Vector Calculus

Let $P(x, y, z)$ be any point in the domain of definition 'R'. Then, we define a scalar function $\phi(P) = \phi(x, y, z)$, which depends on a point P in the space. Ily, we define a vector function $\vec{V}(P) = \vec{v}(x, y, z)$, whose values are vectors.

Also, the corresponding region 'R', where the functions are defined are scalar or vector fields.

Derivatives: A vector $\vec{v}(t)$ is said to be differentiable at a point 't', if the following limit exists:

$$\vec{v}'(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

The vector $\vec{v}'(t)$ is the derivative of $\vec{v}(t)$.

In component form $\vec{v}'(t) = [v_1'(t), v_2'(t), v_3'(t)]$

$\therefore v'(t)$ is obtained by differentiating each component seperately.

Properties:

$$(\vec{u} \cdot \vec{v})' = \vec{u} \cdot \vec{v}' + \vec{u}' \cdot \vec{v}$$

$$(\vec{u} \times \vec{v})' = \vec{u} \times \vec{v}' + \vec{u}' \times \vec{v}$$

$$(c\vec{u})' = c\vec{u}'$$

$$(\vec{u} + \vec{v})' = \vec{u}' + \vec{v}'$$

Eg. If $\vec{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$

$\vec{b} = \sin t\,\hat{i} - \cos t\,\hat{j}$, Find $\frac{d}{dt}(\vec{a}\cdot\vec{b})$ and $\frac{d}{dt}(\vec{a}\times\vec{b})$.

Sol $\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\vec{b}' + \vec{a}'\cdot\vec{b}$

$= 5t^2\cos t + t\sin t + 10t\sin t - \cos t$

$= 5t^2\cos t + 11t\sin t - \cos t$

$$\frac{d}{dt}(\vec{a}\times\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$= \hat{i}(t^3\sin t - 3t^2\cos t) - \hat{j}(t^3\cos t + 3t^2\sin t)$

$+ \hat{k}(5t^2\sin t - t\cos t - 10t\cos t - \sin t)$

$= t^2(t\sin t - 3\cos t)\hat{i} - t^2(t\cos t + 3\sin t)\hat{j}$

$+ [(5t^2 - 1)\sin t - 11t\cos t]\hat{k}$

Note: Derivatives can be found directly by performing dot and cross products first and then differentiating component-wise.

Curves: $C : \vec{r}(t) = \langle t_1, t_2, t_3 \rangle$

$= t_1\hat{i} + t_2\hat{j} + t_3\hat{k}$

Here t is the parameter.

Eg (1) $x^2 + y^2 = 4$, $z = 0$ is a circle in xy plane with centre at $(0,0,0)$ and radius 2

$\vec{r}(t) = \langle 2\cos t, 2\sin t, 0 \rangle$

$= 2\cos t\,\hat{i} + 2\sin t\,\hat{j}$

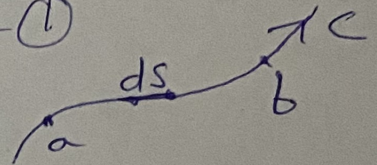
(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0 \Rightarrow \vec{r}(t) = \langle a\cos t, b\sin t, 0 \rangle$

$= a\cos t\,\hat{i} + b\sin t\,\hat{j}$

Arc length (vector function)

Let a curve C be defined by the vector function
$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ in the interval $a \le t \le b$

Then arc length $S = \int_C ds$ — ①



Now $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$$\therefore S = \int_C \sqrt{(dx)^2 + (dy)^2 + (dz)^2}\ \frac{dt}{dt}$$ → multiplying and dividing by dt

$$\Rightarrow S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ .dt \quad \text{— ②}$$

Also $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\therefore r'(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\Rightarrow |r'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \quad \text{— ③}$$

From ① ~~and~~ ② and ③

$$S = \int_C ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |r'(t)|\, dt$$

$$\Rightarrow ds = |r'(t)|\, dt$$

$$\Rightarrow \frac{ds}{dt} = |r'(t)|$$

Q Find the arc length of the vector function

$$\vec{r}(t) = \sqrt{2}\,t\,\hat{i} + e^{t}\hat{j} + e^{-t}\hat{k}, \quad 0 \le t \le 1$$

