Unit-2

Reciprocity Theorem :-

This theorem is most powerful theorem in circuit and field theories both.

The original theorem is due to Rayleigh Helmholtz which was generalised, to include continuous media, by J.R. Carson, and it is also known as Rayleigh Reciprocity Theorem.

Statement :- If an emf is applied to terminals of an antenna no. 1 and current measured at terminals of another antenna no. 2, then an equal current both in amplitude and phase will be obtained at terminals of antenna no. 1 if the same emf is applied to terminals of antenna no. 2.

OR

If a current I_1 at the terminals of antenna no. 1 induces an emf E_{21} at open terminals of antenna no. 2 and current I_2 at the terminals of antenna no. 2 induces an emf E_{12} at open terminals of antenna no. 1, then

$$E_{12} = E_{21} \text{ provided } I_1 = I_2$$

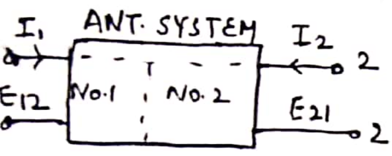
Assumptions :- It is assumed that

1) e.m.f's are of same frequency
2) Medium between the two antennas are linear, passive and isotropic

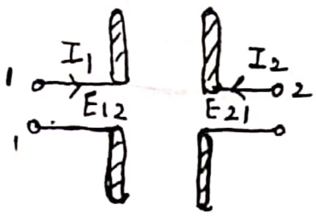
generator and ammeter impedances are equal.

<u>Explanation</u> :- Let i) A transmitter of frequency f and zero impedance be connected to terminals of antenna 2, which is generating a current I_2 and inducing an emf E_{12} at open terminals of antenna no. 1

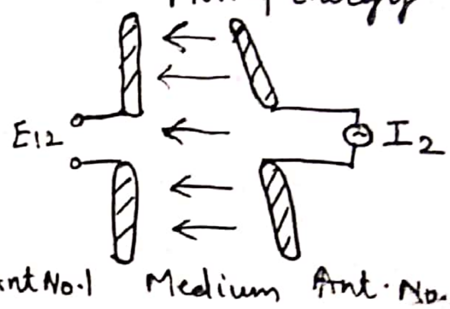
2) Now the same transmitter is transferred to antenna no. 1 which is generating a current I_1 and inducing a voltage E_{21} at open terminals of antenna no. 2



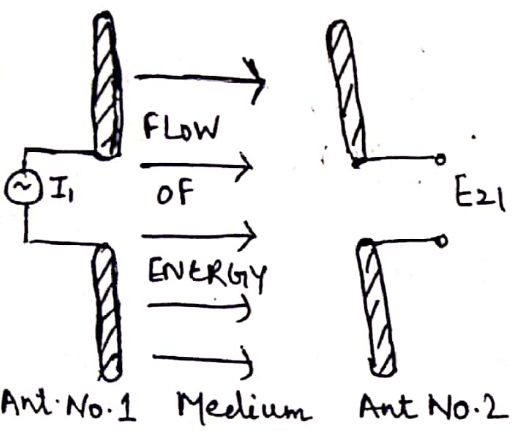
a)



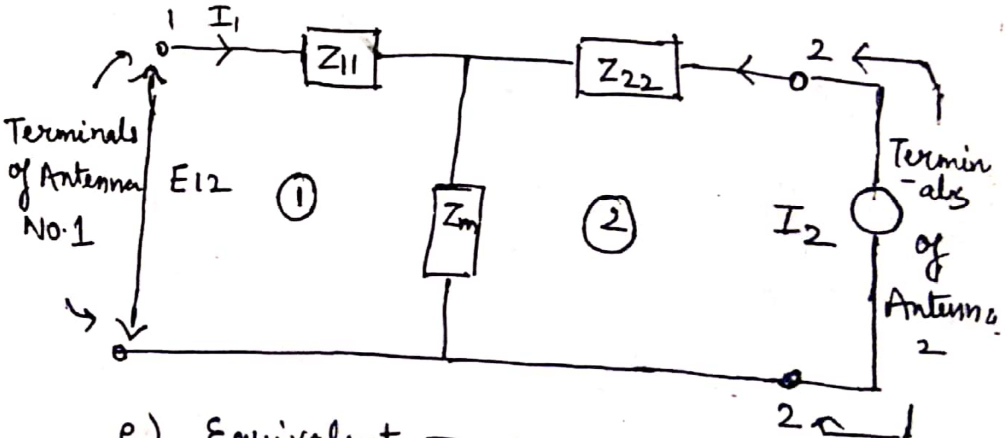
b)



