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Roots of Complex Numbers

$x = \cos\theta + i\sin\theta$

$x^{1/n} = (\cos\theta + i\sin\theta)^{1/n} = \left(\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\right)^{1/n}$

$$(\cos\theta + i\sin\theta)^{1/n} = \left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$$

where, $k = 0, 1, 2, 3, \ldots (n-1)$

Q. (1) → Solve $x^6 + 1 = 0$.

Sol → $x^6 + 1 = 0$

$\therefore x^6 = -1 = -1 + 0i$

$\therefore x^6 = \cos\pi + i\sin\pi$

$\therefore x^6 = [\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]$

(Take $(1/6)^{th}$ power on both side)

$$x = [\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]^{1/6}$$

(According to De-Moivre's Theorem)

$$x = \left[\cos\left(\frac{2k\pi+\pi}{6}\right) + i\sin\left(\frac{2k\pi+\pi}{6}\right)\right] \text{ --- (General eqn for Roots)}$$

$\therefore$ For $k = 0, 1, 2, 3, 4, 5$

[At $k = 0$]

$$x_0 = \left[\cos\left(\frac{0+\pi}{6}\right) + i\sin\left(\frac{0+\pi}{6}\right)\right] = \left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right] = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

[At $k = 1$]

$$x_1 = \left[\cos\left(\frac{3\pi}{6}\right) + i\sin\left(\frac{3\pi}{6}\right)\right] = \left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] = 0 + i(1) \Rightarrow i$$

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[At $k=2$]

$$x_2 = \left[\cos\left(\frac{2(2)\pi+\pi}{6}\right) + i\sin\left(\frac{2(2)\pi+\pi}{6}\right)\right] = \left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$$

$$= -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

[At $k=3$]

$$x_3 = \left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

[At $k=4$]

$$x_4 = \left[\cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right)\right] = 0 + i(-1) = -i$$

[At $k=5$]

$$x_5 = \left[\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right] = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

Q. 2 → find roots common to $x^4+1=0$ and $x^6-i=0$

Sol → $x^4+1=0$

$x^4=-1$

$x^4=-1+0i$

$x^4=(\cos\pi + i\sin\pi)$

Taking $(1/4)^{th}$ power,

$$x = [\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]^{1/4}$$

$$x = \left[\cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right)\right]$$ (Apply De-Moivre's)

where $k=0,1,2,3$

[At $k=0$]

$$x_0 = \left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

[At $k=1$]

$$x_1 = \left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

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[At $k=2$]

$$x_2 = \left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$

[At $k=3$]

$$x_3 = \left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]$$

Also, $x^6 - i = 0 \quad \therefore x^6 = i$

$$\therefore x^6 = 0 + i = \left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$$

Taking $(1/6)^{th}$ power

$$\therefore x = \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{1/6}$$

$$\Rightarrow x = \left[\cos\left(\frac{4k\pi + \pi}{2}\right) + i\sin\left(\frac{4k\pi + \pi}{2}\right)\right]^{1/6}$$

(Apply De-Moivre's)

$$\Rightarrow x = \left[\cos\frac{(4k+1)\pi}{12} + i\sin\frac{(4k+1)\pi}{12}\right]$$

where $k = 0, 1, 2, 3, 4, 5$

[At $k=0$]

$$x_0 = \left[\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right]$$

[At $k=1$]

$$x_1 = \left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$$

[At $k=2$]

$$x_2 = \left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

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[At k = 3]

$$x_3 = \left[\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right]$$

[At k = 4]

$$x_4 = \left[\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right]$$

[At k = 5]

$$x_5 = \left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]$$

$\therefore$ Common roots of $x^4+1=0$ & $x^6+i=0$ are

$$\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right] \;\&\; \left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]$$

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LOGARITHMS OF COMPLEX NUMBERS

$z = x + iy$ — (Cartesian form)

$z = r(\cos\theta + i\sin\theta)$ — (Polar form)

$z = re^{i\theta}$ — (Exponential form)

⇒ Equating Polar & Exponential form.

$r(\cos\theta + i\sin\theta) = re^{i\theta}$

(Take log on both sides)

$\log r(\cos\theta + i\sin\theta) = \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$ ($\log ab = \log a + \log b$)

$= \log r + i\theta \log e$

$= \log r + i\theta$

(r = magnitude $= \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$

θ = Argument $= \tan^{-1}(y/x)$)

$= \log(x^2+y^2)^{1/2} + i\tan^{-1}\left(\frac{y}{x}\right)$

$$\boxed{\log z = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)}$$

This is principle value.

∴ General Value is

$\text{Log}(x+iy) = 2n\pi i + \log(x+iy)$

Results:

① $\boxed{\log i = \frac{i\pi}{2}}$

Put $x=0$; $y=1$

$\log(0+i) = \frac{1}{2}\log(0^2+1^2) + i\tan^{-1}\left(\frac{1}{0}\right)$

$\log i = \frac{1}{2}\log 1 + i\tan^{-1}(\infty)$

$\boxed{\log i = \frac{i\pi}{2}}$

General value

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$$\boxed{\log i = 2n\pi i + \frac{i\pi}{2}}$$

(2) $\boxed{\log i^{i} = -\frac{\pi}{2}}$

$$i \log i = i\left(i\frac{\pi}{2}\right) = i^2\frac{\pi}{2} = -\frac{\pi}{2}$$

(3) $\boxed{\log(x-iy) = \frac{1}{2}\log(x^2+y^2) - i\tan^{-1}(y/x)}$

Q. Find value of $\log(1+i)$.

Sol→ Comparing $\log(1+i)$ with $\log(x+iy)$

$\therefore x = 1,\ y = 1$

$\therefore$ According to formulae

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$$\log(1+i) = \frac{1}{2}\log(1+1) + i\tan^{-1}(1/1)$$

$$\boxed{\log(1+i) = \frac{1}{2}\log 2 + \cancel{i\tan^{-1}\pi/4}\ i\frac{\pi}{4}}$$ Ans

Q. Prove that : $\log\left(\frac{x+iy}{x-iy}\right) = 2i\tan^{-1}(y/x)$

Sol→ According to definition;

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$$\log(x-iy) = \frac{1}{2}\log(x^2+y^2) - i\tan^{-1}(y/x)$$

Subtract ~~Add~~ above two equations

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$\log(x+iy) - \log(x-iy) = 2i\tan^{-1}(y/x)$

$\left(\log\left(\frac{a}{b}\right) = \log a - \log b\right)$

$$\boxed{\log\left(\frac{x+iy}{x-iy}\right) = 2i\tan^{-1}(y/x)}$$

Q. Show that $i\log\left(\frac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x$

Sol→ $\log(x-i) = \frac{1}{2}\log(x^2+1) - i\tan^{-1}(1/x)$

$\log(x+i) = \frac{1}{2}\log(x^2+1) + i\tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \log\left(\frac{x-i}{x+i}\right) = \log(x-i) - \log(x+i)$

$= \left[\frac{1}{2}\log(x^2+1) - i\tan^{-1}\left(\frac{1}{x}\right)\right] - \left[\frac{1}{2}\log(x^2+1) + i\tan^{-1}(1/x)\right]$

$= -2i\tan^{-1}\left(\frac{1}{x}\right)$

$= -2i\left(\frac{\pi}{2} - \tan^{-1}(x)\right)$ $\left(\tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}(x)\right)$

$\therefore \log\left(\frac{x-i}{x+i}\right) = -i(\pi - 2\tan^{-1}(x))$

Multiplying i on both sides

$i\log\left(\frac{x-i}{x+i}\right) = -i^2(\pi - 2\tan^{-1}(x))$

$$\boxed{i\log\left(\frac{x-i}{x+i}\right) = (\pi - 2\tan^{-1}(x))}$$

Hence, Proved.

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Q find the value of $\log(\sin(x+iy))$

Sol→ As $\sin(x+iy) = \sin x \cdot \cos iy + \cos x \sin iy$

$$= \underbrace{\sin x \cosh y}_{R.P.} + i\underbrace{\cos x \sinh y}_{I.P.}$$

Taking log on both sides

$$\log(\sin(x+iy)) = \frac{1}{2}\log\left[(\sin^2 x \cdot \cosh^2 y) + (\cos^2 x \cdot \sinh^2 y)\right] + i\tan^{-1}\left(\frac{\cos x \cdot \sinh y}{\sin x \cdot \cosh y}\right)$$

$\therefore \sin^2 x \cdot \cosh^2 y + \cos^2 x \cdot \sinh^2 y$

$= (1-\cos^2 x)\cosh^2 y + \cos^2 x(\cosh^2 y - 1)$

$= \cosh^2 y - \cos^2 x$

$= \cosh^2 y - \cos^2 x = \left(\frac{1+\cosh 2y}{2}\right) - \left(\frac{1+\cos 2x}{2}\right)$

$= \frac{1}{2}(\cosh 2y - \cos 2x)$

Ans $\therefore \log(\sin(x+iy)) = \frac{1}{2}\log\left(\frac{\cosh 2y - \cos 2x}{2}\right) + i\tan^{-1}(\cot x \cdot \tanh y)$

Q Show that: $\tan\left(i\log\left(\frac{a-bi}{a+bi}\right)\right) = \frac{2ab}{a^2-b^2}$

Sol→ $\log(a-bi) = \frac{1}{2}\log(a^2+b^2) - i\tan^{-1}(b/a)$

$\log(a+bi) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)$

$\therefore \log\left(\frac{a-bi}{a+bi}\right) = \log(a-bi) - \log(a+bi)$

$= -2i\tan^{-1}(b/a)$

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Multiplying by i,

$$i\log\left(\frac{a-bi}{a+bi}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$$

Take tan on both sides.

L.H.S. = $\tan\left(i\log\left(\frac{a-bi}{a+bi}\right)\right) = \tan\left(2\tan^{-1}\left(\frac{b}{a}\right)\right)$

Put $\tan^{-1}\left(\frac{b}{a}\right) = \theta$ $\therefore$ $\frac{b}{a} = \tan\theta$

$$\therefore \tan\left(2\tan^{-1}\left(\frac{b}{a}\right)\right) = \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2(b/a)}{1-(b/a)^2} = \frac{2b/a}{\frac{a^2-b^2}{a^2}} = \frac{2ab}{a^2-b^2} = R.H.S.$$

Hence, proved.

Q Prove that : $\log\left(\frac{1}{1-e^{i\theta}}\right) = \log\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right) + i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$

Sol→ L.H.S. = $\log\left(\frac{1}{1-e^{i\theta}}\right) = \log\left(\frac{1}{1-(\cos\theta+i\sin\theta)}\right) = \log\left(\frac{1}{(1-\cos\theta)-i\sin\theta}\right)$

$\left(1-\cos\theta = 2\sin^2\theta/2 \;,\; \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$

$$= \log\left(\frac{1}{2\sin^2\theta/2 - 2\sin\theta/2\cos\theta/2}\right)$$

$$= \log\left(\frac{1}{2\sin\frac{\theta}{2}(\sin\theta/2-\cos\theta/2)}\right)$$

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$= \log(1) - \log\left(2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)\right)$

$\left(\text{becoz } \log\left(\frac{a}{b}\right) = \log a - \log b\right)$

$\Rightarrow 0 - \left(\log\left(2\sin\frac{\theta}{2}\right) + \log\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)\right)$

$(\text{becoz } \log(ab) = \log a + \log b)$

$\Rightarrow -\log\left(2\sin\frac{\theta}{2}\right) - \log\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)$

Now, $-\log\left(2\sin\frac{\theta}{2}\right) = \log\left(2\sin\frac{\theta}{2}\right)^{-1} = \log\left(\frac{1}{2\sin\theta/2}\right)$

$= \log\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right)$

And; $\log\left(\sin\theta/_2 - i\cos\frac{\theta}{2}\right) = \frac{1}{2}\log\left(\sin^2\frac{\theta}{2} + \cos^2\theta/_2\right) - i\tan^{-1}\left(\frac{\cos\theta/2}{\sin\theta/2}\right)$

$\left(\text{As; } \log(x \pm iy) = \frac{1}{2}\log(x^2+y^2) \pm i\tan^{-1}(y/x)\right)$

$\log\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right) = \frac{1}{2}\log(1) - i\tan^{-1}\left(\cot\frac{\theta}{2}\right)$

$= 0 - i\tan^{-1}\left(\tan\left(\frac{\pi}{2} - \theta/_2\right)\right)$

$= -i\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

$\therefore \log\left(\frac{1}{1-e^{i\theta}}\right) = \log\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right) - \left(-i\left(\frac{\pi}{2} - \theta/_2\right)\right)$

$= \log\left(\frac{1}{2}\operatorname{cosec}\theta/_2\right) + i\left(\frac{\pi}{2} - \theta/_2\right)$

Hence, proved.

