Special theory of relativity

Introduction

The classical mechanics is based entirely on Newton's laws of motion. Earlier it was thought that Newton's second law of motion is universally applicable at all speeds. Later on it was found that it is valid only for the objects moving at low speeds and fails when applied to the objects moving with high velocities comparable with the velocity of light. This led to the development of the remarkable special theory of relativity by Albert Einstein in 1905. According to this theory, everything in the universe is relative; nothing is absolute. Einstein in his new theory extended and generalized Newtonian mechanics. He predicted that old classical mechanics is the limiting case of the new theory.

Frame of reference

In order to specify the location of a point object in space, we require a coordinate system. The location of a point object is expressed in terms of three real numbers called coordinates of that point object with respect to the origin. For complete information about an event, we must know its correct time of occurence. Such a coordinate system with three space coordinates and one time coordinate is called a frame of reference.

The frames of reference are of two types

1. Inertial or unaccelerated frame of reference
2. Non-inertial or accelerated frame of reference.

In inertial frame of reference, Newton's laws of motion hold good. In this frame a body at rest or moving with uniform velocity and not under the influence of any force remain at rest or moving with the same uniform velocity. ie unaccelerated frames are inertial frames. The Earth is an example of inertial frame.

Accelerated frames are called non-inertial. In this frame Newton's law does not hold good. A rotating merry-go-round is an example of non-inertial frame.

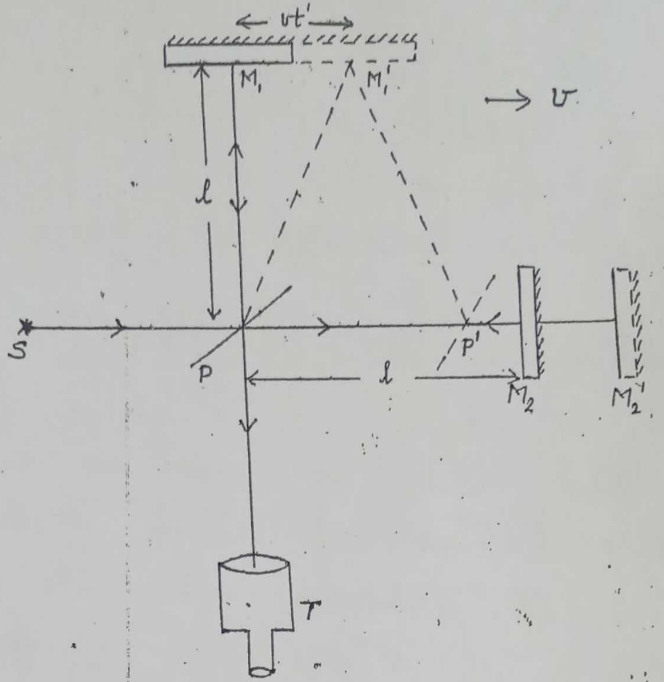
Ether hypothesis

The nineteenth century physicist believed that a material medium is essential for the propagation of light and other waves in free space. They assumed that the entire space of universe including vacuum is filled by a hypothetical light transmitting medium called ether, which is rigid, invisible, massless, perfectly transparent, perfectly non-resistive & having high elasticity and negligible density. All bodies including earth move freely through this hypothetical medium. Thus ether provides a fixed frame of reference called ether frame or rest frame or absolute frame of reference. Scientists were of the view that if the ether hypothesis is correct then it should be possible to determine the absolute velocity of the earth with respect to stationary ether frame.

Michelson - Morley Experiment

To confirm the existence of ether, Michelson and Morley performed an experiment to determine the absolute velocity of the earth with respect to stationary ether. The experimental set up consists of a Michelson interferometer. A beam of light from an extended source S is incident on a semi-silvered glass plate P, inclined at 45° to the beam. The plate splits the light into two parts. The transmitted part falls normally on mirror M_2 which reflects them back to P. Similarly the reflected part falls normally on M_1 and reflects back to P. The two parts of the beam returned to P and directed towards the telescope T and they interfere to form interference fringes.

The two mirrors M_1 and M_2 are at equal distance l from the plate P. If the apparatus is at rest, then the two rays would take equal time to return to P. But the whole apparatus is moving with a velocity v. Due to the motion of the apparatus with earth, the time taken by two rays in their journey would not be same.



