Solution of a system of linear equations

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

In matrix notation, these equations can be written as

$$AX = B$$

where, A, X and B are co-efficient matrix, column matrix of unknowns and column matrix of constants respectively.

A system is said to be homogeneous system of linear equations if $B=O$, otherwise, system is known as a system of non-homogeneous linear equations.

For a system of non-homogeneous linear equations $AX=B$

(i) if $\rho[A:B] \neq \rho(A)$, the system is inconsistent.

(ii) if $\rho[A:B] = \rho(A) =$ no. of unknowns, the system has a unique solution.

(iii) if $\rho[A:B] = \rho(A) <$ no. of unknowns, the system has an infinite no. of solutions.

The matrix $[A:B]$ in which the elements of A & B are written side by side is called the augmented matrix.

For a system of homogeneous linear equations
$AX = O$

(i) $X=O$ is always a solution. This solution in which each unknown has the value zero is called the Null Solution or the Trivial Solution. Thus a homogeneous system is always consistent.

(ii) If $\rho(A)$ = no. of unknowns, the system has only the trivial solution.

(iii) If $\rho(A) <$ no. of unknowns, the system has an infinite number of non-trivial sol^{n}s.

Note:- If A is a non-singular matrix, then the matrix equation $AX=B$ has a unique solution

The given equation is $AX=B$
Write the augmented matrix $[A:B]$.
By E-row operations on A and B, reduce A to a diagonal matrix, thus getting

$$[A:B] \sim \left[\begin{array}{ccc:c} p_1 & 0 & 0 & q_1 \\ 0 & p_2 & 0 & q_2 \\ 0 & 0 & p_3 & q_3 \end{array}\right]$$

Then $x = \dfrac{q_1}{p_1}$, $y = \dfrac{q_2}{p_2}$, $z = \dfrac{q_3}{p_3}$

Problem 1:- Solve the given system of eq^{n}s

$$x+y+z=3$$
$$x+2y+3z=4$$
$$x+4y+9z=6$$

Solution:- Augmented matrix $[A:B]$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]$$

operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right]$$

operating $R_3 \to R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

operating $R_3 \to \frac{1}{2} R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 2R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

operating $R_1 \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore \quad x = 2, \quad y = 1, \quad z = 0$

Problem 2:- Show that the equations

$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

are consistent and hence obtain the solutions for x, y and z.

Solution:- Augmented matrix $[A : B]$

$$\left[\begin{array}{ccc:c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

operating $R_1 \longleftrightarrow R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

operating $R_2 \rightarrow R_2 - 3R_1$, $R_4 \rightarrow R_4 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

operating $R_3 \rightarrow R_3 + 3R_2$, $R_4 \rightarrow R_4 - 2R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 1 & 9 & -35 \\ 0 & -1 & -5 & 19 \end{array}\right]$$

operating $R_1 \rightarrow R_1 - 2R_3$, $R_2 \rightarrow R_2 + 3R_3$, $R_4 \rightarrow R_4 + R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -18 & 74 \\ 0 & 0 & 29 & -116 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 4 & -16 \end{array}\right]$$

operating $R_2 \longleftrightarrow R_3$, $R_4 \longleftrightarrow \frac{1}{4} R_4$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -18 & 74 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

operating $R_1 \rightarrow R_1 + 18R_4$, $R_2 \rightarrow R_2 - 9R_4$,
$R_3 \rightarrow R_3 - 29R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & : & -4 \end{bmatrix}$$

operating $R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & -4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho(A) = \rho(A:B) = 3 =$ no. of unknowns

$\Rightarrow$ The given system of equations is consistent and the unique solution is

$x = 2,\ y = 1,\ z = -4$

Problem 3:- For what values of parameters λ and μ do the system of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (i) no solution

(ii) unique solution

(iii) more than one solution?

Solution:- In matrix notation, the given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Augmented matrix $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$

operating $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix}$$

operating $R_1 \to R_1 - R_2$, $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 & : & 2 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$$

Case I   If $\lambda = 3$, $\mu \neq 10$

$\rho(A) = 2$, $\rho(A:B) = 3$

$\because$ $\rho(A) \neq \rho(A:B)$

$\therefore$ The system has no solution.

Case II   If $\lambda \neq 3$, μ may have any value

$\rho(A) = \rho(A:B) = 3 =$ no. of unknowns

$\therefore$ The system has unique solution.

