UNIT-2 Part-I (Waves & Oscillations)

Waves:- A wave motion is a disturbance of some kind which moves from one place to another by means of a medium.

Oscillation:- oscillation is an effect expressible as a quantity that repeatidly & regularly fluctuate above and below some mean position.

I) **Transverse wave:-** wave motion in which particle of medium vibrate about their mean position at right angle to the direction of propagation.
Eg- Light Waves.

II) **Longitudinal waves:-** Wave motion in which wave vibrate about their mean position along the same line as propagation of wave.
Eg. Sound waves

Simple Harmonic Motion:- If the acceleration of a particle in a periodic motion is always directly proportional to its displacement from its equilibrium position.

Types of SHM:-

I) Linear ~~Harmonic~~ Simple Harmonic Motion:-
If the displacement of a particle Executing SHM is linear is said to be linear simple Harmonic Motion.
Example - Simple Pendulum.

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Angular Simple Harmonic Motion:-

If the Displacement of a Particle Executing SHM is angular. Example- compound Pendulum.

Essential Conditions for SHM:-

1) If f be the linear acceleration and x be the displacement from Equilibrium Position.

The essential condition is -

$$\boxed{f \propto -x}$$

2) If α be the angular Momentum θ be the angular displacement then essential condition is-

$$\boxed{\alpha = -\theta}$$

Time Period:- The smallest time interval during which oscillation repeat itself is called time Period, denoted by T. Its Unit is seconds.

Frequency:- Number of oscillation that a body complete in one second is called frequency of Periodic Motion.

It is reciprocal of time period T and is given by-

$$\boxed{n = \frac{1}{T}}$$

Unit - Hertz represented by Hz.

Amplitude:- Maximum displacement of a body from Its Mean Position.

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Phase :- It is a physical quantity that express the instaneous position and direction of motion of an oscillating system.

Differential Equation of SHM & its solution :-

Let us consider a particle of mass m executing SHM along a straight line with x as displacement from the mean position at any time t. Then from the basic condition of SHM restoring force F will be proportional to displacement and will be directed opposite to it.

Therefore
$$F \propto -x$$
$$\boxed{F = -kx} \quad -(1)$$

k is proportionality constant known as force constant.

If $a = \frac{d^2x}{dt^2}$ be the acceleration at any instant of time

From (1)
$$F = -kx$$
$$ma = -kx$$
$$m\frac{d^2x}{dt^2} = -kx$$
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad -(2)$$

Substituting $\frac{k}{m} = \omega^2$
$$\frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \quad -(3)$$

Equation (1) & (2) are known as differential equation of SHM.

Solution of Differential Equation:-

$$x = A \sin(\omega t + \delta)$$

This Equation gives the displacement of particle Executing SHM at any instant of time.

A → Maximum Displacement of Particle

$$x = A \sin(\omega t + \delta) \quad -(1)$$

Replacing t by $\left(t + \frac{2\pi}{\omega}\right)$, then we have-

$$x = A \sin\left(\omega\left(t + \frac{2\pi}{\omega}\right) + \delta\right)$$

$$= A \sin(\omega t + \delta) \quad -(2)$$

(1) & (2) are same this shows that motion is repeated after an interval of $\frac{2\pi}{\omega}$. So this interval will be time period of SHM given by $T = \frac{2\pi}{\omega}$

Also we have-

$$\omega^2 = \frac{K}{m} \Rightarrow \omega = \sqrt{\frac{K}{m}}$$

Time Period:-

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}}$$

Frequency $n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$

Phase:- The quantity $(\omega t + \delta)$ is known as phase of vibrating particle. If $t=0$ then $(\omega t + \delta) = \delta$, so that initial phase will be δ.

If the particle start from mean position, then $\delta = 0$ for a particle start from extreme position then $\boxed{\delta = \pi/2}$

Velocity + Acceleration :- For a Particle Executing SHM, Expression for Displacement is -

$$x = A \sin(\omega t + \delta)$$

Differentiating it wrt time we get -

$$v = \frac{dx}{dt} = A\omega \cos(\omega t + \delta) \quad \text{—①}$$

$$= A\omega \sqrt{1 - \sin^2(\omega t + \delta)}$$

$$= \omega \sqrt{A^2 - A^2 \sin^2(\omega t + \delta)}$$

$$v = \omega \sqrt{A^2 - x^2}$$

This is the Expression for velocity of Particle at any displacement x.

I) Maximum velocity is obtained by substituting

$$x = 0$$

$$\therefore \boxed{V_{max} = \omega A}$$

because $x = 0$ correspond to Mean Position, so the Particle has Maximum velocity when it is at Mean Position.

