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Numerical solution of ordinary differential equations: (1)

Picard's method: Consider the differential equation of first order

$$\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0$$

We have $\int_{x_0}^{x} \frac{dy}{dx} dx = \int_{x_0}^{x} f(x,y)\, dx$

or $y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\, dx$

or $y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\, dx$ ——— (1)

Then n^{th} approximation is given by

$$\boxed{y_n(x) = y(x_0) + \int_{x_0}^{x} f(x, y_{n-1})\, dx} \; ; \; n = 1, 2, 3, \ldots$$

which is the iteration formula for $y(x)$.

Que Solve $\frac{dy}{dx} = 1 + xy$, given that $y = 1$ when $x = 0$, in the interval $[0, 0.5]$ for $h = 0.1$ correct to two decimal places.

Sol $\frac{dy}{dx} = 1 + xy \; ; \; y(0) = 1 \Rightarrow f(x,y) = 1 + xy, \; x_0 = 0, \; y_0 = y(x_0) = 1$

Picard's iterative formula is

$$y_n(x) = y(x_0) + \int_{x_0}^{x} f(x, y_{n-1})\, dx \; ; \; n = 1, 2, 3, \ldots$$

i.e., $y_n(x) = 1 + \int_0^x [1 + x\, y_{n-1}(x)]\, dx$

$$= 1 + x + \int_0^x x\, y_{n-1}(x)\, dx \; ; \; n = 1, 2, 3, \ldots$$

——— (1)

$\therefore\ y_1(x) = 1+x+\int_0^x x\,y_0(x)\,dx = 1+x+\int_0^x x.1\,dx = 1+x+\frac{x^2}{2}$

$y_2(x) = 1+x+\int_0^x x\,y_1(x)\,dx$

$= 1+x+\int_0^x x\left(1+x+\frac{x^2}{2}\right)dx = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}$

$y_3(x) = 1+x+\int_0^x x\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}\right)dx$

$= 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}$

$y_4(x) = 1+x+\int_0^x x\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}\right)dx$

$= 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}+\frac{x^7}{108}+\frac{x^8}{384}$ and so on.

Now, in $[0, 0.5]$, for $\frac{x^5}{15}$, max. value $= \frac{(0.5)^5}{15} \simeq 0.0021$

$\therefore$ for values correct to two decimal places, from last two approximations, we have

$$y(x) = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}$$

$\therefore\ y(0.1) = 1+(0.1)+\frac{(0.1)^2}{2}+\frac{(0.1)^3}{3}+\frac{(0.1)^4}{8} \approx 1.11$

$y(0.2) = 1+(0.2)+\frac{(0.2)^2}{2}+\frac{(0.2)^3}{3}+\frac{(0.2)^4}{8} \simeq 1.22$

$y(0.3) = 1+(0.3)+\frac{(0.3)^2}{2}+\frac{(0.3)^3}{3}+\frac{(0.3)^4}{8} \simeq 1.35$

$y(0.4) = 1+(0.4)+\frac{(0.4)^2}{2}+\frac{(0.4)^3}{3}+\frac{(0.4)^4}{8} \simeq 1.50$

$y(0.5) = 1+(0.5)+\frac{(0.5)^2}{2}+\frac{(0.5)^3}{3}+\frac{(0.5)^4}{8} \simeq 1.67$

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Taylor series method:

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

To solve it over the interval $[x_0, x_n]$ with step size $h = \frac{x_n - x_0}{n}$,

the Taylor series method of order m is

$$y(x+h) = y(x) + h y'(x) + \frac{h^2}{2!} y''(x) + \cdots + \frac{h^m}{m!} y^{(m)}(x)$$

where all the derivatives y'', y''', ... $y^{(m)}$ have been determined analytically.

Euler's method: Euler's method is the Taylor series method of order 1 and can be written as

$$y(x+h) = y(x) + h f(x, y(x))$$

Note: In Euler's method, the truncation error is large, and the results cannot be computed with much accuracy because only two terms in the Taylor series are used. Consequently, higher-order Taylor series methods are used most often.

Que Using Taylor series method find the solution of the diffrential equation

$$x y' = x - y \; ; \quad y(2) = 2 \text{ at } x = 2.1$$ correct to 5 places of decimals. Compare the value with exact solution.

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Sol Given $xy' = x - y$

On differentiating,

$$xy'' + y' = 1 - y'$$

$$\Rightarrow xy'' + 2y' = 1$$

Again, differentiating,

$$xy''' + y'' + 2y'' = 0$$

$$\Rightarrow xy''' + 3y'' = 0$$

In general, $xy^{(n)} + ny^{(n-1)} = 0$ for $n \geq 3$

Put $x = 2$

$$2y'(2) = 2 - y(2) = 2 - 2 = 0$$

$$\Rightarrow y'(2) = 0$$

$$2y''(2) + 2y'(2) = 1 \Rightarrow y''(2) = \frac{1-0}{2} = \frac{1}{2}$$

$$y^{(n)}(2) = \frac{-ny^{(n-1)}(2)}{2}, \quad n \geq 3$$

$$\therefore y^{(3)}(2) = -\frac{3}{2} \cdot \frac{1}{2} = \frac{-3}{4}$$

$$y^{(4)}(2) = -\frac{4}{2} \cdot y^{(3)}(2) = +\frac{4}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{2}$$ and so on.

By Taylor series method,

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \cdots + \frac{h^m}{m!}y^{(m)}(x)$$

By taking $x = 2$ and $h = 0.1$, we get

$$y(2.1) = y(2) + (0.1)y'(2) + \frac{(0.1)^2}{2!}y''(2) + \frac{(0.1)^3}{3!}y'''(2) + \frac{(0.1)^4}{4!}y^{IV}(2) + \cdots$$

$$= 2 + (0.1) \times 0 + \frac{(0.1)^2}{2} \cdot \frac{1}{2} + \frac{(0.1)^3}{3!}\left(-\frac{3}{4}\right) + \frac{(0.1)^4}{4!}\left(\frac{3}{2}\right)$$

$$= 2 + 0 + 0.0025 - 0.000125 + 0.00000625$$

$\therefore y(2.1) = 2.00238$ correct to 5 decimal places.

Now, given equation is $\frac{dy}{dx} = 1 - \frac{1}{x}y$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x}y = 1 \quad \text{------------ (1)}$$

which is a linear differential equation in y.

$\therefore$ I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\therefore$ Sol. is given by $y.x = \int 1.x\,dx = \frac{x^2}{2} + c$

$$\Rightarrow y = \frac{x}{2} + \frac{c}{x}$$

Now, $y(2) = 2 \Rightarrow 2 = 1 + \frac{c}{2} \Rightarrow c = 2$

$$\therefore y = \frac{x}{2} + \frac{2}{x} = \frac{x^2+4}{2x}$$

$$\Rightarrow y(2.1) = \frac{(2.1)^2+4}{2\times 2.1} = \frac{8.41}{4.2} = 2.002381$$

$\therefore$ $y(2.1) = 2.00238$ is same upto 5 decimal places.

Que Solve $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$ in the range $0 \le x \le 0.2$ using Euler's method.

Sol By Euler's method,

$$y(x+h) = y(x) + hf(x, y(x)) \quad \text{------------ (1)}$$

Here $f(x,y) = y - \frac{2x}{y}$, $y(0) = 1$

We consider $h = 0.1$

$\therefore$ From (1),

By taking $x=0$, $y(0+0.1) = y(0) + hf(0, y(0))$

$$\Rightarrow y(0.1) = 1 + (0.1)\left(y(0) - \frac{2\times 0}{y(0)}\right) = 1 + 0.1 = 1.1 \text{ A}$$

By taking $x = 0.1$, $h = 0.1$, in (1),

$$y(0.1+0.1) = y(0.1) + (0.1) f(0.1, y(0.1))$$

$$= 1.1 + (0.1)\left(y(0.1) - \frac{2\times 0.1}{y(0.1)}\right) = 1.1 + 0.1\left(1.1 - \frac{0.2}{1.1}\right)$$

$$= 1.1918 \text{ A}$$

Que Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's method.

Sol By Euler's method,

$$y(x+h) = y(x) + hf(x, y(x)) \quad \text{——— (1)}$$

Here $f(x, y) = \frac{y-x}{y+x}$, $y(0) = 1$

We consider $h = 0.02$

By equ. (1), $$y(x+0.02) = y(x) + 0.02\left\{\frac{y(x)-x}{y(x)+x}\right\} \quad \text{——— (2)}$$

Take $x = 0$, $$y(0.02) = 1 + 0.02\left(\frac{1-0}{1+0}\right) = 1.02$$

Take $x = 0.02$ in (2), $$y(0.04) = y(0.02) + 0.02\left(\frac{1.02-0.02}{1.02+0.02}\right)$$
$$= 1.02 + 0.02(0.9615) = 1.0392$$

Take $x = 0.04$ in (2), $$y(0.06) = y(0.04) + 0.02\left(\frac{1.0392-0.04}{1.0392+0.04}\right)$$
$$= 1.0392 + 0.02(0.926) = 1.0577$$

Take $x = 0.06$ in (2), $$y(0.08) = 1.0577 + 0.02\left(\frac{1.0577-0.06}{1.0577+0.06}\right)$$
$$= 1.0756$$

Take $x = 0.08$ in (2), $$y(0.10) = 1.0756 + 0.02\left(\frac{1.0756-0.08}{1.0756+0.08}\right)$$
$$= 1.0928$$

Hence the required approximate value of $y = 1.0928$.

