

# 2

## Chapter

# CONTINUOUS PROBABILITY DISTRIBUTION

### INSIDE THIS CHAPTER

- 2.1. Continuous Random Variable
- 2.2. probability density function
- 2.3. Cumulative Distribution (Distribution function)
- 2.4. Expectation of random variable
- 2.5. Variance and standard deviation of continuous Random variable
- 2.6. normal distribution
- 2.7. Exponential Distribution
- 2.8. Constants of Exponential Distribution
- 2.9. Gamma distribution

### 2.1. CONTINUOUS RANDOM VARIABLE

A random variable  $X$  which can take every value in the domain or when its range  $R$  is an interval then  $X$  is continuous random variable.

**Example :**

- 1. Age
- 2. Height
- 3. Weight
- 4. Temperature

### 2.2. PROBABILITY DENSITY FUNCTION

The probability density function of random variable  $X$  is defined as

$$f_x(x) = P(x \leq X \leq x + \delta x)/\delta x$$

for small interval  $(x, x + \delta x)$  of length  $\delta x$  around the point  $x$

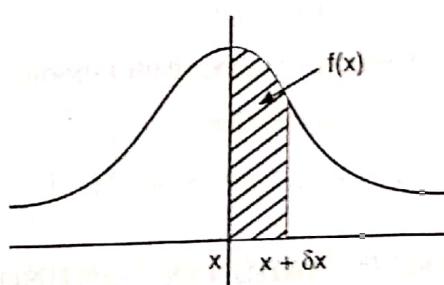


Fig. 2.1

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

which represent the area between the curve  $y = f(x)$ ,  $x$  axis and the ordinate at  $x = a$  and  $x = b$  since total probability is unity.

i.e.,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

The probability density function (p.d.f) of a random variable  $X$  usually denoted by  $f_X(x)$  has following properties.

$$f(x) \geq 0, -\infty < x < \infty$$

$$2. \int_{-\infty}^{\infty} f(x) dx = 1$$

### 2.3. CUMULATIVE DISTRIBUTION (DISTRIBUTION FUNCTION)

If  $X$  is a random variable, then  $P(X \leq x)$  is called the cumulative distribution (c.d.f) or the distribution function and it is denoted by  $F(x)$ .

$$F(x) = P(X \leq x)$$

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(x) dx$$

### 2.4. EXPECTATION OF RANDOM VARIABLE

If  $X$  is a continuous random variable, then the expectation of the random variable  $X$  is defined as

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

The expected value of  $X^2$  is defined as

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$E(X)$  is also called mean of  $X$

#### Properties

1. If  $X$  is random variable and  $a$  is constant then

$$(i) E(a) = a$$

$$(ii) E(aX) = aE(X)$$

$$(iii) E(X - \bar{X}) = 0$$

2. If  $X$  and  $Y$  are two random variables then

$$E(X + Y) = E(X) + E(Y)$$

3.  $E(XY) = E(X)E(Y)$  if  $X$  and  $Y$  are two independent random variable.

4. If  $y = a + bx$ , where  $a$  and  $b$  are constants then

$$E(Y) = E(aX + b) = aE(X) + b$$

### 2.5. VARIANCE AND STANDARD DEVIATION OF CONTINUOUS RANDOM VARIABLE

Variance of  $X$  is defined as

$$\text{Var}(X) = V(X)$$

$$\text{Standard deviation of random variable } x \text{ is denoted by S.D}(x) \text{ and is defined as}$$

$$\text{S.D.}(x) = \sigma = \sqrt{V(X)} = \sqrt{E(X^2) - [E(X)]^2}$$

Example 2.1. A continuous random variables  $X$  has a probability density function defined by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2 & \text{if } -3 \leq x < -1 \\ \frac{1}{16}(6-2x^2) & \text{if } -1 \leq x < 1 \\ \frac{1}{16}(3-x)^2 & \text{if } 1 < x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

Verify that  $f(x)$  is a density function and also find the mean of the random variable  $X$ .

Solution. Since  $f(x)$  is density function,  $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-3}^{-1} f(x) dx + \int_{-1}^{1} f(x) dx + \int_{1}^{3} f(x) dx + \int_{3}^{\infty} f(x) dx \\ &= \int_{-3}^{-1} 0 dx + \int_{-1}^{1} \frac{1}{16}(3+x)^2 dx + \int_{1}^{3} (6-2x^2) dx + \int_{3}^{\infty} \frac{1}{16}(3-x)^2 dx + \int_{3}^{\infty} 0 dx \\ &= \frac{1}{16} \int_{-3}^{-1} (3+x)^2 dx + \int_{1}^{3} (6-2x^2) dx + \frac{1}{16} \int_{1}^{3} (3-x)^2 dx \\ &= \frac{1}{16} \left[ \left[ \frac{(3+x)^3}{3} \right]_{-3}^{-1} + \left[ 6x - \frac{2x^3}{3} \right]_{1}^{3} - \left[ \frac{(3-x)^3}{3} \right]_1^3 \right] \\ &= \frac{1}{16} \left[ \left[ \frac{8}{3} - 0 \right] + \left[ \left( 6 - \frac{2}{3} \right) - \left( -6 + \frac{2}{3} \right) - \left( 0 - \frac{8}{3} \right) \right] \right] = 1 \end{aligned}$$

$$\int_{-\infty}^{\infty} f(x) dx = 1,$$

Hence  $f(x)$  is a density function.

Mean of the random variable  $X$  is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$\begin{aligned} &\approx \frac{1}{16} \int_{-3}^{-1} x(3+x)^2 dx + \frac{1}{16} \int_{-1}^{1} x(6-2x^2) dx + \frac{1}{16} \int_{1}^{3} x(3-x)^2 dx \end{aligned}$$

$$= \frac{1}{16} \int_{-3}^{-1} x(9 + x^2 + 6x) dx + 0 + \frac{1}{16} \int_1^3 x(9 + x^2 - 6x) dx$$

since the integrand of the second integral is odd function.

$$= \frac{1}{16} \int_{-3}^{-1} (9x + x^3 + 6x^2) dx + \frac{1}{16} \int_1^3 (9x + x^3 - 6x^2) dx$$

$$\begin{aligned} &= \frac{1}{16} \left[ \left( \frac{9x^2}{2} + \frac{x^4}{4} + \frac{6x^3}{3} \right) \Big|_{-3}^{-1} + \left[ \frac{9x^2}{2} + \frac{x^4}{4} - \frac{6x^3}{3} \right] \Big|_1^3 \right] \\ &= \frac{1}{16} \left[ \left( \frac{9}{2} - \frac{81}{2} \right) + \left( \frac{1}{4} - \frac{81}{4} \right) + \left( \frac{-6}{3} - \frac{-162}{3} \right) \right] + \left[ \left( \frac{81}{2} - \frac{9}{2} \right) + \left( \frac{81}{4} - \frac{1}{4} \right) - \left( \frac{162}{3} - \frac{1}{3} \right) \right] \\ &= 0 \end{aligned}$$

Therefore, the mean of the random variable  $X$  is zero.

**Example 2.2.** A continuous random variable  $X$  has

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

represents the density, find the mean and standard deviation of  $X$ .

Solution: If  $f(x)$  is density function, then it satisfies

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$\Rightarrow$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^{-1} f(x) dx + \int_{-1}^1 f(x) dx + \int_1^{\infty} f(x) dx \\ &= \int_{-\infty}^{-1} 0 dx + \int_{-1}^1 \frac{1}{2}(x+1) dx + \int_1^{\infty} 0 dx \\ &= \frac{1}{2} \left[ \frac{x^2}{2} + x \right] \Big|_{-1}^1 \end{aligned}$$

$$= \frac{1}{2} \left[ \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{2} + \frac{1}{2} \right) \right]$$

$$= \frac{1}{2} \left[ \frac{1}{3} + \frac{1}{2} \right]$$

$$= \frac{1}{2} \left[ \frac{5}{6} \right]$$

$$= \frac{5}{12}$$

$$= 1$$

### Continuous Probability Distribution

Hence,

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \text{ is a density function.}$$

Mean of the random variable  $X$  is

$$E(X) = \int_{-\infty}^{\infty} xf(x) dx$$

$$= \int_{-\infty}^{-1} 0 dx + \int_{-1}^1 x \cdot \frac{1}{2}(x+1) dx + \int_1^{\infty} 0 dx$$

$$\begin{aligned} &= \frac{1}{2} \int_{-1}^1 (x^2 + x) dx \\ &= \frac{1}{2} \left[ \frac{x^3}{3} + \frac{x^2}{2} \right] \Big|_{-1}^1 \\ &= \frac{1}{2} \left[ \frac{1}{3} + \frac{1}{2} \right] \\ &= \frac{5}{12} \end{aligned}$$

Therefore the mean of the random variable  $X$  is  $\frac{5}{12}$

i. The variance of the random variable  $X$  is

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

Now

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \int_{-1}^1 x^2 \cdot \frac{1}{2}(x+1) dx \\ &= \frac{1}{2} \left[ \frac{x^3}{3} + \frac{x^2}{2} \right] \Big|_{-1}^1 \\ &= \frac{1}{2} \left[ \left( \frac{1}{2} + \frac{1}{3} \right) - \left( \frac{-1}{2} + \frac{1}{3} \right) \right] \\ &= \frac{1}{2} \left[ \left( \frac{3}{2} \right) - \left( -\frac{1}{2} \right) \right] \\ &= \frac{1}{2} \left[ \frac{4}{2} \right] \\ &= \frac{1}{2} \left[ 2 \right] \\ &= 1 \end{aligned}$$

$$= \frac{1}{2} \left[ \frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3} \right]$$

$$\therefore k = \frac{1}{3}$$

Now,  $\text{Var}(X) = E(X^2) - (E(X))^2$

$$\begin{aligned} &= \frac{1}{3} - \left( \frac{1}{3} \right)^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9} \\ &= \frac{1}{81} \left[ \left( \frac{3}{2} \right)^2 - \left( \frac{1}{2} \right)^2 \right] = \frac{1}{81} \left[ \frac{80}{4} \right] \end{aligned}$$

∴ Standard deviation of  $X = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$

**Example 2.3.** If the probability density function

f(x) = \begin{cases} kx^3 & \text{if } 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}

Find the value 'k' and find the probability between  $x = \frac{1}{2}$  and  $x = \frac{3}{2}$ .

**Solution.** From the given data,  $f(x) = \begin{cases} kx^3 & \text{if } 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$

If  $f(x)$  is a density function, then it satisfies  $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_0^3 f(x)dx + \int_0^3 f(x)dx = 1$$

$$\Rightarrow k \cdot \int_0^3 x^3 dx = 1 \Rightarrow \left[ \frac{x^4}{4} \right]_0^3 = 1 \Rightarrow k \left[ \left( \frac{3^4}{4} - 0 \right) \right] = 1$$

$$\Rightarrow k \cdot \frac{81}{4} = 1 \Rightarrow k = \frac{4}{81}$$

$$\therefore k = \frac{4}{81}$$

Now,

$$f(x) = \begin{cases} \frac{4}{81}x^3 & \text{if } 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$(i) P\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right)$$

$$P\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(x)dx$$

$$= \frac{4}{81} \int_{\frac{1}{2}}^{\frac{3}{2}} x^3 dx$$

$$\begin{aligned} &= \frac{1}{81} \left[ \frac{3^4}{4} \right] - \left( \frac{1}{2} \right)^3 = \frac{1}{81} \left[ \frac{80}{4} \right] \\ &= \frac{80}{324} = 0.6617 \end{aligned}$$

**Example 2.4.** Is the function defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{3+2x}{18} & \text{if } 2 \leq x \leq 4 \\ 0 & \text{if } x > 4 \end{cases}$$

a probability density function? Find the probability that a variate having  $f(x)$  as density function will fall in the interval  $(2 \leq X \leq 3)$ .

