Example 3.1 *An electron gas obeys the Maxwell-Boltzman statistics. Calculate average thermal ener (in eV) of an electron in the system at 300 K.* **[GGSIPU, March 2015 (2 mark**

Solution. $$E = \frac{3}{2}k_BT = \frac{3}{2}\times 1.38\times 10^{-23}\times 300\text{ J} = \frac{3\times(1.38\times 10^{-23})\times 300}{2(1.6\times 10^{-19})}\text{ eV} = 0.039\text{ eV}$$

Example 3.2 *At what temperature would one in a thousand of atom in a gas of atom hydrogen be in n = energy level ?*

Solution. For hydrogen $$n(E) = -\frac{13.6}{n^2}\text{ eV}$$

$$n(E_1) = n_1 = -13.6\text{ eV}$$

$$n(E_2) = n_2 = -3.4\text{ eV}$$

$g(E)$ = no. of states formed $= 2n^2$

$$g(E_1) = g_1 = 2 \text{ and } g(E_2) = g_2 = 8$$

For Maxwell-Boltzmann distribution is

$$\frac{n_2}{n_1} = \frac{g_2}{g_2}\frac{e^{-E_2/k_BT}}{e^{-E_1/k_BT}} = \frac{8}{2}e^{-(E_2-E_1)/k_BT}$$

or $$\frac{1}{10^3} = 4e^{-(E_2-E_1)/k_BT} \quad \text{or} \quad e^{(E_2-E_1)/k_BT} = \frac{10^4}{2.5} = 4000$$

Taking logarithms both the sides

$$\frac{E_2 - E_1}{k_BT} = \ln 4000$$

$$k_BT = \frac{(E_2 - E_1)}{\ln 4000} = \frac{10.2\text{ eV}}{8.29}$$

$$T = \frac{10.2\times 1.6\times 10^{-19}}{8.29\times 1.38\times 10^{-23}}\text{ K} = 1.43\times 10^4\text{ K}$$

$$T = 14300\text{ K} = 14300 - 273 = 14027°\text{C}$$

ample 3.3 *How many photons are present in 100 cm^3 of radiation in thermal equilibrium at 1000 K ?*

Solution. The total number of photons per unit volume is given by

$$\frac{N}{V} = \int_0^\infty n(\nu)\, d\nu$$

here $n(\nu)d\nu$ is the number of photons per unit volume with frequencies between ν and $(\nu + d\nu)$.

Since such photons have energies of $h\nu$.

$$n(\nu)d\nu = \frac{E(\nu)d\nu}{h\nu}$$

h $E(\nu)d\nu$ being the energy density given by Planck's formula. Hence the total number of photon he volume V is

$$N = \int_0^\infty \frac{E(\nu)d\nu}{h\nu} = \frac{8\pi V}{c^3}\int_0^\infty \frac{\nu^2 d\nu}{(e^{h\nu/k_B T} - 1)}$$

If we let $\frac{h\nu}{k_B T} = x$, then $\nu = \frac{k_B T x}{h}$ and $d\nu = \left(\frac{k_B T}{h}\right)dx$

So that, $$N = 8\pi V\left(\frac{k_B T}{hc}\right)^3 \int_0^\infty \frac{x^2\, dx}{(e^x - 1)}$$

The definite integral is a standard one equal to 2.404. Inserting numerical values of the other antities with $V = 100\text{ cc} = 100\times 10^{-6}\text{ m}^3$, we get

$$N = 2.03\times 10^{10} \text{ photons.}$$

Example 3.5 *Consider silver in the metallic state with one free electron per atom. Density of silver is 10.5 g/cc and atomic weight is 108.* **[GGSIPU, May 2014 reappear (6 marks)]**

Solution. Here $\frac{N}{V}=\frac{N}{M/\rho}=\frac{6.02\times 10^{26}}{108/(10.5\times 1000)}$

$$=\frac{6.02\times 10^{26}\times 10.5\times 1000}{108}=5.85\times 10^{28}$$

$$\therefore \quad E_F=\frac{h^2}{2m}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3}=\frac{(6.625\times 10^{-34})^2}{2\times(9\times 10^{-31})}\times\left(\frac{3}{8\pi}\times(5.85\times 10^{28})\right)^{2/3}$$

$$=8.92\times 10^{-19}\ \text{J}=5.57\ \text{eV}$$

(b) Electronic Specific H...

