Classification of Partial Differential Equations :-

Consider the Equation

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+F(x,y,u,p,q)=0 \longrightarrow (1)$$

Where A, B, C may be constants or function of x and y.

Now the eqn (1) is

(1) Elliptic if $B^2-4AC<0$

(2) Parabolic if $B^2-4AC=0$

(3) Hyperbolic if $B^2-4AC>0$

Q. (1) Classify the equation

$$2\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+3\frac{\partial^2 u}{\partial y^2}=0$$

Soln Here $A=2, B=4, C=3$

$\therefore B^2-4AC=16-24 = -8<0$

i.e Elliptic

Q.2 $$\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}=0$$

Soln. $A=1, B=4, C=4$

$\therefore B^2-4AC=16-4\times4=0$

i.e Parabolic

Q.3. $$2\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+3\frac{\partial^2 u}{\partial y^2}=0$$

Soln $A=2, B=6, C=3$

$\therefore B^2-4AC=36-4\times2\times3=36-24=12>0$

i.e Hyperbolic.

Classification of Partial Differential Equation :—

In practical problems, the following types of equations are generally used

(1) Wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

(2) One-dimensional heat flow : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

(3) Two-dimensional heat flow : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Consider the equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + F(x, y, u, p, q) = 0 \longrightarrow ①$$

Where A, B, C may be constants or functions of x and y.

Now the eqn ① is

① Elliptic if $B^2 - 4AC < 0$

② Parabolic if $B^2 - 4AC = 0$

③ Hyperbolic if $B^2 - 4AC > 0$

Q.1 Classify the equation

$$2 \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 3 \frac{\partial^2 u}{\partial y^2} = 0$$

Sol: Given eqn compair with $A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + F(x, y, u, p, q) = 0$

$\therefore A = 2, \; B = 4, \; C = 3$

$\therefore B^2 - 4AC = 16 - 24 = -8 < 0$

i.e Given eqn is Elliptic Ans

Q.2 $\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} = 0$

Sol $A = 1, B = 4, C = 4$

$\therefore B^2 - 4AC = 16 - 16 = 0$ i.e Given eqn is Parabolic

Q.3 $2 \frac{\partial^2 u}{\partial x^2} + 6 \frac{\partial^2 u}{\partial x \partial y} + 3 \frac{\partial^2 u}{\partial y^2} = 0$

Soln $A = 2, B = 6, C = 3$

$\therefore B^2 - 4AC = 36 - 24 = 12 > 0$

i.e given eqn is Hyperbolic Ans

Method of Separation of Variable :→

In this method, we assume that the dependent variable is the product of two ~~...~~ functions, each of which involves only one of the independent variable. So two ordinary differential equations are formed.

(1) Using the method of Separation of Variable, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

→(4)

Where $u(x,0) = 6e^{-3x}$

Sol. The given equation is.

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \longrightarrow (1)$$

Let $u = X \cdot T \longrightarrow (2)$

Where X is function of x and T is function of t only

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} X \cdot T$$

$$\frac{\partial u}{\partial x} = T \cdot \frac{\partial X}{\partial x}$$

And $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial t} X \cdot T$

$$\frac{\partial u}{\partial t} = X \cdot \frac{\partial T}{\partial t}$$

Putting these value in eqn (1)

$$\therefore T \cdot \frac{\partial X}{\partial x} = 2X \cdot \frac{\partial T}{\partial t} + X \cdot T$$

$$T \cdot X' = 2X \cdot T' + X \cdot T$$

divid by $X \cdot T$

$$\therefore \frac{X'}{X} = 2\frac{T'}{T} + 1 = c \text{ (say)}$$

Take First condition

$$\frac{X'}{X} = c$$

$$\frac{1}{X}\cdot\frac{dX}{dx} = c$$

$$\frac{dX}{X} = c\,dx$$

on Integrating

$$\operatorname{Log} X = cx + \operatorname{Log} a$$

$$\operatorname{Log}\frac{X}{a} = cx$$

$$\frac{X}{a} = e^{cx}$$

$$X = ae^{cx}$$

Now taking second condition.

$$2\frac{T'}{T} + 1 = c$$

$$2\frac{T'}{T} = c-1$$

$$\frac{T'}{T} = \frac{1}{2}(c-1)$$

$$\frac{1}{T}\cdot\frac{dT}{dt} = \frac{1}{2}(c-1)$$

$$\frac{dT}{T} = \frac{1}{2}(c-1)\,dt$$

on Integrating.

$$\operatorname{Log} T = \frac{1}{2}(c-1)t + \operatorname{Log} b$$

$$\operatorname{Log}\frac{T}{b} = \frac{1}{2}(c-1)t$$

$$\frac{T}{b} = e^{\frac{1}{2}(c-1)t}$$

$$T = be^{\frac{1}{2}(c-1)t}$$

Putting the value of X and T in (2)

$$u = ae^{cx}\cdot be^{\frac{1}{2}(c-1)t}$$

$$u = ab\,e^{cx+\frac{1}{2}(c-1)t}$$

Given $u(x,0) = 6e^{-3x}$

$$\therefore 6e^{-3x} = ab\,e^{cx+\frac{1}{2}(c-1)0}$$

$$6e^{-3x} = ab\,e^{cx} \quad \therefore 6 = ab$$

$$c = -3$$

$$\therefore u = 6e^{-3x+\frac{1}{2}(-3-1)t}$$

$$\boxed{u = 6e^{-3x-2t}} \text{ Ans}$$

...ve the following equation by method of separation of variables

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x \text{ , given that } u=0, \text{ when } t=0$$

and $\frac{\partial u}{\partial t} = 0$, when $x=0$

Sol^n

Let $u = XT \rightarrow (1)$

Where X is function of x and T is function of t only

$\therefore \frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t}$

$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial X}{\partial x}\frac{\partial T}{\partial t}$

Putting in Given eq^n

$\frac{\partial X}{\partial x}\frac{\partial T}{\partial t} = e^{-t}\cos x$

$X'T' = e^{-t}\cos x$

$e^{t}T' = \frac{\cos x}{X'} = -p^2$ say

$\therefore e^{t}T' = -p^2$

$T' = -p^2 e^{-t}$

$\frac{dT}{dt} = -p^2 e^{-t}$

$dT = -p^2 e^{-t} dt$

on Integration

$T = -p^2 \frac{e^{-t}}{-1} + c_1$

$T = p^2 e^{-t} + c_1 \longrightarrow (2)$

And $\frac{\cos x}{X'} = -p^2$

$\frac{\cos x}{p^2} = -\frac{dX}{dx}$

$\frac{-dX}{} = \frac{1}{p^2}\cos x\, dx$

$X = -\frac{1}{p^2}\sin x + c_2 \longrightarrow (3)$

Putting in (1)

$\therefore u = XT$

$u(x,t) = \left(-\frac{1}{p^2}\sin x + c_2\right)\left(p^2 e^{-t} + c_1\right) \rightarrow (4)$

Now using condition

$u=0,\ t=0$

$0 = \left(-\frac{1}{p^2}\sin x + c_2\right)\left(p^2 e^{-0} + c_1\right)$

$0 = p^2 + c_1 \Rightarrow c_1 = -p^2$

Again

$\frac{\partial u}{\partial t} = \left(-\frac{1}{p^2}\sin x + c_2\right)\left(-p^2 e^{-t} + 0\right)$

$0 = \left(-\frac{1}{p^2}\sin 0 + c_2\right)\left(-p^2 e^{-t} + 0\right)$

$0 = c_2\left(-p^2 e^{-t}\right) \Rightarrow c_2 = 0$

$\therefore u(x,t) = \left(-\frac{1}{p^2}\sin x + 0\right)\left(p^2 e^{-t} - p^2\right)$

$= -\frac{\sin x}{p^2}\left(p^2 e^{-t} - p^2\right)$

$= -\frac{\sin x}{\cancel{p^2}}\cancel{p^2}\left(e^{-t} - 1\right)$

$$\boxed{u(x,t) = \left(1 - e^{-t}\right)\sin x}$$

2010, 20

Q. Solve the P.D.E. by separation of variable method

$$u_{xx} = u_y + 2u, \quad u(0,y) = 0, \quad \frac{\partial}{\partial x} u(0,y) = 1 + e^{-3y}$$

