List of Topics

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Unit - IV

Vector Calculus

Topic:
Scalar & Vector Point Functions

Quantity

Scalar	Vector (Denoted by $\vec{R}$ & **R**)
1-Dimensional (• Magnitude)	2-Dimensional ($\vec{R}$ • Magnitude, $\hat{R}$ • Direction)
Eg → Mass = 50 kg	Eg → Force = 50N
Length = 100 m	$\boxed{} \leftarrow$ Force
$f(x,y,z) = 5x+3y+7z$	$\phi(x,y,z) = 5x\hat{i} + 3y\hat{j} + 7z\hat{k}$
$r = \sqrt{x^2+y^2+z^2}$	$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ → Position vector
Scalar Point function	vector point function

Note:
$\hat{R} = \hat{i} + \hat{j} + \hat{k}$
$\hat{R}$ is a unit vector

■ Position Vector

$$\cancel{\boxed{\vec{A} = 2\hat{i} + 3\hat{j} + (-4\hat{k})}}$$

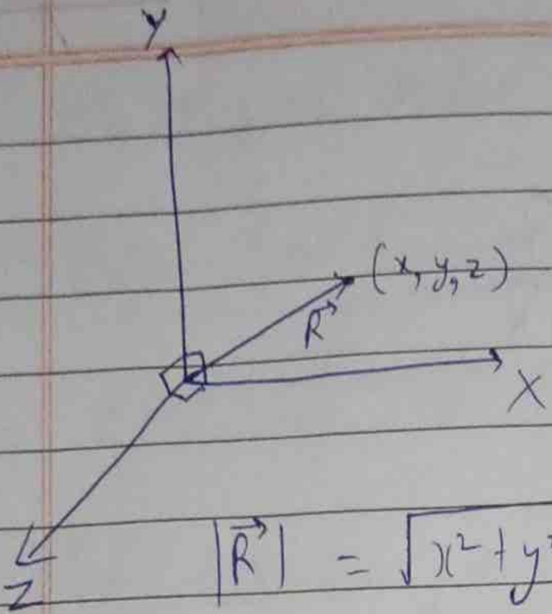
$$\boxed{\vec{r} = \vec{R} = x\hat{i} + y\hat{j} + z\hat{k}}$$ Default

$\hat{i}$ → Unit vector in x axis, +ve direction

$\hat{j}$ → Unit vector y-axis

$\hat{k}$ → Unit vector. z axis

GOOD WRITE



$$|\vec{R}| = \sqrt{x^2 + y^2 + z^2}$$

$$|\vec{R}| = R = \boxed{r = \sqrt{x^2 + y^2 + z^2}}$$ Default

Magnitude

$\vec{A} = 2\hat{i} + 3\hat{j} - 4\hat{k}$

Magnitude $|\vec{A}| = A = \sqrt{2^2 + 3^2 + (-4)^2}$

$A = \sqrt{29}$

Direction

$\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$

$\vec{A} = \dfrac{2}{\sqrt{29}}\hat{i} + \dfrac{3}{\sqrt{29}}\hat{j} - \dfrac{4}{\sqrt{29}}\hat{k}$

Scalar Function

Scalar Function $f(x, y, z)$ is a function defined at each point in a domain D. Its value is real & depends only on the point $P(x, y, z)$.

Eg → (i) $f(x, y, z) = xyz^2$

(ii) $\phi(x, y, z) = x + y + z$

Vector Function

If to each value of a scalar variable t, there corresponds a value of a vector $\vec{r}$, then $\vec{r}$ is called a vector function of scalar variate t.

We write $\vec{r} = \vec{f}(t)$

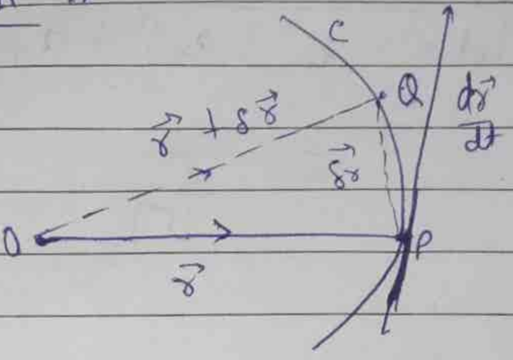
$$\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

GOOD WRITE

- Derivative of a vector function wrt scalar

Let $\vec{r} = \vec{F}(t)$ be a vector function of scalar variable t.
Then derivative of vector $\vec{r}$ wrt. t is $\frac{d\vec{r}}{dt}$

Geometrical interpretation :-



$\frac{d\vec{r}}{dt}$ is a vector along the tangent to the curve at point P.

- General rules for differentiation

If $\vec{a}$, $\vec{b}$ & $\vec{c}$ are vector functions of a scalar t & ϕ is a scalar function of t. Then

(i) $\frac{d}{dt}(\vec{a} \pm \vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$

(ii) $\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{b}$

(iii) $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$

(iv) $\frac{d}{dt}(\phi\vec{a}) = \phi\frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\vec{a}$

(v) $\frac{d}{dt}[\vec{a}\ \vec{b}\ \vec{c}] = \left[\frac{d\vec{a}}{dt}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \frac{d\vec{b}}{dt}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \frac{d\vec{c}}{dt}\right]$

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Q. If a, b are constant vectors; ω is a constant & r is vector function given by r = a cos ωt + b sin ωt.
Show that:
(i) $r \times \frac{dr}{dt} = \omega(a \times b)$

Sol→ $\frac{dr}{dt} = b\omega \cos\omega t - a\omega \sin\omega t$

$r \times \frac{dr}{dt} = (a\cos\omega t + b\sin\omega t) \times (-a\omega\sin\omega t + b\omega\cos\omega t)$

$= 0 + (a \times b)\,\omega\cos^2\omega t - (b \times a)\,\omega\sin^2\omega t$

$= (a \times b)\,\omega\cos^2\omega t + (a \times b)\,\omega\sin^2\omega t$

$= (a \times b)\,\omega(\cos^2\omega t + \sin^2\omega t)$

$= \omega(a \times b)$

$$\boxed{r \times \frac{dr}{dt} = \omega(a \times b)}$$

Hence proved.

(ii) $\frac{d^2r}{dt^2} = -\omega^2 r$

Sol→ $\frac{dr}{dt} = b\omega\cos\omega t - a\omega\sin\omega t$

$\frac{d^2r}{dt^2} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t$

$= -\omega^2(a\cos\omega t + b\sin\omega t)$

$= -\omega^2 r$

$$\boxed{\frac{d^2r}{dt^2} = -\omega^2 r}$$

Hence proved

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Q. If $a = (\sin\theta)\hat{i} + (\cos\theta)\hat{j} + \theta\hat{k}$, $b = (\cos\theta)\hat{i} - (\sin\theta)\hat{j}$ $-3\hat{k}$ & $c = 2\hat{i} + 3\hat{j} - \hat{k}$ find $\frac{d}{d\theta}\{a \times (b \times c)\}$ at $\theta = 0$.

Sol. $b \times c = \begin{vmatrix} \hat{i} & -\hat{j} & \hat{k} \\ \cos\theta & +\sin\theta & 3 \\ 2 & 3 & -1 \end{vmatrix}$

$= \hat{i}(9 + \sin\theta) + \hat{j}(6 - \cos\theta) + \hat{k}(2\sin\theta - 3\cos\theta)$

at $\theta = 0$; $b \times c = 9\hat{i}$

$\Rightarrow \frac{d}{d\theta}\{a \times (b \times c)\}$

$\Rightarrow a \times \frac{d}{d\theta}(b \times c) + \frac{da}{d\theta} \times (b \times c)$

$\Rightarrow a \times \left(b \times \frac{dc}{d\theta} + \frac{db}{d\theta} \times c\right) + \frac{da}{d\theta} \times (b \times c)$

$\Rightarrow a \times b \times \frac{dc}{d\theta} + a \times \frac{db}{d\theta} \times c + \frac{da}{d\theta} \times (b \times c)$

$\frac{da}{d\theta} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} + \hat{k}$ $\Rightarrow \left(\frac{da}{d\theta}\right)_{\theta=0} = i + k$

$\frac{db}{d\theta} = -\sin\theta\,\hat{i} - \cos\theta\,\hat{j}$

$\left(\frac{db}{d\theta}\right)_{\theta=0} = -\hat{j}$

$\left(\frac{dc}{d\theta}\right)_{\theta=0} = 0$

$\Rightarrow 0 + a \times \frac{db}{d\theta} \times c + \frac{da}{d\theta} \times (b \times c)$

$\Rightarrow \{\hat{j} \times (-\hat{j}) \times (2\hat{i} + 3\hat{j} - \hat{k})\} + (\hat{i} + \hat{k}) \times [(\hat{i} - 3\hat{k}) \times (2\hat{i} + 3\hat{j} - \hat{k})]$

$\Rightarrow \hat{j} \times [2\hat{k} + \hat{i}] + (\hat{i} + \hat{k}) \times (9\hat{i} - 5\hat{j} + 3\hat{k})$

$\Rightarrow 2\hat{i} - \hat{k} + (5\hat{i} + 6\hat{j} - 5\hat{k})$

$\Rightarrow 7\hat{i} + 6\hat{j} - 6\hat{k}$ Ans

Topic → Del, ~~Grad~~ Gradient, ~~Di~~ Divergence, Curl

Lec - 1

Gradient of a Scalar Field

Del

∇ or $\vec{\nabla}$ → Del Operator
→ Vector Differential Operator
→ Vector Operator

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

Gradient

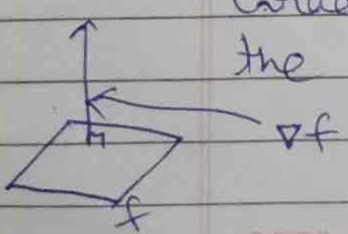
If f be a scalar point function, then ∇f is defined as gradient of f.

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\underbrace{f}_{\text{scalar}}$$

$$\boxed{\text{grad}(f) = \underbrace{\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}_{\text{vector}}}$$

Geometrical Interpretation of Gradient

Gradient of a scalar point function represents the normal vector to that surface.



Q. If $\phi = 3x^2y - y^3z^2$. Find grad (ϕ) at the point $(1, -2, -1)$.

Sol→ The given scalar function is

$$\phi = 3x^2y - y^3z^2$$

$$\text{grad}(\phi) = \nabla(\phi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\left(\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}(6xy - 0) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(0 - 2y^3z)$$

$$\boxed{\text{grad}(\phi) = 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\,\hat{k}}$$

at point $(1, -2, -1)$

$$= 6(1)(-2)\hat{i} + [\cancel{0}3(1) - 3(-2)^2(-1)^2]\hat{j} - 2(-2)^3(-1)\hat{k}$$

$$= -12\hat{i} + (\cancel{0}3 - 12)\hat{j} - 16\hat{k}$$

$$\boxed{\text{grad}(\phi) = -12\hat{i} - 9\hat{j} - 10\hat{k}}$$ Ans

↳ Normal vector to ϕ

→ Gradient (∅)
→ grad (∅)
→ ∇(∅)
→ Normal vector
→ ⊥ vector
→ Normal unit vector
→ ⊥ unit vector

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Lec-2

Q. Find a unit vector Normal to the surface $x^3+y^3+3xyz=3$ at point (1, 2, −1).

Sol→ The equation of the given surface is

$$x^3+y^3+3xyz=3$$
$$x^3+y^3+3xyz-3=0$$

So, let

$$\boxed{\phi = x^3+y^3+3xyz-3}$$

Now,

Normal vector = grad (∅)

$$= \nabla \phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^3+y^3+3xyz-3)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^3+y^3+3xyz-3)+\hat{j}\frac{\partial}{\partial y}(x^3+y^3+3xyz-3)$$
$$+\hat{k}\frac{\partial}{\partial z}(x^3+y^3+3xyz-3)$$

$$= \hat{i}(3x^2+0+3yz-0)+\hat{j}(0+3y^2+3xz-0)+\hat{k}(0+0+3xy-0)$$

$$\boxed{\vec{N} = 3(x^2+yz)\hat{i}+3(y^2+xz)\hat{j}+3xy\hat{k}}$$

at point (1, 2, −1)

$$\vec{N} = 3[1^2+2(-1)]\hat{i}+3[2^2+(1)(-1)]\hat{j}+$$
$$3(1)(2)\hat{k}$$

GOOD WRITE

$$\boxed{\vec{N} = -3\hat{i} + 9\hat{j} + 6\hat{k}} \Rightarrow \text{Normal Vector}$$

Normal unit vector $\Rightarrow \hat{N} = \dfrac{\vec{N}}{|\vec{N}|} = \dfrac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{9+81+36}}$

$$= \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{9+81+36}} \Rightarrow \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{126}}$$

$$= \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}}$$

$$\boxed{\hat{N} = -\frac{1}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{2}{\sqrt{14}}\hat{k}} \text{ Ans}$$

Lec-3
Gradient of a scalar Field

(Directional Derivative)

① Directional Derivative

Directional → Value in a given direction
↳ unit vector
+
Derivative → Gradient

[Pehle derivative karke then directional pe jana]

The value of $\vec{A}$ in the Direction of $\vec{B}$
↓
$\vec{A} \cdot \hat{B}$
↳ Dot Product

The value of $\vec{B}$ in the Direction of $\vec{A}$
↓
$\vec{B} \cdot \hat{A}$
↳ Dot Product

② Greatest Rate of Increase
↳ Always in Normal Direction
↳ Gradient

Q Find the Directional Derivative at $\phi(x, y, z) = x^2yz + 4xz^2$ at point $(1, -2, 1)$ in the direction of vector $(2\hat{i} - \hat{j} - 2\hat{k})$.
What is greatest rate of increase in ϕ at the point $(1, 0, 3)$?