Let c be the velocity of light through ether. The velocity of light with respect to the apparatus along the path PM_2 is $(c-v)$ in the forward trip and is $(c+v)$ in the backward trip. If t_1 be the time taken by the light to go from P to M_2 and then back to P, then

$$t_1 = \frac{l}{c+v} + \frac{l}{c-v} = \frac{l(c-v) + l(c+v)}{c^2-v^2}$$

$$= \frac{2lc}{c^2-v^2} = \frac{2lc}{c^2[1-v^2/c^2]} = \frac{2l}{c[1-v^2/c^2]}$$

$$= \frac{2l}{c}\left[1-v^2/c^2\right]^{-1}.$$

Expanding using binomial theorem and neglecting higher order terms,

$$t_1 = \frac{2l}{c}\left[1+\frac{v^2}{c^2}\right] \quad \text{——} \quad (1)$$

The reflected ray travelling towards M_1 retain its velocity v. If t' is the time taken by the beam in going from P to M_1, the distance travelled is ct'. At the same time mirror M_1 will shift to M_1' travelling a horizontal distance vt'. Then from $\Delta PM_1M_1'$

$$(PM_1')^2 = PM_1^2 + (M_1M_1')^2$$

$$(ct')^2 = l^2 + (vt')^2$$

$$c^2t'^2 = l^2 + v^2t'^2$$

$$(c^2 - v^2)\, t'^2 = l^2 \qquad t'^2 = \frac{l^2}{(c^2 - v^2)} = \frac{l^2}{c^2[1 - v^2/c^2]}$$

$$t' = \frac{l}{c\sqrt{1 - v^2/c^2}} \qquad \text{—— } (2)$$

If t_2 is the total time taken by the light to go from P to M_1' and then back to P', then

$$t_2 = 2t' = \frac{2l}{c\sqrt{1 - v^2/c^2}} = \frac{2l}{c}\left[1 - v^2/c^2\right]^{-1/2}$$

Expanding using binomial theorem and neglecting higher order terms

$$t_2 = \frac{2l}{c}\left[1 + \frac{v^2}{2c^2}\right] \qquad \text{—— } (3)$$

The time difference Δt between the two beams is

$$\Delta t = t_1 - t_2 = \frac{2l}{c}\left[1 + \frac{v^2}{c^2}\right] - \frac{2l}{c}\left[1 + \frac{v^2}{2c^2}\right]$$

$$= \frac{2l}{c} \times \frac{v^2}{2c^2} = \underline{\underline{\frac{lv^2}{c^3}}}$$

The corresponding path difference, $\delta = c\,\Delta t = \frac{lv^2}{c^2}$

If the path difference between the two interfering rays changes by λ, the shifting of one fringe in the field of view is observed. If n be the number of fringes that shift, then

$$n = \frac{\delta}{\lambda} = \frac{lv^2}{c^2\lambda}$$

The whole apparatus, which was placed on a stone floated on mercury was turned through 90°. The rotation of apparatus through 90° introduces a path difference of lv^2/c^2. The total path difference is $2lv^2/c^2$ and a shift of $2lv^2/c^2\lambda$ was expected.

Taking $l = 11\,m$; $v = 3\times10^4\,m/s$

$\lambda = 5500\,\mathring{A}$ & $c = 3\times10^8\,m/s$

the expected fringe shift was

$$n = \frac{2\times 11\times(3\times10^4)^2}{(3\times10^8)^2\times 5500\times10^{-10}} = \underline{0.4}$$

A shift of this magnitude can be easily measured by Michelson - Morley set up. But no fringe shift was observed. ie the relative velocity between the earth and the ether is zero. Thus the hypothesis of the existence of stationary medium was disapproved.

Explanation of negative results

The three main explanations to explain the negative results are

1) Ether - drag hypothesis

According to this hypothesis, ether is not stationary, it moves with the earth in its motion through space. There is no relative motion between the two and hence no shift in fringe arise.

2) Lorentz - Fitzgerald contraction hypothesis

According to this hypothesis, all material bodies are contracted in the direction of motion relative to stationary ether by a factor $\sqrt{1-v^2/c^2}$. The contraction in the interferometer arm could equalize the two times t_1 and t_2, and thus no fringe - shift could be expected.

3) Constancy of speed of light hypothesis

It was proposed that light travels with a constant velocity not with respect to the stationary ether but with respect to the source.

Postulates of Einstein's Special theory of relativity

The formulation of the special theory of relativity is based upon two basic postulates.

1. The principle of equivalence which states that the laws of physics are the same in all inertial frames of reference moving with a constant velocity with respect to one another.
2. The principle of constancy of the speed of light states that the speed of light in free space is always a constant equal to c. Its value is independent of the relative motion of the inertial frames, source and the observer.

Lorentz transformation equations of space and time

Based on the postulates of relativity, new transformation equations were discovered by Lorentz and are known as Lorentz transformation equation. For speeds much smaller than c, these equations reduce to Galilean transformation equation. ∴ Einstein's theory of relativity does not overthrow the classical theory but extends and modifies it.