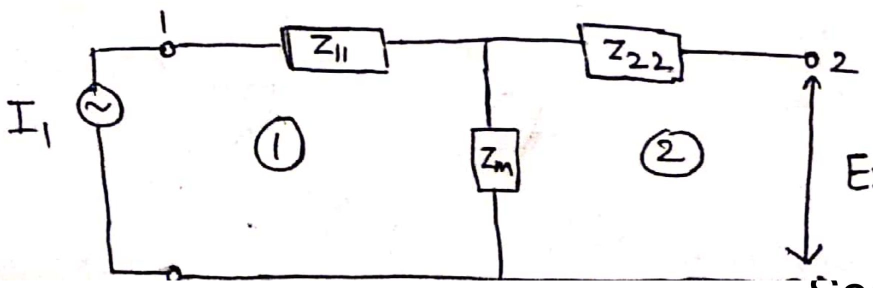
c) Current I_2 inducing and E_{12} emf in antenna 1.



d) Current I_1 inducing an emf E_{21} in antenna no. 2



e) Equivalent T network (As per (c))



f) Equivalent T-N/W (As per (d))

According to statement of Reciprocity Theorem.

$I_1 = I_2$ provided $E_{12} = E_{21}$

Ratio of an emf to current is an impedance

$\therefore \frac{E_{12}}{I_2}$ = Transfer Impedance Z_{12} in case I

$\frac{E_{21}}{I_2}$ = Transfer Impedance Z_{21} in case II

The ratio of voltage (E_1) of one to the current I_2 in second is defined as transfer impedance Z_T or Z_{12} i.e.

$$\boxed{Z_T = Z_{12} = \frac{E_1}{I_2}}$$

Thus from reciprocity it follows that two ratio i.e two impedance are equal ie $\boxed{Z_{12} = Z_{21}}$

This, of course, nothing but mutual Impedance (Z_m) between two antennas, $\therefore$ $\boxed{Z_m = Z_{12} = Z_{21} = \frac{E_{12}}{I_2} = \frac{E_{21}}{I_1}}$ - (1)

$$\boxed{\frac{E_{12}}{I_2} = \frac{E_{21}}{I_1}} \quad -(2)$$

<u>Proof</u> → To prove Reciprocity Theorem for antennas, space b/n antenna no. 1 and antenna no. 2. are replaced by network of linear, passive and bilateral Impedances shown in fig (e) & (f)

Z_{11}, Z_{22} = Self Impedance antenna no. 1 and 2

Z_m = mutual Impedance b/n two antennas

1, 1 = Terminal of antenna no. 1

2, 2 = Terminal of antenna no. 2

Now Apply Kirchoff's mesh Law to fig (e), From loop (2)

$(Z_{22} + Z_m) I_2 - Z_m I_1 = 0$ [$\because$ No voltage source pr... in Loop (2)]

$$\boxed{I_2 = I_1 \frac{Z_m}{(Z_{22}+Z_m)}} \quad - (3)$$

From mesh (1)

$$= (Z_{11}+Z_m) I_1 - Z_m I_2 = E_{12}$$

$$= (Z_{11}+Z_m) I_1 - \frac{Z_m^2 I_1}{(Z_{22}+Z_m)} = E_{12} \text{ from Eq (3)}$$

$$= I_1 \left[\frac{(Z_{11}+Z_m)(Z_{22}+Z_m) - Z_m^2}{(Z_{22}+Z_m)} \right] = E_{12}$$

$$= I_1 \left[\frac{Z_{11} Z_{22} + Z_{11} Z_m + Z_{22} Z_m + Z_m^2 - Z_m^2}{(Z_{22}+Z_m)} \right] = E_{12}$$

$$\boxed{I_1 = \frac{E_{12}(Z_{22}+Z_m)}{Z_{11}Z_{22} + Z_m(Z_{11}+Z_{22})}} \quad - (4)$$

Now substituting Eq (4) in Eq (3)

$$I_2 = \frac{E_{12}(Z_{22}+Z_m) \cdot Z_m}{[Z_{11} \times Z_{22} + Z_m(Z_{11}+Z_{22})](Z_{22}+Z_m)}$$

$$\boxed{I_2 = \frac{E_{12} \cdot Z_m}{Z_{11} \times Z_{22} + Z_m(Z_{11}+Z_{22})}} \quad - (5)$$

Similarly current I_1 can be obtained.

$$\boxed{I_1 = \frac{E_{21} \cdot Z_m}{Z_{22}Z_{11} + Z_m(Z_{22}+Z_{11})}} \quad - (6)$$

Acc. to theorem statement, theorem is proved if we prove

$$E_{12} = E_{21} \text{ if } I_1 = I_2$$

∴ Apply condition $I_1 = I_2$

$$\frac{E_{12} Z_m}{[Z_{11}Z_{22}] + [Z_m](Z_{11}+Z_{12})} = \frac{E_{21} Z_m}{[Z_{11}Z_{22}] + [Z_m](Z_{11}+Z_{22})}$$

$$\boxed{E_{12} = E_{21}}$$

Hence proved.

