.7695$,

$y_0^{iv} = 2y_0''' + 3e^{0.1} = 2\,(35.7695) + 3e^{0.1} = 74.8545$, etc.

$\therefore$ $y(0.2) = 1.5701 + 0.1(6.1557) + \frac{(0.1)^2}{2!}(16.2270) + \frac{(0.1)^3}{3!}(35.7695) + \frac{(0.1)^4}{4!}(74.8545)$

$= 2.3025$ approx. **Ans.**

Analytic Solution

Given equation is linear in y with integrating factor $= e^{-2x}$

$\therefore$ Solution is $ye^{-2x} = \int 3e^x e^{-2x}\,dx + C$ or $y = -3e^x + Ce^{2x}$.

Since $y = 1$ at $x = 0$ $\therefore$ $C = 4$. Hence $y = -3e^x + 4e^{2x}$.

$\therefore$ $y_1 = -3e^{0.1} + 4e^{0.2} = 1.57$ and $y(0.2) = -3e^{0.2} + 4\,e^{0.4} = 2.3031.$

PICARD'S METHOD OF SUCCESSIVE APPROXIMATION:

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ or $dy = f(x, y)dx$ where $y(x_0) = y_0$. ...(1)

Integrating it on both sides between the corresponding limits for x and y, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx \Rightarrow \boxed{y = y_0 + \int_{x_0}^{x} f(x, y)\,dx} \quad \text{...(2)}$$

Equations of the type (2) are called integral equations and will be solved by successive approximations. We get first approximation for y, as $y^{(1)}$, by substituting y_0 for y in the integrand $f(x, y)$.

Thus, $$\boxed{y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx} \quad \text{...(3)}$$

The integral on the right hand side of (3) can be evaluated and the resulting $y^{(1)}$ is substituted for y_0 in the integrand of (2) to obtain the second approximation $y^{(2)}$, as

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

The process is successively continued till the desired accuracy is achieved.

The n^{th} approximation $y^{(n)}$ is given by $\boxed{y^{(n)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(n-1)}\right)dx}$

EXAMPLE 22.3. (a) Employing Picard's method, solve the equation $\frac{dy}{dx} = 2x - y$, given that $y(0) = 0.9$ for $x = 0.2$, $x = 0.4$ and $x = 0.6$ and check the result with exact values. [GGSIPU 2010]

(b) Obtain the Picard's approximation for the initial value problem $\frac{dy}{dx} = xy$, given $y(1) = 2$ correct to three decimal places. [GGSIPU 2013]

SOLUTION: (a) Here $x_0 = 0$, $y_0 = 0.9$, $f(x, y) = 2x - y$. Therefore, by Picard's scheme

$$y^{(1)} = y_0 + \int_0^x f(x, y_0)\, dx = y_0 + \int_0^x (2x - y_0)\, dx = 0.9 + \int_0^x (2x - 0\cdot 9)\, dx = x^2 - 0\cdot 9x + 0\cdot 9.$$

Next
$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx = y_0 + \int_0^x (2x - x^2 + 0\cdot 9x - 0.9)\, dx = 0.9 - 0.9x + 0\cdot 9\frac{x^2}{2} - \frac{1}{3}x^3.$$

and
$$y^{(3)} = y_0 + \int_0^x (2x - y^{(2)})\, dx = 0.9 + \int_0^x \left(2x - 0.9 + 0.9x - 0.9\frac{x^2}{2} + \frac{1}{3}x^3\right) dx$$

$$= 0.9 - 0.9x + 1.45x^2 - 0.15x^3 + \frac{x^4}{12}. \qquad \text{...(1)}$$

Now putting $x = 0.2$, 0.4 and 0.6 in (1), we get

$$y^{(3)}(0.2) = \frac{(0\cdot 2)^4}{12} - \frac{2\cdot 9}{6}(0\cdot 2)^3 + \frac{2\cdot 9}{2}(0.2)^2 - 0.9(0.2) + 0.9 = 0.7742,$$

$$y^{(3)}(0.4) = 0.7432 \quad \text{and} \quad y^{(3)}(0.6) = 0.7884.$$

The **analytic solution** to the given equation is

$$ye^x = \int 2x\, e^x\, dx + C = C + 2(x - 1)e^x.$$

Since $y(0) = 0\cdot 9$ we get $C = (0\cdot 9 + 2)(1) = 2.9$.

$\therefore$ Analytic solution is $y = 2.9e^{-x} + 2(x - 1)$.

at $x = 0\cdot 2$, $y(0\cdot 2) = 2\cdot 9\, e^{-0.2} + 2(0\cdot 2 - 1) = 0.7743$,

at $x = 0\cdot 4$, $y(0\cdot 4) = 2\cdot 9\, e^{-0\cdot 4} + 2(0\cdot 4 - 1) = 0.7439$,

and at $x = 0.6$, $y(0.6) = 2\cdot 9e^{-0\cdot 6} + 2(0.6 - 1) = 0.7915$.

Clearly, at $x = 0.2$ and $x = 0.4$ the numerical solutions are correct upto two decimal places only.

Note that the methods of Taylor and Picard give the solution in series which yields value of y at a particular value of x by direct substitution in the series solution.

(b) $\frac{dy}{dx} = f(x, y) = xy$. Here $x_0 = 1$, $y_0 = 2$. Therefore, by Picard's formula

$$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\, dx, \quad n = 1, 2, 3, \ldots = y_0 + \int_{x_0}^x xy_{n-1}\, dx$$

$$\therefore \; y_1 = y_0 + \int_{x_0}^x x\, y_0\, dx = 2 + \int_1^x 2x\, dx = x^2 + 1.$$

$$y_2 = y_0 + \int_1^x xy_1 dx = 2 + \int_1^x x(1 + x^2)\, dx = 2 + \frac{x^2}{2} + \frac{x^4}{4} - \left(\frac{1}{2} + \frac{1}{4}\right) = \frac{5}{4} + \frac{x^2}{2} + \frac{x^4}{4}.$$

$$y_3 = 2 + \int_1^x x\left(\frac{5}{4} + \frac{x^2}{2} + \frac{x^4}{4}\right) dx = 2 + \frac{5x^2}{8} + \frac{x^4}{8} + \frac{x^6}{24} - \left(\frac{5}{8} + \frac{1}{8} + \frac{1}{24}\right) = \frac{29}{24} + \frac{5x^2}{8} + \frac{x^4}{8} + \frac{x^6}{24} \quad \textbf{Ans.}$$

EULER'S METHOD:

[GGSIPU 2011]

This is a step by step method, quite elementary in nature and should not be used for practical solutions. However, the principle involved is of vital importance. More accurate methods which can be used in practice, are given later. Let $x_0, x_1, x_2, \dots$ be equi-spaced values of x at a spacing h and let $y_0, y_1, y_2, \dots$ be the corresponding values of y on a curve satisfying the differential equation

$$\frac{dy}{dx}=f(x,y), \quad y(x_0)=y_0 \qquad \text{...(1)}$$

Suppose the solution of (1) is $y=\phi(x)$.

For the first step we find the value of y at $x=x_0+h$ and call it y_1, then by (1)

$$y_1=\phi(x_1)=\phi(x_0+h)=\phi(x_0)+h\phi'(x_0)+\frac{h^2}{2!}\phi''(x_0)+\dots$$

$$=y_0+hf(x_0,y_0)+\frac{h^2}{2!}f'(x_0,y_0)+\dots$$

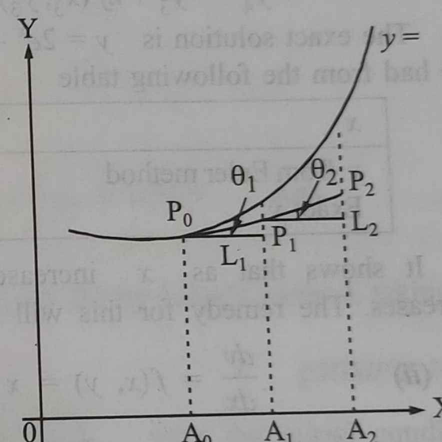
Hence, if h is small, neglecting terms involving h^2 and higher powers of h, we have

$$\boxed{y_1=y_0+hf(x_0,y_0).} \qquad \text{...(2)}$$

On the same lines, for the second step, we take (x_1, y_1) as the starting point and get $y_2=y_1+hf(x_1,y_1)$.

