Interpolation

Interpolation is the art of reading between the lines of the table.

Suppose we are given the following values of $y = f(x)$ for a set of values of x:

x :	x_0	x_1	x_2	x_3	x_n
$y=f(x)$:	y_0	y_1	y_2	y_3	y_n

Thus the process of finding the values of y corresponding to any values of $x = x_i$ between x_0 and x_n is called <u>interpolation</u>. While the process of finding the values of function outside the given range is called <u>extrapolation</u>.

The study of the interpolation is based on the concept of differences of a function.

Finite Difference

The calculus of finite differences deals with the changes that take places in the value of function (dependent variable), due to finite changes in the independent variable.

Let the function $y = f(x)$ is tabulated for the <u>equally spaced values</u> $x = x_0, x_0+h, x_0+2h, \ldots x_0+nh$ giving $y = y_0, y_1, y_2 \ldots y_n$. To determine the values of $f(x)$ for some intermediate values of x. The following types of differences are found useful

(i) ~~Ein~~. Forword Difference

(ii) Backword Difference

(iii) Central Difference.

Note Independent Variable are called argument. Dependent Variable are called entry.

Forward DIFFERENCE

Let $y = f(x)$ be any function have values $f(a), f(a+h)$, $f(a+2h), f(a+3h), \dots, f(a+nh)$ corresponding to the values $x = a, a+h, a+2h, a+3h, \dots a+nh$, thus Forward difference denoted by Δ and defined as

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

$$\Delta^2 f(x) = \Delta[\Delta f(x)] = \Delta[f(x+h) - f(x)]$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x).$$

Similarly

$$\Delta y_0 = y_1 - y_0$$

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_0 = y_2 - y_1 - y_1 + y_0 = y_2 - 2y_1 + y_0$$

BackWard Difference

Let $y = f(x)$ be any function have values $f(a), f(a+h)$, $f(a+2h), f(a+3h), \dots f(a+nh)$, corresponding to the values $x = a, a+h, a+2h, a+3h, \dots a+nh$, then backward difference is denoted by ∇ and defined as

$$\nabla f(x) = f(x) - f(x-h)$$

Forward Difference Table

Value of x (Arguments)	Value of y (Entries)	1'st diff Δ	2'nd diff Δ^2	3'rd diff Δ^3	4th diff Δ^4	5th diff Δ^5
x_0	y_0					
		Δy_0				
x_0+h	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_0+2h	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_0+3h	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_0+4h	y_4		$\Delta^2 y_3$			
		Δy_4				
x_0+5h	y_5					

BackWard Difference Table.

Value of x Arguments	Value of y Entries	Ist diff ∇	2'nd diff ∇^2	3'rd diff ∇^3	4th diff ∇^4	5th diff ∇^5
x_0	y_0					
		∇y_1	$\nabla^2 y_2$			
x_0+h	y_1			$\nabla^3 y_3$		
		∇y_2			$\nabla^4 y_4$	$\nabla^5 y_5$
x_0+2h	y_2		$\nabla^2 y_3$	$\nabla^3 y_4$		
		∇y_3	$\nabla^2 y_4$		$\nabla^4 y_5$	
x_0+3h	y_3			$\nabla^3 y_5$		
		∇y_4	$\nabla^2 y_5$			
x_0+4h	y_4					
		∇y_5				
x_0+5h	y_5					

RELATION BETWEEN OPERATORS.

SHIFT OPERATOR (E-OPERATOR)

It gives a constant increment (say h) to the given function. E-operator is also known as shift operator. E operator defined as

$$E f(x) = f(x+h)$$
$$E^2 f(x) = f(x+2h)$$
$$\vdots$$
$$E^n f(x) = f(x+nh)$$

Note E-operator can operate in both direction i.e Forward and backward.

e.g $E f(x) = f(x+h)$

$E^{1/2} f(x) = f(x+\frac{1}{2}h)$

$E^{-5} f(x) = f(x-5h)$

Relation Between E and Δ OPERATORS OR $E = 1+\Delta$

$\because$ $\Delta f(x) = f(x+h) - f(x)$

$\Delta f(x) = E f(x) - f(x) \Rightarrow \Delta = E - 1$

$$\boxed{E = 1+\Delta}$$

Relation Between E and ∇ OPERATORS OR $E = (1-\nabla)^{-1}$

$\because$ $\nabla f(x) = f(x) - f(x-h)$

$\nabla f(x) = f(x) - E^{-1} f(x)$

$\nabla = 1 - E^{-1} \Rightarrow E^{-1} = 1 - \nabla$

$$\boxed{E = (1-\nabla)^{-1}}$$

AVERAGING OPERATOR

Averaging operator is denoted by μ and defined

$$\boxed{\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]}$$

Relation Between μ and E.

