NEW TOPICS ADDED FROM ACADEMIC SESSION 2021-22 ONWARDS
SECOND SEMESTER
APPLIED MATHEMATICS-II [BS-112]
UNIT - I

Q.1. If $2\cos\theta = x+\frac{1}{x}$, then prove that $\frac{x^{2n}+1}{x^{2n-1}+x}=\frac{\cos n\theta}{\cos(n-1)\theta}$

Ans. We have given $x+\frac{1}{x}=2\cos\theta$

$\Rightarrow x^2-2x\cos\theta+1=0$

$\Rightarrow x=\cos\theta \pm i\sin\theta$

Let $x=\cos\theta+i\sin\theta$

Consider $\frac{x^{2n}+1}{x^{2n-1}+x}$

$$=\frac{(\cos\theta+i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+\cos\theta+i\sin\theta}$$

$$=\frac{\cos 2n\theta+i\sin 2n\theta+1}{\cos(2n-1)\theta+i\sin(2n-1)\theta+\cos\theta+i\sin\theta}$$

$$=\frac{2\cos^2 n\theta+2i\sin n\theta\cos n\theta}{2\cos n\theta\cos(n-1)\theta+2i\sin n\theta\cos(n-1)\theta}$$

$$=\frac{2\cos n\theta[\cos n\theta+i\sin n\theta]}{2\cos(n-1)\theta[\cos n\theta+i\sin n\theta]}$$

$$=\frac{\cos n\theta}{\cos(n-1)\theta}$$

Q.2. If $a_r=\cos\frac{\pi}{2^r}+i\sin\frac{\pi}{2^r}$, then show that $a_1a_2a_3\ldots$ upto $\infty=-1$ **[*IPU, 2007*]**

Ans. Consider LHS $=a_1a_2a_3\ldots\infty$

$$=\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\ldots$$

$$=\cos\left(\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{8}+\ldots\right)+i\sin\left(\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{8}+\ldots\right)$$

$$=\cos\left(\frac{\pi/2}{1-1/2}\right)+i\sin\left(\frac{\pi/2}{1-1/2}\right)$$ [Sum of infinite GP with a = $\pi/2$ & r = 1/2]

$=\cos\pi+i\sin\pi$

$=-1=$ R.H.S

Q.3. Prove that $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\cos n\left(\dfrac{\pi}{2}-\theta\right)+i\sin n\left(\dfrac{\pi}{2}-\theta\right)$ ***[IPU 2007]***

Ans. Let $\alpha=\dfrac{\pi}{2}-\theta \Rightarrow \theta=\dfrac{\pi}{2}-\alpha$

Consider LHS

$$=\left[\frac{1+\sin\left(\frac{\pi}{2}-\alpha\right)+i\cos\left(\frac{\pi}{2}-\alpha\right)}{1+\sin\left(\frac{\pi}{2}-\alpha\right)-i\cos\left(\frac{\pi}{2}-\alpha\right)}\right]^n$$

$$=\left[\frac{1+\cos\alpha+i\sin\alpha}{1+\cos\alpha-i\sin\alpha}\right]^n$$

$$=\left[\frac{2\cos^2\alpha/2+i2\sin\alpha/2\cos\alpha/2}{2\cos^2\alpha/2-i2\sin\alpha/2\cos\alpha/2}\right]^n$$

$$=\left[\frac{2\cos\alpha/2(\cos\alpha/2+i\sin\alpha/2)}{2\cos\alpha/2(\cos\alpha/2-i\sin\alpha/2)}\right]^n$$

$$=\left[\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)\right]^n$$

$$=\left[\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right]^{2n}$$

$$=\cos n\alpha+i\sin n\alpha$$

$$=\cos n\left(\frac{\pi}{2}-\theta\right)+i\sin n\left(\frac{\pi}{2}-\theta\right)$$

Q.4. If $\sin\alpha+\sin\beta+\sin\gamma=0=\cos\alpha+\cos\beta+\cos\gamma$, Prove that

(i) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

Ans. Let $a=\cos\alpha+i\sin\alpha$ ...(1)

$b=\cos\beta+i\sin\beta$ ...(2)

$c=\cos\gamma+i\sin\gamma$ ...(3)

$\therefore a+b+c=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)$

$=0+i\,0$

$\therefore a+b+c=0$ ...(4)

$\Rightarrow a+b=-c$

Cubing both sides, we get

$a^3+b^3+3ab(a+b)=-c^3$

$\Rightarrow a^3+b^3-3abc=-c^3$

$\Rightarrow a^3+b^3+c^3=3abc$ ...(5)

By (5), we get

$(\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\gamma+i\sin\gamma)^3$

$=3(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)$

$\Rightarrow \cos 3\alpha+i\sin 3\alpha+\cos 3\beta+i\sin 3\beta+\cos 3\gamma+i\sin 3\gamma$

$=3[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)]$

Equating real and imaginary parts

(i) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

Q.5. Solve the equation $x^7-x^4+x^3-1=0$ and find all the roots of the equation.

Ans. $x^7-x^4+x^3-1=0$

$\Rightarrow x^4(x^3-1)+1(x^3-1)=0$

$\Rightarrow (x^4+1)(x^3-1)=0$

$x^4+1=0$ and $x^3-1=0$

Now $x^4+1=0$

$\Rightarrow x=(-1)^{1/4}$

$=(\cos\pi+i\sin\pi)^{1/4}$

$\Rightarrow x=[\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)]^{1/4}$

$\Rightarrow x=\cos(2k\pi+\pi)\dfrac{1}{4}+i\sin(2k\pi+\pi)\dfrac{1}{4}$ $\quad k=0,1,2,3$

$\Rightarrow x=\cos(2k+1)\dfrac{\pi}{4}+i\sin(2k+1)\dfrac{\pi}{4}$ $\quad k=0,1,2,3$

Now $\alpha_1=\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}=\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}$

$\alpha_2=\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}=\dfrac{-1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}$

$\alpha_3=\cos\dfrac{5\pi}{4}+i\sin\dfrac{5\pi}{4}=\dfrac{-1}{\sqrt{2}}-\dfrac{i}{\sqrt{2}}$

$\alpha_4=\cos\dfrac{7\pi}{4}+i\sin\dfrac{7\pi}{4}=\dfrac{1}{\sqrt{2}}-\dfrac{i}{\sqrt{2}}$

Let $x^3-1=0$,

$\Rightarrow x=(1)^{1/3}$

$\Rightarrow x=(\cos 0+i\sin 0)^{1/3}$

$\Rightarrow x=(\cos 2k\pi+i\sin 2k\pi)^{1/3}$

$\Rightarrow x=\cos\dfrac{2k\pi}{3}+i\sin\dfrac{2k\pi}{3}$, $k=0,1,2$

$\alpha_5=\cos 0+i\sin 0=1$

$\alpha_6=\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}=\dfrac{-1}{2}+\dfrac{i\sqrt{3}}{2}$

$\alpha_7=\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}=\dfrac{-1}{2}\dfrac{-i\sqrt{3}}{2}$

Q.6. Show that the roots of the equation $(z-1)^5 + z^5 = 0$ are given by

$$z = \frac{1}{2}\left(1 + i\cot\frac{m\pi}{10}\right) \quad m = 1,3,5,7,9$$

Ans. Consider $(z-1)^5 + z^5 = 0$

$$\Rightarrow \left(\frac{z-1}{z}\right)^5 + 1 = 0$$

Let $w = \dfrac{z-1}{z}$

$$\Rightarrow w^5 + 1 = 0$$
$$\Rightarrow w = (-1)^{1/5}$$
$$\Rightarrow w = [\cos\pi + i\sin\pi]^{1/5}$$
$$\Rightarrow w = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/5} \qquad k = 0,1,2,3,4$$
$$\Rightarrow w = \cos(2k+1)\frac{\pi}{5} + i\sin(2k+1)\frac{\pi}{5} \qquad k = 0,1,2,3,4$$
$$\Rightarrow w = \cos\alpha + i\sin\alpha,\ \alpha = (2k+1)\frac{\pi}{5}. \qquad \text{...(1)}$$

As $w = \dfrac{z-1}{z} \Rightarrow z = \dfrac{1}{1-w}$

$$\Rightarrow z = \frac{1}{1-\cos\alpha - i\sin\alpha} \quad \text{(By (1))}$$
$$\Rightarrow z = \frac{1}{2\sin^2\alpha/2 - 2i\sin\alpha/2\cos\alpha/2}$$
$$\Rightarrow z = \frac{1}{2\sin\alpha/2\left(\sin\frac{\alpha}{2} - i\cos\frac{\alpha}{2}\right)}$$
$$\Rightarrow z = \frac{1}{2\sin\alpha/2}\left(\sin\frac{\alpha}{2} + i\cos\frac{\alpha}{2}\right)$$
$$= \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)$$
$$= \frac{1}{2}\left(1 + i\cot(2k+1)\frac{\pi}{10}\right),\ k = 0,1,2,3,4$$

$\therefore$ Roots are

$$z = \frac{1}{2}\left(1 + i\cot\frac{m\pi}{10}\right),\ m = 1,3,5,7,9.$$

Q.7. Show that the nth roots of unity are given 1, λ, λ^2,...λ^{n-1}, where λ = $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and show that the continued product of all the nth roots is $(-1)^{n+1}$ *[IPU, 2004]*

Ans. Let $x^n = 1$

$$\Rightarrow x = (1)^{1/n}$$
$$\Rightarrow x = (\cos 0 + i\sin 0)^{1/n}$$

$$\Rightarrow x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},\ k = 0,1,\ldots n-1$$

Thus roots are

$$x = \cos 0 + i\sin 0,\ \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},$$
$$\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \ldots\ldots \cos\frac{2(n-1)\pi}{n} + i\sin\frac{(2n-1)\pi}{n}$$
$$\Rightarrow x = 1,\ \cos\frac{2\pi}{n} + i\frac{\sin 2\pi}{n}, \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^2 \ldots \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{n-1}$$
$$= 1, \lambda, \lambda^2 .. \lambda^{n-1}$$

where $\lambda = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$

Product of values is

$$= (\cos 0 + i\sin 0)\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)$$
$$\left(\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}\right) \ldots \left[\cos\frac{2(n-1)\pi}{n} i\sin\frac{2(n-1)\pi}{n}\right]$$
$$\cos\left[0 + \frac{2\pi}{n} + \frac{4\pi}{n} + \ldots + \frac{2(n-1)\pi}{n}\right]$$
$$+ i\sin\left[0 + \frac{2\pi}{n} + \frac{4\pi}{n} + \ldots + 2\frac{(n-1)\pi}{n}\right]$$
$$= \cos\left[\frac{2\pi}{n}(1 + 2 + \ldots(n-1))\right] + i\sin\left[\frac{2\pi}{n}(1 + 2 + \ldots(n-1))\right]$$
$$= \cos\left(\frac{2\pi}{n}\cdot\frac{n(n-1)}{2}\right) + i\sin\left(\frac{2\pi}{n}\frac{n(n-1)}{2}\right)$$
$$= \cos(n-1)\pi + i\sin(n-1)\pi$$
$$= (-1)^{n-1} + 0 = (-1)^{n-1}(-1)^2$$
$$= (-1)^{n+1}$$

Q.8. Find the equation whose roots are $2\cos\frac{\pi}{7}, 2\cos\frac{3\pi}{7}, 2\cos\frac{5\pi}{7}$

Ans. Let $y = \cos\theta + i\sin\theta$ such that

$$y + \frac{1}{y} = 2\cos\theta$$

where θ be any of angles

$$\theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{11\pi}{7}, \frac{13\pi}{7}$$

Now $y^7 = (\cos\theta + i\sin\theta)^7$

$= \cos 7\theta + i \sin 7\theta$

$= -1$

$\Rightarrow y^7 + 1 = 0$

$\Rightarrow (y+1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1) = 0$

For the factor $y + 1 = 0$

We get $\theta = \pi$ (Not possible)

We reject the factor $(y + 1)$

$\Rightarrow$ We have

$y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 = 0$

Dividing the throughout by y^3, we get

$$y^3 - y^2 + y - 1 + \frac{1}{y} - \frac{1}{y^2} + \frac{1}{y^3} = 0$$

$$\Rightarrow \left(y^3 + \frac{1}{y^3}\right) - \left(y^2 + \frac{1}{y^2}\right) + \left(y + \frac{1}{y}\right) - 1 = 0$$

$$\Rightarrow \left\{\left(y + \frac{1}{y}\right)^3 - 3\left(y + \frac{1}{y}\right)\right\} - \left\{\left(y + \frac{1}{y}\right)^2 - 2\right\} + y + \frac{1}{y} - 1 = 0$$

Let $x = y + \frac{1}{y} = 2\cos\theta$

$\Rightarrow x^3 - 3x - x^2 + 2 + x - 1 = 0$

$\Rightarrow x^3 - x^2 - 2x + 1 = 0$

Now $\cos\frac{13\pi}{7} = \cos\left(2\pi - \frac{\pi}{7}\right) = \cos\frac{\pi}{7}$

$\cos\frac{11\pi}{7} = \cos\left(2\pi - \frac{3\pi}{7}\right) = \cos\frac{3\pi}{7}$

$\cos\frac{9\pi}{7} = \cos\left(2\pi - \frac{5\pi}{7}\right) = \cos\frac{5\pi}{7}$

Thus roots of equation

$x^3 - x^2 - 2x + 1 = 0$ are

$2\cos\frac{\pi}{7}, 2\cos\frac{3\pi}{7}, 2\cos\frac{5\pi}{7}$.

Q.9. Show that

$128\sin^3\theta\cos^5\theta = -\sin 8\theta - 2\sin 6\theta + 2\sin 4\theta + 6\sin 2\theta$

[*IPU, 2004, 2007*]

Ans. Let $x = \cos\theta + i\sin\theta$

then $x^k = (\cos\theta + i\sin\theta)^k$

$= \cos k\theta + i\sin k\theta$

$$\therefore \left.\begin{aligned} x^k + \frac{1}{x^k} &= 2\cos k\theta \text{ and} \\ x^k - \frac{1}{x^k} &= 2i\sin k\theta \end{aligned}\right\} \text{(A)}$$

Consider $(2i\sin\theta)^3(2\cos\theta)^5$

$$= \left(x - \frac{1}{x}\right)^3\left(x + \frac{1}{x}\right)^5 \text{ [by (A)]}$$

$$\Rightarrow -256\, i\sin^3\theta\cos^5\theta = \left(x - \frac{1}{x}\right)^3\left(x + \frac{1}{x}\right)^3\left(x + \frac{1}{x}\right)^2$$

$$= \left[\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\right]^3\left(x + \frac{1}{x}\right)^2$$

$$= \left(x^2 - \frac{1}{x^2}\right)^3\left(x + \frac{1}{x}\right)^2$$

$$= \left(x^6 - \frac{1}{x^6} - 3x^4.\frac{1}{x^2} + 3x^2.\frac{1}{x^4}\right)\left(x^2 + \frac{1}{x^2} + 2\right)$$

$$= \left(x^8 - \frac{1}{x^8}\right) + 2\left(x^6 - \frac{1}{x^6}\right) + \left(x^4 - \frac{1}{x^4}\right)$$

$$-3\left(x^4 - \frac{1}{x^4}\right) - 6\left(x^2 - \frac{1}{x^2}\right)$$

$= 2i\sin 8\theta + 4i\sin 6\theta + 2i\sin 4\theta - 6i\sin 4\theta - 12i\sin\sin 2\theta$

$= 2i\sin\theta + 4i\sin 6\theta - 4i\sin 4\theta - 12i\sin 2\theta$

Dividing both sides by $(-2i)$

$\Rightarrow 128\sin^3\theta\cos^5\theta = -\sin 8\theta - 2\sin 6\theta + 2\sin 4\theta + 6\sin 2\theta$

Q.10. If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha = e^{i\alpha}$ Prove it

(i) $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$ (ii) $\phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$

Ans. As $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$ ...(1)

By changing i to $(-i)$

$\Rightarrow \tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$ ...(2)

(i) Consider $\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$\Rightarrow \tan 2\theta = \frac{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha}{1 - (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2\cos\alpha}{1 - (\cos^2\alpha - i^2\sin^2\alpha)}$$

$$= \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)}$$

$$= \frac{2\cos\alpha}{0} = \alpha = \tan\frac{\pi}{2}$$

$$\Rightarrow 2\theta = n\pi + \frac{\pi}{2}$$

$$\Rightarrow \theta = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

(ii) Now $\tan 2i\phi = \tan[(\phi + i\phi) - (0 - i\phi)]$

$$= \frac{\tan(\theta+i\phi)-\tan(\theta-i\phi)}{1+\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{\cos\alpha + i\sin\alpha - \cos\alpha + i\sin\alpha}{1+(\cos\alpha+i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2i\sin\alpha}{1+(\cos^2\alpha+\sin^2\alpha)} = \frac{2i\sin\alpha}{2}$$

$= i\sin\alpha$

$\Rightarrow i\tan h\, 2\phi = i\sin\alpha$

$\Rightarrow \tan h\, 2\phi = \sin\alpha$

$$\Rightarrow \frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}} = \sin\alpha$$

$$\Rightarrow \frac{e^{2\phi}+e^{-2\phi}}{e^{2\phi}-e^{-2\phi}} = \frac{1}{\sin\alpha}$$

Apply componendo and dividendo

$$\Rightarrow \frac{2e^{2\phi}}{2e^{-2\phi}} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

$$\Rightarrow e^{4\phi} = \frac{\cos^2\alpha/2+\sin^2\alpha/2+2\sin\alpha/2\cos\alpha/2}{\cos^2\alpha/2+\sin^2\alpha/2-2\sin\alpha/2\cos\alpha/2}$$

$$= \frac{(\cos\alpha/2+\sin\alpha/2)^2}{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2}$$

$$\Rightarrow e^{2\phi} = \frac{\cos\alpha/2+\sin\alpha/2}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$

$$= \frac{1+\tan\alpha/2}{1-\tan\alpha/2}$$

$$\Rightarrow e^{2\phi} = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

Take log both sides

$$\log e^{2\phi} = \log\ \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$\Rightarrow 2\phi = \log\ \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$\Rightarrow \phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

Q.11. If $\tan(A + iB) = x + iy$, prove that

(i) $x^2 + y^2 + 2x\cot 2A = 1$

(ii) $x^2 + y^2 - 2y\cot h\, 2B + 1 = 0$

Ans. $\tan(A + iB) = x + iy$ ...(1)

changing i to $(-i)$, we get

$\tan(A - iB) = x - iy$ ...(2)

(i) Consider $2A = (A + iB) + (A - iB)$

$\Rightarrow \tan 2A = \tan[(A + iB) + (A - iB)]$

$$= \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$\Rightarrow \tan 2A = \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$= \frac{2x}{1-(x^2+y^2)}$$

$$\Rightarrow \frac{1}{\cot 2A} = \frac{2x}{1-(x^2+y^2)}$$

$$\Rightarrow \cot 2A = \frac{1-(x^2+y^2)}{2x}$$

$\Rightarrow 1 - (x^2 + y^2) = 2x\cot 2A.$

$\Rightarrow x^2 + y^2 + 2x\cot 2A = 1$

(ii) Consider $2iB = (A + iB) - (A - iB)$

$\Rightarrow \tan 2iB = \tan[(A + iB) - (A - iB)]$

$$\Rightarrow \tan 2iB = \frac{\tan(A+iB)-\tan(A-iB)}{1+[\tan(A+iB)\tan(A-iB)]}$$

$$\Rightarrow \tan 2iB = \frac{x+iy-(x-iy)}{1+(x+iy)(x-iy)}$$

$$= \frac{2iy}{1+(x^2+y^2)}$$

$$\Rightarrow i\tan h\, 2B = \frac{2iy}{1+x^2+y^2}$$

$$\Rightarrow \frac{1}{\coth 2B} = \frac{2y}{1+x^2+y^2}$$

$\Rightarrow 2y \coth 2B = 1 + x^2 + y^2$

$\Rightarrow x^2 + y^2 - 2y \coth 2B + 1 = 0$

Q.12. Prove that $\cosh^{-1} x = \log (x + \sqrt{x^2-1})$

Ans. Let $\cosh^{-1} x = y$

$\Rightarrow x = \cosh y$

$$\Rightarrow x = \frac{e^y + e^{-y}}{2} = \frac{e^{2y}+1}{2e^y}$$

$\Rightarrow e^{2y} - e^y . 2x + 1 = 0$

This is quadratic equation in e^y

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2-1}$$

Taking positive sign only

$$e^y = x + \sqrt{x^2-1}$$

$$\Rightarrow y = \log (x + \sqrt{x^2-1})$$

$$\Rightarrow \cosh^{-1} x = \log (x + \sqrt{x^2-1})$$

Q.13. Prove that $\tan \left(i \log \frac{a-ib}{a+ib} \right) = \frac{2ab}{a^2-b^2}$

Ans. Let $a + ib = r (\cos \theta + i \sin \theta)$

$= re^{i\theta}$

then $a - ib = r (\cos \theta - i \sin \theta) = re^{-i\theta}$

$$\text{LHS} = \tan\left(i \log \frac{a-ib}{a+ib} \right)$$

$$= \tan\left(i \log \frac{re^{-i\theta}}{re^{i\theta}} \right)$$

$= \tan (i \log (e^{-i\theta} . e^{-i\theta}))$

$= \tan (i \log e^{-2i\theta})$

$= \tan (i(-2i\theta))$

$$= \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \qquad ...(1)$$

Since $a + ib = r \cos \theta + ri \sin \theta$

Equating real and imaginary parts, we get

$a = r \cos \theta, b = r \sin \theta$

$$\Rightarrow r = \sqrt{a^2+b^2} \text{ and } \tan \theta = \frac{b}{a}$$

By (1), we get

$$\text{LHS} = \frac{2(b/a)}{1-\frac{b^2}{a^2}} = \frac{2ab}{a^2-b^2} = \text{RHS}$$

Q.14. Prove that i^i is wholly real and find its principal value. Also show that the values of i^i form a G.P.

Ans. $i^i = e^{i \log_e i}$

$= e^{i (2n\pi i + \log_e i)}$

$= e^{i [2n\pi i + \log_e (\cos \pi/2 + i \sin \pi/2)]}$

$= e^{i (2n\pi i + \log_e i\pi/2)}$

$= e^{i (2n\pi i + i\pi/2)}$

$= e^{i^2 (2n\pi + \pi/2)}$

$= e^{-\pi (2n + 1/2)}$

$= e^{-(4n+1)\pi/2} \qquad ...(1)$

We get

$i^i = e^{-(4n+1)\pi/2}$ which is wholly real.

The principle value of i^i is $e^{-\pi/2}$ putting $n = 0,1,2,...$ successively in (1), we get the values of of i^i are $e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, e^{-13\pi/2},...$ which form a G.P. (with $r = e^{-2\pi}$).

Q.15. Find general value of log (– 3)

Ans. As $-3 = 3(-1) = 3 (\cos \pi + i \sin \pi)$

$= 3e^{i\pi}$

Now $\log (-3) = \text{Log} (3e^{i\pi})$

$= 2n\pi i + \log (3e^{i\pi})$

$= 2n\pi i + \log 3 + i\pi$

$= i (2n\pi + \pi) + \log 3.$

UNIT - IV

Two Dimensional Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = C^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

Laplace Equation in Polar Coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Q.1. Solve the differential equation $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ subject to boundary conditions

(i) u is finite when $r \to 0$

(ii) $u = \Sigma C_n \cos n\theta$ where $r = a$.

Ans. Given $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(1)$

Let (1) has a solution of the form

u(r, θ) = R(r) θ(θ) ...(2)

where R and θ are functions of r and θ respectively.