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Q. Separate into real & imaginary parts $Log(3+4i)$

Sol→ $3+4i = x+iy = r(\cos\theta + i\sin\theta)$

$\therefore x=3;\ y=4;\ r=\sqrt{x^2+y^2} = \sqrt{3^2+4^2} = 5$

Now;

$x = r\cos\theta \quad ; y = r\sin\theta$

$3 = 5\cos\theta \quad ; 4 = 5\sin\theta$

$\cos\theta = \left(\frac{3}{5}\right) \quad ; \sin\theta = \left(\frac{4}{5}\right)$

$\therefore \tan\theta = \frac{4}{3} \quad \therefore \theta = \tan^{-1}\left(\frac{4}{3}\right)$

According to General formulae.

$Log(x+iy) = \log r + 2n\pi i$

$= \frac{1}{2}\log(x^2+y^2) + i(2n\pi + \theta)$

So; $Log(3+4i) = \frac{1}{2}\log(3^2+4^2) + i\left(2n\pi + \tan^{-1}\left(\frac{4}{3}\right)\right)$

$= \log 5 + i\left(2n\pi + \tan^{-1}\left(\frac{4}{3}\right)\right)$

Ans Real Part = $\log 5$

Imaginary Part = $2n\pi + \tan^{-1}\left(\frac{4}{3}\right)$

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Q. Considering only the principle values, prove that the real part of $(1+i\sqrt{3})^{(1+i\sqrt{3})}$ is $2e^{-\pi/\sqrt{3}}\left(\cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)\right)$

Sol → Let $(x+iy) = (1+i\sqrt{3})^{(1+i\sqrt{3})}$

(Take log on both sides)

$$\log(x+iy) = (1+i\sqrt{3})\log(1+i\sqrt{3})$$

$$= (1+i\sqrt{3})\left(\frac{1}{2}\log(1+3) + i\tan^{-1}(\sqrt{3})\right)$$

$$= (1+i\sqrt{3})\left(\frac{1}{2}\log 2^2 + i\frac{\pi}{3}\right)$$

$$= (1+i\sqrt{3})\cdot\left(\log 2 + i\pi/3\right)$$

$$= \log 2 + i\frac{\pi}{3} + i\sqrt{3}\log 2 + i^2\sqrt{3}\frac{\pi}{3}$$

$$= \left(\log 2 - \frac{\pi}{\sqrt{3}}\right) + i\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)$$

(Taking Anti log)

$$(x+iy) = e^{\log 2}\cdot e^{-\pi/\sqrt{3}}\cdot e^{i(\sqrt{3}\log 2 + \pi/3)}$$

$$= 2e^{-\pi/\sqrt{3}}\left(\cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right) + i\sin\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)\right)$$

$$\boxed{\text{Real part} = 2e^{-\pi/\sqrt{3}}\cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)}$$

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Q. Prove that : $\log\left(\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)\right) = i\tan^{-1}(\sinh x)$

Sol → $L.H.S. = \log\left(\tan\left(\frac{\pi}{4}+i\frac{x}{2}\right)\right)$

$$= \log\left(\frac{\tan\pi/4 + \tan i x/2}{1-(\tan\pi/4)\tan(ix/2)}\right)$$

$$= \log\left(\frac{1+i\tanh x/2}{1-i\tanh x/2}\right)$$

$$= \log(1+i\tanh x/2) - \log(1-i\tanh x/2)$$

$$= \frac{1}{2}\log(1+\tanh^2 x/2) + i\tan^{-1}(\tanh x/2) - \frac{1}{2}\log(1+\tanh^2 x/2)$$

$$+ i\tan^{-1}(\tanh x/2)$$

$$= 2i\tan^{-1}\left(\tanh\frac{x}{2}\right)$$

$$= \frac{2i}{2}\tan^{-1}\left(\frac{2\tanh x/2}{1-\tanh^2 x/2}\right)$$

$$= i\tan^{-1}(\sinh x) = R.H.S.$$

Other way

$LHS \Rightarrow \log\left(\tan\left(\frac{\pi}{4}+i\frac{x}{2}\right)\right)$

$$= \log\left(\frac{\tan\pi/4 + \tan ix/2}{1-\tan\pi/4\tan ix/2}\right) \Rightarrow \log\left(\frac{1+i\tanh x/2}{1-i\tanh x/2}\right)$$

$$= \log\left(\frac{1+i\tanh x/2}{1-i\tanh x/2}\times\frac{1+i\tanh x/2}{1+i\tanh x/2}\right)$$

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$$= \log\left(\frac{(1+i\tanh\frac{x}{2})^2}{1-(i\tanh\frac{x}{2})^2}\right)$$

$$= \log\left(\frac{1+(i\tanh\frac{x}{2})^2+2i\tanh\frac{x}{2}}{1-i^2\tanh^2\frac{x}{2}}\right)$$

$$= \log\left(\frac{1+i^2\tanh^2\frac{x}{2}+2i\tanh\frac{x}{2}}{1+\tanh^2 x/2}\right)$$

$$= \log\left(\frac{1-1\tanh^2\frac{x}{2}+2i\tanh\frac{x}{2}}{1+\tanh^2 x/2}\right)$$

$$= \log\left(\frac{1-\frac{\sinh^2 x/2}{\cosh^2 x/2}+\frac{2i\sinh x/2}{\cosh x/2}}{1+\frac{\sinh^2 x/2}{\cosh^2 x/2}}\right)$$

$$= \log\left(\frac{\cosh^2 x/2-\sinh^2 x/2+2i\sinh x/2\cosh x/2}{\cosh^2 x/2+\sinh^2 x/2}\right)$$

$$= \log\left(\frac{1+2i\sinh x/2\cosh x/2}{\cosh^2 x/2+\sinh^2 x/2}\right)$$

$\cosh^2 z-\sinh^2 z=1$
$\cosh^2 z+\sinh^2 z=\cosh 2z$

$$= \log\left(\frac{1+i\sinh x}{\cosh x}\right)$$

$$= \log\left(\frac{1}{\cosh x}+i\tanh x\right)$$

$$= \log(\operatorname{sech} x+i\tanh x)$$

$$= \frac{1}{2}\log(\operatorname{sech}^2 x+\tanh^2 x)+i\tan^{-1}\left(\frac{\tanh x}{\operatorname{sech} x}\right)$$

$$= \frac{1}{2}\log\left(1-\tanh^2 x+\tanh^2 x\right)+i\tan^{-1}(\sinh x)$$

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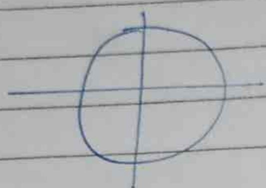
$= i \tan^{-1}(\sinh x) = R.H.S$

Hence, proved

Hyperbolic functions

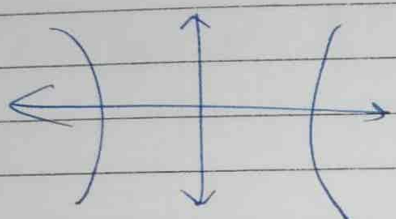
① $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

② $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2}$



imaginary term possible.

Now Hyperbolic



imaginary term not possible

① $\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$

② $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

* Convert hyperbolic to circular / circular to hyperbolic

① $\sin ix = i\sinh x$

→ As, $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

But replace x by ix

$\therefore \sin ix = \dfrac{e^{i(ix)} - e^{-i(ix)}}{2i} = \dfrac{e^{i^2 x} - e^{-i^2 x}}{2i} = \dfrac{e^{-x} - e^{x}}{2i}$

$= -\dfrac{(e^{x} - e^{-x})}{2i}$

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$$\therefore \sin ix = -\frac{(e^{x}-e^{-x})}{2i} \times \frac{i}{i} = \frac{-i(e^{x}-e^{-x})}{-2}$$

$$= \frac{i(e^{x}-e^{-x})}{2}$$

~~sin ix = sin hx~~

$\boxed{\sin ix = i \sinh x}$ & $\boxed{\sin ix = i \sinh x}$

$\cos ix = \cosh x$; $\cosh ix = \cos x$

$\tanh ix = i\tan x$; $\tan x = -i \tanh ix$

* Hyperbolic Identities

① $\sinh(-x) = -\sinh x$

② $\cosh x + \sinh x = \left(\frac{e^{x}+e^{-x}}{2}\right) + \left(\frac{e^{x}-e^{-x}}{2}\right)$

$$= \frac{e^{x}+e^{-x}+e^{x}-e^{-x}}{2} \Rightarrow e^{x}$$

$\therefore$ $\boxed{e^{x} = \cosh x + \sinh x}$

$\boxed{e^{-x} = \cosh x - \sinh x}$

③ $\boxed{\cosh^2 x - \sinh^2 x = 1}$

④ $\text{sech}^2 x + \tanh^2 x = 1$

⑤ $\coth^2 x - \text{cosech}^2 x = 1$

⑤ $\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y$

⑥ $\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y$

⑦ $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \mp \tanh x \tanh y}$

⑧ $\sinh 2x = 2\sinh x \cosh x$

⑨ $\cosh 2x = \cosh^2 x + \sinh^2 x$

$= 2\cosh^2 x - 1$

$= 1 + 2\sinh^2 x$

⑩ $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

Double Angle Formula

Q. If $\tanh x = \frac{1}{2}$; find the value of x & $\sinh 2x$.

Sol→ $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{2}$

$\therefore \dfrac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \dfrac{1}{2}$

$\therefore \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{1}{2}$

$\therefore 2(e^{2x} - 1) = 1(e^{2x} + 1)$

$\therefore 2e^{2x} - 2 = e^{2x} + 1$

$\therefore 2e^{2x} - e^{2x} = 3$

$\therefore e^{2x} = 3$

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Taking log on both sides;

$$2x \log e = \log 3$$

$$2x = \log 3$$

$$\boxed{x = \frac{1}{2} \log 3}$$

Also; $\sinh 2x = \dfrac{e^{2x} - e^{-2x}}{2}$

$$= \frac{e^{2x} - \frac{1}{e^{2x}}}{2}$$

$$= \frac{3 - \frac{1}{3}}{2} \Rightarrow \frac{\frac{9-1}{3}}{2} \Rightarrow \frac{8}{6} = \frac{4}{3}$$

$$\boxed{\sinh 2x = \frac{4}{3}}$$

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Hyperbolic functions

(1) $\sinh 2x = 2\sinh x \cosh x$

(2) $\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cosh^2 x - 1$
$= 1 + 2\sinh^2 x$

(3) $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

(4) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(5) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(6) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

* factorization & Defactorization Formulae

(1) $\sinh x + \sinh y = 2\sinh\left(\dfrac{x+y}{2}\right)\cosh\left(\dfrac{x-y}{2}\right)$

(2) $\sinh x - \sinh y = 2\cosh\left(\dfrac{x+y}{2}\right)\sinh\left(\dfrac{x-y}{2}\right)$

(3) $\cosh x + \cosh y = 2\cosh\left(\dfrac{x+y}{2}\right)\cosh\left(\dfrac{x-y}{2}\right)$

(4) $\cosh x - \cosh y = 2\sinh\left(\dfrac{x+y}{2}\right)\sinh\left(\dfrac{x-y}{2}\right)$

(5) $\sinh(x+y) + \sinh(x-y) = 2\sinh x \cosh y$

(6) $\sinh(x+y) - \sinh(x-y) = 2\cosh x \sinh y$

(7) $\cosh(x+y) + \cosh(x-y) = 2\cosh x \cosh y$

(8) $\cosh(x+y) - \cosh(x-y) = 2\sinh x \sinh y$

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• Derivative & Integration of Hyperbolic Functions

① $\frac{d(\sinh x)}{dx} = \cosh x$

② $\frac{d(\cosh x)}{dx} = \sinh x$

③ $\frac{d(\tanh x)}{dx} = \operatorname{sech}^2 x$

④ $\int \sinh x \, dx = \cosh x$

⑤ $\int \cosh x \, dx = \sinh x$

⑥ $\int \operatorname{sech}^2 x \, dx = \tanh x$

Q.1 → Find the value $\tanh(\log x)$ if $x = \sqrt{3}$.