$$\because \quad \mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-h/2\right)\right]$$

$$\mu f(x) = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$$

CENTRAL DIFFERENCE OPERATOR

Central Difference operator is denoted by δ and defined as

$$\boxed{\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-h/2\right)}$$

Relation Between δ and E

$$\because \quad \delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-h/2\right)$$

$$\delta f(x) = \left(E^{1/2} - E^{-1/2}\right) f(x)$$

$$\delta = E^{+1/2} - E^{-1/2}$$

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

Q. Prove that $\Delta = E - 1$ or $E = 1 + \Delta$

$$\because \quad \Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x) = E f(x) - f(x)$$

$$\Delta f(x) = (E-1) f(x)$$

$$\boxed{\Delta = E - 1}$$

Prove that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$, Where Δ is Forward and ∇ is Backward operator

Given $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$.

$$R.H.S = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}$$

$$= \frac{E-1}{\frac{E-1}{E}} - \frac{\frac{E-1}{E}}{E-1}$$

$$= E - E^{-1}$$

$$= 1+\Delta-(1-\nabla) = \cancel{1}+\Delta-\cancel{1}+\nabla$$

$$= \Delta+\nabla = \underline{L.H.S}$$

Q.2 Prove that

$$(E^{1/2}+E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta$$

$$L.H.S = (E^{1/2}+E^{-1/2})(1+\Delta)^{1/2} \qquad \because 1+\Delta = E$$

$$= (E^{1/2}+E^{-1/2})(E^{1/2})$$

$$= E + E^{0} = E+1 = 1+\Delta+1 = 2+\Delta \quad \underline{Ans}$$

Q.3 Prove that $(1+\Delta)(1-\nabla) = 1$

Sol^n R.H.S $(1+\Delta)(1-\nabla) = E \cdot E^{-1} = E^{0} = 1$

Q.4. Prove that $\nabla - \Delta = -\nabla\Delta$

Sol^n $L.H.S = \nabla - \Delta = (1-E^{-1})(E-1)$

$$= (1-\tfrac{1}{E})(E-1) = \frac{(E-1)(E-1)}{E}$$

$$= (E-1)(1-E^{-1})$$

$$= -(E-1)(E^{-1}-1)$$

$$= -\Delta\nabla$$

$$= \underline{-\Delta\nabla}$$

Q. Prove that $\nabla E = E\nabla = \Delta$

Sol^n We know that

$$\nabla f(x) = f(x) - f(x-h) \longrightarrow (1)$$

Taking shift operator both side.

$$E\nabla f(x) = Ef(x) - Ef(x-h)$$

$$E\nabla f(x) = f(x+h) - f(x-h+h)$$

$$E\nabla f(x) = f(x+h) - f(x)$$

$$E\nabla \cancel{f(x)} = \Delta \cancel{f(x)}$$

$$\boxed{E\nabla = \Delta}$$

Again $Ef(x) = f(x+h)$

Taking Backward operator both side.

$$\nabla E f(x) = \nabla f(x+h)$$

$$= f(x+h) - f(x)$$

$$\nabla E f(x) = \Delta f(x)$$

$$\boxed{\nabla E = \Delta}$$

Hence $\boxed{\nabla E = E\nabla = \Delta.}$

Q. Show that $\Delta \operatorname{Log} f(x) = \operatorname{Log}\left[1 + \frac{\Delta f(x)}{f(x)}\right]$

L.H.S $= \Delta \operatorname{Log} f(x)$

$= \operatorname{Log} f(x+h) - \operatorname{Log} f(x)$

$\because \Delta f(x) = f(x+h) - f(x)$

$f(x+h) = \Delta f(x) + f(x)$

$= \operatorname{Log}\left[\Delta f(x) + f(x)\right] - \operatorname{Log} f(x)$

$= \operatorname{Log}\left[\frac{\Delta f(x) + f(x)}{f(x)}\right]$

$= \operatorname{Log}\left[\frac{\Delta f(x)}{f(x)} + \frac{\cancel{f(x)}}{\cancel{f(x)}}\right] = \operatorname{Log}\left[1 + \frac{\Delta f(x)}{f(x)}\right]$ Ans