However, current distribution in receiving antenna is not same as in transmitting antenna.

Further Reciprocity theorem is equally applicable to two separate the antennas and also for two point of same antenna.

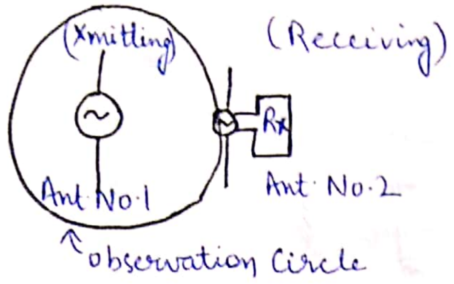
Limitations 1) Although theorem is applicable to radio commn but it fails to be true, only when propagation of radio wave is affected by presence of Earth's magnetic field.

2) It holds good for all radio work (Practical) but for long distance commn through ionosphere, still it is expected to apply results averaged over resonable interval of time in which case it cannot be expected to be exactly correct at every given time.

Application of Reciprocity Theorem

1) Equality of Directional Patterns
2) Equality of Directivities
3) Equality of Effective Lengths
4) Equality of Antenna Impedances.

1) Equality of Directional Patterns



Directional patterns of transmitting & Receiving antennas are identical if all media are linear, isotropic, passive and reciprocity theorem holds good.

Proof - ~~Lev~~ Let antenna 1 is transmitting and 2 is Receiving. Pattern may be either field pattern or power pattern which is proportional to square of field pattern.

Consider field pattern, keep transmitting antenna 1 ie test antenna at centre of observation circle, receiving antenna 2 ie exploring is moved along surface.

Now, if voltage (E) applied to antenna 1 and resulting current (I) at terminals of antenna 2 is measured which will be indication of electric field at location of antenna 2.

If process is reversed ie same voltage (E) applied at 2 (which transmits) and resulting current (I) is measured at test antenna 1 (which receives). This time receiving pattern of test antenna 2 is obtained while previously to that antenna 1.

Thus, it is proved that radiation pattern of test antenna no. 1 (transmitting), observed by moving receiving antenna 2 is identical with radiation pattern obtained when antenna 2 is transmitting and antenna 1 is receiving ie when process is reversed.

· quality of Directivities – $D = 4\pi / \iint f(\theta, \phi)\, d\Omega$

Directivities will be same, whether it is calculated from antenna's transmitting pattern or receiving pattern.

3) Equality of Effective lengths –

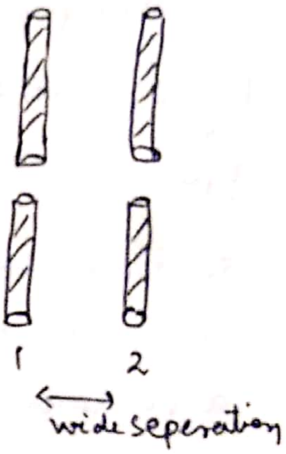
Maximum effective aperture of an antenna is

$$(A_e)_{max} = \left(\frac{\lambda^2}{4\pi}\right) D$$

$\lambda \rightarrow$ wavelength
$D \rightarrow$ Directivity

Value of max. effective aperture is same for an antenna whether its transmitting or receiving.

4) Equality of Antenna Impedance –



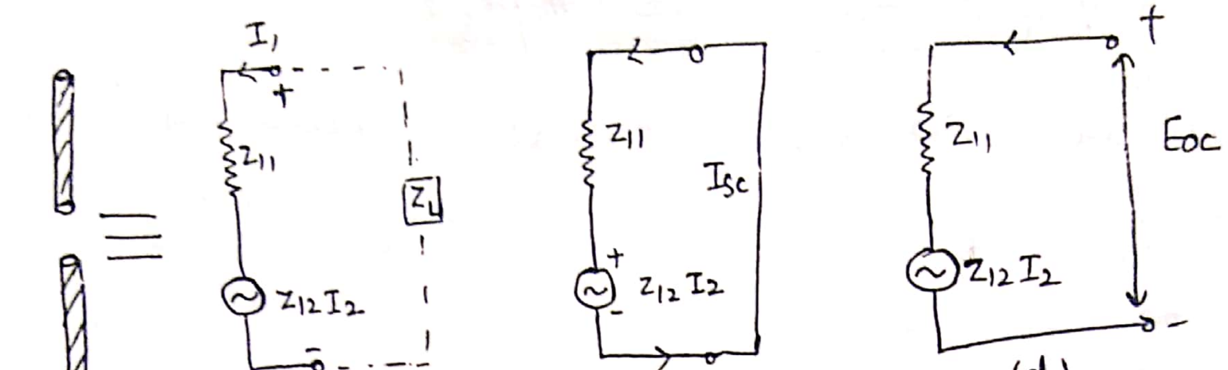
While transmitting, one point of antenna length is excited whereas during reception the entire length of antenna is excited. Hence current distribution during transmission and reception are not same. Receiving antenna is excited depends on direction of incoming wave.