Sol→ The given function is $\phi(x, y, z) = x^2yz + 4xz^2$
Now; Derivative (ϕ) = Gradient $(\phi) = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$\cancel{= \hat{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \hat{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \hat{k}\frac{\partial}{\partial z}(x^2y + 8xz)}$$

$$\Rightarrow \hat{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \hat{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \hat{k}\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$\Rightarrow \hat{i}(2xyz + 4z^2) + \hat{j}(x^2z + 0) + \hat{k}(x^2y + 8xz)$$

$$\boxed{\vec{A} = 2(xyz + 2z^2)\hat{i} + x^2z\hat{j} + (x^2y + 8xz)\hat{k}} \quad -(1)$$

at point $(1, -2, 1)$ in eqn (1)

$$\vec{A} = 2[(1)(-2)(+1) + 2(1)^2]\hat{i} + (1)^2(1)\hat{j} + [(1)(-2) + 8(1)(1)]\hat{k}$$

$$\boxed{\vec{A} = 0\hat{i} + \hat{j} + 6\hat{k}}$$

Now, Given $\vec{B} = 2\hat{i} - \hat{j} - 2\hat{k}$

Direction of $\vec{B} \Rightarrow \hat{B} = \frac{\vec{B}}{|\vec{B}|} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4 + 1 + 4}} \rightarrow \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{9}}$

$$\boxed{\hat{B} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}}$$

Now Directional Derivative of $\vec{A}$ in the direction of $\vec{B} = \vec{A} \cdot \hat{B}$

$$= (0\hat{i} + \hat{j} + 6\hat{k}) \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= (0)\left(\frac{2}{3}\right) + (1)\left(-\frac{1}{3}\right) + (6)\left(\frac{-2}{3}\right)$$

$$= 0 - \frac{1}{3} - \frac{12}{3}$$

$$\boxed{\text{Directional Derivative of } \vec{A} = -\frac{13}{3}}$$

Now, Greatest rate of increase at point $(1, 0, 3)$

$$= 2(0 + 18)\hat{i} + ((1)(3))\hat{j} + (0 + 24)\hat{k}$$

$$= \boxed{36\hat{i} + 3\hat{j} + 24\hat{k}}$$

Lec-4

(Gradient of a Scalar Field)

Q. Find the directional derivative at $\phi(x,yz) = xy^2+yz^3$ at the point $(2,-1,1)$ in the direction of the Normal to the surface $x\log z - y^2+4=0$ at $(2,-1,1)$.

[NIT ; PATNA - 2015]

Sol→ The given 1st equation is

$$\phi(x,y,z) = xy^2+yz^3$$

Now, Derivative (ϕ) = Gradient $(\phi) = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(xy^2+yz^3)$$

$$\boxed{\vec{A} = y^2\hat{i}+(2xy+z^3)\hat{j}+3yz^2\hat{k}} \quad -(1)$$

Put point $(2,-1,1)$ in eqn (1)

$$\boxed{\vec{A} = \hat{i}-3\hat{j}-3\hat{k}}$$

Derivative

The given 2nd equation is

$$x\log z - y^2+4 = 0$$

Let $\boxed{f = x\log z - y^2+4}$

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Normal vector (f) = Gradient (f) = ∇f

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\log z - y^2 + 4)$$

$$\boxed{\vec{B} = (\log z)\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k}} \quad -(2)$$

Put point $(2, -1, 1)$ in eqn (2)

$$\vec{B} = (\log(1))\hat{i} - 2(-1)\hat{j} + \frac{(2)}{(1)}\hat{k}$$

$$\boxed{\vec{B} = 0\hat{i} + 2\hat{j} + 2\hat{k}}$$

↳ Normal Vector

Now; Direction of Normal vector

$$\hat{B} = \frac{\vec{B}}{|\vec{B}|} = \frac{0\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{0^2 + 2^2 + 2^2}}$$

$$= \frac{0\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{8}}$$

$$\boxed{\hat{B} = 0\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}}$$

↳ Direction

Now, Directional Derivative

$$= \vec{A}\cdot\hat{B}$$

$$= (\hat{i} - 3\hat{j} - 3\hat{k})\cdot\left(0\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right)$$

 $\Rightarrow \frac{-6}{\sqrt{2}} \Rightarrow \boxed{-3\sqrt{2}}$ Ans

Lec - 5

Gradient of a Scalar Field

Q. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

i) Show that

$$\nabla r^n = n r^{n-2} \vec{r}$$

Sol → As we know that

$$\boxed{\vec{R} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}}$$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}}$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

Similarly; $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ & $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

⇒ Now, L.H.S.

$$\Rightarrow \nabla r^n$$

$$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n$$

$$\Rightarrow \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}$$

$$\Rightarrow \hat{i}\left(n r^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(n r^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(n r^{n-1}\frac{\partial r}{\partial z}\right)$$

(⇒ Putting the values of $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$ & $\frac{\partial r}{\partial z}$ from above)

$\Rightarrow \hat{i}\left(n r^{n-1}\left(\frac{x}{r}\right)\right)+\hat{j}\left(n r^{n-1}\left(\frac{y}{r}\right)\right)+\hat{k}\left(n r^{n-1}\left(\frac{z}{r}\right)\right)$

$\Rightarrow n r^{n-2}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$

$\Rightarrow n r^{n-2}\vec{r}$

$\Rightarrow$ R.H.S.

L.H.S. = R.H.S.

Hence, proved

ii) Show that; $\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$

Sol→ As we know that

$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$r^2 = x^2+y^2+z^2$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r} \; ; \; \frac{\partial r}{\partial y} = \frac{y}{r} \; \& \; \frac{\partial r}{\partial z} = \frac{z}{r}}$$

Now, L.H.S.

$\Rightarrow \vec{\nabla}\left(\frac{1}{r}\right)$

$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$

$\Rightarrow \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$\Rightarrow \hat{i}\left(-r^{-2}\frac{\partial r}{\partial x}\right)+\hat{j}\left(-r^{-2}\frac{\partial r}{\partial y}\right)+\hat{k}\left(-r^{-2}\frac{\partial r}{\partial z}\right)$

$\Rightarrow \hat{i}\left(\frac{-1}{r^2}\left(\frac{x}{r}\right)\right)+\hat{j}\left(-\frac{1}{r^2}\left(\frac{y}{r}\right)\right)+\hat{k}\left(-\frac{1}{r^2}\left(\frac{z}{r}\right)\right)$

$$\Rightarrow \hat{i}\left(\frac{-x}{r^3}\right) + \hat{j}\left(\frac{-y}{r^3}\right) + \hat{k}\left(\frac{-z}{r^3}\right)$$

$$\Rightarrow \frac{-1}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\Rightarrow -\frac{\vec{r}}{r^3}$$

$\Rightarrow$ R.H.S.

So, L.H.S. = R.H.S.

Hence, proved.

~~Lec-8~~ Extra lecture 8 (Laplacian Operator)

$$(\text{grad } \phi) = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\text{div.}(\text{grad }\phi) = \nabla.(\nabla\phi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$\boxed{\nabla.\nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

Laplacian Operator

- Harmonic function

If $\nabla^2\phi = 0$

then ϕ is Harmonic function.

Lec-1

Divergence & Curl of a Vector field

• Divergence of a vector Field

Definition :-

If $\vec{F} \rightarrow$ vector Point Function
→ Continuous
→ Differentiable

Then Divergence of $\vec{F}$ is defined as

$$\text{div } \vec{F} = \underbrace{\nabla \cdot \vec{F}}_{\text{Dot Product}}$$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$\boxed{\text{div } \vec{F} = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial F_3}{\partial z}}$$ → scalar valued function

vector

Physical Meaning of divergence :-

It gives the Rate of ~~flow~~ Fluid originating per unit volume at any point in vector field.

Solenoidal / Incompressible / Divergence Free vector field

if Divergence (vector field) = 0

$\text{div}(\vec{F}) = 0$

$$\boxed{\nabla \cdot \vec{F} = 0}$$

Q Evaluate $\operatorname{div}(3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$ at point $(1, 2, 3)$.

Sol → $\operatorname{div}(3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$

$\Rightarrow \nabla \cdot (3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$

$\Rightarrow \frac{\partial}{\partial x} 3x^2 + \frac{\partial}{\partial y} 5xy^2 + \frac{\partial}{\partial z} xyz^3$

$\Rightarrow 6x + 10xy + 3xyz^2$

at point $(1, 2, 3)$

$\Rightarrow 6(1) + 10(1)(2) + 3(1)(2)(3)^2$

$\Rightarrow \boxed{80}$ Ans

Curl of a vector Field

Definition

If $\vec{F} \to$ Vector Point Function

→ Continuous

→ Differentiable

Then $\operatorname{curl}(\vec{F})$ is defined as

$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}$

↳ Cross Product

$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$

$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$ ⇒ Vector Valued function

Physical meaning of curl

It gives Angular velocity at any point in vector field.

~~Irrotional~~.

Irrotational Vector Field

if $\text{curl}(\vec{F}) = 0$

$$\boxed{\nabla \times \vec{F} = 0}$$

Irrotational → Angular velocity Zero

Q. Find the curl of $\vec{F} = xy\hat{i} + y^2\hat{j} + zx\hat{k}$ at point $(-2, 4, 1)$.

Sol → $\vec{F} = xy\hat{i} + y^2\hat{j} + zx\hat{k}$

Now $\text{curl}(\vec{F}) = \nabla \times \vec{F}$

$$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(xy\hat{i} + y^2\hat{j} + zx\hat{k}\right)$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & y^2 & zx \end{vmatrix}$$

$$\Rightarrow \hat{i}(0-0) - \hat{j}(z-0) + \hat{k}(0-x)$$

$$\Rightarrow \boxed{-z\hat{j} - x\hat{k}}$$

at point $(-2, 4, 1)$

$$\boxed{\text{curl } \vec{F} = -\hat{j} + 2\hat{k}}$$ Ans

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Extra Questions

Q. If $\vec{F} = 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}$

find $\nabla\cdot\vec{F}$
$\nabla\times\vec{F}$

Sol → Here; $\vec{F} = 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}$

Now; $\nabla\cdot\vec{F}$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}(2xy^3z^4) + \frac{\partial}{\partial y}(3x^2y^2z^4) + \frac{\partial}{\partial z}(4x^2y^3z^3)$$

Ans $\Rightarrow \boxed{\nabla\cdot\vec{F} = 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^2}$

Now; $\nabla\times\vec{F}$

$$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}\right)$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix}$$

Expanding along R_1, we get

$$\Rightarrow \hat{i}\,(12x^2y^2z^3 - 12x^2y^2z^3) - \hat{j}\,(8xy^3z^3 - 8xy^3z^3)$$

$$+ \hat{k}\,(6xy^2z^4 - 6xy^2z^4)$$

$\Rightarrow \boxed{\nabla\times\vec{F} = 0\hat{i} + 0\hat{j} + 0\hat{k}}$ Ans

→ Zero Vector

Lec-2

Divergence & Curl of a vector Field

M.M. Imp.

Q Find div $\vec{F}$ & Curl $\vec{F}$ where $\vec{F} = \text{grad}[x^3+y^3+z^3-3xyz]$.

Sol→ $\vec{F} = \text{grad}\,(x^3+y^3+z^3-3xyz)$

$$\vec{F} = \nabla\,(x^3+y^3+z^3-3xyz)$$

$$\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$\boxed{\vec{F} = 3(x^2-yz)\hat{i}+3(y^2-zx)\hat{j}+3(z^2-xy)\hat{k}}$$

Now; div $\vec{F} = \nabla\cdot\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(3(x^2-yz)\hat{i}+(3y^2-zx))\hat{j}+\hat{k}(3z^2-3xy)\right)$$

$$= \frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3zx)+\frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x+6y+6z$$

$$\boxed{\text{div}\,\vec{F} = 6(x+y+z)}$$ Ans

Now; curl $\vec{F} = \nabla\times\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left[(3x^2-3yz)\hat{i}+(3y^2-3zx)\hat{j}+(3z^2-3xy)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3zx & 3z^2-3xy \end{vmatrix}$$

$$\Rightarrow \hat{i}(-3x+3x)-\hat{j}(-3y+3y)+\hat{k}(-3z+3z)$$

$$\Rightarrow 0\hat{i}+0\hat{j}+0\hat{k}$$

$$\boxed{\text{curl}\,\vec{F} = \vec{0}}$$ Ans

Imp Note

curl(grad F) = 0

$\nabla\times(\nabla F) = 0$

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Q. If $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$
find $\nabla \cdot \vec{R}$ or div $\vec{R}$ or $\nabla \cdot \vec{r}$.

