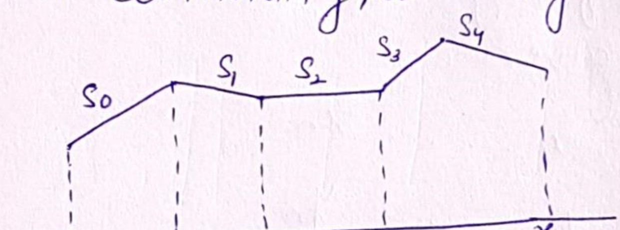
Approximation by Spline function –

A spline function is a function that consists of polynomial pieces joined together with certain smoothness conditions. Let us assume, without loss of generality, that $x_0 < x_1 < \dots < x_n$. Instead of a single polynomial for the entire domain (x_0, x_n) we can approximate the fn by several polynomials defined over subdomains of (x_0, x_n)

A simple example is a polygonal fn. (also called spline fn of degree 1) whose pieces are linear polynomials joined together to acheive continuity, as in fig.



The points $x_0, x_1 \dots x_4$ at which the fn. changes its character are termed as knots. We can define

$$S(x) = \begin{cases} S_0(x) & x_0 < x < x_1 \text{ or } x \in [x_0, x_1] \\ S_1(x) & x \in [x_1, x_2] \\ \vdots & \vdots \\ S_4(x) & x \in [x_3, x_4] \end{cases} \quad -(1)$$

where $S_i(x) = a_i x + b_i \quad \forall\ i = 0, 1, 2, 3, 4.$

because each piece of $S(x)$ is a linear polynomial. Such a fn. is piecewise linear.

If the fn S defined by (1) is continuous, it is called first degree spline.

Outside the interval $[x_0, x_n]$, $S(x)$ is usually defined to be the same fn on the left of x_0 as it is on the leftmost subinterval $[x_0, x_1]$ & the same on right of x_n as it is on the rightmost subinterval $[x_{n-1}, x_n]$.

Continuity of a fn at a point of spline is defined in usual manner.

e.g: $$s(x) = \begin{cases} x & x \in [-1,0] \\ 1-x & x \in (0,1) \\ 2x-2 & x \in [1,2] \end{cases}$$

Can not be a first degree spline, since it is not continuous at $x=0$:

$$\lim_{x \to 0^-} s(x) = 0 \qquad \lim_{x \to 0^+} s(x) = 1$$

The spline fn. of degree 1 can be used for interpolation.

Suppose we are given

x	x_0	x_1 ...	x_n
y	y_0	y_1 ...	y_n

Let $x_0 < x_1 < x_2 .. < x_n$. The pts $(x_0, y_0) ... (x_n, y_n)$ can be represented on a plane. These points can be joined by polygonal lines. The eqn of line joining (x_i, y_i) & (x_{i+1}, y_{i+1}) is

$$y - y_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i) \Rightarrow y = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$$

OR

$$s_i(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$$

where $(y_{i+1} - y_i)/(x_{i+1} - x_i)$ represents the slope of line joining $(x_i\ y_i)$ & (x_{i+1}, y_{i+1})

Second degree splines –

Let Q be a second degree spline then

$$Q(x) = \begin{cases} Q_1(x) & x \in [x_0, x_1] \\ \vdots & \\ Q_n(x) & x \in [x_{n-1}, x_n] \end{cases}$$

where all Q_i's are atmost quadratic fn. Also Q as well as $Q'(x)$ are continuous. i.e Quadratic spline is continuously differentiable

eg:
$$Q(x) = \begin{cases} x^2 & -10 \le x \le 0 \\ -x^2 & 0 \le x \le 1 \\ 1-2x & 1 \le x \le 20 \end{cases}$$

we have all $Q_i(x)$ are quadratic or linear. Also Q_i's are continuous at $x=0$ & $x=1$ (ie at Knots).

Now to check continuity of $Q'(x)$

$$\lim_{x \to 0^-} Q'(x) = \lim_{x \to 0^-} 2x = 0$$

$$\lim_{x \to 0^+} Q'(x) = \lim_{x \to 0^+} (-2x) = 0$$

$\therefore$ continuous at $x=0$

$$\lim_{x \to 1^-} Q'(x) = \lim_{x \to 1^-} (-2x) = -2$$

$$\lim_{x \to 1^+} Q'(x) = \lim_{x \to 1^+} (-2) = -2$$

$\therefore$ continuous at $x=1$

$\therefore$ the fn $Q(x)$ is a spline of degree 2.

