

UNIT-2: Complex Analysis - II (Continuation of unit I...)

1) Laurent Series:

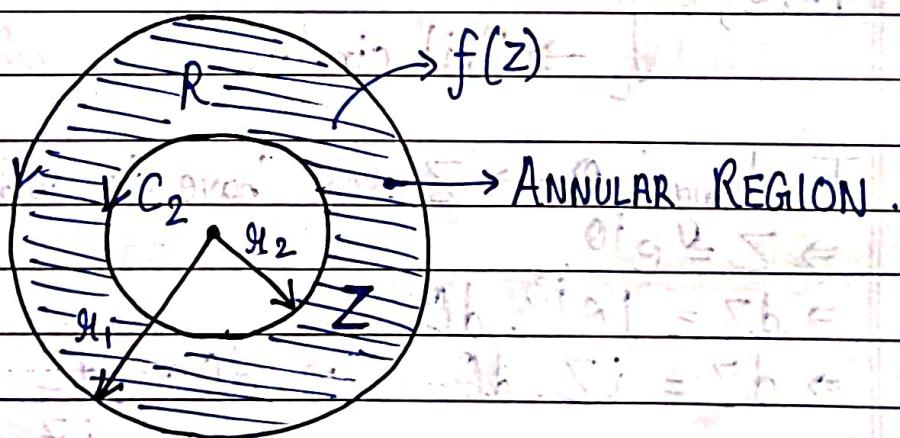
If $f(z)$ is analytic in annular (co-centric) region R bounded by two co-centric circles C_1 and C_2 of radii r_1 and r_2 , where, $r_1 \geq r_2$ and center at a , then for all Z in R we have,

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{General term}} + \underbrace{\sum_{n=0}^{\infty} b_n (z-a)^{-n}}_{\text{Principal term}}$$

here,

$$a_n = \frac{1}{2\pi i} \int_C f(t) dt, \quad n=0,1,2,3,\dots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt, \quad n=1,2,3,4,\dots$$



Note: Some important binomial expansions,

$$1) (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots \dots \infty$$

$$2) (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots \dots \infty$$

2) Residue Integration Method :-

2.1) i) Residue → Residue of an analytic function $f(z)$ at $z=a$ is the coefficient of $\frac{1}{z-a}$, i.e., b_n in the Laurent series expansion (z) for $n=1 \Rightarrow b_1$.

→ Residue of $f(z)$ is denoted by $\text{Res } f(z)$ or $\text{Res}\{f(z)\}; a$ at $z=a$.

Since,

$$\Rightarrow b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{-n+1}} dz.$$

$$\Rightarrow b_1 = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{-1+1}} dz \quad (n=1).$$

or,

$$b_1 = \frac{1}{2\pi i} \int_C f(z) \cdot dz.$$

* Cauchy's Residue Theorem :-

$$\Rightarrow \int_C f(z) \cdot dz = 2\pi i b_1 = 2\pi i [\text{Residue}] = 2\pi i \sum R^+$$

ii) How to find out Residue :-

i) Residue at simple pole at $z=a$.

$$\text{Res } f(z) = \lim_{z \rightarrow a} (z-a) \cdot f(z).$$

2) Residue at a pole of order 'm'.

$$\text{Res } f(z) = \frac{1}{(n-1)!} \lim_{z \rightarrow a} \frac{d}{dz}^{n-1} [(z-a)^n f(z)]_{z=a}$$

3) If $f(z)$ is of the form,

$$f(z) = \frac{\phi(z)}{\psi(z)} \text{ at } z=a.$$

where, $\psi(a) \neq 0$ and, $\phi(a) \neq 0$.

then,

$$\text{Res } f(z) = \frac{\phi(a)}{\psi(a)}$$

4) Residue of $f(z)$ at $z=\infty$

$$\text{Res}(f(z)) = \lim_{z \rightarrow \infty} [-z \cdot f(z)]$$

OR

$$= - \left[\text{coefficient of } \frac{1}{z} \text{ in the expansion of } f(z) \right]$$

Q.1) Determine the residue of the function at each pole:

$$f(z) = \frac{z^2}{(z-1)(z-2)^2}$$

Ans: First we calculate pole,

$$\Rightarrow (z-1)(z-2)^2 = 0$$

$$\Rightarrow z=1, 2, 2.$$

or,

$$z=1 \quad (\text{simple pole}) ; \quad z=2 \quad (\text{order : 2}).$$

- Residue of $f(z)$ at $z=1$ = $\lim_{z \rightarrow 1} (z-1) \cdot \frac{z^2}{(z-1)(z-2)^2}$

$$= \lim_{z \rightarrow 1} \frac{z^2}{(z-2)^2} = \frac{1^2}{(1-2)^2} = \frac{1}{1} = 1 = R_1$$

• Residue of $f(z)$ at $z=2$ (order 2) $= \frac{1}{(2-1)!} \lim_{z \rightarrow 2} \frac{d^{2-1}}{dz^{2-1}} [(z-2)^2 \cdot z^2] / (z-1)(z-2)^2$

$$\begin{aligned} \Rightarrow \text{Res}_{z=2}(f(z)) &= \frac{d}{dz} \left(\frac{z^2}{z-1} \right)_{z=2} \\ &= \left[\frac{2z(z-1) - z^2(1)}{(z-1)^2} \right]_{z=2} \\ &= \left[\frac{2z^2 - 2z - z^2}{(z-1)^2} \right]_{z=2} \\ &= \left[\frac{z(z-2)}{(z-1)^2} \right]_{z=2} \end{aligned}$$

$$\therefore \text{Res}_{z=2} f(z) = \frac{2 \times (2-2)}{(2-1)^2} = 0 = R_2$$

Hence,

$$\sum R^+ = R_1 + R_2 = 1 + 0 = 1 \rightarrow \text{Residue.}$$

Q.2) Find the residue of the tangent at its pole :-

$$f(z) = \frac{\sin z}{\cos z} = \tan z$$

Ans : For pole,

$$\Rightarrow \cos z = 0 = \cos \pi/2$$

$$\therefore z = \frac{\pi}{2}$$

If, $f(z) = \frac{\phi(z)}{\psi(z)}$ at pole $z=a$.

\Rightarrow Residue of $f(z) = \frac{\phi(a)}{\psi'(a)}$, if $\phi(a) \neq 0, \psi'(a) = 0$

$$\Rightarrow \underset{z=a}{\text{Res } f(z)} = \frac{\sin \pi/2}{\cos \pi/2} = \frac{1}{0} = \infty.$$

Q.3) Find the residue of $f(z) = z^2 e^{1/z}$?

$$\text{Ans: } f(z) = z^2 e^{1/z}$$

$$\Rightarrow f(z) = z^2 \left[1 + \frac{1}{(1/z)^{-1}} + \frac{1}{2!} \times \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} + \dots \right]$$

$$\left(\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right)$$

$$\Rightarrow f(z) = z^2 + z + \frac{1}{4} + \frac{1}{6z} + \dots$$

$$\therefore \text{Res } f(z) = -(\text{coefficient of } 1/z \text{ in the expansion}) = \boxed{-1/6}$$

Q.4) Find the integral of $f(z)$ using Cauchy's residue theorem:-

$$f(z) = \int_C \frac{1-2z}{z(z-1)(z-2)} dz$$

where, 'C' is the circle $\rightarrow |z| = 1.5$?

$$\text{Ans} = \int_C \frac{1-2z}{z(z-1)(z-2)} dz = 2\pi i \sum R^+ \text{ (from Residue theorem)}$$

\rightarrow Clearly, poles are : $z=0, 1, 2$.

But,

for the given circle $|z| = 1.5$, pole = 2 lies outside this circle, and only poles '0' and '1' lie inside C.

So, applying residue for $z=0$ and $z=1$ only,

$$\Rightarrow \text{Res } f(z) = \lim_{z \rightarrow 0} (z-0) \cdot \left(\frac{1-2z}{(z)(z-1)(z-2)} \right)$$

$$\Rightarrow \text{Res } f(z) = \lim_{z \rightarrow 0} \frac{1-2z}{(z-1)(z-2)}$$

$$\therefore \text{Res } f(z) = \frac{1}{-1 \times -2} = \boxed{\frac{1}{2}} \rightarrow R_1$$

and,

$$\Rightarrow \text{Res } f(z) = \lim_{z \rightarrow 1} (z-1) \left(\frac{1-2z}{z(z-1)(z-2)} \right)$$

$$\Rightarrow \text{Res } f(z) = \lim_{z \rightarrow 1} \frac{1-2z}{z(z-2)}$$

$$\therefore \text{Res } f(z) = \frac{-1}{1 \times -1} = \boxed{1} \rightarrow R_2$$

hence,

$$\sum R^+ = R_1 + R_2 = \frac{1}{2} + 1 = \frac{3}{2}$$

i.e.,

$$\Rightarrow \int_C f(z) \cdot dz = 2\pi i \sum R^+ = 2\pi i \times \frac{3}{2} = \boxed{3\pi i}$$

2.2) Contour Integration: Residue Integration of Real Integrals :-

→ Integration of these types, can be solved by contour integration method only.

$$1) \int_0^{2\pi} f(z) \cdot dz$$

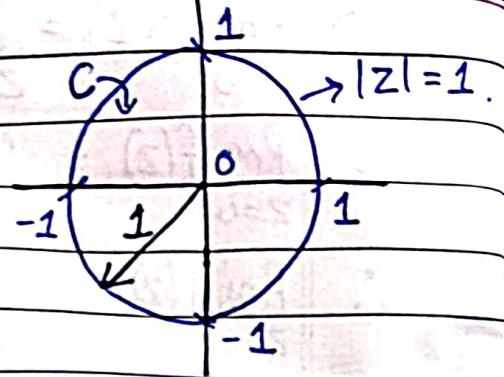
$$2) \int_{-\infty}^{\infty} f(z) \cdot dz$$

Case 1)

Integration round the unit circle $|z|=1$ of the type,

$$I = \int_0^{2\pi} f(\sin \theta, \cos \theta) \cdot d\theta$$

→ For the region 'C' → a unit circle from 0 to 2π ,
 'θ' will change to 'z' by;



Substituting,
 $\Rightarrow z = e^{i\theta}$

$$\Rightarrow z = e^{i\theta}$$

$$\Rightarrow dz = e^{i\theta} \cdot i d\theta = z i d\theta$$

$$\Rightarrow d\theta = \frac{dz}{iz} \quad \text{--- (1)}$$

For trigonometric terms,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2 + 1}{2z} \quad \text{--- (2)}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2 - 1}{2zi} \quad \text{--- (3)}$$

Use the equations (1) - (3) to convert,

$$I = \int_0^{2\pi} f(\theta) \cdot d\theta \quad \text{to} \quad \int_C f(z) \cdot dz.$$

