Q1: Minimize $f(x) = .65 - \left[\frac{.75}{(1+x^2)}\right] - .65 \tan^{-1}\left(\frac{1}{x}\right)$ in the interval $[0,3]$ by fibonacci method using $n=6$.

Sol^n Here $n=6$, $L_0 = 3-0 = 3$ which gives

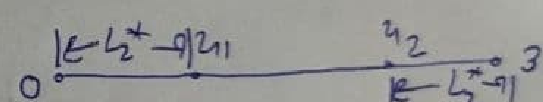
$$L_2^* = \frac{F_{n-2}}{F_n} L_0 = \frac{F_{6-2}}{F_6} L_0 = \frac{F_4}{F_6} L_0$$

$\therefore$ fibonacci series 1, 1, 2, 3, 5, 8, 13

F_0 F_1 F_2 F_3 F_4 F_5 F_6

$$L_2^* = \frac{5}{13}(3) = 1.153846$$

Thus the positions of first two experiments are given by



$x_1 = 0 + L_2^* = 0 + 1.153846$, $\quad x_2 = 3 - L_2^*$

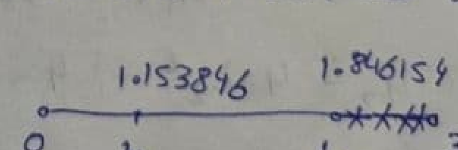
$x_1 = 1.153846$ $\quad = 3 - 1.153846$

$x_2 = 1.846154$

$\therefore f(x_1) = -0.207270$

$f(x_2) = -.115843$

Since, $f(x_1) < f(x_2)$, we can delete interval $[x_2, 3]$



The third experiment is placed at

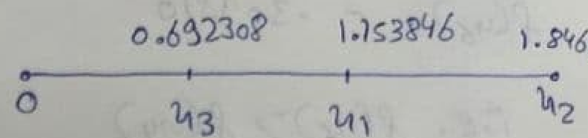
$x_3 = 0 + (x_2 - x_1)$

$x_3 = 0 + (1.846154 - 1.153846)$

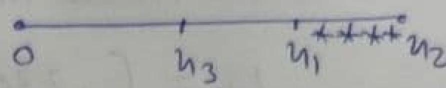
$= .692308$

Here, $f(u_3) = -.291364$

i.e.



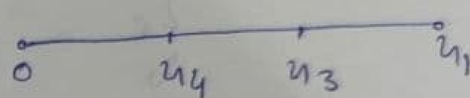
$f(u_1) > f(u_3)$ i.e. we delete interval $[u_1, u_2]$



