CHAPTER 5

TRANSFORM-DOMAIN REPRESENTATION OF SIGNALS: DISCRETE FOURIER TRANSFORM (DFT)

Inside this Chapter

5.1 INTRODUCTION

In previous chapters, we have discussed the representation of sequences and Linear shift Invariant (LSIV) systems in terms of the Fourier transforms and z-transforms. If the sequences are of finite duration, it forms a special case and we shall see that it is possible to develop an alternative Fourier representation, which we refer to as Discrete Fourier Transform (DFT).

As a matter of fact, the frequency analysis of discrete-time signals is usually and conveniently performed on a digital signal processor. Now, this digital signal processor may be a general purpose digital computer or a specially designed digital hardware. We know that the Fouier transform of a discrete time signal $x(n)$ is called Discrete-Time Fourier Transform (DTFT) and it is denoted by $X(e^{j\omega})$. We also know that DTFT $X(e^{j\omega})$ is a continuous function of frequency ω. Therefore, this type of representation is not a computationally convenient representation of a discrete-time signal $x(n)$. Thus, taking one step further, we represent a sequence by samples of its continuous spectrum. This type of frequency-domain representation of a signal is known as **Discrete-Fourier Transform (DFT)**. It is very powerful tool for frequency analysis of discrete-time signals.

It may be noted that the Discrete-Fourier Transform (DFT) is itself a sequence rather than a function of a continuous variable and it corresponds to equally spaced frequency samples of Discrete-Time Fouier Transform (DTFT) of a signal. Also, Fourier series representation of the periodic sequence corresponds to the Discrete Fourier Transform (DFT) of the finite length sequence.

In short, we can say that DFT is used for transforming discrete-time sequence $x(n)$ of finite length into discrete-frequency sequence $X(k)$ of finite length. This means that by using DFT, the discrete-time sequence $x(n)$ is transformed into corresponding discrete-frequency sequence $X(k)$.

Further, in the design of a DSP system, two fundamental tasks are involved. These are :

(*i*) analysis of the input signal, and

(*ii*) design of a processing system to provide the desired output.

In this perspective, the discrete Fourier transform (DFT) and Fast Fourier Transform (FFT) are very important mathematical tools to carry out these types of tasks. Both these transforms can be used to analyse a two-dimensional signal.

5.2 FREQUENCY DOMAIN SAMPLING: THE DISCRETE FOURIER TRANSFORM

Before we discuss discrete Fourier transform (DFT) in detail, let us first introduce the sampling of the Fourier transform of an aperiodic discrete-time sequence. This means that we shall establish the relationship between the sampled Fourier transform and the discrete Fourier transform (DFT).

5.2.1 Frequency-Domain Sampling

According to the definition of DTFT, we write

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

We know that $X(\omega)$ is Fourier transform of discrete time signal $x(n)$.. The range of 'ω' is from 0 to 2π or $-\pi$ to π. Hence, it is not possible to compute $X(\omega)$ on dgital computer. Because in above expression, the range of summation is from $-\infty$ to $+\infty$. However, if we make this range finite then it is possible to do these calculations on digital computer.

When a Fourier transform is calculated only at discrete points then it is called as discrete Fourier transform (DFT).
Now, if we have aperiodic time domain signal then discrete time Fourier transform (DTFT) is obtained. But, DTFT is continous in nature and its range is from $-\infty$ to $+\infty$. Then a finite sequence is obtained by extracting a particular protion from such infinite sequence.

Since, $X(\omega)$ is a continuous-time signal, a discrete-time signal is obtained by sampling $X(\omega)$. A particular sequence which is extracted from infinite sequence is called is **windowed sequence**. This windowed signal is considered as periodic signal. We can obtain periodic extension of this signal. This periodic extension in frequency domain is called is **Discrete Fourier Transform (DFT)**. From this original sequence. $x(n)$ is obtained by performing inverse process which is know as **Inverse Discrete Fourier Transform (IDFT).**

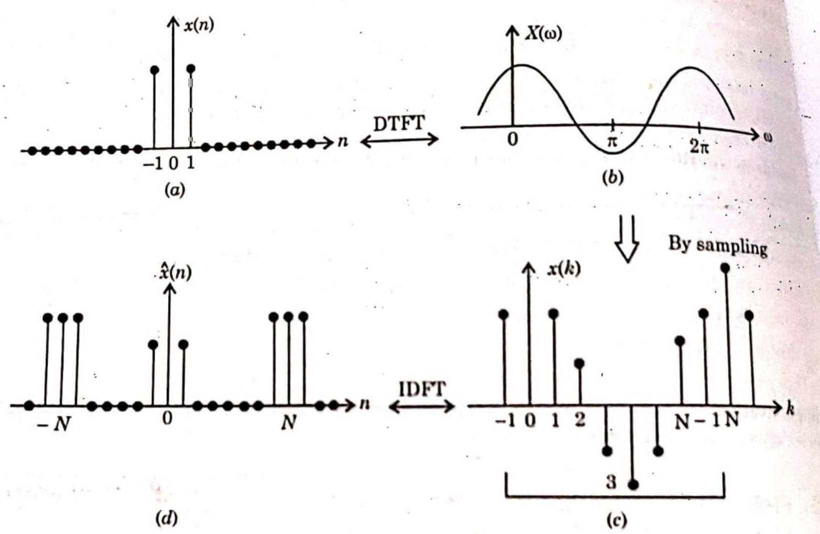

FIGURE 5.1.

This process is explained graphically as shown in figure 5.1. Figure 5.1(a) shows discrete time signal $x(n)$. By taking DTFT of $x(n)$, $X(\omega)$ is obtained as shown in figure 5.1(b). The sampled version of $X(\omega)$ is denoted by $X(k)$ which is called as DFT. It is shown in figure 5.1(c). By performing IDFT, original signal is obtained. It is denoted by $\hat{x}(n)$. It is shown in figure 5.1(d). It is periodic extension of sequence $x(n)$.

Here, N denotes the number of samples of input sequence and the number of frequency points in the DFT output.

5.2.2 Reconstruction of Discrete-Time Signals

We know that aperiodic finite-energy signals have continuous spectra. Let us consider an aperiodic discrete-time signal $x(n)$ with Fourier transform as under:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad ...(5.$$

Now, let us assume that we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ radians between successive samples. Because, $X(\omega)$ is periodic with period 2π, only samples in the fundamental frequency range are essential. For simplicity, let us take N equidistant samples in the interval $0 \le \omega < 2\pi$ with a spacing $\delta\omega = \frac{2\pi}{N}$ as depicted in figure 5.2.

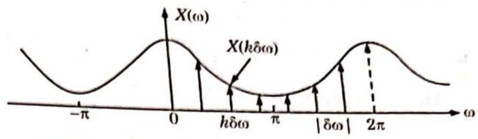

FIGURE 5.2 *Frequency-domain sampling of the Fourier transform.*

First, let us consider the selection of N, the number of samples in the frequency domain. If we evaluate equation (5.1) at

$$\omega = \frac{2\pi k}{N}, \text{ we get}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots N-1 \qquad \ldots(5.2)$$

The above summation may be subdivided into an infinite numbers of summations, where each sum consists of N terms.

Hence, we have

$$X\left(\frac{2\pi}{N}k\right) = \ldots\ldots + \sum_{n=-N}^{-1} x(n)\, e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} + \sum_{n=N}^{2N-1} x(n)\, e^{-j2\pi kn/N} + \ldots\ldots$$

or

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j2\pi kn/N} \qquad \ldots(5.3)$$

In equation (5.3), if we change the index in the inner summation from n to $n - lN$ and interchange the order of the summation, we obtain the following result:

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j2\pi kn/N} \qquad \ldots(5.4)$$

for $\quad k = 0, 1, 2, \ldots N-1$

DO YOU KNOW?

The discrete Fourier Transform (DFT) is a tool for computing the spectra of discrete-time signals.

Also, the signal

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \qquad \ldots(5.5)$$

obtained by the periodic repetition of $x(n)$ every N samples, is clearly periodic with fundamental period N. As a result of this, it my be expanded in a Fourier series as under.

$$x_p(n) = \sum_{k=0}^{N-1} c_k\, e^{j2\pi kn/N}, \quad n = 0, 1, \ldots N-1$$

with the following Fourier coefficients:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots N-1$$

Comparing equations (5.4) and (5.7), we get

$$c_k = \frac{1}{N} X\left(e^{j\frac{2\pi}{N}k}\right), \quad k = 0, 1, \ldots N-1$$

Hence, we have

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(e^{j\frac{2\pi}{N}k}\right) e^{j2\pi kn/N}, \quad n = 0, 1, \ldots N-1 \quad \ldots(5.9)$$

The relationship obtained in equation (5.9) provides the reconstruction of the periodic signal $x_p(n)$ from the samples of the spectrum $X(\omega)$. But it does not mean that we can recover $X(\omega)$ or $x(n)$ from the samples. In fact, to achieve this, we need to consider the relationship between $x_p(n)$ and $x(n)$.

Now, since $x_p(n)$ is the periodic extension of $x(n)$ as given by the following expression:

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \quad \ldots(5.10)$$

then, it is obvious that $x(n)$ can be recovered from $x_p(n)$ if there is no alising in the time-domain. This means that if $x(n)$ is time-limited to less than the period N of $x_p(n)$. Figure 5.3 illustrates this particular situation.

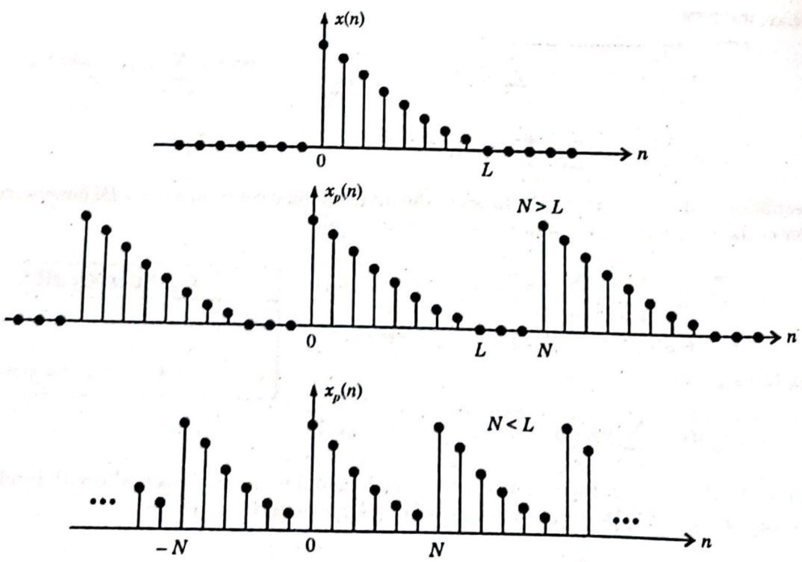

FIGURE 5.3 *Aperiodic sequence $x(n)$ of length L and its periodic extension for $n \geq L$ (no aliasing) and $N < L$ (aliasing).*

In this figure, we have considered a finite-duration sequence $x(n)$; This finite-duration sequence $x(n)$ is nonzero in the interval $0 \leq n \leq L-1$.

It may be observed that

when $N \geq L$, we have

$$x(n) = x_p(n) \quad \text{for} \quad 0 \leq n \leq N-1$$

so that $x(n)$ can be recovered from $x_p(n)$.

On the other hand,

If $N < L$, then it is not possible to recover $x(n)$ from its periodic extension due to time-domain aliasing.

Therefore, we can conclude that the spectrum of an aperiodic discrete-time signal with finite duration L, can be exactly recovered from its samples at frequencies $w_k = \frac{2\pi k}{N}$, if $N \geq L$.

The procedure is to compute $x_p(n)$, $n = 0, 1, \ldots N-1$ with the help of equation (5.9), then, we have

$$x(n) = \begin{cases} x_p(n) & \text{for} \quad 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{...(5.12)}$$

and, finally, $X(\omega)$ may be computed with the help of equation (5.1)

5.3 THE DFT AS A LINEAR TRANSFORMATION

(Important)

(i) Definition of DFT

It is a finite duration discrete frequency sequence which is obtained by sampling one period of Fourier transform. Sampling is done at 'N' equally spaced points over the period points over the period extending from $\omega = 0$ to $\omega = 2\pi$.

(ii) Mathematical Expressions

The DFT of discrete sequence $x(n)$ is denoted by $X(k)$. It is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N} \quad \text{...(5.13)}$$

Here, $k = 0, 1, 2 \ldots N-1$

Since, this summation is taken for N points; it is called as **N-point DFT.**

We can obtain discrete sequence $x(n)$ from its DFT. It is called as inverse discrete fourier transform (IDFT). It is given by,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad \text{...(5.14)}$$

Here, $n = 0, 1, 2, \ldots N-1$

This is called as **N-point IDFT.**

(iii) Twiddle Factor and its Importance

Now, we will define the new term W as,

$$W_N = e^{-j2\pi/N} \quad \text{...(5.15)}$$

This is called as **twiddle factor.** Twiddle factor makes the computation of DFT a bit easy and fast.

Using twiddle factor, we can write equations of DFT and IDFT as under :

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad \text{...(5.16)}$$

Here, $n = 0, 1, 2 \ldots N-1$

and
$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \quad \text{...(5.1}$$

Here, $n = 0, 1, 2, \ldots N-1$

(iv) Linear Transformation

Let us view the DFT and IDFT as linear transformations on sequences $\{x(n)\}$ and ... respectively. Let us define an N-point vector x_N of frequency samples, and an $N \times N$ matrix ...

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{1(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

With these definitions, the N-point DFT may be expressed in matrix form as

$$X_N = W_N x_N$$

where W_N is the matrix of the linear transformation. We observe that W_N is a symmetric matrix. If we assume that the inverse of W_N exists, then the last expression can be inverted by premultiplying both sides by W_n^{-1}. Thus we obtain

$$x_N = W_N^{-1} X_N \quad \text{...(5.18)}$$

But this is just an expression for the IDFT.

In fact, the IDFT can be expressed in matrix form as under

$$x_N = \frac{1}{N} W_N^* X_N \quad \text{...(5.19)}$$

where W_N^* denotes the complex conjugate of the matrix W_N. Comparison of (5.19) with (5.18) leads us to conclude that

DO YOU KNOW?

When DFTs are used to process continuous-time signals ... sampling, several potential error sources may be important. These are aliasing, spectral leakage, and picket-fence effect.

$$W_N^{-1} = \frac{1}{N} W_N^* \quad \text{...(5.20)}$$

which, in turn, implies that

$$W_N W_N^* = N I_N \quad \text{...(5.21)}$$

where I_N is an $N \times N$ identity matrix.

Therefore, the matrix W_N in the transformation is an orthogonal (unitary) matrix. Furthermore, its inverse exists and is given as W_N^*/N. In fact, the existence of the inverse of W_N was established previously from our derivation of the IDFT.

Note : *The DFT and IDFT are computational tools that play a very important role in many digital signal processing applications, such as frequency analysis (spectrum analysis) of signals, power spectrum estimation, and linear filtering. The importance of the DFT and IDFT in such practical applications is due to a large extent on the existences of computationally efficient algorithms, known collectively as fast Fourier transform (FFF) algorithms, for computing the DFT and IDFT.*

5.4 COMPARISON OF DTFT AND DFT

(U.P. Tech., Tutorial Question Bank)

We know that the DTFT is discrete time Fourier transform and is given by,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \text{...(5.}$$

The range of ω is from $-\pi$ to π or 0 to 2π.

Now, we know that discrete Fourier transform (DFT) is obtained by sampling one cycle of Fourier transform. Also, DFT of $x(n)$ is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad ...(5.23)$$

Comparing equations (5.22) and (5.23), we can say that DFT is obtained from DTFT by substituting $\omega = \frac{2\pi k}{N}$.

Hence,

$$X(k) = X(\omega)|_{\omega=\frac{2\pi k}{N}}$$

Few Important Points

By comparing DFT with DTFT, we can write

(i) The continuous frequency spectrum $X(\omega)$ is replaced by discrete Fourier spectrum $X(k)$.

(ii) Infinite summation in DTFT is replaced by finite summation in DFT.

(iii) The continuous frequency variable is replaced by finite number of frequencies located at $\frac{2\pi k}{NT_s}$, where T_s is called as *sampling time*.

5.5 DISCRETE FOURIER TRANSFORM (DFT) OF SOME STANDARD SIGNALS

(GGSIPU, Sem. Exam., May 2005)

In this article, let us obtain DFT of few standard signals in the form of solved examples as follows :

EXAMPLE 5.1 Obtain DFT of unit impulse $\delta(n)$.

Solution : Here $x(n) = \delta(n)$ ...(i)

According to the definition of DFT, we have,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad ...(ii)$$

But $\delta(n) = 1$ only at $n = 0$.

Thus, equation (ii) becomes,

$$X(k) = \delta(0)e^0 = 1$$

Therefore, we can write

$$\delta(n) \xleftrightarrow{DFT} 1$$

This is the standard DFT pair.

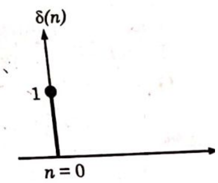

FIGURE 5.4.

EXAMPLE 5.2 Obtain DFT of delayed unit impulse $\delta(n - n_0)$.

Solution : We know that $\delta(n - n_0)$ indicates unit impulse delayed by 'n_0' samples.

Here, $$x(n) = \delta(n - n_0) \quad ...(i)$$

Now, we have, $$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad ...(ii)$$

But, $\delta(n - n_0) = 1$ only at $n = n_0$.

Thus, equation (ii) becomes,

$$X(k) = 1 \,.\, e^{-j2\pi kn_0/N}$$

Hence $\delta(n - n_0) \xleftrightarrow{DFT} e^{-j2\pi kn_0/N}$

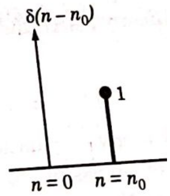

FIGURE 5.

Similarly, we can write,

$$\delta(n + n_0) \xleftrightarrow{DFT} e^{j2\pi k n_0/N}$$

EXAMPLE 5.3 Compute *N*-point DFT of the following exponential sequence:

$$x(n) = a^n u(n) \text{ for } 0 \le n \le N - 1$$

Solution : According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

Here, $x(n) = a^n u(n)$

The multiplication of a^n with $u(n)$ indicates that sequence is positive. Substituting x in equation (i), we obtain

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j2\pi kn/N}$$

or

$$X(k) = \sum_{n=0}^{N-1} (ae^{-j2\pi k/N})^n$$

Now, let us use the following standard summation expression:

$$\sum_{k=N_1}^{N_2} A^k = \frac{A^{N_1} - A^{N_2+1}}{1-A}$$

Here, $N_1 = 0$, $N_2 = N - 1$ and $A = ae^{-j2\pi k/N}$

Therefore,

$$X(k) = \frac{\left(ae^{-j2\pi k/N}\right)^0 - \left(ae^{-j2\pi k/N}\right)^{N-1+1}}{1-ae^{-j2\pi k/N}} = \frac{1-a^N e^{-j2\pi k}}{1-ae^{-j2\pi k/N}}$$

Making use of Euler's identity to the numerator term, we shall have

$$e^{-j2\pi k} = \cos 2\pi k - j \sin 2\pi k$$

But, k is an integer

Therefore, $\cos 2\pi k = 1$ and $\sin 2\pi k = 0$

or $e^{-j2\pi k} = 1 - j0 = 1$

or

$$X(k) = \frac{1-a^N}{1-ae^{-j2\pi k/N}}$$

Hence,

$$a^n u(n) \xleftrightarrow{DFT} \frac{1-a^N}{1-ae^{-j2\pi k/N}}$$

EXAMPLE 5.4 Compute the DFT of following window function:

$$w(n) = u(n) - u(n - N)$$

Solution : According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

The given equation is $x(n) = w(n) = 1$ for $0 \le n \le N - 1$.

