CHAPTER 4

LAPLACE TRANSFORM

4.1. INTRODUCTION

The Laplace transform is one of the mathematical tools used for the solution of linear ordinary integro-differential equations. (Mostly continuous-time system are described by integro-differential equations). In comparison (as shown in figure 4.1.) with the classical method of solving linear integro-differential equations the Laplace transform method has the following two attractive features :

(i) The homogeneous equation and the particular integral of the solution are obtained in one operation.

(ii) The Laplace transform converts the integro-differential equation into an algebraic equation in s (Laplace operator). It is then possible to manipulate the algebraic equation by simple algebraic rules to obtain the expression in suitable forms. The final solution is obtained by taking the inverse Laplace transform.

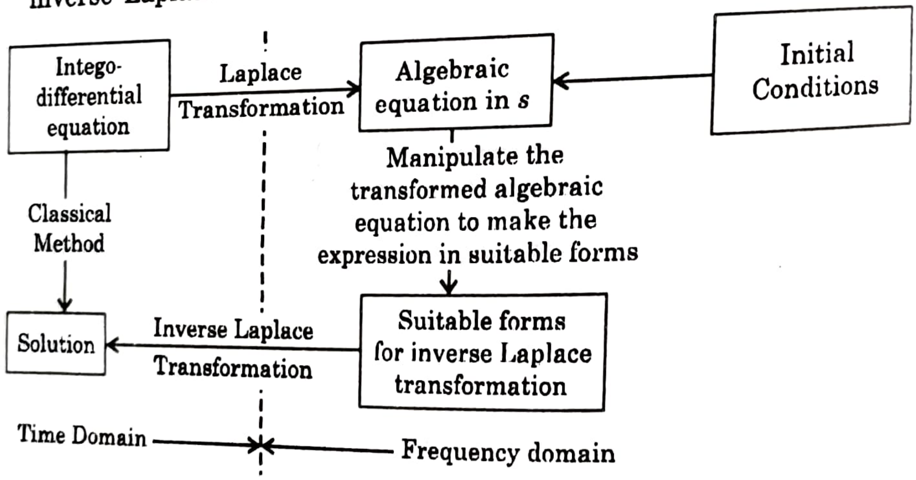


Fig. 4.1. Comparison of Classical method and Laplace transform method

From the 2nd feature of Laplace transform over classical method, Laplace transformation is somewhat similar to logarithmic operation. To find the product or quotient of two number, we find

(i) logarithm of two numbers

(ii) add or subtract

(iii) take antilogarithm to get product or quotient

4.2. DEFINITION OF THE LAPLACE TRANSFORM

The Laplace transform method is a powerful technique for solving circuit problems. We define a Laplace transform as follows :

For the time function $f(t)$ which is zero for $t < 0$ and that satisfy the condition

$$\int_0^{\infty} |f(t)e^{-\sigma t}|\, dt < \infty$$

for some real and positive σ, the Laplace transform of $f(t)$ is defined as

$$\boxed{F(s) = £[f(t)] = \int_0^{\infty} f(t)e^{-st}\, dt}$$

The variable s is referred to as the *Laplace operator*, which is complex variable, *i.e.*, $s = \sigma + j\omega$. And the functions $f(t)$ and $F(s)$ are known as *Laplace transform pair*.

4.3. INVERSE LAPLACE TRANSFORMATION

Given the Laplace transform $F(s)$, the operation of obtaining $f(t)$ is termed the *inverse Laplace transformation* and is denoted by

$$f(t) = £^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\, ds$$

Though one could evaluate the inverse transform of a function F(s) by using above equation, normally the transform table is used to obtain the inverse transformation.

4.4. PROPERTIES OF LAPLACE TRANSFORM

1. Multiplication by a constant

Let k be a constant and $F(s)$ be the Laplace transform of $f(t)$. Then

$$£[kf(t)] = kF(s)$$

2. Sum and Difference

Let $F_1(s)$ and $F_2(s)$ be the Laplace transform of $f_1(t)$ and $f_2(t)$, respectively. Then

$$£[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

3. Differentiation with respect to "t" (Time-Differentiation)

Let $F(s)$ be the Laplace transform of $f(t)$ and let $f(0^+)$* be the value of $f(t)$ as t approaches 0. Then,

$$£\left[\frac{df(t)}{dt}\right] = sF(s) - \lim_{t \to 0} f(t) = sF(s) - f(0^+)$$

Proof :

$$F(s) = £[f(t)] = \int_0^{\infty} f(t) \cdot e^{-st}\, dt$$

Let $\quad f(t) = u \quad$; then $\left[\frac{df(t)}{dt}\right] \cdot dt = du$

and $\quad e^{-st}\, dt = dv \quad$; then $v = -\frac{1}{s}e^{-st}$

* In some books $f(0^-)$ is written instead of $f(0^+)$.

On integrating

$$F(s) = \int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= f(t)\cdot\left(-\frac{1}{s}e^{-st}\right)\Bigg|_0^\infty - \int_0^\infty\left(-\frac{1}{s}e^{-st}\right)\left[\frac{df(t)}{dt}\right]dt$$

($\because$ putting the values)

$$= \frac{1}{s}\cdot f(0^+) + \frac{1}{s}\int_0^\infty e^{-st}\left[\frac{df(t)}{dt}\right]dt = \frac{1}{s}\cdot f(0^+) + \frac{1}{s}£\left[\frac{df(t)}{dt}\right]$$

Therefore,

$$£\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$$

Thus the Laplace transform of the second derivative of $f(t)$ as.

$$£\left[\frac{d^2f(t)}{dt^2}\right] = £\left[\frac{d}{dt}\left\{\frac{df(t)}{dt}\right\}\right] = s\,£\left[\frac{df(t)}{dt}\right] - \frac{df(t)}{dt}\bigg|_{t=0}$$

$$= s[sF(s) - f(0^+)] - f^1(0^+) = s^2F(s) - sf(0^+) - f^1(0^+)$$

(where $f^1(0^+)$ is the value of the first derivative of $f(t)$ as t approaches 0)

In general,

$$£\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^+) - s^{n-2}f^1(0^+) - \ldots\ldots - sf^{n-2}(0^+) - f^{n-1}(0^+)$$

4. Integration by "t" (Time-Integration)

If $£[f(t)] = F(s)$

then, the Laplace transform of the first integral of $f(t)$ is given by

$$£\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Proof :

$$£\left[\int_0^t f(t)\,dt\right] = \int_0^\infty\left[\int_0^t f(t)\,dt\right]e^{-st}\,dt$$

Let

$$u = \int_0^t f(t)\,dt \quad ; \text{ then } du = f(t)\,dt$$

and

$$dv = e^{-st}\,dt \quad ; \text{ then } v = -\frac{1}{s}e^{-st}$$

On integrating

$$£\left[\int_0^t f(t)\,dt\right] = \int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= -\frac{1}{s}e^{-st}\cdot\int_0^t f(t)\,dt\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt$$

$$= 0 - 0 + \frac{1}{s}\,£\,[f(t)] = \frac{F(s)}{s}$$

In general,

$$£\left[\int_0^{t_1}\int_0^{t_2}\cdots\cdots\int_0^{t_n} f(t)\,dt_1\cdot dt_2\cdots\cdots dt_n\right] = \frac{F(s)}{s^n}$$

Note : The Laplace transform of the indefinite integral is given as

$$£\left[\int f(t)\,dt\right] = £\left[\int_0^t f(t)\,dt + f^{-1}(0^+)\right] = \frac{F(s)}{s} + \frac{f^{-1}(0^+)}{s}$$

(where $f^{-1}(0^+)$ is the value of the integral $f(t)$ as t approaches zero)

5. Differentiation with respect to "s" *(Frequency-Differentiation)*

The differentiation in the s-domain corresponds to the multiplication by "t" in the time domain, *i.e.*

$$£[t\cdot f(t)] = -\frac{dF(s)}{ds}$$

Proof :

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\cdot\left(\frac{d}{ds}e^{-st}\right)dt$$

$$= \int_0^\infty f(t)\cdot e^{-st}\cdot(-t)\,dt = -\int_0^\infty t\cdot f(t)\cdot e^{-st}\,dt = -\,£[t\cdot f(t)]$$

6. Integration by "s" *(Frequency-Integration)*

The integration of $F(s)$ in s-domain corresponds to division by "t" in the time domain, *i.e.*

$$£\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\cdot ds$$

Proof :

$$\int_s^\infty F(s)\cdot ds = \int_s^\infty\left[\int_0^\infty f(t)\,e^{-st}\,dt\right]ds = \int_0^\infty f(t)\int_s^\infty e^{-st}ds\,dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt = \int_0^\infty f(t)\cdot\left(0-\frac{e^{-st}}{-t}\right)dt$$

$$= \int_0^\infty f(t)\cdot\frac{e^{-st}}{t}\,dt = £\left[\frac{f(t)}{t}\right]$$

7. Shifting Theorem

[a] *Shifting in time (Time-shifting)*: The Laplace transform of a shifted or delayed function is given as

$$£[f(t-a)\cdot U(t-a) = e^{-as}F(s)$$

Proof :

Let $t-a=y$, then, $dt = dy$

$$£[f(t-a)\cdot U(t-a) = \int_{-a}^\infty f(y)\cdot U(y)\,e^{-s(y+a)}\,dy$$

$$= \int_0^\infty f(y)e^{-s(y+a)}\,dy \qquad \text{(since } U(y)=0\,;\,y<0)$$

$$= e^{-as}\int_0^\infty f(y)e^{-sy}\,dy = e^{-as}.F(s)$$

[b] *Shifting in frequency (Frequency shifting)*: The Laplace transform of e^{-at} times a function is equal to the Laplace transform of that function, with s is replaced by $(s + a)$.

