Topic

Cauchy Euler's Homogeneous Linear differential Eqn with variable coefficient

Standard form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots +$$

$$a_{n-1} x^1 \frac{dy}{dx} + a_n y = f(x)$$

Variable Coefficents

$$Eg \rightarrow x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + 4y = 2x^2$$

$\boxed{\text{Put } x = e^t} \rightarrow$ Constant coefficient

	Given	After changing
Dependent variable	y	y
Independent variable	x	t

- Steps

$$x = e^t$$
$$\log x = \log e^t$$
$$\log x = t \log e$$
$$\boxed{\log x = t}$$

Cauchy form with variable coefficient

$\Downarrow$

Changing of Independent variable (Put $x = e^t$)

$\Downarrow$

Reducing to LDE with constant coefficients

$\Downarrow$

Solving LDE with constant coefficient   C.F. = ?
P.I. = ?

$\Downarrow$

Put $t = \log x$

Imp results

$x\dfrac{dy}{dx} = Dy$

$x^2\dfrac{d^2y}{dx^2} = D(D-1)y$ (Here; $D = \dfrac{d}{dt}$

$x^3\dfrac{d^3y}{dx^3} = D(D-1)(D-2)y$

& so on

Problem 1

Q Solve the differential equation:

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$$

Sol→ The given differential equation is

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2 \quad - (1)$$

It is cauchy Homogeneous ~~Eut~~ Linear Differential equation

So, Put $\boxed{x = e^t} \Rightarrow \boxed{t = \log x}$

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$$x\frac{dy}{dx} = Dy$$

$$\boxed{\text{Here; } D = d/dt}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Put these values in eq (1), we get

$$D(D-1)y - 3Dy + 4y = 2(e^t)^2$$

$$(D^2 - D - 3D + 4)y = 2e^{2t}$$

$$(D^2 - 4D + 4)y = 2e^{2t} \quad -(2)$$

which is a Linear Differential Equation with constant coefficient.

From above symbolic form (2); we get

$$f(D) = D^2 - 4D + 4$$

Now auxilliary equation is written as

Put $f(m) = 0$

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$\boxed{m = 2, 2}$$

Since; values of m are real & repeated

$$\boxed{C.F. = c_1 e^{2t} + c_2 t e^{2t}}$$

independent variable

Good Write

$\Rightarrow$ Now; $P.I = \frac{1}{f(D)} \cdot R.H.S$

$$= \frac{1}{D^2 - 4D + 4} \cdot (2e^{2t})$$

$\boxed{\text{Put } D = 2}$

$$= 2\left[\frac{1}{4-8+4}\right]e^{2t}$$

$$= 2\left[\frac{1}{0}\right]e^{2t} \quad \text{(Case of failure)}$$

$$= 2t\left[\frac{1}{2D-4} \cdot e^{2t}\right]$$

$$= t\left(\frac{1}{D-2}\right) \cdot e^{2t}$$

$\boxed{\text{Put } D = 2}$

$$= t\left[\frac{1}{0} \cdot e^{2t}\right] \quad \text{(Case of failure)}$$

$$\Rightarrow 2t^2\left(\frac{1}{2}\right) \cdot e^{2t}$$

$$\Rightarrow t^2 e^{2t}$$

$$\Rightarrow \boxed{P.I. = t^2 e^{2t}}$$

Now complete solution of eq (2) is given as

$$y = C.F. + P.I.$$

$$y = c_1 e^{2t} + c_2 t e^{2t} + t^2 e^{2t} \quad -(3)$$

Put $t = \log x$ in eq (3), we get

$$\boxed{y = c_1 x^2 + c_2 x^2 \log x + x^2 (\log x)^2} \text{ Ans}$$

MM Imp

Q Solve the differential equation:

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$$

Sol → The given differential equation is ⇒

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right) \quad -(1)$$

which is cauchy homogeneous linear Differential equation.

