18. RESIDUE OF A COMPLEX FUNCTION ①

The coefficient b_1 of $\frac{1}{(z-a)}$ in the Laurent's expansion of $f(z)$ about an isolated singularity $z=a$, is called the residue of $f(z)$ at $z=a$

As defined, $b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$

If z_0 is a simple pole, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{(z-a)}$$

$$\Rightarrow \quad (z-a)f(z) = \sum_{n=0}^{\infty} a_n(z-a)^{n+1} + b_1$$

$$\therefore \quad \lim_{z\to a} (z-a)f(z) = b_1$$

which provides a simple method of calculating the residue of $f(z)$ at its simple pole $z=a$.

If $f(z)$ has a pole of order m at $z=a$, then the Laurent expansion of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

$$\Rightarrow (z-a)^m f(z) = \sum_{n=0}^{\infty} a_n(z-a)^{n+m} + b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + \cdots + b_m$$

Differentiating this equation w.r.t z, ②

(m-1) times and taking limit $z \to a$

$$\Rightarrow \quad b_1 = \frac{1}{(m-1)!} \lim_{z \to a} \left[\frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m \cdot f(z) \right) \right]$$

This gives the residue of $f(z)$ at its pole of ~~of~~ order m at $z = a$.

Ex Determine the poles and residue for :-

(1) $\dfrac{z}{\cos z}$ (2) $z \cos\left(\frac{1}{z}\right)$

(3) $\dfrac{1-e^{2z}}{z^4}$

Solution

(1). $f(z) = \dfrac{z}{\cos z}$

For the singularities, put $\cos z = 0 = \cos (2n+1)\frac{\pi}{2}$

$n \in I$.

$\therefore \quad z = \left(n + \frac{1}{2}\right)\pi$

let $z_0 = \left(n+\frac{1}{2}\right)\pi = n\pi + \frac{\pi}{2}$.

Residue

$$\therefore \lim_{z \to z_0} (z-z_0) f(z) = \lim_{z \to z_0} \frac{(z-z_0)\, z}{\cos z} = \lim_{z \to z_0} \frac{z\left(z - n\pi - \frac{\pi}{2}\right)}{\cos z}.$$

which is $\frac{0}{0}$ form.

$\therefore$ By L' Hospital Rule.

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \frac{2z - n\pi - \frac{\pi}{2}}{-\sin z}$$

$$= \frac{2z_0 - n\pi - \frac{\pi}{2}}{-\sin z_0} = \frac{2\left(n\pi + \frac{\pi}{2}\right) - n\pi - \frac{\pi}{2}}{-\sin\left(n\pi + \frac{\pi}{2}\right)}$$

$$= \frac{n\pi + \frac{\pi}{2}}{-\sin\left(n\pi + \frac{\pi}{2}\right)} = \frac{n\pi + \frac{\pi}{2}}{-\cos n\pi}$$

$$= \frac{\left(n\pi + \frac{\pi}{2}\right)}{-(-1)^n}$$

$$= (-1)^{n+1}\left(n\pi + \frac{\pi}{2}\right)$$

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(2) $f(z) = z\cos\left(\frac{1}{z}\right) = z\left(1 - \frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} - \cdots\right)$

$$= z - \frac{1}{2!\,z} + \frac{1}{4!\,z^3} - \cdots$$

$\therefore$ The ~~pole of f(z) all at~~ essential $z = 0$ only.

~~and residue of f(z) is~~ $\cancel{= \frac{-1}{2!}}$.

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(3). $\dfrac{1-e^{2z}}{z^4}$

$$= \frac{1}{z^4}\left[1-\left(1+2z+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+\frac{(2z)^4}{4!}+\frac{(2z)^5}{5!}+\cdots\right)\right]$$

$$= \frac{1}{z^4}\left[-2z-\frac{4z^2}{2}-\frac{8z^3}{6}-\frac{16z^4}{24}-\frac{32z^5}{120}---\right]$$

$$= \frac{-2}{z^3}-\frac{2}{z^2}-\frac{4}{3z}-\frac{2}{3}-\frac{32z}{120}----$$

$\therefore$ $z=0$ is a pole of order 3.

and Residue $= -\dfrac{4}{3}$.

