Dr. Nishu Gupta

System of Linear Algebraic Equations:

These equations arise in electrical networks, mechanical frameworks etc.

A linear system of m equations in n unknowns $x_1, \dots, x_n$, is a set of equations of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (1)$$

It can be written as $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Homogeneous system: If all the $b_1, b_2, \dots, b_m$ are zero, then (1) is called a homogeneous system.

Nonhomogeneous system: If at least one b_i is not zero, then (1) is called a nonhomogeneous system.

Solution of System (1): A solution of system (1) is a set of numbers $x_1, x_2, \dots, x_n$ that satisfies all the m equations.

Remark: If the system (1) is homogeneous, it has at least the trivial solution $x_1 = 0, x_2 = 0, \dots, x_n = 0$

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Augmented Matrix:

$$[A:B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Elementary Row Operations for Matrices:

(a) Interchange of two rows $R_i \leftrightarrow R_j$ or R_{ij}

(b) Multiplication of a row by a nonzero constant k

$$R_i \to kR_i \quad (k \neq 0)$$

(c) Addition of a constant multiple of one row to another row

$$R_i \to R_i + kR_j \quad (k \neq 0)$$

Consistent and Inconsistent System:

A linear system (1) is called consistent if it has at least one solution (thus, one solution or infinitely many solutions), but inconsistent if it has no solution at all.

System of Equations

- Homogeneous
  - $AX = 0$
  - → Consistent (sol. exist)
- Non-homogeneous
  - $AX = B \; (B \neq 0)$
  - Consistent (sol. exist)
    - if rank $[A:B]$ = rank A
    - Unique solution
      - if rank $[A:B]$ = rank A = no. of unknowns
    - Infinitely many solutions exist
      - if rank $[A:B]$ = rank A $<$ no. of unknowns
  - Inconsistent (no solution exist)
    - if rank $[A:B] \neq$ rank A

Remark: (1) For homogeneous system if

rank $[A:B]$ = rank A = no. of unknowns (n), then only trivial solution exist i.e. $x_1 = 0, x_2 = 0, \ldots, x_n = 0$

(2) In case of infinitely many solutions i.e, when

$$\text{rank}[A:B] = \text{rank } A \; (r \text{ say}) < \text{no. of unknowns } (n)$$

we shall assign arbitrary values to $(n-r)$ unknowns and get the values of remaining r unknowns.

(3) rank of any matrix can be determined by the number of non-zero rows when the matrix is converted into Echelon form. (Already studied in first semester)

Naive Gauss Elimination method and its computational effort:

From the given system of equations (1), we write augmented matrix $[A:B]$. Element a_{11} of first column C_1 is called pivotal element and first equation as pivot equation. For each of the remaining equations $(2 \le i \le m)$, we compute $R_i \to R_i - \left(\frac{a_{i1}}{a_{11}}\right) R_1$

{ Here the quantities $\frac{a_{i1}}{a_{11}}$ are multipliers

The new coefficients of x_1 in the ith equation will be 0 because $a_{i1} - \left(\frac{a_{i1}}{a_{11}}\right) a_{11} = 0 \quad (2 \le i \le m)$

Repeat the process, now taking second equation as pivot equation and second component of second column C_2 to be pivotal element. Go on repeating this until A is changed to the form:

$$\left[\begin{array}{cccc|c} c_{11} & c_{12} & \cdots & c_{1n} & d_1 \\ 0 & c_{22} & \cdots & c_{2n} & d_2 \\ 0 & 0 & \cdots & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots 0 & c_{mn} & d_m \end{array}\right]$$

In equation from, it can be written as

$$\left.\begin{aligned} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \cdots + c_{2n}x_n &= d_2 \\ &\vdots \\ c_{mn}x_n &= d_m \end{aligned}\right\} \text{Forward elimination}$$

Then by back substitutions from equations we get the solution.

The following examples will make the process clear:

Que 1 Solve the linear system of equations

$$6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$$

$$12x_1 - 8x_2 + 6x_3 + 10x_4 = 26$$

$$3x_1 - 13x_2 + 9x_3 + 3x_4 = -19$$

$$-6x_1 + 4x_2 + x_3 - 18x_4 = -34$$ by Gauss elimination method.

Sol The given system of equations can be written as $AX = B$

where $A = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $B = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$

Here the augmented matrix

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$$[A : B] = \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 3 & -14 & -18 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - \frac{1}{2}R_1 \\ R_4 \to R_4 + R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 4 & -13 & -21 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 + \frac{1}{2}R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right] \quad R_4 \to R_4 - 2R_3$$

$\therefore$ The given system is equivalent to

$$6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$$
$$-4x_2 + 2x_3 + 2x_4 = -6$$
$$2x_3 - 5x_4 = -9$$
$$-3x_4 = -3$$

So, we have completed the step of forward elimination.
$\therefore$ By back substitution, we get.

$x_4 = 1$, $x_3 = \frac{5x_4 - 9}{2} = \frac{5-9}{2} = -2$, $x_2 = \frac{1}{4}[2x_3 + 2x_4 + 6]$
$= \frac{1}{4}[-4 + 2 + 6] = 1$

$\therefore$ $x_1 = \frac{1}{6}[2x_2 - 2x_3 - 4x_4 + 16] = \frac{1}{6}[2 + 4 - 4 + 16] = 3$

Hence $\boxed{x_1 = 3,\ x_2 = 1,\ x_3 = -2,\ x_4 = 1}$ A

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Que 2 Solve the linear system of equations

$$x+4y-6z=1$$
$$3x+y-z=2$$
$$2x-3y+5z=1$$

by Gauss-elimination method.

Sol The given system can be written as $AX=B$ where

$$A=\begin{bmatrix}1 & 4 & -6\\ 3 & 1 & -1\\ 2 & -3 & 5\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\quad B=\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$$

Here augmented matrix

$$[A\,\vdots\,B]=\left[\begin{array}{ccc|c}1 & 4 & -6 & 1\\ 3 & 1 & -1 & 2\\ 2 & -3 & 5 & 1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1 & 4 & -6 & 1\\ 0 & -11 & 17 & -1\\ 0 & -11 & 17 & -1\end{array}\right] \quad \text{by } R_2\to R_2-3R_1 \ \& \ R_3\to R_3-2R_1$$

$$\sim\left[\begin{array}{ccc|c}1 & 4 & -6 & 1\\ 0 & -11 & 17 & -1\\ 0 & 0 & 0 & 0\end{array}\right] \quad \text{by } R_3\to R_3-R_2$$

$\therefore$ The given system of equations is equivalent to

$$x+4y-6z=1$$
$$-11y+17z=-1$$

Let $z=k$. Then $y=\dfrac{17k+1}{11}$ and $x=1-4\left(\dfrac{17k+1}{11}\right)+6k$

$$=\frac{11-68k-4+66k}{11}=\frac{7-2k}{11}$$

Hence the solutions are

$x=\dfrac{7-2k}{11}$, $y=\dfrac{17k+1}{11}$, $z=k$ where k is arbitrary. **A**

Since k is arbitrary, the given system of equations has infinitely many solutions

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Ques 3 Solve the linear system of equations

$$3x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + x_2 + x_3 = 0$$
$$6x_1 + 2x_2 + 4x_3 = 6$$

by Gauss elimination method.

Sol The given system can be written as $AX = B$ where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

Here augmented matrix

$$[A ; B] = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \quad \text{by } R_2 \to R_2 - \frac{2}{3}R_1 \;\&\; R_3 \to R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \quad \text{by } R_3 \to R_3 - 6R_2$$

$\therefore$ The given system of equations is equivalent to

$$3x_1 + 2x_2 + x_3 = 3$$
$$-\frac{1}{3}x_2 + \frac{1}{3}x_3 = -2$$
$$0 = 12$$

The false statement $0 = 12$ shows that the system has no solution.

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Que4 Show that the system of equations

$$x_1 + 4x_2 + \alpha x_3 = 6$$
$$2x_1 - x_2 + 2\alpha x_3 = 3$$
$$\alpha x_1 + 3x_2 + x_3 = 5$$

possesses a unique solution when $\alpha = 0$, no solution when $\alpha = -1$, and infinitely many solutions when $\alpha = 1$. Also, investigate the corresponding situation when the right-hand side is replaced by '0's. Also, find the solutions in each case.

Sol: The given system of equations can be written as

$AX = B$ where

$$A = \begin{bmatrix} 1 & 4 & \alpha \\ 2 & -1 & 2\alpha \\ \alpha & 3 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$$

Here augmented matrix

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 2 & -1 & 2\alpha & 3 \\ \alpha & 3 & 1 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & -9 & 0 & -9 \\ 0 & 3-4\alpha & 1-\alpha^2 & 5-6\alpha \end{array}\right] \quad \text{by } R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - \alpha R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & -9 & 0 & -9 \\ 0 & 0 & 1-\alpha^2 & 2(1-\alpha) \end{array}\right] \quad \text{by } R_3 \to R_3 + \frac{(3-4\alpha)}{9} R_2$$

rank $[A:B] = 3$ when $\alpha \neq 1$ otherwise 2.

rank $A = 3$ when $\alpha \neq 1, -1$ otherwise 2.

$\therefore$ When $\alpha \neq -1, 1$ rank $[A:B]$ = rank A = no. of unknowns = 3

$\therefore$ The system of equations has a unique solution.

Hence, for $\alpha = 0$, a unique sol. exists and is given by

$$\left.\begin{aligned} x_1 + 4x_2 &= 6 \\ -9x_2 &= -9 \\ x_3 &= 2 \end{aligned}\right\} \Rightarrow x_3 = 2,\ x_2 = 1,\ x_1 = 2$$

When $\alpha = 1$ rank $[A:B] = 2$ = rank A < no. of unknowns = 3

$\therefore$ The system of equations has infinitely many solutions and is given by

$$\left.\begin{aligned} x_1 + 4x_2 + x_3 &= 6 \\ -9x_2 &= -9 \end{aligned}\right\} \Rightarrow x_2 = 1 \text{ and } x_1 + x_3 = 2$$

Let $x_3 = k$. Then $x_1 = 2-k,\ x_2 = 1,\ x_3 = k$ where k is arbitrary (gives the infinitely many solutions)

When $\alpha = -1$

rank $[A:B] = 3$ but rank $A = 2$

$\because$ rank $[A:B] \neq$ rank A

$\therefore$ No solution exists when $\alpha = -1$.

When the right-hand side is replaced by 0's

The given system of equations will be equivalent to homogeneous system and

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 4 & \alpha & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 0 & 1-\alpha^2 & 0 \end{array}\right]$$

i. In this case,

$$\text{rank}[A:B] = \text{rank } A = 3 \quad \text{when } \alpha \neq -1, 1$$
$$= \text{no. of unknowns } (3)$$

$\therefore$ Only trivial solution $x_1 = 0,\ x_2 = 0,\ x_3 = 0$ exists in this case.

When $\alpha = -1$ or $\alpha = 1$ in both cases

$$\text{rank}[A:B] = \text{rank } A = 2 < \text{no. of unknowns } (3)$$

$\therefore$ Infinitely many solutions exists in this case and is given by

$$x_1 + 4x_2 + \alpha x_3 = 0$$
$$-9x_2 = 0$$

which implies, $x_2 = 0,\ x_1 + \alpha x_3 = 0$

Let $x_3 = k$. Then $x_1 = -k\alpha,\ x_2 = 0,\ x_3 = k$, k is arbitrary.

For $\alpha = -1$, $x_1 = k,\ x_2 = 0,\ x_3 = k$, k is arbitrary.

For $\alpha = 1$ $x_1 = -k,\ x_2 = 0,\ x_3 = k$, k is arbitrary.

Note: Gauss elimination algorithm is unsatisfactory in the following cases:

For example: For the following system

$$\left.\begin{aligned} 0x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2 \end{aligned}\right\} \quad \text{--- (1)}$$

the above algorithm fails as $a_{11} = 0$

If a numerical procedure actually fails for some values of the data, then the procedure is probably untrustworthy for values of the data near the failing values. To test this, consider the system

$$\left.\begin{aligned} \varepsilon x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2 \end{aligned}\right\} \quad \text{--- (2)}$$

in which ε is a small number different from 0.

Now, using the Gaussian elimination algorithm, the system (2) reduces to

$$\left.\begin{aligned} \varepsilon x_1 + x_2 &= 1 \\ (1-\varepsilon^{-1})x_2 &= 2-\varepsilon^{-1} \end{aligned}\right\} \quad \text{--- (3)}$$

By back substitution, $x_2 = \dfrac{2-\varepsilon^{-1}}{1-\varepsilon^{-1}} \approx 1$, $x_1 = \varepsilon^{-1}(1-x_2) \approx 0$

$\therefore$ We conclude that for values of ε sufficiently close to 0, the computer calculates x_2 as 1 and then x_1 as 0. Since the correct solution is $x_1 = \dfrac{1}{1-\varepsilon} \approx 1$, $x_2 = \dfrac{1-2\varepsilon}{1-\varepsilon} \approx 1$

$\therefore$ the relative error in the computed solution for x_1 is extremely large: 100%.

Actually, the Gauss elimination algorithm works well on System ① & System ② if the equations are first permuted:

$$\begin{cases} x_1 + x_2 = 2 \\ 0x_1 + x_2 = 1 \end{cases} \qquad \text{———— ①}$$

and
$$\begin{cases} x_1 + x_2 = 2 \\ \varepsilon x_1 + x_2 = 1 \end{cases} \qquad \text{———— ②}$$

The first system is easily solved and we get by back substitution

$$x_2 = 1$$
$$x_1 = 2 - x_2 = 1$$

Moreover, the second of these systems becomes

$$x_1 + x_2 = 2$$
$$(1-\varepsilon)x_2 = 1 - 2\varepsilon$$

and by back substitution we get the solution

$$x_2 = \frac{1-2\varepsilon}{1-\varepsilon} \approx 1 \quad \text{and} \quad x_1 = 2 - x_2 \approx 1$$

Notice that we do not have to rearrange the equations in the system: it is necessary only to select a different pivot row.

The difficulty is System ② is not due simply to ε being small but rather to its being small relative to other coefficients in the same row.

To verify this, let us consider

$$\left.\begin{aligned} x_1 + \varepsilon^{-1} x_2 &= \varepsilon^{-1} \\ x_1 + x_2 &= 2 \end{aligned}\right\} \quad \text{———} \quad (4)$$

System (4) is mathematically ~~equa~~ equivalent to (2), but Gauss elimination algorithm fails here as

It produces $x_1 + \varepsilon^{-1} x_2 = \varepsilon^{-1}$

$(1-\varepsilon^{-1}) x_2 = 2 - \varepsilon^{-1}$

which gives, $x_2 = \dfrac{2-\varepsilon^{-1}}{1-\varepsilon^{-1}} \approx 1$, $x_1 = \varepsilon^{-1} - \varepsilon^{-1} x_2 \approx 0$

i.e, not the correct solution.

