Vectors

Any ordered n-tuple of numbers is called an n-vector. If $x_1, x_2, \dots, x_n$ be any n numbers then the ordered n-tuple $X = (x_1, x_2, \dots, x_n)$ is called an n-vector.

Let $X = (x_1, x_2, \dots, x_n)$ & $Y = (y_1, y_2, \dots, y_n)$ be two vectors.

(1) If $X = Y$, then $x_i = y_i$ for $i = 1, 2, \dots, n$

(2) $X \pm Y = (x_1 \pm y_1, x_2 \pm y_2, \dots, x_n \pm y_n)$

(3) $kX = (kx_1, kx_2, \dots, kx_n)$ where k is any arbitrary

Linear Dependence and Linear Independence of Vectors

A set of r n-vectors $X_1, X_2, \dots, X_r$ is said to be linearly dependent if there exist r scalars (numbers) $k_1, k_2, \dots, k_r$ not all zero, such that

$$k_1 X_1 + k_2 X_2 + \dots + k_r X_r = 0$$

If $k_i \neq 0$, then $k_i X_i = -k_1 X_1 - k_2 X_2 - \dots - k_r X_r$

$$\Rightarrow X_i = -\frac{k_1}{k_i} X_1 - \frac{k_2}{k_i} X_2 - \dots - \frac{k_r}{k_i} X_r$$

$$\text{or } X_i = \lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_r X_r$$

The vector X_i is called a linear combination of the remaining vectors $X_1, X_2, \dots, X_r$.

If a set of vectors is linearly dependent, then atleast one member of the set can be expressed as a linear combination of the remaining vectors.

A set of r n-vectors $X_1, X_2, X_3, \dots, X_r$ is said to be linearly independent if every relation of the type

$$k_1 X_1 + k_2 X_2 + \dots + k_r X_r = 0 \Rightarrow k_1 = k_2 = \dots = k_r = 0$$

Problem Show that the vectors $x_1=(1,2,4)$, $x_2=(2,-1,3)$, $x_3=(0,1,2)$ and $x_4=(-3,7,2)$ are linearly dependent and find the relation between them.

Solution Consider the relation

$$k_1x_1+k_2x_2+k_3x_3+k_4x_4=0 \quad \text{---(1)}$$

$$k_1(1,2,4)+k_2(2,-1,3)+k_3(0,1,2)+k_4(-3,7,2)=0$$

$$(k_1+2k_2-3k_4,\ 2k_1-k_2+k_3+7k_4,\ 4k_1+3k_2+2k_3+2k_4)$$
$$=(0,0,0)$$

Equating corresponding components on both sides, we have

$$\left.\begin{aligned} k_1+2k_2-3k_4&=0\\ 2k_1-k_2+k_3+7k_4&=0\\ 4k_1+3k_2+2k_3+2k_4&=0 \end{aligned}\right\} \text{---(2)}$$

We have a homogeneous system of 3 linear equations in 4 unknowns.

The co-efficient matrix

$$A=\begin{bmatrix} 1 & 2 & 0 & -3\\ 2 & -1 & 1 & 7\\ 4 & 3 & 2 & 2 \end{bmatrix} \text{ is } 3\times 4$$

$$\rho(A)\le \min\{3,4\}=3$$

Since rank of A is less than the number of unknowns, the homogeneous system has infinitely many non-zero solution. Thus there exist scalars k_1, k_2, k_3, k_4 not all zero, such that

$$k_1x_1+k_2x_2+k_3x_3+k_4x_4=0$$

$\therefore$ The vectors x_1, x_2, x_3, x_4 are linearly dependent.

To find the relation between the given vectors

In matrix notation, the system of equations (2) can be written as

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 0 & 12 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow k_1 + 2k_2 - 3k_4 = 0$

$-5k_2 + 12k_4 = 0$

$k_3 + k_4 = 0$

$\Rightarrow k_1 + \frac{24}{5}k_4 - 3k_4 = 0$, $k_2 = \frac{12}{5}k_4$, $k_3 = -k_4$

$\Rightarrow k_1 = -\frac{9}{5}k_4$, $k_2 = \frac{12}{5}k_4$, $k_3 = -k_4$

For non-zero solutions, $k_4 \neq 0$

Putting these values in (1), we get

$-\frac{9}{5}k_4 x_1 + \frac{12}{5}k_4 x_2 - k_4 x_3 + k_4 x_4 = 0$

$9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$

$\Rightarrow 9x_1 = 12x_2 - 5x_3 + 5x_4$

$\Rightarrow x_1 = \frac{4}{3}x_2 - \frac{5}{4}x_3 + \frac{5}{9}x_4$

Excercise

Are the following vectors linearly dependent? If so, find a relation between them.

1) $x_1 = (1,2,1)$, $x_2 = (2,1,4)$, $x_3 = (4,5,6)$
Sol^n: Yes, $x_3 = 2x_1 + x_2$

2) $x_1 = (2,-1,4)$, $x_2 = (0,1,2)$, $x_3 = (6,-1,16)$
Sol^n: Yes, $x_3 = 3x_1 + 2x_2$

3) $x_1 = (3,2,7)$, $x_2 = (2,4,1)$, $x_3 = (1,-2,6)$
Sol^n: Yes, $x_1 = x_2 + x_3$

4) $x_1 = (2,-1,3,2)$, $x_2 = (1,3,4,2)$, $x_3 = (3,-5,2,2)$
Sol^n: Yes, $2x_1 - x_2 - x_3 = 0$

5) $x_1 = (2,3,1,-1)$, $x_2 = (2,3,1,-2)$, $x_3 = (4,6,2,1)$
Sol^n: Yes, $5x_1 - 3x_2 - x_3 = 0$

6) $x_1 = (1,1,1,3)$, $x_2 = (1,2,3,4)$, $x_3 = (2,3,4,9)$
Sol^n: No

7) $x_1 = (1,1,-1,1)$, $x_2 = (1,-1,2,-1)$, $x_3 = (3,1,0,1)$
Sol^n: Yes, $2x_1 + x_2 - x_3 = 0$