Consider a system of two inertial frames of reference S and S'. S is at rest and S' is moving with a velocity v. Two observers are situated at O and O' and observing any event P. Let us assume that the x-axis of two systems coincide permanently. The event P is determined by coordinates x, y, z, t for an observer on frame S and by x', y', z', t' for an observer on frame S'. The time be counted from the instant when the two origins momentarily coincide. After a time t, S' must move a distance vt. In new transformation, the measurement in the x-direction made in S must be linearly proportional to that made in S'.

$$x' = \gamma(x-vt) \quad \text{——} \quad (1)$$

γ is the proportionality constant.

According to 1st postulate; the equations relating the physical quantities have the same form in both frames of reference.

$$\therefore \quad x = \gamma(x'+vt') \quad \text{——} \quad (2)$$

Substituting (1) in (2)

$$x = \gamma[\gamma(x-vt) + vt']$$

$$\frac{x}{\gamma} = \gamma x - \gamma vt + vt' \qquad vt' = \frac{x}{\gamma} - \gamma x + \gamma vt$$

$$t' = \frac{x}{\gamma v} - \frac{\gamma x}{v} + \gamma t$$

$$t' = \gamma t - \frac{\gamma x}{v}\left[1 - \frac{1}{\gamma^2}\right] \quad \text{——} \quad (3)$$

The value of γ can be evaluated with the help of second postulate. Let a flash of light be emitted from the common origin of S and S' at $t = t' = 0$. The light flash travels with the same velocity c which is same for both the frames. The position of the flash as seen from the two frames after some time is $x = ct$ & $x' = ct'$

Substituting the value of x & x' in (1) & (2)

$$ct' = \gamma(ct - vt) = \gamma t(c-v) \quad \text{——} \quad (4)$$

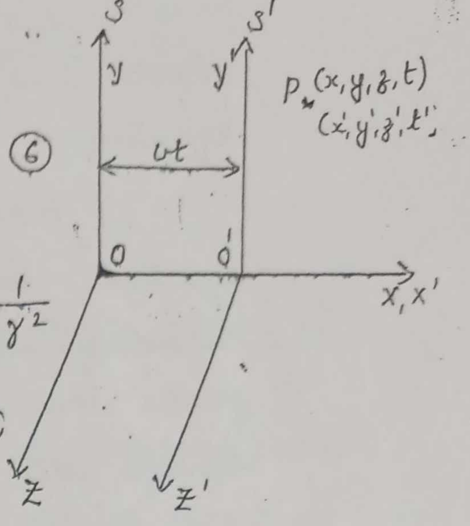
$$ct = \gamma(ct' + vt') = \gamma t'(c+v) \quad \text{——} \quad (5)$$

Multiplying 4 & 5

$$c^2 tt' = \gamma^2 tt'(c-v)(c+v)$$

$$c^2 = \gamma^2 (c^2 - v^2)$$

$$\gamma^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - v^2/c^2}$$



or $$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{—— (6)}$$

$$\gamma^2 = \frac{1}{1 - v^2/c^2} \quad \text{or} \quad 1 - v^2/c^2 = \frac{1}{\gamma^2}$$

$$1 - \frac{1}{\gamma^2} = \frac{v^2}{c^2} \quad \text{—— (7)}$$

Substituting (7) in (8)

$$t' = \gamma t - \frac{\gamma x}{v} \times \frac{v^2}{c^2} = \gamma t - \gamma x . v/c^2$$

ie $$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \quad \text{—— (8)}$$

Similarly $$t = \frac{t' + x'v/c^2}{\sqrt{1 - v^2/c^2}} \quad \text{—— (9)}$$

Substituting (6) in 1 & 2

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad \text{—— (10)}$$

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \quad \text{—— (11)}$$

$$y' = y \;\; \& \;\; z' = z \quad \text{—— (12)}$$

The equations 8 to 12 are known as Lorentz transformation equations for space and time. The important aspect of this equations is that the measurement in space and time are no longer absolute, but depend upon the frame of reference of the observer.

