Unit III

System of Linear Algebraic Eq^n.

Let us consider the following system of eq^{n}s.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

This is called a system of n algebraic eq^{n}s which is linear in $x_1, x_2 \dots x_n$. Here a_{ij} & b_i, $i = 1 \dots n$, $j = 1 \dots n$ are constants & x_i's are unknowns. This system can be written in matrix form as

$$AX = B \quad \text{where}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

A is called coefficient matrix.

Existence of the solution –

The matrix $[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right]$

is called Augmented matrix.

1) If rank(A) = rank(aug A) = no. of unknowns, then solution is unique
2) If rank(A) = rank(Aug A) < no of unknowns, infinitely many solutions exists
3) If rank(A) ≠ rank(Aug A), no solution exists for the system.

Def :

① Symmetric matrix — A matrix A is symmetric if

$$A^T = A \quad \text{or} \quad [a_{ij}] = [a_{ji}]$$

If $A = -A^T$, then matrix is called skew symmetric.

② Lower triangular / upper triangular matrix —
A matrix in which all elements above its main diagonal are zero, its called lower triangular matrix. On the other hand, if all elements below the main diagonal are zero, the matrix is called upper triangular matrix.
If the diagonal elements are '1's, the matrices are called unit lower triangular matrix & unit upper triangular matrix.

3) Orthogonal matrix — A matrix is called orthogonal if

$$A^T = A^{-1} \quad \text{or} \quad AA^T = A^TA = I$$

e.g $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

4) Unitary matrix — A matrix A is unitary if $A^* = A^{-1}$
where $A^* = (\bar{A})^T$

5) Positive definite matrix — A matrix A is said to be positive definite if for any non zero column vector x;

$$x^TAx > 0 .$$

If all the leading principal minors are +ve, the matrix will be positive definite.

Gauss Elimination method —

We consider the eqns

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad — ①$$

Step 1: To eliminate x, from second & third equations

Assuming $a_1 \neq 0$, we eliminate x from the second eq. by subtracting (a_2/a_1) times the first equation from second equation. Similarly we eliminate x from the third equation by eliminating (a_3/a_1) times the first eq. from third equation.

We thus, get the new system

$$② - \frac{a_2}{a_2} \times ①$$

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned} \quad —②$$

$$③ - \frac{a_3}{a_1} \times ①$$

The first eq is called the pivotal eq. & a_1 is called the first pivot.

Step 2: To eliminate y from third equation in ②.:

Assuming $b_2' \neq 0$, we eliminate y from the third eq of ② by subtracting (b_3'/b_2') times the second eq from third eq.

We thus get the new system

$$③' - ②' \times \frac{b_3'}{b_2'}$$

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2 \\ c_3''z &= d_3'' \end{aligned} \quad —③$$

Here the second eq is the pivotal eq & b_2' is the new pivot.

Step 3: To evaluate the unknowns:

The values of x, y, z are found from the reduced system ③ by back substitution.

Observations —

① The method will fail if any one of the pivots a_1, b_2', c_3'' becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non zero.

② The eq with largest coefficient of x should be taken as first eq. Now to choose second eq, we see the largest coefficient of y from remaining eqs. This eq is second equation. Then the coefficient of z will be seen and so on. This procedure is called partial pivoting.

eg: Apply Gauss Elimination method to solve the eqs.

$$x+4y-z=-5$$
$$x+y-6z=-12$$
$$3x-y-z=4$$

Using partial pivoting, we get

$$3x-y-z=4$$
$$x+4y-z=-5$$
$$x+y-6z=-12$$

step 1: Multiply first eq by $\frac{1}{3}$ & subtract from second equation and third eq.

$$3x-y-z=4$$
$$\frac{13}{3}y-\frac{2}{3}z=-\frac{19}{3}$$
$$\frac{4}{3}y-\frac{17}{3}z=-\frac{40}{3}$$

Step 2: Multiply second eq. by $\frac{4}{13}$ & subtract from third eq.

$$3x-y-z=4$$
$$\frac{13}{3}y-\frac{2}{3}z=-\frac{19}{3}$$
$$-\frac{213}{39}z=\frac{-444}{39}$$
$$z=\frac{444}{213}=\frac{148}{71}.$$

$$-\frac{\cancel{243}^{81}}{\cancel{213}_{71}}$$

$$z=\frac{148}{71}$$

Step 3: By Back substitution.