This situation can be resolved by interchanging the two equations in (4)

$$x_1 + x_2 = 2$$

$$x_1 + \varepsilon^{-1} x_2 = \varepsilon^{-1}$$

which produce $x_1 + x_2 = 2$

$(\varepsilon^{-1} - 1) x_2 = \varepsilon^{-1} - 2$

and gives, $x_2 = \dfrac{\varepsilon^{-1}-2}{\varepsilon^{-1}-1} \approx 1$, $x_1 = 2 - x_2 \approx 1$

which is the correct solution.

Pivoting: Permuting the order of the rows of the matrix is called pivoting.

Partial Pivoting: Gaussian elimination with partial pivoting selects the pivot row to be the one with the maximum pivot entry in absolute value from those in the leading column of the reduced submatrix. Two rows are interchanged to move the designated row into the pivot row position.

Complete Pivoting: Gaussian elimination with complete pivoting selects the pivot entry as the maximum pivot entry from all entries in the submatrix. (This complicates things because some of the unknown are rearranged.) Two rows and two columns are interchanged to accomplish this. In practice, partial pivoting is almost as good as complete pivoting and involves significantly less work.

Note: In certain situations, Gaussian elimination with simple partial pivoting may lead to an incorrect solution because here the relative sizes of entries in a row are not considered. In that case we use Gaussian elimination with scaled partial pivoting to get the correct solution.

Scaled Partial Pivoting: In scaled partial pivoting the relative sizes of entries in a row are also considered ~~to~~ while choosing pivot element. So, scaled partial pivoting means selecting the row whose pivot element has the highest relative absolute value.

Que Solve the following system using Gaussian elimination with partial pivoting

$$x_1+x_2+x_3=4$$
$$2x_1+2x_2+3x_3=9$$
$$3x_1+4x_2-2x_3=9$$

Sol Here $[A:B]=\left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 2 & 2 & 3 & 9 \\ 3 & 4 & -2 & 9 \end{array}\right]$

$\sim \left[\begin{array}{ccc:c} 3 & 4 & -2 & 9 \\ 2 & 2 & 3 & 9 \\ 1 & 1 & 1 & 4 \end{array}\right]$ $R_3 \leftrightarrow R_1$ ($\because \max(|1|,|2|,|3|) = 3$ corresponding to R_3 $\therefore$ we make R_3 as pivotal row)

$\sim \left[\begin{array}{ccc:c} 3 & 4 & -2 & 9 \\ 0 & -\frac{2}{3} & \frac{13}{3} & 3 \\ 0 & -\frac{1}{3} & \frac{5}{3} & 1 \end{array}\right]$ $R_2 \to R_2-\frac{2}{3}R_1$, $R_3 \to R_3-\frac{1}{3}R_1$

$\sim \left[\begin{array}{ccc:c} 3 & 4 & -2 & 9 \\ 0 & -2/3 & 13/3 & 3 \\ 0 & 0 & -1/2 & -1/2 \end{array}\right]$ ($\because \max\left(\left|-\frac{2}{3}\right|, \left|-\frac{1}{3}\right|\right)=\frac{2}{3}$ corresponding to R_2, So no change in pivotal row)

$R_3 \to R_3-\frac{1}{2}R_2$

Hence the given system is equivalent to

$$3x_1+4x_2-2x_3=9$$
$$-\frac{2}{3}x_2+\frac{13}{3}x_3=3$$
$$-\frac{1}{2}x_3=-\frac{1}{2}$$

$\therefore$ By back substitution, $x_3=1$, $x_2=\frac{3}{2}\left(\frac{13}{3}-3\right)=2$

$$x_1=\frac{1}{3}(9-4x_2+2x_3)=\frac{1}{3}(9-8+2)=1$$

Hence, $\boxed{x_1=1,\ x_2=2,\ x_3=1}$ **A**

Procedure for scaled partial pivoting:

- We use a scale vector $s = [s_1, s_2, \ldots, s_n]$ in which

$$s_i = \max_{1 \le j \le n} |a_{ij}| \quad (1 \le i \le n)$$

and an Index vector $l = [l_1, l_2, \ldots, l_n]$, initially set $l = [1, 2, \ldots n]$.

Here scale vector s is set once at the beginning of the algorithm while the elements in the index vector are interchanged rather than the rows of the matrix A, which reduces the amount of data movement considerably.

- Now, for step k $(k = 1, 2, \ldots n)$, we find the largest ratio in the set $\left\{ \frac{|a_{l_i,k}|}{s_{l_i}} : k \le i \le n \right\}$

Suppose j to be the first index associated with the largest ratio. Then we interchange l_j with l_k in l and move further with pivotal row l_k as shown in below example:

Que Solve the following system using Gaussian Elimination with scaled partial pivoting

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Sol Here $[A;B] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 3 & 2 & 1 & 4 & 1 \\ 5 & -8 & 6 & 3 & 1 \\ 4 & 2 & 5 & 3 & -1 \end{array}\right]$

scale vector $s = \underset{s_1 \quad s_2 \quad s_3 \quad s_4}{[2, 4, 8, 5]}$ $\left[\because s = [s_1, s_2, s_3, s_4] \text{ where } s_i = \max_{1 \le j \le 4} |a_{ij}| \; (1 \le i \le 4)\right]$

and index vector $l = \underset{l_1, \; l_2, \; l_3, \; l_4}{[1, 2, 3, 4]}$

Now, for step 1 $(k=1)$

$$\text{Max.}\left\{ \frac{|a_{l_i,1}|}{s_{l_i}} : 1 \le i \le 4 \right\} = \text{Max.}\left\{ \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \boxed{\frac{4}{5}} \right\} = \frac{4}{5}$$ which

corresponds to l_4.

$\therefore$ 4th row is the pivotal row. Interchange l_4 and l_1, so $l = [4, 2, 3, 1]$
$(\because l_4 = 4)$

$$\therefore [A:B] \sim \left[\begin{array}{cccc|c} 0 & -1.5 & 0.75 & 0.25 & 1.25 \\ 0 & 0.5 & -2.75 & 1.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 & 2.25 \\ 4 & 2 & 5 & 3 & -1 \end{array}\right]$$ by $R_1 \to R_1 - \frac{1}{4}R_4$, $R_2 \to R_2 - \frac{3}{4}R_4$, $R_3 \to R_3 - \frac{5}{4}R_4$

$s = [2, 4, 8, 5]$, $l = [4, 2, 3, 1]$

$\therefore$ For step 2 $(k=2)$

$$\text{Max}\left\{ \frac{|a_{l_i,2}|}{s_{l_i}} ; 2 \le i \le 4 \right\} = \text{Max.}\left\{ \frac{0.5}{2}, \boxed{\frac{10.5}{8}}, \frac{1.5}{2} \right\} = \frac{10.5}{8}$$

$(\because l_2 = 2, l_3 = 3, l_4 = 1)$

which corresponds to l_3.

$\therefore$ 3rd row is the pivotal row. Interchange l_3 with l_2 in l
$(\because l_3 = 3)$
So, $l = [4, 3, 2, 1]$

$$\therefore [A:B] \sim \left[\begin{array}{cccc|c} 0 & 0 & 0.7857 & 0.3571 & 0.9286 \\ 0 & 0 & -2.7619 & 1.7143 & 1.8571 \\ 0 & -10.5 & -0.25 & -0.75 & 2.25 \\ 4 & 2 & 5 & 3 & -1 \end{array}\right] \quad \text{by } R_1 \to R_1 - \frac{1}{7}R_3, \quad R_2 \to R_2 + \frac{1}{21}R_3$$

$S = [2, 4, 8, 5]$, $l = [4, 3, 2, 1]$

$\therefore$ <u>For step 3 ($k=3$)</u>

$$\operatorname{Max}\left\{\frac{|a_{l_i,3}|}{S_{l_i}} : i = 3, 4\right\} = \operatorname{Max}\left\{\boxed{\frac{2.7619}{4}}, \frac{0.7857}{2}\right\} = \frac{2.7619}{4}$$

which corresponds to l_3

$\therefore$ 2^{nd} row ($\because l_3 = 2$) is the pivotal row.

Interchange l_3 with l_3 we get the same $l = [4, 3, 2, 1]$

$$\therefore [A:B] \sim \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0.8448 & 1.4569 \\ 0 & 0 & -2.7619 & 1.7143 & 1.8571 \\ 0 & -10.5 & -0.25 & -0.75 & 2.25 \\ 4 & 2 & 5 & 3 & -1 \end{array}\right] \quad \text{by } R_1 \to R_1 + \frac{0.7857}{2.7619}R_2$$

$\therefore$ By back substitution (reading the entries in the index vector l from the last to the first, we have the order in which the back substitution is to be performed)

$$x_4 = \frac{1.4569}{0.8448} = 1.7245$$

$$x_3 = \frac{1.8571 - 1.7143 \times 1.7245}{-2.7619} = 0.3980$$

$$x_2 = \frac{2.25 + 0.25 \times 0.3980 + 0.75 \times 1.7245}{-10.5} = -0.3469$$

$$x_1 = (-1 - 3x_4 - 5x_3 - 2x_2)/4 = \frac{-1 - 3 \times 1.7245 - 5 \times 0.3980 + 2 \times 0.3469}{4} = -1.8673$$

Ans

Dr. Nishu Gupta

Que For the following system of equations:

$$2x + 2cy = 2c \quad \text{---(1)}$$
$$x + y = 2 \quad \text{---(2)}$$

where c is a parameter that can take very large numerical values, show that Gaussian elimination with partial pivoting gives the incorrect solution while Gaussian elimination with scaled partial pivoting gives the correct solution $x = 1, y = 1$.

Sol Given system is $AX = B$

where $A = \begin{bmatrix} 2 & 2c \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 2c \\ 2 \end{bmatrix}$

$\therefore$ Augment matrix $[A \vdots B] = \left[\begin{array}{cc:c} 2 & 2c & 2c \\ 1 & 1 & 2 \end{array}\right]$

Gaussian elimination with partial pivoting:

$[A \vdots B] = \left[\begin{array}{cc:c} 2 & 2c & 2c \\ 1 & 1 & 2 \end{array}\right]$ ($\because \max(|2|, |1|) = 2$ corresponding to row 1, so R_1 is the pivotal row)

$\sim \left[\begin{array}{cc:c} 2 & 2c & 2c \\ 0 & 1-c & 2-c \end{array}\right]$ $R_2 \to R_2 - \frac{1}{2} R_1$

$\therefore$ The given system is equivalent to

$$2x + 2cy = 2c$$
$$(1-c)y = 2-c$$

$\therefore$ By back substitution $y = \frac{2-c}{1-c} \approx 1$ and $x = \frac{2c - 2cy}{2} \approx 0$

which is incorrect solution (As correct sol. is $x = y = 1$)

Gaussian elimination with scaled partial pivoting

Here $[A:B] = \begin{bmatrix} 2 & 2c & \vdots & 2c \\ 1 & 1 & \vdots & 2 \end{bmatrix}$

scale vector $s = [s_1, s_2]$ where $s_i = \max\limits_{1 \le j \le 2} |a_{ij}|$

$= [2c, 1]$

and Index vector $l = [l_1, l_2] = [1, 2]$

Now, for step 1 $(k=1)$

$\text{Max}\left\{ \frac{|a_{l_i,1}|}{s_{l_i}} ; i = 1, 2 \right\} = \text{Max}\left\{ \frac{2}{2c}, \frac{1}{1} \right\} = 1$ which is corresponding to l_2

$\therefore$ 2nd row ($\because l_2 = 2$) is the pivotal row. Interchange l_2 and l_1,

so $l = [2, 1]$

$\therefore [A:B] \sim \begin{bmatrix} 0 & 2(c-1) & \vdots & 2(c-2) \\ 1 & 1 & \vdots & 2 \end{bmatrix}$ by $R_1 \to R_1 - 2R_2$

$\therefore$ The given system is equivalent to

$x + y = 2$

$2(c-1)y = 2(c-2)$

$\therefore$ By back substitution,

$y = \frac{c-2}{c-1} \approx 1$

$\therefore x = 2 - 1 = 1$

Hence $x = 1, y = 1$

which is the correct solution.

Gauss-Jordan Method: Jordan's modification to Gauss elimination method is that the elimination is performed not only in the equations below but also in the equations above the pivotal row so that we get a diagonal matrix. In this way, we have the solution without further computation.

Comparing the methods of Gauss and Jordan, we find that the number of operations is essentially $\frac{n^3}{3}$ for Gauss method and $\frac{n^3}{2}$ for Jordan method. Hence, Gauss method should usually be preferred over Jordan method.

Remark: It has the same problems as Gaussian elimination and can be modified to do partial scaled pivoting.

Que Solve

$$10x+y+z=12$$
$$x+10y+z=12$$
$$x+y+10z=12$$

by Gauss-Jordan method.

Sol The augmented matrix for the given system is

$$= \left[\begin{array}{ccc:c} 10 & 1 & 1 & 12 \\ 1 & 10 & 1 & 12 \\ 1 & 1 & 10 & 12 \end{array}\right] \leftarrow \text{pivotal row}$$

$$\sim \left[\begin{array}{ccc:c} 10 & 1 & 1 & 12 \\ 0 & 99/10 & 9/10 & 108/10 \\ 0 & 9/10 & 99/10 & 108/10 \end{array}\right] \begin{array}{l} R_2 \to R_2 - \frac{1}{10}R_1 \quad \leftarrow \text{pivotal row} \\ \& \ R_3 \to R_3 - \frac{1}{10}R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 10 & 0 & 10/11 & 120/11 \\ 0 & 99/10 & 9/10 & 108/10 \\ 0 & 0 & 108/11 & 108/11 \end{array}\right] \begin{array}{l} R_1 \to R_1 - \frac{10}{99} R_2 \\ R_3 \to R_3 - \frac{1}{11} R_2 \\ \leftarrow \text{pivotal row} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 10 & 0 & 0 & 10 \\ 0 & 99/10 & 0 & 99/10 \\ 0 & 0 & 108/11 & 108/11 \end{array}\right] \begin{array}{l} R_1 \to R_1 - \frac{10}{108} R_3 \\ R_2 \to R_2 - \frac{11}{120} R_3 \end{array}$$

Hence the solution of the given system is given by $10x = 10$, $\frac{99}{10} y = \frac{99}{10}$, $\frac{108}{11} z = \frac{108}{11}$

$\therefore$ $\boxed{x = 1,\ y = 1,\ z = 1}$ A

Que Solve the linear system of equations

$$x + 4y - 6z = 1$$

$$3x + y - z = 2$$

$$2x - 3y + 5z = 1$$

by Gauss-Jordan method.