Q.1) Find the integral $f(z)$, $I = f(z) = \int_0^{2\pi} \frac{d\theta}{5 - 4 \sin\theta}$

Ans: Let,

$$\Rightarrow z = e^{i\theta}$$

$$\Rightarrow dz = ie^{i\theta} \cdot d\theta = iz \cdot d\theta$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

and,

$$\Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2 - 1}{2zi}$$

On substituting in the integral,
 we have;

$$\Rightarrow I = \int_C \frac{1}{5 - 4(z^2 - 1)} \cdot \frac{dz}{iz}$$

$$\Rightarrow I = \int_C \frac{2iz}{10iz - 4z^2 + 4} \cdot \frac{dz}{iz} = \int_C \frac{2 \cdot dz}{-4(z^2 - \frac{5}{2}iz - 1)}$$

$$\Rightarrow I = -\frac{1}{2} \int_C \frac{1}{(z^2 - \frac{5}{2}iz - 1)} dz = -\frac{1}{2} \int_C f(z) dz.$$

here,

$$f(z) = \frac{1}{z^2 - \frac{5}{2}iz - 1}$$

- For pole,

$$\Rightarrow z^2 - \frac{5}{2}iz - 1 = 0 \text{ or, } 2z^2 - 5iz - 2 = 0.$$

$$\Rightarrow z = \sqrt{(-5i)^2 - 4(2)(-2)} + 5i / 4$$

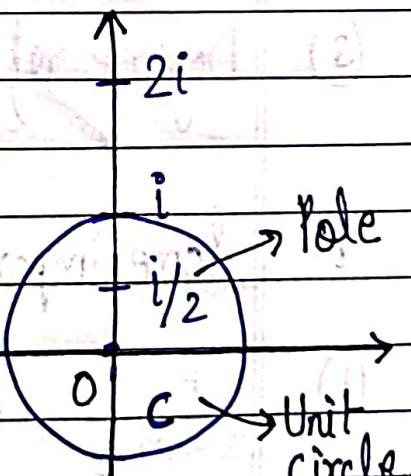
or,

$$\Rightarrow z = 5i \pm \sqrt{-25 + 16} = 5i \pm 3i = i \text{ or } 2i$$

$$\therefore z = i/2 \text{ or } 2i.$$

But,

' $2i$ ' lies outside the region ' C ' so it is not accounted in and, ' $i/2$ ' lies inside the unit circle ' C ' region.



- Residue of $f(z)$ at $z = i/2$.

$$\Rightarrow \text{Res } f(z) = \lim_{z \rightarrow i/2} (z - i/2) \frac{1}{(z - 2i)(z - i/2)}$$

$$\therefore \text{Res } f(z) = \lim_{z \rightarrow i/2} \frac{1}{z - 2i} = \frac{2}{i - 4i} = \boxed{\frac{-2}{3i}}$$

Now, by Cauchy's residue theorem;

$$\Rightarrow \int_C f(z) \cdot dz = 2\pi i \sum R^+ = 2\pi i \times \frac{-2}{3i} = -\frac{4\pi}{3}$$

hence,

$$\Rightarrow I' = -\frac{1}{2} \int_C f(z) \cdot dz$$

$$\therefore I = -\frac{1}{2} \times -\frac{4\pi}{3} = \boxed{\frac{2\pi}{3}}$$

Case 2) Integration of the type, $\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} dx = 2 \int_0^{\infty} \frac{f_1(x)}{f_2(x)} dx$.

* Conditions to solve this integral:

- ① No pole should be on X-axis, i.e.,
 $f_2(x) =$ should have NO real roots.
- ② $f_2(x)$ should have a polynomial degree of '2' atleast.
- ③ Degree of $f_2(x) >$ Degree of $f_1(x)$.

* Some important points to solve this integral :-

- ① If $\lim_{z \rightarrow \infty} z \cdot f(z) = 0$, then $\int_C f(z) \cdot dz = 0$

- ② If $f(z) = 0$, then $\lim_{R \rightarrow \infty} \int_C e^{imz} \cdot f(z) = 0$ JORDAN LEMMA

- ③ If 'C' is the arc of the circle $|z-a| = R$ such that,

$\theta_1 \leq \theta \leq \theta_2$ and $\lim_{z \rightarrow a} f(z) = K$ say, (K : constant)

then, $\lim_{r \rightarrow 0} \int_C f(z) \cdot dz = iK |\theta_2 - \theta_1|$.

(4) If 'C' is the arc of the circle $|z| = R$ such that $\theta_1 \leq \theta \leq \theta_2$ and, $\lim_{z \rightarrow \infty} z \cdot f(z) = K$, then;

$$\lim_{R \rightarrow \infty} \int_C f(z) \cdot dz = iK |\theta_2 - \theta_1|.$$

Q.1) Evaluate the integral $I = \int_0^\infty \frac{\cos mx}{x^2 + 1} \cdot dx$ (Important)

OR
Evaluate the integral $I = \int_{-\infty}^\infty \frac{\sin mx}{x^2 + 1} \cdot dx$

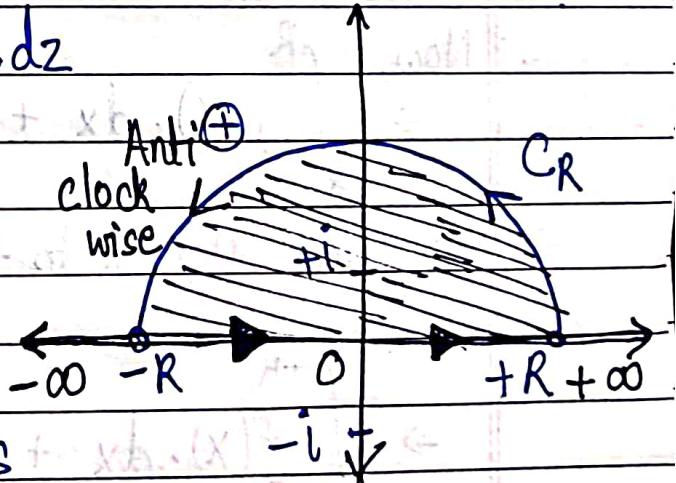
Ans: Consider the integral, $\int_C f(z) \cdot dz$

where, C : is the curve consisting upper half of the circle,

$$|z| = R$$

and the part of the real axis from $-R$ to $+R$ and,

$$f(z) = \frac{e^{imz}}{1+z^2}$$



For pole,

$$\Rightarrow 1+z^2=0$$

$$\Rightarrow z = \pm i$$

But, pole $= -i$ lies outside the region 'C' so the only pole is $= +i$.

Here, $e^{imz} = \cos mz + i \sin mz$.

and,

'x' is replaced by 'z' in the integral I.

Now, residue of $f(z)$ at $z=a$ where $a=i$,

$$\text{Res } f(z) = \lim_{z \rightarrow i} (z-i) \frac{e^{imz}}{1+z^2}$$

$$= \lim_{z \rightarrow i} \frac{(z-i) e^{imz}}{(z+i)(z-i)}$$

$$= \lim_{z \rightarrow i} e^{imz}$$

$$\therefore \text{Res } f(z) = \frac{e^{im(i)}}{i+i} = \frac{e^{-m}}{2i}$$

So, $\int_C f(z) \cdot dz = 2\pi i \sum R^+ = 2\pi i \times \frac{e^{-m}}{2i} = \pi e^{-m}$.

Now,

$$\Rightarrow \int_{-R}^R f(x) \cdot dx + \int_{C_R} f(z) \cdot dz = \pi e^{-m}$$

At limit ∞ , $-R$ to $+R$; $\int_{C_R} f(z) \cdot dz$ will be ALWAYS zero.

$$\Rightarrow \int_{-R}^{+R} f(x) \cdot dx + 0 = \pi e^{-m}$$

Limit $(-R$ to $+R)$ will extend to $(-\infty$ to $+\infty)$;

$$\Rightarrow \int_{-\infty}^{\infty} f(x) \cdot dx = \pi e^{-m}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{1+x^2} dx = \pi e^{-m}.$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{\cos mx + i \sin mx}{1+x^2} dx = \pi e^{-m}$$

or, $\int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2} + i \int_{-\infty}^{\infty} \frac{\sin mx}{1+x^2} = \pi e^{-m} + i(0)$.

On comparing,

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2} = \pi e^{-m}.$$

or,

$$\Rightarrow \int_0^{\infty} \frac{\cos mx}{1+x^2} = \frac{\pi e^{-m}}{2}.$$

and,

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin mx}{1+x^2} = 0.$$

* Explanation: Why will be C_R always zero for limit of Z to ∞ :

- By Jordan's Lemma Theorem,

if $\lim_{z \rightarrow \infty} f(z) = 0$

then,

$$\lim_{R \rightarrow \infty} \int_C e^{imz} \cdot f(z) \cdot dz = 0$$

- In above question,

$$f(z) = \frac{1}{1+z^2}$$

(e^{imz} will be separated out of $f(z)$...)

$$\Rightarrow \lim_{z \rightarrow \infty} \frac{1}{1+z^2} = 0 = \frac{1}{\infty} = 0.$$

$$\text{So, } \lim_{R \rightarrow \infty} \int_C e^{imz} \cdot f(z) \cdot dz = \lim_{R \rightarrow \infty} \int_C e^{imz} (0) \cdot dz = 0.$$

Q.2)

Evaluate the integral:

$$I = \int_0^\infty \frac{dx}{1+x^2}$$

Ans:

Given,

$$I = \int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{1+x^2}$$

here,

$$f(z) = \frac{1}{1+z^2}$$

Pole will be $= +i$

and,

$$\text{Res } f(z) = \lim_{z \rightarrow i} (z-i) \frac{1}{(z+i)(z-i)} = \frac{1}{2i}$$

so, $\int_C f(z) \cdot dz = 2\pi i \sum R^+ = 2\pi i \times \frac{1}{2i} = \pi.$

Now,

$$\int_{-R}^{+R} f(x) \cdot dx + \int_{C_R} f(z) \cdot dz = \pi.$$

By Jordan's Lemma theorem, at limit ∞ in $-R$ to $+R$,

$$\int_{C_R} f(z) \cdot dz = 0$$

$$\Rightarrow \int_{-R}^{+R} f(x) \cdot dx = \pi. \Rightarrow \int_{-\infty}^{\infty} f(x) \cdot dx = \pi.$$

where, $f(x) = \frac{1}{1+x^2}$.