$$\frac{13}{3}y=-\frac{19}{3}+\frac{2}{3}z=-\frac{19}{3}+\frac{2}{3}\times\frac{148}{71}=-\frac{19}{3}+\frac{296}{213}$$
$$=\frac{-1349+296}{213}$$

$$y=-\frac{\cancel{1053}^{81}}{\cancel{213}_{71}}\times\frac{\cancel{3}}{\cancel{13}}=-\frac{81}{71}.$$

$$3x = 4 + z + y$$
$$= 4 + \frac{148}{71} - \frac{81}{71} = \frac{284+148-81}{71}$$

$$3x = \frac{351}{71}$$

$$x = \frac{\cancel{351}^{117}}{71} \times \frac{1}{\cancel{3}} = \frac{117}{71}.$$

$$-5 + \frac{148}{71} + 4 \times \frac{81}{71}$$
$$\frac{-5\times 71 + 148 + 324}{71}$$
$$= -35\,5 + 148 + 324$$

$$-355 \quad \begin{array}{r} 472 \\ 355 \\ \hline 117 \end{array}$$

eg:
$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16.$$

Using partial pivoting
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$
$$2x + y + z = 10$$

Step 1: Elimination of x.
Multiply 1st eq by $\frac{1}{3}$ subtract from second &
" " " by $\frac{2}{3}$ " " third.

$$3x + 2y + 3z = 18$$
$$\frac{10}{3}y + 8z = 10$$
$$-\frac{1}{3}y - z = -2.$$

Step 2: $\left[(iii) + \frac{1}{10}(ii)\right]$, we get

$$3x + 2y + 3z = 18$$
$$\frac{10}{3}y + 8z = 10$$
$$-\frac{1}{5}z = -1$$

Step 3: Using back substitution

$$z = 5$$

$\frac{10}{3}y = 10 - 8z = 10 - 8(5) = 10 - 40 = -30$

$y = -9$

$3x = 18 - 3z - 2y$

$= 18 - 3(5) - 2(-9) = 21$

$x = 7$

Concept of pivoting —

In Gauss elimination method suppose we have to solve

$$2x_2 + 3x_3 = 8$$
$$4x_1 + 6x_2 + 7x_3 = -3$$
$$2x_1 + x_2 + 6x_3 = 5$$

Here $a_{11} = 0$, which creates a problem of division by zero. Same problem arise when a coefficient is very close to zero. The technique of pivoting has been developed to partially avoid these problems.

To overcome this, we determine the largest available coefficient in the column below the pivot element. The row can then be switched so that the largest element is the pivot element. This is called partial pivoting.

It is used to avoid roundoff errors that could be caused when dividing every entry of a row by a pivot value that is relatively small in comparison to its remaining row entries.

Gauss Jordan Method – This is modification of Gauss elimination method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also, ultimately reducing the system to a diagonal matrix form. From this the unknowns x, y, z can be obtained easily.

Let the system be

$$a_{11}x + b_{12}y + c_{13}z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$$

Steps: 1) Make $a_{11} = 1$ using either R_2 or R_3

2) Make a_{21} & a_{31} zero using a_{11}

3) Make $a_{22} = 1$ by division or using R_3

4) Make a_{12} & a_{32} zero using a_{22}.

5) Make $a_{33} = 1$

6) Make a_{13} & a_{23} zero using a_{33}.

Example: Solve the system of Eq. using Gauss Jordan method

$$2x_1 + 4x_2 - 6x_3 = -8$$
$$x_1 + 3x_2 + x_3 = 10$$
$$2x_1 - 4x_2 - 2x_3 = -12$$

Solu:

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -8 \\ 1 & 3 & 1 & 10 \\ 2 & -4 & -2 & -12 \end{array}\right]$$

$R_1 \to R_1/2$ we get

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -4 \\ 1 & 3 & 1 & 10 \\ 2 & -4 & -2 & -12 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -4 \\ 0 & 1 & 4 & 14 \\ 0 & -8 & 4 & -4 \end{array}\right]$$

$a_{22} = 1$ already $\therefore$

$R_1 \to R_1 - 2R_2$

$R_3 \to R_3 + 8R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -11 & -32 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 36 & 108 \end{array}\right]$$

$R_3 \to R_3/36$

$$\left[\begin{array}{ccc|c} 1 & 0 & -11 & -32 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 \to R_1 + 11R_3$

$R_2 \to R_2 - 4R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\therefore \; x_1 = 1 \quad x_2 = 2 \quad x_3 = 3.$

Factorization methods – A square matrix A can be decomposed/factorised into a product of two matrices L & U ie $A = LU$, where L & U are lower & upper triangular matrices respectively.