Interpolating quadratic splines –

Suppose the table of given values is

x	x_0	x_1	...	x_n
y	y_0	y_1	....	y_n

A quadratic spline consists of n quadratic fns

$$x \to a_i x^2 + b_i x + c_i$$

one for each subinterval $[x_i, x_{i+1}]$

Also on each subinterval, the spline fn Q_i must satify

$$Q_i(x_i) = y_i$$

$$Q_i(x_{i+1}) = y_{i+1}$$

Since $Q(x)$ is continuously differentiable on $[x_0, x_n]$

we can put $\quad Z_i = Q_i'(x_i)$

We take up the following formula for Q_i

$$Q_i(x) = \frac{Z_{i+1} - Z_i}{2(x_{i+1} - x_i)} (x - x_i)^2 + Z_i(x - x_i) + y_i$$

To check, if this is correct, we verify that

$$Q_i(x_i) = y_i$$

$$Q_i'(x_i) = Z_i \quad \& \quad Q_i'(x_{i+1}) = Z_{i+1}.$$

Hence $Q(x)$ is continuous as well as $Q_i'(x)$ is continuous. On simplification, we get

$$Z_{i+1} = -Z_i + 2\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right) \qquad 0 \leq i \leq n-1$$

eg: find a quadratic spline interpolant for the data

x	-1	0	$\frac{1}{2}$	1	2	$\frac{5}{2}$
y	2	1	0	1	2	3

assuming $z_0 = 0$

Solu: $Q(x) = \begin{cases} Q_0(x) & x \in [-1, 0] \\ Q_1(x) & x \in [0, \frac{1}{2}] \\ Q_2(x) & x \in [\frac{1}{2}, 1] \\ Q_3(x) & x \in [1, 2] \\ Q_4(x) & x \in [2, \frac{5}{2}] \end{cases}$

where $Q_i(x) = \dfrac{z_{i+1} - z_i}{2(x_{i+1} - x_i)}(x - x_i)^2 + z_i(x - x_i) + y_i$

& $z_{i+1} = -z_i + 2\left(\dfrac{y_{i+1} - y_i}{x_{i+1} - x_i}\right)$

Since $z_0 = 0$

$z_1 = -z_0 + 2\left(\dfrac{y_1 - y_0}{x_1 - x_0}\right) = -0 + 2\left(\dfrac{1-2}{0-(-1)}\right) = 2\left(\dfrac{-1}{1}\right)$
$= -2$

$z_2 = -z_1 + 2\left(\dfrac{y_2 - y_1}{x_2 - x_1}\right) = -(-2) + 2\left(\dfrac{0-1}{\frac{1}{2} - 0}\right)$
$= 2 + 2\left(\dfrac{-1}{1/2}\right) = 2 - 4 = -2$

$z_3 = -z_2 + 2\left(\dfrac{y_3 - y_2}{x_3 - x_2}\right) = -(-2) + 2\left(\dfrac{1-0}{1 - 1/2}\right)$
$= 2 + 2\left(\dfrac{1}{1/2}\right) = 2 + 4 = 6$

$z_4 = -z_3 + 2\left(\dfrac{y_4 - y_3}{x_4 - x_3}\right) = -6 + 2\left(\dfrac{2-1}{2-1}\right) = -6 + 2 = -4$

$z_5 = -z_4 + 2\left(\dfrac{y_5 - y_4}{x_5 - x_4}\right) = -(-4) + 2\left(\dfrac{3-2}{\frac{5}{2} - 2}\right) = 4 + 2\left(\dfrac{1}{1/2}\right)$
$= 4 + 4 = 8$

