CHAPTER

9

Laplace Transformation and Its Application in Electric Circuit Analysis

9.1 INTRODUCTION

Laplace transformation is a very powerful method of solving linear differential equations. As the transient response of an electrical circuit can best be described by a linear differential equation hence, Laplace transformation finds its application in solving the transient behaviour of the electric circuits. Solution of a linear differential equation by Laplace transformation consists essentially *three* steps ; *in the first step the differential is transformed into an algebraic equation. This algebraic equation is then solved conventionally in the next step and in the last step the solution is transformed back in such a way that it becomes the required solution of the differential equation.*

The Laplace transform is also advantageous because this method automatically takes care of *initial conditions* without the necessity of first determining the general solution and then obtain from it a particular solution. Laplace transformation also provides the *direct solution of non-homogeneous differential equations.*

9.2 LAPLACE TRANSFORMATION

The time function being usually designated by *lowercase letter* and the Laplace transformation by *capital letter,* the Laplace transformation of a function $f(t)$ is defined as

$$F(s) = £\, f(t) = \int_0^{\infty} f(t)\, e^{-st}\, dt \qquad \text{...(9.1)}$$

s being the *intermediate or transformed variable.* s is a part of the exponential function and it is described in equation (9.1) that to have Laplace transformation of $f(t)$, the original time function $f(t)$ must be multiplied by (e^{-st}) and then integrated from $t = 0$ to $t = \infty$. Also, $f(t)$ is to be defined for $t > 0$ and $f(t) = 0$ for $t \le 0$.

The operation of $£\, f(t)\, [= F(s)]$ is in the *complex frequency domain* or simply in the s-domain* where s is the complex variable $(\sigma + j\omega)$. In electrical engineering, for most of the function convergence can be assumed provided the real part of the function is positive which, in this context provides $Re[s] > 0$.

* In the time function, t represents time; s in the Laplace transform that must represent the dimension inverse of time *i.e.*, **frequency**. It is this reason that the transformed variable is described as a complex frequency.

Inverse Laplace Transform permits going back in the reverse direction, *i.e.*, from s domain to time domain

i.e., $$\pounds^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\omega}^{\sigma_0 + j\omega} F(s)\, e^{st}\, ds \quad ...(9.2)$$

9.3 (A) LAPLACE TRANSFORM OF A DERIVATIVE $\left[\frac{df(t)}{dt}\right]$

From relation it is evident that

$$\int u\, dv = uv - \int v\, du \quad ...(9.3)$$

Let $u = f(t),\ dv = e^{-st}\, dt$

$\therefore$ $du = f'(t),\ v = -\frac{1}{s}e^{-st}$

Thus integration of equation (9.3) by parts yields

$$F(s) = f(t)\left[-\frac{1}{s}e^{-st}\right]_0^{\infty} - \int_0^{\infty} -\frac{1}{s}e^{-st} f'(t)\, dt$$

$$= -\frac{f(t)}{s}e^{-st}\Big|_0^{\infty} + \frac{1}{s}\int_0^{\infty} f'(t)e^{-st} dt$$

$$= \frac{f(0+)}{s} + \frac{1}{s}\int_0^{\infty} f'(t)\, e^{-st}\, dt$$

or, $$s\,F(s) = f(0+) + \int_0^{\infty} f'(t)\, e^{-st}\, dt$$

$$\therefore \int_0^{\infty} f'(t)\, e^{-st}\, dt = s\,F(s) - f(0+) \quad ...(9.4)$$

Observation of equation (9.4) reveals that this expression gives the expression for the Laplace transformation of the derivative $f(t)$

$$\therefore \quad \pounds\frac{df(t)}{dt} = \pounds\, f'(t)\, e^{-st}\, dt \quad ...(9.5)$$

Thus equation (9.4) can be written as

$$\pounds f'(t) = s\,F(s) - f(0+) \quad ...(9.6)$$

9.3 (B) LAPLACE TRANSFORM OF AN INTEGRAL $\int f(t)\, dt$

$$\pounds\left[\int f(t)\, dt\right] = \int_0^{\infty} \int f(t)\, dt\, e^{-st} dt$$

Let $u = \int f(t)\, dt$

and $dv = e^{-st}\, dt.$

The integration by parts gives us

$$\pounds\left[\int f(t)\, dt\right] = \left[\int f(t)\, dt\left(-\frac{1}{s}e^{-st}\right)\right]_0^{\infty}$$

$$-\int_0^{\infty}\left(-\frac{1}{s}e^{-st}\right) f(t)\, dt$$

$$= \frac{1}{s}\int f(t)\, dt\Big|_{0+} + \frac{1}{s}F(s) \quad ...(9.7)$$

$\left[\int f(t)\Big|_{0+} \text{ gives the value of the integral at } t = 0+\right]$

9.4 LAPLACE TRANSFORM OF COMMON FORCING FUNCTIONS

(*i*) Unit Step function

The most common *driving function* in electrical engineering is the *unit step function* and is denoted as

$$u(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases}$$

i.e., the function has no magnitude for the time $t \le 0$ and has a positive unit magnitude at time $t > 0$ (Fig. 9.1)

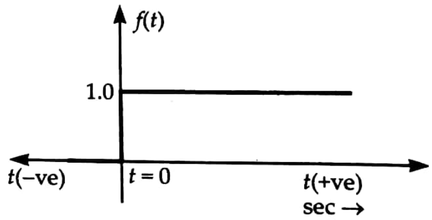

Fig. 9.1 Unit step function occurring at $t = 0$.

The Laplace transformation can be obtained as

$$F(s) = \pounds\, u(t) = \int_0^{\infty} e^{-st}\, dt$$

$$\therefore \quad F(s) = \frac{1}{s} \quad ...(9.8)$$

The unit step function in electric circuits represents the closing of a switch applying a constant voltage V (say) to the circuit. Obviously, its *Laplace transform would be* (V/s).

(*ii*) Exponential function

Exponential function being also very common in electric circuits, it is described by

$$f(t) = e^{-\alpha t} \quad ...(9.9)$$

and is shown in Fig. 9.2.

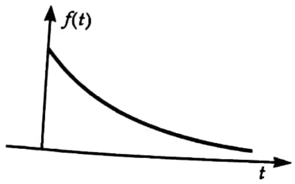

Fig. 9.2 An Exponential function.

Direct application of Laplace transform method gives

$$F(s) = £\, f(t) = £\, e^{-\alpha t}$$

$$= \int_0^\infty e^{-\alpha t}\, e^{-st}\, dt = \frac{1}{s+\alpha} \qquad ...(9.10)$$

[it may be noticed that the time 't' is kept restricted at +ve values only]

(*iii*) Sinusoidal function

The function being extremely common for circuits encountering a.c. source, the *sinusoidal function* (diagrammatically represented in Fig. 9.3) in time domain is given by

$$f(t) = \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \qquad ...(9.11)$$

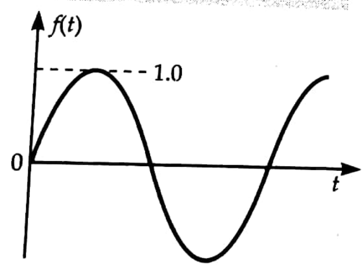

Fig. 9.3 A Sinusoidal function.

Hence, Laplace transformation

$$F(s) = £\, f(t) = £ \sin \omega t$$

or, $$£ \sin \omega t = \frac{1}{2j} \int_0^\infty e^{-(s-j\omega)t} - e^{-(s+j\omega)t}$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

...(9.12)

(*iv*) Cosinusoidal function

This function is given by

$$f(t) = \cos \omega t \qquad ...(9.13)$$

and obviously

$$\cos \omega t = \frac{1}{\omega}\frac{d}{dt}(\sin \omega t) \qquad ...(9.14)$$

The *cosine function* being shown in Fig. 9.4, its Laplace transformation is given by

$$F(s)_{\cos} = £\, f(t) = £(\cos \omega t)$$

$$= \frac{1}{\omega}[s\,F(s) - f(0+)]$$

(Refer to equation 9.4)

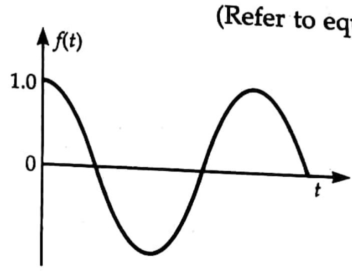

Fig. 9.4 Cosine function.

However, $F(s)$ for sine function is $(\omega / s^2 + \omega^2)$, as has just been derived in equation 9.12.

$$\therefore \quad F(s)_{\cos} = £(\cos \omega t)$$

$$= \frac{1}{\omega}\left[s\frac{\omega}{s^2+\omega^2} - 0\right] = \frac{s}{s^2+\omega^2} \qquad ...(9.15)$$

(*v*) Ramp function

A *ramp function* is given by

$$f(t) = t \qquad ...(9.16)$$

Applying directly the Laplace transform method to the ramp function (Fig. 9.5),

$$F(s) = \int_0^\infty t e^{-st}\, dt$$

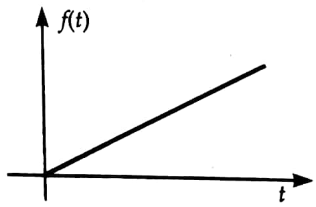

Fig. 9.5 Ramp function.

Integration by parts gives

$$F(s) = -\frac{t\,e^{-st}}{s}\Bigg|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st}\, dt = \frac{1}{s^2}$$

$$\therefore \quad F(s) = \frac{1}{s^2} \qquad ...(9.16(a))$$

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(vi) Parabolic function

The *parabolic function* being given by $f(t) = t$, the Laplace transform of this can be achieved as follows :

$$F(s) = \pounds\, f(t) = \pounds\, t^2 = \pounds \int 2t\, dt$$

[$\because \int 2t\, dt$ in the form of t^2 in terms of an integral]

$$\therefore \quad F(s) = \pounds \int 2t\, dt$$

$$= \frac{F_1(s)}{s} + \frac{f^{-1}(0+)}{s} = \frac{F_1(s)}{s} + 0 \qquad ..(9.17)$$

However,

$$F_1(s) = \pounds\, 2t = \frac{2}{s^2} \qquad ...(9.18)$$

$\therefore$ Utilising equations (9.18) in equation (9.17),

$$F(s) = \frac{2}{s^3}$$

Hence the Laplace transform of a parabolic function (Fig. 9.6) is given by

$$F(s) = \frac{2}{s^3} \qquad ...(9.19)$$

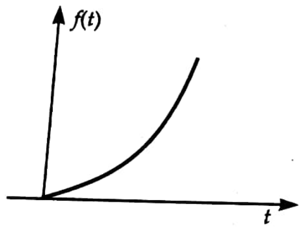

Fig. 9.6 Parabolic function

It may be noted that the Laplace transform of a square function involves a cubic inverse function of *s*. This can be generated as

$$\pounds\, f(t) = \pounds\, t^n = \frac{\underline{|n}}{s^{n+1}}$$

where $f(t) = t^n$.