Cases 1) When $v << c$, then the equations

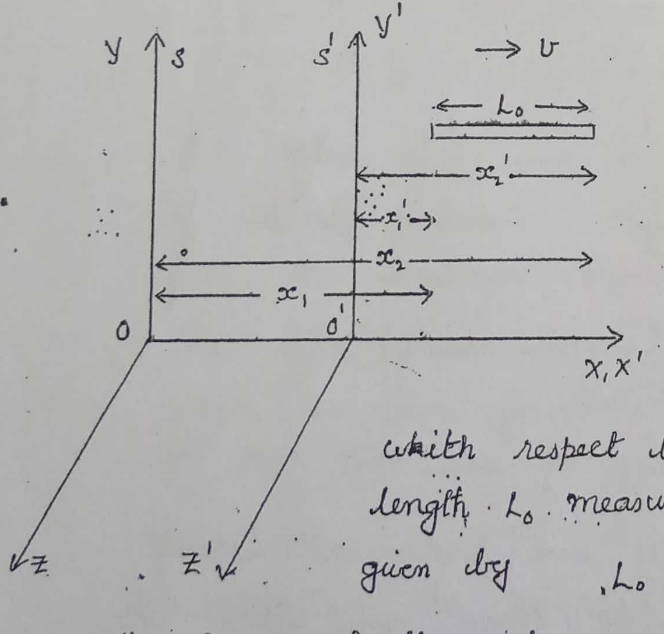
$t = t'$, $x = x' + vt$, $y' = y$ & $z' = z$

which are the classical Galilean transformation equations.

2) When $v > c$, then $t = \infty$. ie this equations limits the maximum velocity of material bodies. According to this, nothing can move with a velocity greater than the velocity of light or v should be always less than c.

Lorentz - Fitzgerald length contraction

According to Fitzgerald length contraction when an object moves with a velocity v relative to a stationary observer, its measured length appeared to be contracted in the direction of motion by a factor $\sqrt{1-v^2/c^2}$.



Consider a frame S' moving with a velocity v, along the +ve x-direction. Let a rod of proper length L_0 be placed, with its length parallel to the x-axis. If x_1' and x_2' are the x coordinates of two end points of the rod with respect to an observer in S'. The proper length L_0 measured by an observer in S' is given by

$$L_0 = x_2' - x_1' \quad \text{——} \quad (1)$$

The length of the rod measured by an observer in stationary frame S is given by

$$L = x_2 - x_1 \quad \text{——} \quad (2)$$

From Lorentz equations $x_2' = \dfrac{x_2 - vt}{\sqrt{1-v^2/c^2}}$ & $x_1' = \dfrac{x_1 - vt}{\sqrt{1-v^2/c^2}}$

Substituting x_2' & x_1' in ①

$$L_0 = \frac{x_2 - vt - x_1 + vt}{\sqrt{1 - v^2/c^2}} = \frac{x_2 - x_1}{\sqrt{1-v^2/c^2}}$$

$$L_0 = \frac{L}{\sqrt{1-v^2/c^2}}$$

$$L = L_0\sqrt{1-v^2/c^2} \quad \text{——— ③}$$

Equation ③ shows that as <u>the rod moves faster, it appears shorter.</u>

If $v = c$, $L = 0$, ie the rod moving with the velocity of light will appear as a point to a stationary observer.

<u>Time Dilation</u>

To derive a relation for time dilation, consider a clock placed in the moving frame S'. The clock is at rest relative to an observer in S' and is moving with a velocity v relative to an observer in stationary frame S. If an observer in S' registered the times of two ticks given by the clock as t_1' and t_2', then the time interval t_0 between the ticks is given by

$$t_0 = t_2' - t_1' \quad \text{——— ①}$$

t_0 is the proper time as it the time interval measured in the coordinate frame in which two events occured. In frame S, these two ticks appeared to occur at different times. An observer in S records the same ticks at times t_1 and t_2. The time interval t is

$$t = t_2 - t_1 \quad \text{——— ②}$$

From Lorentz inverse transformation equation

$$t_2 = \frac{t_2' + x'v/c^2}{\sqrt{1-v^2/c^2}} \quad \& \quad t_1 = \frac{t_1' + x'v/c^2}{\sqrt{1-v^2/c^2}}$$

$$\therefore t = t_2 - t_1 = \frac{t_2' + x'v/c^2 - t_1' - x'v/c^2}{\sqrt{1-v^2/c^2}}$$

$$= \frac{t_2' - t_1'}{\sqrt{1-v^2/c^2}}$$

ie $$t = \frac{t_0}{\sqrt{1-v^2/c^2}}$$

The above equation shows that the time interval appears to be lengthened due to the relative motion by a factor $1/\sqrt{1-v^2/c^2}$ to an observer in frame S. This effect is called time dilation.

Experimental verification of time dilation

Experimental verification of time dilation is found in an experiment on mesons. μ-mesons are created at high altitudes of about 10 km in the earth atmosphere. They are projected towards earth at a high speed of 0.998c. The mesons are unstable having an average life time of 2×10^{-6} sec. In it's lifetime it can travel a distance

$$d = vt = 0.998 \times 3 \times 10^8 \times 2 \times 10^{-6} \cong 600\,m.$$

Then how the μ-mesons can reach the earth surface?

