Gamma Function

If n is positive, then the definite integral

$$\int_0^\infty e^{-x} x^{n-1}\, dx,$$

which is a function of n, is called the Gamma function and is denoted by $\Gamma(n)$.

Thus
$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx, \quad n>0}$$

In particular,
$$\boxed{\Gamma(1) = \int_0^\infty e^{-x}\, dx = \left|-e^{-x}\right|_0^\infty = 1}$$

Reduction formula for $\Gamma(n)$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx = \int_0^\infty x^n e^{-x}\, dx$$

Integrating by parts, we have

$$\Gamma(n+1) = \left| -x^n e^{-x}\right|_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\, dx$$

$$= n\int_0^\infty e^{-x} x^{n-1}\, dx \qquad \left[\because \lim_{x\to\infty} \frac{x^n}{e^x} = 0\right]$$

$$= n\Gamma(n)$$

$$\boxed{\Gamma(n+1) = n\Gamma(n)}$$, which is the reduction formula of $\Gamma(n)$

Note 1. If n is a positive integer, then by repeated application of above formula, where $a^2 = x^2 + b^2$

we get

$$\Gamma(n+1) = n\Gamma(n)$$
$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\Gamma(n-2)$$
$$= n(n-1)(n-2)\ldots 3.2.1\,\Gamma(1)$$
$$= n! \quad \text{since } \Gamma(1) = 1$$

Hence $\boxed{\Gamma(n+1) = n! \text{ when } n \text{ is a positive integer}}$

Note 2. If n is a positive fraction, then by repeated application of above formula, we get

$$\Gamma(n) = (n-1)(n-2)\ldots 1$$

the series of factors being continued so long as the factors remain positive, multiplied by Γ(last factor).

Thus $\Gamma\left(\frac{11}{4}\right) = \frac{7}{4}\Gamma\left(\frac{7}{4}\right) = \frac{7}{4}\cdot\frac{3}{4}\Gamma\left(\frac{3}{4}\right)$

The value of $\Gamma\left(\frac{3}{4}\right)$ can be obtained from the table of gamma functions

Note 3. $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}, \quad n \neq 0$$
$$= \frac{(n+1)\Gamma(n+1)}{n(n+1)}$$
$$= \frac{\Gamma(n+2)}{n(n+1)}, \quad n \neq 0, -1$$
$$= \frac{(n+2)\Gamma(n+2)}{n(n+1)(n+2)}$$
$$= \frac{\Gamma(n+3)}{n(n+1)(n+2)}, \quad n \neq 0, -1, -2$$

where $r^2 = a^2 + b^2$ and $\phi = \tan^{-1} b$

. . . .

$$= \frac{\Gamma(n+k+1)}{n(n+1)\cdots(n+k)}, \quad n \neq 0, -1, -2, \cdots, -k$$

This result defines $\Gamma(n)$ for $n<0$, k being the least positive integer such that $n+k+1>0$

For example, to evaluate $\Gamma(-3.4)$

$$n+k+1>0$$
$$\Rightarrow -3.4+k+1>0$$
$$\Rightarrow k>2.4$$

We choose $k=3$

$$\Gamma(-3.4) = \frac{\Gamma(-3.4+3+1)}{(-3.4)(-2.4)(-1.4)(-0.4)} = \frac{\Gamma(0.6)}{(3.4)(2.4)(1.4)(0.4)}$$

The value of $\Gamma(0.6)$ can be obtained from the table of gamma functions.

Also we observe that $\Gamma(n)$ is infinite when $n=0$ or a negative integer.

Value of $\Gamma\left(\frac{1}{2}\right)$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

Putting $t=x^2$ so that $dt = 2x\,dx$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x^2} \cdot \frac{1}{x} \cdot 2x\,dx = 2\int_0^\infty e^{-x^2}\,dx \quad \text{— (1)}$$

Now we use the following result from double integrals:

If $f(x)$ & $g(y)$ are functions of x & y only, and the limits of integration are constants, then the double integral can be represented as a product of two integrals. Then

$$\int_c^d \int_a^b f(x)g(y)\,dx\,dy = \int_a^b f(x)dx \cdot \int_c^d g(y)\,dy \quad \text{---} \quad (2)$$

In eqⁿ (1) writing y for x, we have

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-y^2} dy \quad \text{---} \quad (3)$$

From (1) & (3), we have

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dx\,dy$$

Changing to polar coordinates with $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$; the region of integration in this integral is the complete positive quadrant, to cover which, r must vary from 0 to ∞ from 0 to $\frac{\pi}{2}$.