Proof :

$$\pounds[e^{-at} f(t)] = \int_0^{\infty} e^{-at} f(t) e^{-st}\, dt$$

$$= \int_0^{\infty} f(t) \cdot e^{-(s+a)t}\, dt = F(s + a)$$

Note : $\pounds[e^{at} f(t)] = F(s - a)$

8. Initial Value Theorem

If the first derivative of $f(t)$ is Laplace transformable, then the initial value of a function $f(t)$ is given as

$$f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty}[s \cdot F(s)]$$

9. Final Value Theorem

The final value of a function $f(t)$ is given as

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0}[s \cdot F(s)]$$

10. Theorem for Periodic Functions

The Laplace transform of a periodic function (wave) with period T is equal to $\frac{1}{1-e^{-Ts}}$ times the Laplace transform of the first cycle of that function (wave).

Proof :

Let $f_1(t)$, $f_2(t)$, $f_3(t)$,..... be the functions describing the first, second, third, ... cycles of a periodic function $f(t)$ whose time period is T. Then

$$f(t) = f_1(t) + f_2(t) + f_3(t) + \ldots..$$

$$= f_1(t) + f_1(t - T)\, U(t - T) + f_1(t - 2T)\, U(t - 2T) + \ldots..$$

Now, let $\pounds[f_1(t)] = F_1(s)$

Therefore, by shifting theorem (property 7[a] of L.T.), we get

$$\pounds[f(t)] = F_1(s) + e^{-Ts}.F_1(s) + e^{-2Ts}\, F_1(s) + \ldots..$$

$$= F_1(s)\, [1 + e^{-Ts} + e^{-2Ts} + \ldots..]$$

$$= \frac{1}{1-e^{-Ts}} \cdot F_1(s)$$

11. Convolution Theorem

Given two functions $f_1(t)$ and $f_2(t)$, which are zero for $t < 0$. If $\pounds[f_1(t)] = F_1(s)$ and $\pounds[f_2(t)] = F_2(s)$, then

$\pounds^{-1}[F_1(s) \cdot F_2(s)] = f_1(t) * f_2(t)$ is called the convolution of $f_1(t)$ and $f_2(t)$ and is equal to

$$\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau \quad \text{or} \quad \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau$$

12. Time-Scaling

If Laplace transform of $f(t)$ is $F(s)$, then

$$\pounds[f(at)] = \frac{1}{a} \cdot F\left(\frac{s}{a}\right)$$

Note: Final value theorem doesnot apply when $f(t)$ is a periodic function.

Table 4.1. Laplace Transform Pairs

S. No.	$f(t)$*	$F(s) = \int_0^\infty f(t) e^{-st}\, dt$
1.	1 or $U(t)$, K	$\frac{1}{s}, \frac{K}{s}$
2.	t, t^n	$\frac{1}{s^2}, \frac{n!}{s^{n+1}}$
3.	$\delta(t)$	1
4.	$e^{\pm at}$	$\frac{1}{s \mp a}$
5.	$t\, e^{\pm at}$	$\frac{1}{(s \mp a)^2}$
6.	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
8.	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
9.	$e^{-at} \cos \omega t$	$\frac{(s+a)}{(s+a)^2 + \omega^2}$
10.	$\sinh at$	$\frac{a}{s^2 - a^2}$
11.	$\cosh at$	$\frac{s}{s^2 - a^2}$
12.	$e^{\pm at} f(t)$	$F(s \mp a)$
13.	$f(t \pm t_0)$	$e^{\pm t_0 s} F(s)$

*All $f(t)$ should be thought of as being multiplied by $U(t)$, *i.e.*, $f(t) = 0$ for $t < 0$.

Example 4.1. Find the Laplace transform of the following standard signals (functions)

(a) The unit step function $U(t)$.
(b) The delayed step function $KU(t-a)$.
(c) The ramp function $Kr(t)$ or $Kt\, U(t)$.
(d) The delayed unit ramp function $r(t-a)$.
(e) The unit impulse function $\delta(t)$.
(f) The unit doublet function $\delta'(t)$.

Solution :

(a)
$$f(t) = U(t)$$

$$F(s) = \int_0^\infty U(t) \cdot e^{-st}\, dt$$

By the definition of $U(t)$ given in chapter 2, we have

$$= \int_0^\infty e^{-st}\, dt = \left. \frac{e^{-st}}{-s} \right|_0^\infty = \frac{1}{s}$$

(b) $$f(t) = KU(t-a)$$

$$F(s) = \int_0^\infty KU(t-a)e^{-st}\,dt$$

By the definition of $U(t-a)$ given in chapter 2, we have

$$= \int_a^\infty K e^{-st}\,dt = K\frac{e^{-st}}{-s}\Big|_a^\infty = K\frac{e^{-as}}{s}$$

(Alternatively we can find directly using property 7[a])

(c) $$f(t) = Kr(t) = Kt\;U(t)$$

By parts

$$F(s) = \int_0^\infty KtU(t)e^{-st}\,dt = \int_0^\infty Kt\,e^{-st}\,dt = K\left[t\cdot\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty 1\cdot\frac{e^{-st}}{-s}\,dt\right]$$

$$= K[0-0] + \frac{K}{s}\int_0^\infty e^{-st}\,dt = -\frac{K}{s^2}e^{-st}\Big|_0^\infty = \frac{K}{s^2}$$

(d) $$f(t) = r(t-a) = (t-a)\;U(t-a)$$

$$F(s) = \int_0^\infty (t-a)U(t-a)e^{-st}\,dt$$

$$= \int_a^\infty (t-a)e^{-st}\,dt = (t-a)\frac{e^{-st}}{-s}\Big|_a^\infty - \int_a^\infty 1\cdot\frac{e^{-st}}{-s}\,dt$$

$$= 0 - 0 + \frac{1}{s}\int_a^\infty e^{-st}\,dt = \frac{1}{s}\cdot\frac{e^{-st}}{-s}\Big|_a^\infty = \frac{e^{-as}}{s^2}$$

(Alternatively we can find directly using property 7[a])

(e) $$f(t) = \delta(t)$$

$$F(s) = \int_0^\infty \delta(t)e^{-st}\,dt$$

By the definition of $\delta(t)$ given in chapter 2, we have

$$F(s) = e^{-st}\Big|_{t=0} = 1 \qquad \text{(Since } \delta(t) = 1 \text{ only at } t = 0)$$

(f) $f(t) = \delta'(t)$ $\qquad F(s) = s.£\,[\delta(t)] = s$

Example 4.2. Find the Laplace transform of the following functions :

(a) The exponential decay function Ke^{-at}
(b) The sinusoidal function $\sin \omega t$.
(c) The cosine function $\cos \omega t$.
(d) $e^{-at}\sin\omega t$
(e) $e^{-at}\cos\omega t$
(f) $e^{-at}\,t\,U(t)$
(g) $\sin h\,at$
(h) $\cos h\,at$

Solution :

(a) $$f(t) = Ke^{-at}$$

$$F(s) = \int_0^\infty K\,e^{-at}\,e^{-st}\,dt = K\int_0^\infty e^{-(s+a)t}\,dt$$

$$= K\cdot\frac{e^{-(s+a)t}}{-(s+a)}\Big|_0^\infty = K\left[0-\left\{-\frac{e^0}{s+a}\right\}\right] = \frac{K}{s+a}$$

(b)
$$f(t) = \sin \omega t$$

$$F(s) = \int_0^\infty \sin \omega t\, e^{-st}\, dt \qquad \left(\because \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)$$

$$= \int_0^\infty \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\cdot e^{-st}\, dt = \frac{1}{2j}\cdot\left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right]$$

$$= \frac{1}{2j}\cdot\left[\frac{s + j\omega - s + j\omega}{(s + j\omega)(s - j\omega)}\right] = \frac{\omega}{s^2 + \omega^2}$$