Sol → As, $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\therefore \tanh(\log x) = \frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}$

$= \frac{e^{\log x} - e^{\log x^{-1}}}{e^{\log x} + e^{\log x^{-1}}}$

$\Rightarrow \frac{x - x^{-1}}{x + x^{-1}} \Rightarrow \frac{x - \frac{1}{x}}{x + \frac{1}{x}}$

$\Rightarrow \frac{x^2 - 1}{x^2 + 1}$

Put $x = \sqrt{3}$

$\therefore \tanh(\log x) = \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$

$$\boxed{\tanh(\log x) = \frac{1}{2}}$$

Q.2 → If $5\sinh x - \cosh x = 5$, find $\tanh x$.

Sol → $5\left(\frac{e^x - e^{-x}}{2}\right) - \left(\frac{e^x + e^{-x}}{2}\right) = 5$

$\therefore \frac{5e^x - 5e^{-x} - e^x - e^{-x}}{2} = 5$

$\therefore 4e^x - 6e^{-x} = 10$

$\therefore 2e^x - 3e^{-x} = 5$

$\therefore 2e^x - \frac{3}{e^x} - 5 = 0$

$\therefore 2e^{2x} - 3 - 5e^x = 0$

$\therefore 2e^{2x} - 5e^x - 3 = 0$

$\therefore 2e^{2x} - 6e^x + 1e^x - 3 = 0$

$\therefore 2e^x(e^x - 3) + 1(e^x - 3) = 0$

$(2e^x + 1)(e^x - 3) = 0$

$\boxed{e^x = -\frac{1}{2}}$ or $\boxed{e^x = 3}$

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$\therefore \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

At $e^x = -\frac{1}{2}$

$\tanh x = \dfrac{(-\frac{1}{2}) - (\frac{1}{-1/2})}{(-\frac{1}{2}) + (\frac{1}{-1/2})}$

$= \dfrac{-\frac{1}{2}+2}{-\frac{1}{2}-2} = \dfrac{3/2}{-5/2} = -\dfrac{3}{5}$ Ans

At $e^x = 3$

$\tanh x = \frac{4}{5}$ Ans

Q. Prove that : $16\cosh^5 x = \cosh 5x + 5\cosh 3x + 10\cosh x$

Hint

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Sol→ L.H.S. $= 16\cosh^5 x$

$= 16\left(\dfrac{e^x + e^{-x}}{2}\right)^5$

$= \dfrac{16}{2^5}(e^x + e^{-x})^5$

$= \dfrac{16}{32}\left(e^{5x} + 5e^{4x}e^{-x} + 10e^{3x}e^{-2x} + 10e^{2x}e^{-3x} + 5e^{x}e^{-4x} + e^{-5x}\right)$

$= \frac{1}{2}\left(e^{5x} + 5e^{3x} + 10e^{x} + 10e^{-x} + 5e^{-3x} + e^{-5x}\right)$

$= \left(\dfrac{e^{5x}+e^{-5x}}{2}\right) + 5\left(\dfrac{e^{3x}+e^{-3x}}{2}\right) + 10\left(\dfrac{e^{x}+e^{-x}}{2}\right)$

$$\left(\left(\frac{e^{nx}+e^{-nx}}{2}\right) = \cosh nx\right)$$

$$= \left(\cosh 5x + 5\cosh 3x + 10\cosh x\right) = \underline{RHS}$$

Hence, Proved.

Q. Express $\sinh^7 x$ in terms of hyperbolic sines of multiples of x.

Sol→ $$\sinh^7 x = \left(\frac{e^x - e^{-x}}{2}\right)^7$$

$$= \frac{1}{2^7}\left(e^{7x} - 7e^{6x}e^{-x} + 21e^{5x}e^{-2x} - 35e^{4x}e^{-3x} + 35e^{3x}e^{-4x} - 21e^{2x}e^{-5x} + 7e^{x}e^{-6x} - e^{-7x}\right)$$

$$= \frac{1}{2^7}\left(e^{7x} - 7e^{5x} + 21e^{3x} - 35e^{x} + 35e^{-x} - 21e^{-3x} + 7e^{-5x} - e^{-7x}\right)$$

$$= \frac{1}{2^7}\left[(e^{7x} - e^{-7x}) - 7(e^{5x} - e^{-5x}) + 21(e^{3x} - e^{-3x}) - 35(e^{x} - e^{-x})\right]$$

$$= \frac{1}{2^6}\left[\left(\frac{e^{7x} - e^{-7x}}{2}\right) - 7\left(\frac{e^{5x} - e^{-5x}}{2}\right) + 21\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 35\left(\frac{e^{x} - e^{-x}}{2}\right)\right]$$

$$\left(\frac{e^{nx} + e^{-nx}}{2} = \cosh nx \;;\; \frac{e^{nx} - e^{-nx}}{2} = \sinh nx\right)$$

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$= \frac{1}{2^6}\left(\sinh 7x - 7\sinh 5x + 21\sinh 3x - 35\sinh x\right)$

$=$ R.H.S.

Hence, Proved.

Q. Prove that: $\cosh^2 x = \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1-\cosh^2 x}}}$

Sol: R.H.S. $= \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1-\cosh^2 x}}}$

(As $\cosh^2 x - \sinh^2 x = 1$)

$= \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{(\sinh^2 x)}}}$

$= \cfrac{1}{1 - \cfrac{1}{1+\operatorname{cosech}^2 x}}$

$= \cfrac{1}{1 - \cfrac{1}{\coth^2 x}}$ $\left(\coth^2 x = 1 + \operatorname{cosech}^2 x\right)$

$= \frac{1}{1 - \tanh^2 x}$

$$= \frac{1}{1 - \frac{\sinh^2 x}{\cosh^2 x}}$$

$$= \frac{1}{\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}} \Rightarrow \frac{\cosh^2 x}{\cosh^2 x - \sinh^2 x}$$

$$\Rightarrow \frac{\cosh^2 x}{1} \quad (\text{As } \cosh^2 x - \sinh^2 x = 1)$$

$\Rightarrow \cosh^2 x =$ L.H.S.

~~Hence~~, Proved

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Q. If $\sin\alpha \cdot \cosh\beta = \frac{x}{2}$, $\cos\alpha \sinh\beta = \frac{y}{2}$. Show that

$$\operatorname{cosec}(\alpha - i\beta) + \operatorname{cosec}(\alpha + i\beta) = \frac{4x}{x^2+y^2}.$$

Sol→ $\operatorname{cosec}(\alpha + i\beta) = \frac{1}{\sin(\alpha + i\beta)}$

$$= \frac{1}{(\sin\alpha \cos i\beta + \cos\alpha \sin i\beta)}$$

$$= \frac{1}{(\sin\alpha \cosh\beta + i\cos\alpha \sinh\beta)}$$

$$= \frac{1}{\frac{x}{2} + \frac{iy}{2}}$$

$$= \frac{1}{\frac{x+iy}{2}} = \frac{2}{x+iy}$$

Similarly; $\operatorname{cosec}(\alpha - i\beta) = \frac{2}{x-iy}$

$$\therefore \operatorname{cosec}(\alpha + i\beta) + \operatorname{cosec}(\alpha - i\beta) = \frac{2}{x-iy} + \frac{2}{x+iy}$$

$$= \frac{2(x+iy) + 2(x-iy)}{(x-iy)(x+iy)}$$

$$= \frac{2x + 2iy + 2x - 2iy}{x^2 - i^2y^2}$$

$$= \frac{4x}{x^2+y^2} = R.H.S$$

= Hence, Proved.

* Separation of R.P. & I.P.

① $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$$\boxed{\sin(x+iy) = \sin x \cdot \cosh y + i\cos x \sinh y}$$

② $\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$

$$\boxed{\cos(x+iy) = \cos x \cosh y - i\sin x \sinh y}$$

Q.1→ If $\sin(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$. Prove that;

$$r^2 = \frac{1}{2}(\cosh 2\phi - \cos 2\theta) \text{ and } \tan\alpha = \tanh\phi \cot\theta$$

Sol→ $\sin(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$

$\therefore \sin\theta \cos i\phi + \cos\theta \sin i\phi = r(\cos\alpha + i\sin\alpha)$

$\therefore \sin\theta \cosh\phi + i\cos\theta \sinh\phi = r\cos\alpha + ir\sin\alpha$

Separating RP & IP

$\therefore \sin\theta \cosh\phi = r\cos\alpha$ — ①

& $\cos\theta \sinh\phi = r\sin\alpha$ — ②

$\therefore$ Squaring & Adding

$r^2\cos^2\alpha + r^2\sin^2\alpha = \sin^2\theta \cosh^2\phi + \cos^2\theta \sinh^2\phi$

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$$r^2(\cos^2\alpha + \sin^2\alpha) = (1-\cos^2\theta)\cosh^2\phi + \cos^2\theta(\cosh^2\phi - 1)$$

$$r^2 = \cosh^2\phi - \cos^2\theta\cosh^2\phi + \cos^2\theta\cosh^2\phi - \cos^2\theta$$

$$r^2 = \cosh^2\phi - \cos^2\theta$$

$$r^2 = \left(\frac{1+\cosh 2\phi}{2}\right) - \left(\frac{1+\cos 2\theta}{2}\right)$$

$$r^2 = \frac{1 + \cosh 2\phi - 1 - \cos 2\theta}{2}$$

$$\boxed{r^2 = \frac{1}{2}(\cosh 2\phi - \cos 2\theta)}$$

Hence, Proved

∴ ~~Equating~~ i.e. Divide eqn (2) by (1)

$$\frac{r\sin\alpha}{r\cos\alpha} = \frac{\cos\theta\sinh\phi}{\sin\theta\cosh\phi}$$

$$\boxed{\tan\alpha = \cot\theta\tanh\phi}$$

Hence, Proved

Q. If $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$. Prove that $\cos 2\theta\cosh 2\phi = 3$.

Soln $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$

$\therefore \sin\theta\cos i\phi + \cos\theta\sin i\phi = \tan\alpha + i\sec\alpha$

$\therefore \sin\theta\cosh\phi + i\cos\theta\sinh\phi = \tan\alpha + i\sec\alpha$

Equating RP & IP

$$\tan\alpha = \sin\theta\cosh\phi \quad -(1)$$
$$\sec\alpha = \cos\theta\sinh\phi \quad -(2)$$

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$(\because \sec^2\theta = 1+\tan^2\theta)$

