3 Chapter

BIVARIATE DISTRIBUTION

INSIDE THIS CHAPTER

3.1. JOINT PROBABILITY

Two random variable X and Y are said to be jointly distributed if they are defined on same probability space. The joint probability function is denoted by $P_{XY}(x, y)$ or $f_{XY}(x, y)$.

3.2. JOINT PROBABILITY MASS FUNCTION

Let X and Y be random variables on a sample space S with respective image sets $X(S) = \{x_1, x_2 \dots x_n\}$ and $Y(S) = \{y_1, y_2 \dots y_m\}$. The function p on $X(S) \times Y(S)$ defined as $p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j)$ is called *joint probability function* of X and Y where $X(S) \times Y(S) = \{x_1, x_2 \dots x_n\} \times \{y_1, y_2 \dots y_m\}$.

The tabular representation is given below:

$x \rightarrow$ \ $y \downarrow$	y_1	y_2	y_3	...	y_j	...	y_m	Total
x_1	p_{11}	p_{12}	p_{13}	...	p_{1j}	...	p_{1m}	p_1
x_2	p_{21}	p_{22}	p_{23}	...	p_{2j}	...	p_{2m}	p_2
x_3	p_{31}	p_{32}	p_{33}	...	p_{3j}	...	p_{3m}	p_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_i	p_{i1}	p_{i2}	p_{i3}	...	p_{ij}	...	p_{im}	p_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_n	p_{n1}	p_{n2}	p_{n3}	...	p_{nj}	...	p_{mn}	p_n
Total	p_1	p_2	p_3	...	p_j	...	p_m	

Also $$\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$

3.3. MARGINAL AND CONDITIONAL PROBABILITY FUNCTION

Suppose the joint distribution of two random variable X and Y is given then the probability distribution of X is determined as follows

$$f_X(x) = p_X(x_i) = P(X = x_i) = p_{i1} + p_{i2} \dots + p_{ij} + \dots + p_{im}$$
$$= \sum_{j=1}^{m} p_{ij} = p_i$$

and is known as *marginal probability function of X*.

Similarly

$$f_Y(x) = p_y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = p_j$$

is called *marginal probability function of Y*.

Also
$$f_{X/Y}(x/y) = P(X = x/\,Y = y_j) = \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)} = \frac{p(x_i, y_j)}{p(y_j)} = \frac{p_{ij}}{p_j}$$

This called *conditional probability function of X* when $Y = y_j$ is given

Similarly
$$f_{Y/X}(y/x) = P(Y = y_j / X = x_i) = \frac{p(x_i, y_j)}{p(x_i)} = \frac{p_{ij}}{p_i}$$

is conditional probability function of Y when $X = x_i$ is given

Note: Two random variables X and Y are said to be *independent* if

$$P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j)$$

otherwise they are *dependent*.

3.4. JOINT PROBABILITY DISTRIBUTION FUNCTION

Let (X, Y) be a two dimensional random variable then their joint distribution function is denoted by $F_{X,Y}(x, y)$ and is defined as

$$F_{X,Y}(x, y) = P(X \le x, Y \le y) \ \forall\ x, y \in R$$

where $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = 1$, in case of continuous random variable

or $\sum_x \sum_y f_{X,Y}(x, y) = 1$, in case of discrete random variable

3.5. PROPERTIES OF JOINT DISTRIBUTION FUNCTION

A joint distribution function of two variables has following properties:

1. If $x_1 < x_2$ and $y_1 < y_2$ then

$$P[x_1 < X \le x_2, y_1 < Y \le y_2] = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \ge 0 \text{ [Rectangle Rule]}$$

2. (*a*) $F(-\infty, y) = \lim_{x \to -\infty} F(x, y) = 0 \ \forall\ y \in R$

(*b*) $F(x, -\infty) = \lim_{y \to -\infty} F(x, y) = 0 \ \forall\ x \in R$

(*c*) $\lim_{\substack{x \to \infty \\ y \to \infty}} F(x, y) = F(\infty, \infty) = 1$

3. $F(x, y)$ is right continuous in each argument *i.e.*,

$$\lim_{h \to 0^+} F(x + h, y) = \lim_{h \to 0^+} F(x, y + h) = F(x, y)$$

4. If the density function $f(x, y)$ is continuous at (x, y) then

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$$

Expectation

Let (X, Y) be two dimensional discrete random variable with joint discrete density function $f_{X,Y}(x, y)$. The expectation of $g(X, Y)$ is denoted by $E(g(X, Y))$ and is defined as

$$E[g(X, Y)] = \sum_x \sum_y g(x, y)\, f_{X,Y}(x, y)$$

In particular
$$E(X) = \Sigma x f_X(x)$$
$$E(Y) = \Sigma y f_Y(y)$$
$$E(XY) = \sum_x \sum_y xy\, f_{X,Y}(x, y)$$

Covariance

Let X and Y be two random variables defined on the same probability space. The covariance of X and Y is defined as

$$\text{Cov}(X, Y) = E(X - E(X))\, E(Y - E(Y))$$
$$= E(XY) - E(X)\, E(Y)$$

Correlation coefficient

The correlation coefficient of two random variable X and Y is defined as

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var } X}\sqrt{\text{Var } Y}} = \frac{\text{Cov}(X, Y)}{\sigma_X\, \sigma_Y}$$

where $\sigma_X > 0$ and $\sigma_Y > 0$

Note: $-1 \le \rho(X, Y) \le 1$

Conditional Expectation

If (X, Y) are joint discrete random variable then conditional expectation of $g(X, Y)$ given $X = x$ is defined as

$$E[g(x, y)/X = x] = \sum_j g(x_i\, y_j)\, f_{Y|X}(y_j \,|\, X)$$

In particular
$$E[Y/X = x] = \sum_j y_j\, f_{Y|X}(y_j \,|\, x) = \sum_j y_j\, P(Y = y_j \,|\, X = x)$$

SOLVED EXAMPLES

Example 3.1. For the following bivariate probability distribution of X and Y find:

(i) $P(X \le 2, Y = 3)$ (ii) $P(X \le 1)$ (iii) $P(Y = 4)$

(iv) $P(Y \le 5)$ (v) $P(X < 2, Y \le 3)$

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution. The marginal distributions are given below:

X \ Y	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	1

(i) $P(X \le 2, Y = 3) = P(X = 0, Y = 3) + P(X = 1, Y = 3) + P(X = 2, Y = 3)$

$$= \frac{1}{32} + \frac{1}{8} + \frac{1}{64} = \frac{2+8+1}{64} = \frac{11}{64}$$

(ii) $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$

(iii) $P(Y = y) = \frac{13}{64}$

(iv) $P(Y \le 5) = + P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) + P(Y = 5)$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} + \frac{13}{64} + \frac{6}{32}$$

$$= \frac{6+6+11+13+12}{64} = \frac{48}{64}$$

(v) $P(X < 2)\ Y \le 3) = + P(X = 0, Y = 1) + P(X = 0, Y = 2) + P(X = 0, Y = 3) + P(X = 1, Y = 1)$
$+ P(X = 1, Y = 2) + P(X = 1, Y = 3)$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \frac{1+2+2+4}{32} = \frac{9}{32}$$

Example 3.2. Let X and Y have joint p.d.f.

X \ Y	−1	0	1
0	b	$2b$	b
1	$3b$	$2b$	b
2	$2b$	b	$2b$

Find marginal distribution of X and Y. Also find conditional distribution of X given $Y = 1$.

Solution.

Y \ X	−1	0	1	$p_Y(y)$
0	b	$2b$	b	$4b$
1	$3b$	$2b$	b	$6b$
2	$2b$	b	$2b$	$5b$
$p_X(x)$	$6b$	$5b$	$4b$	$15b$

Marginal distribution of X

$$P(X = -1) = 6b, \quad P(X = 0) = 5b, \quad P(X = 1) = 4b$$

Marginal distribution of Y

$$P(Y = 0) = 4b, \quad P(Y = 1) = 6b, \quad P(Y = 2) = 5b$$

Conditional distribution of X when $Y = 1$

$$P(X = x/Y = 1) = \frac{P(X = x \cap Y = 1)}{P(Y = 1)}$$

$$= \begin{cases} \dfrac{P(X = -1 \cap Y = 1)}{P(Y = 1)} \\ \dfrac{P(X = 0 \cap Y = 1)}{P(Y = 1)} \\ \dfrac{P(X = 1 \cap Y = 1)}{P(Y = 1)} \end{cases}$$

$$= \begin{cases} \dfrac{3b}{6b} = \dfrac{1}{2} & \text{when} \quad X = -1/Y = 1 \\ \dfrac{2b}{6b} = \dfrac{1}{3} & \text{when} \quad X = 0/Y = 1 \\ \dfrac{b}{6b} = \dfrac{1}{6} & \text{when} \quad X = 1/Y = 1 \end{cases}$$

Example 3.3. **The joint probability distribution of X and Y is given in the following table.**

X \ Y	1	3	9
2	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$
4	$\frac{1}{4}$	$\frac{1}{4}$	0
6	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$

(*a*) **Find the marginal probability distribution of Y.**

(*b*) **Find the conditional distribution of Y given $X = 4$.**

(*c*) **Find covariance of X and Y.**

(*d*) **Are X and Y independent?**

Solution.

X \ Y	1	3	9	$f_X(x)$
2	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{6}{24}$
4	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{2}{4}$
6	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{6}{24}$
$f_Y(y)$	$\frac{4}{8}$	$\frac{8}{24}$	$\frac{2}{12}$	1

(*a*) Marginal probability distribution of Y

$$P(Y=1) = \frac{4}{8} = \frac{1}{2}, \quad P(Y=2) = \frac{8}{24} = \frac{1}{3}, \quad P(Y=3) = \frac{2}{12} = \frac{1}{6}$$

(*b*) Conditional probability of Y when $X = 4$

$$P(Y = y/X = 4) = \frac{P(Y = y \cap X = 4)}{P(X = 4)}$$

$$P(Y = 1/X = 4) = \frac{P(Y = 1 \cap X = 4)}{P(X = 4)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

$$P(Y = 3/X = 4) = \frac{P(Y = 3 \cap X = 4)}{P(X = 4)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

$$P(Y = 9/X = 4) = \frac{P(Y = 9 \cap X = 4)}{P(X = 4)} = 0$$

(*c*) To find Cov (X, Y) we will first find $E(XY)$, $E(X)$ and $E(Y)$

$$E(X) = \Sigma x f_X(x) = 2 \times \frac{6}{24} + 4 \times \frac{2}{4} + 6 \times \frac{6}{24} = 4$$

$$E(Y) = \Sigma y f_Y(y) = 1 \times \frac{4}{8} + 3 \times \frac{8}{24} + 9 \times \frac{2}{12} = 3$$

$$E(X, Y) = \Sigma xy\, f_{X,Y}(x, y)$$

$$= \left(2\times\frac{1}{8} + 4\times\frac{1}{4} + 6\times\frac{1}{8}\right) + \left(6\times\frac{1}{24} + 12\times\frac{1}{4} + 18\times\frac{1}{24}\right)$$

$$+ \left(18\times\frac{1}{12} + 36\times 0 + 54\times\frac{1}{12}\right) = 12$$

$$\text{Cov}(X, Y) = E(XY) - E(X)\,E(Y) = 12 - 12 = 0$$

(*d*)
$$f_{X,Y}(4, 3) = \frac{1}{4} \quad f_X(4) = \frac{2}{4} \quad f_Y(3) = \frac{8}{24}$$

$$f_X(4)\, f_Y(3) = \frac{2}{4} \times \frac{8}{24} = \frac{1}{6}$$

$\therefore$
$$f_{X,Y}(4, 3) \neq f_X(4)\, f_Y(3)$$

$\therefore$ X and Y are not independent.