Que Find by Taylor's series method, the values of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from

$$\frac{dy}{dx} = x^2y - 1, \quad y(0) = 1$$

Sol Given $y' = x^2 y - 1$ $\qquad \therefore y'(0) = -1$

$\therefore$ Differentiating successively, we get

$y'' = 2xy + x^2 y'$ $\qquad y''(0) = 0$

$y''' = 2y + 4xy' + x^2 y''$ $\qquad y'''(0) = 2y(0) = 2$

$y^{IV} = 6y' + 6xy'' + x^2 y'''$ $\qquad y^{IV}(0) = -6$

$y^{V} = 12y'' + 8xy''' + x^2 y^{IV}$ $\qquad y^{V}(0) = 24$ etc.

$y^{VI} = 20y'' + 10xy^{IV} + x^2 y^{V}$ $\qquad y^{VI}(0) = 0$

$\therefore$ By Taylor series,

$$y(x+h) = y(x) + h y'(x) + \frac{h^2}{2!} y''(x) + \cdots + \frac{h^m}{m!} y^{(m)}(x) \quad \text{—— (1)}$$

By taking $x = 0$ and $h = 0.1$,

$$y(0.1) = y(0) + (0.1) y'(0) + \frac{(0.1)^2}{2!} y''(0) + \frac{(0.1)^3}{3!} y'''(0) + \frac{(0.1)^4}{4!} y^{IV}(0) + \frac{(0.1)^5}{5!} y^{V}(0) + \cdots$$

$$= 1 + (0.1)(-1) + \frac{(0.01)}{2} \times 0 + \frac{(0.001)}{6} \times 2 + \frac{(0.0001)}{24}(-6) + \frac{(0.00001)}{120} \times 24$$

$$= 1 - 0.1 + 0.000333 - 0.000025 + 0.000002$$

$y(0.1) = 0.90031$ Ans

By taking $x = 0$ and $h = 0.2$, in (1)

$$y(0.2) = y(0) + (0.2) y'(0) + \frac{(0.2)^2}{2!} y''(0) + \frac{(0.2)^3}{3!} y'''(0) + \frac{(0.2)^4}{4!} y^{IV}(0) + \frac{(0.2)^5}{5!} y^{V}(0) + \cdots$$

$$= 1 + (0.2)(-1) + \frac{(0.04)}{2} \times 0 + \frac{(0.008)}{6} \times 2 + \frac{(0.0016)}{24}(-6) + \frac{0.00032}{120} \times 24 + \frac{0.000064}{720} \times 0$$

$y(0.2) = 1 - 0.2 + 0.002667 - 0.0004 + 0.000064 = 0.80233$ Ans

Runge-Kutta's methods:

(1) The second-order Runge-Kutta method is

$$y(x+h) = y(x) + K$$

where $K = \frac{1}{2}(K_1 + K_2)$

and $\begin{cases} K_1 = hf(x, y) \\ K_2 = hf(x+h, y+K_1) \end{cases}$

This method requires two evaluations of the function f per step. It is equivalent to a Taylor series method of order 2.

(2) One of the most popular single-step methods for solving ODE is the fourth-order Runge-Kutta method

$$y(x+h) = y(x) + K$$

where $K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$.

and $\begin{cases} K_1 = hf(x, y) \\ K_2 = hf\left(x+\frac{h}{2}, y+\frac{K_1}{2}\right) \\ K_3 = hf\left(x+\frac{h}{2}, y+\frac{K_2}{2}\right) \\ K_4 = hf(x+h, y+K_3) \end{cases}$

It needs four evaluations of the function f per step. Since it is equivalent to a Taylor series method of order 4, it has truncation error of order $O(h^5)$. The small number of function evaluations and high-order truncation error account for its popularity.

Note: If nothing is mention about the order of the above methods, then always apply fourth-order R-K method.

Que Solve the differential equation

$$\frac{dy}{dx} = -xy^2$$

$$y(0) = 2$$

at $x = -0.2$, correct to two decimal places using one step of the Runge-Kutta method of order 2.

Sol Here $f(x,y) = -xy^2$, $y(0) = 2$

For one step, we take $h = -0.2$

$\therefore$ By Runge-Kutta method of order 2, we have

$$y(x+h) = y(x) + K \quad \text{———} (1)$$

where $K = \frac{1}{2}(K_1 + K_2)$

and $\begin{cases} K_1 = hf(x,y) \\ K_2 = hf(x+h, y+K_1) \end{cases}$

By taking $x = 0$ and $h = -0.2$,

$K_1 = (-0.2)f(0,2) \quad (\because y(0) = 2)$

$= 0$

$K_2 = (-0.2)f(-0.2, 2)$

$= (-0.2)\{+(0.2).2^2\} = -0.16$

$\therefore K = \frac{1}{2}(K_1 + K_2) = \frac{1}{2}(0 - 0.16) = -0.08$

$\therefore$ By equ. (1), $y(0-0.2) = y(0) + K$

$\Rightarrow y(-0.2) = 2 - 0.08$

Hence, $\boxed{y(-0.2) = 1.92}$ Ans

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Que Apply the fourth order Runge-Kutta method to solve $\frac{dy}{dx} = x^2+y^2$, $y(0)=1$ to find $y(0.2)$ in 2 steps.

Sol By Runge-Kutta method of fourth order,

$$y(x+h) = y(x)+K \quad \text{———} \quad (1)$$

where $K = \frac{1}{6}(K_1+2K_2+2K_3+K_4)$

and $K_1 = hf(x,y)$

$K_2 = hf\left(x+\frac{h}{2}, y+\frac{K_1}{2}\right)$

$K_3 = hf\left(x+\frac{h}{2}, y+\frac{K_2}{2}\right)$

$K_4 = hf(x+h, y+K_3)$

Here $f(x,y) = x^2+y^2$, $y(0)=1$, $h=0.1$ (for two steps)

Step I By taking $x=0$, $y(0)=1$, $h=0.1$,

$K_1 = (0.1)f(0,1) = 0.1$

$$K_2 = (0.1)f\left(0+\frac{0.1}{2}, 1+\frac{0.1}{2}\right) = (0.1)f(0.05, 1.05)$$
$$= (0.1)\{(0.05)^2+(1.05)^2\}$$
$$= 0.1105$$

$$K_3 = (0.1)f\left(0+\frac{0.1}{2}, 1+\frac{0.1105}{2}\right) = (0.1)f(0.05, 1.05525)$$
$$= (0.1)\{(0.05)^2+(1.05525)^2\}$$
$$= 0.11161$$

$$K_4 = (0.1)f(0+0.1, 1+0.11161) = (0.1)f(0.1, 1.11161)$$
$$= (0.1)\{(0.1)^2+(1.11161)^2\}$$
$$= 0.12457$$

$\therefore K = \frac{1}{6}(0.1+2\times0.1105+2\times0.11161+0.12457) = 0.11146$

$\therefore$ By (1), $y(0+0.1) = y(0)+0.11146 = 1.11146$

$\therefore y(0.1) = 1.11146$

Step 2   By taking $x = 0.1$, $y(0.1) = 1.11146$, $h = 0.1$,

$K_1 = (0.1) f(0.1, 1.11146) = (0.1)\{(0.1)^2 + (1.11161)^2\} = 0.12457$

$K_2 = (0.1) f\left(0.1 + \frac{0.1}{2}, 1.11146 + \frac{0.12457}{2}\right)$

$= (0.1) f(0.15, 1.17374) = (0.1)[(0.15)^2 + (1.17374)^2]$

$= 0.14002$

$K_3 = (0.1) f\left(0.1 + \frac{0.1}{2}, 1.11146 + \frac{0.14002}{2}\right)$

$= (0.1) f(0.15, 1.18147) = (0.1)[(0.15)^2 + (1.18147)^2]$

$= (0.1)[0.0225 + 1.39587] = 0.14184$

$K_4 = (0.1) f(0.1 + 0.1, 1.11146 + 0.14184)$

$= (0.1) f(0.2, 1.2533) = (0.1)[(0.2)^2 + (1.2533)^2] = 0.16108$

$\therefore K = \frac{1}{6}(0.12457 + 2 \times 0.14002 + 2 \times 0.14184 + 0.16108) = 0.14156$

$\therefore$ By (1), $y(0.1 + 0.1) = y(0.1) + 0.14156$

$= 1.11146 + 0.14156 = 1.25302$

$\therefore y(0.2) = 1.25302$

Hence $\boxed{y(0.1) = 1.11146, \; y(0.2) = 1.25302}$ A

Predictor-corrector methods: To solve $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$, $y(x_n)=?$
where $x_n = x_0+nh$

Modified Euler's Method: $y_{n+1}^{(p)} = y_n + hf(x_n, y_n)$ ——— (1)

$$y_{n+1}^{(c)} = y_n + \frac{h}{2}\left[f(x_n,y_n)+f(x_{n+1}, y_{n+1}^{(p)})\right] \qquad (2)$$

In this method, the value of y_{n+1} is first estimated by (1) and then used in R.H.S. of equ. (2) giving a best approximation of y_{n+1}. This value of y_{n+1} is again substituted in equ. (2) to find a still better approximation of y_{n+1}.