Therefore,

$$I = \frac{1}{2} \int_{-\infty}^{\infty} f(x) \cdot dx = \frac{1}{2} \times \pi = \boxed{\frac{\pi}{2}}.$$

Q.3) Apply the calculus of residue to evaluate the integral I :

$$I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$$

Ans : Consider the integral, $\int_C f(z) dz$ where, $f(z) = \frac{1}{(1+z^2)^3}$

and C is the curve consisting the upper half C_R of the circle $|z|=R$ and, part of the real axis from $-R$ to $+R$.

Pole of $f(z)$ is,

$$\Rightarrow (1+z^2)^3 = 0$$

$$\Rightarrow [z = +i] \text{ (order 3)}$$

Residue of $f(z)$ of order = 3,

$$= \frac{1}{(n-1)!} \lim_{z \rightarrow a} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n \cdot f(z)] \quad (\text{for order } n).$$

$$= \frac{1}{2!} \times \lim_{z \rightarrow i} \frac{d^2}{dz^2} \left[(z-i)^3 \cdot \frac{1}{(1+z^2)^3} \right]$$

$$= \frac{1}{2} \lim_{z \rightarrow i} \frac{d^2}{dz^2} \left[(z-i)^3 \times \frac{1}{[(z-i)(z+i)]^3} \right]$$

$$= \frac{1}{2} \lim_{z \rightarrow i} \frac{d^2}{dz^2} \left(\frac{1}{(z+i)^3} \right)$$

$$= \frac{1}{2} \lim_{z \rightarrow i} \left[-3(z+i)^{-5} \times -4 \right]$$

$$= \frac{1}{2} \times \lim_{z \rightarrow i} \frac{12}{(z+i)^5}$$

$$= \frac{1}{2} \times \frac{12}{32 \times i^5} = \frac{3}{16i} \quad \left(\because i^5 = i^{4+1} = i \right)$$

Now,

By Residue's theorem;

$$\Rightarrow \int_C f(z) \cdot dz = 2\pi i \sum R^+ = 2\pi i \times \frac{3}{16i} = \boxed{\frac{3\pi}{8}}$$

Hence,

$$= \int_{-R}^R f(x) \cdot dx + \int_{C_R} f(z) \cdot dz = \boxed{\frac{3\pi}{8}}$$

or,

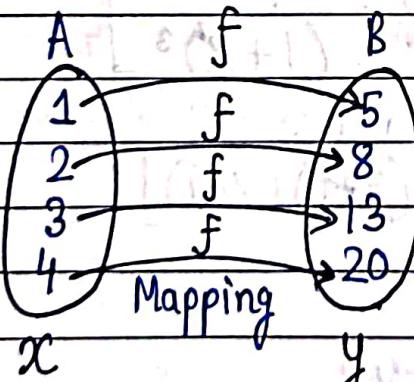
$$I = \int_{-\infty}^{\infty} f(x) \cdot dx = \boxed{\frac{3\pi}{8}}.$$

3) Geometrical Interpretation of Analytic Functions :-

3.1) Conformal Mapping :-

- What is mapping?

→ An operator that is used to map or relate the elements of one set into another set is mapping.



Here,

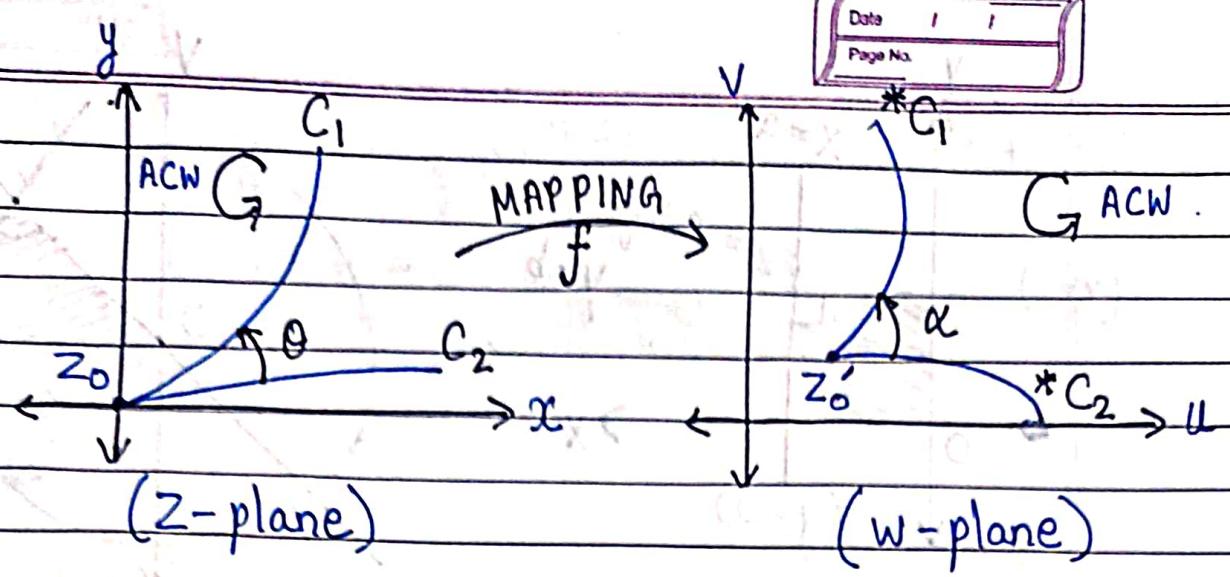
$$y = f(x) = x^2 + 4 \quad \forall x \in N.$$

In this case, element 1 of set 'x' is mapped with element 5 of set 'y' only.

- What is conformal mapping?

→ The mapping of Z-plane into W-plane or W-plane into Z-plane by following relation,

$$\bullet \quad u + iv = w = f(z) = f(x+iy). \quad (w = u + iv) \\ (z = x + iy)$$



- For conformal mapping,

$$\textcircled{1} \quad \theta = \alpha.$$

\textcircled{2} Sense of Rotation must be same.
(here, ACW for both the sketches)

Q.1) Show that the mapping $w = e^z$ is conformal mapping in the whole z -plane?

Ans: We have,

$$\Rightarrow w = e^z.$$

$$\Rightarrow u + iv = \exp(x+iy) = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$$

$$\Rightarrow u + iv = e^x \cos y + i e^x \sin y.$$

$$\text{So, } u = e^x \cos y \text{ and, } v = e^x \sin y.$$

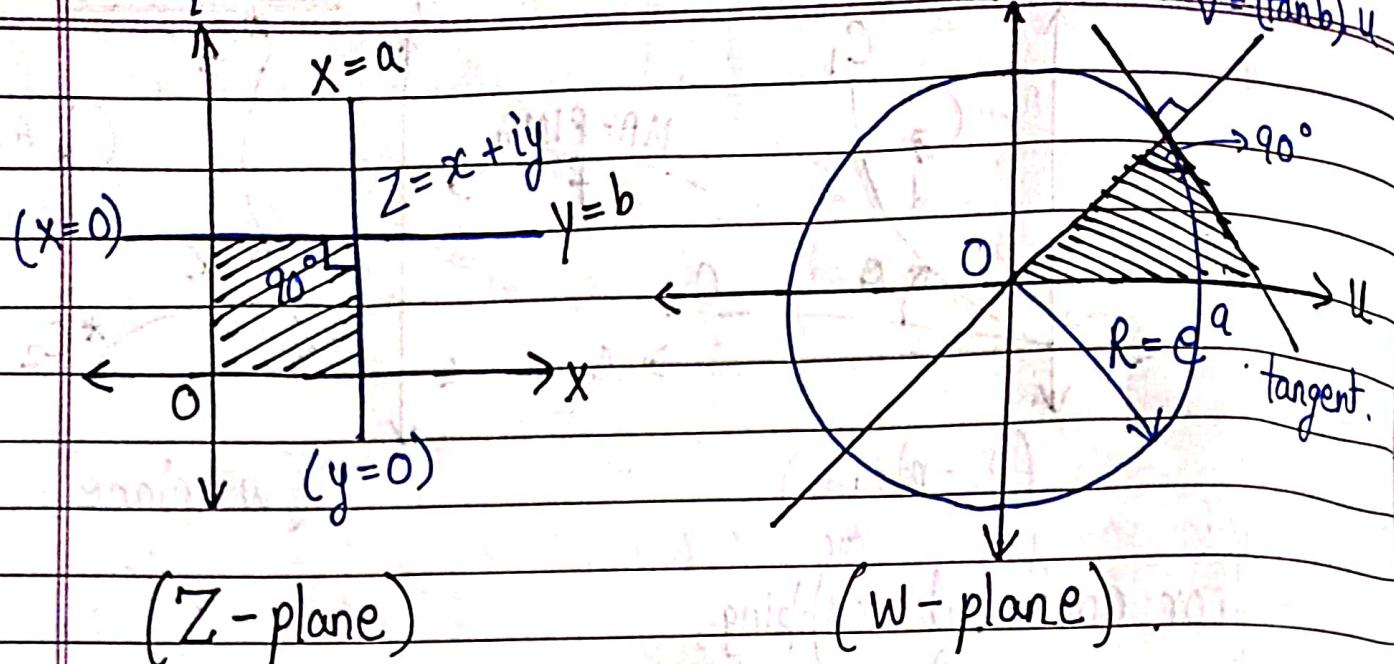
$$\Rightarrow (u)^2 + (v)^2 = e^{2x} (\cos^2 y + \sin^2 y) = e^{2x}.$$

$$\text{or, } u^2 + v^2 = (e^x)^2. \rightarrow \text{A circle.}$$

and,

$$\Rightarrow \frac{u}{v} = \frac{e^x \cos y}{e^x \sin y} = \cot y$$

$$\Rightarrow \frac{v}{u} = \tan y \text{ or, } \Rightarrow v = (\tan y) \cdot u \rightarrow \text{Straight line}$$



- The rectangle in Z-plane is mapped as a triangle in w-plane, where $\theta = 90^\circ$ and sense of rotation is same.
- Both figures having the shaded areas are of equal areas.
- ∴ It is a conformal mapping of $w = e^z$.

Note: Polar coordinates of,

$$w = r e^{i\phi} \quad \text{and,} \quad z = re^{i\theta}.$$

Q.2) Show that the image on the real axis in the Z-plane on the w-plane by the transformation,

$$w = \frac{1}{z+i}$$

is a circle and, find radius 'R' and center of this circle?

