STABILITY THEORY

CONCEPT OF STABILITY

The concept of stability is very important to analyze and design the system. A system is said to be stable if it's input (bounded) produces a bounded output.

Types of systems based on stability :-

1. Absolutely Stable System :- If the system is stable for all range of system component values, then it is known as absolutely stable systems.

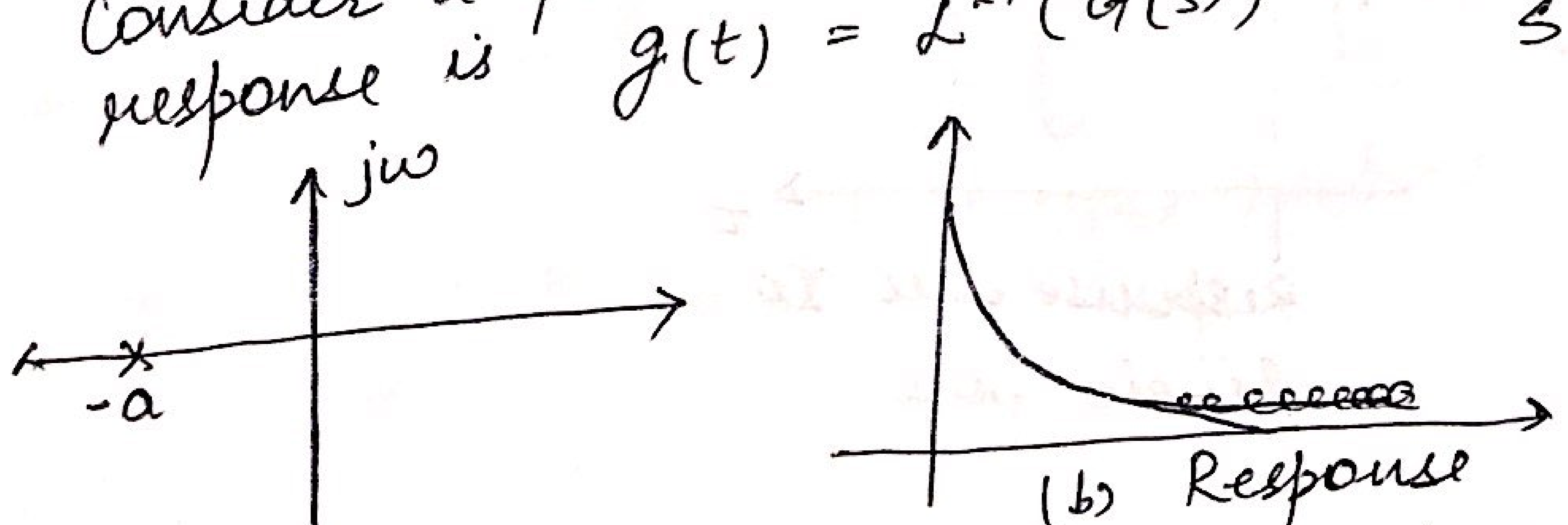
2. Conditionally Stable System :- If the system is stable for a certain range of component values, then it is known as conditionally stable system

3. Marginally Stable system :- If the system is stable by producing an output signal with constant amplitude and constant freq. of oscillations for bounded input, then it is known as marginally stable system.

EFFECT OF LOCATION OF POLES ON STABILITY

(a) Poles on -ve real Axis.

Consider a pole at $s=-a$. The corresponding impulse response is $g(t) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}\frac{k}{s+a} = K \cdot e^{-at}$.



(b) Response

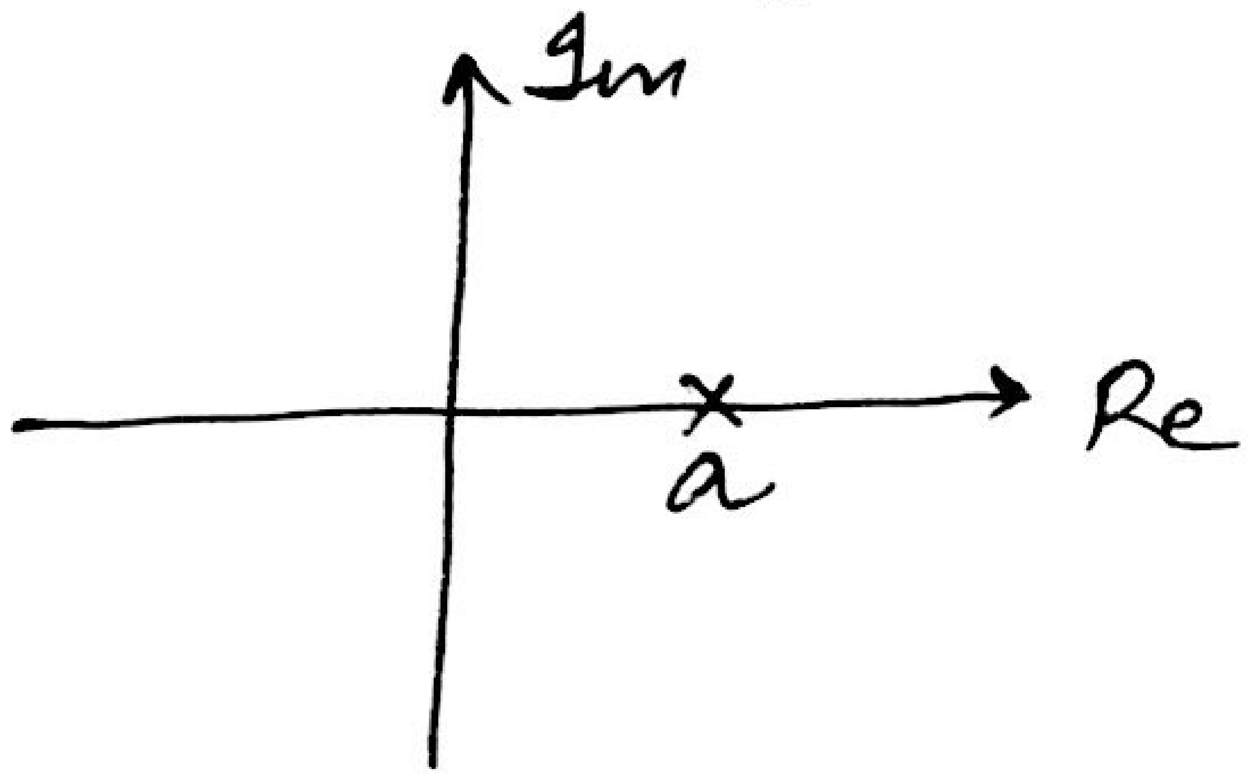
As the time t, increases, the response approaches zero and system is stable.

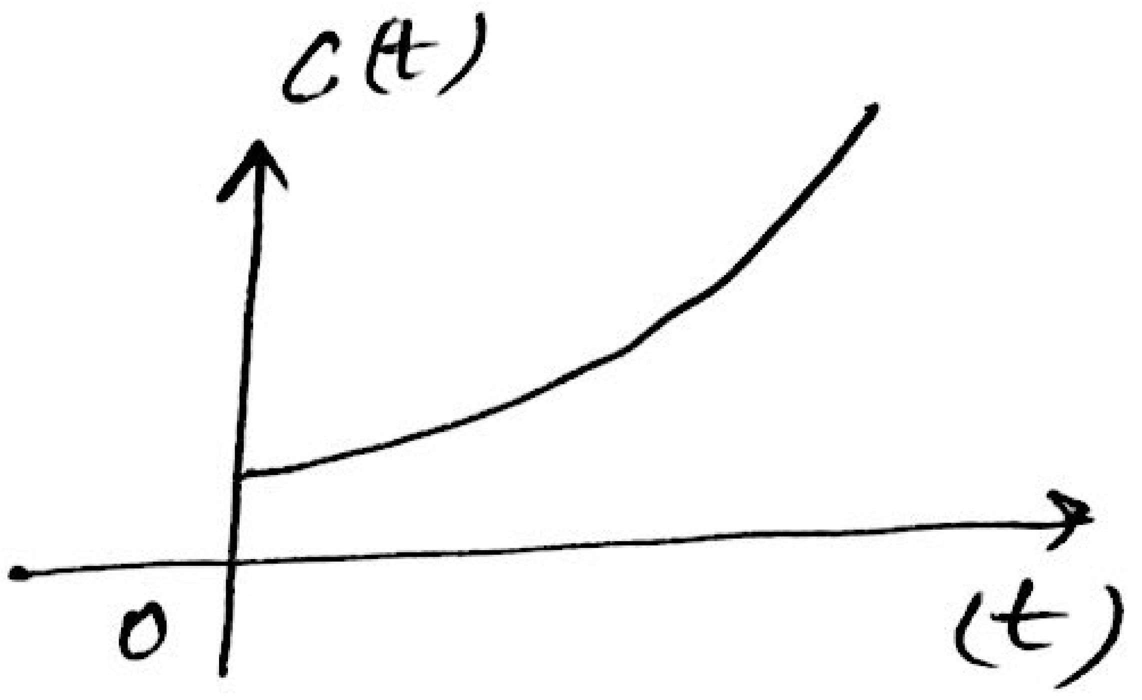
(b) Pole on the real axis

Consider a simple pole $s = a$,

$$g(t) = \mathcal{L}^{-1}\left(\frac{K}{s-a}\right) = K e^{at}$$

The response increases exponentially with time, hence the system is unstable.