Ex Determine the poles of $f(z)=\dfrac{z^2}{(z-1)(z-2)^2}$

and the residue at each pole.

Solution. $f(z)=\dfrac{z^2}{(z-1)(z-2)^2}$. have poles at $z=1, z=2$.

At $z=1$, $f(z)$ has a simple pole

$$\therefore \operatorname*{Res}_{z=1} f(z) = \lim_{z\to1}(z-1)f(z) = \lim_{z\to1}\frac{z^2}{(z-2)^2}$$

$$= \frac{1}{1^2} = 1.$$

At $z=2$, a pole of order 2.

$$\underset{z=2}{\text{Res}}\, f(z) = \lim_{z\to 2} \frac{1}{(2-1)!}\left[\frac{d}{dz}\left((z-2)^2 f(z)\right)\right]$$

$$= \lim_{z\to 2}\left[\frac{d}{dz}\left(\frac{z^2}{(z-1)}\right)\right]$$

$$= \lim_{z\to 2}\left[\frac{(z-1)2z - z^2(1)}{(z-1)^2}\right]$$

$$= \lim_{z\to 2} \frac{z^2-2z}{(z-1)^2} = \frac{4-4}{1^2} = 0.$$

$\therefore$ The residue of $f(z)$ at $z=1$ is 1 & at $z=2$ is 0.

Ex Find the nature and location of singularity of the function $f(z) = z^2 e^{z^{-1}}$. Find its residue.

Solution $f(z) = z^2 e^{z^{-1}} = z^2 e^{1/z}$

$$= z^2\left[1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^3 3!} + \cdots\right]$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{3!\,z} + \cdots$$

$\therefore$ Essential singularity at $z=0$

and Residue $= \dfrac{1}{3!} = \dfrac{1}{6}$.

CAUCHY'S RESIDUE THEOREM.

If $f(z)$ is analytic at all points in a simple closed curve C except for a finite no. of isolated singularities $z_1, z_2, \ldots, z_n$ inside C, then

$$\int_C f(z)\,dz = 2\pi i * [\text{Sum of the residues of } f(z) \text{ at } z_1, z_2, \ldots, z_n].$$

Ex. Evaluate: $\oint_C \frac{dz}{(z-1)(z-2)(z-3)}$, $C: |z| = 4$

Solution. $f(z) = \frac{1}{(z-1)(z-2)(z-3)}$ is analytic inside $C: |z| = 4$ except at the poles $z = 1, 2, 3$ of order 1.

- $\underset{z=1}{\text{Res}}\, f(z) = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{1}{(z-2)(z-3)}$

$$= \frac{1}{(1-2)(1-3)} = \frac{1}{(-1)(-2)} = \frac{1}{2}$$

- $\underset{z=2}{\text{Res}}\, f(z) = \lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} \frac{1}{(z-1)(z-3)}$

$$= \frac{1}{(2-1)(2-3)} = \frac{1}{1(-1)} = -1.$$

- $\operatorname{Res}_{z=3} f(z) = \lim_{z\to 3} (z-3)f(z) = \lim_{z\to 3} \frac{1}{(z-1)(z-2)}$

$$= \frac{1}{(3-1)(3-2)} = \frac{1}{2\cdot 1} = \frac{1}{2}$$

$\therefore$ By Residue theorem

$$\int f(z)\,dz = 2\pi i\left[+\frac{1}{2} + \frac{1}{2} - 1\right] = 0.$$

Ex. If $C : |z| = 3$, prove that

(a) $$\int \frac{(z+3)\,dz}{(z+1)^2(z-2)} = 0.$$

(b) $$\int \frac{z^2}{(z-1)^2(z+2)}\,dz = 2\pi i$$

Solution. (a) $f(z) = \dfrac{z+3}{(z+1)^2(z-2)}$ has a pole of order 1 at $z=2$ and of order 2 at $z=-1$.

Both are lying inside the circle $|z|=3$.

