Dr. Nisha Gupta

Interpolation

Suppose that we have a table of numerical values of a function:

x :	x_0	x_1	----	x_n
y :	y_0	y_1	----	y_n

where $y = f(x)$. Let I be the interval formed by these values of x. If the value of y is required for some $x \in I$, then it is called problem of interpolation and if value of y is required for some x outside the interval I, then it is called problem of extrapolation.

Assumptions for interpolation:

1. There are no sudden jumps or falls in the values of dependent variable during the period under consideration.
2. The rise and fall in the values of dependent variable should be uniform.
3. When we apply calculus of finite differences, we assume that the given set of observations is capable of being expressed in a polynomial form.
4. There will be no consecutive missing values in series.

Polynomial interpolation: A polynomial p for which $p(x_i) = y_i$ when $0 \le i \le n$ is said to interpolate the table

x :	x_0	x_1	------	x_n
y :	y_0	y_1	------	y_n

Here, the points x_i are called nodes or arguments.

We have an important result in the form of a Theorem as

Theorem on Existence of Polynomial Interpolation:

If points $x_0, x_1, \ldots, x_n$ are distinct, then for arbitrary real values $y_0, y_1, \ldots, y_n$ there is a unique polynomial p of degree at most n such that $p(x_i) = y_i$ for $0 \le i \le n$.

Note: (1) When $n=0$ i.e, we are given only one node table

x :	x_0
y :	y_0

then we have the constant polynomial $p(x) = y_0$.

(2) When $n=1$ i.e, we are given

x :	x_0	x_1
y :	y_0	y_1

, then

the polynomial p defined by

$$p(x) = \left(\frac{x-x_1}{x_0-x_1}\right) y_0 + \left(\frac{x-x_0}{x_1-x_0}\right) y_1$$

$$= y_0 + \left(\frac{y_1-y_0}{x_1-x_0}\right)(x-x_0)$$

is a linear polynomial such that $p(x_0) = y_0$ and $p(x_1) = y_1$.

So, this p is used for linear interpolation.

Que Find the polynomial of least degree that interpolates the table:

x	1.4	1.25
y	3.7	3.9

Sol Since only two pair of values of (x, y) are given, therefore we can interpolate at most 1 degree polynomial i.e., linear polynomial by using

$$p(x) = \left(\frac{x-x_1}{x_0-x_1}\right) y_0 + \left(\frac{x-x_0}{x_1-x_0}\right) y_1$$

or $p(x) = y_0 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0)$

$$\Rightarrow p(x) = 3.7 + \left(\frac{3.9 - 3.7}{1.25 - 1.4}\right)(x - 1.4)$$

$$= 3.7 - \frac{4}{3}(x - 1.4) \qquad \underline{A}$$

Note: In our further discussion, we shall study various ways of polynomial interpolation

- Lagrange's Interpolation
- Newton's divided difference interpolation
- Gregory-Newton's Forward Interpolation
- Gregory-Newton's Backward Interpolation

Also, we shall study errors in polynomial interpolation.

Dr. Nishu Gupta

Error in polynomial interpolation:

When a function f is approximated on an interval $[a,b]$ by means of an interpolating polynomial p, the discrepancy between f and p will be (theoretically) zero at each node of interpolation.

A natural expectation is that the function f will be well approximated at all intermediate points and that as the number of nodes increases, this agreement will become better and better. But it is not always true.

For example: Consider the Dirichlet function f,

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$$

If we choose nodes that are rational numbers, then

$$p(x) \equiv 0 \quad \text{and} \quad f(x) - p(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$$

Theorems on Interpolation Errors:

It is possible to assess the errors of interpolation by means of a formula that involves the $(n+1)$st derivative of the function being interpolated. Here is the formal statement:

Theorem 1 Interpolation Errors I: (w/p)

If p is the polynomial of degree at most n that interpolates f at $(n+1)$ distinct nodes $x_0, x_1, \dots, x_n$ belonging to an interval $[a,b]$ and if $f^{(n+1)}$ is continuous, then for each x in $[a,b]$ there is a ξ in (a,b) for which

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i)$$

Note: A special case that often arises is the one in which the nodes are equally spaced.

Lemma 1 Upper Bound Lemma: (w/p)

Suppose that $x_i = a + ih$ for $i = 0, 1, \dots, n$ and that $h = (b-a)/n$. Then for any $x \in [a, b]$

$$\prod_{i=0}^{n} |x - x_i| \le \frac{1}{4} h^{n+1} n!$$

Using this lemma, we can now find a bound on the interpolation error.

Theorem 2 Interpolation Errors II: (w/p)

Let f be a function such that $f^{(n+1)}$ is continuous on $[a,b]$ and satisfies $|f^{(n+1)}(x)| \le M$. Let p be the polynomial of degree $\le n$ that interpolates f at $(n+1)$ equally spaced nodes in $[a,b]$, including the endpoints. Then on $[a,b]$,

$$|f(x) - p(x)| \le \frac{1}{4(n+1)} M h^{n+1}$$

where $h = (b-a)/n$ is the spacing between nodes.

Que: Assess the error if $\sin x$ is replaced by an interpolation polynomial that has ten equally spaced nodes in $[0, 1.6875]$

Sol Here $f(x) = \sin x$, $a = 0$, $b = 1.6875$, $n+1 = 10 \Rightarrow n = 9$

$$\therefore h = \frac{(b-a)}{n} = \frac{1.6875}{9} = (0.1875)$$

Also, $f^{(10)}(x) = -\sin x \Rightarrow |f^{(10)}(x)| \le 1$

$\Rightarrow M = 1$

Dr. Nishu Gupta

$\therefore$ By above theorem,

$$|f(x)-p(x)| \leq \frac{1}{4\times 10} \times 1 \times (0.1875)^{10} = 1.34\times 10^{-9}$$

Hence, $|f(x)-p(x)| \leq 0.134\times 10^{-8}$

$\therefore$ The interpolation polynomial that has ten equally spaced nodes on the interval $[0, 1.6785]$ approximates $\sin x$ to at least eight decimal digits of accuracy.

Note: The error expression in polynomial interpolation can also be given in terms of divided differences:

Theorem 3 Interpolation Errors III (w/p)

If p is the polynomial of degree n that interpolates the function f at nodes $x_0, x_1, \ldots, x_n$, then for any x that is not a node,

$$f(x) - p(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{i=0}^{n} (x - x_i)$$

Note: The following theorem shows that there is a relationship between divided differences and derivatives.

Theorem 4 Divided Differences and Derivatives (w/p)

If $f^{(n)}$ is continuous on $[a,b]$ and if $x_0, x_1, \ldots, x_n$ are any $(n+1)$ distinct points in $[a,b]$, then for some ξ in (a,b),

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

Dr. Nishu Gupta

Corollary 1 Divided Differences Corollary

If f is a polynomial of degree n, then all of the divided differences $f[x_0, x_1, \ldots x_k] = 0$ for $k \geq n+1$.