Sol The given system can be written as $AX = B$ where

$$A = \begin{bmatrix} 1 & 4 & -6 \\ 3 & 1 & -1 \\ 2 & -3 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Here augmented matrix

$$[A ; B] = \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 3 & 1 & -1 & 2 \\ 2 & -3 & 5 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & -11 & 17 & -1 \end{array}\right] \quad \text{by } R_2 \to R_2 - 3R_1 \;\&\; R_3 \to R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{11} & \frac{7}{11} \\ 0 & -11 & 17 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{by } R_1 \to R_1 + \frac{4}{11}R_2,\; R_3 \to R_3 - R_2$$

which cannot be reduced further.

$\therefore$ The given system of equations is equivalent to

$$x + \frac{2}{11}z = \frac{7}{11}$$

$$-11y + 17z = -1$$

So, let $z = k$. Then $y = \dfrac{17k+1}{11}$

and $x = \dfrac{7-2z}{11} = \dfrac{7-2k}{11}$

Hence the solutions are

$x = \dfrac{7-2k}{11}$, $y = \dfrac{17k+1}{11}$ and $z = k$ where k is arbitrary.

Since k is arbitrary, the given system of equations has infinitely many solutions.

Triangular Matrix factorization methods:

Matrix Factorizations: An $n \times n$ system of linear equations can be written in matrix form $AX = B$

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$ is the coefficient matrix.

Our main objective is to show that the naive Gaussian algorithm applied to A yields a factorization of A into a product of two simple matrices, one unit lower triangular:

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}$$

and the other upper triangular:

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

In short, we refer to this as an LU factorization of A

i.e, $\boxed{A = LU}$

Solution of system of linear equations by LU factorization:

Given $AX = B$. When we find L & U s.t. $A = LU$ (as above). Then $LUX = B$. Suppose $UX = Y$. Then $LY = B$. From $LY = B$ we can find Y by back substitutions and then from $UX = Y$ we can find X by back substitutions and equations will be solved.

Theorem LU Factorization Theorem (w/p)

Let $A = [a_{ij}]_{n\times n}$. Assume that the forward elimination phase of naive Gaussian algorithm is applied to A without encountering any 0 divisors. Let the resulting matrix be denoted by

$\tilde{A} = [\tilde{a}_{ij}]_{n\times n}$. If

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & m_{32} & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \\ m_{n1} & m_{n2} & \cdots & m_{n,n-1} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \cdots & \tilde{a}_{1n} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} & \cdots & \tilde{a}_{2n} \\ 0 & 0 & \tilde{a}_{33} & \cdots & \tilde{a}_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots 0 & \tilde{a}_{nn} \end{bmatrix}$$

then $\boxed{A = LU}$. (Here m_{ij}'s, in matrix L are the multipliers that have been used to convert A as upper triangular matrix U)

LDL^T Factorization: This formation can be carried out if A is symmetric and has an ordinary LU factorization with L unit lower triangular matrix.

In the LDL^T factorization, L is unit lower triangular, and D is a diagonal matrix.

To see this $\quad A = LU$ ———— (1)

$\because$ A is symmetric $\Rightarrow A = A^T$

$\Rightarrow LU = (LU)^T$ by (1)

$\Rightarrow LU = U^T L^T \qquad (\because (AB)^T = B^T A^T)$

$\Rightarrow L^{-1}(LU) = L^{-1}(U^T L^T) \quad (\because L^{-1}$ exists being L a unit lower triangular matrix)

$\Rightarrow U = L^{-1} U^T L^T$

$\Rightarrow U (L^T)^{-1} = L^{-1} U^T$

Since the right hand side of this equation is lower triangular and the left side is upper triangular, both sides are diagonal, say D. From the equation $U(L^T)^{-1} = D$, we have

$$U = DL^T$$

$\therefore$ By (1), $\boxed{A = LDL^T}$

<u>Cholesky Factorization:</u> Any symmetric matrix that has an LU factorization in which L is unit lower triangular, has an LDL^T factorization. The Cholesky factorization $A = LL^T$ is a simple consequence of it for the case in which A is symmetric and positive definite.

<u>Note:</u> Recall that a matrix A is symmetric and positive definite if $A = A^T$ and $X^T A X > 0$ for every nonzero vector X.

<u>Theorem</u> <u>Cholesky Theorem on LL^T Factorization</u> (w/p)

If A is a real, symmetric, and positive definite matrix, then it has a unique factorization, $A = LL^T$, in which L is lower triangular with a positive diagonal.

<u>Doolittle factorization and Crout factorization:</u>

Suppose in the factorization $A = LU$ the matrix L is lower triangular and the matrix U is upper triangular.

- When <u>L is unit lower triangular</u>, it is called Doolittle factorization.
- When <u>U is unit upper triangular</u>, it is called Crout factorization.

In the case in which A is symmetric positive definite and $U = L^T$, it is called Cholesky factorization.

Note: The $n \times n$ factorization

$$A = LU$$

where $L = [l_{ij}]$ is lower triangular
and $U = [u_{ij}]$ is upper triangular, can be computed directly after multiplying the matrices L & U and then comparing the coefficients on both sides (by equality of matrices).

Que 1 Use Gauss elimination method to find triangular factorization $A = LU$ of the coefficient matrix A of the system

$$x_1 + 2x_2 + 3x_3 = 14$$
$$2x_1 + 5x_2 + 2x_3 = 18$$
$$3x_1 + x_2 + 5x_3 = 20$$

and hence solve the system.

Sol In matrix form, we have $AX = B$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & -5 & -4 \end{bmatrix} \quad \text{by } R_2 \to R_2 - 2R_1 \quad (m_{21} = 2)$$
$$\& \; R_3 \to R_3 - 3R_1 \quad (m_{31} = 3)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix} \quad \text{by } R_3 \to R_3 + 5R_2 \quad (m_{32} = -5)$$

$\therefore \; A = LU$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix}$

(Here $l_{21} = m_{21}$, $l_{31} = m_{31}$, $l_{32} = m_{32}$)

Now, to solve the system of equations

$$AX = B \Rightarrow LUX = B$$

$$\Rightarrow LY = B, \; UX = Y$$

Now, $LY = B$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

and so
$$\left.\begin{aligned} y_1 &= 14 \\ 2y_1 + y_2 &= 18 \\ 3y_1 - 5y_2 + y_3 &= 20 \end{aligned}\right\} \Rightarrow \begin{aligned} y_1 &= 14 \\ y_2 &= -10 \\ y_3 &= -72 \end{aligned}$$

Then $UX = Y$ implies

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ -72 \end{bmatrix}$$

and so $-24x_3 = -72 \Rightarrow x_3 = 3$

$$x_2 - 4x_3 = -10 \Rightarrow x_2 = 12 - 10 = 2$$

$$x_1 + 2x_2 + 3x_3 = 14 \Rightarrow x_1 = 1$$

Hence, the required solution is $\boxed{x_1 = 1, \; x_2 = 2 \text{ and } x_3 = 3}$

A

Note: Another way to find the triangular factorization $A = LU$ in above question:

In matrix form, we have $AX = B$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$

Let $A = LU$ i.e.,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

and so we have

$$\left.\begin{aligned} 1 &= u_{11} \\ 2 &= l_{21}u_{11} \\ 3 &= l_{31}u_{11} \end{aligned}\right\} \Rightarrow \begin{aligned} u_{11} &= 1 \\ l_{21} &= 2 \\ l_{31} &= 3 \end{aligned}$$

$2 = u_{12} \Rightarrow u_{12} = 2$

$5 = l_{21}u_{12} + u_{22} \Rightarrow u_{22} = 1$

$1 = l_{31}u_{12} + l_{32}u_{22} \Rightarrow l_{32} = -5$

$3 = u_{13} \Rightarrow u_{13} = 3$

$2 = l_{21}u_{13} + u_{23} \Rightarrow u_{23} = -4$

$5 = l_{31}u_{13} + l_{32}u_{23} + u_{33} \Rightarrow u_{33} = -24$

Hence, $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix}$

Now, we can solve as above.

Que Determine the LDL^T factorization of the matrix

$$A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

First we find the LU factorization!

Sol $\wedge$ $A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ (Clearly A is a symmetric matrix)

$$\sim \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

by $R_2 \to R_2 - \frac{3}{4}R_1$ $\left(m_{21} = \frac{3}{4}\right)$

$R_3 \to R_3 - \frac{1}{2}R_1$ $\left(m_{31} = \frac{1}{2}\right)$

$R_4 \to R_4 - \frac{1}{4}R_1$ $\left(m_{41} = \frac{1}{4}\right)$

$$\sim \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

by $R_3 \to R_3 - \frac{2}{3}R_2$ $\left(m_{32} = \frac{2}{3}\right)$

$R_4 \to R_4 - \frac{1}{3}R_2$ $\left(m_{42} = \frac{1}{3}\right)$

$$\sim \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

by $R_4 \to R_4 - \frac{1}{2}R_3$ $\left(m_{43} = \frac{1}{2}\right)$

$\therefore$ $A = LU$

where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

Now, extract the diagonal elements from U and place them into a diagonal matrix D,

$$\therefore U = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = DL^T$$

Clearly, we have $A = LDL^T$ where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/2 & 2/3 & 1 & 0 \\ 1/4 & 1/3 & 1/2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

Note: In the above question, we can also find $A = LU$ by alternative method as for previous question.

Que Determine the Cholesky factorization of the matrix

$$A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Sol First, we examine A is a real, symmetric and positive definite matrix.

Now, $A = LDL^T$ where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/2 & 2/3 & 1 & 0 \\ 1/4 & 1/3 & 1/2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

(Solve previous question completely)

Now, we can write

$$A = LDL^T = (LD^{1/2})(D^{1/2}L^T) = L_1 L_1^T$$

where $L_1 = LD^{1/2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/2 & 2/3 & 1 & 0 \\ 1/4 & 1/3 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3/2 & \sqrt{3}/2 & 0 & 0 \\ 1 & \sqrt{3}/3 & \sqrt{2/3} & 0 \\ 1/2 & \sqrt{3}/6 & \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2.0000 & 0 & 0 & 0 \\ 1.5000 & 0.8660 & 0 & 0 \\ 1.0000 & 0.5774 & 0.8165 & 0 \\ 0.5000 & 0.2887 & 0.4082 & 0.7071 \end{bmatrix}$$

Clearly L_1 is the lower triangular matrix in the Cholesky factorization $A = L_1 L_1^T$.

Note: Another way to find the Cholesky factorization of the matrix $A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Sol Let $A = LL^T$ where $L = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}$. Then

$A = LL^T$ implies

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ 0 & l_{22} & l_{32} & l_{42} \\ 0 & 0 & l_{33} & l_{43} \\ 0 & 0 & 0 & l_{44} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} & l_{11}l_{41} \\ l_{21}l_{11} & l_{21}^2+l_{22}^2 & l_{21}l_{31}+l_{22}l_{32} & l_{21}l_{41}+l_{22}l_{42} \\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & l_{31}^2+l_{32}^2+l_{33}^2 & l_{31}l_{41}+l_{32}l_{42}+l_{33}l_{43} \\ l_{41}l_{11} & l_{41}l_{21}+l_{42}l_{22} & l_{41}l_{31}+l_{42}l_{32}+l_{43}l_{33} & l_{41}^2+l_{42}^2+l_{43}^2+l_{44}^2 \end{bmatrix}$$

Hence, $l_{11}^2 = 4$ $\quad l_{21}^2+l_{22}^2 = 3$ $\quad l_{31}^2+l_{32}^2+l_{33}^2 = 2$

$l_{21}l_{11} = 3$ $\quad l_{31}l_{21}+l_{32}l_{22} = 2$ $\quad l_{41}l_{31}+l_{42}l_{32}+l_{43}l_{33} = 1$

$l_{31}l_{11} = 2$

$l_{41}l_{11} = 1$ $\quad l_{41}l_{21}+l_{42}l_{22} = 1$ $\quad l_{41}^2+l_{42}^2+l_{43}^2+l_{44}^2 = 1$

$\therefore\ l_{11} = 2$, $\frac{9}{4} + l_{22}^2 = 3 \Rightarrow l_{22}^2 = \frac{3}{4} \Rightarrow l_{22} = \frac{\sqrt{3}}{2}$, $1 + \frac{1}{3} + l_{33}^2 = 2 \Rightarrow l_{33} = \sqrt{\frac{2}{3}}$

$l_{21} = \frac{3}{2}$, $\frac{3}{2} + \frac{\sqrt{3}}{2} l_{32} = 2 \Rightarrow l_{32} = \frac{1}{\sqrt{3}}$, $\frac{1}{2} + \frac{1}{6} + \sqrt{\frac{2}{3}}\, l_{43} = 1$

$l_{31} = 1$, $\frac{3}{4} + \frac{\sqrt{3}}{2} l_{42} = 1 \Rightarrow l_{42} = \frac{1}{2\sqrt{3}}$, $\Rightarrow l_{43} = \frac{2/6}{\sqrt{2/3}} = \frac{1}{2}\sqrt{\frac{2}{3}}$

$l_{41} = \frac{1}{2}$, $\frac{1}{4} + \frac{1}{12} + \frac{1}{6} + l_{44}^2 = 1$

$\Rightarrow l_{44} = \frac{1}{\sqrt{2}}$

Hence, $A = LL^T$ where $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3/2 & \sqrt{3}/2 & 0 & 0 \\ 1 & 1/\sqrt{3} & \sqrt{2/3} & 0 \\ 1/2 & 1/2\sqrt{3} & \frac{1}{2}\sqrt{\frac{2}{3}} & 1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} 2.0000 & 0 & 0 & 0 \\ 1.5000 & 0.8660 & 0 & 0 \\ 1.0000 & 0.5774 & 0.8165 & 0 \\ 0.5000 & 0.2887 & 0.4082 & 0.7071 \end{bmatrix}$$

Que Show the Cholesky's factorization and solve the following system of equations

$$4x_1+2x_2+14x_3=14$$

$$2x_1+17x_2-5x_3=-101$$

$$14x_1-5x_2+83x_3=155$$

by Cholesky method.

