CENTROID, CENTRE OF MASS AND CENTRE OF GRAVITY

CENTRE OF GRAVITY ⟶ It is the point through which the resultant of the distributed gravity forces act irrespective of the orientation of the body.

CENTRE OF MASS ⟶ It is the point where the entire mass of a body may be assumed to be concentrated.

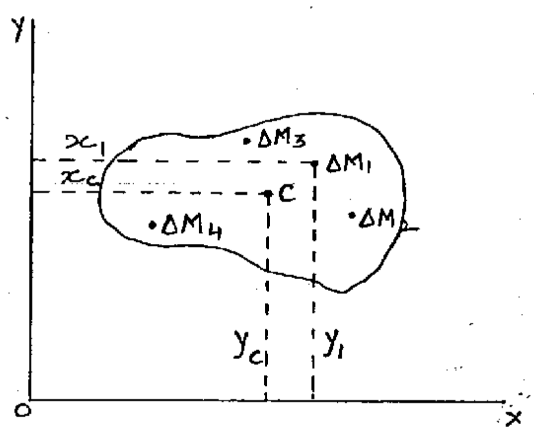
C.G. and C.M. are different only when the gravitational field is not uniform and parallel.

CENTRE OF GRAVITY OF A BODY: DETERMINATION BY THE METHOD OF MOMENTS

Consider a body having mass M. It is composed of number of masses $\Delta M_1, \Delta M_2, \Delta M_3 \ldots\ldots \Delta M_n$ distributed within the body. Let 'n' be the number of masses.

Now

$$M = \Delta M_1 + \Delta M_2 + \Delta M_3 + \ldots + \Delta M_n$$



Distance of these masses with respect to the axes be,

$$(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$$

Let CG of whole mass M lie at a distance (x_c, y_c) with respect to reference axes.

Gravitational force acting on the mass ΔM_1

$$F_1 = \Delta M_1 g$$

Similarly gravitational forces acting on the masses ΔM_2, ΔM_3, ΔM_4 ..... ΔM_n.

$$F_2 = \Delta M_2 g$$
$$F_3 = \Delta M_3 g$$
$$\vdots$$
$$F_n = \Delta M_n g$$

By principle of moments, resultant of parallel forces $F_1, F_2 \ldots F_n$ can be determined.

Moment of resultant of all forces about y-axis = Sum of moments of all forces about y-axis

$$Mg(x_c) = \Delta M_1 g(x_1) + \Delta M_2 g(x_2) + \ldots + \Delta M_n g(x_n)$$

$$x_c = \frac{g[\Delta M_1 x_1 + \Delta M_2 x_2 + \ldots + \Delta M_n x_n]}{Mg}$$

$$\boxed{x_c = \frac{\Delta M_1 x_1 + \Delta M_2 x_2 + \ldots + \Delta M_n x_n}{M}}$$

We know,

$$M = \Delta M_1 + \Delta M_2 + \Delta M_3 + \ldots + \Delta M_n$$

$$\boxed{M = \Sigma \Delta M_i}$$

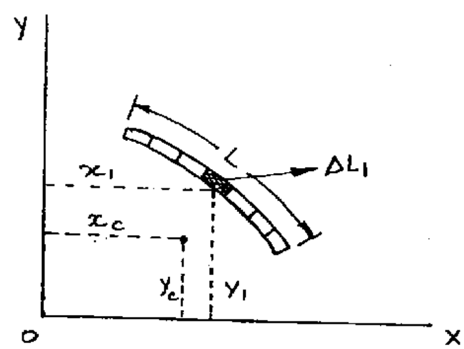
Therefore,

$$\boxed{\begin{aligned} x_c &= \frac{\Sigma(\Delta M_i x_i)}{\Sigma(\Delta M_i)} \\ y_c &= \frac{\Sigma(\Delta M_i y_i)}{\Sigma(\Delta M_i)} \end{aligned}}$$

CONCEPT OF CENTROID.

→ ONE DIMENSIONAL BODY :
[LINE SEGMENT]

Consider a body of shape of curved homogenous wire of uniform cross-section and of length L.