(c) As similar to above case

$$f(t) = \cos \omega t, \text{ then, } F(s) = \frac{s}{s^2 + \omega^2}$$

Alternatively for (b) and (c) we know, that

$$\pounds[e^{-at}] = \frac{1}{s + a}$$

put $a = j\omega$

$$\pounds[e^{-j\omega t}] = \frac{1}{s + j\omega} \qquad \left[\because e^{-j\omega t} = \cos \omega t - j \sin \omega t\right]$$

$$\pounds[\cos \omega t - j \sin \omega t] = \frac{1}{s + j\omega}\cdot\frac{s - j\omega}{s - j\omega}$$ [Multiplying Numerator and Denominator by $(s - j\omega)$]

$$= \frac{s - j\omega}{s^2 + \omega^2}$$

Equating real and imaginaries on both sides, we have

$$\pounds[\cos \omega t] = \frac{s}{s^2 + \omega^2} \text{ and } \pounds[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

(d)
$$f(t) = e^{-at} \sin \omega t$$

$$F(s) = \frac{\omega}{(s + a)^2 + \omega^2}$$ Using property 7[b]

(e)
$$f(t) = e^{-at} \cos \omega t$$

$$F(s) = \frac{s}{(s + a)^2 + \omega^2}$$ Using property 7[b]

(f)
$$f(t) = e^{-at}\, t\, U(t)$$

$$F(s) = \frac{1}{(s + a)^2}$$ Using property 7[b]

(g) We know that

$$\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$$

$$\pounds[\sinh at] = \frac{1}{2}\left[\int_0^\infty e^{at}\cdot e^{-st}\, dt - \int_0^\infty e^{-at} e^{-st}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - a} - \frac{1}{s + a}\right] = \frac{a}{s^2 - a^2}$$

Using property 7[(d)], shifting in time

$$= e^{-t_0 s} \cdot \left(\frac{\omega}{s^2 + \omega^2} \right)$$

Example 4.4. Obtain the Laplace transforms of the waveforms shown in (*i*) figure 2.4, (*ii*) figure 2.7, (*iii*) figure 2.17, (*iv*) figure 2.19, (I.P. Univ., 2001) (*v*) figure 2.20, (*vi*) figure 2.22(*b*), (*vii*) figure 2.22(*c*), (*viii*) figure 2.22(*d*), (*ix*) figure 2.22(*e*).

Solution :

(*i*) $$F(s) = K \frac{e^{t_1 s}}{s}$$

(*ii*) $$F(s) = K \cdot \frac{1}{s^2} - K \frac{e^{-s}}{s^2} - K \frac{e^{-s}}{s} = \frac{K}{s^2}(1 - e^{-s} - se^{-s})$$

(iii) $$V(s) = V_m \cdot \left[\frac{\omega}{s^2+\omega^2} + e^{-\frac{T}{2}s} \cdot \frac{\omega}{s^2+\omega^2}\right] = V_m \cdot \frac{\omega}{s^2+\omega^2}\left(1+e^{-\frac{T}{2}s}\right)$$

(iv) $$F(s) = \frac{1}{s}\left[1 - 2e^{-as} + 2e^{-2as} - 2e^{-3as} + \ldots\right]$$

$$= \frac{1}{s}\left[1 - 2e^{-as}(1 - e^{-as} + e^{-2as} - \ldots)\right] = \frac{1}{s}\left[1 - \frac{2e^{-as}}{1+e^{-as}}\right] = \frac{(1-e^{-as})}{s(1+e^{-as})}$$

$$= \frac{e^{as/2} - e^{-as/2}}{s(e^{as/2} + e^{-as/2})}$$

Alternatively ways : (Using theorem for periodic functions)

$$F(s) = \frac{1}{1-e^{-2as}} \cdot F_1(s) = \frac{1}{1-e^{-2as}} \cdot \left[\frac{1}{s} - \frac{2e^{-as}}{s} + \frac{e^{-2as}}{s}\right]$$

(Since $f_1(t) = U(t) - 2U(t-a) + U(t-2a)$)

$$= \frac{1}{1-e^{-2as}} \cdot \frac{1}{s} \cdot (1-e^{-as})^2 = \frac{1-e^{-as}}{s(1+e^{-as})} = \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

(v) $$F(s) = \frac{1}{s} - \frac{3e^{-2s}}{s} + 4\frac{e^{-3s}}{s} - 2\frac{e^{-5s}}{s}$$

$$= \frac{1}{s}\left(1 - 3e^{-2s} + 4e^{-3s} - 2e^{-5s}\right)$$

(vi) $$F(s) = K\left[\frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}\right] - \frac{K}{s}e^{-2s} = \frac{Ke^{-s}}{s^2}\left(1-e^{-s}\right) - \frac{K}{s}e^{-2s}$$

(vii) $$F(s) = \frac{K}{a} \cdot \frac{1}{s^2}(1-e^{-as})$$

(viii) $$F(s) = \frac{K}{a} \cdot \frac{e^{-as}}{s^2}(1-e^{-as})$$

(ix) $$F(s) = K \cdot \left[\frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}\right] = \frac{K}{s^2}\left[1 - 2e^{-s} + e^{-2s}\right] = \frac{K(1-e^{-s})^2}{s^2}$$

Example 4.5. Obtain the Laplace Transforms of the periodic waveforms as shown in (*i*) figure 2.23 (*ii*) figure 2.27.

Solution : (Using theorem for periodic functions)

(i) $$i'(t) = (2t-2)[U(t) - U(t-2)] + (-2t+6)[U(t-2) - U(t-4)]$$
$$= 2t\,U(t) - 2U(t) - (2t - 2 + 2t - 6)\,U(t-2) + (2t-6)\,U(t-4)$$
$$= 2tU(t) - 2U(t) - 4(t-2)\,U(t-2) + 2(t-4)\,U(t-4) + 2U(t-4)$$

$$I'(s) = £[i'(t)] = \frac{2}{s^2} - \frac{2}{s} - 4\frac{e^{-2s}}{s^2} + 2\frac{e^{-4s}}{s^2} + \frac{2e^{-4s}}{s}$$

$$= \frac{2}{s^2}\left[1 - 2e^{-2s} + e^{-4s}\right] - \frac{2}{s}(1-e^{-4s})$$

$$= \frac{2}{s^2}(1-e^{-2s})^2 - \frac{2}{s}(1-e^{-4s})$$

Since Time period, $T = 4$, Therefore,

$$I(s) = \frac{1}{1-e^{-4s}} \cdot I'(s) = \frac{2}{s^2}\left(\frac{1-e^{-2s}}{1+e^{-2s}}\right) - \frac{2}{s} = \frac{2}{s}\left[\left(\frac{1}{s}\right)\tanh s - 1\right]$$

(ii) $v'(t) = 1.G_{0,1}(t) + (-2)\, G_{1,2}(t)$

$= [U(t) - U(t-1)] - 2[U(t-1) - U(t-2)]$

$= U(t) - 3U(t-1) + 2U(t-2)$

$$V'(s) = \frac{1}{s}(1 - 3e^{-s} + 2e^{-2s}) = \frac{1}{s}(1 - e^{-s})(1 - 2e^{-s})$$

Sinc Time period, $T = 2$, therefore

$$V(s) = \frac{1}{1 - e^{-2s}} \cdot E'(s) = \frac{1}{1 - e^{-2s}} \cdot \frac{1}{s} \cdot (1 - e^{-s})(1 - 2e^{-s}) = \frac{1}{s} \cdot \left(\frac{1 - 2e^{-s}}{1 + e^{-s}}\right)$$

Alternative ways :

$$E(t) = U(t) - 3U(t-1) + 3U(t-2) - 3U(t-3) + \ldots$$

$$E(s) = \pounds[E(t) = \frac{1}{s}\left[1 - 3e^{-s} + 3e^{-2s} - 3e^{-3s} + \ldots\right]$$

$$= \frac{1}{s}\left[1 - \frac{3e^{-s}}{1 + e^{-s}}\right] = \frac{1}{s} \cdot \frac{1 - 2e^{-s}}{1 + e^{-s}}$$

Example 4.6. Obtain the L.T. of the waveform as shown in figure 2.28.

Solution: $f(t) = e^{-t/2}[U(t) - U(t-1) + U(t-2) - U(t-3) + U(t-4) \ldots..]$

$$F(s) = \pounds[f(t)] = \frac{1}{s + \frac{1}{2}} - \frac{e^{-(s+\frac{1}{2})}}{s + \frac{1}{2}} + \frac{e^{-2\left(s+\frac{1}{2}\right)}}{s + \frac{1}{2}} - \frac{e^{-3\left(s+\frac{1}{2}\right)}}{s + \frac{1}{2}} + \ldots..$$

$$= \frac{1}{s + \frac{1}{2}} \cdot \left[1 - e^{-\left(s+\frac{1}{2}\right)} + e^{-2\left(s+\frac{1}{2}\right)} - e^{-3\left(s+\frac{1}{2}\right)} + \ldots..\right]$$

$$= \frac{1}{s + \frac{1}{2}} \cdot \left[\frac{1}{1 + e^{-\left(s+\frac{1}{2}\right)}}\right] = \left(\frac{1}{s + 0.5}\right)\left(\frac{1}{1 + e^{-(s + 0.5)}}\right)$$

Example 4.7. Obtain the Laplace transform of the waveform shown in figure 2.30(c).