Let us assume some value of N.

Let $N = 4$, so we will get 4-point DFT.

Hence,
$$X(k) = \sum_{n=0}^{3} 1.e^{-j2\pi kn/4} \quad ...(ii)$$

The range of k is from 0 to $N-1$. Therefore, in this case, k will vary from 0 to 3.

For $k = 0$, we have $X(0) = \sum_{n=0}^{3} 1.e^{0} = \sum_{n=0}^{3} 1 = 1 + 1 + 1 + 1 = 4$

For $k = 1$, we have $X(1) = \sum_{n=0}^{3} e^{-j2\pi n/4}$

∴ $$X(1) = e^{0} + e^{-j2\pi/4} + e^{-j4\pi/4} + e^{-j6\pi/4}$$

or $$X(1) = 1 + \left(\cos\frac{2\pi}{4} - j\sin\frac{2\pi}{4}\right) + \left(\cos\frac{4\pi}{4} - j\sin\frac{4\pi}{4}\right) + \left(\cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4}\right)$$

or $$X(1) = 1 + (0 - j) + (-1 - 0) + (0 + j)$$

or $$X(1) = 1 - j - 1 + j = 0$$

For $k = 2$, we have, $X(2) = \sum_{n=0}^{3} e^{-j2\pi \times 2n/4} = \sum_{n=0}^{3} e^{-j\pi n}$

∴ $$X(2) = e^{0} + e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi}$$

or $$X(2) = 1 + (\cos\pi - j\sin\pi) + (\cos 2\pi - j\sin 2\pi) + (\cos 3\pi - j\sin 3\pi)$$

or $$X(2) = 1 + (-1 - 0) + (1 - 0) + (-1 - 0) = 1 - 1 + 1 - 1 = 0$$

DO YOU KNOW?

An inverse DFT can be computed using a direct DFT algorithm by conjugating the frequency samples, taking the DFT of these conjugated samples, conjugating the output of the DFT operation, and dividing by N.

For $k = 3$, we have

$$X(3) = \sum_{n=0}^{3} e^{-j2\pi \times 3n/4} = \sum_{n=0}^{3} e^{-j6\pi n/4}$$

∴ $$X(3) = e^{0} + e^{-j6\pi/4} + e^{-j3\pi} + e^{-j9\pi/2}$$

or $$X(3) = 1 + \left(\cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4}\right) + (\cos 3\pi - j\sin 3\pi) + \left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$$

or $$X(3) = 1 + (0 + j) + (-1 - 0) + (0 - j) = 1 + j - 1 - j = 0$$

or $$X(k) = \{4, 0, 0, 0\}$$ **Ans.**

5.6 DETAILED EXPLANATION OF CYCLIC PROPERTY OF TWIDDLE FACTOR

The twiddle factor is denoted by W_N and is given by,
$$W_N = e^{-j2\pi/N} \quad ...(5.24)$$

Now, the discrete time sequence $x(n)$ can be denoted by x_N. Here, N stands for N point DFT. While in case of N point DFT, the range of n is from 0 to $N-1$.

Now, the sequence x_N can be represented in the matrix from as under:

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \cdot \\ \cdot \\ x(N-1) \end{bmatrix}_{N\times 1}$$

This is a $N \times 1$ matrix and n varies from 0 to $N-1$. Now, the DFT of $x(n)$ is denoted by We have denoted $x(n)$ by x_N. Similarly, we can denote $X(k)$ by X_N. In the matrix form, X_k is represented as under:

$$X_k = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \cdot \\ \cdot \\ X(N-1) \end{bmatrix}_{N\times 1}$$

This is also $N \times 1$ matrix and k varies from 0 to $N-1$. Recall the definition of DFT i.e.,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \text{...(5.}$$

We can also represent W_N^{kn} in the matrix form. Further, since k varies from 0 to $N-1$ and n also varies from 0 to $N-1$, therefore, we have

$$W_N^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ \vdots \\ k=N-1 \end{array} \begin{array}{c} \begin{array}{ccccc} n=0 & n=1 & n=2 & \cdots & n=N-1 \end{array} \\ \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}_{N\times N} \end{array} \quad \text{...(5.2}$$

Note that each value is obtained by taking multiplication of k and n.

As an example, if $k = 2$, $n = 2$, then we get $W_N^{kn} = W_N^4$.

Thus, DFT can be represented in the matrix form as,

$$X_N = [W_N]x_N \quad \text{...(5.29)}$$

Similarly, IDFT can be represented in the matrix forms as,

$$x_N = \frac{1}{N}\left[W_N^*\right]X_N \quad \text{...(5.30)}$$

Here, W_N^* is complex conjugate of W_N.

Now, let us show that W_N possess the periodicity property. This means that after some perio the value of W_N repeats. Let us consider 8-point DFT, *i.e.*, $N = 8$.

We have, $\quad W_N = e^{-\frac{j2\pi}{N}}$

Therefore, $\quad W_N^{kn} = e^{-\frac{j2\pi}{N}\times kn}$

But, $N = 8$

Hence,

$$W_8^{kn} = e^{-\frac{j2\pi}{8}\times kn} = e^{-\frac{j\pi}{4}\times kn} \quad ...(5.31)$$

Now, let us obtain W_8^{kn} by substituting different values of kn. This has been shown in Table 5.1.

Here, it may be noted that all calculations have been done by making use of Euler's identity. For example, when $kn = 1$, equation (5.47) becomes,

$$W_8^1 = e^{-j\frac{\pi}{4}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j\,0.707$$

TABLE 5.1.

S.No.	Value of kn	$W_8^{kn} = e^{-j\frac{\pi}{4}\times kn}$	Value of the phasor
1	0	$W_8^0 = e^0$	1
2	1	$W_8^1 = e^{-j\frac{\pi}{4}\times 1} = e^{-j\frac{\pi}{4}}$	$0.707 - j\,0.707$
3	2	$W_8^2 = e^{-j\frac{\pi}{4}\times 2} = e^{-j\frac{\pi}{2}}$	$0 - j\,1$
4	3	$W_8^3 = e^{-j\frac{\pi}{4}\times 3} = e^{-j\frac{3\pi}{4}}$	$-0.707 - j\,0.707$
5	4	$W_8^4 = e^{-j\frac{\pi}{4}\times 4} = e^{-j\pi}$	-1
6	5	$W_8^5 = e^{-j\frac{\pi}{4}\times 5} = e^{-j\frac{5\pi}{4}}$	$-0.707 + j\,0.707$
7	6	$W_8^6 = e^{-j\frac{\pi}{4}\times 6} = e^{-j\frac{3\pi}{2}}$	$0 + j\,1$
8	7	$W_8^7 = e^{-j\frac{\pi}{4}\times 7} = e^{-j\frac{7\pi}{4}}$	$0.707 + j\,0.707$
9	8	$W_8^8 = e^{-j\frac{\pi}{4}\times 8} = e^{-j2\pi}$	1
10	9	$W_8^9 = e^{-j\frac{\pi}{4}\times 9} = e^{-j\frac{9\pi}{4}}$	$0.707 - j\,0.707$
11	10	$W_8^{10} = e^{-j\frac{\pi}{4}\times 10} = e^{-j\frac{5\pi}{2}}$	$0 - j\,1$
12	11	$W_8^{11} = e^{-j\frac{\pi}{4}\times 11} = e^{-j\frac{11\pi}{4}}$	$-0.707 - j\,0.707$

From table 5.1, it may be observed that the value of W_8^0 is same as W_8^8. Similarly same as W_8^9 and W_8^2 is same as W_8^{10}. Since, this is 8-point DFT ($N = 8$), after 8 points, t repeats.

This means that

$$W_8^0 = W_8^8 = W_8^{16} \;....$$

$$W_8^1 = W_8^9 = W_8^{17} \ldots$$
$$W_8^2 = W_8^{10} = W_8^{18} \ldots$$
$$\vdots$$
$$W_8^7 = W_8^{15} = W_8^{23} \ldots$$

This property of twiddle factor is called as ***periodicity property*** *or* ***cyclic property.***

Few Important Points

(i) In Table 5.1, every value of W_N^{kn} can be represented in terms of magnitude and angle.

For example, we have $W_8^0 = e^0$. We know that magnitude and angle can be expressed as, **magnitude** $e^{j\,\text{angle}}$. Thus, for $W_8^0 = 1.e^0$. Here, magnitude is 1 and angle is zero.

(ii) Similarly, we have $W_8^1 = 1 \,.\, e^{-j\frac{\pi}{4}}$, so magnitude is 1 and angle is $-\frac{\pi}{4}$.

(iii) Likewise, we can write the magnitude and angle of each value. The cyclic property of twiddle factor has been illustrated in figure 5.6.

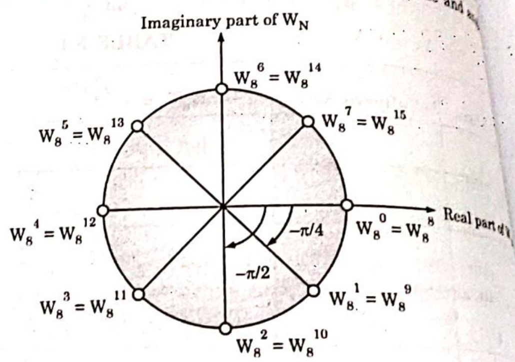

FIGURE 5.6 *Cyclic property of twiddle factor.*

(iv) In figure 5.6, we have drawn unit circle that means a circle having radius equal to 1. Every point is in the clockwise direction because we have negative angles. Here, we have considered 8-point DFT. Therefore, the circle is divided into 8 points. This spacing of DFT or the resolution of DFT is also called as the **bin spacing of DFT output.**

EXAMPLE 5.5 Compute 2-point and 4-point DFT of the following sequence:

$$x(n) = u(n) - u(n-2)$$

Sketch the magnitude of DFT in both the cases.

Solution : First, let us obtain the sequence $x(n)$. It has been represented as shown in figure 5.7.

Thus, from figure (5.7), we get

$$x(n) = \{1, 1\} \quad \ldots(i)$$

(i) Determination of 2-point DFT

For 2-point DFT, $N = 2$

We have, $W_N = e^{-j\frac{2\pi}{N}}$

so that $W_2 = e^{-j\frac{2\pi}{2}} = e^{-j\pi}$

Hence, $W_2^{kn} = e^{-j\pi kn} \quad \ldots(ii)$

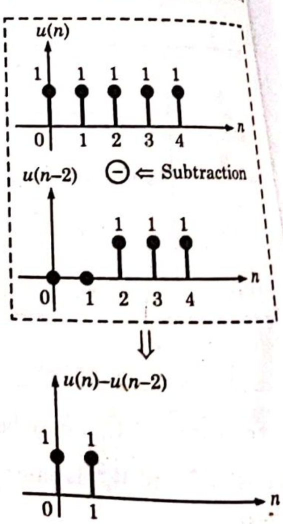

FIGURE 5.7 $x(n) = u(n) - u(n-2)$

We know that n is from 0 to $N-1$. In this case, n is from 0 to 1. Similarly, k is from 0 to $N-1$. In this case, k is from 0 to 1.

Now, the matrix $W_N = W_2^{kn} = e^{-j\pi kn}$ can be written as under:

$$W_2^{kn} = \begin{array}{c c} & \begin{array}{cc} n=0 & n=1 \end{array} \\ \begin{array}{c} k=0 \\ k=1 \end{array} & \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \end{array} \quad \text{...(iii)}$$

According to equation (*ii*), we have

$$W_2^{kn} = e^{-j\pi kn}$$

For $kn = 0$, we have

$$W_2^0 = e^{-j\pi \times 0} = e^0 = 1$$

For $kn = 1$, we have

$$W_2^1 = e^{-j\pi \times 1} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

Substituting all these values in equation (*iii*), we shall get

$$W_2^{kn} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{...(iv)}$$

Also, given sequence is $x(n) = \{1, 1\}$.

In the matrix form, this sequence can be written as,

$$x_N = x(n) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{...(v)}$$

For above sequence, the DFT matrix is given by,

$$X_N = [W_N]x_N$$

Substituting values from equation (*iv*) and (*v*), we get

$$X_N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1\times1)+(1\times1) \\ (1\times1)+(1\times-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Therefore, the 2-point DFT will be

$$X(k) = \{2, 0\} \quad \text{...(vi)}$$

Magnitude plot

We know that magnitude $= \sqrt{(\text{Real part})^2 + (\text{Imaginary part})^2}$

In equation (*vi*), the imaginary part is zero.

Hence, the magnitude at $k = 0$ is 2 and magnitude at $k = 1$ is 0.

This magnitude plot has been shown in figure 5.8.

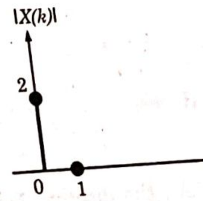

FIGURE 5.8 *Magnitude*

(*ii*) Determination of 4-point DFT

For 4-point DFT, $N = 4$

We have, $W_N = W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$

Therefore, $W_N^{kn} = e^{-j\frac{\pi}{2}kn}$

The range of k and n is from 0 to $N - 1$. *i.e.*, 0 to 3.

Now, the matrix $W_N = W_4 = W_4^{kn}$ can be written as

$$[W_4] = W_4^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \begin{array}{c} \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\ \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array} \quad ...(viii)$$

$\left(e^{-\frac{j2\pi}{4}}\right)$

Making use of equation (*vii*), we obtain

$$W_4^0 = e^{-j\frac{\pi}{2}\times 0} = e^0 = 1$$

$$W_4^1 = e^{-j\frac{\pi}{2}\times 1} = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_4^2 = e^{-j\frac{\pi}{2}\times 2} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

$$W_4^3 = e^{-j\frac{\pi}{2}\times 3} = e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = +j$$

According to cyclic property of DFT, we know that

$$W_4^0 = W_4^4 = 1$$

$$W_4^1 = W_4^5 = W_4^9 = -j$$

$$W_4^2 = W_4^6 = W_4^{10} = -1$$

and $\quad W_4^3 = W_4^7 = W_4^{11} = +j$

Substituting all these values in equation (*viii*), we shall get the matrix $[W_4]$ *i.e.*,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad ...(ix)$$

The given sequence is $x(n) = \{1, 1\}$. Since, the desirable length of this sequence is equal to 4, it is obtained by adding zeros at the end of sequence. This is called as **zero padding.**

Thus, with the help of zero padding, we get

$$x(n) = \{1, 1, 0, 0\}$$

Hence, $$x_N = x_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad ...(x)$$

Now, the discrete Fourier Transform (DFT) is given by

$$X_N = [W_N]x_N$$

or $$X_N = X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+1+0+0 \\ 1-j+0+0 \\ 1-1+0+0 \\ 1+j+0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$$X_4 = \{2,\ 1-j,\ 0,\ 1+j\}$$

or

The above DFT sequence can also be written as under:

$$X_4 = \{\underset{\substack{\uparrow \\ k=0}}{2+j\,0},\ 1-j,\ 0+j\,0,\ 1+j\}$$

Magnitude Plot

The magnitude at different values can be obtained as under:

For $k = 0$, we have

$$|X(k)| = \sqrt{(2)^2+(0)^2} = 2$$

For $k = 1$, we have

$$|X(k)| = \sqrt{(1)^2+(-1)^2} = \sqrt{2} = 1.414$$

For $k = 2$, we have

$$|X(k)| = \sqrt{0+0} = 0$$

For $k = 3$, we have

$$|X(k)| = \sqrt{(1)^2+(1)^2} = \sqrt{2} = 1.414$$

This magnitude plot has been shown in figure 5.9.

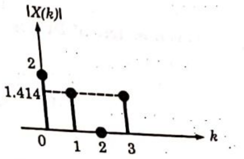

FIGURE 5.9 *Magnitude plot.*

EXAMPLE 5.6 Determine the discrete Fourier Transform (DFT) of four point sequence $x(n) = \{0, 1, 2, 3\}$

Solution. The 4-point DFT in the matrix form is given by,

$$X_4 = [W_4]\,.\,x(n)$$

Thus,

$$X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix} = \begin{bmatrix} 6 \\ 2j-2 \\ -2 \\ -2j-2 \end{bmatrix}$$

> **DO YOU KNOW?**
>
> Under suitable restrictions, the DFT closely approximates the spectrum of a continuous-time signal at a discrete set of frequencies.

Simplifying, we shall get

$$X_4 = \{\underset{\uparrow}{6},\ 2j-2,\ -2,\ -2j-2\} \quad \textbf{Ans.}$$

EXAMPLE 5.7 Compute the length-4 sequence from its DFT which is given by

$$X(k) = \{4,\ 1-j,\ -2,\ 1+j\}$$ *(Expected)*

Solution: We know that the IDFT in matrix form is expressed as

$$\text{IDFT} = x(n) = x_N = \frac{1}{N}\left[W_N^*\right].X_N \quad ...(i)$$

Here, X_N is the given DFT matrix. Also, '*' indicates complex conjugate. To obtain the complex conjugate, we have to change the sign of j term. For example, complex conjugate of $1 - j1$ i[s] $1 + j1$.

Now, we have already obtained the matrix W_4 in previous examples. It is reproduced here

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Therefore, $$\left[W_4^*\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Given matrix of DFT is

$$X_N = X_4 = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

Substituting equations (*iii*) and (*iv*), and substituting $N = 4$ in equation (*i*), we shall have

$$x_N = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

or $$x_N = \frac{1}{4}\begin{bmatrix} 4+1-j-2+1+j \\ 4+j-j^2+2-j-j^2 \\ 4-1+j-2-1-j \\ 4-j+j^2+2+j+j^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4+2+1+1 \\ 4-4 \\ 4+2-2 \end{bmatrix} \quad (\because j^2 = -1)$$

Simplifying, we get

or $$x_N = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

or $x(n) = \{1, 2, 0, 1\}$ **Ans.**

EXAMPLE 5.8 Determine the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ and check the validity of your answer by calculating its IDFT.

Solution: Let us compute 4 point DFT. We have already obtained the matrix for $[W_4]$ in previous example. It is reproduced here

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -j \\ 1 & +j & -1 & -j \end{bmatrix}$$

The given sequence is $x(n) = \{1, 1, 0, 0\}$

The DFT of this sequence has been computed in previous examples. It is $X_N = X(k) = \{2, 1-j, 0, 1+j\}$

Now, let us check this answer by using the expression for IDFT.

The IDFT is given by,

$$x(n) = \frac{1}{N}\left[W_N^*\right].X_N$$

Here,
$$\left[W_N^*\right] = \left[W_4^*\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

and
$$X_N = X_4 = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

Therefore, we have

$$x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2+1-j+0+1+j \\ 2+j+1+0-j+1 \\ 2-1+j+0-1-j \\ 2-j-1+0+j-1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

This means that $x(n) = \{1, 1, 0, 0\}$

But, this is same as the given sequence. Therefore, calculated DFT is correct.