Note: Sometimes we can interpolate the given function by a polynomial of degree less than n, when the data is given at $(n+1)$ nodes.

If such a polynomial exists and is of degree $k < n$, then its $(k+1)$th order divided differences would all be zero.

(See its example after the proof of Newton's Divided Difference Interpolation Formula)

Finite Differences: Suppose $y = f(x)$ and it is given

$x:$	x_0	x_0+h	x_0+2h	---	x_0+nh
$y=f(x):$	$f(x_0)$	$f(x_0+h)$	$f(x_0+2h)$	---	$f(x_0+nh)$

where $x_n = x_0 + nh$ and $y_n = f(x_n) = f(x_0+nh)$

or

$x:$	x_0	x_1	x_2	----	x_n
$y:$	y_0	y_1	y_2	----	y_n

where the nodes are equispaced and h is called interval of differencing.

We define the following operators

(i) **Shift Operator E:**

$$Ef(x) = f(x+h) \quad \text{or} \quad Ey_r = y_{r+1}$$

In general $E^n f(x) = f(x+nh)$ or $E^n y_r = y_{n+r}$

where n can be +ive or −ive and also it can take any fraction value.

$$\therefore \; E^{-1}f(x) = f(x-h), \quad E^{1/2}f(x) = f\left(x+\tfrac{1}{2}h\right)$$

$$\text{or} \quad E^{-1}y_r = y_{r-1}, \quad E^{1/2}y_r = y_{r+\frac{1}{2}} \quad \text{etc.}$$

(ii) **Forward difference operator Δ:**

$$\Delta f(x) = f(x+h) - f(x)$$

or

$$\Delta y_r = y_{r+1} - y_r$$

(iii) **Backward difference operator ∇:**

$$\nabla f(x) = f(x) - f(x-h)$$

or

$$\nabla y_r = y_r - y_{r-1}$$

Relation between these operators:

	E	Δ	∇
E	E	$1+\Delta$	$(1-\nabla)^{-1}$
Δ	$E-1$	Δ	$(1-\nabla)^{-1}-1$
∇	$1-E^{-1}$	$1-(1+\Delta)^{-1}$	∇

Mean value operator

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]$$

The above relationship can be proved as

(i) $\because \Delta f(x) = f(x+h) - f(x)$

$= E f(x) - f(x)$

$\therefore \Delta f(x) = (E-1) f(x)$

$\Rightarrow \boxed{\Delta = E-1}$

$\therefore \boxed{E = 1+\Delta}$

(ii) $\because \nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$\therefore \nabla f(x) = (1-E^{-1}) f(x)$

$\Rightarrow \boxed{\nabla = 1-E^{-1}}$

$\therefore E^{-1} = 1-\nabla$

$\Rightarrow \boxed{E = (1-\nabla)^{-1}}$

Central difference Operator

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

- $\delta = E^{1/2} - E^{-1/2}$
- $\delta = \Delta(1+\Delta)^{-1/2}$
- $\delta = \nabla(1-\nabla)^{-1/2}$
- $E = 1+\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}$
- $\Delta = \frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}$
- $\nabla = -\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}$

(iii) By (i) and (ii)

$E = 1+\Delta$ and $E = (1-\nabla)^{-1}$.

$\therefore 1+\Delta = (1-\nabla)^{-1}$

$\Rightarrow \boxed{\Delta = (1-\nabla)^{-1}-1}$

By (i) and (ii)

$\nabla = 1-E^{-1}$, $E = 1+\Delta$

$\therefore \boxed{\nabla = 1-(1+\Delta)^{-1}}$

Que Show that

(i) $\Delta + \nabla = E - E^{-1}$

(ii) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

(iii) $\Delta\nabla = \nabla\Delta = \Delta - \nabla$

Sol (i) We know that $\Delta = E-1$, $\nabla = 1-E^{-1}$

$\therefore \Delta + \nabla = (E-1) + (1-E^{-1}) = E - E^{-1}$ proved

(ii) We know that $\Delta = E-1$, $\nabla = 1-E^{-1}$

$\therefore \Delta + \nabla = (E-1) + (1-E^{-1}) = E - E^{-1}$ ——— (1)

Now, $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}$

$= \frac{E(E-1)}{(E-1)} - \frac{(E-1)}{E(E-1)}$

$= E - \frac{1}{E} = E - E^{-1}$ ——— (2)

From (1) & (2), $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ proved

(iii) We know that $\Delta = E-1$, $\nabla = 1-E^{-1}$

$\therefore \Delta\nabla = (E-1)(1-E^{-1}) = E-1-1+E^{-1} = E+E^{-1}-2$

$\nabla\Delta = (1-E^{-1})(E-1) = E-1-1+E^{-1} = E+E^{-1}-2$

$\Delta - \nabla = (E-1)-(1-E^{-1}) = E+E^{-1}-2$

Hence $\Delta\nabla = \nabla\Delta = \Delta - \nabla$ proved

Forward differences: The differences

$y_1-y_0, y_2-y_1, ----, y_n-y_{n-1}$ are denoted by $\Delta y_0, \Delta y_1, ----, \Delta y_{n-1}$ respectively and are called first order forward differences i.e., $\Delta y_r = y_{r+1}-y_r$.

Similarly $\Delta^2 y_r = \Delta(\Delta y_r) = \Delta(y_{r+1}-y_r) = \Delta y_{r+1} - \Delta y_r$ are called second order forward differences.

In general, $\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r$ (nth order forward differences)

These differences are shown below in forward difference table.

Forward difference table (Here nodes are equispaced i.e, $x_n = x_0 + nh$)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		Δy_0		
x_1	y_1		$\Delta^2 y_0$	
				$\Delta^3 y_0$
		Δy_1		
x_2	y_2		$\Delta^2 y_1$	
		Δy_2		
x_3	y_3			

Note: $\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3 y_{n-3} + \cdots + (-1)^n y_0$ (1)

Proof We have $\Delta^n y_0 = (E-1)^n y_0$ $(\because \Delta = E-1)$

$$= \left[E^n - {}^nC_1 E^{n-1} + {}^nC_2 E^{n-2} - {}^nC_3 E^{n-3} + \cdots + (-1)^n\right] y_0$$

$$= y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3 y_{n-3} + \cdots + (-1)^n y_0 .$$

Proved $(\because E^n y_r = y_{n+r})$

Dr. Nishu Gupta

Backward differences: The differences

$y_1 - y_0,\ y_2 - y_1, \ldots, y_n - y_{n-1}$ are denoted by

$\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively and are called first order backward differences i.e, $\nabla y_r = y_r - y_{r-1}$

Similarly $\nabla^2 y_r = \nabla(\nabla y_r) = \nabla(y_r - y_{r-1}) = \nabla y_r - \nabla y_{r-1}$ are called second order backward difference.