$$Q_0(x) = \frac{z_1 - z_0}{2(x_1 - x_0)}(x-x_0)^2 + z_0(x-x_0) + y_0$$

$$= \frac{-2-0}{2(0+1)}(x+1)^2 + 0 + 2$$

$$= \frac{-\cancel{2}}{\cancel{2}}(x+1)^2 + 2 = -(x+1)^2 + 2$$

$$Q_1(x) = \frac{z_2 - z_1}{2(x_2 - x_1)}(x-x_1)^2 + z_1(x-x_1) + y_1$$

$$= \frac{-2-(-2)}{2(\frac{1}{2}-0)}(x-0)^2 + (-2)(x-0) + 1$$

$$= -2x+1$$

$$Q_2(x) = \frac{z_3 - z_2}{2(x_3 - x_2)}(x-x_2)^2 + z_2(x-x_2) + y_2$$

$$= \frac{6-(-2)}{2(1-\frac{1}{2})}(x-\tfrac{1}{2})^2 + (-2)(x-\tfrac{1}{2}) + 0$$

$$= \frac{8}{1}(x-\tfrac{1}{2})^2 + (-2)(x-\tfrac{1}{2}) = 8(x-\tfrac{1}{2})^2 - 2(x-\tfrac{1}{2})$$

$$Q_3(x) = \frac{z_4 - z_3}{2(x_4 - x_3)}(x-x_3)^2 + z_3(x-x_3) + y_3$$

$$= \frac{-4-6}{2(2-1)}(x-1)^2 + 6(x-1) + 1$$

$$= \frac{-10}{2}(x-1)^2 + 6(x-1) + 1 = -5(x-1)^2 + 6(x-1) + 1$$

$$Q_4(x) = \frac{z_5 - z_4}{2(x_5 - x_4)}(x-x_4)^2 + z_4(x-x_4) + y_4$$

$$= \frac{8-(-4)}{2(\frac{5}{2}-2)}(x-2)^2 + (-4)(x-2) + 2$$

$$= \frac{12}{1}(x-2)^2 - 4(x-2) + 2 = 12(x-2)^2 - 4(x-2) + 2$$

Natural cubic spline –

The 1st & 2nd degree splines have obvious imperfections due to the discontinuity of 1st & 2nd order derivatives respectively. In 1st order spline the slope of curve abruptly change at knots, whereas in 2nd degree spline the discontinuity is in second order derivative, because of which the curvature changes abruptly. So, whenever more smoothness is needed in approximation, higher degree splines are used. The choice of degree most frequently made for a spline fn is 3. The resulting splines are termed cubic spline, where we join cubic polynomials together in such a way that the resulting fn. has two continuous derivatives everywhere, so, at each knot three continuity conditions are imposed. Since S, S', S'' are continuous; the graph of the fn will appear smooth to the eye. Discontinuities, may occur in the third derivative, but cannot be easily detected visually. Moreover the splines of degree greater than 3 seldom yields any advantage. In general, odd degree splines behave better than even degree splines.

Let us consider the data

x	x_0	x_1	x_2	$\cdots$	x_n
y	y_0	y_1	y_2	$\cdots$	y_n

We define
$$S(x) = \begin{cases} S_0(x) & x \in [x_0, x_1] \\ S_1(x) & x \in [x_1, x_2] \\ \vdots & \\ S_n(x) & x \in [x_{n-1}, x_n] \end{cases}$$

where $S_i(x)$ denotes the cubic polynomial. We have
$$S(x_i) = y_i \quad \forall\ i = 0, 1 \ldots n$$

The continuity conditions are S' & S'' are continuous at Knots, i.e $\lim_{x \to x_i^-} S(x_i) = \lim_{x \to x_i^+} S(x_i)$

$\lim_{x \to x_i^-} S'(x_i) = \lim_{x \to x_i^+} S'(x_i)$

& $\lim_{x \to x_i^-} S''(x_i) = \lim_{x \to x_i^+} S''(x_i)$

In addition, we have to consider

$$S''(x_0) = S''(x_n) = 0$$

The resulting spline function is termed as natural cubic spline.

Using the conditions of continuity, we get

$$S_i(x) = \frac{z_{i+1}}{6h_i}(x - x_i)^3 + \frac{z_i}{6h_i}(x_{i+1} - x)^3 + \left(\frac{y_{i+1}}{h_i} - \frac{h_i}{6} z_{i+1}\right)(x - x_i) + \left(\frac{y_i}{h_i} - \frac{h_i}{6} z_i\right)(x_{i+1} - x)$$

where $x_{i+1} - x_i = h_i \quad \forall \quad 0 \leq i \leq n-1$.