Ans: $w = \frac{1}{z+i} \Rightarrow z+i = \frac{1}{w} \Rightarrow z = \frac{1}{w} - i$

$$\Rightarrow z = \frac{1-iw}{w}$$

$$\Rightarrow x+iy = \frac{1-i(u+iv)}{u+iv} \times \frac{u-iv}{u-iv}$$

$$\Rightarrow x+iy = \left(\frac{1+v-uv}{u+iv} \right) \times \frac{u-iv}{u-iv}$$

$$\Rightarrow x+iy = \left(\frac{u}{u^2+v^2} \right) - i \left(\frac{v}{u^2+v^2} - 1 \right)$$

So,

$$x = \frac{u}{u^2+v^2} \quad \text{and}, \quad y = \frac{1-v}{u^2+v^2}$$

For image on the real axis,

$$y=0$$

$$\Rightarrow \frac{1-v}{u^2+v^2} = 0$$

$$\Rightarrow u^2+v^2 = v$$

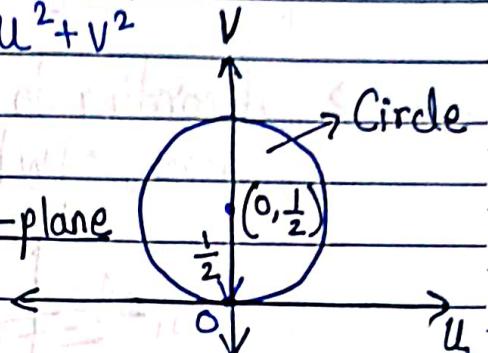
$$\Rightarrow u^2 + v^2 + 0.u + (-1)v + 0 = 0 \rightarrow \text{A circle}$$

∴ General equation of circle,

$$\Rightarrow x^2 + y^2 + 2gx + 2fy + c = 0 \rightarrow C \equiv (-g, -f)$$

i.e.,

$$C \equiv \left(0, \frac{1}{2} \right) \text{ and, } r = \frac{1}{2}$$



Q.3) Under the transformation, $Z = \frac{4}{(w+1)^2}$. Find the region

in the Z -plane corresponding to the interior of circle $|w|=1$?

$$\text{Ans: } \frac{(w+1)^2}{Z} = 4 \Rightarrow w+1 = \frac{2}{\sqrt{Z}} \Rightarrow w = \frac{2}{\sqrt{Z}} - 1.$$

→ On direct substitution, $Z = x+iy$, equation will be hard to solve, so putting polar coordinates,

$$\Rightarrow w = \frac{2}{\sqrt{re^{i\theta}}} - 1 = \frac{2}{\sqrt{r}} \cdot \frac{1}{e^{i\theta/2}} - 1 \quad (\because Z = re^{i\theta})$$

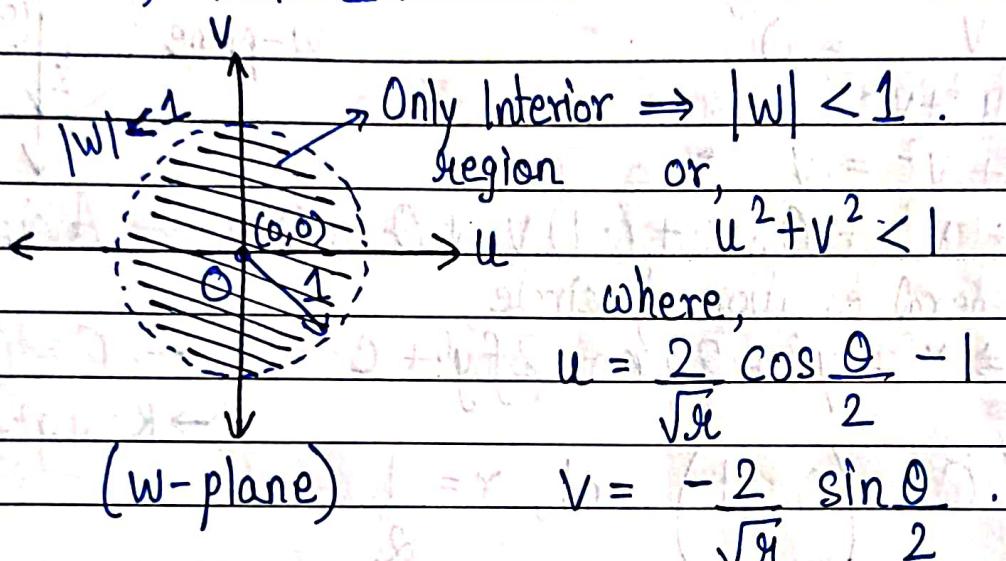
$$\Rightarrow \text{So, } u+iv = \frac{2}{\sqrt{r}} e^{-i\theta/2} - 1 = \frac{2e^{-i\theta/2}}{\sqrt{r}} - \frac{1}{\sqrt{r}}$$

$$\Rightarrow W = \frac{1}{\sqrt{\epsilon}} [2e^{-i\theta/2} - \sqrt{\epsilon}]$$

$$\Rightarrow W = \frac{1}{\sqrt{\epsilon}} [2(\cos \theta - i \sin \theta) - \sqrt{\epsilon}]$$

$$\Rightarrow W = \frac{1}{\sqrt{\epsilon}} \left[\frac{2 \cos \theta}{2} - \sqrt{\epsilon} \right] - i \left[\frac{2 \sin \theta}{2} \right]$$

\Rightarrow According to Question, the region should be interior of circle, $|W| = 1$.



$$\Rightarrow u^2 + v^2 < 1$$

$$\Rightarrow \left(\frac{2 \cos \theta - 1}{\sqrt{\epsilon}} \right)^2 + \left(\frac{-2 \sin \theta}{2} \right)^2 < 1$$

$$\Rightarrow \frac{4 \cos^2 \theta}{\epsilon} + 1 - \frac{4 \cos \theta}{\sqrt{\epsilon}} + \frac{4 \sin^2 \theta}{2} < 1$$

$$\Rightarrow \frac{4}{\epsilon} + 1 - \frac{4 \cos \theta}{\sqrt{\epsilon}} < 1$$

$$\Rightarrow \frac{4}{\epsilon} \left(1 - \frac{\sqrt{\epsilon} \cos \theta}{2} \right) < 0$$

$$\Rightarrow \frac{1}{\epsilon} < \frac{\sqrt{\epsilon} \cos \theta}{2} \Rightarrow \sqrt{\epsilon} > \frac{1}{\cos \theta/2}$$

or,

$$\Rightarrow \frac{1}{\epsilon} > \frac{1}{\cos^2 \theta/2} \Rightarrow \epsilon > \frac{2}{\cos^2 \theta/2}$$

$$\Rightarrow r > 2 \quad (\because 1 + \cos \theta = 2 \cos^2 \theta).$$

$$1 + \cos \theta$$

$$\Rightarrow r(1 + \cos \theta) > 2$$

$$\Rightarrow r + r \cos \theta > 2.$$

$\therefore x = r \cos \theta$ and, $y = r \sin \theta$, i.e.,
on squaring,

$$\Rightarrow x^2 + y^2 = r^2.$$

$$\Rightarrow r = \sqrt{x^2 + y^2}.$$

$$\Rightarrow \sqrt{x^2 + y^2} + x > 2.$$

$$\Rightarrow \sqrt{x^2 + y^2} > 2 - x.$$

$$\Rightarrow x^2 + y^2 > (2 - x)^2.$$

$$\Rightarrow y^2 > (2 - x)^2 - x^2 = 4x - 4x^2 - 4.$$

$$\Rightarrow y^2 > -4(x - 1)$$

\rightarrow A parabola.

(Q.4) Discuss the conformal mapping for $w = z^2$?

Ans : $w = z^2$

$$\Rightarrow u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

So, $u = x^2 - y^2$ and, $v = 2xy$.

Here, $z = (x, y)$; $w = (u, v)$

At $(0, 0)$, $u = 0$ and $v = 0$.

Hence, the origin gets mapped into the origin in w -plane.

At,

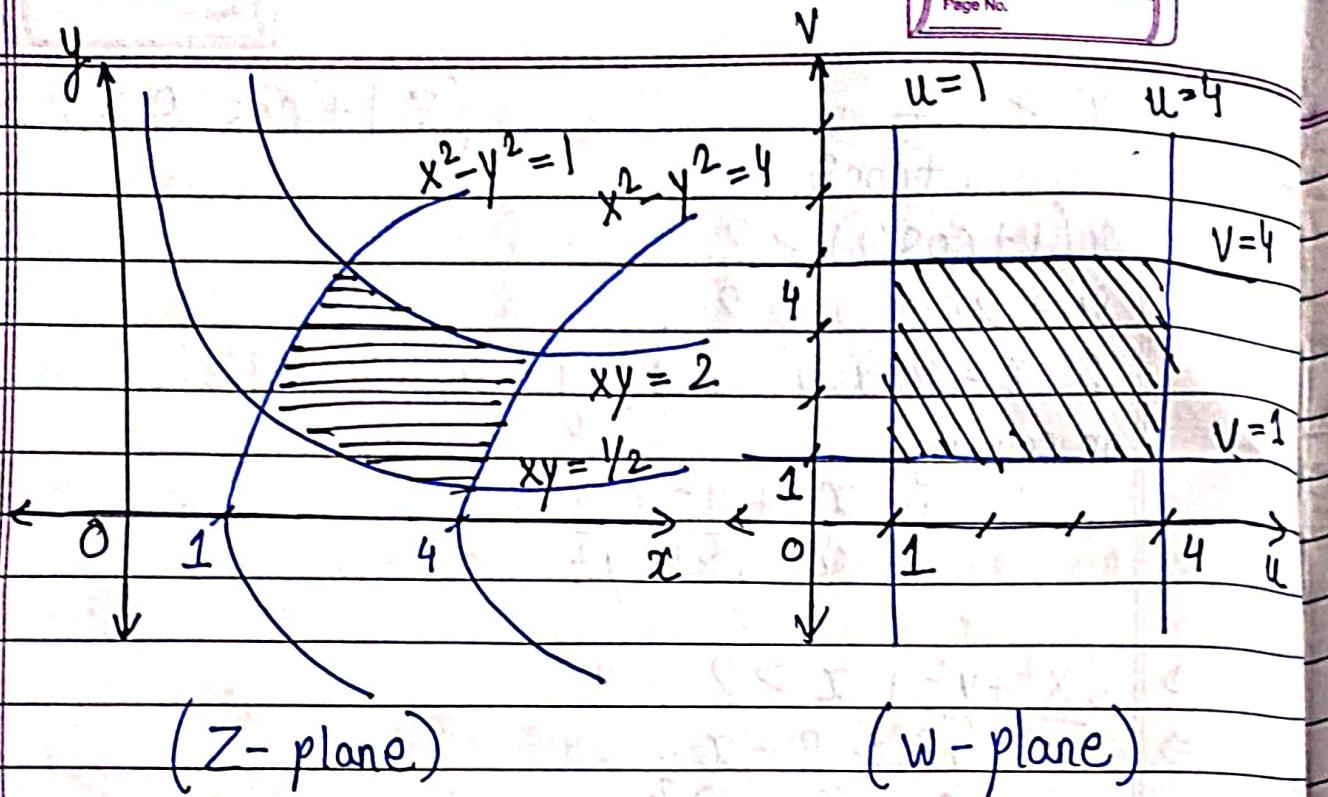
$$u = 1, \quad x^2 - y^2 = 1$$

$$u = 4, \quad x^2 - y^2 = 4$$

$$v = 1, \quad xy = \frac{1}{2}$$

$$v = 4, \quad xy = 2$$

Geometrically,



Q.5) If $u = 2x^2 + y^2$ and, $v = y^2/x$. Show that the curve $u = \text{constant}$ and $v = \text{constant}$ cut orthogonally but the transformation is not conformal?