Put $\boxed{x = e^t} \Rightarrow \boxed{t = \log x}$

$$\boxed{\begin{aligned} x\frac{dy}{dx} &= Dy \\ x^2\frac{d^2y}{dx^2} &= D(D-1)y \\ x^3\frac{d^3y}{dx^3} &= D(D-1)(D-2)y \end{aligned}}$$

$D = \frac{d}{dt}$

Put these value in eqn (1), we get

$$[D(D-1)(D-2)y] + 2[D(D-1)y] + 2y = 10\left(e^t + \frac{1}{e^t}\right)$$

$$[D^3 - 2D^2 - D^2 + 2D + 2D^2 - 2D + 2]y = 10(e^t + e^{-t})$$

$$(D^3 - D^2 + 2)y = 10(e^t + e^{-t}) \quad -(2)$$

which is the required linear differential equation with constant coefficient.

From symbol form ; we get

$$f(D) = D^3 - D^2 + 2$$

for Auxiliary equation, put

$$f(m) = 0$$

$$m^3 - m^2 + 2 = 0 \quad -(3)$$

By (Hit & Trial method.)

put $m = 1 \Rightarrow 1 - 1 + 2 \neq 0$

put $m = -1 \Rightarrow -1 - 1 + 2 = 0$ (satisfy)

By synthetic division

-1	1	-1	0	2
		-1	2	-2
	1	-2	2	0

So, now eq (3) can be written as

$$(m+1)(m^2 - 2m + 2) = 0$$

$[m = -1]$ | $(m^2 - 2m + 2) = 0$

$[m = 1 \pm i]$

$$\boxed{CF = c_1 e^{-t} + e^{t}(c_2 \cos t + c_3 \sin t)}$$

→ Now, PI is written as

$$PI = \frac{1}{f(D)} RHS$$

$$= \frac{1}{D^3 - D^2 + 2}\left(10\left(e^{t} + e^{-t}\right)\right)$$

$$\Rightarrow 10\left(\frac{1}{1-1+2}\right)e^{t} + 10\left(\frac{1}{-1-1+2}\right)e^{-t}$$

$$\Rightarrow \frac{10}{2}e^{t} + 10\left(\left(\frac{1}{0}\right).\bar{e}^{t}\right)$$

$\rightarrow$ case of failure

$$\Rightarrow 5e^{t} + 10t\left(\frac{1}{3D^2-2D}.\bar{e}^{t}\right)$$

$$\Rightarrow 5e^{t} + 10t\left(\frac{1}{3+2}.e^{-t}\right)$$

$$\Rightarrow \boxed{PI = 5e^{t} + 2te^{-t}}$$

$\Rightarrow$ Now the complete solution can be written as:

$y = CF + PI$

$\Rightarrow y = c_1 e^{-t} + e^{t}(c_2 \cos t + c_3 \sin t) + 5e^{t} + 2te^{-t}$ — (4)

Put $t = \log x$ in eq (4), we get

$$y = c_1 e^{-\log x} + e^{\log x}\left(c_2 \cos(\log x) + c_3 \sin(\log x)\right) + 5e^{\log x} + 2(\log x)\,e^{-\log x}$$

Ans $\rightarrow$ $y = \frac{c_1}{x} + x\left[c_2 \cos(\log x) + c_3 \sin(\log x)\right] + 5x + \frac{2}{x}\log x$

Power Series Method

Power series: An indefinite series in powers of $(x-x_0)$ of the form

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad (1)$$

is called power series, where $a_0, a_1, \dots a_n$ are constants & x is variable.

The constant x_0 is called the centre of the power series.

Standard power series:

When the centre of the power series (1) is at the origin i.e. when $x_0 = 0$; then the series

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = \sum_{n=0}^{\infty} a_n x^n$$

is called standard power series.

General solution of LDE of second order.

General form of LDE of II order is

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

$\boxed{y = au + bv}$ is the general solution of given eqn.

Series solution when $x=0$ is an ordinary point (working procedure):

Consider the LDE of second order,

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = 0 \quad \text{--- (1)}$$

(1) Let the soln of given eqn may be taken as
$y = a_0 + a_1 x + a_2 x^2 + \_\_\_ = \sum_{n=0}^{\infty} a_n x^n$

where $a_0, a_1, a_2, \ldots a_n$ are constant coefficient which are to be determined.

