## Multirate Digital Signal Processing :-

A multirate system can increase or decrease the sampling rate of individual signals.
The signal processing that uses more than one sampling rate to perform the desired operations is known as multirate signal processing.

### Advantages of multirate Digital Signal Processing :-

1. Reduced Computational Complexity.
2. Reduced Transmission data rate.
3. Less memory requirements.
4. Lower order filter design.
5. Lower Coefficient sensitivity and noise.

audio processing, Narrowband filtering
video PAL, NTSC

**Decimation:-** Decreasing the sampling rate of a signal is called decimation, which consists of an anti-aliasing LPF followed by downsampling.

### Time domain Characterisation:-

The sampling rate of a signal $x(n)$ is decreased by a factor M if we pick only every $M$th sample of the signal and discarding the rest.
This process is known as downsampling by a factor of M.

Let $x(n)$ be the input and $y(n)$ be the output of a downsampler. Therefore

$$y(n) = x(Mn) \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$x(n) = x(nT_s) \longrightarrow \boxed{\downarrow M} \longrightarrow y(n) = x(nT_s') = x(nMT_s)$$

Input sampling freq. $= F_s = \frac{1}{T_s}$

Output sampling frequency $= F_s' = \frac{F_s}{M} = \frac{1}{MT_s}$

Sampling rate of input signal $= F_s$

Sampling interval of $x(n) = T_s$

Sampling rate of o/p signal $y(n) = F_s' = \frac{F_s}{M}$.

Sampling interval of $y(n) = T_s' = MT_s$

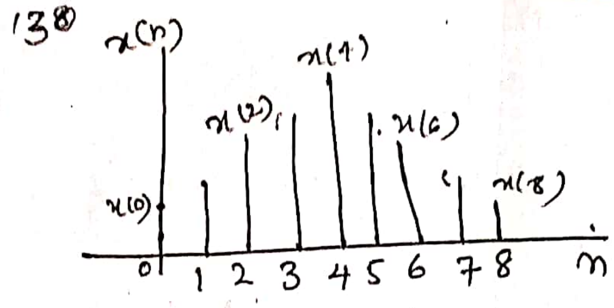
For example, let $x(n)$ be the input signal with $F_s = 10$ samples/s and $T_s = 0.1$ s and downsampling factor $M = 2$.

The downsampled signal is $y(n) = x(2n)$

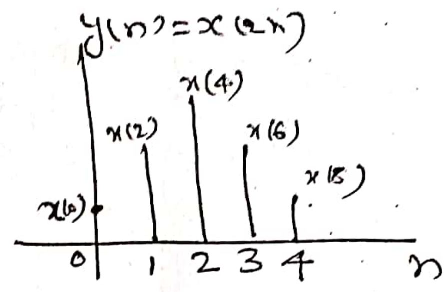
with $F_s' = \frac{F_s}{2} = 5$ samples/s, $T_s' = MT_s = 0.2$ s

$$x(n) = \{\ldots, x(-3), x(-2), x(-1), \underset{\uparrow}{x(0)}, x(1), x(2), x(3), x(4), x(5), x(6), x(7), \ldots\}$$

$$y(n) = x(2n) = \{\ldots, x(-4), x(-2), \underset{\uparrow}{x(0)}, x(2), x(4), x(6) \ldots\}$$




Input signal




Downsampled signal by M=2.

## Frequency domain Characterisation :-

We now derive the relationship between the input and output of a downsampler in z-domain and frequency domain. z-transform of y(n) is given as

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(Mn) z^{-n}$$

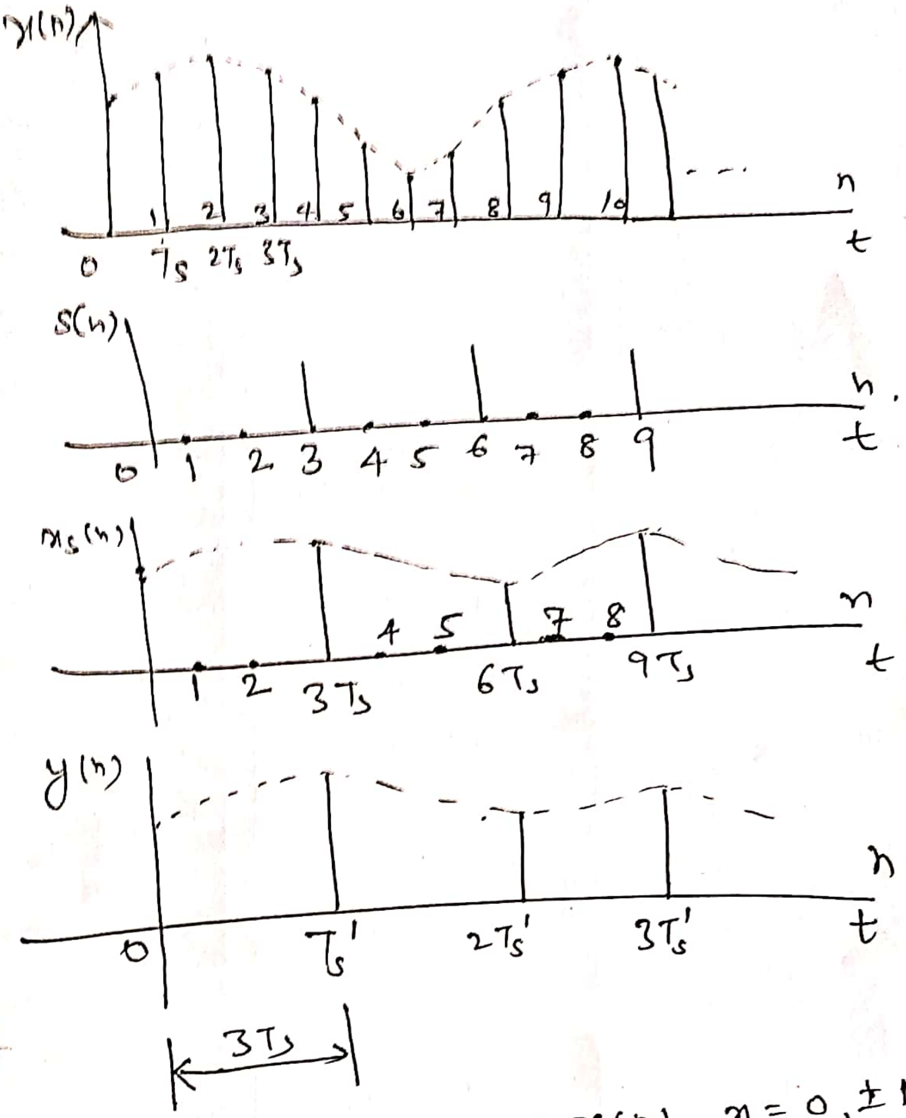
$$Y(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k/M} \quad -①$$

This step is invalid because the downsampler will retain only those samples of $x(k)$ that occurs at $k = 0, \pm M, \pm 2M$ --- and discarding the rest. Therefore, we cannot directly express $Y(z)$ in terms of $X(z)$ using this equation ①.

To overcome this problem, downsampling can be done in two steps:

a) Obtain a discrete time sampled signal $x_s(n)$ by multiplying $x(n)$ by an impulse train $s(n)$ and

b) The downsampled signal is obtained by simply leaving out the M-1 zeros between each sample.




Aliasing

$$x_s(n) = x(n)\,s(n) = \begin{cases} x(n) & n = 0, \pm M, \pm 2M \\ 0 & \text{otherwise.} \end{cases}$$

$$y(n) = x_s(Mn) = x(Mn), \quad n = 0, \pm 1, \pm 2, \ldots$$

Now $$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n} = \sum_{n=-\infty}^{\infty} x_s(Mn) z^{-n}$$

$$Y(z) = \sum_{k=-\infty}^{\infty} x_s(k) z^{-k/M}$$

$$Y(z) = X_s(z^{1/M}) \quad \text{--- (1)}$$

3Q Express $X_s(z)$ in terms of $X(z)$.

$s(n)$ is a periodic impulse train with period M.

$$\therefore \quad s(n) = \sum_{k=-\infty}^{\infty} \delta(n-kM) = \begin{cases} 1 & n=0;\pm M, \pm 2M \\ 0 & \text{otherwise} \end{cases}$$

Using discrete time Fourier series (DTFS), $s(n)$ can be expressed as

$$s(n) = \sum_{k=0}^{M-1} S_k e^{jk\omega_0 n} = \sum_{k=0}^{M-1} S_k e^{j2\pi nk/M}$$

where $\omega_0 = \frac{2\pi}{M}$, and the DTFS coefficients is

$$S_k = \frac{1}{M} \sum_{n=0}^{M-1} s(n) e^{-jk\omega_0 n} = \frac{1}{M} \sum_{n=0}^{M-1} \delta(n) e^{-jk\omega_0 n} = \frac{1}{M}$$

Thus, the periodic sequence $s(n)$ can be expressed as

$$s(n) = \frac{1}{M} \sum_{k=0}^{M-1} e^{j2\pi nk/M} = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{-kn}$$

$$W_M = e^{-j\frac{2\pi}{M}}$$

$$X_s(z) = \sum_{n=-\infty}^{\infty} x_s(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) s(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left( \frac{1}{M} \sum_{k=0}^{M-1} W_M^{-kn} \right) z^{-n}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} \cancel{x(n)} \left( \sum_{n=-\infty}^{\infty} x(n) \left( z W_M^k \right)^{-n} \right)$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X(z W_M^k) \quad \text{—— } (3)$$