- $\operatorname{Res}_{z=2} f(z) = \lim_{z\to 2} \cancel{\frac{1}{(2-1)!}}\,(z-2)f(z) = \lim_{z\to 2} \frac{z+3}{(z+1)^2}$

$$= \frac{5}{3^2} = \frac{5}{9}.$$

- $\operatorname{Res}_{z=-1} f(z) = \lim_{z\to -1} \frac{1}{(2-1)!}\left[\frac{d}{dz}\left((z+1)^2 f(z)\right)\right] = \lim_{z\to -1} \frac{d}{dz}\left(\frac{z+3}{z-2}\right)$

$$= \lim_{z\to -1} \frac{(z-2)(1)-(z+3)(1)}{(z-2)^2}$$

$$= \lim_{z\to -1} \frac{-5}{(z-2)^2} = \frac{-5}{9}$$

$\therefore$ By Residue Theorem,

$$\int f(z)\,dz = 2\pi i\left[\frac{5}{9}-\frac{5}{9}\right] = 0.$$

(b) $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ has a pole of order 2

at $z=1$ and a simple pole at $z=-2$.

and both are lying inside $C: |z|=3$.

- $\underset{z=1}{\text{Res}}\, f(z) = \dfrac{1}{(2-1)!} \lim\limits_{z\to 1} \dfrac{d}{dz}\left((z-1)^2 f(z)\right)$

$$= \lim_{z\to 1} \frac{d}{dz}\left(\frac{z^2}{z+2}\right) = \lim_{z\to 1} \frac{(z+2)2z - z^2(1)}{(z+2)^2}$$

$$= \lim_{z\to 1} \frac{z^2+4z}{(z+2)^2} = \frac{1+4}{(1+2)^2}$$

$$= \frac{5}{9}.$$

- $\underset{z=-2}{\text{Res}}\, f(z) = \lim\limits_{z\to -2} (z+2) f(z)$

$$= \lim_{z\to -2} \frac{z^2}{(z-1)^2} = \frac{4}{(-3)^2} = \frac{4}{9}$$

$\therefore$ By Residue theorem,

$$\int f(z)\,dz = 2\pi i\left(\frac{5}{9}+\frac{4}{9}\right) = 2\pi i.$$

Ex. Evaluate $\displaystyle\int \frac{e^z\,dz}{(z-1)^2(z+2)}$ where $c : |z| = 3$.

Solution. $f(z) = \dfrac{e^z}{(z-1)^2(z+2)}$ ~~are~~ have pole of order 2 at $z=1$ and a pole of order 1 at $z=-2$.

- $$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to1} \frac{1}{(2-1)!}\frac{d}{dz}\left((z-1)^2 f(z)\right)$$

$$= \lim_{z\to1} \frac{d}{dz}\left(\frac{e^z}{z+2}\right)$$

$$= \lim_{z\to1} \frac{(z+2)e^z - e^z(1)}{(z+2)^2} = \frac{3e-e}{9} = \frac{2e}{9}$$

- $$\operatorname*{Res}_{z=-2} f(z) = \lim_{z\to-2} (z+2) f(z)$$

$$= \lim_{z\to-2} \frac{e^z}{(z-1)^2} = \frac{e^{-2}}{9}$$

$$\therefore \int f(z)\,dz = 2\pi i\left(\frac{2e}{9}+\frac{1}{9e^2}\right) = \frac{2\pi i}{9}\left(2e+\frac{1}{e^2}\right)$$

Ex. Evaluate: ① $\int_C \frac{dz}{z^8(z+4)}$; $C : |z| = 2$

② $\int_C \frac{dz}{z^8(z+4)}$; $C : |z+2| = 3$.

Solution ① $f(z) = \frac{1}{z^8(z+4)}$ has a pole of order 8

at $z=0$, which is inside the circle.

$\therefore \underset{z=0}{\text{Res.}}\, f(z) = \lim_{z\to 0} \frac{1}{(8-1)!} \frac{d^7}{dz^7}\left[z^8 f(z)\right]$