Hence Proved

EXAMPLE 5.9 If $y(n) = \dfrac{[x(n) + x(-n)]}{2}$

Find $Y(k)$ if $X(k) = \{0.5, 2 + j, 3 + j2, j, 3, -j, 3 - j2, 2 - j\}$

Solution. We have,

$$y(n) = \frac{[x(n) + x(-n)]}{2} \quad ...(i)$$

Taking DFT of both sides, we obtain

$$Y(k) = \frac{[X(k) + X(-k)]}{2} \quad ...(ii)$$

Given $\quad X(k) = \{0.5, 2 + j, 3 + j2, j, 3, -j, 3 - j2, 2 - j\}$

Therefore, $\quad X(-k) = \{0.5, 2 - j, 3 - j2, -j, +3, j, 3 + j2, 2 + j\}$

Putting these values in equation (*ii*), we get,

$$Y(k) = \frac{1}{2}\{1, 4, 6, 0, 6, 0, 6, 4\}$$

or $\quad Y(k) = \{0.5, 2, 3, 0, 6, 0, 3, 2\}$ **Ans.**

5.7 RELATIONSHIP OF THE DFT TO OTHER TRANSFORMS

In this section, we shall discuss the relationship of DFT with several other transforms.

5.11 RELATIONSHIP TO THE FOURIER SERIES COEFFICIENTS OF A CONTINUOUS-TIME SIGNAL

Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period T_s, ... signal can be expressed in a Fourier Series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t F_0}$$

where (c_k) are the Fourier coefficients. If we sample $x_a(t)$ at a uniform rate $F_s = N/T_s$, ... obtain the discrete-time sequence

$$x(n) \equiv x_a(nT) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 nT} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N}$$

$$= \sum_{k=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} c_{k-lN}\right] e^{j2\pi kn/N}$$

It is clear that (5.48) is in the form of an IDFT formula, where

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \circ N\bar{c}_k$$

and

$$\bar{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN}$$

Hence, the $(\bar{c}_k)$ sequence is an alised version of the sequence (c_k).

5.12 PROPERTIES OF DISCRETE FOURIER TRANSFER (DFT)

The properties of Discrete Fourier Transform (DFT) are quite useful in the practical techniques ... processing signals. At this point, it may be noted that most of the properties of the DFT and ... transform have some similarity since they have some relationship between them.

The properties of Discrete Fourier Transform (DFT) can be listed as under :

1. Periodicity
2. Linearity
3. Shifting property
5. Circular convolution

Now, let us discuss these properties one by one in the subsequent sub-sections.

5.12.1 Periodicity

(i) Statement

This property states that if a discrete-time signal is periodic then its DFT will also be periodic. Also, if a signal or sequence repeats its waveform after N number of samples then it is called ... periodic signal or sequence and N is called the period of signal. Mathematically,

If $X(k)$ is an N-point DFT of $x(n)$ *i.e.*,

If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$, then we have

$$x(n+N) = x(n) \text{ for all values of } n \quad ...(5.3$$

$$X(k+N) = X(k) \text{ for all values of } k \quad ...(5.3$$

(ii) **Proof :**

According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

Replacing k by $k + N$, we obtain

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{(k+N)n} = \sum_{n=0}^{N-1} x(n) W_N^{kn} W_N^{Nn} \quad ...(5.36)$$

We know that W_N is a twiddle factor and it is expressed as

$$W_N = e^{-j\frac{2\pi}{N}} \quad ...(5.37)$$

Also,

$$W_N^{Nn} = \left(e^{-j\frac{2\pi}{N}}\right)^{Nn} = e^{-j\frac{2\pi}{N}.Nn} = e^{-j2\pi n}$$

or

$$W_N^{Nn} = \cos 2\pi n - j \sin 2\pi n$$

Since, n is an integer, therefore, we have

$$\cos 2\pi n = 1 \quad \text{and} \quad \sin 2\pi n = 0 \quad ...(5.38)$$

Hence,

$$W_N^{Nn} = 1$$

Substituting this value in equation (5.37), we have ...(5.39)

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ...(5.40)$$

Comparing equations (5.36) and (5.39), we get

$X(k + N) = X(k)$ **Hence proved.**

5.12.2 Linearity

Linearity properties states that DFT of linear combination of two or more signals is equal to the sum of linear combinatin of DFT of individual signals. Let us consider that $X_1(k)$ and $X_2(k)$ are the N-points DFTs of $x_1(n)$ and $x_2(n)$ respectively, and a and b are arbitrary constants either real or complex-valued, then mathematically

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then,

$$a x_1(n) + b x_2(n) \xleftrightarrow[N]{DFT} a X_1(k) + b X_2(k)$$

Here, a and b are some constants

Proof: According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ...(5.41)$$

Here,

$$x(n) = a x_1(n) + b x_2(n)$$

Therefore,

$$X(k) = \sum_{n=0}^{N-1} [a x_1(n) + b x_2(n)] W_N^{kn} = \sum_{n=0}^{N-1} a x_1(n) W_N^{kn} + \sum_{n=0}^{N-1} b x_2(n) W_N^{kn}$$

Since, a and b are constants, therefore, we can take them out of the summation sign.

Hence,

$$X(k) = a\sum_{n=0}^{N-1} x_1(n) W_N^{kn} + b\sum_{n=0}^{N-1} x_2(n) W_N^{kn} \quad ...(5.42)$$

Comparing equation (6.57) with the definition of DFT, we obtain

$$X(k) = a\,X_1\,(k) + b\,x_2(k)$$

Hence Pro...

5.12.3 Circular Symmetries of a Sequence

(GGSIPU, Sem. Exam., May 2...

We have discussed the periodicity property of discrete Fourier transform (DFT). Let us co... that input discrete time sequence is $x(n)$ then, the periodic sequence is denoted by $x_p(n)$... period of $x_p(n)$ is N which means that after N the sequence $x(n)$ repeats itself.

Now, we can write the sequence $x_p(n)$ as under:

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

Now, let us consider one example.

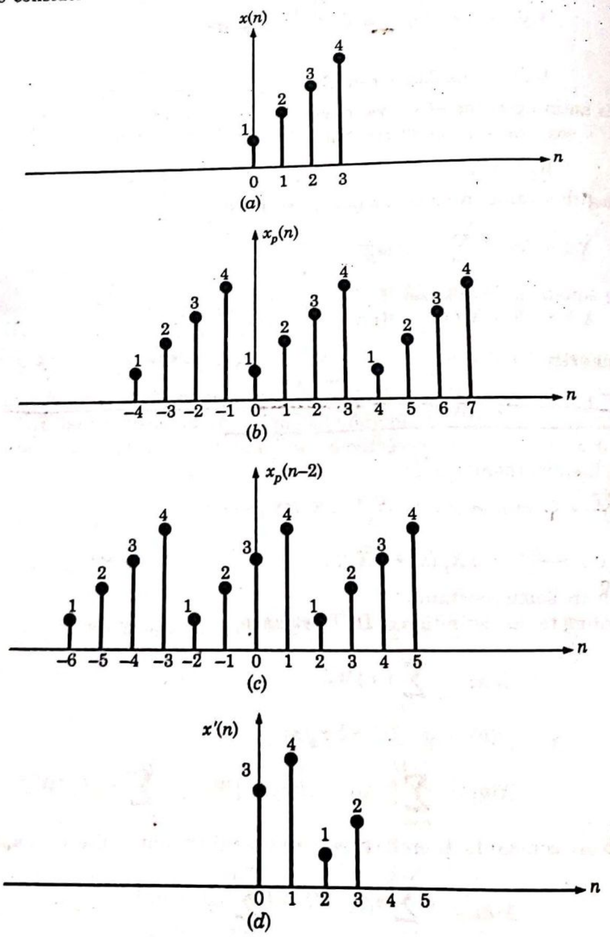

FIGURE 5.10 *Shifting of sequence $x(n)$.*

Let $x(n) = \{1, 2, 3, 4\}$. This sequence is shown in figure 5.10(*a*). The periodic sequence $x_p(n)$ is shown in figure 5.10(*b*). We shall delay the periodic sequence $x_p(n)$ by two samples as shown in figure 5.10(*c*). This sequence is denoted by $x_p(n-2)$. Now, the original signal is present in the range $n = 0$ to $n = 3$. In the same range, we shall write the shifted signal as shown in figure 5.10 (*d*). This signal is denoted by $x'(n)$.

Now, from figure 5.10, we can write every sequence as under:

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

$$x_p(n) = \{\ldots\ 1, 2, 3, 4, \underset{\uparrow}{1}, 2, 3, 4, 1, 2, 3, 4, \ldots\} \quad \ldots(5.44)$$

$$x_p(n-2) = \{\ldots\ 1, 2, 3, 4, 1, 2, \underset{\uparrow}{3}, 4, 1, 2, 3, 4, \ldots\} \quad \ldots(5.45)$$

$$\ldots(5.46)$$

and

$$x'(n) = \{\underset{\uparrow}{3}, 4, 1, 2\} \quad \ldots(5.47)$$

Now, from equation (5.44) and (5.47), we can say that the sequence $x'(n)$ is obtained by **circularly shifting** sequence $x(n)$, by two samples. This means that **$x'(n)$ is related to $x(n)$ by circular shift.**

Notation

This relation of circular shift is denoted by,

$$x'(n) = x(n-k, \text{ modulo } N) \quad \ldots(5.48)$$

It means that divide $(n - k)$ by N and retain the remainder only. We can also use the short hand notation as under :

$$x'(n) = x((n-k))_N \quad \ldots(5.49)$$

Here, k indicates the number of samples by which $x(n)$ is delayed and N indicates N-point DFT. In the present example, the sequence $x(n)$ is delayed by two samples; thus $k = 2$. Because, there are four samples in $x(n)$, this is 4-point DFT. Hence, $N = 4$.

Now, for this example, equation (5.48) becomes,

$$x'(n) = x((n-2))_4 \quad \ldots(5.50)$$

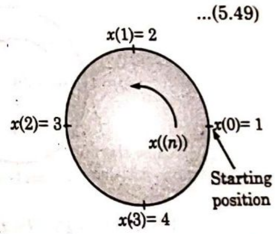

FIGURE 5.11 $x((n))_4$ – *The samples of $x(n)$ are plotted circularly anticlockwise.*

Graphical Representation

The circular shifting of a sequence can be plotted graphically as under:

(i) Circular plot of sequence $x(n)$

Here, we have considered,

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Circular plot of $x(n)$ is denoted by $x((n))_4$. This plot is obtained by writing the samples of $x(n)$ circularly anticlockwise. It is shown in figure 5.11.

(ii) Circular delay by one sample

To delay sequence $x(n)$ circularly by one sample, shift every sample circularly in anticlockwise direction by 1. This is shown in figure 5.12. This operation is denoted by $x((n-1))$.

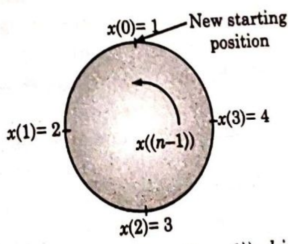

FIGURE 5.12 $x((n-1))$ shi*f*t *every sample by 1 in anticlockwise direction.*

It may be noted that **delay by k samples means shift the sequence circularly in anticlockwise direction by k.**

(iii) Circular advance by one sample

To advance sequence $x(n)$ circularly by one sample shift every sample circularly in clockwise direction by 1 sample. This sequence is denoted by $x((n + 1))$. It is shown in figure 5.13.

It may be noted that **advance by k samples means shift the sequence circularly in clockwise direction by k.**

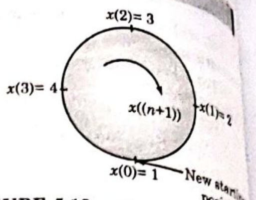

FIGURE 5.13 *$x((n + 1))$ shift every sample by one in clockwise direction.*

(iv) Circularly folded sequence

A circularly folded sequence is denoted by $x((-n))$. We have plotted sequence $x((n))$ in anticlockwise direction. So, folded sequence $x((-n))$ is plotted in clockwise direction. It is shown in figure 5.14.

It may be noted that **circular folding means plot the samples in clockwise direction.**

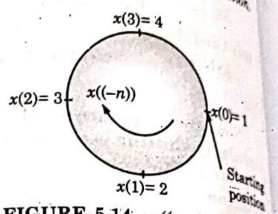

FIGURE 5.14 *$x((-n))$ samples are plotted circularly clockwise.*

Now recall equation (5.61) it is,

$$x'(n) = x((n-2))_4$$

It indicates delay of sequence $x(n)$ by two samples. It is obtained by rotating samples of figure 5.16 in anticlockwise direction by two samples. This sequence is shown in figure 5.15.

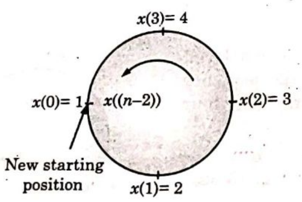

FIGURE 5.15 *Plot of $x((n-2))_4$*

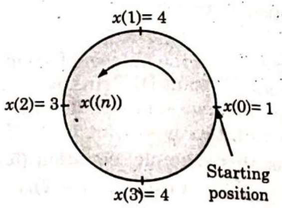

FIGURE 5.16 $x(n) = \{1, 4, 3, 4\}$

(v) Circularly even sequence

The N-point discrete time sequence is circularly even if it is symmetric about the point zero on the circle.

This means that

$$x(N-n) = x(n), \ 1 \le n \le N-1$$

Now, let us consider the sequence

$$x(n) = \{1, 4, 3, 4\}.$$

It has been plotted as shown in figure 5.16.

It may be noted that this sequence is symmetric about point zero on the circle. So, it is circularly even sequence. We can also verify it using mathematical equation,

The sequence is $x(n) = \{1, 4, 3, 4\}$

$\therefore \ x(0) = 1, \quad x(1) = 4, \quad x(2) = 3$ and $x(3) = 4$

We have the following condition for circularly even sequence:

DO YOU KNOW?

The use of window functions to minimize spectral leakage requires windows having transforms with narrow mainlobs and low side-lobs. Useful window functions having transforms with sidelobes lower than those of rectangular windows include triangular, Hanning, Hamming, and Kaiser-Bessel windows.

$$x(N - n) = x(n) \qquad ...(5.51)$$

Here, $N = 4$.

Let us check this condition by substituting different values of n as under :

For $n = 1$, we have

$$x(4 - 1) = x(1) \text{ this means } x(3) = x(1) = 4$$

For $n = 2$, we have

$$x(4 - 2) = x(2) \text{ this means } x(2) = x(2) = 3$$

For $n = 3$, we have

$$x(4 - 3) = x(3) \text{ this means } x(1) = x(3) = 4.$$

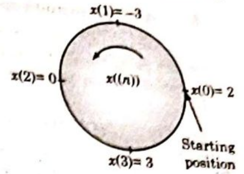

FIGURE 5.17 *Plot of* $x(n) = \{2, -3, 0, 3\}$.

Since for all values of n, equation (5.51) is satisfied, the given sequence is circularly even.

(vi) Circularly odd sequence

A N-point sequence is called circularly odd if it is antisymmetric about point zero on the circle.

This means that

$$x(N - n) = -x(n), \quad 1 \le n \le N - 1$$

Let us consider the sequence,

$$x(n) = \{2, -3, 0, 3\}$$

This sequence has been plotted as shown in figure 5.17.

Here, $x(0) = 2$, $x(1) = -3$, $x(2) = 0$ and $x(3) = 3$.

We have the following condition for circularly odd sequence:

$$x(N - n) = -x(n), \quad \text{for } 1 \le n \le N - 1 \qquad ...(5.64)$$

For $n = 1$, we have

$$x(4 - 1) = -x(1) \text{ this means } x(3) = -x(1)$$

For $n = 2$, we have

$$x(4 - 2) = -x(2) \text{ this means } x(2) = -x(2)$$

For $n = 3$, we have

$$x(4 - 3) = -x(3) \text{ this means } x(1) = -x(3)$$

Thus, for all values of n, equation (5.64) is satisfied. Hence, the sequence is circularly odd.

Summary of Circular Properties

Table 5.2 shows summary of circular property.

TABLE 5.2

S.No.	Sequence	Expression	Explanation
1.	Input sequence	$x((n))$	Plot the samples of $x(n)$ in anti-clockwise direction. Anticlockwise means positive direction.
2.	Circular delay	$x((n - k))$	Shift sequence $x(n)$ in anticlock-wise direction by k samples.
3.	Circular advance	$x((n + k))$	Shift sequence $x(n)$ in clockwise direction by k samples.
4.	Circular folding	$x((-n))$	Plot the samples of $x(n)$ in clockwise direction. Clockwise means negative direction.
5.	Circularly even	$x(N - n) = x(n)$	Sequence is symmetric about the point zero on the circle.
6.	Circularly odd	$x(N - n) = -x(n)$	Sequence is antisymmetric about the point zero on the circle.

5.12.4 Symmetry Properties of DFT

(GGSIPU, Sem. Exam., May 2009)

The symmetry properties of DFT can be derived in a similar way as we derived DTFT symmetry properties. We know that DFT of sequence $x(n)$ is denoted by $X(k)$. Now, if $x(n)$ and $X(k)$ are complex valued sequence then it can be represented as under :

$$x(n) = x_R(n) + jx_I(n), \quad 0 \le n \le N-1 \qquad ...(5.52)$$

and $$X(k) = X_R(k) + jX_I(k), \quad 0 \le k \le N-1 \qquad ...(5.53)$$

Here, R stands for real part and I stands for imaginary part

According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad ...(5.54)$$

Substituting equation (5.65) in equation (5.67), we have

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + jx_I(n)]\, e^{-j2\pi kn/N} \qquad ...(5.55)$$

But according to Euler's identity, we have

$$e^{-j2\pi kn/N} = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)$$

Substituting this value in equation (5.68), we obtain

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + jx_I(n)]\left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

or $$X(k) = \sum_{n=0}^{N-1} \left[x_R(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) - jx_R(n)\sin\left(\frac{2\pi kn}{N}\right) + jx_I(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) - j^2 x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right]$$

Here, $j^2 = -1$ and writing summation for real and imaginary parts separately we get,

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$-j\sum_{n=0}^{N-1} \left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(5.56)$$

Comparing equations (5.69) and (5.66), we can write

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(5.57)$$

and $$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(5.58)$$

Equations (5.70) and (5.71) are obtained by using definition of DFT. Similarly, we can obtain real and imaginary parts of $x(n)$ using definition of IDFT.