(vii) Unit impulse function

The *unit impulse function* given by

$$\delta(t) = \lim_{\Delta t \to 0} \frac{u(t) - u(t - \Delta t)}{\Delta t} \qquad ...(9.20)$$

which is obviously a derivative of the unit step functions *i.e.,*

$$\delta(t) = \frac{d}{dt} u(t) \qquad ...(9.21)$$

$\delta(t)$ has the value zero for $t > 0$ and ∞ at $t = 0$. However, infinity value at $t = 0$ is unrealisable. Let us now introduce a new function

$$g(t) = 1 - e^{-\alpha t} \qquad ...(9.22)$$

It may be observed, that $g(t)$ could approach $\delta(t)$ provided α is alone very large. Again,

$$g'(t) = \frac{d}{dt} g(t) = \alpha\, e^{-\alpha t}$$

and $$\int_0^{\infty} g'(t)\, dt = \int_0^{\infty} \alpha\, e^{-\alpha t} = 1.$$

If we now apply Laplace transform directly,

$$F(\delta) = \pounds\, \delta(t) = \lim_{\alpha \to \infty} \pounds\, g'(t)$$

$$= \lim_{\alpha \to \infty} \pounds\, \alpha\, e^{-\alpha t}$$

$$= \lim_{\alpha \to \infty} \frac{\alpha}{s + \alpha} = 1.$$

This gives that Laplace transform of an unit impulse function is unity.

(viii) Laplace transform of t^n

From definition,

$$F(s) = \int_{0(-)}^{\infty} t^n\, e^{-st}\, dt$$

Integration by parts yields

$$F(s) = -\frac{t^n}{s} e^{-st} \Big|_{0(-)}^{\infty} - \int_{0(-)}^{\infty} -\frac{1}{s} e^{-st}\, n t^{n-1} dt$$

$\int u\, dv = uv - \int v\, du$ and here $u = t^n$, $dv = e^{st}\, dt$ giving $du = nt^{n-1}\, dt$ and $v = -(1/s)\, e^{-st}$

or, $$F(s) = 0 + \frac{n}{s} \pounds \{t^{n-1}\}$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-1}{s} \cdots\cdots \frac{3}{2} \cdot \frac{2}{5} \cdot \frac{1}{5} \cdot \pounds\{t^0\}$$

However,

$$\pounds\{t^0\} = \frac{1}{s}$$

$$\therefore \quad \pounds\{t'\} = \frac{1}{s^2}, \pounds\{t^2\} = \frac{2}{s^3}, \pounds\{t^3\} = \frac{6}{s^4} \text{ \& so on.}$$

$$\therefore \pounds\{t^n u(t)\} = F(s) = \frac{\underline{|n}}{s^{n+1}}$$

$$\left[\text{Also, } \pounds[e^{-\alpha t}\, t^n] = \frac{\underline{|n}}{(s+\alpha)^{n+1}}\right]$$

(ix) Laplace transform of $e^{-\alpha t}\sin\omega t$

Let $f(t) = e^{-\alpha t}\sin\omega t = e^{-\alpha t}\dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}$

$$= \frac{1}{2j}\left[e^{-(\alpha - j\omega)t} - e^{-(\alpha + j\omega)t}\right]$$

$$\therefore \quad F(s) = \frac{1}{2j}\left[\frac{1}{s+(\alpha - j\omega)} - \frac{1}{s+(\alpha + j\omega)}\right]$$

$$= \frac{\omega}{(s+\alpha)^2 + \omega^2}$$

[for $\pounds(e^{-\alpha t}\cos\omega t)$, refer to Example 9.1]

(x) Laplace transform of $\delta(t)$

From definition, $F(s) = \int_{0(-)}^{\infty} \delta(t)e^{-st}\,dt = 1.$

(xi) Laplace transform of $\sinh\theta t$

Let $f(t) = \sinh\theta t = \dfrac{e^{\theta t} - e^{-\theta t}}{2}$

$$\therefore \quad F(s) = \frac{1}{2}\left(\frac{1}{s-\theta} - \frac{1}{s+\theta}\right) = \frac{\theta}{s^2 - \theta^2}\,*$$

(xii) Laplace transform of $\cosh\theta t$

Let $f(t) = \cosh\theta t = \dfrac{e^{\theta t} + e^{-\theta t}}{2}$

$$\therefore \quad F(s) = \frac{1}{2}\left(\frac{1}{s-\theta} + \frac{1}{s+\theta}\right) = \frac{s}{s^2 - \theta^2}.$$

Table 9.1 Laplace Transform of Some Useful Functions

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$u(t)$	$\dfrac{1}{s}$	$e^{-\alpha t}t^n$	$\dfrac{\underline{\lvert n}}{(s+\alpha)^{n+1}}$
$e^{-\alpha t}$	$\dfrac{1}{s+\alpha}$	$e^{-\alpha t}\sin\omega t$	$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$
$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$	$e^{-\alpha t}\cos\omega t$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega^2}$
$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$	$\delta(t)$	1
t	$\dfrac{1}{s^2}$	$\sinh\theta t$	$\dfrac{\theta}{s^2-\theta^2}$
t^2	$\dfrac{2}{s^3}$	$\cosh\theta t$	$\dfrac{s}{s^2-\theta^2}$
t^n	$\dfrac{\underline{\lvert n}}{s^{n+1}}$		

* θ is a constant term.

9.5 INITIAL VALUE AND FINAL VALUE THEOREM

The direct Laplace transform of the derivative of $(df(t)/dt)$ with limit $s \to \infty$ gives

$$\lim_{s\to\infty} \pounds\left[\frac{df(t)}{dt}\right] = \lim_{s\to\infty}\int_0^{\infty} \frac{df(t)}{dt}e^{-st}\,dt$$

t Ki value

$$= \lim_{s\to\infty}\{sF(s) - f(0+)\} \qquad ...(9.23)$$

However, with $s \to \infty$, $e^{-st} \to 0$.

s → value

$$\therefore \quad \lim_{s\to\infty}\{sF(s) - f(0+)\} = 0$$

Since, $f(0+)$ is a constant,

$$\therefore \quad f(0+) = \lim_{s\to\infty}[sF(s)] \qquad ...(9.24)$$

Equation (9.24) permits the evaluation of the initial value of the time domain solution $f(t)$. It is then possible to work directly with the transform version of the solution to obtain *initial values* by performing the operation called for on the right hand side of equation (9.24). Equation (9.24) can be stated as *Initial value theorem*.

In the same way the final value theorem can also be developed from the direct Laplace transform of the derivative, however, the limit is now taken as $s \to 0$.

$$\lim_{s\to 0}\pounds\left[\frac{df(t)}{dt}\right] = \lim_{s\to 0}\int_0^{\infty}\frac{df(t)}{dt}e^{-st}\,dt$$

$$= \lim_{s\to 0}\{sF(s) - f(0+)\}, \qquad ...(9.25)$$

But $\displaystyle \lim_{s\to 0}\int_0^{\infty}\frac{df(t)}{dt}e^{-st}\,dt = \int_0^{\infty} df(t)$

$$= f(\infty) - f(0+) \qquad ...(9.26)$$

However, $f(0+)$ being a constant,

$$f(\infty) - f(0+) = -f(0+) + \lim_{s\to 0}[sF(s)],$$

$$\therefore \quad f(\infty) = \lim_{s\to 0}[sF(s)]$$

i.e.,

$$\lim_{s\to 0}[sF(s)] = \lim_{t\to\infty} f(t) \qquad ...(9.27)$$

Expression (9.27) is the *final value theorem* and is useful where the transform solution of problem is available and the needed information is about the *final* or *steady state* solution.

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9.6 TIME DISPLACEMENT THEOREM

This theorem states that if a function $f(t)$ be Laplace transformable, and if $\pounds f(t) = F(s)$, then

$$\pounds f(t-T) = e^{-sT} F(s), \quad ...(9.28)$$

where, the function $f(t-T)$ is the function $f(t)$ displaced by T. By this theorem, the *displacement in the time domain* becomes multiplication by e^{-sT} in the s-domain.

Example 9.1 *Obtain the Laplace transform of $e^{-\theta t}\cos\omega t$, θ being a constant.*

Solution. By definition, $\pounds f(t) = \int_0^\infty f(t)e^{-st}\,dt$

In this case,

$$f(t) = e^{-\theta t}\cos\omega t$$

$$\pounds f(t) = \pounds[e^{-\theta t}\cos\omega t]$$

$$= \int_0^\infty \cos\omega t\, e^{-(s+\theta)t}\,dt$$

$$= \left[\frac{-(s+\theta)\cos\omega t\, e^{-(s+\theta)t} + e^{-(s+\theta)t}\omega\sin\omega t}{(s+\theta)^2+\omega^2}\right]_0^\infty$$

$$= \frac{s+\theta}{(s+\theta)^2+\omega^2}.$$

Example 9.2 *Obtain the Laplace transformation of $f(t) = 1 - e^{-at}$, a being a constant.*

Solution.

$$\pounds f(t) = \pounds(1-e^{-at}) = \int_0^\infty (1-e^{-at})e^{-st}\,dt$$

$$= \int_0^\infty e^{-st}\,dt - \int_0^\infty e^{-(s+a)t}\,dt$$

$$= \left[-\frac{1}{s}e^{-st} + \frac{1}{s+a}e^{-(s+a)t}\right]_0^\infty$$

$$= \frac{1}{s} - \frac{1}{s+a} = \frac{a}{s(s+a)}.$$

Example 9.3 *Obtain the Laplace transform of the pulse shown in Fig. E9.1. (The function is known as gate function).*

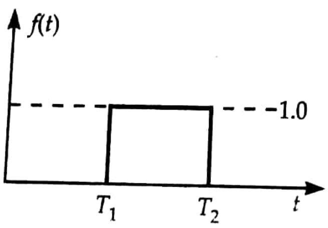


Fig. E9.1

Solution. Here, we utilise the time displacement theorem. The pulse may be considered to have delayed unit step functions as its constituents and as shown in Fig. E9.1(*a*)

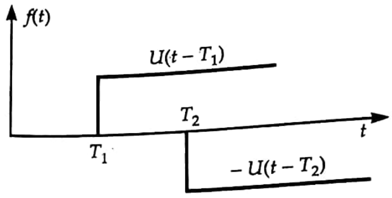


Fig. E9.1 (*a*)

Hence $\quad f(t-T) \equiv u(t-T_1) - u(t-T_2)$

and $\quad \pounds f(t-T) = \pounds u(t-T_1) - \pounds u(t-T_2).$

This gives $\pounds(\text{pulse}) = \dfrac{e^{-sT_1}}{s} - \dfrac{e^{-sT_2}}{s}.$

Example 9.4 *A function is given by*

$$f(t) = Ae^{-\alpha t}\sin(\beta t+\psi).$$

Evaluate : $\lim_{t\to 0} f(t)$

Solution. The given function is

$$f(t) = Ae^{-\alpha t}\sin(\beta t+\psi)$$

as $\quad t\to 0, \ \lim_{t\to 0} f(t) = A\sin\psi.$

Example 9.5 *In the Laplace domain, a function is given by*

$$F(s) = M\left[\frac{(s+\alpha)\sin\theta}{(s+\alpha)^2+\beta^2} + \frac{\beta\cos\theta}{(s+\alpha)^2+\beta^2}\right]$$

Show, by initial value theorem, $\lim_{t\to 0} f(t) = M\sin\theta.$

Solution. As per initial value theorem,

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

However,

$$sF(s) = sM\left[\frac{(s+\alpha)\sin\theta}{(s+\alpha)^2+\beta^2} + \frac{\beta\cos\theta}{(s+\alpha)^2+\beta^2}\right]$$

$$= M\left[\frac{s(s+\alpha)\sin\theta}{(s+\alpha)^2+\beta^2} + \frac{s\beta\cos\theta}{(s+\alpha)^2+\beta^2)}\right]$$

$$= M\left[\frac{\left(1+\frac{\alpha}{s}\right)\sin\theta}{\left(1+\frac{\alpha}{s}\right)^2+\left(\frac{\beta}{s}\right)^2} + \frac{\frac{\beta}{s}\cos\theta}{\left(1+\frac{\alpha}{s}\right)^2+\left(\frac{\beta}{s}\right)^2}\right],$$

(dividing the numerator and denominator by s^2)

$$\therefore \quad \lim_{s\to\infty} sF(s) = M\sin\theta = \lim_{t\to 0} f(t)$$

EXAMPLE 9.6 *A function, in Laplace domain is given by*

$$F(s) = \frac{2}{s} - \frac{1}{s+3}.$$

Obtain its value by Final value theorem in t domain.