Prove that $\Delta = \frac{\delta^2}{2} + \delta\sqrt{1+\frac{\delta^2}{4}}$

Ans $\because \quad \delta = E^{1/2} - E^{-1/2} \longrightarrow (1)$

On squaring both side

$\delta^2 = E + E^{-1} - 2$

$\delta^2 = E + \frac{1}{E} - 2$

$E\delta^2 = E + 1 - 2E$

$(1+\Delta)\delta^2 = (E-1)^2 = \Delta^2$

$\delta^2 + \Delta\delta^2 = \Delta^2$

$\therefore \Delta^2 - \Delta\delta^2 - \delta^2 = 0$

$\therefore \Delta = \frac{\delta^2 \pm \sqrt{\delta^4 + 4\delta^2}}{2}$

$= \frac{\delta^2}{2} \pm \frac{\delta}{2}\sqrt{\delta^2+4}$

$\boxed{\Delta = \frac{\delta^2}{2} + \delta\sqrt{\frac{\delta^2}{4}+1}}$

Q. Prove that $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta$

L.H.S. $= (E^{1/2} + E^{-1/2})(1+\Delta)^{1/2}$

$= (E^{1/2} + E^{-1/2})(E^{1/2})$

$= E + E^0 = E+1 = 1+\Delta+1 = 2+\Delta.$

Q. Prove that $(E^{1/2} + E^{-1/2}) = 2(1+\frac{1}{2}\Delta)(1+\Delta)^{-1/2}$

R.H.S $= 2(1+\frac{1}{2}\Delta)(1+\Delta)^{-1/2}$

$= (2+\Delta)(1+\Delta)^{-1/2}$

$= (2+E-1)(1+E-1)^{-1/2}$

$= (E+1)(E^{-1/2}) = E^{1/2} + E^{-1/2}$ Ans

Q. Show that $\Delta\left[\frac{1}{f(x)}\right] = \frac{-\Delta f(x)}{f(x)\cdot f(x+1)}$, Using h = .

L.H.S = $\Delta\left[\frac{1}{f(x)}\right]$

$= \frac{1}{f(x+h)} - \frac{1}{f(x)}$ If h = 1

$= \frac{1}{f(x+1)} - \frac{1}{f(x)} = \frac{f(x) - f(x+1)}{f(x+1)\cdot f(x)}$

$= \frac{-[f(x+1) - f(x)]}{f(x+1)\cdot f(x)} = \frac{-\Delta f(x)}{f(x)\cdot f(x+1)}$ = R.H.S

Q. Prove that $\mu^2 = 1 + \frac{\delta^2}{4}$

Sol^n We know that

$\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

$\mu^2 = \frac{1}{4}[E^{1/2} + E^{-1/2}]^2$

$= \frac{1}{4}[(E^{1/2})^2 + (E^{-1/2})^2 + 2]$

$= \frac{1}{4}[(E^{1/2})^2 + (E^{-1/2})^2 - 2 + 4]$

$= \frac{1}{4}[(E^{1/2} - E^{-1/2})^2 + 4] = \frac{1}{4}[\delta^2 + 4]$

$\boxed{\mu^2 = 1 + \frac{\delta^2}{4}}$ Ans

Q. Relation Between E, δ, μ or Prove that $E^{1/2} = \mu + \frac{1}{2}\delta$

Sol^n We know that

$\delta = E^{1/2} - E^{-1/2} \longrightarrow (1)$

$\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$.

$2\mu = E^{1/2} + E^{-1/2} \longrightarrow (2)$

Adding (1) and (2)

$\delta + 2\mu = E^{1/2} - E^{-1/2} + E^{1/2} + E^{-1/2} = 2E^{1/2}$

$\therefore \boxed{E^{1/2} = \mu + \frac{\delta}{2}}$ Ans

Prove that $\Delta(1+\Delta)^{-1/2} = \nabla(1-\nabla)^{-1/2}$

R.H.S $= \nabla(1-\nabla)^{-1/2}$

$= (1-E^{-1})(\cancel{1}-\cancel{1}+E^{-1})^{-1/2}$

$= \left(1-\frac{1}{E}\right)(E^{-1})^{-1/2} = \frac{E-1}{E}\cdot E^{1/2}$

$= \frac{E-1}{E^{1/2}} = \frac{\Delta}{(1+\Delta)^{1/2}} = \frac{\Delta}{(1+\Delta)^{1/2}} = \Delta(1+\Delta)^{-1/2}$ Ans

Q. Prove that $\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{E\cdot e^x}{\Delta^2 e^x} = e^x$

Soln

Now $\left(\frac{\Delta^2}{E}\right)e^x = \frac{(E-1)^2}{E}\cdot e^x = \frac{E^2-2E+1}{E}\cdot e^x$

$= (E-2+E^{-1})e^x$

$= e^{x+h}-2e^x+e^{x-h} \longrightarrow (1)$

And $\frac{Ee^x}{\Delta^2 e^x} = \frac{e^{x+h}}{(E-1)^2 e^x} = \frac{e^{x+h}}{(E^2-2E+1)e^x}$