Hence, we have

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1} \left[X_R(k)\cos\left(\frac{2\pi kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(5.59)$$

and $$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1} \left[X_R(k)\sin\left(\frac{2\pi kn}{N}\right) + X_I(k)\cos\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(5.60)$$

Now, let us consider different cases as under:

Case (i) : When $x(n)$ is real valued

Statement : If $x(n)$ is real valued then, we have

$$X(N-k) = X(-k) = X^*(k)$$

Proof

According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ...(5.61)$$

Replacing k by $N-k$, we have

$$X(N-k) = \sum_{n=0}^{N-1} x(n) W_N^{(N-k)n}$$

or

$$X(N-k) = \sum_{n=0}^{N-1} x(n) W_N^{Nn} W_N^{-kn} \quad ...(5.62)$$

Now we have,

twidde factor $\quad W_N = e^{-\frac{j2\pi}{N}}$

Therefore $\quad W_N^{Nn} = \left(e^{-\frac{j2\pi}{N}}\right)^{Nn} = e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n$

Since, n is an integer, $\cos 2\pi n = 1$ and $\sin 2\pi n = 0$

Therefore $\quad W_N^{Nn} = 1$

Thus, equation (5.62) becomes,

$$X(N-k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn} \quad ...(5.63)$$

Comparing equation (5.63) with definition of DFT (equation (5.61)), we get

$$X(N-k) = X(-k) \quad ...(5.64)$$

Now, using equation (5.66), we can write

$$X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn} \quad ...(5.65)$$

Thus, from equations (5.64) and (5.65), we get

$$X(N-k) = X(-k) = X^*(k)$$

Case (ii) : When $x(n)$ is real and even

Statement : When $x(n)$ is real and even which means,

$$x(n) = x(N-n), \text{ then DFT becomes}$$

$$X(k) = X_R(k)$$

Proof

Since imaginary part is zero, substituting $x_I(n) = 0$ in equation (5.57), we get

$$X_R(k) = \sum_{n=0}^{N-1} X_R(n) \cos\left(\frac{2\pi kn}{N}\right)$$

Similarly, IDFT can be written by substituting $X_I(k) = 0$ in equation (5.59) *i.e.*,

$$x_R(n) = \frac{1}{N} \sum_{K=0}^{N-1} X_R(k) \cos\left(\frac{2\pi kn}{N}\right)$$

Case (*iii*) : When $x(n)$ is real and odd

Statement : When $x(n)$ is real and odd which means,

$x(n) = -x(N-n)$ then the DFT becomes,

$$X(k) = -j\sum_{n=0}^{N-1} x_R(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Proof

Because, $x(n)$ is real, let us substitute $x_I(n) = 0$ in equation (5.56). Similarly, $x(n)$ is odd and 'cos' is even function so we can write, $\cos\left(\frac{2\pi kn}{N}\right) = 0$. Thus, first summation in equation (5.56) becomes zero. In the second summation of equation (5.69), substituting $x_I(n) = 0$, we shall have

$$X(k) = -j\sum_{n=0}^{N-1} x_R(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Similarly, IDFT can be written as

$$x(n) = j\frac{1}{N}\sum_{k=0}^{N-1} X_R(k)\sin\left(\frac{2\pi kn}{N}\right)$$

Case (*iv*) : When $x(n)$ is purely imaginary sequence

When x(n) is purely imaginary which means $x_R(n) = 0$ and $x(n) = jx_I(n)$ then substituting $x_R(n) = 0$ in equation (5.70), we get

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n)\sin\left(\frac{2\pi kn}{N}\right)$$

And substiting $x_R(n) = 0$ in equation (5.71), we obtain

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n)\cos\left(\frac{2\pi kn}{N}\right)$$

Symmetry properties can be summarized as shown in figure 5.18.

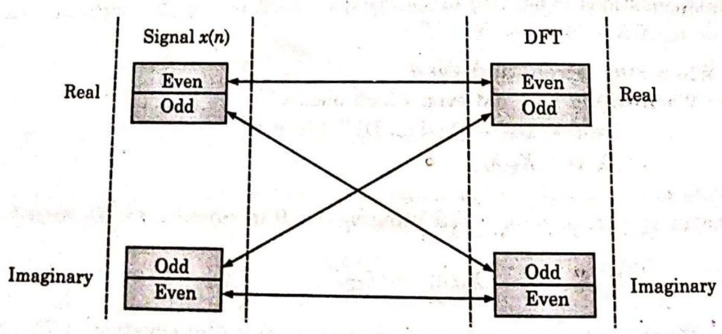

FIGURE 5.18 *Summary of symmetry property.*

Table 5.4 illustrates this summary.

TABLE 5.4

S.No.	N point sequence $x(n)$, $0 \le n \le N-1$	N point DFT
1.	$x^*(n)$	$X^*(N-k)$
2.	$x^*(N-k)$	$X^*(k)$
3.	$x_R[n]$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$
4.	$jx_I[n]$	$X_{ce}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$
5.	$x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$X_R(k)$
6.	$x_{co}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$jX_I(k)$

EXAMPLE 5.10 The first five points of the 8 point DFT of a real valued sequence are {0.25, 0.125 − j0.3018, 0, 0.125 − j0.0518, 0}. Determine the remaining three points.

Solution : Given DFT points are :

$$X(0) = 0.25$$
$$X(1) = 0.125 - j0.3018$$
$$X(2) = 0$$
$$X(3) = 0.125 - j0.0518$$
$$X(4) = 0$$

Given sequence is a real valued sequence. According to the symmetry property we have,

$$X^*(k) = X(N-k)$$

or

$$X(k) = X^*(N-k) \quad ...(i)$$

This is 8 point DFT. Thus, $N = 8$

Therefore, $X(k) = X^*(8-k)$ ...(ii)

Now, we want remaining three samples namely X(5), X(6) and X(7). Substituting $k = 5$ in equation (ii), we have

$$X(5) = X^*(8-5) = X^*(3)$$

We have $X(3) = 0.125 - j0.0518$

Also $X^*(3) = 0.125 + j0.0518$

Hence, $X(5) = 0.125 + j0.0518$

Substiting $k = 6$ in equation (ii), we have

$$X(6) = X^*(8-6) = X^*(2)$$

We have $X(2) = 0$,

Thus $X^*(2) = 0$

Therefore, $X(6) = 0$

Similarly, substituting $k = 7$ in equation (ii), we obtain

$$X(7) = X^*(8-7) = X^*(1)$$

We have $X(1) = 0.125 - j0.3018$

Thus, $X(7) = 0.125 + j0.3018$ **Ans.**

EXAMPLE 5.11 The first five DFT points of real and even sequence $x(n)$ of length eight are given below. Determine remaining three points. $X(k) = \{5, 1, 0, 2, 3, ...\}$

Solution : Given DFT points are

$$X(0) = 5,\ X(1) = 1,\ X(2) = 0,\ X(3) = 2 \text{ and } X(4) = 3.$$

According to symmetry property, we have

$$X^*(k) = X(N - k)$$

Also, $X(k) = X^*(N - k)$

This is 8 point DFT.

Thus, $N = 8$

Therefore, $X(k) = X^*(8 - k)$

Hence $X(5) = X^*(8 - 5) = X^*(3)$

or $X(5) = 2$

Now, $X(6) = X^*(8 - 6) = X^*(2)$

so that $X(6) = 0$

Also, $X(7) = X^*(8 - 7) = X^*(1)$

or $X(7) = 1$

5.12.5 Duality Property

Statement : If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$

then $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} Nx[((-k))_N]$

Proof

Let us consider a discrete time sequence $x(n)$. Its periodic extension is denoted by $x_p(n)$. Now, DFT of x(n) is $X(k)$ and the periodic expension of X(k) is denoted by $X_p(k)$.

This means that

$$x_p(n) = x((n))_N \quad ...(5.66)$$

and

$$X_p(k) = X((k))_N \quad ...(5.67)$$

Thus, we can write

$$x_p(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_p(k) \quad ...(5.68)$$

Now, let us define periodic sequence $x_{1p}(n) = X_p(n)$. One period of this sequence is a finite duration sequence $x_1(n) = x(n)$.

The discrete Fourier series coefficients of $x_{1p}(n)$ are denoted by $X_{1p}(k)$ and $X_{1p}(k) = Nx_p(-k)$.

Thus, DFT of $x_1(n)$ which is denoted by $X_1(k)$ will be

$$X_1(k) = \begin{cases} Nx_p(-k) & \text{for} \quad 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases} \quad ...(5.69)$$

Equation (5.82) can also be written as,

$$X_1(k) = \begin{cases} Nx[((-k))]_N & \text{for} \quad 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases} \quad ...(5.70)$$

Now $X_1(k)$ is DFT of $x_1(n)$ and $x_1(n)$ is a finite duration sequence denoted by $X(n)$.

Therefore, $X(n) \xleftrightarrow[N]{DFT} Nx[((-k)_N]$

5.12.6 Multiplication of Two DFTs and Circular Convolution*

(GGSIPU, Sem. Exam., May 2006)

This property states that the multilication of two DFTs is equivalent to the circular convolution of their sequences in time domain.

Mathematicaly, we have

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then,

$x_1(n) \; Ⓝ \; x_2(n) \xleftrightarrow[N]{DFT} X_1(k) \, . \, X_2(k)$ ...(5.71)

Here, Ⓝ indicates circular convolution.

Let the result of circular convolution of $x_1(n)$ and $x_2(n)$ be $y(m)$ then the circular convolution can also be expressed as,

$$y(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N, \quad m = 0, 1, \ldots N-1 \qquad \text{...(5.72)}$$

Here, the term $x_2((m-n))_N$ indicates the circular convolution.

Proof

Let us consider two discrete time sequences $x_1(n)$ and $x_2(n)$.

The DFT of $x_1(n)$ can be expressed as under:

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi kn}{N}}, \; k = 0, 1, \ldots N-1 \qquad \text{...(5.73)}$$

To avoid the confusion, let us write the DFT of $x_2(n)$ using different index of summation *i.e.*,

$$X_2(k) = \sum_{l=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi kl}{N}}, \; k = 0, 1, \ldots N-1 \qquad \text{...(5.74)}$$

It may be noted that in equation (5.74), instead of n we have used l.

Let us denote the multiplication of two DFTs $X_1(k)$ and $X_2(k)$ by $Y(k)$.

Therefore, $Y(k) = X_1(k) \, . \, X_2(k)$ ...(5.75)

Let IDFT of $Y(k)$ be $y(m)$. Then using definition of IDFT, we have

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k)\, e^{\frac{j2\pi km}{N}} \qquad \text{...(5.76)}$$

Substituting equation (5.75) in equation (5.76), we obtain

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \, . \, X_2(k)\, e^{\frac{j2\pi km}{N}} \qquad \text{...(5.77)}$$

Substituting the values of $X_1(k)$ and $X_2(k)$ from equations (5.73) and (5.74) in equation (5.77), we obtain

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n)\, e^{-\frac{j2\pi kn}{N}}\right]\left[\sum_{l=0}^{N-1} x_2(l)\, e^{-\frac{j2\pi kl}{N}}\right] e^{\frac{j2\pi km}{N}} \qquad \text{...(5.78)}$$

* State and prove the 'circular convolution' property of DFT. (GGSIPU, Sem. Exam., May 2006)

Rearranging the summations and terms in equation (5.91), we get

$$y(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{\frac{-j2\pi kl}{N}} \cdot e^{\frac{-j2\pi kn}{N}} \cdot e^{j2\pi km}\right]$$

Therefore, we have

$$y(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{+j2\pi k(m-n-l)/N}\right]$$

Let us consider the last term of equation (5.79). It may be written as under:

$$e^{j2\pi k(m-n-l)/N} = [e^{j2\pi(m-n-l)/N}]^k$$

Now, let us use the following standard summation expression:

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & \text{for } a = 1 \\ \dfrac{1-a^N}{1-a} & \text{for } a \neq 1 \end{cases}$$

Let, here, $$a = e^{+j2\pi\frac{(m-n-l)}{N}}$$

Now, according to equation (5.81), we shall consider two cases :

Case (i) : When $a = 1$

If $(m-n-l)$ is multiple of N which means,

$(m-n-l) = N, 2N, 3N, \ldots$ then equation (5.95) becomes,

$$a = e^{+j2\pi} = e^{+j2\pi(2)} = e^{+j2\pi(3)} \ldots = 1$$

Thus, when $(m-n-l)$ is multiple of N (this means that $a = 1$), then according to equation (5.81), the third summation in equation (5.79) becomes equal to N.

Case (ii) : When $a \neq 1$

If $a \neq 1$, this means that if $m-n-l$ is not multiple of N then according to equation (5.81), we have

$$\sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a} \quad \ldots(5.83)$$

Substituting equation (5.82) in equation (5.83), we obtain

$$\sum_{k=0}^{N-1} [e^{+j2\pi(m-n-l)}]^k = \frac{1-e^{+j2\pi(m-n-l)}}{1-e^{\frac{+j2\pi(m-n-l)}{N}}} \quad \ldots(5.84)$$

Here, m, n and l are integers.

Hence, $e^{+j2\pi(m-n-l)} = 1$ always. Therefore, R.H.S. of equation (5.84) becomes zero when $a \neq 1$. Therefore to get the result of equation (5.81), we have to consider the condition $a = 1$. This means that when $m-n-l$ is multiple of N. For this condition, we have the result of summation equals to N. Thus, equation (5.79) becomes,

$$y(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot N$$

Therefore, we have

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \quad \ldots(5.85)$$

We have obtained equation (5.98) for the condition $(m - n - l)$ is multiple of N. This condition can be expressed as,

$$m - n - l = -pN$$

Here, p is an integer and an integer can be positive or negative. For simplicity, we have considered negative integer. Now, from equation (5.86), we obtain ...(5.86)

$$l = m - n + pN$$

Substituting this value in equation (5.85), we get, ...(5.87)

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m - n + pN)$$

...(5.88)

Here, we have not considered the second summation of equation (5.85). Because this summation is in terms of l and exponential term is absent in equation (5.88).

Now, the term $x_2(m - n + pN)$ indicates a periodic sequence with period N. This is because p is an integer. This term also indicates that the periodic sequence is delayed by n samples. Further, we know that if a sequence is periodic and delayed then it can be expressed as,

$$x_2(m - n + pN) = x_2((m - n))_N$$

...(5.89)

Here, the R.H.S. term indicates circular shifting of $x_2(n)$. Substituting this value in equation (5.88), we get

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m - n))_N \quad m = 0, 1, \ldots N - 1 \qquad \text{...(5.90)}$$

This is the equation of circular convolution. Since in this equation, the sequence $x_2(n)$ is shifted circularly, this type of convolution is called as **Circular Convolution.**

EXAMPLE 5.12 Given the two sequences of length 4 as under:

$$x(n) = \{0, 1, 2, 3\}$$

$$h(n) = \{2, 1, 1, 2\}$$

Compute the circular convolution.

Solution : According to the definition of circular convolution, we have

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m - n))_N \qquad \text{...(i)}$$

Here, given sequences are $x(n)$ and $h(n)$. The length of sequence is 4 that means $N = 4$. Thus, equation (i) becomes,

$$y(m) = \sum_{n=0}^{3} x(n)h((m - n))_4 \qquad \text{...(ii)}$$

(i) We draw $x(n)$ and $h(n)$ as shown in figure 5.19(a) and (b).

It may be noted that $x(n)$ and $h(n)$ are plotted in anticlockwise direction.

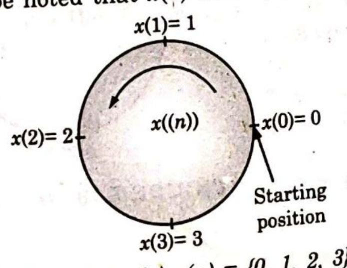

FIGURE 5.19 *(a)* $x(n) = \{0, 1, 2, 3\}$.

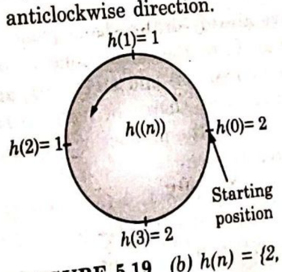

FIGURE 5.19 *(b)* $h(n) = \{2, 1, 1, 2\}$.

Now, let us calculate different values of $y(m)$ by putting $m = 0$ to $m = 3$ in equation (*ii*).

(*ii*) Calculation of $y(0)$

Substituting $m = 0$ in equation (*ii*), we get

$$y(0) = \sum_{n=0}^{3} x(n)h((-n))_4 \qquad ...(iii)$$

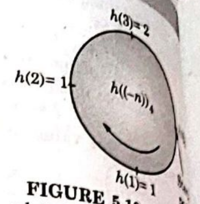

FIGURE 5.19 (*c*) $h((-n))_4$ plotted in clockwise direction

Equation (*iii*) shows that we have to obtain the product of $x(n)$ and $h((-n))$, and then we have to take the summation of product elements. Using graphical method, this calculation is done as follows.

The sequence $h((-n))_4$ indicates circular folding of $h(n)$. This sequence is obtained by plotting $h(n)$ in a clockwise direction as shown in figure 5.19(*c*).

To do the calculations, we plot $x(n)$ and $h((-n))$ on two concentric circles as shown in figure 5.19(*d*). Also, $x(n)$ is plotted on the inner circle and $h((-n))$ is plotted on the outer circle.

Now, according to equation (*ii*), individual values of product $x(n)$ and $h((-n))$ are obtained by multiplying two sequences point by point. Then, $y(0)$ is obtained by adding all product terms.

Therefore, we have

$$y(0) = (0 \times 2) + (1 \times 2) + (1 \times 2) + (3 \times 1) = 0 + 2 + 2 + 3$$

or $\quad y(0) = 7$

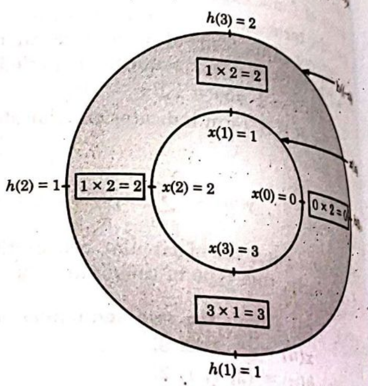

FIGURE 5.19 (*d*) $\sum_{n=0}^{3} x(n)h((-n))_4$

(*iii*) Calculation of $y(1)$

Substituting $m = 1$ in equation (*ii*), we have

$$y(1) = \sum_{n=0}^{3} x(n)h((1-n))_4$$

Here, $h((1-n))_4$ is same as $h((-n+1))_4$. This indicates delay of $h((-n))$ by 1 sample. This is obtained by shifting $h((-n))$ in anticlockwise direction by 1 sample, as shown in figure 5.19(*e*).

We have already drawn the sequence $x(n)$ as shown in figure 5.19(*a*). To do the calculations, according to equation (*iv*), two sequences $x(n)$ and $h((1-n))_4$ are plotted on two concentric circles as shown in figure 5.19(*f*). Also, $y(1)$ is obtained by adding the product of individual terms.

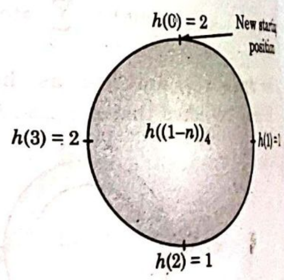

FIGURE 5.19 (*e*). $h((-n+1))_4$

Therefore, we write

$$y(1) = (0 \times 1) + (3 \times 1) + (2 \times 2) + (1 \times 2) = 0 + 3 + 4 + 2$$

or $\quad y(1) = 9$

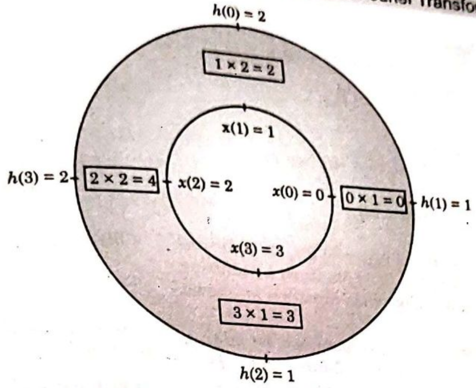

FIGURE 5.19 (f) $y(1) = \sum_{n=0}^{3} x(n)h((1-n))_4.$

(iv) Calculation of $y(2)$

Substituting $m = 2$ in equation (ii), we obtain,

$$y(2) = \sum_{n=0}^{3} x(n)h((2-n))_4 \quad ...(v)$$

Here, $h((2-n))_4$ is same as $h((-n+2))_4$. It indicates delay of $h((-n))_4$ by 2 samples. It is obtained by shifting $h((-n))_4$ by two samples in anticlockwise direction as shown in figure 5.19(g).