SOLUTION. As per final value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

However, $sF(s) = 2 - \frac{s}{s+3}$.

$$\therefore \quad \lim_{s\to 0} sF(s) = 2$$

Hence, $\lim_{t\to\infty} f(t) = 2.$

EXAMPLE 9.7 *A shifted unit step function is expressed as $f(t) = u(t-a)$. Obtain its Laplace transform.*

SOLUTION. $F(s) = \int_a^\infty f(t)e^{-st}\,dt = \int_a^\infty 1.e^{-st}\,dt$

$$= -\frac{1}{s}e^{-st}\Big|_a^\infty$$

$$\therefore \quad \pounds\, u(t-a) = e^{-as}\left(\frac{1}{s}\right).$$

EXAMPLE 9.8 *A scale change is performed to a time-variable function f (t) by introducing t_0 in the time domain where t_0 is a + ve constant. The new function being $f(t/t_0)$, obtain its Laplace transform.*

SOLUTION.

$$\pounds[f(t/t_0)] = \int_0^\infty f(t/t_0)e^{-st}\,dt$$

$$= t_0 \int_0^\infty f(t/t_0)e^{-(t_0\cdot s)t/t_0}\,d(t/t_0)$$

Let $t/t_0 = T.$

$$\therefore \quad \pounds[f(t/t_0)] = \pounds[f(T)]$$

$$= t_0 \int_0^\infty f(T)e^{-t_0 sT}\,dT$$

$$\therefore \quad \pounds[f(t/t_0)] = t_0 F(t_0 s).$$

The result obtained here has been termed as *'Scaling Theorem'*.

EXAMPLE 9.9 *A ramp function occurs at $t = a$. Find its Laplace Transform.*

SOLUTION. Ramp function occurring at $t = 0$ is denoted by $r(t)$ and is the integration of step function. Hence, the ramp function occurring at $t = a$ will be denoted by $r(t-a)$ and will obv. be the integration of shifted step function, *i.e.*,

$$r(t-a) \equiv \int_0^t u(t-a)\,dt.$$

$$\therefore \quad \pounds[r(t-a] = \pounds\int_0^t u(t-a)\,dt$$

$$= \frac{1}{s}\pounds\, u(t-a) = \frac{1}{s}\cdot\frac{1}{s}e^{-as} = \frac{e^{-as}}{s^2}.$$

EXAMPLE 9.10 *An impulse function is given by $\delta(t-t_1)$. Obtain its Laplace transform.*

SOLUTION. With reference to equation (9.2) in the text,

$$\delta(t) = \frac{d}{dt}u(t)$$

$$\therefore \quad \delta(t-t_1) = \frac{d}{dt}u(t-t_1)$$

Hence $\pounds[\delta(t-t_1)] = \pounds\left[\frac{d}{dt}u(t-t_1)\right]$

$$= s\pounds[u(t-t_1)]$$

$$= s\frac{e^{-t_1 s}}{s} = e^{-t_1 s}.$$

$$\therefore \quad \pounds[\delta(t-t_1)] = e^{-t_1 s}.$$

EXAMPLE 9.11 *A half cycle sine wave function is given by $v(t) = \sin \omega t$. Determine its Laplace transform.*

SOLUTION. $v(t) = \sin\omega t = \sin\frac{2\pi t}{T}$

$$\left[\because\ \omega = 2\pi f = \frac{2\pi}{T}\right]$$

This function will be positive for $0 \le t \le \frac{T}{2}$, T being the time period. Also it is observed that the half cycle sine wave has unity amplitude.

Observing Figs. E9.2(*b*) and (*c*), it is noticed that the half cycle sine wave shown in Fig. E9.2 is the combination of two sin waves given by the relations

$$v_1 = \left[\sin 2\pi\frac{t}{T}\right]u(t)$$

and $$v_2 = \left[\sin\frac{2\pi}{T}\left(t - \frac{T}{2}\right)\right]u\left(t - \frac{T}{2}\right)$$

when v_2 is shifted by $\frac{T}{2}$ from v_1.

CHAPTER 9

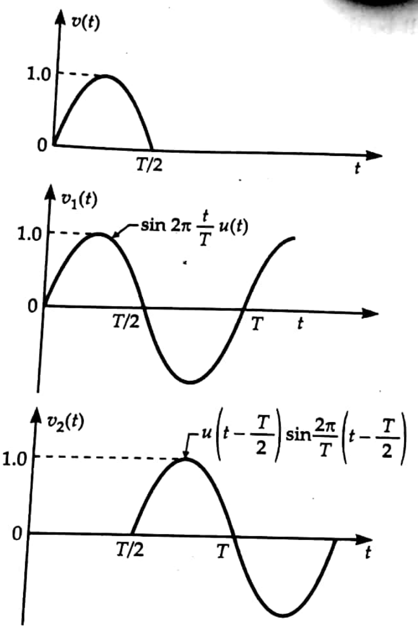


Fig. E9.2

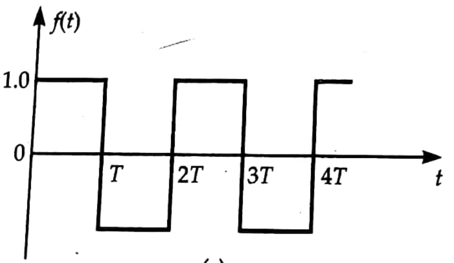
Addition of v_1 and v_2 graphically gives the desired half cycle of the sine wave

$$\therefore\ v(t)=\sin\frac{2\pi t}{T}\cdot u(t)+\sin\frac{2\pi}{T}\left(t-\frac{T}{2}\right)\cdot u\left(t-\frac{T}{2}\right)$$

Utilising Laplace transform methods,

$$V(s)=\frac{\frac{2\pi}{T}}{s^2+\left(\frac{2\pi}{T}\right)^2}\left[1+e^{-\frac{T}{2}\cdot s}\right].$$

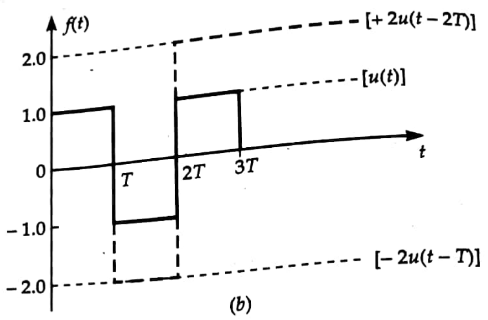
EXAMPLE 9.12 *Obtain the Laplace transform of the square wave train shown in Fig. E9.3(a).*



(*a*)

Fig. E9.3

SOLUTION. Observing the original wave shape (square wave pulse train), it may be seen that this wave shape can be formed by adding different unit step functions and shifted step functions [Fig. E9.3(*b*)].



(*b*)

Fig. E9.3

i.e., $f(t)=u(t)-2\,u(t-T)+2u(t-2T)$
$-2\,u(t-3T)+\ldots\ldots$

$\therefore\ F(s)=\pounds[u(t)-2\,u(t-T)+2\,u(t-2T)$
$-2\,u(t-3T)+\ldots\ldots]$

$$=\frac{1}{s}-\frac{2e^{-Ts}}{s}+\frac{2e^{-2Ts}}{s}-\frac{2e^{-3T}}{s}+\ldots\ldots$$

$$=\frac{1}{s}[1-2e^{-Ts}\,(1-e^{-Ts}+e^{-2Ts}-e^{-3Ts}+\ldots\ldots)]$$

9.7 CONVOLUTION

Convolution of two real functions corresponds to multiplication of their respective functions.

Thus if, $\pounds\ f_1(t)=F_1(s)$ and $\pounds\ f_2(t)=F_2(s)$,
Convolution is defined by

$$\pounds[f_1(t)*f_2(t)]=F_1(s)\,F_2(s) \qquad \ldots(9.29)$$

(the word "convolve" means to revolve continuously). The two functions $f_1(t)$ and $f_2(t)$ are multiplied in such a manner that one is continually moving with time τ (say) relative to the other

i.e., $$f_1(t)*f_2(t)=\int_0^t f_1(t-\tau)\,f_2(\tau)\,d\tau \qquad \ldots(9.30)$$

The statement of the mathematical expression given in expression (9.29) is called *Convolution Theorem.*

Let $\pounds[f_1(t)*f_2(t)]=F(s)$ ...(9.30(*a*))

i.e., $$F(s)=\int_0^\infty [f_1(t)*f_2(t)]\,e^{-st}\,dt$$

$$=\int_{t=0}^{\infty}\left[\int_{\tau=0}^{t} f_1(t-\tau)\,f_2(\tau)\,d\tau\right]e^{-st}\,dt \qquad \ldots(9.31)$$

Again, an unit step function $u(t-\tau)=0$ if $t<\tau$ *i.e.* if $\tau>t$. Thus if we multiply the r.h.s. of expression (9.31) by $u(t-\tau)$, the product will be zero for values of $\tau>t$. The upper limit of integration of the inner integral *i.e.*, of $\int\limits_{\tau=0}^{t}$ can then be changed to $\int\limits_{\tau=0}^{\infty}$ if the inner integral is multiplied by $u(t-\tau)$.