Continuing the process, we have, in general,

$$\boxed{y_{n+1}=y_n+hf(x_n,y_n),\ n=0,1,2,\dots} \qquad \text{...(3)}$$



This formula can be used to determine y_{n+1} when y_n is known. Putting $n = 0, 1, 2, \dots$ in (3) we can successively determine $y_1, y_2, y_3, \dots$ provided the starting point (x_0, y_0) is known.

Let us now interpret the above formulae geometrically. From the adjoining figure, we observe that if, from the given starting point $P_0(x_0, y_0)$ we draw a line P_0P_1 with slope $f(x_0, y_0)$ it meets the ordinate at $A_1(x_0+h, 0)$ in the point P_1 so that

$$L_1P_1=P_0L_1\tan\theta_1=hf(x_0,y_0)$$

$$\therefore \quad y_1=A_1P_1=A_1L_1+L_1P_1=y_0+hf(x_0,y_0) \quad \text{where} \quad \angle P_1P_0L_1=\theta_1. \quad \text{(see figure)}$$

Then from the known point $P_1(x_1, y_1)$ we draw a line P_1P_2 with slope $f(x_1, y_1)$. This meets the ordinate at the point $A_2(x_0 + 2h, 0)$ in the point P_2 so that

$$L_2P_2=P_1L_2\tan\theta_2=hf(x_1,y_1) \quad \text{where} \quad \theta_2=\angle P_2P_1L_2.$$

$$\therefore \quad y_2=A_2P_2=A_2L_2+L_2P_2. \quad \text{Thus,} \quad y_2=y_1+hf(x_1,y_1).$$

Proceeding this way from point to point, the numerical solution by Euler's method represents the sequence of short lines $P_0P_1, P_1P_2, P_2P_3, \dots$ This curve of short lines approximates the analytical solution curve $y=\phi(x)$ as depicted in the above figure where h is quite small. However, if h is not small the curve $P_0P_1P_2\dots$ of short lines deviates significantly from the correct solution curve $y=\phi(x)$ as shown in the above figure. Therefore, h must be small.

EXAMPLE 22.4. (a) Solve for y, the differential equation

$$\frac{dy}{dx} = x + y, \quad y(0) = 1 \quad \text{at} \quad x = 0.0\,(0.2)\,0.8$$

using (*i*) Euler's method. (*ii*) Picard's method.

Also compare with the exact values. [GGSIPU 2011, 2012]

(*b*) Solve by Euler's method $\frac{dy}{dx} = \frac{x - y}{2}$, $y(0) = 1$ over $[0, 3]$ using step size 1/2. [GGSIPU 2013]

SOLUTION: (a) (*i*) Here $x_0 = 0$, $y_0 = 1$, $f(x, y) = x + y$, $h = 0.2$.

We are to find y at $x = 0.2, 0.4, 0.6, 0.8$. Let us write $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$ and $y_1 = y(x_1)$, $y_2 = y(x_2)$, $y_3 = y(x_3)$, and $y_4 = y(x_4)$.

By Euler's scheme

$$\begin{aligned}
y_1 &= y_0 + hf(x_0, y_0) = y_0 + h(x_0 + y_0) = 1 + 0.2(0 + 1.0) = 1.2 \\
y_2 &= y_1 + hf(x_1, y_1) = y_1 + h(x_1 + y_1) = 1.2 + 0.2(0.2 + 1.2) = 1.48 \\
y_3 &= y_2 + hf(x_2, y_2) = y_2 + h(x_2 + y_2) = 1.48 + 0.2(0.4 + 1.48) = 1.856. \\
y_4 &= y_3 + hf(x_3, y_3) = y_3 + h(x_3 + y_3) = 1.856 + 0.2(0.6 + 1.856) = 2.3472.
\end{aligned}$$

The exact solution is $y = 2e^x - x - 1$. Comparison between estimated values and exact values can be had from the following table

x	0	0.2	0.4	0.6	0.8
y from Euler method	1	1.2	1.48	1.856	2.3472
Exact y	1	1.2428	1.5836	2.0442	2.6511

It shows that as x increases, y deviates from exact values and rate of deviation also increases. The remedy for this will be provided in the modified Euler's method to be discussed next.

(*ii*) $\frac{dy}{dx} = f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$ then by Picard's method

$$y_1 = y_0 + \int_0^x (x + y_0)\,dx = 1 + \int_0^x (x + 1)\,dx = 1 + x + \frac{x^2}{2}$$

Taking the second approximation, we get

$$y_2 = y_0 + \int_0^x (x + y_1)\,dx = 1 + \int_0^x \left(x + 1 + x + \frac{x^2}{2}\right) dx = 1 + x + x^2 + \frac{x^3}{6}.$$

Taking third approximation, we have

$$y_3 = y_0 + \int_0^x (x + y_2)\,dx = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Similarly, $$y_4 = y_0 + \int_0^x (x + y_3)\,dx = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

and $$y_5 = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}.$$

Analytically, the solution comes out as

$$y = -1 - x + 2e^x = -1 - x + 2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots\right)$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} + \ldots$$ **Ans.**

which validates the numerical approach. **Hence proved.**

(*b*) Here $x_0 = 0$, $y_0 = 1$, $f(x, y) = (x - y)/2$.

Using Euler's scheme, we have

$$y_{n+1} = y_n + h\,f(x_n, y_n) = y_n + \frac{1}{2}\left(\frac{x_n - y_n}{2}\right) = \frac{x_n}{4} + \frac{3}{4}y_n.$$

$$y\left(\frac{1}{2}\right) = y_1 = \frac{x_0}{4} + \frac{3}{4}y_0 = 0 + \frac{3}{4} = 0.75,$$

$$y(1) = y_2 = \frac{1}{4}(x_1) + \frac{3}{4}y_1 = \frac{1}{4}\left(\frac{1}{2}\right) + \frac{3}{4}\left(\frac{3}{4}\right) = 0.6875,$$

$$y\left(\frac{3}{2}\right) = y_3 = \frac{1}{4}(x_2) + \frac{3}{4}y_2 = \frac{1}{4}(1) + \frac{3}{4}\left(\frac{11}{16}\right) = 0.7656,$$

$$y(2) = y_4 = \frac{1}{4}(x_3) + \frac{3}{4}y_3 = \frac{1}{4}\left(\frac{3}{2}\right) + \frac{3}{4}\left(\frac{49}{64}\right) = 0.9492.$$

Similarly $y\left(\frac{5}{2}\right) = \frac{1}{4}(x_4) + \frac{3}{4}(y_4) = 1.2119$ and $y(3) = \frac{1}{4}(x_5) + \frac{3}{4}y_5 = 1.5339$ **Ans.**

EXAMPLE 22.5. (*a*) Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 1$, compute $y(0.02)$ by Euler's method, taking $h = 0.01$. [*GGSIPU 2010*]

(*b*) Solve the differential equation $\frac{dy}{dx} = \frac{y - x}{y + x}$ with the initial condition $y(0) = 1$ for $x = 0.1$, taking $h = 0.02$. [*GGSIPU 2014, 2015*]

SOLUTION: (*a*) We have $\frac{dy}{dx} = f(x, y) = x^3 + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.01$ $\therefore$ $x_1 = 0.01$ and $x_2 = 0.02$.

Applying Euler's formula, we get $y_1 = y(x_0 + h) = y(0.01)$

$$y_1 = y_0 + h\,f(x_0, y_0) = 1 + 0.01\,(x_0^3 + y_0) = 1 + 0.01\,(0^3 + 1) = 1.01$$

Next, $y_2 = y(0.2) = y_1 + hf(x_1, y_1) = 1.01 + 0.01\,(x_1^3 + y_1) = 1.01 + 0.01\,[(0.01)^3 + 1.01] = 1.0201$

or $y_2 = y\,(0.02) = 1.0201.$ **Ans.**

(*b*) Here $\frac{dy}{dx} = \frac{y - x}{y + x}$ hence $f(x, y) = \frac{y - x}{y + x}$. Also $x_0 = 0$, $y_0 = 1$.

