2. $z = 4.851$ ...

6.14. TEST OF SIGNIFICANCE OF SMALL SAMPLES

When the size of the sample is less than 30, then the sample is called small sample. For such sample it will not be possible for us to assume that the random sampling distribution of a statistic is approximately normal and the values given by the sample data are sufficiently close to the population values and can be used in their place for the calculation of the standard error of the estimate.

t-TEST

6.15. STUDENT'S-t-DISTRIBUTION

This t-distribution is used when sample size is ≤ 30 and the population standard deviation is unknown.

t-statistic is defined as $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}} \sim t(n-1 \text{ d.f.})$ d.f. degrees of freedom where $s = \sqrt{\dfrac{\Sigma(X - \bar{X})^2}{n-1}}$.

The t-table

The t-table given at the end is the probability integral of t-distribution. The t-distribution has a different values for each degrees of freedom and when the degrees of freedom are infinitely large, the t-distribution is equivalent to normal distribution and the probabilities shown in the normal distribution tables are applicable.

Application of t-distribution

Some of the applications of t-distribution are given below:

1. *To test if the sample mean ($\bar{X}$) differs significantly from the hypothetical value μ of the population mean.*
2. *To test the significance between two sample means.*
3. *To test the significance of observed partial and multiple correlation coefficients.*

Critical value of t

The critical value or significant value of t at level of significance α degrees of freedom γ for two tailed test is given by

$$P[|t| > t_\gamma(\alpha)] = \alpha$$
$$P[|t| \leq t_\gamma(\alpha)] = 1 - \alpha$$

The significant value of t at level of significance α for a single tailed test can be got from those of two tailed test by referring to the values at 2α.

6.16. TEST I: t-TEST OF SIGNIFICANCE OF THE MEAN OF A RANDOM SAMPLE

To test whether the mean of a sample drawn from a normal population deviates significantly from a stated value when variance of the population is unknown.

H_0: There is no significant difference between the sample mean $\bar{x}$ and the population mean μ *i.e.*, we use the statistic

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}}, \qquad \text{where } \bar{X} \text{ is mean of the sample.}$$

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \text{ with degrees of freedom } (n-1).$$

At given level of significance α_1 and degrees of freedom $(n - 1)$. We refer to t-table t_α (two tailed or one tailed).

If calculated t value is such that $|t| < t_\alpha$ the null hypothesis is accepted. $|t| > t_\alpha$ H_0 is rejected.

Fiducial limits of population mean

If t_α is the table of t at level of significance α at $(n - 1)$ degrees of freedom.

$$\left| \frac{\bar{X} - \mu}{s/\sqrt{n}} \right| < t_\alpha \text{ for acceptance of } H_0.$$

$$\bar{x} - t_\alpha\, s/\sqrt{n} < \mu < \bar{x} + t_\alpha\, s/\sqrt{n}$$

95% confidence limits (level of significance 5%) are $\bar{X} \pm t_{0.05}\, s/\sqrt{n}$.

99% confidence limits (level of significance 1%) are $\bar{X} \pm t_{0.01}\, s/\sqrt{n}$.

Note. Instead of calculating s, we calculate S for the sample.

Since $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \quad \therefore \quad S^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2. \qquad \left[(n-1)s^2 = nS^2,\ s^2 = \frac{n}{n-1} S^2\right]$

ILLUSTRATIVE EXAMPLES

Example 1. *A random sample of size 16 has 53 as mean. The sum of squares of the derivation from mean is 135. Can this sample be regarded as taken from the population having 56 as mean? Obtain 95% and 99% confidence limits of the mean of the population.*

Sol. H_0: There is no significant difference between the sample mean and hypothetical population mean.

$$H_0: \mu = 56;\ H_1: \mu \neq 56 \qquad \text{(Two tailed test)}$$

$$t: \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t(n - 1 \text{ d.f.})$$

Given: $\bar{X} = 53, \mu = 56, n = 16, \Sigma(X - \bar{X})^2 = 135$

$$s = \sqrt{\frac{\Sigma(X-\bar{X})^2}{n-1}} = \sqrt{\frac{135}{15}} = 3;\ t = \frac{53-56}{3/\sqrt{16}} = \frac{-3\times 4}{3} = -4$$

$|t| = 4$. d.f. $= 16 - 1 = 15$.

Conclusion. $t_{0.05} = 1.753$.

Since $|t| = 4 > t_{0.05} = 1.753$ *i.e.*, the calculated value of t is more than the table value. The hypothesis is rejected. Hence the sample mean has not come from a population having 56 as mean.