Thus the expression (9.30) can be written as

$$F(s)=\int\limits_{t=0}^{\infty}\left[\int\limits_{\tau=0}^{\infty} f_1(t-\tau)\, f_2(\tau)\, u(t-\tau)\, e^{-st}\, d\tau\right] dt$$

As the functions are Laplace transformable,

$$F(s)=\int\limits_{\tau=0}^{\infty}\left[\int\limits_{t=0}^{\infty} f_1(t-\tau)\, f_2(\tau)\, u(t-\tau)\, e^{-st}\, dt\right] d\tau$$

However, $f_2(\tau)$ is not a function of t.

$$\therefore\quad F(s)=\int\limits_{\tau=0}^{\infty} f_2(\tau)\left[\int\limits_{t=0}^{\infty} f_1(t-\tau)\, u(t-\tau) e^{-st}\, dt\right] d\tau$$

But the inner integral is the transform of the displaced time function and this can be expressed as $e^{-st}\, F_1(s)$.

$$\therefore\quad F(s)=\int\limits_{\tau=0}^{\infty} f_2(\tau)\, e^{-s\tau}\, F_1(s)\, d\tau$$

$$=F_1(s)\int\limits_{0}^{\infty} f_2(\tau)\, e^{-s\tau}\, d\tau$$

$$=F_1(s)\, F_2(s) \qquad \text{...(9.32)}$$

$\therefore$ From equation (9.30),

$$£[f_1(t) * f_2(t)]=F(s)$$

$$=F_1(s)\, F_2(s) \qquad \text{...(9.33)}$$

Thus the convolution theorem is proved.

9.8 APPLICATION OF LAPLACE TRANSFORMATION TECHNIQUE IN ELECTRIC CIRCUIT ANALYSIS

Figure 9.7 represents a series *R-L* circuit with imposed voltage being $v(t)$ and circulating current

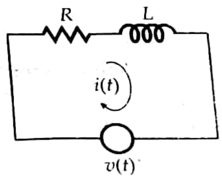


Fig. 9.7 *R-L* series circuit.

being $i(t)$ at any finite value of t after switching. Kirchhoff's law gives

$$Ri(t)+L\frac{di(t)}{dt}=v(t) \qquad \text{...(9.34)}$$

Expression (9.34) gives the loop equation in time domain. Here, we observe that if this time domain equation is converted to Laplace domain,

$$i(t)\rightarrow I(s)$$

[$\because$ we have seen earlier that $£\, f(t)=F(s)$],

$$\frac{di(t)}{dt}\rightarrow[sI(s)-i(0^+)],$$

since, $£\left[\frac{df(t)}{dt}\right]=sF(s)-f(0^+)$ and $v(t)\rightarrow V(s)$.

Hence, Laplace transformation of equation (9.34) gives

$$RI(s)+L[sI(s)-i(0^+)]=V(s) \qquad \text{...(9.35)}$$

The circuit representation of expression in (9.35) is shown in Fig. 9.8. The initial current appears in the circuit shown in s-domain as a voltage source $[Li(0^+)]$. Kirchhoff's voltage law remains valid in s-domain.

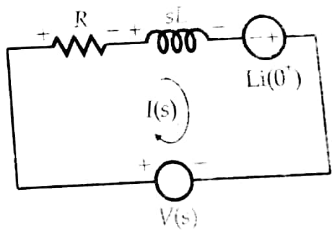
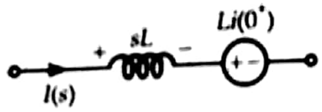
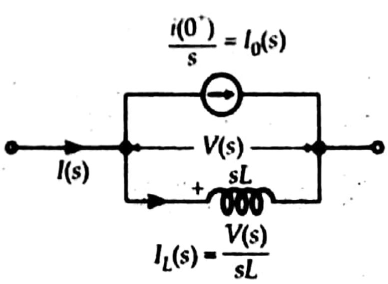


Fig. 9.8 *R-L* circuit in s-doman

Following Table 9.2 exhibits transformation of some conventional time domain and complex frequency (s) domain sources and circuits for ready reference.

Table 9.2

	Time domain	s-domain
1.	Constant voltage source (say a battery terminal voltage) V.	$\dfrac{V}{s}$
2.	Time dependent voltage source, $V = f(t)$.	$V(s)$
3.	Constant current source I.	$\dfrac{I}{s}$
4.	Time dependent current source, $I = f(t)$.	$I(s)$
5.	(a) Resistance element (R)	R
	(b) current is $i(t)$ through R causing drop $v = Ri(t)$.	$V(s) = RI(s)$
6.	(a) Inductance element (L).	Ls
	(b) Initial current is $i(0^+)$, clockwise circuit current being $i(t)$; $v(t) = L\dfrac{di(t)}{dt}$.	$V(s) = [sLI(s) - Li(0^+)]$ (Fig. 9.9)  Fig. 9.9 [If the initial current is $-i(0^+)$, the voltage equation becomes $V(s) = [sLI(s) + Li(0^+)]$ and the equivalent circuit in Laplace domain is [Fig. 9.9(a)].  Fig. 9.9(a)
	(c) Current in the inductor is given by $i(t) = \dfrac{1}{L}\displaystyle\int_0^t v(t)\,dt + i(0^+)$.	$I(s) = \dfrac{1}{L}\cdot\dfrac{V(s)}{s} + \dfrac{i(0^+)}{s} = I_L(s) + I_0(s)$ which gives the following circuit configuration in s-domain (Fig. 9.10) $\dfrac{i(0^+)}{s} = I_0(s)$.  Fig. 9.10

Time domai...	s-domain

7. (*a*) Capacitance element (C) — $1/Cs$

(*b*) Initial voltage is V_0, with +ve polarit... of the capacitor that opposes charging cu... $i(t)$ though it.

$$v(t) = V_0 + \frac{1}{C}\int_0^t i(t)\,dt$$

The equiva... circuit [Fig. 9.11(*a*)] would be, in s domain,

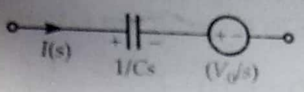


Fig. 9.11(*a*)

(*c*) If the polarity of the initial voltage be $-V_0$ *i.e.*, it assists charging current $i(t)$,

$$v(t) = -V_0 + \frac{1}{C}\int_0^t i(t)\,dt$$

$$V(s) = -\frac{V_0}{s} + \frac{I(s)}{Cs}$$

which gives the equivalent circuit [Fig. 9.11(*b*)] as

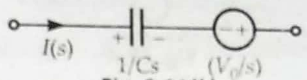


Fig. 9.11(*b*)

(*d*) Current in the capacitor is given by

$$i(t) = C\frac{dv(t)}{dt}$$

$$I(s) = C[V(s)s - V(0^+)] = I_C(s) - I_C(0^+)$$

which gives the equivalent circuit [Fig. 9.11(*c*)] as

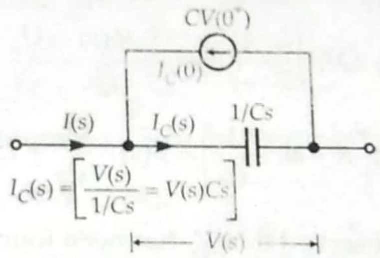


Fig. 9.11 (*c*)

8. Mutual inductance between two coils being M, currents being i_1 and i_2, self-inductances being L_1 and L_2, the voltage drops in each the linked coil is given by

$$v_1(t) = L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt}$$

and $$v_2(t) = L_2\frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt}$$

$$V_1(s) = L_1 sI_1(s) - Li_1(0^+) + MsI_2(s) - Mi_2(0^+) \quad ...(a)$$

$$V_2(s) = L_2 sI_2(s) - Li_2(0^+) + MsI_1(s) - Mi_1(0^+) \quad ...(b)$$

The circuit configuration corresponding to equation (*a*),

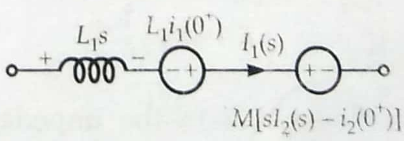


Fig. 9.12 (*a*)

and corresponding to equation (*b*),

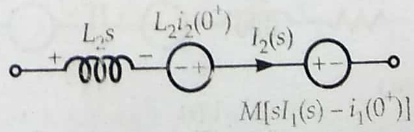


Fig. 9.12 (*b*)

CHAPTER 9

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EXAMPLE 9.13 *A time dependent voltage V (t) is applied to a series connection of RLC network. Find s-domain impedance and current. Assume initial condition of the voltage in inductor to be assisting the input current and that in capacitor opposing the input current. Draw the t-domain and s domain circuits.*

SOLUTION. Using KVL in Fig. E9.4 of the series connection of *RLC* circuit, (in *t*-domain),

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$$

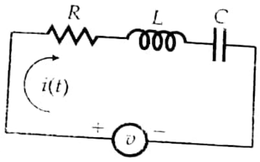


Fig. E9.4

The corresponding Laplace transform is given by,

$$RI(s) + sLI(s) - Li(0^+) + \frac{1}{sC}I(s) + \frac{V_0}{s} = V(s)$$

or,
$$RI(s) + sLI(s) - Li(0^+) + \frac{I(s)}{Cs} + \frac{Q_0}{Cs} = V(s)$$

$$\therefore \quad I(s)\left[R + sL + \frac{1}{Cs}\right] = V(s) - \frac{Q_0}{Cs} + Li(0^+) \quad ...(i)$$

it may be observed that V_0 has been found to be the initial voltage opposing *i(t)*, while $Li(0^+)$ is the drop due to initial current in inductor. Obviously, $V_0 = \frac{Q_0}{C}$ where, Q_0 is the initial charge stored in capacitor of capacitance *C*. Also, in equation (*i*), all the quantities in r.h.s. indicate voltages.

$$\therefore \quad I(s) = \frac{V(s) - \frac{Q_0}{Cs} + Li(0^+)}{\left[R + sL + \frac{1}{Cs}\right]} \quad ...(ii)$$

Here obviously, $Z(s)$ = the impedance of the circuit in *s*-domain $= (R + sL + \frac{1}{Cs})$ while the

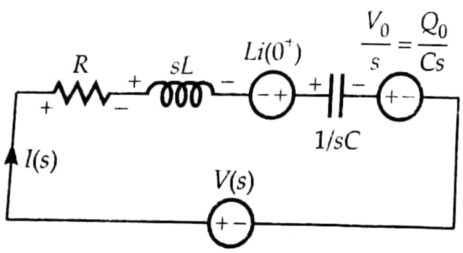


Fig. E9.5

expression of current in *s-domain* is given by expression (*ii*). The *s*-domain circuit has been shown in Fig. E9.5.