Sol: Let $u = XY \longrightarrow (1)$

Where X is function of x only and Y is function of y only

$$\frac{\partial u}{\partial x} = Y \frac{\partial X}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = Y \frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial u}{\partial y} = X \frac{\partial Y}{\partial y}$$

$$\therefore Y \frac{\partial^2 X}{\partial x^2} = X \frac{\partial Y}{\partial y} + 2XY$$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2 = K \text{ (say)}$$

$$\therefore \frac{X''}{X} = K$$

$$X'' - KX = 0$$

$$(D^2 - K)X = 0$$

For C.F A.E is

$$m^2 - K = 0$$

$$m^2 = K$$

$$m = \pm\sqrt{K}$$

$$C.F = c_1 e^{+\sqrt{K}x} + c_2 e^{-\sqrt{K}x}$$

$$\therefore X = c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}$$

And $\frac{Y'}{Y} + 2 = K$

$$\frac{Y'}{Y} = K - 2$$

$$Y' = (K-2)Y$$

$$\frac{dY}{dy} = (K-2)Y$$

$$\frac{dY}{Y} = (K-2)dy$$

$$\log Y = (K-2)y + \log c_3$$

$$\log \frac{Y}{c_3} = (K-2)y$$

$$Y = c_3 e^{(K-2)y}$$

$$\therefore u = (c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x})[c_3 e^{(K-2)y}]$$

$$\therefore u(x,y) = (c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x})\{c_3 e^{(K-2)y}\} \longrightarrow (2)$$

Using condition $u(0,y) = 0$

$$0 = (c_1 e^{\sqrt{K}\cdot 0} + c_2 e^{-\sqrt{K}\cdot 0})\, c_3 e^{(K-2)y}$$

$$0 = (c_1 + c_2)\, c_3 e^{(K-2)y}$$

$$\therefore c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

on putting in (2)

$$u(x,y) = (c_1 e^{\sqrt{K}x} - c_1 e^{-\sqrt{K}x})\{c_3 e^{(K-2)y}\} \longrightarrow (3)$$

$$u(x,y) = c_1 c_3 (e^{\sqrt{K}x} - e^{-\sqrt{K}x})\, e^{(K-2)y}$$

$$\therefore \frac{\partial u}{\partial x} = c_1 c_3 (\sqrt{K} e^{\sqrt{K}x} + \sqrt{K} e^{-\sqrt{K}x})\, e^{(K-2)y}$$

$$1 + e^{-3y} = c_1 c_3 (\sqrt{K} e^{0} + \sqrt{K} e^{-0})\, e^{(K-2)y}$$

$$= 2\sqrt{K}\, c_1 c_3\, e^{(K-2)y}$$

$$1 + e^{-3y} = \sum_{n=1}^{\infty} b_n e^{(K-2)y}$$

$$1 + e^{-3y} = b_1 e^{(K-2)y} + b_2 e^{(K-2)y} + b_3 e^{(K-2)y}, \ldots$$

$$1e^{0y} + e^{-3y} = b_1 e^{(K-2)y} + b_2 e^{(K-2)y} + \ldots$$

$$b_1 = 1,\ K-2 = 0 \Rightarrow b_1 = 1,\ K = 2$$

$$b_2 = 1,\ K-2 = -3 \Rightarrow b_2 = 1,\ K = -1$$

$$b_3 = 0$$

Q. By the method of separation of Variable, solve Q.20

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u, \quad u(x,0) = 4e^{-3x}$$

Sol. The given equation is.

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u \longrightarrow ①$$

Let $u = X \cdot Y \longrightarrow ②$

Where X is function of x only and Y is function of y only.

$$\therefore \frac{\partial u}{\partial x} = Y\frac{\partial X}{\partial x}$$

And $\frac{\partial u}{\partial y} = X\frac{\partial Y}{\partial y}$

Putting these values in ①

$$Y\frac{\partial X}{\partial x} = 2X\frac{\partial Y}{\partial y} + X \cdot Y$$

$$YX' = 2XY' + XY$$

Divid by XY

$$\frac{X'}{X} = 2\frac{Y'}{Y} + 1 = c \text{ (say)}$$

$$\therefore \frac{X'}{X} = c$$

$$\frac{1}{X} \cdot \frac{dX}{dx} = c$$

$$\frac{dX}{X} = c\,dx$$

on Integration

$$\operatorname{Log} X = cx + \operatorname{Log} a$$

$$\operatorname{Log} \frac{X}{a} = cx$$

$$\frac{X}{a} = e^{cx}$$

$$X = a e^{cx}$$

And $2\frac{Y'}{Y} = c - 1$

$$\frac{Y'}{Y} = \frac{1}{2}(c-1)$$

$$\frac{1}{Y} \cdot \frac{dY}{dy} = \frac{1}{2}(c-1)$$

$$\frac{dY}{Y} = \frac{1}{2}(c-1)dy$$

on Int,

$$\operatorname{Log} Y = \frac{1}{2}(c-1)y + \operatorname{Log} b$$

$$\log \frac{Y}{b} = \frac{1}{2}(c-1)y$$

$$\frac{Y}{b} = e^{\frac{1}{2}(c-1)y}$$

$$Y = b\,e^{\frac{1}{2}(c-1)y}$$

Putting the values of X and Y in eqn (2)

$$\therefore u = XY$$

$$= a e^{cx} \cdot b e^{\frac{1}{2}(c-1)y}$$

$$u = ab\,e^{cx+\frac{1}{2}(c-1)y} \longrightarrow (3)$$

Given $u(x,0) = 4e^{-3x}$

$$\therefore u(x,0) = ab\,e^{cx+\frac{1}{2}(c-1)\cdot 0}$$

$$4e^{-3x} = ab\,e^{cx+0}$$

$$4e^{-3x} = ab\,e^{cx}$$

$$\therefore ab = 4, \quad c = -3$$

Putting these value in (3)

$$u = 4e^{-3x+\frac{1}{2}(-3-1)y}$$

$$\boxed{u = 4e^{-3x-2y}} \text{ Ans.}$$

Q. Solve

solve by method of Separation of Variables:

$$4\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=3u,\ u=3e^{-x}-e^{-5x},\ t=0$$

Sol^n Let $u = XT \longrightarrow (1)$

Where X is a function of x only and T is function of t only

$$\therefore \frac{\partial u}{\partial t}=X\frac{\partial T}{\partial t}=XT'$$

$$\frac{\partial u}{\partial x}=T\frac{\partial X}{\partial x}=TX'$$

Putting in (1) given eqn

$$4XT'+TX'=3XT$$

Divid by XT

$$4\frac{T'}{T}+\frac{X'}{X}=3$$

$$4\frac{T'}{T}-3=-\frac{X'}{X}=p^2\ (\text{say})$$

$$\therefore 4\frac{T'}{T}=p^2+3$$

$$\frac{T'}{T}=\frac{p^2+3}{4}$$

$$\frac{T'}{T}=\left(\frac{p^2+3}{4}\right)$$

on Intigration

$$\log T=\left(\frac{p^2+3}{4}\right)t+\log c_1$$

$$\log\frac{T}{c_1}=\left(\frac{p^2+3}{4}\right)t$$

$$T=c_1e^{\left(\frac{p^2+3}{4}\right)t} \longrightarrow (2)$$

And $-\frac{X'}{X}=p^2$

$$\frac{X'}{X}=-p^2$$

$$\frac{1}{X}\frac{dX}{dx}=-p^2$$

$$\frac{dX}{X}=-p^2dx \text{ on Int}$$

$$\log X=-p^2x+\log c_2$$

$$\log\frac{X}{c_2}=-p^2x$$

$$X=c_2e^{-p^2x}$$

$\therefore u = X \cdot T$

$= c_2 e^{-p^2 x} \cdot c_1 e^{\left(\frac{p^2+3}{4}\right)t}$

$u = c_1 c_2 e^{-p^2 x} \cdot e^{\left(\frac{p^2+3}{4}\right)t}$

$u(x,t) = b_n e^{-p^2 x + \left(\frac{p^2+3}{4}\right)t}$

Most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t}$$

$$u(x,0) = \sum_{n=1}^{\infty} b_n e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t}$$

$$\therefore 3e^{-x} - e^{-5x} = \sum_{n=1}^{\infty} b_n e^{-p^2 x + 0}$$

$$3e^{-x} - e^{-5x} = b_1 e^{-p^2 x} + b_2 e^{-p^2 x}$$

$\therefore p^2 = +1,\ b_1 = 3$ and $b_2 = -1,\ p^2 = 5$

$$\therefore u(x,t) = 3e^{-x + \left(\frac{3+1}{4}\right)t} - e^{-5x + \left(\frac{3+5}{4}\right)t}$$

$$u(x,t) = 3e^{-x+t} - e^{-5x+2t}$$

$$\boxed{u(x,t) = 3e^{-x+t} - e^{-5x+2t}}$$ Ans

e the method of separation of Variables to solve the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$$