(2) Differentiate w.r.t. x to get $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$.

(3) Substituting these values in the given eqn

(4) Equating the coefficient of variable powers of x to zero to get recurrence relations of coefficient, which on solving give $a_0, a_1, a_2, \_\_\_$.

(5) Substituting these $a_0, a_1, a_2, \_\_\_$ in the given differential eqn to get the required soln.

Q. Find the power series solution about $x=0$:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right) + 2y = 0$$

Sol→ Given that:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right) + 2y = 0 \quad -(1)$$

Let the solution of eqn (1) be

$$y = \sum_{n=0}^{\infty} a_n x^n \quad -(2)$$

Differentiate equation (2) wrt x

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} a_n n x^{n-1} \quad -(3)$$

again;

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} \quad -(4)$$

Substituting these values in equation (1)

$$\Rightarrow (1-x^2)\left\{\sum_{n=2}^{\infty} a_n (n)(n-1)(x^{n-2})\right\} - 2x\left\{\sum_{n=1}^{\infty} a_n (n) x^{n-1}\right\} +$$

$$2\left\{\sum_{n=0}^{\infty} a_n x^n\right\} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n (n(n-1)) x^{n-2} - \sum_{n=0}^{\infty} a_n (n(n-1)) x^n - 2\sum_{n=0}^{\infty} a_n n x^n +$$

$$2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n \{n(n-1) + 2n - 2\} x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n (n^2 + n - 2) x^n = 0 \quad -(5)$$

Equating to 0 the coefficient of x^n.

$$a_{n+2}(n+2)(n+1) - a_n(n^2+n-2) = 0$$

$$a_{n+2} = \frac{(n^2+n-2)}{(n+2)(n+1)} a_n$$

$$a_{n+2} = \frac{\cancel{(n+2)}(n-1)}{\cancel{(n+2)}(n+1)} a_n$$

$$\boxed{a_{n+2} = \frac{(n-1)}{(n+1)} a_n} \quad (6)$$

→ Now put

$n=0$ in eqn (6) ; $\boxed{a_2 = -a_0}$

$n=1$; $\boxed{a_3 = 0}$

$n=2$; $a_4 = \frac{(2-1)}{(2+1)} a_2 = \frac{1}{3} a_2 \Rightarrow \boxed{a_4 = \frac{-a_0}{3}}$

$n=3$; $a_5 = \frac{(3-1)}{(3+1)} a_3 = \frac{2}{4} a_3 \Rightarrow a_5 = \frac{2}{4} \times 0 \Rightarrow \boxed{a_5 = 0}$

$n=4$; $a_6 = \frac{(4-1)}{(4+1)} a_4 = \frac{3}{5}\left(\frac{-a_0}{3}\right) \Rightarrow \boxed{a_6 = -\frac{a_0}{5}}$

$n=5$; $a_7 = \frac{(5-1)}{(5+1)} a_5 \Rightarrow \boxed{a_7 = 0}$

$n=6$; $a_8 = \frac{5}{7} a_6 = \frac{5}{7}\left(\frac{-a_0}{5}\right) \Rightarrow \boxed{a_8 = -\frac{a_0}{7}}$

⇒ Now substituting these coefficent in eq^n (2)

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$\Rightarrow \therefore y = a_0 + a_1 x + (-a_0) x^2 + \left(\frac{-a_0}{3}\right) x^4 + \left(\frac{-a_0}{5}\right) x^6 + \left(\frac{-a_0}{7}\right) x^8 + \ldots$$

Ans ⇒ $\boxed{y = a_1 x + a_0 \left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} + \ldots\right)}$

Legendre Eqn

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

at $x=0$; $P(0) \neq 0$

then $y = \sum_{r=0}^{\infty} a_r x^r$ is the soln of this eqn.