95% confidence limits of the population mean

$$\bar{X} \pm \frac{s}{\sqrt{n}} t_{0.05},\ 53 \pm \frac{3}{\sqrt{16}}(1.725) = 51.706;\ 54.293$$

99% confidence limits of the population mean

$$\bar{X} \pm \frac{s}{\sqrt{n}} t_{0.01},\ 53 \pm \frac{3}{\sqrt{16}}(2.602) = 51.048;\ 54.951.$$

Example 2. *The life time of electric bulbs for a random sample of 10 from a large consignment gave the following data:*

Item	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Life in '000 hrs.	*4.2*	*4.6*	*3.9*	*4.1*	*5.2*	*3.8*	*3.9*	*4.3*	*4.4*	*5.6*

Can we accept the hypothesis that the average life time of bulb is 4000 hrs?

Sol. H_0: There is no significant difference in the sample mean and population mean. *i.e.*, $\mu = 4000$ hrs.

Applying the t-test: $t = \dfrac{\bar{X}-\mu}{s/\sqrt{n}} \sim t(10-1 \text{ d.f.})$

X	4.2	4.6	3.9	4.1	5.2	3.8	3.9	4.3	4.4	5.6
$X-\bar{X}$	−0.2	0.2	−0.5	−0.3	0.8	−0.6	−0.5	−0.1	0	1.2
$(X-\bar{X})^2$	0.04	0.04	0.25	0.09	0.64	0.36	0.25	0.01	0	1.44

$$\bar{X} = \frac{\Sigma X}{n} = \frac{44}{10} = 4.4 \qquad \Sigma(X-\bar{X})^2 = 3.12$$

$$s = \sqrt{\frac{\Sigma(X-\bar{X})^2}{n-1}} = \sqrt{\frac{3.12}{9}} = 0.589;\ t = \frac{4.4-4}{\frac{0.589}{\sqrt{10}}} = 2.123$$

For $\gamma = 9$, $t_{0.05} = 2.26$.

Conclusion. Since the calculated value of t is less than table $t_{0.05}$. $\therefore$ The hypothesis $\mu = 4000$ hrs is accepted.

i.e., The average life time of bulbs could be 4000 hrs.

Example 3. *A sample of 20 items has mean 42 units and S.D. 5 units. Test the hypothesis that it is a random sample from a normal population with mean 45 units.*

(M.D.U. Dec. 2012)

Sol. H_0: There is no significant difference between the sample mean and the population mean.

i.e., $\mu = 45$ units

$H_1: \mu \neq 45$ (Two tailed test)

Given: $n = 20, \overline{X} = 42, S = 5; \gamma = 19$ d.f.

$$s^2 = \frac{n}{n-1} S^2 = \left[\frac{20}{20-1}\right](5)^2 = 26.31 \quad \therefore \quad s = 5.129$$

Applying t-test $\quad t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}} = \dfrac{42-45}{5.129/\sqrt{20}} = -2.615; \; |t| = 2.615$

The tabulated value of t at 5% level for 19 d.f. is $t_{0.05} = 2.09$.

Conclusion. Since $|t| > t_{0.05}$, the hypothesis H_0 is rejected. *i.e.*, there is significant difference between the sample mean and population mean.

i.e., The sample could not have come from this population.

Example 4. *The 9 items of a sample have the following values 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these values differ significantly from the assumed mean 47.5.*

(M.D.U. Dec. 2011)

Sol. $H_0: \mu = 47.5$

i.e., there is no significant difference between the sample and population mean.

$H_1: \mu \neq 47.5$ (two tailed test); Given: $n = 9$, $\mu = 47.5$

X	45	47	50	52	48	47	49	53	51
$X - \overline{X}$	−4.1	−2.1	0.9	2.9	−1.1	−2.1	−0.1	3.9	1.9
$(X - \overline{X})^2$	16.81	4.41	0.81	8.41	1.21	4.41	0.01	15.21	3.61

$$\overline{X} = \frac{\Sigma x}{n} = \frac{442}{9} = 49.11; \; \Sigma(X - \overline{X})^2 = 54.89; \; s^2 = \frac{\Sigma(X - \overline{X})^2}{(n-1)} = 6.86 \quad \therefore \quad s = 2.619$$

Applying t-test $\quad t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}} = \dfrac{49.1 - 47.5}{2.619/\sqrt{8}} = \dfrac{(1.6)\sqrt{8}}{2.619} = 1.7279$

$t_{0.05} = 2.31$ for $\gamma = 8$.

Conclusion. Since $|t| < t_{0.05}$, the hypothesis is accepted *i.e.*, there is no significant difference between their mean.

Example 5. *The following results are obtained from a sample of 10 boxes is biscuits.*

Mean weight content = 490 gm

S.D. of the weight = 9 gm could the sample come from a population having a mean of 500 gm?

Sol. Given: $n = 10,\ \overline{X} = 490;\ S = 9$ gm, $\mu = 500$

$$s = \sqrt{\frac{n}{n-1}S^2} = \sqrt{\frac{10}{9}\times 9^2} = 9.486$$

H_0: The difference is not significant *i.e.*, $\mu = 500$; H_1: $\mu \neq 500$

Applying *t*-test
$$t = \frac{\overline{X}-\mu}{s/\sqrt{n}} = \frac{490-500}{9.486/\sqrt{10}} = -0.333$$

$$t_{0.05} = 2.26 \text{ for } \gamma = 9.$$

Conclusion. Since $|t| = .333 > t_{0.05}$ the hypothesis H_0 is rejected *i.e.*, $\mu \neq 500$

$\therefore$ The sample could not have come from the population having mean 500 gm.