To obtain the values of z_i's, the eqs are

$$h_{i-1} z_{i-1} + 2(h_{i-1} + h_i) z_i + h_i z_{i+1} = 6(b_i - b_{i-1})$$

where $b_i = \frac{1}{h_i}(y_{i+1} - y_i)$ for $0 \leq i \leq n-1$

If we take $2(h_{i-1} + h_i) = u_i$

$6(b_i - b_{i-1}) = v_i$; we obtain a tridiagonal system of linear Eqns.

$$z_0 = 0$$

$$h_{i-1} z_{i-1} + u_i z_i + h_i z_{i+1} = v_i \qquad 1 \leq i \leq n-1$$

$$z_n = 0$$

Solving this tridiagonal system of linear Eqns, we obtain the values of Z_i's. $Z_0 = 0$ & $Z_n = 0$ are obtained due to conditions $S''(x_0) = S''(x_n) = 0$

eg: Derive the eqns of natural cubic spline for the following table:

x	-1 (x_0)	0 (x_1)	1 (x_2)
y	1 (y_0)	2 (y_1)	-1 (y_2)

Solu: we first obtain $h_0 = x_1 - x_0 = 1$

$h_1 = x_2 - x_1 = 1$

Next $b_1 = \frac{1}{h_1}(y_2 - y_1) = \frac{1}{1}(-1-2) = -3$

$b_0 = \frac{1}{h_0}(y_1 - y_0) = \frac{1}{1}(2-1) = 1$

$u_1 = 2(h_0 + h_1) = 2(1+1) = 4$

$v_1 = 6(b_1 - b_0) = 6(-3-0) = -24$

∴ The TD system becomes

$$\begin{cases} Z_0 = 0 \\ h_0 Z_0 + u_1 Z_1 + h_1 Z_2 = v_1 \\ Z_2 = 0 \end{cases}$$

$$\therefore \begin{cases} Z_0 = 0 \\ Z_0 + 4Z_1 + Z_2 = -24 \\ Z_2 = 0 \end{cases}$$

Solving we obtain $Z_0 = 0$, $Z_1 = -6$, $Z_2 = 0$

$\therefore$

$$S_0(x) = \frac{Z_1}{6h_0}(x-x_0)^3 + \frac{Z_0}{6h_0}(x_1-x)^3 + \left(\frac{y_1}{h_0} - \frac{h_0}{6}Z_1\right)(x-x_0) +$$

$$\left(\frac{y_0}{h_0} - \frac{h_0}{6}Z_0\right)(x_1-x)$$

$$= \frac{-\cancel{6}}{\cancel{6}\times 1}(x-(-1))^3 + 0 + \left(\frac{2}{1} - \frac{1.(-\cancel{6})}{\cancel{6}}\right)(x-(-1))$$

$$+ \left(\frac{1}{1} - \frac{1}{6}\times 0\right)(0-x)$$

$$= -(x+1)^3 + 3(x+1) - x \quad ; \quad x \in [-1, 0]$$

$$S_1(x) = \frac{Z_2}{6h_1}(x-x_1)^3 + \frac{Z_1}{6h_1}(x_2-x)^3 + \left(\frac{y_2}{h_1} - \frac{h_1 Z_2}{6}\right)(x-x_1)$$

$$+ \left(\frac{y_1}{h_1} - \frac{h_1 Z_1}{6}\right)(x_2-x)$$

$$= 0 + \frac{(-\cancel{6})}{\cancel{6}\cdot 1}(1-x)^3 + \left(\frac{-1}{1} - \frac{1.0}{6}\right)(x-0)$$

$$+ \left(\frac{2}{1} - \frac{1.(-6)}{6}\right)(1-x)$$

$$= -(1-x)^3 - x + 3(1-x)$$

$$= -(1-x)^3 - x + 3 - 3x \quad ; \quad x \in [0, 1].$$

An alternate form to write Natural cubic spline is

$$S_i(x) = y_i + (x-x_i)\left[B_i + (x-x_i)\left(\frac{z_i}{2} + \frac{1}{6h_i}(x-x_i)(z_{i+1}-z_i)\right)\right]$$

where

$$B_i = -\frac{h_i}{6} z_{i+1} - \frac{h_i z_i}{3} + \frac{1}{h_i}(y_{i+1}-y_i).$$