Sol→ Here, $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

Now, $\nabla \cdot \vec{R}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$\boxed{\nabla \cdot \vec{R} = 3}$$ Ans
(CRAM)

Q. Find $\nabla \times \vec{R}$ or curl $\vec{R}$ or $\nabla \times \vec{r}$.

Sol→ Here, $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

Now, $\nabla \times \vec{R}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

Expending along R_1, we get

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$\nabla \times \vec{R} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\boxed{\nabla \times \vec{R} = \vec{0}}$$ Ans
(CRAM)

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Lec-3

Divergence & Curl of a vector field

M. Imp.

Q. If $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ then Prove that
$\text{div}(r^n \vec{R}) = (n+3)r^n$
& Find the value of n, if $r^n \vec{R}$ is a solenoidal vector.

Sol: As we know that
$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$r = |\vec{R}| = \sqrt{x^2+y^2+z^2}$$

Now, L.H.S. $= \text{div}(r^n \vec{R})$
$$\Rightarrow \nabla \cdot (r^n \vec{R})$$
$$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(r^n (x\hat{i} + y\hat{j} + z\hat{k})\right)$$
$$\Rightarrow \frac{\partial}{\partial x}(x r^n) + \frac{\partial}{\partial y}(y r^n) + \frac{\partial}{\partial z}(z r^n)$$
$$\Rightarrow \left[x\left(n r^{n-1}\frac{\partial r}{\partial x}\right) + r^n(1)\right] + \left[y\left(n r^{n-1}\frac{\partial y}{\partial x}\right) + r^n(1)\right]$$
$$+ \left[z\left(n r^{n-1}\frac{\partial r}{\partial z}\right) + r^n\right]$$
$$\Rightarrow \left[n x r^{n-1}\left(\frac{x}{r}\right) + r^n\right] + \left[n y r^{n-1}\left(\frac{y}{r}\right) + r^n\right] + \left[n z r^{n-1}\left(\frac{z}{r}\right) + r^n\right]$$
$$\Rightarrow \left[n x^2 r^{n-2} + r^n\right] + \left[n y^2 r^{n-2} + r^n\right] + \left[n z^2 r^{n-2} + r^n\right]$$
$$\Rightarrow n r^{n-2}(x^2+y^2+z^2) + 3r^n$$
$$\Rightarrow n r^{n-2}(r^2) + 3r^n \quad \left[\because r = \sqrt{x^2+y^2+z^2}\right]$$
$$\Rightarrow n r^n + 3r^n$$
$$\Rightarrow (n+3)r^n$$
$$\Rightarrow \text{R.H.S.}$$

So, as L.H.S. = R.H.S.
Hence, Proved

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if $r^n \vec{R}$ is solenoidal vector, then

$$\text{div}(r^n \vec{R}) = 0$$

$$(n+3)\, r^n = 0$$

$$n+3 = 0 \Rightarrow \boxed{n=-3} \text{ Ans}$$

$$(\because r^n \neq 0)$$

M. Imp.

Q. If $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$; then prove that $\text{curl}(r^n \vec{R}) = 0$ & find the value of n, if $r^n \vec{R}$ is a irrotational vector.

Sol→ As; we know that

$$\vec{r} = \vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = |\vec{R}| = \sqrt{x^2+y^2+z^2}$$

Now; L.H.S.

$$\text{curl}(r^n \vec{R})$$

$$\Rightarrow \nabla \times (r^n \vec{R})$$

$$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[r^n (x\hat{i} + y\hat{j} + z\hat{k})\right]$$

$$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(x r^n \hat{i} + y r^n \hat{j} + z r^n \hat{k}\right)$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x r^n & y r^n & z r^n \end{vmatrix}$$

(Expanding along R_1, we get)

$$\Rightarrow \hat{i}\left(\frac{\partial}{\partial y}(z r^n) - \frac{\partial}{\partial z}(y r^n)\right) - \hat{j}\left(\frac{\partial}{\partial x}(z r^n) - \frac{\partial}{\partial z}(x r^n)\right) + \hat{k}\left(\frac{\partial}{\partial x}(y r^n) - \frac{\partial}{\partial y}(x r^n)\right)$$

$$\Rightarrow \hat{i}\left(nyz\, r^{n-2} - nyz\, r^{n-2}\right) - \hat{j}\left(nxz\, r^{n-2} - nxz\, r^{n-2}\right) + \hat{k}\left(nxy\, r^{n-2} - nxy\, r^{n-2}\right).$$

$$\Rightarrow 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\Rightarrow \vec{0}$$

$\Rightarrow$ R.H.S.

So, L.H.S. = R.H.S

Hence, Proved

Since; $r^n \vec{R}$ is irrotational.

So; curl $(r^n \vec{R}) = 0$

which is already ~~true~~ for all values of n.

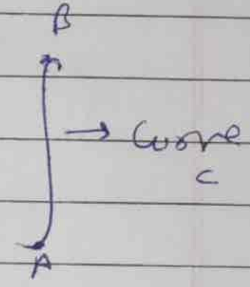
Lec-1
Line Integral
Understanding & its various types & Applications

Vector integral
- Line Integral
- Surface Integral
- Volume Integral

• **Line Integral**

Line integral of a vector point function $\vec{F}(x, y, z)$ continuous along the curve 'C' from point A to B is defined as :-

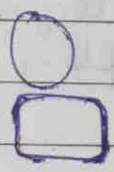
$$\int_C \vec{F} \cdot d\vec{R}$$



$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$d\vec{R} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

If path of integration is a closed curve then line Integral is denoted as

$$\oint \vec{F} \cdot d\vec{R}$$



Other two line integrals are :-

$$\int_C \vec{F} \times d\vec{R} \quad \text{and} \quad \int_C F \, d\vec{R}$$

⇒ Application

1. If $\vec{F}$ is a force applied on a particle moving along the curve AB then the Total work done by $\vec{F}$ during the displacement from A to B is given by

$$\int_A^B \vec{F} \cdot d\vec{R}$$

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(2) Circulation

If $\vec{F}$ represents the velocity of fluid particle then

$\int_C \vec{F}\cdot d\vec{R}$ is also called as circulation.

(3) If $\int_C \vec{F}\cdot d\vec{R} = 0$ $\longrightarrow$ $\vec{F}$ is irrotational.

(VTU Ques 2010, 2011)

Lec - 2

Line Integral

Line integral
Work done
circulation

Q. If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$. Evaluate $\boxed{\int_C \vec{F}\cdot d\vec{R}}$ along the curve C in X - Y plane $y = x^3$ from the ~~plane~~ point (1, 1) to (2, 8).

Sol → The given function is

$$\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$$

⇒ So, now $\int_C \vec{F}\cdot d\vec{R}$

$$\Rightarrow \int_C \left[(5xy - 6x^2)\hat{i} + (2y-4x)\hat{j}\right]\cdot\left[dx\,\hat{i} + dy\,\hat{j}\right]$$

$$\Rightarrow \int_C (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$

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① Ist → dx ② Ist → dy ③ Ist → dx
2nd → dy 2nd → dy 2nd → dx

Three ways

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The given equation of curve is $y = x^3$ from $(1,1)$ to $(2,8)$

$$\int (5x(x^3) - 6x^2)\,dx + \int (2y - 4(x^{1/3}))\,dy$$

Put $x = y^{1/3}$ & $dx = \frac{1}{3} y^{-2/3}\,dy$

$$\int \left[5(y^{1/3})y - 6(y^{1/3})^2\right]\left(\frac{1}{3} y^{-2/3}\,dy\right) + (2y - 4y^{1/3})\,dy$$

Using this, this time

Put $y = x^3$ & $dy = 3x^2\,dx$

$$\int_1^2 (5x(x^3) - 6x^2)\,dx + (2x^3 - 4x)(3x^2\,dx)$$

$$\Rightarrow \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)\,dx$$

$$\Rightarrow \left[5\left(\frac{x^5}{5}\right) - 6\left(\frac{x^3}{3}\right) + 6\left(\frac{x^6}{6}\right) - 12\left(\frac{x^4}{4}\right)\right]_1^2$$

$$\Rightarrow \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2$$

$$\Rightarrow \left[2^5 - 2(2)^3 + 2^6 - 3(2)^4\right] - \left[1^5 - 2(1)^3 + 1^6 - 3(1)^4\right]$$

$$\Rightarrow [32 - 16 + 64 - 48] - (1 - 2 + 1 - 3)$$

$$\Rightarrow 32 + 3 \Rightarrow \boxed{35} \text{ Ans}$$

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Q. Find $\int_C y^2 dx + x^2 dy$ along the curve $x+y=1$ from point $(0,1)$ to $(1,0)$

Sol → The given equation is

$$\int_C y^2\,dx + x^2\,dy \quad \text{--- (1)}$$

Here; equation of curve is

$x + y = 1$

$x = 1-y$

$dx = -dy$

from Point $(0, 1)$ to $(1, 0)$

$\underbrace{\qquad}_{y}$

Put the values of x & dx in eqn (1), we get

$$\int_0^1 y^2(-dy) + (1-y)^2\,dy$$

$$\Rightarrow \int_0^1 -y^2\,dy + (1+y^2-2y)\,dy \quad [\because (A-B)^2 = A^2+B^2-2AB]$$

$$\Rightarrow \int_1^0 (1-2y)\,dy$$

$$\Rightarrow \left[y - 2\left(\frac{y^2}{2}\right)\right]_1^0 \quad \left[\because \int y^n\,dy = \frac{y^{n+1}}{n+1}\right]$$

$$\Rightarrow \left[(y-y^2)\right]_1^0$$

$$\Rightarrow [0-(0)^2] - [1-(1)^2] \Rightarrow 0-0 \Rightarrow 0 \text{ Ans}$$

So, the given $\vec{F}$ is irrotational.

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Lec-3

Line Integral

Q. find the work done in moving a particle with a force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along the curve $x^2 = 4y$ & $3x^3 = 8z$ from $x=0$ to $x=2$.

[IPU Delhi - 2002]

Sol→ The given force field is

$$\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$$

Now, as we know that

$$\text{work done} = \int_C \vec{F} \cdot d\vec{R}$$

$$\Rightarrow \int_C [3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}] \cdot [dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}]$$

$$\Rightarrow \int_C 3x^2\,dx + (2xz - y)\,dy + z\,dz \quad -①$$

⇒ The given equation of curve is

$x^2 = 4y$

$\boxed{x^2/4 = y}$

$\frac{2x\,dx}{4} = dy$

$\boxed{dy = x\,dx/2}$

& $3x^3 = 8z$

$\boxed{\frac{3x^3}{8} = z}$

$\boxed{\frac{9x^2\,dx}{8} = dz}$

⇒ Put the values of y, dy, z & dz in equation ①, we get

$$\Rightarrow \int_0^2 3x^2\,dx + \left[2x\left(\frac{3x^3}{8}\right) - \left(x^2/4\right)\right]\left(x/2\,dx\right) + \left(\frac{3x^2}{8}\right)\left(\frac{9x^2\,dx}{8}\right)$$

$$\Rightarrow \int_0^2 \left(3x^2 + \frac{3x^5}{8} - \frac{x^3}{8} + \frac{27}{64}x^5\right) dx$$

$$\Rightarrow \int_0^2 \left(\frac{51}{64}x^5 - \frac{x^3}{8} + 3x^2\right) dx$$

$$\Rightarrow \left[\frac{51}{64}\left(\frac{x^6}{6}\right) - \frac{1}{8}\left(\frac{x^4}{4}\right) + 3\left(\frac{x^3}{3}\right)\right]_0^2$$

$$\Rightarrow \left[\frac{17}{128}x^6 - \frac{1}{32}x^4 + x^3\right]_0^2$$

$$\Rightarrow \left[\frac{17}{128}(2)^6 - \frac{1}{32}(2)^4 + (2)^3\right] - 0$$

$$\Rightarrow \left[\frac{17}{2} - \frac{1}{2} + 8\right] \Rightarrow \frac{16}{2} + 8 \Rightarrow \underline{16} \text{ Ans}$$

Q. Find the work done by the force $\vec{F} = [2y+3, xz, yz-x]$ when it is moving a particle from the point $(0,0,0)$ to point $(2,1,1)$ along the curve $[2t^2, t, t^3]$.