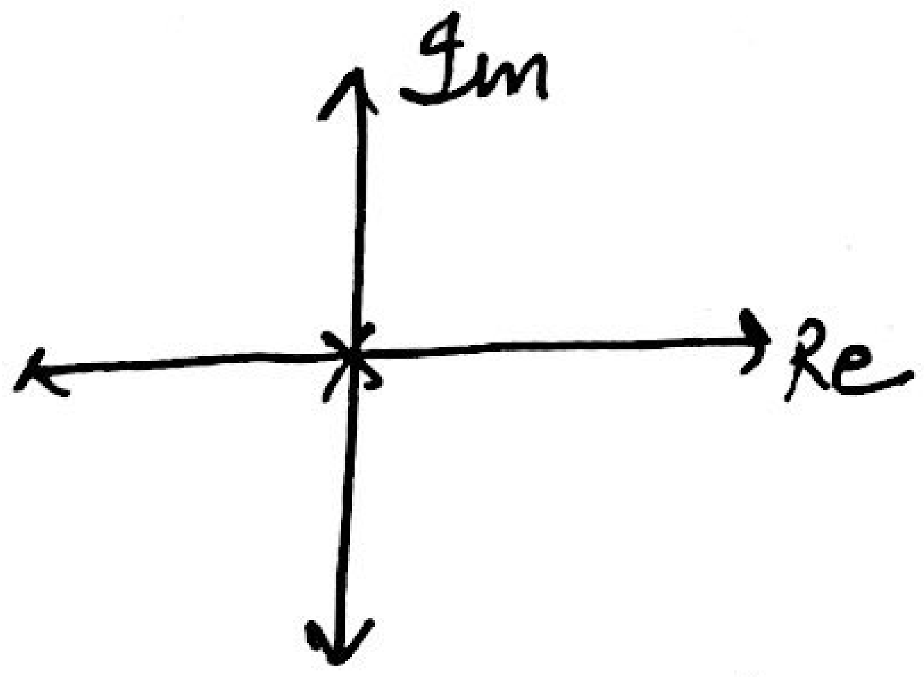
(c) Poles at origin

$$g(t) = \mathcal{L}^{-1}\left\{\frac{K}{s}\right\} = K$$

This is the constant value, hence the system is marginally stable

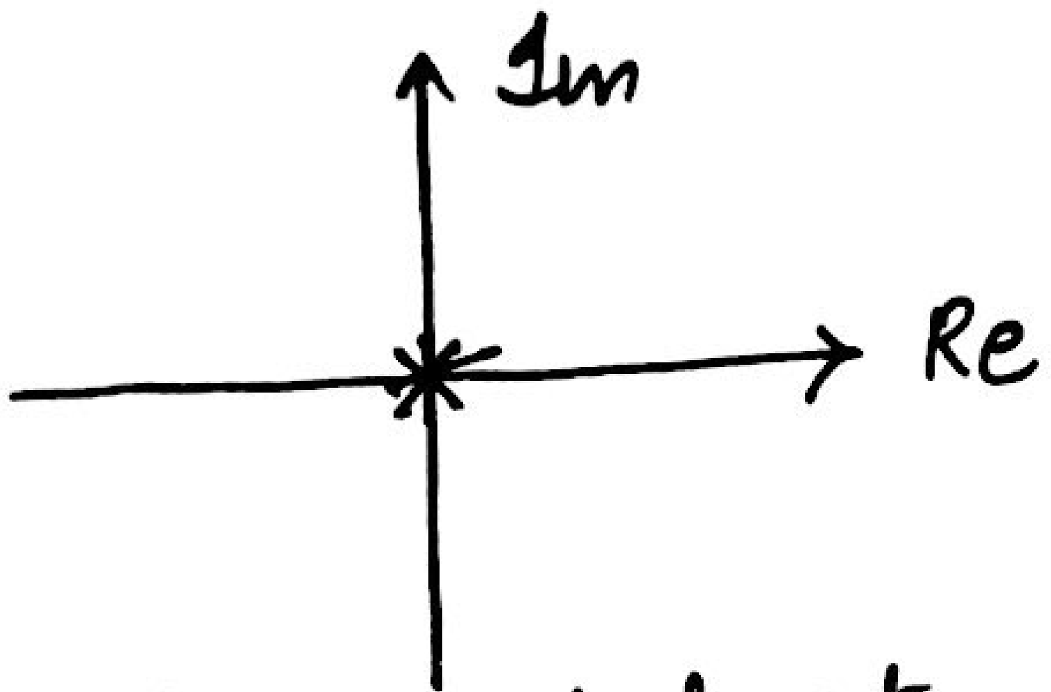
If there are two poles at the origin

$$g(t) = \mathcal{L}^{-1}\left\{\frac{K}{s^2}\right\} = Kt$$

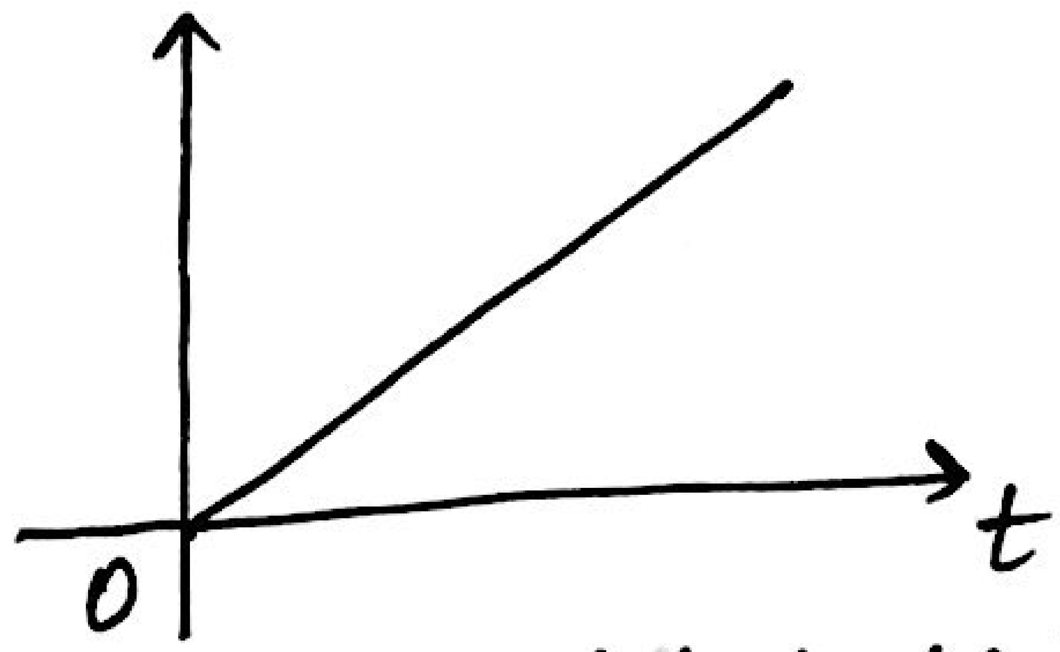


Single pole at origin

Response due to single pole



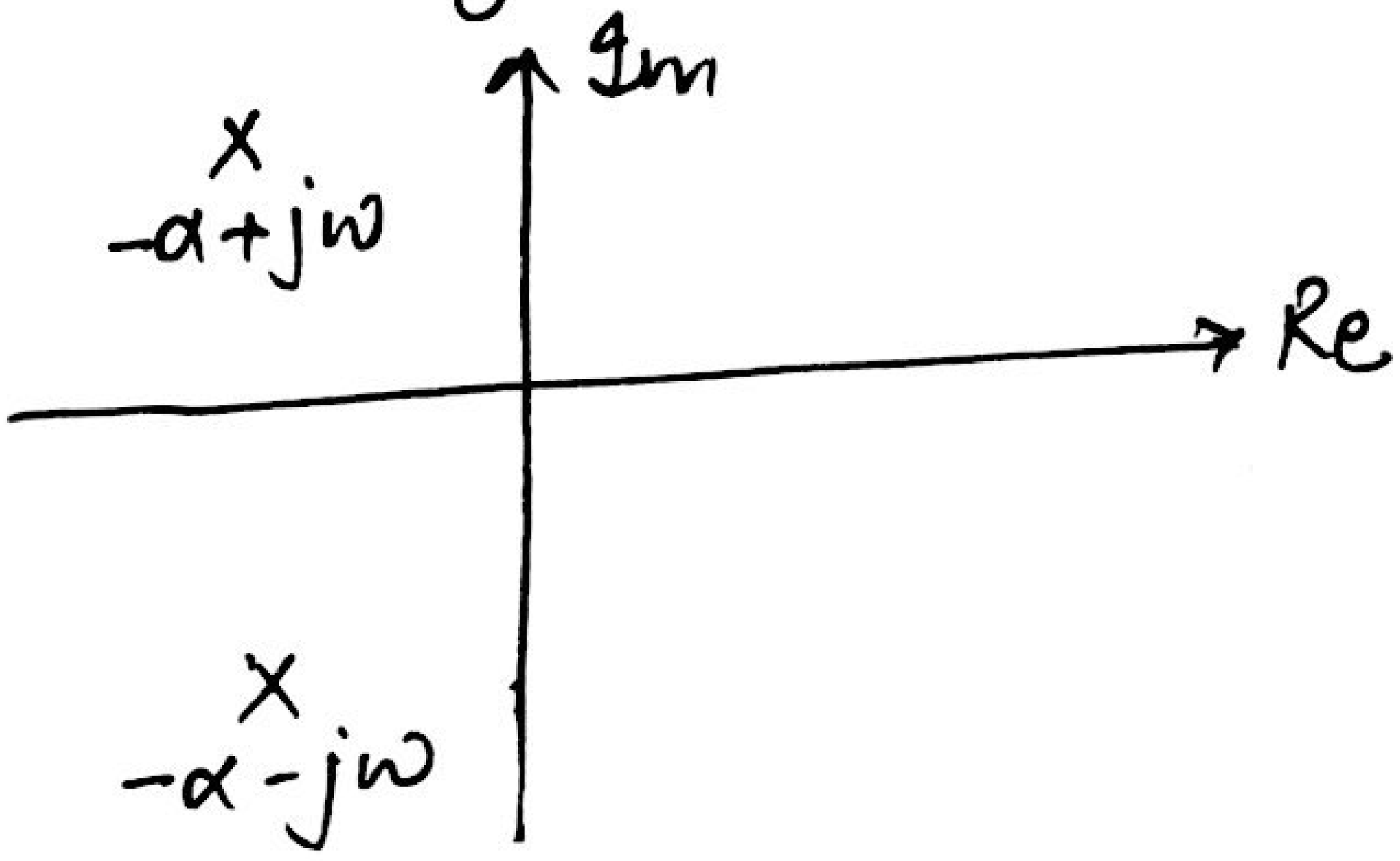
Double pole at Origion

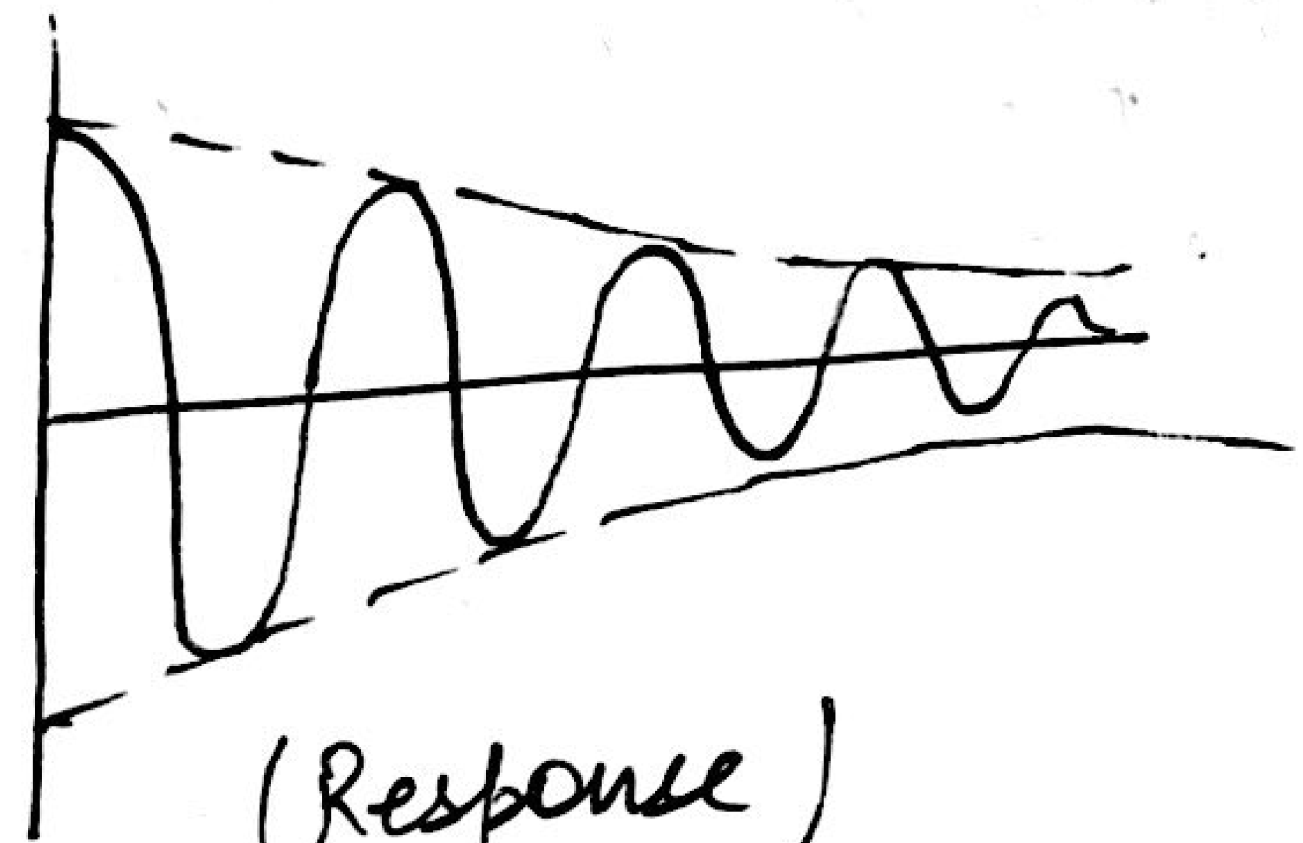


Response (Impulse)

(d:) complex poles in the left half of S-plane

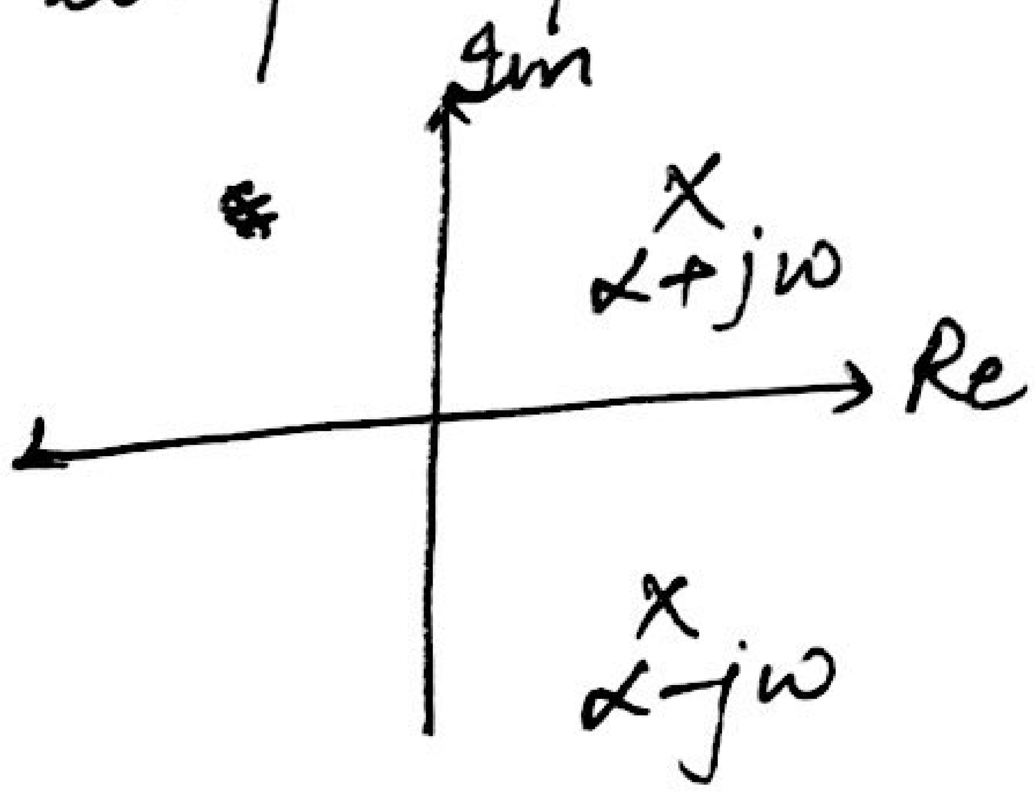
The System is Stable



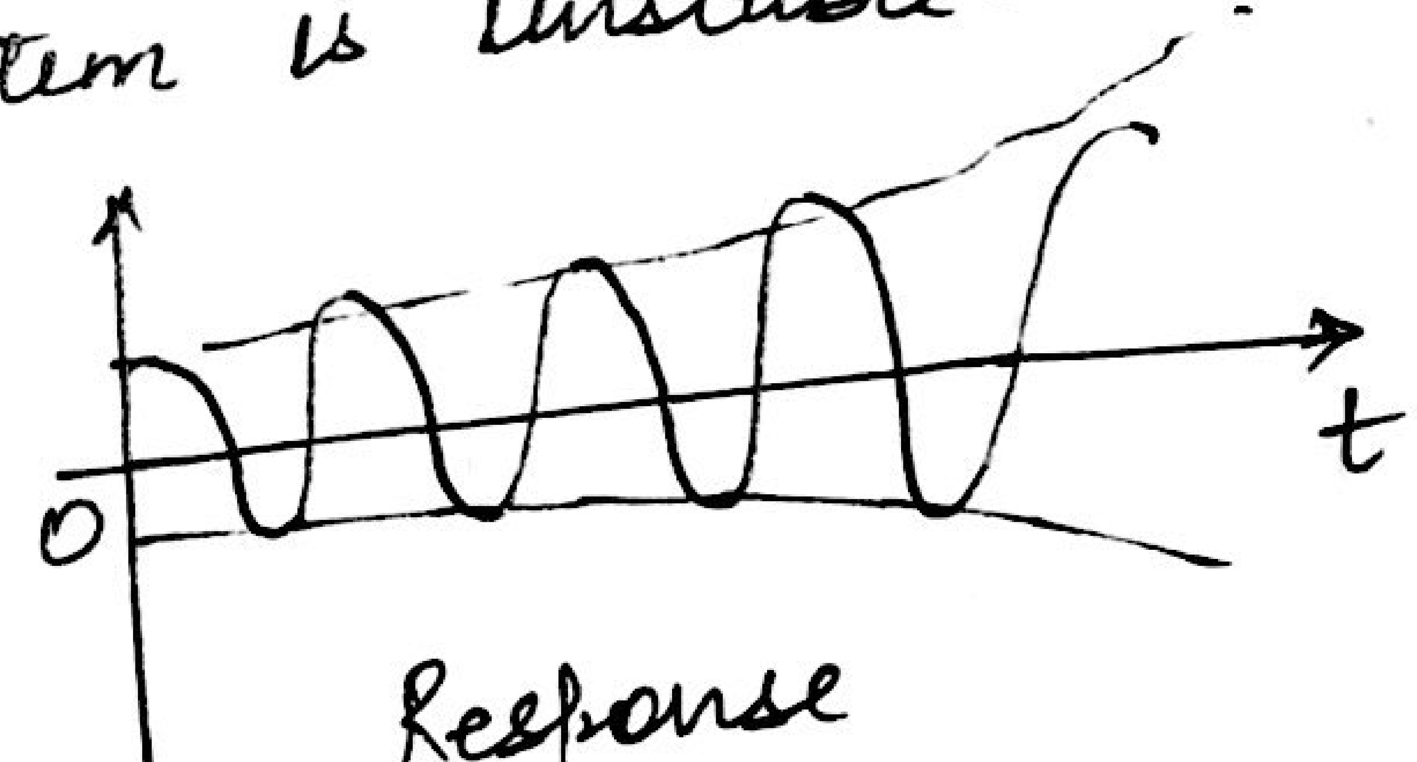


(Response)

(e:) complex poles in the right half of S-plane



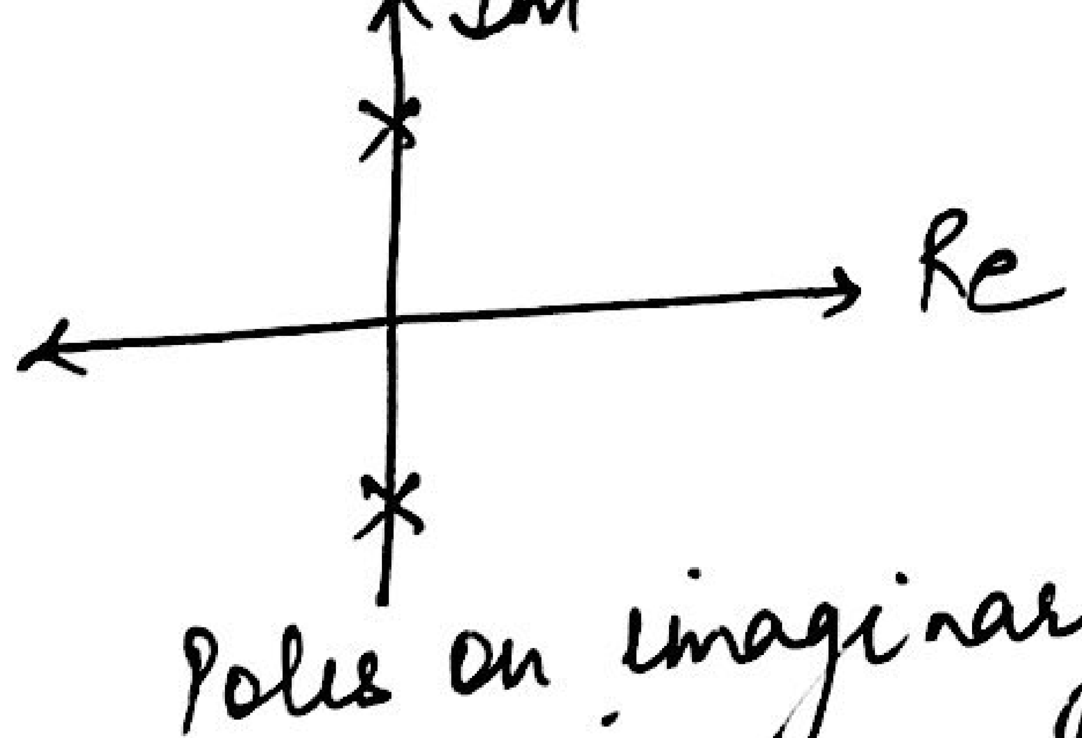
The System is unstable.



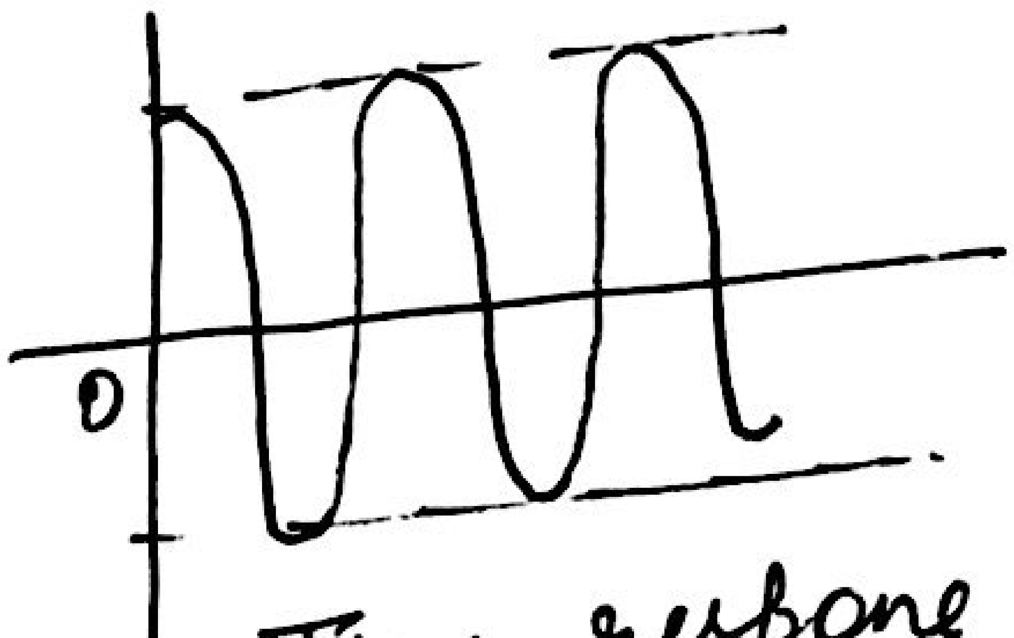
Response

(f:) Poles on $j\omega$-axis

The response is marginally stable



Poles on imaginary axis



Time respone

The Overall transfer func^n is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The characteristic eq^n is :-

$$1+G(s)H(s) = 0$$

The necessary and sufficient cond^n that a feedback system be stable is that all the zeros of the $1+G(s)H(s)=0$ have -ve real part

ASYMPTOTIC STABILITY

Asymptotic stability is known as "zero input stability".
Consider n^{th} order system with zero input and $c(t)$ is the output due to initial cond^{n}s, mathematically

$$c(t) = \sum_{i=0}^{n-1} g_i(t)\, c^i(t_0)$$

$$\text{where } c^i(t_0) = \left.\frac{d^i c(t)}{dt^i}\right|_{t=t_0}$$

If the zero i/p response $c(t)$ subjected to the finite initial cond^{n}s reaches zero as time t approaches infinity the system is said to be zero-input stable or asymptotic stable otherwise system is unstable

$$\lim_{t\to\infty} |c(t)| = 0$$

For Asymptotic stability, all the roots of the characteristic equation must be located in the left half of s-plane.

Necessary But not Sufficient Conditions for Stability

Consider a system with characteristic equation:-

$$a_0 s^m + a_1 s^{m-1} + \dots + a_m = 0$$

(a.) All the coefficients of the eqn should have same sign.

(b.) There should be no missing term

If above two condns are not satisfied the system is unstable.

ROUTH - HURWITZ CRITERION

Routh-Hurwitz Criterion is the algebraic method of determining the location of poles of a characteristic equation w.r.t. the left half & right half of the s-plane without actualy solving the equation.

Consider the following characteristic polynomial

$$a_0 s^n + a_1 s^{n-1} + \dots + a_n = 0 \quad \text{——— (1)}$$

where the coefficients $a_0, a_1, \dots a_n$ are all of the same sign and none is zero.

Step 1:- Arrange the coefficients of eqn (1) as shown in fig:-

Row 1:	a_0	a_2	a_4	----
Row 2:	a_1	a_3	a_5	----

Step 2:- From these two rows form a third row:-

Row 1	a_0	a_2	a_4	---
Row 2	a_1	a_3	a_5	---
Row 3	b_1	b_3	b_5	----

where $b_1 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}$ $\quad b_3 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}$

Step 3:- From 2nd and 3rd Row form a fourth row

Row 1	a_0	a_2	a_4	- - -
Row 2	a_1	a_3	a_5	- - -
Row 3	b_1	b_3	b_5	- -
Row 4	c_1	c_3	c_5	- - -

where $c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$

$c_3 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_5 \end{vmatrix}$

Step 4:- Continue this procedure of forming a new rows

Statement of Routh Hurwitz Criterion

It states that the system is stable if and only if all the elements in the first ~~sign~~ column have the same sign. If all the elements are not of the same sign than the no. of sign changes of the elements in the first col. equals the no. of roots of the characteristic eqn in the right half of the s-plane.

Ques 1. Check the stability of the system whose characteristic eqⁿ is given by :-

$$s^4 + 2s^3 + 6s^2 + 4s + 1 = 0$$

Solⁿ :-

s^4	1	6	1
s^3	2	4	
s^2	4	1	
s^1	3.5		
s^0	1		

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} = 4 \;;\quad c_1 = -\frac{1}{4}\begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 3.5$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 1 \;;\quad d_1 = -\frac{1}{3.5}\begin{vmatrix} 4 & 1 \\ 3.5 & 0 \end{vmatrix} = 1$$

Since, all the coefficients in the 1st col. are of same sign (+ve), the given eqⁿ has no roots with positive real parts. Hence, the system is stable.

