Q

Centroid / C.O.G / C.O.M

The points used to represent entire length, area, volume, mass, gravitational force is called as central point.

Centre of gravity: It is a point through which the resultant of the distributed gravity forces acts irrespective to the orientation of the body.

Centre of Mass: It is the point where the entire mass of a body may be assumed to be concentrated.

For most practical cases they are assumed to be same.

C.O.M & C.O.G are differend only when the gravitational field is not uniform and parallel.

Centroid: The point where the entire length, area or volume is assumed to be concentrated is called the centroid.

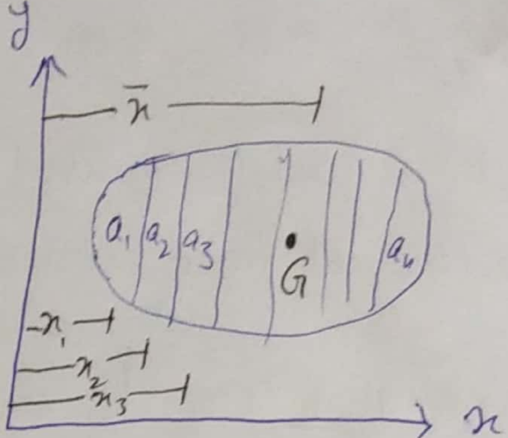
Charactarstics of Centroid / COG / C.O.M

i) A body has only one C.O.G.
ii) Its location does not change even with a change in orientation of the body
iii) It lies in a plane of symmetry.
iv) It is an imaginary point which may occurs outside or inside the body.

① *

I. Location of Centroid/C.O.G.

Consider a body having area A is divided into small strips of area $a_1, a_2, a_3 \dots a_n$ as shown in fig.



Moments of areas of all the strips about y axis.

$a_1x_1 + a_2x_2 + \dots + a_nx_n$ — (i)

moment of total area A about y-axis $= A\bar{x}$ — (ii)

from (i) & (ii)

$$A\bar{x} = \sum ax$$

$$\boxed{\bar{x} = \frac{\sum a.x}{\sum a}}$$

Similarly moment about x axis

$$\boxed{\bar{y} = \frac{\sum a.y}{\sum a}}$$

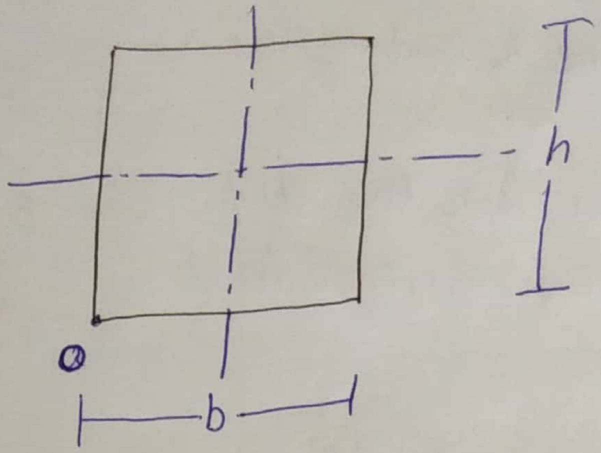
If mass is considered, then

$$\boxed{\bar{x} = \frac{\sum m.x}{\sum m}} \quad ; \quad \boxed{\bar{y} = \frac{\sum my}{\sum m}}$$

Centroid of some lamina (Integration method)

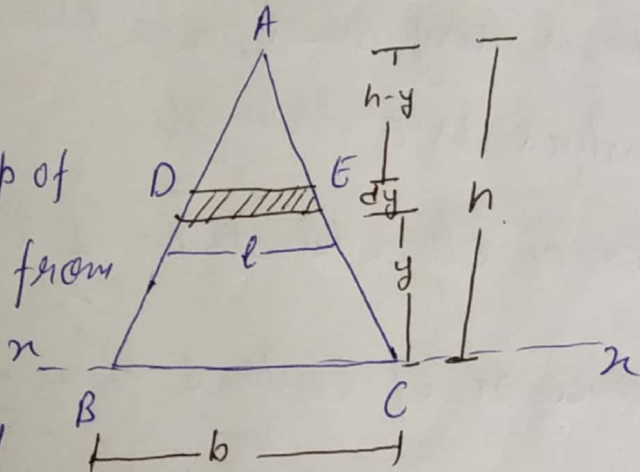
Rectangle

(From point O) $\boxed{\bar{x} = \frac{b}{2}, \bar{y} = \frac{h}{2}}$



Triangle:

Consider an elementary strip of thickness dy at a distance y from base.



Area of elemental part = $l \cdot dy$

Moment about x axis = $l \cdot dy \cdot y$ — (i)

From similarity of triangles

$$\frac{l}{b} = \frac{h-y}{h} \quad ; \quad l = b\left(1 - \frac{y}{h}\right) \quad \text{— (ii)}$$

from (i) and (ii)

Moment of element about x axis = $b\left(1 - \frac{y}{h}\right) \cdot y\, dy$

If $\bar{y}$ be centroid of triangle

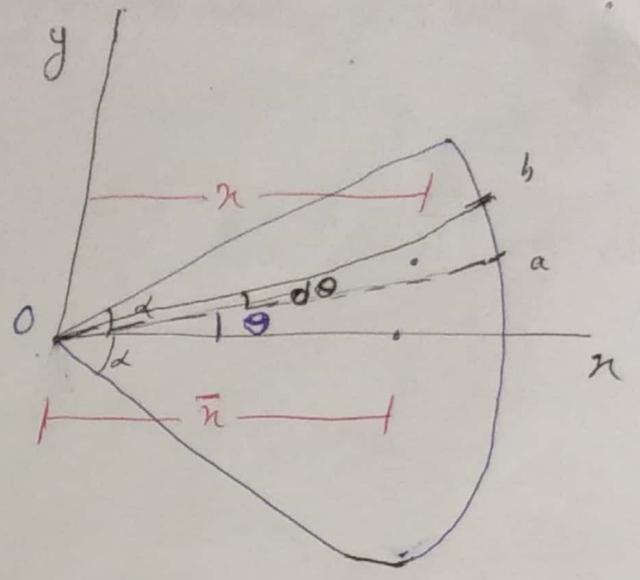
$$\frac{1}{2} bh \cdot \bar{y} = \int_0^h b\left(1 - \frac{y}{h}\right) y\, dy = \frac{bh^2}{6}$$

$$\boxed{\bar{y} = \frac{h}{3}}$$

② *

Sector of a circle

Let radius r and angle 2α.