Example 3.4. **Two tetrahedra with sides numbered 1 to 4 are tossed. Let X denote the number on the downturned face of the first tetrahedron and Y the larger of the downturned numbers. Find**

(*i*) **the joint density function of X and Y,**

(*ii*) **the marginal density function of X and Y,**

(*iii*) **$P[X \leq 2, Y \leq 3]$,**

(*iv*) **the conditional distribution of Y given $X = 2$ and $X = 3$,**

(*v*) **$E[Y/X = 2]$ and $E[Y/X = 3]$,**

(*vi*) **$\rho(X, Y)$**

Solution. (*i*) It is given that X denotes the number of the downturned face of the first tetrahedron and Y the larger of the numbers on the two downturned faces. Then (X, Y) takes the values:

(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)

The joint discrete density function of X and Y is given below:

(x, y)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(2, 2)	(2, 3)	(2, 4)	(3, 3)	(3, 4)	(4, 4)
$f_{X,Y}(x, y)$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{4}{16}$

We can also write the above data as follows:

Y \ X	1	2	3	4	$f_Y(x)$
1	$\frac{1}{16}$	0	0	0	$\frac{1}{16}$
2	$\frac{1}{16}$	$\frac{2}{16}$	0	0	$\frac{3}{16}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	0	$\frac{5}{16}$
4	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{7}{16}$
$f_X(x)$	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{4}{16}$	1

(*ii*) Marginal distribution of X. From the above table, we get

$$f_X(1) = \frac{4}{16},\ f_X(2) = \frac{4}{16},\ f_X(3) = \frac{4}{16},\ f_X(4) = \frac{4}{16}$$

Marginal distribution of Y. From the above table, we get

$$f_Y(1) = \frac{1}{16},\ f_Y(2) = \frac{3}{16},\ f_Y(3) = \frac{5}{16},\ f_Y(4) = \frac{7}{16}$$

(*iii*) $P[X \le 2, Y \le 3] = P[X = 1, Y = 1] + P[X = 1, Y = 2] + P[X = 1, Y = 3] + P[X = 2, Y = 2]$
$+ P[X = 2, Y = 3]$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{2}{16} + \frac{1}{16} = \frac{6}{16} = \frac{3}{8}$$

(*iv*) Conditional distribution of Y given $X = 2$. We have

$$P[Y = y \cap X = 2] = P[X = 2]\, P[Y = y/X = 2]$$

$\therefore$ $$P[Y = y \mid X = 2] = \frac{P[Y = y \cap X = 2]}{P[X = 2]}$$

Now $$P[Y = 2 \mid X = 2] = \frac{P[Y = 2 \cap X = 2]}{P[X = 2]} = \frac{2/16}{4/16} = \frac{1}{2},$$

$$P[Y = 3 \mid X = 2] = \frac{P[Y = 3 \cap X = 2]}{P[X = 2]} = \frac{1/16}{4/16} = \frac{1}{4},$$

$$P[Y = 4 \mid X = 2] = \frac{P[Y = 4 \cap X = 2]}{P[X = 2]} = \frac{1/16}{4/16} = \frac{1}{4}$$

Conditional distribution of Y given $X = 3$. We have

$$P[Y = 3 \mid X = 3] = \frac{P[Y = 3 \cap X = 3]}{P[X = 3]} = \frac{3/16}{4/16} = \frac{3}{4}$$

$$P[Y = 4 \mid X = 3] = \frac{P[Y = 4 \cap X = 3]}{P[X = 3]} = \frac{1/16}{4/16} = \frac{1}{4}$$

(*v*) By definition

$$E[Y \mid X = 2] = \Sigma y_j\, P[Y = y_j \mid X = 2]$$
$$= 2P[Y = 2 \mid X = 2] + 3P[Y = 3 \mid X = 2] + 4P[Y + 4 \mid X = 2]$$
$$= 2 \times \frac{1}{2} + 3 \times \frac{1}{4} + 4 \times \frac{1}{4} = \frac{11}{4} \quad \text{[Use part (iv)]}$$

$$E[Y \mid X = 3] = \Sigma y_j\, P[Y = y_j \mid X = 3]$$
$$= 3P[Y = 3 \mid X = 3] + 4P[Y = 4 \mid X = 3]$$
$$= 3 \times \frac{3}{4} + 4 \times \frac{1}{4} = \frac{13}{4} \quad \text{[Use part (iv)]}$$

(*vi*) Now we shall find $\rho(X, Y)$ *i.e.*, the correlation coefficient between X and Y. From the table, we have

$$E[X] = \Sigma x\, f_X(x) = 1 \times \frac{4}{16} + 2 \times \frac{4}{16} + 3 \times \frac{4}{16} + 4 \times \frac{4}{16} = \frac{5}{2}$$

$$E[X^2] = \Sigma x^2 f_X(x) = 1^2 \times \frac{4}{16} \times 2^2 \times \frac{4}{16} + 3^2 \times \frac{4}{16} + 4^2 \times \frac{4}{16} = \frac{15}{2}$$

$$E[Y] = \Sigma y\, f_Y(y) = 1 \times \frac{1}{16} + 2 \times \frac{3}{16} + 3 \times \frac{5}{16} + 4 \times \frac{7}{16} = \frac{25}{8}$$

$$E[Y^2] = \Sigma y^2 f_Y(y) = 1^2 \times \frac{1}{16} + 2^2 \times \frac{3}{16} + 3^2 \times \frac{5}{16} + 4^2 \times \frac{7}{16} = \frac{85}{8}$$

$$E[XY] = \Sigma xy\, f_{X,Y}(x, y) = \left(1 \times \frac{1}{16} + 2 \times \frac{1}{16} + 3 \times \frac{1}{16} + 4 \times \frac{1}{16}\right)$$
$$+ \left(4 \times \frac{2}{16} + 6 \times \frac{1}{16} + 8 \times \frac{1}{16}\right) + \left(9 \times \frac{3}{16} + 12 \times \frac{1}{16}\right) + 16 \times \frac{4}{16} = \frac{135}{16}$$

$\therefore$ $$\text{Var}[X] = E[X^2] - \{E[X]\}^2 = \frac{15}{2} - \frac{25}{4} = \frac{5}{4}$$

$$\text{Var}[Y] = E[Y^2] - \{E[Y]\}^2 = \frac{85}{8} - \left(\frac{25}{8}\right)^2 = \frac{55}{64}$$

$$\text{Cov}(X, Y) = E[X, Y] - E[X]\, E[Y] = \frac{135}{16} - \frac{5}{2} \times \frac{25}{8} = \frac{5}{8}$$

Hence, $$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = \frac{5/8}{\sqrt{5/4 \times 55/64}} = \frac{2}{\sqrt{11}}$$

Example 3.5. Three fair coins are tossed. Let X denote the number of heads on the first two coins, and let Y denote the numbers of tails on the last two coins.