Sol The given system of equations can be written as $AX=B$

where $A=\begin{bmatrix}4 & 2 & 14\\ 2 & 17 & -5\\ 14 & -5 & 83\end{bmatrix}$, $X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$, $B=\begin{bmatrix}14\\ -101\\ 155\end{bmatrix}$

Let $A=LL^T$ i.e,

$$\begin{bmatrix}4 & 2 & 14\\ 2 & 17 & -5\\ 14 & -5 & 83\end{bmatrix}=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}\begin{bmatrix}l_{11} & l_{21} & l_{31}\\ 0 & l_{22} & l_{32}\\ 0 & 0 & l_{33}\end{bmatrix}$$

$$\therefore \begin{bmatrix}4 & 2 & 14\\ 2 & 17 & -5\\ 14 & -5 & 83\end{bmatrix}=\begin{bmatrix}l_{11}^2 & l_{11}l_{21} & l_{11}l_{31}\\ l_{21}l_{11} & l_{21}^2+l_{22}^2 & l_{21}l_{31}+l_{22}l_{32}\\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & l_{31}^2+l_{32}^2+l_{33}^2\end{bmatrix}$$

$\therefore$ $l_1^2=4$, $l_{11}l_{21}=2$, $l_{11}l_{31}=14$

$l_{21}l_{11}=2$, $l_{21}^2+l_{22}^2=17$, $l_{21}l_{31}+l_{22}l_{32}=-5$

$l_{31}l_{11}=14$, $l_{31}l_{21}+l_{32}l_{22}=-5$, $l_{31}^2+l_{32}^2+l_{33}^2=83$

$\therefore$ $l_{11}=2$, $l_{21}=1$, $l_{31}=7$

$l_{21}=1$, $l_{22}=4$, $l_{32}=-3$

$l_{31}=7$, $l_{32}=-3$, $l_{33}=5$

Hence, $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}$ in $A = LL^T$ factorization.

Now, $AX = B \Rightarrow LL^T X = B$
$\Rightarrow LY = B,\ L^T X = B$

Now, $LY = B$ given

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

$$\therefore \left.\begin{aligned} 2y_1 &= 14 \\ y_1 + 4y_2 &= -101 \\ 7y_1 - 3y_2 + 5y_3 &= 155 \end{aligned}\right\} \Rightarrow \begin{aligned} y_1 &= 7 \\ y_2 &= -27 \\ y_3 &= 5 \end{aligned}$$

Then $L^T X = B$ implies

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}$$

$$\therefore \begin{aligned} 2x_1 + x_2 + 7x_3 &= 7 \\ 4x_2 - 3x_3 &= -27 \\ 5x_3 &= 5 \end{aligned}$$

Hence, $x_3 = 1,\ x_2 = -6,\ x_1 = 3$

$\therefore$ The solution is, $\boxed{x_1 = 3,\ x_2 = -6,\ x_3 = 1}$

Que Solve the above problem by Doolittle method.

Sol The given system of equations can be written as $AX=B$

where $A = \begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$

Let $A = LU$ where L is unit lower triangular matrix and U is unit upper triangular matrix

i.e, $\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$

$\therefore$ $u_{11} = 2$, $u_{12} = 1$, $u_{13} = 4$

$l_{21}u_{11} = 8$, $l_{21}u_{12}+u_{22} = -3$, $l_{21}u_{13}+u_{23} = 2$

$l_{31}u_{11} = 4$, $l_{31}u_{12}+l_{32}u_{22} = 11$, $l_{31}u_{13}+l_{32}u_{23}+u_{33} = -1$

$\therefore$ $u_{11} = 2$, $u_{12} = 1$, $u_{13} = 4$

$l_{21} = 4$, $u_{22} = -7$, $u_{23} = -14$

$l_{31} = 2$, $l_{32} = -\frac{9}{7}$, $u_{33} = -27$

$\therefore$ $A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -9/7 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{bmatrix}$

Now, $AX = B \Rightarrow LUX = B$

$\Rightarrow LY = B,\ UX = Y$

Now, $LY = B$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -9/7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 &= 12 \\ 4x_1 + y_1 &= 20 \\ 2x_1 - \tfrac{9}{7}y_1 + z_1 &= 33 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= 12 \\ y_1 &= -28 \\ z_1 &= -27 \end{aligned}$$

Then $UX = Y$ implies

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -28 \\ -27 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2x + y + 4z &= 12 \\ -7y - 14z &= -28 \\ -27z &= -27 \end{aligned}\right\} \Rightarrow z = 1,\ y = 2,\ x = 3$$

Hence $\boxed{x = 3,\ y = 2,\ z = 1}$ A

Que Solve the following system of equations by Crout's method:

$$2x+y+4z=12$$
$$8x-3y+2z=20$$
$$4x+11y-z=33$$

Sol The given system of equations can be written as $AX=B$

where $A=\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B=\begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$

Let $A=LU$ where U is unit upper triangular matrix and L is lower triangular matrix.

i.e,
$$\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$$\therefore \left.\begin{aligned} & l_{11}=2,\ l_{21}=8,\ l_{31}=4 \\ & l_{11}u_{12}=1,\ l_{21}u_{12}+l_{22}=-3 \\ & l_{31}u_{12}+l_{32}=11 \\ & l_{11}u_{13}=4 \\ & l_{21}u_{13}+l_{22}u_{23}=2 \\ & l_{31}u_{13}+l_{32}u_{23}+l_{33}=-1 \end{aligned}\right\} \Rightarrow \begin{aligned} & l_{11}=2,\ l_{21}=8,\ l_{31}=4 \\ & u_{12}=\frac{1}{2},\ l_{22}=-7,\ l_{32}=9 \\ & u_{13}=2,\ u_{23}=2 \\ & 8+18+l_{33}=-1 \Rightarrow l_{33}=-27 \end{aligned}$$

$\therefore\ A = LU$

where $L = \begin{bmatrix} 2 & 0 & 0 \\ 8 & -7 & 0 \\ 4 & 9 & -27 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $AX = B \Rightarrow LUX = B$

$\Rightarrow LY = B,\ UX = Y$

Now, $LY = B$ gives

$$\begin{bmatrix} 2 & 0 & 0 \\ 8 & -7 & 0 \\ 4 & 9 & -27 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2x_1 &= 12 \\ 8x_1 - 7y_1 &= 20 \\ 4x_1 + 9y_1 - 27z_1 &= 33 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= 6 \\ y_1 &= 4 \\ z_1 &= 1 \end{aligned}$$

Then $UX = Y$ implies

$$\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x + \frac{y}{2} + 2z &= 6 \\ y + 2z &= 4 \\ z &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} z &= 1 \\ y &= 2 \\ x &= 3 \end{aligned}$$

Hence, $\boxed{x = 3,\ y = 2,\ z = 1}$ A

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Eigenvalue Problem:

Eigenvalue problem concern the solution of the system

$$AX = \lambda X \quad \text{------------} \quad (1)$$

where A is a given square matrix and vector X and scalar λ are unknown.

Clearly $X = 0$ is a solution of (1). But this is of no interest, and we want to find solution vector $X \neq 0$ of (1), called eigenvectors of A.

We can find eigenvectors only for certain values of the scalar λ; these values λ for which an eigenvector exists are called the eigenvalues of A.

Note: • eigenvalues are also called characteristic values or latent roots and eigenvectors are also called characteristic vectors.

• Revise properties of eigenvalues from first semester.

Power method: In many engineering problems, it is required to compute the numerically largest eigenvalue and the corresponding eigenvector. In such cases, we use the iterative power method. The procedure will be clear by the following questions: (Here we find dominant eigenvalue λ such that $|\lambda|$ is greater than the absolute values of the other eigenvalues) [* Here we repeat the process till $X^{(n)} - X^{(n-1)}$ becomes negligible]

Que Find the largest eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ using power method. Take $[1,0,0]^T$ as initial eigen vector.

Sol Let the initial approximation to the required eigen vector be $X = [1,0,0]^T$. Then

Take numerically largest value outside

$$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

So the first approximation to the eigenvalue is 2 and the corresponding eigenvector $X^{(1)} = [1,-0.5,0]^T$

Now $$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.43 \\ 1.86 \end{bmatrix} = -3.43\begin{bmatrix} -0.87 \\ +1 \\ -0.54 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.87 \\ +1 \\ -0.54 \end{bmatrix} = \begin{bmatrix} -2.74 \\ +3.41 \\ -2.08 \end{bmatrix} = 3.41\begin{bmatrix} -0.80 \\ +1 \\ -0.61 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.80 \\ 1 \\ -0.61 \end{bmatrix} = \begin{bmatrix} -2.6 \\ 3.41 \\ -2.22 \end{bmatrix} = 3.41 \begin{bmatrix} -0.76 \\ 1 \\ -0.65 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

Clearly $\lambda^{(5)} = \lambda^{(6)} = 3.41$ and $X^{(5)} = X^{(6)} \approx \begin{bmatrix} -0.8 \\ 1 \\ -0.6 \end{bmatrix}$

Hence the largest eigenvalue is 3.41 and the corresponding eigenvector is $[-0.8, 1, -0.6]^T$.

Approximation by Spline Function:

Spline function: A spline function is a function that consists of polynomial pieces joined together with certain smoothness conditions.

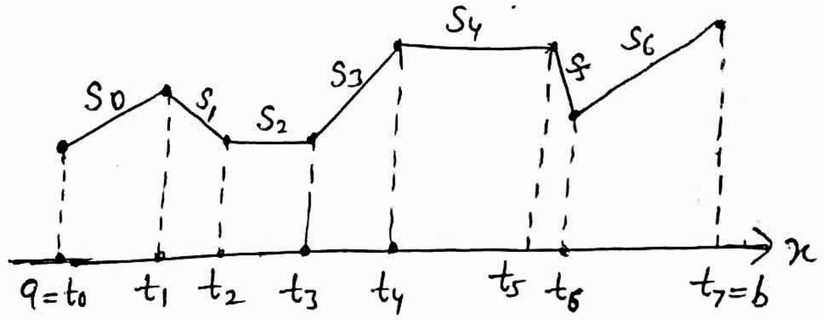
First degree spline: A function S is called a spline of degree 1 if:

(1) The domain of S is an interval $[a, b]$.

(2) S is continuous on $[a, b]$.

(3) There is a partition of the interval
$$a = t_0 < t_1 < \cdots < t_n = b$$ such that S is a linear polynomial on each subinterval $[t_i, t_{i+1}]$.

First-degree spline function



Thus,
$$S(x) = \begin{cases} S_0(x) & x \in [t_0, t_1] \\ S_1(x) & x \in [t_1, t_2] \\ \vdots & \\ S_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

where $S_i(x) = a_i(x) + b_i$

Note: To find the value of $S(x)$ at a specific x, first we find the interval that contains x and then using the appropriate linear function for that interval.

Second degree Splines: A function Q is called a spline of degree 2 (quadratic spline) if:

(1) The domain of Q is an interval $[a, b]$.

(2) Q and Q' are continuous on $[a, b]$.

(3) There are points t_i (called knots) such that $a = t_0 < t_1 < \cdots < t_n = b$ and Q is a polynomial of degree at most 2 on each subinterval $[t_i, t_{i+1}]$.

Ex 1: Determine whether this function is a first-degree spline function:

$$S(x) = \begin{cases} x & x \in [-1, 0] \\ 1-x & x \in (0, 1) \\ 2x-2 & x \in [1, 2] \end{cases}$$

Sol Clearly $S(x)$ is a piecewise linear function but not a spline of degree 1 because it is discontinuous at $x = 0$.

$\because$ Here $\lim\limits_{x \to 0+} S(x) = \lim\limits_{x \to 0} (1 - x) = 1$

but $\lim\limits_{x \to 0-} S(x) = \lim\limits_{x \to 0} x = 0$

Ex 2: Determine whether the following function is a quadratic spline:

$$Q(x) = \begin{cases} x^2 & (-10 \leq x \leq 10) \\ -x^2 & (0 \leq x \leq 1) \\ 1-2x & (1 \leq x \leq 20) \end{cases}$$

Sol The function is obviously piecewise quadratic (≤ 2 degree).

We can check the continuity of Q and Q' at the interior knots as:

$$\lim_{x\to 0^-} Q(x) = \lim_{x\to 0} x^2 = 0, \quad \lim_{x\to 0^+} Q(x) = \lim_{x\to 0} (-x^2) = 0$$

$$\lim_{x\to 1^-} Q(x) = \lim_{x\to 1} (-x^2) = -1, \quad \lim_{x\to 1^+} Q(x) = \lim_{x\to 1} (1-2x) = -1$$

$$\lim_{x\to 0^-} Q'(x) = \lim_{x\to 0} (2x) = 0, \quad \lim_{x\to 0^+} Q'(x) = \lim_{x\to 0} (-2x) = 0$$

$$\lim_{x\to 1^-} Q'(x) = \lim_{x\to 1} (-2x) = -2, \quad \lim_{x\to 1^+} Q'(x) = \lim_{x\to 1} (\cancel{1}-2\cancel{x}) = -2$$

Hence, $Q(x)$ is a quadratic spline.

Natural Cubic Splines:

The general definition of spline functions of arbitrary degree is as follows:

Spline of Degree k:

A function S is called a spline of degree k if:

(1) The domain of S is an interval $[a, b]$

(2) $S, S', S'', \ldots, S^{(k-1)}$ are all continuous functions on $[a, b]$.

(3) There are points t_i (called knots of S) such that $a = t_0 < t_1 < \cdots < t_n = b$ and such that S is a polynomial of degree at most k on each subinterval $[t_i, t_{i+1}]$.

The spline of degree 3 is known as cubic spline.

Linear Spline Interpolation: A linear spline is a type of spline interpolation where every curve connecting two consecutive points is a linear function.

Suppose given

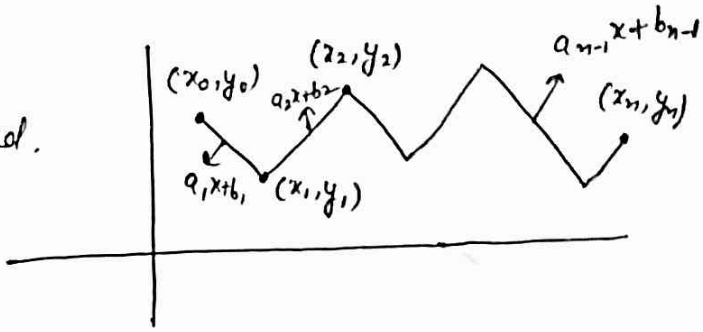
$$x: \; x_0 \quad x_1 \quad x_2 \; \cdots \; x_n$$

$$y: \; y_0 \quad y_1 \quad y_2 \; \cdots \; y_n$$

where $x_0 < x_1 < x_2 < \cdots < x_n$.

Note: If data is not given in ascending order, then rearrange it accordingly.

Thus, for data with $(n+1)$ pairs of points, n spline-functions are needed. These linear splines are of type



$$S(x) = \begin{cases} S_0(x), & x_0 \leq x \leq x_1 \\ S_1(x), & x_1 \leq x \leq x_2 \\ \vdots & \\ S_{n-1}(x), & x_{n-1} \leq x \leq x_n \end{cases}$$

where $S_i(x) = y_i + \dfrac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$, (Equ. of line passing through (x_i, y_i) and (x_{i+1}, y_{i+1}))

$i = 0, 1, 2, \ldots, n-1$

Que Fit the data in the given table with first order splines

x:	3	4.5	7	9
y:	2.5	1	2.5	0.5

Also, evaluate $f(5)$.