$= \frac{e^{x+h}}{e^{x+2h}-2e^{x+h}+e^x} = \frac{\cancel{e^h}\cdot e^x}{\cancel{e^h}(e^{x+h}-2e^x+e^{x-h})}$

$\frac{E\cdot e^x}{\Delta^2 e^x} = \frac{e^x}{e^{x+h}-2e^x+e^{x-h}} \longrightarrow (2)$

$\therefore \left(\frac{\Delta^2}{E}\right)e^x\cdot\frac{Ee^x}{\Delta^2 e^x} = \cancel{(e^{x+h}-2e^x+e^{x-h})}\cdot\frac{e^x}{\cancel{e^{x+h}-2e^x+e^{x-h}}} = e^x$

Ans

Q. Prove that $\mu\delta = \frac{1}{2}(\Delta+\nabla)$

Sol^n We know that

$$\mu\delta = \frac{1}{2}(E^{1/2}+E^{-1/2})\cdot(E^{1/2}-E^{-1/2})$$

$$= \frac{1}{2}[E-E^{-1}] = \cancel{\frac{1}{2}[E-\frac{1}{E}]}$$

$$= \frac{1}{2}[1+\Delta-(1-\nabla)] = \frac{1}{2}[\cancel{1}+\Delta-\cancel{1}+\nabla]$$

$$= \frac{1}{2}[\Delta+\nabla] \text{ Ans}$$

Q. Prove that

$$\nabla^2 = h^2D^2 - h^3D^2 + \frac{7}{12}h^4D^4 + \dots$$

Sol^n We know that

$$E = e^{hD} \quad \text{and} \quad \nabla = 1 - E^{-1}$$

$$\therefore \nabla = 1 - e^{-hD}$$

$$= 1 - \left[1 - hD + \frac{h^2D^2}{\underline{|2}} - \frac{h^3D^3}{\underline{|3}} + \dots\right]$$

$$= \cancel{1} - \cancel{1} + hD - \frac{h^2D^2}{\underline{|2}} + \frac{h^3D^3}{\underline{|3}} - \dots$$

$$\nabla = hD - \frac{h^2D^2}{\underline{|2}} + \frac{h^3D^3}{\underline{|3}} \dots$$

Hence

$$\nabla^2 = \left[hD - \frac{h^2D^2}{\underline{|2}} + \frac{h^3D^3}{\underline{|3}} \dots\right]^2$$

$$= h^2D^2 - \frac{2h^3D^3}{\underline{|2}} + \frac{h^4D^4}{2.2} + \frac{2h^4D^4}{\underline{|3}} + \dots$$

$$\nabla^2 = h^2D^2 - h^3D^3 + \frac{7}{12}h^4D^4 \dots \text{ Ans}$$

Q. Prove that $D = \frac{1}{h}\,\text{Log}\,E$

Sol^n We know that $E = e^{hD}$

$$\therefore hD = \text{Log}\,E$$

$$\boxed{D = \frac{1}{h}\,\text{Log}\,E}$$

...on between E and D or Prove that $E = e^{hD}$

Sol^n

By Taylor's Theorem we know that

$$y(x+h) = y(x) + h\,y'(x) + \frac{h^2}{\underline{|2}}\,y''(x) + \cdots$$

$$y(x+h) = y(x) + h\,D\,y(x) + \frac{h^2}{\underline{|2}}\,D^2 y(x) + \cdots$$

$$E\,y(x) = \left[1 + hD + \frac{h^2}{\underline{|2}}D^2 + \cdots\right]y(x)$$

$$E = 1 + hD + \frac{h^2D^2}{\underline{|2}} + \cdots$$

$$\boxed{E = e^{hD}}$$

Q. Prove that $\Delta \equiv \frac{\delta^2}{2} + \delta\sqrt{1+\frac{\delta^2}{4}}$

Sol^n We know that $\delta = E^{1/2} - E^{-1/2} \longrightarrow (1)$

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] \longrightarrow (2)$$

And $\mu = \sqrt{1+\frac{\delta^2}{4}}$

$$R.H.S = \frac{\delta^2}{2} + \delta\sqrt{1+\frac{\delta^2}{4}} = \frac{1}{2}\left(E^{1/2}-E^{-1/2}\right)^2 + \left(E^{1/2}-E^{-1/2}\right)\sqrt{1+\frac{\left(E^{1/2}-E^{-1/2}\right)^2}{4}}$$

$$= \frac{1}{2}\left[E+E^{-1}-2\right] + \left(E^{1/2}-E^{-1/2}\right)\sqrt{\frac{4+E+E^{-1}-2}{4}}$$

$$= \frac{1}{2}\left[E+E^{-1}-2\right] + \left(E^{1/2}-E^{-1/2}\right)\frac{\sqrt{\left(E^{1/2}+E^{-1/2}\right)^2}}{2}$$