Consider an elementary strip that subtends an angle $d\theta$ at the centre.

Length of $ab = r \cdot d\theta$

As $d\theta$ is very small, the elementary strip can be considered as a triangle.

Area of elemental strip $= \frac{1}{2} (r)(r\,d\theta) = \frac{r^2 d\theta}{2}$ - (i)

Distance x of centroid from y axis. (elemental part)

$x = \frac{2}{3} r \cdot \cos\theta$ - (ii)

moment of area of elemental strip about y axis

from (i) and (ii)

$= \frac{1}{3} r^3 \cos\theta \, d\theta$ - (iii)

If $\bar{x}$ be centroid then, moment of entire lamina

$\bar{x} \cdot \int_{-\alpha}^{+\alpha} \frac{r^2 d\theta}{2}$ - (iv)

From (iii) and (iv)

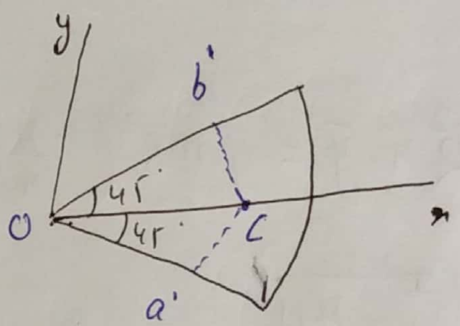
$$\bar{x} \cdot \int_{-\alpha}^{\alpha} \frac{r^2 d\theta}{2} = \int_{-\alpha}^{\alpha} \frac{1}{3} r^3 \cos\theta \, d\theta$$

$$\bar{x} \cdot \frac{r^2}{2} [\theta]_{-\alpha}^{\alpha} = \frac{r^3}{3} [\sin\theta]_{-\alpha}^{\alpha}$$

$$\bar{x} \cdot \frac{r^2}{2} (2\alpha) = \frac{r^3}{3} \cdot 2 \sin\alpha$$

$$\bar{x} = \frac{2r \cdot \sin\alpha}{3 \quad \alpha}$$

$$\boxed{\bar{x} = \frac{2r\sin\alpha}{3\alpha}}$$



Special Case:

Semi circle

$2\alpha = 180°$

$\alpha = \frac{\pi}{2}$

$$\bar{x} = \frac{2r \cdot \sin \pi/2}{3\frac{\pi}{2}}$$

$$\boxed{\bar{x} = \frac{4r}{3\pi}}$$

Quadrant of a circle

$2\alpha = 90°$

$\alpha = \pi/4$

$$\boxed{\bar{x} = \sqrt{2}\,\frac{4r}{3\pi}}$$

$$Oa' = \sqrt{2} \cdot \frac{4r}{3\pi} \cdot \cos 45° = Ob'$$

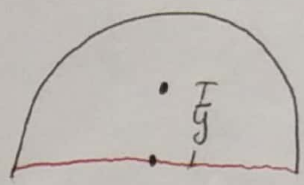
$$\boxed{Oa' = Ob' = \frac{4r}{3\pi}}$$

Scanned by CamScanner

③ *

Circular Arc

$$\boxed{\bar{y} = \frac{2r}{\pi}}$$

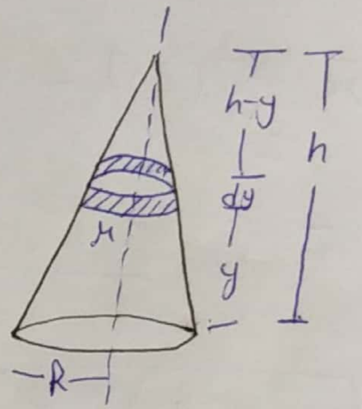


Solid right circular cone

Volume of element $= \pi r^2 dy$

$$\frac{1}{3}\pi R^2 h \cdot \bar{y} = \int_0^h \pi R^2 \left(1 - \frac{y}{h}\right)^2 y\,dy = \pi R^2 \cdot \frac{h^2}{12}$$

$$\boxed{\bar{y} = \frac{h}{4}}$$

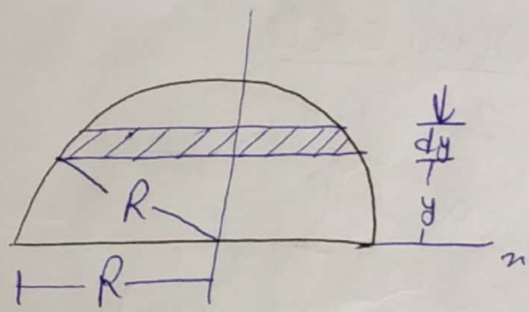


Solid Hemisphere

1 Volume of element $= \pi(R^2 - y^2) \cdot dy$

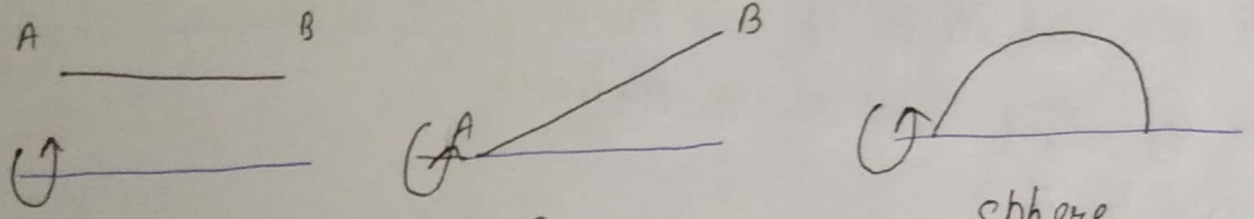
$$\therefore \frac{2}{3}\pi R^3 \cdot \bar{y} = \int_0^R \pi (R^2 - y^2)\, y\,dy = \pi \frac{R^4}{4}$$