This procedure is repeated till two consecutive iterated values of y_{n+1} agree.

The equation (1) is therefore called the predictor while equation (2) serves as a corrector of y_{n+1}.

$\therefore$ The above method is also known as predictor-corrector Euler's method.

Que Using Euler's modified method, find $y(0.4)$ correct to 3D from the equation $\frac{dy}{dx} = x + |\sqrt{y}|$ with initial condition $y=1$ at $x=0$ and $h=0.2$.

Sol Here $f(x,y) = x+|\sqrt{y}|$, $x_0=0$, $y_0=1$, $h=0.2$

$\therefore$ $y_1^{(p)} = y_0 + hf(x_0,y_0) = 1+0.2(0+|\sqrt{1}|) = 1.2$

$$y_1^{(c)} = y_0+\frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(p)})\right]$$

$$= 1+\frac{(0.2)}{2}\left[1+(0.2+|\sqrt{1.2}|)\right] \qquad (\because x_1 = x_0+h = 0+0.2 = 0.2)$$

$$= 1.22954$$

$\therefore |y_1^{p} - y_1^{(c)}| = |1.2 - 1.229541| = 0.0295 \nleq 0.0005$ $\longrightarrow$ no need to show (only you can examine)

$$y_1^{cc} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(c)})\right]$$

$$= 1 + \frac{0.2}{2}\left[1 + \left(0.2 + |\sqrt{1.229541}|\right)\right] = 1.23088$$

$\therefore |y_1^{c} - y_1^{cc}| = |1.22954 - 1.23088| = 0.00134$ $\longrightarrow$ no need to show

$$y_1^{ccc} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{cc})\right]$$

$$= 1 + \frac{0.2}{2}\left[1 + \left(0.2 + |\sqrt{1.23088}|\right)\right] = 1.23095$$

$\therefore |y_1^{cc} - y_1^{ccc}| = |1.23088 - 1.23095| = 0.00007 \leq 0.0005$

$\therefore \boxed{y_1 = y(0.2) = 1.231}$

$$f(x_1, y_1) = 0.2 + |\sqrt{1.23095}| = 1.30948$$

Now, $y_2^{(p)} = y_1 + h f(x_1, y_1)$

$$= 1.23095 + 0.2 \times 1.30948 = 1.49285$$

$$y_2^{(c)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(p)})\right]$$

$\because x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$$= 1.23095 + \frac{0.2}{2}\left[1.30948 + \left(0.4 + |\sqrt{1.49285}|\right)\right]$$

$$= 1.52408$$

$$y_2^{cc} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(c)})\right]$$

$$= 1.23095 + \frac{0.2}{2}\left[1.30948 + \left(0.4 + |\sqrt{1.52408}|\right)\right] = 1.52535$$

$$y_2^{ccc} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{cc})\right]$$

$$= 1.23095 + \frac{0.2}{2}\left[1.30948 + \left(0.4 + |\sqrt{1.52535}|\right)\right] = 1.52540$$

$\because |y_2^{cc} - y_2^{ccc}| = |1.52535 - 1.52540| = 0.00005 \leq 0.0005$

Hence $\boxed{y_2 = y(0.4) = 1.525}$ correct to 3D

A

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Adams-Bashforth-Moulton method: This is a predictor-corrector method which is also a multi-step method.

Let the differential equation be $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ with h fixed and $x_n = x_0 + nh$. Suppose $f_n = f(x_n, y_n)$ and $y_n = y(x_n)$. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are given, then we can find $y(x_n)$ using

Adams-Bashforth-Moulton predictor and corrector formulae

$$y^p_{n+1} = y_n + \frac{h}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right) \quad \text{——— (1)}$$

and $$y^c_{n+1} = y_n + \frac{h}{24}\left(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right); \; f_{n+1} = f(x_{n+1}, y^p_{n+1}) \quad \text{——— (2)}$$

In this method, the value of y_{n+1} is first predicted by (1) and then used in R.H.S. of equ.(2) after finding f_{n+1} and then corrected by equation (2). This value of y^c_{n+1} is again substituted in R.H.S. of equ.(2) to find a still better approximation of y_{n+1} and we denote it by y^{cc}_{n+1}. This procedure is repeated till two consecutive iterative values of y_{n+1} agree.

Note: If here the values at previous steps i.e., (x_1, y_1), (x_2, y_2) and (x_3, y_3) are not given, then we compute these values by some single-step method such as the fourth-order Runge-Kutta method, Taylor series method etc.

Que Solve initial value problem $\frac{dy}{dx} = 1 + xy^2$, $y(0) = 1$ for $x = 0.4$ correct to 3D by using Adams-Bashforth-Moulton method when it is given that

$x =$	0.1	0.2	0.3
$y =$	1.105	1.223	1.355

$h = 0.1$

Sol Given

	x_0	x_1	x_2	x_3	x_4
x :	0	0.1	0.2	0.3	0.4
$y=f(x)$:	1	1.105	1.223	$1.355 = y_3$	$? = y_4$
$\therefore f(x,y) = 1+xy^2$:	1	1.12210	1.30013	1.55081	
	f_0	f_1	f_2	f_3	

We know that by Adams-Bashforth-Moulton method:

$$y_{n+1}^{(p)} = y_n + \frac{h}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right) \quad \text{—— (1)}$$

and $$y_{n+1}^{(c)} = y_n + \frac{h}{24}\left(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right) \quad \text{—— (2)}$$

where $f_{n+1} = f(x_{n+1}, y_{n+1}^{(p)})$ in equ.(2)

Put $n = 3$ in (1) & (2),

$$y_4^{(p)} = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right] \quad \text{—— (3)}$$

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9f_4 + 19f_3 - 5f_2 + f_1\right] \quad \text{—— (4)}$$

$\therefore$ By equ.(3), $y_4^{(p)} = 1.355 + \frac{0.1}{24}\left[55 \times 1.55081 - 59 \times 1.30013 + 37 \times 1.12210 - 9\right]$

$= 1.52627$

By (4), $y_4^{(c)} = 1.355 + \frac{0.1}{24}\left[9 \times 1.93180 + 19 \times 1.55081 - 5 \times 1.30013 + 1.12210\right]$

$\left(\because f_4 = f(x_4, y_4^{(p)}) = f(0.4, 1.52627) = 1 + (0.4)(1.52627)^2 = 1.93180\right)$

$\therefore y_4^{(c)} = 1.52780$

$\therefore |y_4^{(p)} - y_4^{(c)}| = |1.52627 - 1.52780| = 0.00153 \nleq 0.0005$

Now, $y_4^{cc} = 1.355 + \frac{0.1}{24}\left[9\{1 + (0.4)(1.52627)^2\} + 19 \times 1.55081 - 5 \times 1.30013 + 1.12210\right]$

$= 1.52787$

$\therefore |y_4^{c} - y_4^{cc}| = |1.52780 - 1.52787| = 0.00007 \leq 0.0005$

$\therefore \boxed{y(0.4) = y_4 = 1.528}$ (correct to 3D)

Milne's method: It is also a predictor-corrector method. This is a multi-step method because the previous informations at more than one point are required.

Let the diff. equation be $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ with h fixed and $x_n = x_0 + nh$. Suppose $f_n = f(x_n, y_n)$ and $y_n = y(x_n)$. When $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are given, then we can find $y(x_n)$ using

Milne's predictor and corrector formulae

$$y^p_{n+1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n) \quad (1)$$

and

$$y^c_{n+1} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}); \quad f_{n+1} = f(x_{n+1}, y^p_{n+1}) \quad (2)$$

In this method, the value of y_{n+1} is first predicted by (1) and then used in R.H.S. of equ. (2) after finding f_{n+1} and corrected by equ. (2). This value of y^c_{n+1} is again substituted in (2) to find a still better approximation of y_{n+1} and we denote it by y^{cc}_{n+1}. This procedure is repeated till two consecutive iterated values of y_{n+1} agree.

Note: If the values at previous steps are not given i.e., (x_1, y_1), (x_2, y_2) and (x_3, y_3) are not given, then we compute these values by some single-step method such as the fourth-order Runge-Kutta method, Taylor's series method etc.