(a) Find the joint distribution of X and Y.
(b) Find the conditional distribution of Y given that $X = 1$.
(c) Find Cov (X, Y)
(d) Find $\rho(X, Y)$

Solution. Eight (2^3) possible outcomes of the experiment are:

	HHH	HHT	HTH	HTT	THH	TTT	TTH	THT
X:	2	2	1	1	1	0	0	1
Y:	0	1	1	2	0	2	1	1

(a) The joint distribution of X and Y is given below:

Y \ X	0	1	2	$f_Y(y)$
0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}=\frac{1}{4}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8}=\frac{1}{2}$
2	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{2}{8}=\frac{1}{4}$
$f_X[X]$	$\frac{2}{8}=\frac{1}{4}$	$\frac{4}{8}=\frac{1}{2}$	$\frac{2}{8}=\frac{1}{4}$	1

(b) To find the conditional distribution of Y given that $X = 1$

i.e., to find $\quad f_{Y|X}(y \mid X) = \dfrac{f(1,2)}{f_x(1)} = 2f(1, y)$, since $f_X(1) = \dfrac{1}{2}$

Hence,

$$f_{Y|X}(X=1) = \begin{cases} 2\times\frac{1}{8}=\frac{1}{4} & \text{at } y=0, \\ 2\times\frac{2}{8}=\frac{1}{2} & \text{at } y=1, \\ 2\times\frac{1}{8}=\frac{1}{4} & \text{at } y=2, \end{cases}$$

(c) $E[X] = 0\times\frac{1}{4}+1\times\frac{1}{2}+2\times\frac{1}{4} = 1$. Similarly, $E[Y] = 1$.

$$E[XY] = 1\times\frac{2}{8}+2\times\frac{1}{8}+2\times\frac{1}{8}+4\times 0 = \frac{6}{8}=\frac{4}{3}$$

Hence, $\quad \text{cov}(X, Y) = E[XY] - E[X]\,E[Y] = \frac{3}{4}-1=-\frac{1}{4}$

$$E[X^2] = 0^2\times\frac{1}{4}+1^2\times\frac{1}{2}+2^2\times\frac{1}{4}=\frac{3}{2}$$

Also $\quad \text{Var}[X] = E[X^2]-\{E[X]\}^2 = \frac{3}{2}-1=\frac{1}{2}$. Also var $[Y] = \frac{1}{2}$

$\therefore$

Hence, $\quad \rho(X, Y) = \dfrac{\text{cov}(X, Y)}{\sigma_X\sigma_Y} = \dfrac{-1/4}{1/2} = -0.5$

Example 3.6. Consider a sample of size 2 drawn without replacement from an urn containing three balls, numbered 1, 2 and 3. Let X be the smaller of the two numbers drawn and Y the larger.

(a) Find the joint discrete density function of X and Y.
(b) Find the conditional distribution of Y given $X = 1$.
(c) Find $\rho(X, Y)$.

Solution. (a) (X, Y) takes the values:

(1, 2) (1, 3) (2, 1) (2, 3) (3, 1) (3, 2)

The distributions of X and Y are given below:

X:	1	1	1	2	1	2
Y:	2	3	2	3	3	3

(a) The joint discrete density function of X and Y is given below:

Y \ X	1	2	$f_Y(y)$
2	$\frac{2}{6}$	0	$\frac{2}{6}$
3	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{4}{6}$
$f_X(x)$	$\frac{4}{6}$	$\frac{2}{6}$	1

(b) The conditional distribution of Y given $X = 1$ is given by

$$f_{Y/X}(y/X=1) = \frac{f(1,y)}{f_X(1)} = \frac{f(1,y)}{4/6} = \frac{3}{2}f(1,y)$$

$$= \begin{cases} \frac{3}{2}\times\frac{2}{6}=\frac{1}{2}, & y=2 \\ \frac{3}{2}\times\frac{2}{6}=\frac{1}{2}, & y=3. \end{cases}$$

$\therefore \quad P[Y=2 \mid X=1] = P[Y=3 \mid X=1] = \frac{1}{2}$

(c)

$$E[X] = 1\times\frac{4}{6}+2\times\frac{2}{6}=\frac{4}{3},\ E[X^2] = 1^2\times\frac{4}{6}+2^2\times\frac{2}{6}=2,$$

$$E[Y] = 2\times\frac{2}{6}+3\times\frac{4}{6}=\frac{8}{3},\ E[Y^2] = 2^2\times\frac{2}{6}+3^2\times\frac{4}{6}=\frac{22}{3}$$

$$E[XY] = 2\times\frac{2}{6}+3\times\frac{2}{6}+4\times 0+6\times\frac{2}{6}=\frac{11}{3}$$

$$\text{Cov}(X, Y) = E[XY]-E[X]\,E[Y] = \frac{11}{3}-\frac{4}{3}\times\frac{8}{3}=\frac{1}{9}$$

$\therefore$

$$\text{var}[X] = E[X^2]-\{E[X]\}^2-2-\left(\frac{4}{3}\right)^2=\frac{2}{9}$$

Now

$$\text{var}[Y] = E[Y^2]-\{E[Y]\}^2 = \frac{22}{3}-\left(\frac{8}{3}\right)^2=\frac{2}{9}$$

Hence,

$$\rho(X, Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}} = \frac{1/9}{\sqrt{2/9}\sqrt{2/9}}=\frac{1}{9}\times\frac{9}{2}=\frac{1}{2}=0.5$$

Example 3.7. **X and Y are two random variable having joint density function $= \frac{1}{27}(2x+y)$, where x and y can assume only integer values 0, 1 and 2. Find conditional distribution of Y for $X = x$.**

Solution. $f(x, y) = \frac{1}{27}(2x+y)\ x = 0, 1, 2; y = 0, 1, 2$

The joint probability distribution is given below

X \ Y	*0*	*1*	*2*	$f_X(x)$
0	0	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{3}{27}$
1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$	$\frac{9}{27}$
2	$\frac{4}{27}$	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{15}{27}$
$f_Y(y)$	$\frac{6}{27}$	$\frac{9}{27}$	$\frac{12}{27}$	1