$$\therefore \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} \cdot r\,dr\,d\theta = 4\int_0^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^\infty d\theta$$

$$= 2\int_0^{\pi/2} d\theta = \pi$$

Hence

$$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

Question 1:- Prove that $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}$, where a, n are positive. Deduce that (i) $\int_0^\infty e^{-ax} x^{n-1} \cos bx\,dx = \frac{\Gamma(n)}{r^n}\cos n\theta$

(ii) $\int_0^\infty e^{-ax} x^{n-1} \sin bx\,dx = \frac{\Gamma(n)}{r^n}\sin n\theta$

where $r^2 = a^2+b^2$ and $\theta = \tan^{-1}\frac{b}{a}$.

Solution :- Put $ax = z$ so that $dx = \frac{dz}{a}$

$$\therefore \int_0^\infty e^{-ax}x^{n-1}dx = \int_0^\infty e^{-z}\left(\frac{z}{a}\right)^{n-1}\cdot\frac{dz}{a} = \frac{1}{a^n}\int_0^\infty e^{-z}z^{n-1}dz$$

$$= \frac{\Gamma(n)}{a^n}.$$

Deduction Replacing a by $(a+ib)$, we get

$$\int_0^\infty e^{-(a+ib)x}x^{n-1}dx = \frac{\Gamma(n)}{(a+ib)^n} \quad — (1)$$

Now $e^{-(a+ib)x} = e^{-ax}\cdot e^{-ibx} = e^{-ax}(\cos bx - i\sin bx)$

Also, putting $a = r\cos\theta$ and $b = r\sin\theta$

so that $r^2 = a^2+b^2$ & $\theta = \tan^{-1}\frac{b}{a}$

$$(a+ib)^n = (r\cos\theta + ir\sin\theta)^n$$

$$= r^n(\cos\theta + i\sin\theta)^n$$

$$= r^n(\cos n\theta + i\sin n\theta) \quad \text{[De Moivre's Theorem]}$$

$\therefore$ From (1), we have

$$\int_0^\infty e^{-ax}(\cos bx - i\sin bx)x^{n-1}dx$$

$$= \frac{\Gamma(n)}{r^n(\cos n\theta + i\sin n\theta)} = \frac{\Gamma(n)}{r^n}(\cos n\theta + i\sin n\theta)^{-1}$$

$$= \frac{\Gamma(n)}{r^n}(\cos n\theta - i\sin n\theta)$$

Now equate real and imaginary parts on the two sides, we get

$$\int_0^\infty e^{-ax}x^{n-1}\cos bx\, dx = \frac{\Gamma(n)}{r^n}\cos n\theta$$

$$\int_0^\infty e^{-ax}x^{n-1}\sin bx\, dx = \frac{\Gamma(n)}{r^n}\sin n\theta$$

Question 2:- Show that

$$\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}, \text{ where } p>0, q>0.$$

Solution:- Put $\log\frac{1}{y} = x$ so that $\frac{1}{y} = e^x$

$y = e^{-x}$ and $dy = -e^{-x}dx$

$$\therefore \int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \int_\infty^0 e^{-(q-1)x} \cdot x^{p-1}(-e^{-x})dx$$

$$= \int_0^\infty e^{-qx} x^{p-1} dx$$

$$= \int_0^\infty e^{-t} \cdot \left(\frac{t}{q}\right)^{p-1} \cdot \frac{dt}{q}, \text{ where } qx = t$$

$$= \frac{1}{q^p}\int_0^\infty e^{-t} t^{p-1} dt = \frac{\Gamma(p)}{q^p}.$$

Question 3:- Evaluate $\int_{-\infty}^\infty e^{-x^2}dx$.

Solution:- Let $I = \int_{-\infty}^\infty e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx$

put $x = \sqrt{t}$

$$I = 2\int_0^\infty e^{-t}\frac{1}{2\sqrt{t}}dt = \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Question 4:- Evaluate $\int_0^\infty \frac{x^a}{a^x} dx$.

Solution:- Let $I = \int_0^\infty \frac{x^a}{a^x} dx$. substituting $a^x = e^t$

or $x\log a = t$. so that $dx = \frac{dt}{\log a}$, we get

$$I = \int_0^\infty e^{-t}\left(\frac{t}{\log a}\right)^a \frac{dt}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t} t^{a+1-1} dt = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$