According to equation (iv), the value of $y(2)$ is obtained adding individual product terms as shown in figure 5.19(h).

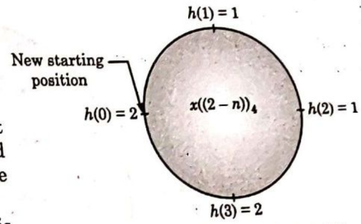

FIGURE 5.19 (g) $h((-n+2))_4.$

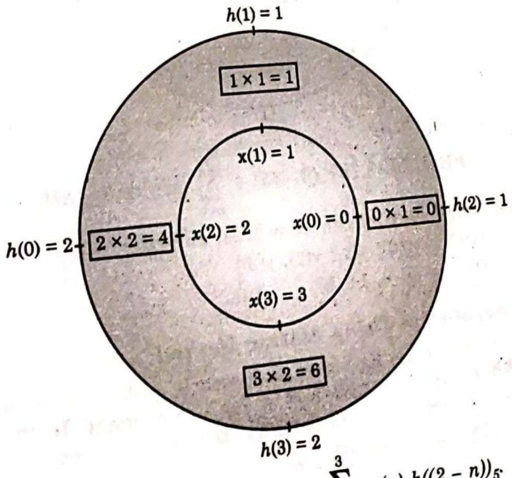

FIGURE 5.19 (h) $y(2) = \sum_{n=0}^{3} x(n)\, h((2-n))_5.$

Therefore, we have

$$y(2) = (0 \times 1) + (3 \times 2) + (2 \times 2) + (1 \times 1)$$

or $$y(2) = (0 + 6 + 4 + 1)$$

or $$y(2) = 11$$

Step V : Calculation of $y(3)$

Substituting $m = 3$ in equation (*ii*), we get

$$y(3) = \sum_{n=0}^{3} x(n)h((3 - n))_4 \quad ...(iv)$$

Here, $h((3 - n))_4$ is same as $h((-n + 3))_4$. It indicates delay of $h((-n))_4$ by 3 samples. It is obtained by shifting $h((-n))_4$ by 3 samples in anticlockwise direction as shown in figure 5.19(*i*).

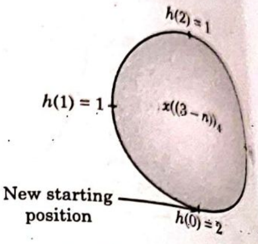

FIGURE 5.19 (*i*). $h((-n +$

According to equation (*v*), $y(3)$ is obtained by adding individual product terms as shown in figure 5.19(*j*) *i.e.*,

$$y(3) = (0 \times 2) + (3 \times 2) + (2 \times 1) + (1 \times 1)$$

or $$y(3) = 0 + 6 + 2 + 1$$

or $$y(3) = 9$$

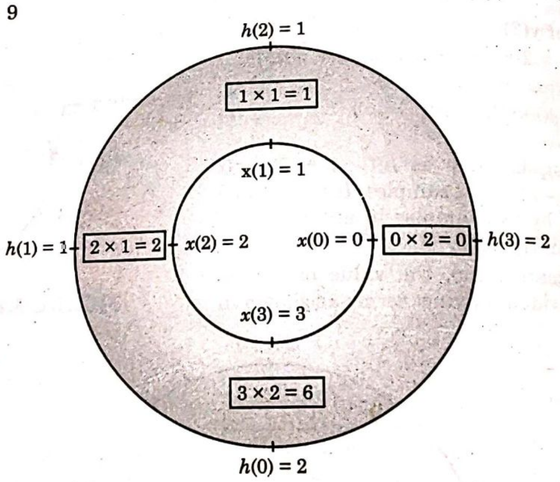

FIGURE 5.19 (*j*). $y(3) = \sum_{n=0}^{3} x(n)h((3 - n))_4$

Now, the resultant sequence $y(m)$ can be written as under:

$$y(m) = \{y(0), y(1), y(2), y(3)\}$$

or $$y(m) = \{7, 9, 11, 9\}.$$ **Ans.**

5.12.6.1. Circular Convolution Using Matrix Method

The graphical method which we have just discussed is quite tedious, especially when m samples are present. While the matrix method is more convenient. In the matrix method, a sequence is repeated via circular shifing of samples. It is represented as under :

we have $y(m) = x(n) \; Ⓝ \; h(n) = h(n) \; Ⓝ \; x(n)$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N-2) \\ y(N-1) \end{bmatrix} = \begin{bmatrix} h(0) & h(N-1) & h(N-2) & \cdots & h(2) & h(1) \\ h(1) & h(0) & h(N-1) & \cdots & h(3) & h(2) \\ h(2) & h(1) & h(0) & \cdots & h(4) & h(3) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ h(N-2) & h(N-3) & h(N-4) & \cdots & h(0) & h(N-1) \\ h(N-1) & h(N-2) & h(N-3) & \cdots & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-2) \\ x(N-1) \end{bmatrix}$$

EXAMPLE 5.13 **Determine the following sequence**

$$y(n) = x(n) \textcircled{N} h(n)$$

where $x(n) = \{1, 2, 3, 1\}$
$\uparrow$

and $h(n) = \{4, 3, 2, 2\}$
$\uparrow$

Solution : we have, $y(m) = x_2(n) \textcircled{N} x_1(n) = h(n) \textcircled{N} x(n)$

Using matrix method, we shall have

Here, $x(0) = 1$, $x(1) = 2$, $x(3) = 3$, $x(3) = 1$

and $h(0) = 4$, $h(1) = 3$, $h(2) = 2$, $h(3) = 2$

Here, $N = 4$

In the matrix form, we have

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} h(0) & h(3) & h(2) & h(1) \\ h(1) & h(0) & h(3) & h(2) \\ h(2) & h(1) & h(0) & h(3) \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 & 3 \\ 3 & 4 & 2 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} (4\times1)+(2\times2)+(2\times3)+(3\times1) \\ (3\times1)+(4\times2)+(2\times3)+(2\times1) \\ (2\times1)+(3\times2)+(4\times3)+(2\times1) \\ (2\times1)+(2\times2)+(3\times3)+(4\times1) \end{bmatrix} = \begin{bmatrix} 4+4+6+3 \\ 3+8+6+2 \\ 2+6+12+2 \\ 2+4+9+4 \end{bmatrix}$$

Hence, we have

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

Therefore, $y(m) = x(n) \textcircled{N} h(n) = \{17, 19, 22, 19\}$ **Ans.**

EXAMPLE 5.14 Use the four point DFT and IDFT to determine the circular conv... following sequences : (Very Im...

$$x_1(n) = (\underset{\uparrow}{1}, 2, 3, 1)$$

$$x_2(n) = (\underset{\uparrow}{4}, 3, 2, 2)$$

Solution : The four point DFT of $x_1(n)$ is $X_1(k)$ and it is given by,

$$X_1(k) = [W_4]x_{1N}$$

We have,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Therefore, $$X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2+3+1 \\ 1-2j-3+j \\ 1-2+3-1 \\ 1+2j-3-j \end{bmatrix} = \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

Therefore, $X_1(k) = \{7, -2-j, 1, -2+j\}$

Similarly, $X_2(k) = [W_4]x_{2N}$

Also, $$X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

Therefore, $$X_2(k) = \begin{bmatrix} 4+3+2+2 \\ 4-3j-2+2j \\ 4-3+2-2 \\ 4+3j-2-2j \end{bmatrix} = \begin{bmatrix} 11 \\ 2-j \\ 1 \\ 2+j \end{bmatrix}$$

or $X_2(k) = \{11, 2, -j, 1, 2, +j\}$

Now, according to property of circular convolution, we have

$$x_1(n) \; N \; x_2(n) = X_1(k) \,.\, X_2(k) = X_3(k)$$

or $X_3(k) = \{7, -2-j, 1, -2+j\} \,.\, \{11, +2-j, 1, 2+j\}$

or $X_3(k) = \{77, -5, 1, -5\}$

Let the result of $x_1(n)$ N $x_2(n)$ be sequence $x_3(n)$. It is obtained by computing IDFT of X_3... According to the definition of IDFT, we have,

$$x_3(n) = \frac{1}{N}\left[W_N^*\right] \,.\, X_N$$

Hence, $$x_3(n) = \frac{1}{4}\left[W_4^*\right] \,.\, X_{3N}$$

or
$$x_3(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}\begin{bmatrix} 77 \\ -5 \\ 1 \\ -5 \end{bmatrix}$$

or
$$x_3(n) = \frac{1}{4}\begin{bmatrix} 77-5+1-5 \\ 77-5j-1+5j \\ 77+5+1+5 \\ 77+5j-1-5j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 68 \\ 76 \\ 88 \\ 76 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

Considering only real part, approximately sequence $x_3(n)$ can be written as under:

$x_3(n) = \{17, 19, 22, 19\}$ **Ans.**

EXAMPLE 5.15 **Compute the 8-point circular convolution for following sequences:**

$$x_1(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

$$x_2(n) = \sin\left(\frac{3\pi n}{8}\right) \quad 0 \le n \le 7$$

Solution : First, let us obtain sequence $x_2(n)$ by substituting values of n from $n = 0$ to $n = 7$.

For $n = 0$, we have

$$x_2(0) = \sin\left(\frac{3\pi\times 0}{8}\right) = 0$$

For, $n = 1$, we have

$$x_2(1) = \sin\left(\frac{3\pi\times 1}{8}\right) = 0.92$$

For $n = 2$, we have

$$x_2(2) = \sin\left(\frac{3\pi\times 2}{8}\right) = 0.707$$

For $n = 3$, we have

$$x_2(3) = \sin\left(\frac{3\pi\times 3}{8}\right) = -0.38$$

For $n = 4$, we have

$$x_2(4) = \sin\left(\frac{3\pi\times 4}{8}\right) = -1$$

For $n = 5$, we have

$$x_2(5) = \sin\left(\frac{3\pi\times 5}{8}\right) = -0.38$$

For $n = 6$, we have

$$x_2(6) = \sin\left(\frac{3\pi\times 6}{8}\right) = 0.707$$

For $n = 7$, we have

$$x_2(7) = \sin\left(\frac{3\pi\times 7}{8}\right) = 0.92$$

Thus, sequence $x_2(n)$ is expressed as

$$x_2(n) = \{0, 0.92, 0.707, -0.38, -1, -0.38, 0.707, 0.92\}$$

Now, we have

$$y(m) = x_1(n) \, \textcircled{N} \, x_2(n)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.92 \\ 0.707 \\ -0.38 \\ -1 \\ -0.38 \\ 0.707 \\ 0.92 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0+0+0+0+0-0.38+0.707+0.92 \\ 0.+0.92+0+0+0+0+0.707+0.92 \\ 0+0.92+0.707+0+0+0+0+0.92 \\ 0+0.92+0.707-0.38-0+0+0+0 \\ 0+0.92+0.707-0.38-1+0+0+0 \\ 0+0+0.707-0.38-1-0.38+0+0 \\ 0+0+0-0.38-1-0.38+0.707+0 \\ 0+0+0+0-1-0.38+0.707+0.92 \end{bmatrix} = \begin{bmatrix} 1.247 \\ 2.547 \\ 2.547 \\ 1.247 \\ 0.247 \\ -1.053 \\ -1.053 \\ 0.247 \end{bmatrix}$$

Hence, $y(m) = \{\underset{\uparrow}{1.247}, 2.547, 2.547, 1.247, 0.247, -1.053, -1.053, 0.247\}$

Ans.

EXAMPLE 5.16 Determine the circular convolution of following sequences and compare the results with linear convolution.

$$x(n) = \{1, 1, 1, 1, -1, -1, -1, -1\}$$

and $$h(n) = \{0, 1, 2, 3, 4, 3, 2, 1\}$$

(Expected

Solution : We have

$$y(m) = x(n) \, \textcircled{N} \, h(n)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0-1-2-3-4+3+2+1 \\ 0+1-2-3-4-3+2+1 \\ 0+1+2-3-4-3-2+1 \\ 0+1+2+3-4-3-2-1 \\ 0+1+2+3+4-3-2-1 \\ 0-1+2+3+4+3-2-1 \\ 0-1-2+3+4+3+2-1 \\ 0-1-2-3+4+3+2+1 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ -8 \\ -4 \\ 4 \\ 8 \\ 8 \\ 4 \end{bmatrix}$$

Therefore, $y(m) = x(n) \; Ⓝ \; h(n) = \{\underset{\uparrow}{-4}, -8, -8, -4, 4, 8, 8, 4\}$ ...(i)

Now, let us obtain linear convolution of given sequences as shown in figure 5.20.

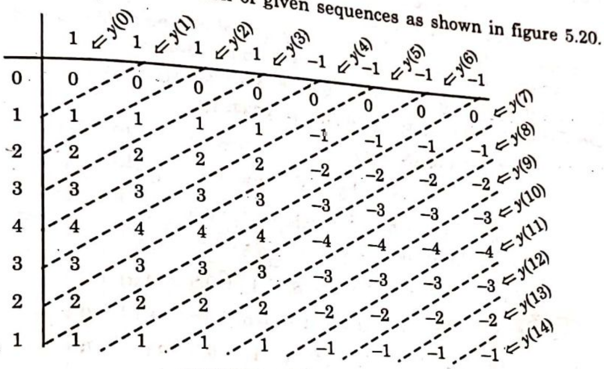

FIGURE 5.20 $x(n) \otimes h(n)$

Thus, $y(m) = x(n) \otimes h(n)$

or $y(m) = \{0, 1, 3, 6, 10, 11, 9, 4, -4, -9, -11, -10, -6, -3, -1\}$ ...(ii)

From equations (i) and (ii) we can conclude that the results of circular convolution and linear convolution are not same. **Ans.**

5.12.6.1. Linear Convolution Versus Circular Convolution

Let us note few important points regarding linear convolution & circular convolution.

(i) The results of circular and linear convolution are not same

Basically, circular convolution, $y(m)$ contains same number of samples as that of $x(n)$ and $h(n)$. In example (5.17), $x(n)$ and $h(n)$ have 8 samples. The circular convolution is given by equation (i). It also contains 8 samples.

Also, in example (5.17), the linear circular is given by equation (ii). Here, the number of samples are 15. We will use the following notations to indicate numbers of samples.

L = Number of samples in $x(n)$

M = Number of samples in $h(n)$

N = Number of samples in the result of linear convolution.

Hence, for the linear convolution, we can write the equation, ...(iii)

$$N = L + M - 1$$

Here, $L = M = 8$ or $N = 15$

That is why the results of linear and circular convolution are not same.

(ii) Steps to obtain same results from linear and circular convolution

To obtain the same results from both convolutions, the following steps are used:

(i) Using equation (*iii*), we calculate the value of N that means number of samples in linear convolution. Let us say, it is 15.

(ii) By doing zero padding, we make the length of every sequence equal to 15. This means that in this case, we need to add seven zeros in $x(n)$ as well as $h(n)$.

(iii) Then, we perform the circular convolution. The result of circular convolution and linear convolution will be same. **Ans.**

EXAMPLE 5.17 Determine the circular convolution of the following sequences and compare the results with linear convolution:

$$x(n) = \{1, 0.5, 1, 0.5, 1, 0.5, 1, 0.5\}$$
$$h(n) = \{0, 1, 2, 3\}$$

Solution : In case of circular convolution, the length of $x(n)$ and $h(n)$ should be same. Thus, we have to add four zeros in $h(n)$ to make its length equal to 8.

Hence, $h(n) = \{0, 1, 2, 3, 0, 0, 0, 0\}$

Now, circular convolution of $x(n)$ and $h(n)$ is given by,

$$y(m) = x(n) \; Ⓝ \; h(n)$$

$$y(m) = \begin{bmatrix} 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0+0.5+2+1.5+0+0+0+0 \\ 0+1+1+3+0+0+0+0 \\ 0+0.5+2+1.5+0+0+0+0 \\ 0+1+1+3+0+0+0+0 \\ 0+0.5+2+1.5+0+0+0+0 \\ 0+1+2+3+0+0+0+0 \\ 0+0.5+2+1.5+0+0+0+0 \\ 0+1+1+3+0+0+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 4 \\ 5 \\ 4 \\ 5 \\ 4 \\ 5 \end{bmatrix}$$

Hence, $y(m) = \{4, 5, 4, 5, 4, 5, 4, 5\}$

Comparison with linear convolution

$x(n)$ has 8 samples and $h(n)$ has 4 samples. Therefore, linear convolution will have 8 + 4 − 1 = 11 samples. Thus, results of linear and circular convolution will not be same. We can verify this by obtaining linear convolution of $x(n)$ and $h(n)$.

5.12.6.2. Comparison of Ci

In this section, we shall
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$y(n)$

where, $x(n$
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In the frequency-dor

If the sequence $y(n$
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5.12.6.2. Comparison of Circular Convolution with Linear Convolution

In this section, we shall carry out the comparison between linear convolution and circular convolution.

Let us consider a finite duration sequence $x(n)$ of length N_1 which is given as input to a system with impulse response $h(n)$ of length N_2.

Then the output is expressed as

$$y(n) = x(n) \otimes h(n) = \sum_{k=0}^{N_1-1} x(k)h(n-k), \; n = 0, 1, ..., N_1 + N_2 - 1$$

$$y(n) = \sum_{k=0}^{N_1-1} h(k)x(n-k), \; n = 0, 1, ..., N_1 + N_2 - 1$$

where, $x(n) = 0$ for $n < 0$ and $n \geq N_1$

$h(n) = 0$ for $n < 0$ and $n \geq N_2$

In the frequency-domain, we have

$$Y(\omega) = H(\omega)\, X(\omega) \qquad ...(5.91)$$

If the sequence $y(n)$ has to be represented uniquely in the frequency-domain by samples of its spectrum $Y(\omega)$, then

$$Y(k) = Y(\omega)\big|_{\omega = 2\pi k/N}, \; k = 0, 1, ... N-1$$

or $$Y(k) = X(\omega)\, H(\omega)\big|_{\omega = 2\pi k/N}, \; k = 0, 1, ..., N-1$$

or $$Y(k) = X(k).H(k) \qquad ...(5.92)$$

Here, $X(k)$ and $H(k)$ are the N-point DFTS of the sequence $x(n)$ and $h(n)$ respectively.

Also, we know that since $y(n)$ can be obtained by

$$y(n) = IDFT[X(k) \cdot H(k)] \qquad ... (5.93)$$

Therefore, the N-point circular convolution of $x(n)$ with $h(n)$ must be equivalent to the linear convolution of sequence $x(n)$ with $h(n)$.

> **Note :** *In the general, the linear convolution of two sequences $x(n)$ and $h(n)$ with lengths N_1 and N_2 respectively would provide a sequence with a length of $N \geq N_1 + N_2 - 1$. When $x(n)$ and $h(n)$ have a duration less than N_1 for implementing linear convolution using circle. Convolution, $N_2 - 1$ and $N_1 - 1$ zeros are padded (i.e., added) at the end of $x(n)$ and $h(n)$ respectively to increase their length to N.*

5.13 ADDITIONAL DFT PROPERTIES

imp

5.13.1 Time Reversal of a Sequence

This property of DFT states that if a sequence is circularly folded, its DFT is also circularly folded.