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Putting (3) in (2) we get

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{1/M} W_M^{k}\right)$$

Substituting $z = e^{j\omega}$, we get

$$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\frac{\omega}{M}} e^{-j\frac{2\pi k}{M}}\right)$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j(\omega - 2\pi k)/M}\right)$$

for $M = 3$, we get

$$Y(e^{j\omega}) = \frac{1}{3} \sum_{k=0}^{2} X\left(e^{j(\omega-2\pi k)/3}\right)$$

$$= \frac{1}{3}\left[X\left(e^{j\frac{\omega}{3}}\right) + X\left(e^{j\left(\frac{\omega-2\pi}{3}\right)}\right) + X\left(e^{j\left(\frac{\omega-4\pi}{3}\right)}\right)\right]$$

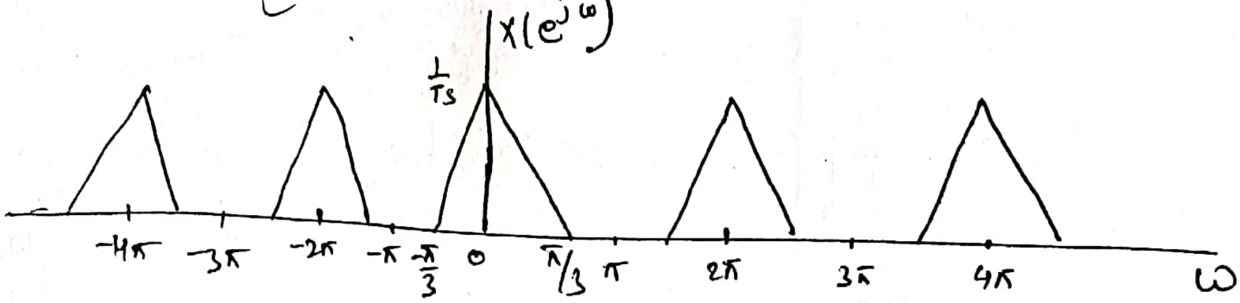
This topic is continued after the completion of <u>3rd Unit.</u>

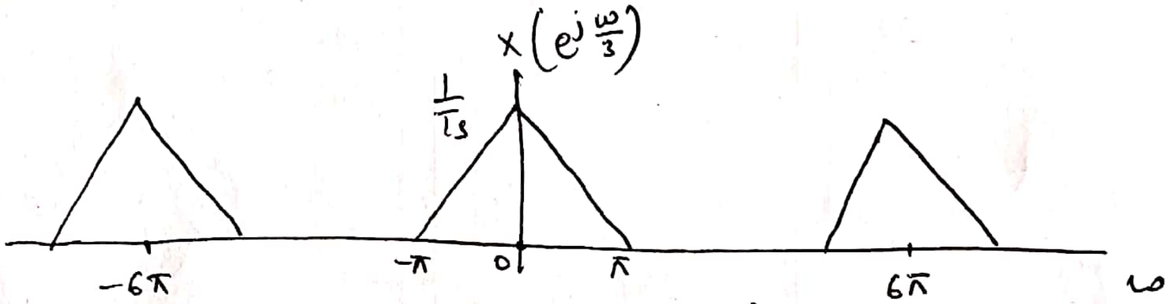
## Decimation Continued :-

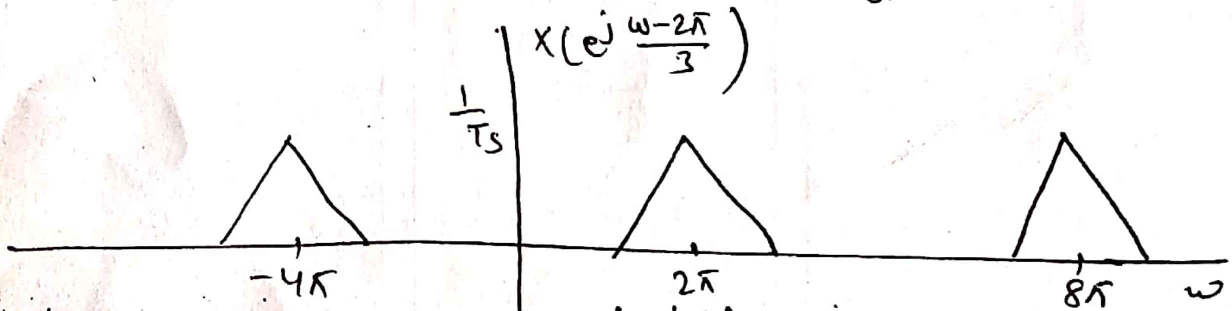
For $M = 3$

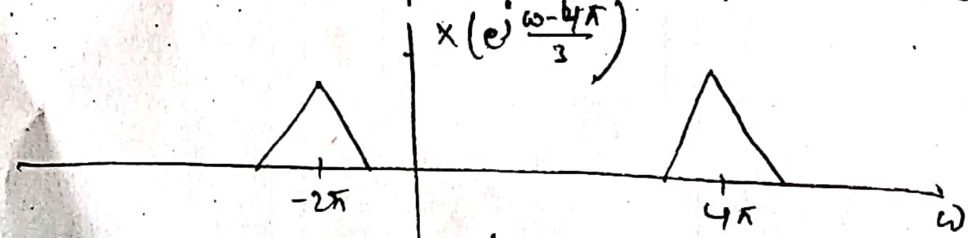
$$Y(e^{j\omega}) = \frac{1}{3}\left[\sum_{k=0}^{2} X\left(e^{j(\omega-2\pi k)/3}\right)\right]$$

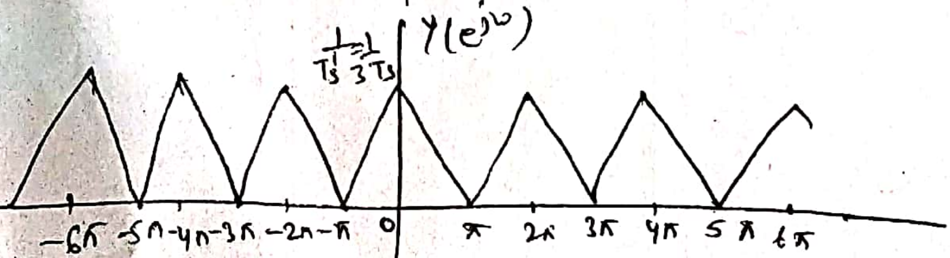
$$= \frac{1}{3}\left[X\left(e^{j\frac{\omega}{3}}\right) + X\left(e^{j\frac{\omega-2\pi}{3}}\right) + X\left(e^{j\frac{\omega-4\pi}{3}}\right)\right]$$



Aliasing Effect :- We can observe from figure of the downsampled signal that if the signal $x(n)$ is not bandlimited to $\frac{\pi}{3}$, then the stretched version $X(e^{j\frac{\omega}{3}})$ will overlap with its shifted replica. This overlapping is known as aliasing.

As an example of downsampling with aliasing is shown below:

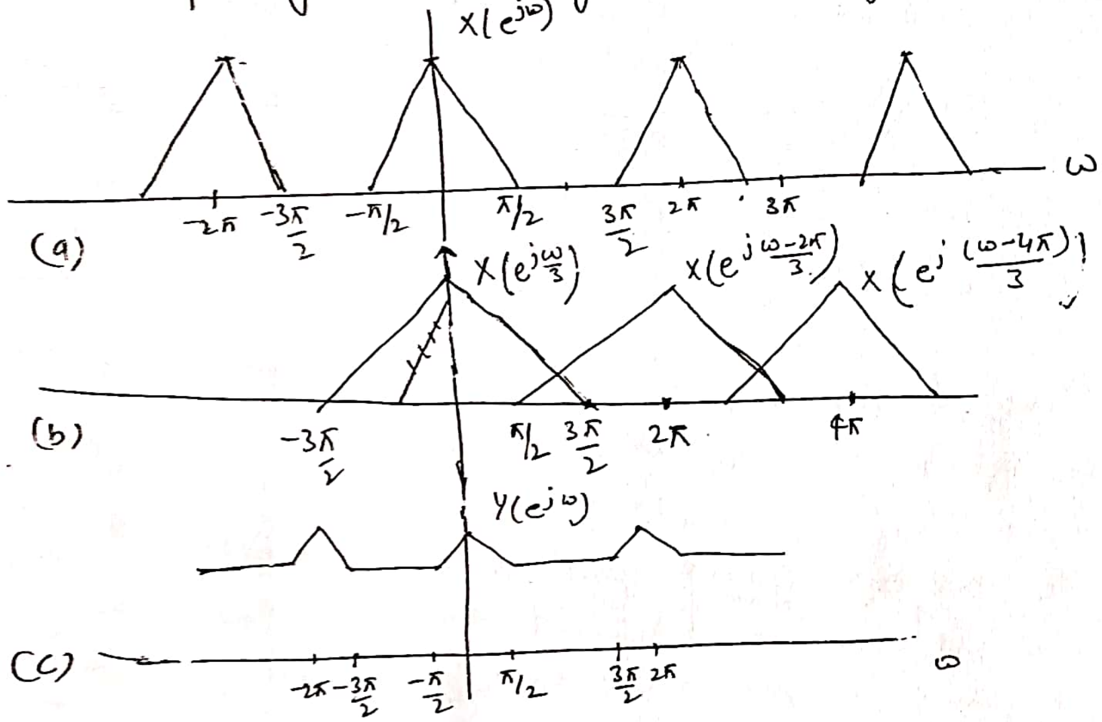



Fig. a shows the spectrum of the signal $x(n)$, which is bandlimited to $|\omega| \leq \frac{\pi}{2}$. (b) shows downsampling by $M=3$. Spectrum of downsampled signal is shown in fig. c.

According to Nyquist sampling theorem, aliasing can be avoided if the sampling frequency $F_s \geq 2 f_m$ i.e twice the maximum freq. present in the signal or the maximum frequency component of the signal is less than the folding frequency, i.e $F < \frac{F_s}{2}$.