$$= \frac{1}{2}\left[E+E^{-1}-2\right] + \frac{\left(E^{1/2}-E^{-1/2}\right)\left(E^{1/2}+E^{-1/2}\right)}{2}$$

$$= \frac{E}{2} + \cancel{\frac{1}{2}E^{-1}} - 1 + \frac{E}{2} - \cancel{\frac{E^{-1}}{2}} = \frac{E}{2} + \frac{E}{2} - 1 = E - 1$$

$$= \cancel{1} + \Delta - \cancel{1}$$

$$= \Delta \quad \text{Ans}$$

Q. Prove that $\mu = \frac{2+\Delta}{2\sqrt{1+\Delta}}$

$$R.H.S = \frac{2+\Delta}{2\sqrt{1+\Delta}} = \frac{2+E-1}{2\sqrt{1+E-1}} = \frac{E+1}{2\sqrt{E}}$$

$$= \frac{1}{2}\left[\frac{E}{\sqrt{E}} + \frac{1}{\sqrt{E}}\right] = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$$

$$= \mu \quad \text{Ans}$$

Q. Prove that $\Delta - \nabla = \delta^2$

Solⁿ $\Delta - \nabla = \delta E^{1/2} - \delta E^{-1/2} = \delta\left[E^{1/2} - E^{-1/2}\right]$

$$= \delta \cdot \delta = \delta^2.$$

Q. Constract the forward difference table, given that

x : 5 10 15 20 25 30

y : 9962 9848 9659 9397 9063 8660

and Point out the value of $\Delta^2 y_{10}$, $\Delta^4 y_5$

Solⁿ

Forward difference Table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
5	9962					
		-114				
10	9848		-75			
		-189		2		
15	9659		-73		-1	
		-262		1		3
20	9397		-72		2	
		-334		3		
25	9063		-69			
		-403				
30	8660					

$\therefore \Delta^4 y_5 = 3$, $\Delta^2 y_{10} = -73$.

Ans

If $u_0 = 3,\ u_1 = 12,\ u_2 = 81,\ u_3 = 2000,\ u_4 = 100$.
Calculate $\Delta^4 u_0$.

Sol^n

x	u	Δu_0	$\Delta^2 y_0$	$\Delta^3 u_0$	$\Delta^4 u_0$
0	3				
		9			
1	12		60		
		69		1790	
2	81		1850		-7459
		1919		-5669	
3	2000		-3819		
		-1900			
4	100				

Hence $\Delta^4 u_0 = -7459$ Ans

Q. Express y_4 in terms of successive forward finite differences.

Sol^n We know that $y_n = E^n y_0$ And $E = 1 + \Delta$

$y_4 = E^4 y_0 = (1+\Delta)^4 y_0$

$= (1 + 4\Delta + 6\Delta^2 + 4\Delta^3 + \Delta^4) y_0$

$= y_0 + 4\Delta y_0 + 6\Delta^2 y_0 + 4\Delta^3 y_0 + \Delta^4 y_0$ Ans

$=$ [illegible]

Q. Find $\Delta\left[\dfrac{1}{x(x+4)(x+8)}\right]$ Using differencing unit as 1.

Sol^n $\Delta\left[\dfrac{1}{x(x+4)(x+8)}\right] = \dfrac{1}{(x+1)(x+5)(x+9)} - \dfrac{1}{x(x+4)(x+8)}$

Ans

Q. Evaluate $\Delta^2\left[\dfrac{5x+12}{x^2+5x+6}\right]$

Sol^n

$$\Delta^2\left[\frac{5x+12}{x^2+5x+6}\right]$$

First we solve $\dfrac{5x+12}{x^2+5x+6}$ By using Partial Fraction

$$\frac{5x+12}{(x+2)(x+3)} = \frac{2}{x+2} + \frac{3}{x+3}$$

$$\therefore \Delta^2\left[\frac{5x+12}{x^2+5x+6}\right] = \Delta^2\left[\frac{2}{x+2}\right] + \Delta^2\left[\frac{3}{x+3}\right]$$

$$= \Delta\left[\frac{2}{x+3} - \frac{2}{x+2}\right] + \Delta\left[\frac{3}{x+4} - \frac{3}{x+3}\right]$$

$$= \frac{2}{x+4} - \frac{2}{x+3} - \frac{2}{x+3} + \frac{2}{x+2} + \frac{3}{x+5} - \frac{3}{x+4} - \frac{3}{x+4} + \frac{3}{x+3}$$