By Euler's method $y_{n+1} = y_n + hf(x_n, y_n)$, $n = 0, 1, 2, \ldots$.

Taking $h = 0.02$ we get $x_1 = 0.02$, $x_2 = 0.04$, $x_3 = 0.06$. $x_4 = 0.08$, and $x_5 = 0.10$.

Using Euler's formula, $y_1 = y(0.02) = y_0 + hf(x_0, y_0) = 1 + (0.02)\left(\frac{1 - 0}{1 + 0}\right) = 1.02$

Next, $y_2 = y\,(0.04) = y_1 + hf\,(x_1, y_1) = 1.02 + 0.02\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right) = 1.0392$

and $$y_3 = y(0.06) = y_2 + h\left(\frac{y_2 - x_2}{y_2 + x_2}\right) = 1.0392 + 0.02\left(\frac{1.0392 - 0.04}{1.0392 + 0.04}\right) = 1.0577.$$

Similarly, $$y_4 = y(0.08) = 1.0577 + 0.02\left(\frac{1.0577 - 0.06}{1.0577 + 0.06}\right) = 1.0756$$

and $$y_5 = y(0.1) = 1.0756 + 0.02\left(\frac{1.0756 - 0.08}{1.0756 + 0.08}\right) = 1.0928. \quad \textbf{Ans.}$$

MODIFIED EULER'S METHOD:

[GGSIPU 2012]

In the Euler's method the curve of solution in the subinterval (A_0, A_1) is approximated by the tangent at P_0 so that at P_1, we have (See figure given in Eulers method)

$$y_1 = y_0 + h f(x_0, y_0) \quad ...(1)$$

and then the slope of the curve of solution through P_1 is $f(x_0 + h, y_1)$ which was calculated and the tangent at P_1 to the curve P_1Q was drawn meeting the ordinate through A_2 in P_2 $(x_0 + 2h, y_2)$ (Refer to the figure in Euler's method). According to the modified Euler scheme we find a better approximation of y_1 as $y_1^{(1)}$ by taking the slope of the curve as mean of the slopes of tangents at P_0 and P_1, thus,

$$\boxed{y_1^{(1)} = y_0 + h\left[\frac{f(x_0, y_0) + f(x_0 + h, y_1)}{2}\right]} \quad ...(2)$$

As the slope of tangent at P_1 is not known, we take y_1 as obtained in (1) by Euler's method and insert it on the right hand side of (2) to get the first modified value of y_1 as $y_1^{(1)}$. The equation (1) is therefore rightly called as **predictor** of y while (2) serves as the **corrector of** y_1. The corrector is then again applied, i.e., insert $y_1^{(1)}$ as y_1 in (2) and we get a still better value of y_1 as $y_1^{(2)}$ given by

$$\boxed{y_1^{(2)} = y_0 + h\left[\frac{f(x_0, y_0) + f(x_0 + h, y_1^{(1)})}{2}\right]}$$

The process is repeated until two consecutive values of y agree within specified degree of accuracy. Once y_1 is obtained to the desired degree of accuracy, then y corresponding to the point $A_2(x_0 + 2h, 0)$ is found from the predictor $y_2 = y_1 + hf(x_0 + h, y_1)$

and a better approximation $y_2^{(1)}$ is obtained from the corrector

$$\boxed{y_2^{(1)} = y_1 + \frac{h}{2}\,[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]}$$

The step is repeated till we get y_2 correct to desired degree of accuracy. Next, we compute y_3 as above and so on. This predictor – corrector method is of great importance in practical problems.

EXAMPLE 22.6. (*a*) Employing modified Euler's method, solve the equation

$$\frac{dy}{dx} = \log(x + y); \quad y(1) = 2, \quad \text{for } x = 1.2 \text{ and } x = 1.4$$

correct to three decimal places. [AKTU 2019; GGSIPU 2014]

(*b*) Use modified Euler's formula to obtain $y(0.2)$, $y(0.4)$ and $y(0.6)$ correct to three decimal places given that $\dfrac{dy}{dx} = y - x^2$, $y(0) = 1$. [GGSIPU 2010, 2013]

SOLUTION: (a) To compute $y(1.2)$ we have $x_0 = 1$, $y_0 = 2$, $h = 0.2$ and $f(x, y) = \log(x + y)$.

By the predictor formula

$$y_1 = y(1.2) = y_0 + hf(x_0, y_0) = y_0 + h\log(x_0 + y_0) = 2 + 0.2\log(1 + 2) = 2.2197.$$

Then applying the corrector formula once, we have

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)] = 2 + \frac{0.2}{2}[\log(1 + 2) + \log(1{\cdot}2 + 2{\cdot}2197)]$$

$$= 2 + 0.1\log 10.2591 = 2.2328$$

Again applying the corrector formula, gives

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

$$= 2 + 0.1[\log(1 + 2) + \log(1{\cdot}2 + 2{\cdot}2328)] = 2.2332.$$

Once again applying the corrector, we get

$$y_1^{(3)} = 2 + 0.1[\log(1 + 2) + \log(1{\cdot}2 + 2{\cdot}2332)] = 2.2332.$$

Since $y_1^{(2)}$ and $y_1^{(3)}$ tally upto third place of decimal, we have $y(1.2) = 2.2332$

Next, to compute $y(1.4)$ we take $x_1 = 1.2$, $y_1 = 2.2332$, $h = 0.2$.

By the predictor formula $y_2 = y(1.4) = y_1 + hf(x_1, y_1) = 2.2332 + 0.2\log(1.2 + 2.2332) = 2.47917.$

After getting the predicted value, we apply the corrector formula and get

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_1 + h, y_2)]$$

$$= 2.2332 + \frac{0.2}{2}[\log(1.2 + 2.2332) + \log(1.4 + 2.47917)]$$

$$= 2.2332 + 0.1\log(3.4332 \times 3.87917) = 2.4924.$$

Again applying the corrector formula, we have

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_1 + h, y_2^{(1)})]$$

$$= 2.2332 + 0.1\left[\log(1.2 + 2.2332) + \log(1.4 + 2.4924)\right] = 2.4924$$

Since there is no change in the values of $y_2^{(1)}$ and $y_2^{(2)}$ we can take $y(1.4) = 2.4924$. **Ans.**

(b) $\frac{dy}{dx} = f(x, y) = y - x^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $x_1 = 0.2$.

By Euler's method $y_1 = y(0.2) = y_0 + hf(x_0, y_0) = 1 + 0.2(1 - 0) = 1.2$

By Euler's modified method we have first improved value of y_1 as

$$y_1^{(1)} = y_0 + h\left\{\frac{f(x_0, y_0) + f(x_1, y_1)}{2}\right\}$$

$$= 1 + \frac{0.2}{2}\{(y_0 - x_0^2) + (y_1 - x_1^2)\} = 1 + 0.1\{1 - 0 + 1.2 - (0.2)^2\} = 1.216.$$

Second improved value of y_1, is given by

$$y_1^{(2)} = y_0 + h\left\{\frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}\right\} = 1 + 0.1\{1 - 0 + 1.216 - (0.2)^2\} = 1.2176.$$

Third improved value of y_1 is given by

$$y_1^{(3)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(2)})\} = 1 + 0.1(1 - 0 + 1.2176 - 0.04) = 1.21776.$$

Thus, the value of $y(0.2)$ correct to four decimal places $= 1.21776$ **Ans.**

Next, for computing $y(0.4)$ we take $x_1 = 0.2,\ x_2 = 0.4,\ h = 0.2,\ y_1 = 1.21776$

By Euler's method $y_2 = y(0.4) = y_1 + hf(x_1, y_1) = y_1 + h(y_1 - x_1^2)$

$$= 1.21776 + 0.2[1.21776 - .004] = 1.4533$$

First improved value of $y_2 = y(0.4)$ by modified Euler's method is given by

$$y_2^{(1)} = y_1 + \frac{h}{2}\{f(x_1, y_1) + f(x_2, y_2)\}$$

Similarly, $y_2^{(2)} = 1.21776 + 0.1[1.21776 - (0.2)^2 + 1.4533 - (0.4)^2] = 1.4648$

and $y_2^{(3)} = 1.21776 + 0.1(1.21776 - 0.2^2 + 1.4648 - 0.4^2) = 1.45264$

and $y_2^{(3)} = 1.2172 + 0.1\,[1.2172 - (0.2)^2 + 141724 - (0.4)^2] = 1.4660$

$\therefore$ Final value of $y_2 = 1.4660$. Similarly we can obtain $y_3(0.6)$. **Ans.**

EXAMPLE 22.7. Evaluate $y(0.4)$ using modified Euler's method from $y' - y = e^x$, $y(0) = 0$. [GGSIPU 2012]

SOLUTION: $y' = \text{slope} = y + e^x$. Here $h = 0.1,\ y_0 = 0,\ x_0 = 0$.