Q. Solve this legendre eqn

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$$

Sol: $P(x) = 1-x^2$; $Q(x) = x$; $R(x) = 4$

at $x = 0$; $P(x) \neq 0$

$x = 0$ (ordinary point)

Let solution of eqn $y = \sum_{r=0}^{r} a_r x^r$

$$\frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r x^{r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r r(r-1) x^{(r-2)}$$

$$\Rightarrow (1-x^2)\sum_{r=2}^{\infty} a_r(r(r-1))x^{r-2} - x\sum_{r=1}^{\infty} a_r r x^{r-1}$$

$$+ 4\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\Rightarrow \sum_{r=2}^{\infty} a_r(r(r-1))x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1)x^r -$$

$$\sum_{r=0}^{\infty} a_r r x^r + 4\sum_{r=0}^{\infty} a_r x^r = 0$$

Good Write

$$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r [r(r-1)+r-4] x^r = 0$$

$$\Rightarrow \sum_{r=2}^{\infty} a_r (r(r-1)) x^{r-2} - \sum_{r=0}^{\infty} a_r (r^2-4) x^r = 0$$

(Put $r = r+2$)

$$\Rightarrow \sum_{r=0}^{\infty} a_{r+2}(r+2)(r+1)x^r - \sum_{r=0}^{\infty} a_r(r^2-4)x^r = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} [a_{r+2}(r+2)(r+1) - a_r(r^2-4)] x^r = 0$$

$$a_{r+2} = \frac{(r-2)\cancel{(r+2)}}{(r+1)\cancel{(r+2)}} a_r$$

$$\boxed{a_{r+2} = \left(\frac{r-2}{r+1}\right) a_r}$$

Put $r = 0$

$$\boxed{a_2 = -2a_0}$$

$r = 1$

$$\boxed{a_3 = -\frac{1}{2} a_1}$$

$r = 2$

$$\boxed{a_4 = 0}$$

$r = 3$

$$\boxed{a_5 = \frac{1}{4} a_3 = -\frac{1}{8} a_1}$$

$r = 4$

$$\boxed{a_6 = \frac{2}{5} a_4 = 0}$$

$r = 5$

$$\boxed{a_7 = \frac{3}{6} a_5 = -\frac{1}{48} a_1}$$

$r = 6$

$$\boxed{a_8 = 0}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y = a_0 + a_1 x - 2a_0 x^2 - \frac{1}{2} a_1 x^3 + 0 - \frac{1}{8} a_1 x^5$$

Ans $\rightarrow$ $y = a_0(1 - 2x^2) + a_1\left(x - \frac{1}{2}x^3 - \frac{1}{8}x^5 + \cdots\right)$

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Solve the legendre eqn.

Q. $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

Sol → at $x=0$; $P(x) \neq 0$

i.e. $x=0$ (ordinary point)

i.e. soln of eqn $\left(y = \sum_{r=0}^{r} a_r x^r\right)$

$\frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r x^{r-1}$; $\frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r (r(r-1)) x^{r-2}$

$(1-x^2)\sum_{r=2}^{\infty} a_r(r(r-1))x^{r-2} - 2x\sum_{r=1}^{\infty} a_r r(x^{r-1}) + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$

$\sum_{r=2}^{\infty} a_r(r(r-1))x^{r-2} - \sum_{r=0}^{\infty} a_r[r(r-1)+2r-n(n+1)]x^r = 0$

(Put $r = r+2$)

$\sum_{r=0}^{\infty} a_{r+2}(r+2)(r+1)x^r - \sum_{r=0} a_r[r^2+r-n(n+1)]x^r = 0$

$\sum_{r=0}^{\infty}\left[a_{r+2}(r+2)(r+1) - a_r[r(r+1)-n(n+1)]\right]x^r = 0$

$a_{r+2}(r+2)(r+1) - a_r(r(r+1)-n(n+1)) = 0$

$$a_{r+2} = \left(\frac{r(r+1)-(n(n+1))}{(r+2)(r+1)}\right)a_r$$

Put $r=0$

$a_2 = \frac{-n(n+1)}{2}a_0$

$r=1$

$a_3 = \frac{2-n(n+1)}{6}a_1$

$= \frac{-(n-1)(n+2)}{6}a_1$

$r=2$

$a_4 = \frac{6-n(n+1)}{12}a_2$

$= \frac{-(6-n(n+1))n(n+1)}{24}a_0$

$a_4 = \frac{(n+2)(n+3)n(n+1)}{6}a_0$

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$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$y = a_0\left(1 + \frac{a_2}{a_0}x^2 + \dots\right) + a_1\left(x + \frac{a_3}{a_1}x^3 + \dots\right)$$