In general, $\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}$ (nth order backward differences)

These differences are shown below in backward difference table.

Backward difference table (Here nodes are equispaced i.e. $x_n = x_0 + nh$)

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1		
			$\nabla^2 y_2$	
x_1	y_1			$\nabla^3 y_3$
		∇y_2		
			$\nabla^2 y_3$	
x_2	y_2			
		∇y_3		
x_3	y_3			

Note: $\nabla^n y_n = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3 y_{n-3} + \cdots + (-1)^n y_0$ ②

Proof We have $\nabla^n y_n = (1 - E^{-1})^n y_n$ $\quad (\because \nabla = (1 - E^{-1}))$

$$= (1 - {}^nC_1 E^{-1} + {}^nC_2 E^{-2} - {}^nC_3 E^{-3} + \cdots + (-1)^n E^{-n}) y_n$$

$$= y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3 y_{n-3} + \cdots + (-1)^n y_0$$

$(\because E^n y_r = y_{n+r})$

Proved

Note: $\Delta^n y_0 = \nabla^n y_n$ (from ① & ②) (see ① on previous page)

Que Given that $y_5 = 4,\ y_6 = 3,\ y_7 = 4,\ y_8 = 10$ and $y_9 = 24$.
Find the value of $\Delta^2 y_6$ and $\Delta^4 y_5$ by using difference table.

Sol Forward Difference table for given data is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	4				
		-1			
6	3		$2 = \Delta^2 y_5$		
		1		$3 = \Delta^3 y_5$	
7	4		$5 = \Delta^2 y_6$		$0 = \Delta^4 y_5$
		6		$3 = \Delta^3 y_6$	
8	10		$8 = \Delta^2 y_7$		
		14			
9	24				

$\therefore$ From the table, $\Delta^2 y_6 = 5$ and $\Delta^4 y_5 = 0$

Note: We can find the required values without difference table as,

$$\Delta^2 y_6 = (E-1)^2 y_6 \qquad (\because \Delta = E-1)$$
$$= (E^2 - 2E + 1)\, y_6$$
$$= y_8 - 2y_7 + y_6 = 10 - 2\times 4 + 3 = 10 - 8 + 3 = 5 \ \underline{A}$$

and
$$\Delta^4 y_5 = (E-1)^4\, y_5$$
$$= (E^4 - 4E^3 + 6E^2 - 4E + 1)\, y_5$$
$$= y_9 - 4y_8 + 6y_7 - 4y_6 + y_5$$
$$= 24 - 4\times 10 + 6\times 4 - 4\times 3 + 4 = 24 - 40 + 24 - 12 + 4 = 0 \ \underline{A}$$

Note: If $f(x)$ is a polynomial of degree n, i.e.,
$$f(x) = \sum_{k=0}^{n} a_k x^k$$
then $\Delta^n f(x)$ is constant and is equal to $n!\, a_n h^n$ and
so $\Delta^m f(x) = 0$ when $m > n$

Hence if n pairs of values of x & y are given, then

$$\Delta^n y = 0$$

Que: Find the missing values in the following table

x:	0	5	10	15	20	25
y:	6	10	—	17	—	31

Sol Let $y(10) = a$, $y(20) = b$

Forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	6				
		4			
5	10		$a-14$		
		$a-10$		$41-3a$	
10	a		$27-2a$		$6a+b-102$
		$17-a$		$3a+b-61$	
15	17		$a+b-34$		$143-4a-4b$
		$b-17$		$82-a-3b$	
20	b		$48-2b$		
		$31-b$			
25	31				

As four entries are given, so 4^{th} order differences are zero

$$\therefore\ 6a+b=102 \quad \text{——— (1)}$$

$$4a+4b=143 \Rightarrow a+b = \frac{143}{4} \quad \text{——— (2)}$$

Subtract (2) from (1), $5a = 102 - \frac{143}{4} = \frac{408-143}{4} = \frac{265}{4}$

$$\Rightarrow a = \frac{53}{4} = 13.25$$

$\therefore$ From equ. (1), $b = 102 - 6 \times 13.25 = 102 - 79.50 = 22.50$

Hence $a = 13.25$, $b = 22.50$ **A**

Que Using the method of separation of symbols, show that

(i) $$\Delta^n u_{x-n} = u_x - n u_{x-1} + \frac{n(n-1)}{2!} u_{x-2} - \frac{n(n-1)(n-2)}{3!} u_{x-3} + \cdots + (-1)^{n} n\, u_{x-n+1} + (-1)^n u_{x-n}$$

(ii) $$u_0 + u_1 x + u_2 x^2 + \cdots \infty = \frac{u_0}{(1-x)} + \frac{x \Delta u_0}{(1-x)^2} + \frac{x^2 \Delta^2 u_0}{(1-x)^3} + \cdots \infty$$

Sol (i) $$\Delta^n u_{x-n} = (E-1)^n u_{x-n} \qquad (\because \Delta = E-1)$$

$$= \left[E^n - {}^nC_1 E^{n-1} + {}^nC_2 E^{n-2} - {}^nC_3 E^{n-3} + \cdots + (-1)^n\right] u_{x-n}$$

$$= u_x - n u_{x-1} + \frac{n(n-1)}{2!} u_{x-2} - \frac{n(n-1)(n-2)}{3!} u_{x-3} + \cdots + (-1)^n u_{x-n}$$

$$(\because E^r y_n = y_{n+r})$$

Hence Proved

(ii) R.H.S. $$= \frac{u_0}{(1-x)} + \frac{x \Delta u_0}{(1-x)^2} + \frac{x^2 \Delta^2 u_0}{(1-x)^3} + \cdots \infty$$

$$= \frac{1}{(1-x)} \left[1 + \frac{x\Delta}{(1-x)} + \frac{x^2 \Delta^2}{(1-x)^2} + \cdots\right] u_0$$

$$= \frac{1}{(1-x)} \left[1 + \frac{x(E-1)}{(1-x)} + \frac{x^2 (E-1)^2}{(1-x)^2} + \cdots\right] u_0$$

$$= \frac{1}{(1-x)} \cdot \frac{1}{\left[1 - \frac{x(E-1)}{(1-x)}\right]} \cdot u_0 \qquad \text{(Sum of Infinite G.P.)}$$

$$= \frac{1}{(1-x)} \left[1 - \frac{x(E-1)}{(1-x)}\right]^{-1} u_0$$

$$= \frac{1}{(1-x)} \frac{(1-xE)^{-1}}{(1-x)^{-1}} u_0 = (1-xE)^{-1} u_0$$

$$= (1 + xE + x^2E^2 + \cdots) u_0$$

$$= u_0 + x u_1 + x^2 u_2 + \cdots \qquad (\because E^r u_0 = u_r)$$

$$= \text{L.H.S.}$$

Hence Proved.