Que: Solve initial value problem $\frac{dy}{dx} = 1 + xy^2$, $y(0) = 1$ for $x = 0.4$, correct to 3D by using Milne's method, It is given that

$x = 0.1 \quad 0.2 \quad 0.3$

$y = 1.105 \quad 1.223 \quad 1.355$

Sol

Given

	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$	$x_4 = 0.4$
	$y_0 = 1$	$y_1 = 1.105$	$y_2 = 1.223$	$y_3 = 1.355$	$y_4 = ?$
$f(x,y) = 1 + xy^2$	$f_0 = 1$	$f_1 = 1.12210$	$f_2 = 1.30013$	$f_3 = 1.55081$	

By Milne's method:

$$y^p_{n+1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n) \quad \text{—— (1)}$$

and $$y^c_{n+1} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}) \;;\; f_{n+1} = f(x_{n+1}, y^p_{n+1}) \quad \text{—— (2)}$$

put $n = 3$ in eq.(1), $y^p_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$ (Here $h = 0.1$)

$1 + \frac{4 \times 0.1}{3}(2 \times 1.12210 - 1.30013 + 2 \times 1.55081)$

$= 1.56580$

Put $n = 3$ in equ.(2),

$y^c_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$; $f_4 = f(x_4, y^p_4) = f(0.4, 1.56580)$

$\therefore y^c_4 = 1.225 + \frac{0.1}{3}\{1.30013 + 4 \times 1.55081 + f(0.4, 1.56580)\}$

$= 1.225 + \frac{0.1}{3}\{1.30013 + 4 \times 1.55081 + 1 + 0.4 \times (1.56580)^2\}$

$= 1.54114$

$\therefore |y^p_4 - y^c_4| = |1.56580 - 1.54114| = 0.02466 \nless 0.0005$

$\therefore$ We proceed to the next step as

$$y_4^{cc} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f(0.4, 1.54114)\right)$$

$$= 1.225 + \frac{0.1}{3}\left\{1.30013 + 4\times 1.55081 + 1 + 0.4\times(1.54114)^2\right\}$$

$$= 1.54011$$

$$\therefore \left|y_4^{c} - y_4^{cc}\right| = |1.54114 - 1.54011| = 0.00103 \nleq 0.0005$$

$$\therefore y_4^{ccc} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f(0.4, 1.54011)\right)$$

$$= 1.225 + \frac{0.1}{3}\left\{1.30013 + 4\times 1.55081 + 1 + 0.4\times(1.54011)^2\right\}$$

$$= 1.54007$$

So, $\left|y_4^{cc} - y_4^{ccc}\right| = |1.54011 - 1.54007| = 0.00004 \leq 0.0005$

$\therefore \boxed{y_4 = y(0.4) = 1.540}$ (correct to 3D)

Dr. Nishu Gupta,

Numerical Solution of Partial Differential Equations

A differential equation that involves more than one independent variable is called a partial differential equation.

The general second order linear partial differential equation is of the form $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$

$$\text{or} \quad A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \quad \text{—} ①$$

where A, B, C, D, E, F, G are all functions of x and y.

Equations of the above form can be classified into three types:

elliptic if $B^2 - 4AC < 0$

parabolic if $B^2 - 4AC = 0$

hyperbolic if $B^2 - 4AC > 0$

Examples:

(1) The Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is elliptic.

(2) The heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is parabolic.

(3) The wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is hyperbolic.

Que Classify the following equation

$$(1+x^2)\frac{\partial^2 u}{\partial x^2} + (5+2x^2)\frac{\partial^2 u}{\partial x \partial t} + (4+x^2)\frac{\partial^2 u}{\partial t^2} = 0$$

Sol Comparing the given equation with ①,

$$A = (1+x^2), \quad B = 5+2x^2, \quad C = 4+x^2$$

$$\therefore B^2 - 4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2)$$

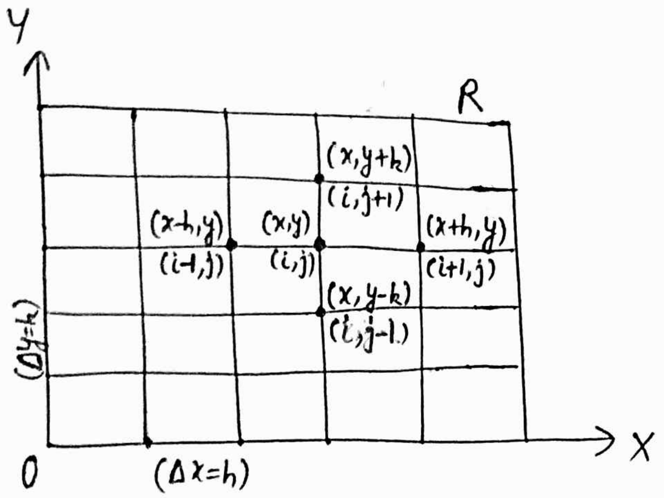
$$= 25 + 20x^2 + 4x^4 - 4(4+5x^2+x^4) = 9 > 0$$

$\therefore$ The given equation is hyperbolic.

Dr. Nishu Gupta

Finite-Difference Method: A principal approach to the numerical solution of such type of problems is the finite-difference method. It proceeds by replacing the partial derivatives in the equation by finite differences.

Consider a rectangular region R in the xy-plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in the figure. The points of intersection of the dividing lines are called mesh points, nodal points or grid points.



Then we have the finite difference approximations for the partial derivatives w.r.to x,

$$\frac{\partial u}{\partial x} \approx \frac{u(x+h,y)-u(x,y)}{h} = \frac{u(x,y)-u(x-h,y)}{h}$$

$$= \frac{u(x+h,y)-u(x-h,y)}{2h}$$

and $$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x-h,y)-2u(x,y)+u(x+h,y)}{h^2}$$

Writing $u(x,y) = u(ih, jk)$ as simply $u_{i,j}$, the above approximations become

$$u_x \approx \frac{u_{i+1,j}-u_{i,j}}{h} \quad \text{or} \quad u_x \approx \frac{u_{i,j}-u_{i-1,j}}{h} \quad \text{or} \quad u_x \approx \frac{u_{i+1,j}-u_{i-1,j}}{2h}$$

and $$u_{xx} \approx \frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}$$

Similarly we have approximations for the derivatives w.r.t. y:

$$u_y \approx \frac{u_{i,j+1}-u_{i,j}}{k} \quad \text{or} \quad u_y \approx \frac{u_{i,j}-u_{i,j-1}}{k} \quad \text{or} \quad u_y \approx \frac{u_{i,j+1}-u_{i,j-1}}{2k}$$

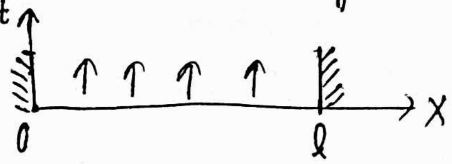
and $$u_{yy} \approx \frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{k^2}$$

Replacing the partial derivatives, in any partial differential equation by their corresponding difference approximations, we obtain the finite-difference analogues of the given equation.

Solution of Parabolic Equations:

The simplest example of parabolic equation is one-dimensional heat equation $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{——— (1)}$$

Its solution gives the temperature $u(x,t)$ at a distance x units of length from one end of a thermally insulated bar after t seconds of heat conduction.



In this problem, the temperatures at the ends of a bar of length l are often known for all time. Thus, the boundary conditions are known. Also, the temperature distribution along the bar is known at some particular instant. This instant is usually taken as zero-time and the temperature distribution is called the initial condition. The solution gives u for all values of x between 0 and l and values of t from 0 to ∞.

Using the difference formulas in equ. (1), we get

$$\frac{u_{i,j+1}-u_{i,j}}{k} = c^2 \cdot \frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2} \qquad \text{——— (2)}$$

Putting $\sigma = \frac{c^2 k}{h^2}$, we get

$$u_{i,j+1} = u_{ij} + \sigma\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

or $$\boxed{u_{i,j+1} = \sigma u_{i-1,j} + (1-2\sigma)u_{i,j} + \sigma u_{i+1,j}} \qquad \text{——— (3)}$$

where $\sigma = \frac{c^2 k}{h^2}$

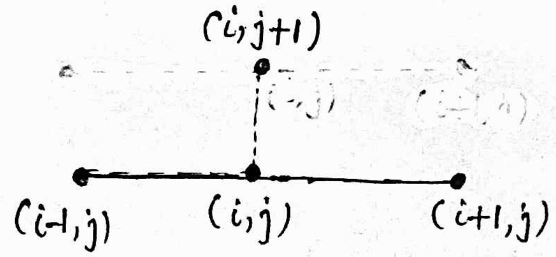
Equation (2) creates the $(j+1)$th row across the grid assuming that approximations in the jth row are known. This formula is therefore, called a <u>2-level formula</u>. This formula is also called <u>explicit formula</u> This formula is valid only for $0 < \sigma \le \frac{1}{2}$. or <u>(Schmidt-method)</u>

In particular when $\sigma = \frac{1}{2}$, equ. (3) reduces to

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right) \qquad \text{——— (4)}$$

which is called Bender-Schmidt method.

<u>Note:</u> The locations of the four points in equ. (3) are shown as:



<u>Heat Equations:</u>
<u>Explicit stencil</u>

Crank - Nicolson method: An alternative procedure of the implicit type is based on a simple variant of equation (2):

$$\frac{u_{i,j}-u_{i,j-1}}{k} = c^2 \cdot \frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2} \quad \text{------} (5)$$

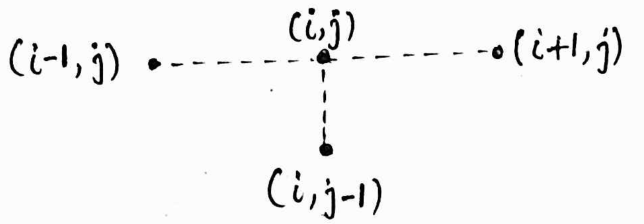
Putting $\frac{h^2}{c^2k} = s$,

$$s\,(u_{i,j}-u_{i,j-1}) = u_{i-1,j}-2u_{i,j}+u_{i+1,j}$$

or
$$\boxed{-u_{i-1,j}+r\,u_{i,j}-u_{i+1,j} = s\,u_{i,j-1}} \quad \text{------} (6)$$

where $r = 2+s$ and $s = \frac{h^2}{c^2k} = \frac{1}{\sigma}$

Note: The locations of the four points in equ. (6) are shown as:



Crank-Nicolson method: Implicit stencil

The equation (6), results in a tridiagonal system of equations to be solved.