II) At extreme Position

At Extreme Position $x = A$

on differentiating ① wrt t we get -

$$f = \frac{dv}{dt} = \frac{d}{dt}(A\cos(\omega t + \delta))$$

$$= A\omega^2 \sin(\omega t + \delta)$$

$$\boxed{f = -\omega^2 x} \quad (\text{using } x = A\sin(\omega t + \delta))$$

This is Standard Equation of SHM

$$\boxed{f_{max} = \omega^2 A}$$

$$\boxed{f_{min} = 0} \quad \text{at Mean Position}$$

Del operator:-

∇ operator is denoted by ∇ and is treated as a vector in Cartesian coordinates.

it is written as.

$$\vec{\nabla} = \hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}$$

Gradient of Scaler Field :-

Let $\phi(r)$ be a Scaler field $\vec{r}$ is the position of vector $(\vec{r} = x\hat{i} + y\hat{j} + z\hat{k})$

where (x, y, z) are the coordinates.

ϕ be a function of three coordinates.

then

$$\boxed{\text{grad}\,\phi = \nabla \cdot \phi}$$

or
$$\text{grad}\,\phi = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)\phi$$

$$\boxed{\text{grad}\,\phi = \left(\hat{i}\frac{d\phi}{dx} + \hat{j}\frac{d\phi}{dy} + \hat{k}\frac{d\phi}{dz}\right)}$$

Curl of a vector

Circulation of a closed field around a closed path. is given by curl of a vector

Mathematically

$$\boxed{\text{Curl of } \vec{F} = \nabla \times \vec{F}}$$

ie cross product between ∇ & $\vec{F}$

If $\boxed{\nabla \times \vec{F} = 0}$ then field is irrotational.

Numericals problem

① If $r = x\hat{i} + y\hat{j} + z\hat{k}$ find $\nabla \cdot r$

$$\nabla \cdot r = \left(i\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \left(\frac{dx}{dx} + \frac{dx}{dx} + \frac{dx}{x}\right)$$

$$= 1 + 1 + 1 = 3$$

② If $\phi = 4x^3y^2z^4$ find $\nabla\phi$

$$\nabla\phi = \left[\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dx} + \hat{k}\frac{d}{dz}\right](4x^3y^2z^4)$$

$$= \left[i\frac{d}{dx}(4x^3y^2z^4) + \hat{j}\left(\frac{d}{dx}(4x^3y^2z^4)\right) + k\frac{d}{dz}(4x^3y^2z^4)\right)$$

$$= 12x^2y^2z^4\hat{i} + 8x^3yz^4\hat{j} + 16x^3y^2z^3\hat{k}$$

Q3 If $A = 2xy + 2y + z^2$ find $\nabla \cdot A$

$$\nabla \cdot A = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + k\frac{d}{dz}\right)(2xy + 2y + z^2)$$

$$= \hat{i}\frac{d}{dx}(2xy + 2y + z^2) + \hat{j}\left(\frac{d}{dy}(2xy + 2y + z^2) + \hat{k}\left(\frac{d}{dz}(2xy + 2y + z^2)\right)\right.$$

$$= \hat{i}(2y) + \hat{j}(2x + 2) + \hat{k}(2z)$$

Q Find the curl of a field $F_1 = x\hat{i} + y\hat{j} + z\hat{k}$

prove that it is rotational or irrotational.

Sol: Curl of a field $F_1 = \nabla \times F_1$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ x & y & \hat{z} \end{vmatrix}$$

where $\nabla = \hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}$

$$= \hat{i}\left(\frac{dz}{dy} - \frac{dy}{dz}\right) + \hat{j}\left(\frac{dz}{dx} - \frac{dx}{dz}\right) + k\left(\frac{dy}{dx} - \frac{dx}{dy}\right)$$

$\Rightarrow$ $\nabla \times F_1 \neq 0$

So field is ~~ir~~rotational.

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Gauss Divergence Theorm:-

It States that, the volume integral of the divergence of Vector field A taken over any volume V bounded by a closed surface S is equal to surface integral of A taken over the surface S.