Solⁿ The given equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \longrightarrow (1)$$

Let $u = XY \longrightarrow (2)$

Where X is a function of x only and Y is a function of y only

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial (X \cdot Y)}{\partial x}$$

$$\frac{\partial u}{\partial x} = Y \frac{\partial X}{\partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = Y \cdot \frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial u}{\partial y} = X \frac{\partial Y}{\partial y}$$

Putting these Value in (1)

$$Y \cdot \frac{\partial^2 X}{\partial x^2} = X \frac{\partial Y}{\partial y} + 2 X \cdot Y$$

divid by X·Y

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{Y} \cdot \frac{\partial Y}{\partial y} + 2$$

$$\frac{1}{X} \cdot X'' = \frac{1}{Y} Y' + 2 = -p^2 \text{ (say)}$$

Take $\frac{X''}{X} = -p^2$

$$X'' = -p^2 X$$

$$\frac{d^2 X}{dx^2} = -p^2 X$$

$$\frac{d^2 X}{dx^2} + p^2 X = 0$$

Which is ordinary diff. eqⁿ

$\therefore$ A.E. is $m^2 + p^2 = 0$

$$m^2 = -p^2$$

$$m = \pm ip$$

C.F. $= C_1 \cos px + C_2 \sin px$ And P.I. $= 0$

$$\therefore X = C_1 \cos px + C_2 \sin px$$

And take relation

$$\frac{1}{Y}Y' + 2 = -p^2$$

$$\therefore \frac{Y'}{Y} = -(p^2+2)$$

$$\frac{Y'}{Y} = -(p^2+2)$$

$$\cancel{\frac{dY}{dy} = -(p^2+2)y}$$

$$\frac{1}{Y} \cdot \frac{dY}{dy} = -(p^2+2)$$

$$\frac{dY}{Y} = -(p^2+2)dy$$

on Inti

$$\text{Log } Y = -(p^2+2)y + \text{Log } a$$

$$\text{Log } \frac{Y}{a} = -(p^2+2)y$$

$$\frac{Y}{a} = e^{-(p^2+2)y}$$

$$Y = ae^{-(p^2+2)y}$$

hence $u = X \cdot Y$

$$u = [C_1 \cos px + C_2 \sin px]\, ae^{-(p^2+2)y}$$

$$\boxed{u = [C_1 \cos px + C_2 \sin px][C_3 e^{-(p^2+2)y}]}$$ Ans.

...tion Of A Stretched String, One Dimensional ...ve Equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Solution of Wave Equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$

Let $y = X.T \longrightarrow (2)$ be a solution of (1)

Where X is a function of x and T is a function of t only.

$$\therefore \frac{\partial y}{\partial x} = T \cdot \frac{\partial X}{\partial x}$$

$$\frac{\partial^2 y}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$$

And $\frac{\partial y}{\partial t} = X \frac{\partial T}{\partial t}$

$$\frac{\partial^2 y}{\partial t^2} = X \frac{\partial^2 T}{\partial t^2}$$

Substituting these values in (1)

$$\therefore X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$X T'' = c^2 T X''$$

Divid by $X.T$

$$\frac{T''}{T} = c^2 \frac{X''}{X}$$

$$\therefore \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} \longrightarrow (3)$$

Now. L.H.S. of (3) is function of x only and R.H.S. is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduces to a constant say p^2. Then equation (3) Leads to ordinary linear differential equation.

$$\therefore \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = p^2 \text{ (say)}$$

Case 1 When p^2 is Positive

$$\therefore \frac{X''}{X} = p^2$$

$$\therefore X'' - p^2 X = 0$$

which is Linear ordinary diff eqn.

$\therefore$ A.E. is $m^2 - p^2 = 0$

$$m^2 = p^2$$

$$m = \pm p$$

$C.F = c_1 e^{px} + c_2 e^{-px}$

$P.I = 0$

$\therefore \quad X = C.F + P.I$

$X = c_1 e^{px} + c_2 e^{-px}$

And $\frac{T''}{T} = c^2 p^2$

$T'' - c^2 p^2 T = 0$

$\therefore$ A.E. is $m^2 - c^2 p^2 = 0$

$m = \pm cp$

$C.F. = c_3 e^{cpt} + c_4 e^{-cpt}$

$\therefore T = c_3 e^{cpt} + c_4 e^{-cpt}$

$\therefore$ complete solution

$y = X T$

$y = [c_1 e^{px} + c_2 e^{-px}][c_3 e^{cpt} + c_4 e^{-cpt}] \longrightarrow (4)$

<u>Case 2</u> - When p^2 is negative

$\therefore \quad \frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T} = -p^2$

$X'' = -p^2 X$

$X'' + p^2 X = 0$

A.E. is $m^2 + p^2 = 0$

$m^2 = -p^2$

$m = \pm ip$

$C.F = c_5 \cos px + c_6 \sin px$

$X = c_5 \cos px + c_6 \sin px$

And $\frac{1}{c^2} \cdot \frac{T''}{T} = -p^2$

$T'' = -p^2 c^2 T$

$T'' + p^2 c^2 T = 0$

$\therefore$ A.E. is $m^2 + c^2 p^2 = 0$

$m = \pm icp$

$\therefore C.F = c_7 \cos cpt + c_8 \sin cpt$

$\therefore T = c_7 \cos cpt + c_8 \sin cpt$

$\therefore$ solution is $y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \longrightarrow (5)$

When p^2 is zero.

$$\therefore \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = 0$$

$$\therefore \frac{X''}{X} = 0$$

$$\therefore X'' - 0X = 0$$

$\therefore$ A.E. is $m^2 = 0$

$m = 0, 0$

$\therefore$ C.F $= c_9 + c_{10}x$

$\therefore X = c_9 + c_{10}x$

And $\frac{1}{c^2}\frac{T''}{T} = 0$

$\therefore T'' - 0T = 0$

A.E. is $m^2 = 0$

$m = 0, 0$

C.F. $= c_{11} + c_{12}t$

$T = c_{11} + c_{12}t$

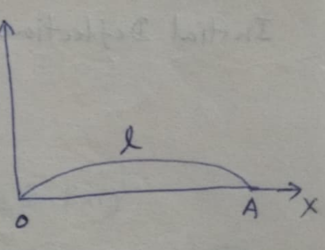
$\therefore$ solution is

$$y = X.T$$

$$y = (c_9 + c_{10}x)(c_{11} + c_{12}t) \longrightarrow (7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of problem. Here we dealing with the wave motion, so $p^2 < 0$ i.e. p^2 is negative

Now applying boundary condition.