Mathematicaly,

If $$x(n) \xleftrightarrow[N]{DFT} X(k)$$

then $$x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N$$

$$= X(N-k)$$

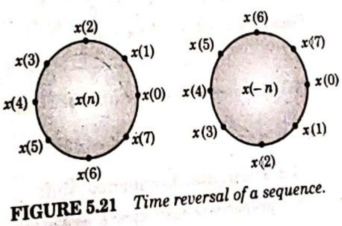

FIGURE 5.21 *Time reversal of a sequence.*

Hence, reversing the N-point sequence in time is equivalent to reversing the DFT values. reversal of a sequence $x(n)$ is illustrated in figure 5.21.

Proof :

From the definition of the DFT, we have

$$\text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{-j2\pi kn/N}$$

If we change the index from n to $m = N - n$, then, we conwrite

$$\text{DFT}\{x(N-n)\} = \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi k(N-m)/N} = \sum_{m=0}^{N-1} x(m)\, e^{j2\pi km/N}$$

or $$\text{DFT}\{x(N-n)\} = \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi m(N-k)/N} = X(N-k)$$

It may be noted that $X(N-k) = X((-k))_N$, for $0 \le k \le N-1$ **Hence Proved**

5.13.2 Circular Time Shift of a Sequence

This property states that shifting the sequence in time domain by 'l' samples is equivalent to multiplying the sequence in frequency domain by W_N^{kl} or $e^{-j2\pi kl/N}$

Mathematically

If $$x(n) \xleftrightarrow[N]{DFT} X(k)$$

then $$x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\, e^{-j2\pi kl/N}$$

or $$x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\, W_N^{kl}$$

Proof : According to the definition of IDFT, we have

$$x(n) = \text{IDFT}\,\{X(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

Hence, $$\text{IDFT}\left\{X(k)W_N^{kl}\right\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \cdot W_N^{kl}$$

or $$\text{IDFT}\left\{X(k)W_N^{kl}\right\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-k(n-l)}$$

Now, we have, $x(n) \xleftrightarrow[N]{DFT} X(k)$

Comparing R.H.S. of equations (5.109) and (5.110), we have

$$x(n-l) \xleftrightarrow[N]{DFT} X(k)\, W_N^{kl}$$

The sequence is circular and DFT is periodic in nature, so, we can write,

$$x((n-l))_N \xleftrightarrow[N]{DFT} W_N^{kl}\, X(k)$$

5.13.3 Circular Frequency Shift

This property is also known as **Quadrature Modulation Theorem.**

This property states that multiplication of sequence $x(n)$ by $e^{\pm j2\pi kl/N}$ is equivalent to the circular shift of DFT in time domain by 'l' samples.

Mathematically,

If $\quad x(n) \xleftrightarrow[N]{DFT} X(k)$, then

$$x(n)e^{j2\pi ln/N} \xleftrightarrow[N]{DFT} X((k-l))_N = X(k+l)$$

or $$x(n)e^{-j2\pi ln/N} \xleftrightarrow[N]{DFT} X((k+l))_N = X(k-l)$$

EXAMPLE 5.18 A four point sequence $x(n) = \{1, 2, 3, 4\}$ has DFT $X(k)$, $0 \le k \le 3$. Without performing DFT or IDFT, determine the signal values which has DFT $X(k-1)$.

Solution : According to the circular frequency shifting property of DFT, we have

$$x(n) \cdot e^{-j2\pi ln/N} \xleftrightarrow[N]{DFT} X((k+l))_N = X(k-l)$$

Here, $l = 1$.

Let the signal whose DFT is $X(k-1)$ be denoted by $x_1(n)$.

Therefore, $\quad x_1(n) = x(n)\, e^{-j2\pi.1.n/4}$

Since $N = 4$ in this case.

The given sequence is $x(n) = \{1, 2, 3, 4\}$

$\therefore \quad x(0) = 1, \;\; x(1) = 2, \;\; x(2) = 3 \;$ and $\; x(3) = 4$

Let us find the sequence $x_1(0)$ as under :

For $n = 0$, we have

$$x_1(0) = x(0) \cdot e^0 = 1$$

For $n = 1$, we have

$$x_1(1) = x(1)\, e^{-\frac{j2\pi}{4}} = 2e^{-\frac{j\pi}{2}} = 2\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right]$$

or $$x_1(1) = -2j$$

For $n = 2$, we have

$$x_1(2) = x(2)\, e^{-\frac{j4\pi}{4}} = 3e^{-j\pi} = 3[\cos \pi - j \sin \pi]$$

or $$x_1(2) = -3$$

For $n = 3$, we have

$$x_1(3) = x(3)\, e^{-\frac{j6\pi}{4}} = 4e^{-\frac{j3\pi}{2}} = 4\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$$

or $$x_1(3) = 4j$$

or $$x_1(n) = \{1, -2j, -3, 4j\} \quad \textbf{Ans.}$$

EXAMPLE 5.19 Given a real finite length sequence,

$$x(n) = \{4, 3, 2, 1, 0, 0, 1, 1\}$$

(i) $y(n)$ is a sequence related to $x(n)$ such that,

$Y(k) = W_8^{4k} X(k)$ where $X(k)$ is 8 point DFT of $x(n)$. Obtain $y(n)$

(ii) Also obtain finite length sequence $q(n)$ related to $x(n)$ such that its 8 point DFT is $Q(k) = R_e\{X(k)\}$. ...(i)

Solution : (i) Given that $Y(k) = W_8^{4k} X(k)$

According to circular time shifting property, we have

$$x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\,W_N^{kl}$$

Here, $N = 8$ and $l = 4$

Thus, comparing equations (*i*) and (*ii*), we have

$$y(n) = x((n-4))_8$$

This means that $y(n)$ represents circular delay of sequence $x(n)$ by 4 samples. It is represented in figure 5.22.

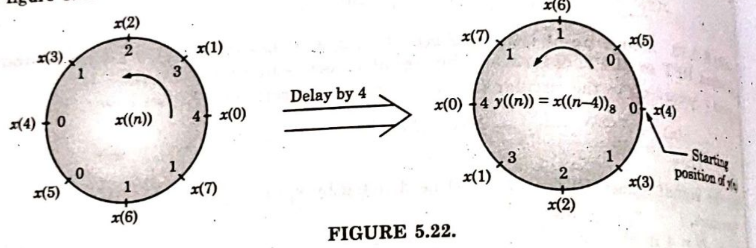

FIGURE 5.22.

Therefore, $y(n) = \{0, 0, 1, 1, 4, 3, 2, 1\}$

(*ii*) Given that $Q(k) = R_e\{X(k)\}$

Let $X(k) = M(k) + jN(k)$

Hence, $M(k)$ represents real part of $X(k)$.

Now, $X^*(k) = M(k) - jN(k)$

Adding equations (*iv*) and (*v*), we obtain

$$X(k) + X^*(k) = 2M(k)$$

or

$$M(k) = \frac{X(k)+X^*(k)}{2}$$

Taking IDFT of both sides, we obtain

$$m(n) = \frac{x(n)+x^*(-n)}{2}$$

As $x^*(-n) \xleftrightarrow[N]{DFT} X^*(k)$

We have, $x(n) = \{4, 3, 2, 1, 0, 0, 1, 1\}$

Therefore, $x^*(n) = \{4, 3, 2, 1, 0, 0, 1, 1\}$

$x^*(-n)$ represents circular folding of $x^*(n)$. It is shown in 5.23.

Hence, $x^*(-n) = \{4, 1, 1, 0, 0, 1, 2, 3,\}$ ...(*ix*)

Substituting equations (*viii*) and (*ix*) in equation (*vii*), we find sequence $m(n)$ as under :

For $n = 0$, we have

$$m(0) = \frac{x(0)+x^*(0)}{2} = \frac{4+4}{2} = 4$$

For $n = 1$, we have

$$m(1) = \frac{x(1)+x^*(1)}{2} = \frac{3+1}{2} = 2$$

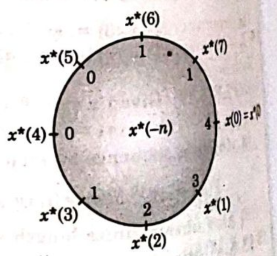

FIGURE 5.23.

For $n = 2$, we have

$$m(2) = \frac{x(2)+x^*(2)}{2} = \frac{2+1}{2} = \frac{3}{2}$$

For $n = 3$, we have

$$m(3) = \frac{x(3)+x^*(3)}{2} = \frac{1+0}{2} = \frac{1}{2}$$

For $n = 4$, we have

$$m(4) = \frac{x(4)+x^*(4)}{2} = \frac{0+0}{2} = 0$$

For $n = 5$, we have

$$m(5) = \frac{x(5)+x^*(5)}{2} = \frac{0+1}{2} = \frac{1}{2}$$

For $n = 6$, we have

$$m(6) = \frac{x(6)+x^*(6)}{2} = \frac{1+2}{2} = \frac{3}{2}$$

For $n = 7$, we have

$$m(7) = \frac{x(7)+x^*(7)}{2} = \frac{1+3}{2} = 2$$

Therefore, we get

$$m(n) = q(n) = \left\{4, 2, \frac{3}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 2\right\}$$ **Ans.**

5.13.4 Multiplication of Two Sequences

The multiplication of two sequences in time domain is equivalent to its circular convolution in the frequency domain.

Mathematically, we have

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then,

$$x_1(n) \,.\, x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N}[X_1(k) \; \textcircled{N} \; X_2(k)]$$

5.13.5 Circular Correlation

The circular cross correlation of two sequences in time domain is equivalent to the multiplication of DFT of one sequence with the complex conjugate DFT of other sequence.

Mathematically,

If $\quad x(n) \xleftrightarrow[N]{DFT} X(k)$,

and $\quad y(n) \xleftrightarrow[N]{DFT} Y(k)$, then,

$$r_{xy}(l) \xleftrightarrow[N]{DFT} R_{xy}(k) = X(k)\,Y^*(k)$$

5.13.6 Complex Conjugate Property

The DFT of complex conjugate of the sequence is equal to the complex conjugate of DFT of that sequence, with the sequence delayed by k samples in the frequency domain.

Mathematically: If $x(n) \xleftrightarrow[N]{DFT} X(k)$, then

$$x^*(n) \xleftrightarrow[N]{DFT} X^*((-k))_N = X^*(N-k)$$

EXAMPLE 5.20 DFT of a sequence $x(n)$ is given by

$$X(k) = \{4,\ 1+2j,\ j,\ 1-3j\}$$

Using DFT property only, determine DFT of $x^*(n)$ if $x^*(n)$ is complex conjugate of

Solution : According to complex conjugate property, we have

If $\quad x(n) \xleftrightarrow[N]{DFT} X(k)$ then

$$x^*(n) \xleftrightarrow[N]{DFT} X^*(-k)_N$$

We have, $\quad X(k) = \{4,\ 1+2j,\ j,\ 1-3j\}$

Also, $\quad X^*(k) = \{4,\ 1-2j,\ -j,\ 1+3j\}$

But, DFT of $x^*(n)$ is $X^*(-k)_N$

Here, $X^*(-k)_N$ indicates circular folding of $X^*(k)$. This means that sequence $X^*(-k)$ obtained by plotting the samples of $X^*(k)$ in clockwise direction.

Therefore, $\quad X^*(-k)_N = \{4,\ 1+3j,\ -j,\ 1-2j\}$

This is the DFT of $x^*(n)$. **Ans.**

EXAMPLE 5.21 DFT of a sequence $x(n)$ is given by,

$$X(k) = \{6,\ 0,\ -2,\ 0\}$$

(i) Determine $x(n)$

(ii) Plot $x_1(n)$ if $X_1(k)$ is $X(k).e^{-j2\pi k/2}$

(iii) Determine circular autocorrelation of x(n) using DFT and IDFT only.

Solution : (i) According to the definition of IDFT, we have

$$x(n) = \frac{1}{N}\left[W_N^*\right] . X_N$$

Here, $\quad \left[W_N^*\right] = \left[W_4^*\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$

Hence, $\quad x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}$

or $\quad x(n) = \frac{1}{4}\begin{bmatrix} 6+0-2+0 \\ 6+0+2+0 \\ 6+0-2+0 \\ 6+0+2+0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

or $\quad x(n) = \{1,\ 2,\ 1,\ 2\}$ **Ans.**

(ii) Given that $X_1(k) =$

According to circular t

$x((n-l))$

The given term can b

$X(k)e^{-j2\pi k/2}$

Comparing equations

$x((n-2$

This means that x_1

Here, $x((n-2))_N$ in

in figure 5.24.

$x(2) = 1$

(iii) According t

$r_{xx}(l)$

We have

Since, $X^*(k)$ is

Also, $\quad X(k)$

Now, $r_{xx}(l)$ m

or

or

or

(*ii*) Given that $X_1(k) = X(k)e^{-j2\pi k/2}$

According to circular time shift property, we have

$$x((n-l))_N \xleftrightarrow[N]{DFT} X(k)e^{-j2\pi kl/N}$$

The given term can be expressed as under :

$$X(k)e^{-j2\pi k/2} = X(k)\,e^{-\frac{j2\pi k.2}{4}} \qquad \text{...(i)}$$

Comparing equations (*i*) and (*ii*), we obtain

$$x((n-2))_N \xleftrightarrow[N]{DFT} X_1(k) = X(k)\,e^{-\frac{j2\pi k.2}{4}} \qquad \text{...(ii)}$$

This means that $x_1(n) = x((n-2))_N$

Here, $x((n-2))_N$ indicates circular delay of $x(n)$ by 2 samples. It has been plotted as shown in figure 5.24.

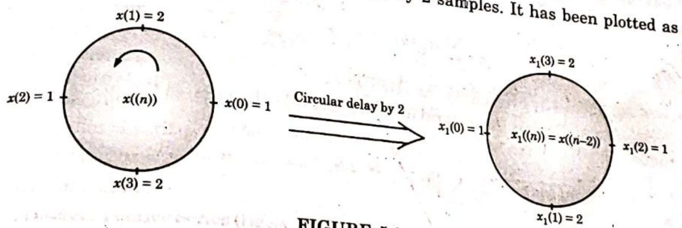

FIGURE 5.24.

(*iii*) According to circular correlation property, we have

$$r_{xx}(l) \xleftrightarrow[N]{DFT} = R_{xx}(k) = X(k)\,.\,X^*(k)$$

We have $\qquad X(k) = \{6,\ 0,\ -2,\ 0\}$

Since, $X^*(k)$ is complex conjugate of $X(k)$, therefore, we have

$$X^*(k) = \{6,\ 0,\ -2,\ 0\}$$

Also, $\qquad X(k)\,.\,X^*(k) = \{36,\ 0,\ 4,\ 0\}$

Now, $r_{xx}(l)$ may be obtained by taking IDFT of $X(k).X^*(k)$, *i.e.*,

$$r_{xx}(l) = \frac{1}{4}\left[W_4^*\right].X(k).X^*(k)$$

or
$$r_{xx}(l) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 36 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

or
$$r_{xx}(l) = \frac{1}{4}\begin{bmatrix} 36+0+4+0 \\ 36+0-4+0 \\ 36+0+4+0 \\ 36+0-4+0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 40 \\ 32 \\ 40 \\ 32 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 10 \\ 8 \end{bmatrix}$$

or
$$r_{xx}(l) = \{10,\ 8,\ 10,\ 8\} \quad \textbf{Ans.}$$

5.13.7 Parseval's Theorem

For complex-valued sequences $x(n)$ and $y(n)$, in general, if

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

and

$$y(n) \xleftrightarrow[N]{DFT} Y(k)$$

then

$$\sum_{n=0}^{N-1} x(n)\,y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,Y^*(k)$$

Proof:

We have

$$\sum_{n=0}^{N-1} x(n)\,y^*(n) = \tilde{r}_{xy}(0)$$

and

$$\tilde{r}_{xy}(l) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{R}_{xy}(k)\,e^{j2\pi kl/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,Y^*(k)e^{j2\pi kl/N}$$

Hence (5.99) follows by evaluating the IDFT at $l = 0$.

The expression in (5.99) is the general form of Parseval's theorem. In the special case when $y(n) = x(n)$, equation (5.99) reduces to

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

which expresses the energy in the finite-duration sequence $x(n)$ in terms of the frequency components $\{X(k)\}$.

5.13.8 Summary of DFT Properties

Table 5.5 shows the summary of DFT properties.

TABLE 5.5

S. No.	Name of DFT Property	Expression in time domain	Expression in frequency domain
1.	Periodicity	$x(n) = x(n + N)$	$X(k) = X(k + n)$
2.	Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
3.	Time Reversal	$x(N - n)$	$X(N - k)$
4.	Circular time shift	$x((n - l))_N$	$X(k)e^{-j2\pi kl/N}$
5.	Circular frequency shift	$x(n)e^{j\pi 2ln/N}$	–
6.	Circular convolution	$x_1(n)$ Ⓝ $x_2(n)$	$X_1(k)X_2(k)$
7.	Circular correlation	$x(n)$ Ⓝ $y^*(-n)$	$X(k)\,Y^*(k)$
8.	Multiplication of two sequences	$x_1(n)\,x_2(n)$	$\frac{1}{N}X_1(k)$ Ⓝ $X_2(k)$
9.	Complex conjugate	$x^*(n)$	$X^*(N - k)$
10.	Parseval's theorem	$\sum_{n=0}^{N-1} x(n)\,y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)\,Y^*(k)$

5.14 LINEAR FILTERING TECHNIQUES BASED ON DFT : LINEAR CONVOLUTION USING DFT

As a matter of fact, linear filtering is same as linear convolution. In this article, we shall discuss how the linear convolution is obtained using DFT.

We know that the linear convolution of $x(n)$ and $h(n)$ is given by,

$$y(n) = x(n) \otimes h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

or

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \qquad ...(5.101)$$

If we obtain the Fourier transform of $x(n)$ and $h(n)$ then we shall get $X(\omega)$ and $H(\omega)$. We know that convolution is equivalent to multiplication in the frequency domain. Therefore, by multiplying $X(\omega)$ and $H(\omega)$, we get $Y(\omega)$.

This means that

$$Y(\omega) = X(\omega)\cdot H(\omega) \qquad ...(5.102)$$

If we take inverse Fourier transform of equation (5.102) then we will get the sequenc $y(n)$. This sequence will be same as the linear convolution of $x(n)$ and $h(n)$. However, we cannot use Fourier transform to obtain linear convolution because of the following two reasons:

(i) In Fourier transform, ω is continuous function of frequency. Hence, the computation cannot be done on digital computers. Because for the digital signal processors, discrete-time signals in place of continuous-time signals are required.

(ii) If we use DFT, then the computation will be more efficient because of the avaiability of Fast Fourier Transform (FFT) algorithms. Therefore, we must use DFT to obtain the linear filtering operation.

DO YOU KNOW?

In applying the DFT to signal analysis, the sampling theorem and the desired resolution imose a constraint between total signal duration, T, signal band-width, W, and the number of DFT points, N. This relationship is $\frac{1}{T} \geq \frac{2W}{N}$

In case of DFT, we know that the multiplication of two DFTs in frequency domain is equvalent to the circular convolution, *i.e.*,

$$X(k).H(k) = x(n) \; Ⓝ \; h(n) \qquad ...(5.103)$$

However, in this case, we want linear convolution (linear filtering) and not the circular convolution.

Earlier, we have studied that if we adjust the length of two sequences, x(n) and h(n), then, the same result can be obtained using linear convolution and circular convolution.