Ans: If transformation is not conformal \rightarrow Not Analytic \rightarrow Equations of $S = w$ not satisfied.

We have

$$u = 2x^2 + y^2 - \textcircled{1} \text{ and, } v = \frac{y^2}{x} - \textcircled{2}$$

$$\Rightarrow 2x^2 + y^2 = C_1$$

$$\Rightarrow \frac{y^2}{x} = C_2 \Rightarrow y^2 = C_2 x$$

]} C_1 and C_2 are constants.

\therefore Both curves cut orthogonally, so;

$$\Rightarrow 2y \cdot \frac{dy}{dx} = 0 - 4x^3 = -4x^3$$

$$\Rightarrow y \frac{dy}{dx} = -2x \quad \Rightarrow \frac{dy}{dx} = -\frac{2x}{y}$$

and.

$$\Rightarrow 2y \cdot \frac{dy}{dx} = C_2 \Rightarrow y \cdot dy = \frac{C_2}{2} dx \Rightarrow \frac{dy}{dx} = \frac{C_2}{2y}$$

\therefore They cut orthogonally at 90° ,
 \Rightarrow product of the two slopes = -1.
 $\Rightarrow m_1 \times m_2 = -1$.

i.e.,

$$\Rightarrow -2x \times C_2 = -1.$$

$$\Rightarrow xC_2 = \frac{y^2}{y} \quad \text{(from ③)} \Rightarrow \frac{y^2}{x} = C_2 = v \quad (\text{from ②}).$$

\therefore v is constant.

also,

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{y} = \frac{C_2}{2y} \Rightarrow x = -\frac{C_2}{4} = k \quad (k: \text{constant})$$

so putting in eq -①,

$$\Rightarrow 2x^2 + y^2 = C_1 = u.$$

or,

$$\Rightarrow u = 2x^2 + y^2 = 2x^2 + xC_2 \quad (\text{from ③})$$

i.e.,

$$\Rightarrow u = 2\left(\frac{-C_2}{4}\right)^2 + \left(\frac{-C_2}{4}\right)C_2$$

$$\Rightarrow u = 2k^2 + KC_2$$

$$\Rightarrow u = C_1 = \text{constant}$$

\therefore u is constant.

Q.6) Discuss the mapping $w = z^{-1}$, and show that the circle in z -plane map into the circle in the w -plane. Find the center and radius in w -plane?

$$\text{Ans: } \Rightarrow w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \Rightarrow x+iy = \frac{u-iw}{u^2+w^2}$$

$$\Rightarrow x+iy = \frac{u}{u^2+w^2} + i \frac{-w}{u^2+w^2}$$

On comparing,

$$x = \frac{u}{u^2 + v^2} \quad \text{and, } y = \frac{-v}{u^2 + v^2}$$

\therefore General equation of circle is,

$$\Rightarrow x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow \left(\frac{u}{u^2 + v^2}\right)^2 + \left(\frac{-v}{u^2 + v^2}\right)^2 + 2g\left(\frac{u}{u^2 + v^2}\right) + 2f\left(\frac{-v}{u^2 + v^2}\right) + c = 0$$

$$\Rightarrow \frac{u^2 + v^2}{(u^2 + v^2)^2} + 2\left(\frac{gu - fv}{u^2 + v^2}\right) + c = 0$$

$$\Rightarrow 1 + 2gu - 2fv + cu^2 + cv^2 = 0.$$

$$\Rightarrow cu^2 + cv^2 + 2gu - 2fv + 1 = 0.$$

or,

$$\Rightarrow u^2 + v^2 + \frac{2g}{c}u - \frac{2f}{c}v + \frac{1}{c} = 0. \rightarrow \text{A circle in } w\text{-plane}$$

here,

$$\Rightarrow C \equiv \left(-\frac{g}{c}, \frac{f}{c}\right)$$

and,

$$\Rightarrow R (\text{radius}) = \sqrt{\frac{g^2}{c^2} + \frac{f^2}{c^2} - \frac{1}{c^2}} = \frac{1}{c} \sqrt{g^2 + f^2 - 1}$$

Q.7)

Show that the transformation $w = \cosh z$, maps the lines parallel to X -axis in the Z -plane into the hyperbolae in w -plane? Further the lines parallel to Y -axis in the Z -plane maps as ellipse into w -plane?

$$\text{Ans: } 2w = e^z + e^{-z} \quad \left(\because \cosh z = \frac{e^z + e^{-z}}{2} \right)$$

$$\Rightarrow 2w = e^{x+iy} + e^{-(x+iy)}$$

$$\Rightarrow 2w = e^x (\cos y + i \sin y) + e^{-x} (\cos y - i \sin y).$$

$$\Rightarrow 2w = (e^x \cos y + e^{-x} \cos y) + i(e^x \sin y - e^{-x} \sin y).$$

$$\Rightarrow u + iv = \left(\frac{e^x + e^{-x}}{2}\right) \cos y + i \left(\frac{e^x - e^{-x}}{2}\right) \sin y.$$

So,

$$u = \cos y \cdot \cosh x \quad \text{and}, \quad v = \sin y \cdot \sinh x.$$

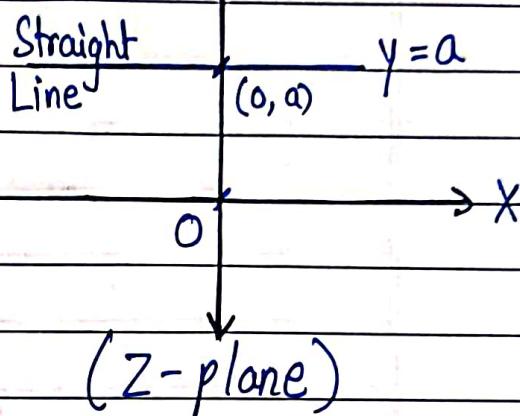
a) Lines parallel to X-axis \rightarrow

$$\Rightarrow y = \text{Constant} = a$$

So,

$$u = \cos a \cdot \cosh x$$

$$v = \sin a \cdot \sinh x$$



$$\therefore \cosh^2 x - \sinh^2 x = 1$$

$$\Rightarrow \left(\frac{u}{\cos a}\right)^2 - \left(\frac{v}{\sin a}\right)^2 = 1.$$

$$\Rightarrow \left|\frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a}\right| = 1.$$

\rightarrow Hyperbola in w-plane.

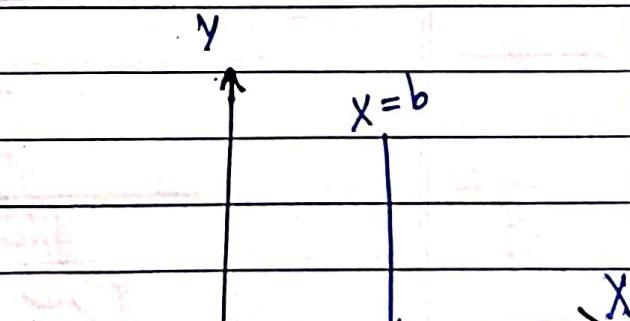
b) Lines parallel to Y-axis \rightarrow

$$\Rightarrow x = \text{constant} = b$$

So,

$$u = \cosh a \cdot \cos y$$

$$v = \sinh a \cdot \sin y$$



$$\therefore \sin^2 y + \cos^2 y = 1$$

$$\Rightarrow \left(\frac{v}{\sinh a}\right)^2 + \left(\frac{u}{\cosh a}\right)^2 = 1.$$

$$\Rightarrow \left|\frac{u^2}{\cosh^2 a} + \frac{v^2}{\sinh^2 a}\right| = 1$$

\rightarrow Ellipse in w-plane.

3.2) Linear And Inverse Transformation :-

Here,

$w = az + b \rightarrow$ Linear transformation or Linear function where a, b are real and complex constants.

$$w = f(z) = az + b$$

- ① Translation \rightarrow Shape, size and orientation same
- ② Magnification \rightarrow Image can be magnified, same or diminished
- ③ Rotation \rightarrow The angle of rotation will be determined...

$$\Rightarrow w = az + b = a \left(z + \frac{b}{a} \right) = aw_1$$

$$\Rightarrow w = |a|e^{i\theta} \left(z + \frac{b}{a} \right)$$

$$\textcircled{1} \text{ Translation: } w_1 = z + \frac{b}{a} = z + C \quad (C = \frac{b}{a})$$

$$\textcircled{2} \text{ Magnification: } w_2 = aw_1 = a \left(z + \frac{b}{a} \right)$$

$$\textcircled{3} \text{ Rotational: } w = e^{i\theta} w_2 = e^{i\theta} \cdot aw_1 = ae^{i\theta} \left(z + \frac{b}{a} \right)$$

- Q.1) Let a rectangular region OABC with vertices O(0,0), A(1,0), B(1,2) and C(0,2) is defined in Z -plane. Find the image of this figure in w -plane under the mapping,

$$w = z + (2+i)$$

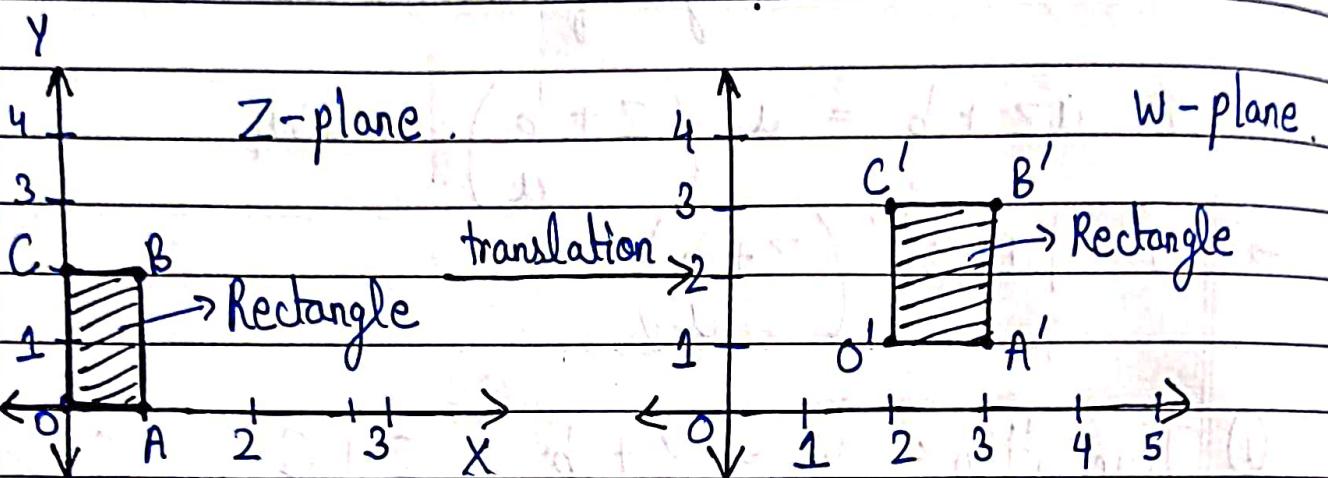
Ans: $w = z + (2+i)$ (here, $(2+i)$ is a complex constant)
 $\Rightarrow u+iv = (x+2) + i(y+1)$.