[$Z(s)$ has also been termed as *operational impedance*]

EXAMPLE 9.14 *A two mesh network is shown in Fig. E9.6. Obtain the expression for $I_1(s)$ and $I_2(s)$ when the switch is closed.*

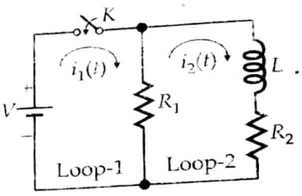


Fig. E9.6

SOLUTION. Utilising KVL *in Loop* 1 and *Loop* 2, we get

$$R_1[i_1(t) - i_2(t)] = V \quad ...(i)$$

and
$$R_2 i_2(t) + R_1 i_2(t) - R_1 i_1(t) + L\frac{di_2(t)}{dt} = 0 \quad ...(ii)$$

Utilising Laplace Transform of equations (*i*) and (*ii*),

$$R_1[I_1(s) - I_2(s)] = \frac{V}{s} \quad ...(iii)$$

$$I_2(s)[R_2 + R_1 + sL] - R_1 I_1(s) = Li(0^+) = 0 \quad ...(iv)$$

Expressions (*iii*) & (*iv*) give the Laplace transform of equations (*i*) & (*ii*). Let us show these equations in matrix form,

$$\begin{bmatrix} (R_1) & (-R_1) \\ (-R_1) & (R_1 + R_2 + sL) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{V}{s} \\ 0 \end{bmatrix}$$

assuming the initial current $i(0^+)$ to be zero through the inductance

$$\therefore \quad \mathbf{I_1(s)} = \frac{\mathbf{V}}{\mathbf{s}}\left[\frac{\mathbf{R_1 + R_2 + sL}}{\mathbf{R_1(R_2 + sL)}}\right]$$

$$\mathbf{I_2(s)} = \frac{\mathbf{V}}{\mathbf{s}}\,\frac{\mathbf{1}}{\mathbf{(R_2 + sL)}}.$$

EXAMPLE 9.15 *Find the loop currents in Laplace domain in matrix form.*

SOLUTION. For loop 1 (Fig. E9.7), using of KVL gives

$$V = i_1(t)[R_1 + R_2] - i_2(t).\,R$$

9.10(A) STEP RESPONSE OF *RL* SERIES CIRCUIT

In this article, the step response of a *R-L* circuit, initially de-energized, has been considered [Fig. 9.13(*a*)]. The governing equation is given by

$$Eu(t) = Ri(t) + L\frac{di}{dt} \qquad \text{...(9.39)}$$

[$\because$ E exists only when s is closed *i.e.*, the voltage is $E\,u(t)$ *i.e.*, the unit step function.]

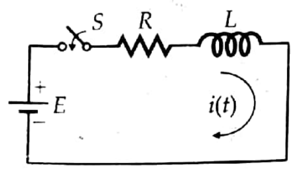


Fig. 9.13 (*a*) Circuit of with step response.

The Laplace transform of equation (9.39) is given by

$$\frac{E}{s} = RI(s) + L[sI(s) - i(0)] \qquad \text{...[9.39(}a\text{)]}$$

It may be noted that $i(0)$ indicates the information regarding the initial value of the current which is zero here as the circuit has been assumed to be in the de-energized condition.

$\therefore$ Equation [9.39(*a*)] becomes

$$\frac{E}{s} = I(s)[R + sL]$$

or,
$$I(s) = \frac{E/L}{s\left(s + \frac{R}{L}\right)} = \frac{A}{s} + \frac{B}{s + \frac{R}{L}}$$

[expressing in the form of partial fraction]

where $A = \left[\dfrac{E/L}{s + \frac{R}{L}}\right]_{s=0} = \dfrac{E}{R}$

$$B = \left[\frac{E/L}{s}\right]_{s=-\frac{R}{L}} = -\frac{E}{R}$$

Thus, $I(s) = \dfrac{E/R}{s} + \dfrac{(-E/R)}{s + \frac{R}{L}}$

$$\therefore \quad i(t) = \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}.t}.$$

[taking inverse Laplace transform]

9.10(B) STEP RESPONSE OF *RC* SERIES CIRCUIT

Let us assume that the capacitance of the circuit is initially having a charge Q_0 owing to the flow of some previously applied current. Let the switch K be now closed. The governing equation of the circuit [Fig. 9.13(*b*)] is then given by

$$Eu(t) = Ri(t) + \frac{1}{C}\int i(t)\,dt \qquad \text{...(9.40)}$$

[Supply voltage being applied at the moment of switching and is a step response voltage with magnitude E (d.c.)]

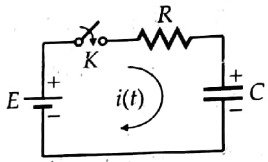


Fig. 9.13 (*b*) Step response of the *RC* circuit.

The Laplace equation is then given by

$$\frac{E}{s} = RI(s) + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{i(0)}{s}\right] \qquad \text{...[9.40(}a\text{)]}$$

In the above equation, the term $i(0)$ indicates the current in the circuit in initial condition when $i(0) \equiv \int_{-\infty}^{0} i\,dt = Q_0$.

$$\therefore \quad \frac{E}{s} = RI(s) + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{Q_0}{s}\right]$$

i.e.,
$$\frac{E}{s} - \frac{Q_0}{C.s} = I(s)\left[R + \frac{1}{C.s}\right]$$

$$I(s) = \frac{\frac{1}{s}\left(E - \frac{Q_0}{c}\right)}{R + \frac{1}{Cs}} = \frac{E - \frac{Q_0}{C}}{s\left(R + \frac{1}{Cs}\right)}$$

or,
$$I(s) = \frac{E - \frac{Q_0}{C}}{R\left(s + \frac{1}{Rc}\right)} = \frac{E - \frac{Q_0}{C}}{R}\left[\frac{1}{s + \frac{1}{RC}}\right] \qquad \text{...[9.40(}b\text{)]}$$

$$= \left(\frac{E}{R} - \frac{Q_0}{R_C}\right)\left[\frac{1}{s + \frac{1}{RC}}\right]$$

$$\therefore \quad i(t) = \left(\frac{E}{R} - \frac{Q_0}{RC}\right)e^{-\frac{1}{RC}.t}$$

[taking inverse Laplace transforms]

Also, we can write $I(s) = \dfrac{E - V_{in}}{s\left(\dfrac{1}{sC} + R\right)}$ from [9.40(*b*)] since V_{in} is the initial voltage stored in the capacitor given by (Q_0 / C)

$$\therefore \quad V_c(s) = \frac{V_{in}}{s} + \frac{I(s)}{Cs} = \frac{V_{in}}{s} + \frac{E - V_{in}}{s^2 C\left[\dfrac{1}{sC} + R\right]}$$

$$\therefore \quad V_c(t) = V_{in} + E - V_{in} - (E - V_{in})\, e^{-t/RC}$$

$$= E - (E - V_{in})\, e^{-t/RC} \text{ volt.}$$

9.11(A) STEP CURRENT RESPONSE OF A *RL* PARALLEL CIRCUIT

In Fig. 9.14(*a*), the constant current source of magnitude I is applied across parallel contribution of R and L. Obviously,

$$RI_R = L\frac{dI_L}{dt} \qquad ...(9.41)$$

and

$$I = I_R + I_L \qquad ...[9.41(a)]$$

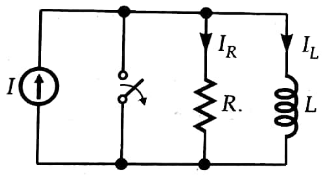


Fig. 9.14 (*a*) Step input to parallel *R-L* circuit.

Multiplying [9.41(*a*)] by *R*, we can write

$$RI = RI_R + RI_L$$

Using (9.41) we find

$$RI = L\frac{dI_L}{dt} + RI_L \qquad ...[9.41(b)]$$

This expression represents the governing differential equation of the circuit shown in Fig. 9.14A just after switching of current source. The solution* is given by

$$i_L = i\,(1 - e^{(-R/L)t})$$

and

$$i_R = i - i_L = ie^{(-R/L)t}$$

9.11(B) STEP CURRENT RESPONSE OF *RC* PARALLEL CIRCUIT

Following application of constant current source I to the parallel RC circuit by opening the switch in the circuit of Fig. 9.14(*b*), we can write, (at $t = 0+$)

$$I = i_R + i_C \qquad ...(9.42)$$

and

$$Ri_R = \frac{1}{C}\int_0^t i_c\, dt \qquad ...[9.42(a)]$$

(We assume the switch is opened at $t = 0$ while the circuit was in steady state at $t = 0-$)

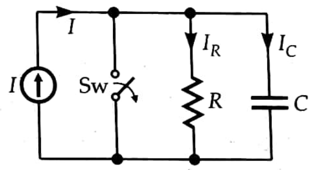


Fig. 9.14 (*b*) Step current input to prallel *R-C* circuit.

Multiplying equation (9.42) by *R*, we get

$$RI = Ri_R + Ri_C$$

$$= \frac{1}{C}\int_0^t i_C\, dt + Ri_C$$

[using equation 9.42(*a*)]

* We can write equation [9.41(*b*)] as

$$\frac{I(s)}{s} \cdot R = I_L(s)\,(R + Ls)$$

or

$$I_L(s) = I(s)\frac{R}{s(R + Ls)} = I(s)\frac{(R/L)}{s\left(\dfrac{R}{L} + s\right)} = I(s)\left[\frac{A_0}{s} + \frac{B_0}{s + \dfrac{R}{L}}\right]$$

Here,

$$A_0 = \left.\frac{(R/L)}{\left(s + \dfrac{R}{L}\right)}\right|_{s=0} = 1 \quad ; \qquad B_0 = \left.\frac{R/L}{s}\right|_{s = -\frac{R}{L}} = -1$$

$$\therefore \quad I_L(s) = I(s)\left[\frac{1}{s} - \frac{1}{s + R/L}\right]$$

$$\therefore \quad i_L(t) = i(t)\,[1 - e^{-(R/L).t}] \qquad ...[9.41(c)]$$

Taking Laplace transformation, we can write

$$R\frac{I(s)}{s} = R\,I_C(s) + \frac{1}{Cs}I_C(s)$$

$$= I_C(s)\left[R + \frac{1}{Cs}\right]$$

or $$I_C(s) = \frac{I(s).R}{\left(R+\frac{1}{Cs}\right)s} = I(s)\left[\frac{1}{s+\frac{1}{RC}}\right]$$

Taking Laplace inverse we can write

$$\therefore \quad i_C(t) = i(t)[e^{-1/RC.t}] \qquad ...(9.42)$$

Expression [9.42(*b*)] represents the capacitor current at $t = 0+$ while the resistor current at $t = 0+$ is given by

$$i_R(t) = i(t) - i_C(t)$$

$$= i(t)(1 - e^{-t/RC})$$

$$= I(1 - e^{-1/RC.t}) \qquad ...[9.42(c)]$$

(Since the magnitude of current for the constant current source is *I*).