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After downsampling by a factor of M, the new sampling interval becomes $T_s' = MT_s$, and the new sampling rate $F_s' = \frac{F_s}{M}$. Hence, the folding frequency after downsampling becomes $\frac{F_s'}{2} = \frac{F_s}{2M}$ i.e after downsampling the new folding frequency is decreased by M.

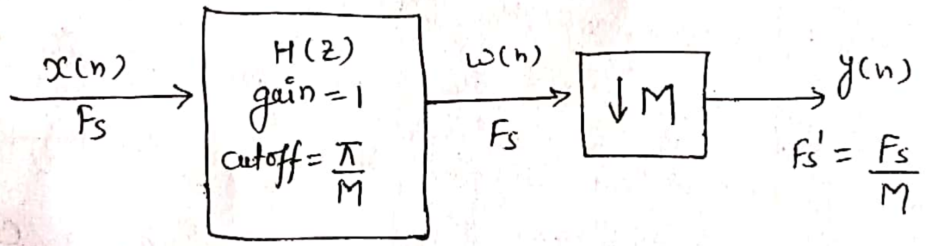
* Thus, aliasing occur if $F > \frac{F_s'}{2}$. Aliasing can be avoided if $F \leq \frac{F_s'}{2}$ i.e $F \leq \frac{F_s}{2M}$, or $\Omega = 2\pi F = \frac{\pi F_s}{M}$.

The equivalent digital frequency is

$$\omega = \Omega T_s = \frac{\pi F_s T_s}{M} = \frac{\pi}{M}$$

Thus, aliasing due to factor M downsampling can be avoided if $x(n)$ is bandlimited to $|\omega| \leq \frac{\pi}{M}$

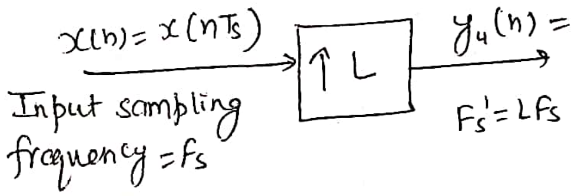
* To prevent aliasing due to downsampling, the digital LPF is used to bandlimit the input signal before applying it to the input of downsampler. This is shown in below Figure.



**Interpolation :-** Increasing the sampling rate of a signal is called interpolation, which consist of upsampling followed by anti-imaging low-pass filtering.

**Time-domain Characterization :-**

The sampling rate of a signal $x(n)$ is increased by a factor $L$ if we insert $L-1$ zero-valued samples between every two consecutive samples, which are known as upsampling by a factor of $L$. Here $L$ is integer.




$$y_u(n) = x(nT_s') = \begin{cases} x\left(n\frac{T_s}{L}\right), & n = 0, \pm L, \pm 2L \\ 0, & \text{otherwise} \end{cases}$$

The output $y_u(n)$ is given as

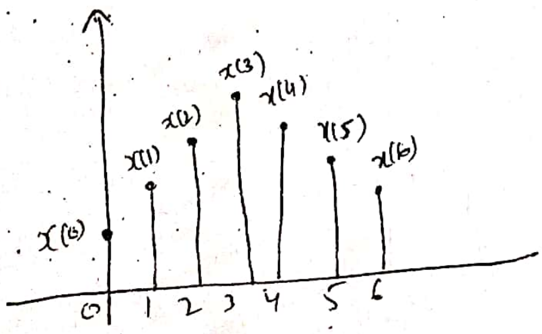
$$y_u(n) = \begin{cases} x\left(\frac{n}{L}\right) & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

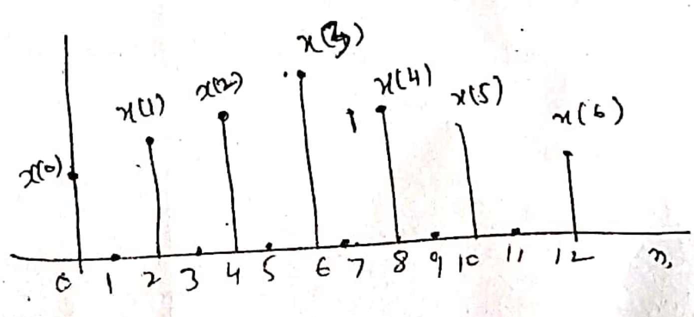
Sampling rate of the input signal $= F_s$

Sampling interval of $x(n) = T_s$

Sampling rate of the output signal $y_u(n) = F_s' = LF_s$

Sampling ~~rate~~ interval of $y_u(n) = T_s' = \frac{T_s}{L}$.



## Frequency domain Characterization :-

$$Y_u(z) = \sum_{n=-\infty}^{\infty} y_u(n) z^{-n} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{L}\right) z^{-n}$$

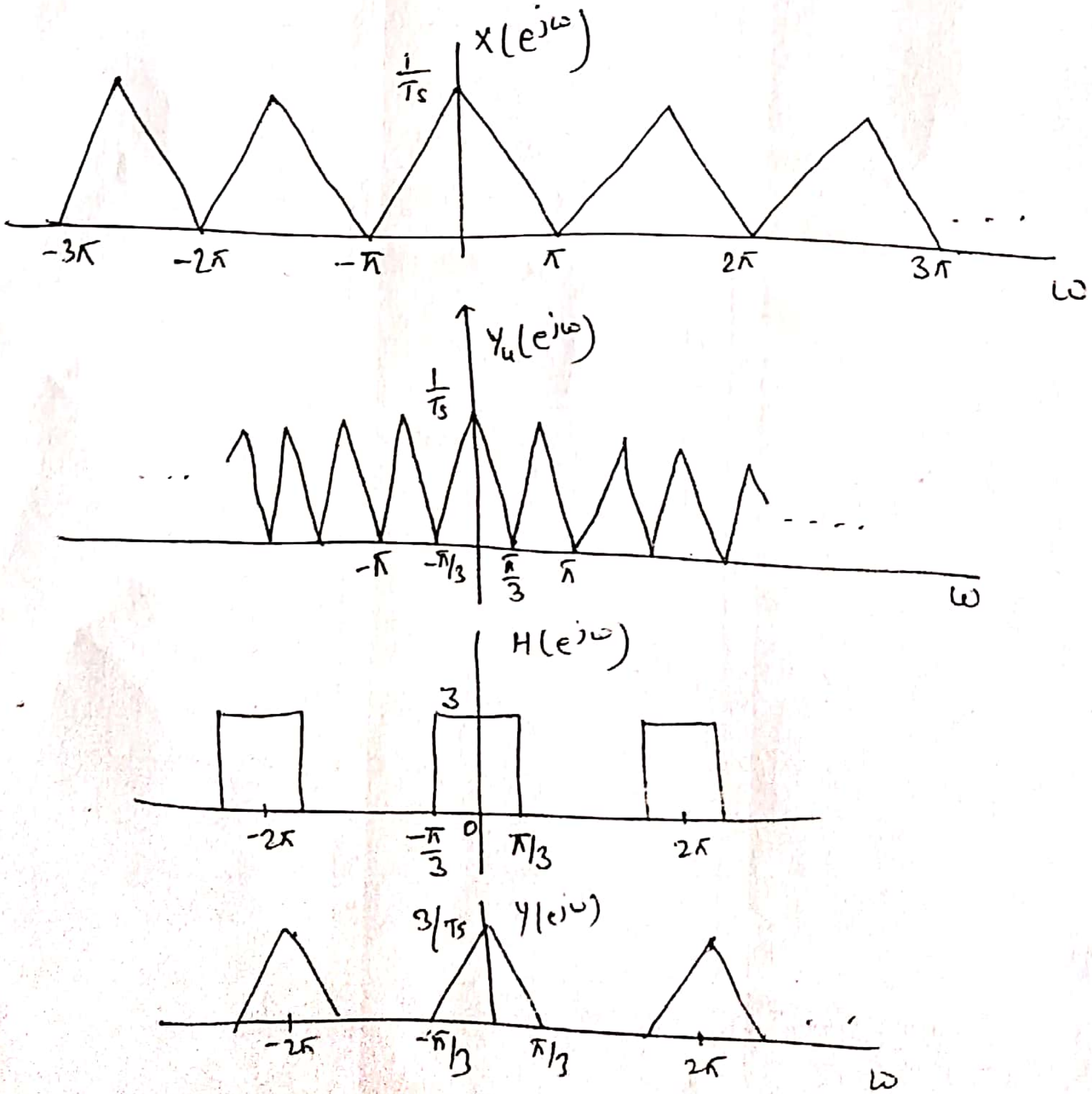
$$= \sum_{r=-\infty}^{\infty} x(r) z^{-Lr} = X(z^L)$$

Putting $z = e^{j\omega}$, we get

$$Y_u(e^{j\omega}) = X(e^{j\omega L})$$

For $L=3$, we get

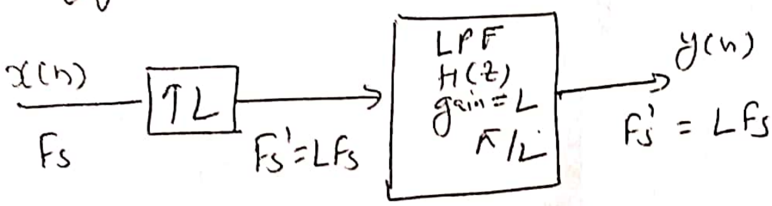
$$Y_u(e^{j\omega}) = X(e^{j3\omega})$$



Upsampling compresses each spectral component; therefore aliasing do not exist in upsampling.

* Also the upsampled signal spectrum $Y_u(e^{j\omega})$ has three spectral component in the interval $(-\pi, \pi)$. However $X(e^{j\omega})$ has only one spectral component. These $(L-1) = (3-1) = 2$ extra spectral components are called images and the phenomenon is called <u>imaging</u>.

