Solved Examples.

Ques: Prove that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$.

Sol: Let $a+ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$

then $a-ib = r(\cos\theta - i\sin\theta) = re^{-i\theta}$

where, $r^2 = a^2+b^2, \quad \theta = \tan^{-1}(b/a)$

$$\therefore \log\left(\frac{a-ib}{a+ib}\right) = \log\left(\frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta+i\sin\theta)}\right) = \log\left(\frac{e^{-i\theta}}{e^{i\theta}}\right)$$

$$= \log e^{-2i\theta} = -2i\theta$$

$$\therefore i\log\left(\frac{a-ib}{a+ib}\right) = i\times -2i\theta = \underline{2\theta} = 2\tan^{-1}(b/a)$$

$$= \tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{b}{a}\right)$$

$$= \tan^{-1}\left[\frac{\frac{b}{a}+\frac{b}{a}}{1-\frac{b}{a}\cdot\frac{b}{a}}\right]$$

$$= \tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$$

$$\therefore \tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan\left[\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)\right]$$

$$= \frac{2ab}{a^2-b^2}$$

For:

$$\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\tan\theta}{a^2-b^2}$$

Ans.

(b) $\sin\left(i\log\left(\frac{a-b}{a+b}\right)\right) = \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$= \frac{2(b/a)}{1+(b/a)^2}$

$= \frac{2ab}{a^2+b^2}$ Ans.

(c) $\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{a^2-b^2}{a^2+b^2}$

LHS. $\cos[i(-2\theta i)] = \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$= \frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{a}{b}\right)^2} = \frac{a^2-b^2}{a^2+b^2}$

Ques!. Show that $i\log\left(\frac{x-i}{x+i}\right) = \pi - 2\tan^{-1}(x)$.

Sol:. Here

$\log\left(\frac{x-i}{x+i}\right) = \log(x-i) - \log(x+i)$

$= \frac{1}{2}\log\cancel{(1+x^2)} - i\tan^{-1}\left(\frac{1}{x}\right) - \frac{1}{2}\log\cancel{(1+x^2)}$

$- i\tan^{-1}\left(\frac{1}{x}\right)$

$= -2i\tan^{-1}\left(\frac{1}{x}\right)$

$\therefore i\log\left(\frac{x-i}{x+i}\right) = i\left(-2i\tan^{-1}\left(\frac{1}{x}\right)\right) = 2\tan^{-1}\left(\frac{1}{x}\right)$

$= 2\left[\frac{\pi}{2} - \cot^{-1}\left(\frac{1}{x}\right)\right] = \pi - 2\cot^{-1}\left(\frac{1}{x}\right)$

$= \pi - 2\tan^{-1}(x).$

Q: Prove that $\log\left(\frac{1}{1-e^{i\theta}}\right) = \log\left(\frac{1}{2}\operatorname{cosec}(\theta/2)\right) + i\,\cancel{\tan}\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$

Sol: Here

$$1-e^{i\theta} = 1-(\cos\theta + i\sin\theta) = r(\cos\phi - i\sin\phi) \text{ (let)}$$

$$= 1-\cos\theta - i\sin\theta = r(\cos\phi - i\sin\phi) = re^{-i\phi}$$

Where, $r = \left[(1-\cos\theta)^2 + (-\sin\theta)^2\right]^{1/2}$

$$= \left[1+\cos^2\theta - 2\cos\theta + \sin^2\theta\right]^{1/2} = \left[2(1-\cos\theta)\right]^{1/2}$$

$$= \left[4\sin^2\frac{\theta}{2}\right]^{1/2} = 2\sin\frac{\theta}{2}$$

$$\tan\phi = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{2\sin^2\theta/2} = \cot\left(\frac{\theta}{2}\right)$$

$$\tan\phi = \tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$\therefore \phi = \left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$\therefore \log\left(\frac{1}{1-e^{\theta i}}\right) = -\log(1-e^{i\theta}) = -\log(r\cdot e^{-i\phi})$$

$$= -\left[\log r + \log e^{-i\phi}\right]$$

$$= -\left[\log r - i\phi\right] = -\log r + i\phi$$

$$= \log\left(\frac{1}{r}\right) + \phi i$$

$$= \log\left(\frac{1}{2\sin\theta/2}\right) + i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$= \log\left(\frac{1}{2}\cdot\operatorname{cosec}\frac{\theta}{2}\right) + i\left(\frac{\pi}{2}-\frac{\theta}{2}\right).$$

Ques: Prove that $\log(1+\cos 2\theta + i\sin 2\theta)$
$$= \log(2\cos\theta) + i\theta.$$

Sol: LHS. $\log(1+\cos 2\theta + i\sin 2\theta)$

$$= \log(2\cos^2\theta + 2i\sin\theta\cdot\cos\theta)$$

$$= \log[2\cos\theta(\cos\theta + i\sin\theta)]$$

$$= \log(2\cos 2\theta) + \log(\cos\theta + i\sin\theta)$$

$$= \log(2\cos\theta) + \log e^{i\theta}$$

$$= \log(2\cos\theta) + i\theta\log e.$$

$$= \log(2\cos\theta) + i\theta \qquad \because \log e = 1$$

$$\therefore \operatorname{Log}(1+\cos 2\theta + i\sin 2\theta) = 2n\pi i + \log(2\cos\theta) + i\theta.$$

— **Ans.**

Ques: Evaluate $\operatorname{Log}(1+i)$.