Alternative Version of the Crank-Nicolson Method:

Another version of the Crank-Nicolson method is obtained as follows: (by taking central differences at $(i, j-\frac{1}{2})$) in equ.(5)

$$\frac{u_{i,j}-u_{i,j-1}}{k} = \frac{c^2}{h^2}\left[u_{i-1,j-\frac{1}{2}}-2u_{i,j-\frac{1}{2}}+u_{i+1,j-\frac{1}{2}}\right] \quad \text{------} (7)$$

Since the u values are known only at integer multiples of k

$\therefore$ we replace $u_{i,j-\frac{1}{2}} \approx \frac{u_{i,j}+u_{i,j-1}}{2}$

So, we have from (7),

$$\frac{u_{i,j}-u_{i,j-1}}{k} = \frac{c^2}{2h^2}\left[u_{i-1,j}+u_{i-1,j-1}-2\left(u_{ij}+u_{i,j-1}\right)+\left(u_{i+1,j}+u_{i+1,j-1}\right)\right]$$

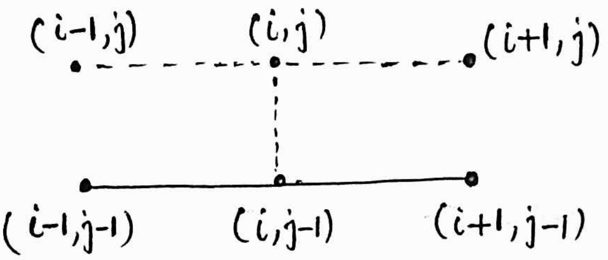
or $$2s\left[u_{ij}-u_{i,j-1}\right] = u_{i-1,j}+u_{i-1,j-1}-2\left(u_{ij}+u_{i,j-1}\right)+u_{i+1,j}+u_{i+1,j-1}$$

where $s = \dfrac{h^2}{c^2k}$

or $$\boxed{-u_{i-1,j}+2(1+s)\,u_{ij}-u_{i+1,j} = u_{i-1,j-1}+2(s-1)\,u_{i,j-1}+u_{i+1,j-1}} \quad (7)$$

where $s = \dfrac{h^2}{c^2k} = \dfrac{1}{\sigma}$

Note: The location of the six points in this equation are shown as:

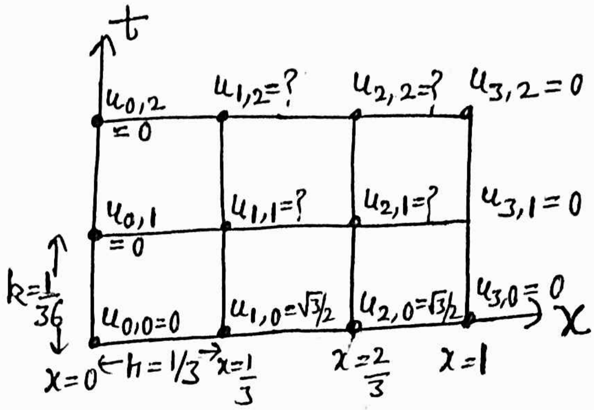


Crank-Nicolson method:

Alternative stencil

The equation (7), results in a tridiagonal system of equations to be solved.

Que Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(0,t) = u(1,t) = 0$ and $u(x,0) = \sin \pi x$, $0 \leq x \leq 1$ using explicit method (Schmidt method) for $h = 1/3$ & $k = 1/36$ up to two levels only.



Sol: Here $c^2 = 1$, $h = \frac{1}{3}$, $k = \frac{1}{36}$

$$\therefore \sigma = \frac{c^2 k}{h^2} = \frac{9}{36} = \frac{1}{4}$$

Also, $u(0,t) = 0$

$$\Rightarrow u_{0,0} = u_{0,1} = u_{0,2} = 0$$

& $u(1,t) = 0$

$$\Rightarrow u_{3,0} = u_{3,1} = u_{3,2} = 0.$$

Also, $u(x,0) = \sin \pi x$, $0 \leq x \leq 1$

$$\Rightarrow u_{1,0} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

and $u_{2,0} = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

The explicit method is given by

$$u_{i,j+1} = \sigma u_{i-1,j} + (1-2\sigma) u_{i,j} + \sigma u_{i+1,j}$$

$$\Rightarrow \quad u_{i,j+1} = 0.25(u_{i-1,j} + u_{i+1,j}) + 0.5\, u_{i,j} \quad \text{——— (1)}$$

<u>For first time level</u>, put $j=0$, in (1)

$$\therefore \quad u_{i,1} = 0.25(u_{i-1,0} + u_{i+1,0}) + 0.5\, u_{i,0}$$

$i = 1 \Rightarrow u_{1,1} = 0.25(u_{0,0} + u_{2,0}) + 0.5\, u_{1,0} = 0.25\left(0 + \frac{\sqrt{3}}{2}\right) + 0.5 \times \frac{\sqrt{3}}{2}$

$$= \frac{0.75 \times \sqrt{3}}{2} = 0.6495$$

$i = 2 \Rightarrow u_{2,1} = 0.25(u_{1,0} + u_{3,0}) + 0.5\, u_{2,0}$

$$= 0.25\left(\frac{\sqrt{3}}{2} + 0\right) + 0.5 \times \frac{\sqrt{3}}{2} = 0.6495$$

<u>For second time level</u> put $j = 1$ in (1)

$$u_{i,2} = 0.25(u_{i-1,1} + u_{i+1,1}) + 0.5\, u_{i,1}$$

$i = 1 \Rightarrow u_{1,2} = 0.25(u_{0,1} + u_{2,1}) + 0.5\, u_{1,1} = 0.25(0 + 0.6495) + 0.5 \times 0.6495 = 0.4871$

$i = 2 \Rightarrow u_{2,2} = 0.25(u_{1,1} + u_{3,1}) + 0.5\, u_{2,1} = 0.25(0.6495 + 0) + 0.5 \times 0.6495 = 0.4871$

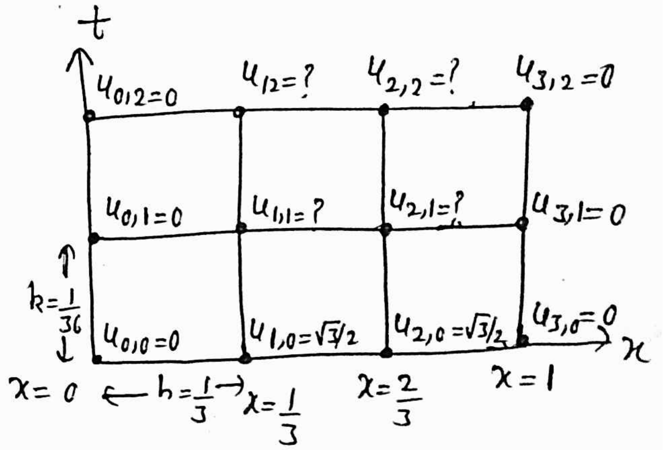
Que Solve the above problem using Crank-Nicolson method (Alternative version) for $h = 1/3$ & $k = 1/36$ up to two levels only.

Sol Crank-Nicolson method (Alternative version) is given by

$$-u_{i-1,j} + 2(1+s)u_{i,j} - u_{i+1,j} = u_{i-1,j-1} + 2(s-1)u_{i,j-1} + u_{i+1,j-1} \quad \text{--- (1)}$$

where $s = \dfrac{h^2}{c^2 k} = \dfrac{1}{\sigma}$

Here $c^2 = 1$, $h = \frac{1}{3}$, $k = \frac{1}{36}$ $\Rightarrow s = \frac{1}{9} \times 36 = 4$



Also $u(0,t) = 0$

$\Rightarrow u_{0,0} = u_{0,1} = u_{0,2} = 0$

& $u(1,t) = 0$

$\Rightarrow u_{3,0} = u_{3,1} = u_{3,2} = 0$

Also, $u(x,0) = \sin \pi x$, $0 \leq x \leq 1$

$\Rightarrow u_{1,0} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

and $u_{2,0} = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore$ From (1), we have

$$-u_{i-1,j} + 10u_{i,j} - u_{i+1,j} = u_{i-1,j-1} + 6u_{i,j-1} + u_{i+1,j-1} \quad \text{--- (2)}$$

For first time level put $j=1$ in (2)

$$\therefore -u_{i-1,1} + 10u_{i,1} - u_{i+1,1} = u_{i-1,0} + 6u_{i,0} + u_{i+1,0} \quad \text{--- (3)}$$