When $y = 0, x = 0$
When $y = 0, x = l$ $\Big] \longrightarrow (8)$

Here we have solution from (5)

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \longrightarrow (9)$$

Use $y = 0, x = 0$

$$0 = (c_5 + c_6 \cdot 0)(c_7 \cos cpt + c_8 \sin cpt)$$

$$0 = c_5 [c_7 \cos cpt + c_8 \sin cpt]$$

$$\therefore c_5 = 0$$

Putting in

And use condition $x=l,\ y=0$

$\therefore 0 = \cancel{\ }$

$0 = (c_5 \cos pl + c_6 \sin pl)(c_7 \cos pct + c_8 \sin pct)$

Putting $c_5 = 0$

$0 = c_6 \sin pl\,(c_7 \cos cpt + c_8 \sin cpt)$

Which is satisfied when $\sin pl = 0$

$\sin pl = \sin n\pi$

$\therefore pl = n\pi$

$p = \frac{n\pi}{l}$, for $n = 1, 2, 3 \ldots$

$\therefore$ Hence solution of the wave equation satisfies the boundary condition is

$$y = c_6\left[c_7 \cos \frac{n\pi ct}{l} + c_8 \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l}$$

$$y = \left[c_6 c_7 \cos \frac{n\pi ct}{l} + c_6 c_8 \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l}$$

$$y = \left[a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l}$$

$$\boxed{y = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l}}$$

Initial Velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} =$

Initial Deflection $y(x,0) =$

tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y = b\sin^3\frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x,t)$

Solⁿ. The equation of string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$$

And solution of eqⁿ (1) is

$$y(x,t) = (C_1\cos cpt + C_2\sin cpt)(C_3\cos px + C_4\sin px) \longrightarrow (2)$$

Boundary conditions are

$$y(0,t) = 0 \longrightarrow (3)$$

$$y(l,t) = 0 \longrightarrow (4)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \longrightarrow (5)$$ [Initial Transverse velocity of any point of the string is zero]

$$y(x,0) = b\sin^3\frac{\pi x}{l} \longrightarrow (6)$$

Applying boundary condition in (2)

From (2)

$\cancel{y(0,0) = C_1}$

$$y(0,t) = (C_1\cos cpt + C_2\sin cpt)(C_3\cos p\cdot 0 + 0)$$

$$0 = (C_1\cos cpt + C_2\sin cpt)C_3$$

$$C_3 = 0$$

∴ from (2)

$$y(x,t) = (C_1\cos cpt + C_2\sin cpt)(0 + C_4\sin px)$$

$$y(x,t) = (C_1\cos cpt + C_2\sin cpt)(C_4\sin px \longrightarrow (7)$$

Again $y(l,t) = (C_1\cos cpt + C_2\sin cpt)\, C_4\sin pl$

$$0 = (C_1\cos cpt + C_2\sin cpt)\, C_4\sin pl$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

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Hence from (7)

$$y(x,t) = \left[c_1 \cos \frac{n\pi c t}{l} + c_2 \sin \frac{n\pi c t}{l}\right] c_4 \sin \frac{n\pi x}{l}$$

$$\therefore \frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi c t}{l} + c_2 \cos \frac{n\pi c t}{l}\right] c_4 \sin \frac{n\pi x}{l}.$$

At $t=0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi c \cdot 0}{l} + c_2 \cos 0\right] c_4 \sin \frac{n\pi x}{l}$$

$$0 = \frac{n\pi c}{l}\left[0 + c_2\right] c_4 \sin \frac{n\pi x}{l}$$

$$0 = c_2$$

$\therefore$ from (8)

$$y(x,t) = \left[c_1 \cos \frac{n\pi c t}{l} + 0\right] c_4 \sin \frac{n\pi x}{l}$$

$$= c_1 c_4 \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi c t}{l}$$

$$y(x,t) = a_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi c t}{l}$$

Most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi c t}{l} \longrightarrow (9)$$

But given condition

$$y(x,0) = b \sin^3 \frac{\pi x}{l}$$

Putting $t=0$ in 9

$$\therefore y(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cdot \cos 0$$

$$b \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$\frac{b}{4}\left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right] = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$\frac{b}{4}\left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right] = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + a_3 \sin \frac{3\pi x}{l} \cdots$$

$$\therefore \frac{b}{4} 3 \sin \frac{\pi x}{l} = a_1 \sin \frac{\pi x}{l}$$

$$\therefore a_1 = \frac{3}{4} b$$

And $a_2 = 0$, ~~And~~

(b) $\sin\frac{3\pi x}{l} = a_3 \sin\frac{3\pi x}{l}$

$\therefore -\frac{b}{4} = a_3$

$\therefore$ from (9)

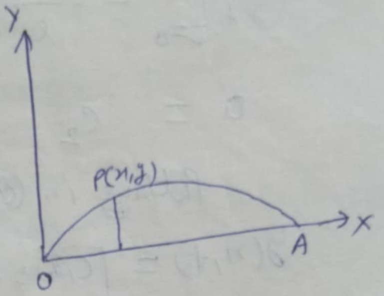
$$y(x,t) = a_1 \sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} + a_2 \sin\frac{2\pi x}{l}.\cos\frac{2\pi ct}{l} + a_3 \sin\frac{3\pi x}{l}.\cos\frac{3\pi ct}{l}$$

$$\boxed{y(x,t) = \frac{3}{4} b \sin\frac{\pi x}{l}.\cos\frac{\pi ct}{l} + 0 - \frac{b}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}}$$

which is required displacement.

string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin \frac{\pi x}{l}$ from which it is released at a time $t=0$. Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x,t) = a \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi c t}{l}\right)$$



Sol^n → Let us consider an string tightly stretched between two point O and A. Let y be the displacement at the point $P(x,y)$ at any time.

We know that the equation of string OR wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$$

And solution of eq^n (1) is

$$y(x,t) = (C_1 \cos cPt + C_2 \sin cPt)(C_3 \cos Px + C_4 \sin Px) \longrightarrow (2)$$

As the end points of the string are fixed, for all time

$$y(0,t) = 0 \longrightarrow (3) \quad \{\text{Boundary conditions}\}$$

$$y(l,t) = 0 \longrightarrow (4)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \longrightarrow (5) \quad [\text{Initial Transverse velocity of any point of the string is zero}]$$

And given $y(x,0) = a \sin \frac{\pi x}{l} \longrightarrow (6)$

Applying boundary condition in eq^n (2)

$$\therefore y(0,t) = (C_1 \cos cPt + C_2 \sin cPt)(C_3 + C_4 \cdot 0)$$

$$0 = C_3 (C_1 \cos cPt + C_2 \sin cPt)$$

$$C_3 = 0 \quad \text{Putting in (2)}$$

$$y(x,t) = (C_1 \cos cPt + C_2 \sin cPt)\, C_4 \sin P(x) \longrightarrow (7)$$

$$\therefore y(l,t) = (C_1 \cos cPt + C_2 \sin cPt)\, C_4 \sin Pl$$

$$0 = C_4 \sin Pl\,(C_1 \cos cPt + C_2 \sin cP)$$

$$\therefore 0 = \sin Pl$$

$$\sin n\pi = \sin Pl$$

$$\therefore P = \frac{n\pi}{l}$$

9

equation (7) becomes

$$y(x,t) = \left[C_1 \cos\frac{n\pi ct}{l} + C_2 \sin\frac{n\pi ct}{l}\right] C_4 \sin\frac{n\pi x}{l}$$

$$\therefore \frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-C_1 \sin\frac{n\pi ct}{l} + C_2 \cos\frac{n\pi ct}{l}\right] C_4 \sin\frac{n\pi x}{l}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{n\pi c}{l}\left[0 + C_2 \cos\frac{n\pi c \cdot 0}{l}\right] C_4 \sin\frac{n\pi x}{l}$$

$$0 = C_2$$

Putting in (8)

$$y(x,t) = \left[C_1 \cos\frac{n\pi ct}{l} + 0\right] C_4 \sin\frac{n\pi x}{l}$$

$$= C_1 C_4 \cos\frac{n\pi ct}{l} \cdot \sin\frac{n\pi x}{l}$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi ct}{l} \cdot \sin\frac{n\pi x}{l} \longrightarrow (9)$$

But given $y(x,0) = a \sin\frac{\pi x}{l}$

$\therefore$ From (9)

$$y(x,0) = b \cos\frac{n\pi c}{l} \cdot 0 \cdot \sin\frac{n\pi x}{l}$$

$$a \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a \sin\frac{\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \cdots$$

$$\therefore a = b_1$$

Hence Putting in (9)

$$y(x,t) = b_1 \cos\frac{\pi ct}{l} \cdot \sin\frac{\pi x}{l}$$

$$\boxed{y(x,t) = a \cos\frac{\pi ct}{l} \cdot \sin\frac{\pi x}{l}}$$ Ans.

(B) 2. Solve completely the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, representing the vibrations of a string of length l, fixed at both ends, given that $y(0,t)=0$, $y(l,t)=0$, $y(x,0)=f(x)$ and $\frac{\partial}{\partial t} y(x,0) = 0$, $0<x<l$.