Sol Given

x:	$x_0 = 3$	$x_1 = 4.5$	$x_2 = 7$	$x_3 = 9$
y:	$y_0 = 2.5$	$y_1 = 1$	$y_2 = 2.5$	$y_3 = 0.5$

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We know that linear splines are given by

$$S(x) = \begin{cases} S_0(x), & 3 \le x \le 4.5 \\ S_1(x), & 4.5 \le x \le 7 \\ S_2(x), & 7 \le x \le 9 \end{cases}$$

($\because$ Here are 4 set of points $\therefore$ 3 linear splines exist)

where $S_i(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$; $i = 0, 1, 2$

$\therefore S_0(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) = 2.5 + \frac{1 - 2.5}{4.5 - 3}(x - 3)$

$= 2.5 - (x - 3) = 5.5 - x$

$S_1(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) = 1 + \frac{2.5 - 1}{7 - 4.5}(x - 4.5)$

$= 1 + 0.6(x - 4.5) = 0.6x - 1.7$

$S_2(x) = y_2 + \frac{(y_3 - y_2)}{(x_3 - x_2)}(x - x_2) = 2.5 + \frac{0.5 - 2.5}{9 - 7}(x - 7)$

$= 2.5 - (x - 7) = 9.5 - x$

Hence $S(x) = \begin{cases} 5.5 - x, & 3 \le x \le 4.5 \\ 0.6x - 1.7, & 4.5 \le x \le 7 \\ 9.5 - x, & 7 \le x \le 9 \end{cases} = f(x)$

$\therefore f(5) = 0.6 \times 5 - 1.7 = 3.0 - 1.7 = 1.3$ Ans

Note: If in above question fitting is not asked in the examination, Only question is find f(7) by using linear spline, then we can find only $S_1(x)$ as 5 lies in [4.5, 7].

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Interpolation:

Quadratic Splines: A quadratic spline is a type of spline interpolation where every curve connecting two consecutive points is a quadratic function.

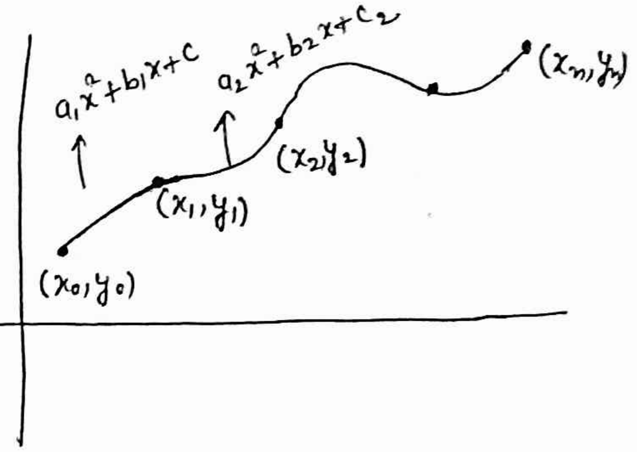
Suppose given

$$x: \; x_0 \quad x_1 \quad x_2 \; \cdots \; x_n$$
$$y: \; y_0 \quad y_1 \quad y_2 \; \cdots \; y_n$$

where $x_0 < x_1 < x_2 < \cdots < x_n$.

Note: If data is not given in ascending order, then rearrange it accordingly.

Thus, for data with $(n+1)$ pairs of points, n spline functions are needed. These quadratic splines are:



$$Q(x) = \begin{cases} Q_0(x) = a_1x^2 + b_1x + c_1, & x_0 \le x \le x_1 \\ Q_1(x) = a_2x^2 + b_2x + c_2, & x_1 \le x \le x_2 \\ \vdots & \vdots \\ Q_{n-1}(x) = a_nx^2 + b_nx + c_n, & x_{n-1} \le x \le x_n \end{cases}$$

There are three unknown coefficients in every quadratic spline (a_i, b_i, c_i); thus there is a need of $3n$ linear equations to solve for these unknowns. These linear equations are derived in the following steps:

1. All given points are parts of the spline function i.e., each spline going through two consecutive data points. There is a total of $2n$ linear equations derived on this step. ($\because Q(x_i) = y_i$; $0 \le i \le n$)
2. The first derivatives of two splines sharing an intermediate point are equal. There is a total of $(n-1)$ linear equations derived on this step.

3. The second derivative of the 1st spline is assumed to be zero. There is only one equation derived on this step, and that is

$$2a_1 = 0$$

Because of this assumption, the first spline is actually a line.

The method is clear with the following example:

Que Determine the value of $f(7)$ for a given set of data below using quadratic spline interpolation.

x:	1	5	8	10
$f(x)$:	12	-26	-14	37

Sol The given problem has 4 set of points. Also, the data is already arranged in ascending order of x. Thus, there is a need of $n = 4-1 = 3$ quadratic functions to perform interpolation. These three functions are

$$f(x) = \begin{cases} a_1 x^2 + b_1 x + c_1 & , \ 1 \le x \le 5 \\ a_2 x^2 + b_2 x + c_2 & , \ 5 \le x \le 8 \\ a_3 x^2 + b_3 x + c_3 & , \ 8 \le x \le 10 \end{cases}$$

The system has $3n = 3(3) = 9$ number of unknowns. Thus, there is a need of 9 linear equations to solve these unknowns.

Given $f(1) = 12 \Rightarrow a_1 + b_1 + c_1 = 12$ ——— (1)

$f(5) = -26 \Rightarrow 25a_1 + 5b_1 + c_1 = -26$ ——— (2)

$f(5) = -26 \Rightarrow 25a_2 + 5b_2 + c_2 = -26$ ——— (3)

$f(8) = -14 \Rightarrow 64a_2 + 8b_2 + c_2 = -14$ ——— (4)

$f(8) = -14 \Rightarrow 64a_3 + 8b_3 + c_3 = -14$ ——— (5)

$f(10) = 37 \Rightarrow 100a_3 + 10b_3 + c_3 = 37$ ——— (6)

Also, $f'(5) = 10a_1+b_1 = 10a_2+b_2$ ——(7)

and $f'(8) = 16a_2+b_2 = 16a_3+b_3$ ——(8)

$\left\{\begin{array}{ll} \because f'(x) = 2a_1x+b_1, & 1\le x\le 5 \\ = 2a_2x+b_2, & 5\le x\le 8 \\ = 2a_3x+b_3, & 8\le x\le 10 \end{array}\right.$

and $f''(x) = 0$ for $1\le x\le 5$

$\Rightarrow 2a_1 = 0 \Rightarrow a_1 = 0$ ——(9)

From equ. (1) & (9) $b_1+c_1 = 12$ ——(10)

From equ. (2) & (9), $5b_1+c_1 = -26$ ——(11)

From equ. (10) and (11), $4b_1 = -38 \Rightarrow b_1 = -9.5$

$\therefore$ From equ. (10), $c_1 = 12-b_1 = 12+9.5 = 21.5$

$\therefore \boxed{a_1 = 0,\ b_1 = -9.5,\ c_1 = 21.5}$

$\therefore$ From equ. (7), $10a_2+b_2 = -9.5$ ——(12)

& From equ. (3) and equ. (4),

$39a_2+3b_2 = 12$ (Subtracting (3) from (4))

$\Rightarrow 13a_2+b_2 = 4$ ——(13)

Subtract equ. (12) from (13) we get

$3a_2 = 13.5 \Rightarrow a_2 = 4.5$

$\therefore$ From equ. (13), $b_2 = 4-13a_2 = 4-58.5 = -54.5$

$\therefore$ From equ. (3), $25\times 4.5 - 5\times 54.5 + c_2 = -26$

$112.5 - 272.5 + c_2 = -26$

$\Rightarrow c_2 = -26+272.5-112.5 = 134$

$\therefore \boxed{a_2 = 4.5,\ b_2 = -54.5,\ c_2 = 134}$

$\therefore$ From equ. (8), $16a_3+b_3 = 16\times 4.5 - 54.5 = 17.5$ ——(14)

Subtract equ. (5) from equ. (6),

$36a_3+2b_3 = 51$

$\Rightarrow 18a_3+b_3 = 25.5$ ——(15)

Subtracting equ.(14) from equ.(15), we get

$$2a_3 = 8 \Rightarrow a_3 = 4$$

$\therefore$ From equ.(14), $\quad 64 + b_3 = 17.5 \Rightarrow b_3 = 17.5 - 64 = -46.5$

$\therefore$ From equ.(5), $\quad c_3 = -14 - 64 \times 4 - 8 \times (-46.5)$

$$c_3 = -14 - 256 + 372 = 102$$

Hence, $\boxed{a_3 = 4,\ b_3 = -46.5,\ c_3 = 102}$

Thus, quadratic splines are

$$f(x) = \begin{cases} -9.5x + 21.5 & , \ 1 \leq x \leq 5 \\ 4.5x^2 - 54.5x + 134 & , \ 5 \leq x \leq 8 \\ 4x^2 - 46.5x + 102 & , \ 8 \leq x \leq 10 \end{cases}$$

Hence, $\boxed{f(7) = 4.5(7)^2 - 54.5(7) + 134 = -27}$ Ans

Note: We can also solve equations (1) to (9) by Gauss elimination method or by Gauss Jordan method by writing it into matrix notation.

Cubic Spline Interpolation: A cubic spline is a type of spline interpolation where every curve connecting two consecutive points is a cubic function.

Suppose given x: x_0 x_1 x_2 --- x_n

y: y_0 y_1 y_2 ---- y_n

where $x_0 < x_1 < x_2 < \cdots < x_n$ (otherwise arrange the data in ascending order)

Thus, for data with $(n+1)$ pairs of points, n spline functions are needed.

These cubic splines are:

$$S(x) = \begin{cases} S_0(x) = a_1x^3 + b_1x^2 + c_1x + d_1, & x_0 \le x \le x_1 \\ S_1(x) = a_2x^3 + b_2x^2 + c_2x + d_2, & x_1 \le x \le x_2 \\ \vdots & \vdots \\ S_{n-1}(x) = a_nx^3 + b_nx^2 + c_nx + d_n, & x_{n-1} \le x \le x_n \end{cases}$$

There are four unknown coefficients in every cubic spline (a_i, b_i, c_i, d_i); thus there is a need of $4n$ linear equations to solve for these unknowns. These linear equations are derived in the following steps:

1. All given points are parts of the spline function i.e, each spline going through two consecutive data points. There is a total of $2n$ linear equations derived on this step.

2. The first derivatives of two splines sharing an intermediate point are equal. There is a total of $(n-1)$ linear equations derived on this step.

3. The second derivatives of two splines sharing an intermediate point are equal. There is a total of $(n-1)$ linear equations derived on this step.

4. The second derivative of the first and last spline functions is assumed to be zero at x_0 and x_n respectively. There is a total of 2 linear equations derived on this step. i.e,
$$S_0''(x_0) = 0 \text{ and } S_{n-1}''(x_n) = 0$$

The method is clear with the following example:

Que: Fit the natural cubic interpolating spline for the following table

x:	-1	0	1
y:	1	2	-1

Also, find the value of y at $x = \frac{1}{2}$.

Sol The given problem has $\underset{=(n+1)}{3}$ set of points. Also, the data is already arranged in ascending order of x. Thus, there will be $\underset{(=n)}{2}$ cubic functions to perform interpolation. Hence the natural cubic spline $S(x)$ is given by

$$S(x) = \begin{cases} S_0(x) = a_1x^3 + b_1x^2 + c_1x + d_1 \ ; & -1 \le x \le 0 \\ S_1(x) = a_2x^3 + b_2x^2 + c_2x + d_2 \ ; & 0 \le x \le 1 \end{cases}$$

The system has $4n = 4(2) = 8$ number of unknowns. Thus there is a need of 8 linear equations to solve these unknowns,

$$\because S(-1) = 1 \Rightarrow S_0(-1) = 1 \Rightarrow -a_1 + b_1 - c_1 + d_1 = 1 \quad \text{---} (1)$$
$$S(0) = 2 \Rightarrow S_0(0) = 2 \Rightarrow d_1 = 2 \quad \text{---} (2)$$
$$S(0) = 2 \Rightarrow S_1(0) = 2 \Rightarrow d_2 = 2 \quad \text{---} (3)$$
$$S(1) = -1 \Rightarrow S_1(1) = -1 \Rightarrow a_2 + b_2 + c_2 + d_2 = -1 \quad \text{---} (4)$$

Also $S_0'(0) = S_1'(0)$

$\Rightarrow \quad c_1 = c_2$ ——— (5) $\qquad \left\{ \because S_0'(x) = 3a_1x^2 + 2b_1x + c_1 \atop S_1'(x) = 3a_2x^2 + 2b_2x + c_2 \right.$

Now, $S_0''(0) = S_1''(0)$ $\qquad \left\{ \because S_0''(x) = 6a_1x + 2b_1 \atop S_1''(x) = 6a_2x + 2b_2 \right.$

$\Rightarrow 2b_1 = 2b_2$

$\Rightarrow b_1 = b_2$ ——— (6)

Finally, $S_0''(-1) = 0$ and $S_1''(1) = 0$

$\Rightarrow \quad -6a_1 + 2b_1 = 0$ and $6a_2 + 2b_2 = 0$

$\Rightarrow \quad b_1 = 3a_1$ ——— (7)

and $b_2 = -3a_2$ ——— (8)

From (6), (7) & (8), $a_1 = -a_2$ ——— (9)

$\therefore$ From (1), (2), (3), (4), we have

$-a_1 + b_1 - c_1 = -1$ ——— (10)

$-a_1 + b_1 + c_1 = -3 \qquad (\because a_2 = -a_1,\ b_2 = b_1,\ c_2 = c_1)$

Subtracting these two equations, we get $c_1 = -1$

$\therefore c_2 = -1 \quad (\because c_2 = c_1)$

Now, by equ. (10), $\quad -a_1 + b_1 + 1 = -1$

$\Rightarrow -a_1 + b_1 = -2$

$\Rightarrow -a_1 + 3a_1 = -2 \qquad (\because$ by (7), $b_1 = 3a_1)$

$\Rightarrow \quad 2a_1 = -2 \Rightarrow a_1 = -1$

$\therefore$ By equ. (9), $a_2 = 1$

$\therefore b_1 = -3,\ b_2 = -3 \qquad$ (by (7) & (8))

Hence, $a_1 = -1,\ b_1 = -3,\ c_1 = -1,\ d_1 = 2,\ a_2 = 1,\ b_2 = -3,\ c_2 = -1,\ d_2 = 2$

$$\therefore S(x) = \begin{cases} -x^3 - 3x^2 - x + 2 \ ; & -1 \le x \le 0 \\ x^3 - 3x^2 - x + 2 \ ; & 0 \le x \le 1 \end{cases}$$

Read the topic of

B-Splines and Approximation

from these pages

(Mathematical Methods by

S.R.K. Iyengar & R.K. Jain)

Appendix 3 (Chapter 4)
B – Splines

B-Splines is an important tool in computer graphics, solution of differential equations and various other areas of engineering and science. *B*-splines form an important class of spline functions. Most other spline functions can be written as a linear combination of B-splines. That is, B-splines form a basis for some spline spaces. It may be mentioned that cubic splines derived in the previous sub-section are global functions. If any knot t_i is changed, then we need to repeat the entire computation. However, *B*-splines are local functions. If any knot t_i is changed, then we need to change only B-splines connected to that knot.