$\therefore$ Squaring & Subtracting

$\therefore \sec^2\alpha - \tan^2\alpha = \cos^2\theta \sinh^2\phi - \sin^2\theta \cosh^2\phi$

$\therefore 1 = \cos^2\theta \sinh^2\phi - \sin^2\theta \cosh^2\phi$

$\therefore \left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\cosh 2\phi - 1}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cosh 2\phi}{2}\right) = 1$

$\frac{(1+\cos 2\theta)(\cosh 2\phi - 1)}{4} - \frac{(1-\cos 2\theta)(1+\cosh 2\phi)}{4} = 1$

$\Rightarrow [(1+\cos 2\theta)(\cosh 2\phi - 1)] - [(1-\cos 2\theta)(1+\cosh 2\phi)] = 4$

$\Rightarrow (\cosh 2\phi - 1 + \cos 2\theta \cosh 2\phi - \cos 2\theta) - (1 + \cosh 2\phi - \cos 2\theta - \cos 2\theta \cosh 2\phi) = 4$

$\Rightarrow \cosh 2\phi - 1 + \cos 2\theta \cosh 2\phi - \cos 2\theta - 1 - \cosh 2\phi + \cos 2\theta + \cos 2\theta \cdot \cosh 2\phi = 4$

$\Rightarrow -2 + 2\cos 2\theta \cosh 2\phi = 4$

$\Rightarrow 2\cos 2\theta \cosh 2\phi = 4+2$

$\Rightarrow \cos 2\theta \cosh 2\phi = \frac{6}{2}$

$\Rightarrow \boxed{\cos 2\theta \cosh 2\phi = 3}$

Hence, Proved

Q. If $2\cosh\left(\alpha + i\frac{\pi}{4}\right) = x + iy$. Prove that $x^2 - y^2 = 2$.

Sol → $2\cosh\left(\alpha + i\frac{\pi}{4}\right) = x + iy$

$(\cosh\theta = \cos i\theta)$

$\therefore 2\cos i\left(\alpha + i\frac{\pi}{4}\right) = x + iy$

$\therefore 2\cos\left(i\alpha - \frac{\pi}{4}\right) = x + iy$

$\therefore 2\left(\cos i\alpha \cos\frac{\pi}{4} + \sin i\alpha \sin\frac{\pi}{4}\right) = x + iy$

$\therefore 2\left((\cosh\alpha)\frac{1}{\sqrt{2}} + i(\sinh\alpha)\frac{1}{\sqrt{2}}\right) = x + iy$

$\therefore \frac{2}{\sqrt{2}}(\cosh\alpha + i\sinh\alpha) = x + iy$

$\Rightarrow \sqrt{2}(\cosh\alpha + i\sinh\alpha) = x + iy$

⇒ Equating RP & IP

$x = \sqrt{2}\cosh\alpha$ & $y = \sqrt{2}\sinh\alpha$

(Squaring & subtracting)

$\Rightarrow x^2 - y^2 = 2(\cosh^2\alpha - \sinh^2\alpha)$

$\Rightarrow \boxed{x^2 - y^2 = 2}$ (As; $\cosh^2\alpha - \sinh^2\alpha = 1$)

Hence, proved

Powers of Complex No.

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$z^c = ?$

Let $z^c = \omega$

$\log z^c = \log \omega$

$\log w = c \log z$

$\boxed{w = e^{c \log z}}$

$\log z = \log|z| + i \arg(z)$

$z = 0 + i$

$|z| = \sqrt{0^2 + 1^2}$

$= \sqrt{1} = 1$

$\theta = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{1}{0}\right)$

$= \pi/2$

Principal $\arg(z) = \frac{\pi}{2} + 2k\pi$

$\mathrm{Arg}(z) = \pi/2$

i^i

So; $\boxed{w = e^{i \log(i)}}$

$w = e^{i(\log(i) + i\arg(i))}$

$w = e^{i(\log |i| + i(\pi/2 + 2k\pi))}$

$i^i = \omega = e^{i(i(\pi/2 + 2k\pi))}$

$w = e^{-(\pi/2 + 2k\pi)}$

$\mathrm{Ln}(1+i)$

$= \log(1+i) + i\,\mathrm{Arg}(1+i)$

$= \log\sqrt{2} + i\frac{\pi}{4}$

$(1+i)^{2-i}$

$(1+i)^{2-i} = e^{(2-i)\mathrm{Ln}(1+i)}$

$= e^{(2-i)(\log\sqrt{2} + \pi/4\, i)}$

$= e^{(2\log\sqrt{2} + \pi/2\, i - i\log\sqrt{2} + \pi/4)}$

$= e^{2\log\sqrt{2} + \pi/4} \cdot e^{i(\pi/2 - \log\sqrt{2})}$

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Q. Principal value of $(-1)^{\left(-\frac{2i}{\pi}\right)}$.

Sol → $z = (-1)^{(-2i/\pi)}$

$\log z = \log (-1)^{\left(-\frac{2i}{\pi}\right)}$

$= \left(-\frac{2i}{\pi}\right) \log(-1)$

$= \left(-\frac{2i}{\pi}\right)(\log 1 + i\pi)$

$= -\frac{2i}{\pi}(\pi i)$

$\Rightarrow -2i^2 \Rightarrow 2$

$\boxed{\log z = 2}$

$\boxed{z = e^2}$ Ans

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Zeroes & Singularity of an Analytic function

Zeroes

The values of z for which the analytic function f(z) becomes zero; is called zero of f(z).

Note: ① For
$$f(z) = \sum_{n=0}^{m} a_n (z-z_0)^n$$

If $a_0 = a_1 = a_2 = \cdots = a_{m-1} = 0$ & $a_n \neq 0$. Then

$$f(z) = a_m (z - z_0)^m$$

Put f(z) = 0 for zeroes; we get

$$a_m (z - z_0)^m = 0$$
$$(z-z_0)^m = 0 \quad (\because a_m \neq 0)$$

Then $z = z_0$ is called zero of f(z) of order m.

② To obtain zero of f(z) put no. of f(z) is equal to zero.
③ The zero of order 1 is called simple zero.

Eg → 1 → Let $f(z) = \frac{(z-1)(z-2)^4}{z}$

for zero of f(z) put f(z) = 0

$$\Rightarrow \frac{(z-1)(z-2)^4}{z} = 0$$
$$\Rightarrow (z-1)(z-2)^4 = 0$$
$$\Rightarrow \boxed{z = 1} \quad \text{or} \quad (z-2)^4 = 0$$
$$\boxed{z = 2}$$

Hence, z = 1 is a zero of order 1 or simple zero.
And z = 2 is a zero of order 4.

Singularity of an analytic function

[Singular Point]
A point at which a function $f(z)$ is not analytic is known as singular point or singularity of the function.

Eg → If $f(z) = \frac{1}{(z-1)(z-2)}$ then $f(z) \to \infty$ at $z=1$; $z=2$

then $f(z)$ is not analytic at $z=1$ & $z=2$.
→ $z=1$ & $z=2$ are called singular point.

[Isolated & non-isolated singularity]
If $z=a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding at the point $z=a$, then $z=a$ is said to be an isolated singularity of the function $f(z)$ otherwise it is called non-isolated.

Eg → 1 → Let $f(z) = \frac{z+1}{z(z-2)}$ $f(z) \to \infty$ at $z=0$, $z=2$.

→ $z=0$ & $z=2$ are the singularity of $f(z)$.
There is no other singularities of $f(z)$ in the ~~of $z=0$, $z=2$.~~ neighbourhood of $z=0$, $z=2$.

Hence, $z=0$, $z=2$ are isolated singularity of $f(z)$.

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Eg-2 →

Let $f(z) = \cot\frac{\pi}{z} = \frac{1}{\tan\frac{\pi}{z}}$

Clearly $f(z)$ is not analytic when $\tan\frac{\pi}{z} = 0$

$\Rightarrow \tan\frac{\pi}{z} = \tan n\pi$

$\Rightarrow \frac{\pi}{z} = n\pi$

$\Rightarrow z = \frac{1}{n} \quad (n = 1, 2, 3, \ldots)$

Then $z = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ & $z = 0$ (at ∞)

$\therefore z = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ are isolated singularities of $f(z)$

& $z = 0$ is non-isolated becoz in the small neighbourhood of $z = 0$ there are infinite no. of other singularities $z = 1/n$ where n is large.

(Types of Singularity)

Let $f(z)$ be analytic within a domain D except at $z = z_0$ which is an isolated singularity. Then expansion of $f(z)$ by Laurent's series.

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic Part of } f(z)} + \underbrace{\sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}}_{\text{Principal Part of } f(z)}$$

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① Removable Singularity

If no terms are in Principal Part all $b_n = 0$.

Eg→ $f(t) = \frac{\sin t}{t} = \frac{1}{t}\left(t - \frac{t^3}{\lfloor 3} + \frac{t^5}{\lfloor 5} ----\right)$

$= 1 - \frac{t^2}{\lfloor 3} + \frac{t^4}{\lfloor 5} ......$

$\left.\begin{aligned} f(t) &= \frac{\sin t}{t} \quad (t \neq 0) \\ &= 1 \quad (t = 0) \end{aligned}\right\}$ $\lim_{t \to t_0} f(t)$ = finite exist

② Essential Singularity

Infinite terms in Principal Part.

$f(t) = e^{1/t} = 1 + \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + ----$

$t = 0$ ✓ Isolated Essential Singularity

③ Pole

Finite no. of terms in Principal Part. ($t = t_0$)

$f(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n + \frac{b_1}{(t - t_0)} + \frac{b_2}{(t - t_0)^2} + - - + \frac{b_m}{(t - t_0)^m}$

at $t = t_0$ ⇒ Pole of order m.

(Simple Pole)

Q.1 [Tell singularity.]

Q.1 → $f(t) = \sin\left(\frac{1}{t}\right)$

$t = 0$.

Sol → $\sin\left(\frac{1}{t}\right) = \frac{1}{t} - \frac{1}{\lfloor 3}\left(\frac{1}{t^3}\right) + \frac{1}{\lfloor 5}\left(\frac{1}{t^5}\right) - \cdots$

at $t = 0$ → Isolated essential singularity.

Q. 2 → $f(t) = \frac{e^{1/t}}{t^2}$ at $t = 0$

Sol → $= \frac{1}{t^2}\left(1 + \frac{1}{t} + \frac{1}{\lfloor 2\, t^2} + \frac{1}{\lfloor 3\, t^3} + \cdots\right)$

$= \frac{1}{t^2} + \frac{1}{t^3} + \frac{1}{\lfloor 2\, t^4} + \cdots$

→ Isolated Essential Singularity.

Q.3 → $f(t) = \cot t$

at $t = \infty$

Sol → $\cot t = \frac{\cos t}{\sin t}$

$\sin t = 0 = \sin n\pi$

$t = n\pi$

$n = 1, 2, 3, 4, \cdots$

So, $t = \infty$ → non Isolated Essential Singularity.

Q 4 → $f(t) = \operatorname{cosec}\left(\frac{1}{t}\right)$ at $t = 0$

Sol → $f(t) = \operatorname{cosec}\left(\frac{1}{t}\right) = \frac{1}{\sin(1/t)}$

$\sin\left(\frac{1}{t}\right) = 0 = \sin n\pi$

$\frac{1}{t} = n\pi$

$t = \frac{1}{n\pi}$

$t = 0$ (non Isolated Essential Singularity)

(5) $f(t) = \frac{1-e^{t}}{1+e^{t}}$ $(\cancel{e^t = -1})$

Sol →

$$\frac{1}{1+e^{t}}\left(1-1-t-\frac{t^2}{\lfloor 2} - \dots\right)$$

$$t = \infty$$

(essential non-isolated singularity)

Q. Let $f(z) = \frac{\sin(z-a)}{z-a}$.

Sol → $\Rightarrow \frac{1}{z-a}\left((z-a) - \frac{(z-a)^3}{\lfloor 3} + \frac{(z-a)^5}{\lfloor 5} - \dots\right)$

$$\Rightarrow \left(1 - \frac{(z-a)^2}{\lfloor 3} + \frac{(z-a)^4}{\lfloor 5} - \dots\right)$$

Here all b_n's are zero ; i.e. ; no terms in Principal Part

Hence ; $z = a$ is Removable singularity of $f(z)$.