Factorial polynomial: The factorial polynomial of degree n (n is a +ive integer) is defined as

$$x^{(n)} = x(x-h)(x-2h)\cdots(x-\overline{n-1}\,h)$$

and $x^{(0)} = 1$

where h is the interval of differencing.

Note: (1) $x^{(n)} = (x-\overline{n-1}\,h)\,x^{(n-1)}$

(2) Also, we define $x^{(-1)} = \dfrac{1}{x+h}$

$$x^{(-n)} = \frac{1}{(x+h)(x+2h)\cdots(x+nh)} = \frac{1}{(x+nh)^{(n)}},$$

$n = 1, 2, 3, \ldots$

(3) $\boxed{\Delta x^{(n)} = nh\,x^{(n-1)}}$ for positive and negative integer n

To see this for positive integer n,

$$\Delta x^{(n)} = \Delta\left[x(x-h)(x-2h)\cdots(x-\overline{n-1}\,h)\right]$$

$$= (x+h)\,x\,(x-h)\cdots(x-\overline{n-2}\,h) - x(x-h)(x-2h)\cdots(x-\overline{n-1}\,h)$$

$$(\because \Delta f(x) = f(x+h) - f(x))$$

$$= x(x-h)\cdots(x-\overline{n-2}\,h)\left[(x+h)-(x-\overline{n-1}\,h)\right]$$

$$= x^{(n-1)}\left[\cancel{x}+\cancel{h}-\cancel{x}+nh-\cancel{h}\right]$$

$$= nh\,x^{(n-1)}$$

Similarly we can verify it for negative integer n.

Also, $\boxed{\Delta (x+a)^{(n)} = nh\,(x+a)^{(n-1)}}$

(4) If we are given $\Delta f(x)$ in factorial notation and we want to find $f(x)$, then we use $\Delta^{-1} x^{(n)} = \dfrac{x^{(n+1)}}{(n+1)h}$ to get $f(x)$.

Que Evaluate (i) $\Delta^3\left(\frac{5x+12}{x^2+5x+6}\right)$ (ii) $\Delta^2\left(\frac{1}{x(x+4)(x+8)}\right)$

Sol (i) $\Delta^3\left(\frac{5x+12}{x^2+5x+6}\right) = \Delta^3\left(\frac{5x+12}{(x+2)(x+3)}\right)$

$= \Delta^3\left(\frac{2}{x+2}+\frac{3}{x+3}\right)$ (by suppression method)

$= \Delta^3\left[2(x+1)^{(-1)}+3(x+2)^{(-1)}\right]$

$= 2.(-1)(-2)(-3)(x+1)^{(-4)}+3(-1)(-2)(-3)(x+2)^{(-4)}$

$= \frac{-12}{(x+2)(x+3)(x+4)(x+5)} - \frac{9}{(x+3)(x+4)(x+5)(x+6)}$

$= \frac{-12(x+6)-9(x+2)}{(x+2)(x+3)(x+4)(x+5)(x+6)}$

$= \frac{-21x-90}{(x+2)(x+3)(x+4)(x+5)(x+6)}$ A

(ii) $\Delta^2\left(\frac{1}{x(x+4)(x+8)}\right)$

$= \Delta^2(x-4)^{(-3)} = (-3)(-4).4^2(x-4)^{(-5)}$

$= \frac{192}{x(x+4)(x+8)(x+12)(x+16)}$ A

$\boxed{\text{Or}}$

Other method $\Delta\left(\frac{1}{x(x+4)(x+8)}\right) = \frac{1}{(x+4)(x+8)(x+12)} - \frac{1}{x(x+4)(x+8)}$

$= \frac{-12}{x(x+4)(x+8)(x+12)}$

$\therefore \Delta^2\left(\frac{1}{x(x+4)(x+8)(x+12)}\right) = -12\,\Delta\left(\frac{1}{x(x+4)(x+8)(x+12)}\right)$

$= -12\left[\frac{1}{(x+4)(x+8)(x+12)(x+16)} - \frac{1}{x(x+4)(x+8)(x+12)}\right]$

$$= -12 \left[\frac{x-(x+16)}{x(x+4)(x+8)(x+12)(x+16)} \right]$$

$$= \frac{192}{x(x+4)(x+8)(x+12)(x+16)} \quad \text{Ans}$$

Que Express the polynomial $f(x) = 2x^3+3x^2-5x+4$ in factorial notation and find its successive differences. Also obtain a function whose first order finite difference is $f(x)$.

Sol
$$f(x) = 2x^3+3x^2-5x+4$$
$$= 2x(x-1)(x-2)+9x^2-9x+4$$
$$= 2x(x-1)(x-2)+9x(x-1)+4$$

$2x(x^2-3x+2) = 2x^3-6x^2+4x$

$9x(x-1) = 9x^2-9x$

$$\therefore \; f(x) = 2x^{(3)}+9x^{(2)}+4$$

Now, $\Delta f(x) = 6x^{(2)}+18x^{(1)} = 6x(x-1)+18x = 6x^2+12x$

$\Delta^2 f(x) = 12x^{(1)}+18 = 12x+18$

$\Delta^3 f(x) = 12$

$\Delta^n f(x) = 0 \;, \; n \in N, \; n > 3$

Let $g(x)$ be a function such that $\Delta g(x) = f(x)$. Then

$$g(x) = \Delta^{-1} f(x)$$
$$= \Delta^{-1}[2x^{(3)}+9x^{(2)}+4x^{(0)}]$$
$$= \frac{2}{4}x^{(4)} + \frac{9}{3}x^{(3)} + 4x^{(1)} + c$$
$$= \frac{1}{2}x(x-1)(x-2)(x-3)+3x(x-1)(x-2)+4x+c$$
$$= \frac{1}{2}(x^4-6x^3+11x^2-6x)+3(x^3-3x^2+2x)+4x+c$$
$$= \frac{1}{2}x^4-\cancel{3x^3}+\frac{11}{2}x^2-3x+\cancel{3x^3}-9x^2+6x+4x+c$$
$$= \frac{1}{2}x^4-\frac{7}{2}x^2+7x+c = \frac{1}{2}(x^4-7x^2+14x)+c \quad \text{Ans}$$

where c is arbitrary constant

Note:

Other method to write f(x) in factorial notation

Let $f(x) = 2x^3 + 3x^2 - 5x + 4$

$$= Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D \quad \text{———} (1)$$

Then $2x^3 + 3x^2 - 5x + 4 = Ax(x-1)(x-2) + Bx(x-1) + Cx + D$

$$= A(x^3 - 3x^2 + 2x) + B(x^2 - x) + Cx + D$$

Equating like coefficients on both sides,

$$\left.\begin{array}{l} A = 2 \\ -3A + B = 3 \\ 2A - B + C = -5 \\ D = 4 \end{array}\right\} \Rightarrow \begin{array}{l} A = 2, \ B = 9 \\ C = -5 - 4 + 9 = 0 \\ \& \ D = 4 \end{array}$$