Let the interval $[a, b]$ be subdivided into *n* parts such that $a = t_0 < t_1 < t_2 < ... < t_n = b$. The nodal points $t_0, t_1, ... t_n$ are called *knots*. (Often for theoretical purposes, B-splines are considered over an infinite set of points $.... < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < ....$). Denote a B-spline by the notation $B_i^k(x)$, where *k* is the degree of the spline and *i* corresponds to the knot t_i. Some books use the notation $B_{k,i}(x)$.

We write the approximating *B*-spline function over the whole interval $[t_0, t_n]$ as a linear combination of the corresponding basis *B*-spline functions. The coefficients are determined by satisfying the data. However, we can construct an approximating function for a given data in terms of *B*-splines which may not satisfy the interpolation conditions, that is, the curve of the approximating function may not pass through any of the data points.

Now, we define *B*-splines.

B-spline of degree zero

Consider the interval $[t_i, t_{i+1}]$. On this interval, *B*-spline of degree zero is defined as a constant as follows

$$B_i^0(x) = 1, \text{ for } t_i \le x < t_{i+1}, \qquad \text{(A3.1)}$$
$$= 0, \text{ else where.}$$

$B_i^0(x)$ is plotted in Fig.1.

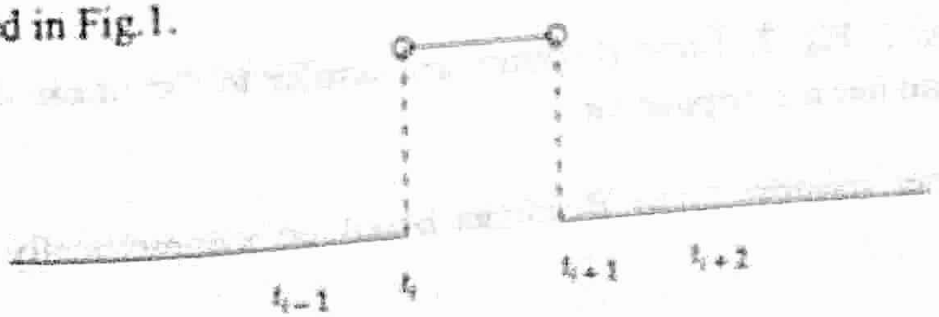


Fig.1. *B*-spline of degree 0.

B-spline of degree zero, $B_i^0(x)$ has the following properties:

(i) On each subinterval, $B_i^0(x)$ is of degree zero.

(ii) $B_i^0(x) \geq 0$ for all i and x. $B_i^0(x) = 0$, outside the interval $[t_i, t_{i+1}]$.

(iii) $B_i^0(x)$ is continuous from the right over all intervals.

(iv) Sum of $B_i^0(x)$ over all intervals is 1 for all x.

(v) The interval $[t_i, t_{i+1}]$ is called the *support* of $B_i^0(x)$.

It is easy to verify the property (iv). Fix a value for x and determine the interval in which it lies. If it lies in the interval $[t_j, t_{j+1}]$, then by property (ii), $\sum_i B_i^0(x) = B_j^0(x) = 1$.

Suppose that the knots are chosen as above. Then, any spline approximation of degree zero for a given data can be written in terms of *B*-splines $B_i^0(x)$, that is, $B_i^0(x)$ form a basis for such a spline approximation.

B-spline of degree one (linear B-spline)

Consider the interval $[t_i, t_{i+2}]$, that is, the knots t_i, t_{i+1}, t_{i+2}. On this interval, *B*-spline of degree one (linear spline) is defined as follows

$$
\begin{aligned}
B_i^1(x) &= \frac{x - t_i}{t_{i+1} - t_i}, \quad \text{for } t_i \leq x < t_{i+1} \\
&= \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}}, \quad \text{for } t_{i+1} \leq x < t_{i+2} \\
&= 0, \quad \text{else where.}
\end{aligned}
\tag{A3.2}
$$

Consider the case of uniform spacing of knots, $t_{i+1} - t_i = h$, $i = 0, 1, 2, \ldots, n-1$. Setting $u = [(x - t_i)/h]$, we obtain

$$
\begin{aligned}
B_i^1(u) &= u, \quad \text{for } 0 \leq u < 1 \\
&= 2 - u, \quad \text{for } 1 \leq x < 2 \\
&= 0, \quad \text{else where..}
\end{aligned}
\tag{A3.3}
$$

u

$B_i^1(x)$ is plotted in Fig. 2. These B-splines are similar to the linear shape functions used in piecewise linear interpolation.

Remark We can consider linear *B*-splines based on symmetrically placed points t_{i-1}, t, t_{i+1}.

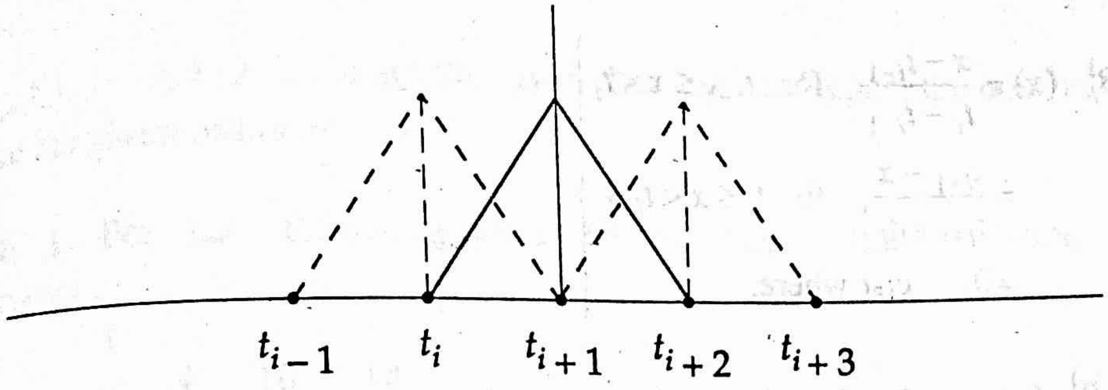


Fig. 2. Linear *B*-spline (uniform spacing).

The linear *B*-spline has the following properties:

(i) $B_i^1(x) \geq 0$ for all i and x. $B_i^1(x) = 0$, outside the interval $[t_i, t_{i+2}]$.

(ii) $B_i^1(x)$ is continuous and differentiable at all points in $[t_i, t_{i+2}]$ except at knots t_i, t_{i+1}, t_{i+2}.

(iii) Sum of $B_i^1(x)$ over all intervals is 1 for all x.

(iv) The interval $[t_i, t_{i+2}]$ is called the *support* of $B_i^1(x)$.

To verify the property (iii), fix a value of x and find the interval in which it lies. Let it lie in $[t_j, t_{j+1}]$. Then, by definition, $B_i^1(x) = 0$, for all x except when $i = j$ or $i = j - 1$. For this fixed value of x, using (A3.2), the contribution from B_{j-1}^1 and B_j^1 gives

$$\sum_i B_i^1(x) = B_{j-1}^1(x) + B_j^1(x) = \frac{t_{j+1} - x}{t_{j+1} - t_j} + \frac{x - t_j}{t_{j+1} - t_j} = 1.$$

Let us construct linear *B*-spline approximation.

Linear *B*-spline approximation

The linear *B*-splines with non-uniform and uniform knot spacing $(u = (x - t_i)/h)$ are defined as

$$B_i^1(x) = \begin{cases} \dfrac{x - t_i}{t_{i+1} - t_i}, & \text{for } t_i \leq x < t_{i+1} \\[2ex] \dfrac{t_{i+2} - x}{t_{i+2} - t_{i+1}}, & \text{for } t_{i+1} \leq x < t_{i+2} \\[2ex] 0, & \text{else where.} \end{cases} \qquad B_i^1(x) = B_i^1(u) = \begin{cases} u, & \text{for } 0 \leq u < 1 \\ 2 - u, & \text{for } 1 \leq u < 2 \\ 0, & \text{else where.} \end{cases}$$

$$B_{i-1}^1(x) = \left.\begin{aligned} &\frac{x - t_{i-1}}{t_i - t_{i-1}}, \quad \text{for } t_{i-1} \le x < t_i \\ &= \frac{t_{i+1} - x}{t_{i+1} - t_i}, \quad \text{for } t_i \le x < t_{i+1} \\ &= 0, \quad \text{else where.} \end{aligned}\right\}$$

$$\begin{aligned} B_{i-1}^1(x) = B_{i-1}^1(u) &= u + 1, \quad \text{for } -1 \le u < 0 \\ &= 1 - u, \quad \text{for } 0 \le u < 1 \\ &= 0, \quad \text{else where.} \end{aligned}$$

$$B_{i+1}^1(x) = \left.\begin{aligned} &\frac{x - t_{i+1}}{t_{i+2} - t_{i+1}}, \quad \text{for } t_{i+1} \le x < t_{i+2} \\ &= \frac{t_{i+3} - x}{t_{i+3} - t_{i+2}}, \quad \text{for } t_{i+2} \le x < t_{i+3} \\ &= 0, \quad \text{else where.} \end{aligned}\right\}$$

$$\begin{aligned} B_{i+1}^1(x) = B_{i+1}^1(u) &= u - 1, \quad \text{for } 1 \le u < 2 \\ &= 3 - u, \quad \text{for } 2 \le u < 3 \\ &= 0, \quad \text{else where.} \end{aligned}$$

Note that the expressions for $B_{i-1}^1(u)$, $B_{i+1}^1(u)$ can be obtained by replacing u by $u + 1$ and $u - 1$ in $B_i^1(u)$ respectively.

The linear B-spline approximation over $[t_0, t_n]$ is written as (Fig. 3)

$$S(x) = \sum_{i=-1}^{n-1} \alpha_i B_i^1(x) = \alpha_{-1} B_{-1}^1(x) + \alpha_0 B_0^1(x) + \alpha_1 B_1^1(x) + \ldots + \alpha_{n-1} B_{n-1}^1(x). \quad \text{(A3.4)}$$

We have $n + 1$ unknowns and $n + 1$ data values. Setting $S(t_i) = f_i$, we can determine uniquely, the values of the coefficients α_i.

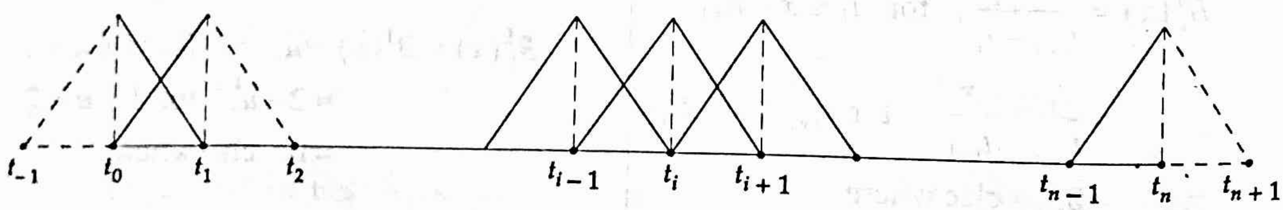


Fig. 3. Linear *B*-spline approximation.

At $t = t_i$, only two basis functions $B_{i-1}^1(t_i), B_i^1(t_i)$ contribute. Therefore,

$$\begin{aligned} S(t_i) = f_i &= \alpha_{i-1} B_{i-1}^1(t_i) + \alpha_i B_i^1(t_i) \\ &= \alpha_{i-1}(1) + \alpha_i(0) = \alpha_{i-1}. \qquad i = 0, 1, 2, \ldots, n \end{aligned}$$

where (t_i, f_i), $i = 0, 1, 2, \ldots, n$ are the given data values. Hence, the coefficients α_i in (A3.4) are the given ordinates.

Example 1 For the following data fit an approximating linear *B*-spline approximation.

x	1	2	3
y	6	10	14

Solution Let $t_0 = 1, t_1 = 2, t_2 = 3, y_0 = 6, y_1 = 10, y_2 = 14$. Using (A3.4), we obtain the spline approximation as

$$S(x) = \alpha_{-1}\, B_{-1}^1(x) + \alpha_0\, B_0^1(x) + \alpha_1\, B_1^1(x)$$
$$= 6\, B_{-1}^1(x) + 10\, B_0^1(x) + 14\, B_1^1(x).$$

Let us now write the contributions of B_i^1 on the given intervals. We have $h = 1$ and $u = x - 1$.

	On (1,2)	On (2, 3)
$B_{-1}^1(x)$	$2 - x$	0
$B_0^1(x)$	$x - 1$	$3 - x$
$B_1^1(x)$	0	$x - 2$

Let us verify the result. We have

$$S(1) = 6(1) + 10(0) + 14(0) = 6.$$
$$S(2) = 6(0) + 10(1) + 14(0) = 10.$$
$$S(3) = 6(0) + 10(0) + 14(1) = 14.$$

Higher degree *B*-splines

All the higher degree *B*-splines can be computed using a recursion formula starting with $B_i^0(x)$. The recursion formula for *k*th degree spline is given by

$$B_i^k(x) = \frac{x - t_i}{t_{i+k} - t_i}\, B_i^{k-1}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}\, B_{i+1}^{k-1}(x), \quad k \geq 1. \qquad \text{(A3.5)}$$

Denote $V_i^k(x) = \dfrac{x - t_i}{t_{i+k} - t_i}$. Then, $1 - V_{i+1}^k(x) = 1 - \dfrac{x - t_{i+1}}{t_{i+k+1} - t_{i+1}} = \dfrac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}$.

Eq. (A3.5) can be written as

$$B_i^k(x) = V_i^k(x) B_i^{k-1}(x) + [1 - V_{i+1}^k(x)] B_{i+1}^{k-1}(x). \qquad \text{(A3.6)}$$

The functions $V_i^k(x)$ and $1 - V_{i+1}^k(x)$ are ratios of lengths. They are similar to the local or natural coordinates used in linear interpolation.

For uniform spacing of knots, $t_{i+1} - t_i = h$, (A3.5) simplifies to

$$B_i^k(x) = \frac{1}{kh}\left[(x - t_i) B_i^{k-1}(x) + (t_{i+k+1} - x) B_{i+1}^{k-1}(x)\right]$$

$$= \frac{1}{k}\left[uB_i^{k-1}(x) + (k+1-u)B_{i+1}^{k-1}(x)\right] \tag{A3.7}$$

where $u = [(x - t_i)/h]$.