By (1), we get

$$R''\theta+\frac{1}{r}R^{1}\theta+\frac{1}{r^{2}}R\theta''=0$$

$$\Rightarrow\frac{r^{2}R''+rR'}{R}=\frac{-\theta''}{\theta} \quad ...(3)$$

Since r and θ are independent (3) is true only if each side is equ ɔ same constant.

$$\Rightarrow\frac{r^{2}R''+rR'}{R}=n^{2},\ \frac{-\theta''}{\theta}=n^{2}$$

$\Rightarrow r^2R'' + rR' - n^2 R = 0$ ...(4)

and $\theta'' + n^2\theta = 0$ ...(5)

By (1) $\frac{r^{2}d^{2}R}{d\theta^{2}}+\frac{rdR}{d\theta}-n^{2}R=0$ (Linear Hom eqⁿ)

Let $r = e^z$

$\Rightarrow z = \log r$

$\Rightarrow D \equiv r\frac{d}{dr}=\frac{d}{dz}$

$\Rightarrow [D(D-1) + D - n^2]\, R = 0$

$\Rightarrow (D^2 - n^2)R = 0$ ...(6)

AE $D^2 - n^2 = 0 \Rightarrow D = \pm n$

$\therefore$ Solⁿ of (6) is

$R = A_n e^{nz} + B_n e^{-nz}$

$\Rightarrow R = A_n r^n + B_n r^{-n}$

(5) $\Rightarrow \theta'' + n^2\theta = 0$

$\Rightarrow (D^2 \cdot\ ^{-2})\theta = 0$

AE Ɪ $^{2} = 0$

$\Rightarrow D = \pm in$

$\therefore \theta = C_n \cos n\theta + D_n \sin n\theta$

Thus complete Solⁿ is

$U_n(r, \theta) = (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta)$ ...(7)

By principal of superposition, general solution of (7) is

$$u(r,\theta)=\sum_{n}(A_{n}r^{n}+B_{n}r^{-n})(C_{n}\cos n\theta+D_{n}\sin n\theta) \quad ...(8)$$

B.C is finite when $r \to 0$.

$\therefore B_n = 0$ in (8)

Now (8) reduces to

$u(r, \theta) = \Sigma A_n r^n (C_n \cos n\theta + D_n \sin n\theta)$

$\Rightarrow u(r, \theta) = \Sigma r^n (E_n \cos n\theta + F_n \sin n\theta)$ ...(9)

where $E_n = A_nC_n$ and $F_n = A_nD_n$.

At $r = a$, $u = \Sigma C_n \cos n\theta$

$\therefore$ (9) $\Rightarrow$

$\Sigma C_n \cos n\theta = \Sigma a^n (E_n \cos n\theta + F_n \sin n\theta)$

$\Rightarrow F_n = 0$

and $a^nE_n = C_n$

$\Rightarrow E_n = \frac{C_n}{a^n}$.

$\therefore$ (9) becomes

$$u(r,\theta)=\Sigma C_{n}\left(\frac{r}{a}\right)^{n}\cos n\theta.$$

Q.2. Potential u(r, θ) in the exterior of a unit sphere satisfies the equation $\frac{\partial}{\partial r}\left(r^{2}\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0$ obtain the expression for u(r, θ) if

(i) $u(1, \theta) = \cos 2\theta$

(ii) $u(1, \theta) = \cos 3\theta - 1$

Ans. We know that potential u satisfies a Laplace equation

Since we have a spherical surface, u is supposed to satisfy

$$\frac{\partial^{2}u}{\partial r^{2}}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^{2}}\frac{\partial^{2}u}{\partial\theta^{2}}+\frac{\cot\theta}{r^{2}}\frac{\partial u}{\partial\theta}+\frac{1}{r^{2}\sin^{2}\theta}\frac{\partial^{2}u}{\partial\phi^{2}}=0$$

Due to symmetrical problem $\frac{\partial^{2}u}{\partial\phi^{2}}=0$

$$\frac{\partial^{2}u}{\partial r^{2}}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^{2}}\frac{\partial^{2}u}{\partial\theta^{2}}+\frac{\cot\theta}{r^{2}}\frac{\partial u}{\partial\theta}=0$$

$$\Rightarrow r^{2}\frac{\partial^{2}u}{\partial r^{2}}+2r\frac{\partial u}{\partial r}+\frac{\partial^{2}u}{\partial\theta^{2}}+\cot\theta\frac{\partial u}{\partial\theta}=0 \quad ...(1)$$

We require potential u(r, θ) at a point (r, θ) outside the given unit sphere. On physical grounds, potential at infinity is zero

$$\therefore \lim_{r\to\infty}u(r,\theta)=0 \quad ...(2)$$

Given $\mu(1, \theta) = \cos 2\theta$ ...(3)

Let solⁿ of (1) be

$u(r, \theta) = R(r)\, \theta(\theta)$

Put value in (1) and divide by Rθ

$$r^{2}\frac{R''}{R}+2r\frac{R'}{R}+\frac{\theta''}{\theta}+\cot\theta\frac{\theta'}{\theta}=0$$

$$\Rightarrow\frac{1}{\theta}(\theta''+\cot\theta')=\frac{-1}{R^{2}}(r^{2}R''+2rR') \quad ...(4)$$

Since r and θ are independent (4) can only be true if each side is equal to a constant say $-n(n+1)$

$$\Rightarrow r^2R'' + 2rR' - n(n+1)R = 0 \quad ...(5)$$

and $\theta'' + \cot\theta\,\theta' + n(n+1)\theta = 0 \quad ...(6)$

Putting $\mu = \cos\theta$ in (6), we get

$$\Rightarrow (1-\mu^2)\frac{d^2\theta}{d\mu^2} - 2\mu\frac{d\theta}{d\mu} + n(n+1)\theta = 0 \quad ...(7)$$

which is Legendre's equation

Solution of (5) and (7) are

$$R = A_n r^n + \frac{B_n}{r^{n+1}} \quad ...(8)$$

$$\theta(\theta) = C_n P_n \cos\theta + D_n \theta_n \cos\theta \quad ...(9)$$

If $\theta(0)$ is to be finite along polar axis $\theta = 0 \Rightarrow D_n = 0$

We take $A_n = 0$ in (8), otherwise $R(r) \to \infty$ which controdicts (2)

$$\therefore R(r) = \frac{B_n}{r^{n+1}}$$

$$\theta(\theta) = C_n P_n \cos\theta$$

$$\therefore u(r,\theta) = \frac{B_n}{r^{n+1}} . C_n P_n \cos\theta$$

$$= E_n r^{-(n+1)} P_n \cos\theta$$

with $E_n = B_n C_n$.

Consider more general solution by superposition

$$u(r,\theta) = \sum_{n=0}^{\infty} u_n(r,\theta)$$

$$= \sum_{n=0}^{\infty} E_n r^{-(n+1)} P_n \cos\theta \quad ...(10)$$

Putting r = 1 in (10) and using (3), we get

$$\cos 2\theta = \sum_{n=0}^{\infty} E_n P_n \cos\theta$$

$$\Rightarrow 2\cos^2\theta - 1 = \sum_{n=0}^{\infty} E_n P_n \cos\theta$$

$$\Rightarrow 2\mu^2 - 1 = \sum_{n=0}^{\infty} E_n P_n(\mu) \quad ...(11)$$

which is Legendre series expansion of $2\mu^2 - 1$

$$\Rightarrow E_n = \frac{2n+1}{2}\int_{-1}^{1}(2\mu^2-1)P_n(\mu)du \quad ...(12)$$

Putting n = 0 in (12) as $P_0(\mu) = 1$

$$E_0 = \frac{1}{2}\int_{-1}^{1}(2\mu^2-1)\mu d\mu = \frac{-1}{3} \quad ...(13)$$

Putting n = 1 in (12) and $P_1(\mu) = \mu$

$$E_1 = \frac{3}{2}\int_{-1}^{1}(2\mu^2-1)\mu d\mu = 0 \quad ...(14)$$

Putting n = 2 in (12) & $P_2(\mu) = \frac{3\mu^2-1}{2}$

$$E_2 = \frac{5}{2}\int_{-1}^{1}\frac{(2\mu^2-1)(3\mu^2-1)}{2}d\mu \quad ...(14\text{ A})$$

$$= \frac{5}{4}\int_{-1}^{1}(6\mu^4 - 5\mu^2 + 1)d\mu$$

$$\Rightarrow E_2 = \frac{5}{4}\left(\frac{6\mu^5}{5} - \frac{5\mu^3}{3} + \mu\right)_{-1}^{1}$$

$$= \frac{4}{3}.$$

Again $\int_{-1}^{1}\mu^m P_n(\mu)d\mu = 0$ if $m < n$ ...(15)

$$\Rightarrow \int_{-1}^{1} Pn(\mu)d\mu = 0 \text{ for } n \ge 1 \quad ...(16)$$

$P_n(\mu)$ is a polynomial of degree n ...(17)

In view of (15), (16) and (17), (12)

$E_n = 0$ for each $n \ge 3$ ...(18)

Using (13), (14), (14A), (18), (10) reduces to.

$$u(r,\theta) = \frac{E_0 P_0(\cos\theta)}{r} + \frac{E_2 P_2(\cos\theta)}{r^2}$$

$$\Rightarrow u(r,\theta) = \frac{-1}{3r} + \frac{4}{3r^3} \times \frac{3\cos^2\theta - 1}{2}$$

$$\Rightarrow u(r,\theta) = \frac{2(3\cos^2\theta - 1) - r^2}{3r^3}$$

which is required potential.

UNIT - I

Q.1. Prove that the function $f(z)$ defined by $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ $(z \neq 0)$, $f(0) = 0$ is continuous and the C-R equations are satisfied at the origin yet $f(o)$ does not exist **(IPU-2015)**

Ans.
$$f(z) = \frac{\left(x^3 - y^3\right) + i\left(x^3 + y^3\right)}{x^2 + y^2}, z \neq 0$$

Let $\quad f(z) = u + iv$

where
$$u = \frac{x^3 - y^3}{x^2 + y^2},$$
$$V = \frac{x^3 + y^3}{x^2 + y^2}$$

At $\quad z \neq 0, x \neq 0, y \neq 0$

$\therefore$ u and v are rational functions of x and y with non – zero denominators. Thus u and v and $f(z)$ being polynomials are continous at $z \neq 0$.

for $z = 0$

Let $\quad x = r\cos\theta, y = r\sin\theta$

$$\Rightarrow \quad u = \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= \frac{r^3\left(\cos^3\theta - \sin^3\theta\right)}{r^3\left(\cos^2\theta + \sin^2\theta\right)}$$
$$u = r(\cos^3\theta - \sin^3\theta)$$

Similarly $\quad v = r(\cos^3\theta + \sin^3\theta)$

when $\quad z \to 0$

$x, y \to 0$ and $r \to 0$

$$\therefore \quad \lim_{z\to 0} u = \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

similarly $\quad \lim_{z\to 0} v = 0$

$$\therefore \quad \lim_{z\to 0} f(z) = 0 = f(0)$$

$\Rightarrow f(z)$ is continuous at $z = 0$

Hence $f(z)$ is continuous for all values of z.

At origin (0,0), we have

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,o) - u(0,0)}{x} = \lim_{x\to 0}\frac{x - 0}{x} = 1$$
$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0}\frac{x - 0}{x} = 1$$
$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y) - u(o,o)}{y} = \lim_{y\to 0}\frac{-y - 0}{y} = -1$$
$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y) - v(o,o)}{y} = \lim_{y\to 0}\frac{y - 0}{y} = 1$$
$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

Hence C–R equations are satisfied at origin

Now
$$f'(0) = \lim_{z\to 0}\frac{f(z) - f(0)}{z - 0}$$
$$= \lim_{z\to 0}\frac{\left(x^3 - y^3\right) + i\left(x^3 + y^3\right) - 0}{\left(x^2 + y^2\right)(x + iy)}$$

Let $z \to 0$ along path $\quad x = y^3$

$$f'(0) = \lim_{y\to 0}\frac{\left(y^9 - y^3\right) + i\left(y^9 + y^3\right)}{\left(y^6 + y^2\right)\left(y^3 + iy\right)}$$
$$= \lim_{y\to 0}\frac{y^3\left[\left(y^6 - 1\right) + i\left(y^6 + 1\right)\right]}{y^2\left(y^4 + 1\right).y\left(y^2 + i\right)}$$
$$= \lim_{y\to 0}\frac{\left(y^6 - 1\right) + i\left(y^6 + 1\right)}{\left(y^4 + 1\right)\left(y^2 + i\right)}$$
$$= \frac{-1 + i}{1 \times i} = \frac{i - 1}{i} = \frac{i^2 - i}{i^2}$$
$$= 1 + i$$

Let $z \to 0$ along path $y = x$

$$f'(o) = \lim_{x\to 0}\frac{\left(x^3 - x^3\right) + i\left(x^3 + x^3\right)}{\left(x^2 + x^2\right)(x + ix)}$$
$$= \lim_{x\to 0}\frac{2i\,x^3}{2x^3(1+i)} = \frac{i}{1+i}\frac{(1-i)}{(1-i)}$$
$$= \frac{i - i^2}{1+1} = \frac{1+i}{2}$$

$\therefore$ since limits are different for 2 paths.

$\therefore f'(0)$ does not exist.

Q.2. Evaluate $\int_c |z|\,dz$ where c is the left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$. **(IPU-2015)**

Ans. Circle $\quad |z| = 1$

$$\therefore \quad z = e^{i\theta} \Rightarrow dz = i\,e^{i\theta}\,d\theta.$$

since left half is considered

$\therefore \theta$ varies from $\dfrac{3\pi}{2}$ to $\dfrac{\pi}{2}$

$$\therefore \quad \int_c |z|\,dz = \int_{3\pi/2}^{\pi/2} i e^{i\theta} d\theta$$

$$= \frac{i}{i}\left|e^{i\theta}\right|_{3\pi/2}^{\pi/2}$$

$$= e^{i\pi/2} \quad e^{i3\pi/2}$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$= i + i = 2i$$

Q.3. Show that the function $v(x, y) = e^x \sin y$ is harmonic. Find its conjugate harmonic and the corresponding analytic function. **(IPU-2015)**

Ans. $V = e^x \sin y$

$$\frac{\partial V}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 V}{\partial x^2} = e^x \sin y$$

$$\frac{\partial V}{\partial y} = e^x \cos y, \quad \frac{\partial^2 V}{\partial y^2} = -e^x \sin y$$

On adding $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = e^x \sin y - e^x \sin y$

$$= 0$$

$\therefore$ V is harmonic

let $f'(z) = u + iV$

Then $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial V}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$f'(z) = e^x \cos y + ie^x \sin y$$

Replace x by z and y by 0

$$f'(z) = e^z$$

On integrating, we get

$$f(z) = \int e^z\, dz$$

$$= e^z + c$$

$$f(z) = e^{x+iy} + c$$

$$= e^x . e^{iy} + c.$$

$$f(z) = e^x (\cos y + i \sin y) + c.$$

$$\therefore \quad u = e^x \cos y + c.$$

Q.4. Prove that the function sinh z is analytic and find its derivative. **(IPU-2016)**

Ans. Let $f(z) = \sinh z.$

$$= \sinh (x + iy)$$

$$= \sinh x \cos y + i \cosh x \sin y.$$

$$= u + iv$$

Here $u = \sinh x \cos y, \; v = \cosh x \sin y.$

Now, $\dfrac{\partial u}{\partial x} = \cosh x \cos y, \; \dfrac{\partial v}{\partial x} = \sinh x \sin y.$

$$\frac{\partial u}{\partial y} = -\sinh x \sin y, \; \frac{\partial v}{\partial y} = \cosh x \cos y.$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

C-R equations are satisfied.

All derivatives are continuous

Thus f(z) is analytic.

Now $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$= \cosh x \cos y + i \sinh x \sin y.$$

$$= \cosh (x + iy) = \cosh z.$$

Q.5. Obtain the Taylor series expansion of $f(z) = \dfrac{1}{z^2 + (1 + 2i)z + 2i}$ about $z = 0$. Also find its radius of convergence. **(IPU-2016)**

Ans. $f(z) = \dfrac{1}{z^2 + (1 + 2i)z + 2i} = \dfrac{1}{(z+1)(z+2i)}$

It has poles at $z = -1, \; -2i$

Now $\dfrac{1}{(z+1)(z+2i)} = \dfrac{1}{(z+1)(-1+2i)} + \dfrac{1}{(1-2i)(z+2i)}$

$$= \frac{1}{(-1+2i)}(1+z)^{-1} + \frac{1}{(1-2i)}\frac{1}{2i}\left(1 + \frac{z}{2i}\right)^{-1}$$

$$= \frac{1}{(-1+2i)}(1 - z + z^2 - z^3 + \ldots) + \frac{1}{(4+2i)}\left(1 - \frac{z}{2i} + \frac{z^2}{(2i)^2} + \ldots\right)$$

$$f(z) = \frac{1}{(-1+2i)}\sum_{n=0}^{\infty}(-1)^n z^n + \frac{1}{(4+2i)}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{(2i)^n}$$

This is taylor's series expansion.

Centre of circle is at $z = 0$ and distance of singularity are $z = -1$ and $z = -2i$ and distance from the centre are 1 and 2 respectively.

Hence, a circle is drawn with centre at $z = 0$ and radius less than unity, then with in the circle $|z| < |$.

$\therefore$ given function is analytic and radius of convergence is $|z| < |$.

Q.6. Show that the function z|z| is not analytic anywhere. **(IPU-2017)**

Ans. Let $f(z) = z\,|z|$

$$f(z) = (x + iy)\sqrt{x^2 + y^2}$$

Here $u = x\sqrt{x^2+y^2}$, $v = y\sqrt{x^2+y^2}$

Now
$$\frac{\partial u}{\partial x} = \sqrt{x^2+y^2} + \frac{x.2x}{2\sqrt{x^2+y^2}}$$
$$= \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}}$$
$$= \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$
$$\frac{\partial u}{\partial y} = \frac{x.2y}{2\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$
$$\frac{\partial v}{\partial x} = \frac{y.2x}{2\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$
$$\frac{\partial v}{\partial y} = \sqrt{x^2+y^2} + \frac{y.2y}{2\sqrt{x^2+y^2}}$$
$$= \sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}} = \frac{2y^2+x^2}{\sqrt{x^2+y^2}}$$

Now $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq \frac{-\partial v}{\partial x}$

$\therefore$ C – R equations are not satisfied

Thus $f(z) = z|z|$ is not analytic anywhere.

Q.7. Determine analytic function $f(z) = u + iv$ in terms of z, if $v = \log(x^2+y^2) + x - 2y$. **(IPU-2018)**

Ans. As
$$V = \log(x^2+y^2) + x - 2y$$
$$\Rightarrow \quad \frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1$$
$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2$$

As $w = u + iv$ is analytic, then
$$w' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \text{[by C-R equations]}$$
$$= \left(\frac{2y}{x^2+y^2} - 2\right) + i\left(\frac{2x}{x^2+y^2} + 1\right)$$

Replacing x by z and y by 0, we get
$$\frac{dw}{dz} = -2 + i\left(\frac{2z}{z^2} + 1\right)$$
$$= -2 + i\left(\frac{2}{z} + 1\right)$$
$$\Rightarrow \quad \frac{dw}{dz} = (i-2) + \frac{2i}{z}$$

On integrating w.r.t 'z', we get
$$w = (i-2)z + 2i\log z + c.$$

Q.8. If $f(\alpha) = \int_c \frac{3z^2+7z+1}{z-\alpha}\, dz$, where c is the circle $x^2+y^2 = 4$, find the value of $f(3)$, $f'(1-i)$ and $f''(1-i)$.

Ans. Given circle is $x^2+y^2 = 4$ or $|z| = 2$

The point $z = 3$ lies outside the circle $|z| = 2$

while $z = 1 - i$ i.e. $(1, -1)$ lie inside the circle $|z| = 2$.

Now $f(3) = \oint_c \frac{3z^2+7z+1}{z-3}dz$ and $\frac{3z^2+7z+1}{z-3}$ is analytic everywhere within c.

$\therefore$ By Cauchy integral theorem
$$f(3) = \oint_c \frac{3z^2+7z+1}{z-3}dz = 0$$
$$\Rightarrow \quad f(3) = 0$$

Now, let $\phi(z) = 3z^2 + 7z + 1$ which is analytic any where

$\therefore$ By Cauchy integral formula
$$\phi(\alpha) = \frac{1}{2\pi i}\oint_c \frac{3z^2+7z+1}{z-\alpha}dz \text{, a is a point within c.}$$
$$\Rightarrow \quad 2\pi i\,\phi(\alpha) = \oint_c \frac{3z^2+7z+1}{z-\alpha}dz = f(\alpha)$$
$$\Rightarrow \quad f(a) = 2\pi i(3\alpha^2 + 7\alpha + 1)$$
$$\Rightarrow \quad f'(\alpha) = 2\pi i(6\alpha + 7)$$

Now
$$f'(1-i) = 2\pi i[6(1-i) + 7]$$
$$= 2\pi i(6 - 6i + 7) = 2\pi(13i + 6)$$
$$f''(a) = 2\pi i \times 6$$
$$f''(1-i) = 12\pi i.$$

Q.9. Show that the limit of the function $f(z) = \frac{\text{Re}(z)}{|z|}$, $z \neq 0$ and $f(z) = 0$, $z = 0$ as $z \to 0$ does not exist. **(IPU-2019)**

Ans. To show $\lim_{z\to 0}\frac{\text{Re}(z)}{|z|}$ does not exist

Let
$$z = x + iy$$
$$\Rightarrow \quad f(z) = \lim_{z\to 0}\frac{x}{\sqrt{x^2+y^2}}$$

Let $z \to 0$ along path $y = mx$
$$\Rightarrow \quad \lim_{z\to 0}\frac{\text{Re}(z)}{|z|} = \lim_{\to}\frac{}{\sqrt{x^2 \quad m^2x^2}}$$

$$= \lim_{x\to 0}\frac{1}{\sqrt{1+m^2}} = \frac{1}{\sqrt{1+m^2}}$$

Since its value depends on m

$\therefore$ For different values of m, we have different limits.

Thus, limit does not exist.

Q.10. Prove that the function $e^x(\cos y + i \sin y)$ is analytic and find its derivative. **(IPU-2019)**

Ans. Let $f(z) = e^x(\cos y + i\sin y)$

Here, $u = e^x \cos y, v = -e^x \sin y.$

$$\frac{\partial u}{\partial x} = e^x \cos y, \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y; \frac{\partial v}{\partial y} = e^x \cos y$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ C – R equations are satisfied. Since e^x, $\cos y$, $\sin y$ are continuous functions:

$\therefore$ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous function satisfying C – R equations.