$= \lim_{z\to 0} \frac{1}{7!} \frac{d^7}{dz^7}\left[\frac{z^8}{z^8(z+4)}\right]$

$= \frac{1}{7!} \lim_{z\to 0} \frac{d^7}{dz^7}\left(\frac{1}{z+4}\right)$

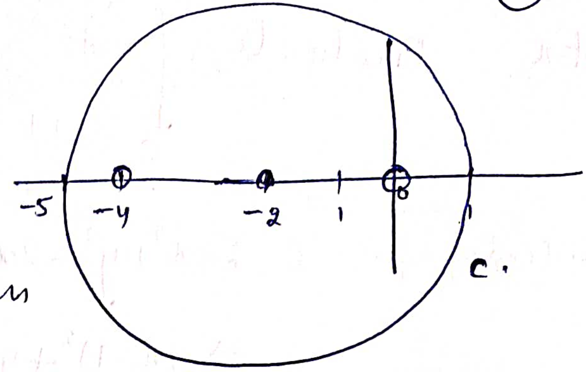
$= \frac{1}{7!} \lim_{z\to 0} \frac{(-1)^7 \cdot 7!}{(z+4)^8}$

$= \frac{-1}{4^8}$.

$\therefore$ By Residue theorem,

$\int f(z)\,dz = 2\pi i\left(\frac{-1}{4^8}\right) = \frac{-2\pi i}{4^8}$.

(2) $f(z) = \dfrac{1}{z^8(z+4)}$



$|z+2| = 3.$

Both the singularities are lying inside the given circle.

$f(z)$ has a simple pole at $z=-4$ and a pole of order 8 at $z=0$.

- $\displaystyle \operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{1}{(8-1)!}\left[\frac{d^7}{dz^7}\left(z^8 f(z)\right)\right]$

$$= \lim_{z\to 0} \frac{1}{7!}\frac{d^7}{dz^7}\left(\frac{1}{(z+4)}\right) = \frac{1}{7!}\lim_{z\to 0}\frac{(-1)\cdot 7!}{(z+4)^8}.$$

$$= \frac{-1}{4^8}.$$

- $\displaystyle \operatorname{Res}_{z=-4} f(z) = \lim_{z\to -4}\frac{1}{z^8} = \frac{1}{4^8}.$

$\therefore$ By Residue theorem, $\displaystyle \int \frac{dz}{z^8(z+4)} = 2\pi i\left(\frac{-1}{4^8} + \frac{1}{4^8}\right)$

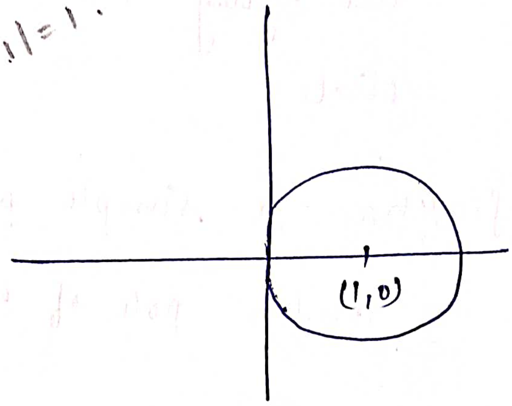
$$= 0$$

Ex Evaluate $\displaystyle\int_C \frac{dz}{z^4+1}$ where $C : x^2+y^2 = 2x$.

Solution. $C : x^2+y^2-2x = 0$

$\Rightarrow (x-1)^2+y^2 = 1$

$|z-1| = 1$.



Now $z^4+1 = 0$

$\Rightarrow z^4 = -1 = \cos\pi + i\sin\pi$

$\Rightarrow z = (\cos\pi + i\sin\pi)^{1/4}$.

$= \cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}$, $n = 0, 1, 2, 3$.

$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$, $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$, $\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$,

$\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$.

$= \dfrac{\pm 1 \pm i}{\sqrt{2}}$.

The poles, $z = \dfrac{1+i}{\sqrt{2}}$ and $z = \dfrac{1-i}{\sqrt{2}}$ lies inside C.

$\left|\frac{1+i}{\sqrt{2}} - 1\right| = \cancel{\sqrt{\frac{1}{2}+\frac{1}{2}+1}}$

$= \sqrt{\left(\frac{1}{\sqrt{2}}-1\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$

$= \sqrt{\frac{1}{2}+1-\frac{2}{\sqrt{2}}+\frac{1}{2}}$

$= \sqrt{2-\frac{2}{\sqrt{2}}} = \sqrt{0.58} < 1$.