$\therefore$ $Y_u(e^{j\omega})$ is passed through a LPF anti-imaging filter with cutoff frequency $\frac{\pi}{L} = \frac{\pi}{3}$ as shown in the figure.




## <u>Sampling Rate conversion by a rational factor $\frac{L}{M}$</u>

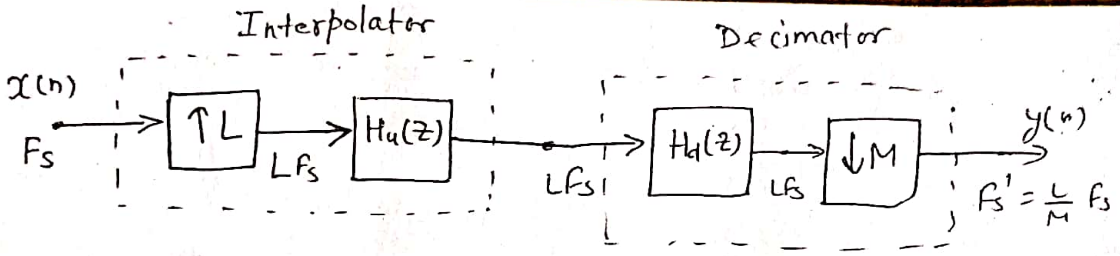
Sampling rate conversion $\frac{L}{M}$ is achieved by a cascading factor of L interpolator and a factor of M decimator. The sampling rate of the output signal $y(n)$ is given as.

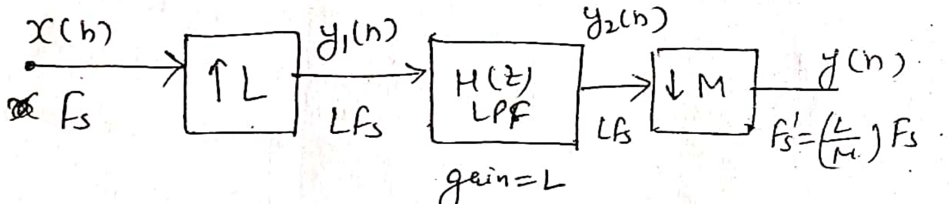
$$F_s' = \left(\frac{L}{M}\right) F_s$$

This scheme (interpolation followed by decimation) is preferred for the following reasons:

a) The interpolation is done first in order to preserve original spectral characterstics of $x(n)$

b) The two LPF's $H_u(z)$ and $H_d(z)$ can be combined into a single LPF because they operate at the same sampling rate.




$$\text{Cutoff} = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$$

In time domain, the relations are as follows

$$y_1(n) = x\left(\frac{n}{L}\right)$$

$$y_2(n) = y_1(n) * h(n) = \sum_{k=-\infty}^{\infty} y_1(k) h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x\left(\frac{k}{L}\right) h(n-k) = \sum_{r=-\infty}^{\infty} x(r) h(n-rL)$$

The output $y(n) = y_2(Mn) = \sum_{r=-\infty}^{\infty} x(r) h(nM - rL)$

<u>For. Example</u>, if $\frac{L}{M} = 2/3$

$$x(n) = \{5, 3, 7, -1, 2, -6, 4, 9, 2\}$$

$$y_1(n) = x\left(\frac{n}{2}\right) = \{\underset{\uparrow}{5}, 0, 3, 0, 7, 0, -1, 0, 2, 0, -6, 0, 4, 0, 9, 0, 2\}$$

$$y_2(n) = y_1(n) * h(n) = \{5, I_1, 3, I_2, 7, I_3, -1, I_4, 2, \ldots\}$$

$$y(n) = y_2(3n) = \{5, I_2, -1, I_5 \ldots\}$$

where $I_1, I_2, I_3, \ldots$ are the interpolated values.

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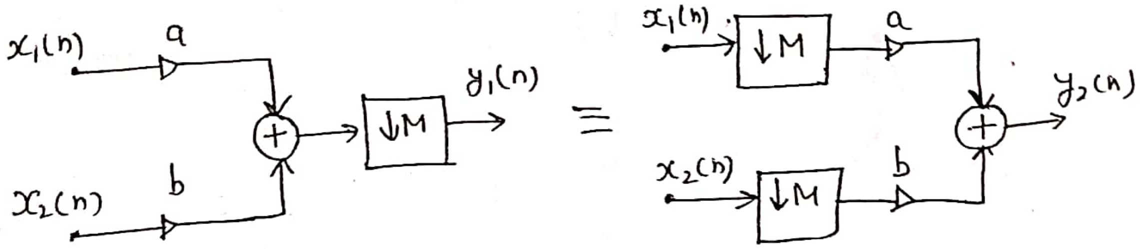
# IDENTITIES

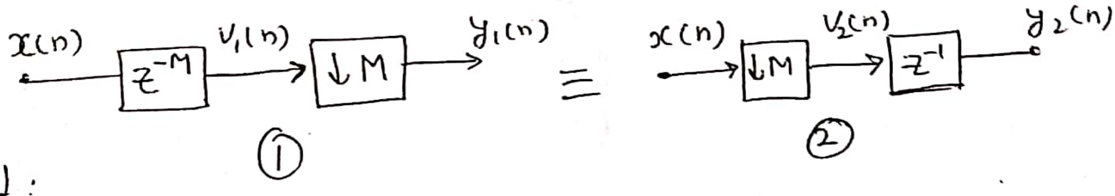
## Identities for the Downsampling :-

1) Down sampler commute with addition and multiplicat-ion.




$$y_1(n) = y_2(n) = a\,x_1(Mn) + b\,x_2(Mn)$$

2)




Proof:

Consider fig 1, we get

$$V_1(z) = z^{-M} X(z)$$

$$Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} V_1\left(z^{1/M} W_M^k\right)$$

$$Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1}\left(z^{-M/M} W_M^{-kM}\right) X\left(z^{1/M} W_M^k\right)$$

$$Y_1(z) = z^{-1}\,\frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M} W_M^k\right)$$

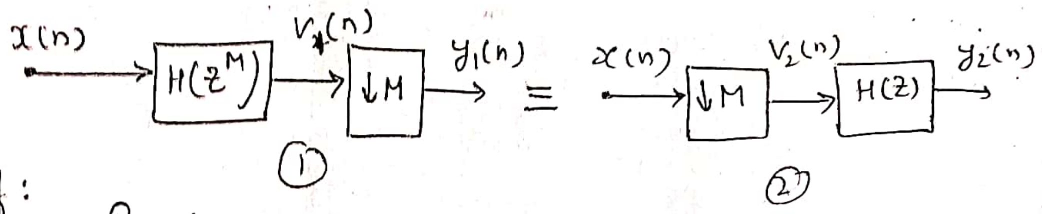
Now consider fig 2, we get

$$V_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M} W_M^k\right)$$

$$Y_2(z) = z^{-1} V_2(z) = z^{-1}\,\frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M} W_M^k\right)$$

Hence $\boxed{Y_1(z) = Y_2(z)}$

3.




Proof:

Consider figure ①, we get

$$V_1(z) = H(z^M)\,X(z)$$

$$Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} V_1\left(z^{1/M} W_M^k\right)$$

$$Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} H\left(z^{M/M} \underbrace{W_M^{kM}}_{1}\right) X\left(z^{1/M} W_M^k\right)$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M} W_M^k\right) H(z)$$

Now, consider, fig. ②, we get.

$$V_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M} W_M^k\right)$$

$$Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M} W_M^k\right) H(z)$$

Hence $\underline{Y_1(z) = Y_2(z)}$

## Identities for Upsampling :-

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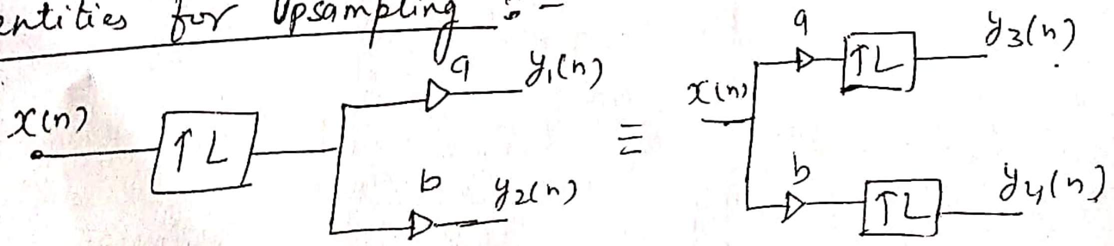


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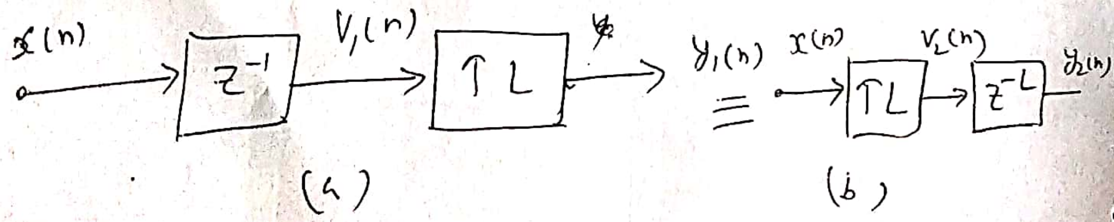