or,

$$u = x+2 \text{ and, } v = y+1$$

$$u = x + 2, \quad v = y + 1$$

(x, y)	(u, v)
$O(0,0)$	$O'(2,1)$
$A(1,0)$	$A'(3,1)$
$B(1,2)$	$B'(3,3)$
$C(0,2)$	$C'(2,3)$
$(Z\text{-plane})$	$(W\text{-plane})$



Q.2)

Let a rectangular region ABCD with vertices $A(2,1)$, $B(3,1)$, $C(3,3)$, $D(2,3)$ be defined in Z -plane.
Find the image in w -plane if $w = 2z$?

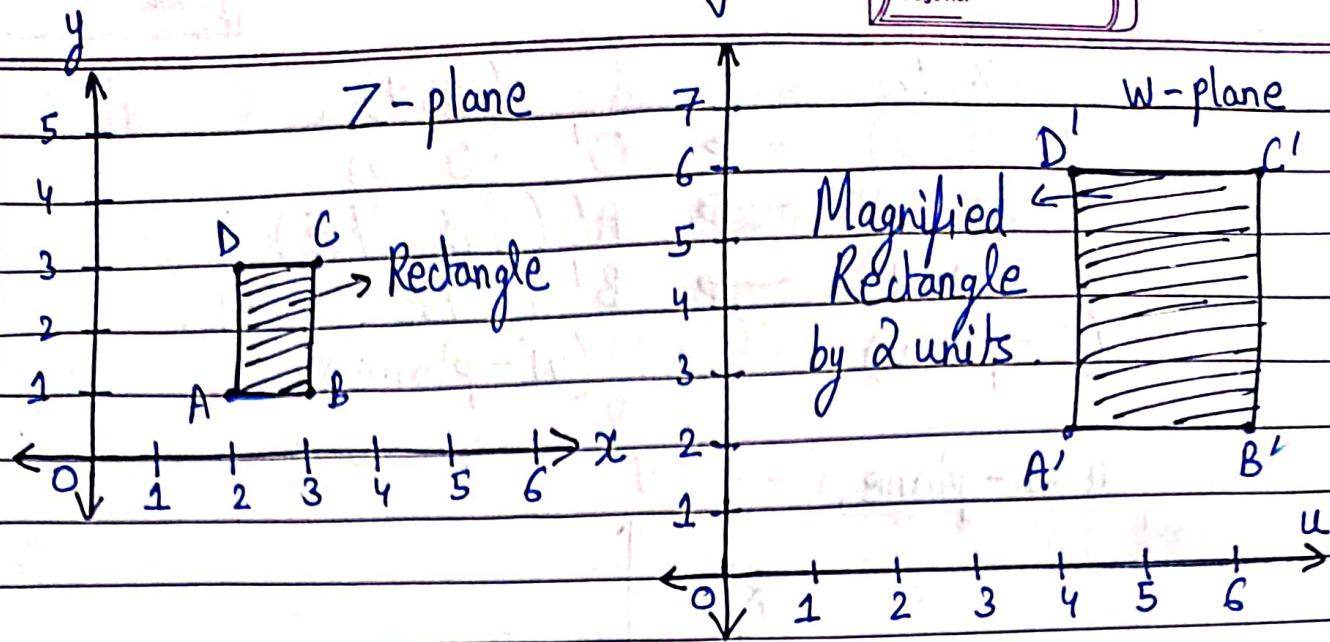
$$\text{Ans: } u + iv = 2(x + iy) = 2x + i(2y).$$

or,

$$(x, y) \rightarrow u = 2x \text{ and } v = 2y$$

(x, y)	(u, v)
$A(2,1)$	$A'(4,2)$
$B(3,1)$	$B'(6,2)$
$C(3,3)$	$C'(6,6)$
$D(2,3)$	$D'(4,6)$
$(Z\text{-plane})$	$(W\text{-plane})$

- The image will be magnified by relation $w = 2z$.



Q.3) Consider the transformation, $w = z e^{\frac{i\pi}{4}}$. Determine the region R' in w-plane corresponding to triangular region R bounded by the lines $x=0$, $y=0$ and $x+y=1$.

$$\text{Ans: } w = e^{\frac{i\pi}{4}} \cdot z = \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \cdot z$$

$$\Rightarrow u+iv = (1+i)\sqrt{2}(x+iy)$$

$$\Rightarrow u+iv = \frac{(x-y)}{\sqrt{2}} + i \frac{(x+y)}{\sqrt{2}}$$

So,

$$u = \frac{x-y}{\sqrt{2}} \quad \text{and} \quad v = \frac{x+y}{\sqrt{2}}$$

$$\sqrt{u^2 + v^2} = \sqrt{\frac{(x-y)^2}{2} + \frac{(x+y)^2}{2}} = \sqrt{x^2 + y^2}$$

→ Region 'R' is bounded by:

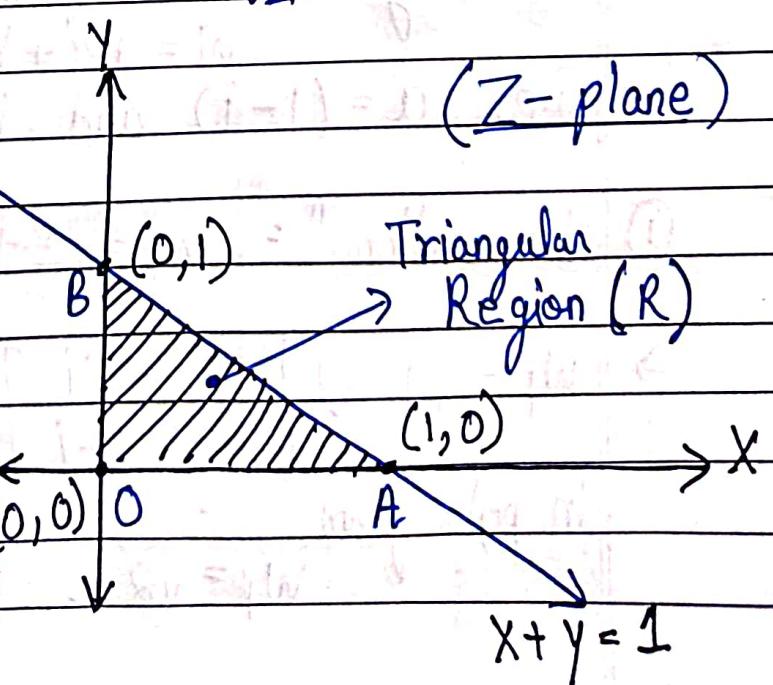
$$\textcircled{1} \quad x=0 \rightarrow y\text{-axis}$$

$$\textcircled{2} \quad y=0 \rightarrow x\text{-axis}$$

$$\textcircled{3} \quad x+y=1 \quad \text{that passes}$$

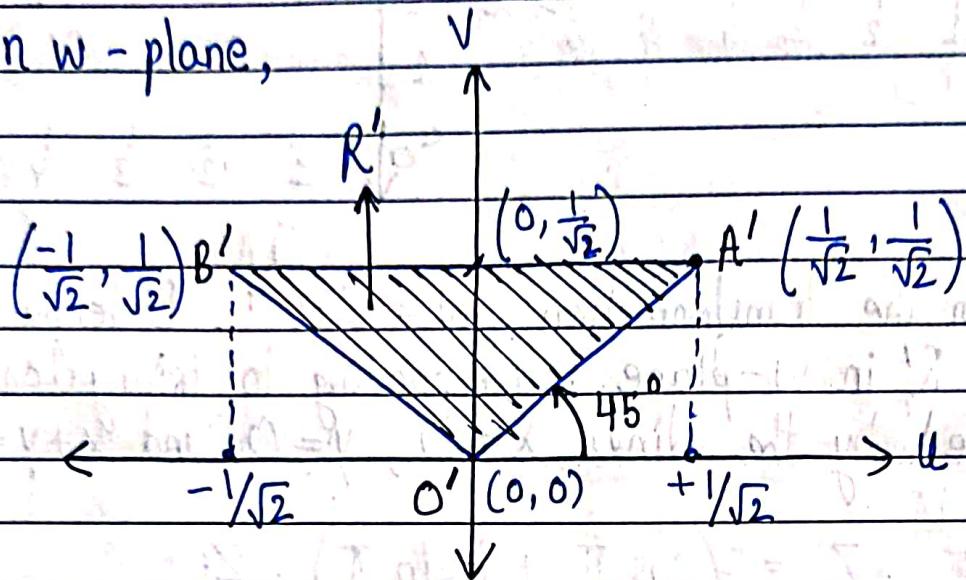
through $(0,1)$ & $(1,0)$, $(0,0)$

a straight line.



(x, y)	(u, v)
$O(0, 0)$	$O'(0, 0)$
$A(1, 0)$	$A'(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
$B(0, 1)$	$B'(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
$(z\text{-plane})$	$(w\text{-plane})$

In w -plane,



- Q.4) Let a rectangular region with vertices $OABC$ where,
 $O(0, 0)$, $A(1, 0)$, $B(1, 2)$ and $C(0, 2)$ in z -plane.
Find the image of this region in w -plane under the mapping,

$$w = (1-i)z - 2i$$

Ans: Converting ' w ' in the form of ,

$$w = az + b$$

i.e., $a = (1-i)$ and $b = -2i$.

