Cayley Hamilton Theorem

Every square matrix satisfies its characteristic equation.

i.e., if the characteristic equation of the nth order square matrix A is

$$|A-\lambda I| = (-1)^n\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \dots + k_n = 0$$

then $(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \dots + k_n I = 0$ — ①

Note 1 :- $A^{-1} = -\frac{1}{k_n}\left[(-1)^n A^{n-1} + k_1 A^{n-2} + \dots + k_{n-1} I\right]$

Thus Cayley Hamilton theorem gives another method for computing the inverse of a matrix. Since this method expresses the inverse of a matrix of order n in terms of $(n-1)$ powers of A, it is the most suitable for computing inverses of large matrices.

Note 2 :- If m be a positive integer such that $m>n$, then multiplying ① by A^{m-n}, we get

$$(-1)^n A^m + k_1 A^{m-1} + k_2 A^{m-2} + \dots + k_{n-1} A^{m-n+1} + k_n A^{m-n} = 0$$

Showing that any positive integral power A^m $(m>n)$ of A is linearly expressible in terms of those of lower degree.

Problem 1:- Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Hence compute A^{-1}

Solution:- The characteristic equation of A is

$$|A - \lambda I| = 0 \quad \text{i.e.,} \quad \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \quad \text{(on simplification)}$$

To verify Cayley-Hamilton theorem, we have to show that

$$A^3 - 6A^2 + 9A - 4I = 0 \quad \text{—— (1)}$$

Now $A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$$A^3 = A^2 \times A = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+ 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

This verifies Cayley-Hamilton theorem.

Now, multiplying both sides of (1) by A^{-1}, we have

$$A^2-6A+9I-4A^{-1}=0$$

$$\Rightarrow 4A^{-1}=A^2-6A+9I$$

$$\Rightarrow 4A^{-1}=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-6\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}+9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}.$$

Problem2:- Using Cayley-Hamilton theorem, find A^8 if $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$.

Solution:- The characteristic equation of A is

$|A-\lambda I|=0$ i.e., $\begin{vmatrix}1-\lambda & 2\\ 2 & -1-\lambda\end{vmatrix}=0$

$$-(1-\lambda^2)-4=0$$

$$\lambda^2-5=0 \quad -(1)$$

By Cayley-Hamilton theorem. A satisfies its characteristic equation (1)

$$\therefore \quad A^2-5I=0$$

$$A^2=5I$$

$$\Rightarrow (A^2)^4=(5I)^4$$

$$\Rightarrow A^8=5^4I^4$$

$$\Rightarrow A^8=625I$$

Problem3:- Find the characteristic equation of the matrix $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$ and, hence, find the matrix represented by $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$.

Solution:- The characteristic equation of A is

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley-Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{———} \quad (1)$$

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$

$= A^2 + A + I$ (Using (1))

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Problem 4:- Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ and use it to find A^{-1} & A^4.

Solution:- The characteristic equation of A is

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

This means, we are to verify

$$A^3 - 5A^2 + 9A - I = 0 \quad \text{———} \quad (1)$$

Now $A^2 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$

$$A^3-5A^2+9A-I = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$+9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence the theorem is verified.

Now, multiplying (1) throughout by, A^{-1}, we get

$$A^2-5A+9I-A^{-1}=O$$

$$\therefore \; A^{-1} = A^2-5A+9I = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Next, to find A^4 we multiply (1) by A & get

$$A^4-5A^3+9A^2-A=O$$

$$\therefore \; A^4 = 5A^3-9A^2+A = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{bmatrix}$$

Exercise

1) Verify Cayley-Hamilton theorem for the matrix and find A^{-1}, when A is

(i) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ Sol^n $\begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & -6 \\ 3 & 4 & -2 \end{bmatrix}$ Sol^n $\frac{1}{67}\begin{bmatrix} -22 & -16 & 37 \\ 18 & 7 & -12 \\ 3 & -10 & -2 \end{bmatrix}$

(iii) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ Sol^n $\frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ Sol^n $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

(v) $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ Sol^n $\frac{1}{50}\begin{bmatrix} -8 & 20 & -7 \\ -40 & 50 & -10 \\ 22 & -30 & 13 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$ Sol^n $\frac{1}{7}\begin{bmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{bmatrix}$

2) Using Cayley-Hamilton theorem, find A^6 if $A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}$. Sol^n $729I$

3) Show that the matrix $A = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}$ satisfies Cayley Hamilton theorem.

4) If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

Sol^n $-4A + 5I$