If no. 2 is quite away from 1, mutual impedance b/n two is neglected while no. 1 antenna is transmitting and thus, self impedance of no 1 is obtained from –

$$E_1 = Z_{11} I_1 + Z_{12} I_2$$
$$= Z_{11} I_1 + 0\, I_2 \qquad \boxed{Z_{12} = 0}$$

$Z_{11} = \frac{E_1}{I_1}$ – Self Impedance of antenna 1

During receiving mutual Impedance (Z_{12}) b/n two antenna which provides coupling. If equivalent antenna 1 drawn under load (Z_L) then $Z_{12} I_2$ acts as voltage generator (Source)



(a) (b) (c) (d)

a) - Receiving antenna 1 b) - antenna 1 under loadd condⁿ.

c) - Receiving antenna 1 under open circuited

d) Receiving antenna no. under short circuited

Now Assume no. 2 antenna away from no. 1 so change in Z_L does not causes any change in current I_2 of voltage source $Z_{12} I_2$, then voltage source act as ideal zero impedance, constant voltage generator fig (b).

fig (c) and (d) are circuit of receiving antenna under open & short circuited conditions.

In such condⁿ, receiving impedance and transmitting impedan-ce are equal as antenna 1 have terminal behaviour of a voltage generator with internal impedance Z_{11}

$$E = Z_{11} I_1 + Z_{12} I_2$$

$$E_{oc} = Z_{11} 0 + Z_{12} I_2 \quad \because \text{ open acted cond}^n\ I_1 = 0$$

$$\boxed{E_{oc} = Z_{12} I_2}$$

$$E = Z_{11} I_1 + Z_{12} I_2$$

$$0 = Z_{11} I_{sc} + Z_{12} I_2 \quad \because \text{ short circuited cond}^n\ E = 0,\ I_1 \equiv I_{sc}$$

$$\boxed{I_{sc} = \frac{-Z_{12} I_2}{Z_{11}}}$$

Let us assume within linear and isotropic medium, there exist two sets of sources $J_1 M_1$ and $J_2 M_2$ which are allowed to radiate simultaneously or individually inside same medium at same frequency, produce fields $E_1 H_1$ and E_2, H_2. Sources and fields satisfy

$$-\nabla\cdot(E_1 \times H_2 - E_2 \times H_1) = E_1 \cdot J_2 + H_2\cdot M_1 - E_2\cdot J_1 - H\cdot M_2$$

Called as Lorentz Reciprocity Theorem in differential form.

Taking volume Integral of above, use divergence theorem on left side

$$-\oint\!\!\oint (E_1 \times H_2 - E_2 \times H_1)\cdot ds' = \iiint_V (E_1\cdot J_2 + H_2\cdot M_1 - E_2\cdot J_1 - H_1\cdot M_2)\,dv'$$

Lorentz Reciprocity theorem in integral form.

Duality Theorem

When two Equations that describe behaviour of two different variables are of same mathematical form their solutions will also be identical, The variables in the two equations that occupy identical positions are known as dual quantities and solution of one can be formed by symmetric interchange of symbol to other. The concept is known as duality Theorem

Comparing Equations → $H_A = \frac{1}{\mu}(\nabla \times A)$, $\nabla \times E_A = -j\omega\mu H_A$, $\nabla \times H_A = J + j\omega\varepsilon E_A$, $\nabla^2 A + K^2 A = -\mu J$ and so on ....

They are to each other dual Eq^n and their variables dual quantity, knowing solution to one set (ie $J \neq 0, M = 0$), solution to other set ($J = 0, M \neq 0$) can be formed by proper Interchange of quantities. Tables 1 and 2 shown below for electric & magnetic sources, Duality used in manner to explain motion of magnetic charges given rise to magnetic currents, when compared to their dual quantities of moving electric charges creating electric currents.