Solution: $i(t) = 1.5(1 - e^{-4t})\, U(t) - 1.5\, [1 - e^{-4(t - 0.1)}]\, U(t - 0.1)$

$$I(s) = 1.5\left[\frac{1}{s} - \frac{1}{s+4}\right] - 1.5\left[\frac{e^{-0.1s}}{s} - \frac{e^{-0.1s}}{s+4}\right]$$

$$= 1.5\,(1 - e^{-0.1s})\left(\frac{1}{s} - \frac{1}{s+4}\right) = \frac{6(1 - e^{-0.1s})}{s(s+4)}$$

Example 4.8. Obtain the Laplace transform of the waveform shown in figure 2.31.

Solution: $v(t) = K[t^2\, U(t) - 4\,(t-1)\, U(t-1) - (t-2)^2\, U(t-2)]$

$$V(s) = K\left[\frac{2}{s^3} - \frac{4e^{-s}}{s^2} - \frac{2e^{-2s}}{s^3}\right] = K \cdot \frac{2}{s^3}\, [1 - 2\, se^{-s} - e^{-2s}]$$

Example 4.9. Obtain the Laplace transform of the periodic, rectified half-sine wave as shown in figure 4.3.

Example 4.19. Determine Laplace Transform of the following wave shown in figure 4.7.

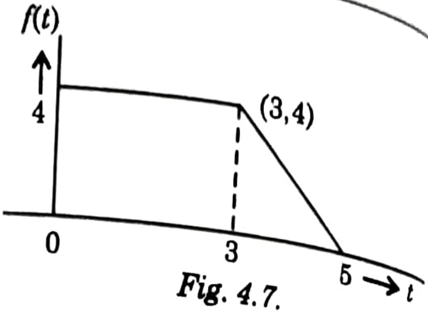


Fig. 4.7.

Solution: From the wave form shown in figure 4.7,

$$f(t) = 4G_{0,3}(t) + (-2t + 10)\, G_{3,5}(t)$$
$$= 4\,[U(t) - U(t-3)] - 2\,(t-5)\,[U(t-3) - U(t-5)]$$
$$= 4\,U(t) - 4U(t-3) - 2(t-5+2-2)\,U(t-3) + 2(t-5)\,U(t-5)$$
$$= 4\,U(t) - 2(t-3)\,U(t-3) + 2(t-5)\,U(t-5)$$

Therefore, Laplace transform of the waveform is

$$F(s) = \pounds\,[f(t)]$$
$$= \frac{4}{s} - \frac{2e^{-3s}}{s^2} + \frac{2\,e^{-5s}}{s^2}$$

Example 4.20. Find the Laplace transforms of the following functions :

(a) $e^{-at}\,U(t)$

(b) $e^{-at}\,U(t-b)$

(c) $e^{-a(t-b)}\,U(t-b) = e^{ab}\,.\,e^{-at}\,U(t-b)$

(d) $e^{-a(t-b)}\,U(t-c) = e^{ab}\,.e^{-at}\,U(t-c).$

Solution : (a) $$F(s) = \int_0^\infty f(t)\,e^{-st}dt = \int_0^\infty e^{-at}.U(t)\,e^{-st}dt = \frac{1}{s+a}$$

(Alternatively we can find directly using properly 7 [b])

(b) $$F(s) = \int_0^\infty e^{-at}\,U(t-b)\,e^{-st}dt = \int_b^\infty e^{-(s+a)t}\,dt$$

$$= \left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_b^\infty = \frac{e^{-(s+a)b}}{(s+a)}$$

(Alternatively we can find directly using property 7[a] and property 7[b])

(c) $$F(s) = \int_0^\infty e^{-a(t-b)}\,U(t-b)\,e^{-st}dt$$

$$= e^{ab}\int_b^\infty e^{-(s+a)t}dt = e^{ab}\,.\left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_b^\infty$$

$$= e^{ab}\,.\,\frac{e^{-(s+a)b}}{s+a} = \frac{e^{-bs}}{s+a}$$

(Alternatively we can find directly using property 7[a])

(d) $$F(s) = \int_0^\infty e^{-a(t-b)}\,U(t-c)\,e^{-st}dt = e^{ab}\int_c^\infty e^{-at}.\,e^{-st}dt$$

$$e^{ab}\,.\left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_c^\infty = e^{ab}\,.\,\frac{e^{-(s+a)c}}{s+a}$$

CHAPTER 5

CIRCUIT ANALYSIS BY LAPLACE TRANSFOR

5.1. INTRODUCTION

The circuit analysis in time domain were presented in the third chapter u the classical method. Having introduced the definition of Laplace transforma and learned to obtain the Laplace transform of many functions in previous chap we shall now be exposed to the remarkable power of the Laplace transform a mathematical tool to find the circuit responses in terms of voltages and curre subject to any arbitrary input functions.

5.2. SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

The Laplace transformation is used to determine the solution of integ differential equation. A differential equation of the general form

$$a_0 \frac{d^n i}{dt^n} + a_1 \frac{d^{n-1} i}{dt^{n-1}} + \ldots\ldots. + a_{n-1} \frac{di}{dt} + a_n i = v(t)$$

becomes, as a result of the Laplace transformation, an algebraic equation whi may be solved for the unknown as

$$I(s) = \frac{£[v(t)] + \text{initial condition terms}}{a_o s^n + a_1 s^{n-1} + \ldots\ldots\ldots + a_{n-1} s + a_n}$$

where $\qquad I(s) = £\,[i(t)]$

$I(s)$ is a ratio of two polynomials in s. Let the numerator and denominat polynomials be designated $P(s)$ and $Q(s)$, respectively, as

$$I(s) = \frac{P(s)}{Q(s)}$$

Note that $Q(s) = 0$ is the characteristic equation. If the transform ter $P(s)/Q(s)$ can now be found in a table of transform pairs, the solution $i(t)$ can b written directly. In general, however, the transform expression for $I(s)$ must b broken into simpler terms before any practical transform table can be used.

Next, we factor the denominator polynomial, $Q(s)$,

$$Q(s) = a_0 s^n + a_1 s^{n-1} + \ldots.. + a_{n-1} + a_n$$
$$\equiv a_0(s + s_1) \ldots..(s + s_n)$$

or very compactly,

$$Q(s) = a_0 \prod_{j=1}^{n} (s + s_j)$$

Where Π indicates a product of factors, and $s_1, s_2, \ldots.. s_n$ are the n roots o the characteristic equation $Q(s) = 0$. Now the possible forms of these roots ar discussed as :

(i) Partial Fraction Expansion When All the Roots of $Q(s)$ are Simple :

If all roots of $Q(s) = 0$, are simple, then

$$I(s) = \frac{P(s)}{(s+s_1)(s+s_2).....(s+s_n)} = \frac{K_1}{s+s_1} + \frac{K_2}{s+s_2} + ... + \frac{K_n}{s+s_n}$$

where the K's are real constants called **residues**. Any of the residues K_1, K_2, ..., K_n can be found by multiplying $I(s)$ by the corresponding denominator factor and setting $(s + s_j)$ equal to zero, *i.e.*

$s = -s_j$.As

$$K_j = \left[(s+s_j)\frac{P(s)}{Q(s)}\right]_{s=-s_j}$$

(ii) Partial Fraction Expansion When Some Roots of $Q(s)$ are of Multiple Order :

If a root of $Q(s) = 0$, is if multiplicity r, then

$$I(s) = \frac{P(s)}{(s+s_1)^r Q_1(s)} = \frac{K_{11}}{s+s_1} + \frac{K_{12}}{(s+s_1)^2} + ... + \frac{K_{1r}}{(s+s_1)^r} + ...$$

The following equations may be used for the evaluation of coefficients of repeated roots.

$$K_{1r} = (s+s_1)^r \cdot I(s)\Big|_{s=-s_1}$$

$$K_{1(r-1)} = \frac{d}{ds}\left[(s+s_1)^r \cdot I(s)\right]\Big|_{s=-s_1}$$

$$K_{1(r-2)} = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+s_1)^r \cdot I(s)\right]\Big|_{s=-s_1}$$

$$K_{11} = \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}\left[(s+s_1)^r \cdot I(s)\right]\Big|_{s=-s_1}$$

(iii) Partial Fraction Expansion When two roots of $Q(s)$ are of Complex Conjugate Pair :

If two roots of $Q(s) = 0$, which form a complex conjugate pair, then

$$I(s) = \frac{P(s)}{(s+\alpha+j\omega)(s+\alpha-j\omega).Q_1(s)} = \frac{K_1}{(s+\alpha+j\omega)} + \frac{K_1^*}{(s+\alpha-j\omega)} + ...$$

Where $\quad K_1 = (s+\alpha+j\omega)\cdot I(s)\big|_{s=-(\alpha+j\omega)}$

and K_1^* is the complex conjugate of K_1. An expression of the type shown above is necessary for each pair of complex conjugate roots.