$$= \frac{3}{x+5} - \frac{4}{x+4} - \frac{1}{x+3} + \frac{2}{x+2}$$

Ans

Q. Prove that $\Delta^4 e^{3x+5} = (e^3-1)^4 \cdot e^{3x+5}$

Sol^n We know that

$$\Delta e^{3x+5} = e^{3(x+1)+5} - e^{3x+5}$$

$$\Delta e^{3x+5} = (e^3-1)\, e^{3x+5}$$

$$\therefore \Delta^2 e^{3x+5} = \Delta[\Delta e^{3x+5}]$$

$$= \Delta (e^3-1) e^{3x+5}$$

$$= (e^3-1)\, \Delta e^{3x+5} = (e^3-1)(e^3-1)\, e^{3x+5}$$

$$\Delta^2 e^{3x+5} = (e^3-1)^2 e^{3x+5}$$

Similarly $\Delta^4 e^{3x+5} = (e^3-1)^4\, e^{3x+5}$

Ans

Prove that $1+\delta^2\mu^2 = \left[1+\frac{\delta^2}{2}\right]^2$

$\underline{\underline{Sol^n}}$ We know that

$$\delta f(x) = f(x+\tfrac{h}{2}) - f(x-\tfrac{h}{2}) = E^{1/2} - E^{-1/2}$$

And $\mu f(x) = \frac{1}{2}[f(x+\frac{h}{2}) + f(x-\frac{h}{2})]$

$$= \frac{1}{2}[E^{1/2} + E^{-1/2}]$$

$$\therefore \quad \delta\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}][E^{1/2} - E^{-1/2}]$$

$$\delta\mu = \frac{1}{2}[E - E^0 + E^0 - E^{-1}] = \frac{1}{2}[E - E^{-1}]$$

$$\delta^2\mu^2 = \frac{1}{4}(E - E^{-1})^2$$

$$\therefore 1+\delta^2\mu^2 = 1+\frac{1}{4}(E-E^{-1})^2 = 1+\frac{1}{4}(E^2-2+E^{-2})$$

$$= \frac{1}{4}[E+E^{-1}]^2$$

And $\left[1+\frac{\delta^2}{2}\right]^2 = \left[1+\frac{1}{2}(E^{1/2} - E^{-1/2})^2\right]^2$

$$= \left[1+\frac{1}{2}(E+E^{-1}-2)\right]^2$$

$$= \left[\cancel{1}+\frac{E+E^{-1}}{2}-\cancel{1}\right]^2$$

$$= \underline{\underline{\frac{1}{4}[E+E^{-1}]^2}}$$

$\boxed{R.H.S = L.H.S}$ A

Q Find the value of $\left(\frac{\Delta^2}{E}\right)x^2$

Solution $\left(\frac{\Delta^2}{E}\right)x^2 = \frac{(E-1)^2}{E}\cdot x^2 = \frac{E^2+1-2E}{E}x^2 = (E+E^{-1}-2)x^2$

$$= Ex^2 + E^{-1}x^2 - 2x^2$$

$$= (x+1)^2 + (x-1)^2 - 2x^2$$

$$= \cancel{x^2}+1+\cancel{2x}+\cancel{x^2}+1-\cancel{2x}-\cancel{2x^2}$$

$$= \underline{\underline{2}}$$

Q Show that $\nabla \log f(x) = -\log\left[1 - \frac{\nabla f(x)}{f(x)}\right]$

L.H.S $\nabla \operatorname{Log} f(x) = \operatorname{Log} f(x) - \operatorname{Log} f(x-h)$

$= \operatorname{Log} f(x) - \operatorname{Log} E^{-1} f(x)$

$= \operatorname{Log} \frac{f(x)}{E^{-1} f(x)}$

$= -\operatorname{Log} \frac{E^{-1} f(x)}{f(x)} = -\operatorname{Log} \frac{(1-\nabla) f(x)}{f(x)}$

$= -\operatorname{Log} \frac{f(x) - \nabla f(x)}{f(x)}$

$= -\operatorname{Log} \left[\frac{f(x)}{f(x)} - \frac{\nabla f(x)}{f(x)}\right]$

$= -\operatorname{Log} \left[1 - \frac{\nabla f(x)}{f(x)}\right]$

Q Evaluate

(i) $\Delta \operatorname{Tan}^{-1} x$ (ii) $\Delta^2 \cos 2x$

(i) $\Delta \operatorname{Tan}^{-1} x = \operatorname{Tan}^{-1}(x+h) - \operatorname{Tan}^{-1} x$

$= \operatorname{Tan}^{-1}\left[\frac{x+h-x}{1+x(x+h)}\right] = \operatorname{Tan}^{-1}\left[\frac{h}{1+hx+x^2}\right]$ Ans

