CM (UNIT 3-4)

→ EXISTENCE OF SOLUTIONS

First we will find the Rank of the Matrix,

First apply Gauss Elimination Method Steps (both) to matrix and then no. of completely non-zero rows is the Rank of matrix.

Now, if $P(A) \neq P(A|B) \Rightarrow$ No Solution

$R = \underbrace{P(A)}_{\text{Rank of 'A' matrix}} = \underbrace{P(A|B)}_{\text{Augmented Matrix}} \Rightarrow$ Solution

$R = n$ ⇒ Unique Solution

$R < n$ ⇒ Infinitely many solution

I. GAUSS ELIMINATION METHOD

Q. $x - y + 2z = 3$ — ①

$x + 2y + 3z = 5$ — ②

$3x - 4y - 5z = -13$ — ③

A. write it in form of matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix}$$

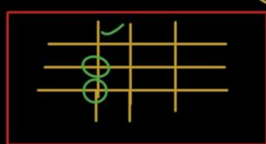
→ $AX = B$

$$(A : B) = \left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 3 & -4 & -5 & -13 \end{array}\right]$$

→ Now, we will apply row transformation converting 'O' to zero using '✓'.

∴ In above case we will convert 3 & 1 into 0 using '1'.

Step 1 →

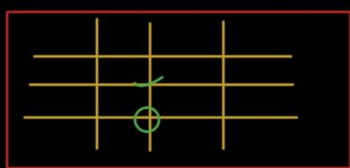


$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & -1 & -11 & -22 \end{array}\right]$$

Step 2 →



$R_3 \rightarrow 3R_3 + R_2$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -32 & -64 \end{array}\right]$$

Now, we will write it again in eq. form → $x - y + 2z = 3$ — ①

Now, we will write it again in eq. form ⇒ $x - y + 2z = 3$ — ①

$3y + z = 2$ — ②

$-32z = -64$ — ③

⇒ $\boxed{z = 2}$ ⇒ $\boxed{y = 0}$ ⇒ $\boxed{x = -1}$

II. GAUSS JORDAN METHOD

Q. $x + 4y + 9z = 16$ — ①

$2x + y + z = 10$ — ②

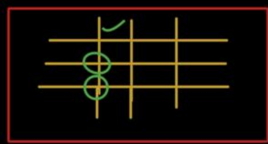
$3x + 2y + 3z = 18$ — ③

A. write it in form of matrix ⇒
$$\begin{bmatrix} 1 & 4 & 9 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \\ 18 \end{bmatrix}$$

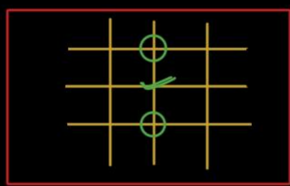
$$\rightarrow AX = B$$

$$(A : B) = \begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 2 & 1 & 1 & : & 10 \\ 3 & 2 & 3 & : & 18 \end{bmatrix}$$

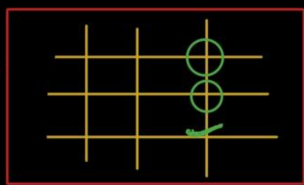
Step 1 →



Step 2 →



Step 3 →



$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$
$$= \begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 0 & -7 & -17 & : & -22 \\ 0 & -10 & -24 & : & -30 \end{bmatrix}$$

$R_3 \rightarrow 7R_3 - 10R_2$, $R_1 \rightarrow 7R_1 + 4R_2$
$$= \begin{bmatrix} 7 & 0 & -5 & : & 24 \\ 0 & -7 & -17 & : & -22 \\ 0 & 0 & 2 & : & 10 \end{bmatrix}$$

$R_1 \rightarrow 2R_1 + 5R_3$, $R_2 \rightarrow 2R_2 + 17R_3$
$$= \begin{bmatrix} 14 & 0 & 0 & : & 98 \\ 0 & -14 & 0 & : & 126 \\ 0 & 0 & 2 & : & 10 \end{bmatrix}$$