Proof: From fig ⓐ

$$V_1(z) = z^{-1} X(z)$$

$$Y_1(z) = V_1(z^L) = z^{-L} X(z^L) \quad \text{— (A)}$$

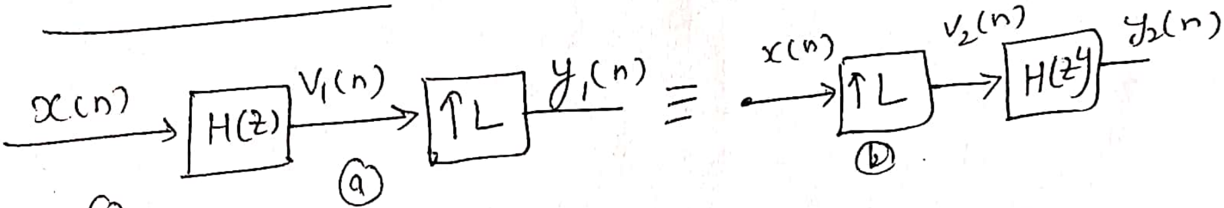
From fig. ⓑ, we get.

$$V_2(z) = X(z^L)$$

$$Y_2(z) = z^{-L} X(z^L) \quad \text{— (B)}$$

Hence (A) = (B)

6)




from ⓐ

$$V_1(z) = H(z) X(z)$$

$$Y_1(z) = V_1(z^L) = H(z^L) X(z^L)$$

from ⓑ $V_2(z) = X(z^L)$

$$Y_2(z) = H(z^L) X(z^L)$$

Exa

## Example ①. Use noble Identities to solve the system

$$x(n) \to \boxed{\uparrow 4} \to \boxed{1+z^{-4}} \to \boxed{1-z^{-4}} \to \boxed{\downarrow 2} \to \boxed{1+z^{-2}} \to \boxed{\downarrow 2} \to y(n)$$

$$x(n) \to \boxed{\uparrow 4} \to \boxed{1+z^{-4}} \to \boxed{1-z^{-4}} \to \boxed{1+z^{-4}} \to \boxed{\downarrow 2} \to \boxed{\downarrow 2} \to y(n)$$

$$x(n) \to \boxed{1+z^{-1}} \to \boxed{1-z^{-1}} \to \boxed{1+z^{-1}} \to \boxed{\uparrow 4} \to \boxed{\downarrow 4} \to y(n)$$

$$x(n) \to \boxed{1+z^{-1}-z^{-2}-z^{-3}} \to y(n)$$

②

$$x(n) \to \boxed{z^{-4}} \to \boxed{\downarrow 2} \to \boxed{\uparrow 3} \to \boxed{\downarrow 2} \to y(n)$$

$$x(n) \to \boxed{z^{-4}} \to \boxed{\uparrow 3} \to \boxed{\downarrow 4} \to y(n)$$

$$x(n) \to \boxed{\uparrow 3} \xrightarrow{v_1(n)} \boxed{z^{-12}} \xrightarrow{v_2(n)} \boxed{\downarrow 4} \to y(n)$$

$$V_1(z) = X(z^3)$$

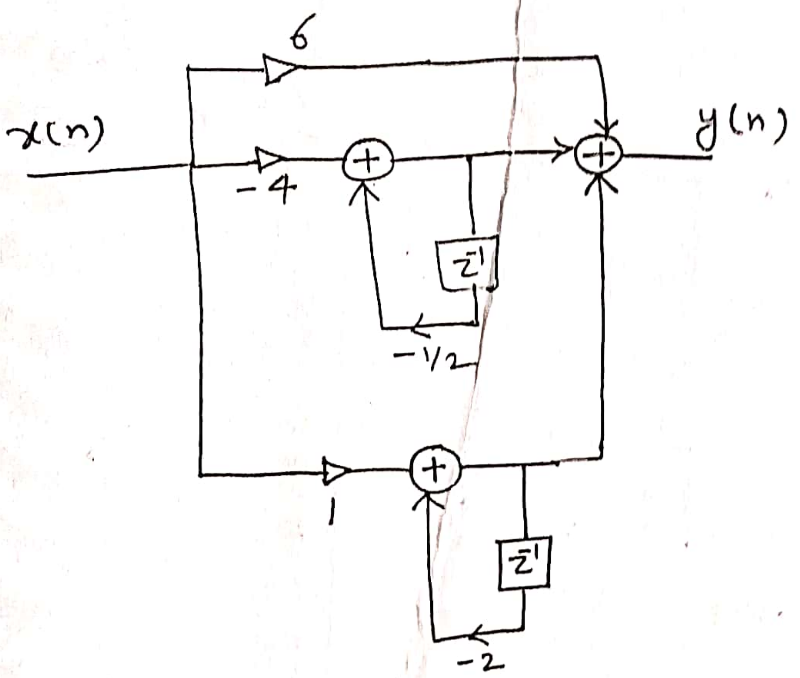
$$V_2(z) = z^{-12} V_1(z) = z^{-12} X(z^3)$$

$$Y(z) = \frac{1}{4} \sum_{k=0}^{3} V_2(z^{1/4} W_4^k)$$

$$= \frac{1}{4} \sum_{k=0}^{3} z^{-12/4} \underbrace{W_4^{-12k}}_{1} X(z^{3/4} W_4^{3k})$$

$$= \frac{1}{4} \sum_{k=0}^{3} z^{-3} X(z^{3/4} W_4^{3k})$$

The parallel realisation of this T.F is shown as




## Lattice Structure of IIR system :-

Let us consider an all-pole system with system function

$$H(z) = \frac{1}{1+\sum_{k=1}^{N} a_N(k)\, z^{-k}} = \frac{1}{A_N(z)} \quad \text{—} (1)$$

The difference equation for this IIR system is

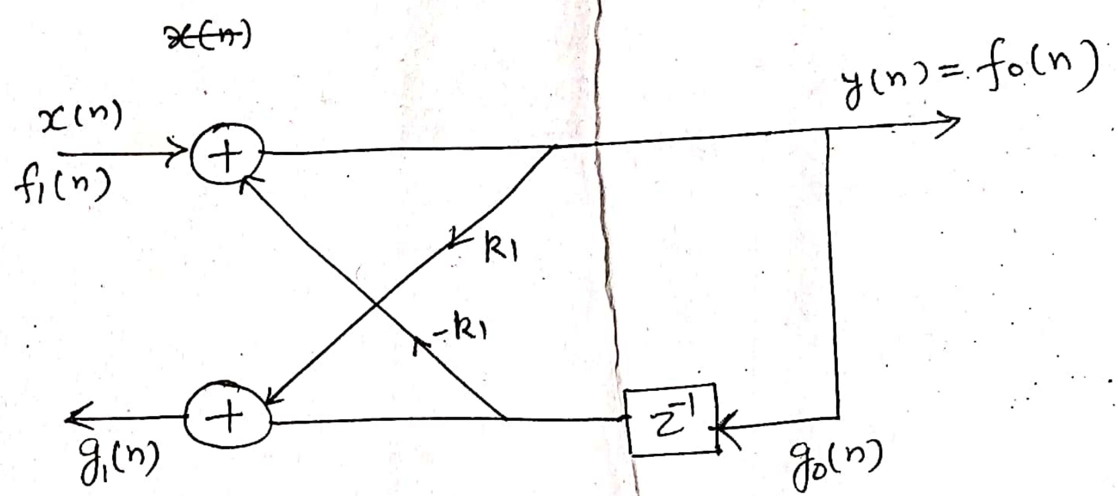
$$y(n) = -\sum_{k=1}^{N} a_N(k)\, y(n-k) + x(n) \quad \text{—} (2)$$

or $$x(n) = y(n) + \sum_{k=1}^{N} a_N(k)\, y(n-k) \quad \text{—} (3)$$

For $N=1$

$$x(n) = y(n) + a_N(1)\, y(n-1) \quad \text{—} (4)$$

Equation (4) can be realised in lattice structure as shown below




From this we obtain

$$x(n) = f_1(n)$$

$$y(n) = f_0(n) = f_1(n) - k_1\, g_0(n-1)$$

$$y(n) = x(n) - k_1\, y(n-1)$$

$$\Rightarrow \quad x(n) = y(n) + k_1\, y(n-1) \quad \text{—— (5)}$$

$$g_1(n) = k_1 f_0(n) + g_0(n-1)$$

$$g_1(n) = k_1\, y(n) + y(n-1) \quad \text{—— (6.)}$$

By comparing equation (4) and (5) we get :-

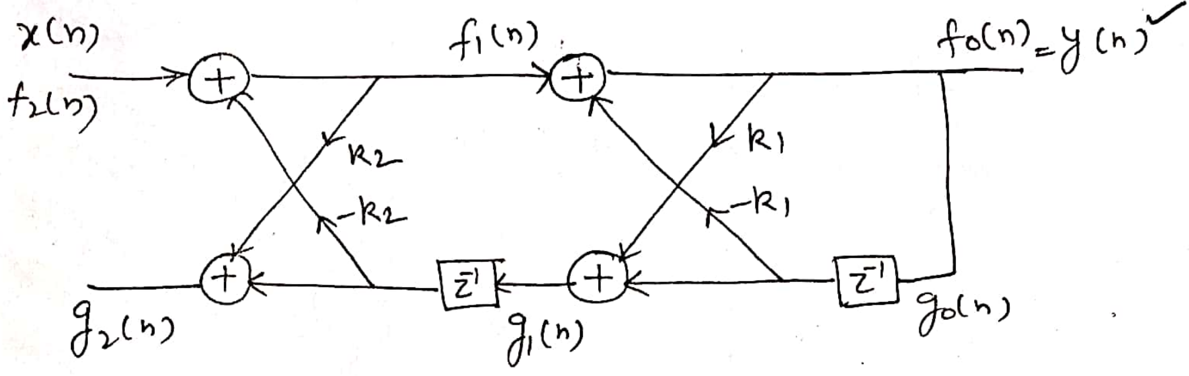
$$k_1 = a_1(1) \quad \text{—— (7)}$$

Now, let us consider for $N = 2$.