EXAMPLE 9.22 *In Fig. E9.8, the battery voltage is applied for a steady state period. Obtain the complete expression for the current after closing the switch K. Assume $R_1 = 1\,\Omega$, $R_2 = 2\,\Omega$, L = 1 H, E = 10 V.*

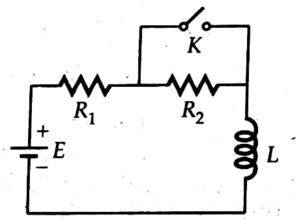


Fig. E9.8

SOLUTION. As soon as the switch is closed, the current $i(t)$ is given by (KVL),

$$Eu(t) = R_1\,i(t) + L\frac{di(t)}{dt} \qquad ...(i)$$

Laplace transformation yields,

$$\frac{E}{s} = R_1 I(s) + sLI(s) - Li(0+)$$

$$\therefore \quad \frac{E}{s} + Li(0+) = I(s)[R_1 + sL]$$

$$\therefore \quad I(s) = \frac{E + sLi(0+)}{s(R_1 + sL)} = \frac{\frac{E}{L} + si(0+)}{s\left(s+\frac{R_1}{L}\right)} \qquad ...(ii)$$

However, partial fraction gives,

$$I(s) = \frac{A}{s} + \frac{B}{s+\frac{R_1}{L}}$$

$$\therefore \quad A = \left.\frac{\frac{E}{L} + si(0+)}{s+\frac{R_1}{L}}\right|_{s=0} = \frac{E}{R_1} = \frac{10}{1} = 10\text{ A}$$

$$B = \left.\frac{\frac{E}{L} + si(0+)}{s}\right|_{s=-\frac{R_1}{L}} = -\frac{E}{R_1} + i(0+)$$

However, the current just before the switching is

$$i(0-) = i(0+) = \frac{E}{R_1 + R_2} = \frac{10}{3} = 3.33\text{ A}$$

$$\therefore \quad B = -\frac{E}{R_1} + i(0+) = -\frac{10}{1} + 3.33 = -6.67\text{ A}$$

$$\therefore \quad I(s) = \frac{10}{s} - \frac{6.67}{s+\frac{R_1}{L}} = \frac{10}{s} - \frac{6.67}{s+1}$$

$$\therefore \quad i(t) = [10 - 6.67e^{-t}]\text{ A.}$$

EXAMPLE 9.23 *A capacitor of 5 μF being charged initially to 10 V is connected to a resistance of 10 kΩ and is allowed to discharge through it by switching of a switch K. Find the expression of discharging current.*

SOLUTION. The circuit is drawn as shown in Fig. E9.9. As soon as the switch *K* is closed, the capacitor, being charged initially to 10 V, starts discharging. Let the discharge current be $i(t)$. Application of KVL in the discharging loop gives

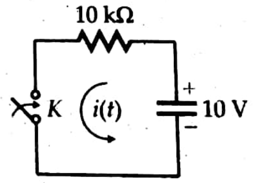


Fig. E9.9

$$0 = Ri(t) + \frac{1}{C}\int i(t)\,dt$$

Laplace transformation gives,

$$0 = RI(s) + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{Q_0}{s}\right]$$

or $$-\frac{Q_0}{Cs} = I(s)\left[R + \frac{1}{Cs}\right]$$

or, $$I(s) = -\frac{Q_0 / Cs}{\left(R + \frac{1}{Cs}\right)}$$

$$= -\frac{Q_0}{CR\left(s + \frac{1}{RC}\right)} = -\frac{10}{R\left(s + \frac{1}{RC}\right)}$$

$$\left[\because \frac{Q_0}{C} = 10\text{ V}\right]$$

$\therefore$ $$i(t) = -\frac{10}{R}(e^{-t/RC})$$

$$= -\frac{10}{10 \times 10^3} e^{-t/10\times 10^3 \times 5 \times 10^{-6}}$$

$$= -\mathbf{10^3 e^{-20t}}\text{ A.}$$

9.12 IMPULSE RESPONSE OF SERIES *RC* NETWORK

Figure 9.15 represents a series RC network excited at $t = 0$ by an *impulse function* of magnitude E.

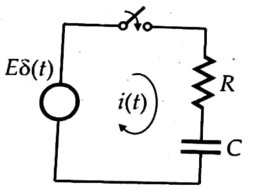


Fig. 9.15 Series *RC* network under the influence of impulse function

Applying KVL,

$$E\delta(t) = Ri(t) + \frac{1}{C}\int i(t)\,dt \qquad ...(9.43)$$

Taking Laplace transform of eqn. (9.43),

$$E = RI(s) + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{Q_0}{s}\right] \qquad ...(9.44)$$

If we assume that initially the capacitor is discharged, $Q_0 = 0$

$\therefore$ Equation (9.44) becomes

$$E = RI(s) + \frac{1}{C}\left[\frac{I(s)}{s} + 0\right]$$

$$= \left(R + \frac{1}{Cs}\right) I(s) = \frac{R}{s}\left(s + \frac{1}{RC}\right) I(s)$$

$\therefore$ $$I(s) = \frac{E}{R}\frac{s}{\left(s + \frac{1}{RC}\right)} = \frac{E}{R}\left[1 - \frac{1/RC}{s + \frac{1}{RC}}\right]$$

$$= \frac{E}{R}\left[1 - \frac{1/T}{s + \frac{1}{T}}\right] \qquad ...[9.44(a)]$$

($\because$ $RC = T$, the time constant)

Hence, the inverse of equation (9.45) gives

$$i(t) = \frac{E}{R}\delta(t) - \frac{E}{RT}e^{-\frac{1}{T}\cdot t}\text{ A.}$$

9.13 IMPULSE RESPONSE OF SERIES *RL* NETWORK

Figure 9.16 represents a series R-L circuit being excited by an *impulse function* $E\,\delta(t)$. Let us assume that the circuit is initially de-energized. Using KVL,

$$E\,\delta(t) = Ri(t) + L\frac{di(t)}{dt} \qquad ...(9.45)$$

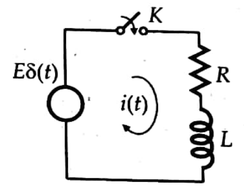


Fig. 9.16 *R-L* circuit excited by impulse function.

Taking the Laplace transform of equation (9.45),

$$E = RI(s) + L[sI(s) - i(0+)] \qquad ...(9.46)$$

However, following the initial condition, $i(0+) = 0$. Thus from equation (9.46),

$$E = RI(s) + Ls\,I(s)$$

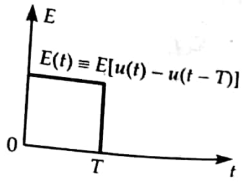
or, $$I(s) = \frac{E}{R + Ls} = \frac{E}{L\left(s + \frac{R}{L}\right)} = \frac{E}{L}\cdot\frac{1}{\left(s + \frac{R}{L}\right)}$$

Taking the inverse of Laplace transformation,

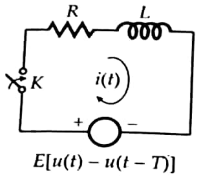
$$i(t) = \frac{E}{L}e^{-\frac{R}{L}\cdot t}.$$

9.14 RESPONSE OF *RL* CIRCUIT WITH PULSE INPUT

A pulse $E(t)$ as shown in Fig. 9.17(a) is applied to a RL circuit [Fig. 9.17(b)] at $t = 0$ by switching a switch K. Application of KVL in the loop of Fig. 9.17(b) yields



(*a*) Pulse input



(*b*) Pulse input applied to *RL* series circuit

Fig. 9.17

$$E(t) = Ri(t) + L\frac{di(t)}{dt}$$

or, $$E[u(t) - u(t-T)] = R.i(t) + L\frac{di(t)}{dt} \quad ...(9.47)$$

The Laplace transform of equation (9.47) gives

$$£\, E\, u(t) - £\, u(t-T) = £\left[Ri(t) + L\frac{di(t)}{dt}\right]$$

or, $$\frac{E}{s} - \frac{E}{s}e^{-Ts} = RI(s) + L[sI(s) - i(0+)] \quad ...(9.48)$$

Assuming the circuit to the initially de-energized, $i(0+) = 0$. Thus equation (9.48) becomes

$$\frac{E}{s}(1-e^{-Ts}) = RI(s) + LsI(s)$$

or, $$I(s) = \frac{E(1-e^{-Ts})}{s(R+Ls)} = (1-e^{-Ts})\frac{E}{L}\frac{1}{s\left(s+\frac{R}{L}\right)} \quad ...(9.49)$$

However, by partial fraction,

$$\frac{E/L}{s\left(s+\frac{R}{L}\right)} = \frac{A}{s} + \frac{B}{s+\frac{R}{L}},$$

when $$A = \left[\frac{E/L}{s+\frac{R}{L}}\right]_{s=0} = \frac{E}{R};$$

$$B = \left[\frac{E/L}{s}\right]_{s=-\frac{R}{L}} = -\frac{E}{R}$$

Thus, equation (9.49) becomes

$$I(s) = (1-e^{-Ts})\left[\frac{E/R}{s} + \frac{(-E/R)}{s+\frac{R}{L}}\right]$$

$$= \frac{E/R}{s} - \frac{E/R}{s}e^{-Ts} - \frac{E/R}{s+\frac{R}{L}} + \frac{E/R}{s+\frac{R}{L}}e^{-Ts}$$

Thus the inverse Laplace transform is given by

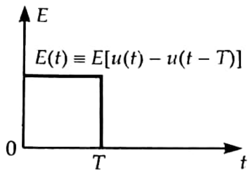
$$i(t) = \frac{E}{R}u(t) - \frac{E}{R}u(t-T) - \frac{E}{R}e^{-\frac{R}{L}t} + \frac{E}{R}e^{-\frac{R}{L}t}\, u(t-T)$$

9.15 PULSE RESPONSE OF SERIES *RC* CIRCUIT

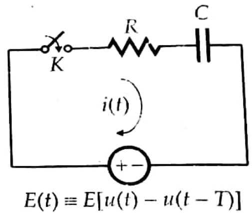
A pulse $E(t)$ [Fig. 9.18(*a*)] is applied to *RC* circuit through a switch *K*, the circulating current being $i(t)$ [Fig. 9.18(*b*)]. We assume the initial charge to be zero in the capacitor. Application of KVL in the loop of Fig. 9.18(*b*) yields

$$E(t) = Ri(t) + \frac{1}{C}\int i(t)\,dt$$

or, $$E[u(t) - u(t-T)] = Ri(t) + \frac{1}{C}\int i(t)\,dt \quad ...(9.50)$$



(*a*) Pulse input



(*b*) Pulse input in *RC* circuit

Fig. 9.18

Taking the Laplace Transform,

$$RI(s) + \frac{1}{C}\left[\frac{Q_0}{s} + \frac{I(s)}{s}\right] = \frac{E}{s}[1-e^{-Ts}] \quad ...(9.51)$$

But $Q_0 = 0$ [following the initial condition]

Thus equation (9.51) becomes,

$$RI(s) + \frac{1}{C}\left[\frac{I(s)}{s}\right] = \frac{E}{s}(1-e^{-Ts})$$

$$\therefore \quad I(s) = \frac{E}{s}\frac{1-e^{-Ts}}{R+\frac{1}{Cs}} = \frac{E}{R}\frac{1-e^{-Ts}}{s+\frac{1}{RC}}$$

$$= \frac{E}{R}\left[\frac{1}{s+\frac{1}{RC}} - \frac{e^{-Ts}}{s+\frac{1}{RC}}\right] \quad ...(9.52)$$

Taking the inverse of equation (9.52),

$$i(t) = \frac{E}{R}\left[e^{-\frac{t}{RC}} - e^{-\frac{t}{RC}}u(t-T)\right].$$

EXAMPLE 9.24 *In the circuit of Fig. E9.10 obtain the expression for the current i(t) when the switch is moved from position (1) to position (2) at t = 0.*

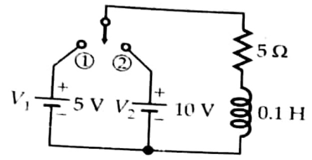


Fig. E9.10

SOLUTION. Assuming the switch to be at position (2), application of KVL in Fig. E9.11 yields

$$5i(t)+0.1\frac{di(t)}{dt}=10.$$

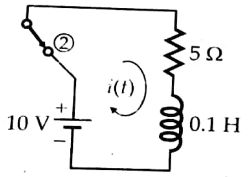
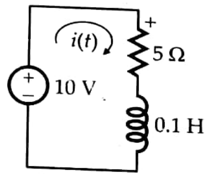


Fig. E9.11



(at $t=0+$)
Fig. E9.12

Taking Laplace transform of the above equation

$$5I(s)+0.1[sI(s)-i(0+)]=\frac{10}{s} \qquad ...(a)$$

$i(0+)$ being the current in the *R-L* circuit just after switching.