now, fourth experiment is placed at

$$u_4 = 0 + (u_1 - u_3) = 0 + (1.153846 - 0.692308)$$

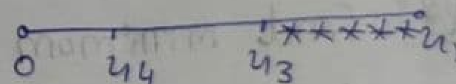
$$u_4 = 0.461538$$



$$f(u_4) = -.309681$$

Here, $f(u_4) < f(u_3)$

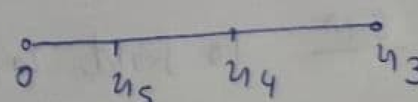
So deleted interval is $(u_3, u_1]$



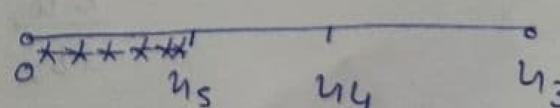
Fifth experiment is placed at

$$u_5 = 0 + (u_3 - u_4) = 0.230770$$

$$f(u_5) = -0.23678$$



i.e. $f(u_5) > f(u_4)$ we ~~double~~ delete interval $[0, u_5]$

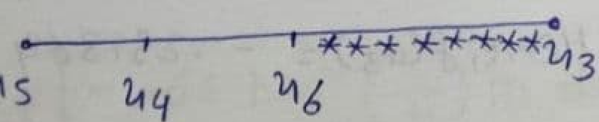


now, sixth experiment is placed at

$$u_6 = u_3 + (u_5 - u_4)$$

$$= 0.692308 + 0.230770 - 0.46153 = 0.461540$$

$f(x_6) = -.309810$



i.e. $f(x_6) > f(x_4)$

delete interval $(x_6, x_3]$

we obtain final interval of uncertainty $L_6 = [x_5, x_6]$

$$L_6 = [.230770, .461540]$$

The ratio of final to initial interval of uncertainty is

$$\frac{L_6}{L_0} = \frac{0.461540 - 0.230770}{3} = .076923$$

This value can be compared to the value $\frac{1}{F_6}$

$$= \frac{1}{13} = .076923$$ Ans

Q2: Find minimum of $f(x) = x^2 + 2x$ within interval $(-3, 4)$ using fibonacci method, obtain minimum value within 5% of exact value.

Soln To find x, we have

$$\frac{\text{Length of final of interval of uncertainty}}{2 \times \text{Length of initial of interval of uncertainty}} \le \frac{5}{100}$$

$$\frac{L_n}{2L_0} \le \frac{5}{100} = \frac{1}{20}$$

$$\frac{L_n}{L_0} \le \frac{1}{10}$$

Note:

$$\frac{1}{F_n} \leq \frac{1}{10}$$

$$L_n = \frac{F_{n(j-1)}}{F_n} L_0$$

$$\boxed{j=n}$$

$$F_n \geq 10$$

$$\frac{L_n}{L_0} = \frac{F_{n-n+1}}{F_n}$$

Fibonacci series

$$\frac{L_n}{L_0} = \frac{F_1}{F_n} \quad \therefore F_1 = 1$$

1, 1, 2, 3, 5, 8, 13

$F_0\ F_1\ F_2\ F_3\ F_4\ F_5\ F_6$

$$13 \geq 10 \text{ i.e. for } n=6$$

$$\boxed{\frac{L_n}{L_0} = \frac{1}{F_n}}$$

Here, $f(x) = x^2 + 2x$, $[-3, 4]$

$n = 6$

$L_0 = 4 + 3 = 7$

$$L_2^* = \frac{F_{n-2}}{F_n} L_0 = \frac{F_{6-2}}{F_6}(7) = \frac{F_4}{F_6}(7)$$