EXERCISE 6.6

1. Find the student's t for the following variable values in a sample of eight: $-4, -2, -2, 0, 2, 2, 3, 3$ taking the mean of the universe to be zero. *(M.D.U. Dec. 2010)*
2. Ten individuals are choosen at random from a normal population of students and their marks found to be 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. In the light of these data discuss the suggestion that mean mark of the population of students is 66.
3. The following values gives the lengths of 12 samples of egyptian cotton taken from a consignment: 48, 46, 49, 46, 52, 45, 43, 47, 47, 46, 45, 50. Test if the mean length of the consignment can be taken as 46.
4. A sample of 18 items has a mean 24 units and standard deviation 3 units. Test the hypothesis that it is a random sample from a normal population with mean 27 units.
5. A random sample of 10 boys had the I.Q's 70, 120, 110, 101, 88, 83, 95, 98, 107 and 100. Do these data support the assumption of a population mean I.Q of 160?
6. A filling machine is expected to fill 5 kg of powder into bags. A sample of 10 bags gave the following weights. 4.7, 4.9, 5.0, 5.1, 5.4, 5.2, 4.6, 5.1, 4.6 and 4.7. Test whether the machine is working properly. *(M.D.U. Dec. 2010)*
7. A machinist is making engine parts with axle-diameter of 0.7 unit. A random sample of 10 parts shows mean diameter 0.742 unit with a standard deviation of 0.04 unit. On the basis of this sample, what would you say that the work in inferior?

Answers

1. $t = 0.27$	2. accepted	3. accepted
4. rejected	5. accepted	6. accepted
7. rejected.		

6.17. TEST II: t-TEST FOR DIFFERENCE OF MEANS OF TWO SMALL SAMPLES (From a Normal Population)

This test is used to test whether the two samples $x_1, x_2, \ldots\ldots, x_{n_1}, y_1, y_2, \ldots\ldots, y_{n_2}$ of sizes n_1, n_2 have been drawn from two normal populations with mean μ_1 and μ_2 respectively under the assumption that the population variance are equal. $(\sigma_1 = \sigma_2 = \sigma)$.

H_0: The samples have been drawn from the normal population with means μ_1 and μ_2 *i.e.*, H_0: $\mu_1 \neq \mu_2$.

Let $\overline{X}, \overline{Y}$ be their means of the two sample.

Under this H_0 the test of statistic t is given by $t = \dfrac{(\overline{X} - \overline{Y})}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t(n_1 + n_2 - 2 \text{ d.f.})$

Note 1. If the two sample standard deviations s_1, s_2 are given then we have $s^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$.

Note 2. If $n_1 = n_2 = n$, $t = \dfrac{\overline{X} - \overline{Y}}{\sqrt{\dfrac{s_1^2 + s_2^2}{n-1}}}$ can be used as a test statistic.

Note 3. If the pairs of values are in some way associated (correlated) we can't use the test statistic as given in note 2. In this case we find the differences of the associated pairs of values and apply for single mean *i.e.*, $t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}}$ with degrees of freedom $n - 1$.

The test statistic is $t = \dfrac{\overline{d}}{s/\sqrt{n}}$ or $t = \dfrac{\overline{d}}{s/\sqrt{n-1}}$, where $\overline{d}$ is the mean of paired difference

i.e., $d_i = x_i - y_i$

$\overline{d_i} = \overline{X} - \overline{Y}$, where (x_i, y_i) are the paired data $i = 1, 2, \ldots\ldots, n$.

ILLUSTRATIVE EXAMPLES

Example 1. *Two samples of sodium vapour bulbs were tested for length of life and the following results were got:*

	Size	*Sample mean*	*Sample S.D.*
Type I	*8*	*1234 hrs*	*36 hrs*
Type II	*7*	*1036 hrs*	*40 hrs*

Is the difference in the means significant to generalise that type I is superior to type II regarding length of life.

Sol. H_0: $\mu_1 = \mu_2$ *i.e.*, two types of bulbs have same lifetime.

H_1: $\mu_1 > \mu_2$ *i.e.*, type I is superior to Type II

$$s^2 = \frac{n_1 s_2^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{8 \times (36)^2 + 7(40)^2}{8 + 7 - 2} = 1659.076 \quad \therefore \quad s = 40.7317$$

The t-statistic
$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{1234 - 1036}{40.7317\sqrt{\dfrac{1}{8} + \dfrac{1}{7}}} = 18.1480 \sim t(n_1 + n_2 - 2 \text{ d.f.})$$

$t_{0.05}$ at d.f. 13 is 1.77 (One tailed test)

Conclusion. Since calculated $|t| > t_{0.05}$ H_0 is rejected *i.e.*, H_1 is accepted.