Mathematically

$$\iiint_V \operatorname{div} A \, dv = \iint_S A \cdot ds$$

Stokes Theorm:-

It State that the surface integral of a curl of a vector field A taken over any surface S is equal to the line integral of A around a closed curve.

$$\iint_S (\operatorname{curl} A) \, ds = \oint A \cdot dl$$

OR

$$\iint_S (\nabla \times A) \, ds = \oint A \cdot dl.$$

Physical significance & derivation of Maxwell's Equations

To derive Maxwell Eqnⁿ we need to study following terms -

1) ## Gauss law of Electrostatics:-

Gauss law states that Electric flux through any closed surface is Equal to net charge Enclosed by the surface divided by Permitivity of vaccum.

$$\phi = \frac{Q}{\epsilon_0} \text{ or } \frac{q}{\epsilon_0}$$

Gauss law of Electrostatics (Integral Form)

The number of lines of force passing through a small area element ds is given by.

$$d\phi = E \cdot ds = E ds \cos\theta$$

This is known as Electric flux of the field over the elementary surface ds.

For a closed surface flux of field is given by.

$$\phi = \oiint E \cdot ds$$

In Integral form

Total outward flux of Electric field over a closed surface is equal to $\frac{1}{\epsilon_0}$ the times the total Net charge contained in a volume enclosed by the surface

$$\boxed{\oiint E \cdot ds = q/\epsilon_0}$$

E - Electric field Intensity ds - Surface Element.

Gauss law of Electrostatics: (Differential Form)

Let E be the Electric field ~~of~~ at the centre of an elementary area ds on the closed surface S. Let ρ be the volume charge density of volume Enclosed by surface S. ρdV may be considered as a point charge contributing electric flux to the elementary area ds.

$$\Rightarrow q = \rho dV$$

Then, Total flux of whole surface -

$$\oint_S E \cdot ds = \frac{1}{\epsilon_0} \int \rho dV \qquad \text{--(1)}$$

According to divergence theorm we have -

$$\oint_S E \cdot ds = \oint_V (\nabla \cdot E) dV \qquad \text{--(2)}$$

Equating (1), (2) we have

$$\oint_V (\nabla \cdot E) dv = \frac{1}{\epsilon_0} \int \rho dV$$

$$\oint_V (\nabla \cdot E) dv = \int_V \frac{\rho}{\epsilon_0} dV \qquad \text{--(3)}$$

Since dv is arbitany volume Element (3) is valid for any volume

Hence $\boxed{\nabla \cdot E = \frac{\rho}{\epsilon_0}}$

OR $\text{div } E = \frac{\rho}{\epsilon_0}$

Gauss Law in Magnetostatics:-

I) Differential form:-

Since in magnetic field lines are continuous the magnetic field entering any region is equal to magnetic flux leaving it. So net flux over a volume is zero

Mathematically $\boxed{\nabla \cdot B = 0}$

This is Differential form of gauss law.

II) Integral form:-

Net flux over a closed surface is -

$$\phi = \oint B \cdot ds$$

Applying divergence theorem we have -

$$\oint_S B \cdot ds = \int_V (\nabla \cdot B)\, dv = 0$$

$$\boxed{\oint B \cdot ds = 0}$$

This is integral form of gauss law.

Faraday Law:- 1) Magnetic flux through a closed loop of conducting wire varies in time with EMF.

$$\oint E \cdot dl = -\frac{d\phi}{dt} = \oint_S \cancel{\frac{d(B \cdot ds)}{dt}}$$

$$= \oint_S \frac{dB \cdot d\vec{s}}{dt}$$

ii) Negative time rate of variation of Magnetic flux linked with circuit is equal to EMF induced within it.

$$\boxed{\nabla \times E = -\frac{dB}{dt}}$$

Ampere's circuital law:-

This law state that line integral of Magnetic Field (B) around any closed Path or loop is equal to μ_0 times the total current enclosed by the loop.

$$\boxed{\oint B \cdot dl = \mu_0 I}$$

Maxwell Equation (Differential Form)

First

I) $\nabla \cdot D = \rho$ (Gauss law of Electrostatics)

OR

$\nabla \cdot E = \rho / \epsilon_0$ (using $D = \epsilon_0 E$)

Second

II) $\nabla \cdot B = 0$ (Gauss law of Magnetostatics)

Third

III) $\nabla \times E = -\frac{dB}{dt}$ (Faraday law)

Fourth

IV) $\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$ (Modified Ampere Circuital law)

OR

$\nabla \times H = \left(J + \frac{dD}{dt} \right)$ (using $B = \mu_0 H$)

Maxwell Equation (Integral Form)

First

I) $\oint E \cdot ds = \frac{q}{\epsilon_0}$

Second

II) $\oint B \cdot ds = 0$

Third

III) $\oint E \cdot dl = \frac{-d\Phi_B}{dt}$

Fourth

IV) $\oint B \cdot dl = \mu_0 \left(J + \frac{dD}{dt} \right) \cdot ds$

where

D- Displacement or Electric Displacement
(Coulomb/m2)