① Translational : $w_1 = z + b$

$$\Rightarrow w_1 = (1-i) \left[z - \frac{2i}{1-i} \right]$$

In polar form,

$$w_1 = \sqrt{2} e^{\frac{i\pi}{4}} \left(z - \frac{2i}{1-i} \right)$$

This ' w ' = $\sqrt{2} e^{i\pi/4} \left(z - 2i \over 1-i \right)$ has linear transformation.

i.e., $w_1 = z - 2i \over 1-i$ (for translation).

$$\Rightarrow u_1 + iv_1 = (x+iy) - \frac{2i(1+i)}{2}$$

$$\Rightarrow u_1 + iv_1 = x+iy - i+1 = (x+1) + i(y-1).$$

or,
 $u_1 = (x+1)$ and, $v_1 = (y-1)$

$$(x, y) \rightarrow (u, v)$$

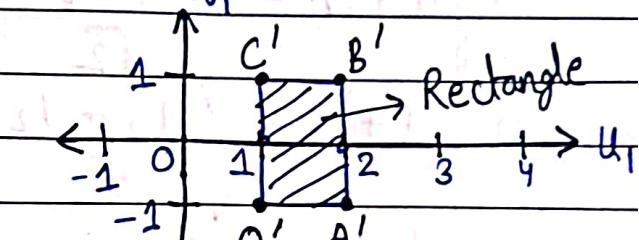
$$O(0, 0) \rightarrow O'(1, -1)$$

$$A(1, 0) \rightarrow A'(2, -1)$$

$$B(1, 2) \rightarrow B'(2, 1)$$

$$C(0, 2) \rightarrow C'(1, 1)$$

$$(z\text{-plane}) \rightarrow (w\text{-plane})$$



$$(w\text{-plane})$$

② Magnification: $w_2 = a w_1$,

$$\Rightarrow w_2 = \sqrt{2}(u_1 + iv_1)$$

$$\Rightarrow u_2 + iv_2 = \sqrt{2}u_1 + i\sqrt{2}v_1$$

or,

$$u_2 = \sqrt{2}u_1 = \sqrt{2}(x+1) \quad \text{and, } v_2 = \sqrt{2}v_1 = \sqrt{2}(y-1)$$

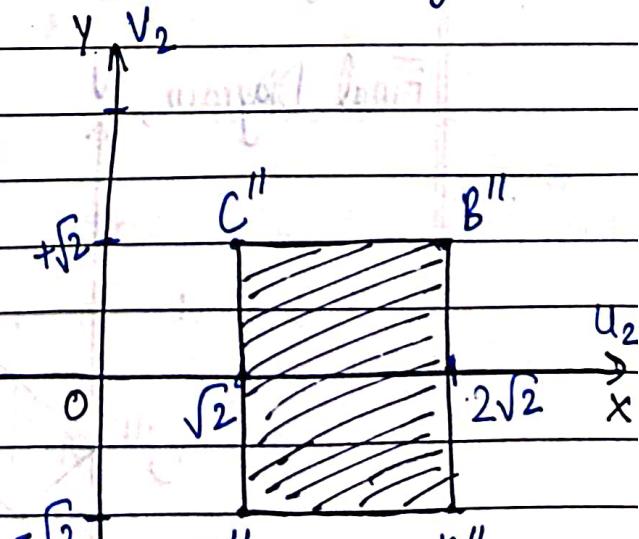
$$(x, y) \rightarrow (u, v)$$

$$O(0, 0) \rightarrow O''(\sqrt{2}, -\sqrt{2})$$

$$A(1, 0) \rightarrow A''(2\sqrt{2}, -\sqrt{2})$$

$$B(1, 2) \rightarrow B''(2\sqrt{2}, \sqrt{2})$$

$$C(0, 2) \rightarrow C''(\sqrt{2}, \sqrt{2})$$



The image is a magnified rectangle by ' $\sqrt{2}$ ' factor.

i.e.,

$$w_2 = \sqrt{2} \cdot w_1$$

$$(w\text{-plane})$$

③ Rotational:

$$\Rightarrow W = \sqrt{2} e^{-\frac{i\pi}{4}} [Z + (1-i)]$$

or,

$$\Rightarrow W = e^{-\frac{i\pi}{4}} W_2.$$

i.e.,

$$\Rightarrow u + iv = e^{-\frac{i\pi}{4}} (u_2 + iv_2).$$

$$\Rightarrow u + iv = \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) (u_2 + iv_2)$$

$$\Rightarrow u + iv = \frac{(1-i)}{\sqrt{2}} (u_2 + iv_2)$$

$$\Rightarrow u + iv = \left(\frac{u_2 + v_2}{\sqrt{2}} \right) + i \left(\frac{v_2 - u_2}{\sqrt{2}} \right)$$

Here,

$$u = u_2 + v_2 \quad (\text{and}, \quad v = \frac{v_2 - u_2}{\sqrt{2}})$$

w-plane (u_2, v_2)

w-plane (u, v)

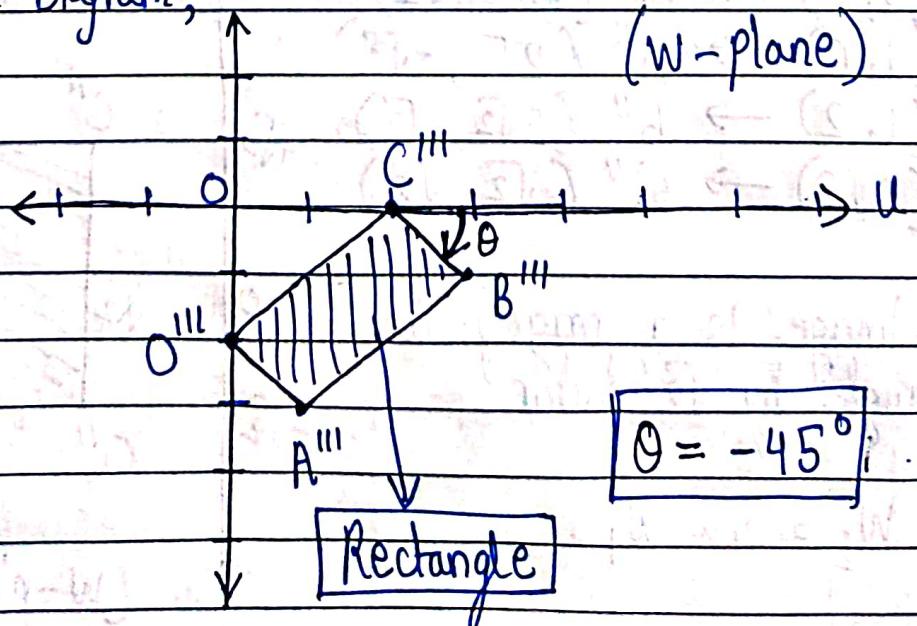
$$O'' (\sqrt{2}, -\sqrt{2}) \rightarrow O''' (0, -2)$$

$$A'' (2\sqrt{2}, -\sqrt{2}) \rightarrow A''' (1, -3)$$

$$B'' (2\sqrt{2}, \sqrt{2}) \rightarrow B''' (3, -1)$$

$$C'' (\sqrt{2}, \sqrt{2}) \rightarrow C''' (2, 0).$$

Final Diagram, ∇



3.3)

Inversion and Reflection :-

$$W = \frac{1}{Z} \quad (\text{general form})$$

In polar form,

$$\vec{Z} = r e^{i\theta}$$

$$W = R e^{i\phi}$$

$$\Rightarrow R e^{i\phi} = \frac{1}{r e^{i\theta}}$$

$$\text{if } \theta = -\phi,$$

$$\Rightarrow R \exp(i\phi) = \frac{1}{r} \exp(i\phi)$$

$$\therefore R = \frac{1}{r}$$

Q.1) Find the image of circle in Z -plane on the w -plane by the transformation $W = 1/Z$?

Ans : In Z -plane,

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

whose,

$$C = (-g, -f) \quad \text{and} \quad R = \sqrt{g^2 + f^2 - c}$$

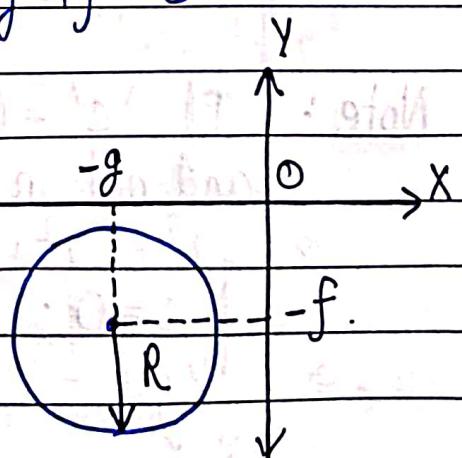
$$\Rightarrow Z = \frac{1}{w} \quad (\text{given})$$

$$\Rightarrow x + iy = \frac{u - iv}{u^2 + v^2}, \quad (w = u + iv)$$

$$\Rightarrow x + iy = \frac{u}{u^2 + v^2} + i \left(\frac{-v}{u^2 + v^2} \right)$$

So, $x = \frac{u}{u^2 + v^2}$ and, $y = \frac{-v}{u^2 + v^2}$. (Z-plane)

$$x = \frac{u}{u^2 + v^2} \quad \text{and,} \quad y = \frac{-v}{u^2 + v^2}$$



$$\Rightarrow x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 + 2g\left(\frac{+u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0$$

$$\Rightarrow \frac{1}{u^2+v^2} + 2gu - 2fv + c = 0.$$

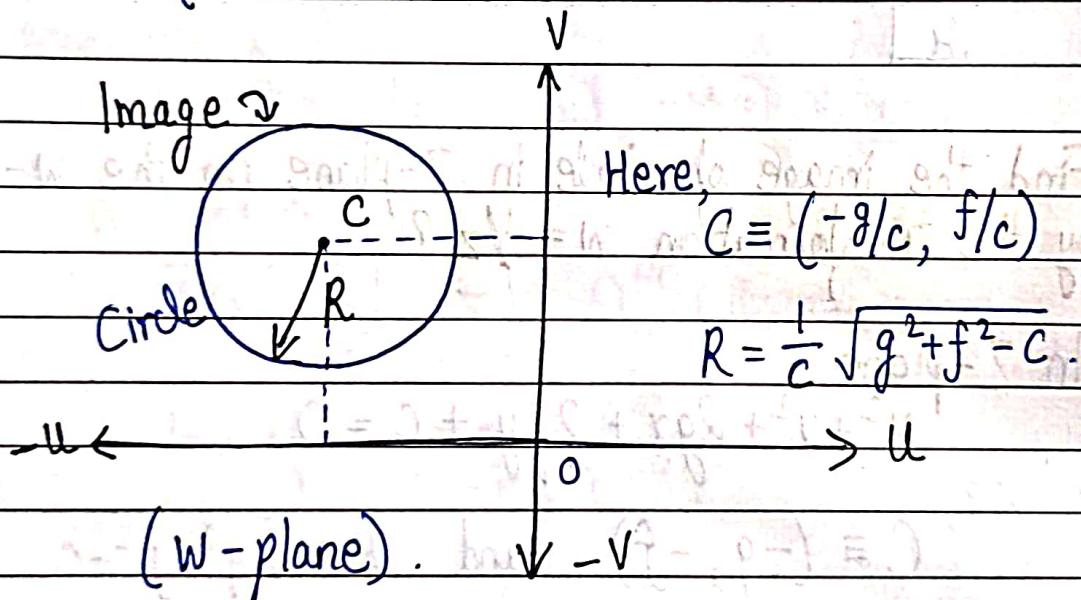
$$\Rightarrow cu^2 + cv^2 + 2gu - 2fv + 1 = 0. \quad \text{--- (1)}$$

or,

$$\Rightarrow u^2 + v^2 + \frac{2gu}{c} - \frac{2fv}{c} + \frac{1}{c} = 0 \quad \text{--- (2)}.$$