Ques 2. Determine the stability $2s^4 + 5s^3 + 5s^2 + 2s + 1 = 0$

s^4	2	5	1
s^3	5	2	
s^2	4.2	1	
s^1	0.809		
s^0	1		

From the Routh table :-

No. of sign changes in 1st col. = 0

No. of poles on right hand side of s-plane = 0

Hence, the system is stable

Ques 3. Check the stability of system

(a.) $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

$$\begin{array}{llll} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & \\ s^2 & 1 & 5 & \\ s^1 & -6 & & \\ s^0 & 5 & & \end{array}$$

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 1 \qquad c_1 = -\frac{1}{1}\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} = -6$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} = 5 \qquad d_1 = \frac{-1}{-6}\begin{vmatrix} 1 & 5 \\ -6 & 0 \end{vmatrix} = 5$$

from the above table :-
No. of sign change in 1st col. = 2
No. of poles in right half of s-plane = 2
Hence the system is unstable.

(b.) $s^5 + 6s^4 + 3s^3 + 2s^2 + s + 1 = 0$

$$\begin{array}{llll} s^5 & 1 & 3 & 1 \\ s^4 & 6 & 2 & 1 \\ s^3 & 2.67 & 0.83 & \\ s^2 & 0.135 & 1 & \\ s^1 & -18.95 & & \\ s^0 & 1 & & \end{array}$$

No. of sign change in 1st col. = 2
No. of poles on right half of s-plane = 2
Hence, system is unstable.

SPECIAL CASES

1.7 If a 1st col term in any row is zero, but the remaining terms are not zero or there is no remaining term, then multiply the original eqn by factor $(s+a)$ where 'a' is any +ve real no.
(Simplest value is taken as $a = 1$)

Ques. Investigate the stability

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

s^5	1	2	3
s^4	1	2	5
s^3	0		
s^2			
s^1			
s^0			

Now, multiply the characteristic eqn by $(s+1)$

$$(s+1)(s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5) = 0$$

$$s^6 + 2s^5 + 3s^4 + 4s^3 + 5s^2 + 8s + 5 = 0$$

s^6	1	3	5	5
s^5	2	4	8	
s^4	1	1	5	
s^3	2	-2		
s^2	2	5		
s^1	-7			
s^0	1			

No. of sign change in 1st col = 2
No. of roots in right half of s-plane = 2
Hence, system is unstable.

Case 2 :- When any one row of Routh table is zero.

Eg :- $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	
s^2			
s^1			
s^0			

From the above table it is clear that third row is zero. Hence the Auxilary polynomial $A(s)$

$$A(s) = 2s^4 + 48s^2 - 50$$

$$\frac{dA(s)}{ds} = 8s^3 + 96s$$

s^5	1	24	-25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	112.6		
s^0	-50		

No. of sign change in 1st col = 1
No. of roots in right half of s-plane = 1

Roots of eqn formed by the auxilary polynomial
$2s^4 + 48s^2 - 50 = 0$
are also roots of the original eqn
$(s+1)(s-1)(s+j5)(s-j5)(s+2) = 0$

The roots of the auxiliary eqn are also the roots of the characterstic eqn because auxiliary eqn is the part of characterstic eqn.

Ques. Investigate the Stability

$$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

Solution

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	0	0	0	0
s^2				
s^1				
s^0				

Auxiliary eqn $A(s) = 2s^4 + 12s^2 + 16 = 0$

$$\frac{d(A(s))}{ds} = 8s^3 + 24s$$

s^6	1	8	20	16
s^5	2	12	16	
s^4	2	12	16	
s^3	8	24		
s^2	6	16		
s^1	2.67			
s^0	16			

NO sign change in 1st col. Hence the System is Stable

$2s^4 + 12s^2 + 16 = 0$

Put $s^2 = x$

$2x^2 + 12x + 16 = 0$

or $x^2 + 6x + 8 = 0$

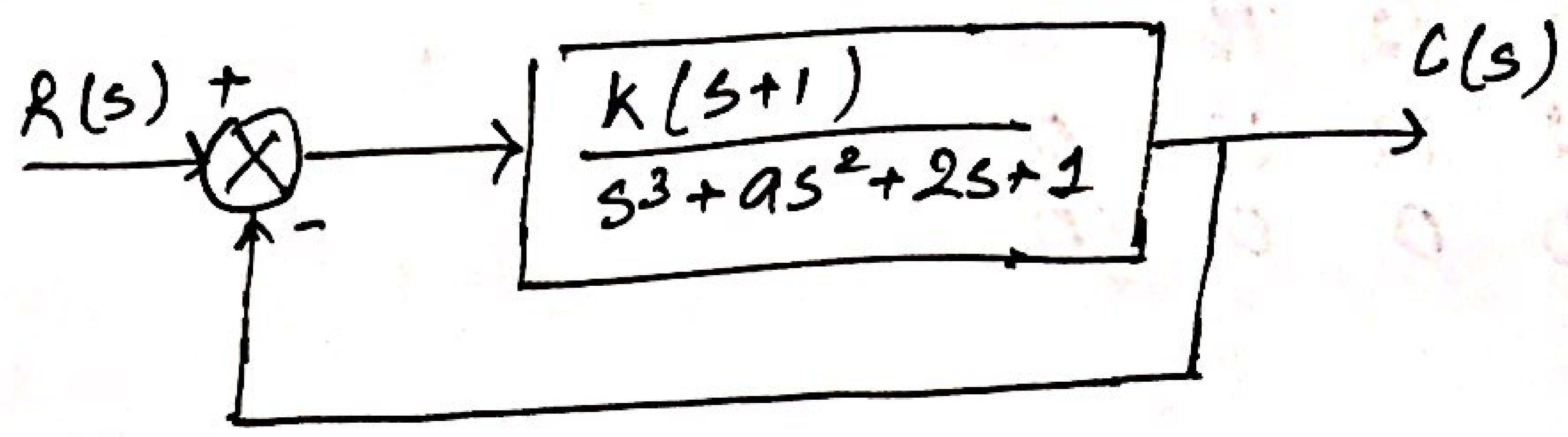
$(x+2)(x+4) = 0$

$x + 2 = 0$ $x + 4 = 0$

$s = \pm j\sqrt{2}$ $s = \pm j2$

Since the roots are non-repeated on imaginary axis, hence the system is marginally stable

Ques. A system oscillates with freq. ω, if it has poles at $s = \pm j\omega$ and no poles in right half s-plane. Determine the value of k and a so that the system shown in fig. oscillates at a freq. of 2rad/sec.



The characteristic eqⁿ is $1 + G(s)H(s) = 0$

$$1 + \frac{k(s+1)}{s^3+as^2+2s+1} \cdot 1 = 0$$

$$s^3 + as^2 + (k+2)s + k+1 = 0$$

s^3	1	$k+2$
s^2	a	$k+1$
s^1	$k+2 - \frac{k+1}{a}$	
s^0	$k+1$	

The system will have sustained oscillation when

$$k+2 - \frac{(k+1)}{a} = 0$$

$$\therefore \quad a = \frac{k+1}{k+2}$$

The Auxiliary eqⁿ of row 2

$$as^2 + (k+1) = 0$$

$$s^2 = \frac{-k+1}{a}$$

$$s = j\omega$$

$$(j\omega)^2 = \frac{-k+1}{a}$$

$\omega = 2$ rad/sec

$$a = \frac{k+1}{4}$$

$$\therefore \frac{K+1}{4} = \frac{K+1}{K+2}$$

$$\therefore K = 2$$

$$\Rightarrow a = 0.75 \text{ Ans.}$$

Ques. The open loop transfer func^n of a unity feedback control system is

$$G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$$

By applying Routh criterion, discuss the stability of the closed loop system as a func^n of K. Determine the value of K which will cause sustained oscillations in the closed loop system. What are the possible oscillation freq.

The characteristic eq^n is

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{(s+2)(s+4)(s^2+6s+25)} \cdot 1 = 0$$

$$s^4 + 12s^3 + 69s^2 + 198s + (200+K) = 0$$

s^4	1	69	200+K
s^3	12	198	
s^2	52.5	200+K	
s^1	$198 - \frac{12(200+K)}{52.5}$		
s^0	200+K		

System will be stable if

$$200 + K > 0 \text{ or } K > -200$$

$$198 - \frac{12(200+K)}{52.5} > 0 \text{ or } K < 666.25$$

Oscillations will occur when $K = 666.25$

$$\therefore 52.5s^2 + (200 + 666.25) = 0$$

$$52.5 s^2 = -866.25$$

$$s^2 = -16.5$$

$$s = \pm j\,4.06$$

$\omega = 4.06$ rad/sec Ans.

Ques. A feedback system has an open loop transfer funcn

$$G(s)H(s) = \frac{K e^{-s}}{s(s^2+5s+9)}$$

Determine by use Routh criterion, the max. value of K for the closed loop system to be stable.

For low freq. $e^{-s} = (1-s)$

$$\therefore G(s)H(s) = \frac{K(1-s)}{s(s^2+5s+9)}$$

Characteristic eqn is $1+G(s)H(s) = 0$

$$s^3 + 5s^2 + (9-K)s + K = 0$$

s^3	1	$9-K$
s^2	5	K
s^1	$9-\frac{6K}{5}$	
s^0	K	

for Stability; $K > 0$

$9 - \frac{6K}{5} > 0$ or $K < 7.5$

Hence, the range of K is $\boxed{0 < K < 7.5}$ Ans.

Ques. The characteristic eqn of feedback control system is $s^4 + 20s^3 + 15s^2 + 2s + K = 0$

(a.) Determine the range of K for the system to be stable

(b.) Can the system be marginally stable? If so, find the required value of K and the freq. of sustained oscillation.

s^4	1	15	K
s^3	20	2	
s^2	14.9	K	
s^1	$\frac{29.8 - 20K}{14.9}$		
s^0	K		

(a.) for stability $K > 0$

$$29.8 - 20K > 0$$
$$K < 1.49$$

Hence, range $0 < K < 1.49$

(b.) for Marginally stable, $K = 1.49$

Auxillary equation, $A(s) = 14.9s^2 + 1.49$

$$14.9s^2 = -1.49$$
$$s^2 = -0.1$$
$$s = \pm j0.316$$
$$\omega = 0.316 \text{ rad/sec.}$$

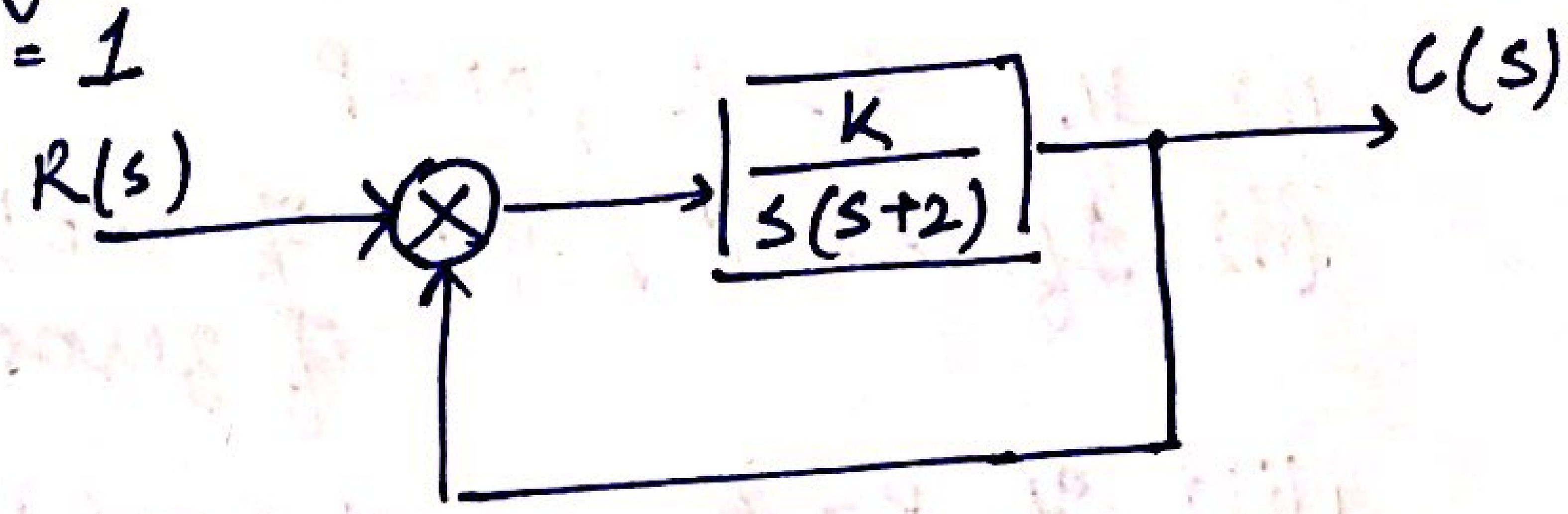
∴ freq. of sustained oscillation = 0.316 rad/sec.

ROOT LOCUS TECHNIQUE

The Root Locus is the path of the roots of the characterstic equation traced out in the S-plane as a system parameter is changed. The locus of the roots of the characteristic equation when gain is varied from zero to infinity.

Consider a unity feedback system as shown in fig.

$G(s) = \frac{K}{s(s+2)}$, $H(s) = 1$



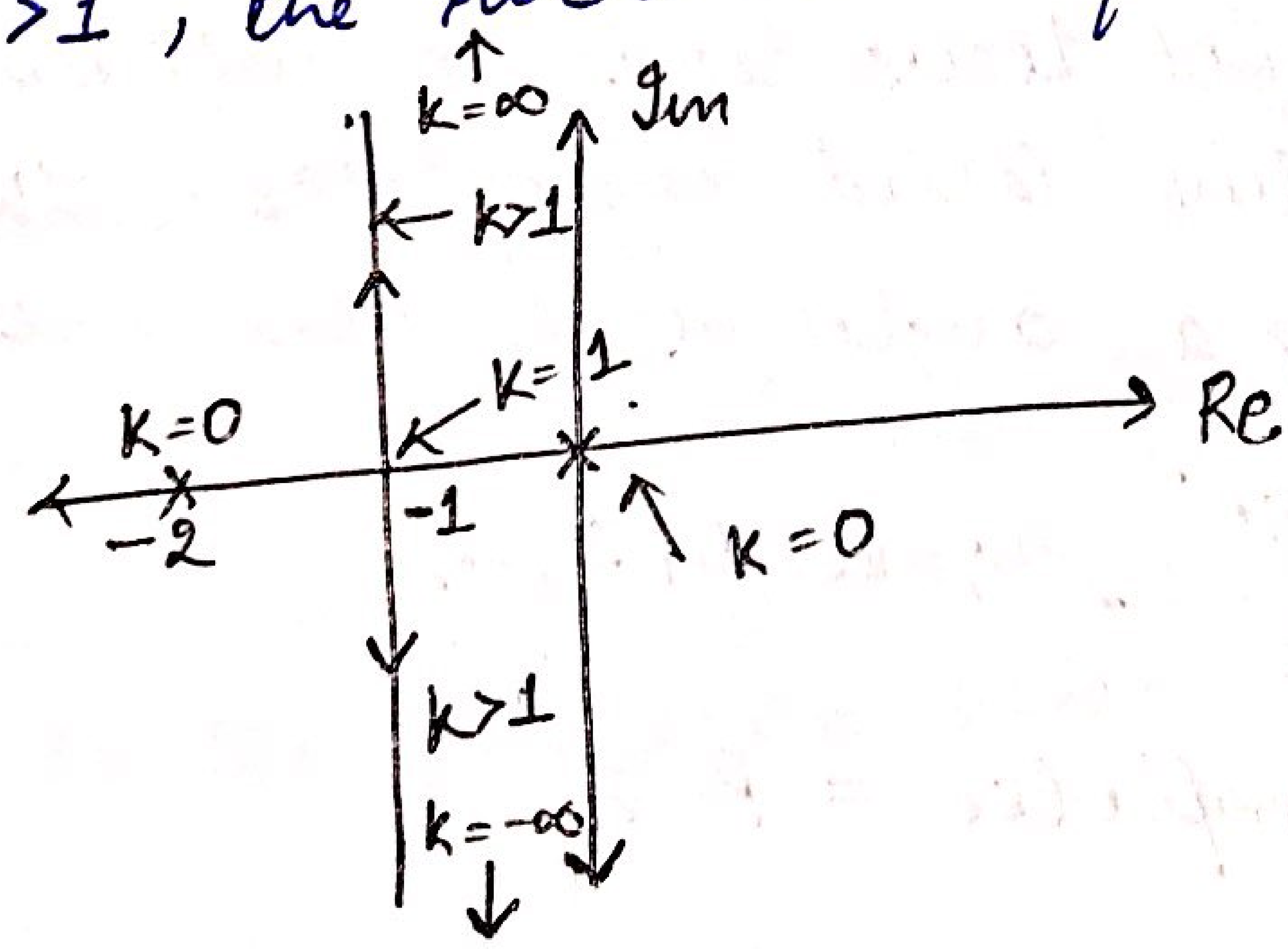
$1 + G(s)H(s) = 0$

$s^2 + 2s + K = 0$

$s_1 = -1 + \sqrt{1-K}$, $s_2 = -1 - \sqrt{1-K}$

As K, is varied, the two roots gives the locii in S-plane

1. when $0 < K < 1$, the roots are real and distinct.
2. when $K = 0$, $s_1 = 0$ and $s_2 = -2$. These are also the open pole loops.
3. when $K = 1$, both roots are real and equal
4. when $K > 1$, the roots are complex conjugate



RULES FOR CONSTRUCTION OF ROOT LOCII

1. The root locus is symmetrical about the real axis
2. The root loci starts from an open loop pole with $K=0$

Eg:- $G(s)H(s) = \frac{K(s+3)}{(s+2)}$

The root loci starts from $s = -2$

3.7 The root loci will terminate ... zeros or on infinity with $K = \infty$.

Eg:- $G(s)\,H(s) = \dfrac{k(s+3)}{(s+2)}$

The root loci will terminate at $s = -3$.

4.7 If N = no. of root loci
P = no. of poles
Z = no. of zeros

(i) If $P > Z$; $N = P$
(ii) If $Z > P$, no. of root loci will be equal to no. of zeros $(N = Z)$
(iii) If $P = Z$, no. of root loci $= P = Z$.

5.7 Root Locii on Real Axis

Any point on the real axis is a part of the root locus if and only if the no. of poles & zeros to its right is odd.