$i = 1 \Rightarrow \quad -u_{0,1} + 10u_{1,1} - u_{2,1} = u_{0,0} + 6u_{1,0} + u_{2,0}$ (by (3))

$$\Rightarrow 10u_{1,1} - u_{21} = 0 + \frac{6\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0$$

$$\Rightarrow 10u_{11} - u_{21} = \frac{7\sqrt{3}}{2} \quad \text{--- (4)}$$

$i=2 \Rightarrow \quad -u_{1,1}+10u_{2,1}-u_{3,1} = u_{1,0}+6u_{2,0}+u_{3,0} \quad$ (by ③)

$$\Rightarrow -u_{1,1}+10u_{2,1} = \frac{\sqrt{3}}{2}+\frac{6\sqrt{3}}{2}+0+0$$

$$\Rightarrow -u_{1,1}+10u_{2,1} = \frac{7\sqrt{3}}{2} \quad \text{———} (5)$$

Subtract (5) from (4), we get $\quad 11u_{11}-11u_{21}=0 \Rightarrow u_{11}=u_{21}$

Hence, $\boxed{u_{1,1}=u_{2,1}=\frac{7\sqrt{3}}{18}} = 0.6736$

For second time level put $j=2$ in ②,

$$-u_{i-1,2}+10u_{i,2}-u_{i+1,2} = u_{i-1,1}+6u_{i,1}+u_{i+1,1} \quad \text{———} (6)$$

$i=1$ in ⑥ $\Rightarrow \quad -u_{0,2}+10u_{1,2}-u_{2,2} = u_{0,1}+6u_{11}+u_{2,1}$

$$\Rightarrow \quad 10u_{1,2}-u_{2,2} = 6\times\frac{7\sqrt{3}}{18}+\frac{7\sqrt{3}}{18} = \frac{49\sqrt{3}}{18} \quad \text{———} (7)$$

$i=2$ in ⑥ $\Rightarrow \quad -u_{1,2}+10u_{2,2}-u_{3,2} = u_{1,1}+6u_{2,1}+u_{3,1}$

$$\Rightarrow \quad -u_{1,2}+10u_{2,2} = \frac{49\sqrt{3}}{18} \quad \text{———} (8)$$

Solving equations ⑦ & ⑧, we get

$$\boxed{u_{1,2}=u_{2,2}=\frac{49\sqrt{3}}{162}} = 0.5239$$

Que Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0,t) = u(1,t) = 0$ and $u(x,0) = \sin \pi x$, $0 \le x \le 1$ using Schmidt method (Take $h = 0.2$ and $\sigma = \frac{1}{2}$).

Sol Since $h = 0.2$ and $\sigma = \frac{1}{2}$, $c^2 = 1$

$$\therefore \quad \sigma = \frac{c^2 k}{h^2} \Rightarrow k = (0.2)^2 \cdot \frac{1}{2} = 0.02$$

Since $\sigma = \frac{1}{2}$, we use Bender-Schmidt method

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right) \qquad (1)$$

We have $u(0,0) = 0$, $u(0.2,0) = \sin \frac{\pi}{5} = 0.5875$

$u(0.4,0) = \sin \frac{2\pi}{5} = 0.9511$, $u(0.6,0) = \sin \frac{3\pi}{5} = 0.9511$

$u(0.8,0) = \sin \frac{4\pi}{5} = 0.5875$, $u(1,0) = \sin \pi = 0$

The values of u at the mesh points can be obtained by using the recurrence relation (1) as shown in table below:

$x \rightarrow$	0	0.2	0.4	0.6	0.8	1.0
$j \backslash i$	0	1	2	3	4	5
$t \downarrow$ 0	0	0.5878	0.9511	0.9511	0.5878	0
1	0	0.4756	0.7695	0.7695	0.4756	0
2	0	0.3848	0.6225	0.6225	0.3848	0
3	0	0.3113	0.5036	0.5036	0.3113	0
4	0	0.2518	0.4074	0.4074	0.2518	0
5	0	0.2037	0.3296	0.3296	0.2037	0

Hyperbolic Equations: The wave equation

(1) —— $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is the simplest example of hyperbolic p.d.e's. This equation governs the vibration of a string (transverse vibration in a plane) or the vibration in a rod (longitudinal vibration).

If equation (1) is used to model the vibrating string, then $u(x,t)$ represents the deflection at time t of a point on the string whose coordinate is x when the string is at rest.

To pose a definite model problem, we suppose that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

s.t. $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$; $0 \leq x \leq 1$ —— (2)

and $u(0,t) = \phi(t)$, $u(1,t) = \psi(t)$ ——— (3)

Now, we solve (1) by using the principle of numerical ~~integral~~ solution. Choosing step sizes h and k for x and t, respectively, and using the familiar approximations for derivatives, we have from eqn. (1),

$$\frac{1}{k^2}\left[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right] = \frac{c^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

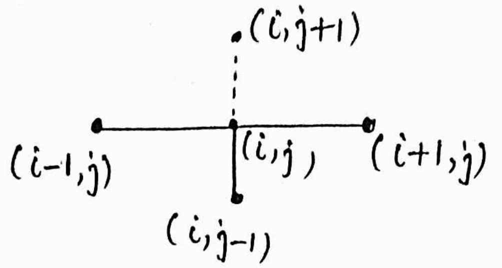
Now, by taking $\sigma = \frac{c^2 k^2}{h^2}$ and rearranging,

$$\boxed{u_{i,j+1} = \sigma u_{i+1,j} + 2(1-\sigma)u_{i,j} + \sigma u_{i-1,j} - u_{i,j-1}} \quad \text{——— (4)}$$

where $\sigma = \frac{c^2k^2}{h^2} = r^2$ where $r = \frac{ck}{h}$

which is called Wave-Equation Explicit method.

The locations of the five points in equation (4) are shown as



Wave equation:
Explicit stencil

Using equ.(4), we can find the value at $(j+1)$ level across the grid assuming that approximations in both levels j and $(j+1)$ are known.

Now, $u(0,t) = \phi(t) \Rightarrow u_{0,j} = \phi(t)$; $j = 1,2,3,\ldots$

$u(l,t) = \psi(t) \Rightarrow u_{n,j} = \psi(t)$, $j = 1,2,3,\ldots$ (where $l = nh$)

Now, $u_t(x,0) = g(x)$

$$\Rightarrow \frac{u_{i,1} - u_{i,0}}{k} = g(x) \Rightarrow u_{i,1} - u_{i,0} = k\,g(ih) \qquad (5)$$

and $u(x,0) = f(x) \Rightarrow u_{i,0} = f(ih)$ ——— (6)

Now, (5) and (6) gives the values of u on the first two rows $j=0$ and $j=1$.

and all other values can be find by using equ.(4).

Note: When $r=1$ i.e., $\sigma = 1$, then equation (4) reduces to

$$\boxed{u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}}$$

(2) For $r = 1$ i.e, $\sigma = 1$, the solution of (4) is stable and coincides with the solution of equation (1).

(3) For $r < 1$, the solution is stable but inaccurate.

(4) For $r > 1$, the solution is unstable.

(5) The formula (4) converges for $k \leq h$.

Que Use the finite difference method to solve the wave equation for a vibrating string

$$u_{tt}(x,t) = 4u_{xx}(x,t) \quad \text{for } 0 \leq x \leq 4 \text{ and } 0 \leq t \leq 2$$

with the boundary conditions

$$u(0,t) = u(4,t) = 0, \; t > 0$$

and the initial conditions

$$\left.\begin{aligned} u(x,0) &= x(4-x) \\ u_t(x,0) &= 0 \end{aligned}\right\} \; 0 \leq x \leq 4$$

Sol We have $c^2 = 4$. We take $h = 1$ and $k = 0.5$. Then

$$\sigma = r^2 = \frac{c^2 k^2}{h^2} = \frac{4 \times \frac{1}{4}}{1} = 1$$

$\therefore$ The difference equation for the problem is

$$\boxed{u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}} \quad \text{———} \quad (1)$$

$u(0,t) = 0 \Rightarrow u_{0,j} = 0$

$u(4,t) = 0 \Rightarrow u_{4,j} = 0$ $\qquad (\because nh = 4)$
$\Rightarrow n = 4 \; (\because h = 1)$

$u(x,0) = x(4-x)$

$\Rightarrow u_{i,0} = ih(4-ih) = i(4-i) \quad (\because x = ih$ & $h = 1)$

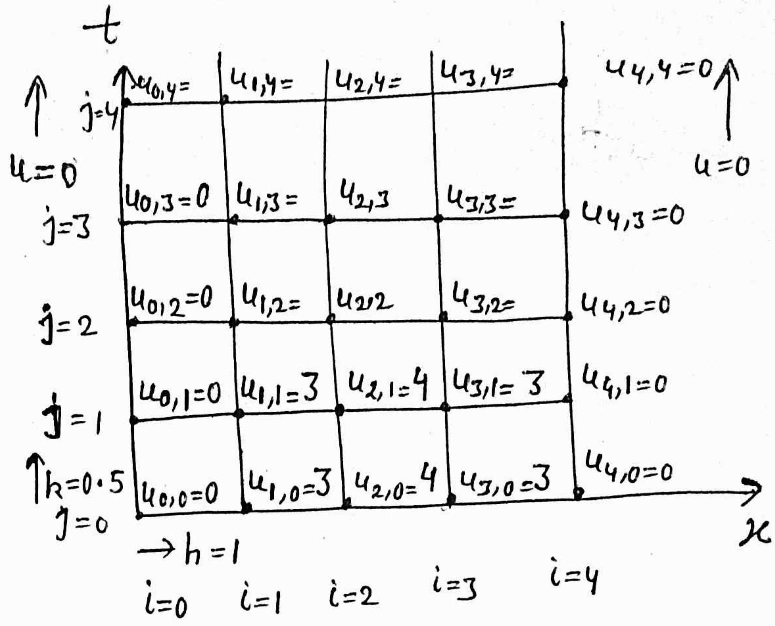
$\Rightarrow u_{0,0} = 0, \; u_{1,0} = 3, \; u_{2,0} = 4, \; u_{3,0} = 3, \; u_{4,0} = 0$