Miscellaneous Solved Numerical Problems

Problem 3.1 *A gas has only two particles A and B. Show that with the help of diagrams, how these two particles can be arranged in three series 1, 2, 3 using (i) Maxwell-Boltzmann, (ii) Bose-Einstein and (iii) Fermi-Dirac statistics.*

Solution. (*i*) **Maxwell-Boltzmann statistics :**

The two particles are distinguishable.

There is no limit to the number of particles in any one state.

The total number of ways $= 3^2 = 9$.

States	Possible distribution in various states								
1	A	B	–	–	A	B	AB	–	–
2	B	A	A	B	–	–	–	–	–
3	–	–	B	A	B	A	–	AB	AB

(*ii*) **Bose-Einstein statistics :**

If A and B are quantum particles, they are indistinguishable. Thus they have to be given the same name, say A.

There is no limit to the number of particles in any one state.

The total number of ways $= 6$

States	Possible distribution in various states					
1	A	A	–	AA	–	–
2	A	–	A	–	AA	–
3	–	A	A	–	–	AA

(*iii*) **Fermi-Dirac statistics :**

The particles are indistinguishable and not more than one particle can be in any one state.

The total number of ways $= 3$

States	Possible distribution in various states		
1	A	A	–
2	A	–	A
3	–	A	A

Problem 3.2 *Show that on increasing temperature, the number of atoms in excited state increases.*

Solution. $n(E) = g(E).\, f(E)$

For Maxwell-Boltzmann distribution

$$n(E_i) = \frac{g(E_i)}{e^{(\alpha + E_i/k_B T)}} = g(E_i) A e^{-E_i/k_B T}$$

Then $n(E_1) = g(E_1)\, A e^{-E_1/k_B T}$

and $n(E_2) = g(E_2)\, A e^{-E_2/k_B T}$

$$\Rightarrow \quad \frac{n(E_2)}{n(E_1)} = \frac{g(E_2)}{g(E_1)} e^{-(E_2 - E_1)/k_B T}$$

On increasing temperature $\Rightarrow n(E_2) > n(E_1)$

Solution. (*i*) **Maxwell-Boltzmann statistics :**

The two particles are distinguishable.

There is no limit to the number of particles in any one state.

The total number of ways $=3^2=9$.

States	*Possible distribution in various states*								
1	A	B	–	–	A	B	AB	–	–
2	B	A	A	B	–	–	–	AB	–
3	–	–	B	A	B	A	–	–	AB

(*ii*) **Bose-Einstein statistics :**

If A and B are quantum particles, they are indistinguishable. Thus they have to be given the same name, say A.

There is no limit to the number of particles in any one state.

The total number of ways $=6$

States	*Possible distribution in various states*					
1	A	A	–	AA	–	–
2	A	–	A	–	AA	–
3	–	A	A	–	–	AA

(*iii*) **Fermi-Dirac statistics :**

The particles are indistinguishable and not more than one particle can be in any one state.

The total number of ways $=3$

States	*Possible distribution in various states*		
1	A	A	–
2	A	–	A
3	–	A	A

Problem 3.2 *Show that on increasing temperature, the number of atoms in excited state increases.*

Solution. $$n(E)=g(E).\,f(E)$$

For Maxwell-Boltzmann distribution

$$n(E_i)=\frac{g(E_i)}{e^{(\alpha+E_i/k_BT)}}=g(E_i)Ae^{-E_i/k_BT}$$

Then $$n(E_1)=g(E_1)\,Ae^{-E_1/k_BT}$$

and

$$n(E_2)=g(E_2)\,Ae^{-E_2/k_BT}$$

$\Rightarrow$ $$\frac{n(E_2)}{n(E_1)}=\frac{g(E_2)}{g(E_1)}e^{-(E_2-E_1)/k_BT}$$

On increasing temperature $\Rightarrow$ $n(E_2)>n(E_1)$

Problem 3.5 *The Fermi level in potassium is 2.1 eV at a particular temperature. Calculate the number of free electrons per unit volume in potassium at the same temperature.*

Solution. Given $E_F = 2.1\text{ eV} = 2.1 \times 1.6 \times 10^{-19}\text{ J}, \quad n_c = ?$

