# Implementation of Discrete Time Systems :

(1) Lattice structures for FIR systems (All-zero systems)
(2) Lattice structures for All-Pole IIR systems
(3) Lattice structures for Pole-Zero IIR systems (or Lattice-Ladder structures).

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## Advantages of Lattic structures :

* modular in nature i.e. filter oder can be increased by adding extra stages and calculating the coefficients of new stages alone. The coefficients of the older stages remains same.
* ~~Lattice~~ Less sensitive to coefficient quantization effects. (finite-word length effect) than DFs.
* computationally more efficient than other structures for implementation of wavelet transforms using filter banks.
* ~~En~~ They can simultaneously yield the forward & backward prediction errors in linear prediction.

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## (1) LATTICE STRUCTURES FOR FIR SYSTEMS (All-zero systems)

Consider an FIR filter of order $m$, $0 \le m \le M-1$.

System function $H(z) = \sum_{k=0}^{m} h(k) z^{-k}$, $m = 0, 1, 2, \cdots M-1$.

$\Rightarrow$ $H(z)$ is a $m^{th}$ degree polynomial denoted by $A_m(z)$ with coefficients $h(k) = a_m(k)$ = direct form coefficients.

$$\Rightarrow \quad H(z) = A_m(z) = \sum_{k=0}^{m} a_m(k) z^{-k}.$$

$$= 1 + \sum_{k=1}^{m} a_m(k) z^{-k}, \quad m = 1, 2, \cdots M-1$$

(considering $h(0) = a_m(0) = 1$).

$\Rightarrow$ Also, $A_0(z) = 1$.

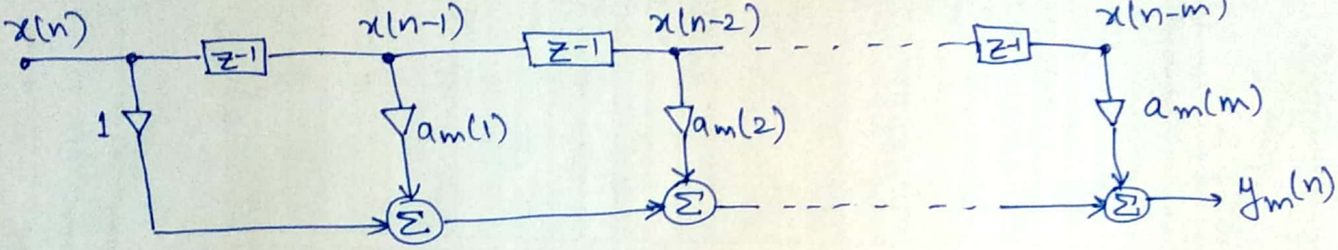
The corresponsing difference equation :

$$H(z) = \frac{Y(z)}{X(z)} = A_m(z) = 1 + \sum_{k=1}^{m} a_m(k) z^{-k} .$$

$$\Rightarrow \quad y(n) = x(n) + \sum_{k=1}^{m} a_m(k) . x(n-k)$$

$$\Rightarrow y(n) = x(n) + a_m^{(1)} x(n-1) + a_m^{(2)} x(n-2) + \cdots + a_m(m) x(n-m)$$

$\Rightarrow$ Drawing the Direct form structure / Realization.




$a_m(1), a_m(2), \cdots, a_m(m)$ are direct form coefficients.

Considering first order filter for realizing Lattice structures,

Difference eq$^n$ : $y_0(n) = x_0(n) + a_1(1) x_0(n-1)$ : $m = 1$

Let $x(n) = f_0(n) = g_0(n)$ : I/Ps to Lattice.