Several examples will illustrate the partial fraction expansion and the evaluation of K's.

Example 5.1. Solve the Differential Equation

$x'' + 3x' + 2x = 0,\ x(0^+) = 2,\ x'(0^+) = -3.$

Solution : Taking the Laplace transform,

$s^2X(s) - sx(0^+) - x'(0^+) + 3sX(s) - 3x(0^+) + 2X(s) = 0$

$(s^2 + 3s + 2)X(s) = sx(0^+) + x'(0^+) + 3x(0^+)$

$(s^2 + 3s + 2)X(s) = 2s + 3$ [By putting initial conditions]

or $$X(s) = \frac{2s+3}{s^2+3s+2} = \frac{2s+3}{(s+1)(s+2)}$$

Hence, all roots of denominator polynomial are simple. Then by partial fraction expansion,

$$X(s) = \frac{2s+3}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

Where $K_1 = (s+1)\cdot X(s)\big|_{s=-1}$

$$= \frac{2s+3}{s+2}\bigg|_{s=-1} = \frac{-2+3}{1+2} = 1$$

and $K_2 = (s+2)\cdot X(s)\,\big|_{s=-2} = \frac{2s+3}{s+1}\bigg|_{s=2} = 1$

The result of the partial fraction expansion is thus,

$$X(s) = \frac{2s+3}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{1}{s+2}$$

Therefore, the solution of given differential equation is

$$x(t) = £^{-1}[X(s)] = £^{-1}\left[\frac{1}{s+1} + \frac{1}{s+2}\right]$$

or $\quad x(t) = e^{-t} + e^{-2t}$

Example 5.2. Find i(t), if $I(s) = \dfrac{1}{s(s+1)^2(s+2)}$.

Solution:

$$I(s) = \frac{1}{s(s+1)^2(s+2)}$$

$$= \frac{K_1}{s} + \frac{K_{21}}{s+1} + \frac{K_{22}}{(s+1)^2} + \frac{K_3}{s+2}$$

Where $K_1 = sI(s)\,|_{s=0} = \frac{1}{2}$, $K_3 = (s+2)\, I(s)\,|_{s=-2} = \frac{1}{2}$

$$K_{22} = (s+1)^2\, I(s)\big|_{s=-1} = -1$$

$$K_{21} = \frac{d}{ds}[(s+1)^2\, I(s)]\,|_{s=-1}$$

$$= \frac{d}{ds}\left[\frac{1}{s(s+2)}\right]\bigg|_{s=-1} = \frac{-(2s+2)}{s^2(s+2)^2}\bigg|_{s=-1} = 0$$

The complete expansion is

$$I(s) = \frac{1}{2s} - \frac{1}{(s+1)^2} + \frac{1}{2(s+2)}$$

Therefore,

$$i(t) = \frac{1}{2} - te^{-t} + \frac{1}{2}e^{-2t}\ A$$

Example 5.3. If $I(s) = \dfrac{s^2+5s+9}{s^3+5s^2+12s+8}$ **; find** $i(t)$.

Solution : $I(s) = \dfrac{s^2+5s+9}{s^3+5s^2+12s+8} = \dfrac{s^2+5s+9}{(s+1)(s^2+4s+8)}$

$$I(s) = \frac{s^2+5s+9}{(s+1)\left[(s+2)^2-(j2)\right]} = \frac{[s^2+5s+9]}{(s+1)(s+2+j2)(s+2-j2)}$$

or $$I(s) = \frac{K_1}{s+1}+\frac{K_2}{(s+2+j2)}+\frac{K_2^*}{(s+2-j2)}$$

$$K_1 = (s+1)\,I(s)\big|_{s=1} = \left.\frac{s^2+5s+9}{s^2+4s+8}\right|_{s=-1} = 1$$

-1

$$K_2 = (s+2+j2)\,I(s)\big|_{-(2+j2)} = -\frac{1}{j4}$$

$$\therefore \quad K_2^* = \frac{1}{j4}$$

Therefore, the complete expansion is

$$I(s) = \frac{1}{s+1}+\frac{-\frac{1}{j4}}{(s+2+j2)}+\frac{\frac{1}{j4}}{(s+2-j2)}$$

Hence, $$i(t) = e^{-t}-\left(\frac{1}{j4}\right)e^{-(2+j2)t}+\left(\frac{1}{j4}\right)e^{-(2-j2)t}$$

$$= e^{-t}-\frac{1}{j4}e^{-2t[e^{-j2t}-e^{j2t}]}$$

or $$i(t) = e^{-t}+\frac{1}{2}e^{-2t}\sin 2t \text{ A}$$

Alternatively, $$I(s) = \frac{s^2+5s+9}{(s+1)(s^2+4s+8)} = \frac{1}{s+1}+\frac{1}{s^2+4s+8}$$

$$= \frac{1}{s+1}+\frac{1}{2}\left[\frac{2}{(s+2)^2+(2)^2}\right]$$

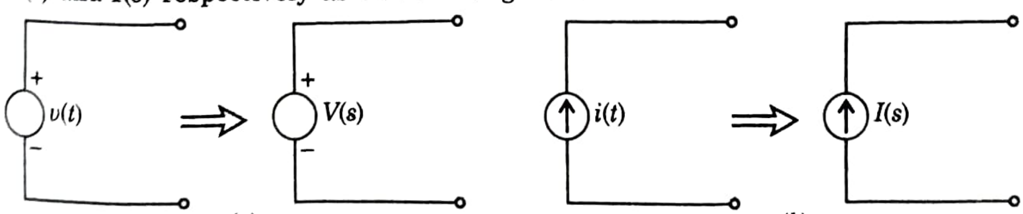
Therefore, $$i(t) = \pounds^{-1}[I(s)] = e^{-t}+\frac{1}{2}e^{-2t}\sin 2t \text{ A}$$

as $$\pounds^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right] = e^{-at}\sin\omega t$$

5.3. TRANSFORMED CIRCUIT COMPONENTS REPRESENTATION

5.3.1. Independent Sources

The sources $v(t)$ and $i(t)$ may be represented by their transformations, namely $V(s)$ and $I(s)$ respectively as shown in figure 5.1.



(a) (b)

Fig. 5.1. Representation of (a) voltage source (b) current source.

5.3.2. Resistance Parameter

By Ohm's law, the v–i relationship for a resistor in t-domain is

$$v_R(t) = Ri_R(t)$$

In the complex-frequency domain (s-domain), above equation becomes

$$V_R(s) = RI_R(s)$$

From above two equations, we observe that the representation of a resistor in t-domain and s-domain are one and the same as shown in figure 5.2.

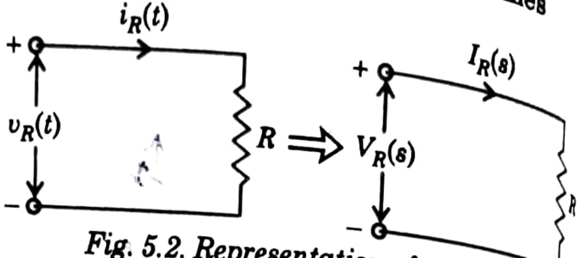


Fig. 5.2. Representation of a resistor.

5.3.3. Inductance Parameter

The v-i relationship for an inductor is

$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{0^+}^{t} v_L(t)\, d(t) + i_L(0^+)$$

The corresponding Laplace transforms are

$$V_L(s) = sL\, I_L(s) - Li_L(0^+)$$

or

$$I_L(s) = \frac{1}{sL}V(s) + \frac{i_L(0^+)}{s}$$

From above equations, we get the transformed circuit representation for the inductor as shown in figure 5.3.