**Solution.** Given  $f(x)$  is

$$f(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{3+2x}{18} & \text{if } 2 \leq x \leq 4 \\ 0 & \text{if } x > 4 \end{cases}$$

If  $f(x)$  is a density function, then it satisfies  $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^2 0 dx + \int_2^4 \frac{3+2x}{18} dx + \int_4^{\infty} 0 dx \\ &= \int_2^4 \frac{3+2x}{18} dx = \frac{1}{18} \left[ 3x + 2x^2 \right]_2^4 \end{aligned}$$

$$\begin{aligned} &= 0 + \frac{1}{18} [12 + 2 \cdot 16] - \left( \frac{1}{18} [6 + 8] \right) + 0 = \frac{1}{18} (4 \cdot 21) + \frac{1}{18} (4^2 - 2^2) \\ &= \frac{1}{3} + \frac{2}{3} = 1 \end{aligned}$$

$\int_{-\infty}^{\infty} f(x)dx = 1$

$$f(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{3+2x}{18} & \text{if } 2 \leq x \leq 4 \\ 0 & \text{if } x > 4 \end{cases}$$

Hence,  $f(x)$  is a density function

$$\Rightarrow k\left[\frac{x^2}{2}\right]_0^2 + 2k[x]_2^4 + k\left[6(x)_4^6 - \left(\frac{x^2}{2}\right)_4^6\right] = 1$$

$$\Rightarrow k\left[\frac{4}{2}\right] + 2k[2] + k\left[6(2) - \left(\frac{36}{2} - \frac{16}{2}\right)\right] = 1$$

$$\Rightarrow 2k + 4k + k(12 - 10) = 1$$

$$\Rightarrow 8k = 1 \Rightarrow k = \frac{1}{8}$$

Now  $P(2 \leq X \leq 3) = \int_2^3 f(x)dx$

$$= \int_2^3 \frac{3+2x}{18} dx$$

$$= \frac{3}{18}[x]_2^3 + \frac{2}{18}\left[\frac{x^2}{2}\right]_2^3$$

$$= \frac{1}{6}[3-2] + \frac{1}{18}[3^2 - 2^2]$$

$$= \frac{1}{6} + \frac{5}{18} = \frac{8}{18} = \frac{4}{9}$$

$$= 0.44$$

**Example 2.5.** A continuous random variable  $X$  has the probability density function

$$f(x) = \begin{cases} \frac{kx}{2k} & \text{for } 0 \leq x < 2 \\ \frac{-kx+6k}{2} & \text{for } 2 \leq x < 4 \\ 0 & \text{for } 4 \leq x < 6 \\ 0 & \text{elsewhere} \end{cases}$$

Find  $k$  and mean of the density function.

**Solution.** If  $f(x)$  is a density function, then it satisfies  $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^2 f(x)dx + \int_2^4 f(x)dx + \int_4^6 f(x)dx + \int_6^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_0^6 f(x)dx + \int_0^2 f(x)dx + \int_2^4 f(x)dx + \int_4^6 f(x)dx + \int_6^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_0^6 0dx + \int_0^2 \frac{kx}{2k} dx + \int_2^4 \frac{-kx+6k}{2} dx + \int_4^6 0dx + \int_6^{\infty} 0dx = 1$$

$$\Rightarrow \frac{1}{2} \int_0^2 x dx + \frac{1}{2} \int_2^4 (-kx+6k) dx = 1$$

$$\Rightarrow \frac{1}{2} \left[ \frac{x^2}{2} \right]_0^2 + \frac{1}{2} \left[ -\frac{kx^2}{2} + 6kx \right]_2^4 = 1$$

$$\Rightarrow \frac{1}{2} \left[ \frac{4}{2} \right] + \frac{1}{2} \left[ -\frac{16}{2} + 24k \right] = 1$$

$$\Rightarrow \frac{1}{2} \left[ 2 + 16k \right] = 1$$

$$\Rightarrow 1 + 16k = 2$$

$$\Rightarrow 16k = 1$$

$$\Rightarrow k = \frac{1}{16}$$

Therefore the mean of random variable  $X$  is 3.

**Example 2.6.** If  $X$  is a continuous random variable and  $K$  is a constant then prove that:

- $V(X+k) = V(X)$ ,
- $V(Xk) = k^2 V(X)$ .

**Solution.** Since

$$V(X+k) = E[(X+k)^2] - [E(X+k)]^2$$

$$= E[X^2 + 2Xk + k^2] - [E(X) + k]^2$$

$$= E[X^2] + 2E[Xk] + k^2 - [E(X)]^2 - 2E[X]k - k^2$$

$$= E[X^2] + 2E[X]k + k^2 - E[X]^2 - 2E[X]k - k^2$$

$$= E[X^2] - E[X]^2$$

$$\begin{aligned}
 &= E[X^2 + 2kX + k^2] - [E(X) + k]^2 \\
 &= E[X^2] + 2kE(X) + k^2 - [E(X)]^2 + 2k[E(X)] + k^2 \\
 &= E[X^2] + 2kE(X) + k^2 - [E(X)]^2 - 2k[E(X)] - k^2
 \end{aligned}$$

$$\begin{aligned}
 E[X^2] - [E(X)]^2 &= Var[X] \\
 Var[X+k] &= Var[X] \\
 \therefore Var[Xk] &= E[(Xk)^2] - [E(Xk)]^2 \\
 &= E[k^2 X^2] - [kE(X)]^2 \\
 &= k^2 E[X^2] - k^2 [E(X)]^2 \\
 &= k^2 [E(X^2) - [E(X)]^2] \\
 &= k^2 Var[X]
 \end{aligned}$$

$$\begin{aligned}
 Var[Xk] &= k^2 Var[X]
 \end{aligned}$$

**Example 2.7.** If the probability density function of a random variable is given by

$$f(x) = \begin{cases} k(1-x^2) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value of  $k$  and the probabilities that a random variable will take on a value between 0.1 and 0.2, (ii) greater than 0.5, (iii) Mean and (iv) Variance.

**Solution.** Since  $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\begin{aligned}
 &\Rightarrow \int_{-\infty}^0 f(x)dx + \int_0^1 f(x)dx + \int_1^{\infty} f(x)dx = 1 \\
 &\Rightarrow 0 + \int_0^1 k(1-x^2)dx + 0 = 1 \\
 &\Rightarrow \left[ k\left[1 - \frac{x^3}{3}\right] \right]_0^1 = 1 \Rightarrow k\left[1 - \frac{1^3}{3}\right] = 1 \\
 &\Rightarrow \frac{k}{3} = 1 \Rightarrow k = \frac{3}{2}
 \end{aligned}$$

Now the probability density function of the random variable  $X$  is

$$f(x) = \begin{cases} \frac{3}{2}(1-x^2) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$(i) P(0.1 < X < 0.2) = \int_{0.1}^{0.2} f(x)dx = \int_{0.1}^{0.2} \frac{3}{2}(1-x^2)dx = \frac{3}{2} \int_{0.1}^{0.2} (1-x^2)dx$$

$$= \frac{3}{2}(0.0977) = 0.1466$$

$$\begin{aligned}
 (ii) P(X > 0.5) &= \int_{0.5}^{\infty} f(x)dx = \int_{0.5}^{\infty} \frac{3}{2}(1-x^2)dx \\
 &= \frac{3}{2} \left[ x - \frac{x^3}{3} \right]_{0.5}^{\infty} = \frac{3}{2} \left[ \left( 1 - \frac{0.5^3}{3} \right) - \left( 0.5 - \frac{(0.5)^3}{3} \right) \right] \\
 &= \frac{3}{2} (0.208) = 0.3125
 \end{aligned}$$

(iii) Mean of the random variable  $X$  is

$$\begin{aligned}
 E(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^0 x \cdot 0 dx + \int_0^1 x \cdot \frac{3}{2}(1-x^2)dx + \int_1^{\infty} x \cdot 0 dx \\
 &= \frac{3}{2} \int_0^1 x(1-x^2)dx = \frac{3}{2} \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{2} \left[ \left( \frac{1}{2} - \frac{1}{4} \right) - 0 \right] \\
 &= \frac{3}{2} \times \frac{1}{4} = \frac{3}{8} = 0.375
 \end{aligned}$$

(iv) Variance

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 \cdot \frac{3}{2}(1-x^2)dx$$

$$\begin{aligned}
 &= \frac{3}{2} \int_0^1 (x^2 - x^4)dx \\
 &= \frac{3}{2} \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\
 &= \frac{3}{2} \left[ \left( \frac{1}{3} - \frac{1}{5} \right) - 0 \right] \\
 &= \frac{3}{2} \left( \frac{2}{15} \right) = 0.2
 \end{aligned}$$

∴ The variance of the random variable  $X$  is

$$\begin{aligned}
 Var(X) &= E(X^2) - [E(X)]^2 \\
 &= 0.2 - (0.375)^2 = 0.0594
 \end{aligned}$$

**Example 2.8.** A continuous random variable has the pdf,

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probabilities that it will take on a value: (i) between 1 and 3 and (ii) greater than 0.5.



$$= \frac{1}{2} \left[ \begin{matrix} e^{-x} \\ -1 \end{matrix} \right] = \frac{1}{2} (e^{-x} - e^0)$$

$$= \frac{1}{2} (1 - e^{-1}) = 0.491$$

**Example 2.10.** A continuous random variable has the pdf  $f(x) = \begin{cases} k \cdot x^2 e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Determine the constant  $k$ , find the mean and variance.

**Solution.** Since  $f(x)$  is a density function,  $\int f(x)dx = 1$

$$\int f(x)dx = \int_0^{\infty} f(x)dx + \int_0^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_0^{\infty} kx^2 e^{-x} dx = 1 \Rightarrow k \int_0^{\infty} x^2 e^{-x} dx = 1$$

$$\Rightarrow k \left[ x^2 \left( \frac{e^{-x}}{-1} \right) - 2x \left( \frac{e^{-x}}{-1} \right) + 2 \left( \frac{e^{-x}}{-1} \right)_0 \right] = 1$$

$$\Rightarrow k[(x^2 \cdot 0 - 2x \cdot 0 + 2 \cdot 0) - (0 - 0 - 2 \cdot 1)] = 1$$

$$\Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$$

$$f(x) = \frac{1}{2} x^2 e^{-x} \text{ for } x \geq 0$$

Mean of the random variable  $X$  is

$$E(X) = \int x f(x)dx = \int_0^{\infty} x f(x)dx + \int_0^{\infty} x f(x)dx$$

$$= 0 + \frac{1}{2} \int_0^{\infty} x^3 e^{-x} dx$$

$$= \frac{1}{2} \left[ x^3 \left( \frac{e^{-x}}{-1} \right) - 3x^2 \left( \frac{e^{-x}}{-1} \right) + 6x \left( \frac{e^{-x}}{-1} \right) - 6 \left( \frac{e^{-x}}{-1} \right)_0 \right]$$

$$= \frac{1}{2} [(x^3 \cdot 0 - 3x^2 \cdot 0 + 6x \cdot 0 - 6 \cdot 0) - (0 - 0 - 6 \cdot 1)] = 3$$

Therefore the mean of the random variable  $X$  is 3.