$$\boxed{\bar{y} = \frac{3R}{8}}$$



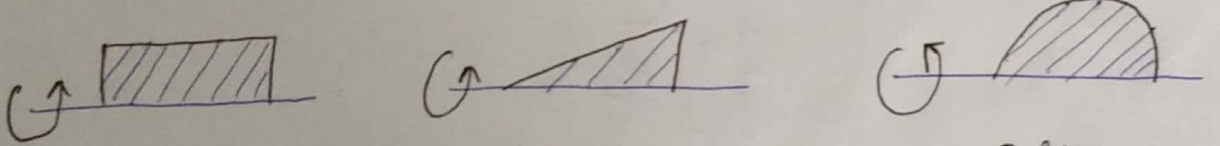
Pappus - Guldinus Theorems

When a plane curve rotates about a fixed ~~point~~ axis, it generates a surface called surface of revalution.



Cylinder Cone Sphere

When a plane area is rotated about a fixed axis, it generates a body called the body of revalution.



Solid Cylinder. Solid cone Solid sphere

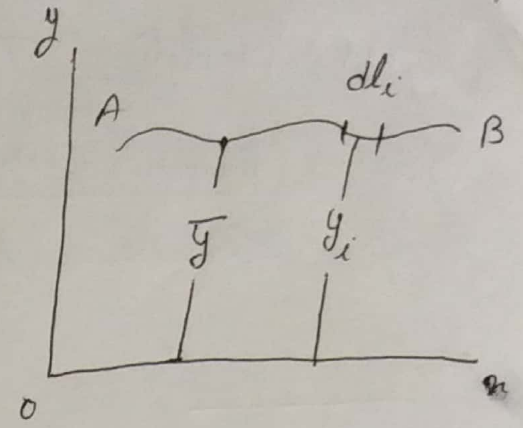
Theorem 1. The area of the surface generated by rotating any plane curve about a non-intersecting axis in its plane is equal to the product of the length of the curve and the distance travelled by its centroid.

(4)*

Let AB be any plane curve of length l that lies in the xy plane and does not intersect the x axis.

Consider an elemental length dl_i.

Let y-co-ordinate of its mid point be y_i and that of the entire curve be $\bar{y}$.



When this curve is rotated about the x-axis, then

area generated by elemental length $= 2\pi y_i \, dl_i$

area generated by entire line $= \int 2\pi y_i \, dl_i$

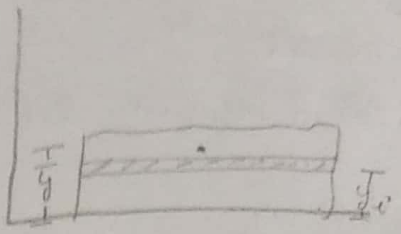
$$\text{area generated} = 2\pi \int y \cdot dl$$
$$= 2\pi \bar{y} \cdot l$$
$$l \cdot 2\pi \bar{y}$$

Area generated = (Length of the generating curve) x (Distance travelled by centroid of the curve)

- Theorem-II

The volume of the solid generated by rotating any plane figure about a non intersecting axis in its plane is equal to the product of area of the figure and the distance travelled by its centroid.

Consider an elemental strip



Volume generated by elemental area $= 2\pi y_i \, dA_i$

Volume generated by entire area $= \int 2\pi y_i \, dA_i$

Volume generated $= 2\pi \cdot A\bar{y}$

$$\boxed{\text{Volume generated} = A \cdot 2\pi\bar{y}}$$

= (Area of figure) × (distance travelled by centroid)

Centroid of semi circular arc

$$4\pi r^2 = \pi r \cdot 2\pi\bar{y}$$

$$\boxed{\bar{y} = \frac{2r}{\pi}}$$

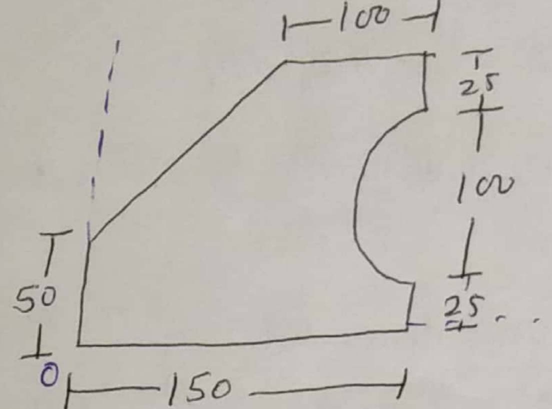
Centroid of a quater circle

$$\frac{2}{3}\pi r^3 = \frac{\pi r^2}{4} \cdot 2\pi\bar{y}$$

$$\boxed{\bar{y} = \frac{4r}{3\pi}}$$

⑤*

Q Locate the centroid of the area shown in fig.
All dimensions are in mm.