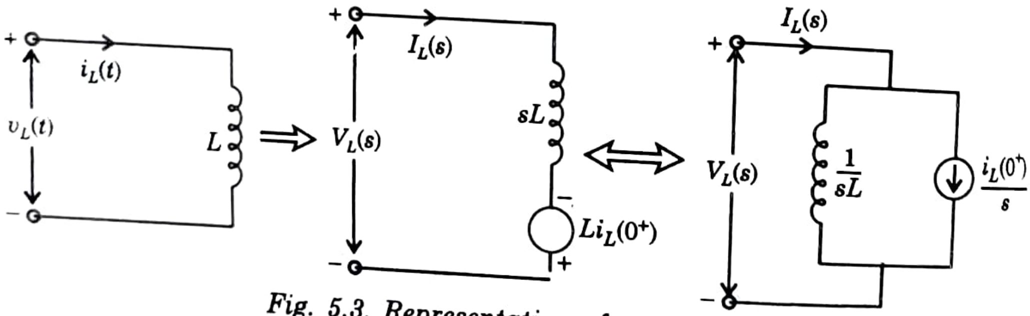


Fig. 5.3. Representation of an inductor.

5.3.4. Capacitance Parameter

For a capacitor, the v–i relationship is

$$i_c(t) = C\frac{dv_c(t)}{dt}$$

or

$$v_c(t) = \frac{1}{C}\int_{0^+}^{t} i_c(t)\, dt + v_c(0^+)$$

The corresponding Laplace Transform are

$$I_c(s) = s\, C\, V_c(s) - C\, v_c(0^+)$$

or

$$V_c(s) = \frac{1}{sC}I_c(s) + \frac{v_c(0^+)}{s}$$

From above equations, we get the transformed circuit representation for the capacitor as shown in figure 5.4.

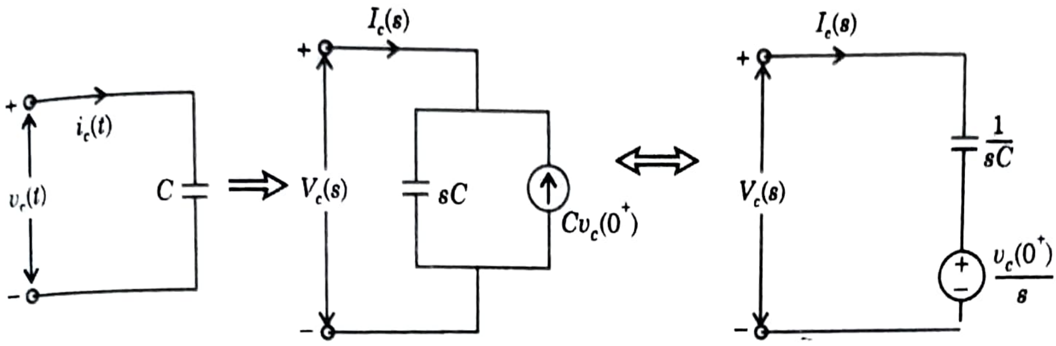


Fig. 5.4. Representation of a capacitor.

Example 5.4. Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + \frac{3dy(t)}{dt} + 2y(t) = 5\,U(t)$$

Where $U(t)$ is unit-step function. The initial conditions are $y(0^+) = -1$ and $\frac{dy}{dt}(0^+) = 2$. Determine $y(t)$ for $t \geq 0$.

Solution : Taking Laplace transform on both sides of given differential equation :

$$s^2Y(s) - sy(0^+) - y'(0^+) + 3sY(s) - 3y(0^+) + 2Y(s) = \frac{5}{s}$$

Substituting the values of initial conditions and solving for $Y(s)$, we get

$$s^2Y(s) + s - 2 + 3sY(s) + 3 + 2Y(s) = \frac{5}{s}$$

or $$Y(s) = \frac{-s^2 - s + 5}{s(s^2 + 3s + 2)} = \frac{-s^2 - s + 5}{s(s+1)(s+2)}$$

Expanded by partial fraction expansion,

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

$$K_1 = sY(s)\,|_{s=0} = \frac{5}{2}$$

$$K_2 = (s+1)\,Y(s)\,|_{s=-1} = \frac{-1+1+5}{(-1)(1)} = -5$$

$$K_3 = (s+2)\,Y(s)\,|_{s=-2} = \frac{-4+2+5}{(-2)(-1)} = \frac{3}{2}$$

Therefore, $$Y(s) = \frac{5}{2s} - \frac{5}{s+1} + \frac{3}{2(s+2)}$$

Hence, taking the inverse Laplace transform, we get the complete solution as

$$y(t) = \frac{5}{2} - 5e^{-t} + \frac{3}{2}e^{-2t} \quad ; t \geq 0$$

Example 5.5. Consider the R–L circuit with $R = 4\ \Omega$ and $L = 1H$ excited by a 48V d.c. source as shown in figure 5.5 (a). Assume the initial current through the inductor is 3A. Using the Laplace transform determine the current $i(t)$; $t \geq 0$. Also draw the s-domain representation of the circuit.

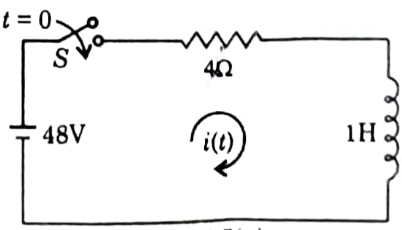


Fig. 5.5(a).

Solution : Applying *KVL*,

$$R(i) + L\frac{di(t)}{dt} = 48$$

Taking Laplace transform

$$RI(s) + L[sI(s) - i_L(0^+)] = \frac{48}{s}$$

$$4I(s) + 1.[sI(s) - 3] = \frac{48}{s}$$

or
$$I(s) = \frac{3s+48}{s(s+4)}$$

Applying the partial fraction expansion, we get

$$I(s) = \frac{3s+48}{s(s+4)} = \frac{K_1}{s} + \frac{K_2}{s+4}$$

where $K_1 = s.I(s)\,|_{s=0} = \left.\frac{3s+48}{s+4}\right|_{s=0} = 12$

and $K_2 = (s+4).I(s)\,|_{s=-4}$

$$= \left.\frac{(3s+48)}{s}\right|_{s=-4} = -9$$

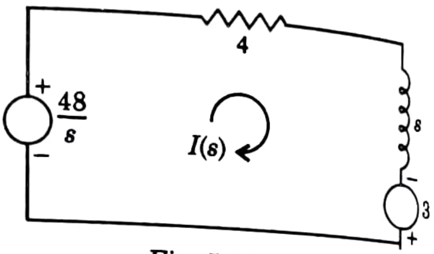


Fig. 5.5(b).

Then, $I(s) = \frac{12}{s} - \frac{9}{s+4}$

or $\quad i(t) = \mathcal{L}^{-1}[I(s)] = 12 - 9e^{-4t}$ **A**

And the s-domain representation is shown in figure 5.5(b).

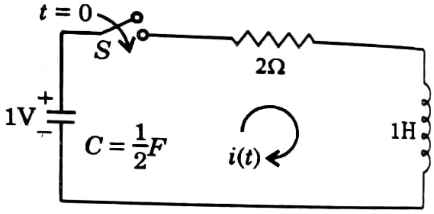


Fig. 5.6(a).

Example 5.6. Consider a series R-L-C circuit with the capacitor initially charged to voltage of 1 V as indicated in figure 5.6 (a). Find the expression for $i(t)$. Also draw the s-domain representation of the circuit.

Solution : The differential equation for the current $i(t)$ is

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt + v_c(0^+) = 0$$

and the corresponding transform equation is

$$L[sI(s) - i(0^+)] + RI(s) + \frac{1}{Cs}I(s) + \frac{v_c(0^+)}{s} = 0$$

The parameters have been specified on $C = \frac{1}{2}F$, $R = 2\Omega$, $L = 1H$, and $v_c(0^+) = -1\,V$ (with the given polarity). The initial current $i(0^+) = 0$, because initially inductor behaves as a open circuit. The transform equation $I(s)$ then becomes

$$sI(s) + 2I(s) + \frac{2}{s}I(s) - \frac{1}{s} = 0$$

or
$$I(s) = \frac{1}{s^2 + 2s + 2}$$

or, Completing the square,

$$I(s) = \frac{1}{(s+1)^2+1}$$

Therefore, $i(t) = \mathcal{L}^{-1}[I(s)]$

or $\quad i(t) = e^{-t} \sin t \text{ A}$

The s–domain representation is shown in figure 5.6(*b*).

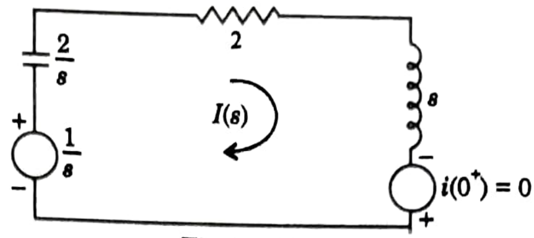


Fig. 5.6(b).

Example 5.7. Consider the R-C parallel circuit as shown in figure 5.7 (a) with $R = 0.5\Omega$ and $C = 4$F excited by d.c. current source of 10 A. Determine the voltage across the capacitor by applying Laplace transformation. Assume the initial voltage across the capacitor as 2V. Also draw the s-domain representation of the circuit.