6.7 Asymptotes :-

The branches of root locus tends to infinity along a set of straight line called asymptotes. These asymptotes making an angles with real axis is

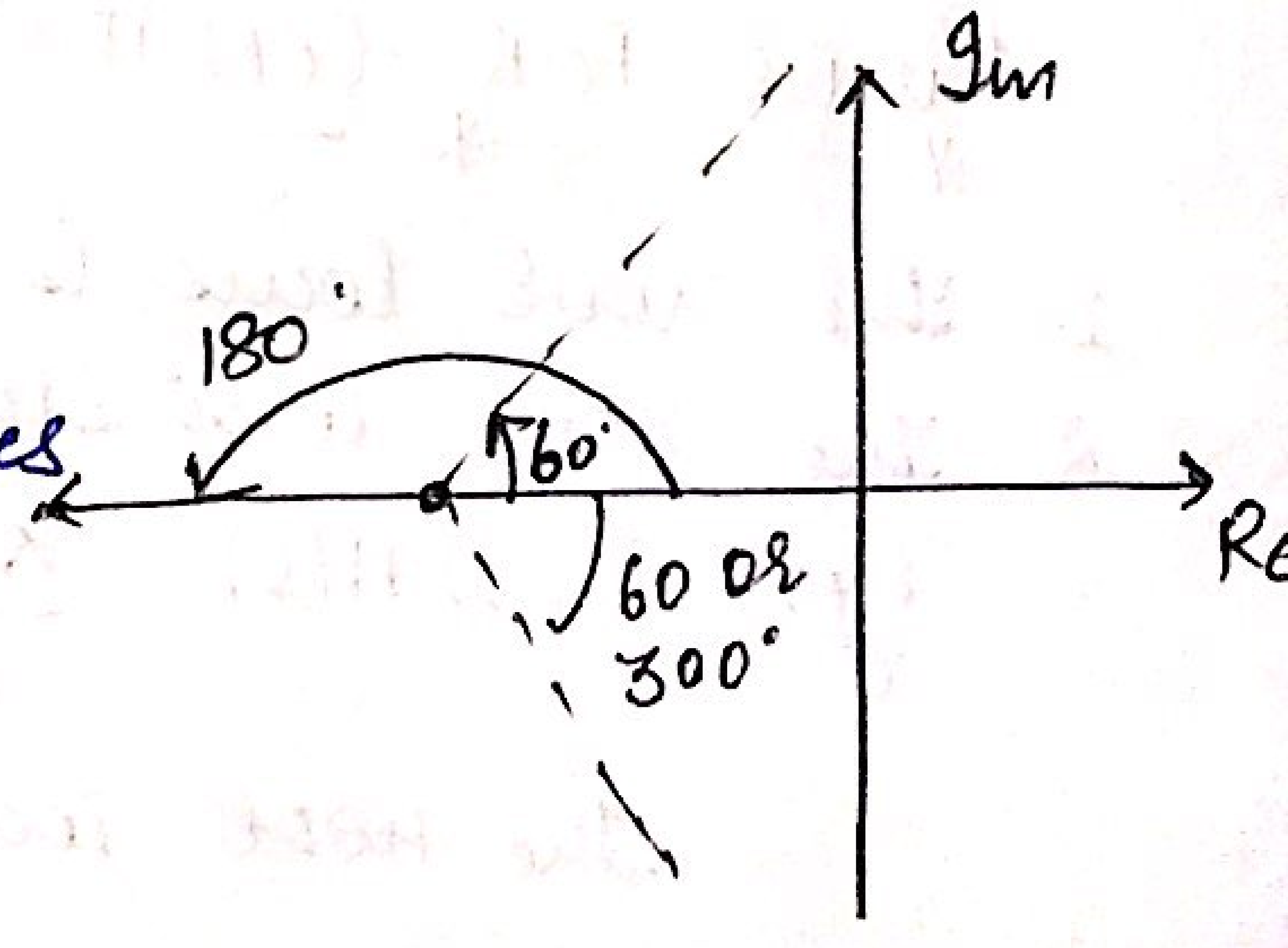
$$\phi = \frac{(2K+1)\,180^\circ}{P-Z} \; ; \; K = 0, 1, 2, \ldots$$

Total no. of asymptotes $= P - Z$

Eg:- If $G(s)\,H(s) = \dfrac{K}{s(s^2+6s+10)}$

$P = 3$
$Z = 0$
$P - Z = 3 - 0$ = No. of Asymptotes

$K = 0$; $\phi_1 = 60^\circ$
$K = 1$; $\phi_2 = 180^\circ$
$K = 2$; $\phi_2 = 300^\circ$



7.) Centroid of Asymptotes :-

The pt. of intersection of asymptotes with real axis is called Centroid of asymptotes (σ_A) and is

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P - Z}$$

for $G(s)H(s) = \frac{K}{s(s^2+6s+10)}$

$s_1 = 0$; ~~$s_2 = -3 \pm j1$~~

$s_2 = -3 + j1$ and $s_3 = -3 - j1$

No. of poles = 3 = P

No. of zeros = 0 = Z

$$\sigma_A = \frac{0 - 3 + j1 - 3 - j1 - 0}{3 - 0}$$

$$= \frac{-6}{3} = -2.$$

The Centroid is shown by pt. 'a' in Step 6.

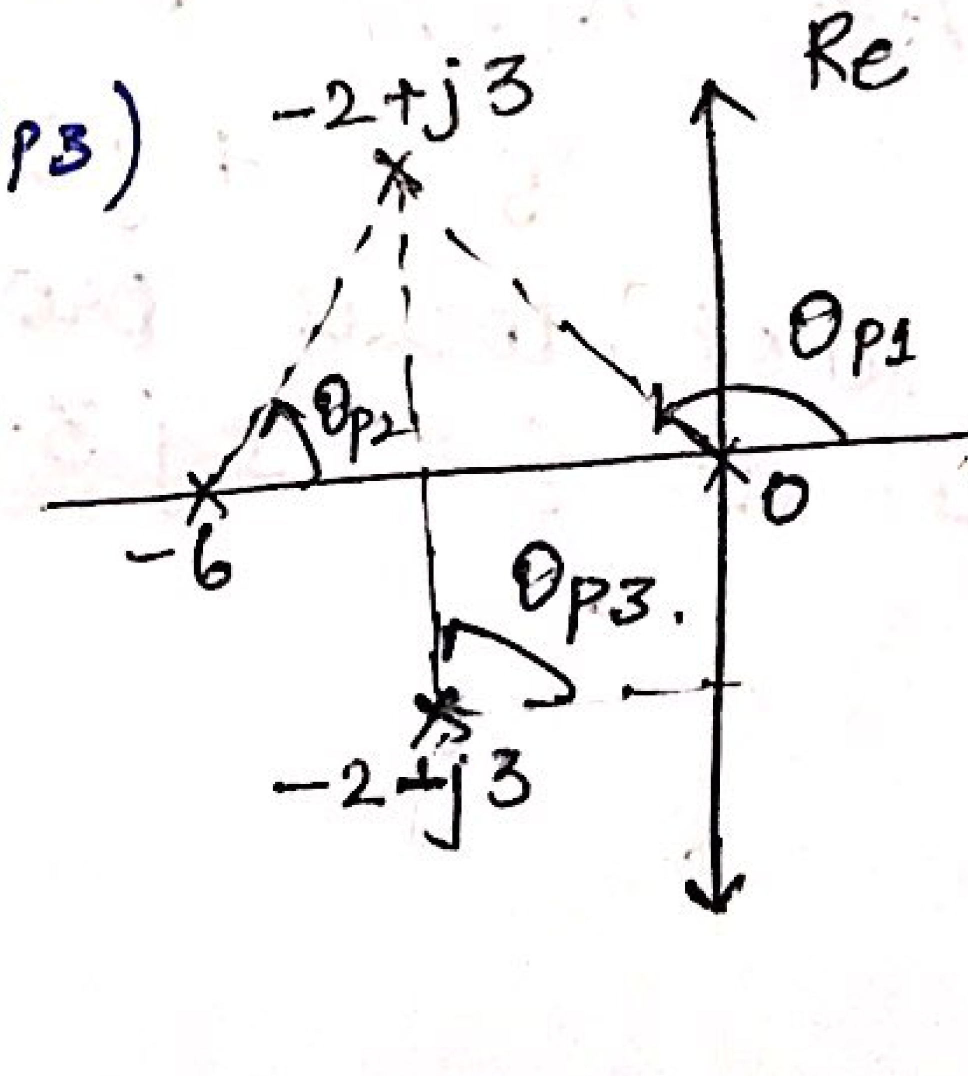
8.) Angle of departure and angle of arrival. (ϕ_d from a complex pole)

ϕ_d (Angle of departure) = 180° − Sum of angles of vectors drawn to this poles from other poles + Sum of angles of vectors drawn to this pole from the zeros)

ϕ_a (Angle of arrival) = 180° − Sum of angles of Vectors drawn to this zero from other zeros + Sum of angles of vectors drawn to this zero from poles.

Eg :- $G(s)H(s) = \frac{K}{s(s+6)(s^2+4s+13)}$

$\phi_d = 180° - (\phi_{p1} + \theta_{p2} + \theta_{p3})$



9.7 Breakaway point

The breakaway or break in pts. can be determined from the roots of:-

$$\frac{dK}{ds} = 0$$

Eg:- If $G(s)H(s) = \frac{K}{s(s^2+6s+10)}$

$$1+G(s)H(s) = 0$$

$$s(s^2+6s+10)+K=0$$

$$K = -s^3-6s^2-10s$$

$$\frac{dK}{ds} = -3s^2-12s-10 = 0$$

$s_1 = -1.1835$ and $s_2 = -2.815$ are the breakaway points.

10.7 Intersection of root locus branches with $j\omega$-axis can be determined through Routh-Hurwitz criterion.

Eg:- $G(s)H(s) = \frac{K}{s(s^2+6s+10)}$

$1+G(s)H(s)=0$; $s^3+6s^2+10s+K=0$

s^3	1	10
s^2	6	K
s^1	$\frac{60-K}{6}$	
s^0	K	

when $K=60$, we get a zero row

A.E. $A(s) = 6s^2+K$

$$s = \pm j3.16$$

The root locus crosses the imaginary axis at $s = \pm j3.16$ for $K=60$.

Scanned with CamScanner

Ques. Sketch the root locus as k varies from zero to infinity if $G(s) = \frac{k}{s(s+4)(s+5)}$; $H(s) = 1$

$$G(s) = \frac{K}{s(s+4)(s+5)}$$

1.) Poles are at $s = 0, -4$ and -5
No. of poles $= P = 3$
No. of zeros $= Z = 0$

2.) The root locus exists b/w $s=0$ and $s=-4$ and to the left of -5

3.) No. of root loci $P = 3$
$Z = 0$

$\therefore$ No. of root loci $(N) = 3$.

4.) Centroid of Asymptotes

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P-Z}$$

$$= -3$$

5.) Angle of Asymptotes :-

$$\phi = \left(\frac{2K+1}{P-Z}\right) 180^\circ$$

$K=0$; $\phi_1 = 60^\circ$
$K=1$; $\phi_2 = 180^\circ$
$K=3$; $\phi_3 = 300^\circ$

6.) Calculation of breakaway point :-

$$1 + G(s) H(s) = 0$$

$$s^3 + 9s^2 + 20s + K = 0$$

$$K = -s^3 - 9s^2 - 20s$$

$$\frac{dK}{ds} = 0$$

$$-3s^2 - 18s - 20 = 0$$

$$s_1 = -1.4725$$

$$s_2 = -4.5275$$

Since -4 to -5 is not the segment of root locus

∴ We consider -1.4725 as the breakaway point.

7.7 Point of intersection with $j\omega$-axis

$$s^3 + 9s^2 + 20s + K = 0$$

s^3	1	20
s^2	9	K
s^1	$\frac{180-K}{9}$	
s^0	K	

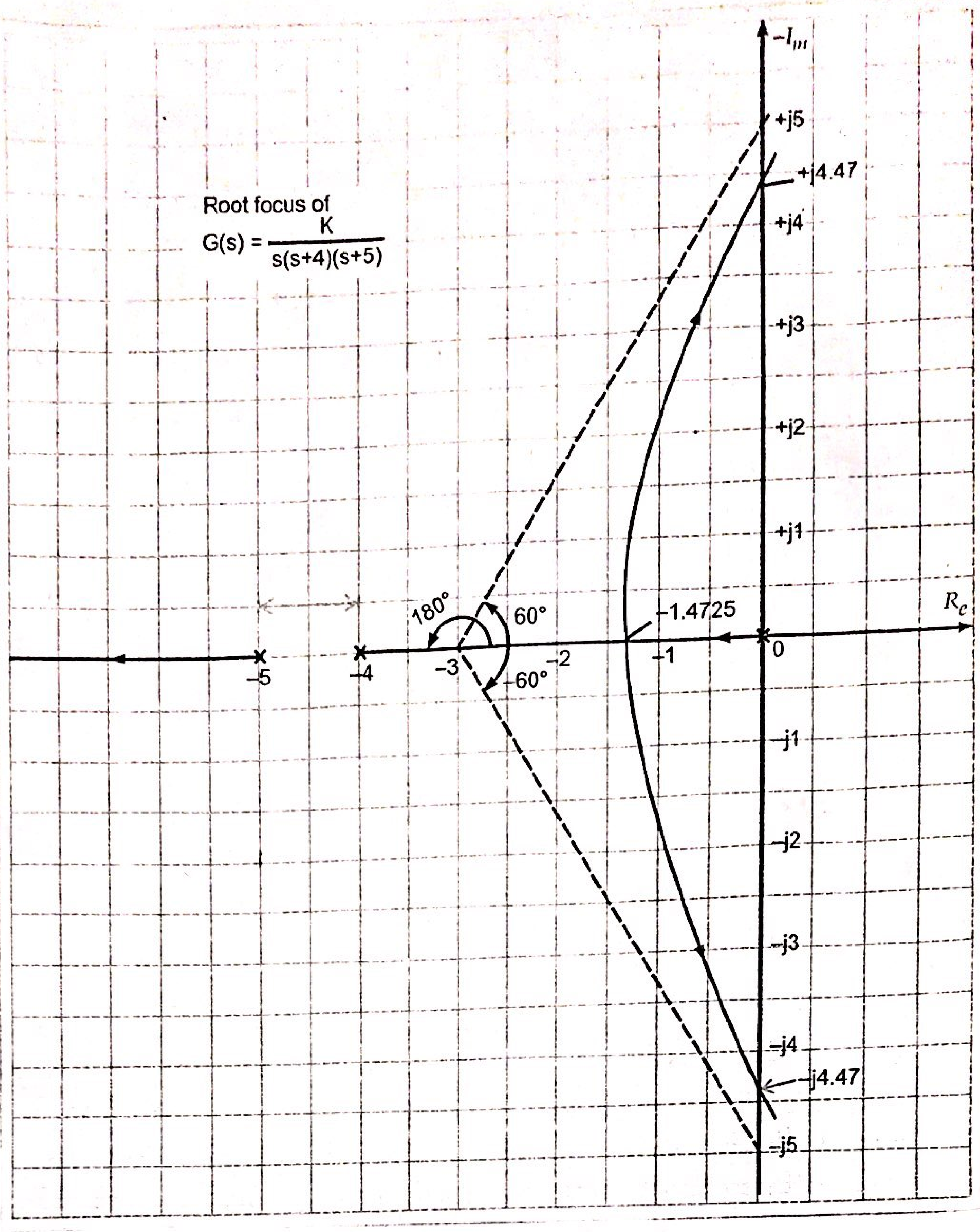
for $K = 180$, the auxillary eq^n is :-

$$A(s) = 9s^2 + K$$

$$9s^2 + K = 0$$

$$9s^2 + 180 = 0$$

$$s = \pm j\,4.47$$



Ques 2. Consider a unity feedback control system with the following feedforward transfer func^n

$$G(s) = \frac{K}{s(s^2+4s+8)}$$

Plot root locii for the system.

Solution:-

1.) Plot the poles and zeros

$$s^2+4s+8=0$$

$$s_2, s_3 = -2 \pm j2$$

$$s_1 = 0$$

$\therefore$ Three poles are at $s_1 = 0$

$$s_2 = -2+j2$$

$$s_3 = -2-j2$$

2.7 Since there is only one pole at $s=0$, the entire left half of the real axis is the part of the root locus.

3.7 No. of poles $P = 3$

No. of zeros $Z = 0$

$\therefore$ No. of root loci $= N = P = 3$.

4.7 Centroid of Asymptotes

$$\sigma_A = \frac{\text{Sum of } \cancel{\text{zeros}} \text{ poles} - \text{Sum of zeros}}{P-Z}$$

$$\sigma_A = -\frac{4}{3} = -1.33$$

5.7 Angle of asymptotes :-

$$\phi = \frac{2k+1}{P-Z} \times 180^\circ$$

$k=0; \quad \phi_1 = 60^\circ$

$k=1; \quad \phi_2 = 180^\circ$

$k=2; \quad \phi_3 = 300^\circ$

6.7 Breakaway point

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s^2+4s+8)} = 0$$

$$s^3 + 4s^2 + 8s + k = 0$$

$$k = -(s^3 + 4s^2 + 8s)$$

$$\frac{dk}{ds} = -(3s^2 + 8s + 8) = 0$$

$$s = -1.33 \pm j\,0.943.$$

Since, at $s = -1.33 \pm j\,0.943$, the angle cond^n is not satisfied. Hence there is no breakaway point.