ρ- Charge Density (Coulom/m3)

B- Magnetic Induction (wb/m2)
or Flux Density

H- Magnetic Field Intensity (Amp/m)

J- Current Density (I/A) (Current/Ampere)

First Maxwell Equation:-

$$\nabla \cdot E = \rho/\epsilon_0$$

Physical significance:-

→ It is based on gauss law of Electrostatics

→ Net Electric flux through a closed surface is Equal to $\frac{1}{\epsilon_0}$ the total charge Enclosed by the surface.

$$\nabla \cdot E = \frac{q}{\epsilon_0} \qquad \oint_S E \cdot ds = \frac{1}{\epsilon_0}\int_V q \cdot dv$$

(Differential Form) (Integral Form)

→ It relate Electric flux with charge.

→ charge acts a a source or sink for the line of Electric force

Derivation:- Consider a surface bounded by a volume V in a medium having charge density as ρ.

then $$q = \int_V \rho dV$$

using Gauss law of Electrostatics we have

$$\oint E \cdot ds = \frac{q}{\epsilon_0}$$

Subustitutting $q = \int_V \rho dV$

then

$$\oint E \cdot ds = \frac{1}{\epsilon_0}\int \rho dV \quad -②$$

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From Gauss theorm we have -

$$\oint_S E\cdot ds = \int_V (\nabla\cdot E)\, dv \quad -(3)$$

(Note Gauss Divergence theorm converts Surface Integral to volume Integral)

Equating (2), (3) we get.

$$\int_V (\nabla\cdot E)\, dv = \frac{1}{\epsilon_0}\int_V \rho\, dv$$

$$\int_V \left(\nabla\cdot E - \frac{\rho}{\epsilon_0}\right) dv = 0$$

$$\Rightarrow \boxed{\nabla\cdot E = \rho/\epsilon_0}$$

↓

Maxwell (I) Equation.

~~Maxwell's Second Equation~~:-

Maxwell's Second Equation:-

$$\nabla \cdot B = 0$$

Significance:-

- It is based on Gauss law of Magnetostatics.
- It states that magnetic flux through any closed surface is zero

Differential form

$$\nabla \cdot B = 0$$

Integral form

$$\oint B \cdot ds = 0$$

- Time independent Equation
- Magnetic flux is zero
- According to this Equation, isolated magnetic poles don't exist.

Derivation:- we have by Gauss law of Magnetostatics

$$\oint B \cdot ds = 0 \quad -(1)$$

By Gauss divergence theorm we have.

$$\oint_S B \cdot ds = \int_V (\nabla \cdot B)\, dv \quad -(2)$$

(Note:- By Gauss divergence theorm we have Surface Integral Equal to volume Integral)

Equating (1), (2)

$$\int_V (\nabla \cdot B)\, dv = 0$$

$$\boxed{\nabla \cdot B = 0} \rightarrow \text{Maxwell Second Equation}$$

Maxwell's third Equation:-

$$\nabla \times E = -\frac{dB}{dt}$$

Significance:-

→ This Equation represent Faraday law of Electro-Magnetic Induction.

Differential form

$$\nabla \times E = -\frac{dB}{dt}$$

Integral form

$$\oint E \cdot dl = -\frac{d\phi}{dt}$$

→ It is time dependent Equation.

→ Relates space variation of E with variation of B

→ Time variation of Magnetic field generates Electric field.

→ Negative sign Justify Lenz law.

Derivation:- we have by Faraday law.

$$\oint_c E \cdot dl = -\frac{d\phi}{dt}$$

$$= -\frac{d(B \cdot ds)}{dt} \qquad (\phi = B \cdot ds) \quad -①$$

By Stokes theorm we have.

$$\oint_c E \cdot dl = \int_s (\nabla \times E) \cdot ds \quad -②$$

Equating ① ②

$$\oint_s (\nabla \times E) ds = -\frac{d}{dt}\int_s B ds$$

$$\oint_s (\nabla \times E) ds = -\int \frac{dB}{dt} \cdot ds$$

$$\Rightarrow \boxed{\nabla \times E = -\frac{dB}{dt}} \quad - \text{Maxwell Third Eqn}$$

Maxwell Fourth Equation

$$\text{curl } H = J + \frac{dD}{dt}$$

Significance

→ It represent Modified form of Amper's law.