Obviously $i(0)$ at $t=0^+$ (*i.e.*, at the instant just after switching) will be equal to the value of $i(0)$ at $t=0^-$ (*i.e.* at the instant just before switching) as the inductance will not allow the current to flow through it immediately with change of switching status.

However, $i(0)$ at $t=0^-$ is given by

$$\frac{\text{voltage source } V_1}{\text{resistance } 5\,\Omega}=1\text{ A}$$

as the current at $t=0^-$ means the switch at position (1) and the circuit in steady state, inductance being of no effect.

Thus, $i(0+)=1$ A.

Thus, equation (*a*) becomes,

$$5\,I(s)+0.1[s\,I(s)-1]=\frac{10}{s}$$

or,
$$I(s)[5+0.1\,s]-0.1=\frac{10}{s}$$

$$\therefore \quad I(s)=\frac{(10/s)+0.1}{5+0.1\,s}$$

$$=\frac{10+0.1s}{s(0.1\,s+5)}=\frac{s+100}{s(s+50)}$$

However,

$$\frac{s+100}{s(s+50)}=\frac{A}{s}+\frac{B}{s+50}.$$

$$A=\left.\frac{s+100}{s+50}\right|_{s=0}=2$$

$$B=\left.\frac{s+100}{s}\right|_{s=-50}=-1$$

$$\therefore \quad I(s)=\frac{2}{s}+\frac{(-1)}{s+50}=\frac{2}{s}-\frac{1}{s+50}$$

$$\therefore \quad i(t)=2-e^{-50t}\text{ A.}$$

9.16(A) STEP RESPONSE OF *RLC* SERIES CIRCUIT

In the circuit shown in Fig. 9.19(*a*), a unit step voltage $u(t)$ is applied across an *RLC* series circuit having zero initial conditions $[V_c(0^-)=0;\, I_L(0^-)=0]$.

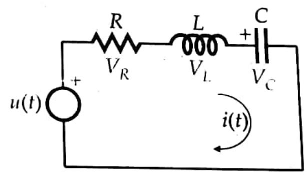


Fig. 9.19 (*a*) *Series RLC* circuit excited by step voltage.

In Laplace domain, the current through the circuit is given by

$$I(s)=\frac{1/s}{R+Ls+\dfrac{1}{Cs}}$$

$$\left[\because \pounds\, u(t)=\frac{1}{s};\ Z(s)=R+Ls+\frac{1}{Cs}\right]$$

$$=\frac{1}{Rs+Ls^2+\dfrac{1}{C}}=\frac{1/L}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}}$$

$$=\frac{1/L}{(s+\alpha)(s+\beta)} \qquad ...(9.53)$$

$$\text{where, } \alpha,\beta=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}} \qquad ...(9.54)$$

It may be noted here that there are *three* possible solutions for $I(s)$:

Case A. Both α and β are real and not equal $\left[\dfrac{R}{2L}>\dfrac{1}{\sqrt{LC}}\right]$

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Here $I(s)=\dfrac{1/L}{(s+\alpha)(s+\beta)}=\dfrac{K_1}{s+\alpha}+\dfrac{K_2}{s+\beta}$.

where $K_1=\left.\dfrac{1/L}{(s+\beta)}\right|_{s=-\alpha}=\dfrac{1}{L(\beta-\alpha)}$

$K_2=\left.\dfrac{1/L}{(s+\alpha)}\right|_{s=-\beta}=\dfrac{1}{L(\alpha-\beta)}$,

Thus, $I(s)=\dfrac{1/L}{(s+\alpha)(s+\beta)}$

$$=\frac{1/L(\beta-\alpha)}{(s+\alpha)}+\frac{1/L(\alpha-\beta)}{(s+\beta)} \qquad ...(9.55)$$

$$\therefore \quad i(t)=\left[\frac{1}{L(\beta-\alpha)}e^{-\alpha t}+\frac{1}{L(\alpha-\beta)}e^{-\beta t}\right]A \qquad ...(9.56)$$

Case B. α and β are equal to each other $[\alpha=\beta=\gamma]$

i.e., $\dfrac{R}{2L}=\dfrac{1}{\sqrt{LC}}$

$$I(s)=\frac{1/L}{(s+\gamma)^2} \quad \text{when } \alpha=\beta=\gamma$$

$$\therefore \quad i(t)=\frac{1}{L}te^{-\gamma t}\ A \qquad ...(9.57)$$

Case C. $\left[\dfrac{R}{2L}\right]<\dfrac{1}{\sqrt{LC}}$ and $\alpha=\beta^*$

Let $\alpha=-A_0+jB$ and $\beta=-A_0-jB$.

$$\left[\text{Here } |A_0|=\frac{R}{2L} \text{ and } |B|=\sqrt{\frac{1}{LC}-\left[\frac{R}{2L}\right]^2}\right]$$

Using partial fraction expansion,

$$I(s)=\frac{1/L}{(s+A_0-jB)(s+A_0+jB)}$$

$$=\frac{K_3}{(s+A_0-jB)}+\frac{K_3^*}{(s+A_0+jB)}$$

Here $K_3=(s+A_0-jB)\,I(s)\big|_{s=-A_0+jB}$

$$K_3=\left.\frac{1/L}{(s+A_0+jB)}\right|_{s=-A_0+jB}=\frac{1}{(j2B)L}$$

i.e., $K_3^*=-\dfrac{1}{(j2B)L}$

$$\therefore \quad I(s)=\frac{1/(j2B)L}{(s+A_0-jB)}-\frac{1/(j2BL)}{(s+A_0+jB)}$$

$$\therefore \quad i(t)=\frac{1}{j2BL}e^{-A_0t}e^{jBt}-\frac{1}{j2BL}e^{-A_0t}e^{-jBt}$$

$$=\frac{e^{-A_0t}}{BL}\left[\frac{e^{jBt}-e^{-jBt}}{2j}\right]$$

or, $$i(t)=\left[\frac{e^{-A_0t}}{BL}\sin Bt\right] \qquad ...(9.58)$$

Equations (9.56), (9.57) and (9.58) represent $i(t)$ at $t=0^+$ for the given circuit.

9.16(B) STEP CURRENT RESPONSE OF *RLC* PARALLEL CIRCUIT

The constant current source I is applied to the parallel RLC circuit by opening the switch (Fig. 9.19(b)). Let V be the voltage at node (x).

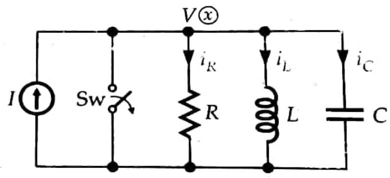

Fig. 9.19 (b) Step current applied to parallel RLC circuit.

Using nodal analysis for node (x), we can write

$$I=\frac{V}{R}+\frac{1}{L}\int v\,dt+C\frac{dv}{dt}$$

Taking Laplace transformation,

$$\frac{I(s)}{s}=\frac{V(s)}{R}+\frac{1}{L}\frac{V(s)}{s}+sCV(s)$$

[We assume switching at $t=0$ and zero initial condition for L, R and C at $t=0-$ when the switch was closed]

i.e., $$\frac{I(s)}{s}=V(s)\left[\frac{1}{R}+\frac{1}{Ls}+Cs\right]$$

$$=V(s)\left[\frac{1}{RC}s+\frac{1}{LC}+s^2\right]\frac{C}{s} \qquad ...(9.59)$$

Assuming the current source to be a unit step function, we can write for equation (9.59),

$$V(s)=\frac{1/C}{s^2+\frac{1}{RC}.s+\frac{1}{LC}} \qquad ...(9.60)$$

For convenience we assume

$$\frac{1}{RC}=2\xi\omega_0 \quad \text{and} \quad \frac{1}{LC}=\omega_0^2$$

$\therefore$ The expression of $V(s)$ can now be written as

$$V(s) = \frac{1/C}{s^2 + 2\,\xi\,\omega_0 \,.\, s + \omega_0^2}$$

$$= \frac{1/C}{s^2 + 2\,\xi\,\omega_0\, s + (\xi\,\omega_0)^2 + \omega_0^2 - (\xi\omega_0)^2}$$

$$= \frac{1}{C\omega_d} \cdot \frac{\omega_d}{(s + \xi\omega_0)^2 + \omega_d^2}$$

where $\omega_d = \sqrt{\omega_0^2 - (\xi\omega_0)^2} = \omega_0\sqrt{1 - \xi^2}$

Also, $\omega_d = \sqrt{\omega_0^2 - (\xi\omega_0)^2} = \sqrt{\frac{1}{LC} - \left(\frac{1}{2\,RC}\right)^2}$

Hence, in time domain,

$$v(t) = \frac{1}{\omega_d\, C}\, e^{-\xi\,\omega_0 \cdot t} \sin \omega_d t \qquad ...(9.61)$$

$$\therefore \quad i_R = \frac{v(t)}{R} = \frac{1}{\omega_d\, CR}\, e^{-\xi\,\omega_0 t} \sin \omega_d t \qquad ...(9.62)$$

and $\quad i_L = \frac{1}{L}\int_0^t v(t)\,dt$

$$= \frac{1}{L}\int_0^t \frac{1}{\omega_d \,.\, C}\, e^{-\xi\,\omega_0 \cdot t} \sin \omega_d t \,.\, dt$$

$$= \frac{1}{\omega_d LC}\int_0^t e^{-\xi\omega_0 \cdot t} \sin \omega_d \,.\, t\, dt$$

$$= \frac{1}{\omega_d \,.\, LC}\left[\frac{e^{-\xi\,\omega_0 \cdot t}}{\sqrt{(\xi\omega_0)^2 + \omega_d^2}} \times \sin(\omega_d t - \phi)\right] \qquad ...(9.63)$$

where $\phi = -\tan^{-1}\frac{\omega_d}{\xi\,\omega_0}$.

$$i.e., \quad i_L(t) = \frac{1}{\omega_d LC}\left[\frac{e^{-\xi\,\omega_0 \cdot t}}{\sqrt{(\xi\omega_0)^2 + \omega_d^2}} \times \sin\left(\omega_d t + \tan^{-1}\frac{\omega_d}{\xi\omega_0}\right)\right] \qquad ...(9.64)$$

Again, $\quad i_C(t) = C\frac{dv(t)}{dt}$

Let $\xi\omega_0 = a$ and $\omega_d = b$

$$\therefore i_C = C\left(\frac{1}{bC}\right)[-ae^{-at}\sin b\,.\,t + be^{-at}\cos bt]$$

$$= \frac{1}{b}\,e^{-at}\,[b\cos bt - a\sin bt]$$

$$= \frac{1}{b}\,e^{-at}\,[\sqrt{a^2 + b^2}\,]\sin(bt - \phi)$$

where $\phi = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{\omega_d}{\xi\omega_0}$

$$\therefore \quad i_C(t) = \frac{e^{-\xi\omega_0 t}}{\omega_d}[\sqrt{(\xi\omega_0)^2 + \omega_d^2}\,]$$

$$\sin\left(\omega_d t - \tan^{-1}\frac{\omega_d}{\xi\,\omega_0}\right) \qquad ...(9.65)$$

EXAMPLE 9.25 *Fig. E9.13 represents a parallel RLC circuit. The switch is suddenly opened at t = 0. Assuming no charge on the capacitor and no current in the inductor before switching find the voltage across the switch.*

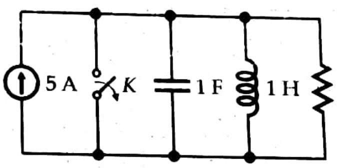

Fig. E9.13

SOLUTION. Let the strength of the current source be I (= 5 A), the current through the capacitor be $i_C(t)$, current through the inductor be $i_L(t)$ and current through the resistor be $i_R(t)$ just after switching K [it may be noted that before switching off K, as K was closed, no current passed through R, L or C and hence there was no energy stored in L or C during steady state condition. The circuit components R, L and C will be energized only after K is off.]