Sol^n. Given equ^n $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$

We know that sol^n of (1) is

$$y(x,t) = [c_1 \cos cpt + c_2 \sin cpt][c_3 \cos px + c_4 \sin px] \longrightarrow (2)$$

Given Boundary conditions are

$y(0,t)=0 \longrightarrow (3)$

$y(l,t)=0 \longrightarrow (4)$

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \longrightarrow (5)$ [Initial Transverse velocity of any point of any string is zero]

And $y(x,0) = f(x) \longrightarrow (6)$

Applying boundary condition in (2)

$\therefore y(0,t) = [c_1 \cos cpt + c_2 \sin cpt][c_3 \cos 0 + c_4 \sin 0]$

$0 = [c_1 \cos cpt + c_2 \sin cpt]\, c_3 + 0$

$0 = c_3$

Putting in (2)

$$y(x,t) = [c_1 \cos cpt + c_2 \sin cpt][c_4 \sin px] \longrightarrow (7)$$

Again Using second boundary condition

$y(l,t) = [c_1 \cos cpt + c_2 \sin cpt][c_4 \sin pl]$

$0 = [c_1 \cos cpt + c_2 \sin cpt]\, c_4 \sin pl$

Which is satisfied when $\sin pl = 0$

$\sin pl = \sin n\pi$

$p = \frac{n\pi}{l}$ for $n = 1, 2, 3, \ldots$

$$\therefore y(x,t) = \left[c_1 \cos \frac{cn\pi t}{l} + c_2 \sin \frac{cn\pi t}{l}\right] c_4 \sin \frac{n\pi x}{l} \longrightarrow (8)$$

$$\therefore \frac{\partial y}{\partial t} = \left[-c_1 \sin \frac{cn\pi t}{l} \cdot \left(\frac{cn\pi}{l}\right) + c_2 \cos \frac{cn\pi t}{l}\left(\frac{cn\pi}{l}\right)\right] c_4 \sin \frac{n\pi x}{l}$$

$$\therefore \left(\frac{\partial y}{\partial t}\right)_{t=0} = \left[-C_1 \cdot 0 + C_2\left(\frac{cn\pi}{l}\right)\right] C_4 \sin\frac{n\pi x}{l}$$

$$0 = C_2$$

Putting in 8

$$y(x,t) = \left[C_1 \cos\frac{n\pi t c}{l} + 0\right] C_4 \sin\frac{n\pi x}{l}$$

$$y(x,t) = C_1 C_4 \cos c\frac{n\pi t}{l} \cdot \sin\frac{n\pi x}{l} \longrightarrow (9)$$

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{cn\pi t}{l} \cdot \sin\frac{n\pi x}{l} \longrightarrow (10)$$

But Given $y(x,0) = f(x)$

$$\therefore y(x,0) = \sum_{n=1}^{\infty} a_n \cos\frac{cn\pi}{l} \cdot 0 \cdot \sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

a_n can be Calculated Using Fourier Sine Series as

$$\boxed{a_n = \frac{2}{l}\int_0^l f(x) \cdot \sin\frac{n\pi x}{l}\, dx}$$

Hence complete solⁿ is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{cn\pi t}{l} \cdot \sin\frac{n\pi x}{l}}$$ Ans

2. If a string of length l is initially at rest in equilibrium position and each of its point is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\frac{\pi x}{l}$, find the displacement $y(x,t)$

Solⁿ. We know that the equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$$

And solution of eqⁿ (1) is

$$y(x,t) = (c_1\cos cpt + c_2\sin cpt)(c_3\cos px + c_4\sin px) \longrightarrow (2)$$

And Boundary conditions are

$$y(0,t) = 0 \longrightarrow (3)$$

$$y(l,t) = 0 \longrightarrow (4)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\frac{\pi x}{l} \longrightarrow (5)$$

$$y(x,0) = 0 \longrightarrow (6)$$

Using first Boundary condition

$$\therefore y(0,t) = (c_1\cos cpt + c_2\sin cpt)(c_3 + 0)$$

$$0 = c_3$$

Putting in (2)

$$y(x,t) = (c_1\cos cpt + c_2\sin cpt)(c_4\sin px) \longrightarrow (7)$$

Using second Boundary condition.

$$y(l,t) = (c_1\cos cpt + c_2\sin cpt)\,c_4\sin pl$$

$$0 = (c_1\cos cpt + c_2\sin cpt)\,c_4\sin pl$$

Which satisfies only $\sin pl = 0 \Rightarrow pl = n\pi$

$$p = \frac{n\pi}{l}$$

Putting in (7)

$$y(x,t) = \left[c_1\cos\frac{cn\pi}{l}t + c_2\sin\frac{cn\pi}{l}t\right]c_4\sin\frac{n\pi x}{l} \longrightarrow (8)$$

Now using fourth Boundary condition

$$y(x,0) = \left[c_1\cos\frac{cn\pi}{l}.0 + c_2\sin\frac{cn\pi}{l}.0\right]c_4\sin\frac{n\pi x}{l}$$

$$0 = c_1c_4\sin\frac{n\pi x}{l} \Rightarrow c_1 = 0$$

Putting in (8)

$$y(x,t) = \underbrace{c_2c_4}_{b_n}\sin\frac{cn\pi}{l}t.\sin\frac{n\pi x}{l} \longrightarrow (9)$$

Now using Third Boundary condition,

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n\cos\frac{cn\pi}{l}t\left(\frac{cn\pi}{l}\right).\sin\frac{n\pi x}{l}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} b_n\frac{cn\pi}{l}.\sin\frac{n\pi x}{l}$$

$$b \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \frac{cn\pi}{l} \cdot \sin \frac{n\pi x}{l}$$

$$\frac{b}{4}\left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right] = b_1 \frac{c\pi}{l} \sin \frac{\pi x}{l} + b_2 \frac{2c\pi}{l} \sin \frac{2\pi x}{l} + b_3 \frac{3c\pi}{l} \sin \frac{3\pi x}{l}$$

$$\therefore \frac{3b}{4} = b_1 \frac{c\pi}{l} \Rightarrow b_1 = \frac{3bl}{4\pi c}, \quad b_2 = 0$$

And

$$-\frac{b}{4} = b_3 \frac{3c\pi}{l} \Rightarrow b_3 = -\frac{bl}{12c\pi}, \quad b_4 = 0, \; b_5 = 0$$

Putting these value in (9)

$$y(x,t) = b_1 \sin \frac{c\pi}{l} t \cdot \sin \frac{\pi x}{l} + b_2 \sin \frac{2c\pi}{l} t \cdot \sin \frac{2\pi x}{l} + b_3 \sin \frac{3c\pi}{l} t \cdot \sin \frac{3\pi x}{l}$$

$$\boxed{y(x,t) = \frac{3bl}{4\pi c} \sin \frac{c\pi}{l} t \cdot \sin \frac{\pi x}{l} + 0 - \frac{bl}{12\pi c} \sin \frac{3\pi c}{l} t \cdot \sin \frac{3\pi x}{l}}$$ Ans

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Q. A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t=0$, find the displacement of any point on the string at a distance of x from one end at time t.