Sol → The given force field is :

$$\vec{F} = [2y+3, xz, yz-x]$$

$$\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$$

As; we know that

$$\text{work done} = \int_C \vec{F} \cdot d\vec{R}$$

$$\Rightarrow \int_C (2y+3)\,dx + (xz)\,dy + (yz-x)\,dz \quad -(1)$$

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⇒ Now given equation of curve is

$$[2t^2, t, t^3]$$

⇒ $x = 2t^2$ | $y = t$ | $z = t^3$

$dx = 4t\,dt$ | $dy = dt$ | $dz = 3t^2 dt$

Now points are $(0,0,0)$ to $(1,1,1)$

If $y = 0 \Rightarrow t = 0$
$y = 1 \Rightarrow t = 1$

⇒ Put the values of x, dx, y, dy, z, & dz in equation ①, we get

$$\Rightarrow \int_0^1 (2(t)+3)(4t\,dt) + (2t^2)(t^3)(dt) + (t(t)^3 - (2t^2))(3t^2 dt)$$

$$\Rightarrow \int_0^1 (8t^2 + 12t + 2t^5 + 3t^6 - 6t^4)\,dt$$

$$\Rightarrow \left[8\left(\frac{t^3}{3}\right) + 12\left(\frac{t^2}{2}\right) + 2\left(\frac{t^6}{6}\right) + 3\left(\frac{t^7}{7}\right) - 6\left(\frac{t^5}{5}\right)\right]_0^1$$

$$\Rightarrow \left[\frac{8}{3}t^3 + 6t^2 + \frac{1}{3}t^6 + \frac{3}{7}t^7 - \frac{6}{5}t^5\right]_0^1$$

$$\Rightarrow \left[\frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}\right] - 0$$

$$\Rightarrow \frac{280 + 630 + 35 + 45 - 126}{105} \Rightarrow \frac{864}{105} \Rightarrow \boxed{\frac{288}{35}} \text{ Ans}$$

Lec-4
Line Integral

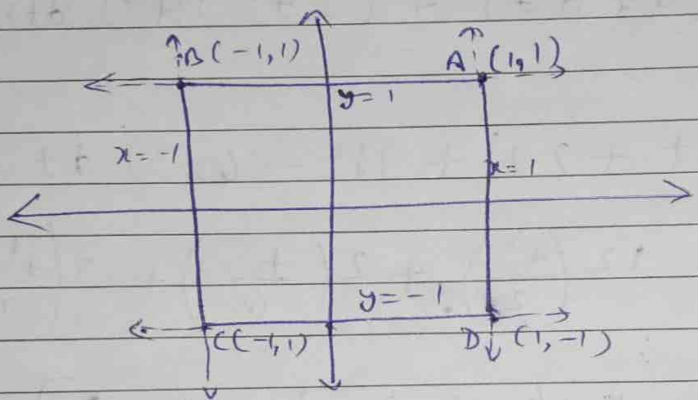
Q Evaluate $\int_c (x^2 + xy)\,dx + (x^2+y^2)\,dy$

where c is a square formed by the lines $x = \pm 1$ & $y = \pm 1$.

(IPU Delhi-2002)

(V imp)

Sol → The general equation of curves is
$x = \pm 1$ & $y = \pm 1$



⇒ Now, as we know that the line integral over square is given as :-

$$\oint_{ABCD} \vec{F}\cdot d\vec{R} = \int_{AB} \vec{F}\cdot d\vec{R} + \int_{BC} \vec{F}\cdot d\vec{R} + \int_{CD} \vec{F}\cdot d\vec{R} + \int_{DA} \vec{F}\cdot d\vec{R} \quad \text{—(1)}$$

Along the ~~cur~~ curve AB

Equation :- $y = 1 \Rightarrow dy = 0$

Points :- $A(1,1)$ to $B(-1,1)$

$\underbrace{\hphantom{A(1,1) \text{ to } B(-1}}_{x}$

$$= \int_{AB} \vec{F}\cdot d\vec{R}$$

$$= \int_{AB} (x^2 + xy)\,dx + (x^2+y^2)\,dy$$

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$$\Rightarrow \int_{1}^{-1} (x^2 + x(-1))dx + (x^2 + 1^2)(0)$$

$$\Rightarrow \int_{+1}^{-1} (x^2 + x)\, dx$$

$$\Rightarrow \left[\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\right]_1^{-1}$$

$$\Rightarrow \left(\frac{-1}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{2}\right)$$

$$\Rightarrow -\frac{1}{3} + \frac{1}{2} - \frac{1}{3} - \frac{1}{2} \Rightarrow -2/3$$

$\Rightarrow$ Along the curve BC

Equation: $x = -1 \Rightarrow dx = 0$

Points: $B(-1, \underbrace{1) \text{ to } C\ (-1, -1}_{y})$

$$\Rightarrow \int_{BC} \vec{F} \cdot d\vec{R}$$

$$\Rightarrow \int_{BC} (x^2 + xy)dx + (x^2 + y^2)dy$$

$$\Rightarrow \int_{+1}^{-1} 0 + [(-1)^2 + y^2]dy$$

$$\Rightarrow \int_{1}^{-1} (1 + y^2)\, dy$$

$$\Rightarrow \left[\left(y + \frac{y^3}{3}\right)\right]_1^{-1}$$

$$\Rightarrow \left(-1 - \frac{1}{3}\right) - \left(1 + \frac{1}{3}\right) \Rightarrow -2 - \frac{2}{3} \Rightarrow \boxed{-\frac{8}{3}}$$

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⇒ Along the curve CD

Eqn ÷ $y = -1 \Rightarrow dy = 0$

Points ÷ $C(-1, -1)$ to $D(1, -1)$

$\underbrace{-1 \to 1}_{x}$

$$\Rightarrow \int_{CD} \vec{F} \cdot d\vec{R}$$

$$\Rightarrow \int_{CD} (x^2 + xy)\,dx + (x^2 + y^2)\,dy$$

$$\Rightarrow \int_{-1}^{1} [x^2 + x(-1)]\,dx + 0$$

$$\Rightarrow \int_{-1}^{1} (x^2 - x)\,dx$$

$$\Rightarrow \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1}$$

$$\Rightarrow \left(\frac{1}{3} - \frac{1}{2}\right) - \left(-\frac{1}{3} - \frac{1}{2}\right)$$

$$\Rightarrow \frac{1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = \frac{2}{3}$$

⇒ Along the curve DA :

Equation ÷ $x = 1 \Rightarrow dx = 0$

Points $= D(+1, -1)$ to $A(1, 1)$

$\underbrace{-1 \to 1}_{y}$

$$\Rightarrow \int_{DA} \vec{F} \cdot d\vec{R}$$

$$\Rightarrow \int_{DA} (x^2 + xy)\,dx + (x^2 + y^2)\,dy$$

$$\Rightarrow \int_{-1}^{1} 0 + [1^2 + y^2]\, dy$$

$$\Rightarrow \int_{-1}^{1} (1+y^2)\, dy$$

$$\Rightarrow \left[\left(y + \frac{y^3}{3}\right)\right]_{-1}^{1}$$

$$\Rightarrow \left(1+\frac{1}{3}\right) - \left(-1-\frac{1}{3}\right)$$

$$\Rightarrow 2 + \frac{2}{3} \Rightarrow \frac{8}{3}$$

⇒ Put the ~~&~~ values of all curve parts in eqn ① ;
we get

$$\Rightarrow \left(-\frac{2}{3}\right) + \left(-\frac{8}{2}\right) + \left(\frac{2}{3}\right) + \left(\frac{8}{3}\right)$$

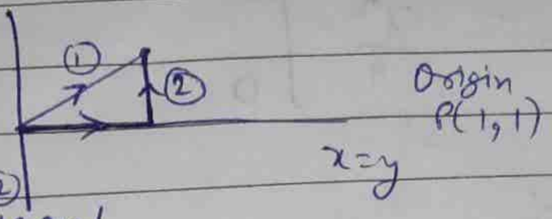
$$\Rightarrow \boxed{0} \text{ Ans}$$

Line Integral

Path Independence

Note: Line Integral may or may not depend on path of integration.

$\boxed{\text{If independent; } \vec{F} \text{ is conservative.}}$



$\vec{F} = x^2\hat{i} - xy\hat{j}$

Ans: if Line integral by path ① = Line integral by path ②

then $\vec{F}$ is independent; means $\vec{F}$ is conservative.

Otherwise $\vec{F}$ is not independent; means $\vec{F}$ is non-conservative.

(Topic → Double Integration)

Q. $\int_{x=0}^{1}\int_{y=0}^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

$\int \frac{1}{a^2+y^2}\,dy = \frac{1}{a}\tan^{-1}\frac{y}{a}$

Sol → $\int_{x=0}^{1}\int_{y=0}^{\sqrt{1+x^2}} \frac{dx\,dy}{(\sqrt{1+x^2})^2+y^2}$

$\Rightarrow \int_{x=0}^{1}\left(\frac{1}{\sqrt{1+x^2}}\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right)_0^{\sqrt{1+x^2}} dx$

$\Rightarrow \int_{x=0}^{1}\frac{\tan^{-1}(1)}{\sqrt{1+x^2}}\,dx$

$\Rightarrow \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$

$\Rightarrow \frac{\pi}{4}\left(\log|x+\sqrt{1+x^2}|\right)_0^1$

$\Rightarrow \frac{\pi}{4}\left[\log|1+\sqrt{2}| - \log|1|\right]$

$\Rightarrow \boxed{\frac{\pi}{4}\log|1+\sqrt{2}|}$ Ans

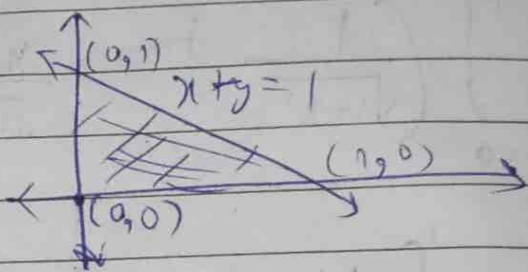
Q. Evaluate $\iint xy \, dx \, dy$ where the following region of integration is

(i) $x + y \leq 1$ in positive quadrant

(ii) $x^2 + y^2 = a^2$ in the positive quadrant.

Sol→ (i) $x + y \leq 1$

x	1	0
y	0	1



$$\Rightarrow \int_{x=0}^{1} \int_{y=0}^{1-x} xy \, dx \, dy$$

$$\Rightarrow \int_{x=0}^{1} x \left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$\Rightarrow \frac{1}{2} \int_0^1 x(1-x)^2 \, dx \Rightarrow \frac{1}{2} \int_0^1 x(1-2x+x^2) \, dx$$

$$\Rightarrow \frac{1}{2} \int_0^1 (x - 2x^2 + x^3) \, dx$$

$$\Rightarrow \frac{1}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

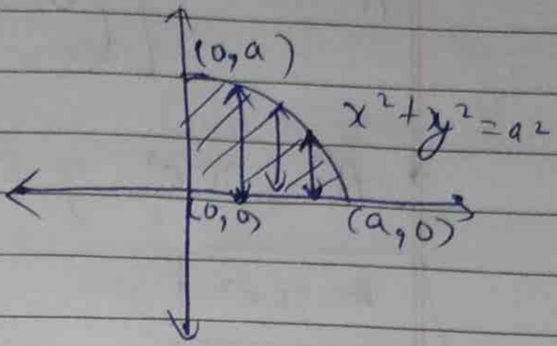
$$\Rightarrow \frac{1}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] \Rightarrow \frac{1}{2}\left[\frac{6-8+3}{12}\right]$$

$$\Rightarrow \frac{1}{24}$$ Ans

(ii)

$x^2 + y^2 = a^2$

$y = \sqrt{}$



$$\Rightarrow \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy \, dy \, dx$$

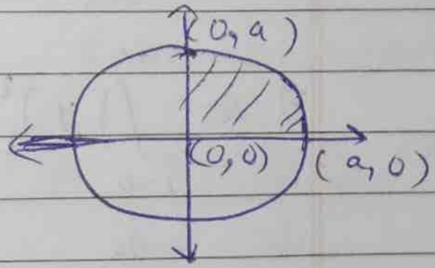
$$\Rightarrow \int_{x=0}^{a} \frac{x}{2}\left[y^2\right]_0^{\sqrt{a^2-x^2}} dx$$

$$\Rightarrow \frac{1}{2}\int_{x=0}^{a} x(a^2-x^2)\, dx \Rightarrow \frac{1}{2}\int_0^a (xa^2 - x^3)\, dx$$

$$\Rightarrow \frac{1}{2}\left[\frac{x^2a^2}{2} - \frac{x^4}{4}\right]_0^a \Rightarrow \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] \Rightarrow \frac{a^4}{8} \text{ Ans}$$