$$= \frac{5}{13} \times 7$$

1, 1, 2, 3, 5, 8, 13

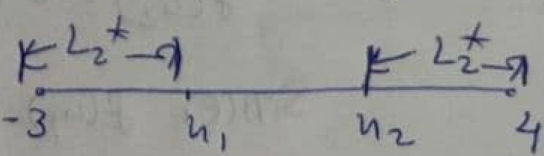
$F_0\ F_1\ F_2\ F_3\ F_4\ F_5\ F_6$

$$L_2^* = 2.6923$$

Thus, the position of first two experiments are given by

$x_1 = -3 + L_2^* = -3 + 2.6923$

$x_1 = -.3076$



$x_2 = 4 - L_2^* = 4 - 2.6923 = 1.3077$

$f(x_1) = -.52072$

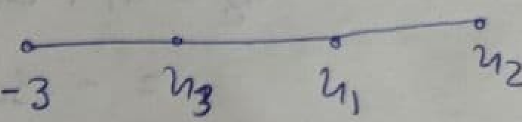
$f(x_2) = 4.3255$

Here $f(u_1) < f(u_2)$, we can delete interval $[u_2, 4]$

The third experiment is placed at,

$$u_3 = -3 + (u_2 - u_1) = -3 + (1.3077 + 0.3076)$$

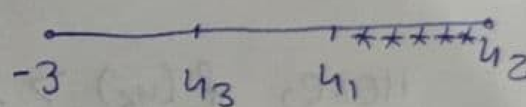
$$u_3 = -1.3846$$



$$f(u_3) = -.8520$$

since $f(u_1) > f(u_3)$

Delete interval $[u_1, u_2]$



now, fourth experiment is placed at

$$u_4 = -3 + (u_1 - u_3)$$

$$= -3 + (-.3076) + (1.3846) = -1.923$$

$$f(u_4) = -.14807$$

Here, $f(u_4) > f(u_3)$

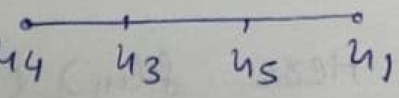
i.e. Deleted interval is $[-3, u_4]$

now fifth experiment is placed at

$$u_5 = u_1 + (u_4 - u_3)$$

$$= -.3076 + (-1.923 + 1.3846)$$

$$u_5 = -.8462$$

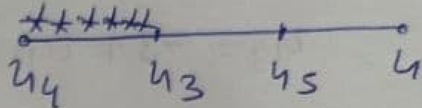


$f(x_5) = -0.976384 \approx -.97634$

$f(x_3) > f(x_5)$ i.e. Delete interval is

$(x_4, x_3]$

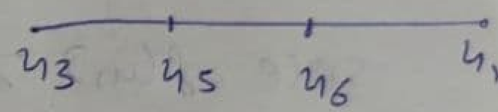
Now sixth experiment is placed at



$x_6 = x_1 + (x_3 - x_5)$

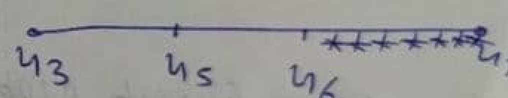
$= -.3076 + (1.3846 + .846)$

$x_6 = -.8462$



$f(x_6) = -.97634$

Here, $f(x_6) > f(x_5)$



Deleted interval is $[x_6, x_1]$

So, we obtain the final interval of uncertainty

$L_6 = [x_3, x_6] = [-1.384, -.8462]$

The ratio of final and initial uncertainty is,

$$\frac{L_6}{L_0} = \frac{-.8462 + 1.384}{4 - (-3)} = .07694$$

This value can be compared with value $\frac{1}{F_6} = .07694$

Q3: Minimize function $f(x) = x^3 + \frac{54}{x}$ in region $[0,5]$ by taking $n=3$

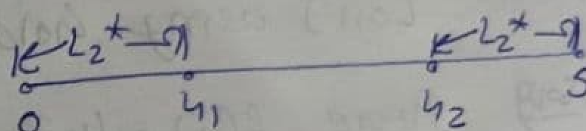
Sol^n Here $n=3$, $L_0 = 5-0 = 5$

which gives, $L_2^* = \frac{F_{n-2}}{F_n} L_0 = \frac{F_{3-2}}{F_3} L_0$

$= \frac{F_1}{F_3} L_0$

1, 1, 2, 3, 5, 8
F_0 F_1 F_2 F_3 F_4 F_5

$L_2^* = \frac{1}{3}(5) = 1.6666$



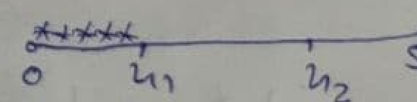
$x_1 = 0 + L_2^* = 0 + 1.6666$
$= 1.6666$

$x_2 = 5 - L_2^* = 5 - 1.6666 = 3.333$

$\therefore f(x_1) = f(1.6666) = (1.6666)^2 + \frac{54}{1.6666} = 35.17749$

$f(x_2) = f(3.333) = (3.333)^2 + \frac{54}{3.333} = 27.3110$

Here, $f(x_1) > f(x_2)$
So, we can delete interval $[0, x_1]$



now, third experiment is placed at

$x_3 = 5 + (x_1 - x_2) = 5 + (1.666 - 3.333)$
$= 3.333$ [Hence, $x_2 = x_3$ Process is stoped]

So find interval of uncertainty $L_3 = [x_1, x_3]$

$L_3 = [1.6666, 3.333]$

The ratio of final to initial interval of uncertainty

$$\frac{L_3}{L_0} = \frac{3.333 - 1.666}{5} = \frac{1.66667}{5} = .33334$$

This value can be compared with value $\frac{1}{F_3}$

$$= \frac{1}{3} = .3333 \quad \text{Ans//}$$

Q4: Minimize $f(x) = 4x^3 + x^2 - 7x + 14$ within interval $[0,1]$ using Golden section method.