As per the modified Euler's method we have the following table:

x	$y' = y + e^x$	*Mean slope*	*Old y + 0.1 (mean slope) = New y*
0	0 + 1 = 1		0 + 0.1 (1) = 0.1
0.1	0.1 + 1.1052 = 1.2052	$\frac{1}{2}(1+1.2052) = 1.1026$	0 + 0.1 (1.1026) = 0.11026
0.1	0.11026 + 1.1052 = 1.21546	$\frac{1}{2}(1+1.21546) = 1.1077$	0 + 0.1 (1.1077) = 0.11077
0.1	0.11077 + 1.1052 = 1.21597	$\frac{1}{2}(1+1.21597) = 1.10798$	0 + 0.1 (1.10798) = 0.11079
0.1	0.11079 + 1.1052 = 1.21599	—	0.11079 + 0.1 (1.21599) = 0.2324
0.2	0.2324 + 1.2214 = 1.4538	$\frac{1}{2}(1.21597 + 1.4538) = 1.33488$	0.11079 + 0.1 (1.33488) = 0.2443
0.2	0.2443 + 1.2214 = 1.4657	$\frac{1}{2}(1.21597 + 1.4657) = 1.3408$	0.11079 + 0.1 (1.3408) = 0.24487
0.2	0.24487 + 1.2214 = 1.46627	$\frac{1}{2}(1.21597 + 1.46627) = 1.34112$	0.11079 + 0.1 (1.34112) = 0.2449
0.2	0.2449 + 1.2214 = 1.4663	$\frac{1}{2}(1.21597 + 1.4663) = 1.34113$	0.11079 + 0.1 (1.34113) = 0.2449
0.2	1.4663	—	0.2449 + 0.1 (1.4633) = 0.3915

x	$y' = y + e^x$	Mean slope	Old y + 0.1 (mean slope) = New y
0.3	0.3915 + 1.34985 = 1.74138	$\frac{1}{2}(1.4663 + 1.74138) = 1.6038$	0.2449 + 0.1 (1.6038) = 0.4053
0.3	0.4053 + 1.34985 = 1.75513	$\frac{1}{2}(1.4663 + 1.75513) = 1.6107$	0.2449 + 0.1 (1.6107) = 0.40597
0.3	0.40597 + 1.34985 = 1.7558	$\frac{1}{2}(1.4663 + 1.7558) = 1.6111$	0.2449 + 0.1 (1.6111) = 0.4060
0.3	0.4060 + 1.34985 = 1.7558	—	0.2449 + 0.1 (1.7558) = 0.42048
0.4	0.42048 + 1.4918 = 1.91228	$\frac{1}{2}(1.7558 + 1.91228) = 1.83340$	0.4060 + 0.1 (1.8340) = 0.5894
0.4	0.5894 + 1.4918 = 2.0812	$\frac{1}{2}(1.7558 + 2.0812) = 1.9185$	0.4060 + 0.1 (1.9185) = 0.5978
0.4	0.5978 + 1.4918 = 2.0896	$\frac{1}{2}(1.7558 + 2.0896) = 1.9227$	0.4060 + 0.1 (1.9227) = 0.5982
0.4	0.5982 + 1.4918 = 2.0900	$\frac{1}{2}(1.7558 + 2.0900) = 1.9229$	0.4060 + 0.1 (1.9229) = 0.59829

$\therefore \quad y(0.4) = 0.59829.$ **Ans.**

RUNGE-KUTTA METHOD (R.K. METHOD).

Actually Euler's method as well as modified Euler's method suffer greatly by the big constraint involved, that h has to be quite small and therefore lack in efficiency in practical problems. The Runge-Kutta method are developed to give greater accuracy and they have the advantage of requiring only function values at some selected points on the subinterval.

It is named after two German mathematicians Carl Runge and Wilhelm Kutta. The method is very simple as it avoids the computation of higher order derivatives which the Taylor series method may involve. Actually in this method the derivatives are replaced by extra values of the given function $f(x, y)$.

CLASSICAL RUNGE-KUTTA METHOD OF ORDER FOUR:

Let the equation be $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$ by this method, we have then

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \text{ where}$$

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

and

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

This method tallies with the Taylor's expansion formula upto term of h^4 and is most widely used as it gives fairly accurate values.

EXAMPLE 22.8. Using Runge-Kutta method of fourth order determine $y(0.1)$ and $y(0.2)$ correct to four decimal places given that $\frac{dy}{dx} = y - x$ where $y(0) = 2$ and $h = 0.1$. [GGSIPU 2013]

SOLUTION: $\frac{dy}{dx} = f(x, y) = y - x$, $x_0 = 0$, $y_0 = 2$, $h = 0.1$. To obtain $y(0.1)$, we have

$$K_1 = hf(x_0, y_0) = 0.1\,(y_0 - x_0) = 0.2$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1\left[y_0 + \frac{K_1}{2} - \left(x_0 + \frac{h}{2}\right)\right] = 0.1\,(2.0 + 0.1 - 0.05) = 0.205$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1\left[y_0 + \frac{K_2}{2} - \left(x_0 + \frac{h}{2}\right)\right] = 0.1[2.1025 - 0.05] = 0.20525$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1\,[y_0 + K_3 - (x_0 + h)] = 0.1\,[2.205 - 0.1] = 0.2105$$

$$\therefore \quad K = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] = \frac{1}{6}\,[0.2 + 0.41 + 0.4105 + 0.2105] = 0.205$$

$$\therefore \quad y(0.1) = y_0 + K = 2.205.$$

Next to obtain $y(0.2)$, take $x_0 = 0.1$, $y_0 = 2.205$, $h = 0.1$ and get

$$K_1 = hf(x_0, y_0) = 0.1\,(2.205 - 0.1) = 0.2105$$

$$K_2 = hf\left(x_0 + \frac{h}{2} + y_0 + \frac{K_1}{2}\right) = 0.1\,[2.205 + 0.10525 - (0.1 + 0.05)] = 0.2160$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1[2.205 + 0.108 - (0.1 + 0.05)] = 0.2163$$

$$K_4 = 0.1\,[(x_0 + h, y_0 + K_3) = 0.1\,[2.205 + 0.2163 - (0.1 + 0.1)] = 0.22213$$

$$\therefore \quad K = \frac{1}{6}[0.2105 + 0.4320 + 0.4326 + 0.22213] = \frac{1}{6}[1.29723] = 0.2162$$

$$\therefore \quad y(0.2) = y_0 + K = 2.205 + 0.2162 = 2.4212$$ **Ans.**

EXAMPLE 22.9. (a) Solve the equation $\frac{dy}{dx} = -y$, $y(0) = 1$ for values of y at $x = 0.1$ and $x = 0.2$ using Runge-Kutta method of order four.