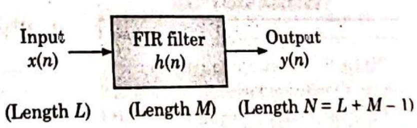

FIGURE 5.25 *FIR filter*

Let us consider an FIR filter having impulse response $h(n)$ as shown in figure 5.25.*

Here, $x(n)$ = Input sequence having length L

Therefore, $x(n) = \{0, 1, 2, \ldots, L-1\}$

$h(n)$ = Impulse response of filter having length M.

Therefore, $h(n) = \{0, 1, 2, \ldots, M-1\}$

Here, the linear convolution of $x(n)$ and $h(n)$ produces the output sequence $y(n)$ and the length of $y(n)$ is,

$$N = L + M - 1 \qquad ...(5.104)$$

* FIR filters will be discussed in details in chapter 7.

In this case, both the sequences $x(n)$ and $h(n)$ are finite. Hence linear convolution finite. Thus, equation (5.101) becomes,

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$$

Now, if we adjust the length of $x(n)$ and $h(n)$ equal to N and if we perform the convolution of $x(n)$ and $h(n)$ then, the result will be same as linear convolution. The length of $x(n)$ and $h(n)$ can be made equal to N by adding required number of zeros in $x(n)$ and $h(n)$. This is known as **zero padding**. This means that we have to increase the length of $x(n)$ by M and length of $h(n)$ by L points to make the total length $N = L + M - 1$. Then, we can obtain the DFTs of $x(n)$ and $h(n)$, that is, $X(k)$ and $H(k)$. The multiplication of these two DFTs yields $Y(k)$, *i.e.*,

$$Y(k) = X(k) \cdot H(k)$$

Now, by taking IDFT of $Y(k)$, the output sequence $y(n)$ may be obtained. Here, the filtering may be obtained using DFTs.

5.14.1 Evaluation of Linear Filtering using DFT

(i) First, we calculate the value of N using $N = L + M - 1$. Here, L represents number of samples in $x(n)$ and M represents number of samples in $h(n)$.

(ii) By adding zeros, we make the length of $x(n)$ and $h(n)$ equal to N.

(iii) We calculate DFT of $x(n)$ that means $X(k)$.

(iv) We calculate DFT of $h(n)$ that means $H(k)$.

(v) Then, we multiply $X(k)$ and $H(k)$ to get $Y(k) = X(k) \cdot H(k)$.

(vi) Lastly, we obtain IDFT of $Y(k)$ that means $y(n)$.

EXAMPLE 5.22 Determine the response of FIR filter using DFT if

$$x(n) = \{\underset{\uparrow}{1}, 2\} \text{ and } h(n) = \{\underset{\uparrow}{2}, 2\}$$

Solution: (i) Here, length of $x(n) = L = 2$.

and length of $h(n) = M = 2$

Therefore, $N = L + M - 1 = 2 + 2 - 1 = 3$

Hence, we have to calculate 3-point DFT. We shall compute standard 4-point DFT *i.e.*, $N = 4$.

(ii) We will make length of $x(n)$ and $h(n)$ equal to 4 by adding zeros at the end.

Hence, $x(n) = \{\underset{\uparrow}{1}, 2, 0, 0\}$

and $h(n) = \{\underset{\uparrow}{2}, 2, 0, 0\}$

(iii) Calculation of $X(k)$

Let us calculate DFT of $x(n)$, $X(k)$ using matrix method.

We have $X(k) = W_N \cdot x_N$

Earlier, we have obtained the matrix for twiddle factor W_4. It is,

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

We have input ma

Substituting equ

Therefore,

(iv) Calculatio

We have

or

(v) Now, Y(

Therefore,

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(vi) Nov

We hav

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means tha

We have input matrix

$$x_N = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad ...(v)$$

Substituting equations (*iv*) and (*v*) in equation (*iii*), we have

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2+0+0 \\ 1-2j+0+0 \\ 1-2+0+0 \\ 1+2j+0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1-2j \\ -1 \\ 1+2j \end{bmatrix}$$

Therefore, $X(k) = \{3, 1 - 2j, -1, 1 + 2j\}$ ...(*vi*)

(*iv*) Calculation of $H(k)$

We have $H(k) = W_N \cdot h_N$

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+2+0+0 \\ 2-2j+0+0 \\ 2-2+0+0 \\ 2+2j+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2-2j \\ 0 \\ 2+2j \end{bmatrix}$$

or $H(k) = \{\underset{\uparrow}{4}, 2 - 2j, 0, 2 + 2j\}$ ...(*vii*)

(*v*) Now, $Y(k) = X(k) \cdot H(k)$

Therefore, $Y(k) = \{\underset{\uparrow}{3}, 1 - 2j, -1, 1 + 2j\} \cdot \{\underset{\uparrow}{4}, 2 - 2j, 0, 2 + 2j\}$

Hence,

$$Y(0) = X(0) \cdot H(0) = 3 \times 4 = 12$$

$$Y(1) = X(1) \cdot H(1) = (1 - 2j)(2 - 2j)$$

$$= 2 - 2j - 4j + 4j^2 = 2 - 6j - 4 = -2 - 6j$$

$$Y(2) = X(2) \cdot H(2) = (-1)(0) = 0$$

$$Y(3) = X(3) \cdot H(3) = (1 + 2j)(2 + 2j)$$

$$= 2 + 2j + 4j + 4j^2 = 2 + 6j - 4 = -2 + 6j$$

Thus, sequence $Y(k)$ will be

$$Y(k) = \{Y(0), Y(1), Y(2), Y(3)\}$$

Hence, $Y(k) = \{12, (-2 - 6j), 0, (-2 + 6j)\}$

(*vi*) Now, let us obtain $y(n)$ by taking IDFT of $Y(k)$.

We have, $y(n) = \frac{1}{N} W_N^* Y(k)$

Here, W_N^* is complex conjugate of W_N, which is obtained by changing the sign of j term. This means that

$$W_N^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Therefore, $y(n)$ will be given by

$$y(n) = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 12 \\ -2-6j \\ 0 \\ -2+6j \end{bmatrix}$$

$$y(n) = \frac{1}{4}\begin{bmatrix} 12-2-6j+0-2+6j \\ 12-2j+6+0+2j+6 \\ 12+2+6j+0+2-6j \\ 12+2j-6+0-2j-6 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 24 \\ 16 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 0 \end{bmatrix}$$

Therefore,
$$y(n) = \{\underset{\uparrow}{2}, 6, 4, 0\}$$

This is the required response of the filter.

Verification of Result

Let us verify the result by calculating linear convolution of $x(n)$ and $h(n)$. It has been shown in figure 5.26.

Hence, $$y(n) = \{\underset{\uparrow}{2}, 6, 4\}$$

Therefore, we conclude that the two results are same.

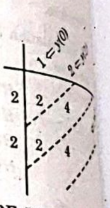

FIGURE 5.26 $x(n) \circledast h(n)$

5.14.2 Linear Filtering of Long Data Sequences

(GGSIPU, Sem. Exam., May ...)

In real time applications, the input data sequence is very long. It is not possible to do calculations for such long data sequences This is because, the digital computer has limited memory.

Hence, long input data sequence is broken into small length sequences (blocks). The computation of each block is done separately. Then all processed blocks are fitted one after other to get the final output.

There are two methods to obtain such computation. These are as under:

(i) Overlap Save method

(ii) Overlap Add method

5.14.2.1. Overlap Save Method

In this method, the filtering of each block is done separately. Let 'M' be the impulse response of filter. Here, input data sequence is very long. This data sequence is divided into blocks. Let N be the length of each block.

Forming different blocks

In this method, each block of length N is formed by taking L samples of the current segment and $M-1$ samples from the previous segment.

Thus, $N = L + M - 1$.

For the first block, $x_1(n)$, there are no previous segments. So, this block is padded with $M-1$ zeros. And, the first L samples are taken from the input sequence.

Therefore, $$x_1(n) = \{\underbrace{0,0,0,\ldots 0}_{(M-1)\text{zeros}},\ \underbrace{x(0), x(1) \ldots x(L-1)}_{\text{First } L \text{ samples from input sequence } x(n)}\} \qquad \ldots(5.10\ldots)$$

Let the long input sequence is,

$$x(n) = \{1, 2, 3, 4, 5, -1, 2, -3, 4, 5, 1, 2, 3, 4, 5, \ldots\}$$

Let $M = 4$ and $L = 5$

So, $M - 1 = 3$ and $N = M + L - 1 = 4 + 5 - 1 = 8$

Thus, we can write the first sequence $x_1(n)$ as,

$$x_1(n) = \{\underbrace{0,0,0,}_{\text{Three zeros}} \underbrace{1,2,3,4,5}_{\text{First five samples of } x(n)}\}$$

Now, $x_2(n)$ is formed by taking last $M - 1$ samples from $x_1(n)$ and next L samples from input sequence $x(n)$. Here, $M - 1 = 3$. Hence, last 3 samples of $x_1(n)$ are 3,4,5.

Therefore, $x_2(n) = \{\underbrace{3,4,5,}_{(M-1)\text{ samples of } x_1(n)} \underbrace{-1,2,-3,4,5}_{\text{Next } L \text{ samples of } x(n)}\}$

Hence, general equation of $x_2(n)$ becomes,

$$x_2(n) = \{\underbrace{x(L-M+1)\dots x(L-1),}_{(M-1)\text{ samples from } x_1(n)} \underbrace{x(L), x(L+1)\dots x(2L-1)}_{\text{Next } L \text{ samples of } x(n)}\} \quad \text{...(5.108)}$$

Similarly, $x_3(n)$ is formed by taking last $M - 1$ samples from $x_2(n)$ and next L samples from $x(n)$.

Therefore,

$$x_3(n) = \{\underbrace{x(2L-M+1\dots\dots x(2L-1),}_{(M-1)\text{ samples from } x_2(n)} \underbrace{x(2L), x(2L+1)\dots\dots x(3L-1)}_{\text{Next L samples of } x(n)}\} \quad \text{...(5.109)}$$

Likewise the different blocks of length N are formed now, we have the length of each block equals to N and $N = L + M - 1$. We have assumed that the length of impulse response is M. This lenght must be same as the length of each block (N). This is done by adding $(L - 1)$ zeros in the sequence $h(n)$.

Therefore, $h(n) = \{\underbrace{h(0), h(1)\dots\dots h(M-1),}_{\text{M samples of } h(n)} \underbrace{0,0,0\dots\dots(L-1 \text{ zeros})}_{(L-1)\text{ zeros}}\}$ ...(5.110)

The N point DFT of $h(n)$ is $H(k)$. Similarly, N point DFT of each block is computed separately and it is stored in the memory. Let the DFT of m^{th} input block be $X_m(k)$. Now, the corresponding output is obtained by taking multiplication of $H(k)$ and $X_m(k)$. Let this output be $\hat{Y}_m(k)$.

$$\hat{Y}_m(k) = H(k).\, X_m(k) \quad \text{...(5.111)}$$

Now, the corresponding time domain sequence is obtained by taking IDFT of $\hat{Y}_m(k)$. Let this sequence be $y_m(n)$.

Hence, $y_m(n) = \text{IDFT}\left\{\hat{Y}_m(k)\right\}$ ...(5.112)

Similarly, the time domain sequence corresponding to the DFT of each input block is obtained.

To avoid loss of data due to aliasing

Let us observe each input block $x_1(n)$, $x_2(n)$........ . Here, each input block consists of initial '$M - 1$' samples taken from the previous block. This means that there is overlap of sequences. While the last L samples of each block are the actual input samples.

Now, bacause of the circular shift of DFT and because of the overlapping of input data blocks; there is alising in the initial $M - 1$ samples in the corresponding output block. To avoid this, aliasing first $M - 1$ samples of $y_m(n)$ are discarded. This means that after computing time domain sequences, initial '$M - 1$' samples are discarded. Then each black is fitted one after other to obtain the final output. The overlap save method is described as shown in figure 5.27.

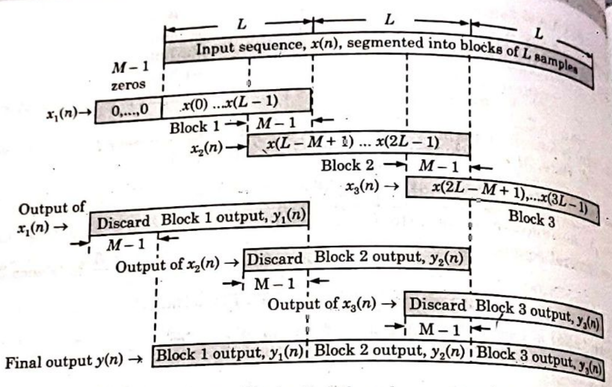

FIGURE 5.27 *Illustration of overlap save method.*

5.14.2.2. Overlap Add Method

In the overlap add method, the size of each input data block is N. Each input data block is formed by taking L samples from the input sequence and adding $M-1$ zeros at the end of the input sequence.

Therefore, $$x_1(n) = \underbrace{\{x(0),........x(L-1),}_{\text{First } L \text{ samples of input sequence } x(n)} \underbrace{0,0.......0\}}_{(M-1)\text{ zeros}} \quad ...(5.1)$$

Also, $x_2(n)$ is formed by taking next L samples from long input sequence $x(n)$ and padding $M-1$ zeros at the end.

Therefore, $$x_2(n) = \underbrace{\{x(L),x(L+1).........x(2L-1),}_{\text{Next } L \text{ samples of input sequence } x(n)} \underbrace{0,0......0\}}_{(M-1)\text{ zeros}} \quad ...(5.1)$$

Similarly, we have

$$x(3) = \underbrace{\{x(2L),x(2L+1).......x(3L-1)}_{\text{Next } L \text{ samples of input sequence } x(n)} \underbrace{0,0.....0\}}_{(M-1)\text{ zeros}} \quad ...(5.1)$$

Impulse response of the filter is $h(n)$ and its lenght is M. This length must be again made equal to N. Therefore, $L-1$ zeros are added to form the sequence $h(n)$.

Thus, we have

$$h(n) = \underbrace{\{h(0),h(1)........h(M-1),}_{M \text{ samples of impulse sequence}} \underbrace{0,0......0\}}_{\substack{(L-1)\text{ zeros to make} \\ \text{total length} = N}} \quad ...(5.1)$$

Now, the DFT of each input data block is computed separately. Similarly, the DFT of $h(n)$ is computed that is, $H(k)$. Now, the output of m^{th} block is obtained by multiplying DFT of m^{th} input block by $\hat{H}(k)$.

Therefore, $$Y_m(k) = H(k).\ X_m(k) \quad ...(5.1)$$

The time domain sequence $Y_m(n)$ is obtained by taking IDFT of $Y_m(k)$. Here, we are performing N point DFT. Hence, the length of each output is N. The different output sequences are as under :

$$y_1(n) = \{(y_1(0), y_1(1) \ldots y_1(L-1), y_1(L), y_1(L+1) \ldots y_1(L+M-1)\}$$
$$y_2(n) = \{y_2(0), y_2(1) \ldots y_2(L-1), y_2(L), y_2(L+1) \ldots y_2(L+M-1)\}$$

Similarly, all output sequences are obtained

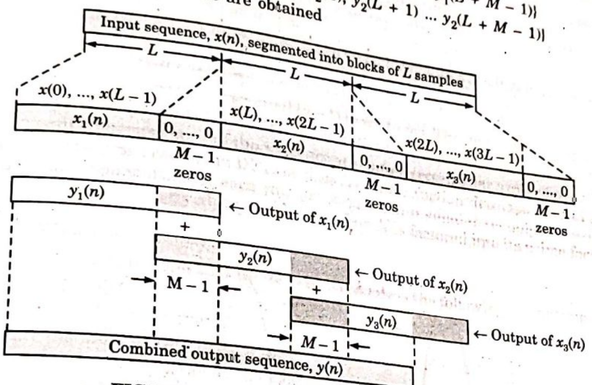

FIGURE 5.28 *Illustration of overlap add method.*

In this method, each input sequence is padded with $M-1$ zeros at the end to make the length $N = L + M - 1$. So, last $M-1$ samples of each output block must be overlapped and added to the first $M-1$ samples of succeeding block as shown in figure 6.28. That is why, this method is called as ***overlap add method.***

Here '$M - 1$' samples are not discarded because there is no aliasing effect.

5.15 FREQUENCY ANALYSIS OF SIGNALS USING DISCRETE FOURIER TRANSFORM (DFT)

The frequency analysis is also called as spectrum analysis. To do the frequency analysis, time domain signal should be first converted into the frequency domain.

If input signal is analog then first it is passed through the antialiasing filter. Then this signal is sampled at the rate f_s; where f_s is the sampling frequency. To avoid the aliasing, f_s should be greater than or equals to $2W$.. Here, W is the maximum frequency of input signal.

In practical cases, the time interval of signal is maintained at T_o and $T_o = LT$. Here, L = number of samples and T is the sampling interval.

Suppose we want to analyze the long input sequence $x(n)$. Now, to limit the time interval of sequence means this input sequence is multiplied by a rectangular window. A rectangular window is denoted by $W_R(n)$ and it is defined as,

$$W_R(n) = \begin{cases} 1 & \text{for } 0 \le n \le L-1 \\ 0 & \text{elsewhere} \end{cases}$$

This rectangular window is shown in figure 5.29.

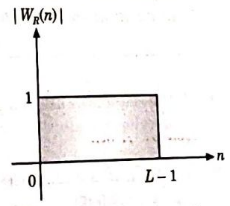

FIGURE 5.29 *Rectangular window*

Since, output of $W_R(n)$ is zero after the interval $L-1$, multiplying $x(n)$ by $W_R(n)$ produces signal $\hat{x}(n)$ only in the range 0 to $L-1$.

Therefore, $\hat{x}(n) = x(n)\, W_R(n)$

Then, by taking DFT of equation (6.118), we obtain

$$\hat{X}(k) = \sum_{n=0}^{L-1} [x(n)\, W_R(n)]\, e^{-j2\pi kn/N}$$

Here, we are considering only 'L' samples of input signal and not the complete input signal. The spectrum will be more accurate if the value of 'L' is large.

Spectral Leakage

As shown in figure 5.30, the magnitude spectrum is not localized to a single frequency, but it is spreadout over the entire frequency range. That means that the power of a signal is spread over the entire frequency range. This leakage of the power is called as spectral leakage which is taking place because of '*windowing*' of input sequence.

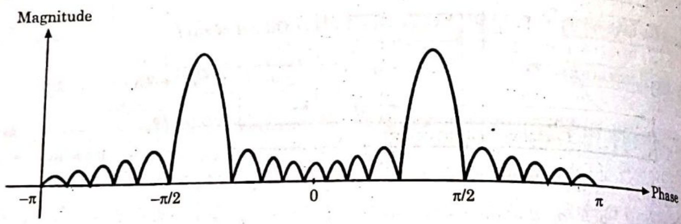

FIGURE 5.30 *Magnitude spectrum.*

5.15.1 Advantages and Limitations of Spectrum Analysis Using DFT

1. Advantages or Salient Features

(*i*) Fast processing of DFT can be done using FFT algorithms.

(*ii*) Estimation of power spectrum can be done.

(*iii*) Calculation of harmonics can also be done.

(*iv*) The resolution can be improved by increasing the number of samples in the calculation of DFT.

2. Limitations

(*i*) The frequency spectrum of entire input signal is not obtained because of windowing.

(*ii*) The leakage of power (*i.e.*, spectral leakage) takes place.