→ It is time dependent Equation

→ It shows that Magnetic field can be generated by Current density vector and time variation

Differential form

$$\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$$

Integral form

$$\oint_C B \cdot dl = \mu_0 \int_S \left(J + \frac{dD}{dt} \right)$$

Derivation:- By Amper's law we have.

$$\oint B \cdot dl = \mu_0 I$$

$$\oint (\cancel{\mu_0} H) dl = \cancel{\mu_0} I$$

$$\oint H \cdot dl = I$$

$$I = \int_S J \cdot ds$$

$$\Rightarrow \int_C H \cdot dl = \int_S J \cdot ds \quad -①$$

By Stokes theorm we have.

$$\int_C H \cdot dl = \int_S (\nabla \times H) ds \quad -②$$

Equating ①, ②

$$\int_S (\nabla \times H) ds = \int_S J \cdot ds$$

$$\int_S (\nabla \times H - J) ds = 0$$

$$\Rightarrow \nabla \times H - J = 0 \quad \Rightarrow \boxed{\text{Curl } H = J} \quad -③$$

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using The equation of time varying field.

$$\operatorname{div} J + \frac{d\rho}{dt} = 0$$

$$\operatorname{div} J = -\frac{d\rho}{dt} \quad -(4)$$

Maxwell added a current density J_d to original current density J ie

$$C = J + J_D$$

using (3) $\quad \operatorname{curl} H = J + J_D \quad -(6)$

$$\operatorname{div}(\operatorname{curl} \vec{H}) = \operatorname{div}(J + J_D)$$

$$\operatorname{div}(J + J_D) = 0$$

$(\operatorname{div}(\operatorname{curl} \vec{H}) = 0$ for time varying field$)$

$$\Rightarrow \operatorname{div} J + \operatorname{div} J_D = 0$$

$$\operatorname{div} J_D = -\operatorname{div} J$$

using (4) $\quad \operatorname{div} J_D = +d\rho/dt \quad -(5)$

But $\boxed{\operatorname{div} D = \rho}$ By maxwell (I) eqn

$$\operatorname{div} J_D = \frac{d}{dt}(\operatorname{div} D)$$

$$= \operatorname{div}\left(\frac{dD}{dt}\right)$$

$$\boxed{J_D = \frac{dD}{dt}}$$

using (6) $\quad \operatorname{curl} H = J + J_D$

$$\boxed{\operatorname{curl} H = J + \frac{dD}{dt}}$$

Maxwell's (4) Equation

Conversion of Maxwell Equation from Differential to Integral Form!

① First Equation :-

$$\operatorname{div} E = \rho/\epsilon_0 \quad -①$$

Integrating above Equation over the volume V then-

$$\int_V \operatorname{div}\vec{E}\, dv = \int_V (\rho/\epsilon_0)\, dv$$

Applying divergence theorm $\oint_S E\cdot dS = \int_V \operatorname{div} E\, dV$

Using ① $\oint_S E\cdot dS = \frac{1}{\epsilon_0}\int_V \rho\, dV$

$$\boxed{\oint_S E\cdot dS = \frac{q}{\epsilon_0}}$$

which is simple gauss law.

② Second Equation.

$$\operatorname{div} B = 0 \quad \text{or} \quad \nabla\cdot B = 0$$

Taking integral both sides.

$$\oint B\cdot dS = 0$$

ie Magnetic flux through a closed surface is always zero.

This implies absence of Magnetic Monopoles.

③ Third Equation we have -

$$\nabla \times E = -dB/dt$$

Taking Integral both Sides.

$$\oint_S (\nabla \times E)\, ds = -\oint_S \frac{dB}{dt} \cdot ds \quad -①$$

Applying Stoke's theorm.

$$\int_C E \cdot dl = \oint_S (\nabla \times E)\, ds \quad -②$$

Substituting in ①

$$\oint E \cdot dl = -\int \frac{dB}{dt} \cdot ds$$

$$= -\frac{d}{dt}\int B \cdot S = -\frac{d\phi_B}{dt}$$

④ Fourth Equation.

$$\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$$

On Integrating both sides.

$$\oint (\nabla \times B) \cdot ds = \mu_0 \oint_S \left(J + \frac{dD}{dt} \right) \cdot ds$$

Using Stokes theorm we get.

$$\oint_S (\nabla \times B)\, ds = \mu_0 \oint_S \left(J + \frac{dD}{dt} \right) \cdot ds$$

$$\boxed{\oint B \cdot dl = \mu_0 \int_S \left(J + \frac{dD}{dt} \right) \cdot ds}$$

$\downarrow$

Integral form of Fourth Equation

Electromagnetic Wave Equation:- (Part 3)

I) In Free space

II) In Dielectric Medium

III) In conducting Medium

Imp

1) Propagation of Electromagnetic Wave Equation in Free Space

It can be derived by using Maxwell's Equation.