Ans $$y = a_0\left(1 - \frac{n(n+1)}{2}x^2 + \dots\right) + a_1\left(x - \frac{(n-1)(n+2)}{6}x^3 \dots\right)$$

Topic : Legendre's Polynomial

Rodrigues Formula

$$\boxed{P_n(X) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n}$$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1) \rightarrow \boxed{x^2 = \frac{1}{3}(2P_2(x) + P_0(x))}$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$\boxed{x^3 = \frac{1}{5}(2P_3(x) + 3P_1(x))}$$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$\boxed{x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)}$$

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Q. Express $x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre's Polynomial.

(By using Rodrigues formula)

Sol. $x^4 + 3x^3 - x^2 + 5x - 2 =$

$$\frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x) + \frac{3}{5}\left(2P_3(x) + 3P_1(x)\right)$$

$$- \frac{1}{3}\left(2P_2(x) + P_0(x)\right) + 5P_1(x) - 2P_0(x)$$

$$\Rightarrow \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) + \left(\frac{4}{7} - \frac{2}{3}\right) P_2(x) +$$

$$14 P_1(x) + \left(\frac{1}{5} - \frac{1}{3} - 2\right) P_0(x)$$

Ans. $\frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) - \frac{32}{15} P_0(x)$

Bessels

Lec → 1

Bessel's Eqn & its ~~so~~ Solution

Step-1 $x^2y'' + xy' + (x^2 - n^2)y = 0$ — (1)

Step-2

Assuming

$$y(x) = \sum_{r=0}^{\infty} a_r x^{m+r}$$

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + a_4 x^{m+4} + \dots$$

Step 3 → Substituting ~~y~~ y and its derivatives into (1)

$$a_0(m^2 - n^2)x^m + a_1[(m+1)^2 - n^2]x^{m+1} +$$

$$[a_2\{(m+2)^2 - n^2\} + a_0]x^{m+2} + \dots$$

Step 4 → Equate lowest degree term to zero

→ $a_0(m^2 - n^2) = 0$ $\quad m = \pm n$

→ $a_1((m+1)^2 - n^2) = 0$ $\quad (a_1 = 0,\ a_3 = 0,\ a_5 = 0)$

→ $a_2 = \dfrac{-a_0}{(m+2)^2 - n^2} = \dfrac{-a_0}{(m+2+n)(m+2-n)}$

The soln is

$$Y = a_0 x^m \left[1 - \frac{x^2}{(m+2+n)(m+2-n)} + \frac{x^4}{(m+2+n)(m+2-n)(m+4-n)(m+4+n)}\right]$$

General Sol n

$$Y = AY_1 + BY_2 \quad \text{where } Y_1 = Y_{m=n}$$
$$Y_2 = Y_{m=-n}$$

$$Y_1^{m=n} = a_0 x^m \left[1 - \frac{x^2}{2.2.(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} - \cdots \right]$$

$$Y_2^{m=-n} = a_0 x^m \left[1 - \frac{x^2}{2.2.(-n+1)} + \frac{x^4}{2.4.2^2(-n+1)(-n+2)} - \cdots \right]$$

~~Bessel's fn of~~

Lec - 2

Bessel's function - first order & Half order

Learn $\boxed{a_0 = \frac{1}{2^m \Gamma(n+1)}}$

Put value of a_0 in Y_1 & Y_2 then they will be converted to $J_n(x)$ & $J_{-n}(x)$ respectively.