Q. Let $f(z) = \sin \frac{1}{z-a}$

Sol → $\frac{1}{z-a} - \frac{1}{\lfloor 3}\left(\frac{1}{z-a}\right)^3 + \frac{1}{\lfloor 5}\left(\frac{1}{z-a}\right)^5 - \dots$

$\Rightarrow f(z)$ has infinite no. of terms in Principal part.

Hence, $z = a$ is isolated essential singularity.

Q. $f(z) = \cos \frac{1}{z}$

Sol→ $1 - \frac{1}{\lfloor 2 \, z^2} + \frac{1}{\lfloor 4 \, z^4} - \ldots$

$\Rightarrow 1 - \frac{1}{\lfloor 2 (z-0)^2} + \frac{1}{\lfloor 4 (z-0)^4} - \ldots$

at $z=0$

(Isolated Essential Singularity)

Q: Let $f(z) = \frac{\sin(z-a)}{(z-a)^4}$

Sol→ $f(z) = \frac{1}{(z-a)^4}\left((z-a) - \frac{(z-a)^3}{\lfloor 3} + \frac{(z-a)^5}{\lfloor 5} - \ldots\right)$

$\Rightarrow \frac{1}{(z-a)^3} - \frac{1}{\lfloor 3 (z-a)} + \left(\frac{1}{\lfloor 5}(z-a) - \frac{1}{\lfloor 7}(z-a)^3 + \ldots\right)$

$\therefore$ $f(z)$ has finite no. of terms (only two terms) in Principal Part

Then $z = a$ is a pole of $f(z)$.

Q: Discuss the nature of singularity of.

$f(z) = \frac{z - \sin z}{z^3}$ at $z = 0$

Sol→ $\frac{1}{z^3}\left(z - \left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} - \frac{z^7}{\lfloor 7} + \ldots\right)\right)$

$\Rightarrow \frac{1}{\lfloor 3} - \frac{z^2}{\lfloor 5} + \frac{z^4}{\lfloor 7} - \ldots$

$\therefore$ $z = 0$ is Removable Singularity

Date..............

Order of Pole

If the principal part of $f(z)$ contains m number of terms at $z = z_0$. Then $z = z_0$ is called a pole of order m.

If $m = 1$ then pole is called simple Pole.

Note: To find pole put denominator of $f(z)$ is equal to 0.

Eg→ $f(z) = \dfrac{e^z}{(z-1)(z-3)^4}$

Sol→ for pole ; put $(z-1)(z-3)^4 = 0$

$z = 1,\ z = 3, 3, 3, 3$

Hence, $z = 1$ is pole of order One or simple pole

& $z = 3$ is a pole of order 4.

Zeroes	Vs	Poles
① Zeroes are infinite		① Poles are infinite.
② Limit point of zeroes ⇓ Isolated essential singularity		② Limit Point of Poles. ⇓ Non-isolated essential singularity.

Date........

Q. Find out zero & discuss nature of singularity

$$f(z) = \frac{z-2}{z^2} \sin\frac{1}{z-1}$$

Sol. $f(z) = \frac{(z-2)\sin(1/z-1)}{z^2}$

Zeroes of f(z) are given by (Numerator of f(z)) = 0

$\Rightarrow (z-2)\sin\left(\frac{1}{z-1}\right) = 0$

$z = 2 \;;\; \sin\frac{1}{z-1} = \sin n\pi$

$z = 2 \;;\; \frac{1}{z-1} = n\pi$

$z = 2 \;;\; z = 1 + \frac{1}{n\pi}$

$(n = 0, \pm 1, \pm 2, \ldots)$

$\therefore z = 2 \;;\; z = 1 + \frac{1}{n\pi} \;;\; n \in I$ are zeroes of order one.

Since z = 1 is a limit point of zeroes.

$z = 1 + \frac{1}{n\pi}$

z = 1 is isolated essential singularity.

Now poles of f(z) are

$z^2 = 0 \Rightarrow z = 0, 0$

Hence, z = 0 is a pole of order = 2.

Q. Discuss singularity of $\frac{\cot \pi z}{(z-a)^2}$ at $z=a$ & $z=\infty$

Sol → Let $f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{(\sin \pi z)(z-a)^2}$

for poles; $\sin \pi z \, (z-a)^2 = 0$

$\Rightarrow (z-a)^2 = 0$, $\sin \pi z = \sin n\pi$

$\Rightarrow z = a, a$; $z = n$ ($n \in I$)

Hence; $z=a$ is pole of order 2.

Clearly ; $z=\infty$ is limit point of poles $z=n$.

Hence, $z=\infty$ is a non - isolated essential singularity

Q Singularity of $\frac{1}{1-e^z}$ at $z = 2\pi i$.

Sol → Let $f(z) = \frac{1}{1-e^z}$ (for pole $1 - e^z = 0$

$e^z = 1 = e^{2n\pi i}$

$(e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi)$

$z = 2n\pi i$ ($n \in I$)

Since, $z=\infty$ is a limit point of pole $z = 2n\pi i$

$z = \infty$ is non isolated essential singularity

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Infinity

Let $f(z)$ be a analytic function in D except $a_1, a_2, \ldots a_n$ then

$$\underset{z=\infty}{\text{Res}}\, f(z) + \underset{z=a_1}{\text{Res}}\, f(z) + \ldots \underset{z=a_m}{\text{Res}}\, f(z) = 0$$

$$\Rightarrow \underset{z=\infty}{\text{Res}}\, f(z) = -\left[\underset{z=a_1}{\text{Res}}\, f(z) + \ldots\ldots + \underset{z=a_n}{\text{Res}}\, f(z)\right]$$

$$= -\left[\sum_{i=1}^{n} \underset{z=a_i}{\text{Res}}\, f(z)\right]$$

or

$$\boxed{\underset{z=\infty}{\text{Res}}\, f(z) = \underset{z=0}{\text{Res}}\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right)\right)}$$

Q Evaluate the residue of $f(z) = \dfrac{z^2}{(z-1)(z-2)}$ at $z = \infty$.

Soln → Given that $f(z) = \dfrac{z^2}{(z-1)(z-2)}$

$f(z)$ have two poles at $z = 1$ & $z = 2$.

$$\underset{z=1}{\text{Res}}\, f(z) = \lim_{z\to 1} \frac{(z-1)\, z^2}{(z-1)(z-2)} = -1$$

$$\underset{z=2}{\text{Res}}\, f(z) = \lim_{z\to 2} \frac{(z-2)\, z^2}{(z-1)(z-2)} = 4$$

Then

$$\operatorname*{Res}_{z=\infty} f(z) = -\left[\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z)\right]$$

$$= -(-1+4) = -3$$

If $z=\infty$ is isolated singularity of $f(z)$ then $\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$

Q. find Residue of $f(z) = z\cos\left(\frac{1}{z}\right)$ at $z=\infty$.

Sol → $f(z) = z\cos\left(\frac{1}{z}\right)$

$f\left(\frac{1}{z}\right) = \frac{1}{z}\cos z$

$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^3}\cos z$

$\operatorname*{Res}_{z=0} \frac{1}{z^3}\cos z$

$\Rightarrow z\left(1 - \frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} - \dots\right)$

$\Rightarrow z - \frac{1}{2z} + \frac{1}{24}\left(\frac{1}{z^3}\right) - \dots$

$\rightarrow \operatorname*{Res}_{z=0} f(z) = \text{coff } \frac{1}{z} = -\frac{1}{2}$

$\Rightarrow \operatorname*{Res}_{z=\infty} f(z) = -R = +\frac{1}{2}$ Ans

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Q. find Residue of $f(z) = e^{1/z}$ at $z = \infty$.

Sol → $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots$

$\text{Res}_{z=0}\ e^{1/z} = \text{coff } \frac{1}{z} = 1$

$\text{Res}_{z=\infty} f(z) = -\text{Res}_{z=0} f(z) = (-1)$ Ans

Q. $f(z) = z^2 \sin(1/z)$ at $z = \infty$ is.

Sol → $z^2 \left(\frac{1}{z} - \frac{1}{3!z^3} + \ldots \right)$

$z - \frac{1}{6}\left(\frac{1}{z}\right) + \ldots$

$\text{Res}_{z=0} f(z) = -1/6$

$\text{Res}_{z=\infty} f(z) = -(-1/6) \Rightarrow (1/6)$ Ans

Q. Evaluate Residue of $f(z) = \frac{z^3}{z^2 - 1}$ at $z = \infty$.

Sol →

$R_1 = \lim_{z \to 1} (z-1) \frac{z^3}{(z+1)(z-1)} = \frac{1}{2}$

$R_2 = \lim_{z \to -1} \frac{(z+1)\ z^3}{(z+1)(z-1)} = 1/2$

$\text{Res}_{z=\infty} f(z) = -(R_1 + R_2) = -\left(\frac{1}{2} + \frac{1}{2}\right) = (-1)$ Ans

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Complex Integration

Line Integral

$\int_C f(z)\,dz \Rightarrow$ Line Integral

$\oint_C f(z)\,dz \Rightarrow$ Contour Integral (closed)

(contour)

$$\int_C f(z)\,dz$$

$f(z) = u + iv$

$dz = du + idv$

$$\Rightarrow \int_C f(z)\,dz = \int_C (u+iv)\,(du + idv)$$

$$\Rightarrow \int_C (udu + iudv + ivdu + i^2 vdv)$$

$$\Rightarrow \int_C (udu - vdv) + i\,(udv + vdu)$$

$$\int_C f(z)\,dz = \int_C (u + iv)\,(dx + idy)$$

$z = x + iy$

$dz = dx + idy$

$$\int_C f(z)\,dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

Q.1 →

Evaluate $\int_c |z|\,dz$, where c is left half of unit circle $|z|=1$ from $z=-i$ to $z=i$.

Sol → $f(z) = |z| = \sqrt{x^2+y^2} = r$

$z = x+iy = re^{i\theta}$

$\therefore \sqrt{x^2+y^2} = r$

method →
① f(z)
② dz
③ Limits

As, $|z| = r = 1$

$\therefore z = e^{i\theta}$

$\therefore dz = e^{i\theta}\cdot i\,d\theta$

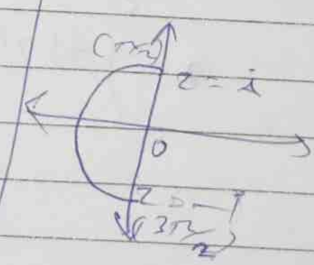
Hence, $\int_c f(z)\,dz = \int_c 1\cdot e^{i\theta}\cdot i\,d\theta$

Jb contour circle hota hai consider fn & derivative in exponential form of complex no.

$= i\int_c e^{i\theta}\,d\theta$

$= i\int_{\theta=3\pi/2}^{\pi/2} e^{i\theta}\,d\theta$

$= i\left[\frac{e^{i\theta}}{i}\right]_{3\pi/2}^{\pi/2}$



$= e^{i(\pi/2)} - e^{i(3\pi/2)}$

$\left(e^{i\theta} = \cos\theta + i\sin\theta\right)$

$= \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] - \left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right]$

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$= i(1) - i(-1) = 2i$ Ans

Q.2→ Evaluate : $\int_C \frac{2z+3}{z} dz$ where c is

① upper half of circle, $|z| = 2$.