Doolittle's method –

Consider the eqns.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad (1)$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Then we have $LUX = B$

Taking $UX = V$ we get

$$LV = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$v_1 = b_1$$
$$l_{21}v_1 + v_2 = b_2$$
$$l_{31}v_1 + l_{32}v_2 + v_3 = b_3$$

Solving this for v_1, v_2, v_3 we get $UX = V$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

from this, we obtain x_1, x_2, x_3.

Now to compute L & U, we have $A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\therefore$ $u_{11} = a_{11}$ $\quad u_{12} = a_{12}$ $\quad u_{13} = a_{13}$

$l_{21} u_{11} = a_{21}$ $\quad l_{21} u_{12} + u_{22} = a_{22}$ $\quad l_{21} u_{13} + u_{23} = a_{23}$

$l_{21} = \dfrac{a_{21}}{a_{11}}$ $\quad u_{22} = a_{22} - l_{21} u_{12}$ $\quad u_{23} = a_{23} - l_{21} u_{13}$.

Now IIIrd row

$l_{31} u_{11} = a_{31}$ $\quad l_{31} u_{12} + l_{32} u_{22} = a_{32}$ $\quad l_{31} u_{13} + l_{32} u_{23} + u_{33} = a_{33}$

$l_{31} = \dfrac{a_{31}}{u_{11}}$ $\quad l_{32} = \dfrac{a_{32} - l_{31} u_{12}}{u_{22}}$ $\quad u_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23}$

Thus we compute the elements of L & U in the following order -

1) first row of U
2) first Column of L
3) second row of U
4) second column of L
5) Third row of U.

eg: Solve the eqns

$$3x + 2y + 7z = 4$$
$$2x + 3y + z = 5$$
$$3x + 4y + z = 7$$

Solu: Let $A = LU$

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

R_1 of U : $u_{11} = 3$ $\quad u_{12} = 2$ $\quad u_{13} = 7$

C_1 of L : $l_{21} u_{11} = 2$ $\quad l_{31} u_{11} = a_{31} = 3$

$l_{21} = 2/3$ $\quad l_{31} = 3/3 = 1$

R_2 of U : $l_{21} u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = 3 - \frac{2}{3} \times 2 = \frac{5}{3}$

$l_{21} u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = 1 - \frac{2}{3} \times 7 = -11/3$

C_2 of L : $l_{32} = \frac{a_{32} - l_{31} u_{12}}{u_{22}} \Rightarrow l_{32} = \frac{4 - 1 \times 2}{5/3} = \frac{6}{5}$

R_3 of U : $u_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23}$

$= 1 - 1 \times 7 - \frac{6}{5} \times \left(\frac{-11}{3}\right) = 1 - 7 + \frac{22}{5} = -\frac{8}{5}$.

$\therefore$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

writing $UX = V$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

solving $v_1 = 4$ $\quad \frac{2}{3} v_1 + v_2 = 5$ or $v_2 = 7/3$

$v_1 + \frac{6}{5} v_2 + v_3 = 7$ or $v_3 = 1/5$

Hence original system becomes.

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}.$$

$$3x + 2y + 7z = 4$$

$$\frac{5}{3}y - \frac{11}{3}z = \frac{7}{3}$$

$$-\frac{8}{5}z = \frac{1}{5}$$

By back substitution, we get

$$z = -1/8 \quad y = 9/8 \quad \& \quad x = 7/8$$

Crout's Algorithm – In crout's Algorithm the matrix A can be expressed as the product of lower & upper triangular matrices as. $A = LU$, where 2

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

The system of eqs $AX = B$ can be now written as

We consider $LUX = B$

$UX = Y$, where Y is an unknown matrix.

Then $LY = B$

After finding Y, we write $UX = Y$ & solve for X.

eg:

$$5x_1 + 4x_2 + x_3 = 3.4$$

$$10x_1 + 9x_2 + 4x_3 = 8.8$$

$$10x_1 + 13x_2 + 15x_3 = 19.2$$

Solve the system of eqns using crout's method.