$\Rightarrow$ Area by Double integration

$$\boxed{A = \iint dx\, dy}$$



Eg $\Rightarrow$ $x^2 + y^2 = a^2$

$$A = 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} dy\, dx$$

$$A = 4\int_{x=0}^{a} [y]_0^{\sqrt{a^2-x^2}} dx \Rightarrow 4\int_{x=0}^{a} \sqrt{a^2-x^2}\, dx$$

$$A = 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

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$$\Rightarrow A = 4\left[\left(0 + \frac{a^2}{2}\sin^{-1} 1\right) - 0\right]$$

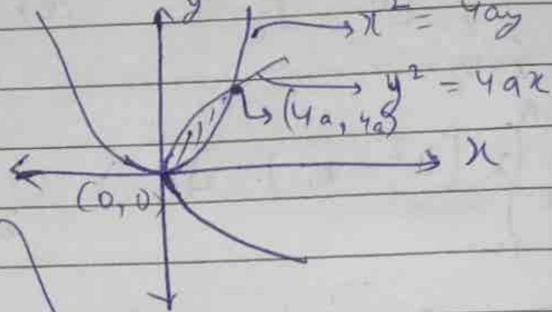
$$\Rightarrow A = \pi \frac{a^2}{2}\frac{\pi}{2} = \pi a^2 \text{ Ans}$$

$$\boxed{A = \pi a^2}$$

Q. Find the area of Region by Double Integration $y^2 = 4ax$ & $x^2 = 4ay$.

Sol→ $\boxed{x = 4a ; y = 4a}$

Dono ke saath x & y ka value nikala



$$\Rightarrow A = \iint dx\, dy$$

$$\Rightarrow A = \int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dx\, dy$$

$$\Rightarrow A = \int_{x=0}^{4a} [y]_{x^2/4a}^{2\sqrt{ax}}\, dx$$

$$\Rightarrow A = \int_{x=0}^{4a}\left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$\Rightarrow A = \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\right]_0^{4a}$$

$$\Rightarrow A = \left(4\sqrt{a}\,\frac{4a\sqrt{4a}}{3} - \frac{(4a)^3}{12a}\right)$$

$$\Rightarrow A = \frac{32a^2}{3} - \frac{64a^2}{12}$$

$$\Rightarrow A = \left(\frac{32}{3}a^2 - \frac{16}{3}a^2\right)$$

$$\Rightarrow \boxed{A = \frac{16}{3}a^2} \text{ Ans}$$

GOOD WRITE

Topic -> Green's Theorem

Lec - 1

Statement :

It gives the relationship b/w Line integral & a double integral over a plane region R, bounded by that closed curve C.

$$\underbrace{\oint_C M\,dx + N\,dy}_{\text{Line Integral}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}_{\text{Double Integral}}$$

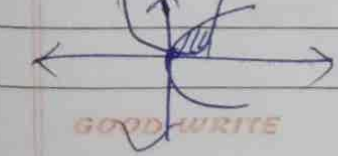
Here M & N are function of x & y. And they have continuous partial derivative.

[* Green's Theorem is special case of Kelvin Stoke's Theorem]

Benefits of Green's Theorem

* L.H.S. = Line integral depends upon no. of lines in closed curve

R.H.S. = Double Integral depends upon no. of regions that is generally one in Green's Theorem questions.

closed curves	No. of Line integral (LHS)	No. of Double integral (RHS)
Triangle	3	1
Square	4	1
Rectangle	4	1
	2	1.

Verify means L.H.S. & R.H.S. dono verify karte hai;
using mein generally R.H.S. use karte hai

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Lec - 2

Q. Verify Green's Theorem for

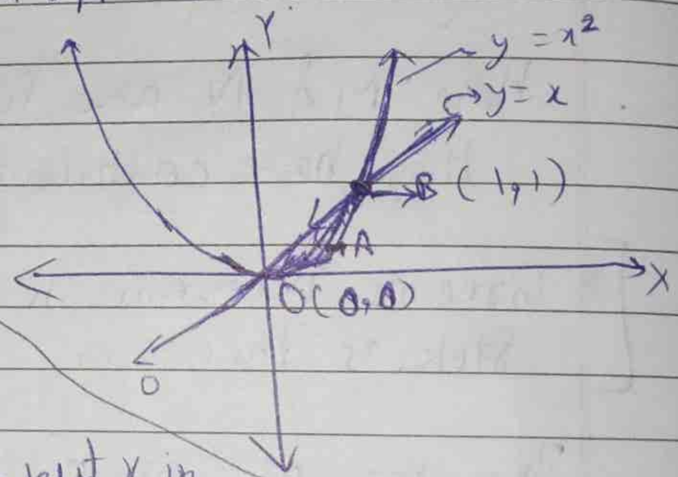
$\oint_C (xy - y^2)\,dx + x^2\,dy$ where C is bounded

by curves $y = x$ & $y = x^2$.

Sol→ The given curves are:

$y = x$ → Denotes a line — ①

$y = x^2$ → Denotes an upper parabola — ②



Points of intersection of eq^n ① & ②

$x = x^2 \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow \boxed{x = 0,\ x = 1}$

So, put x in eq^n ①

$y = 0\ ;\ y = 1$

Points of intersection are (0, 0) & (1,1)

→ Now as we know that by Green's Theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

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L.H.S.

$$\Rightarrow \oint_C (xy-y^2)\,dx + x^2\,dy = \int_{OAB} (xy-y^2)\,dx + x^2\,dy + \int_{BO} (xy-y^2)\,dx + x^2\,dy \quad -(3)$$

$\Rightarrow$ Now along curve OAB

Equation ; $\boxed{y = x^2}$

$dy = 2x\,dx$

Points: $O(0,0)$ to $B(1,1)$
$\underbrace{\qquad\qquad}_{x}$

we get

$$\Rightarrow \int_0^1 [x(x^2) - (x^2)^2]\,dx + x^2(2x\,dx)$$

$$\Rightarrow \int_0^1 (x^3 - x^4 + 2x^3)\,dx$$

$$\Rightarrow \int_0^1 (3x^3 - x^4)\,dx \Rightarrow \left[3\left(\frac{x^4}{4}\right) - \frac{x^5}{5}\right]_0^1$$

$$\Rightarrow \left[\frac{3}{4} - \frac{1}{5}\right] \Rightarrow \frac{11}{20}$$

$\Rightarrow$ Now along curve BO

Equation : $y = x$; $dy = dx$

Points : $B(1,1)$ to $O(0,0)$
$\underbrace{\qquad\qquad}_{x}$

GOOD WRITE

$\Rightarrow$ We get,

$$\Rightarrow \int_0^1 \left(x(x^2) - (x^2)^2\right) dx + x^2(2x\,dx)$$

$$\Rightarrow \int_0^1 (x^3 - x^4 + 2x^3)$$

$$\Rightarrow \int_1^0 (x(x) - x^2)dx + x^2 dx$$

$$\Rightarrow \int_1^0 x^2 dx \Rightarrow \frac{1}{3}[x^3]_1^0 \Rightarrow \boxed{\frac{-1}{3}}$$

$\Rightarrow$ Put the values of line integrals parts in eqn (3) we get

$$L.H.S. = \oint_C (xy - y^2)dx + x^2 dy = \left(\frac{11}{20} - \frac{1}{3}\right)$$

$$L.H.S \Rightarrow \frac{33-20}{60} \Rightarrow \frac{13}{60}$$ Ans

Now, RHS

$$\Rightarrow \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\Rightarrow \iint_R \left(2x - (x - 2y)\right) dx\,dy$$

$$\Rightarrow \int_0^1 \int_y^{\sqrt{y}} (x + 2y)\, dx\, dy$$

$$\Rightarrow \int_0^1 \left[\frac{x^2}{2} + 2yx\right]_y^{\sqrt{y}} dy$$

$$\Rightarrow \int_0^1 \left(\left(\frac{y}{2} + 2y\sqrt{y}\right) - \left(\frac{y^2}{2} + 2y^2\right)\right) dy$$

$$\Rightarrow \int_0^1 \left(\frac{y}{2} + 2y^{3/2} - \frac{5}{2}y^2\right) dy$$

$$\Rightarrow \left[\frac{y^2}{4} + \frac{4y^{5/2}}{5} - \frac{5}{6}y^3\right]_0^1$$

$$\Rightarrow \left(\frac{1}{4} + \frac{4}{5} - \frac{5}{6}\right) \Rightarrow \frac{13}{60} = R.H.S.$$

$\Rightarrow$ So, L.H.S. = R.H.S.

So, Green's Theorem Verified.

Lec-3 (Green's Theorem)

Q. Verify Green's Theorem in a plane for

$$\oint_C (3x - 8y^2)\,dx + (4y - 6xy)\,dy$$ where c is

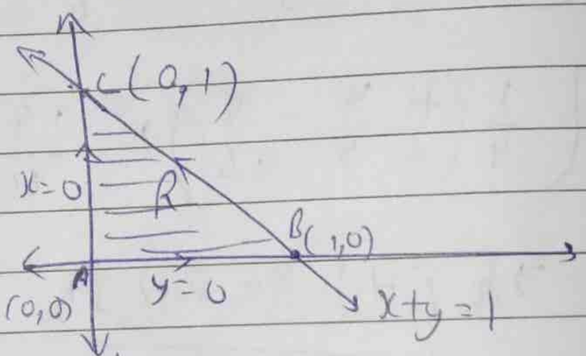
the boundary of region defined by $x = 0,\ y = 0$ & $x + y = 1$.

Sol. The given eqns are:

$y = 0 \rightarrow$ Represents x-axis

$x = 0 \rightarrow$ Represents y-axis

$x + y = 1 \rightarrow$ Represents a line which cuts x-axis at $(1,0)$ & y-axis at $(0,1)$.



$\Rightarrow$ Now, as we know that by Green's Theorem:

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\Rightarrow$ Now LHS

$$\oint_{ABC} M\,dx + N\,dy = \int_{AB} M\,dx + N\,dy + \int_{BC} M\,dx + N\,dy + \int_{CA} M\,dx + N\,dy \quad \text{--- (1)}$$

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Along the line AB

Eqn: $y = 0$; $dy = 0$ Points: $A(0,0)$ to $B(1,0)$

$$\Rightarrow \int_0^1 (3x - 0)\, dx + 0$$

$$\Rightarrow \frac{3}{2}[x^2]_0^1 \Rightarrow \frac{3}{2}$$

Along line CA

Eqn: $x = 0$; $dx = 0$; Points: $C(0,1)$ to $A(0,0)$

$$\Rightarrow \int_1^0 0 + (4y - 0)\, dy$$

$$\Rightarrow \frac{4}{2}[y^2]_1^0 \Rightarrow \boxed{-2}$$

Along the line BC

Eqn $\Rightarrow x + y = 1$; $\boxed{\begin{array}{l} x = 1 - y \\ dx = -dy \end{array}}$ Points: $B(1,0)$ to $C(0,1)$

$$\Rightarrow \int_0^1 (3(1-y) - 8y^2)(-dy) + (4y - 6(1-y)y)\, dy$$

$$\Rightarrow \int_0^1 (+3y - 3 + 8y^2 + 4y - 6y + 6y^2)\, dy$$

$$\Rightarrow \int_0^1 (14y^2 + y - 3)\, dy \Rightarrow \left[14\left(\frac{y^3}{3}\right) + \frac{y^2}{2} - 3y\right]_0^1$$

$$\Rightarrow \frac{14}{3} + \frac{1}{2} - 3 \Rightarrow \frac{13}{6}$$

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⇒ Now the values of line integral of curve parts in eq^n (1), we get

$$\Rightarrow \frac{3}{2} + \frac{13}{6} + \left(\frac{-2}{1}\right)$$

$$\Rightarrow \boxed{5/3}$$

Now; R.H.S.

$$\Rightarrow \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$\Rightarrow \int_0^1 \int_0^{1-y} (0-6y) - (0-16y)\, dx\, dy$$

$$\Rightarrow \int_0^1 \int_0^{1-y} 10y\, dx\, dy \Rightarrow \int_0^1 10y\, [x]_0^{1-y}\, dy$$

$$\Rightarrow \int_0^1 10y(1-y)\, dy \Rightarrow 10\int_0^1 (y-y^2)\, dy$$

$$\Rightarrow 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 \Rightarrow 10\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$\Rightarrow \frac{10}{6} = \boxed{\frac{5}{3}}$$

⇒ So; as L.H.S. = R.H.S. = 5/3

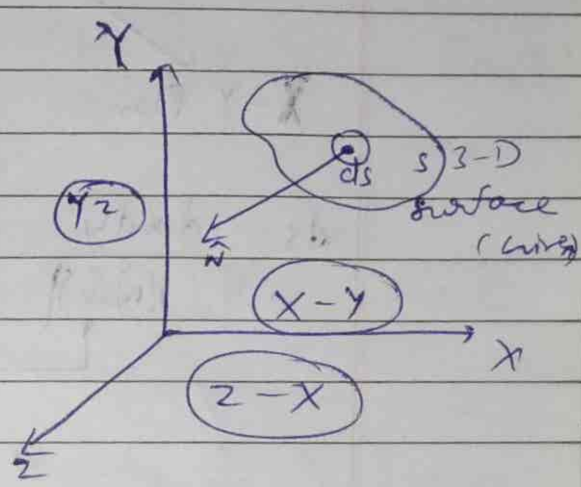
Hence; Green's Theorem verified

Topic →
Surface Integral

Lec – 1

Statement:
Consider a continuous vector point function $\vec{F}$ and 3-D surface S, then surface Integral is given as

$$\int_S \vec{F}\cdot d\vec{S} \text{ or } \int_S \underbrace{\vec{F}\cdot\hat{N}}_{\text{given}}\, ds$$



Note: Double Integral for 2-D surface only.

