Diagonalization

If a square matrix A of order n has n linearly independent eigen vectors, then a matrix B can be found such that $B^{-1}AB$ is a diagonal matrix.

[We prove the result for a square matrix of order 3. The proof can be easily extended to matrices of higher order.]

Let A be a square matrix of order 3. Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$ be the corresponding eigen vectors.

Since eigen vectors are non-trivial solutions of the matrix equation $AX = \lambda X$, we have

$$AX_1 = \lambda_1 X_1, \quad AX_2 = \lambda_2 X_2, \quad AX_3 = \lambda_3 X_3$$

Let $$B = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = [X_1 \ X_2 \ X_3]$$

Consider $$AB = A[X_1 \ X_2 \ X_3]$$
$$= [AX_1 \ AX_2 \ AX_3]$$
$$= [\lambda_1 X_1 \ \lambda_2 X_2 \ \lambda_3 X_3]$$
$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$= BD$, where D is the diagonal matrix $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$\Rightarrow AB = BD$

$\Rightarrow B^{-1}AB = B^{-1}BD$

$\Rightarrow B^{-1}AB = ID$

$\Rightarrow B^{-1}AB = D$ which proves the theorem.

Note:- The matrix B which diagonalises A is called the modal matrix of A & is obtained by grouping the eigen vectors of A into a square matrix. The diagonal matrix D is called the spectral matrix of A and has the eigen values of A as its diagonal elements.

Calculation of Powers of a Matrix

Given a square matrix A, it is quite tedious to find A^n (n being a positive integer) when n is large. On the other hand, it is quite easy to obtain any positive integral power of a diagonal matrix D, since

$$D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Rightarrow D^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

Therefore, diagonalisation of a matrix is useful for finding positive integral powers of a square matrix.

Let A be a square matrix. Then we know

that a non-singular matrix $\bar{B}$ can be found such that

$$B^{-1}AB = D \qquad \text{——— (1)}$$

where D is a diagonal matrix.

$$D^2 = (B^{-1}AB)(B^{-1}AB)$$
$$= B^{-1}ABB^{-1}AB$$
$$= B^{-1}AIAB$$
$$= B^{-1}A^2B$$

Similarly, $D^3 = B^{-1}A^3B$

In general, $D^n = B^{-1}A^nB$ ——— (2)

To find A^n from (2), pre-multiply (2) by B and post-multiply by B^{-1}. Then

$$BD^nB^{-1} = BB^{-1}A^nBB^{-1}$$

$$\Rightarrow \quad \boxed{A^n = BD^nB^{-1}}$$

Problem 1:- Show that the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ is diagonalizable. Hence find P such that $P^{-1}AP$ is a diagonal matrix.

Solution:- The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ (On simplification)

After solving it, we get

$$\lambda = 1, 2, 3$$

Since the matrix A has three distinct eigen values, it has three linearly independent eigen vectors and hence A is diagonalizable.

When $\lambda_1 = 1$, the corresponding eigen vector is given by

$$(A-I)X_1 = 0 \quad \text{or} \quad \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_1 \rightarrow R_1 + R_2$

$$\sim \begin{bmatrix} 0 & 1 & 1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \rightarrow \frac{1}{2} R_2$ & $R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get only two independent equations $y_1 + z_1 = 0$ & $-x_1 + z_1 = 0$, since rank of co-eff matrix is 2.

$$\Rightarrow y_1 = -z_1 \quad \& \quad x_1 = z_1$$

$$\Rightarrow \frac{x_1}{1} = \frac{y_1}{-1} = \frac{z_1}{1}$$

eigen vector $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

When $\lambda_2 = 2$, the corresponding eigenvector is given by

$$(A-2I)X_2 = 0 \quad \text{or} \quad \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_1 \rightarrow R_1 - R_3$ & $R_2 \rightarrow R_2 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 + 2R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we get two independent equations $x_2 - z_2 = 0$ & $y_2 = 0$, since rank of co-efficient matrix is 2.

$\Rightarrow x_2 = z_2$ & $y_2 = 0$ giving $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

When $\lambda_3 = 3$, the corresponding eigenvector is given by

$$(A - 3I)X_3 = 0 \text{ or } \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 + 2R_1$ & $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we get two independent equations $y_3 - z_3 = 0$ or $-2x_3 = 0$, since rank of co-efficient matrix is 2.