☆ $$\boxed{J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}}$$

Put value $-n$

$$\boxed{J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}}$$

Proof

Bessel's function of Half order

$$J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\left(\frac{3}{2}+r\right)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2r}$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma\frac{3}{2}} \left[1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 5} - \cdots\right]$$

$$= \frac{1}{\sqrt{x}\,\sqrt{2}\,\frac{1}{2}\,\Gamma\frac{1}{2}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \boxed{\sqrt{\frac{2}{\pi x}}\sin x} \quad \left(\text{As } \Gamma\tfrac{1}{2} = \sqrt{\pi}\right)$$

So, $\boxed{J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\sin x}$

Similarly: $\boxed{J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\cos x}$

Lec - 3

(A) Properties of Bessel's function

1. $$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

or $$\int_0^x x^n J_{n-1}(x)\,dx = x^n J_n(x)$$

2. $$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

3. $J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$

4. $J_n'(x) = -J_{n+1}(x) + \frac{n}{x} J_n(x)$

5. $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) + J_{n+1}(x)\right]$ [By adding 3 & 4]

6. $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$ [~~By 4 - 3~~]

Lec-4
Bessel's Eqn & fn (Numericals)

Q Express $J_{3/2}$ in terms of sine & cosine function.

Sol → $J_{1/2} = \sqrt{\frac{2}{\pi x}} \sin x$; $J_{-1/2} = \sqrt{\frac{2}{\pi x}} \cos x$

(Remember)

(Apply property 6.)

$\frac{2n}{x} J_n = J_{n-1} + J_{n+1}$

Put $n = 1/2$

$\frac{1}{x} J_{1/2} = J_{1/2 - 1} + J_{1/2 + 1} = J_{-1/2} + J_{3/2}$

$J_{3/2} = \frac{1}{x} J_{1/2} - J_{-1/2} = \sqrt{\frac{2}{\pi x}}\left[\frac{1}{x}(\sin x) - \cos x\right]$

$J_{3/2} = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$ Ans

Extra → [for $J_{-3/2}$ Put $n = -\frac{1}{2}$]

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Q. Express $J_{5/2}$ in terms of sine & cosine function.

Sol→ $J_{1/2} = \sqrt{\frac{2}{\pi x}}\sin x$

$J_{3/2} = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$ (Remember)

[Apply property (VI)]

$$\left[\frac{2n}{x}J_n = J_{n-1} + J_{n+1}\right]$$

(Put $n = 3/2$)

$$\frac{3}{x}J_{3/2} = J_{1/2} + J_{5/2}$$

$$J_{5/2} = \frac{3}{x}J_{3/2} - J_{1/2}$$

$$J_{5/2} = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\left(\frac{\sin x}{x} - \cos x\right) - \sin x\right]$$

$$J_{5/2} = \sqrt{\frac{2}{\pi x}}\left[\frac{3 - x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$$ Ans

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Q. Express J_5 in terms of J_1 & J_2.

Sol: Apply property (VI)

$$\frac{2n}{x} J_n = J_{n-1} + J_{n+1}$$

⇒ Put $n = 4$;

$$J_5 = \frac{8}{x} J_4 - J_3 \quad - (1)$$

⇒ Put $n = 3$;

$$J_4 = \frac{6}{x} J_3 - J_2 \quad - (2)$$

⇒ Put $n = 2$;

$$J_3 = \frac{4}{x} J_2 - J_1 \quad - (3)$$

⇒ Put eqn (2) in (1)

$$J_5 = \frac{8}{x}\left(\frac{6}{x} J_3 - J_2\right) - J_3$$

$$\boxed{J_5 = \frac{48}{x^2} J_3 - \frac{8}{x} J_2 - J_3} \quad - (4)$$

⇒ Put eq (3) in (4)

$$J_5 = \frac{48}{x^2}\left(\frac{4}{x} J_2 - J_1\right) - \frac{8}{x} J_2 - \left(\frac{4}{x} J_2 - J_1\right)$$

$$\boxed{J_5 = \left(\frac{192}{x^3} - \frac{12}{x}\right) J_2 + \left(1 - \frac{48}{x^2}\right) J_1}$$ Ans

— X —