$$x(n) = f_2(n) \quad \text{- - - - - — 8.a}$$

$$y(n) = x(n) - a_2(1)\, y(n-1) - a_2(2)\, y(n-2) \quad \text{(8)}$$

8.b




$$f_2(n) = x(n) \quad \text{— } (9)$$

$$f_1(n) = f_2(n) - k_2 g_1(n-1) \quad \text{— } (10)$$

$$g_2(n) = k_2 f_1(n) + g_1(n-1) \quad \text{— } (11)$$

$$f_0(n) = f_1(n) - k_1 g_0(n-1) \quad \text{— } (12)$$

$$g_1(n) = k_1 f_0(n) + g_0(n-1) \quad \text{— } (13)$$

$$y(n) = f_0(n) = g_0(n) \quad \text{— } (14)$$

$$y(n) = f_1(n) - k_1 g_0(n-1)$$

$$= f_2(n) - k_2 g_1(n-1) - k_1 g_0(n-1)$$

$$y(n) = f_2(n) - k_2 [k_1 f_0(n-1) + g_0(n-2)] - k_1 g_0(n-1)$$

using equation (14), we get

$$\cancel{y(n) = x(n) - k_1 k_2 y(n-1) - k_1 y(n-1) +}$$

$$y(n) = x(n) - k_1 k_2 y(n-1) - k_2 y(n-2) - k_1 y(n-1)$$

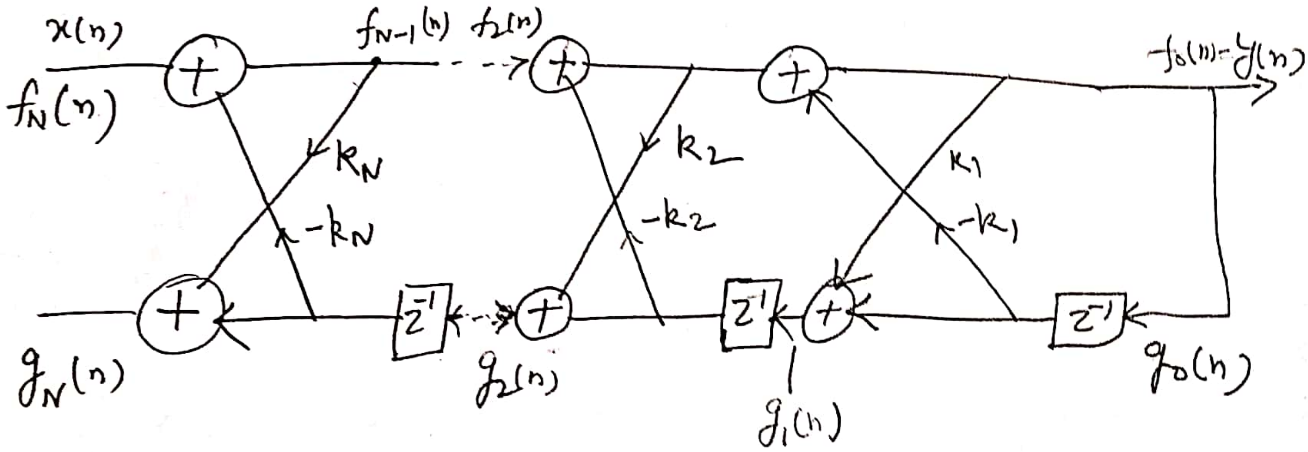
or $$y(n) = x(n) - k_1(1+k_2) y(n-1) - k_2 y(n-2) \quad \text{— } (15)$$

similarly $g_2(n) = k_2 y(n) + k_1(1+k_2)y(n-1) + y(n-2)$ — (16)

comparing equation (8.b) and (15)

$$\left.\begin{aligned} a_2(0) &= 1 \\ a_2(1) &= k_1(1+k_2) \\ a_2(2) &= k_2 \end{aligned}\right\} \quad (17)$$

For a N-stage IIR filter realized in lattice structure as shown below:




From this we get

$$f_N(n) = x(n) \quad \text{— (18)}$$

$$f_{m-1}(n) = f_m(n) - k_m g_{m-1}(n-1), \quad m = N, N-1 \ldots) \quad \text{— (19)}$$

$$g_m(n) = k_m f_{m-1}(n) + g_{m-1}(n-1), \quad m = N, N-1 \ldots) \quad \text{— (20)}$$

$$y(n) = f_0(n) = g_0(n) \quad \text{— (21)}$$

## Conversion from lattice structure to Direct form

For $N=3$ we can write the difference equation for an IIR filter as

$$y(n) = x(n) - a_3(1)\,y(n-1) - a_3(2)\,y(n-2) - a_3(3)\,y(n-3) \qquad -(22)$$

and from lattice structure, we can write

$$y(n) = f_0(n) = g_0(n) \qquad -(23)$$

$$x(n) = f_3(n) = f_2(n) + k_3\, g_2(n-1) \qquad -(24)$$

$$g_3(n) = k_3 f_2(n) + g_2(n-1) \qquad -(25)$$

Substituting $m=2$ in eqn. (19) & (20) we get

$$f_2(n) = f_1(n) + k_2\, g_1(n-1) \qquad -(26)$$

$$g_2(n) = k_2 f_1(n) + g_1(n-1)$$

$$\therefore\ g_2(n-1) = k_2 f_1(n-1) + g_1(n-2) \qquad -(27)$$

Substituting equation (26) and (27) in eqn (24)

$$x(n) = f_1(n) + K_2\, g_1(n-1) + K_3\left[k_2 f_1(n-1) + g_1(n-2)\right]$$

$$x(n) = f_0(n) + k_1\, g_0(n-1) + K_2\left[K_1 f_0(n-1) + g_0(n-2)\right] + K_2 K_3\left[f_0(n-1) + K_1\, g_0(n-2)\right] + K_3\left[K_1 f_0(n-2) + g_0(n-3)\right] \qquad -(28)$$

Using eqn. 21 in 28, we get

$$x(n) = y(n) + [K_1(1+K_2) + K_2K_3]\,y(n-1) + [K_2 + K_1K_3(1+K_2)]\,y(n-2) + K_3\,y(n-3) \quad -(29)$$

$$x(n) = y(n) + [a_2(1) + a_2(2)a_3(3)]\,y(n-1) + [a_2(2) + a_2(1)a_3(3)]\,y(n-2) + a_3(3)\,y(n-3)$$

where $a_2(1) = K_1(1+K_2)$ $\quad -(30)$

$$\therefore x(n) = y(n) + a_3(1)\,y(n-1) + a_3(2)\,y(n-2) + a_3(3)\,y(n-3) \quad -(31)$$

comparing 30 & 31 we get.

$$a_3(0) = 1$$
$$a_3(1) = a_2(1) + a_2(2)\,a_3(3)$$
$$a_3(2) = a_2(2) + a_2(1)\,a_3(3)$$
$$K_3 = a_3(3)$$

In general

$$a_m(0) = 1$$
$$a_m(k) = a_{m-1}(k) + a_m(m)\,a_{m-1}(m-k)$$
$$a_m(m) = K_m \quad -(32)$$

Eqn. 32 can be used to convert lattice structure to direct form.

## Conversion from direct form to Lattice structure

For a 3-stage IIR system

$$x(n) = f_3(n) = f_2(n) + k_3\, g_2(n-1) \quad — (33)$$

$$g_3(n) = k_3\, f_2(n) + g_2(n-1) \quad —$$

$$\therefore \quad g_2(n-1) = g_3(n) - k_3\, f_2(n) \quad — (34)$$

Substituting eqn. (34) into (33) we get

$$f_3(n) = f_2(n) + k_3\left[g_3(n) - k_3 f_2(n)\right]$$

## Lattice Structures

Let $A_m(z)$ be a polynomial of degree $m$ given by

$$A_m(z) = a_m(0) + a_m(1)z^{-1} + a_m(2)z^{-2} + \cdots a_m(m)z^{-m} \quad \text{--- (1)}$$

or

$$A_m(z) = \sum_{k=0}^{m} a_m(k) z^{-k} \quad \text{--- (1)}$$

where $a_m(k)$, $0 \le k \le m$ are the coefficients of polynomial $A_m(z)$.

$A_m(z)$ can be treated as the system function of an FIR filter of order $m$ with impulse response coefficients

$$h_m(k) = a_m(k)$$

let $a_m(0) = 1$

$$\therefore A_m(z) = 1 + \sum_{k=1}^{m} a_m(k) z^{-k} \quad \text{--- (2)}$$

* The reciprocal or reverse polynomial $B_m(z)$ of $A_m(z)$ is the polynomial with coefficients in reverse order.

(14)

$$B_m(z) = a_m(m) + a_m(m-1)z^{-1} + \cdots + a_m(1)z^{-(m-1)} + a_m(0)z^{-m}$$

$$B_m(z) = \sum_{k=0}^{m} a_m(m-k)\, z^{-k}$$

$$B_m(z) = \sum_{k=0}^{m} b_m(k)\, z^{-k}$$

where $b_m(k)$, $0 \le k \le m$ are the coefficients of reverse polynomial.

$\therefore$ $b_m(k) = a_m(m-k)$, $0 \le k \le m$

where $b_m(m) = a_m(0) = 1$

let $A_m(z) = Z[a_m(k)]$

$B_m(z) = Z[b_m(k)]$

$\therefore$ $a_m(k) \longleftrightarrow A_m(z)$

$b_m(k) \longleftrightarrow B_m(z)$

$a_m(k+m) \longleftrightarrow z^{m} A_m(z)$

$a_m(m-k) \longleftrightarrow z^{-m} A_m(z^{-1})$

$b_m(k) \longleftrightarrow z^{-m} A_m(z^{-1})$

$\therefore$ $\boxed{B_m(z) = z^{-m} A_m(z^{-1})}$

## Advantages of lattice Structures :-

1. Lattice structures are modular in nature. i.e, the filter order can be increased by adding extra stages.
2. Lattice structures are less sensitive to coefficient quantization effects than direct form filter ~~coefficient~~ structure.
3. Lattice structure are computationally more efficient than other filter structures for implementation of wavelet transform using filter banks.