Solⁿ We know that the vibrations of a stretched string are given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$$

And solⁿ of (1) are

$$y(x,t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \longrightarrow (2)$$

And Boundary conditions are

$$y(0,t) = 0 \longrightarrow (3)$$

$$y(l,t) = 0 \longrightarrow (4)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \longrightarrow (5)$$

$$y(x,0) = k(lx - x^2) \longrightarrow (6)$$

Using first Boundary condition

$$\therefore y(0,t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 + 0)$$

$$0 = c_3 \Rightarrow c_3 = 0 \quad \text{Putting in (2)}$$

$$y(x,t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \longrightarrow (7)$$

Using Second Boundary condition.

$$y(l,t) = (c_1 \cos cpt + c_2 \sin cpt)(c_4 \sin pl)$$

$$0 = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin pl$$

$$0 = \sin pl \Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$$

Putting in (7)

$$\cancel{y(x,t) = c_1 \cos cpt + c_2 \sin c}$$

$$y(x,t) = \left(c_1 \cos \frac{cn\pi}{l} t + c_2 \sin \frac{cn\pi}{l} t\right) c_4 \sin \frac{n\pi x}{l} \longrightarrow (8)$$

Now using Third boundary condition

$$\frac{\partial y}{\partial t} = \left[-c_1 \sin \frac{cn\pi}{l} t \left(\frac{cn\pi}{l}\right) + c_2 \cos \frac{n\pi c}{l} t \left(\frac{cn\pi}{l}\right)\right] c_4 \sin \frac{n\pi x}{l}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 + c_2 \left(\frac{cn\pi}{l}\right) c_4 \sin \frac{n\pi x}{l}$$

$$0 = c_2 \quad \text{putting in (8)}$$

$$y(x,t) = \left[c_1 \cos \frac{cn\pi}{l} t + 0\right] c_4 \sin \frac{n\pi x}{l}$$

$$y(x,t) = c_1 c_4 \cos \frac{cn\pi}{l} t \cdot \sin \frac{n\pi x}{l}.$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \cos \frac{cn\pi}{l} t \cdot \sin \frac{n\pi x}{l} \longrightarrow (9)$$

Using Last boundary condition

$$y(x,0) = \sum_{n=1}^{\infty} b_n \cdot 1 \sin \frac{n\pi x}{l}$$

$$k(lx-x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \longrightarrow (10)$$

obviously (9) represents the expansion of $f(x)$ in the form of a fourier sine series, therefore

$$b_n = \frac{2}{l}\int_0^l f(x) \cdot \sin \frac{n\pi x}{l}$$

$$= \frac{2}{l}\int_0^l k(lx-x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}k\left[\left\{(lx-x^2)\left(-\cos \frac{n\pi x}{l}\right)\cdot \frac{l}{n\pi}\right\}_0^l + \int_0^l (l-2x) \cos \frac{n\pi x}{l} \cdot \frac{l}{n\pi}\, dx\right.$$

$$= \frac{2}{l}k\left[(0+0) + \frac{l}{n\pi}\left[\left\{(l-2x)\sin \frac{n\pi x}{l} \cdot \frac{l}{n\pi}\right\}_0^l + \int_0^l 2 \sin \frac{n\pi x}{l} \cdot \frac{l}{n\pi}\, dx\right]\right]$$

$$= \frac{2}{l}k \cdot \frac{l}{n\pi}\left[\left\{-l \sin \frac{n\pi l}{l} \cdot \frac{l}{n\pi}\right\} - 2\frac{l^2}{n^2\pi^2}\left(\cos \frac{n\pi x}{l}\right)_0^l\right]$$

...mensional Heat Flow $\boxed{\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}}$

...ution of The Heat Equation :—

The Heat Equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

Let $\quad u = X \cdot T \longrightarrow (2)$

be a solution of (1), where X is a function of x only and T is a function of t only.

Then $\dfrac{\partial u}{\partial t} = X \cdot \dfrac{\partial T}{\partial t}$

$$\frac{\partial u}{\partial x} = T \frac{\partial X}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$$

Substituting these values in (1).

$$X \cdot \frac{\partial T}{\partial t} = c^2 \cdot T \frac{\partial^2 X}{\partial x^2}$$

$$X \cdot T' = c^2 T X''$$

Divid by $X \cdot T$

$$\frac{T'}{T} = c^2 \frac{X''}{X}$$

$$\therefore \frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T'}{T} \longrightarrow (3)$$

Now L.H.S. of (3) is function of x only and the R.H.S. is function of t only. Since x and t are independent variables, this can hold only when both sides reduce to a constant say p^2. The equation (3) leads to the ordinary differential equations.

$$\therefore \frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T'}{T} = p^2 \longrightarrow (4)$$

Case 1 — when p^2 is positive

$$\therefore \frac{X''}{X} = p^2$$

$$X'' - p^2 X = 0$$

Which is ordinary diff. eqn.

$\therefore$ A.E. is $m^2 - p^2 = 0$

$m = \pm p$

C.F. $= c_1 e^{px} + c_2 e^{-px}$

$$\frac{T'}{T} = c^2 p^2$$

$$\frac{1}{T}\frac{dT}{dt} = c^2 p^2$$

$$\frac{dT}{T} = c^2 p^2 \, dt$$

on Intg

$$\log T = c^2 p^2 t + \log c_3$$

$$\log \frac{T}{c_3} = c^2 p^2 t$$

$$\therefore T = c_3 e^{c^2 p^2 t}$$

$$\therefore u = X \cdot T = [c_1 e^{px} + c_2 e^{-px}] \cdot c_3 e^{c^2 p^2 t} \longrightarrow (5)$$

Case. 2 → p^2 is negative

$\therefore \frac{X''}{X} = -p^2$

$X'' + p^2 X = 0$

ordinary diff. eqn.

$\therefore$ A.E. is. $m^2 + p^2 = 0$

$m^2 = -p^2$

$m = \pm ip$

$C.F = (c_1 \cos px + c_2 \sin px)$

$\therefore X = c_1 \cos px + c_2 \sin px$

$\frac{1}{c^2} \cdot \frac{T'}{T} = -p^2$

$\frac{T'}{T} = -p^2 c^2$

$\frac{1}{T} \cdot \frac{dT}{dt} = -p^2 c^2$

$\frac{dT}{T} = -p^2 c^2 dt$

on Int

$\log T = -p^2 c^2 t + \log c_3$

$\frac{T}{c_3} = e^{-p^2 c^2 t}$

$T = c_3 e^{-p^2 c^2 t}$

$\therefore$ solution is $u = X \cdot T$

$u = [c_1 \cos px + c_2 \sin px] \cdot c_3 e^{-p^2 c^2 t} \longrightarrow$ (6)

Case 3 → p^2 is zero.

$\therefore \frac{X''}{X} = 0$

$X'' = 0$

$\therefore X = c_1 + c_2 x$

$\frac{1}{c^2} \cdot \frac{T'}{T} = 0$

$T' = 0$

$\frac{dT}{dt} = 0$

$T = c_3$

$\therefore$ solution is $u = X \cdot T$

$u = (c_1 + c_2 x) \cdot c_3 \longrightarrow$ (7)

of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. since u ~~dec~~ decreases as time t ~~increases~~ increases, the only suitable solution of the heat equation is

$$\boxed{u = [c_1 \cos px + c_2 \sin px] \cdot c_3 e^{-p^2 c^2 t}}$$

2006

Q. Determine the solution of one dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, subject to the boundary conditions $u(0,t) = 0$, $u(l,t) = 0$, $(t > 0)$ and the initial conditions $u(x,0) = x$, l being the length of the bar.

Solⁿ. Given Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

As we know that By Separation of Variable method. its solⁿ given as

~~u = XT → (2) be the solⁿ of (1)~~

$$u(x,t) = [C_1 \cos px + C_2 \sin px]\, C_3 e^{-c^2p^2t} \longrightarrow (2)$$

Boundary Condition are

$$u(0,t) = 0 \longrightarrow (3)$$

$$u(l,t) = 0 \longrightarrow (4)$$

$$u(x,0) = x \longrightarrow (5)$$

Now using First condition i.e $u(0,t) = 0$

so putting $x = 0$ in (2)

$$\therefore u(0,t) = (C_1 + 0)\, C_3 e^{-c^2p^2t}$$

$$0 = C_1 \quad \text{putting in (2)}$$

$$\therefore u(x,t) = [C_2 \sin px]\, C_3 e^{-c^2p^2t} \longrightarrow (6)$$

Now using second condition i.e $u(l,t) = 0$

i.e putting $x = l$ in (6), we get

$$u(l,t) = [C_2 \sin pl]\, C_3 e^{-c^2p^2t}$$

$$0 = \sin pl$$

$$\sin n\pi = \sin pl \Rightarrow pl = n\pi$$

$$p = \frac{n\pi}{l}, \quad n = 1, 2, 3 \ldots$$

$\therefore$ Putting the value of p in (6)

$$u(x,t) = \left[C_2 \sin\left(\frac{n\pi x}{l}\right)\right] C_3 e^{-c^2 \frac{n^2\pi^2}{l^2} \cdot t} \longrightarrow (7)$$