B spline –

Let us consider an infinite set of knots t_i defined as

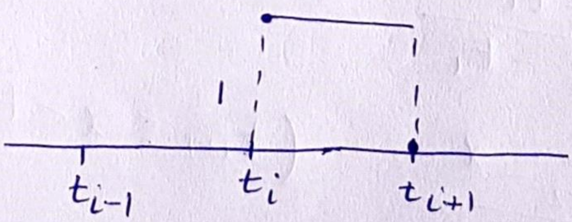
$$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots$$

$$\lim_{i\to\infty} t_i = \infty = -\lim_{i\to\infty} t_{-i}$$

We define B_i^0 spline (i.e B spline of degree 0) as

$$B_i^0(x) = \begin{cases} 1 & t_i \le x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

The graph of B_i^0 is as follows



obviously B_i^0 is discontinuous. However, it is continuous from the right at all points even where the jump occurs.

Thus

$$\lim_{x\to t_i^+} B_i^0(x) = 1 = B_i^0(t_i)$$

$$\lim_{x\to t_{i+1}^+} B_i^0(x) = 0 = B_i^0(t_{i+1})$$

Since B_i^0 is a piecewise constant fn., it is a spline of degree 0. Also

$$B_i^0(x) \geq 0 \quad \forall x \ \& \ \forall i$$

$$\sum_{i=-\infty}^{\infty} B_i^0(x) = 1 \quad \forall x.$$

With the fns B_i^0 as a starting point, we now generate all higher degree B splines by a simple recursive definition:

$$B_i^k(x) = \left(\frac{x - t_i}{t_{i+k} - t_i}\right) B_i^{k-1}(x) + \left(\frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}\right) B_{i+1}^{k-1}(x) \qquad (k \geq 1)$$

where $k = 1, 2, 3 \ldots$ & $i = 0, \pm 1, \pm 2 \ldots$

Hence B_i^1 is defined as

$$B_i^1(x) = \left(\frac{x - t_i}{t_{i+1} - t_i}\right) B_i^0(x) + \left(\frac{t_{i+2} - x}{t_{i+2} - t_{i+1}}\right) B_{i+1}^0(x)$$

$$= \begin{cases} 0 & (x \geq t_{i+2} \text{ or } t_i \geq x) \\ \dfrac{x - t_i}{t_{i+1} - t_i} & (t_i \leq x < t_{i+1}) \\ \dfrac{t_{i+2} - x}{t_{i+2} - t_{i+1}} & (t_{i+1} \leq x < t_{i+2}) \end{cases}$$

Also $\displaystyle\sum_{i=-\infty}^{\infty} B_i^1(x) = 1 \quad \forall x$

The fns B_i^k defined above are called B splines of degree k.

Interpolation & approximation by B splines —

we are given

$$x : t_0 \; t_1 \; \ldots \; t_n$$

$$y : y_0 \; y_1 \; \ldots \; y_n$$

We want to find coefficients A_i s.t

$$S(x) = \sum_{i=-\infty}^{\infty} A_i B_{i-k}^{k}(x)$$

& $$S(t_i) = y_i \qquad (0 \leq i \leq n)$$

for zero degree spline, it is very simple

$$S(x) = \sum_{i=0}^{\infty} y_i B_i^0(x)$$

$$\because \; B_i^0(t_j) = \delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$$

for degree 1 spline, we have

$$S(x) = \sum_{i=0}^{n} y_i B_{i-1}^{1}(x)$$

$$\because \; B_{i-1}^{1}(t_j) = \delta_{ij}$$

for degree 2 spline,

$$S(x) = \sum_{i=-\infty}^{\infty} A_i B_{i-k}^{2}(x)$$

where

$$A_j(t_{j+1}-t_j)+A_{j+1}(t_j-t_{j-1})=y_j(t_{j+1}-t_{j-1}) \qquad (0\le j\le n)$$

This is a system of $(n+1)$ linear eqns in $(n+2)$ unknowns $A_0, A_1, A_2 \ldots A_{n+1}$, that can be solved recursively.