The actual life time of μ-meson is 2×10^{-6} sec in it's own frame of reference. For an observer on the earth surface it's life time will be lengthened due to relativity effect.

$$t = \frac{t_0}{\sqrt{1-v^2/c^2}} = \frac{2 \times 10^{-6}}{\sqrt{1 - \frac{(0.998c)^2}{c^2}}} \cong 3.17 \times 10^{-5}\,sec.$$

In this dilated lifetime, μ-meson can travel a distance

$$d = 3.17 \times 10^{-5} \times 0.998 \times 3 \times 10^8 \cong 10\ km.$$

This explains the presence of μ-mesons on the earth surface despite their brief lifetime.

Relativistic addition of velocities

Suppose a frame of reference S' is moving with velocity v relative to stationary frame S along the +ve direction of x-axis. If a particle moves a distance dx in a time interval dt in S^*, then the velocity of the particle as measured by an observer in S is

$$u = dx/dt \quad \text{—— ①}$$

For an observer in S', the distance and time will be dx' and dt', and the velocity will appear as

$$u' = dx'/dt' \quad \text{—— ②}$$

From Lorentz transformation equations

$$x' = \frac{x - vt}{\sqrt{1-v^2/c^2}} \qquad t' = \frac{t - xv/c^2}{\sqrt{1-v^2/c^2}}$$

Differentiating these equations

$$dx' = \frac{dx - vdt}{\sqrt{1-v^2/c^2}} \qquad dt' = \frac{dt - vdx/c^2}{\sqrt{1-v^2/c^2}} \quad \text{—— ③}$$

Substituting dx' & dt' in ②

$$u' = \frac{dx - vdt}{dt - vdx/c^2} = \frac{(dx/dt) - v}{1 - \frac{v\,dx/dt}{c^2}} = \frac{u - v}{1 - uv/c^2} \quad \text{—— ④}$$

Equation ④ represents the relativistic velocity addition formula. Similarly the velocity of the particle in frame S can be obtained by replacing v by $-v$ in eqn. ④

$$u = \frac{u' + v}{1 + vu'/c^2} \quad \text{—— ⑤}$$

Cases When $u' = c$, ie the moving particle is a photon moving with the velocity of light

$$u = \frac{c + v}{1 + cv/c^2} = \frac{(c+v)c}{(c+v)} = c$$

ie the velocity of light c is same in all inertial frame of reference.

If $u' = c$ and $v = c$, then

$$u = \frac{c+c}{1+c^2/c^2} = c$$

ie the addition of any velocity to the velocity of light simply reproduces the velocity of light. Or the velocity of light is the maximum attainable velocity in nature.

Einstein's mass-energy relation

Several experiments showed that the mass is convertible into energy and energy into mass and the conversion factor between two is the square of speed of light.

Consider a particle of mass m acted upon by a force F in the same direction as it's velocity. The increase in energy of the particle is defined in terms of work. If the force F displaces the particle through a distance dx then work done dW is stored as its kinetic energy dK.

$$dK = dW = F.ds \quad \text{—— (1)}$$

According to Newton's law of motion, force is the rate of change of momentum

$$F = \frac{d}{dt}(p) = \frac{d}{dt}(mv) \quad \text{—— (2)}$$

According to the theory of relativity mass varies with velocity.

$$\therefore \; F = m\frac{dv}{dt} + v\frac{dm}{dt} \quad \text{—— (3)}$$

Putting (3) in (1)

$$dW = dK = v\frac{dm}{dt}ds + m\frac{dv}{dt}ds$$

$$dK = mv\,dv + v^2 dm \qquad \text{—— (4)}$$

But $$m = \frac{m_0}{\sqrt{1-v^2/c^2}} = m_0\left(1-v^2/c^2\right)^{-1/2} \qquad \text{—— (5)}$$

where m_0 is the rest mass of the particle

Differentiating eqn. (5)

$$dm = m_0\left(\frac{-1}{2}\right)\left[1-\frac{v^2}{c^2}\right]^{-3/2}\left(\frac{-2v\,dv}{c^2}\right)$$

$$= \frac{m_0 v\,dv}{c^2\left[1-v^2/c^2\right]^{3/2}} \qquad m_0 = m\left[1-v^2/c^2\right]^{1/2}$$

$$dm = \frac{m\left[1-v^2/c^2\right]^{1/2} v\,dv}{c^2\left[1-v^2/c^2\right]^{3/2}} = \frac{mv\,dv}{c^2\left[1-v^2/c^2\right]} = \frac{mv\,dv}{(c^2-v^2)}$$

$$\therefore\ mv\,dv = (c^2-v^2)\,dm \qquad \text{—— (6)}$$

Substituting (6) in (4)

$$dK = (c^2-v^2)\,dm + v^2 dm = c^2 dm$$

If the change in kinetic energy be K, when the mass changes from m_0 to m

$$K = \int dK = \int_{m_0}^{m} c^2 dm = c^2(m-m_0)$$

or $$\underline{K = (m-m_0)c^2} \qquad \text{—— (7)}$$

Equn. (7) is the relativistic expression for kinetic energy of a particle. The total energy is the sum of kinetic energy of motion and rest mass energy.

$$E = (m-m_0)c^2 + m_0c^2 = \underline{\underline{mc^2}} \qquad \text{—— (8)}$$

Equation ⑧ is the well known mass-energy relation.