The Fermi energy is given $E_F = \dfrac{h^2}{2m}\left(\dfrac{3n_c}{8\pi}\right)^{2/3}$

$$\therefore \qquad n_c = \left(\frac{2m}{h^2}E_F\right)^{3/2} \times \frac{8\pi}{3} = \left[\frac{2 \times 9.1 \times 10^{-31} \times 2.1 \times 1.6 \times 10^{-19}}{(6.625 \times 10^{-34})^2}\right]^{3/2} \times \frac{8 \times 3.14}{3}$$

$$= (5.579 \times 10^{18}) \times 1.047 = 1.379 \times 10^{28}\text{ electrons/m}^3$$

Problem 3.6 *The density of zinc is $7.13 \times 10^3\,kg/m^3$ and its atomic weight is 65.4. Calculate the Fermi energy and the mean energy at T = 0 K.*

Solution. Given : $\rho = 7.13 \times 10^3\text{ kg m}^{-3}, \quad M = 65.4$

Since we know that

$$E_F = \frac{h^2}{2m}\left(\frac{3n_c}{8\pi}\right)^{2/3} \quad \text{or} \quad E_F = 3.65 \times 10^{-19}\, n_c^{2/3}\text{eV} \quad \text{(On putting the value } h, m, \pi)$$

and

$$n_c = \frac{2\rho N}{M} = \frac{2 \times \text{density} \times \text{Avogadro's number}}{\text{Molecular weight}}$$

$$n_c = \frac{2 \times 7.13 \times 6.023 \times 10^{26}}{65.4} = 1313 \times 10^{26}$$

$$\therefore \qquad E_F = 3.65 \times 10^{-19} \times (1313 \times 10^{26})^{2/3} = 11.1\text{ eV}$$

and Mean energy $(\overline{E}) = \dfrac{3}{5}E_F = \dfrac{3}{5} \times 11.1\text{ eV} = 6.66\text{ eV}$

Problem 3.7 *At what temperature can we expect a 10% probability that electrons in a metal will have an energy which is 1% above E_F ? The Fermi energy of the metal is 5.5 eV.* **[GGSIPU, May 2014 (4.5 marks)]**

Solution. Given : $f(E) = 10\%, \; E = E_F + 1\%\text{ of }E_F, \; E_F = 5.5\text{ eV}, \; T = ?$

$$f(E) = \frac{1}{e^{(E-E_F)/k_BT} + 1}$$

$$E = 5.5 + \frac{5.5}{100} = 5.5 + 0.555\,; \; E - E_F = 0.555.$$

$$0.1 = \frac{1}{\left(\exp.\dfrac{0.555 \times 1.6 \times 10^{-19}}{1.38 \times 10^{-23}T}\right) + 1} = \frac{1}{\exp.\left(\dfrac{637.7}{T}\right) + 1} \quad \text{or} \quad T = 290.2\text{ K}$$

Problem 3.8 *Suppose that the maximum temperature in an atomic bomb explosion is 10^7 K. What is the corresponding wavelength of maximum energy ?*

Solution. $\qquad \lambda_{max}T = 0.289\text{ cm K}$

$$\lambda_{max} = \frac{0.2892}{T} = \frac{0.2892}{10^7} \approx 2.9 \times 10^{-8}\text{ cm} \approx 2.9\text{ Å}$$

Problem 3.9 *Calculate the surface temperature of the sun and moon given that $\lambda_m = 4573$ Å and $14\,\mu$ respectively, λ_m being the wavelength of the maximum intensity of emission.*

Solution. We know that $\lambda_{max} T = 0.2892$

For Sun $\qquad \lambda_{max} = 4573 \times 10^{-10}\,\text{m}$ or $4573 \times 10^{-8}\,\text{cm}$

$$4573 \times 10^{-8} \times T = 0.2892 \quad \text{or} \quad T = 6324\,\text{K} \quad \text{or} \quad 6051^\circ\text{C}$$

For moon $\qquad \lambda_{max} = 14 \times 10^{-6}$ m or 14×10^{-4} cm

$$\therefore \quad 1400 \times 10^{-6} \times T = 0.2892$$

$$T = \frac{0.2892 \times 10^6}{1400} = 206.57\,\text{K}$$

Problem 3.10 *Verify that rms speed of an ideal gas molecular is about 9% greater than its average speed.*