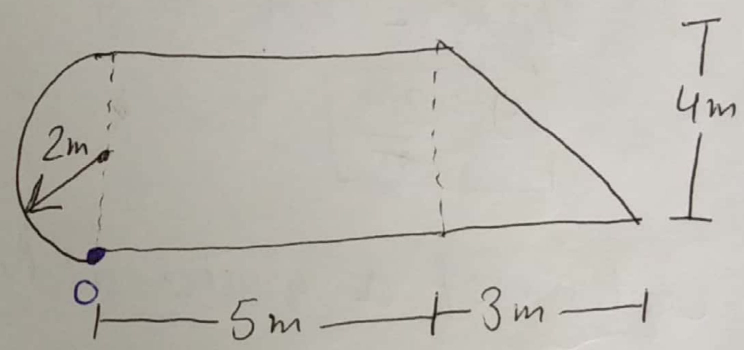


From point O.

$\bar{x} = 70.94$ mm

$\bar{y} = 68.52$ mm

Q Locate the centroid of area shown in fig

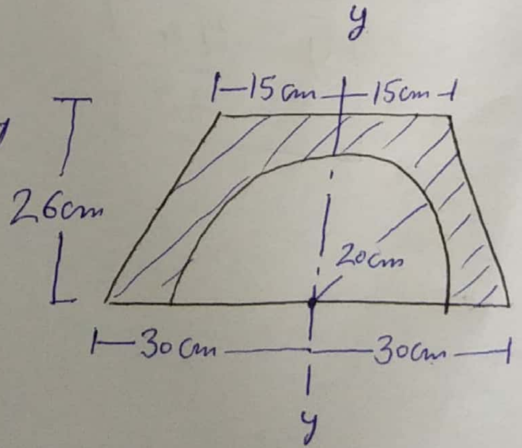


From point O

$\bar{x} = 2.5$ m

$\bar{y} = 1.875$ m

Q Locate the position of centroid of the plane shaded area as shown in fig

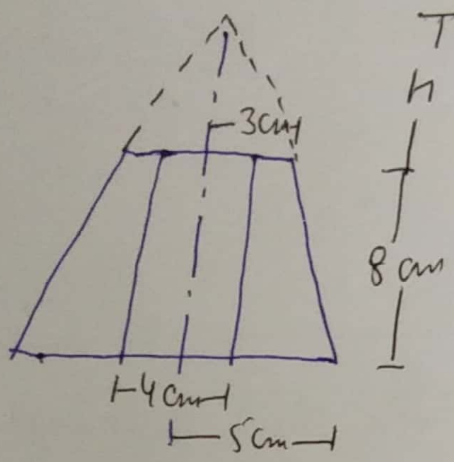


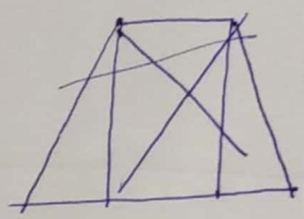
Sol. As the given lamina is symmetrical about x-axis, centroid lie on yy axis.

$\bar{y} = 15.11$ cm

Q- The frustum of a right circular cone has a bottom radius 5 cm, top radius 3 cm and height 8 cm. A co-axial cylindrical hole of 4 cm dia. is made througout the frustum. Locate the position of C.O.G. of the remaining solid.

Sol-



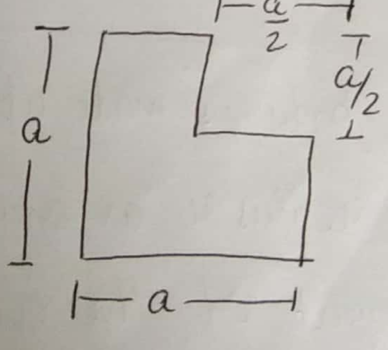


from similarly of triangle

$$\frac{5}{3} = \frac{8+h}{h} \quad ; \quad h = 12\,\text{cm}$$

$$\boxed{\bar{y} = 3.139\,\text{cm from base}}$$

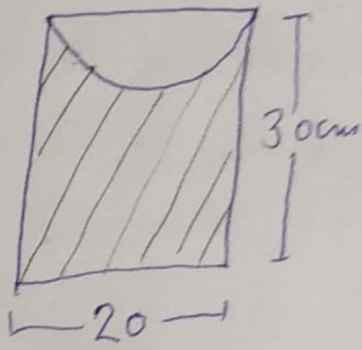
Q Find the centroid of the cross sectional area as shown in fig (2017) (12.5)



(6)*

Q Determine the M.O.I of shaded area about Centroidal x and y axis. (2016)

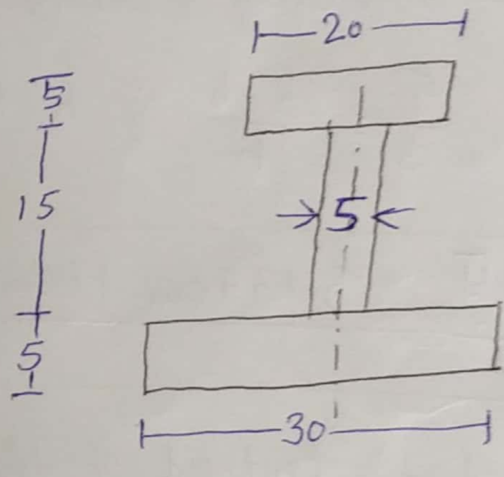
Sol.



Q Determine the centroid of the cross-sectional area of unequal I section as shown in fig.

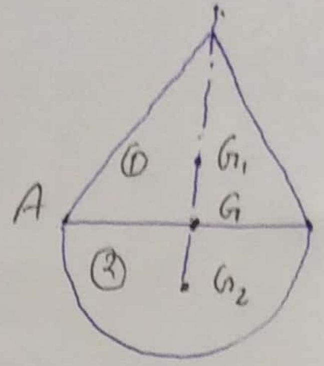
Dimensions = Cm.

$\boxed{\bar{y} = 10.96 \text{ cm from base}}$



Q A body is made up of a hemisphere and a cone each of radius r as shown in fig. what should be the greatest height of the cone so that the combined body of the hemisphere and the cone may stand upright?