② lower half of circle, $|z| = 2$

③ whole circle in anticlockwise direction.

Ans→ As, $z = x + iy = r e^{i\theta}$

$r = |z| = \sqrt{x^2 + y^2}$

$z = 2e^{i\theta}$

$dz = 2e^{i\theta} i d\theta = 2i e^{i\theta} d\theta$

As $f(z) = \frac{2z + 3}{z}$

$= \frac{2(2e^{i\theta}) + 3}{(2e^{i\theta})}$

$= \frac{4e^{i\theta}}{2e^{i\theta}} + \frac{3}{2e^{i\theta}}$

$f(z) = 2 + \frac{3}{2} e^{-i\theta}$

① Upper half of circle $|z| = 2$

$\int_C f(z) dz = \int_{\theta=0}^{\pi} \left(2 + \frac{3}{2} e^{-i\theta}\right) (2i e^{i\theta} d\theta)$

$$= 2i \int_0^{\pi} \left(2e^{i\theta} + \frac{3}{2} e^{-i\theta} e^{i\theta}\right) d\theta$$

$$= 2i \int_0^{\pi} \left(2e^{i\theta} + \frac{3}{2}\right) d\theta$$

$$= 2i \left[\frac{2e^{i\theta}}{i} + \frac{3}{2}\theta\right]_0^{\pi}$$

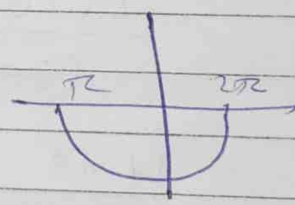
$$= 2i\left[\left(\frac{2}{i}e^{i\pi} + \frac{3}{2}\pi\right) - \left(\frac{2}{i}e^{0} + \frac{3}{2}(0)\right)\right]$$

$$= 2i\left[\left(\frac{2}{i}(\cos\pi + i\sin\pi) + \frac{3}{2}\pi\right) - \left(\frac{2}{i}\right)\right]$$

$$= 2i\left[-\frac{2}{i} + \frac{3}{2}\pi - \frac{2}{i}\right] = 2i\left[-\frac{4}{i} + \frac{3}{2}\pi\right]$$

② Lower half of circle $|z| = 2$

$$\int_c f(z)\,dz = \int_{\theta=\pi}^{2\pi} \left(2 + \frac{3}{2} e^{-i\theta}\right)\left(2ie^{i\theta}\right) d\theta$$



$$= 2i\int_{\pi}^{2\pi}\left(2e^{i\theta} + \frac{3}{2}\right) d\theta$$

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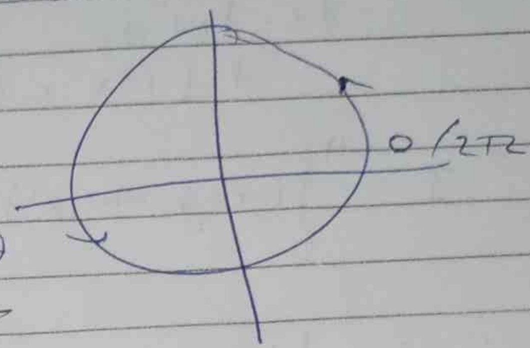
(3) whole circle in anticlockwise direction.

$\theta = 0$ to 2π



$$\int_C f(z)\,dz = \int_0^{2\pi} \left(2 + \frac{3}{2} e^{-i\theta}\right)(2ie^{i\theta})\,d\theta$$

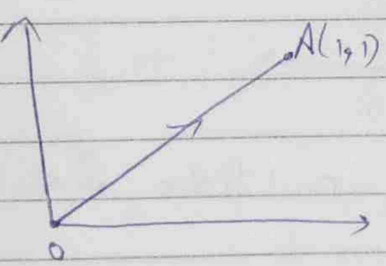
(further solve self

Q. Evaluate integral $\int_0^{1+i} (x - y + ix^2)\,dz$

i) along line from $z = 0$ to $z = 1+i$.

ii) along the real axis from $z = 0$ to $z = 1$ & then along line parallel to imaginary axis from $z = 1$ to $z = 1+i$.

Sol → (1)



$\because z = x + iy = 1+i$

Let OA be line from $z = 0$ to $1+i$

Equation of line ; $y = x$

$\boxed{dy = dx}$

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As; $z = x + iy$

$dz = dx + i\,dy$

$= dx + i\,dx = dx(1+i)$

$$I = \int_0^{1+i} (x - y + ix^2)\, dz$$

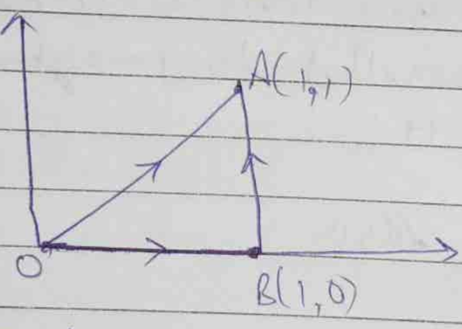
$$\therefore I = \int_0^1 (x - x + ix^2)(1+i)\, dx$$

$$= \int_0^1 ix^2 (1+i)\, dx$$

$$= \int_0^1 (i-1)x^2 dx = (i-1)\left[\frac{x^3}{3}\right]_0^1$$

$$= (i-1)\left(\frac{1}{3}\right) = \frac{i-1}{3} \text{ Ans}$$

(2)



Here, the first contour is OB and then BA.

$\therefore$ On OB ; $y = 0$ $\therefore dy = 0$

As, $z = x + iy$

$dz = dx + i\,dy$

$dz = dx$

$$\therefore \int (x - y + ix^2)\, dz = \int_0^1 (x + ix^2)\, dx$$

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$= \frac{1}{2} + i\frac{1}{3}$

On BA ; $x = 1$ $\therefore dx = 0$

$\therefore dz = dx + i dy$

$\therefore dz = i dy$

$\therefore \int (x - y + ix^2)\, dz = \int_0^1 (1 - y + i)\, i dy$

$= \int_0^1 [(-1 + i) - iy]\, dy$

$= \left[(-1+i)y - \frac{iy^2}{2} \right]_0^1$

$= -1 + \frac{i}{2}$

Total length of contour

$I = \frac{1}{2} + i\left(\frac{1}{3}\right) - 1 + i/_2$

$\boxed{I = -\frac{1}{2} + \frac{5}{6} i}$ Ans

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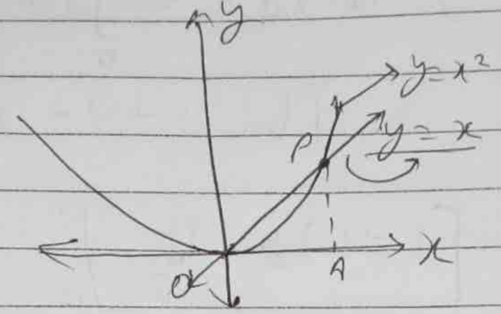
(1) Evaluate $\int_0^{1+i} (x^2 + iy)\,dz$; along path

(i) $y = x$
(ii) $y = x^2$

Is line integral independent of path?



Sol →

(1) Along path $y = x$

$\therefore dy = dx$

Also ; $z = x + iy$

$dz = dx + i\,dy$

$= dx + i\,dx = dx(1+i)$

$\therefore$ Substituting all terms

$$\int_0^{1+i} (x^2 + iy)\,dz = \int_0^1 (x^2 + ix)(1+i)\,dx$$

$$= (1+i)\left(\frac{x^3}{3} + i\frac{x^2}{2}\right)_0^1$$

$$= (1+i)\left(\frac{1}{3} + i\left(\frac{1}{2}\right)\right)$$

$$= (1+i)\left(\frac{2+3i}{6}\right) = \frac{-1+5i}{6}$$ Ans

Date..............

② Along path

$y = x^2$

$\therefore dy = 2x\,dx$

Also; $z = x + iy$

$dz = dx + i\,dy$

$= dx + i\,2x\,dx$

$= dx\,(1 + 2ix)$

$$\therefore \int_0^{1+i} (x^2 + iy)\,dz = \int_0^1 (x^2 + ix^2)(1 + 2ix)\,dx$$

$$= \int_0^1 x^2(1+i)(1+2ix)\,dx$$

$$= (1+i)\int_0^1 x^2(1+2ix)\,dx$$

$$= (1+i)\int_0^1 (x^2 + 2ix^3)\,dx$$

$$= (1+i)\left[\frac{x^3}{3} + 2i\frac{x^4}{4}\right]_0^1$$

$$= (1+i)\left(\frac{1}{3} + \frac{2i}{4}\right) = (1+i)\left(\frac{1}{3} + \frac{i}{2}\right) \text{ Ans}$$

$$= \frac{-1 + 5i}{6}$$

$\therefore$ Two line integrals are equal.

③ Consider integral along 3rd path i.e. from OA to AP.

$$\therefore \int_{OA} (x^2 + iy)\, dz = \int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

Now $\int_{AP} (x^2 + iy)\, dz$

$$= \int_0^1 (1+iy)\, i\, dy$$

$z = x+iy$
$dz = dx + i dy$

$$= i\left(y + \frac{iy^2}{2}\right)_0^1$$

$$= i\left(1 + \frac{i}{2}\right) = i - \frac{1}{2}$$

$$\therefore \int_0^{1+i} (x^2 + iy)\, dz = \frac{1}{3} + i - \frac{1}{2} = \frac{-1}{6} + i$$

∴ Third integral is not equal to first two.

So, they are not Independent.

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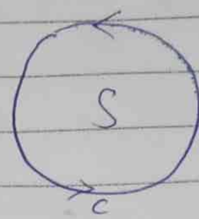
• Cauchy's Theorem

If $f(z)$ is an analytic function and $f'(z)$ is continuous within and on the closed curve C then $\int_C f(z)\,dz = 0$.



$f(z) = u + iv$; $z = x + iy$

$dz = dx + i\,dy$

$\therefore \int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$

$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \quad — (1)$

$= \int_S \left(-\frac{dv}{dx} - \frac{du}{dy} \right) dx\,dy + i \int_S \left(\frac{du}{dx} - \frac{dv}{dy} \right) dx\,dy$

$\left(\text{By Green's Theorem} \quad \int_C (M\,dx + N\,dy) = \int_S \left(\frac{dN}{dx} - \frac{dM}{dy} \right) dx\,dy \right)$

$\Rightarrow$ Since $f(z)$ is analytic $\Rightarrow$ C-R equations holds.

$\Rightarrow \frac{du}{dx} = \frac{dv}{dy}$, $\frac{du}{dy} = -\frac{dv}{dx}$

$f'(z) = \frac{du}{dx} + i\frac{dv}{dx}$

$f'(z) = \frac{du}{dy} + i\frac{dv}{dy}$

$\int_C f(z)\,dz = \int_S \left(-\frac{dv}{dx} + \frac{dv}{dx} \right) dx\,dy + i \int_S \left(\frac{du}{dx} - \frac{du}{dx} \right) dx\,dy$

$= 0$.

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Q. Evaluate $\int_C \frac{z^2}{z-1}\,dz$ where c is circle $|z| = 1/2$.