Table 1 :- Dual Eq^n for Electric (J) and Magnetic (M) current sources.

Electric Sources $(J \neq 0, M = 0)$	Magnetic Sources $(J = 0, M \neq 0)$
$\nabla \times E_A = -j\omega\mu H_A$	$\nabla \times H_F = j\omega\epsilon E_F$
$\nabla \times H_A = J + j\omega\epsilon E_A$	$-\nabla \times E_F = M + j\omega\mu H_F$
$\nabla^2 A + k^2 A = -\mu J$	$\nabla^2 \cdot F + k^2 F = -\epsilon M$
$A = \frac{\mu}{4\pi} \iiint_V J \frac{e^{-jkR}}{R} dv'$	$F = \frac{\epsilon}{4\pi} \iiint_V M \frac{e^{-jkR}}{R} dv'$
$H_A = \frac{1}{\mu} \nabla \times A$	$E_F = -\frac{1}{\epsilon} \nabla \times F$
$E_A = -j\omega A - j\frac{1}{\omega\mu\epsilon} \nabla(\nabla \cdot A)$	$H_F = -j\omega F - j\frac{1}{\omega\mu\epsilon} \nabla(\nabla \cdot F)$

Table 2 :- Dual Quantity for Electric (J) and Magnetic (M) Current Sources.

Electric Sources $(J \neq 0, M = 0)$	Magnetic Sources $(J = 0, M \neq 0)$
E_A	H_F
H_A	$-E_F$
J	M
A	F
ϵ	μ
μ	ϵ
K	K
η	$1/\eta$
$1/\eta$	η

UNIT-2

Far field Radiation

Fields radiated by antennas of finite dimensions are spherical waves.

$$A = \hat{a}_r A_r(r,\theta,\phi) + \hat{a}_\theta A_\theta(r,\theta,\phi) + \hat{a}_\phi A_\phi(r,\theta,\phi) \quad (1)$$

Neglecting higher order terms of $\frac{1}{r^n}$ reduces to

$$A \simeq \left[\hat{a}_r \hat{A}_r(\theta,\phi) + \hat{a}_\theta \hat{A}_\theta(\theta,\phi) + \hat{a}_\phi \hat{A}_\phi(\theta,\phi)\right] \frac{e^{-jkr}}{r}, \quad r \to \infty \quad (2)$$

Now $E = \frac{1}{r}\left[-j\omega e^{-jkr}\left[\hat{a}_r(0) + \hat{a}_\theta A'_\theta(\theta,\phi) + \hat{a}_\phi A'_\phi(\theta,\phi)\right]\right\} + \frac{1}{r^2}\{\cdots\} + \cdots \quad (3)$

Similary

$H = \frac{1}{r}\left\{j\frac{\omega}{\eta} e^{-jkr}\left[\hat{a}_r(0) + \hat{a}_\theta A'_\phi(\theta,\phi) - \hat{a}_\phi A'_\theta(\theta,\phi)\right]\right\} + \frac{1}{r^2}\{\cdots\} + \cdots$

$\eta = \sqrt{\mu/\epsilon}$ is intrinsic Impedance of medium.

Far-field Region

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega A_\theta \\ E_\phi &\simeq -j\omega A_\phi \end{aligned}\right\} \Rightarrow \boxed{E_A \simeq -j\omega A}$$

$$\left.\begin{aligned} H_r &\simeq 0 \\ H_\theta &\simeq +j\frac{\omega}{\eta} A_\phi = -\frac{E_\phi}{\eta} \\ H_\phi &\simeq -j\frac{\omega}{\eta} A_\theta = +\frac{E_\theta}{\eta} \end{aligned}\right\} \Rightarrow \boxed{H_A \simeq \frac{\hat{a}_r}{\eta} \times E_A = -j\frac{\omega}{\eta} \hat{a}_r \times A}$$

Far fields due to Magnetic Source M (Potential F)

$$\left.\begin{aligned} H_r &\simeq 0 \\ H_\theta &\simeq -j\omega F_\theta \\ H_\phi &\simeq -j\omega F_\phi \end{aligned}\right\} \Rightarrow H_F \simeq -j\omega F$$

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega \eta F_\phi = \eta H_\phi \\ E_\phi &\simeq +j\omega \eta F_\theta = -\eta H_\theta \end{aligned}\right\} \Rightarrow \boxed{E_F = -\eta \hat{a_r} \times H_F = j\omega \eta \hat{a_r} \times F}$$

The corresponding far-zone E and H field components are othogonal to each other and form TEM mode fields

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