$\therefore$ Type I is definitely superior to type II.

here $\overline{X} = \sum_{i=1}^{n_1} \frac{X_i}{n_i}$, $\overline{Y} = \sum_{j=1}^{n_2} \frac{Y_j}{n_2}$; $s^2 = \frac{1}{n_1 + n_2 - 2} [\Sigma(X_i - \overline{X})^2 + (Y_j - \overline{Y})^2]$

an unbiased estimate of the population variance σ^2.

follows t distribution with $n_1 + n_2 - 2$ degrees of freedom.

Example 2. *Samples of sizes 10 and 14 were taken from two normal populations with .D. 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of he two populations are the same at 5% level.*

Sol. $H_0: \mu_1 = \mu_2$ *i.e.*, the means of the two populations are the same.

$H_1: \mu_1 \neq \mu_2$.

Given $\overline{X}_1 = 20.3, \overline{X}_2 = 18.6; n_1 = 10, n_2 = 14, s_1 = 3.5, s_2 = 5.2$

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{10(3.5)^2 + 14(5.2)^2}{10 + 14 - 2} = 22.775 \quad \therefore \quad s = 4.772$$

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{20.3 - 18.6}{\left(\sqrt{\frac{1}{10} + \frac{1}{14}}\right) 4.772} = 0.8604$$

The value of t at 5% level for 22 d.f. is $t_{0.05} = 2.0739$.

Conclusion. Since $|t| = 0.8604 < t_{0.05}$ the hypothesis is accepted. *i.e.*, there is no ignificant difference between their means.

Example 3. *The height of 6 randomly chosen sailors are in inches are 63, 65, 68, 69, 71 nd 72. Those of 9 randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Test whether the sailors are on the average taller than soldiers.*

Sol. Let X_1 and X_2 be the two samples denoting the heights of sailors and soldiers.

Given the sample size $n_1 = 6$, $n_2 = 9$, $H_0: \mu_1 = \mu_2$.

i.e., the mean of both the population are the same.

$H_1: \mu_1 > \mu_2$ (one tailed test)

Calculation of two sample mean:

X_1	63	65	68	69	71	72
$X_1 - \overline{X}_1$	– 5	– 3	0	1	3	4
$(X_1 - \overline{X}_1)^2$	25	9	0	1	9	16

$$\overline{X}_1 = \frac{\Sigma X_1}{n_1} = 68; \quad \Sigma(X_1 - \overline{X}_1)^2 = 60$$

X_2	61	62	65	66	69	70	71	72	73
$X_2 - \overline{X}_2$	– 6.66	– 5.66	– 2.66	1.66	1.34	2.34	3.34	4.34	5.34
$(X_2 - \overline{X}_2)^2$	44.36	32.035	7.0756	2.7556	1.7956	5.4756	11.1556	18.8356	28.5156

$$\overline{X}_2 = \frac{\Sigma X_2}{n_2} = 67.66; \quad \Sigma(X_2 - \overline{X}_2)^2 = 152.0002$$

$$s^2 = \frac{1}{n_1 + n_2 - 2}\,[\Sigma(X_1 - \overline{X}_1)^2 + \Sigma(X_2 - \overline{X}_2)^2]$$

$$= \frac{1}{6+9-2}\,[60 + 152.0002] = 16.3077 \quad \Rightarrow \quad s = 4.038$$

Under H_0,

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{68 - 67.666}{4.0382\sqrt{\frac{1}{6} + \frac{1}{9}}} = 0.3031 \sim t(n_1 + n_2 - 2 \text{ d.f.})$$

The value of t at 10% level of significance ($\because$ the test is one tailed) for 13 d.f. is 1.77.

Conclusion. Since $|t| = 0.3031 < t_{0.05} = 1.77$ the hypothesis H_0 is accepted.

i.e., there is no significance difference between their average.

i.e., the sailors are not on the average taller than the soldiers.

Example 4. *A certain stimulus administered to each of 12 patients resulted in the following increases of blood pressure: 5, 2, 8, – 1, 3, 0, – 2, 1, 5, 0, 4, 6. Can it be concluded that the stimulus will in general be accompanied by an increase in blood pressure?*

(M.D.U. Dec. 2011)

Sol. To test whether the mean increase in blood pressure of all patients to whom the stimulus is administered will be positive. We have to assume that this population is normal with mean μ and S.D. σ which are unknown.

$$H_0: \mu = 0; \; H_1: \mu_1 > 0$$

The test statistic under H_0

$$t = \frac{\overline{d}}{s/\sqrt{n-1}} \sim t(n - 1 \text{ degrees of freedom})$$

$$\overline{d} = \frac{5+2+8+(-1)+3+0+6+(-2)+1+5+0+4}{12} = 2.583$$

$$s^2 = \frac{\Sigma d^2}{n} - \overline{d}^2 = \frac{1}{12}\,[5^2 + 2^2 + 8^2 + (-1)^2 + 3^2 + 0^2 + 6^2 + (-2)^2 + 1^2 + 5^2 + 0^2 + 4^2] - (2.583)^2$$

$$= 8.744 \quad \therefore \quad s = 2.9571$$

$$t = \frac{\overline{d}}{s/\sqrt{n-1}} = \frac{2.583}{2.9571/\sqrt{12-1}} = \frac{2.583\sqrt{11}}{2.9571} = 2.897 \sim t(n - 1 \text{ d.f.})$$

Conclusion. The tabulated value of $t_{0.05}$ at 11 d.f. is 2.2.