$\displaystyle \operatorname*{Res}_{z=\frac{1+i}{\sqrt{2}}} f(z) = \lim_{z\to\frac{1+i}{\sqrt{2}}} \frac{\left(z-\frac{1+i}{\sqrt{2}}\right)}{z^4+1}$ $\quad\left(\frac{0}{0}\right)$

By L'Hospital Rule.

$\displaystyle = \lim_{z\to\frac{1+i}{\sqrt{2}}} \frac{1}{4z^3}$.

$$= \frac{1}{4\left(\frac{1+i}{\sqrt{2}}\right)^3} = \frac{2\sqrt{2}}{4\left(1+i^3+3i+3i^2\right)}$$

$$= \frac{2\sqrt{2}}{4(1-i+3i-3)} = \frac{2\sqrt{2}}{4(2i-2)}$$

$$= \frac{\sqrt{2}}{4(i-1)}$$

$$= \frac{\sqrt{2}(1+i)}{-4(2)}$$

$$= \frac{-1-i}{4\sqrt{2}}.$$

- $$\operatorname*{Res}_{z=\frac{1-i}{\sqrt{2}}} f(z) = \lim_{z\to\frac{1-i}{\sqrt{2}}} \frac{z-\left(\frac{1-i}{\sqrt{2}}\right)}{z^4+1} = \lim_{z\to\frac{1-i}{\sqrt{2}}} \frac{1}{4z^3}$$

$$= \frac{2\sqrt{2}}{4(1-i)^3} = \frac{\sqrt{2}}{2(1-i^3-3i+3i^2)}$$

$$= \frac{\sqrt{2}}{2(1+i-3i-3)} = \frac{\sqrt{2}}{2(-2-2i)}$$

$$= \frac{\sqrt{2}}{-4(1+i)}$$

$$= \frac{\sqrt{2}(1-i)}{-8}$$

$$= \frac{-(1-i)}{4\sqrt{2}}.$$

$\therefore$ By residue theorem,

$$\int f(z)\,dz = 2\pi i\left[\frac{-1}{4\sqrt{2}} - \cancel{\frac{i}{4\sqrt{2}}} - \frac{1}{4\sqrt{2}} + \cancel{\frac{i}{4\sqrt{2}}}\right] = \frac{-2\pi i}{2\sqrt{2}}$$

$$= -\frac{\pi i}{\sqrt{2}}.$$

Ex Evaluate the integral $\int \frac{e^z - 1}{z(z+1)} dz$

where $c : |z| = 2$

Solution: $\int \frac{e^z-1}{z(z+1)} dz = \int e^z - 1 \left(\frac{1}{z} - \frac{1}{z+1} \right) dz.$

$$= \int \frac{e^z-1}{z} dz - \int \frac{e^z-1}{z+1} dz$$

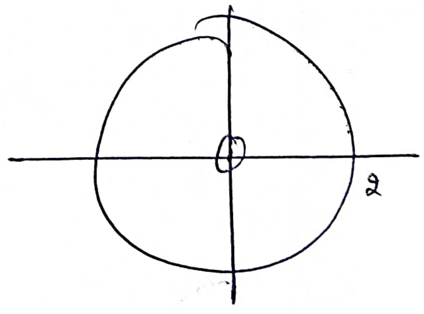
$$= 2\pi i \left[e^z - 1 \right]_{z=0} - 2\pi i \left[e^z - 1 \right]_{z=-1}$$

$$= 2\pi i \left[0 \right] - 2\pi i \left[e^{-1} - 1 \right].$$

$$= 2\pi i \left(1 - \frac{1}{e} \right).$$

Ex. $\int_C \frac{dz}{z^3(z+4)}$ where $c : |z| = 2$.

Solution $f(z) = \frac{1}{z+4}$



$$\therefore \int \frac{f(z)}{z^3} dz = \frac{2\pi i}{2!} \left(\frac{d^2}{dz^2} \left(\frac{1}{z+4} \right) \right)_{z=0}$$

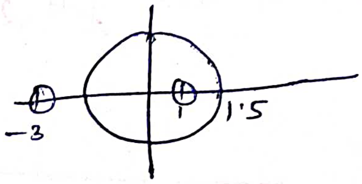
$$= \pi i \left[\frac{(-1)(-2)}{(z+4)^3} \right]_{z=0} = \frac{2\pi i}{4^3} = \frac{\pi i}{32}.$$

Ex. Evaluate $\oint_C \frac{e^z dz}{(z-1)(z+3)^2}$

where (1) $C : |z| = 3/2$.