Using $B_i^0(x)$, we can compute $B_i^1(x)$ which is a spline polynomial of degree 1. Using $B_i^1(x)$, we can compute $B_i^2(x)$ which is a polynomial of degree 2, etc.

For higher order splines, we use the following knots.

$B_i^2(x)$: $t_i, t_{i+1}, t_{i+2}, t_{i+3}$. (Quadratic B-spline).

$B_i^3(x)$: $t_i, t_{i+1}, t_{i+2}, t_{i+3}, t_{i+4}$. (Cubic B-spline).

Cubic B-spline $B_i^3(x)$ has the following properties.

(i) On each interval, the spline is a polynomial of degree 3.

(ii) The spline and its first two derivatives are continuous over the entire interval.

(iii) $B_i^3(x) \geq 0$ for all i and x. $B_i^3(x) = 0$ outside the interval $[t_i, t_{i+4}]$.

(iv) $\sum_i B_i^3(x) = 1$.

Remark There are various other notations used to represent B-splines. For example, B_j^k or $B_{k,j}$ is used to denote a spline of degree $k - 1$. Also, $B_{k,j}$ may represent a spline at t_j using the required number of points in the backward direction. For a spline of degree 3, say at t_5, we may use the points t_5, t_4, t_3, t_2, t_1.

Let us derive a few B-splines using the formula (A3.5).

In the following expressions, $u = [(x - t_i)/h]$.

We have

$$\left.\begin{aligned} B_i^0(x) &= 1, \quad \text{for } t_i \leq x < t_{i+1}, \\ &= 0, \quad \text{else where.} \end{aligned}\right\} \qquad \left.\begin{aligned} B_{i+1}^0(x) &= 1, \quad \text{for } t_{i+1} \leq x < t_{i+2}, \\ &= 0, \quad \text{else where.} \end{aligned}\right\}$$

$$\begin{aligned} B_i^1(x) &= \frac{x - t_i}{t_{i+1} - t_i} B_i^0(x) + \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} B_{i+1}^0(x) \\ &= \frac{x - t_i}{t_{i+1} - t_i}, \quad \text{for } t_i \leq x < t_{i+1} \\ &= \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}}, \quad \text{for } t_{i+1} \leq x < t_{i+2} \\ &= 0, \quad \text{else where.} \end{aligned}$$

$$B_{i+1}^1(x) = \frac{x - t_{i+1}}{t_{i+2} - t_{i+1}} B_{i+1}^0(x) + \frac{t_{i+3} - x}{t_{i+3} - t_{i+2}} B_{i+2}^0(x)$$

$$= \frac{x - t_{i+1}}{t_{i+2} - t_{i+1}}, \quad \text{for } t_{i+1} \le x < t_{i+2}$$

$$= \frac{t_{i+3} - x}{t_{i+3} - t_{i+2}}, \quad \text{for } t_{i+2} \le x < t_{i+3}$$

$$= 0, \quad \text{else where.}$$

$$B_i^2(x) = \frac{x - t_i}{t_{i+2} - t_i} B_i^1(x) + \frac{t_{i+3} - x}{t_{i+3} - t_{i+1}} B_{i+1}^1(x)$$

$$= \frac{(x - t_i)^2}{(t_{i+2} - t_i)(t_{i+1} - t_i)}, \quad \text{for } t_i \le x < t_{i+1}$$

$$= \frac{(x - t_i)(t_{i+2} - x)}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} + \frac{(t_{i+3} - x)(x - t_{i+1})}{(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+1})}, \quad \text{for } t_{i+1} \le x < t_{i+2} \qquad \text{(A3.8)}$$

$$= \frac{(t_{i+3} - x)^2}{(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})}, \quad \text{for } t_{i+2} \le x < t_{i+3}$$

$$= 0, \quad \text{elsewhere.}$$

$$B_{i+1}^2(x) = \frac{(x - t_{i+1})^2}{(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+1})}, \quad \text{for } t_{i+1} \le x < t_{i+2}$$

$$= \frac{(x - t_{i+1})(t_{i+3} - x)}{(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})} + \frac{(t_{i+4} - x)(x - t_{i+2})}{(t_{i+4} - t_{i+2})(t_{i+3} - t_{i+2})}, \quad \text{for } t_{i+2} \le x < t_{i+3} \qquad \text{(A3.9)}$$

$$= \frac{(t_{i+4} - x)^2}{(t_{i+4} - t_{i+2})(t_{i+4} - t_{i+3})}, \quad \text{for } t_{i+3} \le x < t_{i+4}$$

$$= 0, \quad \text{elsewhere.}$$

For uniform spacing of knots, we obtain

$$B_i^2(u) = [u^2/2], \quad \text{for } 0 \le u < 1,$$

$$= -u^2 + 3u - (3/2), \quad \text{for } 1 \le u < 2, \qquad \text{(A3.10)}$$

$$= [(3-u)^2/2], \quad \text{for } 2 \le u < 3,$$

$$= 0, \quad \text{else where.}$$

We can also obtain these expressions directly from (A3.7) as

$$B_i^2(x) = \frac{1}{2}\left[uB_i^1(u) + (3-u)B_{i+1}^1(u)\right]$$

$$B_{i+1}^2(u) = [(u-1)^2/2], \quad \text{for } 1 \le u < 2,$$

$$= -(u-1)^2 + 3(u-1) - (3/2), \quad \text{for } 2 \le u < 3, \qquad \text{(A3.11)}$$

$$= [(4-u)^2/2], \quad \text{for } 3 \le u < 4,$$

$$= 0, \quad \text{else where.}$$

since $(1/2)[(u-1)(3-u) + (4-u)(u-2)] = -(u-1)^2 + 3(u-1) - (3/2)$.

$$B_{i-1}^2(u) = [(u+1)^2/2], \quad \text{for } -1 \le u < 0,$$
$$= -(u+1)^2 + 3(u+1) - (3/2), \quad \text{for } 0 \le u < 1, \tag{A3.12}$$
$$= [(2-u)^2/2], \quad \text{for } 1 \le u < 2,$$
$$= 0, \quad \text{else where.}$$

$$B_{i-2}^2(u) = [(u+2)^2/2], \quad \text{for } -2 \le u < -1,$$
$$= -(u+2)^2 + 3(u+2) - (3/2), \quad \text{for } -1 \le u < 0, \tag{A3.13}$$
$$= [(1-u)^2/2], \quad \text{for } 0 \le u < 1,$$
$$= 0, \quad \text{else where.}$$

Again, the expressions for $B_{i-2}^2(u), B_{i-1}^2(u)$, $B_{i+1}^2(u)$ can also be obtained by replacing u by $u + 2$, $u + 1$ and $u - 1$ in $B_i^2(u)$ respectively. $B_i^2(u)$ is a bell shaped curve (Fig. 4).

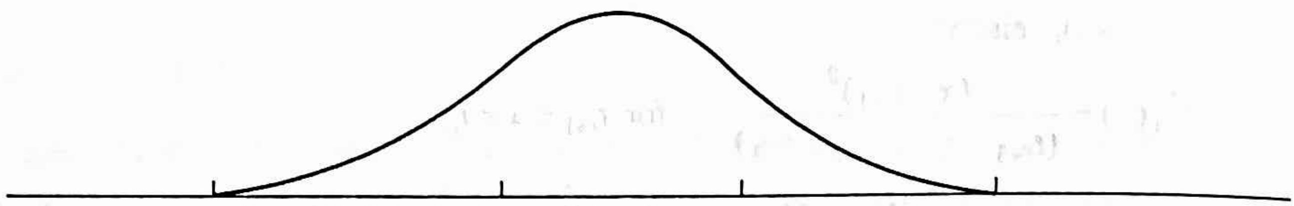


Fig. 4. Quadratic *B*-spline.

Quadratic *B*-spline approximation

Consider constructing a spline approximation using quadratic basis splines. The spline over (t_0, t_n) is written as (Fig. 5)

$$S(x) = \sum_{i=-2}^{n-1} \alpha_i \, B_i^2(x). \tag{A3.14}$$

There are $n + 2$ parameters, α_i, $i = -2, -1, 0, 1, \ldots, n - 1$, to be determined. The interpolation conditions $S(t_i) = f_i$, $i = 0, 1, \ldots, n$ gives $n + 1$ equations. An extra condition is required to obtain α_i uniquely. We can use the condition $S'(t_0) = f_0'$, or $S'(t_n) = f_n'$ to obtain an extra equation.

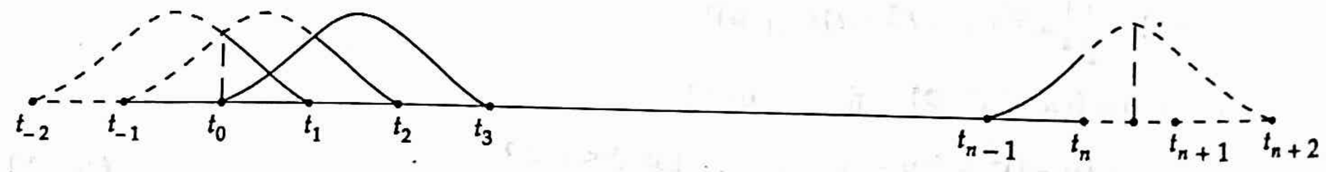


Fig. 5. Quadratic B-spline approximation.

Now, at $t = t_i$, only two basis functions $B_{i-1}^2(t_i) = (1/2)$ and $B_{i-2}^2(t_i) = (1/2)$ contribute a non-zero value, since $B_i^2(t_i) = 0$, (these are contributions at $u = 0$, on the interval $0 \le u < 1$).

The interpolating condition $S(t_i) = f_i$ gives

$$\frac{1}{2}\alpha_{i-2} + \frac{1}{2}\alpha_{i-1} = f_i, \quad i = 0, 1, 2, \ldots, n.$$

If the condition $S'(t_0) = f_0'$ is used, we get

$$(B_{-2}^2)'(t_0)\alpha_{-2} + (B_{-1}^2)'(t_0)\alpha_{-1} = -\frac{1}{h}\alpha_{-2} + \frac{1}{h}\alpha_{-1} = f_0'$$

since, on $0 \le u < 1$,

$$(B_{-2}^2)'(u) = -[(1-u)/h], \; (B_{-2}^2)'(0) = -(1/h),$$

$$(B_{-1}^2)'(u) = [-2(u+1)+3]/h, \; (B_{-1}^2)'(0) = (1/h).$$

In matrix form, we get the system of equations as

$$\begin{bmatrix} -1 & 1 & & & & & \\ 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & 1 & 1 & & & \\ .. & .. & .. & .. & .. & .. & .. \\ & & & & & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ .. \\ \alpha_{n-1} \end{bmatrix} = \begin{bmatrix} hf_0' \\ 2f_0 \\ 2f_1 \\ 2f_2 \\ .. \\ 2f_n \end{bmatrix} \tag{A3.15}$$

We solve this system and substitute the values of α_i in (A3.14) to obtain the quadratic B-spline approximation.

Example 2 For the following data values fit a quadratic B-spline

x	0	1	2	3
$f(x)$	1	6	31	94

Use the condition $f_0' = 1$.

Solution We have $h = 1$. The spline approximation in $(t_0, t_n) \to (0, 3)$ is given by

$$S(x) = \sum_{i=-2}^{2} \alpha_i B_i^2(x)$$

$$= \alpha_{-2}B_{-2}^2(x) + \alpha_{-1}B_{-1}^2(x) + \alpha_0 B_0^2(x) + \alpha_1 B_1^2(x) + \alpha_2 B_2^2(x).$$

Using (A3.8)-(A3.13), we obtain the contributions of the B-splines as follows.

	On (0, 1)	On (1, 2)	On (2, 3)
$B_{-2}^2(x)$	$0.5(1-x)^2$	0	0
$B_{-1}^2(x)$	$-(x+1)^2 + 3(x+1) - 1.5$	$0.5(2-x)^2$	0
$B_0^2(x)$	$0.5x^2$	$-x^2 + 3x - 1.5$	$0.5(3-x)^2$

$B_1^2(x)$	0	$0.5(x-1)^2$	$-(x-1)^2+3(x-1)-1.5$
$B_2^2(x)$	0	0	$0.5(x-2)^2$

Using the condition $f_0' = 1$, the system of equations (A3.15) become

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 12 \\ 62 \\ 188 \end{bmatrix}.$$

The first two equations give $\alpha_{-2} = 0.5, \alpha_{-1} = 1.5$. From the remaining equations, we get

$$\alpha_0 = 12 - \alpha_{-1} = 10.5, \quad \alpha_1 = 62 - \alpha_0 = 51.5, \quad \alpha_2 = 188 - \alpha_1 = 136.5.$$

The quadratic B-spline approximation is given by

$$S(x) = 0.5\, B_{-2}^2(x) + 1.5\, B_{-1}^2(x) + 10.5\, B_0^2(x) + 51.5\, B_1^2(x) + 136.5\, B_2^2(x).$$

Let us verify that this spline satisfies the given data.

$$S(0) = 0.5\, B_{-2}^2(0) + 1.5\, B_{-1}^2(0) + 10.5\, B_0^2(0) + 51.5\, B_1^2(0) + 136.5\, B_2^2(0)$$
$$= 0.5(0.5) + 1.5(0.5) + 10.5(0) = 1.0.$$

$$S(1) = 0.5\, B_{-2}^2(1) + 1.5\, B_{-1}^2(1) + 10.5\, B_0^2(1) + 51.5\, B_1^2(1) + 136.5\, B_2^2(1)$$
$$= 0 + 1.5(0.5) + 10.5(0.5) = 6.0.$$

$$S(2) = 0.5\, B_{-2}^2(2) + 1.5\, B_{-1}^2(2) + 10.5\, B_0^2(2) + 51.5\, B_1^2(2) + 136.5\, B_2^2(2)$$
$$= 0 + 0 + 10.5(0.5) + 51.5(0.5) + 136.5(0) = 31.0.$$

$$S(3) = 0.5\, B_{-2}^2(3) + 1.5\, B_{-1}^2(3) + 10.5\, B_0^2(3) + 51.5\, B_1^2(3) + 136.5\, B_2^2(3)$$
$$= 0 + 0 + 0 + 51.5(0.5) + 136.5(0.5) = 94.0.$$

Cubic *B*-spline

For computing $B_i^3(x)$, we require the expressions for $B_i^2(x)$ and $B_{i+1}^2(x)$. Cubic B-spline is given by

$$B_i^3(x) = \frac{x - t_i}{t_{i+3} - t_i} B_i^2(x) + \frac{t_{i+4} - x}{t_{i+4} - t_{i+1}} B_{i+1}^2(x),$$

In terms of $u = [(x - t_i)/h]$, we obtain

$$B_i^3(u) = \frac{1}{3}[uB_i^2(x) + (4-u)B_{i+1}^2(x)],$$
$$= (u^3/6), \quad \text{for } 0 \le u < 1,$$
$$= [-3(u-1)^3 + 3(u-1)^2 + 3(u-1) + 1]/6, \quad \text{for } 1 \le u < 2, \qquad \text{(A3.16)}$$
$$= [-3(3-u)^3 + 3(3-u)^2 + 3(3-u) + 1]/6, \quad \text{for } 2 \le u < 3,$$
$$= [(4-u)^3/6], \quad \text{for } 3 \le u < 4,$$
$$= 0, \quad \text{else where}$$

since, on $1 \le u < 2$,

$$\frac{1}{3}\left[u\left\{-u^2 + 3u - \frac{3}{2}\right\} + \frac{(4-u)}{2}(u-1)^2\right] = \frac{1}{6}\left[-3(u-1)^3 + 3(u-1)^2 + 3(u-1) + 1\right]$$

and on $2 \le u < 3$,

$$\frac{1}{3}\left[\frac{u(3-u)^2}{2} + (4-u)\left\{-(u-1)^2 + 3(u-1) - \frac{3}{2}\right\}\right] = \frac{1}{6}\left[-3(3-u)^3 + 3(3-u)^2 + 3(3-u) + 1\right]$$

Note that

$$B_i^3(t_i) = B_i^3(u=0) = 0, B_i^3(t_{i+1}) = B_i^3(u=1) = 1/6,$$
$$B_i^3(t_{i+2}) = B_i^3(u=2) = 2/3, \text{ and } B_i^3(t_{i+3}) = B_i^3(u=3) = 1/6.$$

The graph of $B_i^3(u)$ is a bell shaped curve (Fig. 6). It is easy to verify that $B_i^3(u)$ is continuous and has continuous first and second derivates on [0, 4]. Further, $\left[\frac{d}{du}B_i^3(u)\right] = 0$, at $u = 2$. The tangent to the graph of $B_i^3(u)$ at the middle node $u = 2$ is parallel to the axis.