$\because \quad |t| > t_{0.05}$, H_0 is rejected.

i.e., the stimulus does not increase the blood pressure. The stimulus in general will be accompanied by increase in blood pressure.

Example 5. *Memory capacity of 9 students was tested before and after a course of meditation for a month. State whether the course was effective or not from the data below (in same units)*

Before	*10*	*15*	*9*	*3*	*7*	*12*	*16*	*17*	*4*
After	*12*	*17*	*8*	*5*	*6*	*11*	*18*	*20*	*3*

Sol. Since the data are correlated and concerned with the same set of students we use paired *t*-test.

H_0: Training was not effective $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$ (Two tailed test).

Before training (X)	*After training (Y)*	$d = X - Y$	d^2
10	12	– 2	4
15	17	– 2	4
9	8	1	1
3	5	– 2	4
7	6	1	1
12	11	1	1
16	18	– 2	4
17	20	– 3	9
4	3	1	1
		$\Sigma d = -7$	$\Sigma d^2 = 29$

$$\bar{d} = \frac{\Sigma d}{n} = \frac{-7}{9} = -0.7778;\ s^2 = \frac{\Sigma d^2}{n} - (\bar{d})^2 = \frac{29}{9} - (-0.7778)^2 = 2.617$$

$$t = \frac{\bar{d}}{s/\sqrt{n-1}} = \frac{-0.7778}{\sqrt{2.6172}/\sqrt{8}} = \frac{-0.7778 \times \sqrt{8}}{1.6177} = -1.359$$

The tabulated value of $t_{0.05}$ at 8 d.f. is 2.31.

Conclusion. Since $|\ t\ | = 1.359 < t_{0.05}$, H_0 is accepted *i.e.*, training was not effective in improving performance.

Example 6. *The following figures refer to observations in live independent samples.*

Sample I	*25*	*30*	*28*	*34*	*24*	*20*	*13*	*32*	*22*	*38*
Sample II	*40*	*34*	*22*	*20*	*31*	*40*	*30*	*23*	*36*	*17*

Analyse whether the samples have been drawn from the populations of equal means.

Sol. H_0: The two samples have been drawn from the population of equal means. *i.e.*, there is no significant difference between their means

i.e.,

$$\mu_1 = \mu_2$$

H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

Given n_1 = Sample I size = 10; n_2 = Sample II size = 10

To calculate the two sample mean and sum of squares of deviation from mean. Let X_1 be the sample I and X_2 be the sample II.

X_1	25	30	28	34	24	20	13	32	22	38
$X_1 - \overline{X}_1$	−1.6	3.4	1.4	7.4	−2.6	−6.6	−13.6	5.4	4.6	11.4
$(X_1 - \overline{X}_1)^2$	2.56	11.56	1.96	54.76	6.76	43.56	184.96	29.16	21.16	129.96
X_2	40	34	22	20	31	40	30	23	36	17
$X_2 - \overline{X}_2$	10.7	4.7	−7.3	−9.3	1.7	10.7	0.7	−6.3	6.7	−12.3
$(X_2 - \overline{X}_2)^2$	114.49	22.09	53.29	86.49	2.89	114.49	0.49	39.67	44.89	151.29

$$\overline{X}_1 = \sum_{i=1}^{10} \frac{X_1}{n_1} = 26.6 \qquad \overline{X}_2 = \sum_{i=1}^{10} \frac{X_2}{n_2} = \frac{293}{10} = 29.3$$

$$\Sigma(X_1 - \overline{X}_1)^2 = 486.4 \qquad \Sigma(X_2 - \overline{X}_2)^2 = 630.08$$

$$s^2 = \frac{1}{n_1 + n_2 - 2}[\Sigma(X_1 - \overline{X}_1)^2 + \Sigma(X_2 - \overline{X}_2)^2]$$

$$= \frac{1}{10 + 10 - 2}[486.4 + 630.08] = 62.026 \qquad \therefore \quad s = 7.875$$

Under H_0 the test statistic is given by

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{26.6 - 29.3}{7.875\sqrt{\frac{1}{10} + \frac{1}{10}}} = -0.7666 \sim t(n_1 + n_2 - 2 \text{ d.f.})$$

$$|t| = 0.7666.$$

Conclusion. The tabulated value of t at 5% level of significance for 18 d.f. is 2.1. Since the calculated value $|t| = 0.7666 < t_{0.05}$, H_0 is accepted.

i.e., there is no significant difference between their means.

i.e., the two samples have been drawn from the populations of equal means.

