Joint Probability Distributions

In general, if X and Y are two random variables, the probability distribution that defines their simultaneous behavior is called a joint probability distribution.

Note:

If X and Y are 2 discrete random variables, this distribution can be described with a joint probability **mass** function. If X and Y are continuous, this distribution can be described with a joint probability **density** function.

Two Discrete Random Variables:

If X and Y are discrete, with ranges R_X and R_Y , respectively, the joint probability mass function is,

$$p(x, y) = P(X = x \text{ and } Y = y),\ x \in R_X,\ y \in R_Y.$$

In the discrete case

The function $f(x, y)$ is a **joint probability distribution** or **probability mass function** of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y),
2. $\sum_x \sum_y f(x, y) = 1$,
3. $P(X = x, Y = y) = f(x, y)$.

Two Continuous Random Variables:

If X and Y are continuous, the joint probability density function is a function f(x,y) that produces probabilities:

$$P[(X,Y) \in A] = \iint\limits_{A} f(x,y)\,dydx$$

in the continuous case,

The function $f(x, y)$ is a **joint density function** of the continuous random variables X and Y if

1. $f(x, y) \geq 0$, for all (x, y),
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$,
3. $P[(X, Y) \in A] = \int \int_A f(x, y)\, dx\, dy$, for any region A in the xy plane.

4) $P(a \leq x \leq b, c \leq y \leq d) = \int_a^b \int_c^d f(x, y)\, dy\, dx$

Example:

Suppose we have the following joint mass function

X \ Y	-2	0	5
1	0.15	K	0.20
3	0.20	0.05	0.15

Find the value of k?

Answer:

Using

$$\sum_{x}\sum_{y} f(x,y) = 1$$

We get,

$$0.15 + 0.20 + k + 0.05 + 0.20 + 0.15 = 1$$
$$0.75 + k = 1$$
$$k = 1 - 0.75 = 0.25$$

Example:

Suppose we have the following joint density function

$$f(x,y)=\begin{cases}\dfrac{6-x-y}{8} & 0\le x\le 2 \quad , \quad 2\le y\le 4\\ 0 & O.W.\end{cases}$$

1) Prove that $f(x,y)$ is a joint probability function?

2) Calculate $P\left(X\le\frac{2}{3}, Y\le\frac{5}{2}\right)$

Answer:

1) $f(x, y) \geq 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) = \int_{2}^{4}\int_{0}^{2} \frac{6 - x - y}{8} dx dy$$

$$= \frac{1}{8}\int_{2}^{4} \left[\int_{0}^{2}(6 - x - y) dx\right] dy$$

$$= \frac{1}{8}\int_{2}^{4} \left[6x - \frac{x^2}{2} - yx\right]_{0}^{2} dy$$

$$= \frac{1}{8}\int_{2}^{4} \left[\left(6(2) - \frac{(2)^2}{2} - y(2)\right) - 0\right] dy$$

$$= \frac{1}{8}\int_{2}^{4}(10 - 2y) dy$$

$$= \frac{1}{8}\left[10y - y^2\right]_{2}^{4} = \left[\left(10(4) - (4)^2\right) - \left(10(2) - (2)^2\right)\right]$$

$$= \frac{1}{8}(40 - 16) - (20 - 4) = \frac{1}{8}(8) = 1$$

2) $P\left(x \le \frac{2}{3}, y \le \frac{5}{2}\right) = \int_0^{\frac{2}{3}}\int_2^{\frac{5}{2}}\left(\frac{6 - x - y}{8}\right)dydx$

$$\vdots$$

$$\vdots$$

$$= \frac{41}{288} = 0.142$$

The marginal distributions

The **marginal distributions** of X alone and of Y alone are

$$g(x) = \sum_y f(x,y) \quad \text{and} \quad h(y) = \sum_x f(x,y)$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

for the continuous case.