The variance of the random variable  $X$  is

$$E(X^2) = \int_0^{\infty} x^2 f(x)dx$$

$$= \int_0^{\infty} x^2 f(x)dx + \int_0^{\infty} x^2 f(x)dx = 0 + \frac{1}{2} \int_0^{\infty} x^4 e^{-x} dx$$

$$= \frac{1}{2} \left[ x^4 \left( \frac{e^{-x}}{-1} \right) - 4x^3 \left( \frac{e^{-x}}{-1} \right) + 12x^2 \left( \frac{e^{-x}}{-1} \right) - 24x \left( \frac{e^{-x}}{-1} \right) + 24 \left( \frac{e^{-x}}{-1} \right)_0 \right]$$

$$= \frac{1}{2} [(x^4 \cdot 0 - 4x^3 \cdot 0 + 12x^2 \cdot 0 - 24x \cdot 0 + 24 \cdot 0) - (0 - 0 - 0 - 24 \cdot 1)]$$

$$= 12$$

$$V(X) = E(X^2) - \{E(X)\}^2 = 12 - (3)^2 = 3$$

∴ The variance of the random variable  $X$  is 3.

**Example 2.11.** Let  $F(x)$  be the distribution function of a random variable  $X$  given by

$$F(x) = \begin{cases} cx^3 & \text{when } 0 \leq x \leq 3 \\ 1 & \text{when } x > 3 \\ 0 & \text{elsewhere} \end{cases}$$

If  $P(X = 3) = 0$ , Determine: (i) c, (ii) mean and (iii)  $P(X > 1)$ .

**Solution.** Since

$$f(x) = \frac{d}{dx} F(x), \text{ then}$$

$$f(x) = \begin{cases} 3cx^2 & \text{when } 0 \leq x \leq 3 \\ 0 & \text{when } x > 3 \\ 0 & \text{when } x < 0 \end{cases}$$

$$f(x) = \begin{cases} 3cx^2 & \text{when } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Since  $f(x)$  is a density function,  $\int f(x)dx = 1$  and the range of  $x$  is 0 to 3

$$\Rightarrow \int_0^3 f(x)dx + \int_0^3 f(x)dx + \int_0^3 f(x)dx = 1$$

$$\Rightarrow \int_0^3 0 \cdot dx + \int_0^3 3cx^2 dx + \int_0^3 0 \cdot dx = 1$$

$$3c \int_0^3 x^2 dx = 1$$

$$\frac{3c}{3} \left[ \frac{x^3}{3} \right]_0^3 = 1$$

$$\therefore c(27 - 0) = 1$$

$$\therefore 27c = 1 \Rightarrow c = \frac{1}{27}$$

$$f(x) = \begin{cases} 3 \cdot \frac{1}{27} x^2 & \text{when } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Let,

$$f(x) = \begin{cases} \frac{1}{9} x^2 & \text{when } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Now the mean,

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{9} \int_0^3 x \cdot x^2 dx = \frac{1}{9} \int_0^3 x^3 dx$$

$$= \frac{1}{9} \left[ \frac{x^4}{4} \right]_0^3$$

$$= \frac{1}{9} \left[ \frac{1}{4} (81 - 0) \right] = \frac{9}{4}$$

$$E(X) = \frac{9}{4} = 2.25$$

$$P(X > 1) = \int_1^{\infty} f(x)dx = \int_1^3 f(x)dx + \int_3^{\infty} f(x)dx = \int_1^3 f(x)dx + \int_3^{\infty} 0 \cdot dx$$

$$= \frac{1}{9} \int_1^3 x^2 dx = \frac{1}{9} \left[ \frac{x^3}{3} \right]_1^3 = \frac{1}{9} \left[ \frac{1}{3} (27 - 1) \right]$$

$$= 0.9629$$

$$P(X > 1) = 0.9629$$

### EXERCISE 2.1

Determine: (i)  $f(x)$ , (ii) mean at  $D = 1$ .

$\{1 \quad \text{if } x > 3$

$$\text{Ans. } f(x) = \begin{cases} 4k(x - D)^3 & \text{if } x > 3 \\ 0 & \text{otherwise} \end{cases}$$

## 2.5. NORMAL DISTRIBUTION

Normal distribution is the most popular and commonly used distribution. It was discovered by Moivre in 1733 after 20 years when Bernoulli gave Binomial distribution. This distribution is a limit case of Binomial distribution when neither  $p \rightarrow q$  is too small and  $n$ , the number of trials becomes infinitely large i.e.,  $n \rightarrow \infty$ . In fact any quantity whose variation depends on random cause will be distributed according to the normal distribution whereas in Binomial and Poisson distribution  $X$  assume values like  $0, 1, 2, \dots$  and thus these distribution are discrete distributions. Cases when the variables can assume any value between 0 and 1 are classified under the continuous variate, for example, in case of height, weight etc.

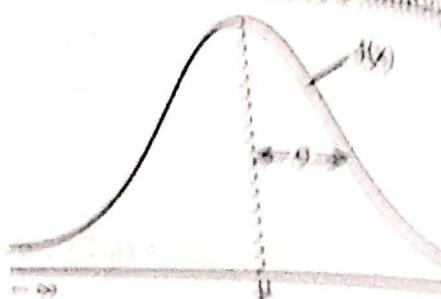


Fig. 2.2.

The continuous random variable  $X$  is said to have a normal distribution, if its probability density function is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty, \sigma > 0$$

where  $\mu$  and  $\sigma$  are parameters of the normal distribution.

### 2.5.1. Mean of the Normal Distribution

The normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put  $\frac{x-\mu}{\sigma} = z$  then  $dx = \sigma dz$  and  $x = \mu + \sigma z$  limits are unaltered

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} dz$$

$$\begin{aligned}
 &= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\mu} e^{-\frac{1}{2}t^2} dt + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\mu} te^{-\frac{1}{2}t^2} dt \\
 &= \frac{\mu}{\sqrt{2\pi}} 2 \int_0^{\mu} e^{-\frac{1}{2}t^2} dt + 0
 \end{aligned}$$

(As  $te^{-\frac{1}{2}t^2}$  is an odd and  $e^{-\frac{1}{2}t^2}$  is even function)

$$\begin{aligned}
 &= \frac{2\mu}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{2}} \\
 &= \mu \\
 E(X) &= \mu
 \end{aligned}$$

$\therefore$  The mean of the normal distribution is ' $\mu$ '.

### 2.6.2. Variance of the Normal Distribution

Since  $\text{Var}(X) = E(X^2) - [E(X)]^2 = E(X - \mu)^2$

$$\text{Now } E(X - \mu)^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^{\infty} (x - \mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Put  $\frac{x-\mu}{\sigma} = z$  then  $dx = \sigma dz$  limits are unaltered

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{1}{2}z^2} \sigma dz = \frac{\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-\frac{1}{2}z^2} dz \quad (\because z^2 e^{-\frac{1}{2}z^2} \text{ is an even function})$$

Put  $\frac{z^2}{2} = t$  then  $zdz = dt$  and  $z = \sqrt{2t}$  also limits are unaltered

$$\begin{aligned}
 &= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} 2te^{-t} \frac{dt}{\sqrt{2t}} \\
 &= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{2t} e^{-t} dt = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}} e^{-t} dt = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{\frac{3}{2}-1} dt \\
 &= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \quad \because \int_0^{\infty} e^{-t} t^{\frac{3}{2}-1} dt = \sqrt{\left(\frac{3}{2}\right)} \\
 &= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \sqrt{\pi} \quad \because \sqrt{\frac{1}{2}} = \sqrt{\pi} \\
 &= \sigma^2 \quad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}
 \end{aligned}$$

Variance of normal distribution is  $\sigma^2$ .

2.6.3. Median of the Normal Distribution then

Suppose 'M' is median of normal distribution then

$$\int_{-\infty}^M f(x) dx = \frac{1}{2}$$

$$\int_{-\infty}^M f(x) dx = \frac{1}{2} \Rightarrow \int_{-\infty}^{\mu} f(x) dx + \int_{\mu}^M f(x) dx$$

Now take,

$$\int_{-\infty}^{\mu} f(x) dx = \frac{1}{2} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} dx + \int_{\mu}^M f(x) dx = \frac{1}{2} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} dx$$

$$= -\frac{1}{\sigma^2 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}$$

$$\text{Consider, } \int_{-\infty}^{\mu} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} dx$$

$$= 0$$

$$\text{Put } \frac{x-\mu}{\sigma} = z \text{ then } dz = \sigma dx$$

Also limits are when  $x = -\infty$  then  $z = -\infty$   
Also when  $x = \mu$  then  $z = 0$

$$\int_{-\infty}^{\mu} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} dx = \int_{-\infty}^0 \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} z^2} \sigma dz$$

$$= \frac{1}{2\pi} \int_0^{\infty} e^{-\frac{1}{2} z^2} dz$$

by symmetry property

$$\therefore \int_{-\infty}^{\mu} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} dx = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{2}} = \frac{1}{2}$$

From (ii) and (iii)

$$\frac{1}{2} + \int_{\mu}^M f(x) dx = \frac{1}{2}$$

$\therefore \int_{\mu}^M f(x) dx = 0$ . This implies that  $\mu = M$

Hence, the median of the normal distribution is ' $\mu$ '.

2.6.4. Mode of the Normal Distribution

Mode is the value of  $x$  at which  $f(x)$  has maximum value.

Now the probability density function of ' $x$ ' is

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}$$

$-\infty \leq X \leq \infty$

$$f'(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \left\{ -\frac{1}{2} \cdot 2 \left( \frac{x-\mu}{\sigma} \right) \right\}$$

$$= -\frac{1}{\sigma^2 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} (x-\mu)$$

(when  $x = \mu$ )

and

$$f''(x) = (x-\mu) \left\{ -\frac{1}{\sigma^2 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \right\} \left\{ -\frac{1}{2} \cdot 2 \left( \frac{x-\mu}{\sigma} \right) \right\} + \left\{ \frac{-1}{\sigma^2 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \right\}$$

which is -ve at  $x = \mu$ .

$\therefore f(x)$  has maximum value, when the value of  $x = \mu$ .

$\therefore$  The mode of the normal distribution is  $\mu$ .

Note: For the normal distribution, the mean median, and mode are equal i.e.,

Mean = Median = Mode

### 2.6.5. Properties of the Normal Distribution

1. The normal probability curve with mean  $\mu$  and standard deviation  $\sigma$  is given by

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}, -\infty \leq X \leq \infty$$

2. The curve is bell-shaped and symmetrical about the line  $x = \mu$ .

3. Mean, median and mode of the normal distribution of coincide the normal distribution is unimodal.

4.  $f(x)$  decreases rapidly as  $x$  increases.

5.  $X$ -axis is an asymptote to the curve (the tangent to the curve at  $\infty$  is  $X$ -axis)

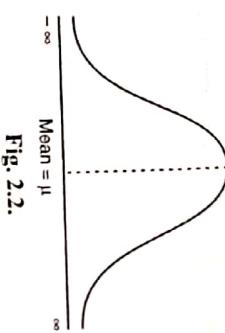


Fig. 2.2.

6. The maximum probability occurs at the point  $x = \mu$  and is  $\frac{1}{\sigma \sqrt{2\pi}}$ .

7. Mean deviation about mean =  $\frac{4}{5}\sigma$ .

8. Since  $f(x)$ , being the probability, can never be negative, so that no portion of the curve lies below the  $X$ -axis.