7.7 Point of intersection with imaginary axis :-

The characteristic eq^n is :-

$$s^3 + 4s^2 + 8s + K = 0$$

s^3	1	8
s^2	4	K
s^1	$\frac{32-K}{4}$	
s^0	K	

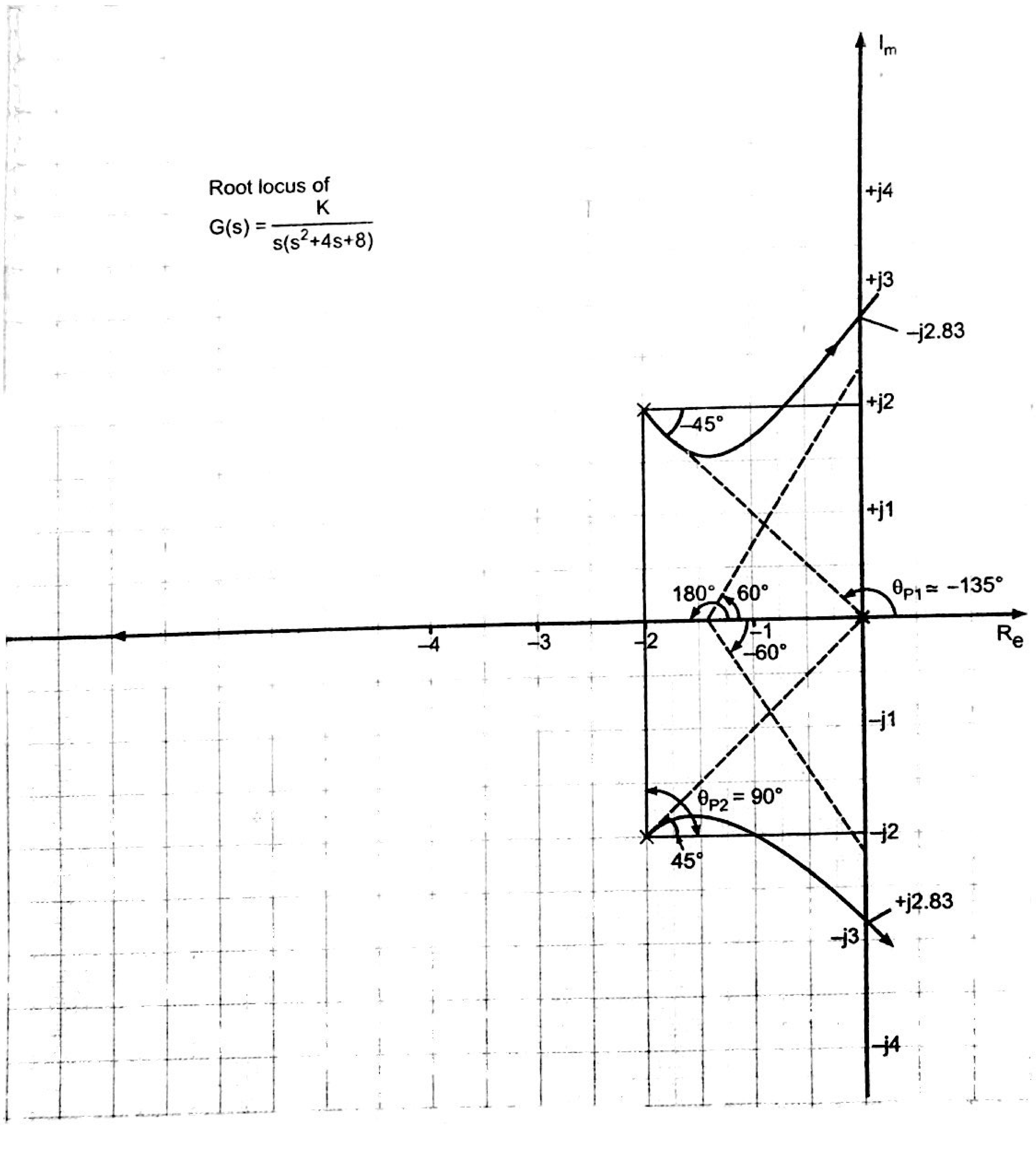
For sustained oscillation $32-K=0$ or $K=32$

AE, $A(s) = 4s^2 + K$

$$4s^2 + 32 = 0$$

$$s = \pm j2.83$$

8.7 The angle of departure, $\phi_d = 180° - (135° + 90°) = -45°$.



Ques. Sketch the root Locii for

$$G(s) = \frac{k(s+1)}{s^2(s+3.6)}$$

and $H(s) = 1$.

Solution:-

Step 1:- Plot the poles and zeros

poles are at $s = 0, 0, -3.6$

zeros are at $s = -1$.

Step 2:- The segment b/w $s=-1$ and -3.6 is the part of the root locus.

Step 3:- Centroid of asymptotes :-

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P-Z}$$

$$= \frac{0+0-3.6+1}{3-1} = -1.3.$$

Step 4:- Angle of asymptotes

$$\phi = \frac{2k+1}{P-Z} \times 180^\circ$$

$k=0$; $\phi_1 = 90^\circ$

$k=1$; $\phi_2 = 270^\circ$.

Step 5:- Breakaway point

Characteristic eqn $1+G(s)H(s) = 0$

$$1 + \frac{k(s+1)}{s^2(s+3.6)} = 0$$

$$k = -\frac{s^3+3.6s^2}{s+1}$$

$$\frac{dk}{ds} = \frac{(s+1)(3s^2+7.2s) - (s^3+3.6s^2)}{(s+1)^2} = 0$$

$$s^3 + 3.3s^2 + 3.6s = 0$$

$$s(s^2 + 3.3s + 3.6) = 0$$

$$s = 0 \text{ and } s = \frac{-3.3 \pm \sqrt{(3.3)^2 - 4\times1\times3.6}}{2}$$

$$s = 0; \quad s = -1.65 \pm j0.936.$$

Point $s = 0$ is the actual breakaway point.

Step 6:- Point of intersection

The characteristic eq^n is:- $s^3 + 3.6s^2 + ks + k = 0$

s^3	1	K
s^2	3.6	K
s^1	0.72K	
s^0	K	

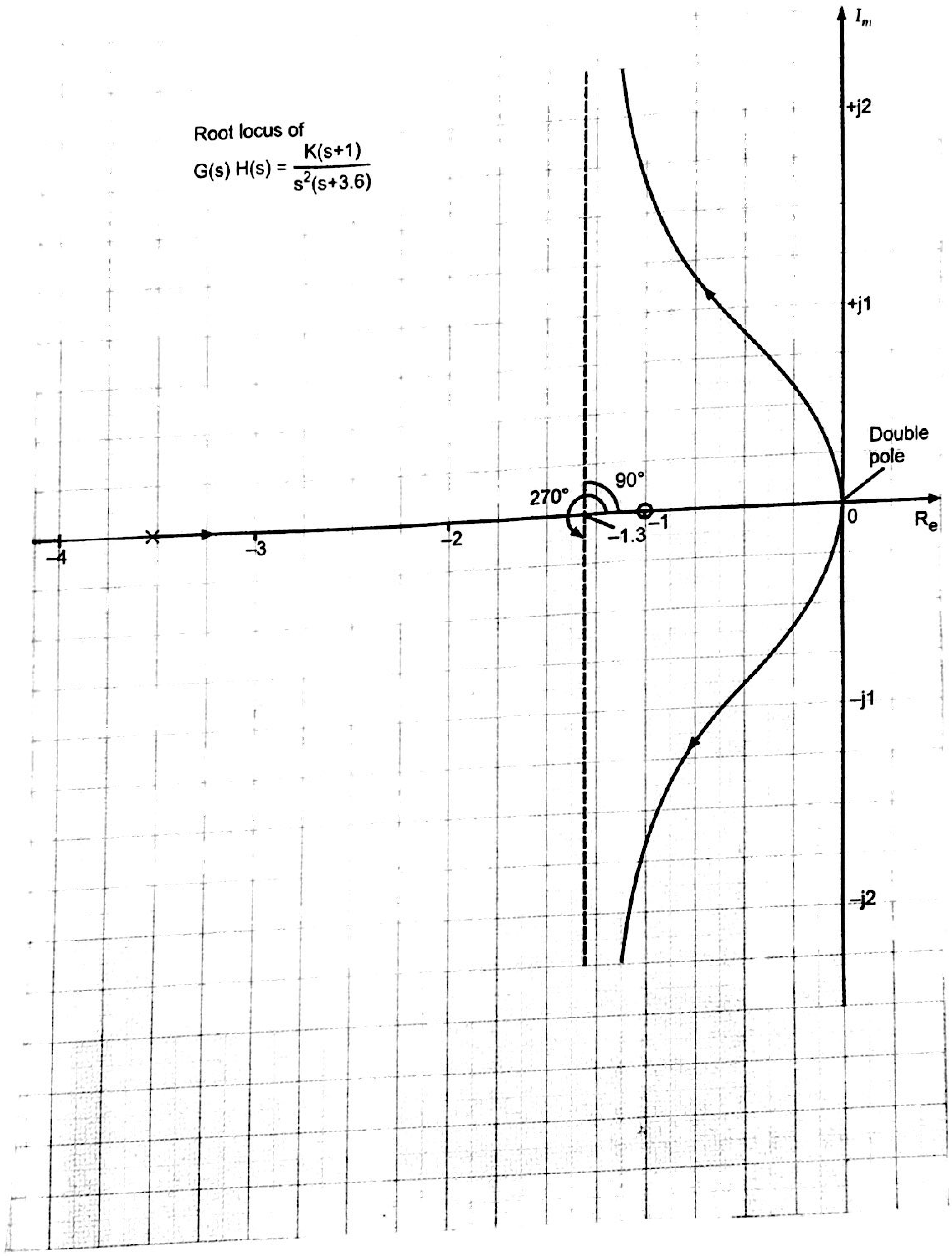
for sustained Oscillation $K = 0$

$$A(s) = 3.6s^2 + K$$

$$3.6s^2 + 0 = 0$$

$$s^2 = 0$$

Root locus branches doesnot crosses the $j\omega$-axis



Scanned with CamScanner

Ques For a unity feedback system the open loop transfer func^n is given by :-

$$G(s) = \frac{K}{s(s+2)(s^2+6s+25)}$$

(a.) Sketch the root locus for $0 \le K \le \infty$

(b.) At what value of 'K' the system becomes unstable

(c.) At this point of instability determine the freq. of oscillation of the system.

Solution :-

1.) Poles are at $s_1 = 0$, $s_2 = -2$, $s_3 = -3+j4$, $s_4 = -3-j4$

No. of poles = 4

No. of zeros = 0

2.) The segment on the real axis b/w $s=0$ and $s=-2$ is the part of the root locus.

3.) No. of root locii

No. of poles $P = 4$

No. of zeros $Z = 0$

$\therefore$ No. of root locii $= N = P = 4$.

4.) Centroid of Asymptotes :-

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P-Z}$$

$$= -2$$

5.) Angle of Asymptotes :-

$$\phi = \left(\frac{2k+1}{P-Z}\right) 180$$

$k=0$; $\phi_1 = 45°$

$k=1$; $\phi_2 = 135°$

$k=2$; $\phi_3 = 225°$

$k=3$; $\phi_4 = 315°$

6.7 Calculation of Breakaway point :-

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+2)(s^2+6s+25)} = 0$$

$$K = -(s^4 + 8s^3 + 37s^2 + 50s)$$

$$\frac{dK}{ds} = 0$$

$$4s^3 + 24s^2 + 74s + 50 = 0$$

put values of s b/w 0 and -2 because the portion b/w 0 and -2 is the part of root locus

$$\therefore s = -0.8981$$

7.7 Point of intersection on jω-axis.

$$s^4 + 8s^3 + 37s^2 + 50s + K = 0$$

s^4	1	37	K
s^3	8	50	
s^2	30.75	K	
s^1	$\frac{1537.5 - 8K}{30.75}$		
s^0	K		

$$\frac{1537.5 - 8K}{30.75} = 0$$

$$K = 192.18$$

$$\therefore A.E,\ A(s) = 30.75s^2 + K = 0$$

$$\therefore s = \pm j2.5$$

8.7 Angle of departure

$$\phi_d = 180° - (104° + 90° + 127°)$$

$$= -141°$$

The range of value of stability is $0 < K < 192.18$

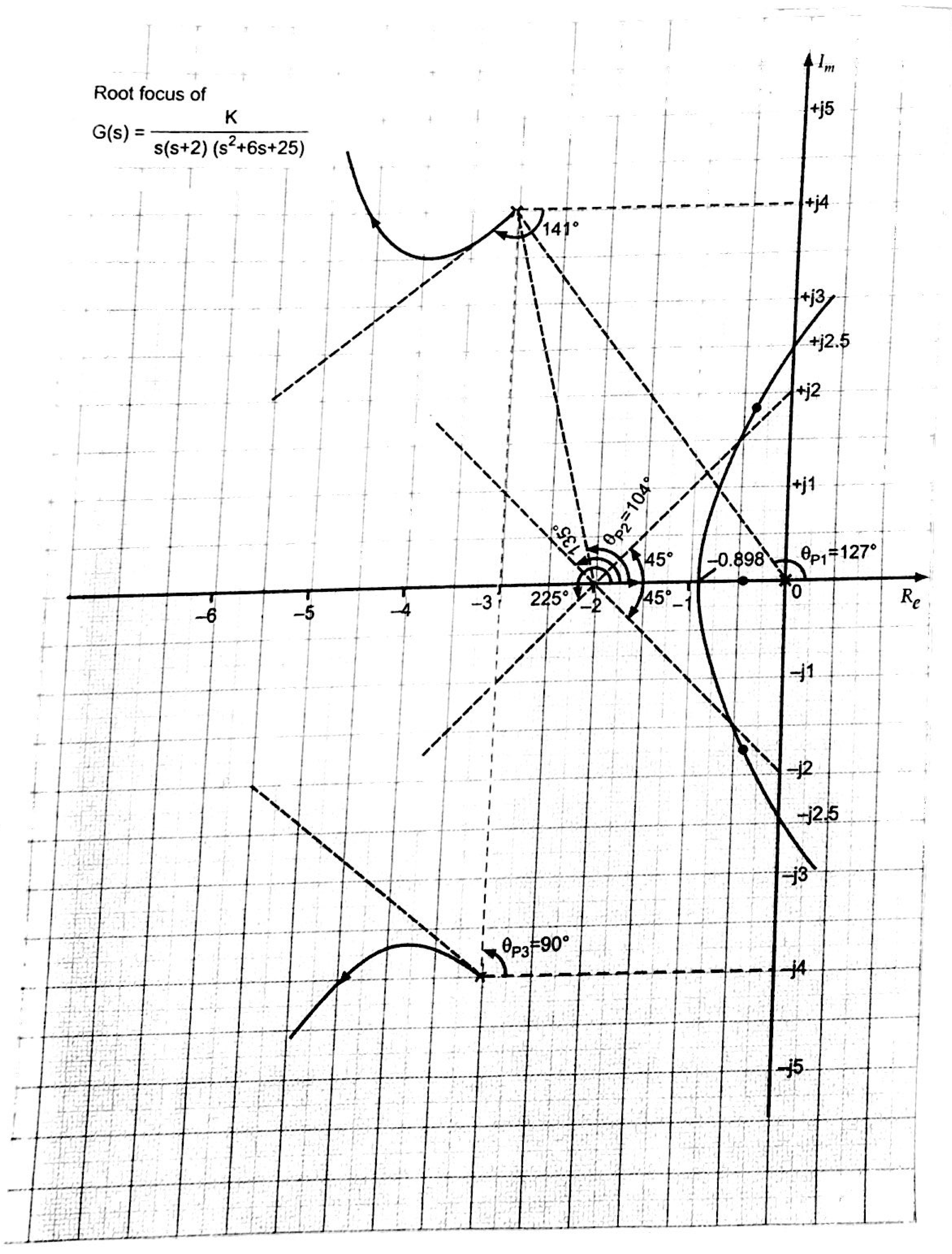
The closed loop system becomes unstable for $K < 0$ and $K > 192.18$

(C) At this pt. of instability

$A(s) = 30.75 s^2 + 192.18 = 0$

$s = \pm j 2.5$

$(\omega = 2.5 \text{ rad/sec})$



FREQUENCY DOMAIN ANALYSIS

The magnitude and phase response or relationship b/w sinusoidal input and steady state output of a system is known as frequency response. It is independent of the amplitude and phase of the input signal.

If input signal is :-

$$r(t) = X \sin \omega t$$

the output can be written as :-

$$c(t) = Y \sin(\omega t + \theta)$$

For a stable system, the sinusoidal output of the same freq. as input but the amplitude and phase of the output will be diff. from the input.

$$G(j\omega) = M \angle \phi$$

where 'M' is the ratio of amplitudes of o/p and i/p sinusoids and is called the gain or magnitude ratio. 'ϕ' is the angle by which o/p leads the input. M & ϕ are the func^n of angular freq. ω.

ADVANTAGES OF FREQUENCY DOMAIN ANALYSIS

1.) Freq. response tests are simple to perform.

2.) Transfer func^n can be obtained from from freq response of the system.

3.) Freq response methods can also be used to determine the stability of the system

4.) Those systems which do not have rational transfer func^n, freq. response can be precisely applied to them.

DISADVANTAGES OF FREQ. DOMAIN ANALYSIS

1.) Obtaining freq. response practically is fairly time consuming.

2.) These methods are applied to linear systems.

3.) For high time constants, freq. response method is not convienent.

EFFECT OF ADDING A POLE AND A ZERO

* When a pole is added in the forward path for a second order system:-
  - (i) Bandwidth decreases
  - (ii) Rise time (t_r) increases
  - (iii) Resonant peak (M_r) increases
  - (iv) System becomes less stable.

* When a zero is added in the forward path:-
  - (i) B.W. increases
  - (ii) Less t_r
  - (iii) t_s increases
  - (IV) System becomes non-stable.

POLAR PLOT

The polar plot of a sinusoidal transfer func^n $G(j\omega)$ is a plot of magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as 'ω' is varied from zero to infinity.

The polar plot, therefore is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity.

The advantage of using a polar plot that it depicts the freq. response char. of a system over the entire freq range in a single plot.

The disadvantage is that the plot doesnot indicate the contributions of each individual factor of the open loop transfer funcⁿ.

PROCEDURE TO SKETCH THE POLAR PLOT

1.) Determine the transfer funcⁿ $G(s)$ of the system

2.) Put $s = j\omega$ in the transfer funcⁿ to obtain $G(j\omega)$

3.) At $\omega = 0$ and $\omega = \infty$ calculate $|G(j\omega)|$ by $\lim_{\omega \to 0} |G(j\omega)|$ and $\lim_{\omega \to \infty} |G(j\omega)|$

4.) Calculate the phase angle of $G(j\omega)$ at $\omega = 0$ and $\omega = \infty$. by $\lim_{\omega \to 0} \angle G(j\omega)$ and $\lim_{\omega \to \infty} \angle G(j\omega)$

5.) Rationalise the funcⁿ $G(j\omega)$ and seperate the real and imaginary parts.

6.) Equate the Imaginary part $Im|G(j\omega)|$ to zero and determine the frequencies at which plot intersects the real axis and calculate the value $G(j\omega)$ at the point of intersection by substituting the determined value of freq. in the expression of $G(j\omega)$.

7.) Equate the real part $Re|G(j\omega)|$ to zero and determine the freq. at which plot intersects at imaginary axis and calculate the value of $G(j\omega)$ at the point of intersection by substituting the determined value of freq. in the rationalized expression of $G(j\omega)$.

8.) Sketch the polar plot with help of above information.

TYPE ZERO SYSTEM

$$G(s) = \frac{K}{(1+sT_1)(1+sT_2)}$$

1) put $s = j\omega$

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$

$$G(j\omega) = \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

2) $$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{K}{\sqrt{(1+\omega T_1)^2}\sqrt{(1+\omega T_2)^2}} = K.$$

$$\lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to \infty} \frac{K}{\sqrt{(1+\omega T_1)^2}\sqrt{(1+\omega T_2)^2}} = 0.$$

3) $$\lim_{\omega \to 0} \angle G(j\omega) = \lim_{\omega \to 0} -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

$$= 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -180^\circ$$

4) Sepereating real and imaginary parts of

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)} \frac{(1-j\omega T_1)(1-j\omega T_2)}{(1-j\omega T_1)(1-j\omega T_2)}$$

$$G(j\omega) = \frac{K(1-\omega^2 T_1 T_2)}{1+\omega^2 T_1^2+\omega^2 T_2^2+\omega^4 T_1^2 T_2^2} - j\frac{K\omega(T_1+T_2)}{1+\omega^2 T_1^2+\omega^2 T_2^2+\omega^4 T_1^2 T_2^2}$$

Equating the real part to zero:-

$$\omega^2 = \frac{1}{T_1 T_2} \quad \text{or} \quad \omega = \frac{1}{\sqrt{T_1 T_2}} \quad \text{and} \quad \omega = \pm\infty.$$

Scanned with CamScanner

The freq. at which plot intersects imaginary axis is $\frac{1}{\sqrt{T_1 T_2}}$.