Maxwell's Equation in general form are given by.

$$\left.\begin{aligned} &\text{I)} \quad \nabla\cdot E = \rho/\epsilon_0 \\ &\text{II)} \quad \nabla\cdot B = 0 \\ &\text{III)} \quad \nabla\times E = -\frac{dB}{dt} \\ &\text{IV)} \quad \nabla\times B = \mu_0\left(J + \frac{dD}{dt}\right) \end{aligned}\right\} \text{— (A)}$$

Now in vaccum (Free Space) we have.

$$\boxed{D = \epsilon_0 E}, \quad \boxed{B = \mu_0 H}$$

conductivity $\boxed{\sigma = 0}$ (ie vacuum is non-conducting)

No free electron are there ie $\boxed{J=0}$, $\boxed{\rho=0}$

Using these conditions in (1) we get.

$$\left.\begin{aligned} &\text{I)} \quad \nabla\cdot E = 0 \\ &\text{II)} \quad \nabla\cdot H = 0 \\ &\text{III)} \quad \nabla\times E = \cancel{-\frac{dB}{dt}} -\mu_0\frac{dH}{dt} \\ &\text{IV)} \quad \nabla\times H = \epsilon_0\frac{dE}{dt} \end{aligned}\right\} \text{— (B)}$$

Now

I) Equation for Electric field vector -

Taking curl on both side of (III) of (B)

$$\nabla \times (\nabla \times E) = \nabla \times \left(-\mu_0 \frac{dH}{dt}\right)$$

$$= -\mu_0 \frac{d}{dt} (\nabla \times H)$$

or $\nabla \cdot (\nabla \cdot E) - \nabla^2 E = -\mu_0 \frac{d}{dt} (\nabla \times H)$

(using $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$)

Using (I, IV) of B

$$0 - \nabla^2 E = -\mu_0 \epsilon_0 \frac{d^2 E}{dt^2}$$

$$\Rightarrow \boxed{\nabla^2 E = \mu_0 \epsilon_0 \frac{d^2 E}{dt^2}} \quad \text{--- (1)}$$

This is general Equation of Electromagnetic waves in term of Electric field vector $\vec{E}$.

II) Equation for Magnetic field vector.

Taking curl on both side of Equation (IV) of (B) we have.

$$\nabla \times (\nabla \times H) = \epsilon_0 \left(\nabla \times \frac{dE}{dt}\right)$$

$$\Rightarrow \nabla \times (\nabla \times H) = \epsilon_0 \left(\frac{d}{dt} (\nabla \times E)\right)$$

or $\nabla (\nabla \cdot H) - \nabla^2 H = \epsilon_0 \left(\frac{d}{dt} (\nabla \times E)\right)$

Using II, III of (B)

$$0 - \nabla^2 H = -\mu_0 \epsilon_0 \frac{d^2 H}{dt^2}$$

$$\Rightarrow \boxed{\nabla^2 H = \mu_0 \epsilon_0 \frac{d^2 H}{dt^2}} \quad \text{--- (2)}$$

This is Electromagnetic wave Equation in term of magnetic field.

Wave velocity:- The Electromagnetic wave Equation for E & H (Electric & Magnetic vectors) are written as-

$$\nabla^2 E - \mu_0 \epsilon_0 \frac{d^2 E}{dt^2} = 0$$

$$\nabla^2 H - \mu_0 \epsilon_0 \frac{d^2 H}{dt^2} = 0$$

These Equation can be written by using x-coordinates-

$$\frac{d^2 E_x}{dx^2} - \mu_0 \epsilon_0 \frac{d^2 E_x}{dt^2} = 0 \quad \text{— (3)}$$

$$\frac{d^2 H_x}{dx^2} - \mu_0 \epsilon_0 \frac{d^2 H_x}{dt^2} = 0 \quad \text{— (4)}$$

These Equations are of type-

$$\frac{d^2 y}{dx^2} - \frac{1}{c^2}\frac{d^2 y}{dt^2} = 0 \quad \text{— (5)}$$

Equation (5) is general Equation of wave travelling with velocity c.