It is a equation of circle,

$$C = \left(-\frac{g}{c}, \frac{f}{c}\right); R = \frac{1}{c} \sqrt{g^2 + f^2 - c}$$



Note: If 'c' = 0 in above question then put $c=0$ in eq -① and not in eq -②. So,

$$\Rightarrow cu^2 + cv^2 + 2gu + 1 - 2fv = 0$$

$$\text{If } c=0;$$

$$\Rightarrow 2gu - 2fv + 1 = 0.$$

or,

$$\Rightarrow u = \frac{f}{g}v - \frac{1}{2g} \rightarrow (\text{A straight line of form: } y = mx + c)$$

$$m = f/g.$$

Q.2)

Find the image in the w -plane of the disc in z -plane,

$$|z - |z-1|| \leq 1 \rightarrow |z-1| \leq 1$$

under the mapping,

$$w = \frac{1}{z}$$

Ans :-

$$\text{Given, } w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$\Rightarrow |z - |z-1|| \leq 1$$

$$\Rightarrow \left| \frac{1}{w} - \left| \frac{1}{w} - 1 \right| \right| \leq 1$$

$$\Rightarrow \left| \frac{1}{w} - 1 \right| - \left| \frac{1}{w} \right| \geq |1|$$

$$\Rightarrow \frac{|1-w|}{|w|} \geq \left| 1 + \frac{1}{w} \right|$$

$$\Rightarrow \frac{|1-w|}{|w|} \geq \left| w + 1 \right|$$

$$\Rightarrow \left| \frac{1}{w} - 1 \right| \leq 1. \quad (\text{for } |z-1| \leq 1)$$

$$\Rightarrow |1-w| \leq |w|$$

$$\Rightarrow |1-(u+iv)| \leq |u+iv|$$

or,

$$\Rightarrow \sqrt{(1-u)^2 + v^2} \leq \sqrt{u^2 + v^2} \quad (\text{w-plane})$$

$$\Rightarrow (1-u)^2 + v^2 \leq u^2 + v^2$$

$$\Rightarrow 1 \leq 2u$$

$$\Rightarrow u \geq \frac{1}{2}$$

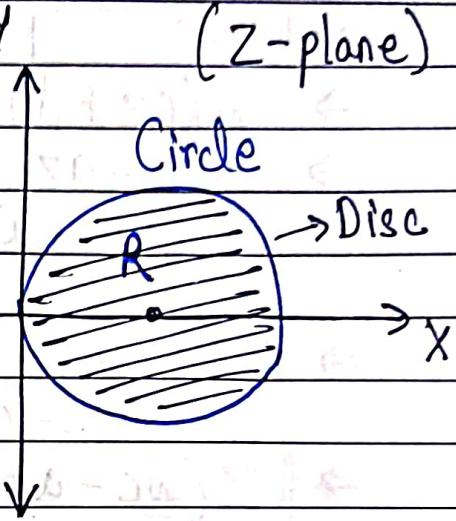
$$u = \frac{1}{2}$$

$$R'$$

$$u$$

$$\frac{1}{2}$$

The line $u = \frac{1}{2}$ will also be counted in for the region 'R' as, $u \geq \frac{1}{2}$.



3.4)

Bilinear Transformation :-

$$\Rightarrow W = \frac{az + b}{cz + d}, \text{ where } ad - bc \neq 0.$$

- also called as functional transformation or Möbius transformation.
- $ad - bc \neq 0$ is called determinant of this transformation.

$$\Rightarrow W(cz + d) = az + b.$$

$$\Rightarrow WCz - az = b - wd.$$

$$\Rightarrow Z(wc - a) = b - wd.$$

or

$$\Rightarrow Z = \frac{b - wd}{wc - a}$$

$$\Rightarrow Z(wc - a) + wd - b = 0. \quad \text{--- (1)}$$

$$\Rightarrow W(cz + d) - az - b = 0 \quad \text{--- (2)}$$

here,

equations (1) and (2) both are linear in Z and w , hence both are bilinear.

also,

$$W = \frac{a(z + b)}{c(z + d)} \quad \text{and,} \quad Z = \frac{d(b - w)}{c(w - a)}$$

- The point $Z = -d/c$ where $c \neq 0 \rightarrow w = \infty$, it is a critical point.

- The point $w = a/c$ where $c \neq 0 \rightarrow z = \infty$, it is a critical point.

Q.1) Find the fixed point of the transformation where,

$$w = \frac{2z - 5}{z + 4}$$

$$\Rightarrow w(z + 4) = 2z - 5$$

$$\Rightarrow wz - 2z = -5 - 4w$$

Ans : $z = f(w)$

$$\Rightarrow z = \frac{5 + 4w}{2-w} \quad (\text{replace } w \text{ by } z)$$

or,

$$\text{or, } \Rightarrow z = \frac{2z - 5}{z + 4} \quad \Rightarrow z = \frac{-5 + 4w}{2-w}$$

$$\Rightarrow z^2 + 4 = 2z - 5$$

$$\Rightarrow z^2 - 2z + 9 = 0 \quad \Rightarrow z = -1 \pm 2i$$

→ The points which coincide with their transformation are called fixed points.

or,

$$\Rightarrow z = \frac{az + b}{cz + d}$$

$$\Rightarrow cz^2 - (a-d)z - b = 0$$

$$\Rightarrow z = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c} \rightarrow 2 \text{ fixed points.}$$

- For one fixed point, put $D = 0$;
 $(a-d)^2 + 4bc = 0$.

3.5) Cross Ratio :-

→ Let z_1, z_2, z_3 and z_4 are distinct points then the cross ratio will be,

$$= (z_1 - z_2)(z_3 - z_4) / (z_2 - z_3)(z_4 - z_1)$$

$$\begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \\ \textcircled{1} (z_1 - z_2) & \textcircled{2} (z_2 - z_3) & \textcircled{3} (z_3 - z_4) & \textcircled{4} (z_4 - z_1) \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \text{numerator} & & & \text{denominator} \end{array}$$

In terms of w ,

$$= \frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)}.$$

Q.1) Find the bilinear transformation which maps the point:

Z	w	
z_1	0	$-i$
z_2	-1	0
z_3	i	∞

Ans:

$$\begin{matrix} w = az + b \\ cz + d \end{matrix}$$

$$\Rightarrow \frac{z - z_1}{z_1 - z_2} \cdot \frac{z_2 - z_3}{z_3 - z} = \frac{w - w_1}{w_1 - w_2} \cdot \frac{w_2 - w_3}{w_3 - w}$$

$$\Rightarrow \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} = \frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)}$$

$$\Rightarrow \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} = \frac{(w - w_1)w_3(w_2/w_3 - 1)}{(w_1 - w_2)w_3(1 - w/w_3)}$$

$$\Rightarrow \frac{(z - 0)(-1 - i)}{(0 + 1)(i - z)} = \frac{(w + i)(0 - 1)}{(-i - 0)(1 - 0)}$$

$$\Rightarrow \frac{z(-1 - i)}{i - z} = \frac{(w + i)}{i}$$

Q.2) Find the bilinear transformation which maps the point:

(a)

	Z_1	Z_2	Z_3	W
	1	i	-1	w_1
	$-i$	0	$-i$	w_2
	-1	$-i$	-1	w_3

Ans:

For,

$$Z : Z_1, Z_2, Z_3 = W : w_1, w_2, w_3$$

$$\Rightarrow \frac{(Z-Z_1)(Z_2-Z_3)}{(Z_1-Z_2)(Z_3-Z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$\Rightarrow \frac{(Z-1)(-i+1)}{(1+i)(-1-z)} = \frac{(w-i)(0+i)}{(i-0)(-i-w)}$$

$$\Rightarrow \frac{(Z-1)(1-i)}{(1+i)(-1-z)} = \frac{(w-i)i}{i(-w-i)}$$

$$\Rightarrow \frac{(Z-1)}{-(z+1)} \times \frac{(1-i)^2}{2} = \frac{w-i}{-(w+i)} \times \frac{(w-i)}{(w-i)}$$

or,

$$\Rightarrow \frac{i-w}{i+w} = \frac{(Z-1)(i-1)}{(z+1)(i+1)} = \frac{(i-1)z + (1-i)}{(i+1)z + 1 + i}$$

$$\Rightarrow \frac{i-w}{i+w} = \frac{(i-1)z + (1-i)}{(i+1)z + (1+i)}$$

$$\Rightarrow \frac{(i-w) - (i+w)}{(i-w) + (i+w)} = \frac{[(i-1)z + (1-i)] - [(i+1)z + (1+i)]}{[(i-1)z + (1-i)] + [(i+1)z + (1+i)]}$$

$$\Rightarrow \boxed{W = \frac{iz-1}{iz+1}}.$$

(b) Also show that the transformation maps the region outside the region circle $|z|=1$, into the half plane, $\operatorname{Re}(w) \geq 0$.

$$\text{Ans: } \because w = \frac{i(z-1)}{iz+1} \quad \textcircled{1}$$

$$\Rightarrow u + iv = \frac{i(x+iy-1)}{i(x+iy)+1} \rightarrow \text{now solve and separate } u \text{ & } v.$$

or,

$$\Rightarrow |w| = \left| \frac{i(z-1)}{iz+1} \right| \quad (\text{taking modulus on both sides of eq-1})$$

or,

$$\Rightarrow |z| = \left| \frac{i(w+1)}{w-1} \right| = 1. \quad (\because |z|=1)$$

$$\Rightarrow \left| \frac{w+1}{w-1} \right| = 1.$$

$$\Rightarrow |w+1| = |w-1|$$

$$\Rightarrow \sqrt{(u+1)^2 + v^2} = \sqrt{(u-1)^2 + v^2}$$

$$\Rightarrow u^2 + 1 + 2u = u^2 + 1 - 2u$$

$$\Rightarrow 4u \geq 0. \quad (\because |z| \geq 1).$$

$$\Rightarrow [u \geq 0].$$