Case III   If $\lambda = 3$, $\mu = 10$

$\rho(A) = \rho(A:B) = 2 <$ no. of unknowns

$\therefore$ The system has an infinite no. of solutions

Problem 4   Solve the system of equations

$$x_1 + 3x_2 + 2x_3 = 0$$
$$2x_1 - x_2 + 3x_3 = 0$$
$$3x_1 - 5x_2 + 4x_3 = 0$$
$$x_1 + 17x_2 + 4x_3 = 0$$

Solution   In matrix notation, the given system of equations can be written as $AX = 0$

where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

operating $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$$

operating $R_3 \to R_3 - 2R_2$, $R_4 \to R_4 + 2R_2$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

operating $R_1 \to R_1 + 2R_2$

$$\sim \begin{bmatrix} 1 & -11 & 0 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 <$ no. of unknowns

$\Rightarrow$ The system has an infinite no. of non-trivial solutions given by

$$x_1 - 11x_2 = 0$$
$$-7x_2 - x_3 = 0$$

i.e., $x_1 = 11k$, $x_2 = k$, $x_3 = -7k$, where k is arbitrary.

Problem 5 Show that the system of equations

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$
$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$
$$-x_1 + 2x_2 = \lambda x_3$$

can possess a non-trivial solution only if $\lambda = 1$ or -3. Obtain the general solution in each case.

Solution The given system of equation is

$$(2-\lambda)x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - (3+\lambda)x_2 + 2x_3 = 0$$
$$-x_1 + 2x_2 - \lambda x_3 = 0$$

In matrix notation, it can be written as $AX = 0$

where $A = \begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

For non-trivial solution, $|A| = 0$

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 + \lambda^2 - 5\lambda + 3 = 0$ (on simplification)

$\Rightarrow (\lambda - 1)^2(\lambda + 3) = 0$

$\lambda = 1$ or -3

When $\lambda = 1$, the equations become

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - 4x_2 + 2x_3 = 0$$
$$-x_1 + 2x_2 - x_3 = 0$$

which are identical.

The given system is equivalent to a single equation

$$x_1 - 2x_2 + x_3 = 0$$

Taking $x_2 = t$, $x_3 = s$, we get $x_1 = 2t - s$

$\therefore x_1 = 2t - s$, $x_2 = t$, $x_3 = s$

which give an infinite no. of non-trivial solutions, t and s being the parameters.

When $\lambda = -3$, the equations become

$$5x_1 - 2x_2 + x_3 = 0$$
$$2x_1 + 2x_3 = 0$$
$$-x_1 + 2x_2 + 3x_3 = 0$$

Solving the first two, we have

$$\frac{x_1}{-4} = \frac{x_2}{2-10} = \frac{x_3}{4} \text{ or } x_1 = \frac{x_2}{2} = \frac{x_3}{-1}$$

$\therefore x_1 = t$, $x_2 = 2t$, $x_3 = -t$

which give an infinite no. of non-trivial solutions, t being the parameter.

Exercise

1) Examine if the following equations are consistent and solve them if they are consistent:

(i) $2x-3y+7z=5$
$3x+y-3z=13$
$2x+19y-47z=32$

Sol^n Inconsistent

(ii) $2x+6y+11=0$
$6x+20y-6z+3=0$
$6y-18z+1=0$

Sol^n Inconsistent

(iii) $x+2y+z=3$
$2x+3y+2z=5$
$3x-5y+5z=2$
$3x+9y-z=4$

Sol^n consistent
$x=-1,\ y=1,\ z=2$

(iv) $5x+3y+7z=4$
$3x+26y+2z=9$
$7x+2y+10z=5$

Sol^n consistent
$x=\frac{1}{11}(7-16k),\ y=\frac{1}{11}(3+k),$
$z=k$, where k is arbitrary

2) For what values of a and b do the equations
$x+2y+3z=6$
$x+3y+5z=9$
$2x+5y+az=b$ have

(i) no solution
(ii) a unique solution
(iii) more than one solution?

Sol^n (i) $a=8,\ b\neq 15$
(ii) $a\neq 8$, b may have any value
(iii) $a=8,\ b=15$

3) Investigate for consistency the following equations and if possible, find the solutions

$4x-2y+6z=8$
$x+y-3z=-1$
$15x-3y+9z=21.$

Sol^n consistent
$x=1,\ y=3k-2,$
$z=k$, where k is arbitrary

4) Solve the system of equations

$$x+y+z+w=0$$
$$x+3y+2z+w=0$$
$$2x+z-w=0$$

Sol^n $x=k,\ y=k,$
$z=-2k,\ w=0,$
where k is arbitrary