Solu: we write

$$\begin{bmatrix} 5 & 4 & 1 \\ 10 & 9 & 4 \\ 10 & 13 & 15 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

We have $l_{11} = 5 \quad l_{21} = 10 \quad l_{31} = 10$

$l_{11} u_{12} = 4$

$\therefore u_{12} = \frac{4}{5}$

$l_{21} u_{12} + l_{22} = 9$

$l_{22} = 9 - l_{21} u_{12}$
$= 9 - 10 \times \frac{4}{5}$
$= 1$

$l_{31} u_{12} + l_{32} = 13$

$l_{32} = 13 - l_{31} u_{12}$
$= 13 - 10 \times \frac{4}{5}$
$= 5$

$l_{11} u_{13} + 0 = 1$

$u_{13} = \frac{1}{l_{11}} = \frac{1}{5}$

$l_{21} u_{13} + l_{22} u_{23} = 4$

$l_{22} u_{23} = 4 - l_{21} u_{13}$
$= 4 - 10 \times \frac{1}{5}$

$l_{22} u_{23} = 2$

$u_{23} = \frac{2}{l_{22}} = 2$

$l_{31} u_{13} + l_{32} u_{23} + l_{33} = 15$

$l_{33} = 15 - l_{31} u_{13} - l_{32} u_{23}$
$= 15 - 10 \times \frac{1}{5} - 5 \times 2$
$= 3$

$$\therefore \quad A = \begin{bmatrix} 5 & 0 & 0 \\ 10 & 1 & 0 \\ 10 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4/5 & 1/5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $LY = B$

$$\begin{bmatrix} 5 & 0 & 0 \\ 10 & 1. & 0 \\ 10 & 5 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 8.8 \\ 19.2 \end{bmatrix}$$

$5y_1 = 3.4 \quad y_1 = 0.68$

$10y_1 + y_2 = 8.8 \Rightarrow y_2 = 2$

$10y_1 + 5y_2 + 3y_3 = 19.2 \Rightarrow y_3 = 0.80$

$\therefore$ The given system can be written as $UX = Y$

$$\begin{bmatrix} 1 & 4/5 & 1/5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.68 \\ 2 \\ 0.80 \end{bmatrix}$$

$x_3 = 0.80 \qquad x_2 + 2x_3 = 2 \Rightarrow x_2 = 0.40$

$x_1 + \frac{4}{5} x_2 + \frac{1}{5} x_3 = 0.68 \Rightarrow x_1 = 0.20.$

Cholesky's method—

This method deals with a special case when the given matrix is symmetric & positive definite. If A is symmetric matrix, it can be expressed as product of two matrices L & L^T where L is a lower triangular matrix i.e

$$A = LL^T \text{ or } (U^TU). \quad \text{i.e}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

for the system $AX = B$, we get

$$LL^TX = B$$

We take $L^TX = Y$ so that

$$LY = B.$$

Solving for $LY = B$, we get the unknown vector Y

Using Y, we get $L^TX = Y$ & solve for X.

eg: solve

$$x_1 - x_2 + 2x_3 = 17$$
$$-x_1 + 5x_2 - 4x_3 = 31$$
$$2x_1 - 4x_2 + 6x_3 = -5$$

solu: Here $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{bmatrix}$. We take $A = LL^T$

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$l_{11}^2 = 1$ $\qquad$ $l_{11} l_{21} = -1$ $\qquad$ $l_{11} l_{31} = 2$

$\therefore l_{11} = 1$ $\qquad$ $l_{21} = -\frac{1}{1} = -1$ $\qquad$ $l_{31} = \frac{2}{1} = 2$

$l_{21} l_{11} = -1$

$l_{21} = -1$

$l_{21}^2 + l_{22}^2 = 5$

$l_{22}^2 = 5 - (-1)^2$

$= 4$

$l_{22} = 2$

$l_{21} l_{31} + l_{22} l_{32} = -4$

$l_{32} = \frac{-4 - l_{21}\cdot}{l_{22}}$

$= \frac{-4 + 1 \times 2}{2}$

$= \frac{-4+2}{2} = -1$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 6 \quad \therefore l_{33} = 1$

$\therefore$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{L^T}$$

Now

$$LL^T X = B$$

Let $L^T X = Y$ then $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 17 \\ 31 \\ -5 \end{bmatrix}$$

$y_1 = 17$

$-y_1 + 2y_2 = 31 \Rightarrow y_2 = 24$

$2y_1 - y_2 + y_3 = -5 \Rightarrow y_3 = -15$

Now $L^T X = Y$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ 24 \\ -15 \end{bmatrix}$$

$x_1 - x_2 + 2x_3 = 17 \Rightarrow x_1 = 51.5$

$2x_2 - x_3 = 24 \Rightarrow x_2 = 4.5$

$x_3 = -15$

By back substitution.