on comparing (3), (4), (5) we conclude that E & H travel in free space in term of wave with velocity c given by

$$\frac{1}{c^2} = \mu_0 \epsilon_0$$

$$\Rightarrow \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

But $\mu_0 = 4\pi \times 10^{-7} NA^{-2}$, $\epsilon_0 = 8.85 \times 10^{-12} C^2 N^{-1} M^{-2}$

$$\Rightarrow \quad c = \frac{1}{\sqrt{4\pi \times 10^{-7} NA^{-2} \times 8.85 \times 10^{-12} C^2 N^{-1} M^{-2}}}$$

$$= 2.99792 \times 10^8 MS^{-1} = 3 \times 10^8 MS^{-1}$$

ie velocity of Electromagnetic wave in free space equal to velocity of light

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② **Electromagnetic Wave Equation in Dielectric:-**

Maxwell's Equation in General form is given by-

1) $\nabla \cdot E = \rho / \epsilon_0$

11) $\nabla \cdot B = 0$

111) $\nabla \times E = -dB/dt$

1V) $\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$

For dielectric conductivity $\boxed{\sigma = 0}$ and $\boxed{J = \sigma E = 0}$

When Dielectric is not charged then $\boxed{\rho = 0}$

Then-

1) $\nabla \cdot E = 0$

11) $\nabla \cdot H = 0$

111) $\nabla \times E = -\mu_0 \frac{dH}{dt}$

1V) $\nabla \times H = \epsilon_0 \frac{dE}{dt}$

Now Replacing μ_0, ϵ_0 by μ + ϵ respectively in Equation ③ + ④ we get.

$$\nabla^2 E = -\mu \epsilon \frac{d^2 E}{dt^2}$$

$$\nabla^2 H = -\mu \epsilon \frac{d^2 H}{dt^2}$$

or
$$\frac{d^2 E_x}{dx^2} - \mu \epsilon \frac{d^2 E_x}{dt^2} = 0$$

$$\frac{d^2 H_x}{dx^2} - \mu \epsilon \frac{d^2 H_x}{dt^2} = 0$$

Comparing with general wave Equation.

$$\frac{d^2 y}{dx^2} - \frac{1}{c^2} \frac{d^2 y}{dt^2} = 0$$

$\Rightarrow \frac{1}{c^2} = \mu \epsilon$

$\Rightarrow \boxed{c = \frac{1}{\sqrt{\mu \epsilon}}}$ - wave velocity

Electromagnetic Wave Equation in conducting Medium:-

Maxwell Equation in conducting medium with permeability μ, Permitivity ϵ and conductivity σ can be written as-

I) $\nabla \cdot E = 0$ — ①

II) $\nabla \cdot H = 0$ — ②

III) $\nabla \times E = -\mu \frac{dH}{dt}$ — ③

IV) $\nabla \times H = J + \epsilon \frac{dE}{dt}$

$= \sigma E + \epsilon \frac{dE}{dt}$ — ④

Taking Curl of Equation ③ we have-

$$\nabla \times (\nabla \times E) = \nabla \times \left(-\mu \frac{dH}{dt}\right)$$

$$= -\mu \left(\nabla \times \frac{dH}{dt}\right)$$

$$= -\mu \left(\frac{d}{dt}(\nabla \times H)\right)$$

$$= -\mu \left(\frac{d}{dt}\left(\sigma E + \epsilon \frac{dE}{dt}\right)\right) \quad (\text{using ④})$$

$$\nabla \times (\nabla \times E) = -\mu\sigma \frac{dE}{dt} - \mu\epsilon \frac{d^2E}{dt^2}$$

$$\Rightarrow \nabla \cdot (\nabla \cdot E) - \nabla^2 E = -\mu\sigma \frac{dE}{dt} - \mu\epsilon \frac{d^2E}{dt^2}$$

$$0 - \nabla^2 E = -\mu\sigma \frac{dE}{dt} - \mu\epsilon \frac{d^2E}{dt^2}$$

$$\nabla^2 E = \mu\sigma \frac{dE}{dt} + \mu\epsilon \frac{d^2E}{dt^2}$$

In case of Non-Conducting Medium $\boxed{\sigma = 0}$ then

$$\boxed{\nabla^2 E = \mu\epsilon \frac{d^2E}{dt^2}}$$

Now taking curl of Equation (4) we obtain

$$\nabla \times (\nabla \times H) = \nabla \times \left(\sigma E + \epsilon \frac{dE}{dt}\right)$$

$$= \sigma (\nabla \times E) + \epsilon \frac{d(\nabla \times E)}{dt}$$

(using (3)) $$= \sigma \left(-\mu \frac{dH}{dt}\right) + \epsilon \frac{d}{dt}\left(-\mu \frac{dH}{dt}\right)$$