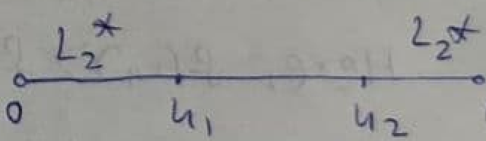
Sol: Here $f(x) = 4x^3 + x^2 - 7x + 14$

$$L_0 = 1 - 0 = 1$$

In Golden section method,

$$L_2^* = \frac{1}{\gamma^2} L_0, \quad \text{where } \gamma = 1.618$$

and $\frac{1}{\gamma} = 0.618$



$$L_2^* = \frac{1}{(1.618)^2} \times 1 = .3819$$

$$x_1 = 0 + L_2^* = 0 + .3819 = .3819$$

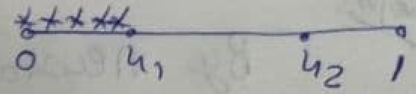
$$x_2 = 1 - L_2^* = 1 - .3819 = .6181$$

$$f(x_1) = 4(.3819)^3 + (.3819)^2 - 7(.3819) + 14 = 11.6953$$

$$f(x_2) = 4(.6181)^3 + (.6181)^2 - 7(.6181) + 14 = 10.999$$

Here, $f(u_1) > f(u_2)$ we can delete interval $(0, u_1)$

The third experiment is placed at



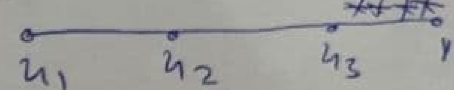
$u_3 = 1 + (u_3 - u_2)$

$= 1 + .3819 - .6181 = .7638$

$f(u_3) = 4(.7638)^3 + (.7638)^2 - 7(.7638) + 14$

$= 11.0192$

Here $f(u_3) > f(u_2)$



So, we delete interval $(u_3, 1)$

now, fourth experiment is placed at

$u_4 = u_1 + (u_3 - u_2) = .3819 + .7638 - .6181$

$= .5278$

$f(u_4) = 11.1721$

Here, $f(u_4) > f(u_2)$ i.e. deleted interval is (u_1, u_4)

now, fifth experiment is

$u_5 = u_3 + (u_4 - u_2)$

$= .7638 + .5278 - .6181$

$u_5 = .6736$

$f(u_5) = 4(.6736)^3 + (.6736)^2 - 7(.6736) + 14$

$= 10.9611$

Qs: Minimize $f(u_1, u_2) = u_1 - u_2 + 2u_1^2 + 2u_1u_2 + u_2^2$ taking starting point as $x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by newton method.

Soln

By newton method, we find x_2 as

$$\therefore \; x_{i+1} = x_i - [J_i]^{-1} \nabla f_i$$

$$x_2 = x_1 - [J_1]^{-1} \nabla f_1 \quad — \; (1)$$

Here,
$$J_1 = \begin{bmatrix} \frac{\partial^2 f}{\partial u_1^2} & \frac{\partial^2 f}{\partial u_1 \partial u_2} \\ \frac{\partial^2 f}{\partial u_2 \partial u_1} & \frac{\partial^2 f}{\partial u_2^2} \end{bmatrix}$$

and
$$\nabla f_1 \text{ (gradient of } f_1) = \begin{bmatrix} \frac{\partial f}{\partial u_1} \\ \frac{\partial f}{\partial u_2} \end{bmatrix}_{x_1}$$