Sol.



The Composite body will stand upright if the position of COG coincides with Common base i.e lies on AB

$$\boxed{h = 1.732\,m}$$

$$(V_1+V_2)\cdot 0 = V_1\cdot GG_1 + V_2\, GG_2$$

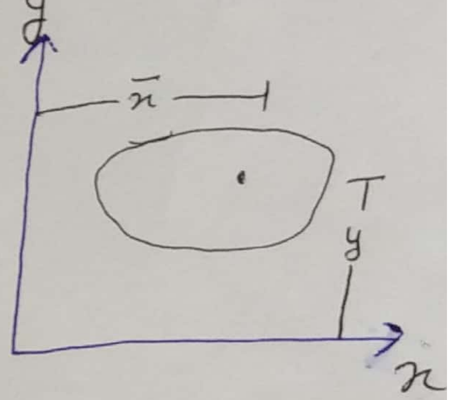
Moment of Inertia

It is a quantity expressing a body's tendency to resist angular acceleration, which is the sum of the products of the mass/area of each particle in the body with the square of ~~the square of~~ its distance from the axis of rotation.

It is also known as second moment of area/mass

$$I_x = A \cdot y^2 = \int dA \cdot y^2$$

$$I_y = A x^2 = \int dA \cdot x^2$$



Unit = mm^4 [Area M.O.I]
= $kg\,m^2$ [Mass M.O.I]

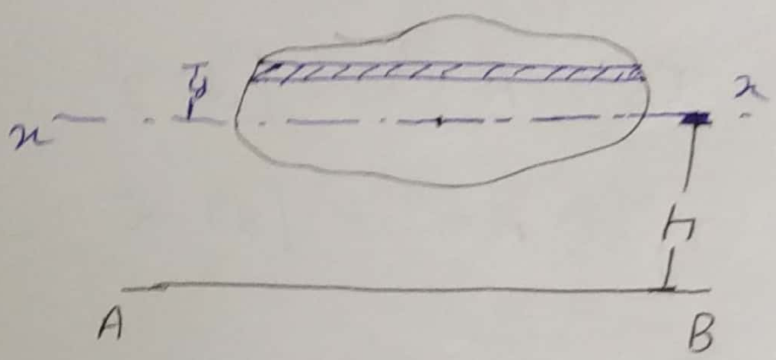
Radius of gyration: It is defined as the distance from reference axis to a point where whole area of a given lamina can be considered to be concentrated and produce same M.O.I with respect to given axis

$$I = AK^2$$

$$K = \sqrt{\frac{I}{A}}$$

Parallel axis theorem:- Acc to this theorem " The M.O.I of a plane lamina about any axis is equal to the sum of its M.O.I about a parallel axis through its C.O.G and product of its area (mass) and the square of the distance between two axes.

$$I_{AB} = I_{xx} + Ah^2$$



Proof - Consider an elemental components of area dA parallel to x axis located at distance y from x-axis.

Consider M.O.I of elementary compound about axis AB

$$= dA \cdot (y+h)^2$$

M.O.I of entire lamina

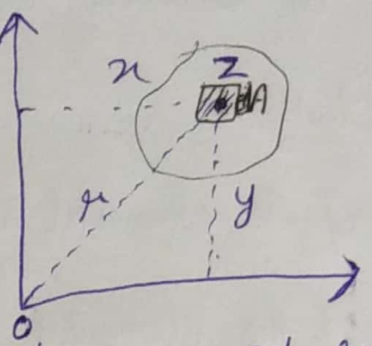
$$I_{AB} = \sum dA (h+y)^2$$

$$= \sum dA h^2 + \sum dA \cdot y^2 + \sum dA . 2hy$$

$$= h^2 \sum dA + \sum dA \cdot y^2 + 2h \sum dA \cdot y$$

$$\boxed{I_{AB} = h^2 . A + I_{xx}}$$

Perpendicular Axis theorem



It states that "The M.O.I of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the MOI of the lamina about the two axes at right angles to each other and intersecting each other at the point where the perpendicular axis passes through it."

Proof: MOI of the elemental component about axis OZ.

$$dA\,r^2 = dA\,(x^2+y^2)$$

$$dA\,r^2 = dA\,x^2 + dA\,y^2$$

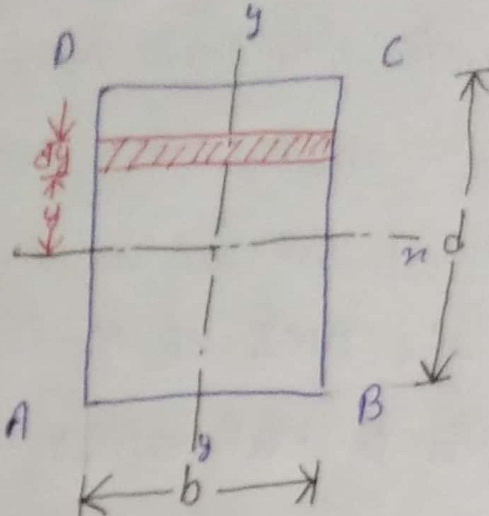
M.O.I of entire lamina

$$\sum dA \cdot r^2 = \sum dA \cdot x^2 + \sum dA \cdot y^2$$

$$\boxed{I_{zz} = I_{yy} + I_{xx}}$$

MOI of lamina of diff. Shapes

1) Rectangle



Area of the elemental strip = b·dy

MOI of the elemental component about $xx = dA \cdot y^2$

$= b \cdot dy \cdot y^2$

$= by^2 \, dy$

M.O.I of the entire lamina about the xx axis

$$I_{xx} = \int_{-d/2}^{d/2} by^2 \cdot dy$$

$$\boxed{I_{xx} = \frac{bd^3}{12}} \quad ; \quad \boxed{I_{yy} = \frac{db^3}{12}}$$