$\omega = \frac{1}{\sqrt{T_1 T_2}}$ and $\omega = \infty$ (for the values of freq.)

when $\omega = \frac{1}{\sqrt{T_1 T_2}}$, $G(j\omega) = \frac{k\sqrt{T_1 T_2}}{T_1 + T_2} \angle -90^\circ$

$\omega = \infty$ $\quad G(j\omega) = 0 \angle -180^\circ$

5.7 Equating imaginary part to zero.

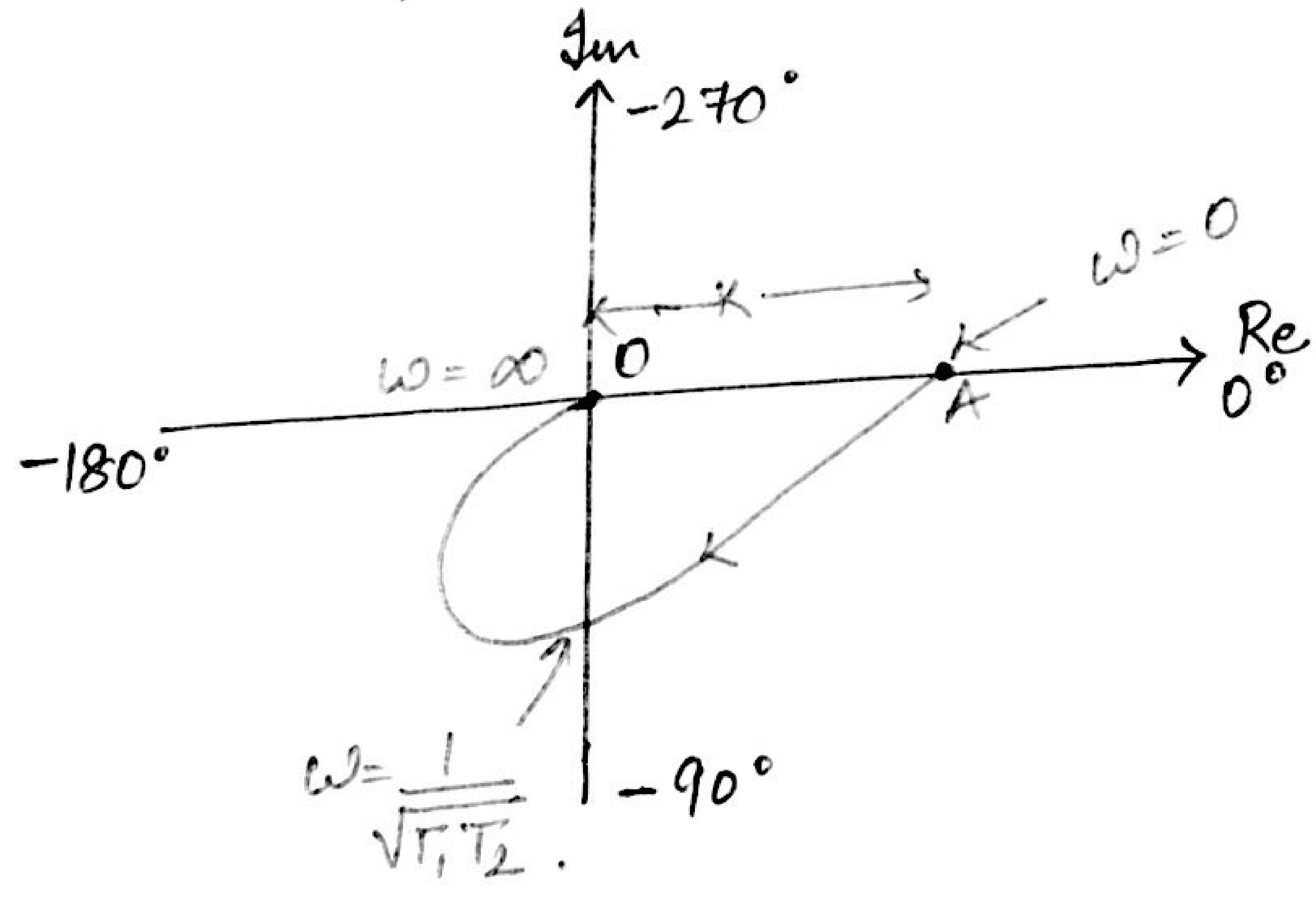
$$\frac{k\omega(T_1 + T_2)}{1 + \omega^2 T_1^2 + \omega^2 T_2^2 + \omega^4 T_1 T_2} = 0$$

$\omega = 0$ and $\pm\infty$

when $\omega = 0$; $|G(j\omega)| = k$ $\quad \angle G(j\omega) = 0^\circ$

$\omega = \infty$; $|G(j\omega)| = 0$ $\quad \angle G(j\omega) = 0^\circ$.

$$G(s) = \frac{k}{(1+sT_1)(1+sT_2)}$$



TYPE ONE SYSTEM

$$G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$$

1.) Put $s = j\omega$

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$

$$G(j\omega) = \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} \angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

2.) $$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} = \infty$$

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

3.) $$\lim_{\omega \to 0} \angle G(j\omega) = \lim_{\omega \to 0} \angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

$$= -90^\circ$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = \lim_{\omega \to \infty} \angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

$$= -270^\circ.$$

4.) ~~[illegible]~~ Seprating real and imaginary parts.

$$G(j\omega) = \frac{K}{j\omega(1+\omega T_1)(1+\omega T_2)}$$

$$= \frac{-\omega K(T_1+T_2)}{\omega + \omega^3(T_1^2+T_2^2+\omega^2 T_1^2 T_2^2)} + \frac{j(K\omega^2 T_1 T_2 - K)}{\omega + \omega^3(T_1^2+T_2^2+\omega^2 T_1^2 T_2^2)} \quad \text{--- (1)}$$

Equating the imaginary part equal to zero

$$\frac{K\omega^2 T_1 T_2 - K}{\omega + \omega^3(T_1^2+T_2^2+\omega^2 T_1^2 T_2^2)} = 0$$

$$\omega = \frac{1}{\sqrt{T_1 T_2}} \quad \text{and} \quad \omega = \pm\infty$$

The freq. at the point of intersection on real axis is $\frac{1}{\sqrt{T_1T_2}}$. Now calculate the value of $G(j\omega)$ at this point.

Put, $\omega = \frac{1}{\sqrt{T_1 T_2}}$ in eqn ① :-

$$G(j\omega) = \frac{K_1 T_1 T_2}{T_1 + T_2} \quad \angle G(j\omega) = 0°$$

put $\omega = \infty$, in eqn ① :-

$$G(j\omega) = \infty ; \quad \angle G(j\omega) = 0°$$

57 Equate the real part to zero.

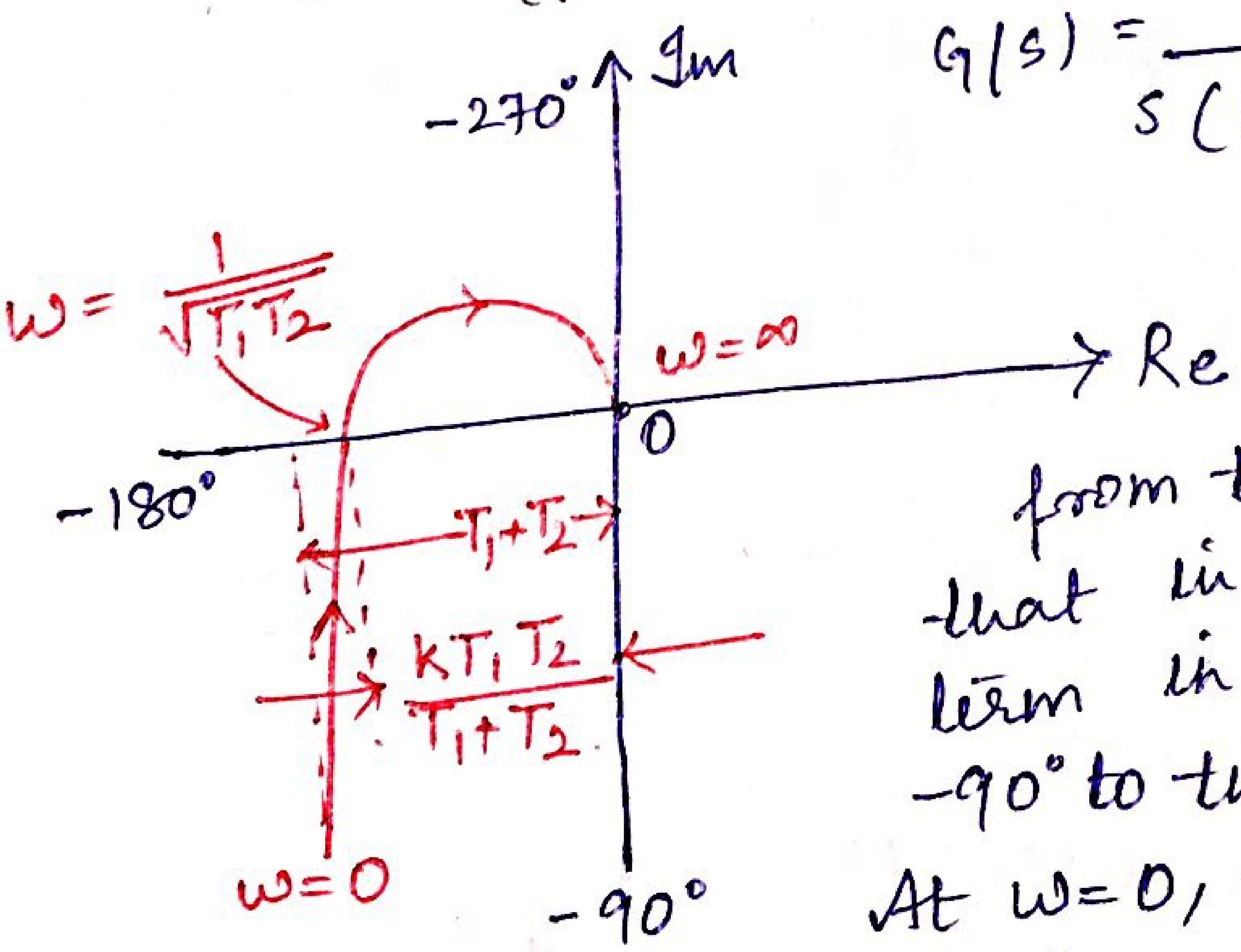
$$\frac{-\omega k\,(T_1 + T_2)}{\omega + \omega^3 (T_1^2 + T_2^2 + \omega^2 T_1 T_2)} = 0$$

$$\therefore \quad \omega = \infty.$$

For +ve values of freq. the polar plot intersects imaginary axis at $\omega = \infty$.

$$\therefore \quad G(j\omega) = 0 \angle -270°$$

$$G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$$



from the polar plot it is clear that in type one system the $j\omega$ term in denominator contributes $-90°$ to the total phase angle.
At $\omega = 0$, the magnitude is infinity and phase angle is $-90°$. At $\omega = \infty$, the magnitude becomes zero and curve converges to origion.

TYPE TWO SYSTEM

$$G(s) = \frac{k}{s^2(1+sT_1)}$$

put $s = j\omega$

$$G(j\omega) = \frac{k}{(j\omega)^2(1+j\omega T_1)}$$

$$= \frac{k}{-\omega^2\sqrt{1+(\omega T_1)^2}} \angle -180^\circ - \tan^{-1}\omega T_1$$

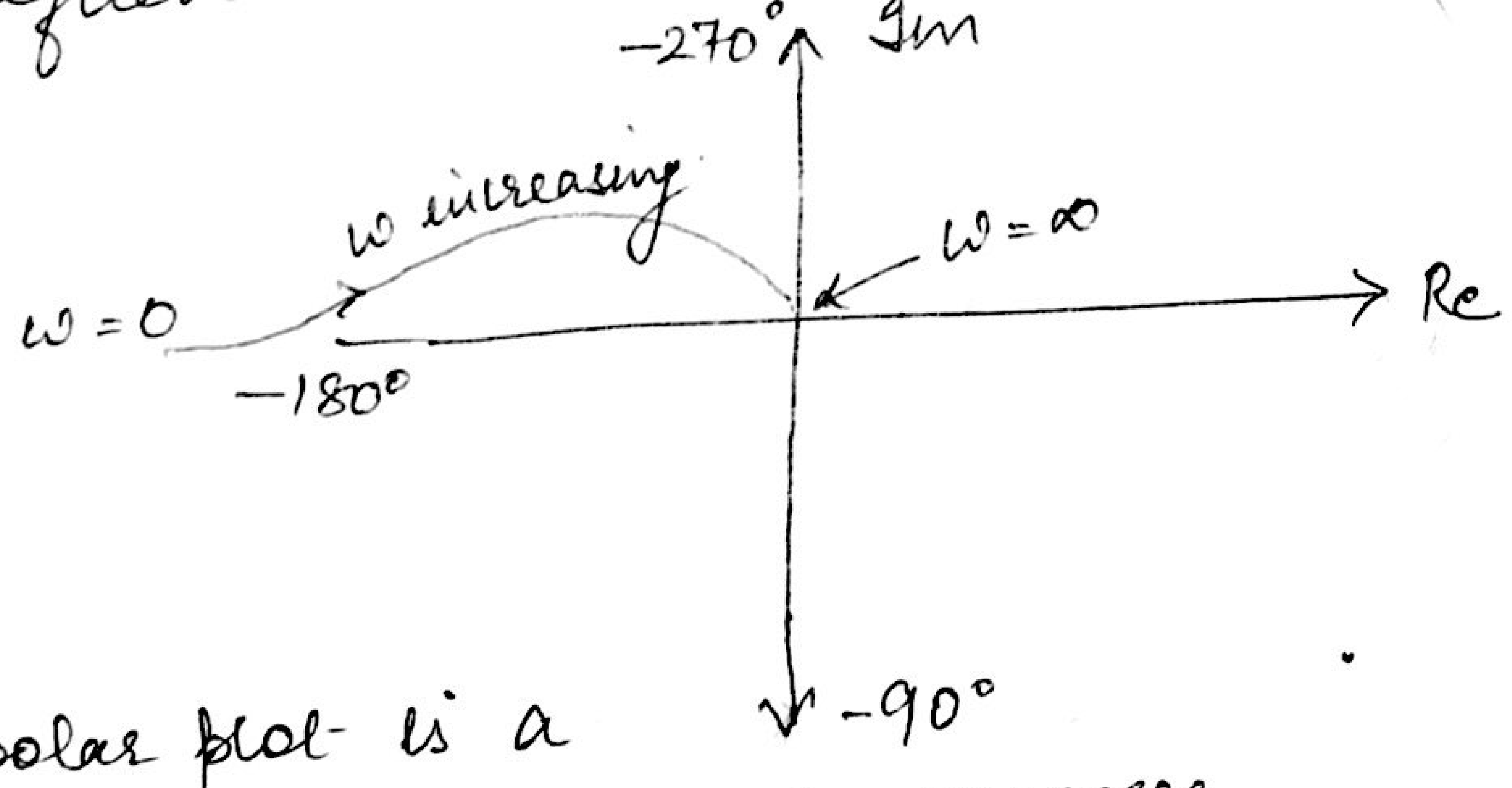
$$\lim_{\omega \to 0} |G(j\omega)| = \lim_{\omega \to 0} \frac{-k}{\omega^2\sqrt{1+(\omega T_1)^2}} = \infty$$

$$\lim_{\omega \to \infty} |G(j\omega)| = \lim_{\omega \to \infty} \frac{k}{-\omega^2\sqrt{1+(\omega T_1)^2}} = 0$$

$$\lim_{\omega \to 0} \angle -180^\circ - \tan^{-1}\omega T_1 = -180^\circ$$

$$\lim_{\omega \to \infty} \angle -180^\circ - \tan^{-1}\omega T_1 = -270^\circ.$$

The presence of s^2 in the denominator contributes a constant -180° to the angle of $G(j\omega)$ for all frequencies.



The polar plot is a smooth curve whose angle decreases continously from -180° to -270°.

INTRODUCTION OF ADDITIONAL POLE

$$G(s) = \frac{K}{s^2(1+sT_1)(1+sT_2)}$$

Put $s = j\omega$

$$G(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}$$

$$G(j\omega) = \frac{K}{-\omega^2\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} \angle -180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

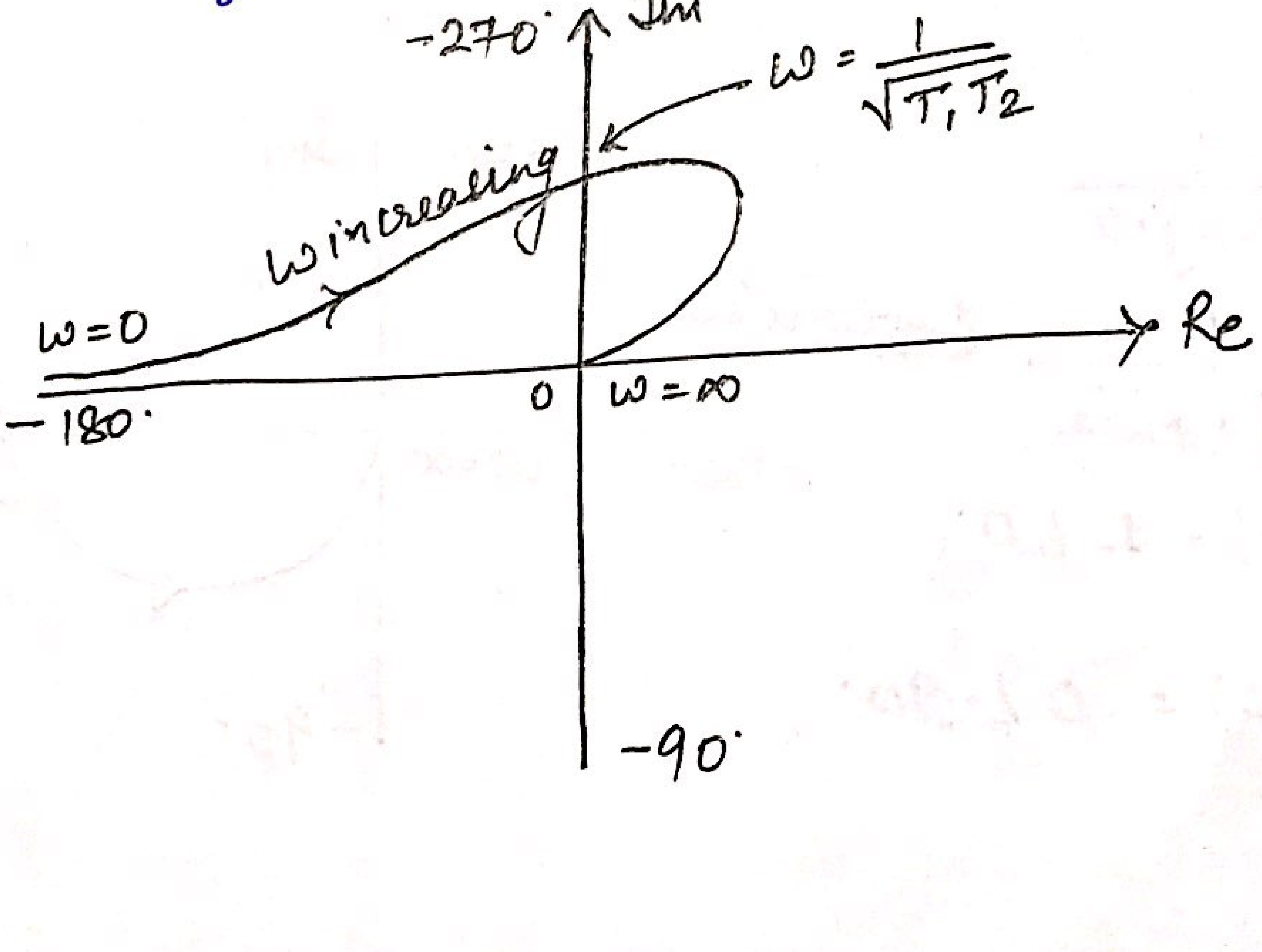
$$\lim_{\omega \to 0} |G(j\omega)| = \infty$$