Now, $u_t(x,0) = 0$

$$\Rightarrow \frac{u_{i,1} - u_{i,0}}{k} = 0 \Rightarrow u_{i,1} = u_{i,0} \quad \text{———} \quad (2)$$

which shows that entries in the second row are same as those of the first row. Thus, the grid for the solution is as shown here

$x = ih$
$t = jk$



For $j=2$ by equ.(1), (Put $j=1$)

$$u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0}$$

$\therefore\ i=1 \Rightarrow u_{1,2} = u_{2,1} + u_{0,1} - u_{1,0}$
$= 4+0-3 = 1$

$i=2 \Rightarrow u_{2,2} = u_{3,1} + u_{1,1} - u_{2,0}$
$= 3+3-4 = 2$

$i=3 \Rightarrow u_{3,2} = u_{4,1} + u_{2,1} - u_{3,0}$
$= 0+4-3 = -1$

Hence $u_{1,2} = 1$
$u_{2,2} = 2$
$u_{3,2} = 1$

Similarly, (for $j=3$) $u_{1,3} = 0+2-3 = -1$
$u_{2,3} = 1+1-4 = -2$
$u_{3,3} = 2+0-3 = -1$

(for $j=4$) $u_{1,4} = 0+(-2)-1 = -3$
$u_{2,4} = -1-1-2 = -4$
$u_{3,4} = -1-1-1 = -3$

Elliptic Equations: The Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ is the simplest example of elliptic

p.d.e. Its finite difference analog is

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0 \qquad (1)$$

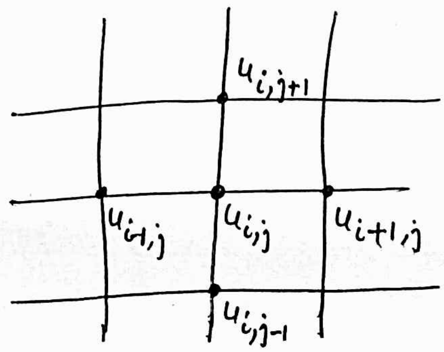
If we consider uniform spacing in both directions i.e, if we consider square mesh, then $h = k$. So, equation (1) implies

$$u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = 0$$

$$\Rightarrow \boxed{u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)} \qquad (2)$$

which is known as standard five point formula.

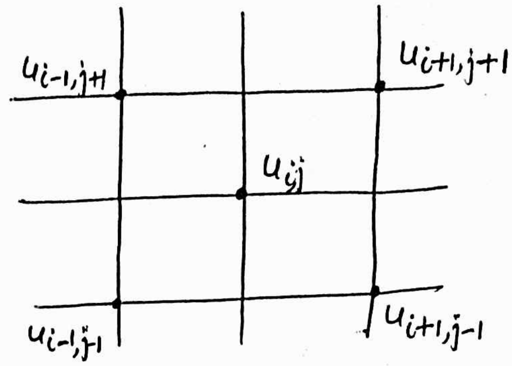
It can be shown as



- If we rotate the coordinate axes through $45°$, then the Laplace equation remains invariant. In fact,
  $X = x\cos\theta + y\sin\theta$, $Y = x\sin\theta - y\cos\theta$ where $\theta = 45°$, then $u_{XX} + u_{YY} = 0$. Therefore, we may use the function values at the diagonal points in place of the neighbouring points. Then we may use

$$\boxed{u_{i,j} = \frac{1}{4}\left(u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}\right)} \quad \text{——— (3)}$$

This formula is called the diagonal five point formula.

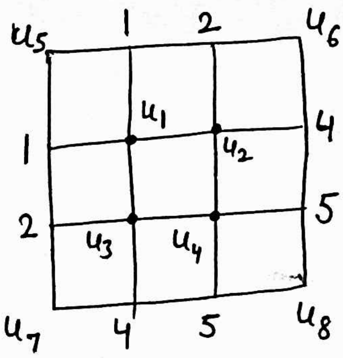


Gauss-Seidel iterative method: Let $u_{i,j}^{(n)}$ be the nth iterative value of $u_{i,j}$. Then the iterative procedure to solve equation (2) is

$$\boxed{u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right]} \quad \text{——— (4)}$$

This method uses the latest iterative values available and scans the mesh point systematically from left to right along successive rows.

Que: Solve Laplace equation $u_{xx} + u_{yy} = 0$ for the square meshes with the boundary values shown in Figure below.



Sol Using diagonal five point formula, we have

$$u_1 = \frac{1}{4}[2+2+u_4+u_5] \quad \text{——— (1)}$$

$$u_2 = \frac{1}{4}[1+5+u_3+u_6] \quad \text{——— (2)}$$

$$u_3 = \frac{1}{4}[1+5+u_2+u_7] \quad \text{——— (3)}$$

$$u_4 = \frac{1}{4}[4+4+u_1+u_8] \quad \text{——— (4)}$$

If we use standard five point formula, we have

$$u_1 = \frac{1}{4}[1+1+u_2+u_3] \quad \text{——— (5)}$$

$$u_2 = \frac{1}{4}[2+4+u_1+u_4] \quad \text{——— (6)}$$

$$u_3 = \frac{1}{4}[2+4+u_1+u_4] \quad \text{——— (7)}$$

$$u_4 = \frac{1}{4}[5+5+u_2+u_3] \quad \text{——— (8)}$$

From equation (6) and (7), $u_2 = u_3$. Therefore expressions (5), (6), (7) and (8) reduce to

$$u_1 = \frac{1}{4}[2+2u_2]$$

$$u_2 = \frac{1}{4}[6+u_1+u_4]$$

$$u_3 = \frac{1}{4}[6+u_1+u_4]$$

$$u_4 = \frac{1}{4}[10+2u_2]$$

If we start with the approximation $u_2 = 0$, then

$$u_1 = \frac{1}{2}, \; u_2 = 0, \; u_3 = 0, \; u_4 = \frac{5}{2}$$

Then, by Gauss-Seidel's method, we have

First iteration: $u_1^{(1)} = \frac{1}{4}[2+2u_2] = \frac{1}{2} = 0.5$

$$u_2^{(1)} = \frac{1}{4}[6+u_1^{(1)}+u_4] = \frac{1}{4}\left[6+\frac{1}{2}+\frac{5}{2}\right] = \frac{9}{4} = 2.25$$

$$u_3^{(1)} = \frac{1}{4}[6+u_1^{(1)}+u_4] = 2.25$$

$$u_4^{(1)} = \frac{1}{4}[10+2u_2^{(1)}] = \frac{1}{4}[10+4.5] = \frac{14.5}{4} = 3.625$$

Second Iteration:

$$u_1^{(2)} = \frac{1}{4}\left[2 + 2u_2^{(1)}\right] = \frac{1}{4}\left[2+4.50\right] = \frac{6.50}{4} = 1.625$$

$$u_2^{(2)} = \frac{1}{4}\left[6+u_1^{(2)}+u_4^{(1)}\right] = \frac{1}{4}\left[6+1.625+3.625\right] = \frac{11.250}{4} = 2.8125$$

$$u_3^{(2)} = \frac{1}{4}\left[6+u_1^{(2)}+u_4^{(1)}\right] = 2.8125$$

$$u_4^{(2)} = \frac{1}{4}\left[10+2u_2^{(2)}\right] = \frac{1}{4}\left[10+5.6250\right] = \frac{15.6250}{4} = 3.90625$$

Third iteration:

$$u_1^{(3)} = \frac{1}{4}\left[2+2u_2^{(2)}\right] = \frac{1}{4}\left[2+2\times 2.8125\right] = 1.9062$$

$$u_2^{(3)} = \frac{1}{4}\left[6+u_1^{(3)}+u_4^{(2)}\right] = \frac{1}{4}\left[6+1.9062+3.9062\right] = 2.9531$$

$$u_3^{(3)} = \frac{1}{4}\left[6+u_1^{(3)}+u_4^{(2)}\right] = 2.9531$$

$$u_4^{(3)} = \frac{1}{4}\left[10+2u_2^{(3)}\right] = \frac{1}{4}\left[10+2\times 2.9531\right] = 3.9766$$

Fourth iteration:

$$u_1^{(4)} = \frac{1}{4}\left[2+2u_2^{(3)}\right] = \frac{1}{4}\left[2+2\times 2.9531\right] = 1.9766$$

$$u_2^{(4)} = \frac{1}{4}\left[6+u_1^{(4)}+u_4^{(3)}\right] = \frac{1}{4}\left[6+1.9766+3.9766\right] = 2.9883$$

$$u_3^{(4)} = \frac{1}{4}\left[6+u_1^{(4)}+u_4^{(3)}\right] = 2.9883$$

$$u_4^{(4)} = \frac{1}{4}\left[10+2u_2^{(4)}\right] = \frac{1}{4}\left[10+2\times 2.9883\right] = 3.9942$$