9. A linear function of independent normal variates is also normal variate.

10. The point of inflection of the curve are at  $x = \mu \pm \sigma$ .

(i) Area of the normal curve between  $(\mu - \sigma)$  and  $(\mu + \sigma)$  is 0.6826

i.e.,  $P(\mu - \sigma < X < \mu + \sigma) = 0.6826$

(ii) Area of the normal curve between  $\mu - 2\sigma$  and  $\mu + 2\sigma$  is 0.9544

$$\text{P}(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$$

i.e., area of the normal curve between  $\mu = 10$  and  $\mu + 2\sigma$  is 0.9544

(iii) Area of the normal curve between  $\mu = 10$  and  $\mu + 3\sigma$  is 0.9973

$$\text{P}(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

i.e.,

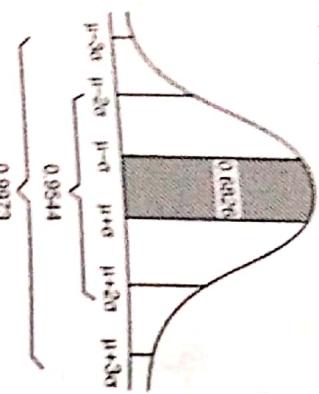


Fig. 2.3

### 2.6.6. Normal Distribution as a Limiting Form of Binomial Distribution

Normal distribution is another limiting form of the binomial distribution under the following conditions:

- (i) The number of trials  $n$ , is indefinitely large, i.e.,  $n \rightarrow \infty$
- (ii) Neither ' $p$ ' nor ' $q$ ' is very small.

For large  $n$ , the calculation of binomial probabilities is very difficult. In such a case we can use normal distribution and compute the required probability.

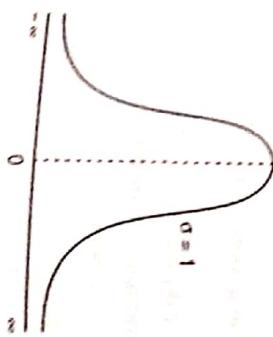


Fig. 2.4.

Suppose the number of successes ranges from  $x_1$  to  $x_2$ , then the probability of getting  $x_1$  to  $x_2$  successes is given by

$$\sum_{x=x_1}^{x_2} C_x p^x q^{n-x}$$

Now consider two cases.

**Case 1:**  $p = q = \frac{1}{2}$  and  $n$  is not large. The mean of the binomial distribution =  $np$  and standard

$$\text{deviation} = \sqrt{npq}.$$

### Continuous Probability Distribution

The corresponding mean and standard deviation of the normal distribution are  $\mu$  and  $\sigma$  respectively. Since we know that

Let  $z_1$  and  $z_2$  be the values of  $Z$ , corresponding to  $x_1$  and  $x_2$  of  $X$  respectively. Then

$$P(x_1 < X < x_2) = P(z_1 < Z < z_2) = \int_{z_1}^{z_2} \Phi(z) dz$$

And the value can be computed from the standard normal distribution.

**Case 2:**  $p \neq q$  for large  $n$ .

We can approximate the binomial to the normal curve and calculate the corresponding probabilities.

For any success  $x$  which lies in the interval  $\left(x - \frac{1}{2}, x + \frac{1}{2}\right)$  then  $z_1$  can be calculated corresponding to the lower limit of the  $x_1$  class and  $z_2$  to the limit of the  $x_2$  class.

$$\therefore z_1 = \frac{x_1 - \frac{1}{2}}{\sigma} = \frac{x_1 - \frac{1}{2} - np}{\sqrt{npq}}$$

and

$$z_2 = \frac{\left(x_2 + \frac{1}{2}\right) - \mu}{\sigma} = \frac{x_2 + \frac{1}{2} - np}{\sqrt{npq}}$$

$\therefore$  The required probability =  $\int_{z_1}^{z_2} \Phi(z) dz$ ,

and this value can be computed from standard normal distribution.

#### Constant of Normal Distribution

The mean of the normal distribution is  $\bar{x}$

The standard deviation of the normal distribution is  $\sigma$

$$\mu_2 = \sigma^2, \quad \mu_3 = 0 \quad \text{and} \quad \mu_4 = 3\sigma^4$$

$$\beta_1 \text{ or moment coefficient of skewness, } \beta_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = 0$$

$$\beta_2 \text{ or moment coefficient of kurtosis, } \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\sigma^4}{\sigma^4} = 3$$

### SOLVED EXAMPLES

- Example 2.12.** If  $\mu = 50$  and  $\sigma = 10$  find : (i)  $P(50 \leq X \leq 80)$ , (ii)  $P(60 \leq X \leq 70)$ , (iii)  $P(30 \leq X \leq 40)$ , (iv)  $P(40 \leq X \leq 60)$ . Use Table : Area under the normal curve.

$$\text{Solution. Standard normal variate } z = \frac{X - \mu}{\sigma} = \frac{X - 50}{10}$$

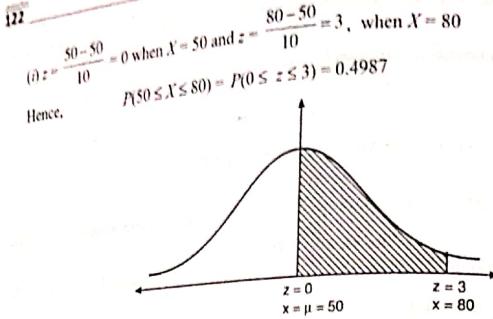


Fig. 2.5.

(ii)  $P(60 \leq X \leq 70) = P(1 \leq z \leq 2) = \text{Area from } z = 1 \text{ to } z = 2$   
 $= (\text{Area from } z = 0 \text{ to } z = 2) - (\text{Area from } z = 0 \text{ to } z = 1)$   
 $= 0.4772 - 0.3413 = 0.1359$

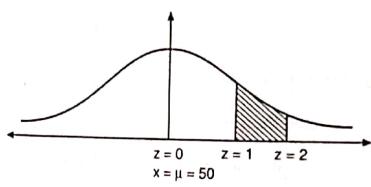


Fig. 2.6.

(iii)  $P(30 \leq X \leq 4) = P(-2 \leq z \leq -1)$   
 Due to symmetry, area between  $z = -1$  to  $z = -2$  will be the same as between  $z = -1$  to  $z = 1$ , which is the same as in (ii) i.e., 0.1359.

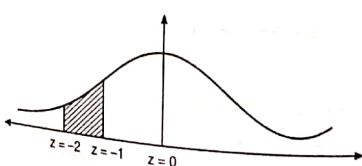


Fig. 2.7.

(iv)  $P(40 \leq X \leq 60) = P(-1 \leq z \leq 1)$   
 $= \text{area between } z = -1 \text{ to } z = 1$   
 $= \text{twice the area between } z = 0 \text{ to } z = 1$   
 $= 2 \times 0.3413 = 0.6826$

### CONTINUOUS PROBABILITY DISTRIBUTION

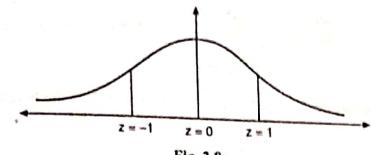


Fig. 2.8.

**Example 2.13.** In a normal distribution 31% of items are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution.

**Solution.** Let  $\mu$  be the mean and  $\sigma$  be the standard deviation of the distribution

$$\text{Normal variate } z = \frac{x - \mu}{\sigma}$$

As 31% of items are under 45 and 8% over 64

$$\therefore \text{At } x = 45, \text{ we have } z = \frac{45 - \mu}{\sigma} = z_2 \text{ (say)}$$

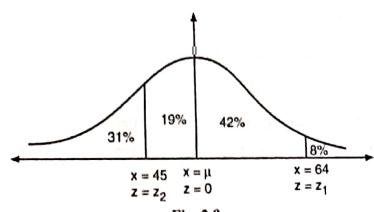


Fig. 2.9.

Area from  $z = 0$  to  $z = z_1$  is 42% i.e., 0.42. The value of  $z_1$  corresponding to area 0.42 from the table 1 is 1.405

$$\therefore 1.405 = \frac{64 - \mu}{\sigma} \quad \dots(i)$$

$$\text{Similarly at } x = 45, |z| = \frac{|45 - \mu|}{\sigma} = |z_2|$$

Area between  $x = 45$  (i.e.,  $z = z_2$ ) to  $x = \mu$  ( $z = 0$ ) is the same numerically as between  $z = 0$  to  $z = z_1$  which is 19%.

Now the value of normal variate  $z_2$  corresponding to area 0.19 is 0.495

$$\therefore \frac{|45 - \mu|}{\sigma} = |0.495|$$

$$\therefore \frac{\mu - 45}{\sigma} = 0.495 \quad \dots(ii)$$

$$\text{From (i) and (ii)} \quad \frac{64 - \mu}{\sigma} = \frac{1.405}{0.495} = 2.8 \\ \mu = 50 \text{ and } \sigma = 10.$$

**Example 2.14.** A sample of 100 dry battery cells tested to find the length of life produced following results:

$$\mu = 12 \text{ hours}, \sigma = 3 \text{ hours}$$

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life:

- (a) more than 15 hours
- (b) less than 6 hours
- (c) between 10 and 14 hours.

**Solution.** Let  $X$  denotes the length of life of dry battery cells.

$$\text{Also } z = \frac{X - \mu}{\sigma} = \frac{X - 12}{3}$$

$$(a) \text{ When } X = 15, \quad z = \frac{15 - 12}{3} = 1$$

$$\therefore P(X > 15) = P(z > 1)$$

Area of the right of  $z = 1$  is  $0.5 - 0.3413 = 0.1587$

$\therefore$  Percentage of battery cells having life more than 15 hours  
 $= 0.1587 \times 100 = 15.87\%$

$$(b) \text{ When } X = 6, \quad z = \frac{6 - 12}{3} = -2$$

$$\therefore P(X < 6) = P(z < -2)$$

$\therefore$  Percentage of battery cells have life less than 6 hours  
 $= 0.0028 \times 100 = 2.28\%$

$$(c) \text{ When } X = 10, \quad z = \frac{10 - 12}{3} = -0.67$$

$$\text{When } X = 14, \quad z = \frac{14 - 12}{3} = 0.67$$

$$P(10 < X < 14) = P(-0.67 < z < 0.67) = 2P(0 < Z < 0.67)$$

Area between  $X = 10$  to  $X = 14$  is twice the area of the left of  $z = 0.67$

$$= 2 \times 0.2437 = 0.4974$$

$\therefore$  Percentage of battery cells having life span between 10 hours and 14 hours = 49.74%

#### CONTINUOUS PROBABILITY DISTRIBUTION

**Example 2.15.** The customer accounts of certain department store have an average balance of ₹ 120 and a standard deviation of ₹ 40. Assuming that the account balances are normally distributed:

- (i) What proportion of the account is over ₹ 150?
- (ii) What proportion of account is between ₹ 100 and ₹ 150?
- (iii) What proportion of account is between ₹ 60 and ₹ 90?

**Solution.** Standard normal variate  $z = \frac{X - \mu}{\sigma}$

$$(i) \text{ When } X = 150, \quad z = \frac{150 - 120}{40} = 0.75$$

To find out the proportion of accounts greater than ₹ 150, we refer to right of  $z = 0.75$

The area to the right of  $z = 0$  is 0.5

Less the area between  $z = 0$  and  $z = 0.75$  is 0.2735.

The area of the right to  $z = 0.75$  is 0.2266.