EXERCISE 6.7

1. The mean life of 10 electric motors was found to be 1450 hrs with S.D. of 423 hrs. A second sample of 17 motors chosen from a different batch showed a mean life of 1280 hrs with a S.D. of 398 hrs. Is there a significant difference between means of the two samples.

2. The marks obtained by a group of 9 regular course students and another group of 11 part time course students in a test are given below

Regular	56	62	63	54	60	51	67	69	58		
Part-time	62	70	71	62	60	56	75	64	72	68	66

Examine whether the marks obtained by regular students and part-time students differ significantly at 5% and 1% level of significance.

Answers

1. accepted	2. rejected	3. accepted	4. accepted
5. accepted	6. rejected	7. rejected	8. accepted
9. rejected	10. rejected.		

6.18. CHI-SQUARE (χ^2) TEST

When a coin is tossed 200 times, the theoretical considerations lead us to expect 100 heads and 100 tails. But in practice, these results are rarely achieved. The quantity χ^2 (a Greek letter, pronounced as chi-square) describes the magnitude of discrepancy between theory and observation. If $\chi = 0$, the observed and expected frequencies completely coincide. The greater the discrepancy between the observed and expected frequencies, the greater is the value of χ^2. Thus χ^2 **affords a measure of the correspondence between theory and observation.**

If O_i $(i = 1, 2, \ldots\ldots, n)$ is a set of observed (experimental) frequencies and E_i $(i = 1, 2, \ldots\ldots, n)$ is the corresponding set of expected (theoretical or hypothetical) frequencies, then, χ^2 is defined as

$$\chi^2 = \sum_{i=1}^{n} \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

where $\Sigma O_i = \Sigma E_i = N$ (total frequency) and degrees of freedom (d.f.) $= (n - 1)$.

Note. (*i*) If $\chi^2 = 0$, the observed and theoretical frequencies agree exactly.

(*ii*) If χ^2 0 they do not agree exactly.

6.19. DEGREE OF FREEDOM

While comparing the calculated value of χ^2 with the table value, we have to determine the degrees of freedom.

If we have to choose any four numbers whose sum is 50, we can exercise our independent choice for any three numbers only, the fourth being 50 minus the total of the three numbers selected. Thus, though we were to choose any four numbers, our choice was reduced to three because of one condition imposed. There was only one restraint on our freedom and our degrees of freedom were $4 - 1 = 3$. If two restrictions are imposed, our freedom to choose will be further curtailed and degrees of freedom will be $4 - 2 = 2$.

In general, the number of degrees of freedom is the total number of observations less the number of independent constraints imposed on the observations. Degrees of freedom (d.f.) are usually denoted by ν (the letter 'nu' of the Greek alphabet).

Thus, $\nu = n - k$, where k is the number of independent constraints in a set of data of n observations.

Note 1. For a $p \times q$ contingency table (p columns and q rows).

$$\nu = (p - 1)(q - 1)$$

Note 2. In the case of a contingency table, the expected frequency of any class

$$= \frac{\text{Total of row in which it occurs} \times \text{Total of col. in which it occurs}}{\text{Total number of observations}}$$

Note 3. χ^2 test is one of the simplest and the most general test known. It is applicable to a very rge number of problems in practice which can be summed up under the following heads :

(*i*) as a test of goodness of fit.

(*ii*) as a test of independence of attributes.

(*iii*) as a test of homogeneity of independent estimates of the population variance.

(*iv*) as a test if the hypothetical value of the population variance σ^2.

(*v*) as a list to the homogeneity of independent estimates of the population correlation coefficient.

20. CONDITIONS FOR APPLYING χ^2 TEST

Following are the conditions which should be satisfied before χ^2 test can be applied.

(*a*) N, the total number of frequencies should be large. It is difficult to say what constitutes largeness, but as an arbitrary figure, we may say that **N should be at least 50,** however, few the cells.

(*b*) No theoretical cell-frequency should be small. Here again, it is difficult to say what constitutes smallness, but 5 should be regarded as the very minimum and **10 is better.** If small theoretical frequencies occur (*i.e.*, < 10), the difficulty is overcome by grouping two or more classes together before calculating (O – E). **It is important to remember that the number of degrees of freedom is determined with the number of classes after regrouping.**

(*c*) The constraints on the cell frequencies, if any, should be linear.

Note. If any one of the theoretical frequency is less than 5, then we apply a corrected given by Yates, which is usually known as 'Yates correction for continuity', we add 0.5 to the cell frequency which is less than 5 and adjust the remaining cell frequency suitably so that the marginal total is not changed.