Sol $\vec{r}(t) = \langle \sqrt{2}t, e^{t}, e^{-t} \rangle$

$r'(t) = \langle \sqrt{2}, e^{t}, -e^{-t} \rangle$

$$S = \int_a^b |r'(t)|\,dt = \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}\,dt$$

$$= \int_0^1 \sqrt{(e^{t}+e^{-t})^2}\,dt = \int_0^1 (e^{t}+e^{-t})\,dt$$

$$= \left[e^{t} - e^{-t}\right]_0^1 = e^{1} - e^{-1} = e - \frac{1}{e} = \frac{e^2-1}{e}$$

Q Find the arc length of the portion of the curve $x = \cos t$, $y = \sin t$, $z = \sqrt{3}t$ from $(3,0,0)$ to $(-3,0,\sqrt{3}\pi)$

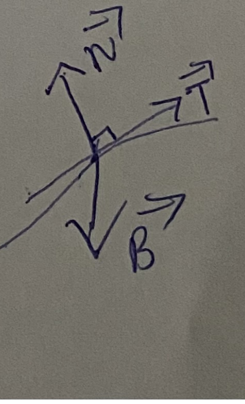
Sol $\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3}t \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, \sqrt{3} \rangle$

$$S = \int_0^{\pi} \sqrt{\sin^2 t + \cos^2 t + 3}\,dt \qquad \begin{pmatrix} \sqrt{3}t = 0 \Rightarrow t = 0 \\ \sqrt{3}t = \sqrt{3}\pi,\ t = \pi \end{pmatrix}$$

Tangent, Normal and Binormal vectors

· Unit Tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$



Unit Normal vector $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$

Binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

Eg $\vec{r}(t) = \langle 3\sin t, 3\cos t, 4t \rangle$

Find unit tangent, unit normal and binormal vectors.

Sol. $\vec{r}'(t) = \langle 3\cos t, -3\sin t, 4 \rangle$

$$|\vec{r}'(t)| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = 5$$

$$\therefore \vec{T}(t) = \frac{r'(t)}{|r'(t)|} = \left\langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5} \right\rangle$$

unit Tangent vector

~~Also~~ $\cancel{|\vec{T}(t)| = \sqrt{\frac{9}{25}\cos^2 t + \frac{9}{25}\sin^2 t + \frac{16}{25}}}$

Also, $T'(t) = \left\langle -\frac{3}{5}\sin t, -\frac{3}{5}\cos t, 0 \right\rangle$

$$|T'(t)| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t} = \frac{3}{5}$$

$\therefore$ Unit Normal vector $\vec{N}(t) = \dfrac{T'(t)}{|T'(t)|}$

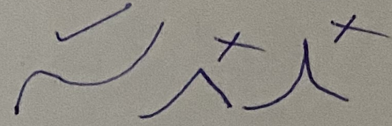
$$= \langle -\sin t, -\cos t, 0 \rangle$$

Again $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\overset{\sin t}{\cancel{\cos t}} & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix}$

$$= \hat{i}\left(\frac{4}{5}\cos t\right) - \hat{j}\left(\frac{4}{5}\sin t\right) + \hat{k}\left(-\frac{3}{5}\cos^2 t - \frac{3}{5}\sin^2 t\right)$$

$$= \left\langle \frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5} \right\rangle$$

Curvature: Curvature is the measure how fast a curve is changing its direction at a point. Curvature is defined only for smooth curves, i.e, the curve which has no corners or cusps.



$\therefore$ The curvature $K(S)$ is defined as the magnitude of the rate of change of unit tangent vector with respect to arc length. $\therefore$ $K(S) = \left|\frac{d\vec{T}}{dS}\right| = \left|\frac{d\vec{T}/dt}{dS/dt}\right|$

$$= \left|\frac{T'(t)}{r'(t)}\right| \quad , \because dS/dt = r'(t)$$

Alternate formulae for curvature

(1) $K(S) = \left|\frac{d\vec{T}}{dS}\right|$

(2) $K(t) = \left|\frac{T'(t)}{r'(t)}\right|$, Here $\vec{T}$ is the unit tangent vector

(3) $K(t) = \frac{|r'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$

Note: (1) Curvature is zero at any point on a st. line.