⇒ $14x = 98$ ⇒ $\boxed{x = 7}$

$-14y = 126$ ⇒ $\boxed{y = -9}$

$2z = 10$ ⇒ $\boxed{z = 5}$

LU DECOMPOSITION METHOD

Let
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Here $AX = B$ — ①

⇒ $A = LU = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}\begin{bmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{bmatrix}$

Lower △

Upper △

LU DECOMPOSITION METHOD

Let
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Here $AX = B$ — ①

$$\Rightarrow A = LU = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}\begin{bmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{bmatrix}$$

Lower Δ

Upper Δ

Putting $A = LU$ in ①, we get $L(UX) = B$

$$L\,Y = B$$

I. DOLITTLE'S METHOD

$\boxed{A = LU}$ where
$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}$$

II. CROUT'S FACTORISATION METHOD

$\boxed{A = LU}$ where
$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}$$

III. CHOLESKY METHOD → only useful for symmetric method

$\boxed{A = L\underbrace{L^T}_{U}}$ where
$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \quad \text{and} \quad U\ (L^T) = \begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$$

POWER METHOD

Q. Find the numerically larger Eigen value of the matrix by power method.
$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$$

A. Let us take initial value as $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Iteration 1 :-
$$AX_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 14 \end{bmatrix}$$
→ Taking largest value common

∴ At the end of iteration 1, the largest eigen value = 14 and

$$\Rightarrow 14\begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

iteration 1, the largest eigen value = 14 and eigen vector = $\begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}$ $\Rightarrow 14 \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}$

<u>Iteration 2</u> :-

$$AX_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 6.5 \end{bmatrix} \Rightarrow 6.5 \begin{bmatrix} 0.07692 \\ 0.15384 \\ 1 \end{bmatrix}$$

<u>Iteration 3</u> :-

$$AX_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.07692 \\ 0.15384 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.6454 \\ -0.07696 \\ 5.92304 \end{bmatrix} \Rightarrow 5.92304 \begin{bmatrix} 0.07692 \\ 0.15384 \\ 1 \end{bmatrix}$$

$\therefore$ Largest Eigen Value = 5.92304

Eigen Vector = $\begin{bmatrix} 0.07692 \\ 0.15384 \\ 1 \end{bmatrix}$

And solve so on!

☆ we will solve till the given iteration (if mentioned) or if not given then will solve till 2 consecutive readings of Eigen value.

PICARD METHOD

$$\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx}$$

Q. Solve by picard method (upto 3rd approximation) $\frac{dy}{dx} = x + y^2$; $y(0) = 0$

Also find $y(0.1)$.

A. $f(x, y) = x + y^2$ | $x_0 = 0 \quad y_0 = 0$ ①

$\left[\because y(0) = 0 \Rightarrow \text{whole } y \text{ value} = 0 \text{ and } y \text{ of } x \text{ value} = 0 \right]$

Now, in Picard formula, put $n = 0$

we have,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= y_0 + \int_{x_0}^{x} (x + y_0^2)\, dx \Rightarrow y_1 = 0 + \int_0^x (x + 0)\, dx \quad [\text{using } ①]$$

$$= y_0 + \int_{x_0}^{x} (x+y_0)^2 dx \Rightarrow y_1 = 0 + \int_0^x (x+0) dx \; [\text{using } 1]$$

$$\Rightarrow y_1 = \left(\frac{x^2}{2}\right)_0^x = \frac{x^2}{2}$$

Now, Put $n=1$, we have $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1) dx$

$$= y_0 + \int_{x_0}^{x} (x+y_1)^2 dx = 0 + \int_0^x \left(x + \left[\frac{x^2}{2}\right]^2\right) dx$$

$$= \frac{x^2}{2} + \frac{x^5}{20}$$

Put $n=2$ and solve futher till mentioned no. of approximations.