Eigen Value problem –

Power method – This method computes the numerically largest eigen value & the corresponding eigen vector of a matrix.

Let $X_1, X_2 \dots X_n$ be eigen vectors of a matrix A, corresp-onding to eigen values $\lambda_1, \lambda_2 \dots \lambda_n$; then any arbitrary column vector can be written as

$$X = k_1 X_1 + k_2 X_2 + \dots + k_n X_n$$
$$AX = k_1 AX_1 + k_2 AX_2 + \dots + k_n AX_n$$
$$= k_1 \lambda_1 X_1 + k_2 \lambda_2 X_2 + \dots + k_n \lambda_n X_n$$

Then $A^r X = k_1 \lambda_1^r X_1 + k_2 \lambda_2^r X_2 + \dots + k_n \lambda_n^r X_n$

(If $\lambda_1, \lambda_2 \dots \lambda_n$ are eigen values of A, then $\lambda_1^p, \lambda_2^p \dots \lambda_n^p$ are eigen values of A^p).

As r increases, & if $|\lambda_1| > |\lambda_2| > \dots |\lambda_n|$, then the contribution of term $k_1 \lambda_1^r X_1$ to the sum on the right increases. So every time we increase r, the vector X becomes nearer to X_1 (if X_1 is an eigen vector then $(k_1 X_1)$ is also eigen vector). Finally we divide by k_1. The resulting vector is largest eigen vector.

To start the process, we take up a column vector X which is as near the solution as possible & evaluate AX which is written as $\lambda^{(1)} X^{(1)}$ after normalization. This gives the first approximation $\lambda^{(1)}$ to the eigen value & $X^{(1)}$ to the eigen ~~value~~ vector. Similarly, we evaluate $AX^{(1)} = \lambda^{(2)} X^{(2)}$ which gives IInd approximation. We repeat this process till $[X^{r} - X^{r-1}]$ becomes negligible. Then $\lambda^{(r)}$ will be the largest eigen value & $X^{(r)}$, the corresponding eigen vector.

Note : Since we have

$$A^{-1}X = \frac{1}{\lambda}X$$

This equation, if we use, will yields the smallest eigen value.

eg: Determine the largest eigen value & the corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solu: Let the initial vector be $[1,0,0]'$. Then

$$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$$

So the first approximation to the eigen value is 2 & eigen vector is $X^{(1)} = [1, -0.5, 0]'$

Now $$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

Repeating the process taking $X^{(2)} = \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$

$$AX^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.428 \end{bmatrix}$$

$$X^{(3)} = \begin{bmatrix} 1 \\ -1 \\ 0.428 \end{bmatrix}.$$

$$AX^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0.428 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3.428 \\ 1.856 \end{bmatrix} = 3.428 \begin{bmatrix} 0.875 \\ -1 \\ 0.541 \end{bmatrix}$$

$$X^{(4)} = \begin{bmatrix} 0.875 \\ -1 \\ 0.541 \end{bmatrix}$$

$$AX^{(4)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.875 \\ -1 \\ 0.541 \end{bmatrix} = \begin{bmatrix} 2.75 \\ -3.416 \\ 2.082 \end{bmatrix} = 3.416 \begin{bmatrix} 0.80 \\ -1 \\ 0.609 \end{bmatrix}$$

$$|||y \quad X^{(5)} = \begin{bmatrix} 0.80 \\ -1 \\ 0.609 \end{bmatrix} \qquad X^{(6)} = \begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} \ldots\ldots.$$

$$AX^{(5)} = \underbrace{\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix}}_{X^{(6)}} \underbrace{0.341}_{\lambda^{(6)}} \qquad AX^{(6)} = \underbrace{3.\bar{41}}_{\lambda^{(7)}} \underbrace{\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix}}_{X^{(7)}}$$

$\lambda^{(6)}$ & $\lambda^{(7)}$ are same & $X^{(6)}$ & $X^{(7)}$ are approximately same so we take largest eigen value as 3.41 & corresponding eigen vector as $\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix}$.