As per KCL

$$i(t) = i_R(t) + i_L(t) + i_C(t)$$

or, $$5 = \frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt}$$

[assuming the voltage across the switch after its opening to be v]

or, $$5 = v + \int_{-\infty}^{t} v\,dt + \frac{dv}{dt}.$$

or, $$5 = v + \int_{-\infty}^{0} v\,dt + \int_{0}^{t} v\,dt + \frac{dv}{dt} \qquad ...(i)$$

CHAPTER 9

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Taking Laplace transform of (*i*),

$$V(s) + \pounds\left[\int_{-\infty}^{0} v\,dt\right] + \frac{V(s)}{s} + [sV(s) - v(0+)] = \frac{5}{s}$$

However, following zero initial condition,

$$V(s) + \frac{V(s)}{s} + sV(s) = \frac{5}{s}$$

$$V(s)\left[1 + \frac{1}{s} + s\right] = \frac{5}{s}$$

or, $$V(s) = \frac{5}{s\left(1 + \frac{1}{s} + s\right)} = \frac{5}{(s + 1 + s^2)}$$

$$= \frac{5}{(s - 0.5 + j\sqrt{0.75})(s - 0.5 - j\sqrt{0.75})}$$

$$= \frac{A}{(s - 0.5 + j\sqrt{0.75})} + \frac{B}{(s - 0.5 - j\sqrt{0.75})}$$

$$A = V(s)(s - 0.5 + j\sqrt{0.75}\,\Big|_{s = 0.5 - j\sqrt{0.75}}$$

$$= \frac{5}{(s - 0.5 - j\sqrt{0.75})}\Bigg|_{s = 0.5 - j\sqrt{0.75}}$$

$$= \frac{5}{-2j\sqrt{0.75}} = \frac{j\,10\sqrt{0.75}}{4 \times 0.75} = j\,2.887$$

$$B = V(s)(s - 0.5 - j\sqrt{0.75}\,\Big|_{s = 0.5 + j\sqrt{0.75}}$$

$$= \frac{5}{(s - 0.5 + j\sqrt{0.75})}\Bigg|_{s = 0.5 + j\sqrt{0.75}}$$

$$= \frac{5}{2j\sqrt{0.75}} = -\frac{10\,j\sqrt{0.75}}{4 \times 0.75} = -j\,2.887$$

$$\therefore\quad V(s) = \frac{j\,2.887}{s - 0.5 + j\sqrt{0.75}} + \frac{-j\,2.887}{s - 0.5 - j\sqrt{0.75}}$$

Inverse of $v(s)$ then gives

$$v(t) = j\,2.887\,(e^{(0.5 - j\sqrt{0.75})t}) - j\,2.887\,(e^{(0.5 + j\sqrt{0.75})t})\text{ V}$$

Example 9.26 *Function in s-domain is given by*

$$F(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)}$$

Find (i) $f(t)$ and (ii) the steady state solution using final value theorem.

Solution. $$F(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)}$$

$$= \frac{P}{s} + \frac{Q}{(s + 2)} + \frac{R}{s + 1}$$

However,

$$P = \frac{s^2 + 3s + 1}{(s + 2)(s + 1)}\Bigg|_{s = 0} = \frac{1}{2}$$

$$Q = \frac{s^2 + 3s + 1}{s(s + 1)}\Bigg|_{s = -2} = -\frac{1}{2}$$

$$R = \frac{s^2 + 3s + 1}{s(s + 2)}\Bigg|_{s = -1} = 1$$

$$\therefore\quad F(s) = \frac{1/2}{s} + \frac{(-1/2)}{s + 2} + \frac{1}{s + 1}\text{ giving}$$

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} + e^{-t}$$

Here the expression $f(t)$ has steady state solution value $1/2$ and the transient part $\left(-\frac{1}{2}e^{-2t} + e^{-t}\right)$.

However, to obtain the steady state value by final value theorem,

$$F(s)\text{ at }\infty = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \cdot s$$

$$= \lim_{s \to 0} \frac{s^2 + 3s + 1}{(s + 1)(s + 2)} = \frac{1}{2}.$$

$\therefore$ Final value of the function $= \frac{1}{2}$.

Example 9.27 *A time domain network is shown in Fig. E9.14. Find the loop currents (following switching) in Laplace domain as well as in time domain when $r_1 = r_2 = r_3 = 1\,\Omega$, $L = 1$ H and $C = 1$ F. Initial potential in the capacitor is e_0 though the initial currents through the inductor and capacitor are zero.*

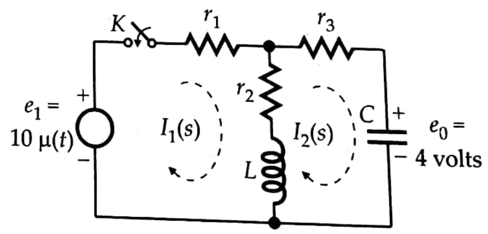

Fig. E9.14

Solution. Let the loop currents be $I_1(s)$ and $I_2(s)$ as shown in Fig. E9.14. In time domain the application of KVL in the loops after switching of the switch K yields

$$r_1 i_1(t) + r_2[i_1(t) - i_2(t)] + L\frac{d}{dt}[i_1(t) - i_2(t)] = e_1 \quad ...(1)$$

and $$r_2[i_2(t) - i_1(t)] + L\frac{d}{dt}[i_2(t) - i_1(t)] + r_3 i_2(t) + \frac{1}{C}\int i_2\, dt = -e_0 \quad ...(2)$$

On rearrangement, these equations become,

$$(r_1 + r_2) i_1(t) - r_2 i_2(t) + L\frac{di_1(t)}{dt} - L\frac{di_2(t)}{dt} = e_1 \quad ...(1A)$$

and $$-r_2 i_1(t) - L\frac{di_1(t)}{dt} + L\frac{di_2(t)}{dt} + (r_2 + r_3) i_2(t) + \frac{1}{C}\int i_2\, dt = -e_0 \quad ...(2A)$$

Taking the Laplace transform of three equations, we get,

$$(r_1 + r_2) I_1(s) - r_2 I_2(s) + sLI_1(s) - Li_1(0^+) - sLI_2(s) + Li_2(0^+) = E_1(s) \quad ...(1B)$$

and $$-r_2 I_2(s) - sLI_1(s) + Li_1(0^+) + sLI_2(s) - Li_2(0^+) + (r_2 + r_3) I_2(s) + \frac{1}{sC} I_2(s) = -E_0(s) \quad ...(2B)$$

On rearrangement, equations (1B) and (2B) become

$$(r_1 + r_2 + sL) I_1(s) - (r_2 + sL) I_2(s) = E_1(s) + L[i_1(0^+) - i_2(0^+)] \quad ...(1C)$$

and $$-(r_2 + sL) I_1(s) + \left(r_2 + r_3 + sL + \frac{1}{Cs}\right) I_2(s) = -E_0(s) + L[i_2(0^+) - i_1(0^-)] \quad ...(2C)$$

However,

$$r_1 = r_2 = r_3 = 1\,\Omega,\ L = 1\,H,\ C = 1\,F,\ e_0 = 4\,V$$

and $i_1(0^+) = i_2(0^+) = 0$.

$\therefore$ Equations (1C) and (2C) become

$$(s+2) I_1(s) - (s+1) I_2(s) = \frac{10}{s} \quad ...(1D)$$

and $$-(s+1) I_1(s) + \frac{s^2 + 2s + 1}{s} I_2(s) = -\frac{4}{s} \quad ...(2D)$$

$$\therefore I_1(s) = \frac{\begin{vmatrix} (10/s) & -(s+1) \\ -4/s & \dfrac{s^2+2s+1}{s} \end{vmatrix}}{\begin{vmatrix} (s+2) & -(s+1) \\ -(s+1) & \dfrac{s^2+2s+1}{s} \end{vmatrix}} = \frac{\dfrac{2}{s^2}(3s+5)(s+1)}{\dfrac{2}{s}(s+1)^2}$$

$$= \frac{3s+5}{s(s+1)}$$

$$I_2(s) = \frac{\begin{vmatrix} s+2 & 10/s \\ -(s+1) & -4/s \end{vmatrix}}{\dfrac{2}{s}(s+1)^2} = \frac{3s+1}{(s+1)^2}$$

Now

$$I_1(s) = \frac{3s+5}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$A = \left.\frac{3s+5}{s+1}\right|_{s=0} = 5$$

$$B = \left.\frac{3s+5}{s}\right|_{s=-1} = -2$$

$$\therefore I_1(s) = \frac{5}{s} - \frac{2}{s+1} \text{ giving } i_1(t) = 5 - 2e^{-t}\ \text{A}$$

Similarly,

$$I_2(s) = \frac{3s+1}{(s+1)^2} = \frac{3}{s+1} - \frac{2}{(s+1)^2}$$

or, $$i_2(t) = 3e^{-t} - 2te^{-t}\ \text{A}$$

EXAMPLE 9.28 *In the circuit of Fig. E9.15 steady state exists when switch K is in position 1. At t = 0, it is moved to position 2. Obtain the expression for current using Laplace transformation.*

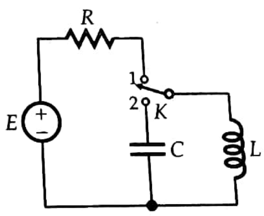

Fig. E9.15

SOLUTION. When the switch is at 1, steady state is reached and the current through the circuit is

$$I_{in} = \frac{E}{R} \quad ...(1)$$

[it may be noted here that at the time of switching, i.e., at $t = 0$, the circuit current is thus (E/R)]