(b) Solve $\frac{dy}{dx} = \log(x + y)$, $y(0) = 2$ for $y(0.3)$ using Runge-Kutta method of fourth order in two stops. [GGSIPU 2014]

SOLUTION: (a) By R. K. Method of order 4. Let us first find $y(0.1)$

$$K_1 = hf(x_0, y_0) = -hy_0 = -0.1,$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = -h\left(y_0 + \frac{K_1}{2}\right) = -0.1[1 - 0.05] = -0.095,$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = -h\left(y_0 + \frac{K_2}{2}\right) = -0.1(1 - 0.0475) = -0.09525,$$

and $$K_4 = hf(x_0 + h, y_0 + K_3) = -h\,(y_0 + K_3) = -0.1(1 - 0.09525) = -0.090475.$$

$\therefore \quad y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 1 + \frac{1}{6}(-0.1 - 0.190 - 0.1905 - 0.090475) = 0.9048375.$

Next to find $y(0.2)$ we start with $x_0 = 0.1,\ y_0 = 0.9048375$

$K_1 = hf(x_0, y_0) = 0.1(-y_0) = -0.09048375,$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = -h\left(y_0 + \frac{K_1}{2}\right) = -0.1(0.9048 - 0.04524) = -0.08596,$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = -h\left(y_0 + \frac{K_2}{2}\right)$

$= -0.1(0.9048375 - 0.0430928) = -0.08618578$

$K_4 = hf(x_0 + h, y_0 + K_3) = -0.1(y_0 + K_3) = -0.1(0.9048375 - 0.08618) = -0.0818651$

Thus, and $K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = -0.0861$

$\therefore \quad y(0.2) = y_1 + K = 0.9048375 - 0.0861 = 0.8187375.$ **Ans.**

(b) $\frac{dy}{dx} = f(x, y) = \log(x + y),\ x_0 = 0,\ y_0 = 2.$ Take $h = 0.15$ for first step.

$K_1 = hf(x_0, y_0) = 0.15\log(0 + 2) = 0.103972$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.15\log\left[0.075 + 2 + \frac{1}{2}(0.103972)\right] = 0.113207.$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.15\log\left[0.075 + 2 + \frac{1}{2}(0.113207)\right] = 0.11353.$

$K_4 = hf(x_0 + h, y_0 + K_3) = 0.15\log[0.15 + 2 + 0.11353] = 0.12254$

$K = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] = \frac{1}{6}[0.10397 + 2(0.113207) + 2(0.11353) + 0.12254] = 0.11333$

$\therefore \quad y(0.15) = y_0 + K = 2.11333.$

Next to compute $y(0.3)$ we take $x_0 = 0.15,\ y_0 = 2.11333,\ h = 0.15,$ then

$K_1 = hf(x_0, y_0) = 0.15\log(x_0 + y_0) = 0.15\log(0.15 + 2.11333)$

$= 0.15\log 2.26333 = 0.1225$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.15\log\left(0.15 + 0.075 + 2.11333 + \frac{1}{2}(0.1225)\right)$

$= 0.15\log(2.39958) = 0.131292$

$K_3 = h\log\left(x_0 + \frac{h}{2} + y_0 + \frac{K_2}{2}\right) = 0.15\log\left[0.225 + 2.11333 + \frac{1}{2}(0.131292)\right] = 0.13157$

$K_4 = h\log(x_0 + h + y_0 + K_3) = 0.15\log(0.3 + 2.11333 + 0.13157)$

$= 0.15\log(2.5449) = 0.140114$

$\therefore \quad K = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] = \frac{1}{6}[0.2626 + 0.5257] = 0.131387$

$\therefore \quad y(0.3) = y_0 + K = 2.24472.$ **Ans.**

Example 22.10. (a) Apply the fourth order Rung-Kutta method to solve $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$. Take step size $h = 0.1$ and determine approximations to $y(0.1)$ and $y(0.2)$ correct to four decimal places. [GGSIPU 2010, 2015]

(b) Rung-Kutta method of 4th order to solve $\frac{dy}{dx} = 1 + y^2, y(0) = 0$ to find $y(0.6)$ in three steps. [AKTU 2019; GGSIPU 2014]

Solution: (a) $\frac{dy}{dx} = f(x, y) = y^2 + x^2$, $x_0 = 0$, $y_0 = 1$. Let us first compute $y(1.1)$.

Then
$$K_1 = hf(x_0, y_0) = 0.1(x_0^2 + y_0^2) = 0.1(0+1) = 0.1$$
$$K_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = h\left[\left(x_0 + \frac{h}{2}\right)^2 + \left(y_0 + \frac{K_1}{2}\right)^2\right]$$
$$= 0.1\left[0.05^2 + 1.05^2\right] = 0.1105$$
$$K_3 = h\,f\left(x_0 + \frac{h}{2}, y + \frac{K_2}{2}\right) = 0.1[0.05^2 + (1.05525)^2] = 0.1(0.0025 + 1.11355) = 0.11605$$
$$K_4 = h\,f(x_0 + h, y_0 + K_3) = 0.1[(0 + 0.1)^2 + (1 + 0.116)^2] = 0.12554$$
$$\therefore \quad y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1 + \frac{1}{6}\{0.1 + 2(0.1105) + 2(0.11605) + 0.12554\}$$
$$= 1 + \frac{1}{6}[0.1 + 0.2210 + 0.2321 + 0.12554] = 1.1131.$$

Next, to compute $y(0.2)$, we take $x_0 = 0.1$, $y_0 = 1.1131$, $h = 0.1$ then
$$K_1 = h\,f(x_0, y_0) = 0.1(0.1^2 + 1.1131^2) = 0.1239$$
$$K_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1[0.15^2 + 1.1751^2] = 0.140336$$
$$K_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1[(0.1 + 0.05)^2 + (1.1131 + 0.0702)^2] = 0.1423$$
$$K_4 = h\,f(x_0 + h, y_0 + K_3) = 0.1[(0 + 0.1)^2 + (1 + 0.1423)^2] = 0.131485$$
$$\therefore \quad K = \frac{1}{6}[0.1239 + 2(0.1403) + 2(0.1423) + 0.1348] = 0.1373$$

Hence $y(0.2) = y_0 + K = 1.1131 + 0.1373 = 1.2504$ **Ans.**

(b) $\frac{dy}{dx} = f(x, y) = 1 + y^2$. Here $x_0 = 0$ and $y_0 = 0$. Take $h = 0.2$

Then
$$K_1 = hf(x_0, y_0) = h(1 + y_0^2) = 0.2(1 + 0) = 0.2$$
$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = h\left[1 + \left(y_0 + \frac{K_1}{2}\right)^2\right] = 0.2[1 + (0 + 0.1)^2] = 0.202$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = h\left[1 + \left(y_0 + \frac{K_2}{2}\right)^2\right] = 0.2[1 + (0 + 0.101)^2] = 0.20204$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = h[1 + (y_0 + K_3)^2] = 0.2[1 + (0 + 0.20204)^2] = 0.208164$$

$$\therefore \quad y(0.2) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0 + \frac{1}{6}(0.2 + 0.404 + 0.40408 + 0.208164) = 0.202707$$

Next, to find $y(0.4)$ we have $x_0 = 0.2$, $y_0 = 0.2027$ and $h = 0.2$, then

$$K_1 = h(1 + y_0^2) = 0.2(1 + 0.2027^2) = 0.2082$$

$$K_2 = h\left[1 + \left(y_0 + \frac{K_1}{2}\right)^2\right] = 0.2[1 + (0.2027 + 0.1041)^2] = 0.21882$$

$$K_3 = h\left[1 + \left(y_0 + \frac{K_2}{2}\right)^2\right] = 0.2[1 + (0.2027 + 0.1094)^2] = 0.21948$$

$$K_4 = h[1 + (y_0 + K_3)^2] = 0.2[1 + (0.2027 + 0.21948)^2] = 0.235647$$

$$\therefore \quad y(0.4) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= 0.2027 + \frac{1}{6}[0.2082 + 2(0.21882 + 0.21948) + 0.2355647] = 0.42277.$$