Sol: Here

$$\operatorname{Log}(1+i) = 2n\pi i + \log(1+i)$$

$$= 2n\pi i + \frac{1}{2}\log(1+1) + i\tan^{-1}\left(\frac{1}{1}\right)$$

$$= 2n\pi i + \log\sqrt{2} + i\frac{\pi}{4} \qquad \text{Ans.}$$

rcise (b) To prove that

(a) $\log(1+e^{i\theta}) = \log\left(2\cos\frac{\theta}{2}\right) + i\frac{\theta}{2}, \; -\pi < \theta < \pi.$

(b) $\log\left(\frac{1}{1+e^{i\theta}}\right) = \log\left(\frac{1}{2}\sec\frac{\theta}{2}\right) - i\frac{\theta}{2}$

Ques: Separate into real and imaginary part of $\text{Log}(4+3i)$.

Sol: Let

$$4+3i = r(\cos\theta + i\sin\theta) \quad \text{——(1)}$$

Equating real and imaginary parts

$$r\cos\theta = 4, \qquad r\sin\theta = 3$$

Squaring and adding $r^2 = 16+9 = 25 \quad \therefore r = 5$

and $\tan\theta = \left(\frac{3}{4}\right) \quad \therefore \theta = \tan^{-1}\left(\frac{3}{4}\right)$.

$$\therefore \text{Log}(4+3i) = \text{Log}(r(\cos\theta + i\sin\theta)) = \text{Log}(re^{i\theta})$$
$$= 2n\pi i + \log(re^{i\theta})$$
$$= 2n\pi i + \log r + \log(e^{\theta i})$$
$$= 2n\pi i + \log r + \theta i \cdot \log e$$
$$= 2n\pi i + \log r + \theta i$$
$$= \log 5 + 2n\pi i + i\tan^{-1}\left(\frac{3}{4}\right)$$

$$\therefore \text{Re}[\text{Log}(4+3i)] = \log 5$$
$$\text{Im}[\text{Log}(4+3i)] = 2n\pi + \tan^{-1}\left(\frac{3}{4}\right)$$

Ques:. Prove that $\log(1+re^{i\theta}) = \frac{1}{2}\log(1+2r\cos\theta + r^2) + i\tan^{-1}\left(\frac{r\sin\theta}{1+r\cos\theta}\right)$.

Deduce that $\log(1+\cos\theta + i\sin\theta) = \log\left(2\cos\frac{\theta}{2}\right) + i\frac{\theta}{2}$.

Sol:. We have

$$\log(1+re^{i\theta}) = \log(1+r(\cos\theta + i\sin\theta))$$
$$= \log[(1+r\cos\theta) + i(r\sin\theta)]$$
$$= \frac{1}{2}\log((1+\cos\theta\cdot r)^2 + (r\sin\theta)^2] + i\tan^{-1}\left(\frac{r\sin\theta}{1+\cos\theta}\right)$$
$$= \frac{1}{2}\log[1+r^2\cos^2\theta + 2r\cos\theta + r^2\sin^2\theta] + i\tan^{-1}\left(\frac{r\sin\theta}{1+\cos\theta}\right)$$
$$= \frac{1}{2}\log[1+2r\cos\theta + r^2] + i\tan^{-1}\left(\frac{r\sin\theta}{1+\cos\theta}\right) \quad \text{—— (1)}$$

$$\log(1+\cos\theta + i\sin\theta) = \log[1+e^{i\theta}]$$

Putting $r=1$ in (1), we get

$$\log(1+\cos\theta + i\sin\theta) = \frac{1}{2}\log(1+2\cos\theta+1] + i\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$
$$= \frac{1}{2}\log[2(1+\cos\theta)] + i\tan^{-1}\left[\frac{\cancel{2}\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{\cancel{2}\cos^2\theta/2}\right]$$
$$= \frac{1}{2}\cdot\log\left[4\cdot\cos^2\frac{\theta}{2}\right] + i\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$
$$= \frac{1}{2}\cdot 2\log\left[2\cos\frac{\theta}{2}\right] + i(\theta/2)$$
$$= \log\left(2\cos\frac{\theta}{2}\right) + i\left(\frac{\theta}{2}\right).$$

Q: Find the real and imaginary parts of $\text{Log}[(1+i)\text{Log}\, i]$

Sol: We have

$$\text{Log}\, i = \log 1 + i\,\pi/2 = i\pi/2$$

Now $$(1+i)\,\text{Log}(i) = \frac{1}{2}(1+i)\pi i = (-1+i)\pi/2$$

Therefore,

$$\text{Log}[(1+i)\text{Log}(i)] = \text{Log}\left(-\frac{\pi}{2} + i\frac{\pi}{2}\right)$$

$$= \log\left[\left(-\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2}\right)^2\right]^{1/2} + i\tan^{-1}\left(\frac{\pi/2}{-\pi/2}\right)$$

$$= \log\left(\frac{\pi}{\sqrt{2}}\right) + \frac{3\pi}{4}i$$

Ques:- Find the general and the principal values of

(i) $\log(1+\sqrt{3}i)$ (ii) $\log(1-\sqrt{3}i)$ (iii) $\log(-1)$.

Sol:- Solution:-

$z = 1+\sqrt{3}i$

$|z| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$

$\arg(z) = \theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\tan \pi/3)$

$= \pi/3$

Therefore

$\log(1+\sqrt{3}i) = \log|z| + i\left(\frac{\pi}{3}+2n\pi\right)$

n is any integer

and $\text{Log}(1+\sqrt{3}i) = \log 2 + i\pi/3$