Relativistic relation between energy and momentum

The relativistic total energy is

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} \quad — ①$$

Momentum of the particle $p = mv$

$$v = p/m \quad — ②$$

Substituting ② in ①

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{p^2}{m^2c^2}}} = \frac{m_0 c^2}{\sqrt{1-\frac{p^2c^2}{m^2c^4}}}$$

$$= \frac{m_0 c^2}{\sqrt{1-p^2c^2/E^2}}$$

$$E^2\left[1-\frac{p^2c^2}{E^2}\right] = m_0^2c^4$$

$$E^2 - p^2c^2 = m_0^2c^4$$

$$E^2 = m_0^2c^4 + p^2c^2$$

$$E = \sqrt{m_0^2c^4+p^2c^2}$$

Relativistic relation between kinetic energy & momentum

The relativistic expression for kinetic energy

$$K = E - m_0c^2$$

$$E = \sqrt{m_0^2c^4+p^2c^2}$$

$$\therefore K = \sqrt{m_0^2c^4+p^2c^2} - m_0c^2$$

or $K = \left[\left(1+\frac{p^2}{m_0^2c^2}\right)^{1/2} - 1\right] m_0c^2$

If $v << c$, then

$$K \cong m_0c^2\left[\left[1+\frac{1}{2}\frac{p^2}{m_0^2c^2}+\cdots\right]-1\right]$$

$$\cong m_0c^2 \times \frac{1}{2}\frac{p^2}{m_0^2c^2}$$ neglecting higher order terms of binomial expansion

$$\therefore K = \frac{p^2}{2m_0}$$

As $v << c$, $m_0 = m$

$$\therefore K \cong \frac{p^2}{2m}$$

ie for small velocities, the relativistic kinetic energy expression reduces to classical expression.

1. Distinguish between inertial and non-inertial frame of references. Give examples of each.

Inertial frame of reference: Bodies moving in this frame obey Newton's law of motion. eg: Earth

Non-inertial frame of reference: Accelerated frames are called non-inertial frame. In this frame Newton's law does not hold good. eg: rotating merry-go-round.

2. Discuss the negative result of Michelson-Morley experiment and how this result were interpreted.

Michelson-Morley experiment was performed to prove the existence of ether. Theoretical calculations show that if there is a relative velocity between the ether and the earth a fringe shift of 0.2 would occur. No fringe shift was detected and that was the negative result. This shows that the relative velocity between earth and ether is zero and the hypothesis of existence of ether medium was disapproved.

The three main interpretations to explain the negative results are

1) Ether drag hypothesis

2) Lorentz-Fitzgerald contraction hypothesis

3) Constancy of speed of light

3. Give the two basic postulates of special theory of relativity.

The two basic postulates of special theory of relativity are

1. The principle of equivalence which states that the laws of Physics are the same in all inertial frames of reference moving with a constant velocity with respect to one another.

2) The principle of constancy of speed of light states that the speed of light in free space is always a constant equal to c. Its value is independent of the relative motion of the inertial frames, source and the observer.

4. Why do we not observe the effect of time dilation in day to day life?

According to time dilation, $t = \frac{t'}{\sqrt{1-v^2/c^2}}$; t will be greater than t' only when the object is moving with a velocity comparable with the velocity of light. In day to day life phenomena the velocity of all moving objects are much small compared to c and the factor $\frac{1}{\sqrt{1-v^2/c^2}}$ is approximately equal to 1. Then $t = t'$ and time dilation effect was not observed.

5. Does light or photon have mass? If no, then how they have momentum?

Photons have zero rest mass. The relativistic formula for momentum is $p^2 = \left(\frac{E}{c}\right)^2 + (m_0 c^2)^2$.

For photons, $m_0 = 0$

$$p^2 = \frac{E^2}{c^2}$$

$$p = \frac{E}{c}$$

6. Show that no signal can travel faster than light.

According to Lorentz transformation equation

$$t' = \frac{t - xv/c^2}{\sqrt{1-v^2/c^2}}$$

$\therefore$ When $v = c$, $t' = \infty$ which is not possible. This shows that no material body can achieve a velocity greater than or equal to the velocity of light.