Next to find $y(0.6)$, we have $x_0 = 0.4$, $y_0 = 0.42277$, and $h = 0.2$ then

$$K_1 = h[1 + y_0^2] = 0.2[1 + 0.42277^2] = 0.235747$$

$$K_2 = h\left[1 + \left(y_0 + \frac{K_1}{2}\right)^2\right] = 0.2[1 + (0.42277 + 0.11787)^2] = 0.25846$$

$$K_3 = h\left[1 + \left(y_0 + \frac{K_2}{2}\right)^2\right] = 0.2[1 + (0.42277 + 0.12923)^2] = 0.26094$$

$$K_4 = h[1 + (y_0 + K_3)^2] = 0.2[1 + (0.42277 + 0.26094)^2] = 0.293492$$

$$\therefore \quad y(0.6) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.42277 + \frac{1}{6}[0.235747 + 2(0.25846 + 0.26094) + 0.293492)]$$

$$= 0.42277 + \frac{1}{6}(1.56804) = 0.6841 \qquad \textbf{Ans.}$$

MILNE'S PREDICTOR-CORRECTOR METHOD:

The methods discussed upto now are single step methods because they use only the values of the unknown from the last step computed. As pointed out earlier the modified Euler's method which is a predictor-corrector method, is also single step method. The method of Milne's predictor-corrector is a multistep method because in this method if we know the values of $y(x_0)$, $y(x_0 + h)$, $y(x_0 + 2h)$ and $y(x_0 + 3h)$, we can calculate the value of $y(x_0 + 4h)$ directly, h being small.

Suppose, we are to solve $\dfrac{dy}{dx} = f(x, y),\ y(x_0) = y_0,$ numerically.

Let us write $\quad y_1 = y(x_0 + h),\quad y_2 = y(x_0 + 2h),\quad y_3 = y(x_0 + 3h).$

By Newton's forward difference interpolation formula, we have

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \quad \text{where } p = \frac{x - x_0}{h}. \qquad \ldots(1)$$

Replacing y by y' and retaining terms upto $\Delta^3 y_0$, we get

$$y' = y_0' + p\Delta y_0' + \frac{p(p-1)}{2!}\Delta^2 y_0' + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0'.$$

$$\therefore \quad \frac{dy}{dx} = f(x, y) = f(x_0, y_0) + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0. \qquad \ldots(2)$$

Hence we can write $\displaystyle\int_{y_0}^{y_4} dy = \int_{x_0}^{x_4} f(x, y)\, dx.$ $\qquad \ldots(3)$

$$\therefore \quad y_4 = y_0 + \int_{x_0}^{x_4}\left[f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0\right]dx \quad (\text{since } x = x_0 + ph)$$

$$= y_0 + h\int_0^4\left[f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0\right]dp \quad (\text{since } x = x_0 + ph)$$

$$= y_0 + h\left[p f_0 + \frac{p^2}{2}\Delta f_0 + \left(\frac{p^3}{6} - \frac{p^2}{4}\right)\Delta^2 f_0 + \left(\frac{p^4}{24} - \frac{p^3}{6} + \frac{p^2}{6}\right)\Delta^3 f_0\right]_{p=0}^{4}$$

$$= y_0 + h\left[4 f_0 + 8(f_1 - f_0) + \frac{20}{3}(f_2 - 2f_1 + f_0) + \frac{8}{3}(f_3 - 3f_2 + 3f_1 - f_0)\right]$$

$$= y_0 + \frac{4h}{3}\left[3f_0 + 6(f_1 - f_0) + 5(f_2 - 2f_1 + f_0) + 2(f_3 - 3f_2 + 3f_1 - f_0)\right]$$

or $\boxed{y_4 = y_0 + \dfrac{4h}{3}(2f_1 - f_2 + 2f_3)}$ (on neglecting fourth and higher order difference) $\qquad \ldots(4)$

The formula (4) is used to **predict** the value of y_4 when those of y_0, y_1, y_2, y_3 are known **and hence called predictor formula.**

Now, to obtain a corrector formula, let us consider the relation $\displaystyle y_2 = y_0 + \int_{x_0}^{x_2} f(x, y)\, dx$ $\qquad \ldots(5)$

Using the relation (2) here, we get

$$y_2 = y_0 + h\int_0^2\left[f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \ldots\right]dp = y_0 + h\left[p f_0 + \frac{p^2}{2}\Delta f_0 + \left(\frac{p^3}{6} - \frac{p^2}{4}\right)\Delta^2 f_0 + \ldots\right]_0^2$$

$$= y_0 + h\left[2 f_0 + 2\Delta f_0 + \left(\frac{8}{6} - 1\right)\Delta^2 f_0\right] \quad \text{(neglecting third and higher order differences)}$$

$$= y_0 + h\left[2 f_0 + 2(f_1 - f_0) + \frac{1}{3}(f_2 - 2f_1 + f_0)\right] = y_0 + \frac{h}{3}[f_0 + 4f_1 + f_2]. \qquad \ldots(6)$$

From (6) we can write a relation for y_4, as

$$\boxed{y_4 = y_2 + \frac{h}{3}\,[f_2 + 4f_3 + f_4]}$$ which is called the **Corrector formula.** ...(7)

The difficulty in using the formula (7) is that we need y_4 to calculate f_4. Therefore we first predict y_4 from (4) and find f_4 for using in (7). The **generalized predictor formula** is

$$\boxed{y_{n+1} = y_{n-3} + \frac{4h}{3}\,(2f_{n-2} - f_{n-1} + 2f_n)}$$

and the **generalized corrector formula** is written as

$$\boxed{y_{n+1} = y_{n-1} + \frac{h}{3}\,[f_{n-1} + 4f_n + f_{n+1}].}$$

The advantage of this method is in its accuracy. The disadvantage of Milne's method is that it requires at least four tabular values to start the solution. To obtain these values we need some other method like Taylor's method, Picard's method or Runge-Kutta method.

EXAMPLE 22.11. (*a*) Solve the differential equation $\frac{dy}{dx} = 2e^x - y$ at $x = 0.5$ using Milne's predictor- corrector formulae, given that $y(0.1) = 2$.

(*b*) Solve the initial value problem $\frac{dy}{dx} = 1 + xy^2$, $y(0) = 1$ for $x = 0.4$ by Milne's method; given that

x	0.1	0.2	0.3
y	1.105	1.223	1.355

[*GGSIPU 2011, 2013*]

SOLUTION: (*a*) Here $x_0 = 0.1$, $y_0 = 2$ and $f(x, y) = 2e^x - y$. To evaluate $y(0.5)$ by Milne's method we divide the interval $(0.1, 0.5)$ into four parts, therefore $h = 0.1$. To apply Milne's method we have to first, find the values of y_1, y_2, y_3. Let us adopt Taylor series method for that, then

$$y(x) = y_0 + (x - x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \frac{(x-x_0)^3}{3!}\,y_0''' + \frac{(x-x_0)^4}{4!}\,y_0^{iv} + \dots \quad ...(1)$$

Here $y_0 = 2$, $y_0' = \left(2e^x - y\right)_{(x_0, y_0)} = 2e^{x_0} - y_0 = 2e^{0.1} - 2 = 0.2103$

$$y_0'' = \left[2e^x - y'\right]_{(x_0, y_0)} = 2e^{x_0} - y_0' = 2e^{0.1} - 0.2103 = 2.0000$$

$$y_0''' = \left[2e^x - y''\right]_{(x_0, y_0)} = 2e^{x_0} - y_0'' = 2e^{0.1} - 2.0000 = 0.2103$$

$$y_0^{iv} = \left[2e^x - y'''\right]_{(x_0, y_0)} = 2e^{x_0} - y_0''' = 2e^{0.1} - 0.2103 = 2.0000$$

Substituting these values in (1) , we get

$$y(x) = 2 + (x - 0.1)(0.2103) + \frac{(x-0.1)^2}{2!}(2.0000) + \frac{(x-0.1)^3}{3!}(0.2103) + \frac{(x-0.1)^4}{4!}(2.0000) + \dots \quad ...(2)$$

Putting $x = 0.2, 0.3$ and 0.4 in (2) , we get $y_1 = 2.0310$, $y_2 = 2.0825$, $y_3 = 2.1548$.