Fifth iteration:

$$u_1^{(5)} = \frac{1}{4}\left[2+2u_2^{(4)}\right] = \frac{1}{4}\left[2+2\times 2.9883\right] = 1.9942$$

$$u_2^{(5)} = \frac{1}{4}\left[6+u_1^{(5)}+u_4^{(4)}\right] = \frac{1}{4}\left[6+1.9942+3.9942\right] = 2.9971$$

$$u_3^{(5)} = \frac{1}{4}\left[6+u_1^{(5)}+u_4^{(4)}\right] = 2.9971$$

$$u_4^{(5)} = \frac{1}{4}\left[10+2u_2^{(5)}\right] = \frac{1}{4}\left[10+2\times 2.9971\right] = 3.9986$$

We observe that the values of 4th and 5th iterations agree up to one decimal place. Hence,

$$u_1 = 2.0, \quad u_2 = 3.0, \quad u_3 = 3.0 \quad u_4 = 4.0$$

$\therefore$ From equ. (1), (2), (3) and (4),

$$2 = \frac{1}{4}[2+2+4+u_5] \Rightarrow u_5 = 0$$

$$3 = \frac{1}{4}[1+5+3+u_6] \Rightarrow u_6 = 3$$

$$3 = \frac{1}{4}[1+5+3+u_7] \Rightarrow u_7 = 3$$

$$4 = \frac{1}{4}[4+4+2+u_8] \Rightarrow u_8 = 6$$

Hence, $u_1 = 2.0,\ u_2 = 3.0,\ u_3 = 3.0,\ u_4 = 4.0$
$u_5 = 0,\ u_6 = 3,\ u_7 = 3,\ u_8 = 6$ Ans

Another Example of Elliptic Partial Differential Equation:

Poisson's Equation: $u_{xx} + u_{yy} = f(x,y)$ ——— (1)

where $f(x,y)$ is a given function of x and y.

Replacing the derivatives by difference expressions at the points $x = ih$, $y = jh$. Thus, we have

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} = f(ih, jh)$$

(by taking square mesh)

or $$\boxed{u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} = h^2 f(ih, jh)}$$

——— (2)

$$\Rightarrow u_{ij} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f(ih, jh)\right] \quad \text{——— (3)}$$

Dr. Nisha Gupta

Que Solve the Poisson's equation

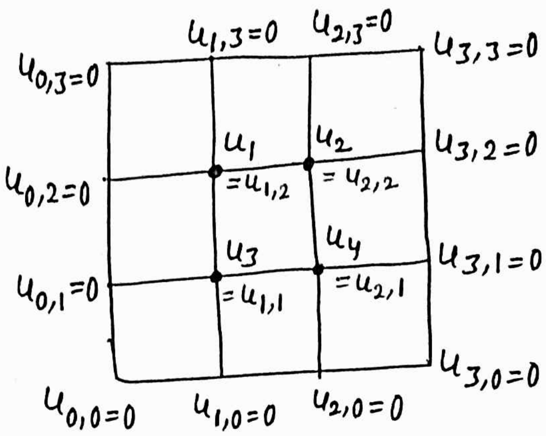
$$u_{xx} + u_{yy} = -10(x^2+y^2+10)$$

over the square with sides $x = 0 = y$, $x = 3 = y$ with $u = 0$ on the boundary and mesh length 1.

Sol According to given conditions

Here $f(x,y) = -10(x^2+y^2+10)$

and $h = 1$



According to standard formula (2), we have

$$u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} = h^2 f(ih, jh)$$

i.e, $u_{ij} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f(ih,jh)\right]$

For $i=1, j=2$, we have $u_1 = \frac{1}{4}\left[0 + u_2 + 0 + u_3 - 1.f(1,2)\right]$

$$\Rightarrow \quad u_1 = \frac{1}{4}(u_2 + u_3 + 150)$$

For $i=1, j=1$, we have $u_3 = \frac{1}{4}\left[0 + u_4 + u_1 + 0 - f(1,1)\right]$

$$\Rightarrow \quad u_3 = \frac{1}{4}(u_1 + u_4 + 120)$$

For $i=2, j=1$, we have

$$u_4 = \frac{1}{4}\left[u_3 + 0 + u_2 + 0 - f(2,1)\right]$$

$$\Rightarrow \quad u_4 = \frac{1}{4}(u_2 + u_3 + 150). \quad \text{So, } u_1 = u_4$$

For $i=2, j=2$, we have

$$u_2 = \frac{1}{4}\left[u_1 + 0 + 0 + u_4 - f(2,2)\right]$$

$$= \frac{1}{4}(u_1 + u_4 + 180)$$

We observe that $u_1 = u_4$. Therefore

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150)$$

$$u_2 = \frac{1}{4}(2u_1 + 180)$$

$$u_3 = \frac{1}{4}(2u_1 + 120)$$

We start with $u_2 = u_3 = 0$ and use Gauss-Seidel's method to improve the values. We have

First iteration:

$$u_1^{(1)} = \frac{1}{4}(0 + 0 + 150) = 37.5$$

$$u_2^{(1)} = \frac{1}{4}(2u_1^{(1)} + 180) = \frac{1}{4}[2 \times 37.5 + 180] = 63.75$$

$$u_3^{(1)} = \frac{1}{4}(2u_1^{(1)} + 120) = \frac{1}{4}[2 \times 37.5 + 120] = 48.75$$

Second iteration:

$$u_1^{(2)} = \frac{1}{4}[u_2^{(1)} + u_3^{(1)} + 150] = \frac{1}{4}[63.75 + 48.75 + 150] = 65.625$$

$$u_2^{(2)} = \frac{1}{4}[2u_1^{(2)} + 180] = \frac{1}{4}[2 \times 65.625 + 180] = 77.8125$$

$$u_3^{(2)} = \frac{1}{4}[2u_1^{(2)} + 120] = \frac{1}{4}[2 \times 65.625 + 120] = 62.8125$$

Third iteration:

$$u_1^{(3)} = \frac{1}{4}[u_2^{(2)} + u_3^{(2)} + 150] = \frac{1}{4}[77.8125 + 62.8125 + 150] = 72.65625$$

$$u_2^{(3)} = \frac{1}{4}[2u_1^{(3)} + 180] = \frac{1}{4}[2 \times 72.65625 + 180] = 81.328125$$

$$u_3^{(3)} = \frac{1}{4}[2u_1^{(3)} + 120] = \frac{1}{4}[2 \times 72.65625 + 120] = 66.328125$$

Fourth iteration:

$$u_1^{(4)} = \frac{1}{4}[u_2^{(3)} + u_3^{(3)} + 150] = \frac{1}{4}[81.328125 + 66.328125 + 150] = 74.4140625$$

$$u_2^{(4)} = \frac{1}{4}[2u_1^{(4)} + 180] = \frac{1}{4}[2 \times 74.4140625 + 180] = 82.2070312 5$$

$$u_3^{(4)} = \frac{1}{4}[2u_1^{(4)} + 120] = \frac{1}{4}[2 \times 74.4140625 + 120] = 67.20703125$$

Fifth iteration:

$$u_1^{(5)} = \frac{1}{4}\left[u_2^{(4)} + u_3^{(4)} + 150\right] = \frac{1}{4}\left[82.2070125 + 67.20703125 + 150\right] = 74.8535$$

$$u_2^{(5)} = \frac{1}{4}\left[2u_1^{(5)} + 180\right] = \frac{1}{4}\left[2\times 74.8535 + 180\right] = 82.4268$$

$$u_3^{(5)} = \frac{1}{4}\left[2u_1^{(5)} + 120\right] = \frac{1}{4}\left[2\times 74.8535 + 120\right] = 67.4268$$

Sixth Iteration:

$$u_1^{(6)} = \frac{1}{4}\left[u_2^{(5)} + u_3^{(5)} + 150\right] = \frac{1}{4}\left[82.4268 + 67.4268 + 150\right] = 74.9634$$

$$u_2^{(6)} = \frac{1}{4}\left[2u_1^{(6)} + 180\right] = \frac{1}{4}\left[2\times 74.9634 + 180\right] = 82.4817$$

$$u_3^{(6)} = \frac{1}{4}\left[2u_1^{(6)} + 120\right] = \frac{1}{4}\left[2\times 74.9634 + 120\right] = 67.4817$$

Seventh iteration:

$$u_1^{(7)} = \frac{1}{4}\left[u_2^{(6)} + u_3^{(6)} + 150\right] = \frac{1}{4}\left[82.4817 + 67.4817 + 150\right] = 74.99085$$

$$u_2^{(7)} = \frac{1}{4}\left[2u_1^{(7)} + 180\right] = \frac{1}{4}\left[2\times 74.99085 + 180\right] = 82.495425$$

$$u_3^{(7)} = \frac{1}{4}\left[2u_1^{(7)} + 120\right] = \frac{1}{4}\left[2\times 74.99085 + 120\right] = 67.495425$$

The values obtained by sixth and seventh iterations are nearby equal. Hence the solution correct to one decimal place is

$$\boxed{u_1 = 75.0, \quad u_2 = 82.5, \quad u_3 = 67.5, \quad u_4 = 75.0}$$

$(\because u_4 = u_1)$