Hence $f(z)$ is analytic every where

Now $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i\, e^x \sin y$$

$$= e^x(\cos y + i \sin y) = e^x.e^{iy} = e^{x+iy}.$$

Q.11. Determine the analytic function $w = u + iv$, if $v = \log(x^2 + y^2) + x - 2y$. **(IPU-2019)**

Ans. $$v = \log(x^2+y^2) + x - 2y.$$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2$$

As $w = u + iv$ is analytic

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \left[\frac{2y}{x^2+y^2} - 2\right] + i\left[\frac{2x}{x^2+y^2} + 1\right]$$

Replacing x by z and y by 0

$$f'(z) = -2 + i\left(\frac{2z}{z^2} + 1\right) = -2 + i\left(\frac{2}{z} + 1\right)$$

$$f'(z) = (i-2) + \frac{2i}{z}$$

On integrating we get

$$f(z) = (i-2)z + 2i\log z + c.$$

Q.1. Evaluate $\int_c \frac{\sin^2 z}{\left(z - \frac{\pi}{6}\right)^3} dz$, where c is the circle $|z| = 1$. **(IPU-2015)**

Ans. It has singularities at $z = \frac{\pi}{6}$ of order 3. It lies within the circle $|z| = 1$

$$\therefore \quad f^n(a) = \frac{n!}{2\pi i}\int_{c_0} \frac{f(z)}{(z-a)^{n+1}} dz$$

Here $$a = \frac{\pi}{6}, n = 2, f(z) = \sin^2 z$$

$$f'(z) = 2\sin z \cos z$$

$$f''(z) = 2[\cos^2 z - \sin^2 z] = 2\cos 2z$$

Now $$f''\left(\frac{\pi}{6}\right) = \frac{2!}{2\pi i}\int_c \frac{\sin^2 z}{(z-\pi/6)^3} dz$$

$$1 = \frac{1}{\pi i}\int_c \frac{\sin^2 z}{(z-\pi/_6)^3} dz$$

$$\Rightarrow \quad \int \frac{\sin^2 z}{(z-\pi/_6)^3} dz = \pi i$$

Q.2. Obtain the residue of $f(z) = \frac{(z+1)^3}{(z-1)^3}$ at its pole. **(IPU-2015)**

Ans. $$f(z) = \frac{(z+1)^3}{(z-1)^3}$$

It has poles at $z = 1$ of order 3.

$$\text{Res}\{f(z), 1\} = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left\{(z-1)^3\frac{(z+1)^3}{(z-1)^3}\right\}$$

$$= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left\{(z+1)^3\right\}$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left[3(z+1)^2\right]$$

$$= \frac{1}{2}\lim_{z\to 1} 6(z+1) = \frac{1}{2}\times 12 = 6$$

Q.3. Show that when $|z+1|<1$, $\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n$ **(IPU-2015)**

Ans. $$f(z) = \frac{1}{z^2} = \frac{1}{[(z+1)-1]^2} = \frac{1}{[1-(z-1)]^2} \text{ as } |z+1| < 1$$

$$\therefore \quad f(z) = [1-(z+1)]^{-2}$$

$$= 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + \ldots..$$

$$= 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$$

Q.4. Find the image of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Draw a sketch of the transformed region. **(IPU-2015)**

Ans. Let
$$w = \frac{1}{z}$$

or
$$w = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$u + iv = \frac{x-iy}{x^2+y^2}$$

Equating real and imaginary parts

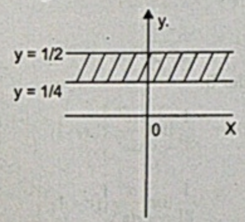
$$u = \frac{x}{x^2+y^2} \quad \ldots(1)$$

$$v = \frac{-y}{x^2+y^2} \quad \ldots(2)$$

On solving (1) and (2), we get

$$y = \frac{-v}{u^2+v^2} \quad \ldots(3)$$

If $y < \frac{1}{2}$, then



(3) becomes
$$\frac{-v}{u^2+v^2} < \frac{1}{z}$$

or $u^2 + v^2 + 2v > 0$

or $u^2 + (v+1)^2 = 1.$

Which represents outer portion of the circle with centre $(0, -1)$ and radius 1.

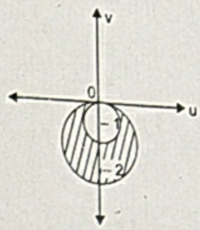
If $y > \frac{1}{4}$, then (3) becomes

$$\frac{-v}{u^2+v^2} > \frac{1}{4} \text{ or}$$

$$u^2 + v^2 + 4v < 0$$

or $u^2 + (v+2)^2 < 2^2$

Which represents the inner portion of circle with centre $(0, -2)$ and radius 2. Hence, the image of the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is the shaded portion in the figure.



Q.5. Evaluate the integral $\oint_c \frac{dz}{(z-1)(z-2)(z-3)}$, c: $|z| = 4$ **(IPU-2015)**

Ans. It has singularity at $z = 1, 2, 3$ which lies inside circle $|z| = 4$.

Res of f(z) at $z = 1$ is

$$\lim_{z\to1}(z-1)f(z) = \lim_{z\to1}(z-1)\frac{1}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z\to1}\frac{1}{(z-2)(z-3)} = \frac{1}{2}$$

$$\text{Res}\{f(z), z=2\} = \lim_{z\to2}(z-2)f(z)$$

$$= \lim_{z\to2}(z-2)\frac{1}{(z-1)(z-2)(z-3)}$$

$$= \frac{1}{-1} = -1$$

$$\text{Res}\{f(z), z=3\} = \lim_{z\to3}(z-3)f(z)$$

$$= \lim_{z\to3}(z-3)\frac{1}{(z-1)(z-2)(z-3)}$$

$$= \frac{1}{2}$$

$\therefore$ By residue theorem

$$\oint_c \frac{dz}{(z-1)(z-2)(z-3)} = 2\pi i\left(\frac{1}{2} - 1 + \frac{1}{2}\right)$$

$$= 0.$$

Q.6. Apply residue theorem to evaluate the integral $\int_0^{2\pi} \frac{d\theta}{2-\sin\theta}$. **(IPU-2015)**

Ans. Put
$$z = e^{i\theta}, \text{ so that}$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \text{ and } d\theta = \frac{dz}{iz}$$

Then $$\int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \oint_c \frac{1}{2-\frac{1}{2i}\left(z-\frac{1}{z}\right)} \frac{dz}{iz}, \ |z| = 1$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \oint_c \frac{1}{2+\frac{iz}{2}-\frac{i}{2z}} \frac{dz}{iz}$$

$$= \oint_c \frac{dz}{\left(\frac{4z+iz^2-i}{2z}\right)} \frac{1}{iz}$$

$$= 2\oint_c \frac{dz}{4iz-z^2+1}$$

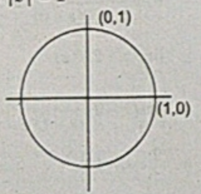
Poles are $$z^2-4iz-1 = 0.$$

$$z = \frac{4i \pm \sqrt{16i^2+4}}{2}$$

$$= \frac{4i \pm 2\sqrt{3i}}{2}$$

$$= 2i \pm \sqrt{3i}$$

$i\left(2-\sqrt{3}\right)$ lie inside the circle $|z| = 1$



Res $\left\{f(z), z = i\left(2-\sqrt{3}\right)\right\}$

$$= \lim_{z \to i(2-\sqrt{3})} \left[z - i\left(2-\sqrt{3}\right)\right] \frac{(-2)}{z^2-4iz-1}$$

$$= \lim_{z \to i(2-\sqrt{3})} \frac{-2}{z-i(2+\sqrt{3})}$$

$$= \frac{-2}{2i-i\sqrt{3}-2i-\sqrt{3i}} = \frac{-2}{-2i\sqrt{3}} = \frac{1}{i\sqrt{3}}$$

$$\Rightarrow \oint_c \frac{-2}{z^2-4iz-1}\,dz = \frac{2\pi i}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \frac{2\pi}{\sqrt{3}}$$

Q.7. Evaluate: $\oint_c \frac{e^z dz}{(z-1)(z+3)^2}$ **where C: $|z| = 10$.** (IPU-2016)

Ans. $$f(z) = \oint_c \frac{e^z}{(z-1)(z+3)^2}$$

It has simple poles at $z = 1$ and pole or order 2 at $z = -3$.
Both singularities lies inside the circle $|z|=10$
By residue theorem

$$\oint_c \frac{e^z dz}{(z-1)(z+3)^2}\,dz = 2\pi i \times \text{sum of residues.} \qquad ...(1)$$

$$\text{Res}\{f(z), 1\} = \lim_{z\to 1}(z-1)\frac{e^z}{(z-1)(z+3)^2} = \frac{e}{16}$$

$$\text{Res}\{f(z), -3\} = \lim_{z\to -3} \frac{d}{dz}\left[(z+3)^2 \cdot \frac{e^z}{(z-1)(z+3)^2}\right]$$

$$= \lim_{z\to -3} \frac{d}{dz}\left(\frac{e^z}{z-1}\right)$$

$$= \lim_{z\to -3} \left[\frac{e^z(z-1)-e^z}{(z-1)^2}\right]$$

$$= \frac{e^{-3}(-4)-e^{-3}}{16} = \frac{-4e^{-3}-e^{-3}}{16} = \frac{-5e^{-3}}{16}$$

Now $$\oint_c \frac{e^z}{(z-1)(z+3)^2}\,dz = \frac{2\pi i}{16}(e-5e^{-3})$$

Q.8. Show that the transformation $w = z+\frac{a^2-b^2}{4z}$ transforms the circle of radius $\frac{a+b}{2}$, centre at the origin, in the z-plane into an ellipse or semi axis a and b in the w-plane. (IPU-2016)

Ans. Here $$w = z+\frac{a^2-b^2}{4z}$$

or $$u+iV = x+iy+\frac{a^2-b^2}{4(x+iy)}$$

$$= (x+iy)+\frac{(a^2-b^2)(x-iy)}{4(x+iy)(x-iy)}$$

$$u+iV = (x+iy)+\frac{(a^2-b^2)(x-iy)}{4(x^2+y^2)} \qquad ...(A)$$

But given that $$x^2+y^2 = \frac{(a+b)^2}{4}$$

$\therefore$ (A) becomes $u+iV\ (x+iy)+\frac{(a^2-b^2)(x-iy)}{(a+b)^2}$

Comparing real and imaginery parts.

$$u = x + \frac{(a^2+b^2)x}{(a+b)^2},\quad V = y - \frac{(a^2+b^2)y}{(a+b)^2}$$

$$\Rightarrow \qquad x = \frac{u(a+b)^2}{(a+b)^2+(a^2-b^2)} \text{ and } y = \frac{V(a+b)^2}{(a+b)^2-(a^2-b^2)}$$

Therefore image of $x^2+y^2 = \frac{(a+b)^2}{4}$ is

$$\frac{u^2(a+b)^4}{(a^2+b^2+2ab+a^2-b^2)^2} + \frac{V^2(a+b)^4}{(a^2+b^2-a^2+b^2+2ab)^2} = \frac{(a+b)^2}{4}$$

$$\Rightarrow \qquad \frac{u^2(a+b)^2}{(2a^2+2ab)^2} + \frac{V^2(a+b)^2}{(2b^2+2ab)^2} = \frac{1}{4}$$

$$\Rightarrow \qquad \frac{u^2(a+b)^2}{4a^2(a+b)^2} + \frac{V^2(a+b)^2}{4b^2(a+b)^2} = \frac{1}{4}$$

$$\Rightarrow \qquad \frac{u^2}{4a^2} + \frac{V^2}{b^2} = 1$$

Which is an ellipse with a and b as semi axis.

Q.9. Evaluate the integral $I = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$ using the residue calculas. (IPU-20

Ans. $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$

put $\qquad z = e^{i\theta},\ \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$

and $\qquad d\theta = \frac{dz}{iz}$

$$\Rightarrow \qquad \int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \oint_C \frac{1}{2+\frac{1}{2i}\left(z-\frac{1}{z}\right)}\frac{dz}{iz},\ C: |z| = 1$$

$$= \oint_C \frac{1}{\left(2+\frac{z^2-1}{2iz}\right)}\frac{dz}{iz}$$

$$= \oint_C \frac{2iz}{4iz+z^2-1}\cdot\frac{dz}{iz}$$

$$= 2\oint_C \frac{1}{z^2+4iz-1}dz,\ C: |z|=1$$

For poles $\qquad = z^2+4iz-1 = 0$

$$\Rightarrow \qquad z = \frac{-4i \pm \sqrt{-16+4}}{2}$$

$$= \frac{-4i \pm 2\sqrt{-3}}{2} = (-2 \pm \sqrt{3})i$$

Only $(-2+\sqrt{3})i$ lies inside circle

residue at $z = \alpha$.

$$\alpha = (-2+\sqrt{3})i,\ \beta = (-2-\sqrt{3})i$$

$$\text{Res}\{f(z),\alpha\} = \lim_{z\to\alpha}(z-\alpha)\frac{2}{(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha}\frac{2}{z-\beta}$$

$$\frac{2}{\alpha-\beta} = \frac{2}{(-2+\sqrt{3})i-(-2-\sqrt{3})i}$$

$$= \frac{2}{2\sqrt{3}i} = \frac{1}{\sqrt{3}i}$$

$\therefore$ By residue theorem

$$\oint_C \frac{2}{z^2+uiz-1}dz = \int_0^{2\pi}\frac{d\theta}{2+\sin\theta} = 2\pi i \times \frac{1}{\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}$$

Q.10. Find the image of $|z| = 1$ under the transformation $w = \frac{i-z}{i+z}$, onto the w-plane. (IPU-2017)

Ans.

$$w = \frac{i-z}{i+z}$$

$$iw + wz = i - z.$$

$$wz + z = i - iw$$

$$z = \frac{i(1-w)}{1+w}$$

As given $|z| = 1$

$$\Rightarrow \qquad \left|\frac{i(1-w)}{1+w}\right| = 1$$

$$\Rightarrow \qquad |i(1-w)| = |1+w|$$

$$\Rightarrow \qquad |i(1-u-iv)| = |1+u+iv|$$

$$\Rightarrow \qquad |i|\,|(1-u)-iv| = |(1+u)+iv|$$

$$\Rightarrow \qquad (1-u)^2+v^2 = (1+u)^2+v^2$$

$$\Rightarrow \qquad 1+u^2-2u = 1+u^2+2u$$

$$\Rightarrow \qquad 4u = 0$$

$$\Rightarrow \qquad u = 0$$

Thus image of $|z| = 1$ in z plane gives $u = 0$ i.e. imaginary axis in w-plane.

Q.11. Expand $f(z) = \frac{1}{(z-1)(z-2)}$, $1 < |z| < 2$. (IPU-2017)

Ans. Consider
$$f(z) = \frac{1}{(z-1)(z-2)}$$
$$= \frac{1}{z-2} - \frac{1}{z-1}$$

As $1 < |z| < 2$

$|z| > 1$ and $|z| < 2$

$$\frac{1}{|z|} < 1 \text{ and } \frac{|z|}{2} < 1$$

$$f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$
$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$
$$= \frac{-1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right)$$

This is a Laurent's series.

Q.12. Using contour integration in complex plane evaluate. **(IPU-2017)**

$$\int_0^{\pi} \frac{d\theta}{3+2\cos\theta}$$

Ans. Let
$$I = \int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} \quad \ldots(1)$$

Let
$$z = e^{i\theta}, \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right), d\theta = \frac{dz}{iz}$$

Consider
$$\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \oint_c \frac{1}{3+\frac{2}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}, C: |z| = 1 = \oint_c \frac{1}{3z+z^2+1}\frac{dz}{i}$$
$$= \frac{1}{i}\oint_c \frac{dz}{z^2+3z+1}$$

Now for poles $z^2 + 3z + 1 = 0$.

$$z = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$$

Let
$$\alpha = \frac{-3+\sqrt{5}}{2}, b = \frac{-3-\sqrt{5}}{2}$$

for $|z| = 1$ only $z = \theta$ lies inside the circle.

$$\text{Res }\{f(z), \theta\} = \lim_{z\to\alpha}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}$$
$$= \lim_{z\to\alpha}\frac{1}{z-\beta}$$
$$= \frac{1}{\alpha-\beta}$$
$$= \frac{2}{-3+\sqrt{5}-(-3-\sqrt{5})}$$
$$= \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$\therefore$ By residue theorem

$$\frac{1}{i}\oint_c \frac{dz}{z^2+3z+1} = 2\pi i \times \frac{1}{i}\cdot\frac{1}{\sqrt{5}} = \frac{2\pi}{\sqrt{5}}$$

By eqn (1)

$$\int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\times\frac{2\pi}{\sqrt{5}} = \frac{\pi}{\sqrt{5}}$$

Q.13. Under the transformation $w = \frac{1}{2}$, $z = 0$, find the image of $|z - 2i| = 2$. **(IPU-2018)**

Ans. Given
$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

To find image of $|z - 2i| = 2$ under transformation. $z = \frac{1}{w}$

$$\Rightarrow \left|\frac{1}{w} - 2i\right| = 2.$$
$$\Rightarrow \left|\frac{1}{u+iv} - 2i\right| = 2$$
$$\Rightarrow |1 - 2i(u+iv)| = 2|u+iv|$$
$$\Rightarrow |(1+2v) - 2iu| = 2|u+iv|$$

On squaring both sides

$$\Rightarrow (1+2v)^2 + 4u^2 = 4(u^2+v^2)$$
$$\Rightarrow 1 + 4v^2 + 4v + 4u^2 = 4u^2 + 4v^2$$
$$\Rightarrow 1 + 4v = 0$$

Thus image of circle $|z - 2i| = 2$ is a straight line $1 + 4v$ in w plane.

Q.14. Prove that if $a > 0$, then $\int_0^{\infty} \frac{1}{x^4+a^4}dx = \frac{\pi\sqrt{2}}{2a^3}$. **(IPU-2018)**

Ans. Given $\int_0^{\infty} \frac{dx}{x^4+a^4}, a > 0$

Let
$$\phi(z) = \frac{1}{z^4+a^4}$$

Poles of $\phi(z)$ are $z^4 + a^4 = 0$

$$\Rightarrow z^4 = -a^4 = a^4(-1)$$

$$z = a(-1)^{1/4} = a(\cos\pi + i\sin\pi)^{1/4}$$

$\Rightarrow$

By De moivre's theorem

$$z = a\left[\cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}\right], n = 0,1,2,3$$

When $$n = 0, z = a\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)a$$

$$= ae^{i\pi/4}$$

When $$n = 1, z = a\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = a\,e^{3i\pi/4}$$

When $$n = 2, z = a\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = a\,e^{5i\pi/4}$$

When $$n = 3, z = a\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = a\,e^{i7\pi/4}$$

Only $z = ae^{i\pi/4}$ and $ae^{i3\pi/4}$ lies in upper half of z – plane

$\therefore$ Residue of $\phi(z)$ at $z = ae^{i\pi/4}$

Res $$\left\{\phi(z), ae^{i\pi/4}\right\} = \lim_{z\to ae^{i\pi/4}}\left(z - ae^{i\pi/4}\right)\frac{1}{z^4 + a^2}\left(\frac{0}{0}\right)$$

$$= \lim_{z\to ae^{i\pi/4}}\frac{1}{4z^3}$$

$$= \frac{1}{4a^3e^{i3\pi/4}}$$

Res $$\left[\phi(z), a\,e^{i3\pi/4}\right] = \lim_{z\to a/e^{i3\pi/4}}\left(z - a\,e^{i3\pi/4}\right)\frac{1}{z^4 + a^4}\left(\frac{0}{0}\right)$$

$$= \lim_{z\to a/e^{i3\pi/4}}\frac{1}{4z^3}$$

$$= \frac{1}{4a^3e^{i9\pi/4}}$$

By Cauchy Residue theorem.

$$\oint_c \phi(z)\,dz = 2\pi i\left[\frac{1}{4a^3e^{3i\pi/4}} + \frac{1}{4a^3e^{i9\pi/4}}\right]$$

$$\Rightarrow \int_{-R}^{R}\phi(x)\,dx + \int_{C_R}\phi(z)\,dz = \frac{\pi i}{2a^3}\left[e^{-3i\pi/4} + e^{i9\pi/4}\right]$$

Consider $$\left|\int_{C_R}\phi(z)\,dz\right| < \left|\int_{C_R}\frac{|dz|}{|z^4 + a^4|}\right|$$

Let $$z = Re^{i\theta}, |dz| = |Ri\,e^{i\theta}\,d\theta| = R\,d\theta.$$

$$\Rightarrow \left|\int_{C_R}\frac{dz}{z^4 + a^4}\right| \le \int_0^{\pi}\frac{1}{R^4 + 1}d\theta.R$$

$$\le \frac{R}{R^4 + 1}\int_0^{\pi}d\theta. \le \frac{\pi R}{R^4 + 1}$$

$$\to 0 \text{ as } R \to \infty.$$

By (1), we have

$$\int_0^{\infty}\frac{dx}{x^4 + a^4} = \frac{\pi i}{2a^3}\left[e^{-3i\pi/4} + e - {}^{9i\pi/4}\right]$$

$$= \frac{\pi i}{2a^3}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{-i}{\sqrt{2}}\right] = \frac{\pi i}{2a^3}\left(\frac{-2i}{\sqrt{2}}\right)$$

$$= \frac{\pi}{\sqrt{2}a^3} = \frac{\pi\sqrt{2}}{2a^3}$$

$$= \int_0^{\infty}\frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{2a^3}$$

Q.15. Find the bilinear transformation which maps 1, i, –1 to i, 0, – 1, respectively. Also find invariant point of this transformation. (IPU-2019)

Ans. Let $z_1 = 1, z_2 = i, z_3 = -1, z_4 = z$ and $w_1 = i, w_2 = 0, w_3 = -i, w_4 = w$.