Applications:

1. ~~film~~ Flux across a surface

If $\vec{F}$ represents the velocity of a fluid particle, then total outward flux of $\vec{F}$ across closed surface S is given by surface integral.

$$\int_S \vec{F}\cdot\hat{N}\, ds$$

2. Solenoidal vector

if $\int_S \vec{F}\cdot\hat{N}\, ds = 0$ (the $\vec{F}$ is a solenoidal vector)

3-D surface

(1) ds → Depends on which plane, the given 3-D surface will be projected

Projected on → Lie in that plane
→ parallel to that plane
→ Your choice

X-Y Plane | Y-Z Plane | X-Z Plane

$ds = \frac{dx\,dy}{|(\hat{N}\cdot\hat{k})|}$ $\quad ds = \frac{dy\,dz}{|\hat{N}\cdot\hat{i}|}$ $\quad ds = \frac{dx\,dz}{|\hat{N}\cdot\hat{j}|}$

Normal unit vector of plane

(2) $\hat{N}$ = Normal unit vector outward to the surface

3-D surface

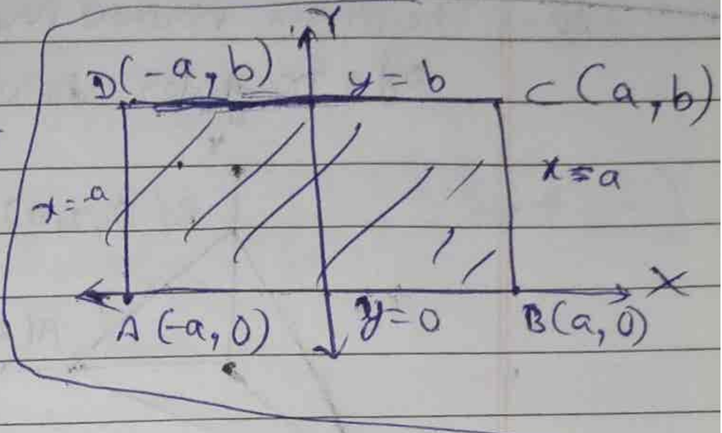
Value of $\hat{N}$

if 3-D surface lie in any plane	if 3-D surface parallel to any plane	if 3-D surface is neither lying nor parallel
$\hat{N}$ = Normal vector of that plane taking outward	$\hat{N}$ = Normal vector of that plane taking outward	$\hat{N} = \frac{grad(\phi)}{\lvert grad(\phi)\rvert}$

(Surface Integral)
Lec-2

Q. If $F = -4y\hat{k}$; then find the surface integral over the rectangle bounded by the lines $x = \pm a$, $y = 0$ & $y = b$.

Sol → The given equations $x = \pm a$, $y = 0$ & $y = b$ form a rectangle.



Now; as we know that surface integral is given as

$$\int_S \vec{F} \cdot \hat{N}\, ds \quad -(1)$$

Now since the given surface (Rectangle) lie in X-Y plane. So, it must be projected in X-Y Plane.

So, $ds = \dfrac{dx\,dy}{|\hat{N} \cdot \hat{k}|}$

$\hat{N}$ = Normal unit vector of plane = $\hat{k}$

$$\int_S \vec{F} \cdot \hat{N}\, ds = \int_0^b \int_{-a}^{a} (-4y\,\hat{k}) \cdot \hat{N} \frac{dx\,dy}{|\hat{N} \cdot \hat{k}|}$$

$$\Rightarrow \int_0^b \int_{-a}^{a} (-4y\hat{k} \cdot \hat{k}) \frac{dx\,dy}{|\hat{k} \cdot \hat{k}|}$$

$$\Rightarrow \int_0^b \int_{-a}^{a} -4y\, dx\,dy \Rightarrow \int_0^b -4y[x]_{-a}^{a}\, dy$$

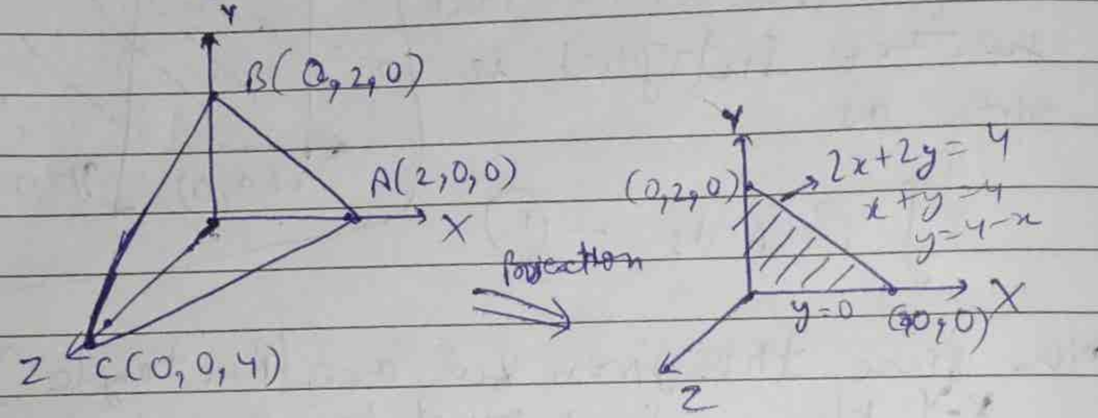
$$\Rightarrow \int_0^b -4y(2a)\, dy \Rightarrow -8a\left[y\right]_0^b \quad -\frac{8}{2}a\,[y^2]_0^b$$

$$\Rightarrow \boxed{-4ab^2} \text{ Ans}$$

Lec-3
(Surface Integral)

Q. Find the surface integral for the function $\vec{F} = x\hat{i} + (z^2 - 2x)\hat{j} - xy\hat{k}$ over the triangular surface with vertices (2,0,0), (0,2,0) & (0,0,4).

Sol → The given vertices are (2,0,0), (0,2,0) & (0,0,4) of a triangular surface.



Equation of triangular surface is given by intercept form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\Rightarrow \frac{x}{2} + \frac{y}{2} + \frac{z}{4} = 1$$

Since the given surface is projected in X-Y Plane. So;

$$ds = \frac{dx\,dy}{|\hat{N}\cdot\hat{k}|}$$

Now, the equation of the surface is

$$2x + 2y + z = 4$$

$$2x + 2y + z - 4 = 0$$

So, let $\boxed{\phi = 2x + 2y + z - 4}$

$\Rightarrow$ So, $\vec{N} = \text{grad}(\phi)$

$\Rightarrow \nabla \phi$

$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (2x + 2y + z - 4)$

$\vec{N} = 2\hat{i} + 2\hat{j} + \hat{k}$

Now; $\hat{N} = \frac{\vec{N}}{|\vec{N}|} = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{4+4+1}} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k} = \hat{N}$

So, now

$\vec{F} \cdot \hat{N} = (x\hat{i} + (z^2 - 2x)\hat{j} - xy\hat{k}) \cdot \left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)$

$$\boxed{\vec{F} \cdot \hat{N} = \frac{2x}{3} + \frac{2(z^2 - 2x)}{3} - \frac{1}{3}xy}$$

$\Rightarrow$ Now; surface integral is given as

$\Rightarrow \int_S \vec{F} \cdot \hat{N} \, ds = \iint \vec{F} \cdot \hat{N} \frac{dx\,dy}{|\hat{N} \cdot \hat{k}|}$

$\Rightarrow \int_0^2 \int_0^{4-x} (2x + 2z^2 - 2zx - xy) \, dx\,dy$

$\Rightarrow \int_0^2 \int_0^{4-x} (2x + 2(4 - 2x - 2y)^2 - 2(4 - 2x - 2y)x - xy) \, dx\,dy$

[$\because$ Equation of surface : $2x + 2y + z = 4$]

$\Rightarrow \int_0^2 \int_0^{4-x} (2x + 8(2 - x - y)^2 - 8 + 4x + 4y - xy) \, dx\,dy$

$\Rightarrow \int_0^2 \int_0^{4-x} (8(2 - x - y)^2 + 6x + 4y - xy - 8) \, dx\,dy$

$\Rightarrow \int_0^2 \left[\frac{8(2 - x - y)^3}{3(-1)} + 6xy + \frac{4y^2}{2} - \frac{xy^2}{2} - 8y\right]_0^{4-x} dx$

$$\Rightarrow \int_0^2 \left[\frac{-8}{3}(2-x-y)^3 + 6xy + 2y^2 - \frac{xy^2}{2} - 8y \right]_0^{4-x} dx$$

$$\Rightarrow \int_0^2 \left[\frac{-8}{3}(2-x-4+x)^3 + 6x(4-x) + 2(4-x)^2 - \frac{x}{2}(4-x)^2 - 8(4-x) \right] dx$$

$$\Rightarrow \int_0^2 \left[\frac{-8}{3}(-2)^3 + 24x - 6x^2 + 2(16+x^2-8x) - \frac{x}{2}(16+x^2-8x) - 32 + 8x \right] dx$$

$$\Rightarrow \int_0^2 \left(\frac{64}{3} + 24x - 6x^2 + 32 + 2x^2 - 16x - 8x - \frac{x^3}{2} + 4x^2 - 32 + 8x \right) dx$$

$$\Rightarrow \int_0^2 \left(-\frac{x^3}{2} + 8x + \frac{16}{3} \right) dx$$

$$\Rightarrow \left[\frac{1}{2\times 4}[x^4] + \frac{8}{2}(x^2) + \frac{16}{3}x \right]^2$$

$$\Rightarrow \int_0^2 \left(\frac{64}{3} + 8x - \frac{x^3}{2} \right) dx$$

$$\Rightarrow \left[\frac{64}{3}x + \frac{8}{2}x^2 - \frac{x^4}{8} \right]_0^2$$

$$\Rightarrow \frac{128}{3} + 16 - \frac{16}{8}$$

$$\Rightarrow \frac{1024 + 384 - 48}{24} \Rightarrow \boxed{\frac{1360}{24}} \text{ Ans}$$

what have done
1. value of $\hat{n}$ = ?
2. Value of ds = ?
3. Extra variable eliminate = ?
4. Limits = ?

Lec - 1

(Topic → Stoke's Theorem)

~~Lec-1~~

Statement

If S be a open surface bounded by a closed curve C & $\vec{F}$ is a continuous differentiable vector point function; then

$$\underbrace{\oint_C \vec{F} \cdot d\vec{R}}_{\text{Line integral}} = \underbrace{\int_S (\text{curl } \vec{F}) \cdot \hat{N} \, ds}_{\text{surface integral}}$$

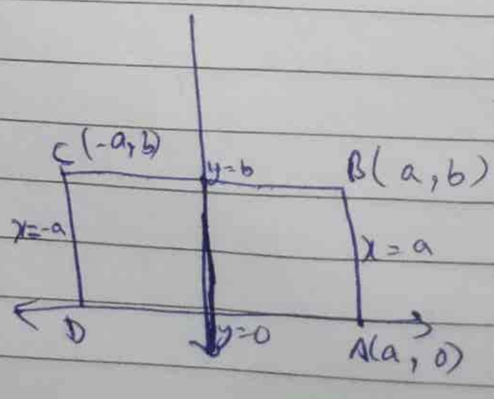
(*) Gives relation b/w Line integral & surface integral.