$\therefore$ From equ. (1), $f(x) = 2x^{(3)} + 9x^{(2)} + 4$ Ans

Que Evaluate $\Delta^3[(1-6x)(6-x)(5-3x)]$

Sol Let $f(x) = (1-6x)(6-x)(5-3x)$

Clearly $f(x)$ is a polynomial of third degree and we know that $\Delta^n x^{(n)} = n(n-1)(n-2)\cdots 1 \cdot h^n = n!\,h^n$

and $\Delta^m x^{(n)} = 0$ if $m > n$

$\therefore \Delta^3[(1-6x)(6-x)(5-3x)]$

$= \Delta^3[(-6)(-1)(-3)x^{(3)} +$ a polynomial of 2^{nd} degree$]$

$= (-6)(-1)(-3) \cdot 3!\, h^3$

$= -108 h^3$

$= -108 \quad (\text{if } h = 1)$ Ans

Que One entry in the following table is incorrect and y is a cubic polynomial in x. Use the difference table to locate and correct the error:

x :	0	1	2	3	4	5	6	7
y :	25	21	18	18	27	45	76	123

Sol Difference table for given data is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	25				
		-4			
1	21		1		
		-3		$2+\varepsilon$	
2	18		$3+\varepsilon$		$4-4\varepsilon$
		$0+\varepsilon$		$6-3\varepsilon$	
3	$18+\varepsilon$		$9-2\varepsilon$		$-6+6\varepsilon$
		$9-\varepsilon$		$0+3\varepsilon$	
4	27		$9+\varepsilon$		$4-4\varepsilon$
		18		$4-\varepsilon$	
5	45		13		$-1+\varepsilon$
		31		3	
6	76		16		
		47			
7	123				

Since the degree of polynomial is 3, the fourth order differences must be zero. Numerical largest fourth order difference is -6 which is corresponding to row of $x=3$. Thus y_3 has error. Let error be ε. Then error propagation is shown in the table.

As 4th order differences are zero

$$\therefore\ -6+6\varepsilon=0 \Rightarrow \varepsilon=1 \Rightarrow y_3=18+1=19$$

When the nodes are equispaced, we use the following formulae for interpolation:

1. Gregory-Newton's Forward Difference Interpolation Formula:

Suppose given

x :	x_0	$x_1 = x_0+h$	$x_2 = x_0+2h$	----	$x_n = x_0+nh$
$f(x)$:	$f(x_0) = y_0$	$f(x_0+h) = y_1$	$f(x_0+2h) = y_2$	---	$f(x_0+nh) = y_n$

where h is the interval of differencing and we want to interpolate $f(x)$ at $x = x_0 + ph$. $\therefore\ p = \dfrac{x-x_0}{h}$

$$\begin{aligned}\therefore\quad f(x) &= f(x_0+ph)\\ &= E^p f(x_0)\\ &= (1+\Delta)^p y_0 \qquad (\because E = 1+\Delta)\\ &= \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right) y_0\end{aligned}$$

$$\therefore\ \boxed{f(x) = f(x_0+ph) \simeq y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)\cdots(p-\overline{n-1})}{n!}\Delta^n y_0}$$

We prefer to use it to interpolate the value of the function near the beginning of set of tabulated value and $0 < p < 1$.

2. Gregory-Newton's Backward Difference Interpolation Formula:

Suppose given

x :	x_0	$x_0+h = x_1$	$x_0+2h = x_2$	-----	$x_0+nh = x_n$
$f(x)$:	$f(x_0) = y_0$	$f(x_0+h) = y_1$	$f(x_0+2h) = y_2$	-----	$f(x_0+nh) = y_n$

where h is the interval of differencing and we want to interpolate $f(x)$ at $x = x_n - ph$ $\quad\therefore\ p = \dfrac{x_n - x}{h}$

$$\therefore f(x) = f(x_n - ph) = E^{-p} f(x_n)$$
$$= (1-\nabla)^{p} f(x_n) \qquad \left\{ \begin{array}{l} \because \nabla = 1 - E^{-1} \\ \Rightarrow E^{-1} = (1-\nabla) \end{array} \right.$$
$$= \left(1 - p\nabla + \frac{p(p-1)}{2!}\nabla^2 - \frac{p(p-1)(p-2)}{3!}\nabla^3 + \cdots\right) y_n$$
$$(\because f(x_n) = y_n)$$

$$\therefore \boxed{f(x) = f(x_n - ph) \simeq y_n - p\nabla y_n + \frac{p(p-1)}{2!}\nabla^2 y_n + \cdots + \frac{(-1)^n p(p-1)\cdots(p-\overline{n-1})}{n!}\nabla^n y_n}$$

We prefer to use it to interpolate the value of the function near the end of set of tabulated value and $0 < p < 1$.

Que The population of a city in the decennial census was as given below. Estimate the population for the year 1895 and 1925.

year x	1891	1901	1911	1921	1931
Population y (in thousands)	46	66	81	93	101

Sol Difference table for given data is

x	y				
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

For $x = 1895$, we shall use Newton's forward difference formula,

$$x_0 = 1891, \quad h = 10, \quad x = 1895$$

$$p = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

$$\therefore \; y(1895) \simeq y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 46 + (0.4)(20) + \frac{(0.4)(-0.6)}{2}(-5) + \frac{(0.4)(-0.6)(-1.6)}{6}(2) + \frac{(0.4)(-0.6)(-1.6)(-2.6)}{24}(-3)$$

$$= 54.8528 \simeq 55 \text{ thousands}$$

A

For $x = 1925$, we shall use Newton's backward difference formula

$$x_n = 1931, \quad h = 10, \quad x = 1925$$

$$\therefore \; p = \frac{x_n - x}{h} = \frac{1931 - 1925}{10} = 0.6$$

$$y(1925) \simeq y_n - p\,\nabla y_n + \frac{p(p-1)}{2!}\nabla^2 y_n - \frac{p(p-1)(p-2)}{3!}\nabla^3 y_n + \frac{p(p-1)(p-2)(p-3)}{4!}\nabla^4 y_n$$

$$= 101 - (0.6)(8) + \frac{(0.6)(-0.4)}{2}(-4) - \frac{(0.6)(-0.4)(-1.4)}{6}(-1) + \frac{(0.6)(-0.4)(-1.4)(-2.4)}{24}(-3)$$

$$= 96.8368 \simeq 97 \text{ thousands}$$

A

Que Given $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$, $\sin 55^\circ = 0.8192$ and $\sin 60^\circ = 0.8660$, find $\sin 52^\circ$ using Newton's forward interpolation formula.