Example:

Suppose we have the following joint mass function

X \ Y	**-2**	**0**	**5**
1	0.15	0.25	0.20
3	0.20	0.05	0.15

Find the marginal distributions of X and Y?

Answer:

Sum Y / X	5	0	2-	
0.6	0.20	0.25	0.15	1
0.4	0.15	0.05	0.20	3
1	0.35	0.30	0.35	Sum

So

The marginal distribution of X

Sum X	3	1	
$f_1(x)$	0.4	0.6	

The marginal distribution of Y

Sum y	5	0	2-	
$f_1(y)$	0.35	0.30	0.35	

Example:

Suppose we have the following joint density function

$$f(x, y) = c(x + y) \quad , \quad 0 \leq x \leq 1, 0 \leq y \leq 2$$

Find the value of c ?

Find the marginal distributions of X and Y?

Answer:

1) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_{0}^{2}\int_{0}^{1} c(x+y)\,dx\,dy = 1$

$\vdots$

$\Rightarrow c = \frac{1}{3} \Rightarrow f(x,y) = \frac{1}{3}(x+y)$

2) $f(x) = \int_{y} f(x,y) = \int_{0}^{2} \frac{1}{3}(x+y)\,dy$

$\vdots$

$\Rightarrow f(x) = \frac{2}{3}(x+1)$

$f(y) = \int_{x} f(x,y) = \int_{0}^{1} \frac{1}{3}(x+y)\,dx$

$\vdots$

$\Rightarrow f(y) = \frac{1}{3}\left(y + \frac{1}{2}\right)$

conditional probability distribution

Let X and Y be two random variables, discrete or continuous. The **conditional distribution** of the random variable Y given that $X = x$ is

$$f(y|x) = \frac{f(x,y)}{g(x)}, \text{ provided } g(x) > 0.$$

Similarly, the conditional distribution of X given that $Y = y$ is

$$f(x|y) = \frac{f(x,y)}{h(y)}, \text{ provided } h(y) > 0.$$

Example:

The joint density for the random variables (X, Y), where X is the unit temperature change and Y is the proportion of spectrum shift that a certain atomic particle produces, is

$$f(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal densities $g(x)$, $h(y)$, and the conditional density $f(y|x)$.

(b) Find the probability that the spectrum shifts more than half of the total observations, given that the temperature is increased by 0.25 unit.

Solution:

(a) By definition,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x}^{1} 10xy^2\,dy$$

$$= \frac{10}{3}xy^3\Big|_{y=x}^{y=1} = \frac{10}{3}x(1-x^3),\ 0 < x < 1,$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{y} 10xy^2\,dx = 5x^2y^2\Big|_{x=0}^{x=y} = 5y^4,\ 0 < y < 1.$$

Now

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = \frac{3y^2}{1-x^3},\ 0 < x < y < 1.$$

(b) Therefore,

$$P\left(Y > \frac{1}{2} \,\middle|\, X = 0.25\right) = \int_{1/2}^{1} f(y \mid x = 0.25)\, dy = \int_{1/2}^{1} \frac{3y^2}{1 - 0.25^3}\, dy = \frac{8}{9}.$$

Statistical Independence

Let X and Y be two random variables, discrete or continuous, with joint probability distribution $f(x, y)$ and marginal distributions $g(x)$ and $h(y)$, respectively. The random variables X and Y are said to be **statistically independent** if and only if

$$f(x, y) = g(x)h(y)$$

for all (x, y) within their range.