Dividing the length of wire into 'n' number of elements of lengths $\Delta L_1, \Delta L_2, \Delta L_3 \dots \Delta L_n$.

Uniform area of cross-section = A

Density of the wire = ρ

Mass M of the wire of length L = $AL\rho$

Mass of an element of length $\Delta L_1 = \Delta M_1$

$$\Delta M_1 = \text{Volume} \times \text{density} = A(\Delta L_1)\rho$$

$$\Delta M_2 = A(\Delta L_2)\rho$$

$$\vdots$$

$$\Delta M_n = A(\Delta L_n)\rho$$

Distances of the centres of these lengths with respect to the axes be,

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

Applying principal of moments

$$x_c = \frac{\Sigma(\Delta M_i x_i)}{\Sigma(\Delta M_i)}$$

$$x_c = \frac{(A\Delta L_1 \rho)x_1 + (A\Delta L_2 \rho)x_2 + \ldots + (A\Delta L_n \rho)x_n}{A\Delta L_1 \rho + A\Delta L_2 \rho + \ldots + A\Delta L_n \rho}$$

$$x_c = \frac{\Delta L_1 x_1 + \Delta L_2 x_2 + \ldots + \Delta L_n x_n}{\Delta L_1 + \Delta L_2 + \ldots + \Delta L_n}$$

$$x_c = \frac{\Sigma \Delta L_i x_i}{\Sigma \Delta L_i}$$

$$y_c = \frac{\Sigma \Delta L_i y_i}{\Sigma \Delta L_i}$$

CG becomes the coordinates of the centroid of the wire ; generally referred to as centroid of a line segment.

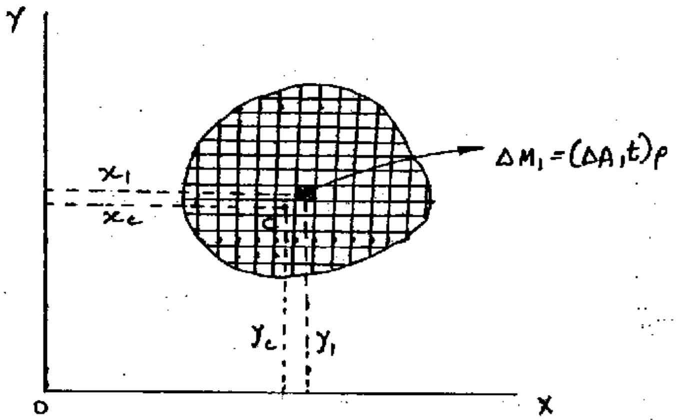
CENTROID OF TWO DIMENSIONAL BODY

Consider a homogenous plate or lamina of uniform thickness t, density ρ and total area A.

Dividing the area of plate into 'n' number of elements of areas $\Delta A_1, \Delta A_2, \ldots \Delta A_n$.

Distances of the centres of these areas with respect to the axes be

$(x_1, y_1), (x_2, y_2) \text{ - - - - - } (x_n, y_n)$



Mass of the plate = $M = At\rho$

Mass of the element = $\Delta M_1 = \Delta A_1 t\rho$

By moments' principle.

$$x_c = \frac{\Sigma \Delta M_i x_i}{\Sigma \Delta M_i}$$

$$x_c = \frac{x_1 \Delta A_1 t\rho + \Delta A_2 t\rho x_2 + \dots + \Delta A_n t\rho x_n}{\Delta A_1 t\rho + \Delta A_2 t\rho + \dots + \Delta A_n t\rho}$$

$$x_c = \frac{\Delta A_1 x_1 + \Delta A_2 x_2 + \dots + \Delta A_n x_n}{\Delta A_1 + \Delta A_2 + \dots + \Delta A_n}$$

$$x_c = \frac{\Sigma A_i x_i}{\Sigma A_i}$$

$$y_c = \frac{\Sigma A_i y_i}{\Sigma A_y.}$$

x_c and y_c are coordinates of centroid of a plate, generally called coordinates of centroid of an area.