$$\because \; f(u_1, u_2) = u_1 - u_2 + 2u_1^2 + 2u_1u_2 + u_2^2$$

$$\frac{\partial f}{\partial u_1} = 1 + 4u_1 + 2u_2$$

$$\frac{\partial^2 f}{\partial u_1^2} = 4 \quad , \quad \frac{\partial^2 f}{\partial u_1 \partial u_2} = 2$$

$$\frac{\partial f}{\partial u_2} = 0 - 1 + 0 + 2u_1 + 2u_2$$

$$\frac{\partial^2 f}{\partial u_2^2} = 2$$

$$J_1 = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

now, we find J_1^{-1}

$$J_1^{-1} = \frac{1}{8-4}\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

$$J_1^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

and $$\nabla f_1 = \begin{bmatrix} \frac{\partial f}{\partial u_1} \\ \frac{\partial f}{\partial u_2} \end{bmatrix}_{x_1} = \begin{bmatrix} 1+4u_1+2u_2 \\ -1+2u_1+2u_2 \end{bmatrix}_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$$\nabla f_1 = \begin{bmatrix} 1+0+0 \\ -1+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore x_2 = x_1 - (J_1)^{-1} \nabla f_1$$

$$x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 + 1/2 \\ -1/2 - 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$$

To see whether or not x_2 is optimum point we evaluate ∇f_2

$$\nabla f_2 = \begin{bmatrix} \frac{\partial f}{\partial u_1} \\ \frac{\partial f}{\partial u_2} \end{bmatrix}_{x_2} = \begin{bmatrix} 1+4u_1+2u_2 \\ -1+2u_1+2u_2 \end{bmatrix}_{\begin{pmatrix} -1 \\ 3/2 \end{pmatrix}}$$

$$= \begin{bmatrix} 1-4+2\times 3/2 \\ -1+2(-1)+2(3/2) \end{bmatrix}$$

$$\nabla f_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As $\nabla f_2 = 0$, x_2 is the optimum point.

now we calculate,

$$x_3 = x_2 - (J_2)^{-1} \nabla f_2 \qquad \because \nabla f_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_3 = x_2 - 0$$

$$x_3 = x_2$$ we stoped the process

Q6: Minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ start from $x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by using steepest descent method.

Soln Iteration first

$$x_2 = x_1 + \lambda_1^* s_1$$

$$X_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

now search direction S_1

$$S_1 = -\nabla f_1$$

Here $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}_{x_1}$

$$\nabla f = \begin{bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{bmatrix}_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Determine step length λ_1 and we minimize

$$f(x_1+\lambda_1^* S_1)$$

$$f\left[\begin{pmatrix}0\\0\end{pmatrix}+\lambda_1^*\begin{pmatrix}-1\\1\end{pmatrix}\right] = f\begin{bmatrix}-\lambda_1^*\\ \lambda_1^*\end{bmatrix}\begin{matrix}\to x_2\\ \to x_1\end{matrix}$$

$$f\left[-2\lambda_1^* + 2\lambda_1^{*2} - 2\lambda_1^{*2} + \lambda_1^{*2}\right]$$

$$= f\left[-2\lambda_1^* + \lambda_1^{*2}\right]$$

now, $\dfrac{\partial f}{\partial \lambda_1^*} = -2+2\lambda_1^* = 0$

$$\therefore \boxed{\lambda_1^* = 1}$$

we obtain x_2,

$$\therefore x_2 = x_1+\lambda_1^* S_1 = \begin{pmatrix}0\\0\end{pmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$x_2 = \begin{bmatrix}-1\\1\end{bmatrix}$$

$$\therefore \nabla f_2 = \begin{bmatrix}\dfrac{\partial f}{\partial x_1}\\[2mm] \dfrac{\partial f}{\partial x_2}\end{bmatrix}_{x_2} = \begin{bmatrix}1+4x_1+2x_2\\ -1+2x_1+2x_2\end{bmatrix}_{\begin{pmatrix}-1\\1\end{pmatrix}}$$

$$= \begin{pmatrix}-1\\-1\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$$

x_2 is not optimum

Iteration 2

$$S_2 = -\nabla f_2 = -\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$\therefore f(u_2 + \lambda_2 S_2) = f\left[\begin{pmatrix}-1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\1\end{pmatrix}\right]$

$$= f \begin{array}{l} \begin{bmatrix} -1+\lambda_2 \\ -1+\lambda_2 \end{bmatrix} \begin{array}{l} \to u_1 \\ \to u_2 \end{array} \end{array}$$