(ii) $\Delta^2 \cos 2x = \Delta[\Delta \cos 2x]$

$= \Delta[\cos 2(x+h) - \cos 2x]$

$= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x]$

$= -2 \sin \frac{2x+4h+2x+2h}{2} \cdot \sin \frac{2x+4h-2x-2h}{2}$

$+ \left[2 \sin \frac{2x+2h+2x}{2} \cdot \sin \frac{2x+2h-2x}{2}\right]$

$= -2 \sin(2x+3h) \sin h + 2 \sin(2x+h) \sin h$ Ans

$= -\cancel{2\sin h}\,[$

$\because \boxed{\cos C - \cos D = -2 \sin \frac{C+D}{2} \cdot \sin \frac{C-D}{2}}$ Ans

ctorial Notation

A Product of the form $x(x-1)(x-2)\cdots(x-n+1)$ is denoted by $[x]^n$ and is called a factorial.

Particularly $[x] = x$

$[x]^2 = x(x-1)$

$[x]^3 = x(x-1)(x-2)$ etc.

Factorial notation helps in finding the successive differences of Polynomial directly by simple rule of differentiation.

Q.1 Express $y = 2x^3 - 3x^2 + 3x - 10$ in factorial notation and hence show that $\Delta^3 y = 12$.

Sol^n Given $y = 2x^3 - 3x^2 + 3x - 10$

First convert Polynomial into factorial notation by using synthetic devision

Let $y = A[x]^3 + B[x]^2 + C[x] + D$ be the factorial notation of the given Polynomial

1	2	-3	3	-10
		2	-1	
2	2	-1	2	
		4		
	2	3		
	2			

$\therefore y = 2[x]^3 + 3[x]^2 + 2[x] - 10 \longrightarrow (1)$

$\therefore \Delta y = 6[x]^2 + 6[x] + 2$

$\Delta^2 y = 12[x] + 6$

$\Delta^3 y = 12$ Ans

Q2 Express $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9$ in factori. Hence show that $\Delta^5 f(x) = 0$

Sol: Given $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9 \longrightarrow (1)$

Let It's Factorial notation is

$f(x) = A[x]^4 + B[x]^3 + C[x]^2 + D[x] + E \longrightarrow (2)$

Now synthetic devision is

1	1	-12	24	-30	9
		1	-11	13	
2	1	-11	13	-17	
		2	-18		
3	1	-9	-5		
		3			
	1	-6			
	1				

$\therefore f(x) = [x]^4 - 6[x]^3 + (-5)[x]^2 - 17[x] + 9$

$f(x) = [x]^4 - 6[x]^3 - 5[x]^2 - 17[x] + 9$

$\Delta f(x) = 4[x]^3 - 18[x]^2 - 10[x] - 17$

$\Delta^2 f(x) = 12[x]^2 - 36[x] - 10$

$\Delta^3 f(x) = 24[x] - 36$

$\Delta^4 f(x) = 24$

$\Delta^5 f(x) = 0$ Ans

Q3 Obtain the function whose first difference is $9x^2 + 11x + 5$

Let $f(x)$ be the required function s.that

$\Delta f(x) = 9x^2 + 11x + 5 \longrightarrow (1)$

First we convert $9x^2+11x+5$ into factorial notation by using synthetic devision.

$$\begin{array}{c|ccc} 1 & 9 & 11 & 5 \\ & & 9 & \\ \hline & 9 & 20 & \\ & & & \\ \hline & 9 & & \end{array}$$

$\therefore 9[x]^2+20[x]+5$

$\therefore \Delta f(x) = 9[x]^2+20[x]+5$

$f(x) = 9\frac{[x]^3}{3}+20\frac{[x]^2}{2}+5[x]$

$f(x) = 3[x]^3+10[x]^2+5[x]$

$f(x) = 3x(x-1)(x-2)+10x(x-1)+5x$

$= 3x(x^2-3x+2)+10(x^2-x)+5x$

$= 3x^3-9x^2+6x+10x^2-10x+5x$

$\boxed{f(x) = 3x^3+x^2+x}$ Ans

Q.4 Express $f(x) = 2x^4+x^3-5x^2+8$ in factorial notation and find Its first and second differences.