(*iii*) If we increase the number of samples to obtain the better accuracy, then the processing time is increased.

SUMMARY

- If the sequences are of finite duration, it forms a special case and we shall see that it is possible to develop and alternative Fourier representation, which we refer to as Discrete Fourier Transform (DFT).
- The frequency analysis of discrete-time signals is usually and conveniently performed on a Digital signal processor.
- The Fouier transform of a discrete time signal $x(n)$ is called Discrete-Time Fourier Transform (DTFT) and it is denoted by $X(e^{j\omega})$. We also know that DTFT $X(e^{j\omega})$ is a continuous function of frequency ω.

- This type of representation is not a computationally convenient representation of a discrete-time signal $x(n)$.
- We represent a sequence by samples of its continuous spectrum. This type of frequency-domain representation of a signal is known as **Discrete-Fourier Transform (DFT)**. It is very powerful tool for frequency analysis of discrete-time signals.
- The Discrete-Fourier Transform (DFT) is itself a sequence rather than a function of a continuous variable and it corresponds to equally spaced frequency samples of Discrete-Time Fouier Transform (DTFT) of a signal.
- Further, in the design of a DSP system, two fundamental tasks are involved. These are :
  (*i*) analysis of the input signal, and
  (*ii*) design of a processing system to provide the desired output.
- The discrete Fourier transform (DFT) and Fast Fourier Transform (FFT) are very important mathematical tools to carry out these types of tasks. Both these transforms can be used to analyse a two-dimensional signal. Also, the FFT algorithms eliminate the redundant calculation and enable to analyse the spectral properties of a signal.
- FFT algorithms are mainly useful in computing the DFT and IDFT and also find applications in linear filtering, digital spectral analysis and correlation analysis.
- The convolution is a mathematical operation or tool which is equivalent to finite impulse response (FIR) filtering. Convolution is of almost importance in digital signal processing (DSP) due to the fact that convolving two sequences in the time-domain is equivalent to multiplying the sequences in the frequency-domain.
- The convolution of these two discrete-time signals is expressed as

$$x(n) = x_1(n) \otimes x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

- Any arbitrary signal $x(n)$ can be resolved into a sum of unit sample sequences.
- For any arbitary signal $x(n)$, we have

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)$$

SHORT QUESTIONS WITH ANSWERS

Q. 1. Define DFT and IDFT.

Ans. DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \quad k = 0, 1, 2, \ldots, N-1$$

IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N} \quad n = 0, 1, 2, \ldots, N-1$$

Q.2. What do yoy mean by twiddle factor? Explain.

Ans. The DFT of discrete sequence $x(n)$ is denoted by $X(k)$. It is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N}$$

Here, $k = 0, 1, 2 \ldots N-1$

Since, this summation is taken for N points; it is called as **N point DFT.**

We can obtain discrete sequence $x(n)$ from its DFT. It is called as inverse discrete fourier tra (IDFT). It is given by,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}$$

Here, $n = 0, 1, 2, \ldots N-1$

This is called as **N point IDFT.**

Now, we will define the new term W as,

$$W_N = e^{-j2\pi/N}$$

This is called as **twiddle factor.** Twiddle factor makes the computation of DFT a bit easy and Using twiddle factor, we can write equations of DFT and IDFT as under :

$$X(k) = \sum_{n=0}^{N-1} X(n)\, W_N^{kn}$$

Here, $n = 0, 1, 2 \ldots N-1$

and
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

Here, $n = 0, 1, 2, \ldots N-1$

Q.3. Compare DFT and DTFT.

Ans. By comparing DFT with DTFT, we can write

(i) The continuous frequency spectrum $X(\omega)$ is replaced by discrete fourier spectrum X
(ii) Infinite summation in DTFT is replaced by finite summation in DFT.
(iii) The continuous frequency variable is replaced by finite number of frequencies loca at $\frac{2\pi k}{NT_s}$, where T_s is sampling time.

Q.4. State Periodicity property of DFT.

Ans. This property states that if a discrete-time signal is periodic then its DFT will also be periodic. Also, if a signal or sequence repeats its waveform after N number of samples then it is called periodic signal or sequence and N is called the period of signal. Mathematically,

If $X(k)$ is an N-point DFT of $x(n)$, then we have

$$x(n + N) = x(n) \text{ for all values of } n$$
$$X(k + N) = X(k) \text{ for all values of } k.$$

Q.5. State Linearity property of DFT.

Ans. Linearity properties states that DFT of linear combination of two or more signals is equal t the sum of linear combinatin of DFT of individual signal. Let us consider that $X_1(k)$ and $X_2(k)$ are the N-points DFTs of $x_1(n)$ and $x_2(n)$ respectively, and a and b are arbitrary constants either real or complex-valued, then mathematically

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then,

$$a\,x_1(n) + b\,x_2(n) \xleftrightarrow[N]{DFT} a\,X_1(k) + b\,X_2(k)$$

Here, a and b are some constants.

Q.6. State Duality Property of DFT.

Ans. Statement : If $\quad x(n) \xleftrightarrow[N]{DFT} X(k)$

then $\quad x(n) \xleftrightarrow[N]{DFT} Nx[((-k))_N]$.

Q.7. Define Time Reve

Ans. This property of DF folded.

Mathematicaly,

If $\quad x(n)$

then $\quad x((-n))_N =$

Hence, reversing th

Q.8. What do you mea

Ans. This property is a

This property stat circular shift of DFT i

Mathematically,

If

$x(n)e$

or $\quad x(n)e$

Q. 9. Carryout a det

Ans. Let us consider a impulse response $h(n)$

Then the output i

where,

In the frequenc

If the sequence $Y(\omega)$, then

or

or

Here, $X(k)$ an

Also, we know

Therefore, the of sequence $x(n)$ u

Q.7. Define Time Reversal of a sequence in DFT.

Ans. This property of DFT states that if a sequence is circularly folded, its DFT is also circularly folded.

Mathematicaly,

If $$x(n) \xleftrightarrow[N]{DFT} X(k)$$

then $$x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$$

Hence, reversing the N-point sequence in time is equivalent to reversing the DFT values.

Q.8. What do you mean by Circular frequency shift?

Ans. This property is also called as **Quadrature Modulation Theorem.**

This property states that multiplication of sequence $x(n)$ by $e^{\pm j2\pi kl/N}$ is equivalent to the circular shift of DFT in time domain by 'l' samples.

Mathematically,

If $$x(n) \xleftrightarrow[N]{DFT} X(k) \text{ then,}$$

$$x(n)e^{j2\pi ln/N} \xleftrightarrow[N]{DFT} X((k-l))_N = X(k+l)$$

or $$x(n)e^{-j2\pi ln/N} \xleftrightarrow[N]{DFT} X((k+l))_N = X(k-l)$$

Q. 9. Carryout a detailed comparison of circular convolution with linear convolution.

Ans. Let us consider a finite duration sequence $x(n)$ of length N_1 which is given as input to a system with impulse response $h(n)$ of length N_2.

Then the output is expressed as

$$y(n) = x(n) \otimes h(n)$$

$$= \sum_{k=0}^{N_1-1} x(k)h(n-k), \quad n = 0, 1, \ldots\ldots, N_1 + N_2 - 1$$

$$y(n) = \sum_{k=0}^{N_1-1} h(k)x(n-k), \quad n = 0, 1, \ldots, N_1 + N_2 - 1$$

where,
$$x(n) = 0 \text{ for } n < 0 \text{ and } n \geq N_1$$
$$h(n) = 0 \text{ for } n < 0 \text{ and } n \geq N_2$$

In the frequency-domain, we have

$$Y(\omega) = H(\omega)\, X(\omega)$$

If the sequence $y(n)$ has to be represented uniquely in the frequency-domain by samples of its spectrum $Y(\omega)$, then

$$Y(k) = Y(\omega)\big|_{\omega = 2\pi k/N},\ k = 0, 1, \ldots N-1$$

or $$Y(k) = X(\omega)\, H(\omega)\big|_{\omega = 2\pi k/N},\ k = 0, 1, \ldots, N-1$$

or $$Y(k) = X(k).H(k)$$

Here, $X(k)$ and $H(k)$ are the N-point DFTS of the sequence $x(n)$ and $h(n)$ respectively.

Also, we know that since $y(n)$ can be obtained by

$$y(n) = IDFT[X(k) \cdot H(k)]$$

Therefore, the N-point circular convolution of x(n) with h(n) must be equivalent to the linear convolution of sequence x(n) with h(n).

Q. 10. What is Praseval's Theorem?

Ans. For complex-valued sequences $x(n)$ and $y(n)$, in general, if

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

and

$$y(n) \xleftrightarrow[N]{DFT} Y(k)$$

then

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)$$

Q.11. Define steps to obtain same results from linear and circular convolution.

Ans. Steps to obtain same results from linear and circular convolution :

To obtain the same results from both convolutions, the following steps are used:

(i) Using equation (iii), we calculate the value of N that means number of samples contained in linear convolution. Let us say, it is 15.

(ii) By doing zero padding, we make the length of every sequence equal to 15. This means in this case, we need to add seven zeros in $x(n)$ as well as $h(n)$.

(iii) Then, we perform the circular convolution. The result of circular convolution and convolution will be same.

Q. 12. Define steps to evaluate of Linear Filtering using DFT.

Ans. Steps to evaluation of Linear Filtering using DFT are as follows :

(i) First, we calculate the value of N using $N = L + M - 1$. Here, L represents number of samples in $x(n)$ and M represents number of samples in $h(n)$.

(ii) By adding zeros, we make the length of $x(n)$ and $h(n)$ equal to N.

(iii) We calculate DFT of $x(n)$ that means $X(k)$.

(iv) We calculate DFT of $h(n)$ that means $H(k)$.

(v) Then, we multiply $X(k)$ and $H(k)$ to get $Y(k) = X(k) \,.\, H(k)$.

(vi) Lastly, we obtain IDFT of $Y(k)$ that means $y(n)$.

Q. 13. Explain about Circular Correlation.

Ans. The circular cross correlation of two sequences in time domain is equivalent to the multiplication of DFT of one sequence with the complex conjugate DFT of other sequence.

Mathematically,

If $\quad x(n) \xleftrightarrow[N]{DFT} X(k),$

and $\quad y(n) \xleftrightarrow[N]{DFT} Y(k)$, then,

$$r_{xy}(l) \xleftrightarrow[N]{DFT} R_{xy}(k) = X(k)\, Y^*(k)$$

Q. 14. Define Multiplication of Two Sequences.

Ans. The multiplication of two sequences in time domain is equivalent to its circular convolution in the frequency domain.

Mathematically, we have

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then,

$$x_1(n) \,.\, x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} [X_1(k) \, \textcircled{N} \, X_2(k)]$$

1. Differentiate betw... of a sequence.
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REVIEW QUESTIONS

1. Differentiate between Discrete-Time Fourier Transform (DTFT) and Discrete Fourier Transform (DFT) of a sequence.
2. Explain how DFT can be used as a linear transformation tool in Digital Signal Processing.
3. Explain various properties of the DFT in brief.
4. Differentiate between circular convolution and linear convolution of two discrete-time sequences.
5. Explain circular convolution of two discrete-time sequences in detail.
6. Explain Parseval's theorem for discrete-time sequences.
7. Explain the relation between DFT and z-transform ?
8. How linear convolution can be performed using the DFT ?
9. In what case resultant of linear convolution and circular convolution will be the same? Discuss.
10. Discuss various problems of pitfalls in using DFT.
11. Write short technical notes on the following :
    (i) Truncation error
    (ii) Aliasing error
    (iii) Spectral leakage
    (iv) Improper use of DFT
    (v) Picket-fence effect.
12. Discuss remedies of various problems of pitfalls in using DFT.
13. Explain fourier transform for discrete time signals and take its important properties.
14. Define DFT and IDFT.
15. Explain Discrete Fourier Series (DFS). What is the relationship between DFT and DFS?
16. Explain circular convolution. What is the difference between circular convolution, linear convolution and periodic convolution?
17. State and prove following properties of DFT.
    (i) Circular time reversal
    (ii) Circular time shift
    (iii) Circular frequency shift
    (iv) Circular correlation
18. Explain how linear convolution can be obtained using DFT and IDFT. Also explain how linear convolution can be obtained using circular convolution.
19. Explain the applications of DFT in linear filtering and spectrum analysis.
20. Explain the relationships between z-transform and DFT and fourier transform.
21. With appropriate diagrams describe:
    (i) Overlap-save method.
    (ii) Overlap-add method.

NUMERICAL PROBLEMS

1. Using matrix method, determine 8 point DFT of sequence $x(n) = \{0, 0, 1, 1, 1, 0, 0, 0\}$
   Ans. $X(k) = \{3, -1.707 + j\,1.707, j, -0.293 + j0.293, 1, -0.293, -j, 0.293 - j - 1.707 + j\,1.707\}$
2. Determine IDFT of $X(k) = \{2, 1 + j, 0, 1 - j\}$ **Ans.** $x(n) = \{1, 0, 0, 1\}$
3. A 4-point DFT of sampled data sequence $\{2, 0, 0, 1\}$ is $\{3, 2 + j, 1, 2 - j\}$. Verify (i) $X(7) = X(3)$ (ii) $X(12) = X(0)$

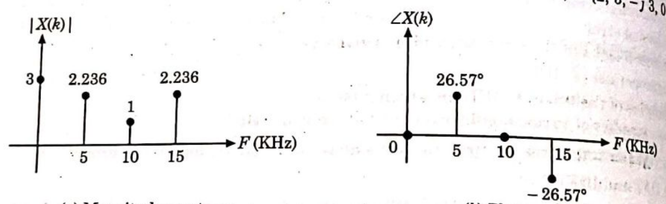
[**Hint :** (*i*) Put N = 4 and $n = 7$ in the equation of IDFT and expand the summation.
(*ii*) Put N = 4 and $k = 12$ in the equation of IDFT and expand the summation.

4. Plot the magnitude and phase spectrum of sampled data sequence {2, 0, 0, 1} which is ... using sampling frequency of 20 kHz. Select N = 4.
[**Hint :** $X(k) = \{3, 2+j, 1, 2-j\} = \{3\angle 0°, 2.236\angle 26.57°, 1\angle 0°, 2.236, \angle -26.57°\}$
To plot the spectrum we have to select frequency on X axis.

Hence, $$F = \frac{1}{NT_S} = \frac{F_S}{N} = \frac{20\text{ kHz}}{4} = 5\text{kHz}$$

$\therefore$ $X(0)$ represents DC component
$X(1)$ represents 5 kHZ component
$X(2)$ represents 10 kHz component
$X(3)$ represents 15 kHz component

5. Calculate DFT of a sequence whose values for one period are given by,
Ans. $X(k) = \{2, 3, -j3, 0, 3, ...$



(*a*) Magnitude spectrum (*b*) Phase spectrum

FIGURE 5.38.

6. First five points of 8 point DFT of a real valued sequence are given by X(0) = 0 X(1) = 2 + j2, X(2) = –j 4, X (3) = 2 – j2, X (4) = 0. Determine remaining points and obtain time-domain sequence x(n).
Ans. $X(5) = 2 + j2$, $X(6) = 0 + j4$, $X(7) = 2 - j2$
and $x(n) = \{1, 1, -1, -1, -1, 1, 1, 1, -1...$

7. Obtain circular convolution of,
$$x_1(n) = \{1, 3, 5, 3\},\ x_2(n) = \{2, 3, 1, 1\}$$
Ans. $y(m) = \{19, 17, 23, 2...$

8. Using DFT and IDFT, obtain circular convolution of,
$$x(n) = \{2, 0, 0, 1\},\ h(n) = \{4, 3, 2, 1\}$$
Ans. $Y(m) = \{11, 8, 5, ...$

9. Find out sequence $x(n) = \{1, 1 - 2, - 2\}$ $Y(n)$ if $h(n) = \{1, 1, 1\}$, $x(n) = \{1, 2, 3, 1\}$, using circular convolution.
[**Hint :** Obtain linear convolution using circular convolution]
Ans. $Y(n) = \{1, 3, 6, 6, 4, 1, 0, 0...$

10. Obtain circular convolution of,
$$x_1(n) = \{1, 2, 3, 4\},\ x_2(n) = \{4, 3, 2, 1\}$$
Ans. $Y(m) = \{24, 22, 24, 30...$

11. Determine response $Y(n)$ of a filter if $x(n) = \{1, 2, 2, 1\}$ and $h(n) = \{1, 2, 3\}$
Ans. $Y(n) = \{1, 4, 9, 11, 8, 3...$

12. The five samples of the 9-point DFT are given as follows :
$X(0) = 23,\ X(1) = 2.242 - j$
$X(4) = -6.379 + j4.121$
$X(6) = 6.5 + j259$
$X(7) = -4.153 + j0.264$

Determine the remai...
[Ans. X(...

13. Compute circular co...
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14. Determine 8-point ...
$x(n)$
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Determine the remaining samples of DFT if the corresponding time domain sequence is real.
[Ans. $X(2) = -4.153 - j0.264$, $X(3) = 6.5 - j2.59$, $X(5) = -6.379 - j4.121$, $X(8) = 2.421 + j$]

13. Compute circular control convolution of the following two sequences :

$$x_1(n) = \left\{\underset{\uparrow}{1}, 2, 0, 1\right\}$$

$$x_2(n) = \left\{\underset{\uparrow}{2}, 2, 1, 1\right\}$$

14. Determine 8-point DFT of the signal.

$$x(n) = \{1,1,1,1,1,1,0,0\}$$

Also sketch its magnitude and phase.

[Ans. $X(k) = \left\{6, -\frac{1}{\sqrt{2}} - j\left(1+\frac{1}{\sqrt{2}}\right), 1-j, \frac{1}{\sqrt{2}} + j\left(1-\frac{1}{\sqrt{2}}\right)\ 0, \frac{1}{\sqrt{2}} + j\left(-1+\frac{1}{\sqrt{2}}\right), 1+j, -\frac{1}{\sqrt{2}} + j\left(1+\frac{1}{\sqrt{2}}\right)\right\}$

$|X(k)| = \{6, 1.847, 1.414, 0.765, 0, 0.765, 0, 0.765, 1.414, 1.847\}$

$\angle X(k) = \{0, -112.5°, -45°, 22.5°, 0°, -22.5°, 45°, 112.5°\}$]

15. Determine N-point DFT of the sequence which is given as, $x(n) = \delta(n - n_0)$.

[Ans. $X(k) = e^{-j2\pi kn_0/N}$]

16. The five samples of the 9-point DFT are given as follows:

$$X(0) = 23, X(1) = 2242 - j$$
$$X(4) = -6.379 + j\,4.121$$
$$X(6) = 65 + j\,259$$
$$X(7) = -4.153 + j\,0.264$$

Determine the remaining samples of DFT if the corresponding time domain sequence is real.
[Ans. $X(2) = -4.153 - j\,0.264$]

$$X(3) = 6.5 - j\,2.59$$
$$X(5) = -6.379 - j\,4.121$$
$$X(8) = 2.42 + j$$

17. Compute circular convolution of the following two sequences:

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 0, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{2}, 2, 1, 1\}$$

[Ans. $x_1(n)\ x_2(n) = \{6, 7, 6, 5\}$]

18. Compute the circular convolution of the following sequences using DFT and IDFT.

$$x_1(n) = \{1, 2, 3, 1\}$$
$$x_2(n) = \{4, 3, 2, 2\}$$

[Ans. $x_1(n)\ x_2(n) = \{17, 19, 22, 19\}$]