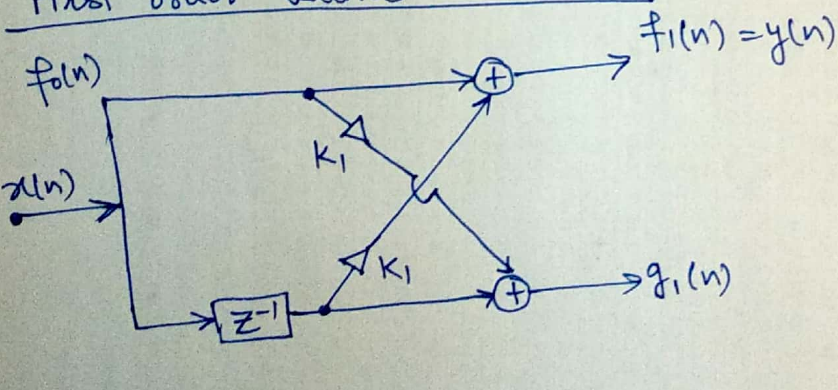
Outputs

$$f_1(n) = f_0(n) + K_1 g_0(n-1)$$
$$= x(n) + K_1 x(n-1)$$

and

$$g_1(n) = K_1 f_0(n) + g_0(n-1)$$
$$= K_1 x(n) + x(n-1)$$

First order Lattice Structure




$K_1$ : reflection coefficient or Lattice coefficient.

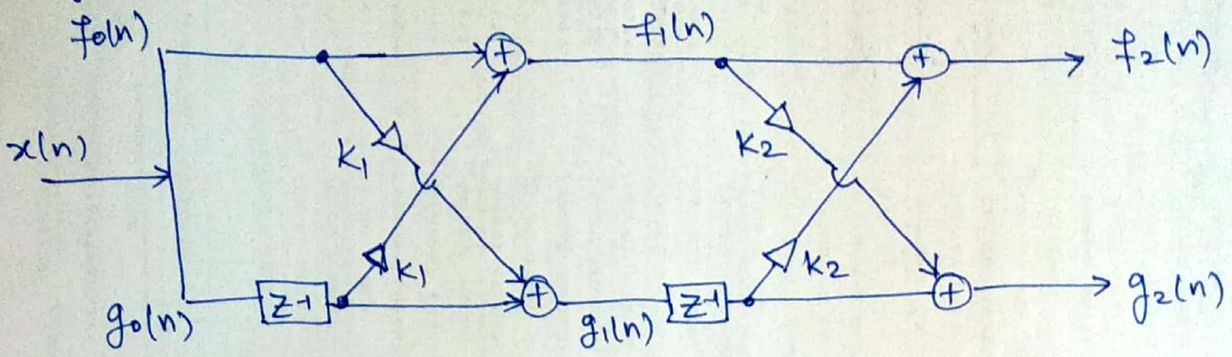
Coefficients of $f_1(n)$ & $g_1(n)$ are reversed ordered.

$f_1(n) \rightarrow \{1, K_1\}$

$g_1(n) \rightarrow \{K_1, 1\}$.

## Second order filter, $m = 2$.

$$y(n) = x(n) + a_2(1)\,x(n-1) + a_2(2)\,x(n-2)$$




$$\Rightarrow \quad x(n) = f_0(n) = g_0(n)$$

$$f_2(n) = f_1(n) + K_2\,g_1(n-1)$$

$$= f_0(n) + K_1\,g_0(n-1) + K_1K_2\,f_0(n-1) + K_2\,g_0(n-2).$$

$$\Rightarrow \quad y(n) = x(n) + K_1x(n-1) + K_1K_2\,x(n-1) + K_2\,x(n-2).$$

$$\Rightarrow f_2(n) = y(n) = x(n) + K_1(K_2+1)\,x(n-1) + K_2\,x(n-2).$$

$$\Rightarrow \quad a_2(1) = K_1(K_2+1), \quad a_2(2) = K_2.$$

$$\Rightarrow \quad \boxed{K_1 = \frac{a_2(1)}{1+a_2(2)} \quad \text{and} \quad K_2 = a_2(2)}$$

~~These~~ ∴ DF coefficients & lattice coefficients are different.

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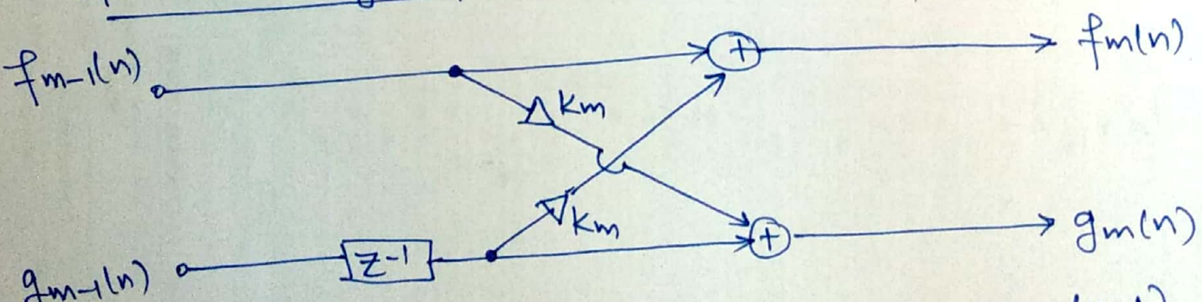
## General stage of lattice structures:




$$\Rightarrow \left\{ \begin{aligned} f_m(n) &= f_{m-1}(n) + K_m\,g_{m-1}(n-1) \\ g_m(n) &= K_m\,f_{m-1}(n) + g_{m-1}(n-1) \end{aligned} \right\} \text{O/P}$$

For $m = (M-1)^{th}$ order filter

$$\underline{y(n) = f_{M-1}(n).}$$

and $\underline{K_m = a_m(m)}$.

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Also $y(n) = f_m(n) = \sum_{k=0}^{m} a_m(k)\, x(n-k)$ , $a_m(0) = 1$.

$\Rightarrow A_m(z) = \dfrac{F_m(z)}{X(z)}$

Since ~~f(n)~~ g(n) is reversed ordered coefficients with f(n)

$$\Rightarrow g_m(n) = \sum_{k=0}^{m} a_m(m-k)\, x(n-k)$$

$$= \sum_{k=0}^{m} b_m(k)\, x(n-k) \qquad \Rightarrow B_m(z) = \frac{G_m(z)}{X(z)}$$

where $b_m(k) = a_m(m-k)$ , $k = 0, 1, 2, \ldots, m$.

with $b_m(m) = a_m(0) = 1$.

Taking Z-Transform.

$$\Rightarrow \boxed{B_m(z) = z^{-m} A_m(z^{-1})}$$

$\Rightarrow B_m(z)$ is called the reversed polynomial of $A_m(z)$

All equations combined in Z-domain.

$$\Rightarrow F_0(z) = G_0(z) = X(z).$$

$$F_m(z) = F_{m-1}(z) + K_m z^{-1} G_{m-1}(z) \,, \quad m = 1, 2, \ldots M-1$$

$$G_m(z) = K_m F_{m-1}(z) + z^{-1} G_{m-1}(z) \,, \quad m = 1, 2, \ldots M-1.$$

Divide these equations by $X(z)$, we get.

$$\Rightarrow A_0(z) = B_0(z) = 1$$

$$A_m(z) = A_{m-1}(z) + K_m . z^{-1} . B_{m-1}(z) \,, \quad m = 1, 2, \ldots M-1$$

$$B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z) \,, \quad m = 1, 2, \ldots M-1$$

In matrix form :

$$\begin{bmatrix} A_m(z) \\ B_m(z) \end{bmatrix} = \begin{bmatrix} 1 & K_m z^{-1} \\ K_m & z^{-1} \end{bmatrix} \begin{bmatrix} A_{m-1}(z) \\ B_{m-1}(z) \end{bmatrix}.$$

## Conversion of Lattice coefficients to Direct-form Filter Coefficients

**Q.** Let lattice coefficients of a 3-stage ($m=3$) FIR lattice structure be $K_1 = 0.1$, $K_2 = 0.2$ and $K_3 = 0.3$. Determine FIR filter coefficients for DF structure.