Solution : Applying *KCL*,

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} = 10$$

Taking Laplace transform

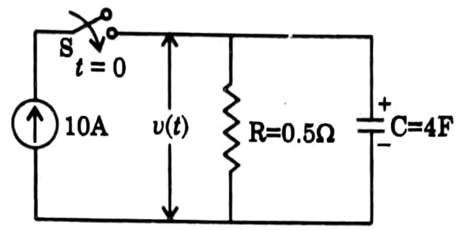


Fig. 5.7(a).

$$\frac{V(s)}{R} + C[sV(s) - v(0^+)] = \frac{10}{s}$$

$$2V(s) + 4[sV(s) - 2] = \frac{10}{s}$$

or $$V(s) = \frac{8s+10}{s(4s+2)}$$

Applying the partial fraction expansion,

$$V(s) = \frac{2s+2.5}{s(s+0.5)} = \frac{K_1}{s} + \frac{K_2}{s+0.5}$$

$$K_1 = sV(s)\,|_{s=0} = 5$$

and $$K_2 = (s+0.5)V(s)\,|_{s=-0.5} = -3$$

or $$V(s) = \frac{5}{s} + \frac{-3}{s+0.5}$$

Therefore, $v(t) = \mathcal{L}^{-1}[V(s)]$

$= 5 - 3e^{-0.5t}$ **V**

The s-domain representation of the circuit is shown in figure 5.7 (*b*).

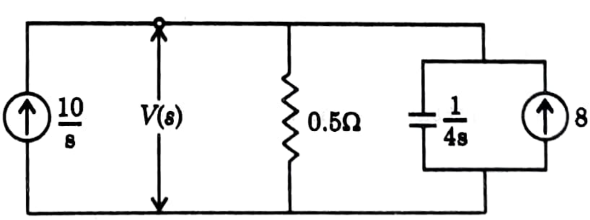


Fig. 5.7(b).

Example 5.8. In the network shown in figure 5.8, the switch S is closed at t = 0. With the network parameter values shown, find the expressions for $i_1(t)$ and $i_2(t)$, if the network is unenergized before the switch is closed.

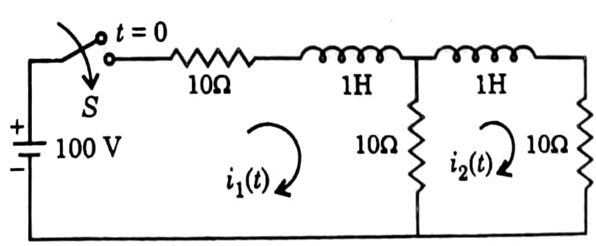


Fig. 5.8.

Solution : Applying *KVL*, Loop equations are

$$\frac{di_1(t)}{dt} + 10i_1(t) + 10[i_1(t) - i_2(t)] = 100$$

or $$\frac{di_1(t)}{dt} + 20i_1(t) - 10i_2(t) = 100 \qquad ...(i)$$

And, $$\frac{di_2(t)}{dt} + 10i_2(t) + 10[i_2(t) - i_1(t)] = 0$$

or $$\frac{di_2(t)}{dt} + 20i_2(t) - 10i_1(t) = 0 \qquad ...(ii)$$

The transform equations of (i) and (ii) may be written as (keeping initial conditions are zero, as given) :

$$(s+20)\, I_1(s) - 10I_2(s) = \frac{100}{s}$$

and $$-10I_1(s) + (s+20)\, I_2(s) = 0$$

Writing in matrix form,

$$\begin{bmatrix} s+20 & -10 \\ -10 & s+20 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{100}{s} \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$I_1(s) = \frac{\Delta_1}{\Delta} \text{ and } I_2(s) = \frac{\Delta_2}{\Delta}$$

Where, $$\Delta_1 = \begin{vmatrix} \frac{100}{s} & -10 \\ 0 & s+20 \end{vmatrix} \ ; \ \Delta_2 = \begin{vmatrix} s+20 & \frac{100}{s} \\ -10 & 0 \end{vmatrix}$$

and $$\Delta = \begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}$$

Therefore, $$\Delta_1 = \frac{100}{s}(s+20) \ ; \ \Delta_2 = \frac{1000}{s}$$

and $$\Delta = (s+20)^2 - 100 = s^2 + 40s + 300$$

So, $$I_1(s) = \frac{100(s+20)}{s(s^2+40s+300)}$$

and $$I_2(s) = \frac{1000}{s(s^2+40s+300)}$$

The partial fraction expansion of above expressions $I_1(s)$ and $I_2(s)$ are

$$I_1(s) = \frac{20/3}{s} + \frac{-5}{s+10} + \frac{-5/3}{s+30}$$

and $$I_2(s) = \frac{10/3}{s} + \frac{-5}{s+10} + \frac{5/3}{s+30}$$

The inverse Laplace transformation give $i_1(t)$ and $i_2(t)$ as

$$i_1(t) = \frac{20}{3} - 5e^{-10t} - \frac{5}{3}e^{-30t}\ A$$

$$i_2(t) = \frac{10}{3} - 5e^{-10t} + \frac{5}{3}e^{-30t}\ A$$

Which is the required solution.

Example 5.9. In the series R-L-C circuit shown in figure 5.9, There is no initial charge on the capacitor. If the switch S is closed at t = 0, determine the resulting current.

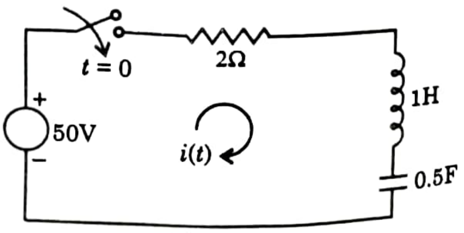


Fig. 5.9.

Solution : The time-domain equation of the given circuit is

$$2i(t) + 1.\frac{di(t)}{dt} + \frac{1}{0.5}\int i(t)\,dt = 50$$

or
$$2i(t) + \frac{di(t)}{dt} + 2\int i(t)\,dt = 50 \qquad (\because v_c(0^-) = 0)$$

Taking Laplase transform,

$$2I(s) + sI(s) - i(0^+) + 2\frac{I(s)}{s} = \frac{50}{5}$$

Because $i(0^+) = 0$, therefore,

$$I(s) = \frac{50}{s^2+2s+2} = \frac{50}{(s+1)^2+1}$$

Therefore,
$$i(t) = \pounds^{-1}[I(s)]$$
$$\boldsymbol{i(t) = 50\,e^{-t}\sin t \ \ \text{A}}$$

Example 5.10. Repeat Example 3.8 as shown in figure 3.11 using Laplace transform.

Solution : Before the switching action takes place,

$$V = i'(t)(R_1 + R_2) + L\frac{di'(t)}{dt}$$

Taking Laplace transform,

$$\frac{V}{s} = I'(s)(R_1 + R_2) + L[s\cdot I'(s) - i'(0^+)]$$

Since $i'(0^+) = 0$, therefore,

$$I'(s) = \frac{V}{s(R_1 + R_2 + Ls)}$$

$$= \frac{\frac{V}{L}}{s\left(s + \frac{R_1+R_2}{L}\right)} = \frac{V}{R_1+R_2}\left[\frac{1}{s} - \frac{1}{s+\frac{R_1+R_2}{L}}\right]$$

$$i'(t) = \pounds^{-1}[I'(s)] = \frac{V}{R_1+R_2}\left(1 - e^{-\frac{R_1+R_2}{L}t}\right)$$

Therefore,
$$i'(\infty) = \frac{V}{R_1+R_2}$$

When switch is closed : R_2 is short circuited. Then

$$V = L\frac{di(t)}{dt} + R_1\,i(t)$$

Taking Laplace transform,

$$\frac{V}{s} = L[sI(s) - i(0^+)] + R_1 I(s)$$

$$i(0^+) = i'(\infty) = \frac{V}{R_1+R_2}$$

$$\frac{V}{s} = (Ls + R_1)\,I(s) - \frac{LV}{R_1+R_2}$$

$$(Ls + R_1)\, I(s) = \frac{V}{s} + \frac{LV}{R_1 + R_2}$$

$$I(s) = \frac{V}{s(Ls + R_1)} + \frac{LV}{(R_1 + R_2)(Ls + R_1)}$$

$$I(s) = \frac{V}{R_1}\left[\frac{1}{s} - \frac{1}{s + \frac{R_1}{L}}\right] + \frac{V}{R_1 + R_2}\cdot\frac{1}{s + \frac{R_1}{L}}$$

$$= \frac{V}{R_1}\cdot\frac{1}{s} - \frac{1}{s + \frac{R_1}{L}}\left\{\frac{V}{R_1} - \frac{V}{R_1 + R_2}\right\}$$

$$= \frac{V}{R_1}\cdot\frac{1}{s} - \frac{VR_2}{R_1(R_1 + R_2)}\cdot\frac{1}{s + \frac{R_1}{L}}$$

Therefore $$i(t) = \pounds^{-1}\,[I(s)] = \frac{V}{R_1} - \frac{VR_2}{R_1(R_1 + R_2)} e^{-\frac{R_1}{L}t}$$

or $$i(t) = \frac{V}{R_1}\left(1 - \frac{R_2}{R_1 + R_2} e^{-\frac{R_1}{L}t}\right) A$$

Example 5.11. Repeat Example 3.10 as shown in figure 3.13 using Laplace transform.