Sol Forward difference table for given data is

x (in degrees)	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$
45	0.7071			
		0.0589		
50	0.7660		−0.0057	
		0.0532		−0.0007
55	0.8192		−0.0064	
		0.0468		
60	0.8660			

For $x = 52$, we choose $x_0 = 50$,

Here $h = 5$

$$\therefore p = \frac{x - x_0}{h} = \frac{52-50}{5} = \frac{2}{5} = 0.4$$

$$\therefore y(52) \simeq y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

$$= 0.7660 + (0.4)(0.0532) + \frac{(0.4)(-0.6)}{2}(-0.0064)$$

$$= 0.7660 + 0.02128 + 0.000768$$

$$= 0.7880$$

Ans

Que For the following data obtain the backward differences interpolation polynomial and estimate $f(x)$ at $x = 0.35$.

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.00	2.28

Sol Backward difference table for given data is

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0	
0.3	1.76		0.04		0
		0.24		0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

For Newton's backward differences interpolation polynomial

$$p = \frac{x_n - x}{h} = \frac{0.5 - x}{0.1} \qquad (\because x_n = 0.5 \text{ and } h = 0.1)$$

$$= 10(0.5 - x) = 5 - 10x = 5(1 - 2x)$$

$$\therefore f(x) \simeq f(x_n) - p\nabla f(x_n) + \frac{p(p-1)}{2!}\nabla^2 f(x_n)$$

$(\because$ other terms vanishes$)$

$$\Rightarrow f(x) = 2.28 - 5(1-2x)(0.28) + \frac{5(1-2x)(4-10x)}{2}(0.04)$$

$$\Rightarrow f(x) = 2.28 - 1.4(1-2x) + 0.2(1-2x)(2-5x)$$

$$\Rightarrow f(x) = (2.28 - 1.4 + 0.4) + 2.8x - 1.8x + 2x^2$$

$$\Rightarrow \boxed{f(x) = 2x^2 + x + 1.28} \quad \text{A}$$

and $f(0.35) = 2 \times (0.35)^2 + (0.35) + 1.28 = 1.875$ A

Que Find the number of men getting wages between ₹ 100 and 150 from the following data:

Wages in ₹	0-100	100-200	200-300	300-400
Frequency	9	30	35	42

Sol Let y denote number of men getting wages below ₹x.
Then difference table is

x (Wages Below ₹)	y (no. of men)	Δy	$\Delta^2 y$	$\Delta^3 y$
100	9			
		30		
200	39		5	
		35		2
300	74		7	
		42		
400	116			

Now, we find $y(150)$ by using Newton's forward interpolation formula:

Here $x = 150$, $x_0 = 100$, $h = 100$

$$\therefore p = \frac{x - x_0}{h} = \frac{150-100}{100} = \frac{50}{100} = \frac{1}{2} = 0.5$$

$$\therefore y(150) \simeq y(100) + p\,\Delta y(100) + \frac{p(p-1)}{2!}\Delta^2 y(100) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y(100)$$

$$= 9 + (0.5)(30) + \frac{(0.5)(-0.5)}{2}(5) + \frac{(0.5)(-0.5)(-1.5)}{6}(2)$$

$$= 9 + 15 - 0.625 + 0.125$$

$$= 23.5 \simeq 24$$

$\therefore$ Number of men getting wages between ₹ 100 and 150

$$= y(150) - y(100) = 24 - 9 = 15 \quad \text{Ans}$$

Dr. Nisha Gupta

Lagrange's Interpolation: Suppose it is given that

x :	x_0	x_1	----	x_n
$f(x)$:	$f(x_0)$	$f(x_1)$	---	$f(x_n)$

where $x_0, x_1, x_2, \ldots, x_n$ are not necessarily equally spaced.

We can interpolate f by the Lagrange form of the interpolation polynomial as
$$p_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i) \qquad \text{——— (1)}$$

where $l_i(x)$ are special polynomials of degree n known as cardinal polynomials for each $i = 0, 1, \ldots, n$ and have the property
$$l_i(x_j) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

We can express each l_i as
$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) \qquad (0 \leq i \leq n)$$

$$\boxed{l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}} \qquad \text{——— (2)}$$

Clearly $l_i(x_i) = 1$

and $l_i(x_j) = 0$ when $i \neq j$ ($\because$ one of the factor will be 0 in the num.)

Note: To write the Lagrange's interpolating polynomial first we find all the cardinal polynomials by eqn. (2) and finally substitute all these values in equation (1).

OR

we can directly remember the form of Lagrange's interpolating polynomial as

$$p_n(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + \dots + l_n(x) f(x_n)$$

i.e.

$$\boxed{p_n(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1) + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} f(x_n)}$$

Note: (1) Lagrange's interpolating polynomial can be written when the values of $f(x)$ are given at equispaced or non-equispaced nodes.

Que: Write out the cardinal polynomials appropriate to the problem of interpolating the following table, and give the Lagrange form of the interpolating polynomial:

x :	$\frac{1}{3}$	$\frac{1}{4}$	1
$f(x)$:	2	-1	7

Sol We have

$$l_0(x) = \frac{(x-\frac{1}{4})(x-1)}{(\frac{1}{3}-\frac{1}{4})(\frac{1}{3}-1)} = -18\left(x-\frac{1}{4}\right)(x-1)$$

$$l_1(x) = \frac{(x-\frac{1}{3})(x-1)}{(\frac{1}{4}-\frac{1}{3})(\frac{1}{4}-1)} = 16\left(x-\frac{1}{3}\right)(x-1)$$

$$l_2(x) = \frac{(x-\frac{1}{3})(x-\frac{1}{4})}{(1-\frac{1}{3})(1-\frac{1}{4})} = +2\left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right)$$

$\therefore$ Interpolating polynomial in Lagrange's form is given by

$$p_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

i.e, $p_2(x) = -36\left(x-\frac{1}{4}\right)(x-1) - 16\left(x-\frac{1}{3}\right)(x-1) + 14\left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right)$

Ans

Que Find the Interpolating polynomial for (0,2), (1,3), (2,12) and (5,147) using Lagrange's interpolation formula.

Sol Given

x :	0	1	2	5
$f(x)$:	2	3	12	147

By Lagrange's interpolation formula

$$p_3(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \times 3$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \times 12 + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \times 147$$

$$= -\frac{1}{5}(x^3 - 8x^2 + 17x - 10) + \frac{3}{4}(x^3 - 7x^2 + 10x)$$

$$- 2(x^3 - 6x^2 + 5x) + \frac{49}{20}(x^3 - 3x^2 + 2x)$$

$$= \left(-\frac{1}{5} + \frac{3}{4} - 2 + \frac{49}{20}\right)x^3 + \left(\frac{8}{5} - \frac{21}{4} + 12 - \frac{147}{20}\right)x^2$$

$$+ \left(-\frac{17}{5} + \frac{15}{2} - 10 + \frac{49}{10}\right)x + 2$$

$$= x^3 + x^2 - x + 2$$

Ans

Note: Remember $(x-a)(x-b)(x-c)$

$$= x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$$

Dr. Nishu Gupta

Que State and Prove Lagrange's interpolation Formula.