**Sol^n** We know, $A_0(z) = B_0(z) = 1$.

$\therefore$ for $m = 1$; $A_1(z) = A_0(z) + K_1 z^{-1} B_0(z)$

$= 1 + 0.1 z^{-1}$

$= a_1(0) + a_1(1) z^{-1}$

Also, $B_m(z) = z^{-m} A_m(z^{-1})$

$\Rightarrow B_1(z) = z^{-1} A_1(z^{-1})$

$= z^{-1}(1 + 0.1 z)$

$= 0.1 + z^{-1}$

for $m = 2$; $A_2(z) = A_1(z) + K_2 z^{-1} B_1(z)$

$= [1 + 0.1 z^{-1}] + 0.2 z^{-1} [0.1 + z^{-1}]$

$= 1 + 0.12 z^{-1} + 0.2 z^{-2}$

$= a_2(0) + a_2(1) z^{-1} + a_2(2) z^{-2}$

Also, $B_2(z) = z^{-2} A_2(z^{-1})$

$= z^{-2}(1 + 0.12 z + 0.2 z^2)$

$= 0.2 + 0.12 z^{-1} + z^{-2}$

for $m = 3$; $A_3(z) = A_2(z) + K_3 z^{-1} B_2(z)$

$= [1 + 0.12 z^{-1} + 0.2 z^{-2}]$

$+ 0.3 z^{-1} [0.2 + 0.12 z^{-1} + z^{-2}]$

$= 1 + 0.18 z^{-1} + 0.236 z^{-2} + 0.3 z^{-3}$

$= a_3(0) + a_3(1) z^{-1} + a_3(2) z^{-2} + a_3(3) z^{-3}$

$\therefore$ DF structure coefficients

$\Rightarrow a_3(0) = 1$, $a_3(1) = 0.18$

$a_3(2) = 0.236$, $a_3(3) = 0.3$.

Draw structure Lattice Also.

## Conversion of DF-filter coefficients to lattice coefficients

we know, $A_m(z) = A_{m-1}(z) + z^{-1} K_m B_{m-1}(z)$ — ①

and $B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z)$.

$\Rightarrow z^{-1} B_{m-1}(z) = B_m(z) - K_m A_{m-1}(z)$ — ②

Substituting eq. ② in ①

$\Rightarrow A_m(z) = A_{m-1}(z) + K_m\left[B_m(z) - K_m A_{m-1}(z)\right]$

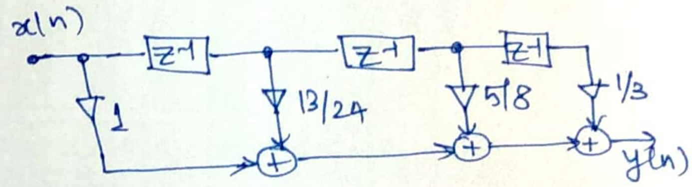
$$\Rightarrow \boxed{A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}} \quad m = M-1, M-2 \ldots\ldots 2, 1.$$

Q. Determine lattice coefficients corresponding to FIR filter with system function:

$$H(z) = 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}$$

Sol$^n$ Now since $H(z) = A_m(z) = A_3(z)$

DF structure



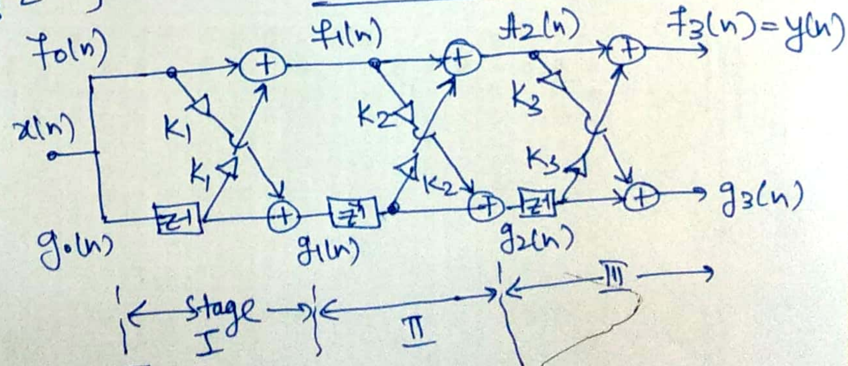

Since we know,

$$\boxed{a_3(3) = K_3 = \frac{1}{3}}$$

we will calculate in Backward order.

$\Downarrow$

Lattice Structure




$B_m(z) = z^{-m} A_m(z^{-1})$

$\Rightarrow B_3(z) = z^{-3}\left[1 + \frac{13}{24} z + \frac{5}{8} z^{+2} + \frac{1}{3} z^{+3}\right]$

$\Rightarrow B_3(z) = \frac{1}{3} + \frac{5}{8} z^{-1} + \frac{13}{24} z^{-2} + z^{-3}$

$$A_2(z) = \frac{A_3(z) - K_3 \cdot B_3(z)}{1 - K_3^2}$$

$$= 1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2}$$

$$\Rightarrow \boxed{a_2(2) = \frac{1}{2} = K_2}$$

and $A_1(z) = 1 + \frac{1}{4} z^{-1} \Rightarrow \boxed{a_1(1) = K_1 = \frac{1}{4}}$