Therefore, 22.66% of account have balance is excess of ₹ 150 (shaded area)

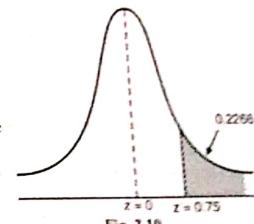


Fig. 2.10.

$$(ii) \text{ When } X = 100, \quad z = \frac{100 - 120}{40} = -0.5$$

$$\text{When } X = 150, \quad z = \frac{150 - 120}{40} = 0.75$$

The area between  $z = -0.5$  and  $z = 0$  is 0.1915.

The area between  $z = 0$  and  $z = 0.75$  is 0.2734

$\therefore$  The total area i.e.,  $z = -0.5$  and  $z = 0$  is 0.1915

The area between  $z = 0$  and  $z = 0.75$  is 0.2734.

$\therefore$  The area total area i.e.,  $z = -0.5$  to  $z = 0.75$  is  $0.1915 + 0.273 = 0.4649$

$\therefore$  The area total area i.e.,  $z = -0.5$  to  $z = 0.75$  is  $0.1915 + 0.273 = 0.4649$

$\therefore$  46.49% of accounts have an average balance of accounts between ₹ 100 and 150.

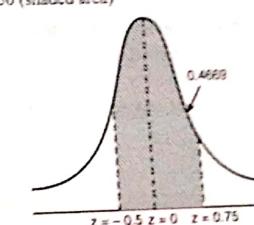


Fig. 2.11

$$(iii) \text{ When } X = 90, \quad z = \frac{90 - 120}{40} = -0.75$$

$$\text{When } X = 60, \quad z = \frac{60 - 120}{40} = -1.5$$

Area between  $z = 0$  to  $z = -1.5$  is 0.4332

Area between  $z = 0$  to  $z = -0.75$  is 0.2734

$\therefore$  The area between  $X = 90$  and  $X = 60$

$(z = -1.5 \text{ to } z = -0.75)$  is  $0.4332 - 0.2734 = 0.1598$

$\therefore$  15.98% of accounts are between ₹ 60 and ₹ 90

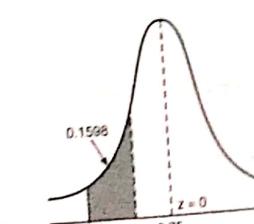


Fig. 2.12.

**Example 2.16.** The average height of soldiers of a country is given as 68.22 inches with variance 10.8 sq. inch. How many soldiers out of 1000 would you expect to be over 72 inches tall? Given that the area under the normal curve between  $z = 0$  to  $z = 0.35$  is 0.1368 and between  $z = 0$  to  $z = 1.15$  is 0.3746.

**Solution.** Standard normal variate

$$z = \frac{x - \mu}{\sigma} = \frac{72 - 68.22}{\sqrt{10.8}} = 1.15$$

$$P(X \geq 72) = P(z \geq 1.15) = \text{Total area to the right of ordinate } z = 0 - (\text{area between } z = 0 \text{ to } z = 1.15) \\ = 0.5 - 0.3746 = 0.1254$$

$$\therefore \text{Total number of soldiers } k \text{ above } 72'' \text{ tall} = 1000 \times 0.1254 = 125$$

**Example 2.17.** In an intelligence test administered to 1000 students the average score was 42 and standard deviation 24. Find: (a) the number of students exceeding a score of 50, (b) the number of students lying between 30 and 54, (c) the value of score exceeded by the top 100 students.

**Solution.** (a) Given  $\mu = 42$ ,  $x = 50$ ,  $\sigma = 24$

$$z = \frac{x - \mu}{\sigma} = \frac{50 - 42}{24} = 0.333$$

Area of the right to ordinate at 0.333 is  $0.5 - 0.1304 = 0.3696$

(b) Expected number of children exceeding a score of 50

$$= 0.3696 \times 1000 = 369.6 \text{ or } 370$$

(b) Standard normal variate for score 30

$$z = \frac{x - \mu}{\sigma} = \frac{30 - 42}{24} = -0.5$$

Standard normal variate for score 54

$$z = \frac{x - \mu}{\sigma} = \frac{54 - 42}{24} = 0.5$$

Area from  $z = 0$  to  $z = 0.5$  is 0.1915

Area from  $z = -0.5$  to  $z = 0$  is 0.1915

$\therefore$  Area from  $z = -0.5$  to  $z = 0.5 = 0.1915 + 0.1915 = 0.3830$

Thus the number of children having score between 30 and 54 =  $0.383 \times 1000 = 383$

(c) Probability of getting top 100 students =  $\frac{100}{1000} = 0.1$

Standard normal variate having 0.1 area of the right = 1.281

$$z = \frac{x - \mu}{\sigma}$$

$$1.281 = \frac{x - 42}{24}$$

$$1.28 \times 24 = x - 42$$

$$x = (1.28 \times 24) + 42 = 72.72 \text{ or } 73$$

**Example 2.18.** The distribution of a random variable is given by

$$f(x) = Ce^{-\frac{1}{20}(9x^2 - 30x)}, -\infty \leq x \leq \infty$$

Find the constant  $C$ , the mean and the variance of the random variable. Find also the upper 5% value of the random variable.

**Solution.**  $P(-\infty \leq x \leq \infty) = 1$  = unit area under the curve

$$y = f(x) \text{ and } x\text{-axis}$$

$$1 = \int_{-\infty}^{\infty} f(x) dx$$

$$1 = Ce^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{(9x-30)^2}{20}} dx$$

Comparing with normal distribution, we have

$$\text{We have } \mu = \frac{5}{3} = 1.667$$

$$2\sigma^2 = \frac{50}{9} \text{ or } \sigma^2 = \frac{25}{9} = 2.778$$

$$\sigma = \frac{5}{3}, C = \frac{1}{\sqrt{2\pi}\sqrt{2.778}} = \frac{1}{\sqrt{2\pi \times 1.667\sqrt{e}}} = 0.145$$

$$\text{Standard normal variate } t = \frac{x - \mu}{\sigma} = \frac{x - \frac{5}{3}}{\frac{5}{3}} \text{ corresponding to 95% area, we have } 1.96 = \frac{x - \frac{5}{3}}{\frac{5}{3}}$$

$$\therefore 1.96 \times \frac{5}{3} + \frac{5}{3} = x \text{ or } x = 4.933$$

**Example 2.19.** Assuming that the diameters of 1000 brass plugs taken consecutively from a machine form a normal distribution with mean 0.7515 cm and standard deviation 0.002 cm. Find the number of plugs likely to be rejected if the approved diameter is  $0.752 \pm 0.004$  cm.

**Solution.**  $\mu = 0.7515, \sigma = 0.002$

Limits of the diameter of non-defective plugs are

$$0.752 - 0.004 = 0.748 \text{ and } 0.752 + 0.004 = 0.756 \text{ cm}$$

At  $x = 0.748$  m the standard normal variate

$$z_1 = \frac{x - \mu}{\sigma} = \frac{0.748 - 0.7515}{0.002} = -1.75$$

$$\text{and at } x = 0.756, z_2 = \frac{0.756 - 0.7515}{0.002} = 2.25$$

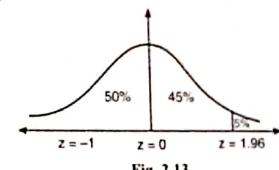


Fig. 2.13.

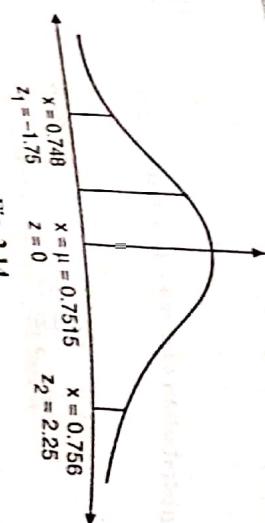


Fig. 2.14.

Area between  $z_1$  and  $z_2$  is numerically equal to the sum of the area between 0 to  $z_1$  and  $z_2$ . From table, when  $z = 1.75$  area is equal to 0.4599 and when  $z = 2.25$  area is equal to 0.4878.

Hence, the required are for non-defective items  $= 0.4599 + 0.4878 = 0.9477$ .

Non-defective plugs are  $100 \times 0.9477 = 94.77 = 95$

Hence, defective plugs are  $1000 - 948 = 52$

$\therefore 52$  plugs are likely to be rejected.

**Example 2.20.** A sales tax officer reported that the average sales of 500 business establishments in a year is ₹ 36,000/and standard deviation of ₹ 10,000/-. Assuming that the sales in these businesses are normally distributed, find: (i) the number of businesses the sales of which are more than 40,000/-, (ii) the percentage of businesses the sales of which are likely to range between ₹ 30,000 and ₹ 40,000.

**Solution.** From the given data

$$N = 500, \mu = 36,000 \quad \text{and} \quad \sigma = 10,000$$

$$\text{The standard normal variate } z = \frac{x - \mu}{\sigma} \Rightarrow z = \frac{x - 36,000}{10,000}$$

$$(i) P(X > 40,000)$$

$$\text{When } x = 40,000 \quad \text{then } z = \frac{40,000 - 36,000}{10,000}$$



Fig. 2.15.

$$\begin{aligned} f(x > 40,000) &\approx P(z > 0.4) \\ &\approx 1 - P(z \leq 0.4) \\ &\approx 1 - P(z \leq 0.4) \\ &\approx 1 - F(0.4) \\ &\approx 1 - 0.6554 \\ &\approx 0.3446 \end{aligned}$$

$\therefore$  The number of businesses the sales which are more than ₹ 40,000/- are

$$\begin{aligned} &\approx N P(X > 40,000) \\ &\approx 500 \times 0.3446 = 172.3 \end{aligned}$$

$$\begin{aligned} (ii) P(30,000 < X < 40,000) &\approx P(-0.6 < z < 0.4) \\ &\approx F(0.4) - F(-0.6) \\ &\approx 0.6554 - 0.2743 \\ &= 0.3811 \end{aligned}$$

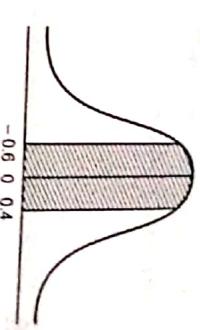


Fig. 2.16.

$\therefore$  The number of business the sales which are likely to range between 30,000 and 40,000 are

$$= N P(30,000 < X < 40,000)$$

$$= 500 \times 0.3811 = 190.55 = 191$$

Out of 500 businesses 191 having sales between 30,000 and 40,000. For 100 businesses 38.2.

$\therefore$  The required percentage is 38.2%.

**Example 2.21.** Suppose 10 percent of probability for a normal distribution  $N(\mu, \sigma^2)$  is below 25 and 5 percent above 90, what are the value of  $\mu$  and  $\sigma$ .