Let factorial notation of given Polynomial is

$f(x) = A[x]^4+B[x]^3+C[x]^2+D$

so by synthetic devision

$$\begin{array}{c|cccc} 1 & 2 & 1 & -5 & 8 \\ & & 2 & 3 & \\ \hline 2 & 2 & 3 & -2 & \\ & & 4 & & \\ \hline & 2 & 7 & & \\ \hline & 2 & & & \end{array}$$

$f(x) = 2[x]^4+7[x]^3-2[x]^2$

$\therefore \cancel{f(x) = 2[x]^4 + 7[x]^3}$

1	2	1	-5	0	8
		2	3	-2	
2	2	3	-2	-2	
		4	14		
3	2	7	12		
		6			
	2	13			
	2				

$\therefore f(x) = 2[x]^4 + 13[x]^3 + 12[x]^2 - 2[x] + 8$

is a Factorial notation of given Polynomial

$\therefore \Delta f(x) = 8[x]^3 + 39[x]^2 + 24[x] - 2$

$\Delta^2 f(x) = 24[x]^2 + 78[x] + 24$

Ans

Missing Term Technique

Suppose n values out of $(n+1)$ values of $y = f(x)$ are given, the values of x being equidistant. Let the unknown value be N. We constract the difference table.

Q.1 Find the missing term in the table :

x : 2 3 4 5 6

y : 45 49.2 54.1 .... 67.4

Sol^n Let the missing term is a

Now difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	45				
		4.2			
3	49.2		.7		
		4.9		$a-59.7$	
4	54.1		$a-59$		$240.2-4a$
		$a-54.1$		$180.5-3a$	
5	a		$121.5-2a$		
		$67.4-a$			
6	67.4				

Here Four terms are given

$\Delta^4 y_0 = 0$

$0 = 240.2 - 4a \Rightarrow a = \frac{240.2}{4} = \underline{\underline{60.05}}$

Q.2 Find the missing term of the following data

x : 1 2 3 4 5

y : 7 ... 13 21 37

Sol^n Here only four value is given i.e $\Delta^4 y_0 = 0$

Let missing term is a

Difference Table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	7				
		$a-7$			
2	a		$20-2a$		
		$13-a$		$3a-25$	
3	13		$a-5$		$38-4a$
		8		$13-a$	
4	21		8		
		16			
5	37				

$\because \quad \Delta^4 y = 38 - 4a = 0$

$4a = 38$

$a = \frac{38}{4} = 9.5$ Ans

Q.3 Find the missing term in the following table

x :	0	1	2	3	4
$f(x)$:	1	3	9	....	81

Solⁿ Here only Four term are given i.e $\Delta^4 y_0 = 0$

$\Delta^4 y_0 = 0$

$(E-1)^4 y_0 = 0$

$(E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$

$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

$81 - 4y_3 + 6\times 9 - 4\times 3 + 1 = 0$

$124 = 4y_3$

$y_3 = \frac{124}{4} = 31$ Ans

Q.2. Find the missing values in the following data

x : 45 50 55 60 65

y : 3 .... 2.0 .... −2.4

Let the missing value be a and b. Then the difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	3			
		$a-3$		
50	a		$5-2a$	
		$2-a$		$b+3a-9$
55	2.0		$b+a-4$	
		$b-2$		$3.6-a-3b$
60	b		$-.4-2b$	
		$-2.4-b$		
65	−2.4			

Here only three entries are given

i.e $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

$\therefore \quad 0 = b + 3a - 9 \implies 3a + b = 9 \rightarrow (1)$

and $3.6 - a - 3b = 0 \Rightarrow a + 3b = 3.6 \rightarrow (2)$

on solving (1) and (2)

$$3a + b = 9$$
$$3a + 9b = 10.8$$
$$-8b = -1.8 \Rightarrow b = \frac{1.8}{8} = .225$$

$$3a = 9 - .225$$

$\boxed{a = 2.925}$ $\boxed{b = .225}$

Find the missing value in the following table

x :	0	5	10	15	20	25
y :	6	10	...	17	...	31

Solⁿ Here Four values are given and two missing terms.

$\therefore\ \Delta^4 y_0 = 0 \longrightarrow (i)$

$\Delta^4 y_1 = 0 \longrightarrow (ii)$

$\because\ \Delta^4 y_0 = 0$

$(E-1)^4 y_0 = 0 \Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$

$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0 \longrightarrow (A)$

And $(E-1)^4 y_1 \Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_1 = 0$

$y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 = 0 \longrightarrow (B)$

Given $y_0 = 6$, $y_1 = 10$, $y_2 = a$ (let), $y_3 = 17$, $y_4 = b$ (let), $y_5 = 31$

Putting in A and B.

$b - 4\times17 + 6(a) - 4(10) + 6 = 0$

$6a + b = 102 \longrightarrow (A)$

$31 - 4(b) + 6(17) - 4(a) + 10 = 0$

$-4b - 4a = -143$

$4a + 4b = 143 \longrightarrow (B)$

on solving (A) and B

$6a + b = 102$

$24a + \cancel{4b} = 408$

$4a + \cancel{4b} = 143$

$- \quad - \quad -$

$20a = 265 \Rightarrow \boxed{a = 13.25}$

$4b = 143 - 4(13.25) =$

$\boxed{b = 22.5}$