DETERMINATION OF CENTROID AND CENTRE OF GRAVITY : INTEGRATION METHOD

If the terms ΔL or ΔA occuring in the expressions of C.G and centroid become infinitesimally small, then the expressions can be written as:

$$x_c = \frac{\int x\,dL}{\int dL} \qquad x_c = \frac{\int x\,dA}{\int dA} \qquad x_c = \frac{\int x\,dm}{\int dm}$$

$$y_c = \frac{\int y\,dL}{\int dL} \qquad y_c = \frac{\int y\,dA}{\int dA} \qquad y_c = \frac{\int y\,dm}{\int dm}$$

The integral $\int x\,dA$ is known as first moment of area with respect to the y-axis. While integral $\int y\,dA$ is known as first moment of area with respect to the x-axis.

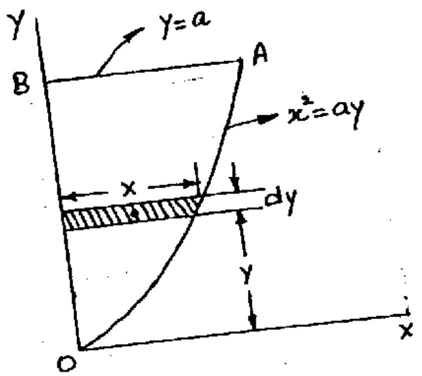
INTEGRATION METHOD

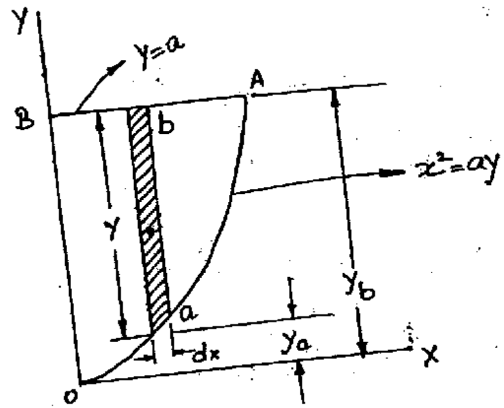
→ CHOICE OF DIFFERENTIAL ELEMENT :

We know,

$$x_c = \frac{\int x\,dA}{\int dA} \quad , \quad y_c = \frac{\int y\,dA}{\int dA}$$

Consider an area OAB bounded by curve $x^2 = ay$ and the straight line $y = a$ as shown below.





Consider a horizontal strip

Area of differential element $dA = x\,dy$

Position of its centroid $\rightarrow \left(\frac{x}{2}, y\right)$

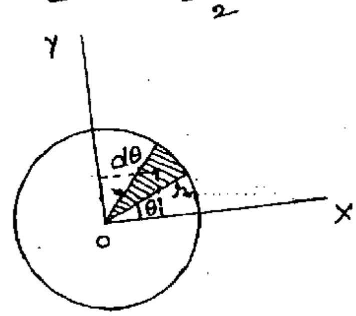
Consider a vertical strip

Area of differential element $dA = y\,dx = (y_b - y_a)\,dx$

Position of centroid $\rightarrow \left[x, \dfrac{y_a + y_b}{2}\right]$

$$\left[\begin{aligned} \frac{y}{2} &= \frac{y_b - y_a}{2} \\ \frac{y}{2} + y_a &= \frac{y_b - y_a}{2} + y_a \\ &= \frac{y_b + y_a}{2} \end{aligned}\right.$$

TRIANGULAR ELEMENT



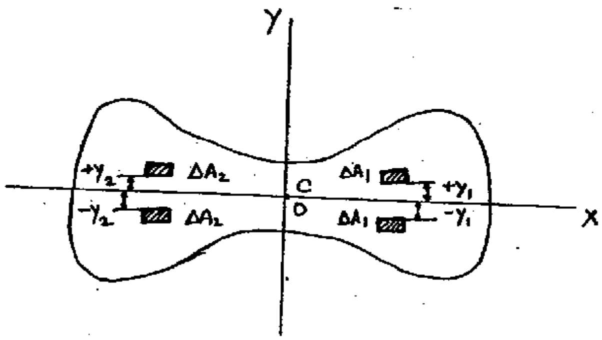
Area of the element, dA

$$= \frac{(r\,d\theta)\,r}{2}$$

Position of its centroid

$$= \left(\frac{2}{3} r \cos\theta, \frac{2}{3} r \sin\theta\right)$$

→ CHOICE OF THE AXES OF REFERENCE :



Consider a body which is symmetrical to both axes i.e. x-axis and y-axis such as a dumbell.