By cross ratio

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}$$

$$\frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{i(-1-w)}{((i-w)-i)}$$

$$\frac{(1-i)(1+z)}{(1+i)(1-z)} = \frac{w+i}{i-w}$$

$$\frac{(1-i)(1+z)}{(1+i)(z-1)} = \frac{w+i}{w-i}$$

By C and D

$$\frac{(1-i)(1+z)}{(1+i)(z-1)} = \frac{(w+i)+w-i}{w+i-w+i}$$

$$\frac{2z-2i}{2-2iz} = \frac{2w}{2i}$$

$$\frac{w}{i} = \frac{z-i}{1-iz}$$

$$w - i\,wz = iz + 1$$

$$w = \frac{iz+1}{1-iz}$$

Put $$w = z$$

$$z = \frac{iz+1}{1-iz}$$

$$z - iz^2 - iz - 1 = 0$$
$$iz^2 + iz - z + 1 = 0$$
$$iz^2 + z(i-1) + 1 = 0$$

Roots are

$$z = \frac{(1-i) \pm \sqrt{(i-1)^2 - 4i}}{2i} = \frac{(1-i) \pm \sqrt{-6i}}{2i} = \frac{(1+i) + \sqrt{6i}}{-2}$$

(IPU-201

Q.16. Expand the function.

$$f(z) = \frac{1}{z^2 - 4z + 3}, \text{ for } 1 < |z| < 3$$

Ans. Here $f(z) = \frac{1}{z^2 - 4z + 3} = \frac{1}{(z-3)(z-1)}$

$$\frac{1}{(z-3)(z-1)} = \frac{A}{z-3} + \frac{B}{z-1}$$

$\Rightarrow$ $1 = A(z-1) + B(z-3)$

for $z = 1, 1 = -2B \Rightarrow B = \frac{-1}{2}$

and $z = 3, 1 = 2A \Rightarrow A = \frac{1}{2}$

$\therefore$ $\frac{1}{(z-3)(z-1)} = \frac{1}{2(z-3)} - \frac{1}{2(z-1)}$

for $1 < |z| < 3$

$\Rightarrow$ $\frac{1}{|z|} < 1$ and $\frac{|z|}{3} < 1$

$\therefore$

$$f(z) = \frac{1}{-6\left(1 - \frac{z}{3}\right)} - \frac{1}{2z\left(1 - \frac{1}{z}\right)}$$

$$= -\frac{1}{6}\left(1 - \frac{z}{3}\right)^{-1} - \frac{1}{2z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{6}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \ldots\right) - \frac{1}{2z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)$$

$$= -\frac{1}{6}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \ldots\right) - \frac{1}{2}\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right)$$

Q.17. Apply calculus of residues to prove that: (IPU-20

$$\int_0^{2\pi} \frac{1}{1 - 2a\sin\theta + a^2}\, d\theta = \frac{2\pi}{1-a^2}, \; (0 < a < 1)$$

Ans. Put $z = e^{i\theta}$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \text{ and } d\theta = \frac{dz}{iz}$$

$\Rightarrow$

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\sin\theta + a^2} = \oint_C \frac{1}{1 - 2a\frac{1}{2i}\left[z - \frac{1}{z}\right] + a^2} \cdot \frac{dz}{iz}$$

$$= \oint_C \frac{dz}{iz\left(1 - \frac{az}{i} + \frac{a}{iz} + a^2\right)}$$

$$= \oint \frac{dz}{iz\left(\frac{z - aiz^2 - ai + a^2 z}{z}\right)}$$

$$= \oint_C \frac{dz}{iz - az^2 + a + ia^2 z}$$

$$= -\oint_C \frac{dz}{az^2 - a - ia^2 z - iz}$$

$$= -\oint_C \frac{dz}{az(z - ia) - i(z - ia)}$$

$$= -\oint_C \frac{dz}{(z - ia)(az - i)}; \text{C} : |z| \; |z|$$

It has simple poles $(a)z = \frac{i}{a}$ and $z = ia$

As $0 < a < 1$; & $|z| = 1$;

$\therefore z = ia$ less inside the circle C.

$$\text{Res. } \{f(z), ia\} = \operatorname*{Lt}_{z \to ia} (z - ai) - \frac{1}{(z - ia)(az - i)}$$

$$= \frac{-1}{ia^2 - i} = \frac{-1}{i[a^2 - 1]}$$

$\therefore$

$$\oint_C \frac{-dz}{(az - i)(z - ia)} = 2\pi i \cdot \frac{-1}{i(a^2 - 1)}$$

$$= \text{——}$$

$\Rightarrow$

$$\oint_C \frac{d\theta}{1 - 20\sin\theta + a^2} = \frac{2\pi}{1 - a^2}$$

Q.1. Define impulse function and periodic functions. Also write their Laplace Transform. (IPU-2015)

Ans. Unit impulse function $\delta(t-a)$ is defined as

$$\delta(t-a) = \begin{cases} \infty, & if \quad t = a \\ 0 & if \quad t \neq a \end{cases}$$

Such that $\int_0^{\infty} \delta(t-a)dt = 1$

Laplace transform of Impulse function

$$L\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\delta(t-a)dt = e^{-sa}$$

Periodic function:

$f(t)$ is said to be a periodic function with period T if $f(t+T) = f(t)$, then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt.$$

Q.2. Find inverse Laplace Transform of $\cot^{-1}\left(\frac{s}{2}\right)$. (IPU-2015)

Ans. Let $f(t) = L^{-1}\left\{\cot^{-1}\left(\frac{s}{2}\right)\right\}$

Now

$$t f(t) = L^{-1}\left\{-\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right\}$$

$$= -L^{-1}\left\{\frac{-1}{1+\frac{s^2}{2^2}}\cdot\frac{1}{2}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{4}{s^2+4}\right\}$$

$$= \frac{1}{2}\sin 2t$$

$$\Rightarrow \quad f(t) = \frac{1}{2t}\sin 2t.$$

Q.3. Solve the differential equation by laplace transform $y''' + 2y'' - y' - 2y = 0$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$. (IPU-2015)

Ans. $y''' + 2y'' - y' - 2y = 0$

Taking laplace both sides, we get $s^3\bar{y} - s^2y(0) - sy'(0) - y''(0) + 2\left\{s^2\bar{y} - sy(0) - y'(0)\right\}$

$$-\{s\bar{y} - y(0)\} - 2\bar{y} = 0$$

$$\Rightarrow \quad s^3\bar{y} - 6 + 2\left\{s^2\bar{y}\right\} - s\bar{y} - 2\bar{y} = 0$$

$$\Rightarrow \quad \left(s^3 + 2s^2 - s - 2\right)\bar{y} = 6$$

$$\Rightarrow \quad \bar{y} = \frac{6}{s^3+2s^2-s-2}$$

$$\Rightarrow \quad \bar{y} = \frac{6}{(s-1)\left(s^2+3s+2\right)}$$

$$\Rightarrow \quad \bar{y} = \frac{6}{(s-1)(s+1)(s+2)}$$

Taking inverse laplace, both sides

$$L^{-1}\{\bar{y}\} = L^{-1}\left\{\frac{6}{(s-1)(s+1)(s+2)}\right\}$$

By partial fraction.

$$\frac{6}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{c}{s+2}$$

$$\Rightarrow \quad 6 = A(s+1)(s+2) + B(s-1)(s+2) + c(s-1)(s+1)$$

for s = 1 $\quad 6A = 6 \Rightarrow A = 1$

for s = −1 $\quad -2B = 6 \Rightarrow B = -3$

for s = −2 $\quad 3c = 6$

$$\Rightarrow \quad c = 2$$

$$\therefore \quad L^{-1}\{\bar{y}\} = L^{-1}\left\{\frac{1}{s-1}\right\} - 3L^{-1}\left\{\frac{1}{s+1}\right\} + 2L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\Rightarrow \quad y = e^t - 3e^{-t} + 2e^{-2t}$$

Q.4. Find inverse laplace transform of $\frac{s^2}{s^4-a^4}$. (IPU-2015)

Ans. Let $\frac{s^2}{s^4-a^4} = \frac{s^2}{\left(s^2+a^2\right)-\left(s^2-a^2\right)}$

Let $\bar{f}(s) = \frac{s}{s^2+a^2}, \bar{g}(s) = \frac{s}{s^2-a^2}$

$$L^{-1}\bar{f}(s) = f(t) = \cos at$$

$$L^{-1}\bar{g}(s) = g(t) = \cos h\, at$$

By convolution theorem

$$L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t \cos au \cos h\, a(t-u)du$$

$$= \int_0^t \cos au\left[\frac{e^{at-au}+e^{au-at}}{2}\right]du$$

$$= \frac{e^{at}}{2}\int_0^t \cos au\, e^{-au}du + \frac{e^{-at}}{2}\int_0^t e^{au}\cos au\, du$$

$$= \frac{e^{at}}{2}\left[\frac{e^{-au}}{2a^2}(-a\cos au + a\sin au)\right]_0^t + \frac{e^{-at}}{2}\left[\frac{e^{au}}{2a^2}(a\cos au + a\sin au)\right]_0^t$$

$$= \frac{e^{at}}{4a^2}\left[e^{-at}(-a\cos at + a\sin at) + a\right] + e^{-at/4a^2}\left[e^{at}(a\cos at + a\sin at) - a\right]$$

$$= \frac{1}{4a}[\sin at - \cos at + e^{at}] + \frac{1}{4a}\left[\cos at + \sin at - e^{-at}\right]$$

$$= \frac{1}{4a}\left[2\sin at + e^{at} - e^{-at}\right]$$

$$= \frac{1}{2a}\sin at + \frac{1}{2a}\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2a}\sin at + \frac{1}{2a}\sin hat$$

Q.5. Find **(IPU-2015)**

(i) $L\left[\frac{\cos at - \cos bt}{t}\right]$ **(ii) $L^{-1}\left[\frac{s+2}{s^2-4s+13}\right]$**

Ans. (i) Here $f(t) = \frac{\cos at - \cos bt}{t}$

$$\therefore \quad L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

Now $L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^{\infty}$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

(ii) $\frac{s+2}{s^2-4s+13} = \frac{s-2+4}{(s-2)^2+3^2}$

$$= \frac{s-2}{(s-2)^2+3^2} + \frac{4}{(s-2)^2+3^2}$$

Now $L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + L^{-1}\left\{\frac{4}{(s-2)^2+3^2}\right\}$

$$= e^{2t}L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + 4e^{2t}L^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t.$$

Q.6. State Laplace Convolution theorem and apply it to find $L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+9)}\right\}$. **(IPU-2015)**

Ans. Convolution theorem.

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$,

then $L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t f(u)g(t-u)du = f - g.$

consider $L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+9)}\right\}$.

Let $L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t = f(t)$

$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t = g(t)$

By convolution theorem.

$$L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+9)}\right\} = \int_0^t \cos u \cos(3t-3u)du$$

$$= \int_0^t 2\cos u\cos(3t-3u)du$$

$$= \frac{1}{2}\int_0^t[\cos(u+3t-3u) + \cos(u-3t+3u)]du$$

$$= \frac{1}{2}\int_0^t \cos[(-2)u+3t] + \cos[4u-3t]du$$

$$= \frac{1}{2}\left[\frac{\sin(3t-2u)}{-2} + \frac{\sin(4u-3t)}{4}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin t}{-2} + \frac{\sin t}{4} - \frac{\sin(3t)}{-2} - \frac{\sin(-3t)}{4}\right]$$

$$= \frac{1}{2}\left[\frac{-2\sin t + \sin t + 2\sin 3t + \sin 3t}{4}\right]$$

$$= \frac{3\sin 3t - \sin t}{8}$$

Q.7. Solve the b.v.p. $y(t) + y''(t) = 6\cos 2t$ with $y = 3$, $\frac{dy}{dt} = 1$ at $t = 0$ using Laplace transforms? **(IPU-2015)**

Ans. Taking laplace transform both sides $s^2\bar{y} - sy(0) - y'(0) + \bar{y} = \frac{6s}{s^2+4}$

$$\Rightarrow \quad s^2\bar{y} - 3s - 1 + \bar{y} = \frac{6s}{s^2+4}$$

$$\Rightarrow \quad (s^2+1)\bar{y} = \frac{6s}{s^2+4} + 3s + 1$$

$$\Rightarrow \quad \bar{y} = \frac{6s}{(s^2+4)(s^2+1)} + \frac{3s}{s^2+1} + \frac{1}{s^2+1}$$

Taking inverse laplace transform.

$$L^{-1}\{\bar{y}\} = 6L^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} + 3L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$y(t) = 6L^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} + 3L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

consider $$L^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\}$$

Let $$\bar{f}(s) = \frac{s}{s^2+4},\ \bar{g}(s) = \frac{1}{s^2+1}$$

$$L^{-1}\{\bar{f}(s)\} = \cos 2t = f(t)$$

$$L^{-1}\{\bar{g}(s)\} = \sin t = g(t)$$

$$L^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} = \int_0^t \cos 2u \sin(t-u)\, du.$$

$$= \frac{1}{2}\int_0^t 2\sin(t-u)\cos 2u\, du$$

$$= \frac{1}{2}\int_0^t [\sin(t+u) + \sin(t-3u)]\, du.$$

$$= \frac{1}{2}\left[-\cos(t+u) - \frac{\cos(t-3u)}{3}\right]$$

$$= \frac{1}{2}\left[-\cos 2t + \frac{\cos 2t}{3} + \cos t - \frac{\cos t}{3}\right]$$

$$= \frac{1}{2}\left[-\frac{2\cos 2t}{3} + \frac{2\cos t}{3}\right]$$

$$= \frac{\cos t - \cos 2t}{3.}$$

$$L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$$

$$L^{-1}\left\{\frac{}{}\right\} = \sin t$$

$\therefore$ $$y(t) = 2[\cos t - \cos 2t] + 3.\cos t + \sin t$$

$$y(t) = 5\cos t - 2\cos 2t + \sin t.$$

Q.8. Find the inverse laplace transform of $\tan^{-1}\left(\frac{2}{s^2}\right)$. (IPU-2015)

Ans. Let $$f(t) = L^{-1}\left\{\tan^{-1}\frac{2}{s^2}\right\}$$

$$\Rightarrow \quad t f(t) = L^{-1}\left\{-\frac{d}{ds}\left(\tan^{-1}\frac{2}{s^2}\right)\right\}$$

$$= L^{-1}\left[\frac{-1}{1+4/s^4}\left(\frac{-4}{s^3}\right)\right]$$

$$= L^{-1}\left[\frac{4s}{s^4+4}\right]$$

$$L^{-1} = \left[\frac{4s}{s^4+4s^2+4-4s^2}\right]$$

$$= L^{-1}\left[\frac{4s}{(s^2+2)^2-4s^2}\right]$$

$$= L^{-1}\left[\frac{4s}{(s^2+2-2s)(s^2+2+2s)}\right]$$

$$= L^{-1}\left[\frac{(s^2+2s+2)-(s^2-2s+2)}{(s^2+2s+2)(s^2-2s+2)}\right]$$

$$= L^{-1}\left[\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right]$$

$$= L^{-1}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{(s-1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^t \sin t - e^{-t}\sin t.$$

$$= 2\sin t\left(\frac{e^t - e^{-t}}{}\right)$$

$$= 2\sin t \sin ht.$$

$$\Rightarrow \quad f(t) = \frac{2\sin t \sin ht}{t}$$

Q.9. Find the $\int_0^\infty t^3 e^{-t} \sin t\, dt$ (IPU-2016)

Ans. To evaluate $\int_0^\infty t^3 e^{-t} \sin t\, dt$.

$$= \int_0^\infty e^{-st}(t^3 \sin t)dt \quad (s=1)$$

$$= L(t^3 \sin t)\ (\text{By } def^n)\ (s=1) - (1)$$

Consider $$L(t^3 \sin t) = (1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{d^2}{ds^2}\left[\frac{2s}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)^2.2 - 2s \times 2(s^2+1).2s}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{2(s^2+1) - 8s^2}{(s^2+1)^3}\right]$$

$$= \frac{d}{ds}\left[\frac{2-6s^2}{(s^2+1)^3}\right]$$

$$= \frac{2\left[(s^2+1)^3(-6s) - (1-3s^2).3(s^2+1)^2.2s\right]}{(s^2+1)^6}$$

$$= \frac{2[-6s(s^2+1) - 6s(1-3s^2]}{(s^2+1)^4}$$

$$= \frac{2[-6s^3 - 6s - 6s + 18s^3]}{(s^2+1)^4} = \frac{24s(s^2-1)}{(s^2+1)^4}$$

By (1) at s = 1 $\quad \int_0^\infty e^{-st} f(t)\, dt = 0$

Q.10. The $\frac{2\pi}{w}$ – periodic function $f(t)$ is given by $f(t) = \begin{cases} a \sin \omega t, & 0<t<\pi/\omega \\ 0, & \pi/w<t<2\pi/w \end{cases}$. Find its laplace transform. **(IPU-2016)**

Ans.

$$L\{f(t)\} = \frac{1}{1-e^{-ST}} \int_0^T e^{-st} f(t) dt.$$

Here $\quad T = \frac{2\pi}{w}$

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{2\pi}{w}s}}\left[\int_0^{\pi/w} a \sin wt\, e^{-st}\, dt + \int_{\pi/w}^{2\pi/w} 0.dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi}{w}s}}\left\{a\left[\frac{e^{-st}(-\sin wt - w\cos wt)}{s^2+w^2}\right]_0^{\pi/w}\right\}$$

$$= \frac{a}{1-e^{-\frac{2\pi}{w}s}}\left[\frac{e^{-\frac{\pi s}{w}}(-w\cos\pi)+w}{s^2+w^2}\right]$$

$$= \frac{a\left[we^{\frac{\pi s}{w}} + w\right]}{(1-e^{-\pi s/w})(1+e^{-\pi s/w})(s^2+w^2)}$$

$$= \frac{aw(1+e^{-\pi s/w})}{(1-e^{-\pi s/w})(1+e^{-\pi s/w})(s^2+w^2)}$$

$$= \frac{aw}{(1-e^{-\pi s/w})(s^2+w^2)}$$

Q.11. If L[$t \sin wt$] = $\frac{2ws}{(s^2+w^2)^2}$, Evaluate L ($wt \cos wt + \sin wt$). **(IPU-2016)**

Ans. Let $\quad f(t) = t \sin wt.$

$\Rightarrow \quad f'(t) = wt \cos wt + \sin wt.$

Consider $\quad L\{f'(t)\} = s\bar{f}(s) - f(0). \quad$ ...(1)

Now $\quad L\{f(t)\} = L\{t \sin wt\} = \frac{2ws}{(s^2+w^2)^2} = \bar{f}(s)$

$f(0) = 0.$

By (1),

$$L\{f'(t)\} = L\{wt \cos wt + \sin wt\}$$

$$= \frac{2ws^2}{(s^2+w^2)^2} - 0 = \frac{2ws^2}{(s^2+w^2)^2}$$

Q.12. A function $f(t)$ is given by $f(t) = \begin{cases} t^2 \text{ for } 0<t<1 \\ 4t \text{ for } t>1 \end{cases}$ Express it in terms of unit step function and then find its laplace transform. **(IPU-2016)**

Ans.

$$f(t) = \begin{cases} t^2, 0<t<1 \\ 4t, t>1 \end{cases}$$

$$\begin{aligned}
f(t) &= u(t-0)t^2 - u(t-1)t^2 + 4u(t-1)t. \\
&= t^2 u(t) - (t-1+1)^2 u(t-1) + 4(t-1+1) u(t-1) \\
&= t^2 u(t) - (t-1)^2 u(t-1) - u(t-1) + 4(t-1) u(t-1) \\
&\quad - 2(t-1) u(t-1) + 4u(t-1) \\
f(t) &= t^2 u(t) - (t-1)^2 u(t-1) + 3u(t-1) + 2(t-1) u(t-1) \\
L\{f(t)\} &= L\{t^2 u(t)\} - L\{(t-1)^2 u(t-1)\} + 3 L\{u(t-1)\} \\
&\quad + 2L\{(t-1) u(t-1)\}
\end{aligned}$$

Now, $\quad g(t) = t^2 \Rightarrow L\{t^2\} = \frac{2!}{s^3}$

$$L\{t\} = \frac{1}{s^2},\ L\{1\} = \frac{1}{s}.$$

By using second shifting property $L^{-1}\left\{e^{-as}\bar{f}(s)\right\} = f(t-a)\, u(t-a)$

Now $L\{f(t)\} = \frac{2}{s^3} - e^{-s}\cdot\frac{2}{s^3} + 3\frac{e^{-s}}{s} + 2\frac{e^{-s}}{s^2}$

Q.13. Solve $\frac{dy}{dt} + y = \cos 2t, y(0) = 1$, using laplace transformation. **(IPU-2016)**

Ans. $\quad \frac{dy}{dt} + y = \cos 2t, \quad y(0) = 1$

Taking laplace both sides

$$S\bar{y} - y(0) + \bar{y} = \frac{s}{s^2+4}$$

$$\Rightarrow \quad (S+1)\bar{y} - 1 = \frac{s}{s^2+4}$$

$$\Rightarrow \quad \bar{y} = \frac{s}{(s^2+4)(s+1)} + \frac{1}{s+1}$$

$$L^{-1}\{\bar{y}\} = L^{-1}\left\{\frac{s}{(s^2+4)(s+1)}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} \quad ...(A)$$

Consider $L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t,\ + L^{-1}\left\{\frac{s}{s+1}\right\} = e^{-t}.$

Now $L^{-1}\left\{\frac{s}{(s^2+4)(s+1)}\right\} = \int_0^t \cos 2u\, e^{-(t-u)} du.$

$$= e^{-t}\int_0^t e^u . \cos 2u\, du.$$

$$= e^{-t}\left[\frac{e^u}{1^2+2^2}(\cos 2u + 2\sin 2u)\right]_0^t$$

$$= e^{-t}\left[\frac{e^t}{5}(\cos 2t + 2\sin 2t) - \frac{1}{5}\right]$$

$$= \frac{\cos 2t + 2\sin 2t}{5} - \frac{e^{-t}}{5}$$

By equation (A), we get

$$y(t) = \frac{\cos 2t + 2\sin 2t}{5} - \frac{e^{-t}}{5} + e^{-t}$$

$$y(t) = \frac{\cos 2t + 2\sin 2t}{5} + \frac{4e^{-t}}{5}$$

Q.14. Find the inverse Laplace Transform of $\frac{5}{s^2} + \frac{(\sqrt{s}-1)^2}{s^2} - \frac{7}{3s+2}$ (IPU-2017)

Ans.

$$\bar{f}(s) = \frac{5}{s^2} + \frac{(\sqrt{s}-1)^2}{s^2} - \frac{7}{3s+2}$$

$$= \frac{5}{s^2} + \frac{s+1-2\sqrt{s}}{s^2} - \frac{7}{3s+2}$$

$$= \frac{5}{s^2} + \frac{1}{s} + \frac{1}{s^2} - \frac{2\sqrt{s}}{s^2} - \frac{7}{3s+2}$$

$$= \frac{6}{s^2} + \frac{1}{s} - 2\frac{1}{s^{3/2}} - \frac{7}{3s+2}$$

$$L^{-1}\{\bar{f}(s)\} = 6L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{1}{s}\right\} - 2L^{-1}\left\{\frac{1}{s^{3/2}}\right\} - 7L^{-1}\left\{\frac{1}{3s+2}\right\}$$

$$= 6t + 1 - 2\frac{t^{1/2}}{\lfloor 3/2}$$

Q.15. Find the Laplace Transform of the function $f(t)$ defined as

$$\mathbf{f(t)} = \begin{cases} 2+t^2 & 0<t<2 \\ 6 & 2<t<3 \\ 2t-5 & 3<t<\infty \end{cases} \quad \textbf{(IPU-2017)}$$

Ans. $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$

$$= \int_0^2 (2+t^2)e^{-st}\,dt + \int_2^3 6e^{-st}\,dt + \int_3^\infty (2t-5)e^{-st}\,dt$$

$$= 2\left|\frac{e^{-st}}{-s}\right|_0^2 + \left[t^2\frac{e^{-st}}{-s} - 2t\frac{e^{-st}}{s^2} + 2\frac{e^{-st}}{-s^3}\right]_0^2$$

$$+6\left|\frac{e^{-st}}{-s}\right|_2^3 + \left[(2t-5)\frac{e^{-st}}{-s} - 2\frac{e^{-st}}{s^2}\right]_3^\infty$$

$$= -\frac{2e^{-2s}}{s} + \frac{2}{s} - \frac{4e^{-2s}}{s} - \frac{4}{s^2}e^{-2s} - \frac{2}{s^3}e^{-2s} + \frac{2}{s^3}$$

$$-\frac{6}{s}e^{-3s} + \frac{6}{s}e^{-2s} + \frac{e^{-3s}}{s} + \frac{2}{s^2}e^{-3s}$$

$$= \frac{2}{s^3}(1-e^{-2s}) + \frac{2}{s^2}(e^{-3s} - 2e^{-2s}) + \frac{2}{s} - \frac{5}{s}e^{-3s}$$

$$= \frac{2}{s^3}(1-e^{-2s}) + \frac{2}{s^2}(e^{-3s} - 2e^{-2s}) + \frac{1}{s}(2-5e^{-3s}).$$

Q.16. Find the Laplace Transform of unit step function $u(t-a)$, also find the Laplace transform of $t^2u\ (t-3)$. (IPU-2017)

Ans. Unit step function is defined as

$$u(t-a) = \begin{cases} 1, & t>a \\ 0, & t<a \end{cases}$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\,dt$$

$$= \int_0^a o.dt + \int_a^\infty e^{-st}\,dt = \int_a^\infty e^{-St}\,dt$$

$$= \left|\frac{e^{-st}}{-s}\right|_a^\infty = o + \frac{e^{-as}}{s} = \frac{e^{-as}}{s}$$

Consider $L\{t^2u\,(t-3)\}$

$$t^2 = t^2 - 6t + 9 + 6t - 9$$

$$= (t-3)^2 + 6(t-3) + 18 - 9$$

$$= (t-3)^2 + 6(t-3) + 9$$

Now $t^2u\,(t-3) = [(t-3)^2 + 6(t-3) + 9]\,u(t-3).$

$$L\{t^2u\,(t-3)\} = L\{(t-3)^2 u(t-3)\} + 6L\{(t-3)u\,(t-3)\} + 9L\{u(t-3)\}$$

$$= f(t-a)\,u(t-a)$$

As $L\{f(t-a)\,u\,(t-a)\} = e^{-as}\,\bar{f}\,(s)$

$$L\{t^2\} = \frac{2}{s^3},\ L\{t\} = \frac{1}{s^2}$$

$$\therefore \quad L\{t^2u(t-3)\} = \frac{2e^{-3s}}{s^3}+\frac{6e^{-3s}}{s^2}+\frac{9}{s}e^{-3s}$$