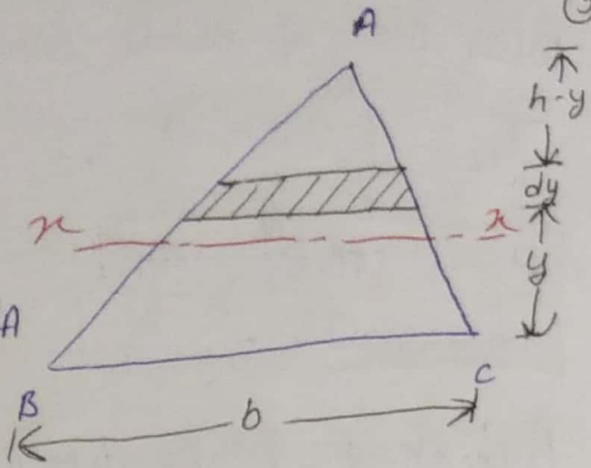
By using parallel axis theorem,

$$I_{AB} = I_{xx} + A \cdot h^2$$

$$I_{AB} = \frac{bd^3}{12} + b \cdot d \cdot \left(\frac{d}{2}\right)^2$$

$$\boxed{I_{AB} = \frac{bd^3}{3}} \qquad \boxed{I_{AD} = \frac{db^3}{3}}$$

Triangular Lamina:



Area of elementary strip = $l \cdot dy$

MoI of this strip about base BC $= y^2 dA$

$= y^2 \cdot l\,dy$

$= y^2 \left(\frac{b-y}{h}\right) b\,dy$

$\left[\frac{l}{b} = \frac{h-y}{h}\right]$

MoI of triangle about base BC

$$I_{BC} = \int_0^h y^2 b\left[1 - \frac{b}{h}\right] dy$$

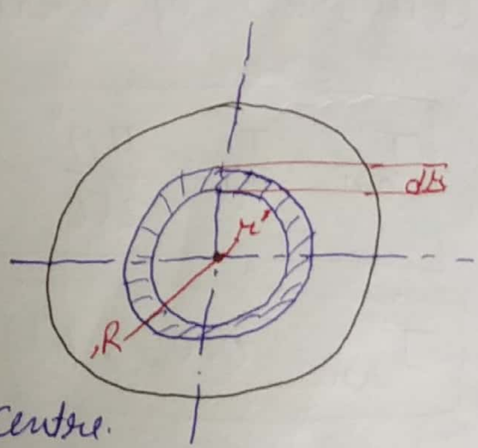
$$\boxed{I_{BC} = \frac{bh^3}{12}}$$

Using parallel axis theorem

$$I_{base} = I_{xx} + A\bar{y}^2$$

$$\boxed{I_{xx} = \frac{bh^3}{36}}$$

Circular lamina



Area of the elemental ring

$dA = 2\pi r \cdot dr$

Polar M.O.I of this element about centre.

$dI_{zz} = (2\pi r)dr \cdot r^2$

Polar MOI of entire lamina

$$I_{zz} = \int_0^R 2\pi r^3\, dr$$

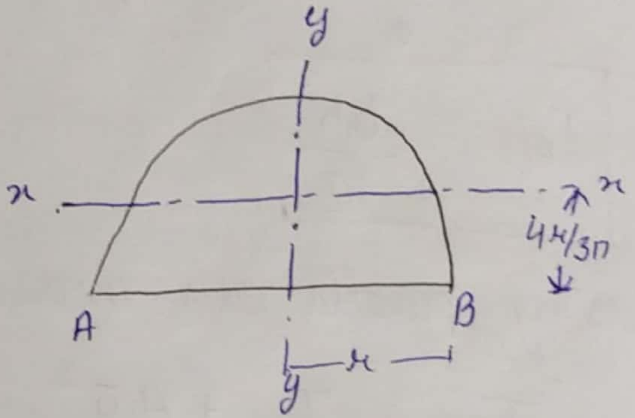
$$\boxed{I_{zz} = \frac{\pi D^4}{32} = \frac{\pi R^4}{2}}$$

By using perpendicular axis theorem,

$$I_{zz} = I_{xx} + I_{yy}$$

$$\boxed{I_{xx} = \frac{\pi D^4}{64} = I_{yy}} = \frac{\pi r^4}{4}$$

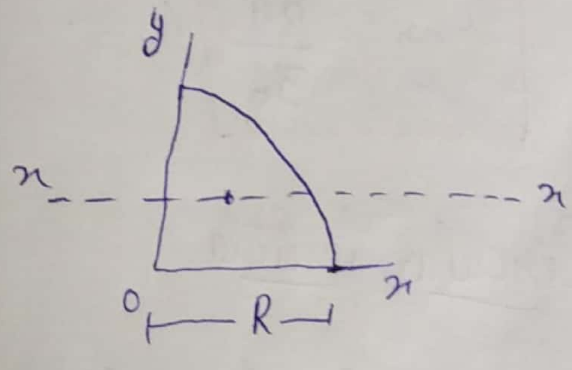
Semicircle



$$\boxed{I_{AB} = I_{yy} = \frac{\pi D^4}{128}} = \frac{\pi r^4}{8}$$

$$\boxed{I_{xx} = 0.11 R^4}$$

Quadrant of circle



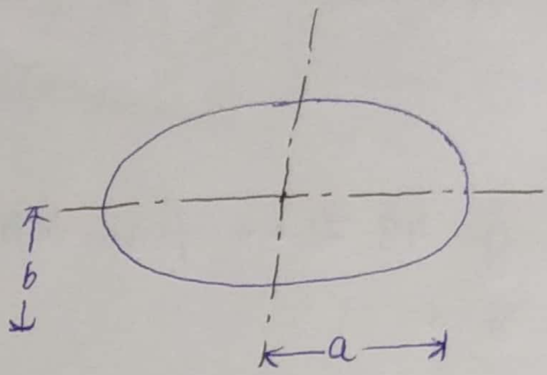
$$\boxed{I_{ox} = I_{oy} = \frac{\pi D^4}{256} = \frac{\pi R^4}{16}}$$

$$\boxed{I_{xx} = 0.055 R^4}$$

Ellipse:

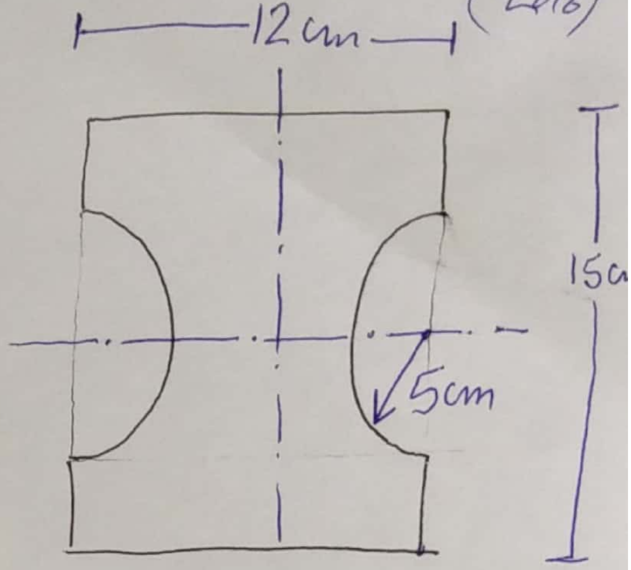
$$\boxed{I_{xx} = \frac{\pi a b^3}{4}}$$

$$\boxed{I_{yy} = \frac{\pi b a^3}{4}}$$



Q Determine I_{xx} and I_{yy} of the cross-section of given beam

(2016)



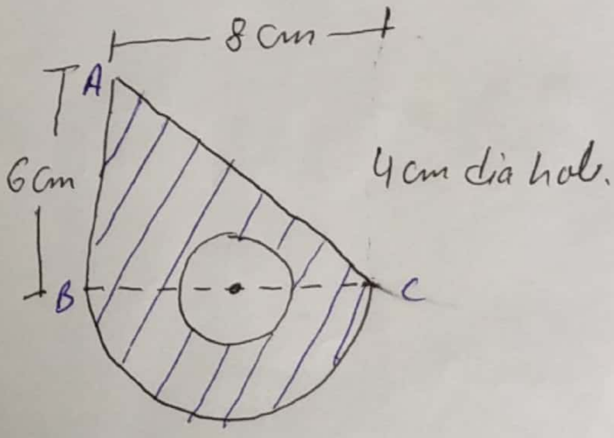
Sol.

I_{xx} = I_{xx} of rectangle − I_{xx} of circular part

= 2884.13 cm^4

I_{yy} = 839.74 cm^4

Q Find the MOI about the centroidal horizontal axis of the area shown in fig.



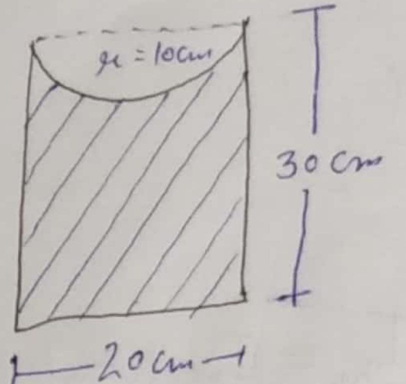
Sol.

$\bar{y}$ = 0.145 cm (from base BC)

I = 231.15 cm^4

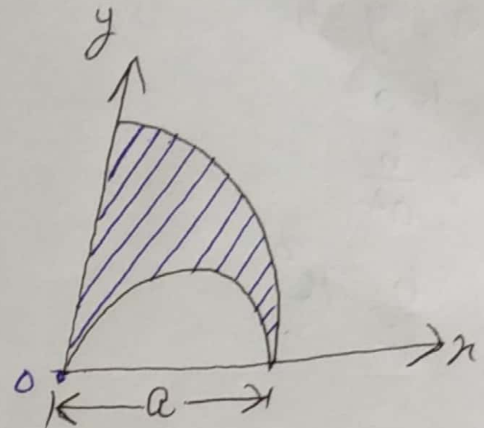
Q Determine the MOI of the shaded area about centroidal vertical and horizontal axes. (2016)

Sol:



Additional Questions

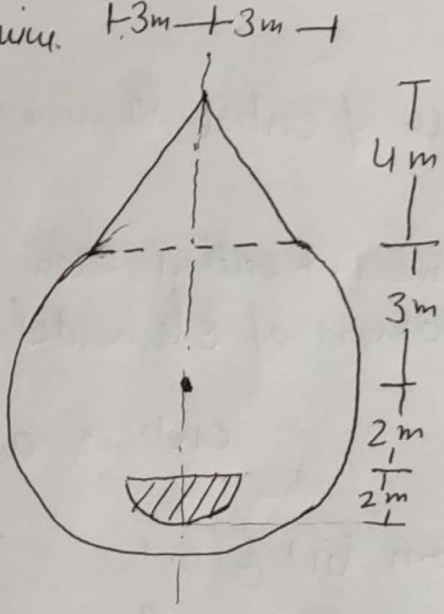
Q Locate the centroid of the shaded area obtained by removing a semi circle of diameter a from a quadrant of a circle of radius a.



Sol.