TAYLOR SERIES METHOD

$$\boxed{y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \ldots\ldots}$$

Q. Solve $\frac{dy}{dx} = x+y$ by Taylor Series Method. Start from $x=1$, $y=0$ and carry to $x = 1.2$ with $h = 0.1$.

A. $\frac{dy}{dx} = f(x,y) = x+y$

Also, $x_0 = 1$, $y_0 = 0$, $h = 0.1$

$\therefore x_1 = x_0 + h = 1.1$ and $y' = x+y$ (given)

$\because y' = x+y$	$y_0' = 1+0 = 1$
$\Rightarrow y'' = 1+y'$	$y_0'' = 1+1 = 2$
$\Rightarrow y''' = y''$	$y_0''' = 2$
$\Rightarrow y^{iv} = y'''$ and so on	$y_0^{iv} = 2$

By Taylor Series, $y_1 = y(x_1) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots\ldots$

$$y(1.1) = 0 + (0.1)(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4 \times 2}{24} \ldots\ldots$$

$$y(1.1) = 0.11034 \simeq \boxed{0.1103} = y_1$$

$$y(1.1) = 0 + (0.1)(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}\times 2 \ldots$$

$$y(1.1) = 0.11034 \simeq \boxed{0.1103} = y_1$$

Now, $x_1 = 1.1$, $y_1 = 0.1103$, $h = 0.1$

$\therefore x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

$y_1' = x_1 + y_1 = 1.1 + 0.1103 = 1.210$

$y_1'' = 1 + y_1' = 1 + 1.2103 = 2.2103$

$y_1''' = y_1'' = 2.2103$

$y_1^{iv} = y_1''' = 2.2103$

By Taylor Series, $y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \ldots$

$$= 0.24276 \simeq \boxed{0.2428} = y_2$$

EULER METHOD

Euler's Method is also called Runge Kutta method of first order whereas Euler's Modified Method is called Runge Kutta method of second order.

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

Q. Solve $\frac{dy}{dx} = x + y$; $\underbrace{y = 1 \text{ at } x = 0}$. Find approx. value of y for $x = 0.1$

$h = \frac{0.1}{5} = 0.02$ ($5 \rightarrow$ Let)

A.

$x_0 = 0$

$x_1 = 0.02$

$x_2 = 0.04$

$x_3 = 0.06$

$x_4 = 0.08$

$x_5 = 0.10$

$x_0 = 0$

$y_0 = 1$

$f(x, y) = x + y$

$y_{n+1} = y_n + h(x_n + y_n)$

Put $n = 0$,

$y_1 = y_0 + h(x_0 + y_0) = 1 + 0.02(0 + 1)$

$= \boxed{1.02}$

Put $n = 1$, $\Rightarrow y_2 = 1.0408$

Put $n = 2$, $\Rightarrow y_3 =$

And so on, we will calculate all 'y' corresponding to x means in this case till y_5 ($\because x_5$ is last iteration).

EULER'S MODIFIED METHOD

EULER's MODIFIED METHOD

$$\boxed{\begin{aligned} y^*_{n+1} &= y_n + h f(x_n, y_n) \\ y_{n+1} &= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right] \end{aligned}}$$

Q. Given $\frac{dy}{dx} = x^2 + y$ with $y(0) = 1$, find $y\underbrace{(0.02)}_{x_1 = 0.02}$ and $y\underbrace{(0.04)}_{x_2 = 0.04}$.

$\Rightarrow x_0 = 0$, $y_0 = 1$

A. $f(x,y) = x^2 + y$

$\Rightarrow h = x_2 - x_1 = 0.0$

Put $n=0$, we have

$$y_1^* = y_0 + h f(x_0, y_0)$$
$$= y_0 + h(x_0^2 + y_0) = 1 + 0.02(0^2 + 1)$$
$$= 1.02$$

$$\Rightarrow \quad y_1 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^*)\right)$$
$$= y_0 + \frac{h}{2}\left(x_0^2 + y_0 + x_1^2 + y_1^*\right)$$
$$= 1 + 0.01\left(0^2 + 1 + (0.02)^2 + 1.02\right) = \boxed{1.0202}$$

Similarly, we will calculate y_2 as well and then question complete!!