**Solution.**

$$P(x \leq 35) = \frac{10}{100} = 0.1 \quad \dots (i)$$

$$P(x > 90) = \frac{5}{100} = 0.05 \quad \dots (ii)$$

The standard normal variate  $z = \frac{x - \mu}{\sigma}$

$$\text{When } x = 35 \quad \text{then } z = \frac{35 - \mu}{\sigma}$$

$$\text{When } x = 90 \quad \text{then } z = \frac{90 - \mu}{\sigma}$$

$$\text{From eqn. (i)} P(x \leq 35) = 0.1$$

$$\begin{aligned} &\Rightarrow P\left(z \leq \frac{35 - \mu}{\sigma}\right) = 0.1 \\ &\Rightarrow \left(\frac{35 - \mu}{\sigma}\right) = -1.28 \text{ from table} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \frac{35 - \mu}{\sigma} = -1.28 \sigma \\ &\Rightarrow \mu = 1.28 \sigma + 35 \end{aligned}$$

$$\begin{aligned} &\text{From eqn. (ii)} P(x > 90) = 0.05 \\ &\Rightarrow P\left(z > \frac{90 - \mu}{\sigma}\right) = 0.05 \\ &\Rightarrow \left(\frac{90 - \mu}{\sigma}\right) = -1.645 \\ &\Rightarrow 90 - \mu = -1.645 \end{aligned}$$

$$\begin{aligned} \text{When } x = 90 \quad \text{then } z = \frac{90 - \mu}{\sigma} \\ \therefore 90 - \mu = -1.645 \\ \therefore \mu = 90 + 1.645 = 91.645 \end{aligned}$$

$$\Rightarrow \mu - 1.645 \sigma = 90$$

From eqns. (iii) and (iv)

$$\begin{aligned}\mu &= 157.87 \\ \sigma &= 150.68\end{aligned}$$

## EXERCISE 2.2

- Students of a class were given an aptitude test. Their marks were found to be normally distributed with mean 60 and standard deviation 5. What percentage of students scored more than 60 marks? [Ans. 50]
- Find the value of  $z$  in each of the cases
  - area between  $z = 0$  and  $z = 0.3770$ .
  - area of the left of  $z$  is 0.8621.

[Ans. (i)  $\pm 1.6$ , (ii) 1.09]
- In a sample of 1000 cases, the mean of a certain test is 14 and standard deviation is 2.5. Assuming the distribution to be normal, find:
  - how many students score between 12 and 15?
  - how many score above 18?
  - how many score below 8?
  - how many score 16?

[Hint : (iv) Take marks between 15.5 and 16.5]  
[Ans. (i) 444, (ii) 55, (iii) 8, (iv) 116]
- Find the area under the normal curve in each of the following case :
  - $z = 0$  and  $z = 1.2$
  - $z = -0.68$  and  $z = 0$
  - $z = -0.46$  and  $z = 2.21$
  - $z = 0.81$  and  $z = 1.94$

[Ans. (i) 0.3849, (ii) 0.2518, (iii) 0.6637, (iv) 0.1828]  
[Hints: Use standard curve table]
- Students of a class were given a mechanical aptitude test. These marks were found to be normally distributed with mean 60 and standard deviation 5. What percent of students scored: (i) more than 60 marks, (ii) less than 56 marks, (iii) between 45 and 65 marks.  
[Ans. (i) 50%, (ii) 21.21%, (iii) 84%]
- The mean height of 500 students is 151 cm and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students's heights lie between 120 and 155 cm.  
[Ans. 294]
- Fit a normal curve

$$P(x) = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}; -\infty \leq x \leq \infty$$

Given the following data :

$x$	8.60	8.59	8.57	8.56	8.55	8.54	8.53	8.52
$f$	2	3	4	10	8	4	1	1

## 2.7. EXPONENTIAL DISTRIBUTION

A random variable  $X$  is said to have an exponential distribution with parameter  $\lambda > 0$ , if its probability density function is given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The cumulative distribution function  $F(x)$  is given by  $F(x) = P(X \leq x) = \int_{-\infty}^x \lambda e^{-\lambda x} dx$

$$\Rightarrow F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \\ 1, & x = \infty \end{cases}$$

**Remark:** Sometimes exponential distribution is defined by the p.d.f.

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad x > 0$$

0, otherwise

**Graph of Exponential Probability Density Function**

For  $f(x) = \lambda e^{-\lambda x}$

$x$	0	1	2	.....	$\infty$
$f(x)$	$\lambda$	$\lambda e^{-\lambda}$	$\lambda e^{-2\lambda}$	.....	0

when  $\lambda = 1, f(x) = 1, e^{-1}, e^{-2}, \dots, 0$   
 $\lambda = 2, f(x) = 2, 2e^{-2}, 2e^{-4}, \dots, 0$

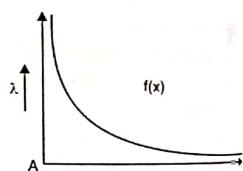


Fig. 2.17.

**Graph of Distribution Function**

For  $F(x) = 1 - e^{-\lambda x}$

$x$	0	1	2	.....	$\infty$
$F(x)$	0	$1 - e^{-\lambda}$	$1 - e^{-2\lambda}$	.....	1

when  $\lambda = 1, F(x) = 0, 1 - e^{-1}, 1 - e^{-2}, \dots, 1$   
 $\lambda = 2, F(x) = 0, 1 - e^{-2}, 1 - e^{-4}, \dots, 1$

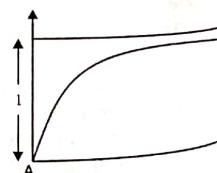


Fig. 2.18.

## 2.8. CONSTANTS OF EXPONENTIAL DISTRIBUTION

### 1. Moments About Origin

$$\mu_1^1 \text{ or Mean} = E(X) = \int_{-\infty}^{\infty} xf(x) dx$$

$$\int_0^{\infty} x \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} x e^{-\lambda x} dx$$

$$= \lambda \cdot \frac{[x]}{\lambda^2}$$

$$= \frac{1}{\lambda}.$$

$$\left[ \because \int_0^{\infty} x^{n-1} e^{-\lambda x} dx = \frac{[x]}{\lambda^n} \right]$$

### CONTINUOUS PROBABILITY DISTRIBUTION

$$\begin{aligned} \mu'_2 &= E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \lambda \cdot \int_0^{\infty} x^2 e^{-\lambda x} dx \\ &= \lambda \cdot \frac{[x]}{\lambda^3} = \frac{2}{\lambda^2}. \end{aligned}$$

Similarly,

$$\mu'_3 = E(X^3) = \frac{6}{\lambda^3}$$

and

$$\mu'_4 = E(X^4) = \frac{24}{\lambda^4}.$$

### 2. Moments About Mean

$$\mu_1 = 0 \quad (\text{always})$$

$$\mu_2 = \mu'_2 - \mu_1^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} = \text{variance } (\sigma^2)$$

$\therefore$  Standard deviation,

$$\sigma = \sqrt{\mu_2} = \sqrt{\text{var}(X)} = \frac{1}{\lambda}.$$

$$\begin{aligned} \mu_3 &= \mu'_3 - 3\mu'_2 \mu_1 + 2(\mu_1^3) \\ &= \frac{6}{\lambda^3} - \frac{3 \times 2}{\lambda^2} \times \frac{1}{\lambda} + 2 \left(\frac{1}{\lambda}\right)^3 \\ &= \frac{6}{\lambda^3} - \frac{6}{\lambda^3} + \frac{2}{\lambda^3} = \frac{2}{\lambda^3}, \\ \mu_4 &= \mu'_4 - 4\mu'_3 \mu_1 + 6\mu'_2 (\mu_1^2) - 3(\mu_1^4) \\ &= \frac{24}{\lambda^4} - 4 \cdot \frac{6}{\lambda^3} \cdot \frac{1}{\lambda} + 6 \times \frac{2}{\lambda^2} \times \left(\frac{1}{\lambda}\right)^2 - 3 \left(\frac{1}{\lambda}\right)^4 \\ &= \frac{24}{\lambda^4} - \frac{24}{\lambda^4} + \frac{12}{\lambda^4} - \frac{3}{\lambda^4} = \frac{9}{\lambda^4}. \end{aligned}$$

### 3. Beta and Gamma Coefficients

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\left(\frac{2}{\lambda^3}\right)^2}{\left(\frac{1}{\lambda^2}\right)^3} = \frac{\left(\frac{4}{\lambda^6}\right)}{\left(\frac{1}{\lambda^6}\right)} = 4$$

$$\gamma_1 = \sqrt{\beta_1} = 2$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\frac{9}{\lambda^4}}{\left(\frac{1}{\lambda^2}\right)^2} = 9$$

$$\gamma_2 = \beta_2 - 3 = 9 - 3 = 6$$

**4. Mean Deviation About Mean**

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \\ 0, \quad \text{otherwise}$$

Then

Hence, mean deviation about mean,

$$\text{M.D.} = E|X - E(X)|$$

$$= \int_0^\infty \left| x - \frac{1}{\lambda} \right| \lambda e^{-\lambda x} dx$$

$$= \int_0^\infty |\lambda x - 1| e^{-\lambda x} dx$$

$$= \frac{1}{\lambda} \int_0^\infty |t - 1| e^{-t} dt \quad \text{where } t = \lambda x$$

$$= \frac{1}{\lambda} \left[ \int_0^1 (1-t)e^{-t} dt + \int_1^\infty (t-1)e^{-t} dt \right]$$

$$= \frac{1}{\lambda} [e^{-1} + e^{-1}] = \frac{2}{\lambda} e^{-1}.$$

**5. Moment Generating Function**

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$M(t) = E(e^{tx}) = \int_0^\infty \lambda e^{-\lambda x} \cdot e^{tx} dx$$

$$= \int_0^\infty \lambda e^{-(\lambda-t)x} dx$$

$$= \frac{\lambda}{\lambda-t} (e^{-(\lambda-t)x})_0^\infty = \frac{\lambda}{\lambda-t}$$

$$= \left( \frac{\lambda-t}{\lambda} \right)^{-1} = \left( 1 - \frac{t}{\lambda} \right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} + \frac{t^4}{\lambda^4} + \dots$$

$$\mu'_r = \text{coefficient of } \frac{t^r}{r!} \text{ in } M_X(t) = \frac{r!}{\lambda^r}.$$

In particular,

$$\mu'_1 = \frac{1}{\lambda}, \mu'_2 = \frac{2!}{\lambda^2}, \mu'_3 = \frac{3!}{\lambda^3}, \mu'_4 = \frac{4!}{\lambda^4} \text{ and so on.}$$

**6. Cumulant Generating Function**

$$f(x) = \lambda e^{-\lambda x}, x \geq 0$$

$$M_X(t) = \left( 1 - \frac{t}{\lambda} \right)^{-1}$$

$$\Rightarrow$$

$$K_X(t) = \log M_X(t) = -\log \left( 1 - \frac{t}{\lambda} \right)$$

$$= \frac{t}{\lambda} + \frac{1}{2} \left( \frac{t^2}{\lambda^2} \right) + \frac{1}{3} \left( \frac{t^3}{\lambda^3} \right) + \dots + \frac{1}{r} \left( \frac{t^r}{\lambda^r} \right) + \dots$$

$$K_1 = \frac{1}{\lambda}, K_2 = \frac{1}{\lambda^2}, K_3 = \frac{2}{\lambda^3}, \dots, K_r = \frac{(r-1)!}{\lambda^r}.$$

## SOLVED EXAMPLES

**Example 2.22.** A power supply unit for a computer component is assumed to follow an exponential distribution with a mean life of 1200 hours. What is the probability that the component will

(i) fail in the first 300 hours?

(ii) survive more than 1500 hours?

(iii) last between 1200 hours and 1500 hours?