* Stoke's Theorem → for 3-D surfaces

Green's Theorem → For 2-D surfaces
↳ Special case of Stoke's Theorem

Q. Verify Stoke's Theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$ & $y = b$.

Sol, The given eqns of line are
$x = \pm a$, $y = 0$, $y = b$



As, we know that by Stoke's Theorem.

$$\int_{ABCD} \vec{F} \cdot d\vec{R} = \int_{ABCD} (\text{curl } \vec{F}) \cdot \hat{N}\, ds$$

Now LHS :→

$$\int_{ABCD} \vec{F} \cdot d\vec{R} = \int_{AB} \vec{F} \cdot d\vec{R} + \int_{BC} \vec{F} \cdot d\vec{R} + \int_{CD} \vec{F} \cdot d\vec{R} + \int_{DA} \vec{F} \cdot d\vec{R} \quad -(1)$$

(1) Now on line AB

Equation ÷ $x = a \Rightarrow dx = 0$

Points ÷ $\underbrace{A(a, 0) \text{ to } B(a, b)}_{y}$

$$\int_{AB} \vec{F} \cdot d\vec{R} = \int_{AB} (x^2 + y^2)\, dx - 2xy\, dy$$

$$\Rightarrow \int_0^b (0 - 2ay)\, dy \Rightarrow -\frac{2a}{2}\left[y^2\right]_0^b$$

$$\Rightarrow \boxed{-ab^2}$$

(2) Now on line BC

Equation ÷ $y = b \Rightarrow dy = 0$

Points ÷ $\underbrace{B(a, b) \text{ to } C(-a, b)}$

$$\int_{BC} (x^2 + y^2)\, dx - 2xy\, dy = \int_a^{-a} (x^2 + b^2)\, dx$$

$$\left[\frac{x^3}{3} + b^2 x\right]_a^{-a} \Rightarrow \boxed{-\frac{2a^3}{3} - 2ab^2}$$

Similarly

$$\Rightarrow \int_{CD} \vec{F} \cdot d\vec{R} = -ab^2$$

$$\Rightarrow \int_{DA} \vec{F} \cdot d\vec{R} = \frac{2a^3}{3}$$

$\Rightarrow$ Now, put all these values in eqn ①, we get

LHS
$$\int_{ABCD} \vec{F} \cdot d\vec{R} = (-ab^2) + \left(\frac{-2a^3}{3} - 2ab^2\right) + (-ab^2) + \left(\frac{2a^3}{3}\right)$$

$$= \underline{-4ab^2}$$

Now RHS

$$\int_{ABCD} (\text{Curl } \vec{F}) \cdot \hat{N}\, ds$$

$$\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$\hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-2y - (0+2y))$$

$$\underline{\text{Curl}\vec{F} = -4y\hat{k}}$$

Since surface ABCD lie in XY plane; so it must be projected in XY plane $\Big\} \Rightarrow \hat{N} = \hat{k}$, $\Rightarrow ds = \dfrac{dx\,dy}{|(\hat{N} \cdot \hat{k})|}$

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$$\Rightarrow \iint_{ABCD} (-4y\hat{k}) \cdot (\hat{N}) \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\Rightarrow \int_0^b \int_{-a}^{a} (-4y\hat{k}) \cdot \hat{k} \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|}$$

$$\Rightarrow \int_0^b \int_{-a}^{a} (-4y\,dx)\,dy \Rightarrow \int_0^b -4xy\,[x]_{-a}^{a}\,dy$$

$$\Rightarrow \int_0^b -4y\,[a+a]\,dy = -8a\left[\frac{y^2}{2}\right]_0^b$$

$$\Rightarrow \underline{-4ab^2} \text{ Ans}$$

So, as L.H.S. = R.H.S. $= -4ab^2$

Hence, verified.

GOOD WRITE

Lec-2

Q. Use Stoke's Theorem to evaluate

$$\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$$

where c is the boundary of triangle with vertices $(2,0,0)$, $(0,3,0)$ & $(0,0,6)$.

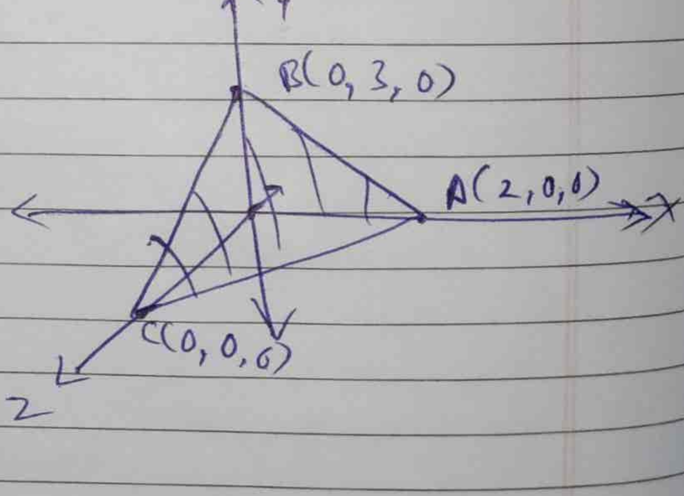
Sol → As, we know that by Stokes Theorem

$$\oint_C \vec{F}\cdot d\vec{R} = \int_S (\text{Curl } \vec{F})\cdot \hat{N}\, ds$$

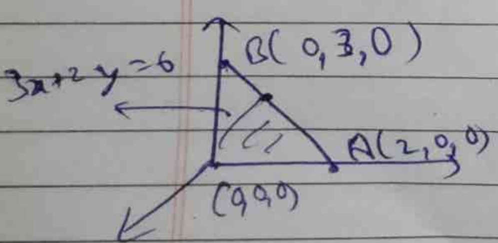
As we known that eqn of triangular surface is given as $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$\boxed{\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1}$$

$$\boxed{3x+2y+z=6}$$



Projection



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Curl $\vec{F} = \nabla \times \vec{F}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix}$$

(→ Expanding along R_1)

$\Rightarrow \hat{i}\,(1+1) - \hat{j}\,(0-0) + \hat{k}\,(0-1)$

$\Rightarrow \boxed{\text{Curl } \vec{F} = 2\hat{i} + \hat{k}}$

Now since the given triangular surface ~~neither~~ neither lie nor || to any plane ÷

$\hat{N} =$.

$$ds = \frac{dx\,dy}{|\hat{N}\cdot\hat{k}|}$$

$$\hat{N} = \frac{\text{grad } \phi}{|\text{grad } \phi|}$$

Now; eqn of surface is :

$3x + 2y + z = 6$

$3x + 2y + z - 6 = 0$

Now let

$\phi = 3x + 2y + z - 6$

$\vec{N} = \text{grad } \phi = \nabla\phi = 3\hat{i} + 2\hat{j} + \hat{k}$

$$\hat{N} = \frac{\vec{N}}{|\vec{N}|} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{9+4+1}} = \frac{3}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}$$

Now using ~~RHS~~ RHS of Stoke's Theorem

$$\Rightarrow \int_S (\text{Curl}\,\vec{F}).\hat{N}\, ds$$

$$\Rightarrow \cancel{\int_0^2 \int_0^{\frac{6-3x}{2}} \left(\frac{6}{\sqrt{14}}\right) + 0}.$$

$$\Rightarrow \int_0^2 \int_0^{\frac{6-3x}{2}} (2\hat{i}+\hat{k}).\left(\frac{3}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}\right) \frac{dx\,dy}{|\hat{N}.\hat{k}|}$$

$$\Rightarrow \int_0^2 \int_0^{\frac{6-3x}{2}} \left(\frac{6}{\sqrt{14}} + 0 + \frac{1}{\sqrt{14}}\right) \frac{dx\,dy}{(1/\sqrt{14})}$$

$$\Rightarrow \int_0^2 \int_0^{\frac{6-3x}{2}} 7\, dx\,dy \Rightarrow \int_0^2 7[y]_0^{\frac{6-3x}{2}} dx$$

$$\Rightarrow \int_0^2 7\left(\frac{6-3x}{2}\right) dx$$

$$\Rightarrow \frac{7}{2}\int_0^2 (6-3x)\,dx \Rightarrow \frac{7}{2}\left[6x - \frac{3x^2}{2}\right]_0^2$$

$$\Rightarrow \frac{7(12-6)}{2} \Rightarrow \frac{42}{2} \Rightarrow 21 \text{ Ans}$$

(Topic → Triple Integral)

Statement

Let the function $f(x, y, z)$ be a function of three independent variable x, y & z be continuous & defined at every point of a finite closed three dimensional region V.

Dividing region V into n elementary subregions of volume $\Delta V_1, \Delta V_2, \dots, \Delta V_n$ then

$$S_n = f(x_1, y_1, z_1)\Delta V_1 + f(x_2, y_2, z_2) V_2 + \dots + f(x_n, y_n, z_n)\Delta V_n$$

$$\Rightarrow \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_n$$

Then limit of this sum when $n \to \infty$ and the dimensions of each subregions $\Delta V_k \to 0$ in introduced as triple integral of function $f(x, y, z)$ over region V. The triple integral can be defined as

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(X_k, y_k, Z_k)\Delta V_k = \iiint_V f(x, y, z)\, dV$$

$$\Rightarrow \iiint f(x, y, z)\, dx\, dy\, dz$$

Evaluation of Triple Integrals

The integral $\iiint f(x, y, z)\, dx\, dy\, dz$ over 3D region V is evaluated by following limits for volume

$$\iiint f(x, y, z)\, dx\, dy\, dz = \int_{x=a}^{b} \int_{y=y(x)}^{y_2(x)} \int_{z=z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dx\, dy\, dz$$

$$\boxed{x^2 + y^2 + z^2 = a^2}$$

$$z = \pm\sqrt{a^2 - x^2 - y^2}$$

$$y = \pm\sqrt{a^2 - x^2}$$

$$x = \pm a$$

$$V = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dx\, dy\, dz$$

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Q. Evaluate: $\int_{x=0}^{3}\int_{y=0}^{2}\int_{z=0}^{1}(x+y+z)\,dx\,dy\,dz$

Sol: $\int_{x=0}^{3}\int_{y=0}^{2}\left[xz+yz+\frac{z^2}{2}\right]_0^1 dx\,dy$

$\Rightarrow \int_{x=0}^{3}\int_{y=0}^{2}\left(x+y+\frac{1}{2}\right)dy\,dx$

$\Rightarrow \int_{x=0}^{3}\left[xy+\frac{y^2}{2}+\frac{y}{2}\right]_0^2 dx$

$\Rightarrow \int_{x=0}^{3}(2x+2+1)\,dx$

$\Rightarrow \int_{x=0}^{3}(2x+3)\,dx \Rightarrow \cancel{\int (x^2+3x}\left[x^2+3x\right]_0^3$

$\Rightarrow 9+9 \Rightarrow 18$ Ans

Q. Evaluate: $\int_{y=0}^{1}\int_{z=y^2}^{1}\int_{x=0}^{1-z} z\,dy\,dz\,dx$

Sol: $\int_{y=0}^{1}\int_{z=y^2}^{1}\int_{x=0}^{1-z}(z\,dx)\,dy\,dz$

$\Rightarrow \int_{y=0}^{1}\int_{z=y^2}^{1} z\,(x)_0^{1-z}\,dy\,dz$

$$\Rightarrow \int_{y=0}^{1}\int_{z=y^2}^{1} (z - z^2)\,dz\,dy$$

$$\Rightarrow \int_{y=0}^{1} \left[\frac{z^2}{2} - \frac{z^3}{3}\right]_{y^2}^{1} dy$$

$$\Rightarrow \int_{y=0}^{1} \left(\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right)\right) dy$$

$$\Rightarrow \int_{y=0}^{1} \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy$$

$$\Rightarrow \left[\frac{y}{6} - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1 \Rightarrow \frac{1}{6} - \frac{1}{10} + \frac{1}{21}$$

$$\Rightarrow \frac{24}{210} \Rightarrow \boxed{\frac{4}{35}} \text{ Ans}$$

Q. Evaluate: $\int_{1}^{e}\int_{0}^{\log y}\int_{1}^{e^x} \log z \, dy\, dx\, dz$

Sol: $\int_{1}^{e}\int_{0}^{\log y}\int_{1}^{e^x} (\log z)\, dz\, dy\, dx$

$$\Rightarrow \int_{1}^{e}\int_{0}^{\log y} (z \log z - z)_1^{e^x}\, dy\, dx$$

$$\Rightarrow \int_{1}^{e}\int_{0}^{\log y} ((e^x x - e^x) + 1)\, dy\, dx \Rightarrow \int_{1}^{e}\int_{0}^{\log y} ((x-1)e^x + 1)\, dx\, dy$$

$$\Rightarrow \int_1^e \left[(x-1)e^x - e^x + x\right]_0^{\log y} dy$$

$$\Rightarrow \int_1^e \left(\left((\log y - 1)y - y + \log y\right) - (-1-1)\right) dy$$

$$\Rightarrow \int_1^e \left((y+1)\log y - 2y + 2\right) dy$$

$$\Rightarrow \left[\log y\left(\frac{y^2}{2} + y\right) - \int \frac{1}{y}\left(\frac{y^2}{2} + y\right) dy - y^2 + 2y\right]_1^e$$

$$\Rightarrow \left[\left(\frac{y^2}{2} + y\right)\log y - \frac{y^2}{4} - y - y^2 + 2y\right]_1^e$$

$$\Rightarrow \frac{e^2}{2} + e - \frac{e^2}{4} - e - e^2 + 2e + \frac{1}{4} + 1 + 1 - 2$$

$$\Rightarrow \left(\frac{1}{2} - \frac{1}{4} - 1\right)e^2 + 2e + \frac{1}{4}$$

$$\Rightarrow \boxed{-\frac{3}{4}e^2 + 2e + \frac{1}{4}} \text{ Ans}$$

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Q.1→ Find volume of sphere $x^2+y^2+z^2=a^2$ by triple integral.