Also $f_0 = y_0' = 2e^{0.1} - 2 = 0.2103$,

$f_1 = y_1' = 2e^{0.2} - 2.0310 = 0.4184$,

$f_2 = y_2' = 2e^{0.3} - 2.0825 = 0.6172$,

and $f_3 = y_3' = 2e^{0.4} - 2.1548 = 0.8289$

Now, using predictor formula

$$y_4 = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3] = 2 + \frac{0.4}{3}[2(0.4178) - 0.6172 + 2(0.8289)] = 2.25016$$

For applying corrector formula, we need f_4.

$$f_4 = y'_4 = 2e^{0.5} - y_4 = 2e^{0.5} - (2.2516) = 1.0458$$

Using the values of y_2, f_2, f_3, f_4 in the corrector formula, we get

$$y_4 = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4] = 2.0825 + \frac{0.1}{3}[0.6172 + 4(0.8289) + 1.0458] = 2.24845 \quad \textbf{Ans.}$$

(*b*) Here $f(x, y) = 1 + xy^2$, $(x_0, y_0) = (0, 1)$, $(x_1, y_1) = (0.1, 1.105)$

$(x_2, y_2) = (0.2, 1.223)$, $(x_3, y_3) = (0.3, 1.355)$, $h = 0.1$

$$f_0 = f(x_0, y_0) = 1 + x_0 y_0^2 = 1,$$
$$f_1 = f(x_1, y_1) = 1 + x_1 y_1^2 = 1 + 0.1(1.105)^2 = 1.122$$
$$f_2 = 1 + x_2 y_2^2 = 1 + 0.2(1.223)^2 = 1.299,$$
$$f_3 = 1 + x_3 y_3^2 = 1 + 0.3(1.355)^2 = 1.550$$

Hence, by Milne's predictor formula, we have

$$y_P(0.4) = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) = 1 + \frac{0.4}{3}[2(1.122) - 1.299 + 2(1.550)] = 1.539.$$

Next, applying Milne's corrector formula, we get

$$y_C(0.4) = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \quad (\text{Here } f_4 = 1 + 0.4\, y_4^2(0.4))$$

$$= 1.223 + \frac{0.1}{3}[1.299 + 4(1.550) + 1 + 0.4(1.539)^2]$$

$$= 1.223 + \frac{0.1}{3}[1.299 + 6.20 + 1.947] = 1.223 + 0.315 = 1.538$$

then new $f_4 = 1 + 0.4(1.538)^2 = 1.9462.$

Again applying the corrector formula, we get

$$y_C(0.4) = 1.223 + \frac{0.1}{3}[1.299 + 4(1.550) + 1.9462] = 1.5378. \quad \textbf{Ans.}$$

EXAMPLE 22.12. (a) Apply Milne's method to find the solution of the differential equation $\frac{dy}{dx} = x + y$, $y(0) = 1$ in the interval $[0, 0.4]$ in steps of $h = 0.1$. It is given that $y(0.1) = 1.1103$, $y(0.2) = 1.2428$, $y(0.3) = 1.3997$. [*GGSIPU 2011, 2013*]

(b) Given $\frac{dy}{dx} = \frac{1}{2}(1 + x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$ and $y(0.3) = 1.21$, evaluate $y(0.4)$ by Milne-Predictor corrector method. [*GGSIPU 2006*]

SOLUTION: (a) $\frac{dy}{dx} = f(x, y) = x + y$. Here $x_0 = 0$, $y_0 = 1$, $f_0 = x_0 + y_0 = 1$,

$x_1 = 0.1$, $y_1 = 1.1103$, $x_2 = 0.2$, $y_2 = 1.2428$, $x_3 = 0.3$, $y_3 = 1.3997$.

hence $f_1 = x_1 + y_1 = 0.1 + 1.1103 = 1.2103,$

$f_2 = x_2 + y_2 = 0.2 + 1.2428 = 1.4428$

and $f_3 = x_3 + y_3 = 0.3 + 1.3997 = 1.6997$

By Milne's Predictor formula, we have

$$(y_4)_P = y_0 + \frac{4}{3}h[2f_1 - f_2 + 2f_3] = 1 + \frac{4}{3}(0.1)[2(1.2103) - 1.4428 + 2(1.6997)] = 1.5836$$

Now, apply the corrector formula $(y_4)_C = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$

Here $\qquad f_4 = x_4 + (y_4)_P = 0.4 + 1.5836 = 1.9836$

$\therefore \qquad (y_4)_C = 1.2428 + \frac{0.1}{3}[1.4428 + 4(1.3997) + 1.9836] = 1.5435$ **Ans.**

(b) Given equation is $\frac{dy}{dx} = f(x, y) = \frac{1}{2}(1 + x^2)\, y^2$ and we have $y(0) = 1,\ y(0.1) = 1.06,$ $y(0.2) = 1.12,\ y(0.3) = 1.21,\quad h = 0.1.$

Here
$$f_1 = \frac{1}{2}(1 + 0.1^2)(1.06)^2 = \frac{1}{2}(1.134836)$$
$$f_2 = \frac{1}{2}(1 + 0.2^2)(1.12)^2 = 0.652288$$
$$f_3 = \frac{1}{2}(1 + 0.3^2)(1.21)^2 = \frac{1}{2}(1.595869)$$

Using predictor formula $(y_4)_p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$ ... (1)

$$\therefore \qquad (y_4)_p = 1 + \frac{4(0.1)}{3}[1.134836 - 0.652288 + 1.595869] = 1.277122$$

Therefore $\qquad f_4 = \frac{1}{2}(1 + 0.4^2)\, 1.277122^2 = 0.9460$

Now to find first corrected value of y_4, we use the corrector formula

$$(y_4)_C = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \qquad \text{... (2)}$$

and get $\qquad y_4 = 1.12 + \frac{0.1}{3}[0.652288 + 3.191738 + 0.9460] = 1.279667$

To have more accurate value of y_4 we find value of f_4 corresponding to this $(y_4)_C$ as

$f_4 = \frac{1}{2}(1 + 0.4^2)(1.272825)^2$ and use it to find second corrected value of y_4 from formula (2) above and get $y_4 = 1.2797$

EXAMPLE 22.13. Apply Runge-Kutta method of fourth order to find $y(0.1), y(0.2)$ and $y(0.3)$ from $\frac{dy}{dx} = 3x + \frac{y}{2}$ given that $y(0) = 1$, taking $h = 0.1$ and then find $y(0.4)$.

[GGSIPU 2012]

SOLUTION: $\frac{dy}{dx} = f(x, y) = 3x + \frac{y}{2},\ h = 0.1,\ x_0 = 0,\ y_0 = 1.$

By Runge-Kutta method of fourth order to find $y(0.1)$ we have

$$K_1 = hf(x_0, y_0) = 0.1\left(3x_0 + \frac{1}{2}y_0\right) = 0.05$$

$$K_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right) = 0.1\left[3\left(x_0 + \frac{h}{2}\right) + \frac{1}{2}\left(y_0 + \frac{K_1}{2}\right)\right]$$

$$= 0.1\left[3(0.05) + \frac{1}{2}(1.025)\right] = 0.06625$$

$$K_3 = hf\left(x_0+\frac{h}{2},\, y_0+\frac{K_2}{2}\right) = 0.1\left[3\left(x_0+\frac{h}{2}\right)+\frac{1}{2}\left(y_0+\frac{K_2}{2}\right)\right]$$

$$= 0.1\left[3\,(0.05)+\frac{1}{2}(1.033125)\right] = 0.06665$$

and
$$K_4 = hf\,(x_0+h,\, y_0+K_3) = h\left[3(x_0+h)+\frac{1}{2}(y_0+K_3)\right]$$

$$= 0.1\left[3\,(0.1)+\frac{1}{2}(1.06665)\right] = 0.08333.$$

$$\therefore \quad y\,(0.1) = y_0+\frac{1}{6}(K_1+2K_2+2K_3+K_4) = 1+\frac{1}{6}[0.05+2\,(0.06625+0.06665)+0.08333]$$

$$= 1+0.0665 = 1.0665.$$

Next, to obtain $y\,(0.2)$ we take $y_0 = 1.0665$, $x_0 = 0.1$ and $h = 0.1$.