**Solution.** Given that  $\lambda = 1/1200$

(i) The probability that the component will fail in the first 300 hours is

$$P(X \leq 300) = 1 - e^{-\frac{300}{1200}}$$

$$= 1 - e^{-0.25}$$

$$= 1 - 0.7788 = 0.2212$$

(ii) The probability that the component will survive more than 1500 hours is

$$P(X \geq 1500) = e^{-1500/1200}$$

$$= e^{-1.25} = 0.2865$$

(iii) The probability that the component will last between 1200 hours and 1500 hours is given as  $P(1200 \leq X \leq 1500)$

$$P[\text{Inside}] = 1 - P[\text{Outside}]$$

$$= 1 - [P(X \leq 1200) + P(X \geq 1500)]$$

$$\text{Now, } P(X \leq 1200) = 1 - e^{-\frac{1200}{1200}} = 0.6321$$

$$P(X \geq 1500) = e^{-\frac{1500}{1200}} = 0.2865$$

and

$$P[\text{Inside}] = 1 - [0.6321 + 0.2865] = 0.0813$$

Hence,

$$P(1200 \leq X \leq 1500) = 0.0813.$$

**Example 2.23.** A random variable has an exponential distribution with probability density function given by

$$f(x) = \begin{cases} 3e^{-3x}, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0. \end{cases}$$

What is the probability, that  $X$  is not less than 4? Find mean and standard deviation. Show that

$$\text{Coefficient of variation} = \left( \frac{\text{mean}}{\text{S.D.}} \right) = 1.$$

**Solution.**  $P(X \geq 4) = P(X \geq 4) = \int_4^\infty 3e^{-3x} dx = [-e^{-3x}]_4^\infty = e^{-12}.$

$$\text{Mean} = E(X) = \int_0^\infty x 3e^{-3x} dx = 3 \int_0^\infty x e^{-3x} dx$$

$$= 3 \cdot \frac{\Gamma(2)}{3^2} = \frac{1}{3}.$$

$$\text{Variance} = E[X - E(X)]^2$$

$$= E(X^2) - [E(X)]^2$$

$$= \int_0^\infty x^2 \cdot 3e^{-3x} dx - \left( \frac{1}{3} \right)^2$$

$$= 3 \cdot \frac{\Gamma(3)}{3^3} - \frac{1}{9} = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

$$\therefore \text{Standard deviation, S.D.} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Hence,

$$\frac{\text{Mean}}{\text{S.D.}} = \frac{1/3}{1/3} = \frac{1}{3} \times \frac{3}{1} = 1.$$

**Example 2.24.** The sales tax,  $X$ , of a shopkeeper has an exponential distribution with  $p.d.f.$

$$f(x) = \frac{1}{4} e^{-x/4}, \quad x \geq 0$$

If sales tax is levied at the rate of 5% what is the probability that his sales exceed ₹ 10,000?

**Solution.** Sales tax on ₹ 10,000 = ₹ 10,000  $\times \frac{5}{100}$  = ₹ 500

i.e., If sales exceed ₹ 10,000, tax will exceed ₹ 500. Hence, the required probability is

$$P(X > 500) = \int_{500}^\infty \frac{1}{4} e^{-x/4} dx$$

$$= \left[ \frac{1}{4} \left( -\frac{1}{4} e^{-x/4} \right) \right]_{500}^\infty$$

$$= e^{-\frac{500}{4}} = e^{-125}$$

**Example 2.25.** If families are selected at random in a certain thickly populated area and their annual income in excess of ₹ 4000 is treated as a random variable having an exponential distribution

$$f(x) = \frac{1}{2000} e^{-\frac{x}{2000}}$$

for  $x > 0$ , what is the probability that 3 out of 4 families selected in the area have income in excess of ₹ 5000?

**Solution.** ₹ 5000 – ₹ 4000 = ₹ 1000

$$P(X > 1000) = \int_{1000}^\infty \frac{1}{2000} e^{-\frac{x}{2000}} dx = \left[ e^{-\frac{x}{2000}} \right]_{1000}^\infty$$

$$= e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

$$P(X < 1000) = 1 - \frac{1}{\sqrt{e}}$$

∴

$$\text{Hence, required probability} = {}^4C_3 \left( \frac{1}{\sqrt{e}} \right) \left( 1 - \frac{1}{\sqrt{e}} \right) = \frac{4(\sqrt{e}-1)}{e^2}$$

**Example 2.26.** The income tax of a man is exponentially distributed with the probability density function given by

$$f(x) = \frac{1}{3} e^{-\frac{1}{3}x}, \quad \text{for } x > 0$$

$$= 0, \quad \text{for } x \leq 0.$$

What is the probability that his income will exceed ₹ 17,000 assuming that the income tax is levied at the rate of 15% on the income above ₹ 15,000?

**Solution.** If the income exceeds ₹ 17,000 then income tax will exceed by 15% of (17000 – 15000), i.e., exceeds by  $\frac{15}{100} \times 2000$  i.e., exceeds by ₹ 300.

Hence, the required probability

$$= P(X > 300) = \int_{300}^\infty \frac{1}{3} e^{-\frac{1}{3}x} dx$$

$$= \frac{1}{3} \left[ -3e^{-\frac{1}{3}x} \right]_{300}^\infty = e^{-\frac{300}{3}} = e^{-100}.$$

**Example 2.27.** If  $X$  has a uniform distribution in  $[10, 1]$  with probability density function

$$f_x(x) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the distribution (probability density function) of  $-2 \log x$ .

**1.8** Now the distribution function  $G$  of  $y$  will be defined as follows.

Solution. Suppose  $y = -2 \log X$ . Now the distribution function  $G$  of  $y$  will be defined as follows.

$$\begin{aligned} G(y) &= P(Y \leq y) = P(-2 \log X \leq y) \\ &= P\left(\log X \geq -\frac{1}{2}y\right) = P(X \geq e^{-y/2}) = 1 - P(X \leq e^{-y/2}) \\ &= 1 - \int_0^{e^{-y/2}} f(x) dx = 1 - \int_0^{e^{-y/2}} 1 dx = 1 - e^{-y/2}. \end{aligned}$$

$$g(y) = (d/dy)G(y) = \frac{1}{2}e^{-y/2}$$

$g(y)$  is the p.d.f. of  $Y$ .

**Example 2.28.** A random variable  $X$  has p.d.f.

$$f(x) = \frac{1}{\sigma} e^{-x/\sigma}, \quad 0 \leq x \leq \infty.$$

A continuous random variable  $X$  is said to follow gamma distribution with parameter  $\alpha$  and  $\lambda$  if its p.d.f. is

$$f(x) = \begin{cases} \frac{\alpha^\lambda e^{-\alpha x} x^{\lambda-1}}{\Gamma(\lambda)}, & x \geq 0, \alpha, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

**Remarks:**

1. The function  $f(x)$  represents a probability density function since

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_0^{\infty} f(x) dx + \int_0^{\infty} f(x) dx \\ &= \int_0^{\infty} f(x) dx \\ &= \int_0^{\infty} \frac{\alpha^\lambda e^{-\alpha x}}{\Gamma(\lambda)} x^{\lambda-1} dx \\ &= \frac{\alpha^\lambda}{\Gamma(\lambda)} \int_0^{\infty} e^{-\alpha x} x^{\lambda-1} dx \\ &= \frac{\alpha^\lambda}{\Gamma(\lambda)} \cdot \frac{\overline{(\lambda)}}{\alpha^\lambda} = 1 \quad \left[ \because \int_0^{\infty} e^{-ax} x^{\lambda-1} dx = \frac{\overline{(\lambda)}}{a^\lambda} \text{ (gamma function)} \right] \end{aligned}$$

$\therefore r$ th cumulant = coefficient of  $\frac{t^r}{r!}$  in  $K_x(t) = (r-1)! \sigma^r$ .

### EXERCISE 2.3

- Define exponential distribution. Give its distribution function.
- Find mean and variance for the exponential distribution.

$$\boxed{\text{Ans. Mean} = \frac{1}{\lambda}, \text{ Variance} = \frac{1}{\lambda^2}}$$

- What is exponential distribution? Obtain  $\mu_2$ ,  $\mu_3$ ,  $\mu_4$ ,  $\beta_1$  and  $\beta_2$ .
- Obtain moment generating function for the exponential distribution

$$f(x) = \theta e^{-\theta x}, \quad x \geq 0$$

Hence, find mean and variance.

5. For the exponential distribution defined by  $y = Ce^{-Cx}$ , where  $C > 0$  and  $0 \leq x \leq \infty$ , prove that this connection?

6. For the exponential distribution  $dF = e^{-x} dx$ ,  $x > 0$ , find  $\mu_2$ ,  $\mu_3$  and  $\mu_4$ . Also find the deviation about  $a$ . Find for what value of  $a$ , this deviation is minimum. State the theorem this connection?

$$\boxed{\text{Ans. } \mu_2 = 1, \mu_3 = 2, \mu_4 = 9; e^{-a} = \frac{1}{2} \Rightarrow a = \text{Median}}$$

**7. Derive the following forgetful property of the exponential distribution**

$$P[X > (s+t)/X > t] = P[X > t], \text{ for any } s, t > 0.$$

8. Let p.d.f. of  $X$  be  $g(x) = e^{-x}$ ,  $x \geq 0$  and p.d.f. of  $y$  be  $h(y) = e^{-y}$ ,  $y \geq 0$ . Show that p.d.f. of  $Z = X + Y$  is

$$f(Z) = Ze^{-Z}, \quad Z \geq 0$$

If  $X$  and  $Y$  are independent random variables.

### 2.9. GAMMA DISTRIBUTION

A continuous random variable  $X$  is said to follow gamma distribution with parameter  $\alpha$  and  $\lambda$  if its p.d.f. is

## 4. M.G.F. of Gamma Distribution

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x) dx$$

$$\begin{aligned} &= \frac{a^\lambda}{(\lambda)} \int_0^\infty e^{-ax} x^{\lambda-1} dx \\ &= \frac{a^\lambda}{(\lambda)} \frac{\Gamma(\lambda+1)}{a^{\lambda+1}} \left[ t; \int_0^\infty e^{-ax} x^{\lambda-1} dx \right] = \boxed{\frac{a^\lambda}{(\lambda)}} \text{ (gamma function)} \end{aligned}$$

$$\begin{aligned} &= \frac{a^\lambda \cdot \Gamma(\lambda+1)}{(\lambda) a^\lambda} \\ &= \frac{\lambda \cdot \Gamma(\lambda+1)}{(\lambda) \lambda^{\lambda+1} \cdot a} \\ &= \frac{\Gamma(\lambda+1)}{(\lambda) a} = \frac{\lambda \cdot \cancel{\Gamma(\lambda)}}{(\lambda) a} = \frac{\lambda}{a} \end{aligned}$$

$\therefore \Gamma(\lambda+1) = \lambda \Gamma(\lambda)$

$$\begin{aligned} &= \frac{\Gamma(\lambda+1)}{(\lambda) a} = \frac{\lambda \cdot \Gamma(\lambda)}{(\lambda) a} = \frac{\lambda}{a} \\ &= \frac{\Gamma(\lambda+1)}{(\lambda) a} = \frac{\lambda \cdot \Gamma(\lambda)}{(\lambda) a} = \frac{\lambda}{a} \end{aligned}$$

$\therefore \Gamma(\lambda+1) = \lambda \Gamma(\lambda)$

$$\boxed{E(X) = \frac{\lambda}{a}}$$

## 3. Variance of Gamma Distribution

$$\text{Var}(X) = \int_0^\infty x^2 f(x) dx - [E(x)]^2$$

$$\begin{aligned} \text{Var}(X) &= \frac{a^\lambda}{(\lambda)} \int_0^\infty x^{\lambda+1} e^{-ax} dx - \left( \frac{\lambda}{a} \right)^2 \\ &= \frac{a^\lambda}{(\lambda)} \int_0^\infty \left( \frac{t}{a} \right)^{\lambda+1} e^{-t/a} \frac{dt}{a} - \frac{\lambda^2}{a^2} \end{aligned}$$