The marginal probability distribution of X is

$$f_X(x) = \sum_y f(x, y)$$

$$f_X(0) = \frac{3}{27},\ f_X(1) = \frac{9}{27},\ f_X(2) = \frac{15}{27}$$

The condition distribution of Y for $X = x$ is given by

$$f_{Y/X}(y/x) = \frac{f(x, y)}{f_X(x)}$$

and is obtained in table below:

X \ Y	*0*	*1*	*2*
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$
2	$\frac{4}{15}$	$\frac{5}{15}$	$\frac{6}{15}$

EXERCISE 3.1

1. Two tetrahedral with sides numbered 1 to 4 are tossed. Let X denote the smaller of down turned face of two tetrahedro and Y the larger of two.
   (a) Find joint density function of X and Y
   (b) Find $P(X \geq 2, Y \geq 2)$
   (c) Find mean and variance of X and Y.
   (d) Find the conditional distribution of Y given X for each possible value of X.
   (e) Find correlation coefficient of X and Y.

   [**Ans.** (b) $\frac{9}{16}$ (c) $\overline{X} = \frac{15}{8}$, $\overline{Y} = \frac{25}{8}$, variance $X = \frac{55}{64}$ variance $Y = \frac{55}{64}$ (e) 0.45]

2. Three coins are tossed. Let X denote the number of heads on first two and Y denote the number of heads on last two.
   (a) Find the joint distribution of X and Y.
   (b) Find $E[Y/X = 1]$ (c) Find $\rho(X, Y)$.

   [**Ans.** (a)

Y \ X	*0*	*1*	*2*
0	$\frac{1}{8}$	$\frac{1}{8}$	0
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$

(b) 1 (c) 0.5]

3. Consider a sample of size 2 drawn without replacement from an urn containing three balls, numbered 1, 2 and 3. Let X be the number on the first ball drawn and Y the larger of the two numbers drawn.
(a) Find joint discrete density function of X and Y.
(b) Find $P[X = 1 \mid Y = 3]$
(c) Find cov $P[X, Y]$
(d) Find $\rho[X, Y]$

[**Ans.** (a)

Y \ X	1	2	3
2	$\frac{1}{6}$	$\frac{1}{6}$	0
3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$

(b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) $\frac{\sqrt{3}}{4}$]

4. An urn contain four balls: Two of the balls are numbered with 1 and other two are numbered with 2. Two balls are drawn from the urn without replacement. Let X denote the smaller of the numbers on drawn balls and Y the larger:
(a) Find joint density of X and Y
(b) Find marginal distribution of Y
(c) Find cov (X, Y)

[**Ans.** (a)

Y \ X	1	2
1	$\frac{1}{6}$	0
2	$\frac{4}{6}$	$\frac{1}{6}$

(b) $\frac{5}{6}$ (c) $\frac{1}{5}$

5. *The joint density of X and Y is*

$$f(x_1 y) = \frac{x+2y}{27}$$

where x and y can assume only integer values 0, 1, 2. Find conditional distribution of Y for $X = x$.

$$\left[\textbf{Ans. } f_{Y/X}(y/x) = \frac{x+2y}{3x+6}\, x,\ y = 1, 2, 3\right]$$

6. Two discrete random variable X and Y have

$$P[X=0, Y=0] = \frac{2}{9},\quad P[X=0, Y=1] = \frac{1}{9}$$

$$P[X=1, Y=0] = \frac{1}{9},\quad P[X=1, Y=1] = \frac{5}{9}$$

Examine whether X and Y are independent. [**Ans.** No]

3.6. JOINT CONTINUOUS DENSITY FUNCTION

A two dimensional random variable (X, Y) is said to be continuous iff there exist a function $f_{X,Y}(x, y) \geq 0$ such that

$$F_{X,Y}(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u, v)\, du dv$$

The function $f_{X,Y}(x, y)$ or $f(x, y)$ is called *joint probability density function* which satisfy following properties

(i) $f(x, y) \geq 0$ for all $(x, y) \in R^2$ (ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx dy = 1$

Note: $F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$

3.7. MARGINAL AND CONDITIONAL PROBABILITY DENSITY FUNCTION

If X and Y are jointly continuous random variable with joint probability density function $f_{X,Y}(x, y)$ then $f_X(x)$ and $f_Y(y)$ are called *marginal probability density function* and are defined as

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

The *conditional probability function* of Y given $X = x$ is defined as

$$f_{Y/X}(y/x) = \frac{f_{X,Y}(x_1 y)}{f_X(x)} \quad \text{if } f_X(x) > 0$$

Similarly $$f_{X/Y}(x/y) = \frac{f_{X/Y}(x, y)}{f_Y(y)} \quad \text{if } f_Y(y) > 0$$

3.8. CONDITIONAL CUMULATIVE DISTRIBUTION

If X and Y are joint continuous random variables with joint probability density function $f_{X_1Y}(x_1 y)$ then *conditional cumulative distribution* of Y, given $X = x$ is defined as

$$F_{Y/X}(y \mid x) = \int_{-\infty}^{y} f_{Y/X}(z \mid x)\, dz$$

Similarly $$F_{X/Y}(x \mid y) = \int_{-\infty}^{x} f_{x/y}(z \mid y)\, dz$$

for all y such that $f_Y(y) > 0$.

Note: If X and Y are independent then

$$f_{X,Y}(x, y) = f_x(x) f_y(y)$$

$$\frac{f_{X,Y}(x, y)}{f_x(x)} = f_y(y)$$

$$f_{Y|X}(y|x) = f_Y(y)$$

Expectation

Let (X, Y) be a two dimensional continuous random variable with joint probability density function $f(X, Y)$. The expectation of $g(X, Y)$ denoted as $E[g(X, Y)]$

$$E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)\, f(x, y)\, dxdy$$

In particular,

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x, y)\, dxdy$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x, y)\, dxdy$$

$$E(XY) = \int_{-\infty}^{\infty} xyf(x, y)\, dxdy$$

Joint Moment Generating Function

The joint moment generating function of a two-dimensional random variable (X,Y), denoted by $m_{X,Y}(t_1, t_2)$, is defined as

$$m_{X,Y}(t_1, t_2) = E[e^{t_1 X + t_2 Y}],$$

where t_1 and t_2 are real parameters.