Symmetry about x-axis →

Let any element of area ΔA_1 at a distance y_1 from x-axis be considered. Due to symmetry a similar element of area ΔA_1 excist at a distance y_1 below x-axis.

If sum of the moments of all the elements of the area above the x-axis are taken then, it will cancel with the sum of the moments of similar elements of the area lying below the x-axis.

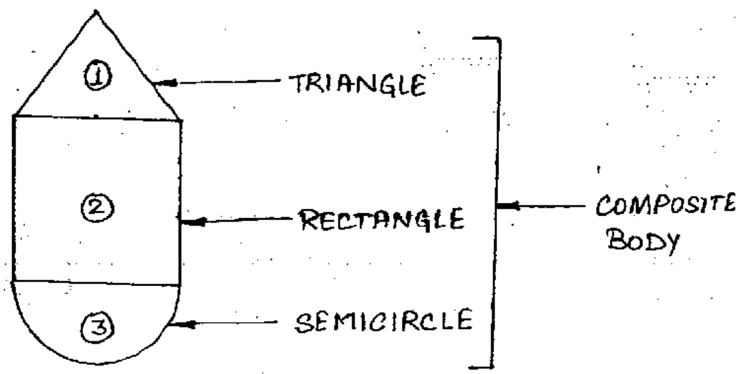
Hence,

$y_c = 0$ → Centroid lies on the x-axis.

$x_c = 0$ → Centroid lies on the y-axis.

CENTROID OF A COMPOSITE PLANE FIGURE

A composite area or a curve is one which is conside to be made up of several components that represent familiar geometric shapes (eg rectangular, circle, ellipse etc) and for which the positions of individual centroid are known.

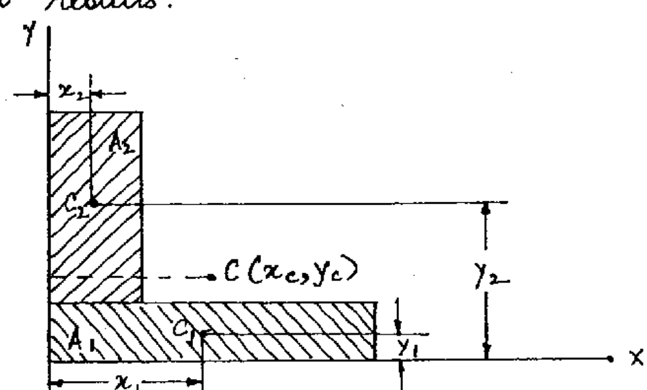


To find the centroid of this area, divide the area into different components and not into infinitesimal elements. Then, use the equation

$$x_c = \frac{\Sigma \Delta A_i x_i}{\Sigma \Delta A_i} \quad , \quad y_c = \frac{\Sigma \Delta A_i y_i}{\Sigma \Delta A_i}$$

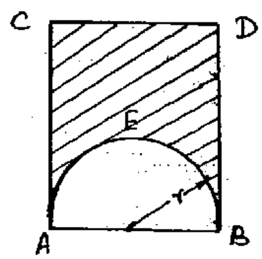
The positions of centroid of the components can be taken as standard results.

EXAMPLE :

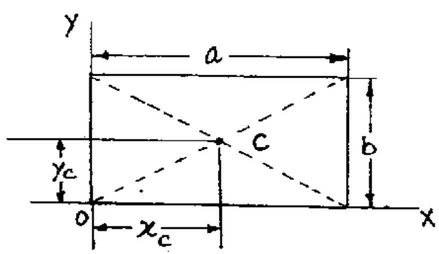
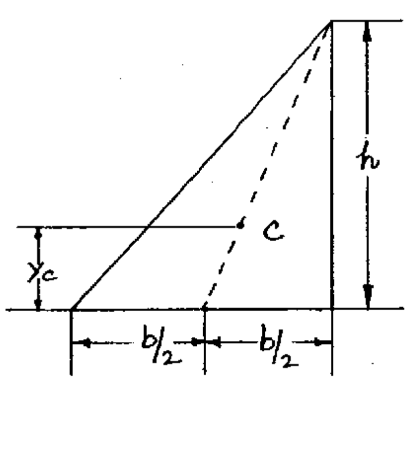