7. Define proper length and proper time.

Proper frame is one which moves with a velocity with respect to a stationary frame. In proper frame (s') the observed body is at rest. The length of a rod as measured by an observer (O') in proper frame (s') is called proper length (L').

$$L = L'\sqrt{1-v^2/c^2}$$

L is the length measured by the observer (O) in stationary frame (s).

The time recorded by a clock in the proper frame (s') is called proper time (t').

$$t = \frac{t'}{\sqrt{1-v^2/c^2}}$$

t is the time of the same event as recorded by an observer (O) in stationary frame (s).

8. What are massless particles? Show that the massless particles can exist only if they move with the speed of light.

Massless particles are particles with zero rest mass. According to the relativistic relation, $E^2 = p^2c^2 + m_0^2c^4$.

For massless particles, $m_0 = 0$

$$\therefore E^2 = p^2c^2$$

$$E = pc$$

But $E = mc^2$ and $p = mv$

$$mc^2 = mvc$$

$$\therefore \underline{v = c}$$

This shows that massless particles will travel with velocity equal to velocity of light. eg: photons, neutrinos.

Special Theory of Relativity

1. Show that $x^2+y^2+z^2 = c^2t^2$ is invariant under Lorentz transformations.

Inverse Lorentz transformations are

$$x = \frac{x'+vt'}{\sqrt{1-v^2/c^2}} \; ; \; y = y' \; ; \; z = z' \; , \; t = \frac{t'+x'v/c^2}{\sqrt{1-v^2/c^2}}$$

Substituting these values in $x^2+y^2+z^2 = c^2t^2$

$$\left[\frac{x'+vt'}{\sqrt{1-v^2/c^2}}\right]^2 + y'^2 + z'^2 = c^2\left[\frac{t'+x'v/c^2}{\sqrt{1-v^2/c^2}}\right]^2$$

$$y'^2+z'^2 = \frac{1}{1-v^2/c^2}\left[c^2t'^2+\frac{x'^2v^2}{c^2} - x'^2 - v^2t'^2\right]$$

$$= \frac{1}{1-v^2/c^2}\left[\left(c^2t'^2 - x'^2\right)\left(1-\frac{v^2}{c^2}\right)\right]$$

$$= c^2t'^2 - x'^2$$

or $\quad x'^2+y'^2+z'^2 = c^2t'^2$

Hence $x^2+y^2+z^2 = c^2t^2$ is invariant under Lorentz transformation.

2. A hypothetical train moving with a speed of 0.6c passes through a platform of a small station without being slowed down. The observer on the platform note that the length of the train is just equal to the length of the platform which is 200 m. Find the rest length of the train.

$$L = L'\sqrt{1-v^2/c^2}$$

$$L' = \frac{L}{\sqrt{1-v^2/c^2}}$$

$L = 200\text{ m}$, $v = 0.6c$

$$L' = \frac{200}{\sqrt{1-\left(\frac{0.6c}{c}\right)^2}} = \frac{200}{\sqrt{1-0.36}} = \frac{200}{0.8} = \underline{\underline{250\text{ m}}}$$

3. A person in a train sleeps at 10.00 pm and gets up at 4.00 am by his watch. If the train is moving with a speed of 3×10^7 m/s, how long did he sleep according to the clocks at the stations?

$$t = \frac{t'}{\sqrt{1-v^2/c^2}}$$

$t' = 6$ hours , $v = 3\times10^7$ m/s

$$t = \frac{6}{\sqrt{1-\left(\frac{3\times10^7}{3\times10^8}\right)^2}} = \frac{6}{\sqrt{1-0.001}} = 6.006 \text{ hours}$$

4. A spacecraft is moving relative to the earth. An observer on the earth finds that, according to her clock 3601 s elapse between 1.00 pm and 2.00 pm on the space craft's clock. What is the velocity of the spaceship relative to the earth?

$$t = \frac{t'}{\sqrt{1-v^2/c^2}}$$

$t = 3601$ s , $t' = 3600$ s , $v = ?$

$$3601 = \frac{3600}{\sqrt{1-v^2/c^2}}$$

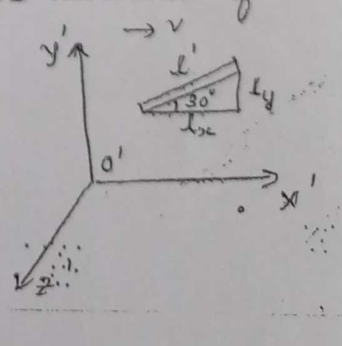
$$1-\frac{v^2}{c^2} = \left(\frac{3600}{3601}\right)^2 = 0.999$$

$$v^2/c^2 = 1-0.999 = 0.001$$

$$\underline{\underline{v = 0.03c}}$$

5. Calculate the length contraction of a metre rod in a frame of reference which is moving with a velocity 0.6c. The rod is inclined at an angle of 30° facing the direction of motion.