Solution. The equation $n(v)\,dv = 4\pi N\left(\dfrac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-\frac{mv^2}{2k_BT}}\,dv$

gives the number of molecules with speeds between v and $(v+dv)$ in a sample of N molecules. To find their average speed $\bar{v}$, we multiply $n(v)\,dv$ by v, integrate over all values of v from 0 to ∞, and then divide by N, we get

$$\bar{v} = \frac{1}{N}\int_0^\infty v\,n(v)dv = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\int_0^\infty v^3 e^{-\frac{mv^2}{2k_BT}}\,dv$$

If we let $a = \dfrac{m}{2k_BT}$, we see that the integral is the standard one

$$\int_0^\infty x^3 e^{-ax^2}\,dx = \frac{1}{2a^2}$$

ans so

$$\bar{v} = \left[4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\right]\left[\frac{1}{2}\left(\frac{2k_BT}{m}\right)^2\right] = \sqrt{\frac{8k_BT}{\pi m}}$$

and we know that

$$v_{rms} = \sqrt{\frac{3k_BT}{m}}$$

Comparing Eqs. (*ii*) and (*iii*), we get : $\quad v_{rms} = \sqrt{\dfrac{3k_BT}{m}} = \sqrt{\dfrac{3\pi}{8}}\,\bar{v} = 1.09\,\bar{v}$

Problem 3.11 *Radiation from the Big Bang has been Doppler-shifted to longer wavelengths by the expansion of the universe and today has a spectrum corresponding to that of a black body at 2.7 K. Find the wavelength at which the energy density of this radiation is a maximum. In what region of the spectrum is this radiation?*

Solution. We know that

$$\lambda_{max} T = 2.892 \times 10^{-3}\,\text{mK}$$

$$\lambda_{max} = \frac{2.892 \times 10^{-3}\,\text{mK}}{T} = \frac{2.892 \times 10^{-3}\,\text{mK}}{2.7\,\text{K}} = 1.1 \times 10^{-3}\,\text{m} = 1.1\,\text{mm}$$

The wavelength is in the microwave region.

Hint : Go through section 3.7.1 at pages 122-123.

3.3 Use the Fermi function to obtain the values of $f(E)$ for $E - E_F = 0.1$ eV. [$k_B = 1.38 \times 10^{-23}\,\text{J K}^{-1}$].

Hint : $f(E) = \dfrac{1}{e^{(E-E_F)/k_BT} + 1} = 0.0205$

3.4 Fermi energy for silver is 5.51 eV. What is the average energy of a free electron at 0 K ?

Hint : $\overline{E} = \dfrac{3}{5} E_F = 3.306$ eV

3.5 Fermi energy for gold is 5.54 eV. Calculate the Fermi temperature, given $k_B = 1.38 \times 10^{-23}\,\text{J K}^{-1}$.

Hint : $T_F = \dfrac{E_F}{k_B} = 6.42 \times 10^4\,\text{eV}$

3.6 Find the rms speed of oxygen molecules at 0°C.

Hint : $v_{\text{rms}} = \sqrt{\dfrac{3k_BT}{m}} = \sqrt{\dfrac{3 \times 1.38 \times 10^{-23} \times 273}{32 \times 1.66 \times 10^{-27}}} = 461\,\text{m/s}$

3.7 What is the wavelength of solar radiation to which human eye is most sensitive ? Assume the temperature of sun to be 5700 K.

Hint : $\lambda_m T = 2.898 \times 10^{-3}\,\text{mK}$, then $\lambda_m = 5070$ Å

3.8 A blackbody at 1373°C has λ_m, the wavelength corresponding to the maximum emission equal to 1.78 micron. Find the temperature of the moon if λ_m for the moon is 14 micron. Assume the moon to be a blackbody.

Hint: $\lambda_m T = C \quad \Rightarrow T = \dfrac{C}{\lambda_m} \quad \Rightarrow \dfrac{T_1}{T_2} = \dfrac{\lambda_{m_1}}{\lambda_{m_2}} \quad \Rightarrow T_2 = \dfrac{\lambda_{m_1}}{\lambda_{m_2}} \times T_1 = \dfrac{1.78}{14} \times (1373 + 273) = 209\,\text{K}.$