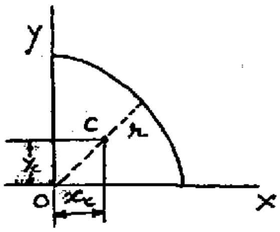
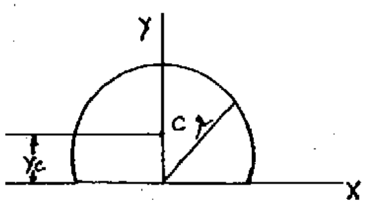
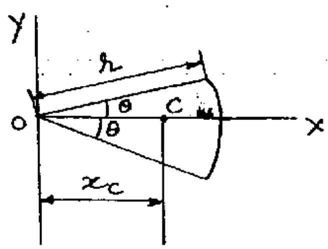
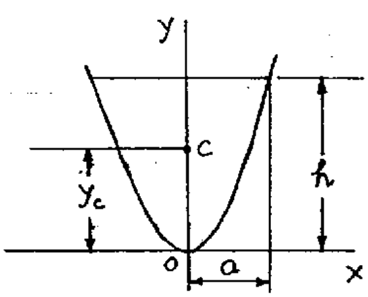
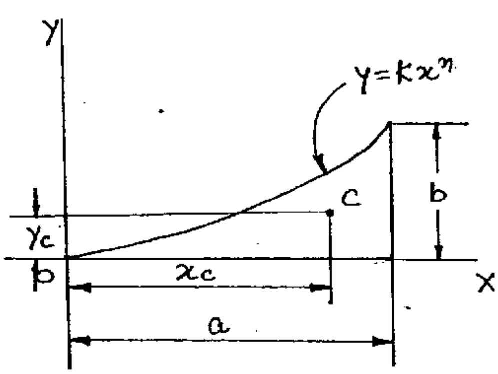
$$x_c = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2} \quad , \quad y_c = \frac{A_2 y_2 + A_1 y_1}{A_1 + A_2}$$

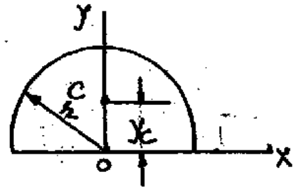
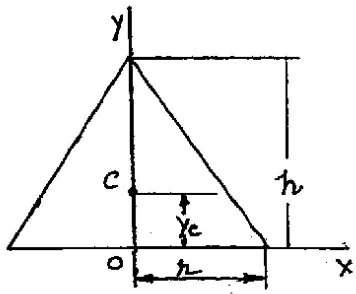
If a hole or a void exist then it is treated as negative area.



In this body, the rectangle ABCD is one of the component and semicircle of radius 'r' having a negative area.

NAME	SHAPE	AREA	x_c	y_c
RECTANGLE		ab	$\frac{a}{2}$	$\frac{b}{2}$
TRIANGLE		$\frac{bh}{2}$	—	$\frac{h}{3}$

NAME	SHAPE	AREA	x_c	y_c
QUARTER-CIRCLE		$\frac{\pi r^2}{4}$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
SEMI-CIRCLE		$\frac{\pi r^2}{2}$	0	$\frac{4r}{3\pi}$
CIRCULAR SECTOR		θr^2	$\frac{2r\sin\theta}{3\theta}$	0
PARABOLA		$\frac{4ah}{3}$	0	$\frac{3h}{5}$
GENERAL SPANDER		$\frac{ab}{n+1}$	$\left(\frac{n+1}{n+2}\right)a$	$\left(\frac{n+1}{2n+1}\right)\frac{b}{2}$

NAME	SHAPE	AREA	x_c	y_c
HEMISPHERE		$\frac{2}{3}\pi r^3$	0	$\frac{3}{8}r$
RIGHT CIRCULAR CONE		$\frac{1}{3}\pi r^2 h$	0	$\frac{h}{4}$