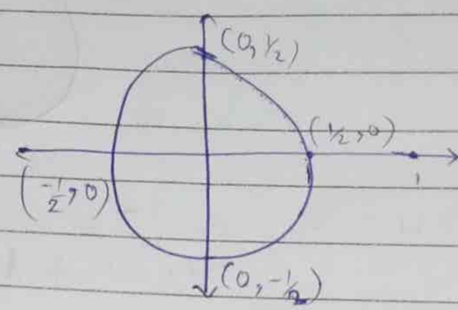
Sol → $|z| = 1/2$

$\Rightarrow |x + iy| = 1/2$

$\Rightarrow \sqrt{x^2+y^2} = (1/2)$

$\Rightarrow x^2+y^2 = (1/2)^2$

Centre $(0,0)$
Radius → $1/2$



Singular point is $z = 1$
which lies outside circle means circle is analytic function. So,

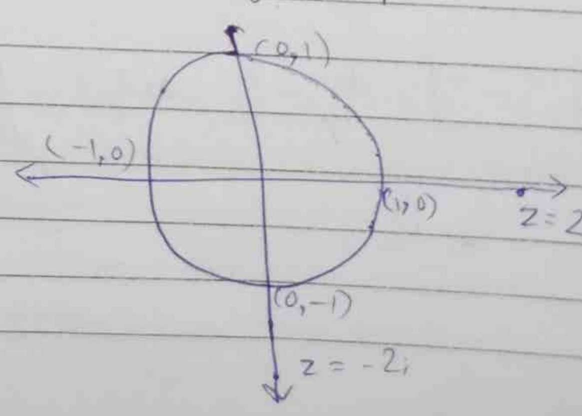
$$\int_C \frac{z^2}{z-1}\,dz = 0 \text{ Ans}$$

Q. Evaluate $\int_C \frac{e^z}{(z+2i)(z-2)}\,dz$ where c is a circle

$|z| = 1.$

Sol → $f(z) = \frac{e^z}{(z+2i)(z-2)} \Big\} z = -2i, 2$ → singular point of $f(z)$

$|z| = 1$
centre $(0,0)$
Radius $= 1$



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Since both singular point of $f(z)$ are outside the circle c.

$\therefore$ By Cauchy Theorem

$$\int_C \frac{e^z}{(z+2i)(z-2)}\,dz = \underline{\underline{0}} \ \underline{\text{Ans}}$$

• <u>Cauchy Integral Formula</u>

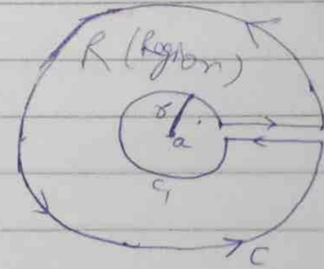
If $f(z)$ is analytic ~~whi~~ within and on closed curve C.
& if a is any point within c at ~~where~~ which function is not analytic then.

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

<u>Proof</u>

Plot a small circle c_1 inside C with radius r and centre "a".
Put $z-a = re^{i\theta}$ $\Rightarrow dz = re^{i\theta}\, i\,d\theta$

Now ;



$$\int_C \frac{f(z)}{z-a}\,dz = \int_C \frac{f(a+re^{i\theta})\, re^{i\theta}\, i\,d\theta}{re^{i\theta}}$$

$$= i\int_C f(a+re^{i\theta})\,d\theta = \underbrace{i\int_R f(a+re^{i\theta})\,d\theta}_{\substack{\shortparallel \\ 0}} + i\int_{C_1} f(a+re^{i\theta})\,d\theta$$

(By Cauchy's Theorem)

$$= i\int_{C_1} f(a+re^{i\theta})\,d\theta$$

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(since r is very small taking $r \to 0$)

$$= i\int_{C_1} f(a)\, d\theta \Rightarrow i \times f(a) \int_{C_1} d\theta$$

$$= i f(a) \int_0^{2\pi} d\theta = i f(a) \times 2\pi = 2\pi i \times f(a)$$

Hence, Proved.

Now;

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

(Differentiate wrt a partially on both side)

$$\int_C \frac{f(z)\, dz}{(z-a)^2} = 2\pi i f'(a)$$

$$\int_C \frac{f(z)\, dz}{(z-a)^3} = \frac{2\pi i}{2} f''(a)$$

Derivative of analytic fn

$$\boxed{\int_C \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{n}(a)}$$

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Note:

$|z| = r$ — $c(0,0)$ Radius r

$|z-a| = r$ — $c(a,0)$ Radius "

$|(z-bi)| = r$ — $c(0,b)$ Radius "

$|z-(a+ib)| = r$ — $c(a,b)$ Radius "

Q. $\int \frac{z^3-6}{3z-i} dz$; $|z|=1$

Sol →

If Put $3z - i = 0$

$z = i/3$

$\boxed{|z| = 1/3}$

$\Rightarrow \frac{1}{3}\int \frac{f(z)}{z-i/3}$

$\Rightarrow \frac{1}{3} 2\pi i \, f(i/3)$

$\Rightarrow \frac{2\pi i}{3}\left((i/3)^3 - 6\right) \Rightarrow \frac{2\pi i}{3}\left(\frac{-i}{27} - 6\right)$

$\Rightarrow \frac{-2\pi i}{3}\left(\frac{i}{27} + 6\right)$ Ans

Q. $\int_C \frac{\cos 2\pi z}{(2z-1)(z-3)} dz$; $|z| = 1$

$\hookrightarrow$

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Sol→ Let $f(z) = \cos 2\pi z$

$$\frac{1}{(2z-1)(z-3)} = \frac{A}{2z-1} + \frac{B}{z-3}$$

$(A = -2/5 \,;\, B = 1/5)$

$$\Rightarrow \int_C f(z)\left(-\frac{2}{5}\left(\frac{1}{2z-1}\right) + \frac{1}{5}\left(\frac{1}{z-3}\right)\right) dz$$

$$\Rightarrow -\frac{2}{5}\int_C \frac{f(z)}{2z-1}\, dz + \frac{1}{5}\int_C \frac{f(z)}{z-3}\, dz$$

$$\Rightarrow -\frac{1}{5}\int_C \frac{f(z)}{z-1/2}\, dz + \underbrace{\frac{1}{5}\int_C \frac{f(z)}{z-3}\, dz}_{\to 0}$$

(By Cauchy Theorem)

$$\Rightarrow -\frac{1}{5}\, 2\pi i\, f\left(\frac{1}{2}\right) + 0$$

$$\Rightarrow \frac{-2\pi i}{5} \cos \pi$$

$$\Rightarrow \frac{2\pi i}{5}$$ Ans

Q. $\int_C \frac{z^3+8}{0.5z-1.5}\, dz \quad ; \; x^2+y^2 = 16$

Sol→ $x^2+y^2 = 16$

$C\,(0,0)$

$R = 4$

$|z| = 4.$

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$$\Rightarrow \frac{1}{0.5}\int_c \frac{z^2+8}{z-\frac{1.5}{0.5}}\,dz$$

$$\Rightarrow 2\int_c \frac{z^2+8}{z-3}\,dz$$

Let $f(z) = z^2+8$

$$\Rightarrow 2\int_c \frac{f(z)}{z-3}\,dz$$

$$\Rightarrow 2\left(2\pi i\ f(3)\right)$$

$$\Rightarrow 4\pi i\ (9+8)$$

$$\Rightarrow 68\pi i$$ Ans

Q. $\int \frac{4-3z}{z(z-1)(z-2)}\,dz \quad ; \quad |z| = 1.5$

Sol → Let $f(z) = 4-3z$

$$\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$(A = \frac{1}{2},\ B = 1,\ C = \frac{1}{2})$

$$\Rightarrow \int_c f(z)\left(\frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}\right)dz$$

$$\Rightarrow \frac{1}{2}\int_c \frac{f(z)}{z-0}\,dz - \int_c \frac{f(z)}{z-1}\,dz + \frac{1}{2}\int_c \frac{f(z)}{z-2}\,dz$$

$= \frac{1}{2}\left(2\pi i f(0)\right) - 2\pi i f(1) + 0$

$= \pi i (4) - 2\pi i (1)$

$\Rightarrow \underline{2\pi i}$ Ans

Q. $\int_C \frac{e^{2z}}{(z+1)^4} dz \; ; |z| = 2.$

Sol→ Let $f(z) = e^{2z}$

$$\int_C \frac{f(z)}{(z+1)^4} dz = \frac{2\pi i f'''(-1)}{3!}$$

$$\Rightarrow \frac{2\pi i \cdot 8 e^{-2}}{6}$$

$$\Rightarrow \boxed{\frac{8\pi i}{3} e^{-2}}$$ Ans

Q. $\int_C \frac{\sin z}{(z - \pi/6)^2} dz \; ; |(z)| = 1$

Sol→ Let $f(z) = \sin z$

$$\int \frac{f(z)}{(z - \pi/6)^2} = \frac{2\pi i \, f'(\pi/6)}{1!}$$

$$\Rightarrow 2\pi i \cos(\pi/6)$$

$$\Rightarrow 2\pi i \frac{\sqrt{3}}{2} \Rightarrow \underline{\sqrt{3}\pi i}$$ Ans

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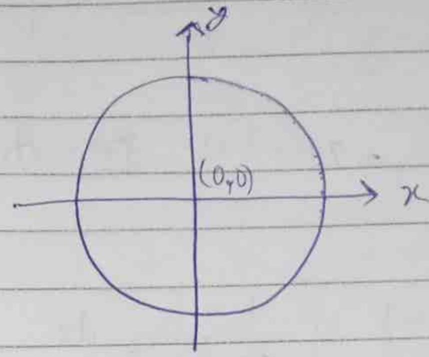
n th order derivatives

Q. Find $\int_C \frac{\cos z}{z^2}\, dz$; where C is $|z| = 1$.

Sol→ $z^2 = 0$

$z = 0 \rightarrow a$



$z = 0 + i0$

$= \boxed{(0, 0)}$

$n = 1$, $a = 0$

$f(z) = \cos z$

$f'(z) = -\sin z$

$f'(0) = -\sin 0$

$= \boxed{0}$

$\int_C \frac{f(z)}{(z-0)^2} = \frac{2\pi i}{\lfloor 1} f'(0)$

$= 2\pi i\,(0)$

$= 0$

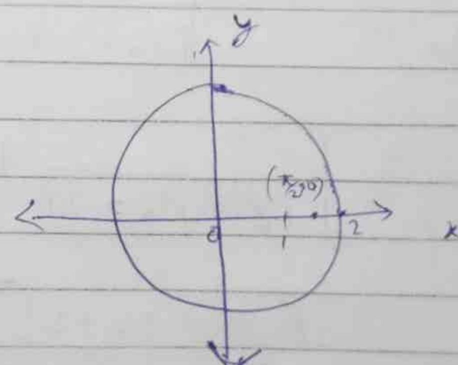
Q. Find $\int \frac{z - 3\cos z}{(z - \pi/2)^2}$; where C is $|z| = 2$.