Solution : Applying *KVL*,

$$1 = Ri(t) + \frac{1}{C}\int_0^t i(t)\, dt + v_c(0^+)$$

Taking Laplace transform and putting $R = 4$, $C = \frac{1}{16}$ and $v_c(0^+) = 9V$, we have

$$\frac{1}{s} = 4I(s) + \frac{16}{s} I(s) + \frac{9}{s}$$

$$-\frac{8}{s} = \left(4 + \frac{16}{s}\right) I(s)$$

$$I(s) = \frac{8}{(4s + 16)} = -\frac{2}{s + 4}$$

$$i(t) = \pounds^{-1}\,[I(s)] = -\,2e^{-4t}$$

Therefore, $$v_c(t) = 16\int_0^t i(t)\, dt + 9 = 16(-2)\int_0^t e^{-4t}\, dt + 9 = 8\cdot\left[e^{-4t}\right]_0^t + 9$$

or $$v_c(t) = 1 + 8e^{-4t}\ \text{V}$$

Example 5.12. Repeat Example 3.13 as shown in figure 3.16 using Laplace transform.

Solutions : With the switch on 1,

$$50 = 40i'(t) + 20\frac{di'(t)}{dt}$$

Taking Laplace transform,

$$\frac{50}{s} = 40\, I'(s) + 20[sI'(s) - i'(0^+)]$$

Since $\quad i'(0^+) = 0$, therefore,

$$I'(s) = \frac{50}{s(40+20s)} = \frac{2.5}{s(s+2)}$$

Using partial fraction expansion,

$$I'(s) = \left(\frac{2.5}{2}\right).\frac{1}{s} + \left(\frac{2.5}{-2}\right).\frac{1}{s+2} = 1.25\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

Therefore, $\quad i'(t) = \mathcal{L}^{-1}[I'(s)] = 1.25(1-e^{-2t})$

as $t \to \infty$

$$i'(\infty) = 1.25 \text{ A}$$

With the switch on 2,

$$10 = 40\, i(t) + 20\frac{di(t)}{dt}$$

Taking Laplace transform,

$$\frac{10}{s} = 40\, I(s) + 20[sI(s) - i(0^+)]$$

As $\quad i(0^+) = i'(\infty) = 1.25$

Therefore, $\quad \dfrac{10}{s} = (40 + 20s)\, I(s) - 20 \times 1.25$

$$I(s) = \frac{\left(\frac{10}{s}+25\right)}{(40+20s)} = \frac{10+25s}{s(40+20s)}$$

or $\quad I(s) = \dfrac{25(s+0.4)}{20s(s+2)} = 1.25\left[\dfrac{s+0.4}{s(s+2)}\right]$

$$= 1.25\left[\frac{\left(\frac{0.4}{2}\right)}{s} + \frac{\left(\frac{-1.6}{-2}\right)}{s+2}\right] = \frac{0.25}{s} + \frac{1}{s+2}$$

Therefore, $\quad i(t) = \mathcal{L}^{-1}[I(s)]$

or $\quad i(t) = 0.25 + e^{-2t}$ A

Example 5.13. Solve for $i(t)$ in circuit as shown in figure 5.10 (a) in which 3F capacitor is initially charged to 20V, the 6-F capacitor to 10V, and the switch is closed at $t = 0$. Also draw the transformed circuit.

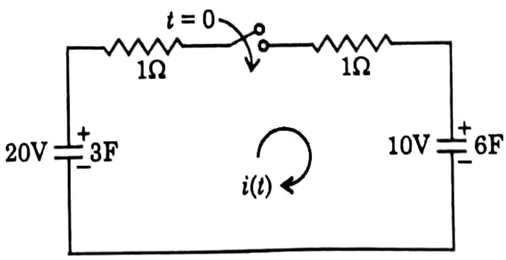


Fig. 5.10(a).

Solution:

$$\frac{1}{3}\int_{-\infty}^{t} i(t)\, dt + 1i(t) + 1i(t)) + \frac{1}{6}\int_{-\infty}^{t} i(t)dt = 0$$

or $$\frac{1}{3}\int_{0}^{t} i(t)dt - 20 + 2i(t) + \frac{1}{6}\int_{0}^{t} i(t)dt + 10 = 0 \qquad ...(i)$$

Taking Laplace transform

$$\frac{1}{3s}I(s) - \frac{20}{s} + 2I(s) + \frac{1}{6s}I(s) + \frac{10}{s} = 0$$

or $$\left(\frac{1}{3s} + \frac{1}{6s}\right)I(s) + 2I(s) = \frac{10}{s} \qquad \ldots(ii)$$

or $$\frac{1}{2s}I(s) + 2I(s) = \frac{10}{s}$$

or $$I(s) = \frac{20}{4s+1}$$

$$= \frac{5}{s+0.25}$$

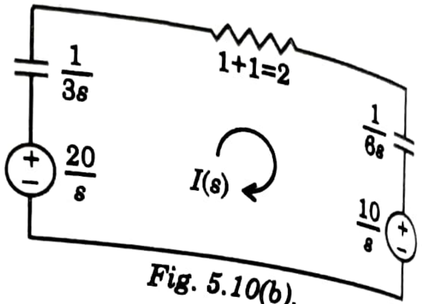


Fig. 5.10(b).

Therefore, $i(t) = \pounds^{-1}[I(s)] = 5\,e^{-0.25\,t}$ A

The transformed circuit from equation (*ii*) is shown in figure 5.10(*b*).

Example 5.14. At $t = 0$, S is closed in the circuit of figure 5.11 find $v_c(t)$ and $i_c(t)$. All initial conditions are zero.

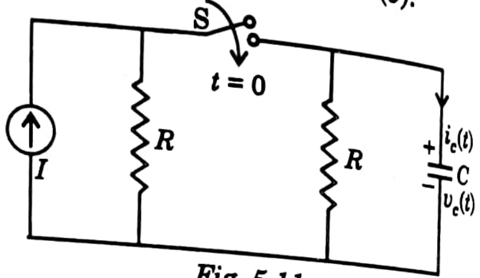


Fig. 5.11.

Solution : Applying KCL,

$$I = \frac{v_c(t)}{R} + \frac{v_c(t)}{R} + C\frac{dv_c(t)}{dt}$$

Taking Laplace transform,

$$\frac{I}{s} = \frac{V_c(s)}{R} + \frac{V_c(s)}{R} + C[sV_c(s) - v_c(0^+)]$$

As $$v_c(0^+) = 0 \text{ (given)}$$

$$V_c(s) = \frac{I}{Cs}\left[\frac{1}{s + \frac{2}{RC}}\right] = \frac{I}{C}\left[\frac{1}{s\left(s + \frac{2}{RC}\right)}\right]$$

$$\text{Therefore, } v_c(t) = \pounds^{-1}[V_c(s)] = \frac{I}{C}\pounds^{-1}\left[\frac{RC}{2}\cdot\frac{1}{s} - \frac{RC}{2}\cdot\frac{1}{s + \frac{2}{RC}}\right]$$

$$v_c(t) = \frac{IR}{2}\left(1 - e^{-\frac{2}{RC}t}\right)$$

And $$i_c(t) = C\frac{d}{dt}[v_c(t)] = C\cdot\frac{IR}{2}\left[0 - \left(\frac{-2}{RC}\right)e^{-\frac{2}{RC}t}\right]$$

or $$i_c(t) = I.e^{-\frac{2}{RC}t}\ \text{A}$$

Example 5.15. In the circuit of figure 5.12, S_1 is closed at $t = 0$, and S_2 is opened at $t = 4$ msec. Determine $i(t)$ for $t > 0$.

(Assume inductor is initially de-energised)

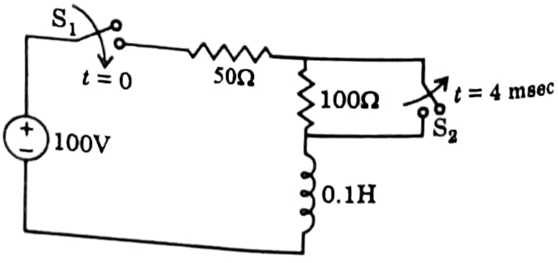


Fig. 5.12.