~~Now using Third condition i.e $u(x,0) = x$~~

~~so putting t=o in (7)~~

~~$\therefore u(x,o) = \left[C_2 \sin \frac{n\pi x}{l}\right] C_3$~~

~~$x = C_3$~~

~~Putting in (7)~~

~~$u(x,t) = x$~~

$$\therefore u(x,t) = C_1 C_3 \, e^{-\frac{n^2\pi^2c^2}{l^2} \cdot t} \cdot \sin \frac{n\pi x}{l}$$

$$u(x,t) = b_n \, e^{-\frac{n^2\pi^2c^2}{l^2} \cdot t} \cdot \sin \frac{n\pi x}{l} \longrightarrow (7)$$

Now using Third Condition.

putting $t=0$

$$u(x,0) = b_n$$

$$x = b_n \sin \frac{n\pi x}{l}$$

In General

$$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \longrightarrow (8)$$

Here b_n is given as by Half range-series

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \left[\left\{ x \frac{\left(-\cos \frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} \right\}_0^l - \left\{ (1) \frac{\left(-\sin \frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}} \right\}_0^l \right]$$

$$= \frac{2}{\cancel{l}} \left[-\frac{\cancel{l}}{n\pi} \{ l \cos n\pi \} + \frac{l^2}{n^2\pi^2} \{ \sin n\pi \} \right]$$

$$b_n = -\frac{2}{n\pi} l \cos n\pi + \frac{2 l^2}{n^2 l} \sin n\pi$$

$$b_n = \frac{-2l \cos n\pi}{n\pi} + 0$$

putting this value of b_n in (7)

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{-2l \cos n\pi}{n\pi} \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2c^2}{l^2} \cdot t}}$$ Ans.

Solve the following boundary value problem which arises in the heat conduction in a rod

Exercis (30.3)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(l,t) = 0,$$
$$u(x,0) = 100\frac{x}{l}$$

2004

OR

An insulated rod of length l has its ends A and B maintained at $0°C$ and $100°C$ respectively until steady state condition prevails. If B is suddenly reduced to $0°C$ and maintained at $0°C$, find the temperature at distance x from A at time t, solve the equation of heat $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$, by method of separation of variables and obtain the solution.

Solⁿ. Given heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

Let $u = X.T \longrightarrow (2)$

be the solution of eqⁿ (1), where X is function of x only and Y is function of y only.

$$\therefore \frac{\partial u}{\partial t} = X.\frac{\partial T}{\partial t}$$

$$\frac{\partial u}{\partial x} = T\frac{\partial X}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$$

Putting these values in (1)

$$\cancel{T\frac{\partial X}{\partial t} = c^2\frac{\partial^2 X}{\partial x^2}}$$

$$X\frac{\partial T}{\partial t} = c^2 T\frac{\partial^2 X}{\partial x^2}$$

$$XT' = c^2 T X''$$

Divid by X.T

$$\frac{T'}{T} = c^2\frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2}\cdot\frac{T'}{T} \longrightarrow (3)$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = -p^2 \text{(say)} \longrightarrow (3)$$

$$\therefore \frac{X''}{X} = -p^2$$

$$X'' + p^2 X = 0$$

which is linear ordinary diff eqⁿ.

A.E. is.

$m^2 + p^2 = 0$

$m^2 = -p^2$

$m = \pm ip$

$C.F = c_1 \cos px + c_2 \sin px$

$P.I = 0$

$\therefore X = C.F + P.I.$

$X = c_1 \cos px + c_2 \sin px$

And $\frac{T'}{T} = -c^2p^2$

$\frac{1}{T}\frac{dT}{dt} = -c^2p^2$

$\frac{dT}{T} = -c^2p^2 dt$

On Integration.

$\log T = -c^2p^2t + \log c_3$

$\log \frac{T}{c_3} = -c^2p^2t$

$\frac{T}{c_3} = e^{-c^2p^2t}$

$T = c_3 e^{-c^2p^2t}$

$\therefore T = c_3 e^{-c^2p^2t}$

$\therefore u = X \cdot T$

$u(x,t) = [c_1 \cos px + c_2 \sin px] c_3 e^{-c^2p^2t} \longrightarrow (4)$

Boundary condition

$u(0,t) = 0 \rightarrow (i)$

$u(l,t) = 0 \rightarrow (ii)$

$u(x,0) = 100\frac{x}{l} \rightarrow (iii)$

Using Boundary Condition.

$u(0,t) = (c_1 + 0) c_3 e^{-c^2p^2t}$

$0 = c_1$ Putting in (4)

$\therefore u(x,t) = [c_2 \sin px] c_3 e^{-c^2p^2t} \longrightarrow (5)$

$u(l,t) = [c_2 \sin pl] c_3 e^{-c^2p^2t}$

$0 = \sin pl$

$\sin n\pi = \sin pl \Rightarrow p = \frac{n\pi}{l}$

$\therefore u(x,t) = c_2 \sin \frac{n\pi x}{l} \cdot c_3 e^{-c^2p^2t}$

$u(x,t) = c_2 \sin \frac{n\pi x}{l} \cdot c_3 e^{-\frac{c^2n^2\pi^2 t}{l^2}} \longrightarrow (6)$

$\therefore$

$u(x,t) = c_2 c_3 \cdot \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2n^2\pi^2}{l^2}t}$

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2n^2\pi^2}{l^2} \cdot t}$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cdot e^0$$

$$100\frac{x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

By Half-range Sine series

$$b_n = \frac{2}{l}\int_0^l 100\frac{x}{l}\cdot \sin\frac{n\pi x}{l}\,dx.$$

$$b_n = \frac{2}{l}\cdot\frac{100}{l}\left[\left(-x\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l + \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi x}{l}\right)_0^l\right]$$

$$= \frac{200}{l^2}\left[\frac{l}{n\pi}(-l\cos n\pi) + \frac{l^2}{n^2\pi^2}(\sin n\pi - 0)\right]$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}\cos n\pi + 0\right]$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}(-1)^n\right]$$

$$b_n = \frac{200}{n\pi}(-1)^{n+1}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty}\frac{200}{n\pi}(-1)^{n+1}\cdot\sin\frac{n\pi x}{l}\cdot e^{-\frac{c^2n^2\pi^2}{l^2}t}$$

$$\boxed{u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin\frac{n\pi x}{l}\cdot e^{-\frac{c^2n^2\pi^2}{l^2}t}}$$ Ans.

2007,09

Q. Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface ~~insulated~~ insulated if the initial temperature is $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

Sol. Let one dimensional Heat Flow eqn is.

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

And boundary condition are

$$u(0,t) = 0 \longrightarrow (2)$$

$$u(2,t) = 0 \longrightarrow (3)$$

And initial Condition

$$u(x,0) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} \longrightarrow (4)$$

We know that soln of (1) is

$$u(x,t) = (c_1\cos px + c_2\sin px)\,c_3 e^{-c^2p^2t} \longrightarrow (A)$$

Using First condition $u(0,t)=0$

$$0 = (c_1 + 0)\,c_3 e^{-c^2p^2t} \Rightarrow c_1 = 0$$

$$\therefore u(x,t) = (c_2\sin px)\,c_3 e^{-c^2p^2t} \longrightarrow (B)$$

Using second condition

$u(2,t) = 0 = (c_2 \sin p.2)\, c_3 e^{-c^2p^2t}$

$0 = \sin 2p \Rightarrow \sin n\pi = \sin 2p$

$2p = n\pi \Rightarrow p = \frac{n\pi}{2}$

$\therefore u(x,t) = c_2c_3 \sin\frac{n\pi x}{2}\, e^{-c^2\frac{n^2\pi^2}{4}.t}$

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}\cdot e^{-\frac{c^2n^2\pi^2}{4}t} \longrightarrow (C)$

Using Third condition

$u(x,0) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

$\therefore \sin\frac{n\pi}{2} + 3\sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}\cdot e^{-\frac{c^2n^2\pi^2}{4}}$

$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}\cdot e^{-0}$

$\sin\frac{x\pi}{2} + 3\sin\frac{5\pi x}{2} = b_1\sin\frac{\pi x}{2} + b_2\sin\frac{2\pi x}{2} + b_3\sin\frac{3\pi x}{2} + b_4\sin\frac{4\pi x}{2}$