$\therefore$ Given function $f[(-1+\lambda_2) - (1+\lambda_2) + 2(-1+\lambda_2)^2 + 2(-1+\lambda_2)(1+\lambda_2) + (1+\lambda_2)^2]$

$f\,(5\lambda_2^2 - 2\lambda_2 - 1)$

so $\frac{\partial f}{\partial \lambda_2} = 0 \Rightarrow \lambda_2 - \lambda_2 = 0 \Rightarrow \lambda_2 = 1/5$

$x_3 = x_2 + \lambda_2 S_2 = \begin{bmatrix}-1\\-1\end{bmatrix} + \frac{1}{5}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-.8\\1.2\end{bmatrix}$

$\nabla f_3 = \begin{Bmatrix} \partial f/\partial u_1 \\ \partial f/\partial u_2 \end{Bmatrix}_{u_3} = \quad = \begin{bmatrix} .2 \\ -.2 \end{bmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$

we proceed to next iteration

<u>Iteration 3</u>

$S_3 = -\nabla f_3 = \begin{bmatrix}-.2\\.2\end{bmatrix}$

$f(x_3 + \lambda_3 S_3) = F\left[\begin{pmatrix}-.8\\.2\end{pmatrix} + \lambda_2 \begin{pmatrix}-.2\\.2\end{pmatrix}\right]$

$$= F \begin{bmatrix} -.8 - .2\lambda_3 \\ .2 + .2\lambda_3 \end{bmatrix} \begin{array}{l} \to u_1 \\ \to u_3 \end{array}$$

putting in function

$$F\left[.8-.2\lambda_3-(.2+.2\lambda_3)+2(-.8-.2\lambda_3)^2\right.$$
$$+2(-.8-.2\lambda_3)(.2+2\lambda_3)$$
$$\left.+(.2+2\lambda_3)^2\right]$$

$\cancel{f(\lambda.8)}$

$\therefore \quad \dfrac{\partial f}{\partial \lambda_3} = 0 \implies$

we get $\lambda_3 = 1$

$\therefore x_4 = x_3 + \lambda_3 S_3$

$$= \begin{pmatrix} -.8 \\ 1.2 \end{pmatrix} + \begin{pmatrix} -.2 \\ .2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1.4 \end{pmatrix} \text{ is}$$

not optemum

$$S_4 = -\nabla f_4 = \begin{Bmatrix} 1 \\ -1.4 \end{Bmatrix} \neq \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$S_4 = -\nabla f_4 = \begin{Bmatrix} 1 \\ -1.4 \end{Bmatrix}$$

AS $F(x_4 + \lambda_4 S_4) = F\left[\begin{pmatrix} -1 \\ 1.4 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1.4 \end{pmatrix}\right]$

$$= F\left[-1+\lambda_4,\ 1.4-\lambda_4(1.4)\right]$$

$$x_5 = x_4 + \lambda_4(S_4) = \begin{bmatrix} -1 \\ 1.4 \end{bmatrix} + \lambda_4 \begin{bmatrix} 1 \\ -1.4 \end{bmatrix}$$

$$x_5 = \begin{bmatrix} -1 \\ 1.3 \end{bmatrix}$$

$\therefore x_5 = x_4$

optemal soln is $x_4 = \begin{pmatrix} -1 \\ 1.4 \end{pmatrix}$ and

$$\nabla f_5 = \begin{bmatrix} -.2 \\ -.2 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$