(2) $C : |z| = 10$.

Solution. (1) $\int \frac{e^z dz}{(z-1)(z+3)^2}$



Let $f(z) = \frac{e^z}{(z+3)^2}$.

$\therefore$ By Cauchy's integral formula.

$$\int \frac{f(z)\,dz}{(z-1)} = 2\pi i \left[\frac{e^z}{(z+3)^2}\right]_{z=1} = 2\pi i\left[\frac{e}{4^2}\right]$$

$$= \frac{\pi i e}{8}.$$

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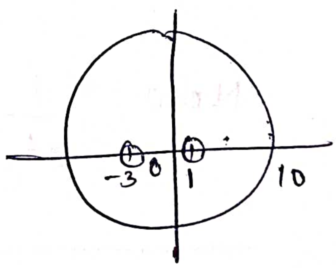
(2). $f(z) = \frac{e^z}{(z-1)(z+3)^2}$ has a pole of order 2 at $z=-3$ and a simple pole at $z=1$.



- $\operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{e^z}{(z+3)^2} = \frac{e}{16}$

- $\operatorname{Res}_{z=-3} f(z) = \lim_{z\to -3} \frac{1}{1!}\left[\frac{d}{dz}\left(\frac{e^z}{z-1}\right)\right]_{z=-3} = \left[\frac{(z-1)e^z - e^z(1)}{(z-1)^2}\right]_{\text{at } z=-3}$

$$= \frac{-4e^{-3} - e^{-3}}{(-4)^2} = \frac{-5e^{-3}}{16}$$

$\therefore$ By Residue theorem

$$= 2\pi i\left[\frac{e}{16} - \frac{5e^{-3}}{16}\right] = \pi i\left[\frac{e}{8} - \frac{5e^{-3}}{8}\right].$$

Ex ① Evaluate $\int \frac{\sin z}{z^6}\,dz$ $\quad c : |z| = 2$

② $\int \frac{dz}{z \sin z}$, $\quad c : |z| = 1$.

③ $\int e^{1/z^2}\,dz$, $\quad c : |z| = 2$.

Solution ① $\int f(z) = \int \frac{\sin z}{z^6}\,dz$.

Now $\frac{1}{z^6}\sin z = \frac{1}{z^6}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots\right]$

$\therefore$ Coefficient of $\frac{1}{z}$ is $\frac{1}{5!} = \frac{1}{120}$.

// Residue.

$\therefore$ By Residue theorem

$$\int f(z)\,dz = 2\pi i * \frac{1}{120} = \frac{\pi i}{60}.$$

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(2) $z \sin z = 0$

$\Rightarrow z = 0$ and $\sin z = 0$

$\sin z = \sin n\pi \qquad n \in I.$

$z = n\pi.$

$z = 0$ is the only pole lying inside the circle $|z| = 2$
(Pole of order 2).

$$\therefore \int \frac{dz}{z\sin z} = \frac{2\pi i}{1!}\left[\frac{d}{dz}\left(\frac{z^2}{z\sin z}\right)\right]_{z=0}.$$

$$= 2\pi i \frac{d}{dz}\left[\frac{z}{\sin z}\right]_{z=0}.$$

$$= 2\pi i\left[\frac{\sin z\cdot 1 - z\cos z}{\sin^2 z}\right]_{z=0}.$$

$\left(\frac{0}{0}\right)$
(L'Hospital Rule)

$$= 2\pi i\left[\frac{\cancel{\cos z} + z\sin z - \cancel{\cos z}}{2\sin z\cos z}\right]_{z=0}.$$

$$= 2\pi i\left[\frac{0}{2\cos 0}\right] = 0.$$

(3) $$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!\,z^4} + \cdots$$

Essential singularity at $z=0$.

$$\underset{z=0}{\text{Residue}}\, f(z) = \text{coefficient of } \frac{1}{z} = 0.$$

$$\therefore \int e^{1/z^2}\, dz = 2\pi i\,[0] = 0.$$