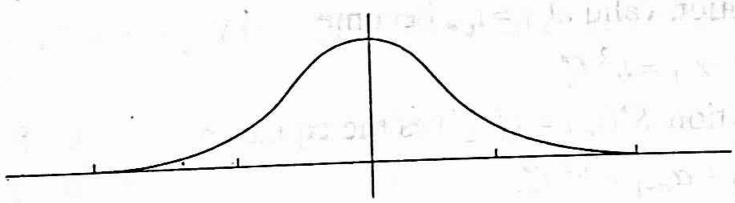


Fig. 6. Cubic B-spline.

The cubic B-spline over (t_0, t_n) is written as

$$S(x) = \sum_{i=-3}^{n-1} \alpha_i\, B_i^3(x)$$
$$= \alpha_{-3}B_{-3}^3(x) + \alpha_{-2}B_{-2}^3(x) + \alpha_{-1}B_{-1}^3(x) + \alpha_0 B_0^3(x) + \ldots + \alpha_{n-1}B_{n-1}^3(x). \qquad \text{(A3.17)}$$

There are $n + 3$ parameters, α_i, $i = -3, -2, -1, 0, 1, \ldots, n-1$, to be determined. The interpolation conditions $S(t_i) = f_i$, $i = 0, 1, \ldots, n$ gives $n + 1$ equations. In order that unique solution exists, we assume two additional conditions at the end points of the interval (t_0, t_n). We may take these conditions as the natural cubic spline conditions $S''(t_0) = 0 = S''(t_n)$.

Alternately, we may derive the values of $S''(t_0), S''(t_n)$ through some computational procedure.

Now, at $t = t_i$, only three basis functions $B_{i-1}^3(t_i) = (1/6)$, $B_{i-2}^3(t_i) = (2/3)$ and $B_{i-3}^3(t_i) = (1/6)$ contribute a non-zero value, since $B_i^3(t_i) = 0$, (these are contributions at $u = 0$, on the interval $0 \le u < 1$).

The interpolating condition $S(t_i) = f_i$ gives

$$\frac{1}{6}\alpha_{i-3} + \frac{2}{3}\alpha_{i-2} + \frac{1}{6}\alpha_{i-1} = f_i, \qquad i = 0, 1, 2, \dots, n.$$

The condition $S''(t_0) = f_0''$ gives the equation

$$(B_{-3}^3)''(t_0)\alpha_{-3} + (B_{-2}^3)''(t_0)\alpha_{-2} + (B_{-1}^3)''(t_0)\alpha_{-1} = f_0''.$$

The expressions for $B_{i-3}^3(u), B_{i-2}^3(u), B_{i-1}^3(u)$ can be obtained by replacing u by $u +$ 3, $u + 2$ and $u + 1$ in $B_i^3(u)$ (eq. A3.16) respectively.

On $0 \le u < 1$, we get

$$B_{i-3}^3(u) = [(1-u)^3/6], \quad B_{i-2}^3(u) = [-3(1-u)^3 + 3(1-u)^2 + 3(1-u) + 1]/6,$$

$$B_{i-1}^3(u) = [-3u^3 + 3u^2 + 3u + 1]/6, \quad B_i^3(u) = u^3/6.$$

$$(B_{-3}^3)''(u) = [(1-u)/h^2], \quad (B_{-3}^3)''(0) = (1/h^2),$$

$$(B_{-2}^3)''(u) = [-3(1-u) + 1]/h^2, \quad (B_{-2}^3)''(0) = (-2/h^2),$$

$$(B_{-1}^3)''(u) = [(-3u + 1)/h^2], \quad (B_{-1}^3)''(0) = (1/h^2),$$

$$(B_0^3)''(u) = [u/h^2], \quad (B_0^3)''(0) = 0.$$

Therefore, the equation valid at $t = t_0$, becomes

$$\alpha_{-3} - 2\alpha_{-2} + \alpha_{-1} = h^2 f_0''.$$

Similarly, the condition $S''(t_n) = f_n''$ gives the equation

$$\alpha_{n-3} - 2\alpha_{n-2} + \alpha_{n-1} = h^2 f_n''.$$

In matrix form, we get the system of equations as

$$\begin{bmatrix} 1 & -2 & 1 & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & 1 & 4 & 1 & \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ & & & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{-3} \\ \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \cdot\cdot \\ \alpha_{n-1} \end{bmatrix} = \begin{bmatrix} h^2 f_0'' \\ 6f_0 \\ 6f_1 \\ 6f_2 \\ \cdot\cdot \\ h^2 f_n'' \end{bmatrix} \qquad \text{(A3.18)}$$

If we eliminate the third element, 1, from the first equation and the $(n+1)$th element, 1, from the last equation then the resulting system of equations is a tri-diagonal system. The interior rows have 1/6, 4/6 and 1/6 (or 1, 4, 1 if multiplied by 6) on the sub-diagonal, diagonal and super-diagonal elements. We solve this system and substitute the values of α_i in (A3.17) to obtain the cubic B-spline approximation.

We can also consider a cubic spline at t_i, using the symmetrically placed knots at $t_{i-2}, t_{i-1}, t_i, t_{i+1}, t_{i+2}$.

Example 3 For the following data values fit a cubic B-spline

x	0	1	2	3	4
$f(x)$	1	1	17	82	257

Use the conditions $f''(0) = 0,\ f''(4) = 192$.

Solution We have $h = 1$. The cubic B-spline approximation in $(t_0, t_n) \to (0, 4)$ is given by (see Fig. 7)

$$S(x) = \alpha_{-3}B_{-3}^3(x) + \alpha_{-2}B_{-2}^3(x) + \alpha_{-1}B_{-1}^3(x) + \alpha_0 B_0^3(x) + \alpha_1 B_1^3(x) + \alpha_2 B_2^3(x) + \alpha_3 B_3^3(x).$$

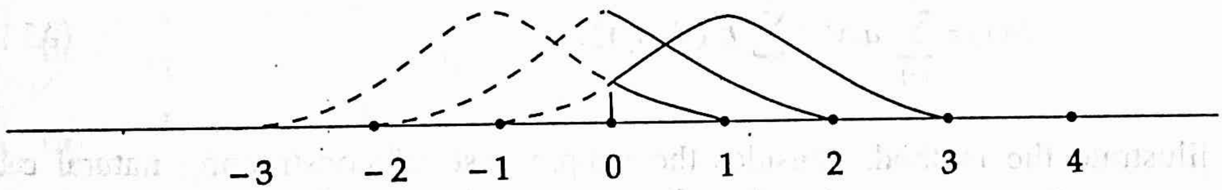


Fig. 7. Example 3.

Using the conditions $f''(0) = 0, f''(4) = 192$, the system of equations (A3.18) become

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{-3} \\ \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 6 \\ 102 \\ 492 \\ 1542 \\ 192 \end{bmatrix}$$

Subtracting the first and second equations, we get $\alpha_{-2} = 1$. Subtracting the sixth and seventh equations, we get $\alpha_2 = 225$. The solutions of the remaining variables are

$$\alpha_{-3} = 3.1785714,\ \ \alpha_{-1} = -1.1785715,\ \ \alpha_0 = 9.7142857,\ \ \alpha_1 = 64.321429,$$

$\alpha_3 = 577.67857$.

The cubic B-spline approximation is given by

$$S(x) = 3.1785714\, B_{-3}^3(x) + B_{-2}^3(x) - 1.1785715\, B_{-1}^3(x) + 9.7142857\, B_0^3(x)$$
$$+ 64.321429\, B_1^3(x) + 225\, B_2^3(x) + 577.67857\, B_3^3(x).$$

The contributions of the cubic B-splines on each interval can be written using (A3.16).

For example, on [0, 1], for $h = 1$, and $t_0 = 0$, we get $u = x$. We have the following contributions.

$B_{-3}^3(x)$: $(1-x)^3/6$. $B_{-2}^3(x)$: $[-3(1-x)^3 + 3(1-x)^2 + 3(1-x) + 1]/6$.
$B_{-1}^3(x)$: $(-3x^3 + 3x^2 + 3x + 1)/6$. $B_0^3(x)$: $(x^3/6)$. $B_1^3(x)$: 0. $B_2^3(x)$: 0. $B_3^3(x)$: 0.

Alternate B-spline base functions

It is possible to use different base functions to construct $S(x)$, an element of the space S_n^k. We describe this procedure below.

Denote,
$$(x - t_i)_+^k = (x - t_i)^k, \quad \text{if } x \ge t_i$$
$$= 0, \quad \text{if } x < t_i.$$

For a cubic spline, we write

$$S(x) = \sum_{i=1}^{k} a_i x^i + \sum_{i=1}^{n-1} b_i (x - t_i)_+^3. \tag{A3.17}$$

To illustrate the method, consider the simple case of constructing natural cubic splines using the integer knots at 0, 1, 2, 3, 4. The spline satisfies the conditions

$$S(0) = 0, S'(0) = 0, S''(0) = 0;\ S(4) = 0, S'(4) = 0, S''(4) = 0; \text{and } S'(2) = 0. \tag{A3.18}$$

On [0, 1], we have

$$S(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3. \tag{A3.19}$$

Now, $S(0) = 0$ gives $a_0 = 0$. $S'(0) = 0$, gives $a_1 = 0$. $S''(0) = 0$, gives $a_2 = 0$.

Hence, $S(x) = a_3 x^3$, where a_3 is arbitrary.

On [0, 2], we have

$$S(x) = a_3 x^3 + b_1 (x-1)_+^3. \tag{A3.20}$$

Setting $S'(2) = 0$, we get $12a_3 + 3b_1 = 0$, or $b_1 = -4a_3$.

Hence,
$$S(x) = a_3[x^3 - 4(x-1)_+^3]. \tag{A3.21}$$

On [0, 3], we have

$$S(x) = a_3[x^3 - 4(x-1)_+^3] + b_2 (x-2)_+^3. \tag{A3.22}$$

On [0, 4], we have

$$S(x) = a_3[x^3 - 4(x-1)_+^3] + b_2 (x-2)_+^3 + b_3 (x-3)_+^3. \tag{A3.23}$$

Using the conditions at $x = 4$, we get

$$S(4) = 0 = (64 - 108)a_3 + 8b_2 + b_3 = -44a_3 + 8b_2 + b_3.$$

$$S'(4) = 0 = (48 - 108)a_3 + 12b_2 + 3b_3 = -60a_3 + 12b_2 + 3b_3.$$

$$S''(4) = 0 = (24 - 72)a_3 + 12b_2 + 6b_3 = -48a_3 + 12b_2 + 6b_3. \qquad \text{(A3.24)}$$

The system of equations (A3.24) is consistent and has the solution $b_2 = 6a_3, b_3 = -4a_3$. Hence, the spline is given by

$$S(x) = a_3[x^3 - 4(x-1)_+^3 + 6(x-2)_+^3 - 4(x-3)_+^3]. \qquad \text{(A3.25)}$$

If we impose one more condition on $S(x)$, then the solution can be obtained uniquely. For example, we may use the condition $S(1) = S(3) = c$, (due to symmetry).

This example shows that the functions $1, x, x^2, x^3, (x-1)_+^3, (x-2)_+^3, (x-3)_+^3$ form a basis for S_k^3. However, it was shown that the use of the basis functions

$$1, x, x^2, \ldots, x^k, (x - t_0)_+^3, (x - t_1)_+^3 \ldots + (x - t_{n-1})_+^3,$$

produces a highly ill-conditioned system of equations for the solution of the coefficients α_j in the cubic B-spline approximation. Hence, one should use the B-splines B_i^k as the basis functions for $S(x)$.

Exercise

1. For the following data, fit a linear B-spline approximation.

(i)

x	0	1	2	3
y	1	5	21	55

(ii)

x	0	1	2	3
y	6	11	18	32

2. For the following data, fit a quadratic B-spline approximation.

(i)

x	0	1	2	3
y	1	5	21	55

Use the condition $y_0' = 1$.

(ii)

x	1	2	3	4
y	5	25	91	269

Use the condition $y_0' = 0$.

[Answers: **1.** (i) $\alpha_{-1} = 1, \alpha_0 = 5, \alpha_1 = 21, \alpha_2 = 55.$ (ii) $\alpha_{-1} = 6, \alpha_0 = 11, \alpha_1 = 18, \alpha_2 = 32..$

2. (i) $\alpha_{-2} = 0.5, \alpha_{-1} = 1.5, \alpha_0 = 8.5, \alpha_1 = 33.5, \alpha_2 = 76.5.$
(ii) $\alpha_{-2} = 5, \alpha_{-1} = 5, \alpha_0 = 45, \alpha_1 = 137, \alpha_2 = 401$]