(2) Radius of curvature (ρ) at any point on the curve is the reciprocal of curvature i.e $\rho = \frac{1}{K}$.

Eg Find the curvature for the curve $\vec{r}(t) = \langle 2t, t^2, -\frac{t^3}{3} \rangle$

sol $r'(t) = \langle 2, 2t, -t^2 \rangle$

$|r'(t)| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2+t^2)^2} = t^2 + 2$

Now $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$$\Rightarrow \vec{T}(t) = \left\langle \frac{2}{t^2+2}, \frac{2t}{t^2+2}, \frac{-t^2}{t^2+2} \right\rangle$$

$$\vec{T}'(t) = \left\langle \frac{-4t}{(t^2+2)^2}, \frac{-2t^2+4}{(t^2+2)^2}, \frac{-4t}{(t^2+2)^2} \right\rangle$$

$$\Rightarrow |\vec{T}'(t)| = \sqrt{\frac{16t^2+4t^4+16-16t^2+16t^2}{(t^2+2)^4}} = \frac{2}{t^2+2}$$

$$\therefore K(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\frac{2}{t^2+2}}{t^2+2} = \frac{2}{(t^2+2)^2}$$

<u>Torsion</u>: The arc rate of rotation of the binormal vector is called torsion and is denoted by τ.

$$\frac{d\vec{B}}{ds} = -\tau\vec{N}$$

$$\Rightarrow \frac{d\vec{B}/dt}{ds/dt} = -\tau\vec{N} \Rightarrow \frac{\vec{B}'(t)}{|\vec{r}'(t)|} = -\tau\vec{N}$$

<u>Eg</u> Find the curvature and torsion of the curve $x = a\cos t$, $y = a\sin t$, $z = bt$

<u>Sol</u> $\vec{r} = \langle a\cos t, a\sin t, bt \rangle$

$$\vec{r}'(t) = \langle -a\sin t, a\cos t, b \rangle$$

$$|\vec{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$$

Again $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$

$$= \left\langle \frac{-a\sin t}{\sqrt{a^2+b^2}}, \frac{a\cos t}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right\rangle$$

$$\vec{T}'(t) = \left\langle -\frac{a\cos t}{\sqrt{a^2+b^2}}, -\frac{a\sin t}{\sqrt{a^2+b^2}}, 0 \right\rangle$$

$$|\vec{T}'(t)| = \sqrt{\frac{a^2\cos^2 t + a^2\sin^2 t}{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\therefore \kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\frac{a}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}$$

Again $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\cos t, -\sin t, 0 \rangle$

Now Binormal vector $\vec{B} = \vec{T} \times \vec{N} = \dfrac{1}{\sqrt{a^2+b^2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & b \\ -\cos t & -\sin t & 0 \end{vmatrix}$

$$\Rightarrow \vec{B} = \frac{1}{\sqrt{a^2+b^2}} \left[\hat{i}(b\sin t) - \hat{j}(b\cos t) + \hat{k}(a\sin^2 t + a\cos^2 t) \right]$$

$$= \frac{b\sin t\,\hat{i} - b\cos t\,\hat{j} + a\hat{k}}{\sqrt{a^2+b^2}}$$

or $\vec{B}(t) = \left\langle \dfrac{b\sin t}{\sqrt{a^2+b^2}}, \dfrac{-b\cos t}{\sqrt{a^2+b^2}}, \dfrac{a}{\sqrt{a^2+b^2}} \right\rangle$

$$\vec{B}'(t) = \left\langle \frac{b\cos t}{\sqrt{a^2+b^2}}, \frac{b\sin t}{\sqrt{a^2+b^2}}, 0 \right\rangle$$

~~Now~~ $\cancel{\dfrac{\vec{B}'(t)}{\vec{r}'(t)} = }$

Now $\dfrac{B'(t)}{|r'(t)|} = -\tau\vec{N}$

$$\Rightarrow \frac{b\cos t}{a^2+b^2}\hat{i} + \frac{b\sin t}{a^2+b^2}\hat{j} = -\tau(-\cos t\,\hat{i} - \sin t\,\hat{j})$$

$$\Rightarrow \tau = \frac{b}{a^2+b^2}$$