Example:

Suppose we have the following joint distribution

$$f(x,y) = \begin{cases} 3e^{-x}e^{-3y} & , \quad x \geq 0,\ y \geq 0 \\ 0 & , \quad O.W. \end{cases}$$

Prove that X and Y are independent?

$$f(x,y) = f(x)\cdot f(y)$$

1) $$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{0}^{\infty} 3e^{-x}e^{-3y}dy = 3e^{-x}\int_{0}^{\infty} e^{-3y}dy$$

$$= 3e^{-x}\left[\frac{e^{-3y}}{-3}\right]_0^{\infty} = -e^{-x}[0-1] = e^{-x} \text{.......................}(1)$$

2) $$f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{\infty} 3e^{-x}e^{-3y}dx = 3e^{-3y}\int_{0}^{\infty} e^{-x}dx$$

$$= 3e^{-3y}\left[-e^{-x}\right]_0^{\infty} = -3e^{-3y}[0-1] = 3e^{-3y} \text{................}(2)$$

From (1) and (2) $\Rightarrow$

$$f(x,y) = f(x)\cdot f(y)$$

Notes:

if X and Y are independent, then

1) $f(x, y) = f(x) \cdot f(y)$

2) $f(x/y) = f(x)$

3) $f(y/x) = f(y)$

Example:

Suppose we have the following joint distribution

$$f(x, y) = k(8 - x - y), \quad 0 \le x \le 4, \; 1 \le y \le 3$$

Find:

1) The value of k

2) $f(x)$, $f(y)$

3) $f(y/x)$, $f(x/y)$

4) $P(x \le 3)$

5) $P(x \le 3 / y \le 2)$

Solution:

1) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_{1}^{3}\int_{0}^{4} k(8-x-y)\,dx\,dy = 1$

$$\Rightarrow k\int_{1}^{3}\left[\int_{0}^{4}(8-x-y)\,dx\right]dy = 1$$

$$\Rightarrow k\int_{1}^{3}\left[8x-\frac{x^2}{2}-xy\right]_{0}^{4} = 1$$

$$\Rightarrow k\int_{1}^{3}\left[8(4)-\frac{4^2}{2}-4y\right]dy = 1$$

$$\Rightarrow k\int_{1}^{3}(-4y+24)\,dy = 1$$

$$\Rightarrow k\left[\frac{-4y^2}{2}+24y\right]_{1}^{3} = 1$$

$$\Rightarrow k(32) = 1 \Rightarrow k = \frac{1}{32} \Rightarrow f(x,y) = \frac{1}{32}(8-x-y)$$

2) $f(x) = \int_{-\infty}^{\infty} f(x, y)dy$

$$\Rightarrow f(x) = \int_{1}^{3} \frac{1}{32}(8 - x - y)dy$$

$$\vdots$$

$$\Rightarrow f(x) = \frac{1}{32}(12 - 2x) \quad , \quad 0 \le x \le 4$$

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

$$\Rightarrow f(y) = \int_{0}^{4} \frac{1}{32}(8 - x - y)dx$$

$$\vdots$$

$$\Rightarrow f(y) = \frac{1}{32}(24 - 4y) \quad , \quad 1 \le y \le 3$$

$$3)\ f(y/x)=\frac{f(x,y)}{f(x)}=\frac{\frac{1}{32}(8-x-y)}{\frac{1}{32}(12-2x)}=\frac{(8-x-y)}{(12-2x)}$$

$$f(x/y)=\frac{f(x,y)}{f(y)}=\frac{\frac{1}{32}(8-x-y)}{\frac{1}{32}(24-4y)}=\frac{(8-x-y)}{(24-4y)}$$

$$4)\ P(x\le 3)=\int_0^3 f(x)dx=\frac{27}{32}$$

5) $P(x \le 3/y \le 2) = \dfrac{P(x \le 3, y \le 2)}{p(y \le 2)}$

$$P(x \le 3, y \le 2) = \int_1^2 \int_0^3 f(x, y)\,dx\,dy = \int_1^2 \int_0^3 \frac{1}{32}(8 - x - y)\,dx\,dy = \frac{30}{64}$$

$$p(y \le 2) = \int_1^2 f(y)\,dy = \int_1^2 \frac{1}{32}(24 - 4y)\,dy = \frac{18}{32}$$

$$P(x \le 3/y \le 2) = \frac{5}{6}$$