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

$$\lim_{\omega \to 0} \angle G(j\omega) = -180^\circ$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -360^\circ$$

Seperate the real and imaginary parts of $G(j\omega)$ The curve intersects the imaginary axis and 'ω' at this pt. is $\frac{1}{\sqrt{T_1 T_2}}$. The angle of curve decreases continously from -180° to -360°



POLAR PLOTS OF SOME STANDARD TYPE FUNCTIONS

1. $G(s) = \frac{1}{s}$

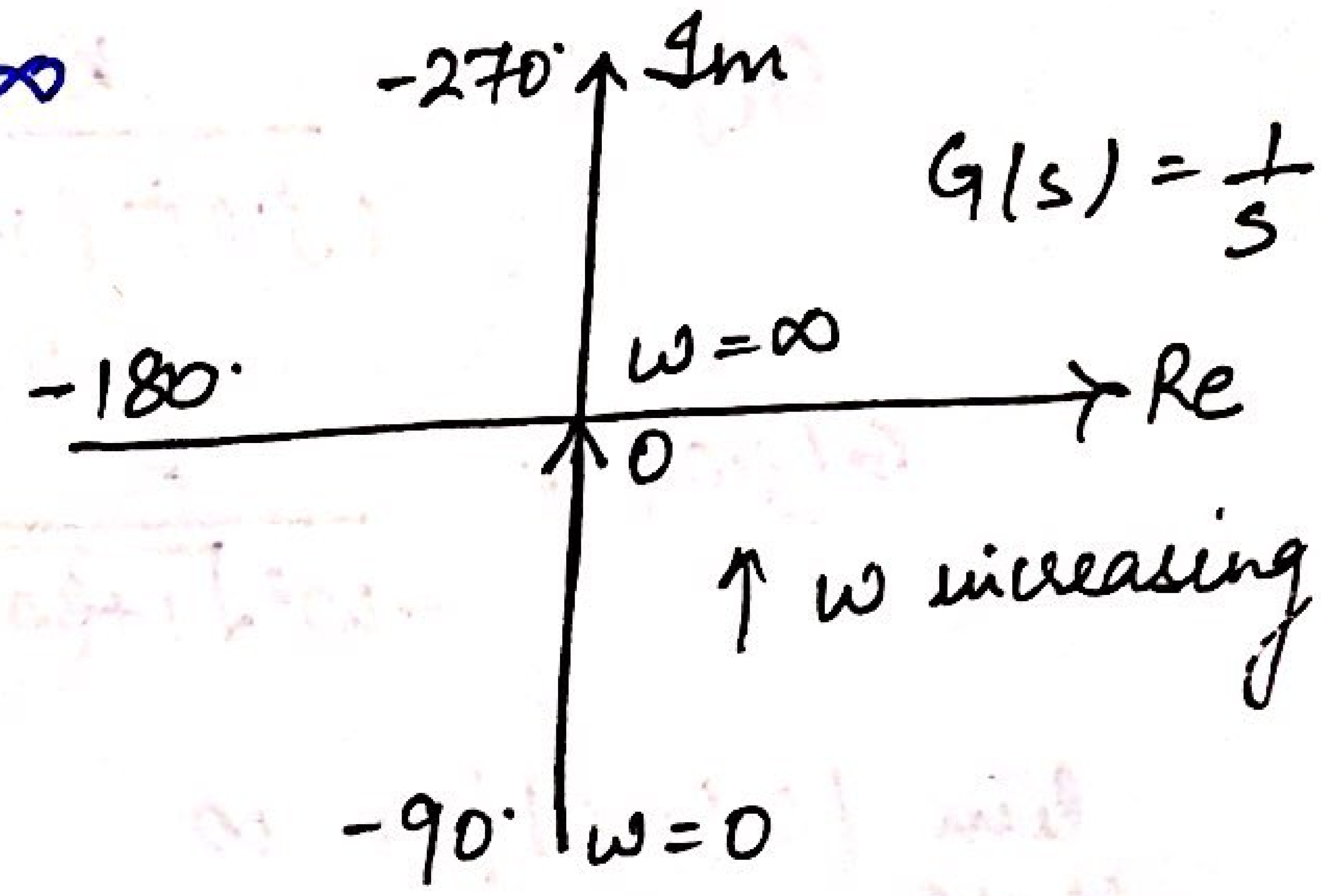
put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega} \angle \tan^{-1}\infty$$

$$G(j\omega) = \frac{1}{\omega} \angle -90^\circ$$

$$\lim_{\omega \to 0} |G(j\omega)| = \infty$$

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$



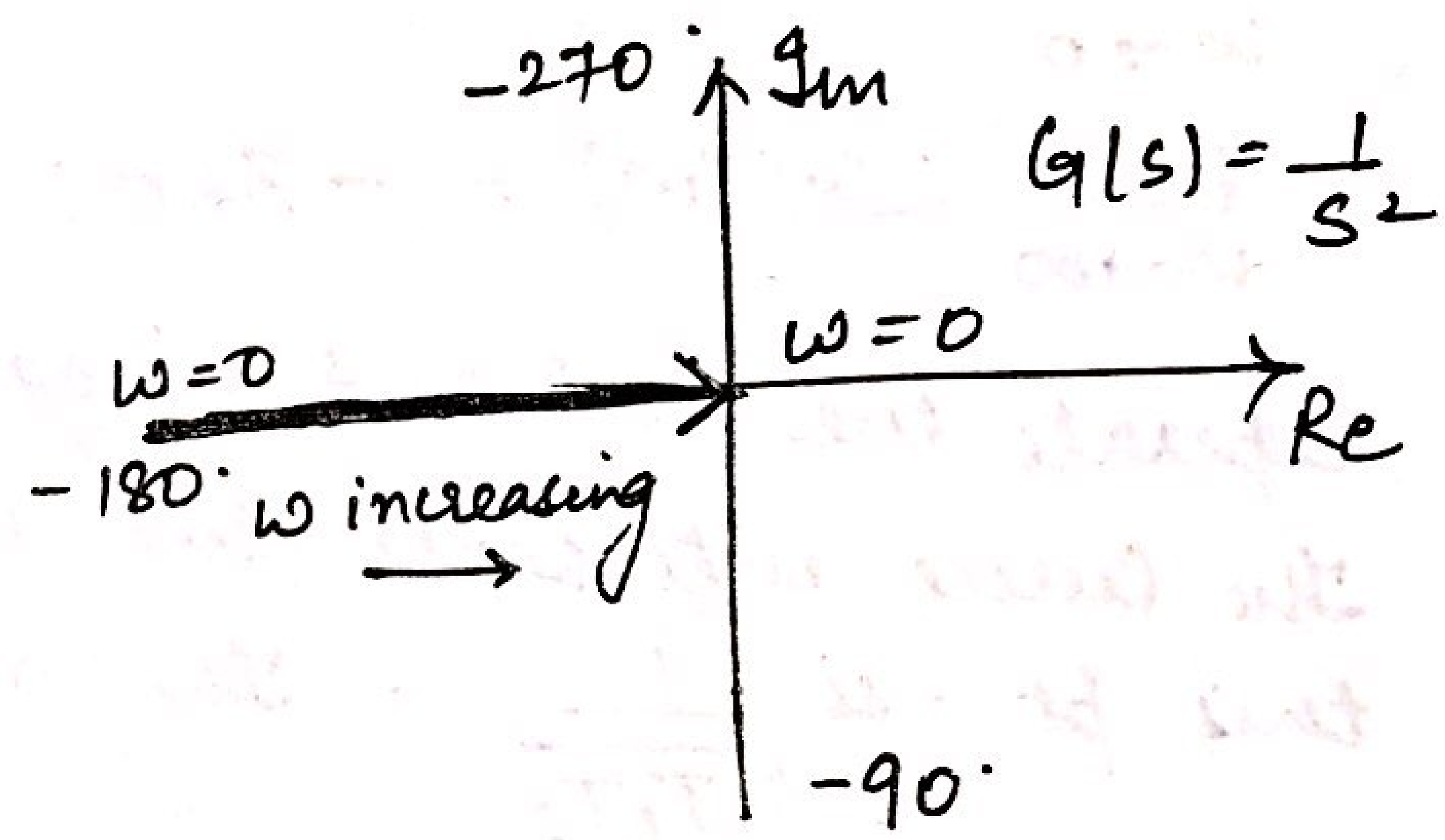
2. $G(s) = \frac{1}{s^2}$

put $s = j\omega$

$$G(j\omega) = \frac{1}{-\omega^2} \angle -180^\circ$$

$\omega = 0$; $G(j\omega) = -\infty$

$\omega = \infty$; $G(j\omega) = 0$



3. $G(s) = \frac{1}{s+1}$

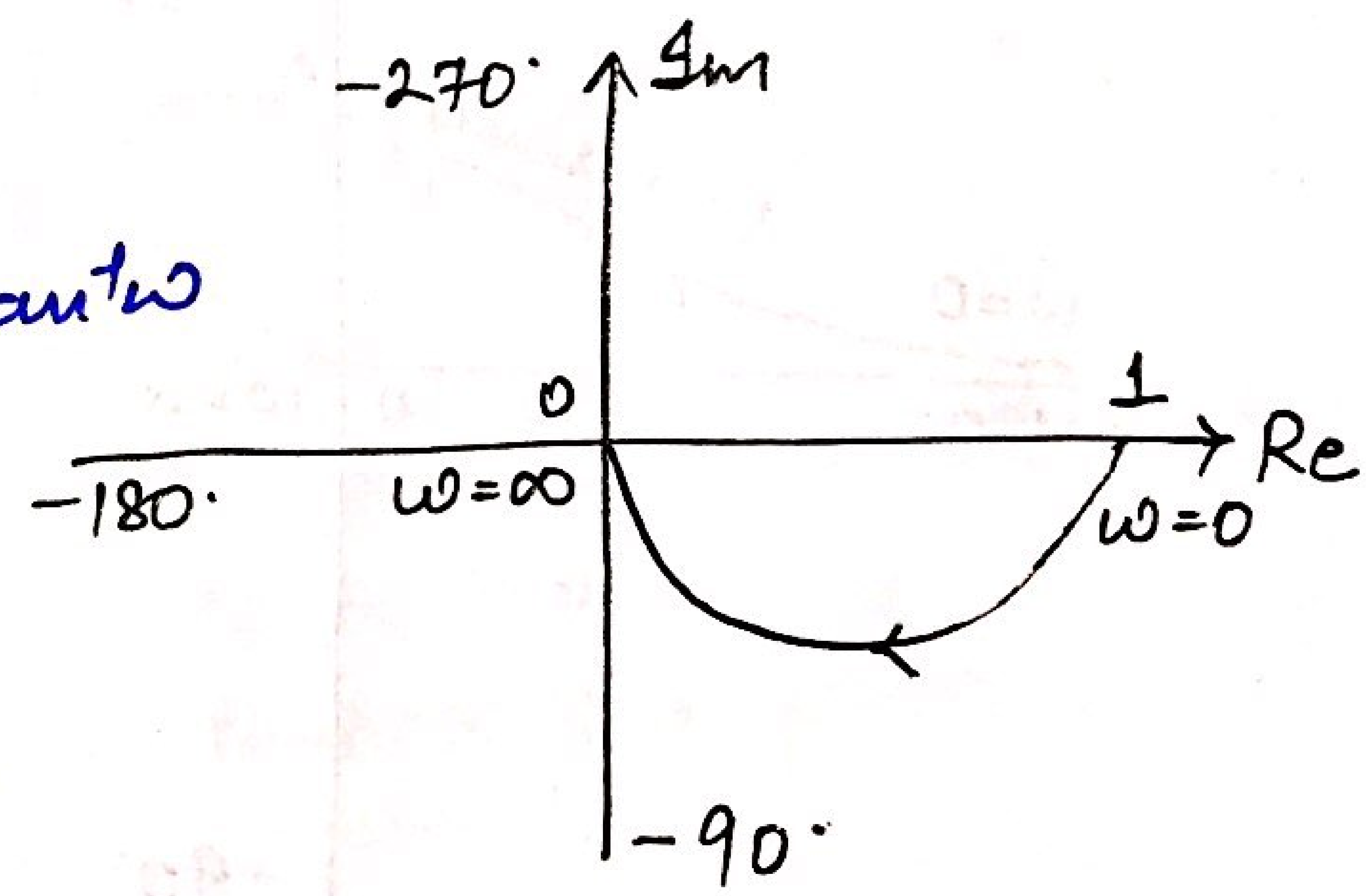
Put $s = j\omega$

$$G(j\omega) = \frac{1}{1+j\omega}$$

$$= \frac{1}{\sqrt{1+\omega^2}} \angle -\tan^{-1}\omega$$

$$\lim_{\omega \to 0} G(j\omega) = 1 \angle 0^\circ$$

$$\lim_{\omega \to \infty} G(j\omega) = 0 \angle -90^\circ$$



Ques 1. Sketch the polar plot for $G(s) = \frac{1}{s(s+1)}$

Solution :-

$$G(s) = \frac{1}{s(s+1)}$$

Put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(j\omega+1)}$$

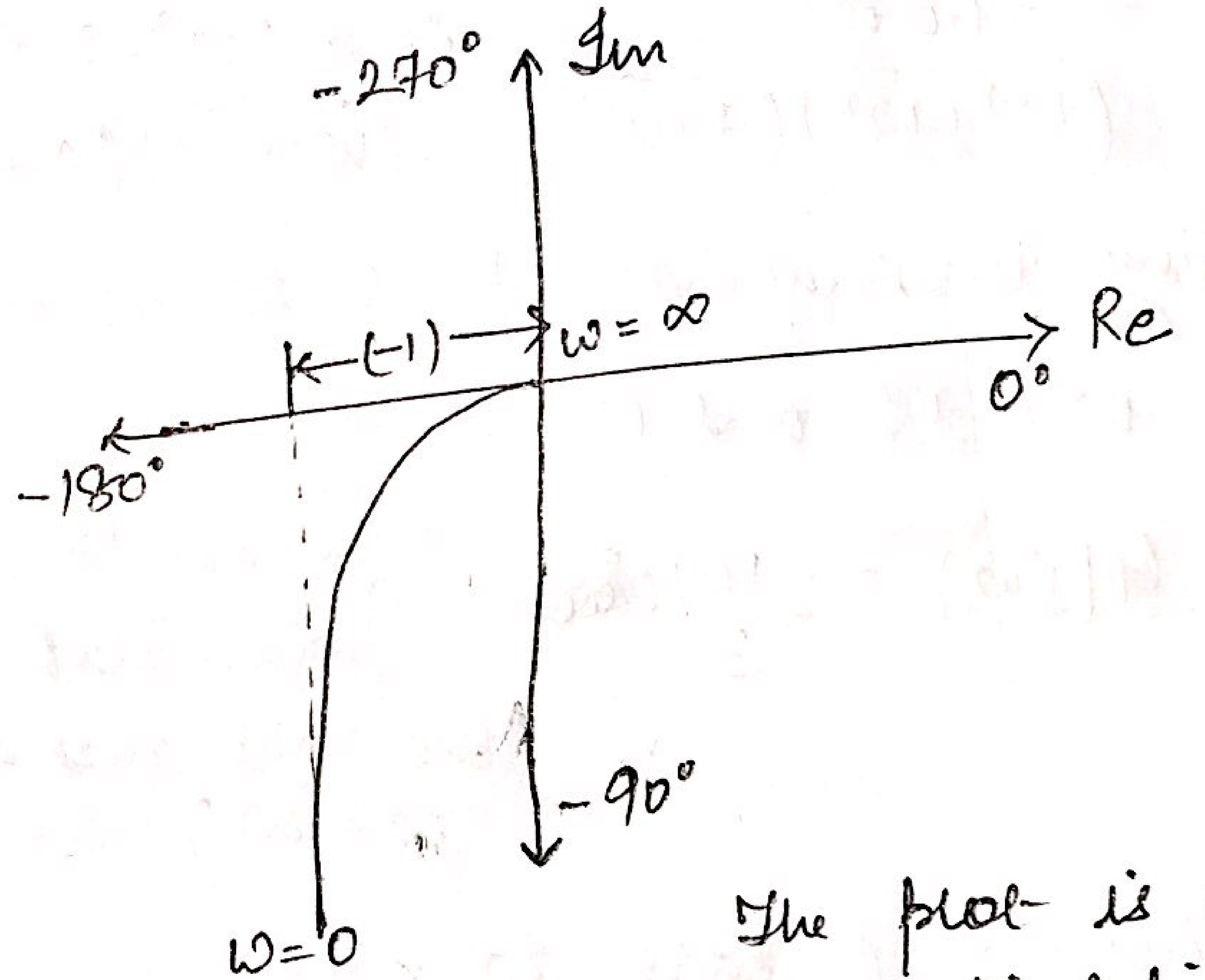
$$G(j\omega) = \frac{1}{\omega\sqrt{\omega^2+1}} \angle -90^\circ - \tan^{-1}\omega$$

$$\lim_{\omega\to 0} |G(j\omega)| = \lim_{\omega\to 0} \frac{1}{\omega\sqrt{1+\omega^2}} = \infty$$

$$\lim_{\omega\to\infty} |G(j\omega)| = \lim_{\omega\to\infty} \frac{1}{\omega\sqrt{1+\omega^2}} = 0$$

$$\lim_{\omega\to 0} \angle G(j\omega) = \lim_{\omega\to 0} \angle -90^\circ - \tan^{-1}\omega = -90^\circ$$

$$\lim_{\omega\to\infty} \angle G(j\omega) = \lim_{\omega\to\infty} \angle -90^\circ - \tan^{-1}\omega = -180^\circ$$



The plot is asymptotic to the vertical line passing through the point (-1,0)