## Lattice Structure for FIR system

### All-zero system :-

Let FIR filter $H(z) = \sum_{k=0}^{m} h(k) z^{-k}$ $\quad m = 0, 1 - - M-1$

$H(z)$ is an m-th ~~order~~ degree polynomial $A_m(z)$ with coefficients $h(k) = a_m(k)$

& $h(0) = a_m(0) = 1$

$$\therefore H(z) = A_m(z) = 1 + \sum_{k=1}^{m} a_m(k) z^{-k}$$

$m = 0 - M-1$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + \sum_{k=1}^{m} a_m(k) z^{-k}$$

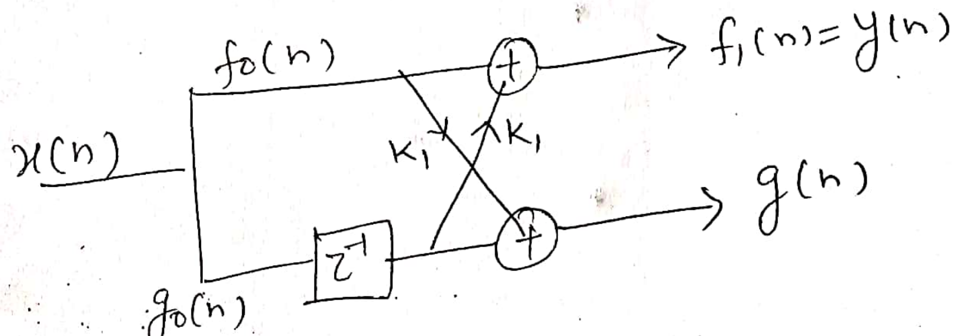
$$\therefore \quad y(n) = x(n) + \sum_{k=1}^{m} a_m(k)\, x(n-k)$$

where $a_m(k)$ are the direct form structure coefficients.

For first order filter ($m=1$)

$$y(n) = x(n) + \cancel{\sum} a_{\cancel{m}}(1)\, x(n-1) \quad \text{--- (3)}$$

The lattice structure for this Ist order system is shown below




Input $f_0(n) = g_0(n) = x(n)$

Output $f_1(n) = f_0(n) + K_1 g_0(n-1) = x(n) + k_1 x(n-1)$

$g_1(n) = K_1 f_0(n) + g_0(n-1) = K_1 x(n) + x(n$

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For second-order $(m=2)$ FIR filter.

$$y(n) = x(n) + \sum_{k=1}^{2} a_2(k)\,x(n-k)$$

$$= x(n) + a_2(1)\,x(n-1) + a_2(2)\,x(n-2)$$

O/Ps

$$f_2(n) = f_1(n) + K_2\,g_1(n-1)$$

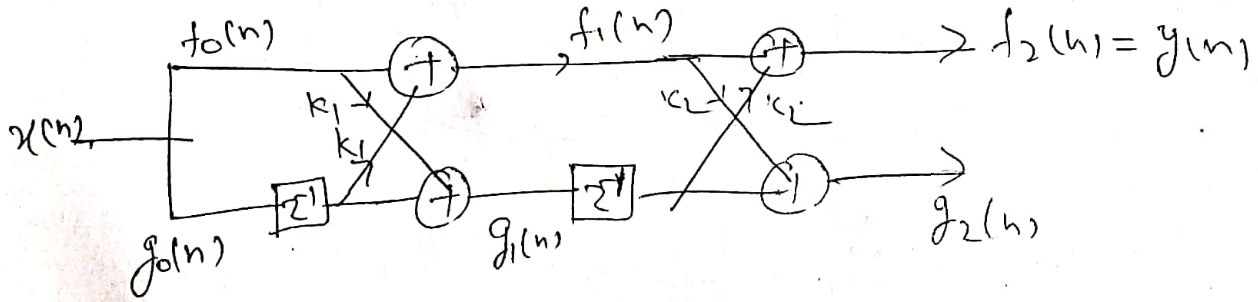
$$g_2(n) = K_2\,f_1(n) + g_1(n-1)$$

$$f_2(n) = \left[x(n) + k_1\,x(n-1)\right] + k_2\left[k_1\,x(n-1) + x(n-2)\right]$$

$$f_2(n) = x(n) + k_1(1+k_2)\,x(n-1) + k_2\,x(n-2)$$

$$a_2(2) = k_2$$

$$a_2(1) = k_1(1+k_2)$$




or $\quad k_2 = a_2(2)$

$$k_1 = \frac{a_2(1)}{1 + a_2(2)}$$

Similarly $(M-1)$th order FIR filter can be implemented by a lattice structure with the following eqns.

$$f_0(n) = g_0(n) = x(n) \quad \text{--- (4)}$$

$$f_m(n) = f_{m-1}(n) + K_m\, g_{m-1}(n-1) \quad \text{--- (5)}$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1) \quad \text{--- (6)}$$

The upper branch gives the filter o/p

$$y(n) = f_m(n) = \sum_{k=0}^{m} a_m(k)\, x(n-k)$$

$$F_m(z) = \sum_{k=0}^{m} a_m(k) z^{-k} X(z) = A_m(z) X(z) \quad \text{--- (7)}$$

$$A_m(z) = \frac{F_m(z)}{X(z)} = \frac{F_m(z)}{F_0(z)}$$

Similarly $\quad g_m(n) = \sum_{k=0}^{m} a_m(m-k)\, x(n-k)$

$$= \sum_{k=0}^{m} b_m(k)\, x(n-k)$$

$$G_m(z) = B_m(z)\, X(z)$$

$$B_m(z) = \frac{G_m(z)}{X(z)} = \frac{G_m(z)}{G_0(z)} \quad \text{--- (8)}$$

But $\quad B_m(z) = z^{-m} A_m(z^{-1})$

Taking Z-Transform of (4), (5) & (6)

$$F_0(z) = G_0(z) = X(z)$$

$$F_m(z) = F_{m-1}(z) + K_m z^{-1} G_{m-1}(z) \quad \text{--- (9)}$$

$$G_m(z) = K_m F_{m-1}(z) + z^{-1} G_{m-1}(z)$$

Now dividing these three equations by $X(z)$ we get.

$$A_0(z) = B_0(z) = 1 \quad \text{— (10)}$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z) \quad \text{— (11)}$$

$$B_m(z) = K_m A_{m-1}(z) + z^{-1} B_m(z) \quad \text{— (12)}$$

<u>Conversion of Lattice Coefficient to Direct form filter Coefficients. :-</u>

we can use the following equations to convert the lattice coefficients, $K_m$ to direct form filter coefficients, $a_m(k)$

$$A_0(z) = B_0(z) = 1 \quad \text{— (13)}$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z) \quad \text{— (14)}$$

$$B_m(z) = z^{-m} A_m(z^{-1}) \quad \text{— (15)}$$

$$A_m(z) = \sum_{k=0}^{m} a_m(k) z^{-k}, \quad m = 1 - M-1 \quad \text{— (16)}$$

Starting from $m=1$, the solution can be obtained recursively.

Example: Given $K_1 = 0.1$
$K_2 = 0.2$
$K_3 = 0.3$

Determine FIR filter coefficients.

Sol:- We know that $A_0(z) = B_0(z) = 1$

for $m=1$

$$A_1(z) = A_0(z) + K_1 B_0(z) z^{-1}$$

$$A_1(z) = 1 + 0.1 z^{-1}$$

$$A_1(z) = a_1(0) + a_1(1) z^{-1}$$

$\therefore$ $a_1(0) = 1$
$a_1(1) = 0.1$

Since $B_m(z) = z^{-m} A_m(z^{-1})$

$$B_1(z) = z^{-1} A_1(z^{-1})$$

$$B_1(z) = z^{-1} [1 + 0.1 z) = 0.1 + z^{-1}$$

$$A_2(z) = A_1(z) + K_2 z^{-1} B_1(z)$$

$$A_2(z) = 1 + 0.1 z^{-1} + 0.2 z^{-1} [0.1 + z^{-1})$$

$$A_2(z) = 1 + 0.12 z^{-1} + 0.2 z^{-2}$$

$a_2(0) = 1$
$a_2(1) = 0.12$, $a_2(2) = 0.2$

$$\therefore \quad B_2(z) = z^{-2} A_2(z^{-1})$$
$$= z^{-2}\left[1 + 0.12z + 0.2z^2\right]$$
$$B_2(z) = 0.2 + 0.12z^{-1} + z^{-2}$$

$$A_3(z) = A_2(z) + K_3 z^{-1} B_2(z)$$

$$A_3(z) = \left[1 + 0.12z^{-1} + 0.2z^{-2}\right] + \cancel{K_3}\, 0.3 z^{-1}\left[0.2 + 0.12z^{-1} + z^{-2}\right]$$

$$A_3(z) = 1 + 0.12z^{-1} + 0.2z^{-2} + 0.06z^{-1} + 0.036z^{-2} + 0.3z^{-3}$$

$$A_3(z) = 1 + 0.18z^{-1} + 0.236z^{-2} + 0.3z^{-3}$$

$$A_3(z) = a_3(0) + a_3(1)z^{-1} + a_3(2)z^{-2} + a_3(3)z^{-3}$$

Hence, the direct form structure coefficients for third order FIR filters are

$a_3(0) = 1$

$a_3(1) = 0.18$

$a_3(2) = 0.236$

$a_3(3) = 0.3$

## Conversion from direct form filter coefficients to Lattice coefficients :-

Since $B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z)$

$$z^{-1} B_{m-1}(z) = B_m(z) - K_m A_{m-1}(z)$$

Therefore :

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)$$

$$A_m(z) = A_{m-1}(z) + K_m [B_m(z) - K_m A_{m-1}(z)]$$

$$A_m(z) = A_{m-1}(z)[1 - K_m^2] + K_m B_m(z)$$

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}, \quad m = M-1, M-2, \dots 1$$

starting from $m = M-1$, the solution can be obtained recursively.