Remark 1. We observe that

$$M_X(t_1) = M_{X,Y}(t_1, 0) = \lim_{t_2 \to 0} M_{X,Y}(t_1, t_2)$$

$$M_Y(t_2) = M_{X,Y}(0, t_2) = \lim_{t_1 \to 0} M_{X,Y}(t_1, t_2)$$

Recall that $M_X(t_1)$, $M_Y(t_2)$ are moment generating functions of X and Y, respectively.

Remark 2. *If X and Y are independent random variables, then*

$$M_{X,Y}(t_1, t_2) = M_X(t_1) \cdot M_Y(t_2).$$

Proof. We have

$$M_{X,Y}(t_1, t_2) = E[e^{t_1 X + t_2 Y}] = E[e^{t_1 X} \cdot e^{t_2 Y}].$$

Taking $g_1(x) = e^{t_1 x}$ and $g_2(y) = e^{t_2 y}$ and using the independence of X and Y, it follows that

$$M_{X,Y}(t_1, t_2) = E[e^{t_1 X}]. E[e^{t_2 Y}]$$

$$= M_X(t_1) \cdot M_Y(t_2).$$

SOLVED EXAMPLES

Example 3.8. Find k so that $f(x, y) = kxy$, $1 \le x \le y \le 2$ will be a joint probability density function.

Solution. We have

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = \int_1^2\int_x^2 kxy\, dx\, dy$$

$$= k\int_1^2 x\left(\int_x^2 y\, dy\right) dx = k\int_1^2 x.\frac{1}{2}(4 - x^2)\, dx$$

$$= \frac{k}{2}\int_1^2 (4x - x^3)\, dx = \frac{k}{2}\left[2x^2 - \frac{x^4}{4}\right]_1^2$$

$$= \frac{k}{2}\left[2(4-1) - \frac{1}{4}(16-1)\right] = \frac{k}{2}\left(6 - \frac{15}{4}\right) = \frac{9k}{8}$$

Hence, $1 = \frac{9k}{8} \Rightarrow k = \frac{8}{9}$.

Example 3.9. Find K so that $f(x, y) = K(x + y)$, $0 < x < 1$ and $0 < y < 1$, is a joint probability density function.

Solution. We have

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = \int_0^1\int_0^1 K(x + y)\, dx\, dy$$

$$= K\int_0^1\left[\left(xy + \frac{1}{2}y^2\right)\right]_0^1 dx = K\int_0^1\left(x + \frac{1}{2}\right) dx$$

$$= K\left(\frac{1}{2} + \frac{1}{2}\right) = K. \text{ Hence } K = 1.$$

Example 3.10. Let the joint p.d.f of X and Y be

$$f(x, y) = (x + y),\ 0 \le x \le 1,\ 0 \le y \le 1$$
$$= 0, \text{ otherwise.}$$

Find $P\left[0 < X < \frac{1}{2}, 0 < Y < \frac{1}{4}\right]$, $E[X]$, $E[Y]$, $E[XY]$, $E[X + Y]$, $\rho[X, Y]$.

Solution. We have

$$P\left[0 < X < \frac{1}{2}, 0 < Y < \frac{1}{4}\right] = \int_0^{1/2}\left[\int_0^{1/4}(x + y)\, dy\right] dx = \int_0^{1/2}\left[xy + \frac{1}{2}y^2\right]_0^{1/4} dx$$

$$= \int_0^{1/2}\left(\frac{1}{4}x + \frac{1}{32}\right) dx = \frac{1}{4}\times\frac{1}{8} + \frac{1}{64} = \frac{3}{64}.$$

Now

$$E[X] = \int_0^1\int_0^1 x f(x, y)\, dx\, dy = \int_0^1\int_0^1 x(x+y)\, dx\, dy$$

$$= \int_0^1 \left[x^2 y + x\left(\frac{1}{2}y^2\right)\right]_0^1 dx = \int_0^1 \left(x^2 + \frac{1}{2}x\right) dx$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Similarly,

$$E[Y] = \int_0^1\int_0^1 y f(x, y)\, dx\, dy = \frac{7}{12}.$$

Now

$$E[XY] = \int_0^1\int_0^1 xy f(x, y)\, dx\, dy \int_0^1\int_0^1 xy(x+y)\, dx\, dy$$

$$= \int_0^1 \left[x^2\left(\frac{1}{2}y^2\right) + x\left(\frac{1}{3}y^3\right)\right]_0^1 dx = \int_0^1 \left(\frac{1}{2}x^2 + \frac{1}{3}x\right) dx$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

$$E[X+Y] = \int_0^1\int_0^1 (x+y) f(x, y)\, dx\, dy = \int_0^1\int_0^1 (x+y)^2\, dx\, dy$$

$$= \int_0^1\int_0^1 (x^2 + 2xy + y^2)\, dx\, dy$$

$$= \int_0^1 \left[x^2 y + xy^2 + \left(\frac{1}{3}y^3\right)\right]_0^1 dx$$

$$= \int_0^1 \left(x^2 + x + \frac{1}{3}\right) dx = \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{7}{6}.$$

We have $\operatorname{cov}[X, Y] = E[XY] - E[X]\,E[Y] = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}.$

Now

$$E[X^2] = \int_0^1\int_0^1 x^2 f(x, y)\, dx\, dy = \int_0^1\int_0^1 x^2(x+y)\, dx\, dy$$

$$= \int_0^1 \left[x^3 y + x^2\left(\frac{1}{2}y^2\right)\right]_0^1 dx$$

$$= \int_0^1 \left(x^3 + \frac{1}{2}x^2\right) dx = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Similarly,

$$E[Y^2] = \frac{5}{12}.$$

Now

$$\operatorname{var}[X] = E[X^2] - (E[X])^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

Similarly,

$$\operatorname{var}[Y] = \frac{11}{144}.$$

Hence,

$$\rho[X, Y] = \frac{\operatorname{cov}[X, Y]}{\sqrt{\operatorname{var}[X]\operatorname{var}[Y]}} = \frac{-\frac{1}{144}}{\sqrt{\frac{11}{144}\times\frac{11}{144}}} = -\frac{1}{11}.$$