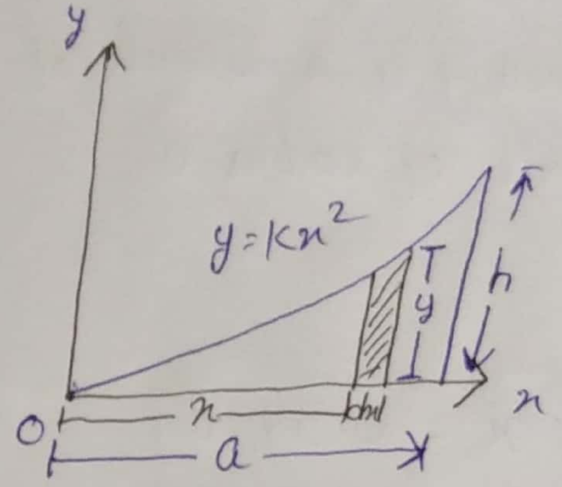
$$\boxed{\begin{aligned} \bar{x} &= 0.349a \\ \bar{y} &= 0.636a \end{aligned}}$$

Q Determine the co-ordinates of the centroid of given lamina. The shaded area is opening in the lamina.



$$\boxed{\bar{y} = 0.907\text{ m (from centre)}}$$

Q Locate the centroid of the parabolic shaded portion as shown in fig



Sol

$x = a,\ y = h$

$h = ka^2$

$k = \frac{h}{a^2}$

$y = \frac{h}{a^2} \cdot x^2$

Consider an elementary strip of height y, thickness dx at a distance x from y axis

Area = $y\,dx = \frac{h}{a^2} \cdot x^2\,dx$

Area of entire lamina = $\int_0^a \frac{h}{a^2} \cdot x^2\,dx = \frac{ah}{3}$ — (i)

Moment of entire area = $\frac{ah}{3} \cdot \bar{x}$ — (ii)

Moment of elemental area about y axis = $y\,dx \cdot x$

" " entire area $= \int_0^a \frac{h}{a^2} \cdot x^2 \cdot dx \cdot x$ — (iii)

from (ii) & (iii)

$\frac{ah}{3}\bar{x} = \int_0^a \frac{h}{a^2} \cdot x^3\,dx$

$\boxed{\bar{x} = \frac{3a}{4}}$

· Moment of elemental area about y axis = $y . dx . \frac{y}{2}$

$$= \left(\frac{h}{a^2} . x^2\right)^2 . \frac{dx}{2}$$

$$= \frac{h^2}{2a^4} . x^4 \, dx$$

$$\frac{ah}{3} . \bar{y} = \int_0^a \frac{h^2}{2a^4} x^4 \, dx$$

$$\boxed{\bar{y} = \frac{3h}{10}}$$

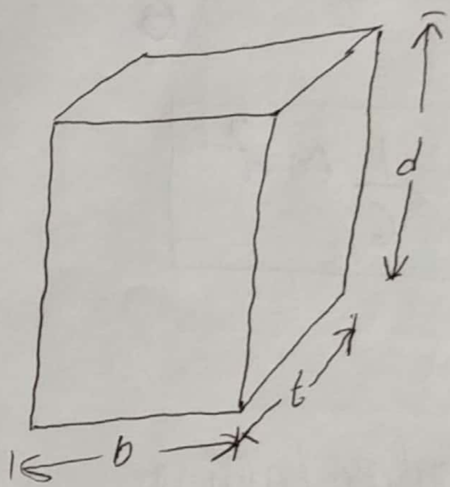
Mass M.O.I.

$$I_M = \rho t I_A$$

I_M : Mass M.O.I

I_A : Area M.O.I

Rectangular plate



$$I_{Ax} : \frac{bd^3}{12}$$

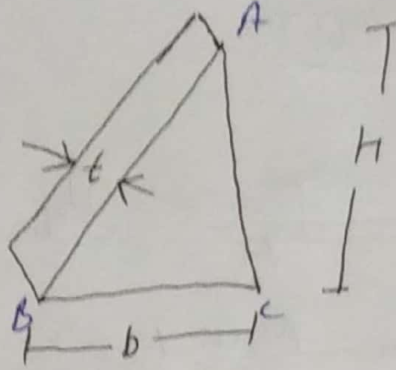
$$I_{Mx} : \frac{\rho t b d^3}{12}$$

$$= \frac{\rho b t d \cdot d^2}{12}$$

$$\boxed{I_{Mx} = \frac{Md^2}{12}} \; ; \; \boxed{I_{My} = \frac{Mb^2}{12}}$$

$$\boxed{I_{Mz} = \frac{M}{12}[d^2 + b^2]}$$

Triangular plate -



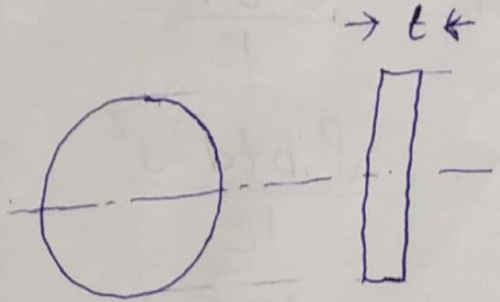
$$I_{(Area)BC} = \frac{bh^3}{12}$$

$$I_{M(BC)} = \frac{\rho t\, bh^3}{12}$$

$$= \rho \cdot \frac{1}{2} bh \cdot t \, \frac{h^2}{6}$$

$$\boxed{I_{M(BC)} = \frac{1}{6} M h^2}$$

<u>Circular Lamina</u>



$$I_{Ax} = I_{Ay} = \frac{\pi R^4}{4}$$

$$I_{Mx} = I_{My} = \frac{\rho t \pi R^4}{4}$$

$$= \frac{\rho \pi R^2 \cdot t \cdot R^2}{4}$$

$$\boxed{I_{Mx} = I_{My} = \frac{MR^2}{4}}$$

$$\boxed{I_{Mz} = \frac{MR^2}{2}}$$