Ques 2. Sketch the polar plot for :-

$$G(s) = \frac{20}{s(s+1)(s+2)}$$

Put $s = j\omega$

$$G(j\omega) = \frac{20}{j\omega(j\omega+1)(j\omega+2)}$$

$$= \frac{20}{\omega\sqrt{\omega^2+1}\sqrt{\omega^2+4}} \angle -90^\circ - \tan^{-1}\omega - \tan^{-1}\omega/2$$

$\lim_{\omega \to 0} |G(j\omega)| = \infty$ $\quad \lim_{\omega \to 0} \angle G(j\omega) = -90^\circ$

$\lim_{\omega \to \infty} |G(j\omega)| = 0$ $\quad \lim_{\omega \to \infty} \angle G(j\omega) = -270^\circ$

Sepereating real and imaginary part of $G(j\omega)$

$$G(j\omega) = \frac{20}{j\omega(1+j\omega)(2+j\omega)}$$

$$G(j\omega) = \frac{-60\omega^2}{(\omega^4+\omega^2)(4+\omega^2)} + j\,\frac{20(\omega^3-2\omega)}{(\omega^4+\omega^2)(4+\omega^2)}$$

Equating the imaginary part to zero

$\omega = \pm\sqrt{2}$ and $\omega = \pm\infty$

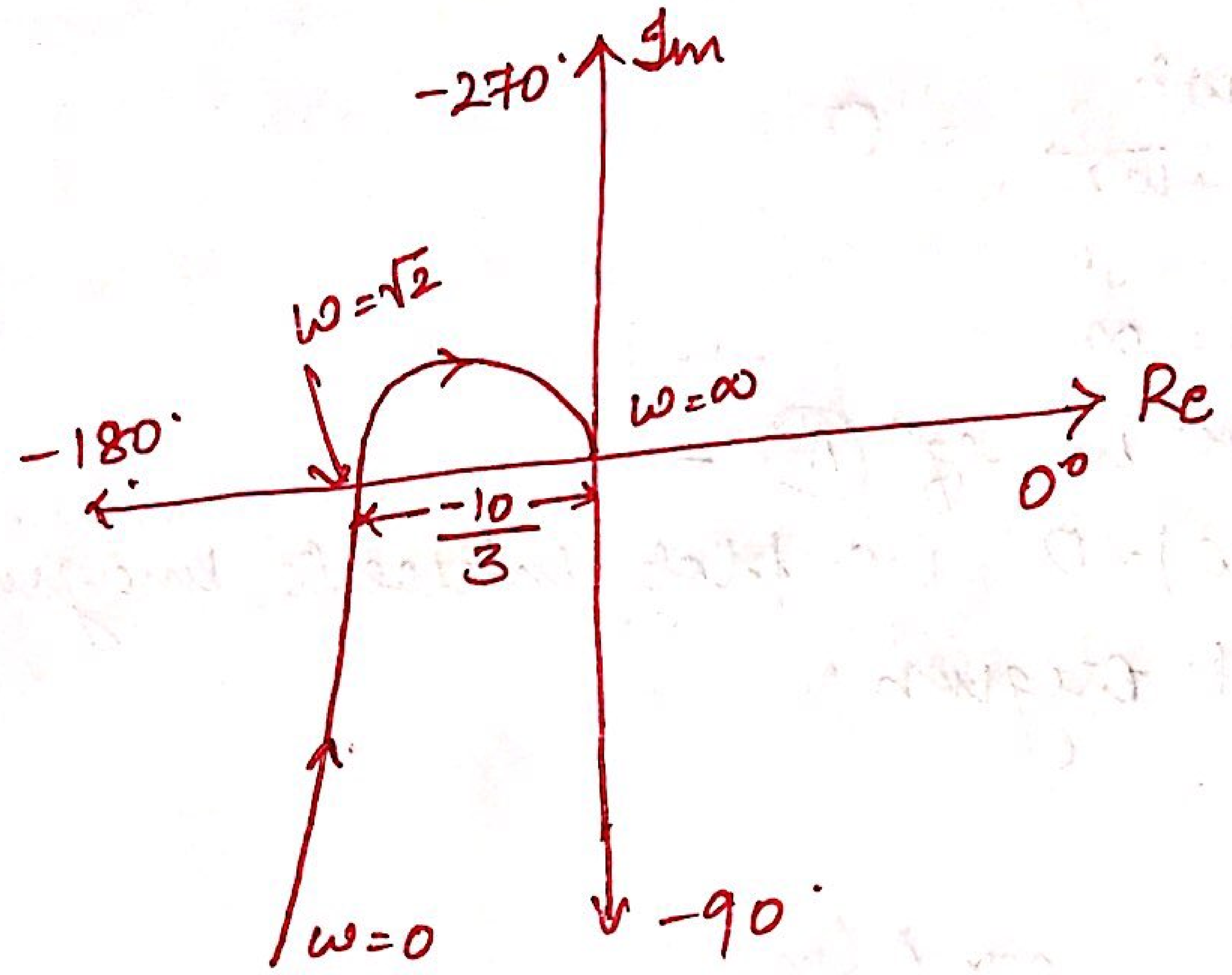
$\therefore G(j\omega) = \frac{-10}{3}\angle 0^\circ$ (for +ve values of freq. polar plot intersects the real axis at $\omega = \sqrt{2}$ and $\omega = \infty$)

Equating real part to zero

$$\frac{-60\omega^2}{(\omega^4+\omega^2)(4+\omega^2)} = 0$$

$\omega = \infty$

The polar plot intersects the imaginary axis at $\omega = \infty$ for $\angle G(j\omega) = -270^\circ$ and $|G(j\omega)| = 0$



Ques. Sketch the polar plot of $G(s) = \dfrac{10}{s(s+1)}$

$$G(s) = \frac{10}{s(s+1)}$$

$$G(j\omega) = \frac{10}{j\omega(j\omega+1)} = \frac{10}{\omega\sqrt{\omega^2+1}} \angle -90^\circ - \tan^{-1}\omega$$

$$\lim_{\omega \to 0} |G(j\omega)| = \infty \qquad \lim_{\omega \to 0} \angle G(j\omega) = -90^\circ$$

$$\lim_{\omega \to \infty} |G(j\omega)| = 0 \qquad \lim_{\omega \to \infty} \angle G(j\omega) = -180^\circ$$

Sepereating real and imaginary parts

$$G(j\omega) = \frac{-10\omega^2}{\omega^4+\omega^2} - j\frac{10\omega}{\omega^4+\omega^2} \quad \text{——— (A)}$$

Equating the imaginary part to zero

$\therefore \omega = \infty$

Put $\omega = \infty$ in eqn (A) :-

$G(j\omega) = 0$, i.e. plot intersects the real axis at Origion

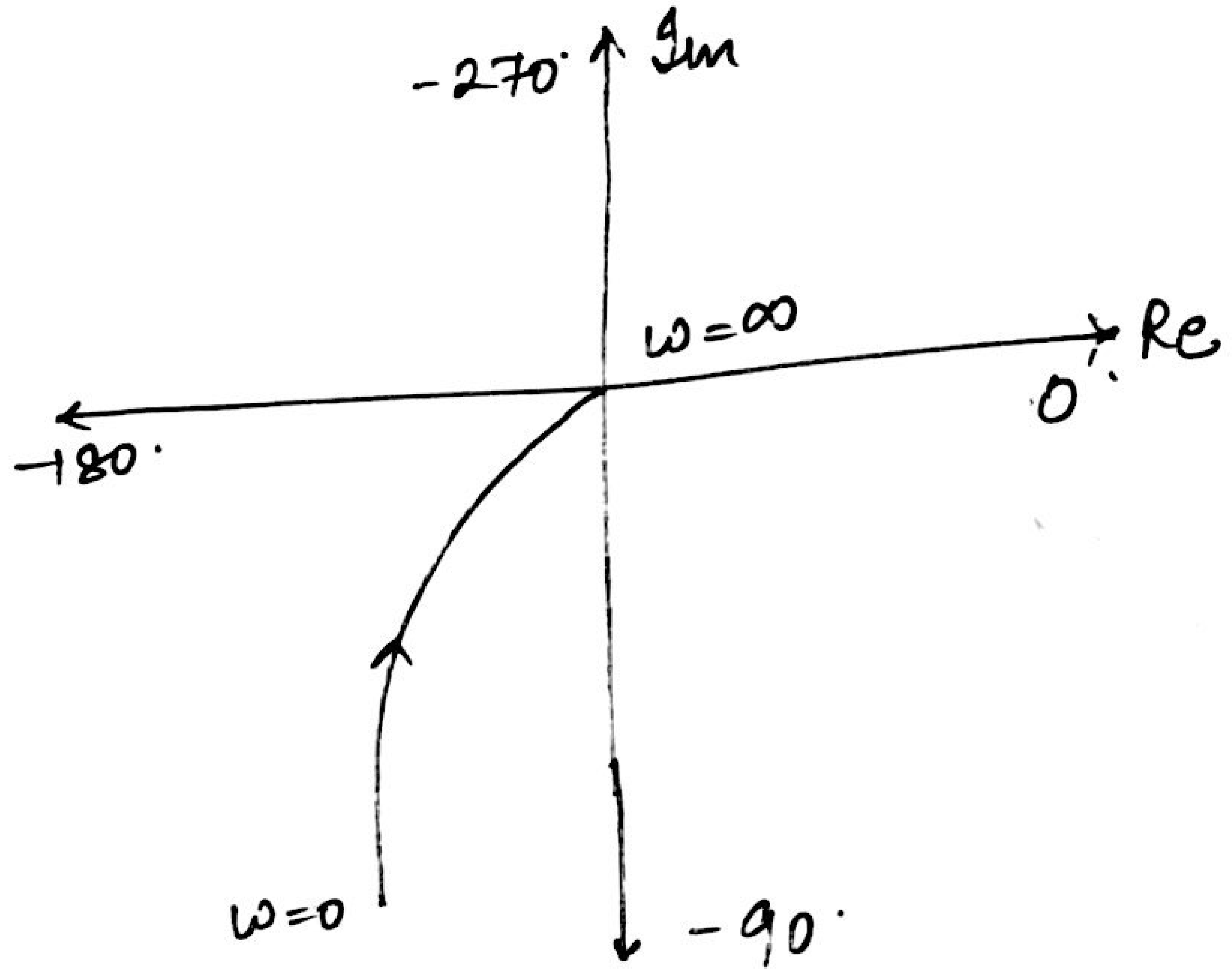
Equating real part to zero

$$\frac{-10\omega^2}{\omega^4 + \omega^2} = 0$$

$$\omega = \infty$$

Put $\omega = \infty$ in eqn (A) :-

$G(j\omega) = 0$, i.e. plot intersects imaginary axis at Origion.



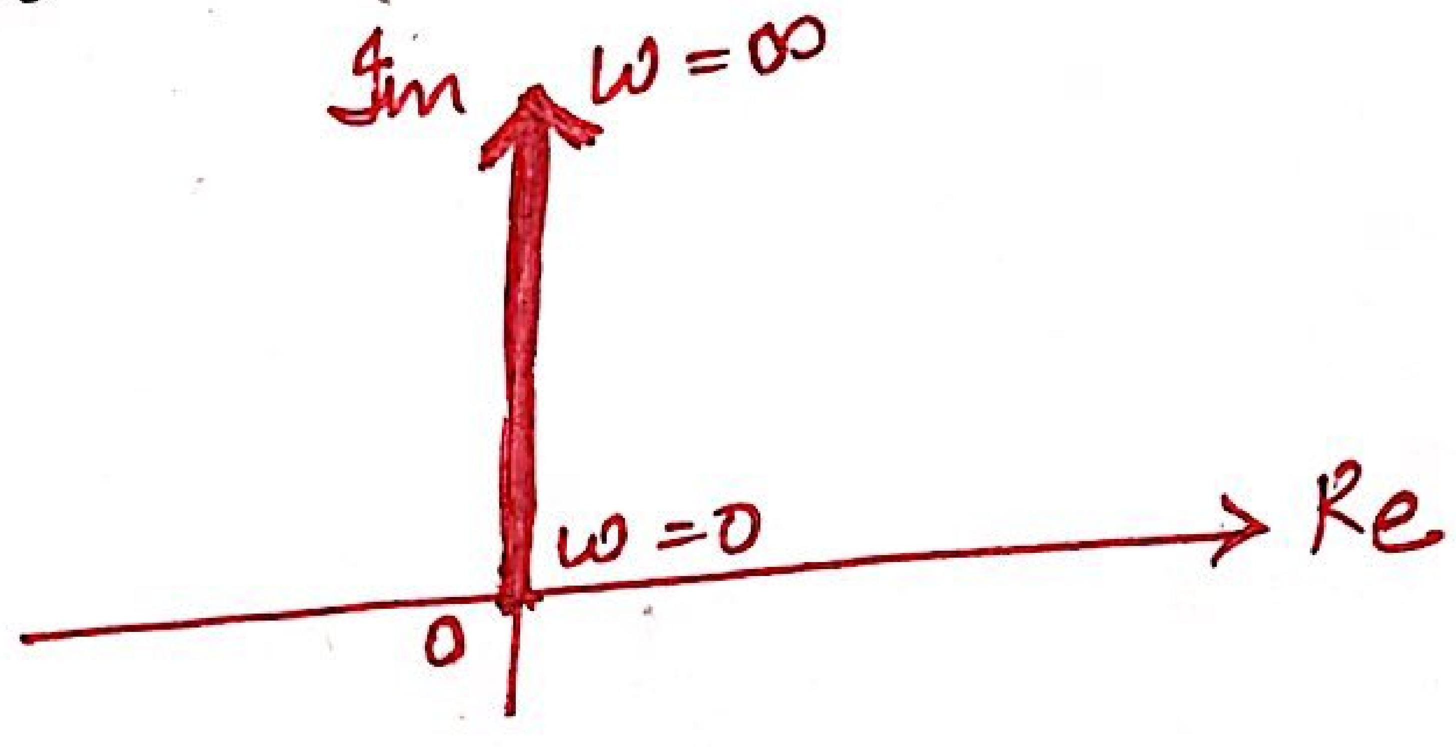
INVERSE POLAR PLOT

The inverse polar plot of $G(j\omega)$ is graph of $\frac{1}{G(j\omega)}$ as a funcⁿ of ω.

if $G(j\omega) = \frac{1}{j\omega}$

then, $G(j\omega)^{-1} = j\omega$

$\lim_{\omega \to 0} |G(j\omega)^{-1}| = 0$ and $\lim_{\omega \to \infty} |G(j\omega)^{-1}| = \infty$



Ques. Sketch the inverse polar plot of $G(s) = \frac{1+ST}{ST}$

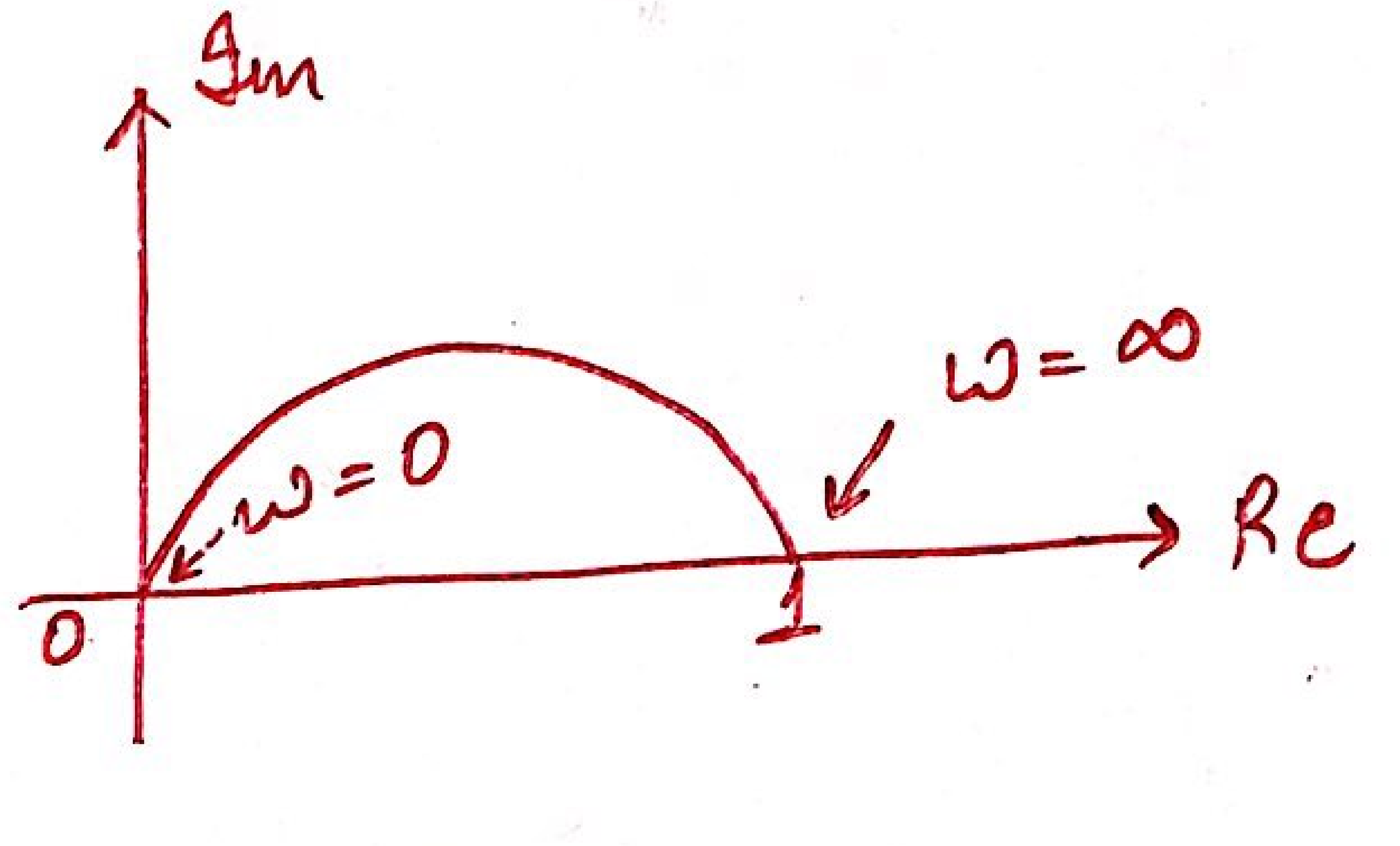
$$G(s)^{-1} = \frac{1}{G(s)} = \frac{ST}{1+ST}$$

Put $s = j\omega$

$$G(j\omega)^{-1} = \frac{1}{G(j\omega)} = \frac{j\omega T}{1+j\omega T}$$

$\lim_{\omega \to 0} |G(j\omega)^{-1}| = 0$ $\quad \lim_{\omega \to 0} \angle G(j\omega)^{-1} = 90°$

$\lim_{\omega \to \infty} |G(j\omega)^{-1}| = 1$ $\quad \lim_{\omega \to \infty} \angle G(j\omega)^{-1} = 0°$



Scanned with CamScanner