Then
$$K_1 = hf\,(x_0, y_0) = h\left[3x_0+\frac{1}{2}y_0\right] = 0.1\left[3(0.1)+\frac{1}{2}(1.0665)\right] = 0.0833$$

$$K_2 = hf\left(x_0+\frac{h}{2},\, y_0+\frac{K_1}{2}\right) = h\left[3\left(x_0+\frac{h}{2}\right)+\frac{1}{2}\left(y_0+\frac{K_1}{2}\right)\right]$$

$$= 0.1\left[3\,(0.1+0.05)+\frac{1}{2}(1.0665+0.04165)\right] = 0.1\,[0.45+0.554] = 0.1004$$

$$K_3 = hf\left(x_0+\frac{h}{2},\, y_0+\frac{K_2}{2}\right) = h\left[3\left(x_0+\frac{h}{2}\right)+\frac{1}{2}\left(y_0+\frac{K_2}{2}\right)\right]$$

$$= 0.1\left[3\,(0.1+0.05)+\frac{1}{2}(1.0665+0.0502)\right] = 0.1\,[0.45+0.5583] = 0.1008$$

and
$$K_4 = hf\,(x_0+h,\, y_0+K_3) = h\left[3\,(x_0+h)+\frac{1}{2}(y_0+K_3)\right]$$

$$= 0.1\left[3\,(0.1+0.1)+\frac{1}{2}(1.0665+0.1008)\right] = 0.1\,[0.6+0.5836] = 0.118365$$

$$\therefore \quad y\,(0.2) = y_0+\frac{1}{6}(K_1+2K_2+2K_3+K_4) = 1.0665+\frac{1}{6}[0.0833+0.2008+0.2016+0.11836]$$

$$= 1.0665+\left(\frac{0.604}{6}\right) = 1.1672.$$

Next, to obtain $y\,(0.3)$ we take $x_0 = 0.2$, $y_0 = 1.1672$, $h = 0.1$

Then
$$K_1 = hf\,(x_0,\, y_0) = h\left(3x_0+\frac{y_0}{2}\right) = 0.1\left[3\,(0.2)+\frac{1}{2}(1.1672)\right] = 0.11836$$

$$K_2 = hf\left(x_0+\frac{h}{2},\, y_0+\frac{K_1}{2}\right) = h\left[3\left(x_0+\frac{h}{2}\right)+\frac{1}{2}\left(y_0+\frac{K_1}{2}\right)\right]$$

$$= 0.1\left[3\,(0.2+0.05)+\frac{1}{2}(1.1672+0.05918)\right] = 0.1\,[0.615+0.6132] = 0.12282$$

$$K_3 = h\left[3\left(x_0+\frac{h}{2}\right)+\frac{1}{2}\left(y_0+\frac{K_2}{2}\right)\right] = 0.1\left[0.615+\frac{1}{2}(1.1672+0.06141)\right]$$

$$= 0.1\,[0.615+0.6143] = 0.12293$$

$$K_4 = h\left[3(x_0+h)+\frac{1}{2}(y_0+K_3)\right] = 0.1\left[3(0.3)+\frac{1}{2}(1.1672+0.12293)\right]$$
$$= 0.1\,[0.9+0.645065] = 0.154506$$

$$\therefore \quad y(0.3) = y_0+\frac{1}{6}(K_1+2K_2+2K_3+K_4)$$
$$= 1.1672+\frac{1}{6}[0.11836+2(0.12282+0.12293)+0.154506] = 1.2945. \qquad \textbf{Ans.}$$

Next to find $y(0.4)$ we use Milne's Predictor-Corrector method, and get

$$(y_4)_p = y_0+\frac{4h}{3}(2f_1-f_2+2f_3)$$

Here $\quad f_1 = 3(0.1)+\frac{1}{2}(1.0665) = 0.83325$

$$f_2 = 3(0.2)+\frac{1}{2}(1.1672) = 1.1836$$

$$f_3 = 3(0.3)+\frac{1}{2}(1.2946) = 1.5473.$$

$$\therefore \quad (y_4)_p = y(0.4)_p = 1+\frac{0.4}{3}[2(0.83325)-1.1836+2(1.5473)] = 1.477.$$

and $\quad (y_4)_c = y(0.4)_c = y_2+\frac{h}{3}(f_2+4f_3+f_4)$

Here $\quad f_4 = 3(0.4)+\frac{1}{2}(1.477) = 1.9385$

$$\therefore \quad (y_4)_c = 1.1672+\frac{0.1}{3}[1.1836+4(1.5473)+1.9385]$$

or $\quad y(0.4) = 1.1672+\frac{0.1}{3}(9.3113) = 1.47758$ **Ans.**

SOLVING SIMULTANEOUS DIFFERENTIAL EQUATIONS BY R.K. METHOD:

Consider the system of simultaneous first order differential equations

$$\frac{dx}{dt} = f_1(t, x, y) \quad \text{and} \quad \frac{dy}{dt} = f_2(t, x, y) \qquad ...(1)$$

where x, y are dependent variables and t is the independent variable and the initial conditions are $x = x_0$, $y = y_0$ when $t = t_0$.

Now starting from (t_0, x_0, y_0) the increments $\Delta x\ (= k)$ in x and $\Delta y\ (= l)$ in y because of the increment $\Delta t\ (= h)$ in t, as per the fourth order Runge-Kutta scheme, we have

$$k_1 = hf_1(t_0, x_0, y_0) \qquad l_1 = hf_2(t_0, x_0, y_0)$$

$$k_2 = hf_1\left(t_0+\frac{h}{2}, x_0+\frac{k_1}{2}, y_0+\frac{l_1}{2}\right) \qquad l_2 = hf_2\left(t_0+\frac{h}{2}, x_0+\frac{k_1}{2}, y_0+\frac{l_1}{2}\right)$$

$$k_3 = hf_1\left(t_0+\frac{h}{2}, x_0+\frac{k_2}{2}, y_0+\frac{l_2}{2}\right) \qquad l_3 = hf_2\left(t_0+\frac{h}{2}, x_0+\frac{k_2}{2}, y_0+\frac{l_2}{2}\right)$$

$$k_4 = hf_1(t_0+h, x_0+k_3, y_0+l_3) \qquad l_4 = hf_2(t_0+h, x_0+k_3, y_0+l_3)$$

$\therefore \quad K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $L = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$\therefore \quad x_1 = x_0 + K$ | $y_1 = y_0 + L$

Having obtained (t_1, x_1, y_1) we can obtain (t_2, x_2, y_2) by repeating the above algorithm once again, starting from (t_1, x_1, y_1). In a similar fashion we can extend Taylor series method or Picard's method to solve the system of equations (1).

Further, we can use the technique of solving simultaneous differential equation TO SOLVE a second order differential equation of the form

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \text{ with the initial conditions } y = y_0 \text{ and } \frac{dy}{dx} = y_0' \text{ at } x = x_0 \quad \text{...(2)}$$

by reducing the equation to a system of simultaneous differential equations of first order, as follows:

Let $\quad \frac{dy}{dx} = z \quad$ so $\quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx}$.

Thus, the equation (2) can be represented as the following system of simultaneous differential equations: $\frac{dz}{dx} = f(x, y, z)$ **and** $\frac{dy}{dx} = z$ **with x as independent variable and y, z as dependent variables and initial conditions as** $y = y_0$ **and** $z = y_0'$ **at** $x = x_0$.

The method will be better illustrated in the following problems.

EXAMPLE 22.14. Solve numerically, the simultaneous differential equations

$$\frac{dy}{dx} = x + z, \quad \frac{dz}{dx} = x - y^2 \quad \text{for } x = 0.1$$

given that $y(0) = 2$ and $z(0) = 1$ ...