Example 3.11. The joint p.d.f. of (X, Y) is given by

$$f(x, y) = 2;\ 0 < x < 1,\ 0 < y < x$$
$$= 0\ ;\ \text{elsewhere},$$

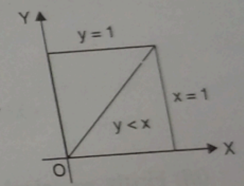
(i) **Find the marginal density functions of X and Y.**

(ii) **Find the conditional density function of Y given $X = x$ and that of X given $Y = y$**

(iii) **Are X and Y independent?**

Solution. It may be observed that

$$f(x, y) \geq 0 \text{ and } \int_0^1\left(\int_0^x 2\, dy\right) dx = \int_0^1 2x\, dx = 1$$



(*i*) The marginal density function of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_0^x 2dy = 2x, 0 < x < 1$$
$$= 0, \text{ elsewhere}.$$

The marginal density function of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_y^1 2dx = 2(1-y), 0 < y < 1$$
$$= 0, \text{ elsewhere}.$$

(*ii*) The conditional density function of Y given $X = x$ is

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x},\ 0 < x < 1.$$

The conditional density function of X given $Y = y$ is

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y},\ 0 < y < 1.$$

(*iii*) $f_X(x)\, f_Y(y) = 2x \times 2(1-y) = 4x(1-y) \neq f(x, y).$

Hence X and Y are not independent.

Example 3.12. If X and Y are two random variables having joint density function.

$$f(x, y) = \frac{1}{8}(6 - x - y);\ 0 < x < 2,\ 2 < y < 4$$
$$= 0, \text{ otherwise}$$

Find (i) $P[X < 1 \cap Y < 3]$, (ii) $P[X + Y < 3]$, (iii) $P[X < 1 \mid Y < 3]$.

Find also the marginal and conditional distributions.

Solution. We have

(i) $P[X < 1 \cap Y < 3] = \int_{-\infty}^{1}\int_{-\infty}^{3} f(x, y)\,dx\,dy$

$$= \int_0^1\int_2^3 \frac{1}{8}(6 - x - y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^1 \left\{(6 - x) - \frac{1}{2}(9 - 4)\right\}dx = \frac{1}{8}\int_0^1 \left(\frac{7}{2} - x\right)dx$$

$$= \frac{1}{8}\left(\frac{7}{2} - \frac{1}{2}\right) = \frac{3}{8}$$

(ii) $P[X + Y < 3] = \int_0^1\int_2^{3-x} \frac{1}{8}(6 - x - y)\,dx\,dy$

$$= \frac{1}{8}\int_0^1 \left[(6 - x)(1 - x) - \frac{1}{2}\{(3 - x)^2 - 4\}\right]dx$$

$$= \frac{1}{16}\int_0^1 (x^2 - 8x + 7)\,dx = \frac{1}{16}\left(\frac{1}{3} - 4 + 7\right) = \frac{5}{24}$$

(iii) Firstly, we calculate $P[Y < 3]$. We have

$$P[Y < 3] = \int_0^2\int_2^3 \frac{1}{8}(6 - x - y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \left\{(6 - x) - \frac{1}{2}(9 - 4)\right\}dx$$

$$= \frac{1}{8}\int_0^2 \left(\frac{7}{2} - x\right)dx = \frac{1}{8}\left(\frac{7}{2} \times 2 - 2\right) = \frac{5}{8}.$$

Now $P[X < 1 \mid Y < 3] = \dfrac{P[X < 1 \cap Y < 3]}{P[Y < 3]} = \dfrac{3/8}{5/8} = \dfrac{3}{5}.$

(iv) The marginal distribution of X is given by

$$f_X(x) = \int_2^4 f(x, y)\,dy = \frac{1}{8}\int_2^4 (6 - x - y)\,dy$$

$$= \frac{1}{8}\left[(6 - x) \times 2 - \frac{1}{2}(16 - 4)\right]$$

$$= \begin{cases} \frac{1}{4}(3 - x), & 0 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal distributions of Y is given by

$$= \int_0^2 \frac{1}{8}(6 - x - y)\,dx = \frac{1}{8}\left[6 \times 2 - \frac{1}{2} \times 4 - 2y\right]$$

$$= \begin{cases} \frac{1}{4}(5 - y), & 2 < y < 4 \\ 0, & \text{otherwise.} \end{cases}$$

The conditional distributions of X and Y are

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} = \frac{6 - x - y}{2(5 - y)},\ 0 < x < 2, 2 < y < 4$$

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(y)} = \frac{6 - x - y}{2(3 - x)},\ 0 < x < 2,\ 2 < y < 4.$$

Example 3.13. The probability density function of a continuous bivariate distribution is given by

$$f(x, y) = x + y, \text{ where } 0 \le x \le 1, 0 \le y \le 1$$
$$= 0, \text{ otherwise.}$$

Find the marginal distributions and the correlation coefficient of x and y.

Solution. The marginal distribution, of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy = \int_0^1 (x + y)\,dy$$

$$= \left[xy + \frac{1}{2}y^2\right]_0^1 = x + \frac{1}{2}.$$

$\therefore$
$$f_X(x) = x + \frac{1}{2},\ 0 \le x \le 1$$
$$= 0, \text{ otherwise}$$

Similarly
$$f_Y(y) = y + \frac{1}{2},\ 0 \le y \le 1$$
$$= 0\ , \text{ otherwise.}$$

Now
$$E[X] = \int_0^1 x f_X(x)\,dx = \int_0^1 x\left(x + \frac{1}{2}\right)dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

By symmetry, $E[Y] = E[X] = \frac{7}{12}$.

Now
$$E[XY] = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1\left(x^2.\frac{1}{2} + x.\frac{1}{3}\right)dx$$
$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

$\therefore$
$$\text{cov}(X, Y) = E[XY] - E[X]\,E[Y] = \frac{1}{3} - \frac{7}{12}\times\frac{7}{12} = -\frac{1}{144}.$$

Now
$$E[X^2] = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2\left(x + \frac{1}{2}\right)dx = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$\therefore$
$$\text{var}[X] = E[X^2] - \{E[X]\}^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

By symmetry, $\text{var}[Y] = \frac{11}{144}$.