21. THE χ^2 DISTRIBUTION

For large sample sizes, the sampling distribution of χ^2 can be closely approximated by a continuous curve known as the chi-square distribution. The probability function of χ^2 distribution is given by

$$f(\chi)^2 = c(\chi^2)^{(\nu/2 - 1)} \; e^{-x^2/2}$$

where $e = 2.71828$, ν = number of degrees of freedom; c = a constant depending only on ν.

Symbolically, the degrees of freedom are denoted by the symbol ν or by d.f. and are obtained by the rule $\nu = n - k$, where k refers to the number of independent constraints.

In general, when we fit a binomial distribution the number of degrees of freedom is one less than the number of classes; when we fit a Poisson distribution the degrees of freedom are less than the number of classes, because we use the total frequency and the arithmetic mean to get the parameter of the Poisson distribution. When we fit a normal curve the number of degrees of freedom are 3 less than the number of classes, because in this fitting we use the total frequency, mean and standard deviation.

If the data is given in a series of "n" number then degrees of freedom = $n - 1$.

In the case of Binomial distribution d.f. = $n - 1$

In the case of Poisson distribution d.f. = $n - 2$

In the case of Normal distribution d.f. = $n - 3$.

6.22. χ^2 TEST AS A TEST OF GOODNESS OF FIT

χ^2 test enables us to ascertain how well the theoretical distributions such as Binomial, Poisson or Normal etc. fit empirical distributions, *i.e.*, distributions obtained from sample data. If the **calculated value of χ^2 is less than the table value** at a specified level (generally 5%) of significance, the **fit is considered to be good** *i.e.*, the divergence between actual and expected frequencies is attributed to fluctuations of simple sampling. If the calculated value of χ^2 is greater than the table value, the fit is considered to be poor.

ILLUSTRATIVE EXAMPLES

Example 1. *The following table gives the number of accidents that took place in an industry during various days of the week. Test if accidents are uniformly distributed over the week.*

Day	*Mon*	*Tue*	*Wed*	*Thu*	*Fri*	*Sat*
No. of accidents	*14*	*18*	*12*	*11*	*15*	*14*

Sol. H_0: Null hypothesis. The accidents are uniformly distributed over the week.

Under this H_0, the expected frequencies of the accidents on each of these days $= \frac{84}{6} = 14$

Observed frequency O_i	14	18	12	11	15	14
Expected frequency E_i	14	14	14	14	14	14
$(O_i - E_i)^2$	0	16	4	9	1	0

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{30}{14} = 2.1428.$$

Conclusion. Table value of χ^2 at 5% level for $(6 - 1 = 5$ d.f.) is 11.09.

Since the calculated value of χ^2 is less than the tabulated value H_0 is accepted *i.e.*, the accidents are uniformly distributed over the week.

Example 2. *A die is thrown 270 times and the results of these throws are given below:*

No. appeared on the die	*1*	*2*	*3*	*4*	*5*	*6*
Frequency	*40*	*32*	*29*	*59*	*57*	*59*

Test whether the die is biased or not.

Sol. Null hypothesis H_0: Die is unbiased.

Under this H_0, the expected frequencies for each digit is $\frac{276}{6} = 46$

To find the value of χ^2

O_i	40	32	29	59	57	59
E_i	46	46	46	46	46	46
$(O_i - E_i)^2$	36	196	289	169	121	169

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{980}{46} = 21.30.$$

Conclusion. Tabulated value of χ^2 at 5% level of significance for $(6-1=5)$ d.f. is 11.09. Since the calculated value of $\chi^2 = 21.30 > 11.07$ the tabulated value, H_0 is rejected.

i.e., Die is not unbiased or Die is biased.

Example 3. *The following table shows the distribution of digits in numbers chosen at random from a telephone directory.*

Digits	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
Frequency	*1026*	*1107*	*997*	*966*	*1075*	*933*	*1107*	*972*	*964*	*853*

Test whether the digits may be taken to occur equally frequently in the directory.

Sol. Null hypothesis H_0: The digits taken in the directory occur equally frequency.

i.e., there is no significant difference between the observed and expected frequency.

Under H_0, the expected frequency is given by $= \dfrac{10{,}000}{10} = 1000$

To find the value of χ^2

O_i	1026	1107	997	996	1075	1107	933	972	964	853
E_i	1000	1000	1000	1000	1000	1000	1107	1000	1000	1000
$(O_i - E_i)^2$	676	11449	9	1156	5625	11449	4489	784	1296	21609

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{58542}{1000} = 58.542.$$

Conclusion. The tabulated value of χ^2 at 5% level of significance for 9 d.f. is 16.919. Since the calculated value of χ^2 is greater than the tabulated value, H_0 is rejected.

i.e., there is significant difference between the observed and theoretical frequency.

i.e., the digits taken in the directory do not occur equally frequently.

Example 4. *Records taken of the number of male and female births in 800 families having four children are as follows:*

No. of male births	*0*	*1*	*2*	*3*	*4*
No. of female births	*4*	*3*	*2*	*1*	*0*
No. of families	*32*	*178*	*290*	*236*	*94*

Test whether the data are consistent with the hypothesis that the binomial law holds and the chance of male birth is equal to that of female birth, namely $p = q = 1/2$.