RUNGE KUTTA METHOD OF FOURTH ORDER

$$\boxed{\begin{aligned} K_1 &= h f(x_n, y_n) \\ K_2 &= h f\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right) \\ K_3 &= h f\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right) \\ K_4 &= h f(x_n + h, y_n + K_3) \\ K &= \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ \Rightarrow y_{n+1} &= y_n + K \end{aligned}}$$

Q. Given $\frac{dy}{dx} = x + y^2$, $\underbrace{y(0) = 1}$, Find $y(0.2)$ where $h = 0.1$.

$\Rightarrow x_0 = 0$ $\quad \because h = 0.1 \Rightarrow x_0 = 0$

A. $f(x,y) = x + y^2$

$y_0 = 1$ $\quad x_1 = 0.1$

Q. Given $\frac{dy}{dx} = x + y^2$, $y(0) = 1$, Find $y(0.2)$ where $h = 0.1$.

$\rightarrow x_0 = 0$, $y_0 = 1$

$\because h = 0.1 \Rightarrow x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$

A. $f(x,y) = x + y^2$

Put $n = 0$,

$$K_1 = h f(x_0, y_0) = h(x_0 + y_0^2) = 0.1(0+1) = 0.1$$

$$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = h\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{K_1}{2}\right)^2\right]$$

$$= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2\right] = 0.11525$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = h\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{K_2}{2}\right)^2\right] = 0.1169$$

$$K_4 = h f(x_0 + h, y_0 + K_3) = h\left[(x_0 + h) + (y_0 + K_3)^2\right] = 0.1347$$

$$\Rightarrow K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right] = 0.1165$$

$$\Rightarrow y_1 = y_0 + K = 1 + 0.1165 = \boxed{1.1165}$$

similarly, we will find y_2 value and that'll be ans $[y(0.2)]$.

MILNE'S PREDICTOR AND CORRECTOR METHOD

Consider $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$; $y(x_1) = y_1$; $y(x_2) = y_2$; $y(x_3) = y_3$

where x_0, x_1, x_2 & x_3 are equi-distant values of x with step size h.

$\therefore$ Milne's Predictor Formula $\rightarrow$ $y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

Milne's Corrector Formula $\rightarrow$ $y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^p)$

Q. Using Milne's Predictor Corrector Method, find y when $x = 0.8$, given

$\frac{dy}{dx} = x - y^2$; $y(0) = 0$; $y(0.2) = 0.02$; $y(0.4) = 0.0795$;

$y(0.6) = 0.1762$.

A. $f(x,y) = x - y^2$

x	y	$f = x - y^2$
$x_0 = 0$	$y_0 = 0$	$f_0 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$f_1 = 0.1996$

Here $h = 0.2$.

On Applying Milne's Predictor Formula, we have

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

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Aa

Q. Using Milne's Predictor Corrector Method, find y when $x = 0.8$, given

$\frac{dy}{dx} = x - y^2$; $y(0) = 0$; $y(0.2) = 0.02$; $y(0.4) = 0.0795$;

$y(0.6) = 0.1762$.

A. $f(x, y) = x - y^2$

x	y	$f = x - y^2$
$x_0 = 0$	$y_0 = 0$	$f_0 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$f_1 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$f_2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$f_3 = 0.5689$
$x_4 = 0.8$	$y_4^p = 0.3049$	$f_4^p = 0.707$
	$y_4^c = 0.3046$	

Here $h = 0.2$.

On Applying Milne's Predictor formula, we have

$$y_4^p = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$= 0 + \frac{4 \times 0.2}{3}\left(2 \times 0.1996 - 0.3937 + 2 \times 5689\right)$$

$$\Rightarrow \boxed{y_4^p = 0.3049}$$

☆ After this put this value in table & find f_4^p.

Now, using Milne's Corrector formula, we have

$$\boxed{y_4^c = 0.3046}$$

ADAM BASHFORTH PREDICTOR & CORRECTOR FORMULA

$$y_4^p = y_3 + \frac{h}{24}\left(55f_3 - 59f_2 + 37f_1 - 9f_0\right)$$

$$y_4^c = y_3 + \frac{h}{24}\left(9f_4^p + 19f_3 - f_2 + f_1\right)$$