The length contracts only in the direction of motion. The component of length along the direction of motion



$$l_x = l'\cos 30\sqrt{1-v^2/c^2}$$

$$= 1\times\cos 30\times\sqrt{1-\left(\frac{0.6c}{c^2}\right)^2} = 0.69$$

$$l_y = l \sin 30 = 0.5$$

$$l = \sqrt{l_x^2 + l_y^2} = \sqrt{(0.69)^2 + (0.5)^2} = \underline{0.85\ m}$$

6. Find the velocity that an electron which must be given so that its momentum is 11 times its rest mass times the speed of light. What is the energy at this speed?

Relativistic momentum, $p = mv = \dfrac{m_0 v}{\sqrt{1 - v^2/c^2}}$

But $p = 11 m_0 c$

$$m_0 c = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

$$v = \underline{2.99 \times 10^8 m/s}$$

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \frac{9.1 \times 10^{-31}}{\sqrt{1 - (0.996)^2}} = 10.22 \times 10^{-30} kg$$

$$E = mc^2 = 10.22 \times 10^{-30} \times (3 \times 10^8)^2 = \underline{9.13 \times 10^{-13} J}$$

7. What is the total energy of a 2.5 MeV electron?

$$\text{Total energy} = \text{Kinetic energy} + m_0 c^2$$

$$= 2.5\ MeV + \frac{9.1 \times 10^{-31} \times (3 \times 10^8)^2}{1.6 \times 10^{-19}}$$

$$= 2.5\ MeV + 0.511\ MeV = \underline{3.011\ MeV}$$

8. A spaceship moving away from the earth with velocity 0.6c fires a rocket whose velocity relative to the spaceship is a) 0.8c away from the earth b) towards the earth. What will be the velocity of the rocket as observed from the earth in the two cases.

$$u = \frac{u' + v}{1 + u'v/c^2}$$

Away from earth : $u' = 0.8c$, $v = 0.6c$

$$u = \frac{0.8c + 0.6c}{1 + \frac{0.8c \times 0.6c}{c^2}} = \frac{1.4c}{1.48} = \underline{0.94c}$$

Towards the earth

$$u' = -0.8c \ , \ v = 0.6c$$

$$u = \frac{-0.8c + 0.6c}{1 + \frac{0.6c \times -0.8c}{c^2}} = \frac{-0.2c}{1 + \frac{-0.48c^2}{c^2}} = \frac{-0.2c}{0.52}$$

$= 0.38c$ towards the earth.

9. Calculate the percentage contraction in the length of rod in a frame of reference moving with velocity $0.8c$ in a direction parallel to its length.

$$L = L'\sqrt{1 - v^2/c^2}$$

$$= L'\sqrt{1 - (0.8c/c)^2} = L' \times 0.6 = \underline{\underline{0.6L'}}$$

Percentage contraction $= \frac{L' - L}{L'} \times 100 = \frac{L' - 0.6L'}{L'} = 40\%$

10. Calculate the amount of work to be done to increase the speed of a electron from $0.8c$ to $0.9c$. Given the rest energy of electron $= 0.5$ MeV.

$$m_0c^2 = 0.5 \text{ MeV} = 0.5 \times 10^6 \times 1.6 \times 10^{-19} = 8 \times 10^{-14} J$$

$v_1 = 0.8c$ and $v_2 = 0.9c$

When velocity is $0.8c$, kinetic energy will be

$$K_1 = m_0c^2\left[\left(1 - v^2/c^2\right)^{-1/2} - 1\right] = 5.33 \times 10^{-14} J$$

When velocity is $0.9c$, kinetic energy will be

$$K_2 = m_0c^2\left[\left(1 - v^2/c^2\right)^{-1/2} - 1\right] = 10.16 \times 10^{-14} J$$

$\therefore$ Work done in increasing the speed from $0.8c$ to $0.9c$ is given by

$$W = K_2 - K_1 = (10.16 \times 5.33) \times 10^{-14} = \underline{\underline{4.83 \times 10^{-14} J}}$$

Note: Relativistic kinetic energy $= (m - m_0)c^2$

$$= \left(\frac{m_0}{\sqrt{1 - v^2/c^2}} - m_0\right)c^2$$

$$= \underline{\underline{m_0c^2\left[\left(1 - v^2/c^2\right)^{-1} - 1\right]}}$$