[∴ substituting  $t = ax$ ]

$$\begin{aligned} &= \frac{a^\lambda}{(\lambda)} \int_0^\infty t^{\lambda+1} e^{-t} \frac{dt}{a} - \frac{\lambda^2}{a^2} \\ &= \frac{a^\lambda}{a^{\lambda+2} \Gamma(\lambda)} \int_0^\infty t^{\lambda+1} e^{-t} dt - \frac{\lambda^2}{a^2} \end{aligned}$$

(∴ by definition of gamma function)

$$F(x) = \int_0^x \frac{a^\lambda e^{-ax} x^{\lambda-1}}{(\lambda)} dx$$

$$= \frac{a^\lambda}{(\lambda)} \int_0^x t^{\lambda-1} e^{-at} dt$$

$$\text{Put } \alpha x = y \Rightarrow dx = \frac{dy}{\alpha}$$

$$= \frac{a^\lambda}{(\lambda)} \int_0^{\alpha x} \left( \frac{y}{\alpha} \right)^{\lambda-1} e^{-y} \frac{1}{\alpha} dy$$

$$= \frac{1}{(\lambda)} \int_0^{\alpha x} y^{\lambda-1} e^{-y} dy$$

## 5. Cumulative Distribution Function of Gamma Distribution

The cdf of  $X \sim \text{gamma}(\alpha, \lambda)$  is

$$\begin{aligned} &= \frac{\Gamma(\lambda+2)}{\alpha^2 \Gamma(\lambda)} - \frac{\lambda^2}{\alpha^2} \\ &= \frac{\Gamma(\lambda+2) - \lambda^2 \Gamma(\lambda)}{\alpha^2 \Gamma(\lambda)} \\ &= \frac{\lambda(\lambda+1)\Gamma(\lambda) - \lambda^2 \Gamma(\lambda)}{\alpha^2 \Gamma(\lambda)} \\ &= \frac{\lambda \Gamma(\lambda)(\lambda+1 - \lambda)}{\alpha^2 \Gamma(\lambda)} \\ &= \frac{\lambda \Gamma(\lambda)(\lambda+1 - \lambda)}{\alpha^2 \Gamma(\lambda)} = \frac{\lambda}{\alpha^2} \end{aligned}$$

This is called incomplete gamma function.

## 6. Special case of Gamma Distribution

Consider the gamma density function,

$$\boxed{\text{Var}(X) = \frac{\lambda}{a^2}}$$

$$f(x) = \frac{a^\lambda e^{-ax} x^{\lambda-1}}{(\lambda)}; \quad x \geq 0 \text{ and } \alpha, \lambda > 0$$

$$f(x) = \frac{\alpha \cdot e^{-\alpha x}}{\Gamma(\alpha)} x^{\alpha-1}, \quad x \geq 0$$

If  $\lambda = 1$  then  
 $\alpha e^{-\alpha x}$   
 $x \geq 0$   
 and this is the density function of exponential distribution as  $\lambda \rightarrow \infty$ .

**7. Limiting form of Gamma Distribution**, then  $E(X) = \frac{\lambda}{\alpha}$  and  $\text{var}(X) = \frac{\lambda}{\alpha^2}$ . Then since  $X_1 + X_2 + \dots + X_K$  is also a gamma variate with parameter  $\lambda_1 + \lambda_2 + \dots + \lambda_K$ .

We know that if  $X$  follows gamma distribution, then

gamma variate is given by

$$Z = \frac{X - \mu}{\sigma} = \frac{X - \lambda}{\sqrt{\lambda}}$$

The m.g.f. of sum  $X_1 + X_2 + \dots + X_K$  is given by

$$M_{X_1 + X_2 + \dots + X_K}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot M_{X_3}(t) \cdots M_{X_K}(t)$$

(since  $X_1, X_2, \dots, X_K$  are independent)

$$\begin{aligned} M_z(t) &= E[e^{zt}] \\ &= E\left[e^{t(X_1/\sqrt{\lambda}) + t(X_2/\sqrt{\lambda}) + \dots + t(X_K/\sqrt{\lambda})}\right] \\ &= e^{-t\sqrt{\lambda}} E[e^{t(X_1/\sqrt{\lambda}) + t(X_2/\sqrt{\lambda}) + \dots + t(X_K/\sqrt{\lambda})}] \\ &= e^{-t\sqrt{\lambda}} M_{X_1}(t/\sqrt{\lambda}) \cdot M_{X_2}(t/\sqrt{\lambda}) \cdots M_{X_K}(t/\sqrt{\lambda}) \\ &= e^{-t\sqrt{\lambda}} \left(1 - \frac{t}{\sqrt{\lambda}}\right)^{-\lambda_1} \left(1 - \frac{t}{\sqrt{\lambda}}\right)^{-\lambda_2} \left(1 - \frac{t}{\sqrt{\lambda}}\right)^{-\lambda_3} \cdots \left(1 - \frac{t}{\sqrt{\lambda}}\right)^{-\lambda_K} \end{aligned}$$

which is the m.g.f. of a gamma variate with parameter  $\lambda_1 + \lambda_2 + \dots + \lambda_K$ .

### SOLVED EXAMPLES

**Example 2.29.** The daily consumption of milk in a city, in excess of 20,000 litres, is approximately distributed as a Gamma variate with parameters  $a = \frac{1}{10,000}$  and  $\lambda = 2$ . The city has a daily stock of 30,000 litres. What is the probability that the stock is insufficient on a particular day?

**Solution.** If the r.v.  $X$  denotes the daily consumption of milk (in litres) in a city, then the r.v.  $Y = X - 20,000$  has a gamma distribution with p.d.f.;

$$g(y) = \frac{1}{(10,000)^2 \Gamma(2)} y^{2-1} e^{-y/10,000} = \frac{e^{-y/10,000}}{(10,000)^2}; \quad 0 < y < \infty$$

where  $0 (\lambda^{-1/2})$  are terms containing  $\frac{1}{2}$  and higher powers of  $\lambda$  in the denominator

$$\lim_{\lambda \rightarrow \infty} K_z(t) = \frac{t^2}{2} \Rightarrow \lim_{\lambda \rightarrow \infty} M_z(t) = e^{t^2/2}$$

which is the m.g.f. of a standard normal variate.

In other words, gamma distribution tends to normal distribution for large values of parameter.

$$p = P(Y > 10,000) = P(Y > 10,000)$$

$$= \int_{10,000}^{\infty} g(y) dy = \int_{10,000}^{\infty} \frac{y e^{-y/10,000}}{(10,000)^2} dy$$

$$= \int_1^\infty x e^{-x} dx$$

[Taking  $z = y/\ln x$ ]

$$P = \left| -ze^{-z} \right|_1^\infty + \int_1^\infty e^{-z} dz = e^{-1} - \left| e^{-z} \right|_1^\infty = e^{-1} + e^{-1} = \frac{2}{e}$$

Integrating by parts,  
Remark. Since  $\lambda = 2$ , the integration is easily done. However, for general values of  $a$  and  $k$ , the integral is evaluated by using tables of incomplete Gamma Integral of the form:

$\int_0^\infty \frac{e^{-x} x^{n-1}}{(n)} dx$ , which have been tabulated for different values of  $\alpha$  and  $n$ .

**Example 2.30.** If  $X \sim N(\mu, \sigma^2)$ , obtain the p.d.f. of  $U = \frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2$ .

**Solution.** Since  $X \sim N(\mu, \sigma^2) \Rightarrow z = \frac{x - \mu}{\sigma} \sim N(0, 1)$  with p.d.f.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right); \quad -\infty < z < \infty.$$

The distribution function  $G(\cdot)$  of  $U = \frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2$  is given by:

$$\begin{aligned} G_U(u) &= P(U \leq u) = P\left[\frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2 \leq u\right] \\ &= P(Z^2 \leq 2u) = P(-\sqrt{2u} \leq Z \leq \sqrt{2u}), \text{ where } Z \sim N(0, 1) \\ &= P(Z \leq \sqrt{2u}) - P(Z \leq -\sqrt{2u}) \\ &= \Phi(\sqrt{2u}) - \Phi(-\sqrt{2u}) \end{aligned}$$

where  $\Phi(\cdot)$  is the distribution function of standard normal variate (SNV).

Differentiating w.r.t  $u$ , the p.d.f.  $g(\cdot)$  of  $U$  is given by:

$$\begin{aligned} g(u) &= \Phi(\sqrt{2u}) \cdot \frac{d}{du} (\sqrt{2u}) - \Phi(-\sqrt{2u}) \cdot \frac{d}{du} (-\sqrt{2u}) \\ &= \frac{1}{\sqrt{2u}} [\Phi(\sqrt{2u}) + \Phi(-\sqrt{2u})] \\ &= \frac{1}{\sqrt{2u}} \cdot 2\Phi(\sqrt{2u}) \end{aligned}$$

[ $\because \Phi(u)$  is even function of  $u$ ]

$$= \frac{\sqrt{2}}{u} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \cdot 2u\right)$$

[From (i)]

$$= \frac{1}{\Gamma(1/2)} e^{-u} \cdot u^{(1/2)-1}, \quad u \geq 0$$

$$\left[ \because \sqrt{\pi} = \Gamma(1/2) \text{ and } u = \frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \geq 0 \right]$$

which is the p.d.f. of gamma distribution with parameter  $\frac{1}{2}$ .

Hence,  $U = \frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2$  is a  $\gamma\left(\frac{1}{2}\right)$  variate.

**Example 2.31.** Show that the mean value of positive square root of a  $\gamma(\mu)$  variate is  $\Gamma\left(\mu + \frac{1}{2}\right) / \Gamma(\mu)$ . Hence, prove that the mean deviation of a normal variate from its mean is  $\sqrt{2/\pi}$ ,

where  $\sigma$  is the standard deviation of the distribution.

**Solution.** Let  $X$  be a  $\gamma(\mu)$  variate. Then  $f(x) = \frac{e^{-x} x^{\mu-1}}{\Gamma(\mu)}$ ;  $\mu > 0, 0 < x < \infty$

$$\therefore E(\sqrt{X}) = \int_0^\infty x^{1/2} f(x) dx = \frac{1}{\Gamma(\mu)} \int_0^\infty e^{-x} x^{\mu+1/2-1} dx = \frac{\Gamma\left(\mu + \frac{1}{2}\right)}{\Gamma(\mu)}$$

If  $X \sim N(\mu, \sigma^2)$ , then  $U = \frac{1}{2} \left( \frac{X - \mu}{\sigma} \right)^2$  is a  $\gamma\left(\frac{1}{2}\right)$  variate. (refer example 2.30)

$|X - \mu| = \sqrt{2} \sigma U^{1/2}$ , where  $U$  is a  $\gamma\left(\frac{1}{2}\right)$  variate.

Hence, mean deviation of  $X$  about mean is given by:

$$E|X - \mu| = E(\sqrt{2} \sigma U^{1/2}) = \sqrt{2} \sigma E(U^{1/2})$$

$$\begin{aligned} &= \sqrt{2} \sigma \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)}{r\left(\frac{1}{2}\right)} = \frac{\sqrt{2}\sigma}{\sqrt{\pi}} = \sigma\sqrt{2/\pi}. \quad [\text{Using (i) with } \mu = \frac{1}{2}] \end{aligned}$$