Q.17. Find inverse Laplace Transform of $\log\left(\frac{s+a}{s+b}\right)$ (IPU-2018)

Ans. Let $L^{-1}\left\{\log\frac{s+a}{s+b}\right\} = f(t)$

then
$$L\{f(t)\} = \bar{f}(s) = \log\frac{s+a}{s+b}$$
$$L\{t\,f(t)\} = (-1)\frac{d}{ds}\left\{\log\frac{s+a}{s+b}\right\}$$
$$= -\frac{d}{ds}[\log(s+a)-\log(s+b)]$$
$$= -\left[\frac{1}{s+a}-\frac{1}{s+b}\right]$$
$$= \frac{1}{s+b}-\frac{1}{s+a}$$
$$t\,f(t) = L^{-1}\frac{1}{s+b}-L^{-1}\frac{1}{s+a}$$
$$= e^{-bt}-e^{-at}$$
$$f(t) = \frac{e^{-bt}-e^{-at}}{t}$$

Q.18. Find the inverse Laplace transform of $\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$ (IPU-2018)

Ans.
$$L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2}\frac{s}{s^2+\pi^2}\right\}+L^{-1}\left\{\frac{\pi e^{-s}}{s^2+\pi^2}\right\}$$

Let
$$L^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = \cos\pi t \text{ and } L^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\} = \sin\pi t$$

By 2nd shifting property
$$L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2}\frac{s}{s^2+\pi^2}\right\}+L^{-1}\left\{\frac{e^{-s}\pi}{s^2+\pi^2}\right\}$$
$$= \cos\pi\left(t-\frac{1}{2}\right)u\left(t-\frac{1}{2}\right)+\sin\pi(t-1)u(t-1)$$
$$= \cos(\pi t-\pi/2)u(t-1/2)+\sin(\pi t-\pi)u(t-1)$$
$$= \sin\pi t\; u(t-1/2)-\sin\pi t\; u(t-1)$$
$$= \sin\pi t[u(t-1/2)-u(t-1)]$$
$$= \sin\pi t,\ \frac{1}{2}<t<1$$

Q.19. If $L\{f(t)\} = \bar{f}(s)$, then prove that $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$ provided integral exists. Hence evaluate $L\left\{\frac{\sin at}{t}\right\}$ (IPU-2018)

Ans. Since
$$\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$$
$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty\left[\int_0^\infty e^{-st}f(t)dt\right]ds$$

On changing order of integration.
$$= \int_0^\infty\int_s^\infty e^{-st}f(t)\,ds\,dt$$
$$= \int_0^\infty\left|\frac{e^{-st}f(t)}{-t}\right|_s^\infty dt$$
$$= \int_0^\infty\frac{e^{-st}}{t}f(t)dt = \int_0^\infty e^{-st}\left(\frac{f(t)}{t}\right)dt$$
$$\int_s^\infty \bar{f}(s)\,ds = L\left\{\frac{f(t)}{t}\right\}$$

Consider
$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

Now
$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty\frac{a}{s^2+a^2}ds$$
$$= a\int_s^\infty\frac{ds}{s^2+a^2} = \frac{a}{a}\left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$
$$= \tan^{-1}\infty-\tan^{-1}\left(\frac{s}{a}\right)$$
$$= \frac{\pi}{2}-\tan^{-1}\frac{s}{a}$$
$$= \cot^{-1}\frac{s}{a}$$

Q.20. Using Laplace Transform $\frac{d^2x}{dt^2}+9x = \cos 2t$, If $x(0)=1$, $x\left(\frac{\pi}{2}\right) = -1$. (IPU-2018)

Ans. Taking Laplace transform, we have

$$L\left\{\frac{d^2x}{dt^2}\right\}+9L\{x\} = L\{\cos 2t\}$$

$$\Rightarrow s^2\bar{x}-sx(0)-x'(0)+9\bar{x} = \frac{s}{s^2+4}$$

$\Rightarrow$ Let $x'(0) = k$

$$\Rightarrow (s^2+9)\bar{x}-s-k = \frac{s}{s^2+4}$$

$$\Rightarrow (s^2+9)\bar{x} = \frac{s}{s^2+4}+s+k$$

$$\Rightarrow \bar{x} = \frac{s}{(s^2+9)(s^2+4)}+\frac{s}{s^2+9}+\frac{k}{s^2+9}$$

Taking inverse laplace transform

$$x(t) = L^{-1}\left\{\frac{s}{(s^2+9)(s^2+4)}\right\}+L^{-1}\left\{\frac{s}{s^2+9}\right\}+KL^{-1}\left\{\frac{1}{s^2+9}\right\}$$

Consider $L^{-1}\left\{\frac{s}{(s^2+9)(s^2+4)}\right\}$

Let $\bar{f}(s) = \frac{1}{s^2+9},\ \bar{g}(s)=\frac{s}{s^2+4}$

$$\Rightarrow L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+9}\right\}=\frac{1}{3}\sin 3t$$

$$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

By convolution theorem

$$L^{-1}\left\{\frac{s}{(s^2+9)(s^2+4)}\right\} = \int_0^t \frac{1}{3}\sin 3u\cos 2(t-u)du$$

$$= \frac{1}{6}\int_0^t \sin(u+2t)+\sin(5u-2t)\,du$$

$$= \frac{1}{6}\left[-\cos(u+2t)-\frac{\cos(5u-2t)}{5}\right]_0^t$$

$$= \frac{1}{6}\left[-\cos 3t-\frac{\cos 3t}{5}+\cos 2t+\cos\frac{2t}{5}\right]$$

$$= \frac{1}{6}\left[-6\frac{\cos 3t}{5}+6\frac{\cos 2t}{5}\right]$$

$$= \frac{\cos 2t}{5}-\frac{\cos 3t}{5}$$

By (1) we get

$$x(t) = \frac{\cos 2t}{5}-\frac{\cos 3t}{5}+\cos 3t+\frac{k}{3}\sin 3t.$$

$$x(t) = \frac{\cos 2t}{5}+\frac{4}{5}\cos 3t+\frac{k}{3}\sin 3t. \quad ...(2)$$

As when $t=\frac{\pi}{2}$ $x=-1$

$$\Rightarrow -1 = \frac{4}{5}\cos\frac{3\pi}{2}+\frac{1}{5}\cos\left(\frac{2\pi}{2}\right)+\frac{k}{3}\sin\frac{3\pi}{2}$$

$$\Rightarrow -1 = \frac{-1}{5}\frac{-k}{3}\Rightarrow\frac{-k}{3}=-1+\tfrac{1}{5}$$

$$\Rightarrow k = 12/5.$$

By (2), we have

$$\therefore x(t) = \frac{\cos 2t}{5}+\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t$$

$$x(t) = \frac{1}{5}[\cos 2t+4\cos 3t+4\sin 3t]$$

Q.21. Using laplace transform, find the solution of $\frac{d^2y}{dx^2}+9y=\cos 2x$, if $y(0)=1, y\left(\frac{\pi}{2}\right)=-1$. **(IPU-2019)**

Ans. $\frac{d^2y}{dx^2}+9y = \cos 2x$

Taking laplace both sides, we get

$$s^2\bar{y}-sy(0)-y'(0)+9\bar{y} = \frac{s}{s^2+4}$$

As $y(0) = 1,$

Let $y'(0) = k$

$$\Rightarrow s^2\bar{y}-s-k+9\bar{y} = \frac{s}{s^2+4}$$

$$\Rightarrow (s^2+9)\bar{y} = \frac{s}{s^2+4}+s+k$$

$$\Rightarrow \bar{y} = \frac{s}{(s^2+4)(s^2+9)}+\frac{s}{s^2+9}+\frac{k}{s^2+9}$$

Taking inverse transform, we get

$$L^{-1}\{\bar{y}\} = L^{-1}\left\{\frac{s}{(s^2-4)(s^2+9)}\right\}+L^{-1}\left\{\frac{s}{s^2+9}\right\}+L^{-1}\left\{\frac{k}{s^2+9}\right\} \quad ...(1)$$

Now, $L^{-1}\left\{\frac{s}{(s^2-4)(s^2+9)}\right\}$

Let

$$\bar{f}(s) = \frac{s}{s^2+4}, \bar{g}(s) = \frac{1}{s^2+9}$$

$\therefore$

$$L^{-1}\{\bar{f}(s)\} = \cos 2t = f(t)$$

$$L^{-1}\{\bar{g}(s)\} = \frac{\sin 3t}{3} = g(t)$$

By convolution theorem

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\} = \frac{1}{3}\int_0^t \cos 2u \sin(3t-3u)du$$

$$= \frac{1}{6}\int_0^t \sin(3t-u)-\sin(5u-3t)du$$

$$= \frac{1}{6}\left[-\frac{\cos(3t-u)}{-1}+\frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{6}\left[\cos 2t+\frac{\cos 2t}{5}-\cos 3t-\frac{\cos 3t}{5}\right]$$

$$= \frac{1}{6}\left[\frac{6\cos 2t}{5}-\frac{6\cos 3t}{5}\right] = \frac{\cos 2t-\cos 3t}{5}$$

By (1)

$$L^{-1}\{\bar{y}\} = \frac{1}{5}(\cos 2t-\cos 3t)+\cos 3t+\frac{k}{3}\sin 3t$$

$$\Rightarrow \quad y(t) = \frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{k}{3}\sin 3t$$

Put

$$y\left(\frac{\pi}{2}\right) = -1$$

We get

$$\Rightarrow \quad -1 = \frac{1}{5}\cos\pi+\frac{4}{5}\cos\frac{3\pi}{2}+\frac{k}{3}\sin\frac{3\pi}{2}$$

$$\Rightarrow \quad -1 = -\frac{1}{5}+\frac{k}{3}$$

$$\Rightarrow \quad k = \frac{12}{5}$$

$\therefore$ required solution is

$$y(t) = \frac{1}{5}(\cos 2t+4\cos 3t+4\sin 3t)$$

Q.22. Evaluate (i) $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$ (ii) $L^{-1}\left\{\frac{se^{-\frac{s}{2}}+\pi e^{-s}}{s^2+\pi^2}\right\}$ (IPU-2019)

Ans. (i) $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$

Let

$$L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

As

$$L^{-1}\left\{\frac{1}{s}\bar{f}(s)\right\} = \int_0^t f(u)du$$

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\,du = -(\cos u)_0^t$$

$$= -(\cos t-1) = 1-\cos t$$

$$\Rightarrow \quad L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos u)du$$

$$= u-\sin u|_0^t$$

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t-\sin t$$

(ii)

$$L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2}\frac{s}{s^2+\pi^2}\right\}+L^{-1}\left\{\frac{\pi e^{-s}}{s^2+\pi^2}\right\}$$

Let

$$L^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = \cos\pi t$$

$$L^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\} = \sin\pi t$$

By 2nd shifting property.

$$L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2}\frac{s}{s^2+\pi^2}\right\}+L^{-1}\left\{\frac{e^{-s}\pi}{s^2+\pi^2}\right\}$$

$$= \cos\pi\left(t-\frac{1}{2}\right)u\left(t-\frac{1}{2}\right)+\sin\pi(t-1)\,u(t-1)$$

$$= \cos\left(\pi t-\frac{\pi}{2}\right)u\left(t-\frac{1}{2}\right)+\sin(\pi t+\pi)\,u(t-1)$$

$$= \sin\pi t\,u\left(t-\frac{1}{2}\right)-\sin\pi t\,u(t-1)$$

$$= \sin\pi t\left[u\left(t-\frac{1}{2}\right)-u(t-1)\right] = \sin\pi t,\ \frac{1}{2}<t<1$$

Q.23 Obtain fourier series for the function $f(x) = x^2 + x, x \in [-\pi, \pi]$ and deduce from it.

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

(IPU-2014)

Sol. Let $f(x) = x^2 + x$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \dots(1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x^2 dx + \int_{-\pi}^{\pi} x dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{x^3}{3}\right|_{-\pi}^{\pi} + \left|\frac{x^2}{2}\right|_{-\pi}^{\pi}\right] = \frac{1}{\pi}\left[\frac{2\pi^3}{3} + 0\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2+x)\frac{\sin nx}{n} - (2x+1)\left(-\frac{\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2} - (1-2\pi)\frac{\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\cos n\pi\right] = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2+x)\left(-\frac{\cos nx}{n}\right) - (2x+1)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi^2+\pi)\frac{\cos n\pi}{n} + \frac{2\cos n\pi}{n^3} + (\pi^2-\pi)\frac{\cos n\pi}{n} - \frac{2}{n^3}\cos n\pi\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi\cos n\pi}{n}\right] = \frac{-2}{n}(-1)^n$$

By equation (1), we have

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^n\cos nx - \frac{2}{n}(-1)^n \sin nx\right]$$

$$= \frac{\pi^2}{3} + \left(-\frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \dots\right) + \left(\frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x \dots\right)$$

Put $x = \pi$

$$\pi^2 + \pi = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$\Rightarrow \frac{2\pi^2}{3} + \pi = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$\Rightarrow \frac{\pi^2}{6} + \frac{\pi}{4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad \dots(1)$$

Put $x = -\pi$

$$\pi^2 - \pi = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$\Rightarrow \frac{\pi^2}{6} - \frac{\pi}{4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad \dots(2)$$

Adding (1) and (2)

$$\frac{2\pi^2}{6} = 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

Q.24. Express $f(x) = x$ as half range cosine series for $0 < x < 2$ (IPU-2014)

Sol.

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{2}\right)$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)dx$$

$$= \int_0^2 x\,dx = \left|\frac{x^2}{2}\right|_0^2 = 2$$

$$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx$$

$$= \left[x\sin\frac{n\pi x}{2} \times \frac{2}{n\pi} - \left(-\cos\frac{n\pi x}{2}\right)\frac{4}{n^2\pi^2}\right]$$

$$= 0 + \cos n\pi \cdot \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2} = [(-1)^n - 1]\frac{4}{n^2\pi^2}$$

$$\therefore \quad f(x) = \frac{2}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi x}{2}$$

$$= 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos\frac{n\pi x}{2}$$

Q.25. Find fourier transform of

$$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$$

Hence show that $\int_0^\infty \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx = \frac{-3\pi}{16}$ **(IPU-201**

Sol. Fourier transform of $f(x)$ is

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}dx$$

$$= \int_{-\infty}^{-1} f(x)e^{isx}dx + \int_{-1}^{1} f(x)e^{isx}dx + \int_{1}^{\infty} f(x)e^{isx}dx$$

$$= \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1}(1-x^2)e^{isx}dx + \int_{1}^{\infty} 0.dx$$

$$= \int_{-1}^{1}(1-x^2)\,e^{isx}dx = \left[(1-x^2)\frac{e^{isx}}{is} - (-2x)\frac{e^{isx}}{i^2 s^2} + (-2)\frac{e^{isx}}{i^3 s^3}\right]_{-1}^{1}$$

$$= \left[\left\{0+\frac{2e^{is}}{i^2s^2} - 2\frac{e^{is}}{i^3s^3}\right\} - \left\{0+2(-1)\frac{e^{-is}}{i^2s^2} - \frac{2e^{-is}}{i^3 s^3}\right\}\right]$$

$$= \frac{-2e^{is}}{s^2} + \frac{2e^{is}}{is^3}\frac{-2e^{-is}}{s^2} - \frac{2e^{-is}}{i\,s^3}$$

$$= \frac{-2}{s^2}\left[e^{is}+e^{-is}\right] + \frac{2}{is^3}(e^{is}-e^{-is})$$

$$= \frac{-4}{s^2}\left[\frac{e^{is}+e^{-is}}{2}\right] + \frac{4}{is^3}\left(\frac{e^{is}-e^{-is}}{2}\right)$$

$$= \frac{-4}{s^2}\cos s + \frac{4}{s^3}\sin s = \frac{-4}{s^3}(s\cos s - \sin s)$$

By inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\{f(x)\}e^{-isx}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-4}{s^3}(s\cos - \sin s)e^{-isx}ds$$

$$= \frac{-2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)(\cos sx - i\sin sx)dx$$

$$= \frac{-2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\,ds + \frac{2}{\pi}i\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\sin sx\,ds$$

since the integrand in first integral is even and in second integral is odd.

$$\Rightarrow \quad f(x) = \frac{-4}{\pi}\int_0^\infty\left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\,ds$$

$$\Rightarrow \quad \int_0^\infty\left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\,ds = -\frac{\pi}{4}f(x)$$

$$= \begin{cases} -\frac{\pi}{4}(1-x^2), & |x|\leq 1 \\ 0, & |x|>1 \end{cases}$$

Putting $x = \frac{1}{2}$

$$\int_0^\infty\left(\frac{s\cos s - \sin s}{s^3}\right)\cos\frac{s}{2}ds = \frac{-\pi}{4}\left(1-\frac{1}{4}\right) = \frac{-3\pi}{16}$$

$$\Rightarrow \int_0^\infty\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}dx = \frac{-3\pi}{16}$$

Q.4. State giving reasons whether $\sin\frac{1}{x}$ can be expanded into Fourier series in the interval $(-\pi, \pi)$. **(IPU-2015)**

Sol. $f(x) = \sin\frac{1}{x}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin\frac{1}{x}dx = \frac{1}{\pi}\left|\frac{-\cos 1/x}{-1/x^2}\right|_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\cos\frac{1}{\pi}}{1/\pi^2} - \frac{\cos^{1/\pi}}{1/\pi^2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin\frac{1}{x}\cos nx\,dx.$$

It can not be expanded as integration is not finite.

$\sin\frac{1}{x}$ is discontinuous in $(-\pi, \pi)$.

Q.5. Find the Fourier transform of the function

$f(x) = e^{-a|x|}, -\infty < x < \infty$ **(IPU-2015)**

Sol. $f(x) = e^{-a|x|}$

$-|x| = x$ for $-\infty < x < 0$

and $-|x| = -x$ for $0 < x < \infty$

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\,e^{isx}dx = \int_{-\infty}^{\infty} e^{-a|x|}e^{isx}dx$$

$$= \int_{-\infty}^{0} e^{ax}e^{isx}dx + \int_0^\infty e^{-ax}e^{isx}dx = \int_{-\infty}^{0} e^{(a+is)x}dx + \int_0^\infty e^{(-a+is)x}dx$$

$$= \left|\frac{e^{(a+is)x}}{a+is}\right|_{-\infty}^{0} + \left|\frac{e^{-(a-is)x}}{-(a-is)}\right|_{0}^{\infty} = \frac{1}{a+is} - \frac{1}{a-is} = \frac{2a}{a^2+s^2}$$

Q.6. Find the fourier series expansion of the following periodic function **(IPU-2016)**

$f(x) = |\sin x|, -\pi < x < \pi$

Sol. Since $f(-x) = |\sin(-x)| = |\sin x| = f(x)$.

$\therefore f(x)$ is an even function.

$\Rightarrow \quad bn = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x|\, dx = \frac{2}{\pi}\int_0^{\pi} |\sin x|\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\, d\,x = \frac{2}{\pi}|(-\cos x)|_0^{\pi}$$

$$= \frac{2}{\pi}(-\cos \pi + 1) = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{\frac{-1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right], n \neq 1$$

$$= \begin{cases} \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right], \text{ if } n \text{ is even} \\ \frac{1}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right], \text{ if } n \text{ is odd} \end{cases}, n \neq 1$$

$$= \begin{cases} \frac{-4}{\pi(n^2-1)}, \text{ if } n \text{ is even} \\ 0, \text{ if } n \text{ is odd} \end{cases}, n \neq 1$$

When $n = 1$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x\, dx$$

$$= \frac{1}{\pi}\left|\frac{-\cos 2x}{2}\right|_0^{\pi} = \frac{1}{2\pi}(-\cos 2\pi + \cos 0) = 0$$

$\therefore$
$$f(x) = \frac{2}{\pi} + \left(\frac{-4}{\pi}\right)\sum_{n=1}^{\infty}\left[\frac{\cos 2n\, x}{(2n)^2-1}\right]$$

Q.26. State Dirichlet's condition for convergence of Fourier series & check whether the func. $f(x) = \frac{1}{3-x}$, $0 < x < 2\pi$ satisfy Dirichlet's conditions or not.

Ans. Dirichlet's condition for uniform convergence of fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where a_o, a_n, b_n are constants are

(i) $f(x)$ is periodic, single valued and finite

(ii) $f(x)$ has a finite number of discontinuities in any one period

(iii) $f(x)$ has a finite number of maxima and minima

Let $f(x) = \frac{1}{3-x}, o < x < 2\pi$

This function is not periodic in $(0, 2\pi)$ and at $t = 3$ function is not defined in the interval $(0, 2\pi)$.

Q.27. Find the fourier series for $f(x) = x^2$ in (0,4) and deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \frac{\pi^2}{6}$

(IPU-2015)

Ans.
$$2l = 4$$

$\Rightarrow$
$$l = 2$$

$$f(x) = x^2 \text{ is an even function}$$

$\therefore$
$$b_n = 0$$

$$f(x) = \frac{a_o}{2} + \sum a_n \cos\frac{n\pi x}{2}$$

$$a_o = \frac{2}{l}\int_o^2 x^2 dx = \int_o^2 x^2 dx = \left|\frac{x^3}{3}\right|_0^2$$

$$a_0 = \frac{8}{3}$$

$$a_n = \frac{2}{l}\int_0^2 x^2 \cos\frac{n\pi x}{1} dx = \int_0^2 x^2 \cos\frac{n\pi x}{2} dx$$

$$= \left[x^2\frac{\sin n\pi x/2}{n\pi/2} - 2x\left(-\frac{\cos n\pi x/2}{n^2\pi^2/4}\right) + 2\left(\frac{-\sin n\pi x/2}{n^3\pi^3/8}\right)\right]_0^2$$

$$= \left[o + \frac{4\times 4}{n^2\pi^2}\cos n\pi - 0 - (0+0-0)\right] = \frac{16}{n^2\pi^2}(-1)^n$$

$\therefore$
$$f(x) = \frac{8}{6} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$$

$$= \frac{4}{3} + \frac{16}{\pi^2}\left[\frac{-1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{2^2}\cos\frac{2\pi x}{2} - \frac{1}{3^2}\cos\frac{3\pi x}{2} + \right]$$

Put $x = 2$

$$2^2 = \frac{4}{3} + \frac{16}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ..\right]$$

$$\left(4 - \frac{4}{3}\right)\frac{\pi^2}{16} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...$$

$\Rightarrow$
$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{8}{3}\times\frac{\pi^2}{16} = \frac{\pi^2}{6}$$

Q.28. Find a series of cosines of multiples of x which will represent $x \sin x$ in the interval (o, π) & show that $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\ldots=\frac{\pi-2}{4}$ (IPU-2015)

Ans. Let $f(x) = \frac{a_o}{2}+\sum_{n=1}^{\infty} a_n \cos nx\, dx$

$$a_o = \frac{2}{\pi}\int_o^{\pi} f(x)dx = \frac{2}{\pi}\int_o^{\pi} x \sin x dx$$

$$= \frac{2}{\pi}\left[x(-\cos x)-1(-\sin x)\right]_0^{\pi} = \frac{2}{\pi}\left[-\pi\cos\pi+o\right]=2$$

$$a_n = \frac{2}{\pi}\int_o^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_o^{\pi} x \sin x \cos nx dx$$

$$= \frac{2}{\pi}\int_o^{\pi} x(\cos nx \sin x)dx = \frac{1}{\pi}\int_o^{\pi} x[\sin(n+1)x-\sin(n-1)x]dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right)-1\left(\frac{-\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{n+1}+\frac{\pi\cos(n-1)\pi}{n-1}\right], \text{ when } n \neq 1$$

$$= -\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1} = (-1)^{n-1}\left[\frac{-1}{n+1}+\frac{1}{n-1}\right]$$

$$= \frac{(-1)^{n-1}.2}{n^2-1}, \text{ when } n \neq 1$$

$\therefore$ $$a_n = \frac{2(-1)^{n-1}}{n^2-1}, \text{ when } n \neq 1$$

when $n = 1$

$$a_1 = \frac{2}{\pi}\int_o^{\pi} x \sin x \cos x dx = \frac{1}{\pi}\int_o^{\pi} x \sin 2x dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-1\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\cos 2\pi\right] = -\frac{1}{2}$$

$\therefore$ $$f(x) = x\sin x = 1-\frac{1}{2}\cos x-2\left(\frac{\cos 2x}{1.3}\frac{-\cos 3x}{2.4}+\frac{\cos 4x}{3.5}\right)$$

Putting $x = \frac{\pi}{2}$

$$\frac{\pi}{2} = 1-2\left(\frac{-1}{1.3}+\frac{1}{3.5}-\frac{1}{5.7}+\ldots\right)$$

$$\frac{\pi}{2} = 1+ 2 \quad 2 \quad 2$$

$\Rightarrow$ $$\frac{2}{1.3}-\frac{2}{3.5}+\frac{2}{5.7}\ldots = \frac{\pi}{2}-1 = \frac{\pi-2}{2}$$

$\Rightarrow$ $$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\!-\!- = \frac{\pi-2}{4}$$

Q.29. Find the fourier transform of the function $f(x) = e^{-a|x|}, -\infty < x < \infty$.