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Sol. H_0: The data are consistent with the hypothesis of equal probability for male and female births. *i.e.*, $p = q = 1/2$.

We use Binomial distribution to calculate theoretical frequency given by:

$N(r) = N \times P(X = r)$

where N is the total frequency. $N(r)$ is the number of families with r male children:

$P(X = r) = {}^nC_r p^r q^{n-r}$

where p and q are probability of male and female birth, n is the number of children.

$$N(0) = \text{No. of families with 0 male children} = 800 \times {}^4C_0 \left(\frac{1}{2}\right)^4 = 800 \times 1 \times \frac{1}{2^4} = 50$$

$$N(1) = 800 \times {}^4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 200; \; N(2) = 800 \times {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 300$$

$$N(3) = 800 \times {}^4C_3 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 200; \; N(4) = 800 \times {}^4C_4 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = 50$$

Observed frequency O_i	32	178	290	236	94
Exp-frequency E_i	50	200	300	200	50
$(O_i - E_i)^2$	324	484	100	1296	1936
$\frac{(O_i - E_i)^2}{E_i}$	6.48	2.42	0.333	6.48	38.72

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 54.433.$$

Conclusion. Table value of χ^2 at 5% level of significance for $5 - 1 = 4$ d.f. is 9.49. Since the calculated value of χ^2 is greater than the tabulated value, H_0 is rejected. *i.e.*, the data are not consistent with the hypothesis that the Binomial law holds and that the chance of a male birth is not equal to that of a female birth.

Note. Since the fitting is binomial, the degrees of freedom $\nu = n - 1$ *i.e.*, $\nu = 5 - 1 = 4$.

Example 5. *Verify whether Poisson distribution can be assumed from the data given below:*

No. of defects	*0*	*1*	*2*	*3*	*4*	*5*
Frequency	*6*	*13*	*13*	*8*	*4*	*3*

Sol. H_0: Poisson fit is a good fit to the data.

$$\text{Mean of the given distribution} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{94}{97} = 2$$

To fit a Poisson distribution we require m. Parameter $m = \bar{x} = 2$.

By Poisson distribution the frequency of r success is

$$N(r) = N \times e^{-m} \cdot \frac{m^r}{r!}, \text{ N is the total frequency.}$$

$$N(0) = 47 \times e^{-2} . \frac{(2)^0}{0!} = 6.36 \approx 6; \; N(1) = 47 \times e^{-2} . \frac{(2)^1}{1!} = 12.72 \approx 13$$

$$N(2) = 47 \times e^{-2} . \frac{(2)^2}{2!} = 12.72 \approx 13; \; N(3) = 47 \times e^{-2} . \frac{(2)^3}{3!} = 8.48 \approx 9$$

$$N(4) = 47 \times e^{-2} . \frac{(2)^4}{4!} = 4.24 \approx 4; \; N(5) = 47 \times e^{-2} . \frac{(2)^5}{5!} = 1.696 \approx 2$$

X	0	1	2	3	4	5
O_i	6	13	13	8	4	3
E_i	6.36	12.72	12.72	8.48	4.24	1.696
$\frac{(O_i - E_i)^2}{E_i}$	0.2037	0.00616	0.00616	0.02716	0.0135	1.0026

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 1.2864.$$

Conclusion. The calculated value of χ^2 is 1.2864. Tabulated value of χ^2 at 5% level of significance for $\gamma = 6 - 2 = 4$ d.f. is 9.49. Since the calculated value of χ^2 is less than that of tabulated value. H_0 is accepted *i.e.*, Poisson distribution provides a good fit to the data.

Example 6. *The theory predicts the proportion of beans in the four groups, G_1, G_2, G_3, G_4 should be in the ratio 9 : 3 : 3 : 1. In an experiment with 1600 beans the numbers in the four groups were 882, 313, 287 and 118. Does the experimental result support the theory?*

Sol. H_0. The experimental result support the theory. *i.e.*, there is no significant difference between the observed and theoretical frequency under H_0, the theoretical frequency can be calculated as follows:

$$E(G_1) = \frac{1600 \times 9}{16} = 900; \; E(G_2) = \frac{1600 \times 3}{16} = 300;$$

$$E(G_3) = \frac{1600 \times 3}{16} = 300; \; E(G_4) = \frac{1600 \times 1}{16} = 100$$

To calculate the value of χ^2.

Observed frequency O_i	882	313	287	118
Exp. frequency E_i	900	300	300	100
$\frac{(O_i - E_i)^2}{E_i}$	0.36	0.5633	0.5633	3.24

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 4.7266.$$

Conclusion. Table value of χ^2 at 5% level of significance for 3 d.f. is 7.815. Since the calculated value of χ^2 is less than that of the tabulated value. Hence H_0 is accepted *i.e.*, the experimental result support the theory.