Ex. Determine the lattice coefficients corresponding to the FIR filter with system function.

$$H(z) = A_3(z) = 1 + 0.18 z^{-1} + 0.236 z^{-2} + 0.3 z^{-3}$$

Sol:-

$$A_3(z) = a_3(0) + a_3(1) z^{-1} + a_3(2) z^{-2} + a_3(3) z^{-3}$$

$$k_3 = a_3(3) = 0.3$$

Since $B_m(z) = A_m(z^{-1}) z^{-m}$

$$B_3(z) = A_3(z^{-1}) z^{-3}$$

$$B_3(z) = 0.3 + 0.236 z^{-1} + 0.18 z^{-2} + z^{-3}$$

$\underline{m=3}$, we get:

$$A_2(z) = \frac{A_3(z) - K_3 B_3(z)}{1 - K_3^2}$$

$$A_2(z) = 1 + 0.12 z^{-1} + 0.2 z^{-2}$$

$$A_2(z) = a_2(0) + a_2(1) z^{-1} + a_2(2) z^{-2}$$

$$K_2 = a_2(2) = 0.2$$

$$A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - K_2^2}$$

$$= 1 + 0.1 z^{-1}$$

$$A_1(z) = a_1(0) + a_1(1) z^{-1}$$

$$\therefore \underline{K_1 = a_1(1) = 0.1}$$

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## Lattice structure for all-pole IIR systems

Let the Nth order all-pole system function H(z) be given by

$$H(z) = \frac{1}{A_N(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}} \quad \text{—(1)}$$

which is the inverse system of the all-zero system.

$$\therefore \quad y(n) = -\sum_{k=1}^{N} a_N(k) y(n-k) + x(n)$$

If we interchange x(n) & y(n) then

$$x(n) = -\sum_{k=1}^{N} a_N(k) x(n-k) + y(n)$$

$$\text{or} \quad y(n) = x(n) + \sum_{k=1}^{N} a_N(k) x(n-k) \quad \text{—(A.)}$$

$$\frac{Y(z)}{X(z)} = H_{a_2}(z) = 1 + \sum_{k=1}^{N} a_N(k) z^{-k} = A_N(z)$$

where $H_{a_2}(z)$ is the system function of an all zero system.

Therefore an all pole system can be obtained from all zero ~~if~~ system by interchanging x(n) & y(n).

Now $f_N(n)$ is the input and $f_0(n)$ is output.

$x(n) = f_N(n)$ and $y(n) = f_0(n)$

Therefore given input $f_N(n) = x(n)$, we must successively find $f_{N-1}(n), f_{N-2}(n) \cdots f_0(n)$.

From all zero system the lattice m-th stage difference equations are

$$f_m(n) = f_{m-1}(n) + K_m g_{m-1}(n-1) \quad \text{— (2)}$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1) \quad \text{— (3)}$$

For all pole equation (2) is rearranged in reverse order and (3) remains unchanged.

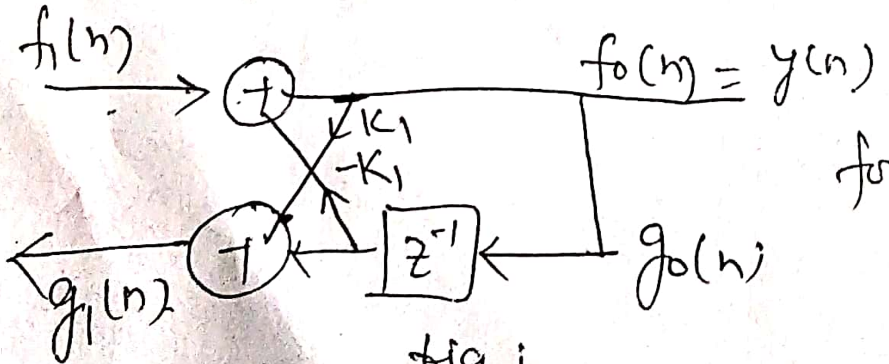
$\therefore$

$$x(n) = f_N(n) \quad \text{— (4)}$$

$$f_{m-1}(n) = f_m(n) - K_m g_{m-1}(n-1) \quad \text{— (5)}$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1) \quad \text{— (6)}$$

$$y(n) = f_0(n) = g_0(n) \quad \text{— (7)}$$




for $m=1$

fig. i

As an example consider a first order all pole lattice filter (N=1) then

$$x(n) = f_1(n) \quad , \quad y(n) = f_0(n) = g_0(n)$$

$$y(n) = f_0(n) = f_1(n) - K_1\, g_0(n-1)$$

$$= x(n) - K_1\, y(n-1) \quad — (8)$$

$$g_1(n) = K_1 f_0(n) + g_0(n-1)$$

$$g_1(n) = K_1 y(n) + y(n-1) \quad — (9)$$

This is represented in figure.1.

Let us consider a second order all-pole lattice filter. The difference equation for N=2 are

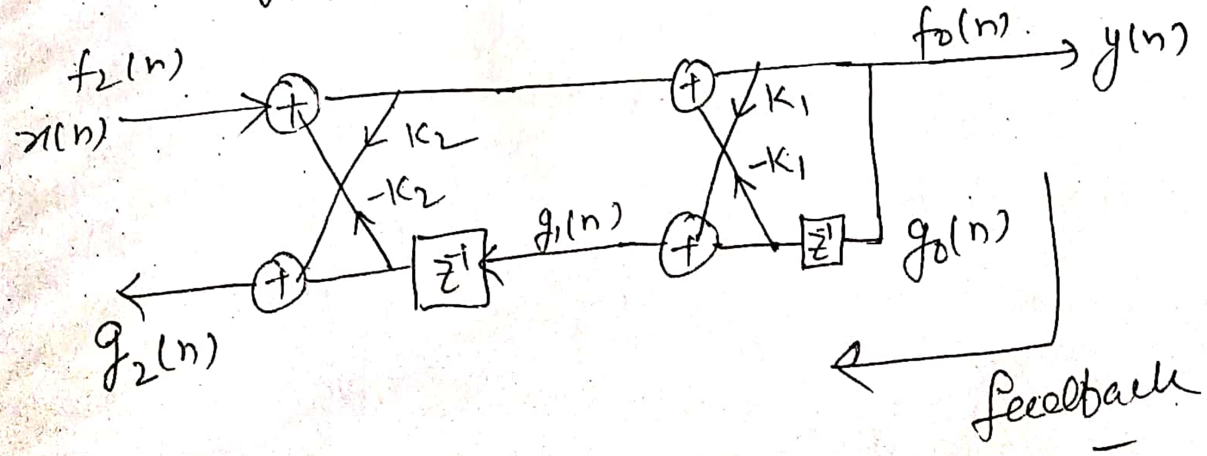
$$x(n) = f_2(n) \quad , \quad y(n) = f_0(n) = g_0(n) \quad — (10)$$

$$f_1(n) = f_2(n) - K_2\, g_1(n-1) \quad — (11)$$

$$g_2(n) = K_2 f_1(n) + g_1(n-1) \quad — (12)$$

$$f_0(n) = f_1(n) - K_1\, g_0(n-1) \quad — (13)$$

$$g_1(n) = K_1 f_0(n) + g_0(n-1) \quad — (14)$$



putting eqn. (11) into (13) we get.

$$f_0(n) = f_2(n) - k_2\, g_1(n-1) - k_1\, g_0(n-1)$$

$$f_0(n) = f_2(n) - k_2\left[k_1 f_0(n-1) + g_0(n-2)\right] - k_1 g_0(n-1)$$

using eqn. (10) we get

$$y(n) = x(n) - k_1(1+k_2)\, y(n-1) - k_2\, y(n-2) \quad \text{— (15)}$$

Similarly

$$g_2(n) = k_2\, y(n) + k_1(1+k_2)\, y(n-1) + y(n-2) \quad \text{— (16)}$$

Using z-transform eqn. (15) & (16) can be written as

$$Y(z) = X(z) - k_1(1+k_2) z^{-1} Y(z) - k_2 z^{-2} Y(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{A_2(z)} = \frac{1}{1 + k_1(1+k_2) z^{-1} + k_2 z^{-2}}$$

$$\frac{G_2(z)}{X(z)} = H_b(z) = B_2(z) = k_2 + k_1(1+k_2) z^{-1} + z^{-2}$$

It has to be noted that the coefficients of $A_2(z)$ and $B_2(z)$ are in reverse order.

In general, for any $N$, the system function for all pole IIR system from input $x(n) = f_N(n)$ to output $y(n) = f_0(n) -$ H

$$H(z) = \frac{Y(z)}{X(z)} = \frac{F_0(z)}{F_N(z)} = \frac{1}{A_N(z)}$$

$$H_b(z) = \frac{G_N(z)}{Y(z)} = B_N(z) = \frac{G_N(z)}{G_0(z)}.$$

where $B_N(z) = z^{-N} A_N(z^{-1})$.

## Schur-cohn Stability Test :-

The schur-cohn stability test provides a method for verifying the stability of LTI system with rational system function without explicitly finding the roots of the denominator polynomial

$$H(z) = \frac{1}{A_N(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}}$$

This all pole system is stable if all the poles, or all the roots of the denominator polynomial $A_N(z)$, lie inside the unit circle. The SC Test states that the polynomial has all its roots inside the unit circle if & only if

$$|K_m| < 1 \quad , \quad m = 1, 2 \ldots N.$$

Ex. Consider an all-pole IIR filter

$$H(z) = \frac{1}{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}}$$

Determine lattice structure. comment on its stability.

Sol.