Ans. $f(x) = e^{-a|x|}$, $-\infty < x < \infty$

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-a|x|}\cdot e^{isx}\,dx = \int_{-\infty}^{0} e^{ax}e^{isx}dx+\int_0^{\infty} e^{-ax}e^{isx}\,dx$$

$$= \int_{-\infty}^{0} e^{(a+is)x}dx+\int_0^{\infty} e^{-(a-is)x}dx$$

$$= \left.\frac{e^{(a+is)x}}{a+is}\right|_{-\infty}^{0}+\left|\frac{e^{-(a-is)x}}{-a+is}\right|_0^{\infty}$$

$$= \frac{1}{a+is}+\frac{1}{a-is} = \frac{2a}{a^2+s^2}.$$

Q.30. Find the fourier series expansion of the following periodic function $f(x) = |x|, -\pi < x < \pi$. Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ (IPU-2016)

Ans. Since $f(-x) = |-x| = |x| = f(x)$

$\therefore f(x)$ is an even function and hence $b_n = 0$.

Let $f(x) = |x| = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}|x|dx = \frac{2}{\pi}\int_0^{\pi} x dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}|x|\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2}-\frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n-1]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)$$

Putting $x = 0$ in above, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\Rightarrow \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Q.31. Find the fourier transform of the function $f(x) = e^{-x^2/2}$. (IPU-2016)

Ans. $F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{isx} \cdot dx$

$$= \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx}\, dx = \int_{-\infty}^{\infty} e^{-x^2/2 + isx}\, dx$$

$$= \int_{-\infty}^{\infty} e^{\frac{1}{2}[x^2 - 2isx]}\, dx = \int_{-\infty}^{\infty} e^{\frac{-1}{2}[(x-is)^2 - i^2 s^2]}\, dx$$

$$= \int_{-\infty}^{\infty} e^{\frac{-1}{2}[(x-is)^2 + s^2]}\, dx = e^{-s^2/2} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-is)^2}\, dx$$

Put $\quad x - is = y \Rightarrow dx = dy$

$$= e^{-s^2/2} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = 2e^{-s^2/2} \int_{0}^{\infty} e^{-y^2/2}\, dy$$

Put $\quad \dfrac{y^2}{2} = t$

$$\Rightarrow \quad F\{f(x)\} = 2e^{-s^2/2} \int_0^{\infty} e^{-t} \frac{t^{\frac{-1}{2}}}{\sqrt{2}}\, dt = \sqrt{2}\, e^{-s^2/2} \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1}\, dt$$

$$= \sqrt{2}\, e^{-s^2/2} \left\lceil \frac{1}{2} \right. = \sqrt{2\pi}\, e^{-s^2/2}$$

Q.32. The temperature distribution $u(x, t)$ in a semi-infinite rod is determined by the P D E. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $0 \le x < \infty$, subject to conditions. (6)

(i) $u(x, 0) = 0, x \ge 0$ (ii) $\dfrac{\partial u}{\partial x} = -\mu$ (a constant), when $x = 0, t > 0$.

Determine the temperature formula. **(IPU-2016)**

Ans. Given $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ ...(1)

Boundary condition $\dfrac{\partial u}{\partial x} = -\mu$, when $x = 0, t > 0$.

Initial condition is $u(x, 0) = 0$.

Take fourier cosine transform both sides of (1),

$$F_C\left(\frac{\partial u}{\partial t}\right) = F_C\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \quad \int_0^{\infty} \frac{\partial u}{\partial t} \cos sx\, dx = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \cos sx\, dx$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^{\infty} u \cos sx\, dx = \left[\left\{\cos sx \cdot \frac{\partial u}{\partial x}\right\}_0^{\infty} - \int_0^{\infty} -s \sin sx \cdot \frac{\partial u}{\partial x}\, dx\right]$$

$$= \mu + s\int_0^{\infty} \sin sx \frac{\partial u}{\partial x}\, dx \qquad \left[\because \frac{\partial u}{\partial x} \to 0 \text{ when } x \to \infty, \frac{\partial u}{\partial x} = -\mu, x = 0\right]$$

$$= \mu + s\left\{(\sin sx \cdot \mu)_0^{\infty} - \int_0^{\infty} s \cos sx \cdot u\, dx\right\}$$

$$= \mu + s\left\{0 - s^2\int_0^{\infty} \cos sx \cdot u\, dx\right\} \qquad [\because u \to 0 \text{ when } x \to \infty]$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^{\infty} u \cos sx\, dx = \mu - s^2 \int_0^{\infty} u \cos sx\, dx$$

$$\frac{d\bar{u}_c}{dt} = \mu - s^2 \bar{\mu}_c$$

where $\quad \bar{u}_c = \bar{u}_c(s, t) = F_c[u(x, t)]$

$$\Rightarrow \quad \frac{d\bar{u}_c}{dt} + s^2 \bar{u}_c = \mu \qquad \text{(Linear different equation)}$$

$$\text{I.F.} = e^{\int s^2 dt} = e^{s^2 t}$$

$\therefore$ Its solution is

$$\bar{u}_c \cdot e^{s^2 t} = \int \mu \cdot e^{s^2 t} dt + c = \mu \cdot \frac{e^{s^2 t}}{s^2} + c$$

$$\Rightarrow \quad \bar{u}_c = \frac{\mu}{s^2} + c e^{-s^2 t} \qquad \text{...(2)}$$

Putting $t = 0$ in (2), we get

$$\bar{u}_c(s, 0) = \frac{\mu}{s^2} + c$$

$$\Rightarrow \quad c = \frac{-\mu}{s^2} + \bar{u}_c(s,0) = \frac{-\mu}{s^2} + F_c[u(x,0)]$$

$$= \frac{-\mu}{s^2} + \int_0^\infty 0 \cdot \cos sx \, dx = \frac{-\mu}{s^2}$$

By (2), $\quad \bar{\mu}_c(s,t) = \frac{\mu}{s^2}(1 - e^{-s^2 t})$

Taking inverse fourier cosine transform, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\mu}{s^2}(1 - e^{-s^2 t}) \cos sx \, ds.$$

Q.33. Find the fourier series for $f(x)$ if $f(x) =$

$$\begin{cases} -\pi, & -\pi < x < 0, \\ x, & 0 < x < \pi \end{cases}$$ **deduce that** $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$ **(IPU-2016)**

Ans. $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_0^\pi x\,dx\right] = \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^\pi\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{-\pi^2}{2\pi} = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_0^\pi x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\pi\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left\{x\frac{\sin nx}{n} - \left(\frac{-\cos nx}{n^2}\right)\right\}_0^\pi\right]$$

$$= \frac{1}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right) = \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{(-1)^n - 1}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_0^\pi x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0} + \left\{-\pi\frac{\cos nx}{n} - \left(\frac{-\sin nx}{n^2}\right)\right\}_0^\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n}\right]$$

$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi}\cos nx + \left(\frac{1 - 2(-1)^n}{n}\right)\sin nx\right].$$

Q.34. Solve the partial differential equation $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $x > 0, t > 0$, subject to the following conditions. **(IPU-2016)**

(*i*) $u(0,t) = 0, t > 0$ (*ii*) u and $\frac{\partial u}{\partial x} \Rightarrow 0$ as $x \to \infty$ (*iii*) $u(x,0) = e^{-x}, x > 0$.

Ans. Given $\quad \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, x > 0, t > 0 \qquad \ldots(1)$

$$\left.\begin{array}{l}\text{Boundary condition } u(0,t) = 0, t > 0 \\ \text{Initial condition } u(x,0) = e^{-x}, x > 0\end{array}\right\} \qquad \ldots(2)$$

and $u \to 0$, $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

Taking fourier sine transform of (1), we get

$$F_S\left\{\frac{\partial u}{\partial t}\right\} = F_S\left\{2\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \quad \int_0^\infty \frac{\partial u}{\partial t}\sin sx\,dx = \int_0^2 2\frac{\partial^2 u}{\partial x^2}\sin sx\,dx$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty u\sin sx\,dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin sx\,dx$$

$$= 2\left[\sin sx\frac{\partial u}{\partial x}\Big|_0^\infty - \int_0^\infty s\cos sx\frac{\partial u}{\partial x}dx\right]$$

$$= -2s\int_0^\infty \cos sx\frac{\partial u}{\partial x}dx\left[as\ \frac{\partial u}{\partial x} \to 0 \text{ as } x \to 0\right]$$

$$= -2s\left[\cos sx - u\big|_0^\infty + \int_0^\infty s\sin sx\cdot u\,dx\right] \qquad \text{[by (1) and (2)]}$$

$$= -2s^2\int_0^\infty \sin sx\cdot u\,dx$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty u\cdot\sin sx\,dx = -2s^2\int_0^\infty u\cdot\sin sx\,dx$$

$$\frac{d\bar{u}_s}{dt} = -2s^2\bar{u}_s$$

where $\bar{u}_s = F_S\{u(x,t)\}$

$$= \int_0^\infty u(x,t)\cdot \sin sx\,dx$$

Now $\dfrac{d\bar{u}_s}{\bar{u}_s} = -2s^2 dt$

$$\Rightarrow \quad \log \bar{u}_s = -2s^2 t + \log A$$

$$\Rightarrow \quad \bar{u}_s = Ae^{-2s^2 t}$$

$$\Rightarrow \quad \bar{u}_s(s,t) = Ae^{-2s^2 t} \qquad \ldots(3)$$

Putting $t = 0$ in (3), we get

$$\bar{u}_s(s,0) = A$$

$$\therefore \quad A = \bar{u}_s(s,0) = F_S\{u(x,0)\}$$

$$= \int_0^\infty u(x,0)\sin sx\,dx$$

$$= \int_0^\infty e^{-x}\sin sx\,dx$$

$$= \frac{e^{-x}}{s^2+1}\left[-\sin sx - s\cos sx\right]_0^\infty$$

$$= 0 + \frac{s}{s^2+1} = \frac{s}{s^2+1}$$

by (3) $\quad \bar{u}_s(s,t) = \dfrac{s}{s^2+1}e^{-2s^2 t}$

Taking inverse sine transform, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \bar{u}_s(s,t)\sin sx\,ds$$

$$\Rightarrow \quad u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1}e^{-2s^2 t}\sin sx\,dx$$

Q.35. Find the Fourier Cosine series of the function

$$F(x) = \begin{cases} 2x, & 0 < x < 1 \\ 2(2-x), & 1 < x < 2 \end{cases}$$ **(IPU-2017)**

Ans. Given $F(x) = \begin{cases} 2x, & 0 < x < 1 \\ 2(2-x), & 1 < x < 2 \end{cases}$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \text{ here } l = 2 \qquad \ldots(i)$$

$$a_o = \frac{2}{l}\int_0^2 f(x)\,dx$$

$$= \frac{2}{2}\left[\int_0^1 2x\,dx + \int_1^2 2(2-x)\,dx\right]$$

$$= \left[x^2\right]_0^1 + \left[2\left(2x - \frac{x^2}{2}\right)\right]_1^2$$

$$= 1 + 2\left(4 - \frac{4}{2} - 2 + \frac{1}{2}\right)$$

$$= 1 + 2\left(2 - \frac{3}{2}\right) = 2$$

$$a_n = \frac{2}{l}\int_0^2 f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\left[\int_0^1 2x\cos\frac{n\pi x}{2}dx + \int_1^2 (4-2x)\cos\frac{n\pi x}{2}\,dx\right]$$

$$= 2\left[x\sin\frac{n\pi x}{2}\cdot\frac{2}{n\pi} + \cos\frac{n\pi x}{2}\cdot\frac{4}{n^2\pi^2}\right]_0^1$$

$$+2\left[(2-x)\sin\frac{n\pi x}{2}\cdot\frac{2}{n\pi} - (-1)\left(-\cos\frac{n\pi x}{2}\right)\cdot\frac{4}{n^2\pi^2}\right]_1^2$$

$$= 2\left[\frac{2}{n\pi}\cdot\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi\right.$$

$$\left. - \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{8}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

Thus

$$F(x) = 1 + \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]\cos\frac{n\pi x}{2}$$

$$= 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]\cos\frac{n\pi x}{2}$$

(IPU-2017)

Q.36. Find fourier cosine transform of e^{-x^2}.

Ans. $F_C\{e^{-x^2}\} = \int_0^\infty e^{-x^2} \cos sx\, dx = I(\text{say})$ ...(i)

Differentiating w.r.t. s, we have

$$\frac{dI}{ds} = -\int_0^\infty x e^{-x^2} \sin sx\, dx = \frac{1}{2}\int_0^\infty (\sin sx)(-2xe^{-x^2})\, dx$$

$$= \frac{1}{2}\left[\left\{\sin sx\, e^{-x^2}\right\}_0^\infty - s\int_0^\infty \cos sx . e^{-x^2} dx\right]$$

$$= -\frac{s}{2}\int_0^\infty e^{-x^2} . \cos sx\, dx$$

$$\frac{dI}{ds} = \frac{-s}{2} I$$

$$\Rightarrow \quad \frac{dI}{I} = \frac{-s}{2} ds$$

On integrating, we get

$$\log I = \frac{-s^2}{4} + \log A$$

$$\Rightarrow \quad I = A e^{-s^2/4} \quad \text{...(ii)}$$

When $s = 0$, from (i), we get

$$I = \int_0^\infty e^{-x^2} dx$$

$$= \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}} \quad (\text{Put } x^2 = t)$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2}\, dt = \frac{1}{2}\sqrt{1/2} = \frac{\sqrt{\pi}}{2}$$

By (ii), we have, $I = A$.

$$\Rightarrow \quad A = \sqrt{\frac{\pi}{2}}$$

Hence $I = F_C\{e^{-x^2}\} = \sqrt{\frac{\pi}{2}}\, e^{-s^2/4}$

Q.37. Solve $\frac{\partial u}{\partial t} = \frac{2\partial^2 u}{\partial x^2}$, if $u(0, t) = 0$, $u(x, 0) = e^{-x}$, $x > 0$, $u(x, t)$ is bounded where $x > 0$, $t > 0$.

(IPU-2017)

Ans. Given $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$ ...(i)

Boundary conditions are $u(0, t) = 0, t > 0$
and $u \to 0, \frac{\partial u}{\partial x} \to 0$ as $x \to \infty$ ...(ii)

Initial conditions are $u(x, 0) = e^{-x}, x > 0$ ...(iii)

Taking fourier sine transform of (i), we get

$$F_S\left\{\frac{\partial u}{\partial t}\right\} = F_S\left\{2\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \quad \int_0^\infty \frac{\partial u}{\partial t} \sin sx\, dx = \int_0^\infty 2\frac{\partial^2 u}{\partial x^2} \sin sx\, dx$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty u \sin sx\, dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx\, dx$$

$$= 2\left[\left(\sin sx \frac{\partial u}{\partial x}\right)_0^\infty - \int_0^\infty s \cos sx \frac{\partial u}{\partial x} dx\right]$$

$$= -2s\int_0^\infty \cos sx \cdot \frac{\partial u}{\partial x} dx \quad [\text{By (ii)}]$$

$$= -2s\left[(u \cdot \cos sx)_0^\infty + \int_0^\infty s . \sin sx\, u\, dx\right] \quad [\text{as } u \to 0 \text{ as } x \to \infty \,\&\, x \to 0]$$

$$= -2s^2\int_0^\infty \sin sx . u dx$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty u . \sin sx\, dx = -2s^2\int_0^\infty u . \sin sx\, dx$$

$$\Rightarrow \quad \frac{d\bar{u}_s}{dt} = -2s^2 \bar{u}_s$$

$$\Rightarrow \quad \frac{d\bar{u}_s}{\bar{u}_s} = -2s^2\, dt$$

$$\Rightarrow \quad \log \bar{u}_s = -2s^2 t + \log A$$

$$\Rightarrow \quad \bar{u}_s = A\, e^{-2s^2 t} \quad \text{...(iv)}$$

$$\Rightarrow \quad \bar{u}_s(s, t) = A\, e^{-2s^2 t}$$

Putting $t = 0$ in (iv), we get

$$\bar{u}_s(s, o) = A$$

$$\therefore \quad A = \bar{u}_s(s, o) = F_S\{u(x, 0)\}$$

$$= \int_0^\infty u(x, o) \sin sx\, dx$$

[by (iii)]

$$= \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \left[\frac{e^{-x}}{s^2+1} (-\sin sx - s \cos sx) \right]_0^\infty$$

$$A = 0 + \frac{s}{s^2+1} = \frac{s}{s^2+1}$$

$$(iv) \Rightarrow \qquad \bar{u}_s(s, t) = \frac{s}{s^2+1} e^{-2s^2 t}$$

Taking inverse sine transform, we get

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \bar{u}_s(s,t) \sin sx \, ds$$

$$\Rightarrow \qquad u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+1} e^{-2s^2 t} . \sin sx \, ds$$

Q.38. Using Fourier integral prove that

$$\int_0^\infty \frac{\lambda \sin(\lambda x)}{\lambda^2 + m^2} d\lambda = \frac{\pi}{2} e^{-mx}, m > 0, x > 0$$ **(IPU-2017)**

Ans. Let fourier sine intergral be

$$f(x) = e^{-mx}, \quad x > 0$$

Now, $$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty e^{-mt} \sin \lambda t \, dt \, d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \left[\frac{e^{-mt}}{\lambda^2 + m^2} (-m \sin \lambda t - \lambda \cos \lambda t) \right]_0^\infty d\lambda$$

$$\Rightarrow \qquad f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \left(\frac{\lambda}{\lambda^2 + m^2} \right) d\lambda$$

$$\Rightarrow \qquad \int_0^\infty \sin \lambda x \frac{\lambda}{\lambda^2 + m^2} d\lambda = \frac{\pi}{2} f(x)$$

$$\Rightarrow \qquad \int_0^\infty \frac{\lambda}{\lambda^2 + m^2} \sin \lambda x \, d\lambda = \frac{\pi}{2} e^{-mx}$$

Q.39. Find the Fourier series expansion of the function $f(x) = 2x - x^2$, $0 < x < 3$. Also sketch the graph of the function. **(IPU-2017)**

Ans. $f(x) = 2x - x^2$

Here $l = 3/2$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \qquad \text{...(i)}$$

$$a_o = \frac{1}{l}\int_0^{2l} f(x)dx = \frac{2}{3}\int_0^3 (2x - x^2)dx$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\left(9 - \frac{27}{3}\right) = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x - x^2)\sin\frac{2n\pi x}{3}\cdot\frac{3}{2n\pi} - \frac{9(2-2x)}{4n^2\pi^2}\left(-\cos\frac{2n\pi x}{3}\right) + (-2)\left(-\sin\frac{2n\pi x}{3}\right)\frac{27}{8n^3\pi^3}\right]_0^3$$

$$a_n = \frac{2}{3}\times\frac{3}{2}\left[\frac{(2x - x^2)}{n\pi}\sin\frac{2n\pi x}{3} + \frac{(2-2x).3}{2n^2\pi^2}\cos\frac{2n\pi x}{3} + \frac{18}{4n^3\pi^3}\sin\frac{2n\pi x}{3}\right]_0^3$$

$$a_n = \left[0 - \frac{6}{n^2\pi^2}\cos 2n\pi + 0 - \frac{3}{n^2\pi^2}\right]$$

$$a_n = \frac{-6}{n^2\pi^2} - \frac{3}{n^2\pi^2} = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$\frac{2}{3}\left[(2x - x^2)\left(-\cos\frac{2n\pi x}{3}\right)\cdot\frac{3}{2n\pi} - (2-2x)\left(-\sin\frac{2n\pi x}{3}\right)\frac{9}{4n^2\pi^2} + (-2)\cos\frac{2n\pi x}{3}\cdot\frac{27}{8n^3\pi^3}\right]_0^3$$

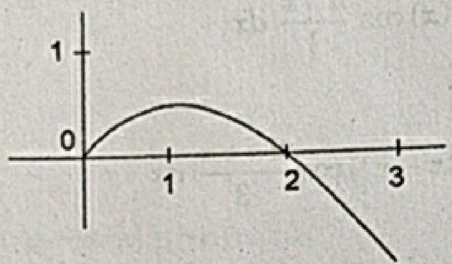
$$a_n = \frac{2}{3}\times\frac{3}{2}\left[\frac{(2x-x^2)}{n\pi}\cos\frac{2n\pi x}{3}+\frac{3(2-2x)}{2n^2\pi^2}\sin\frac{2n\pi x}{3}-\frac{9}{2n^3\pi^3}\cos\frac{2n\pi x}{3}\right]_0^3$$

$$b_n = \left[\frac{3}{n\pi}\cos 2n\pi - 0 - \frac{9}{2n^3\pi^3}\cos 2n\pi + \frac{9}{2n^3\pi^3}\right]$$

$$\Rightarrow \quad b_n = \frac{3}{n\pi}$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} -\frac{9}{n^2\pi^2}\cos\frac{2n\pi x}{3}+\frac{3}{n\pi}\sin\frac{2\pi x}{3}$$

$$\Rightarrow \quad f(x) = \frac{3}{\pi}\sum_{n=1}^{\infty} -\frac{3}{n^2\pi}\cos\frac{2n\pi x}{3}+\frac{1}{n}\sin\frac{2n\pi}{3}$$



Q.40. Find half range cosine series for the function

$$f(x) = \begin{cases} x & , 0 < x < \pi/2 \\ \pi - x & , \pi/2 < x < \pi \end{cases}$$ **(IPU-2017)**

Ans. $$f(x) = \frac{a_o}{2}+\sum_{n=1}^{\infty} a_n \cos nx \quad ...(i)$$

$$a_o = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)dx\right] = \frac{2}{\pi}\left[\frac{x^2}{2}\Big|_0^{\pi/2} + \left(\pi x \frac{\pi^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8}+\pi^2-\frac{\pi^2}{2}-\frac{\pi^2}{2}+\frac{\pi^2}{8}\right] = \frac{2}{\pi}\left[\frac{\pi^2}{4}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left\{\left\{x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right\}_0^{\pi/2} + \left\{(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right.$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2}\cdot\frac{1}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$\therefore \quad f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{[1+(-1)^n]}{n^2}\right\}\cos nx$$

Q.41. Using Fourier transform, solve the heat equation.

$\frac{\partial\theta}{\partial t} = C^2\frac{\partial^2\theta}{\partial x^2}, t > 0$ subject to the condition $\theta(x,0) = \begin{vmatrix}\theta_0, |x| < a\\ 0, |x| > a\end{vmatrix}$.