Sol **Lagrange's Interpolation Formula:**

Suppose it is given that

x :	x_0	x_1	---	x_n
$f(x)$:	$f(x_0)$	$f(x_1)$	---	$f(x_n)$

where $x_0, x_1, x_2, \ldots, x_n$ are not necessarily equally spaced.

Then we can interpolate f by the Lagrange interpolation formule

$$\boxed{\begin{aligned} p_n(x) \simeq f(x) &= \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} \cdot f(x_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1) \\ &+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n) \end{aligned}} \quad \text{①}$$

Proof Since $(n+1)$ pairs of values are given, we can approximate $f(x)$ by a polynomial of degree n i.e,

$$f(x) \simeq p_n(x) = a_0(x-x_1)(x-x_2)\cdots(x-x_n) + a_1(x-x_0)(x-x_2)\cdots(x-x_n) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad \text{②}$$

Now, $p_n(x_k) = f(x_k)$; $k = 0, 1, 2, 3, \ldots, n$

$$\therefore \quad a_k = \frac{f(x_k)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} ;$$

$k = 0, 1, 2, \ldots n$

Substituting in ②, we get the formula ①.

Que Resolve $\dfrac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ into partial fractions by using Lagrange's interpolating formula.

Sol Let $f(x) = 3x^2+x+1$ and $x_0=1,\ x_1=2,\ x_2=3$

$\therefore f(x_0) = f(1) = 3+1+1 = 5$

$f(x_1) = f(2) = 3\times4+2+1 = 15$

$f(x_2) = f(3) = 3\times9+3+1 = 31$

$\therefore$ By Lagrange's Interpolating formula, we have

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

$$\therefore 3x^2+x+1 = \frac{(x-2)(x-3)}{(1-2)(1-3)}\times5 + \frac{(x-1)(x-3)}{(2-1)(2-3)}\times15 + \frac{(x-1)(x-2)}{(3-1)(3-2)}\times31$$

$$\therefore \frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{15}{(x-2)} + \frac{31}{2(x-3)}$$ Ans

Que If $y_1=4,\ y_3=12,\ y_4=19$ and $y_x=7$, find x using suitable interpolation formula.

Sol Given

y	$4=y_0$	$12=y_1$	$19=y_2$
x	$1=x_0$	$3=x_1$	$4=x_2$

. We have to find x for $y=7$.

By Lagrange's interpolation formula

$$x(y) \simeq \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}x_2$$

$$\therefore x(7) \simeq \frac{(7-12)(7-19)}{(4-12)(4-19)}\times1 + \frac{(7-4)(7-19)}{(12-4)(12-19)}\times3 + \frac{(7-4)(7-12)}{(19-4)(19-12)}\times4$$

$$= \frac{(-5)(-12)}{(-8)(-15)} + \frac{(3)(-12)}{(8)(-7)}\times3 + \frac{(3)(-5)}{(15)(7)}\times4 = \frac{1}{2}+\frac{27}{14}-\frac{4}{7}$$

$$= \frac{1}{2}+\frac{19}{14} = \frac{26}{14} = \frac{13}{7}$$

$$= 1.86$$ Ans

Dr. Nisha Gupta

Divided Differences: Suppose, we are given

x :	x_0	x_1	x_2 --- x_n
$f(x)$:	$f(x_0)$	$f(x_1)$	$f(x_2)$ --- $f(x_n)$

The divided differences of order 1 are defined as

$$f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1 - x_0} = \frac{f(x_0)-f(x_1)}{x_0-x_1} = f[x_1, x_0]$$

$$f[x_1, x_2] = \frac{f(x_2)-f(x_1)}{x_2 - x_1}$$

$$\vdots$$

$$f[x_{n-1}, x_n] = \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}} = f[x_n, x_{n-1}]$$

The divided differences of order 2 are defined as

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2]-f[x_0, x_1]}{x_2-x_0}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3]-f[x_1, x_2]}{x_3-x_1}$$

$$\vdots$$

$$f[x_{n-2}, x_{n-1}, x_n] = \frac{f[x_{n-1}, x_n]-f[x_{n-2}, x_{n-1}]}{x_n - x_{n-2}}$$

Similarly, we define nth order divided difference as

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2 \dots, x_n]-f(x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

Note: $f[x_0, x_1, x_2] = f[x_1, x_2, x_0] = f[x_0, x_2, x_1]$ etc.

In general, **Invariance Theorem:** The divided difference $f[x_0, x_1, \dots x_n]$ is invariant under all permutations of the arguments $x_0, x_1, \dots, x_k$.

Dr. Nishu Gupta

Que If $f(x) = \frac{1}{x^2}$ then find $f[a,b,c]$.

Sol $f[a,b] = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b^2}-\frac{1}{a^2}}{b-a} = \frac{a^2-b^2}{a^2b^2(b-a)} = \frac{(a-b)(a+b)}{a^2b^2(b-a)}$

$\therefore f[a,b] = -\frac{(a+b)}{a^2b^2}$

By symmetry $f[b,c] = -\frac{(b+c)}{b^2c^2}$

$$\therefore f[a,b,c] = \frac{f[b,c]-f[a,b]}{c-a} = \frac{-\frac{(b+c)}{b^2c^2}+\frac{(a+b)}{a^2b^2}}{c-a}$$

$$= \frac{-a^2(b+c)+c^2(a+b)}{a^2b^2c^2(c-a)}$$

$$= \frac{-a^2b-a^2c+ac^2+bc^2}{a^2b^2c^2(c-a)}$$

$$= \frac{bc^2-a^2b+ac^2-a^2c}{a^2b^2c^2(c-a)}$$

$$= \frac{b(c^2-a^2)+ac(c-a)}{a^2b^2c^2(c-a)}$$

$$= \frac{b(c+a)+ac}{a^2b^2c^2} = \frac{ab+bc+ca}{a^2b^2c^2} \quad \text{A}$$

Que Prove that the nth divided difference of $\frac{1}{x}$ based on the points $x_0, x_1, x_2, ---, x_n$ is $\frac{(-1)^n}{x_0x_1x_2\cdots x_n}$ i.e.,

$$f[x_0, x_1, x_2, --, x_n] = \frac{(-1)^n}{x_0x_1x_2\cdots x_n}.$$

Sol We have $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$ $\quad [\because f(x) = \frac{1}{x}]$

$$= \frac{\frac{1}{x_1}-\frac{1}{x_0}}{x_1-x_0} = \frac{x_0-x_1}{x_0x_1(x_1-x_0)} = \frac{-1}{x_0x_1}$$

$\therefore$ The result is true for $n=1$.