$+ b_5\sin\frac{5\pi x}{2} + \ldots$

on Comparing the coefficient of sin

$\boxed{b_1 = 1}$, $\boxed{b_5 = 3}$, $b_2 = b_3 = b_4 = \ldots = 0$

$\therefore u(x,t) = b_1\sin\frac{\pi x}{2}\cdot e^{-\frac{c^2\pi^2}{4}} + b_2\sin\frac{5\pi x}{2}\cdot e^{-\frac{c^2 \cancel{4}\pi^2}{4}t} \ldots$

$+ b_5 \sin\frac{5\pi x}{2}\cdot e^{-\frac{25c^2\pi^2}{4}t}, \ldots$

$$\boxed{u(x,t) = \sin\frac{\pi x}{2}\, e^{-\frac{c^2\pi^2}{4}} + 3\sin\frac{5\pi x}{2}\cdot e^{-\frac{25\pi^2c^2t}{4}}}$$

A

..od of length l with ~~is~~ insulated sides is initially at a uniform temperature u. Its ends are suddenly cooled to $0^\circ C$ and are kept at that temperature. Prove that the temperature function $u(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cdot e^{-\frac{c^2\pi^2 n^2}{l^2}t}$$

Where b_n is determined from the equation

$$b_n = \frac{2}{l}\int_0^l u_0(x)\cdot \sin\frac{n\pi x}{l}\,dx$$

Solⁿ.

Let temperature of the bar at any time t at a point x distance from the origin be $u(x,t)$.

Then the eqⁿ of one dimensional heat flow is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

Let $u = X\cdot T \longrightarrow (2)$

Where X is function of x only and T is function of t only.

$\therefore \frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t}$ and $\frac{\partial u}{\partial x} = T\cdot\frac{\partial X}{\partial x}$

$\frac{\partial^2 u}{\partial x^2} = T\cdot\frac{\partial^2 X}{\partial x^2}$

Putting in (1)..

$X\frac{\partial T}{\partial t} = c^2 T\frac{\partial^2 X}{\partial x^2}$

$XT' = c^2 T X''$

Divid by $X\cdot T$

$\frac{T'}{T} = c^2\frac{X''}{X}$

$\therefore \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = -p^2$ say $\longrightarrow (3)$

$\therefore \frac{X''}{X} = -p^2$

$X'' = -p^2 X$

$X'' + p^2 X = 0$

which is ordinary diff. eqⁿ

A.E. is $m^2 + p^2 = 0$

$m^2 = -p^2$

$m = \pm ip$

C.F. $= c_1\cos px + c_2\sin px$

$\frac{1}{c^2}\cdot\frac{T'}{T} = -p^2$

$\frac{T'}{T} = -p^2c^2$

$\frac{1}{T}\frac{dT}{dt} = -p^2c^2$

$\frac{dT}{T} = -p^2c^2\,dt$

On int

$\log T = -p^2c^2t + \log c_3$

$\log\frac{T}{c_3} = -p^2c^2t$

15

$$T = C_3 e^{-p^2c^2t}$$

$$\therefore u = X \cdot T$$

$$u(x,t) = [C_1 \cos px + C_2 \sin px]\, C_3 e^{-p^2c^2t} \longrightarrow (4)$$

Using initially condition

$$\left.\begin{array}{l} u(0,t) = 0 \\ u(l,t) = 0 \end{array}\right] \rightarrow (5)$$

$$\therefore 0 = [C_1 \cos 0 + 0]\, C_3 e^{-p^2c^2t}$$

$$0 = C_1 \Rightarrow C_1 = 0$$

putting in (4)

$$u(x,t) = C_2 \sin px\, C_3 e^{-p^2c^2t} \longrightarrow (6)$$

$$u(l,t) = C_2 \sin pl\, C_3 e^{-p^2c^2t}$$

$$0 = C_2 \sin pl \cdot C_3 e^{-p^2c^2t}$$

$$0 = \sin pl$$

$$\sin n\pi = \sin pl \Rightarrow p = \frac{n\pi}{l}$$

$$\therefore u(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2c^2t}{l^2}}$$

$$u(x,t) = b_n e^{-\frac{n^2\pi^2c^2}{l^2}t} \cdot \sin \frac{n\pi}{l} x \quad \text{where } C_3 C_2 = b_n$$

$$\therefore u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2c^2t}{l^2}}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2c^2}{l^2}t}} \rightarrow (7)$$

Ans.

~~putting t = 0~~

$$\cancel{u(x,0) = \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{l}} \rightarrow (8)$$

An insulated rod of length l has its ends A and B maintained at $0°C$ and $100°C$ respectively until steady state condition prevail. If B is suddenly reduced to $0°C$ and maintained at $0°C$, find the temperature at a distance x from A at time t, solve the equation of heat

(2004, 05)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

by method of separation of variable and obtain the solution.

Sol^n → Given heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

We know that the solution of (1) is

$$u(x,t) = [c_1 \cos px + c_2 \sin px] c_3 e^{-c^2p^2t} \longrightarrow (2)$$

with steady state condition

$$u(0,t) = 0 \longrightarrow (3)$$

$$u(l,t) = 0 \longrightarrow (4)$$

And Initial condition is →

$$u(x,0) = \frac{100}{l} x \longrightarrow (5)$$

Using steady state condition in (2)

$$\therefore u(0,t) = [c_1 \cdot 1 + 0] c_3 e^{-c^2p^2t}$$

$$0 = c_1 \Rightarrow c_1 = 0$$

Putting in (2)

$$u(x,t) = c_2 \sin px \cdot c_3 e^{-c^2p^2t} \longrightarrow (6)$$

$$u(l,t) = c_2 \sin pl \cdot c_3 e^{-c^2p^2t}$$

$$0 = c_2 \sin pl \cdot c_3 e^{-c^2p^2t}$$

$$0 = \sin pt \Rightarrow \sin n\pi = \sin pl$$

$$\therefore p = \frac{n\pi}{l}$$

$\therefore$ Putting in (6)

$$u(x,t) = c_2 \sin \frac{n\pi x}{l} \cdot c_3 e^{-c^2 \cdot \frac{n^2\pi^2}{l^2} t}$$

$$u(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cdot c_3 e^{-\frac{c^2n^2\pi^2}{l^2} t}$$

$$u(x,t) = b_n \sin\frac{n\pi x}{l}\, e^{-\frac{c^2\pi^2 n^2}{l^2}\cdot t}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cdot e^{-\frac{c^2\pi^2 n^2}{l^2} t} \longrightarrow (7)$$

Using initial condition

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cdot 1$$

$$\frac{100}{l}x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Which is half-range sine series for $\frac{100}{l}x$

$$\therefore\ b_n = \frac{2}{l}\int_0^l f(x)\cdot \sin\frac{n\pi x}{l}\,dx$$

$$= \cancel{\frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}dx}$$

$$= \frac{2}{l}\int_0^l \frac{100}{l}x\cdot \sin\frac{n\pi x}{l}dx$$

$$= \frac{200}{l^2}\int_0^l x\sin\frac{n\pi x}{l}dx$$

$$= \frac{200}{l^2}\left[x\cdot\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^l + \int_0^l \frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}dx\right]$$

$$= \frac{200}{l^2}\left[-x\frac{l}{n\pi}\left\{\cos n\pi - \cancel{\cos 0}\right\} + \frac{l^2}{n^2\pi^2}\cdot\left(\sin\frac{n\pi x}{l}\right)_0^l\right]$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}\left\{(-1)^n - \cancel{0}\right\}\right] + 0$$

$$= -\frac{200}{n\pi}(-1)^n$$

$$b_n = (-1)^1\frac{200}{n\pi}(-1)^n = \frac{200}{n\pi}(-1)^{n+1}$$

Putting in (7)

$$u(x,t) = \sum_{n=0}^{\infty}\frac{200}{n\pi}(-1)^{n+1}\cdot \sin\frac{n\pi x}{l}\cdot e^{-\frac{c^2\pi^2n^2}{l^2}\cdot t}$$

$$\boxed{u(x,t) = \frac{200}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\cdot \sin\frac{n\pi x}{l}\cdot e^{-\frac{c^2\pi^2n^2}{l^2}\cdot t}}$$ Ans.