Find the temperature $\theta(x, t)$. **(IPU-2017)**

Ans.
$$\frac{\partial\theta}{\partial t} = C^2\frac{\partial^2\theta}{\partial x^2}, t > 0 \qquad \text{...(i)}$$

Subject to initial condition $\theta(x,0) = \begin{vmatrix}\theta_0, |x| < a\\ 0, |x| > a\end{vmatrix}$ ...(ii)

Taking fourier transform of (i), we get

$$\int_{-\infty}^{\infty}\frac{\partial\theta}{\partial t}e^{isx}\,dx = c^2\int_{-\infty}^{\infty}\frac{\partial^2\theta}{\partial x^2}e^{isx}\,dx$$

$$\Rightarrow \quad \frac{d}{dt}\int_{-\infty}^{\infty}\theta e^{isx}\,dx = c^2\left[e^{isx}\cdot\frac{\partial\theta}{\partial x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\text{is}\, e^{isx}\cdot\frac{\partial\theta}{\partial x}\,dx\right]$$

$$\Rightarrow \quad \frac{d}{dt}\int_{-\infty}^{\infty}\theta e^{isx}\,dx = \text{is}\, c^2\left[-\int_{-\infty}^{\infty}e^{isx}\cdot\frac{\partial\theta}{\partial x}\,dx\right]\left[\begin{matrix}\text{as } \frac{\partial\theta}{\partial x}\to 0\\ \text{as } x\to\pm\infty\end{matrix}\right]$$

$$\Rightarrow \quad = -\text{is}\; c^2\left[\left|e^{isx}\cdot\theta\right|_{-\infty}^{\infty} - \text{is}\int_{-\infty}^{\infty}e^{isx}\cdot\theta\,dx\right]$$

$$= i^2 s^2\cdot c^2\int_{-\infty}^{\infty}\theta\cdot e^{isx}\,dx\left[\begin{matrix}\text{as } \theta\to 0\\ \text{as } x\to\pm\infty\end{matrix}\right]$$

$$\Rightarrow \quad \frac{d}{dt}\int_{-\infty}^{\infty}\theta e^{isx}\,dx = -s^2c^2\int_{-\infty}^{\infty}e^{isx}\theta\,dx$$

$$\Rightarrow \quad \frac{d\bar{\theta}}{dt} = -c^2s^2\,\bar{\theta} \qquad \text{...(iii)}$$

where $\bar{\theta} = \bar{\theta}(s,t) = F[\theta(x,t)] = \int_{-\infty}^{\infty}\theta e^{isx}\,dx$

$$\Rightarrow \qquad \frac{d\bar{\theta}}{\bar{\theta}} = - c^2 s^2 \, dt$$

on integrating, we get

$$\Rightarrow \qquad \log \bar{\theta} = - c^2 s^2 t + \log A$$

$$\Rightarrow \qquad \bar{\theta} = A\, e^{-c^2 s^2 t} \qquad \ldots(i$$

At $t = 0$, (iv) gives

$$\bar{\theta}(s, 0) = A$$

$$\Rightarrow \qquad A = \bar{\theta}(s, 0) = F\{\theta(x, 0)\}$$

$$= \int_{-\infty}^{\infty} \theta(x,0)\, e^{isx}\, dx \qquad \text{[by (ii}$$

$$= \int_{-\infty}^{-a} 0 \cdot e^{isx}\, dx + \int_{-a}^{a} \theta_0\, e^{isx}\, dx + \int_{a}^{\infty} 0 . e^{isx}\, dx$$

$$= \theta_0 \int_{-a}^{a} e^{isx}\, dx = \theta_0 \left[\frac{e^{isx}}{is.}\right]_{-a}^{a}$$

$$= \theta_0 \left[\frac{e^{ias} - e^{-ias}}{is}\right] = \frac{2\theta_0}{s}\left[\frac{e^{ias} - e^{-ias}}{2i}\right]$$

$$= \frac{2\theta_0 \sin as}{s}$$

By (iv), we get

$$\bar{\theta} = \frac{2\theta_0 \sin as}{s}\, e^{-c^2 s^2 t}$$

Taking inverse fourier transform

$$\theta(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\theta_0 \sin as}{s}\, e^{-c^2 s^2 t} \cdot e^{-sx}\, ds$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin as}{s}\, e^{-c^2 s^2 t}(\cos xs - i \sin xs)\, ds$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin as}{s}\, e^{-c^2 s^2 t} \cos xs \, ds - i \int_{-\infty}^{\infty} \frac{\sin as}{s} \sin xs\, e^{-c^2 s^2 t}\, ds$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin as}{s} \cos xs\, e^{-c^2 s^2 t}\, ds$$

(Second integral vanishes as it is an odd function

$$= \frac{\theta_0}{\pi}\int_0^\infty \frac{e^{-c^2s^2t}}{s}\, 2\sin as \cos xs\, ds$$

$$\theta(x,t) = \frac{\theta_0}{\pi}\int_0^\infty e^{-c^2s^2t}\left(\frac{\sin(a+x)s + \sin(a-x)s}{s}\right) ds$$ is required solution.

Q.42. Compute the Fourier cosine series of the function in $f(x) = 1 - \frac{x}{\pi}$ $(0, \pi)$. (IPU-2018)

Ans.
$$f(x) = 1 - \frac{x}{\pi}$$

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi \left(1 - \frac{x}{\pi}\right) dx$$

$$= \frac{2}{\pi}\left[x - \frac{x^2}{2\pi}\right]_0^\pi = \frac{2}{\pi}\left[\pi - \frac{\pi^2}{2\pi}\right] = \frac{2}{\pi}\left[\pi - \frac{\pi}{2}\right]$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{2} = 1$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi \left(1 - \frac{x}{\pi}\right)\cdot \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{x}{\pi}\right)\frac{\sin nx}{n} - \left(\frac{-1}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[0 - \frac{\cos n\pi}{n^2\pi} + \frac{1}{n^2\pi}\right] = \frac{2}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1 - (-1)^n}{n^2}\cos nx$$

Q.43. Find Fourier sine and cosine transform of xe^{-ax}. (IPU-2018)

Ans.
$$F_s\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx$$

$$= \int_0^\infty xe^{-ax}\sin sx\,dx = \int_0^\infty xe^{-ax}\left(\frac{e^{isx} - e^{-isx}}{2i}\right) dx$$

$$= \frac{1}{2i}\int_0^\infty [x\,e^{(is-a)x} - x\,e^{-(is+a)x}]\,dx$$

$$= \frac{1}{2i}\left[\left\{\frac{x\,e^{-(a-is)x}}{-(a-is)} - \frac{e^{(-is+a)x}}{(a-is)^2}\right\}\right.$$

$$\left.-\left\{\frac{xe^{-(is+a)x}}{-(is+a)} - \frac{e^{-(is+a)x}}{(is+a)^2}\right\}\right]_0^\infty$$

$$= \frac{1}{2i}\left[\left\{(0-0) + \frac{1}{(a-is)^2}\right\} - \left\{(0-0) + \frac{1}{(a+is)^2}\right\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{(a-is)^2} - \frac{1}{(a+is)^2}\right] = \frac{1}{2i}\left[\frac{(a+is)^2 - (a-is)^2}{(a+is)^2(a-is)^2}\right]$$

$$= \frac{1}{2i}\left[\frac{a^2 - s^2 + 2ias - a^2 + s^2 + 2ias}{(a^2+s^2)^2}\right]$$

$$= \frac{1}{2i}\frac{4ias}{(a^2+s^2)^2} = \frac{2as}{(s^2+a^2)^2}$$

$$F_C\{f(x)\} = \int_0^\infty f(x)\cos sx\,dx$$

$$= \int_0^\infty xe^{-ax}\left(\frac{e^{isx} + e^{-isx}}{2}\right)dx = \frac{1}{2}\int_0^\infty\left[x\,e^{-(a-is)x} + xe^{-(a+is)x}\right]dx$$

$$= \frac{1}{2}\left[\left\{\frac{xe^{-(a-is)x}}{-(a-is)} - \frac{e^{-(a-is)x}}{(a-is)^2}\right\}\right.$$

$$\left.+\left\{\frac{xe^{-(a+is)x}}{-(a+is)} - \frac{xe^{-(a+is)x}}{(a+is)^2}\right\}\right]_0^\infty$$

$$= \frac{1}{2}\left[\left\{(0-0) + \frac{1}{(a-is)^2}\right\} + \left\{0 - 0 + \frac{1}{(a+is)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{(a+is)^2 + (a-is)^2}{((a-is)(a+is))^2}\right]$$

$$= \frac{1}{2}\left[\frac{a^2 - s^2 + 2ias + a^2 - s^2 - 2ias}{(a^2+s^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2(a^2-s^2)}{(a^2+s^2)^2}\right] = \frac{a^2 - s^2}{(a^2+s^2)^2}$$

Q.44. Express the function $f(x) = \begin{cases} 1 \text{ for } |x| \le 1 \\ 0 \text{ for } |x| > 1 \end{cases}$, as a Fourier Integral.

Hence evaluate $\int_0^\infty \frac{\sin \mu \cos \mu x \, d\mu}{\mu}$.

(IPU-2018)

Ans.
$$f(x) = \begin{cases} 0, & -\infty < x < -1 \\ 1, & -1 < x < 1 \\ 0, & 1 < x < \infty \end{cases}$$

The Fourier integral for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) \cos \lambda (t - x)\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-1}^1 \cos \lambda(t - x)\, dt + \int_{-\infty}^{-1} 0 \cdot dt + \int_1^\infty 0 \cdot dt\right] d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-1}^1 \cos \lambda (t - x)\, dt\, d\lambda \;=\; \frac{1}{\pi}\int_0^\infty \left|\frac{\sin \lambda(t - x)}{\lambda}\right|_{-1}^{1} d\lambda$$

Q.45. Find the Fourier cosine series of the function

$$f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 4, & 2 \le x \le 4 \end{cases}$$

(IPU-2018)

Ans.
$$f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 4, & 2 \le x \le 4 \end{cases}$$

Here $l = 4$

Cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

Here
$$a_0 = \frac{2}{l}\int_0^l f(x)\, dx$$

$$= \frac{2}{4}\left[\int_0^2 x\, dx + \int_2^4 4\, dx\right] = \frac{1}{2}\left[\left|\frac{x^2}{2}\right|_0^2 + |4x|_2^4\right]$$

$$= \frac{1}{2}[2 + 16 - 8] = \frac{10}{2} = 5$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{4}\left[\int_0^2 x \frac{\cos n\pi x}{4}\, dx + \int_2^4 4 \cdot \cos \frac{n\pi x}{4}\, dx\right]$$

$$= \frac{1}{2}\left[\left(x\sin\frac{n\pi x}{4}\cdot\frac{4}{n\pi}+\frac{\cos n\pi x}{4}\cdot\frac{16}{n^2\pi^2}\right)_0^2 + 4\left(\sin\frac{n\pi x}{4}\cdot\frac{4}{n\pi}\right)_2^4\right]$$

$$= \frac{1}{2}\left[\frac{8}{n\pi}\sin\frac{n\pi}{2}+\frac{16}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{16}{n^2\pi^2}-\frac{16}{n\pi}\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{2}\left[\frac{-8}{n\pi}\sin\frac{n\pi}{2}+\frac{16}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{16}{n^2\pi^2}\right]$$

$$= \frac{-4}{n\pi}\sin\frac{n\pi}{2}+\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{8}{n^2\pi^2}$$

$$f(x) = \frac{5}{2}+\frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi}\cos\frac{n\pi}{2}-\frac{2}{n^2\pi}-\frac{1}{n}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi x}{4}$$

UNIT IV

Q. 1. $\frac{\partial u}{\partial x} = \frac{2\partial u}{\partial t} + u$; $u(x, 0) = 6e^{-3x}, x > 0$ **(IPU-2014)**

Ans. The given equation is

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \qquad \dots (1)$$

Let $$u = X(x)\,T(t) \qquad \dots (2)$$

Where X is a function of x only and T is a function of t only.

Then $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(XT) = T\frac{\partial X}{\partial x}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{\partial T}{\partial t}$$

Substituting in equation (1), we get

$$T\frac{dX}{dx} = 2x\frac{dT}{dt} + XT$$

$\Rightarrow$ $$TX' = 2XT' + XT$$

$\Rightarrow$ $$TX' = X(2T' + T)$$

$\Rightarrow$ $$\frac{X'}{X} = 2\frac{T'}{T} + 1 = -p^2 \text{ (say)}$$

(*i*) $$\frac{dX}{dx} + p^2 X = 0$$

$\Rightarrow$ $$\frac{dX}{X} = -p^2 dx$$

On integration, we get $\log X = -p^2 x + \log C_1$

$$X = C_1 e^{-p^2 x} \qquad \dots (3)$$

(*ii*) $$\frac{2T¢}{T} = -(p^2 + 1)$$

$\Rightarrow$ $$\frac{dT}{T} = -\left(\frac{p^2+1}{2}\right)dt$$

On integration, we get $\log T = -\frac{(p^2+1)}{2}t + \log C_2$

$$T = C_2 e^{-\left(\frac{p+1}{2}\right)t} \qquad \dots (4)$$

From (2), (3) and (4) we get

$$u = XT = C_1 C_2 e^{-p^2 x - \left(\frac{p^2+1}{2}\right)t} \qquad \dots (5)$$

$$u(x, 0) = 6e^{-3x} \text{ (given)}$$

From (5), we have

$$6\,e^{-3x} = C_1C_2\,e^{-p^2x}$$
$$C_1C_2 = 6 \text{ and } p^2 = 3$$

Hence the solution is $u(x, t) = 6\,e^{-3x-2t}$

Q. 2. A rod of lenght l with insulated sides is initially a uniform temperature u_0. Its ends are suddenly cooled to 0°C and kept at that temperature. Find the temperature function $u(x, t)$ **(IPU-2014)**

Ans. The temperature function $u(x, t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

let $\qquad u = XT$

where X is a function of x only and T is a function of t only be a solution of (1).

Then $\qquad \frac{\partial u}{\partial t} = XT' \text{ and } \frac{\partial^2 u}{\partial x^2} = X''T$

Substituting is equation (1)

$$XT' = C^2 X'' T$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{1}{C^2}\frac{T'}{T} = -p^2 \text{ (say)}$$

$$\frac{X''}{X} = -p^2$$

$$\Rightarrow \quad X'' + p^2X = 0$$

$$\Rightarrow \quad X = C_1 \cos px + C_2 \sin px$$

and $\qquad \frac{1}{C^2}\frac{T'}{T} = -p^2$

$$\Rightarrow \quad \frac{T'}{T} = -C^2p^2$$

$$\Rightarrow \quad \log T = -C^2p^2t + \log C_3$$

$$\Rightarrow \quad T = e^{-C^2p^2t}$$

Thus the solution of heat eq. 4 (1) is

$$u(x, t) = (C_1 \cos px + C_2 \sin px)\,C_3\,e^{-c^2p^2t} \qquad \ldots(2)$$

Since the ends $x = 0$ and $x = l$ are cooled to 0°C and kept at that temperature throughout, the boundary condition are $u(0,t) = u(l, t) = 0$ for all t.

Also, $u(x, 0) = u_0$ is the initial condition.

Since $u(0,t) = 0$, we have from (2)

$$0 = C_1C_3\,e^{-c^2p^2t} \Rightarrow c_1 = 0$$

$$u(x, t) = C_2\, C_3 \sin px .\ e^{-c^2p^2t} \qquad \ldots (3)$$

Since $\quad u(l, t) = 0$, we have from (3)

$$0 = C_2C_3 \sin pl\ e^{-c^2p^2t}$$

$$\Rightarrow \quad \text{Sin}\, pl = 0 \Rightarrow pl = n\bar{h}$$

$$\Rightarrow \quad p = \frac{n\pi}{l},\ n \text{ being an integer}$$

Solution (3) reduces to

$$u\,(x, t) = b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{C^2n^2\pi^2t}{l^2}} \quad \text{on replacing}$$

$C_2\, C_3$ by b_n.

The most general solution is obtained by adding all such solution for $n = 1, 2, 3, \ldots$

$$u\,(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{C^2n^2\pi^2t}{l^2}} \qquad \ldots (4)$$

Since $u(x, 0) = u_0$

$$\Rightarrow u_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \text{ which is half-range sin}\,e \text{ series for } u_0$$

$$\Rightarrow \quad b_n = \frac{2}{l}\int_0^l u_0 \sin\left(\frac{x\pi x}{l}\right) dx = \begin{cases} 0, \text{ when } n \text{ is even} \\ \dfrac{4u_0}{x\pi}, \text{when } n \text{ is add} \end{cases}$$

Hence the temprature function is

$$u[x, t] = \frac{4u_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-C^2n^2\pi^2t}{l^2}}$$

$$= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\left(\frac{(2n-1)\pi x}{l}\right) e^{\frac{-C^2n^2\pi^2t}{l^2}}$$

Q.3. Solve the differential equation $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$ subject to the conditions $u = \sin t$ at $x = 0$ and $\dfrac{\partial u}{\partial x} = \sin t$ at $x = 0$ (IPU-2014)

Ans.

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \qquad \ldots (1)$$

Let $u = XT$

Where X is a function of x only and T is a function of t only be a solution of (1).

Then
$$\frac{\partial^2 u}{\partial t^2} = XT'' \text{ and } \frac{\partial^2 u}{\partial x^2} = X''T$$

Substrituting in eq. (1)
$$X\,T'' = 2^2 X''\,T$$

$$\Rightarrow \quad \frac{1}{2^2}\frac{T''}{T} = \frac{X''}{X} = -p^2 \text{ (say)}$$

$$\frac{X''}{X} = -p^2$$

$$\Rightarrow \quad X = C_1 \cos px + C_2 \sin px$$

$$\frac{1}{2^2}\frac{T''}{T} = -p^2$$

$$\Rightarrow \quad T = C_3 \cos(2pt) + C_4 \sin(2pt)$$

Thus the solution of equetion (1) is

$$u = (C_1 \cos px + C_2 \sin px)\,(C_3 \cos(2pt) + C_4 \sin(2pt))$$

On putting $x = 0, u = \sin t$ in (2), we get

$$\sin t = C_1 (C_3 \cos(2pt) + C_4 \sin(2pt))$$

$$\Rightarrow \quad C_1C_3 = 0 \text{ and } C_1C_4 = 1,\ 2p = 1 \Rightarrow p = \frac{1}{2}$$

$$\Rightarrow \quad C_3 = 0 \text{ and } C_4 = \frac{1}{C_1}$$

So eqn. (2) reduce to

$$u = \left(C_1 \cos\frac{x}{2} + C_2 \sin\frac{x}{2}\right)\frac{1}{C_1}\sin t$$

$$\Rightarrow \quad u = \left(\cos\frac{x}{2} + \left(\frac{C_2}{C_1}\right)\sin\frac{x}{2}\right)\sin t$$

$$\Rightarrow \quad u = \left(\cos\frac{x}{2} + C_5 \sin\frac{x}{2}\right)\sin t \text{ where } C_5 = \frac{C_2}{C_1}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \left(\frac{-1}{2}\sin\frac{x}{2} + \frac{1}{2}C_5 \cos\frac{x}{2}\right)\sin t$$

On putting $x = 0, \frac{\partial u}{\partial x} = \sin t$ in (4) we get

$$\sin t = \frac{1}{2}C_5 \sin t$$

$$\Rightarrow \quad C_5 = 2$$

Hence the solution to given equation (1) is

$$u = \left(\cos\frac{x}{2} + 2\sin\frac{x}{2}\right)\sin t$$

Q. 4. Write the steady state two dimensional heat flow equation. Find its solution in cartesian coordinates. (IP[...]

Ans. In the steady state two dimensional ...

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

Let $u(x,y) = X(x).\ Y(y)$ be a solution of (1)
When X is a function of x only and y is a function of y only.

$$\frac{\partial^2 u}{\partial x^2} = X''Y \text{ and } \frac{\partial^2 u}{\partial y^2} = XY''$$

Put in equation (1)

$$X''Y = XY''$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{-Y''}{4} = K(\text{say}) \qquad ...(2)$$

(*i*) When K is positive and is equal to p^2, say

$$\Rightarrow \quad X = C_1 e^{px} + C_2 e^{-px} \text{ and } Y = C_3 \cos py + C_4 \sin py$$

(*ii*) When K is negative and is equal to $-p^2$, say

$$\Rightarrow \quad X = C_5 \cos px + C_6 \sin px,\ Y = C_7 e^{py} + C_8 e^{py}$$

(*iii*) When $K = 0$

$$\Rightarrow \quad X = C_9 x + C_{10} \text{ and } Y = C_{11}\, y + C_{12}$$

The various possible solution of (1) are

$$u = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \qquad ...(3)$$

$$u = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py}) \qquad ...(4)$$

$$u = (Cx_9 + C_{10})(C_{11} y + C_{12}) \qquad ...(5)$$

Out of these we take that solution which is consistent with the given boundary conditions.