Sol→ $V = \iiint dx\,dy\,dz$

$$V = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz$$

$$V = 8\int_{0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz$$

$$V = 8\int_{0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} [z]_{0}^{\sqrt{a^2-x^2-y^2}}\, dx\,dy$$

$$V = 8\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2-y^2}\; dx\,dy$$

$$V = 8\int_{0}^{a} \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_{0}^{\sqrt{a^2-x^2}} dx$$

$$V = 8\int_{0}^{a} \left(0 + \frac{a^2-x^2}{2}\sin^{-1}1\right) dx$$

$$V = \cancel{8 \times \frac{\pi}{4} \int_{0}^{a} \left(a^2x - \frac{x^3}{3}\right)} \quad 8 \times \frac{\pi}{4}\int_{0}^{a} (a^2-x^2)\,dx$$

$$V = 2\pi\left[a^2x - \frac{x^3}{3}\right]_{0}^{a} = 2\pi\left(a^3 - \frac{a^3}{3}\right)$$

$$V = 2\pi\left(\frac{2a^3}{3}\right)$$

$$\boxed{V = \frac{4\pi a^3}{3}}$$ Ans

Q.2 → Find value of $\iiint_V \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ where

V is the volume of sphere $x^2+y^2+z^2=1$.

Sol →
$$\int_{x=-1}^{1}\int_{y=\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2-z^2}}$$

$$\Rightarrow 8\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$\Rightarrow 8\int_0^1\int_{y=0}^{\sqrt{1-x^2}}\left(\sin^{-1}1-\sin^{-1}0\right)dx\,dy$$

$$\Rightarrow \frac{8\pi}{2}\int_0^1\int_{y=0}^{\sqrt{1-x^2}} dx\,dy$$

$$\Rightarrow 4\pi\int_0^1 [y]_0^{\sqrt{1-x^2}}\,dx$$

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$$\Rightarrow 4\pi \int_{x=0}^{x=1} \sqrt{1-x^2}\, dx$$

$$\Rightarrow 4\pi \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x \right]_0^1$$

$$\Rightarrow 4\pi \left(0 + \frac{1}{2}\sin^{-1} 1 \right)$$

$$\Rightarrow 4\pi \left(\frac{1}{2} \times \frac{\pi}{2} \right)$$

$\Rightarrow \pi^2$ Ans

Find the volume of the cylinder $x^2+y^2=a^2$ & $x^2+z^2=a^2$.

Sol →

$$V = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx\,dy\,dz$$

$$V = 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} dx\,dy\,dz$$

$$V = 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \left[z\right]_{0}^{\sqrt{a^2-x^2}} dx\,dy$$

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$$\Rightarrow V = 8 \int_{x=0}^{a} \int_{y=0}^{y=\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dx\, dy$$

$$\Rightarrow V = 8 \int_{x=0}^{a} \sqrt{a^2-x^2}\, (y)_0^{\sqrt{a^2-x^2}}\, dx$$

$$\Rightarrow V = 8 \int_{x=0}^{a} \sqrt{a^2-x^2} \left(\sqrt{a^2-x^2}\right) dx$$

$$\Rightarrow V = 8 \int_{x=0}^{a} (a^2-x^2)\, dx$$

$$\Rightarrow V = 8 \left(a^2 x - \frac{x^3}{3} \right)_0^a$$

$$\Rightarrow V = 8 \left(a^3 - \frac{a^3}{3} \right) \Rightarrow V = \frac{16 a^3}{3}$$ Ans

Ans

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Topic
Gauss Divergence Theorem

Statement

The normal surface integral of a vector function F over the boundary of a closed region is equal to the volume integral of div F taken throughout the region.

$$\iint_S \vec{F}\cdot\hat{N}\,ds = \iiint_V \operatorname{div} F\, dv$$

the more form

$$\iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy) =$$

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

Q, Show that $\iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy) = 4\pi a^3$

where the S is sphere $x^2+y^2+z^2=a^2$.

Sol: $F_1 = x$; $F_2 = y$; $F_3 = z$

$$\iint_S x\,dy\,dz + y\,dx\,dz + z\,dy\,dx = \iiint_V (1+1+1)\, dx\,dy\,dz$$

$$\Rightarrow 3\iiint_V dx\,dy\,dz \Rightarrow 3\left(\frac{4}{3}\pi a^3\right)$$ (Volume of sphere $= \frac{4}{3}\pi a^3$)

$$\Rightarrow 4\pi a^3$$

Hence; shown.

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Q. Find $\iint_S \vec{F}\cdot\hat{N}\,ds$ where $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$ where S is surface of the sphere having centre $(3,-1,2)$ & radius 3.

Sol→ $\iint_S \vec{F}\cdot\hat{N}\,ds = \iiint_V \operatorname{div} F\,dV$

$\Rightarrow \operatorname{div} F = \nabla\cdot F \Rightarrow \frac{d}{dx}(2x+3z) + \frac{d}{dy}(-xz-y) + \frac{d}{dz}(y^2+2z)$

$\Rightarrow 2-1+2$

$\Rightarrow 3$

$\Rightarrow \iint_S \vec{F}\cdot\hat{N}\,ds = \iiint_V 3\,dx\,dy\,dz$

$\Rightarrow 3\iiint_V dx\,dy\,dz$

$\Rightarrow 3\left(\frac{4}{3}\pi(3)^3\right)$

$\Rightarrow 12\times 9\pi$

$\Rightarrow \underline{108\pi}$ Ans

GOOD WRITE

Q. Evaluate $\iint_S \vec{F}\cdot\hat{N}\,ds$; where $\vec{F} = x\hat{i} - y\hat{j} + (z^2-1)\hat{k}$ & S is a closed surface bounded by plane $z=0$ & $z=1$ & cylinder $x^2+y^2=4$. Also verify Gauss divergence theorem.

Sol → By ~~volume integral~~ Gauss Divergence Theorem

$$\iint \vec{F}\cdot\hat{N}\,ds = \iiint \operatorname{div} F\,dv$$

$$\operatorname{div} F = (\nabla\cdot\vec{F}) \Rightarrow 1 - 1 + 2z = 2z$$

$$\iiint_V \operatorname{div}\vec{F}\,dv$$

$$\iiint_V 2z\,dx\,dy\,dz \Rightarrow 2\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{1} z\,dz\,dy\,dz$$

$$\Rightarrow 2\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(\frac{z^2}{2}\right)_0^1 dx\,dy$$

$$\Rightarrow \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx\,dy \Rightarrow 4\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} dx\,dy$$

$$\Rightarrow 4\int_{x=0}^{2} \sqrt{4-x^2}\,dx \Rightarrow 4\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$\Rightarrow 4\left(\frac{4}{2}\cdot\frac{\pi}{2}\right) = \underline{4\pi} \quad \underline{\text{Ans}}$$

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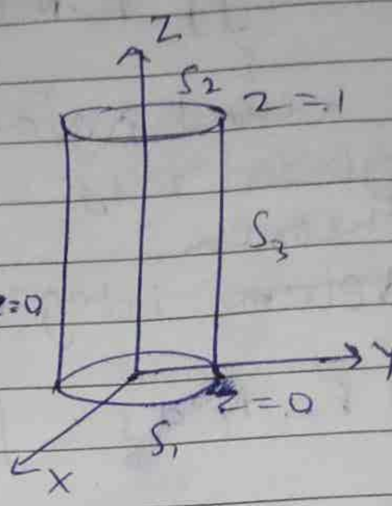
By surface integral

Clearly surface ~~integral~~ consists of three parts:



(i) Base S_1; i.e. $\Rightarrow x^2+y^2=4; z=0$

(ii) Upper face S_2; i.e. $\Rightarrow x^2+y^2=4; z=1$

(iii) curved surface S_3; cylinder $x^2+y^2=4$

$$\iint_S \vec{F}\cdot\hat{N}\,ds = \iint_{S_1} \vec{F}\cdot\hat{N}\,ds + \iint_{S_2} \vec{F}\cdot\hat{N}\,ds + \iint_{S_3} \vec{F}\cdot\hat{N}\,ds \quad -(1)$$

for S_1; $x^2+y^2=4$; $z=0$

$$\vec{F} = x\hat{i} - y\hat{j} - \hat{k}$$

$$\hat{N} = -\hat{k}$$

$$\iint_{S_1} \vec{F}\cdot\hat{N}\,ds = \iint (x\hat{i}-y\hat{j}-\hat{k})\cdot(-\hat{k})\,\frac{dx\,dy}{|(-\hat{k}\cdot-\hat{k})|}$$

$$\iint_{S_1} \vec{F}\cdot\hat{N}\,ds = \iint dx\,dy$$

$\Rightarrow \pi(2)^2$

$\Rightarrow \underline{4\pi}$

$\boxed{x^2+y^2=2^2}$ r

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For S_2; $x^2+y^2=4$; $z=1$; $\hat{N}=k$

$$\vec{F} = x\hat{i} - y\hat{j} + 0\hat{k}$$

$$\iint_{S_2} \vec{F}\cdot\hat{N}\,ds = \iint_{S_2} (x\hat{i} - y\hat{j} + 0\hat{k})\cdot\hat{k}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{k}|}$$

$$= 0$$

For S_3; $\phi = x^2+y^2$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2)$$

$$\text{grad}\,\phi = \nabla\phi = 2x\hat{i} + 2y\hat{j}$$

$$\hat{N} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{2}$$

$$\vec{F}\cdot\hat{N} = (x\hat{i} - y\hat{j} + (z^2-1)\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{2}\right)$$

$$= \left(\frac{x^2-y^2}{2}\right)$$

$$\iint_{S_3} \vec{F}\cdot\hat{N}\,ds = \iint\left(\frac{x^2-y^2}{2}\right)ds$$

$$\Rightarrow \frac{1}{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{1}(4\cos^2\theta - 4\sin^2\theta)\,d\theta\,dz$$

$$\Rightarrow 2\int_{\theta=0}^{2\pi}\int_{z=0}^{1}\cos 2\theta\,d\theta\,dz$$

$\boxed{\begin{array}{l} x^2+y^2=4 \\ \text{Put } x = 2\cos\theta \\ y = 2\sin\theta \end{array}}$

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$$\Rightarrow 2\int_{\theta=0}^{2\pi} [z]_0^1 \cos 2\theta \, d\theta.$$

$$\Rightarrow 2\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$\Rightarrow \frac{0}{2}$$

Putting surface integral of all parts in eqn (1).

$$= 4\pi + 0 + 0$$
$$= 4\pi$$

So, as answer by surface integral & by Gauss divergence Theorem is same.

i.e. 4π

So, Hence, verified.

Topic

AOC: Curvature & Torsion

Curvature

Rate of change of direction of tangent wrt arc length.

Denoted by K (known as kappa)

$$K = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}$$

where $\dot{r} = \frac{dr}{dt}$ or $\frac{dr}{d\theta}$

$\ddot{r} = d^2r/dt^2$ or $\frac{d^2r}{d\theta^2}$

$\dddot{r} = \frac{d^3r}{dt^3}$ or $\frac{d^3r}{d\theta^3}$

Extra

Torsion

Rate of change of direction of unit vector along

y-normal wrt arc length

Denoted by τ (Know as Tau).

$$\tau = \frac{[\dot{r}\ \ddot{r}\ \dddot{r}]}{|\dot{r} \times \ddot{r}|^2}$$

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Q Find the curvature for the curve

$x = a\cos t$; $y = a\sin t$; $z = a t \cot\alpha$

Sol: $\dot{r} = (-a\sin t,\ a\cos t,\ a\cot\alpha)$

$\ddot{r} = (-a\cos t,\ -a\sin t,\ 0)$

$k = \text{curvature} = \dfrac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}$

$$\dot{r} \times \ddot{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\cot\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$\Rightarrow a^2\cot\alpha\sin t\,\hat{i} - a^2\cos t\cot\alpha\,\hat{j} + a^2\hat{k}$

$|\dot{r} \times \ddot{r}| = \sqrt{a^4\cot^2\alpha\sin^2 t + a^4\cos^2 t\cot^2\alpha + a^4}$

$\Rightarrow a^2\sqrt{\cot^2\alpha(1) + 1}$

$\Rightarrow a^2\sqrt{\operatorname{cosec}^2\alpha} \Rightarrow a^2\operatorname{cosec}\alpha$

$|\dot{r}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + a^2\cot^2\alpha}$

$|\dot{r}| = a\sqrt{(1 + \cot^2\alpha)} = a\operatorname{cosec}\alpha$

$k = \dfrac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} = \dfrac{a^2\operatorname{cosec}\alpha}{a^3\operatorname{cosec}^3\alpha}$

$\Rightarrow k = \dfrac{\sin^2\alpha}{a}$ Ans