Suppose the result is true for $n=k$ i.e,

$$f[x_0, x_1, x_2, \ldots, x_k] = \frac{(-1)^k}{x_0 x_1 x_2 \ldots x_k} \quad \text{——— (1)}$$

and $$f[x_1, x_2, \ldots, x_k, x_{k+1}] = \frac{(-1)^k}{x_1 x_2 \ldots x_k \cdot x_{k+1}} \quad \text{——— (2)}$$

$\therefore f[x_0, x_1, x_2, \ldots, x_k, x_{k+1}]$

$$= \frac{f[x_1, x_2, \ldots, x_k, x_{k+1}] - f[x_0, x_1, x_2, \ldots, x_k]}{x_{k+1} - x_0}$$

$$= \frac{\dfrac{(-1)^k}{x_1 x_2 \ldots x_k \cdot x_{k+1}} - \dfrac{(-1)^k}{x_0 x_1 \ldots x_k}}{x_{k+1} - x_0} \quad \text{by (1) and (2)}$$

$$= \frac{(-1)^k (x_0 - x_{k+1})}{x_0 x_1 x_2 \ldots x_k \cdot x_{k+1} \cdot (x_{k+1} - x_0)}$$

$$= \frac{(-1)^{k+1}}{x_0 x_1 x_2 \ldots x_k \cdot x_{k+1}}$$

Hence the result is true for $n = k+1$,

$\therefore$ By the principle of mathematical induction method, the result is true for every positive integer n.

Newton's Divided Difference Interpolation Formula:

Given

x :	x_0	x_1	x_2	....	x_n
$f(x)$:	$f(x_0)$	$f(x_1)$	$f(x_2)$	...	$f(x_n)$

where $x_0, x_1, x_2, \ldots, x_n$ are not necessarily equally spaced.

Then Newton's divided difference interpolating polynomial is of the form

$$\boxed{f(x) \simeq p_n(x) = f(x_0) + (x-x_0) f[x_0,x_1] + (x-x_0)(x-x_1) f[x_0,x_1,x_2] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1}) f[x_0,x_1,\ldots x_n]}$$

Proof We have $f[x, x_0] = \dfrac{f(x)-f(x_0)}{x-x_0}$

$$\therefore \; f(x) = f(x_0) + (x-x_0) f[x, x_0] \qquad \text{——— (1)}$$

Again, $f[x, x_0, x_1] = \dfrac{f[x,x_0] - f[x_0,x_1]}{(x-x_1)}$

$$\therefore \; f[x,x_0] = f[x_0,x_1] + (x-x_1) f[x,x_0,x_1] \qquad \text{——— (2)}$$

Similarly,

$$f[x,x_0,x_1] = f[x_0,x_1,x_2] + (x-x_2) f[x,x_0,x_1,x_2] \qquad \text{——— (3)}$$

$$\vdots$$

$$f[x,x_0,x_1,\ldots x_{n-1}] = f[x_0,x_1,\ldots x_{n-1},x_n] + (x-x_n) f[x,x_0,x_1,\ldots x_n]$$

Multiply eqn. (2) by $(x-x_0)$, eqn. (3) by $(x-x_0)(x-x_1)$, ------- and last equation by $(x-x_0)(x-x_1)\cdots(x-x_{n-1})$ and adding, we get

$$\boxed{f(x) \simeq f(x_0) + (x-x_0) f[x_0,x_1] + (x-x_0)(x-x_1) f[x_0,x_1,x_2] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1}) f[x_0,x_1,\ldots,x_n]}$$

Que Form the divided difference table for the following data

x:	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

and find the interpolating polynomial and estimate the value of $f(3)$.

Sol: Divided difference table for given data is

x	$f(x)$	$f[\,,\,]$ ($\Delta f(x)$)	$f[\,,\,,\,]$ ($\Delta^2 f(x)$)	$f[\,,\,,\,,\,]$ ($\Delta^3 f(x)$)	$f[\,,\,,\,,\,,\,]$ ($\Delta^4 f(x)$)	$f[\,,\,,\,,\,,\,,\,]$ ($\Delta^5 f(x)$)
0	1					
		$\frac{14-1}{1-0}=13$				
1	14		$\frac{1-13}{2-0}=-6$			
		$\frac{15-14}{2-1}=1$		$\frac{-2+6}{4-0}=1$		
2	15		$\frac{-5-1}{4-1}=-2$		$\frac{1-1}{5-0}=0$	
		$\frac{5-15}{4-2}=-5$		$\frac{2+2}{5-1}=1$		$\frac{0-0}{6-0}=0$
4	5		$\frac{1+5}{5-2}=2$		$\frac{1-1}{6-1}=0$	
		$\frac{6-5}{5-4}=1$		$\frac{6-2}{6-2}=1$		
5	6		$\frac{13-1}{6-4}=6$			
		$\frac{19-6}{6-5}=13$				
6	19					

$\therefore$ By Newton's divided difference interpolation formula

$$f(x) \simeq f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3]$$

i.e, $f(x) \simeq 1 + 13(x-0) + (-6)(x-0)(x-1) + 1\cdot(x-0)(x-1)(x-2)$

$$= 1 + 13x - 6(x^2-x) + x(x^2-3x+2)$$

$$= x^3 - 9x^2 + 21x + 1 \quad \underline{A}$$

$$f(3) = (3)^3 - 9\times(3)^2 + 21\times(3) + 1$$

$$= 27 - 81 + 63 + 1 = 91 - 81 = 10 \quad \underline{A}$$

Que Given that $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find $\log_{10} 656$ using interpolation formula.

Dr. Nisha Gupta

Sol: Divided difference table for the given data is

x	$f(x)=\log_{10}x$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
654	2.8156			
		0.00065		
			+0.00001	
658	2.8182			−0.0000039
		0.00070		
			−0.000017	
659	2.8189			
		0.00065		
661	2.8202			

∴ By Newton's Divided difference Interpolation formula,

$$f(x) \simeq f(x_0) + (x-x_0)\, f[x_0, x_1] + (x-x_0)(x-x_1)\, f[x_0, x_1, x_2]$$
$$+ (x-x_0)(x-x_1)(x-x_2)\, f[x_0, x_1, x_2, x_3]$$

i.e, $f(x) \simeq 2.8156 + (x-654)\times 0.00065 + (x-654)(x-658)(0.00001)$
$+ (x-654)(x-658)(x-659)(-0.0000039)$

$\therefore f(656) \simeq 2.8156 + 2\times 0.00065 + 2\times(-2)\times 0.00001$
$+ (2)(-2)(-3)(-0.0000039)$

$= 2.8156 + 0.00130 - 0.00004 + 0.0000468$

$= 2.8169$ Ans