Sol→ $\left(z - \frac{\pi}{2}\right)^2 = 0$

$z - \frac{\pi}{2} = 0 \Rightarrow \boxed{z = \frac{\pi}{2}}$

$\hookrightarrow \pi/2 + i0$

$\boxed{(\pi/2, 0)}$



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$n = 1 \; ; \; a = \pi/2$

$$\int_C \frac{f(z)}{\left(z-\frac{\pi}{2}\right)^2} = \frac{2\pi i}{1!} f'\left(\frac{\pi}{2}\right)$$

$= 2\pi i (4) = \underline{8\pi i}$ Ans

$f(z) = z - 3\cos z$

$f'(z) = 1 + 3\sin z$

$f'(\pi/2) = 1 + 3\sin(\pi/2)$

$= 1 + 3 = 4$

Q. Find $\int \frac{1}{(z^2+4)^2} dz$ where c is $|z - i| = 2$.

Sol→ $(z^2+4)^2 = 0$

$z^2 + 4 = 0$

$z^2 + 2^2 = 0$

$(z+2i)(z-2i) = 0$

$z = -2i \; ; \; z = 2i \Rightarrow a$

$0 - 2i$ → $\boxed{(0, -2)}$

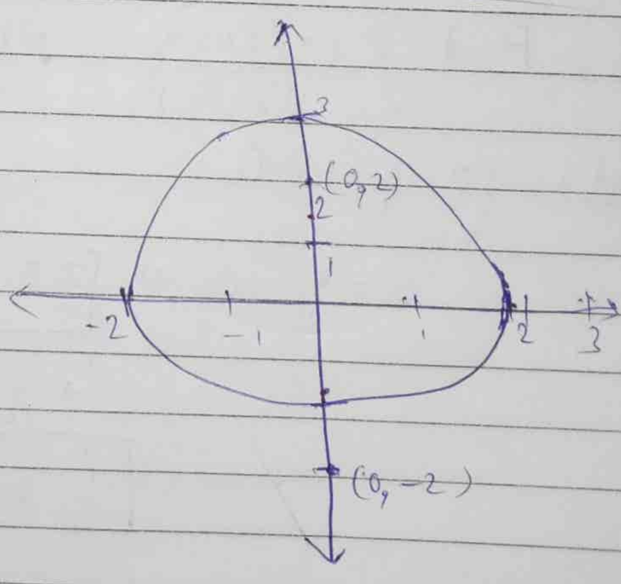
$0 + 2i$ → $\boxed{(0, 2)}$

singular point

Centre $(z - i) = 0$

$z = i = 0 + 1(i) \Rightarrow (0, 1)$

$\boxed{r = 2}$



$$\int_C \frac{1}{(z^2+4)^2} dz = \int \frac{1}{[(z+2i)(z-2i)]^2} dz$$

$$= \int \frac{1}{(z-2i)^2 (z-2i)^2} dz$$

$$= \int_C \frac{\overbrace{1/(z+2i)^2}^{f(z)}}{(z-2i)^2} dz$$

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$n = 1 \; ; \; a = 2i$

$$\int_C \frac{f(z)}{(z-2i)^2}\, dz = \frac{2\pi i}{1!} f'(2i)$$

$$\Rightarrow 2\pi i \left(\frac{1}{32i}\right)$$

$$\Rightarrow \frac{\pi}{16} \text{ Ans}$$

$f(z) = \frac{1}{(z+2i)^2}$

$f'(z) = \frac{-2}{(z+2i)^3}$

$f'(2i) = \frac{-2}{(2i+2i)^3}$

$= \frac{-2}{(4i)^3} = \frac{1}{32i}$

Q. Find $\int \frac{z \cos z}{(z-\pi/2)^2}\, dz$ where C is $|z-1| = 1$.

Sol→ $z - \pi/2 = 0$

$z = \pi/2$

$= \pi/2 + 0i$

$\left(\frac{\pi}{2}, 0\right)$

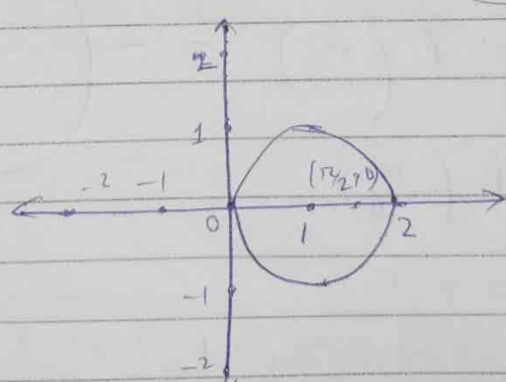
Centre $z - 1 = 0$

$z = 1$

$1 + i0$

$(1, 0)$

$\boxed{r = 1}$



$a = \pi/2 \quad ; \quad n = 1$

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$$\int_C \frac{f(z)}{(z-\pi/2)^2}\, dz = \frac{2\pi i}{1!} f'(\pi/2)$$

$\Rightarrow 2\pi i\left(-\frac{\pi}{2}\right) \Rightarrow -\pi^2 i$ Ans

$f(z) = z\cos z$

$f'(z) = -z\sin z + \cos z\ (1)$

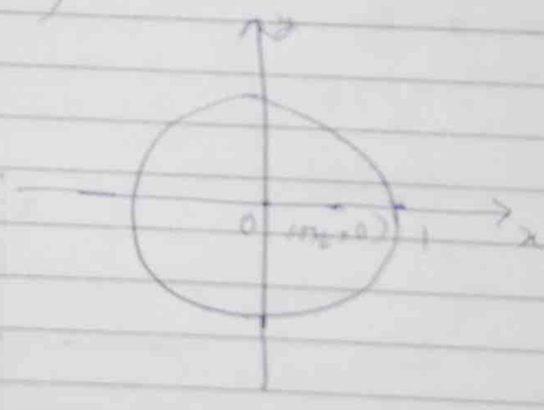
$f'(\pi/2) = -\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2}(1) + 0 \Rightarrow -\frac{\pi}{2}$

Q. find $\int \frac{\sin^6 z}{(z-\pi/6)^3}\, dz$ where c is $|z| = 1$.

Sol→ $z - \pi/6 = 0$

$z = \pi/6 = \pi/6 + i0 \Rightarrow (\pi/6, 0)$



$a = \pi/6$; $n = 2$

$$\int \frac{f(z)}{(z-\pi/6)^3}\, dz = \frac{2\pi i}{2!} f''(\pi/6)$$

$= \pi i f''(\pi/6)$

$= \pi i\left(\frac{21}{16}\right)$

$= \boxed{\frac{21}{16}\pi i}$ Ans

$f(z) = \sin^6 z$

$f'(z) = 6\sin^5 z\cos z$

$f''(z) = 6(\sin^5 z(-\sin z) + \cos z(5\sin^4 z)\cos z)$

$= 6(-\sin^6 z + 5\sin^4 z\cos^2 z)$

$f''(\pi/6) = 6(-\sin^6 \pi/6 + 5\sin^4 \pi/6 \cos^2 \pi/6)$

$= 6\left(-\frac{1}{64} + \frac{5}{16}(3/4)\right) = 6\left(-\frac{1}{64} + \frac{15}{64}\right)$

$\frac{21}{16}$

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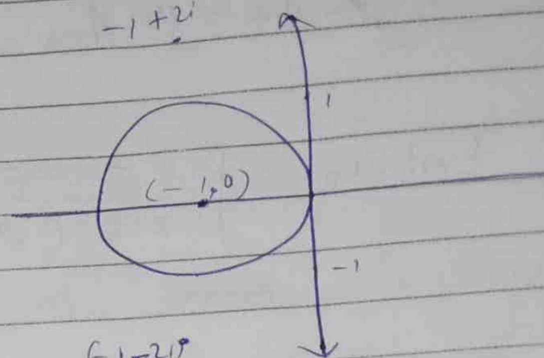
Q. Find integral of $\int_c \frac{z+4}{z^2+2z+5}$ where c is circle

$|z+1| = 1.$



Sol→ Radius = 1

$z+1=0$

$z=-1$

Poles: $z^2+2z+5=0$

$\Rightarrow \frac{-2 \pm \sqrt{4-20}}{2} \Rightarrow \frac{-2 \pm 4i}{2} \Rightarrow -1 \pm 2i$

So, as both points are outside fn & singular point

So,

$$\int_c \frac{z+4}{z^2+2z+5} = 0 \quad \text{Ans}$$

Q. Evaluate $\int_c z^2 \, dz$ using Cauchy Integral Theorem

where c : $|z|=1.$

Sol→ $f(z) = z^2$

$(x+iy)^2 = x^2+i^2y^2+2xiy$

$= \underbrace{x^2-y^2}_{u} + i\underbrace{2xy}_{v}$

$\frac{du}{dx} = 2x \quad \frac{du}{dy} = -2y$

$\frac{dv}{dx} = 2y \quad \frac{dv}{dy} = 2x$

$\boxed{\frac{du}{dx} = \frac{dv}{dy}}$ & $\boxed{\frac{du}{dy} = -\frac{dv}{dx}}$

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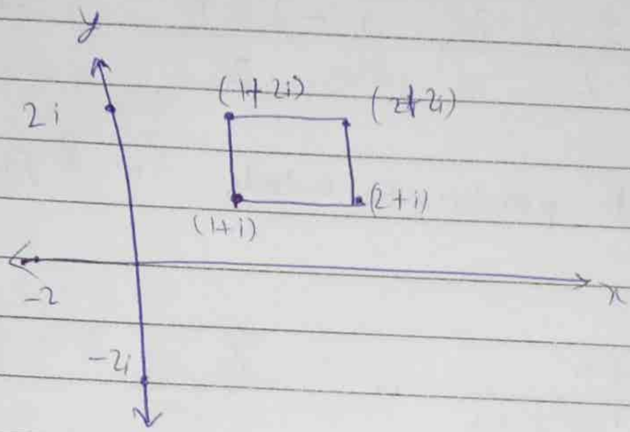
So, as C-R eqn holds
So, $f(z)$ is analytic function.

$\Rightarrow \int_C z^2\, dz = 0$ Ans

Q. Evaluate : $\oint_C \dfrac{2z^2+5}{(z+2)^3(z^2+4)}\, dz$

where c is square with vertices at $1+i$, $2+i$, $2+2i$, $1+2i$.

Sol →



Poles: $(z+2)^3(z^2+4) = 0$

$(z+2)^3 = 0$; $(z^2+4) = 0$

$z = -2$ (Pole of order 3

$z = \pm 2i$ (simple poles)

The obtained poles don't lie inside contour C. Hence by Cauchy Integral theorem.

$$\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)}\, dz = 0 \text{ Ans}$$

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Q. Use cauchy Integral formula ; to evaluate

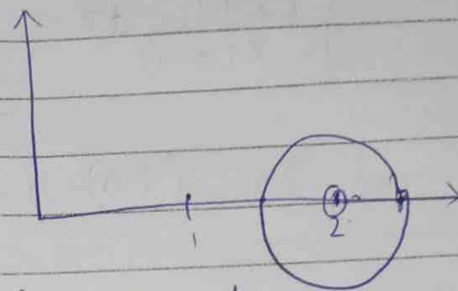
$$\int_C \frac{z}{(z^2-3z+2)}\,dz$$ where C is circle $|(z-2)| = \frac{1}{2}$

Sol→

Poles → $z^2 - 3z + 2 = 0$

$= (z-1)(z-2) = 0$

$z = 1, 2$



There is only one pole at $z = 2$ inside the given circle

$$\int_C \frac{z}{(z-1)(z-2)}\,dz = \int \frac{\overbrace{\left(\frac{z}{z-1}\right)}^{\to f(z)}}{z-2}\,dz$$

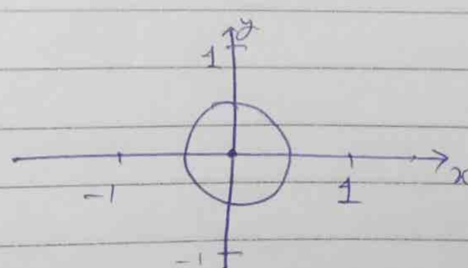
$$\left(= \int_C \frac{f(z)}{z-a} = 2\pi i\, f(a) \right)$$

$$\Rightarrow 2\pi i\, f(2) = 2\pi i \left(\frac{z}{z-1}\right)_{z=2}$$

$$= 2\pi i \left(\frac{2}{2-1}\right) \Rightarrow \underline{4\pi i} \text{ Ans}$$

Q. Evaluate $\int_C \frac{2z+1}{z^2+z}\,dz$ where c is circle $|z| = \frac{1}{2}$.

Sol→



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$\Rightarrow$ Poles $= z^2 + z = 0$

$z(z+1) = 0$

$z = 0, -1$

Only $z=0$, pole is enclosed.

$$\int_C \frac{2z+1}{z(z+1)}\,dz = \int_C \frac{\frac{2z+1}{z+1}}{z}\,dz$$

$$\Rightarrow 2\pi i\, f(0) \Rightarrow 2\pi i \left(\frac{2z+1}{z+1}\right)_{z=0} \Rightarrow \underline{\underline{2\pi i}} \text{ Ans}$$