The correlation coefficient of X and Y is given by
$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X\,\sigma_Y} = \frac{-1/144}{11/144} = -\frac{1}{11}.$$

Example 3.14. Two random variables X and Y have the joint density
$$f(x, y) = 2 - x - y,\ 0 < x < 1,\ 0 < y < 1$$
Find the marginal and conditional density functions of X and Y, and show that $\rho[X, Y] = -1/11$: Find $E[X \mid y]$.

Solution. The marginal density function of X is given by
$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy = \int_0^1 (2 - x - y)\,dy = 2 - x - \frac{1}{2} = \frac{3}{2} - x.$$

Thus
$$f_X(x) = \frac{3}{2} - x,\ 0 < x < 1$$
$$= 0, \text{ otherwise.}$$

By symmetry,
$$f_Y(y) = \frac{3}{2} - y,\ 0 < y < 1$$
$$= 0\ , \text{ otherwise.}$$

The conditional density functions of X and Y are
$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} = \frac{2 - x - y}{\frac{3}{2} - y},\ 0 < x, y < 1;$$
$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)} = \frac{2 - x - y}{\frac{3}{2} - x},\ 0 < x, y < 1.$$

Now
$$E[X \mid y] = \int_{-\infty}^{\infty} x.\frac{f(x, y)}{f_Y(y)}\,dx = \int_0^1 \frac{x(2 - x - y)}{\frac{3}{2} - y}\,dx$$
$$= \frac{1}{\left(\frac{2}{3} - y\right)}\left[1 - \frac{1}{3} - \frac{1}{2}y\right] = \frac{4 - 3y}{9 - 6y},\ 0 < y < 1.$$

Now we shall determine the correlation coefficient of X and Y.
$$E[X] = \int_0^1 x f_X(x)\,dx = \int_0^1 x\left(\frac{3}{2} - x\right)dx = \frac{3}{2}.\frac{1}{2} - \frac{1}{3} = \frac{5}{12}.$$

By symmetry, $E[Y] = E[X] = \frac{5}{12}$.

Now
$$E[XY] = \int_0^1\int_0^1 xy(2 - x - y)\,dx\,dy$$
$$= \int_0^1\left(2x.\frac{1}{2} - x^2.\frac{1}{2} - x.\frac{1}{3}\right)dx = \frac{1}{2} - \frac{1}{6} - \frac{1}{6} = \frac{1}{6}.$$

$\therefore$
$$\text{cov}(X, Y) = E[XY] - E[X]\,E[Y] = \frac{1}{6} - \frac{5}{12}.\frac{5}{12} = -\frac{1}{144}.$$
$$E[X^2] = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2\left(\frac{3}{2} - x\right)dx$$
$$= \frac{3}{2}.\frac{1}{3} - \frac{1}{4} = \frac{1}{4}.$$

$\therefore$
$$\text{var}[X] = E[X^2] - \{E[X]\}^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144}.$$

Hence,
$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X\,\sigma_Y} = \frac{-1//144}{11/144} = -\frac{1}{11}.$$

EXERCISE 3.2

1. If $f(x, y) = e^{-(x+y)},\ x \geq 0,\ y \geq 0$
$= 0$ elesewhere
is joint probability density function of random variable X and Y, find:
(*a*) $P[X < 1]$
(*b*) $P[X > 4]$
(*c*) $P[X + Y < 1]$

[**Ans.** (*a*) $1 - e^{-1}$, (*b*) $\frac{1}{2}$, (*c*) $1 - 2e^{-1}$]

2. The joint distribution of X and Y is

$$f(x, y) = K(x^2 + y^2), \; 0 \le x \le y \le 1$$
$$= 0 \qquad \text{elsewhere}$$

Determine K and find marginal densities of X and Y.

$\left[\textbf{Ans. } K = \frac{1}{3}\right]$

3. The joint probability density function of the two dimensional random variable (X, Y) is given below:

$$f(x, y) = \begin{cases} \frac{8}{9}xy & 1 \le x \le y < 2 \\ 0 & \text{elesewhere} \end{cases}$$

(a) Find marginal density function of X and Y:

(b) Find conditional density function of Y given $X = x$ and conditional density function of X given $Y = y$.

[**Ans.**
$$f_X(x) = \frac{4}{9}x(4 - x^2), \; 1 \le x \le 2$$
$$f_Y(y) = \frac{4}{9}y(y^2 - 1), \; 1 \le y \le 2$$
$$f_{Y/X}(y|x) = \frac{2y}{4 - x^2}, \; x \le y \le 2$$
$$f_{X/Y}(x|y) = \frac{2y}{y^2 - 1}, 1 \le x \le y \text{ .]}$$

4. If X and Y are two random variables having joint density function

$$f(x, y) = \frac{1}{8}(6 - x - y); \quad 0 < x < 2, \, 2 < y < 2$$
$$= 0 \qquad \text{otherwise}$$

Find: (a) $P(X < 1 \cap y < 3)$, (b) $P(X + Y < 3)$, (c) $P(X < 1 \mid 4 < 3)$

[**Ans.** (a) $\frac{3}{8}$, (b) $\frac{5}{24}$, (c) $\frac{3}{5}$]

5. Joint distribution of X and Y is given by

$$f(x, y) = 4xye^{-(x^2 + y^2)}, \; x \ge 0, \, y \ge 0$$

Test whether X and Y are independent. Also find the conditional density of X given $Y = y$.

[**Ans.** Yes, $f(X = x \mid Y = y) = 2xe^{-x^2}$]

3.9. BAYE'S THEOREM

Statement. Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive events accociated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then

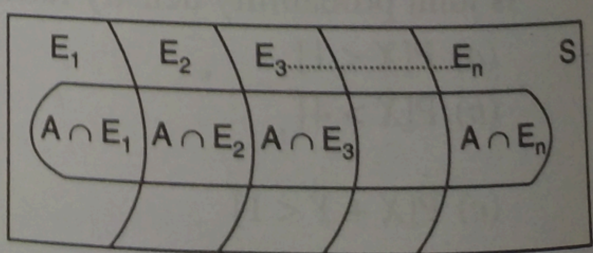


Fig. 3.1