$\Rightarrow x_3 = 0$ & $y_3 = z_3$ giving the eigenvector

$$X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ The modal matrix $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

& $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is the required diagonal matrix.

Problem 2:- Diagonalise the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ & obtain the modal matrix.

Solution:- The characteristic equation of A is

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$\lambda^3 - \lambda^2 - 5\lambda + 5 = 0$ (On simplification)

Solving it, we have $\lambda = 1, \pm\sqrt{5}$

When $\lambda_1 = 1$, the corresponding eigen vector is given by $(A-I)X_1 = 0$

$$-2x_1 + 2y_1 - 2z_1 = 0$$
$$x_1 + y_1 + z_1 = 0$$
$$x_1 - y_1 + z_1 = 0$$

Solving the last two $\frac{x_1}{2} = \frac{y_1}{0} = \frac{z_1}{-2}$

giving the eigen vector $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

When $\lambda = \sqrt{5}$, the corresponding eigen vector is given by $(A-\sqrt{5}I)X_2 = 0$

$$(-1-\sqrt{5})x_2 + 2y_2 - 2z_2 = 0$$
$$x_2 + (2-\sqrt{5})y_2 + z_2 = 0$$
$$-x_2 - y_2 - \sqrt{5}z_2 = 0$$

Solving the last two $\frac{x_2}{6-2\sqrt{5}} = \frac{y_2}{-1+\sqrt{5}} = \frac{z_2}{1-\sqrt{5}}$

or $\frac{x_2}{(\sqrt{5}-1)^2} = \frac{y_2}{\sqrt{5}-1} = \frac{z_2}{1-\sqrt{5}}$

or $\frac{x_2}{\sqrt{5}-1} = \frac{y_2}{1} = \frac{z_2}{-1}$

givining the eigen vector $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \sqrt{5}-1 \\ 1 \\ -1 \end{bmatrix}$

Similarly, the eigen vector corresponding to $\lambda_3 = -\sqrt{5}$ is

$$X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} \sqrt{5}+1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ The modal matrix $B = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

& $B^{-1}AB = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$

is the required diagonal matrix.

Problem 3:- Diagonalise the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and hence find A^4.

Solution:- The characteristic equation of A is

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0$ (On simplification)

$\lambda = -1, 3, 4$ (after solving)

When $\lambda_1 = -1$, the corresponding eigen vector is given by

$$2x_1 + 6y_1 + z_1 = 0$$
$$x_1 + 3y_1 = 0$$
$$4z_1 = 0$$

Solving the last two, $\dfrac{x_1}{3} = \dfrac{y_1}{-1}$, $z_1 = 0$

giving the eigen vector $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

When $\lambda_2 = 3$, the corresponding eigen vector is given by

$$-2x_1 + 6y_2 + z_2 = 0$$
$$x_2 - y_2 = 0$$

Solving $\frac{x_2}{1} = \frac{y_2}{1} = \frac{z_2}{-4}$ giving the eigen vector $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$

When $\lambda_3 = 4$, the corresponding eigen vector is given by

$$-3x_3 + 6y_3 + z_3 = 0$$
$$x_3 - 2y_3 = 0$$
$$z_3 = 0$$

Solving the last two, $\frac{x_3}{2} = \frac{y_3}{1}$, $z_3 = 0$ giving the eigen vector, $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ The modal matrix $B = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$

& $B^{-1}AB = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

is the required diagonal matrix.

To find B^{-1}

$$|B| = \begin{vmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{vmatrix} = 20$$

$$\operatorname{adj} B = \begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$B^{-1} = \frac{\operatorname{adj} B}{|B|} = \frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

Now $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\Rightarrow D^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix}$

$$A^4 = BD^4B^{-1} = \frac{1}{20}\begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 103 & 306 & 82 \\ 51 & 154 & 31 \\ 0 & 0 & 81 \end{bmatrix}$$

Exercise

1) Reduce the matrix $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$ to the diagonal form.

Sol^n $\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$

2) By diagonalising the matrix $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$, find A^4.

Sol^n $\begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$

3) Diagonalise $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ and hence find A^3.

Sol^n $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & -7 & 32 \\ 0 & 8 & -19 \\ 0 & 0 & 27 \end{bmatrix}$

4) Diagonalise the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and hence find A^4

Sol^n $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, $\begin{bmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{bmatrix}$