Q. 5. A tightly stretched string with fixed end point $x = 0$ and $x = l$ is initially in a position given by $u(x) = u_0 \sin^3(\pi x/l)$, if it is released from rest from this position. Find the displacement $u(x,t)$. (IPU-2015)

Ans. The equation of the string is

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

The solution of equation (1) is

$$u(x, t) = (C_1 \cos cpt + C_2 \sin cpt)(C_3 \cos px + C_4 \sin px) \qquad ..(2)$$

Boundary conditions are

$$u(0, t) = 0 \qquad ...(3)$$

$$u(l, t) = 0 \qquad ...(4)$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \qquad ...(5)$$

$$u(x, 0) = u_0 \sin^3\left(\frac{\pi x}{l}\right) \qquad ...(6)$$

Applying boundary condition in (2)

$$u(0, t) = 0 = (C_1 \cos cpt + C_2 \sin cpt)C_3$$

$\therefore$ From (2), $u(x, t) = (C_1 \cos cpt + C_2 \sin cpt)\, C_4 \sin pl$

Again, $\quad u(l, t) = 0 = (C_1 \cos cpt + C_2 \sin cpt)\, C_4 \sin pl$ ...(7)

$\Rightarrow \quad \sin pl = 0 = \sin n\pi,\ x \in \mathrm{I}$

$\Rightarrow \quad p = \dfrac{n\pi}{l}$

From (7)

$$u(x, t) = \left(C_1 \cos \frac{n\pi ct}{l} + C_2 \sin \frac{n\pi ct}{l}\right) C_4 \sin\left(\frac{n\pi x}{l}\right) \quad ...(8)$$

$$\frac{\partial u}{\partial t} = \frac{n\pi c}{l}\left(-C_1 \sin \frac{n\pi ct}{l} + C_2 \cos \frac{n\pi ct}{l}\right) C_4 \mathrm{Sin}\left(\frac{n\pi x}{l}\right)$$

At $\quad t = 0,$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} C_2 C_4 \sin\left(\frac{n\pi x}{l}\right)$$

$\Rightarrow \quad C_2 = 0$

From (8) $\quad u(x, t) = C_1 C_4 \sin\left(\dfrac{n\pi x}{l}\right)\cos\left(\dfrac{n\pi ct}{l}\right) = b_n \sin\left(\dfrac{n\pi x}{l}\right)\cos\left(\dfrac{n\pi ct}{l}\right)$

Most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) \quad ...(9)$$

$$u(x, 0) = u_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow u_0 = \left\{\frac{3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}}{4}\right\} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + \ldots$$

On comparing, we get

$$b_1 = \frac{3u_0}{4}, b_2 = 0, b_3 = -\frac{u_0}{4}, u_4 = u_5 = \quad ...(10)$$

Hence from (9), $u(x, t) = \dfrac{3u_0}{4}\sin\dfrac{\pi x}{l}\cos\dfrac{\pi ct}{l} - \dfrac{u_0}{4}\sin\left(\dfrac{3\pi x}{l}\right)\cos\left(\dfrac{3\pi ct}{l}\right)$

Q. 6. Solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u : u(0, y) = 0$

$\left(\dfrac{\partial u}{\partial x}\right)_{(0,y)} = \text{it } e^{-3y}$ by the method of separation of variables. (IPU-2015)

Ans. Let $\quad U = XY$ ...(1)

Where X is a function of x only and Y is a function of y only.

$$\frac{\partial u}{\partial y} = X\frac{\partial Y}{\partial y} = XY' \text{ and } \frac{\partial^2 u}{\partial x^2} = YX''$$

Put in given eq^n. $YX'' = XY' + 2XY = X(Y' + 2Y)$

$$\Rightarrow \frac{X''}{X} = \frac{Y'}{y} + 2 = \text{k (say)}$$

$$\frac{X''}{X} = k$$

$$\Rightarrow \quad X'' - kX = 0$$

Auxilliary equation is

$$m^2 - k = 0$$

$$m = \pm\sqrt{k}$$

$$C.F. = C_1 e^{\sqrt{kx}} + C_2 e^{-\sqrt{kx}}$$

$$\text{P.I} = 0$$

$$X = C_1 e^{\sqrt{kx}} + C_2 e^{-\sqrt{kx}}$$

and $\qquad \dfrac{Y'}{y} + 2 = k$

$$\frac{Y'}{Y} = k - 2$$

On interpation, we get

$$\log Y = (k-2)y + \log C_3$$

$$Y = C_3 e^{(k-2)y}$$

Hence from (1)

$$u(x, y) = \left(C_1 e^{\sqrt{kx}} + C_2 e^{\sqrt{kx}}\right) C_3 e^{(k-2)y} \qquad \text{...(2)}$$

Applying the condition $u(0, y) = 0$ in (2), we get

$$0 = (C_1 + C_2)\, C_3 e^{(k-2)y}$$

$$\Rightarrow \quad C_1 + C_2 = 0 \Rightarrow C_2 = -C_1$$

From (2), the most general solution is

$$u(x, y) = \sum C_1 C_3 \left(e^{\sqrt{kx}} - \bar{e}^{\sqrt{kx}}\right) e^{(k-2)y}$$

$$\frac{\partial u}{\partial x} = \sum C_1 C_3 \sqrt{k}\,(e^{\sqrt{kx}} + e^{-\sqrt{kx}})\, e^{(k-2)y}$$

$$\left(\frac{\partial u}{\partial n}\right)_{x=0} = 1 + e^{-3y} = \sum C_1 C_3 \sqrt{k}(2) e^{(k-2)y}$$

$$= \sum b_n e^{(k-2)y}$$

Comparing the coefficients, we get

(*i*) $\qquad b_1 = 1, k - 2 = 0$

$$2C_1C_3\sqrt{k} = 1, k = 2$$

$$C_1C_3 = \frac{1}{2\sqrt{2}}$$

(*ii*) $$b_3 = -1, k - 2 = 3$$

$$2C_1C_3\sqrt{k} = 1, k = -1$$

$$C_1C_3 = \frac{1}{2i}$$

hence from (6). the particular solution is

$$u(x, y) = \frac{1}{2\sqrt{2}}\left(e^{\sqrt{2x}} - e^{-\sqrt{2x}}\right) + \frac{1}{2i}(e^{ix} - e^{-ix})e^{-3y}$$

$$\Rightarrow \quad \boxed{u(x, y) = \frac{1}{\sqrt{2}}\sin h\sqrt{2x} + e^{-3y}\sin x}$$

Q.7. Using the method of separation of variable solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u$ where $u(x, 0) = e^{-3x} - 2e^{-x}, x > 0, y > 0$ **(IPU-2017)**

Ans. Given equation is

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u \quad ...(1)$$

Let $U = X(x)\, Y(y) = XY$

$$\therefore \quad \frac{\partial u}{\partial x} = X'Y$$

and $$\frac{\partial u}{\partial y} = XY'$$

Equating (1) becomes.

$$X'Y = 2XY' + XY$$

$$\Rightarrow \quad X'Y = X(2Y' + Y)$$

$$\Rightarrow \quad \frac{X'}{X} = \frac{2Y'}{Y} + 1$$

Since x and y are independent variables

$\therefore$ it is true only when each equation is equal to a constant.

Let $$\frac{X'}{X} = \frac{2Y'}{Y} + 1 = a \text{ (say)}$$

Now $$\frac{X'}{X} = a.$$

$$\Rightarrow \frac{dX}{dx}\cdot\frac{1}{X} = a$$

$$\Rightarrow \quad \frac{dX}{X} = a\,dx$$

$$\Rightarrow \quad \log X = ax + \log c_1$$

$$\Rightarrow \quad X = C_1 e^{ax}$$

and $$\frac{2Y'}{Y} + 1 = a$$

$$\Rightarrow \quad 2\frac{dY}{dy}\cdot\frac{1}{Y} + 1 = a$$

$$\Rightarrow \quad 2\frac{dY}{Y} = (a-1)\text{dy}$$

$$\frac{dY}{Y} = \left(\frac{a-1}{2}\right)dy$$

$$\log Y = \left(\frac{a-1}{2}\right)y + \log C_2$$

$$Y = C_2 e^{\left(\frac{a-1}{2}\right)y}$$

$$U = C_1 C_2 e^{ax} e^{\left(\frac{a-1}{2}\right)y} \qquad \text{...(2)}$$

As given $\quad u(x, 0) = e^{-3x} - 2e^{-x}$

$$\Rightarrow \quad e^{-3x} - 2e^{-x} = C_1 C_2 e^{ax}$$

Comparing two, we get

$$e^{-3x} = C_1 C_2 e^{ax}$$

Here $\quad C_1 C_2 = 1, a = -3$

and $\quad 2e^{-x} = C_1 C_2 e^{ax}$

$$\Rightarrow \quad C_1 C_2 = 2, a = -1$$

Thus eq^n (2) becomes

$$U(x, y) = e^{-3x} e^{-2y} - 2e^{-x} e^{-y}$$

$$\Rightarrow \quad U = e^{-3x-2y} - 2e^{-x-y}$$

Q.8. Solve the boundary value problem $\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}$, given that

$y(0, t) = 0$, $y(5, t) = 0$, $y(x, 0) = 0$ and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sin \pi x$ (IPU-2017)

Ans. Wave equation is $\frac{\partial^2 y}{\partial t^2} = \frac{4\partial^2 y}{\partial x^2}$ ...(1)

Let $\quad y = X(x)\,T(t) = XT$ ...(2)

be the solution of (1)

Then $\quad \frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$

By equation (1), we have

$$XT'' = 4\,X''T$$

$$\Rightarrow \qquad \frac{X''}{X} = \frac{1}{4}\frac{T''}{T} \qquad ...(3)$$

Since LHS of (3) is a function of x only and RHS is of t only. As x and t are independent variables. Thus both sides reduce to a constant say $-a^2$.

$\therefore$ (3) reduces to

$$\frac{X''}{X} = -a^2 \text{ and } \frac{1}{4}\frac{T''}{T} = -a^2$$

$\Rightarrow \quad X'' + a^2 X = 0$

$\Rightarrow \quad (D^2 + a^2)X = 0 \Rightarrow D = \pm ia$

$\therefore \quad X = C_1 \cos ax + C_2 \sin ax$

and $\quad T'' + a^2 4T = 0$

$\Rightarrow \quad (D^2 + 4a^2)T = 0$

$\Rightarrow \quad D = \pm i2a$

$\therefore \quad T = C_3 \cos 2at + C_4 \sin 2at$

Solution is

$$y = (C_1 \cos ax + C_2 \sin ax)(C_3 \cos 2at + C_4 \sin 2at) \qquad ...(4)$$

Now, applying boundary conditions in (4)

$$y(0, t) = 0, y(5, t) = 0$$

$$y(0, t) = C_1 (C_3 \cos 2at + C_4 \sin 2at)$$

$$0 = C_1 (C_3 \cos 2at + C_4 \sin 2at)$$

$\Rightarrow \quad C_1 = 0$

$\therefore$ (4) reduces to

$$y(x, t) = C_2 \sin ax (C_3 \cos 2at + C_4 \sin 2at) \qquad ...(5)$$

Apply $y(5, t) = 0$ in equation (5), we get

$$0 = C_2 \sin 5a (C_3 \cos 2at + C_4 \sin 2at)$$

This is satisfied when

$$\sin 5a = 0$$

$\Rightarrow \quad 5a = n\pi$

$\Rightarrow \quad a = \frac{n\pi}{5}$ where $n = 1, 2, ...$

Solution of wave equation reduces to

$$y(x, t) = C_2\left(C_3 \cos_2 \frac{n\pi t}{5} + C_4 \sin \frac{2n\pi}{5}t\right)\sin \frac{n\pi x}{5}$$

$$\Rightarrow \quad y(x, t) = \left(a_n \cos \frac{2n\pi t}{5} + b_n \sin \frac{2n\pi}{5}t\right)\sin \frac{n\pi x}{5}$$

where $\quad a_n = C_2 C_3$ and $b_n = C_2 C_4$

Adding solution for different values of n, we get

$$y(x, t) = \sum_{n=1}^{\infty}\left(a_n \cos \frac{2n\pi}{5}t + b_n \sin \frac{2n\pi}{5}t\right)\sin \frac{n\pi x}{5} \qquad ...(6)$$

Now, applying initial conditions on (6),

$y(x, 0) = 0$ and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sin \pi x$

$\therefore$ by (6), we get

$$(y, 0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{5}$$

$$\Rightarrow \qquad 0 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{5}$$

$$\Rightarrow \qquad a_n = 0$$

Thus, solution reduces to

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi}{5}t \sin\frac{n\pi x}{5} \qquad ...(7)$$

$$\Rightarrow \qquad \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \cos\frac{2n\pi}{5}t.\frac{2n\pi}{5}\sin\frac{n\pi x}{5}$$

As given $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sin \pi x$

$$\Rightarrow \sin \pi x = \frac{2\pi}{5}\sum_{n=1}^{\infty} nbn \sin\frac{n\pi x}{5}$$

This represents fourier sine series for $\sin \pi x$

$$\Rightarrow \frac{2\pi}{5}nbn = \frac{2}{5}\int_0^5 \sin \pi x \sin\frac{n\pi x}{5}\,dx$$

We solve for $n = 1$.

$$\frac{2\pi}{5}b_1 = \frac{2}{5}\int_0^5 \sin \pi x \sin\frac{\pi x}{5}\,dx \Rightarrow b_1 = \frac{1}{2\pi}$$

$\therefore$ Complete Solution is

$$y = \frac{1}{2\pi}\sin\frac{\pi x}{5}\sin\frac{2\pi t}{5}$$

Q.9. An insulated rod of length l has its end A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t. **(IPU-2017)**

Ans. The temperature function $u(x, t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Prior to temperture change at end B, when $t = 0$, the heat flow was independent of time (steady state) i.e., $\frac{\partial u}{\partial t} = 0$.

When temperature u depends upon x and not on t, (1) $\frac{\partial^2 u}{\partial x^2} = 0$

$$\Rightarrow \qquad \frac{\partial^2 u}{\partial x^2} = 0$$

$\therefore$ General sol^n is $\qquad u = ax + b$ ...(2)

Since $\qquad u = 0$ for $x = 0$

$u = 100$ for $x = l$

$\therefore$ (2) gives

$$0 = b$$

$$\Rightarrow \qquad u = ax$$

and $\qquad 100 = al \Rightarrow a = \frac{100}{l}$

$\therefore$ Initial condition is $u(x, 0) = \frac{100}{l}x$

Boundary conditions for subsequent flow are $u(0, t) = 0$, $u(l, t) = 0$ for all values of t.

Let $\qquad U = X(x)\,T(t) = XT$

$$\Rightarrow \qquad \frac{\partial u}{\partial t} = XT', \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

Put these values in equation (1)

$$XT' = c^2 X''T$$

$$\Rightarrow \qquad \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T}$$

Now LHS is a function of x only and RHS is a function of t only. Since x and t are independent.

$\therefore$ both sides reduce to a constant say 'k':

$\therefore$ (3) gives

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k.$$

Let $\qquad k = -p^2$

$$\Rightarrow \qquad \frac{X''}{X} = -p^2 \Rightarrow X'' + p^2 X = 0$$

Aux. eq^n is $\qquad m^2 + p^2 = 0 \Rightarrow m = \pm ip$

$$\therefore \qquad X = c_1 \cos px + c_2 \sin px$$

Also $\qquad \frac{T'}{T} = -p^2c^2$

$$\therefore \qquad \log T = -p^2c^2t + \log c_3$$

$$\Rightarrow \qquad T = c_3 e^{-p^2c^2t}$$

General solution is

$$u(x, t) = (c_1 \cos px + c_2 \sin px)e^{-p^2c^2t} \qquad \text{...(3)}$$

Using condition $u(0, t) = 0$ in (3), we get

$$u(0, t) = 0 = c_1 e^{-p^2 c^2 t}$$

$$\Rightarrow \quad c_1 = 0$$

equation (3) reduces to

$$u(x, t) = c_2 \sin px \, e^{-p^2 c^2 t} \quad \text{...(4)}$$

Using $\quad u(l, t) = 0$ in (4)

$$\Rightarrow \quad u(l, t) = 0 = c_2 \sin pl e^{-p^2 c^2 t}$$

$$\Rightarrow \quad \sin pl = 0$$

$$\Rightarrow \quad pl = n\pi$$

$$\Rightarrow \quad p = \frac{n\pi}{l}, \qquad n = 1, 2, ...$$

equation (4) reduces to

$$u(x, t) = c_2 \sin \frac{n\pi x}{l} e^{\frac{-n^2\pi^2}{l^2} c^2 t}$$

Adding all such solutions for different values of n, we get

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{\frac{-n^2\pi^2 c^2 t}{l^2}} \quad \text{...(5)}$$

As initial condition is $u(x, 0) = \frac{100x}{l}$

$$\Rightarrow \quad \frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Which is a fourier sine series for $\frac{100x}{l}$

$$\therefore \quad b_n = \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} \, dx$$

$$\Rightarrow \quad b_n = \frac{200}{l^2} \left[x\left(-\cos \frac{n\pi x}{l}\right) \cdot \frac{l}{n\pi} - \left(-\sin \frac{n\pi x}{l}\right) \frac{l^2}{n^2\pi^2} \right]_0^l$$

$$\Rightarrow \quad b_n = \frac{200}{l^2} \left[\frac{-l^2}{n\pi} \cos n\pi \right] = \frac{-200}{n\pi}(-1)^n = \frac{200}{n\pi}(-1)^{n+1}$$

$$\therefore \quad u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 c^2 t}{l^2}}$$

Q.10. Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C. **(IPU-2016)**

Ans. Consider equation $\quad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad$ ...(1)

As solved earlier solution is

$$u(x, 0) = \frac{100x}{l} \quad \text{...(2)}$$

Further B.c's are $\left.\begin{aligned} u(0,t) &= 20 \\ u(l,t) &= 80 \end{aligned}\right\} - (A)\ \forall\, t$

Let the required solution be

$$u(x, t) = u_s(x, t) + u_t(x, t) \qquad ...(3)$$

Let u_s is steady state Solution and u_t is transient solution given by

$$u_t(x, t) = u(x, t) - u_s(x, t) \qquad ...(4)$$

For steady state, solution is gives as

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \quad u_s(x, t) = c_1 x + c_2 \qquad ...(5)$$

Using condition (A)

$$(5) \Rightarrow \quad 20 = c_2$$

$$\therefore \quad u_s(x, t) = c_1 x + 20$$

and

$$80 = c_1 l + 20$$

$$\Rightarrow \quad c_1 = \frac{60}{l}$$

$$\therefore \quad u_s(x, t) = \frac{60x}{l} + 20 \qquad ...(6)$$

At $x = 0$, By (4) $\Rightarrow u_t(0, t) = u(0, t) - u_s(0, t)$

$$= 20 - 20$$

$$= 0. \qquad \text{[By (A) and (6)]}$$

and

$$u_t(l, t) = u(l, t) - u_s(l, t)$$

$$= 80 - 80 = 0$$

Again by (4) $u_t(x, 0) = u(x, 0) - u_s(x, 0)$

$$= \frac{100x}{l} - \frac{60x}{l} - 20$$

$$= \frac{40x}{l} - 20$$

Now $\quad u_t(0, t) = 0,\ u_t(l, t) = 0$

and $\quad u_t(x, 0) = \dfrac{40x}{l} - 20 \qquad ...(7)$

Then Sol^n of (1) is given by

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-n^2\pi^2c^2t/l^2}$$

using (7)

$$u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

This is a fourier sine series for $\left(\dfrac{40x}{l} - 20\right)$

where $$b_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{l}\left[\left(\frac{40x}{l} - 20\right)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - \left(\frac{40}{l}\right)\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right]_0^l$$

$$b_n = \frac{2}{l}\left[-20\cos n\pi . \frac{l}{n\pi} - \frac{20l}{n\pi}\right]$$

$$b_n = \frac{-40}{n\pi}\left((-1)^n + 1\right)$$

$$b_n = \begin{cases} \dfrac{-80}{m\pi}, & \text{n is odd} \\ & \text{n is even} \end{cases}$$

$$b_n = \begin{cases} 0, & \text{n is odd} \\ \dfrac{-80}{m\pi}, & \text{n = 2m (even)} \end{cases}$$

$$u_t(x, t) = \frac{-80}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}\,e^{-\frac{\pi^2 c^2 4m^2 t}{l^2}}$$

Using (3), (6) and (8), we get

$$u(x, t) = \frac{60x}{l} + 20 - \frac{80}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}\,e^{-\frac{4m^2\pi^2c^2t}{l^2}}$$

Q.12. Find the solution of the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which satisfies the conditions **(IPU-2016)**

(*i*) $u \to 0$ as $y \to \infty$ for all x (*ii*) $u = 0$ at $x = 0$ for all y

(*iii*) $u = 0$ at $x = l$ for all y (iv) $u = lx - x^2$ if $y = 0$ for all $x \in (0, l)$

Ans. Given equation is $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(A)$$

The boundary conditions are

$$\left.\begin{aligned} u(0, y) &= 0 \\ u(l, y) &= 0 \end{aligned}\right\}\text{for all y}$$

$$u(x, \infty) = 0\ \forall x$$
$$u(x, 0) = lx - x^2 \quad 0 < x < l$$

The three possible solutions are

(*i*) $u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

(ii) $u(x, y) = (c_5 \cos px + c_6 \sin px)\, c_7 e^{py} + c_8 e^{-py}$

(iii) $u(x,y) = (c_9 x + c_{10})(c_{11} y + c_{12})$

from the condition that $u \to 0$ as $y \to \infty$ for all value of x, solutions (i) and (iii) lead to trivial solutions and hence (ii) is the only suitable one.

i.e $u(x,y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py})$ ...(2)

using boundary conditions $u(0,y) = 0$ in (2) gives

$$0 = A(ce^{py} + De^{-py})$$

$$\Rightarrow \quad A = 0$$

$\therefore$ (2) reduces to

$$u(x,y) = B \sin px\, (Ce^{py} + De^{-py})$$

$$u(x,y) = \sin px\, (C'e^{py} + D'e^{-py}) \quad ...(3)$$

Using the condition $u(l,y) = 0$

$$0 = \sin pl\, (C'e^{py} + D'e^{-py})$$

$\Rightarrow \sin pl = 0$ or $p = \dfrac{n\pi}{l}$, n being an integer.

Also, using $u(x, \infty) = 0$ in (3), we get $C' = 0$

By (3), we get

$$u(x,y) = \sin \frac{n\pi x}{l} . D\, e^{-n\pi y/l}, \; n \text{ is an integer}$$

$\therefore$ General solution is of the form

$$u(x,y) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{l} e^{-ny\pi/l} \quad ...(4)$$

Using, condition $u(x, 0) = lx - x^2$, we get

$$lx - x^2 = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{l}, \; 0 < x < l$$

which is half range sine series

$$\therefore D_n = \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} dx.$$

$$= \frac{2}{l}\left[\left| lx\left(-\cos\frac{n\pi x}{l}\right).\frac{l}{n\pi} - l\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_0^l\right.$$

$$\left. -\left(x^2\left(-\cos\frac{n\pi x}{l}\right).\frac{l}{n\pi} - 2x\left(-\sin\frac{n\pi x}{l}\right).\frac{l^2}{n^2\pi^2} + 2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right|_0^l\right]$$

$$= \frac{2}{l}\left[\frac{-l^3}{n\pi}\cos n\pi + \frac{l^3}{n\pi}\cos n\pi - \frac{2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{4l^2}{n^3\pi^3}\left[-(-1)^n + 1\right]$$

$$D_n = \begin{cases} \dfrac{8l^2}{n^3\pi^3}, n \text{ is odd} \\ 0, n \text{ is even} \end{cases}$$

$$\therefore \qquad u(x,y) = \frac{8l^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)x\pi}{l}e^{(2n-1)\pi,y/l}$$

Q.14. A long rectangular plate of width π cm with insulated surfaces has its temperature equal to zero on both the long sides and one of the short side so that $u(0, y) = 0$, $u(\pi, y) = 0$, $u(x, \infty) = 0$ and $u(x, 0) = kx$. Find the steady state temperature within the plate. **(IPU-2016)**

Ans. Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

$$\left.\begin{array}{l} u(0,y) = 0, 0 < y < \infty \\ u(\pi,y) = 0, 0 < y < \infty \end{array}\right\} B.C.$$

$$u(x,\infty) = 0, 0 < x < \pi$$
$$u(x,\infty) = kx, 0 < x < \pi$$

General solutions of (1) are

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \qquad ...(2)$$
$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad ...(3)$$
$$u(x,y) = (c_1 x + c_2)(c_3 y + c_4) - (4)$$

As solution (2) does not satisfy the boundary conditions for all values of y.

Also (4) does not satisfy.

Thus only possible solution is (3), *i.e.* of the form

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \qquad ...(5)$$

for $\quad u(0,y) = 0$

$\Rightarrow \quad 0 = A\,[Ce^{py} + De^{-py}]$

$\Rightarrow \quad A = 0$

(5) reduces to

$$u(x,y) = B\sin px\,[Ce^{py} + De^{-py}]$$
$$u(x,y) = \sin px\,[C'e^{py} + D'e^{-py}] \qquad ...(6)$$

for $\quad u(\pi, y) = 0$

$\sin p\pi\,(C'e^{py} + D'e^{-py}) = 0.$

$\Rightarrow \quad \sin p\pi = 0$

$\Rightarrow \quad p\pi = n\pi \Rightarrow p = n, \quad n$ being on integer and $\qquad u(x, \infty) = 0.$

$\Rightarrow \qquad c' = 0$

$\therefore$ Reduced solution (6) is,

$u(x,y) \quad = D' \sin nx\, e^{-ny}$, n being an integer.

Adding all solution, we get

$$u = \sum D_n \sin nx e^{-ny} \qquad \text{...(7)}$$

Given $u(x, 0) = kx$.

$$kx = \sum D_n \sin nx$$

This is half range sine series in interval $(0, \pi)$.

$$D_n = \frac{2}{\pi}\int_0^{\pi} kx \sin nx\, dx$$

$$= \frac{2k}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2k}{\pi}\left[-\frac{\pi}{n}\cos n\pi\right]$$

$$= \frac{-2k}{n}(-1)^n$$

$$= \frac{2k}{n}(-1)^{n+1}$$

Thus (7) reduces to

$$u(x,y) \quad = 2k \sum \frac{(-1)^{n+1}}{n} \sin nx e^{-ny}.$$