$$-\nabla^2 H = -\mu \sigma \frac{dH}{dt} - \mu \epsilon \frac{d^2 H}{dt^2}$$

$$\nabla^2 H = \mu \sigma \frac{dH}{dt} + \mu \epsilon \frac{d^2 H}{dt^2}$$

If Medium is non-conducting $\boxed{\sigma = 0}$

$$\Rightarrow \boxed{\nabla^2 H = \mu \epsilon \frac{d^2 H}{dt^2}}$$

Solution of wave Equation:-

$$E(r,t) = E_0 e^{i(k \cdot r - \omega t)}$$

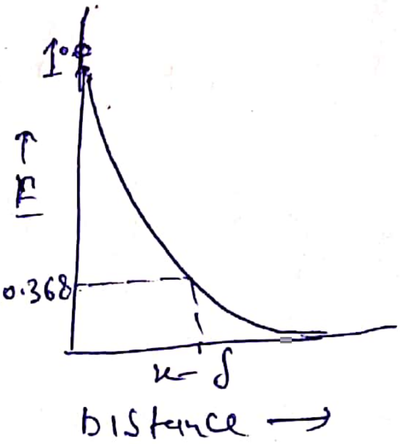
$$H(r,t) = H_0 e^{i(k \cdot r - \omega t)}$$

These two wave Equation Represent solution of wave Equation

Skin depth (Depth of Penetration)

It may be defined as the depth in which strength of Electric field associated with the Electromagnetic waves reduce to 1/e times to its initial value.

The Amplitude of strength of Electric field of an Electromagnetic waves decrease by a factor $e^{-\alpha x}$, where α is attenuation constant.



If depth of Penetration is represented by δ then-

$$x = \delta$$

$$\text{or} \quad \delta = \frac{1}{\alpha}$$

Thus Skin depth or Penetration depth

$$\delta = \frac{1}{\text{Attenuation Constant}}$$

Hence Reciprocal of Attenuation constant is called skin depth or Penetration depth where Attenuation constant is given by -

$$\alpha = \omega\left(\frac{\mu\epsilon}{2}\left(\sqrt{\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)}-1\right)\right)^{1/2}$$

$$\text{Skin Depth } \delta = \frac{1}{\omega\left(\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}\right)-1\right)^{1/2}}$$

POYNTING THEORM:- IMP

The rate of Energy transport per unit area, is called Poynting vector. It is also termed as instaneous energy flux density and is represented by S or P.

$$\boxed{\vec{S} = \vec{E} \times \vec{H}}$$

$\vec{S}$ is perpendicular to both $\vec{E}$ and $\vec{H}$.

Unit — W/m^2

Derivation:- We can calculate the energy density carried by Electromagnetic waves with the help of Maxwell's Equation given below-

i) $\nabla \cdot \vec{B} = 0$ —①

ii) $\nabla \cdot \vec{B} = 0$ —②

iii) $\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$ —③

iv) $\nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$ —④

Now Taking (.) Dot Product of ~~③ with~~ H with ③ and Dot Product of E with ④

$$\vec{H} \cdot (\nabla \times \vec{E}) = -\mu \vec{H} \cdot \frac{d\vec{H}}{dt} \quad —⑤ \quad (B = \mu \vec{H})$$

$$\vec{E} \cdot (\nabla \times \vec{H}) = \vec{E} \cdot \vec{J} + \epsilon \vec{E} \frac{d\vec{E}}{dt} \quad —⑥ \quad (D = \epsilon \vec{E})$$

Substracting Equation ⑤ from Equation ⑥

$$\Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{H}) - \vec{H}(\vec{\nabla} \times \vec{E}) = \vec{E} \cdot \vec{J} + \epsilon \vec{E} \frac{d\vec{E}}{dt} - \left(-\mu \vec{H} \frac{d\vec{H}}{dt}\right)$$

Since $\vec{A} \cdot (\vec{\nabla} \times \vec{B}) - \vec{B} \cdot (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \cdot (\vec{B} \times \vec{A}) = -\vec{\nabla}(\vec{A} \times \vec{B})$

$$\cancel{-\nabla\cdot(E\times H) = J\cdot E + \frac{1}{2}\epsilon\frac{d^2E}{dt^2}}$$

$$\Rightarrow -\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{J}\cdot\vec{E} + \frac{1}{2}\epsilon\frac{dE^2}{dt} + \frac{1}{2}\mu\frac{dH^2}{dt}$$

using
$$\left(\because \epsilon\vec{E}\frac{dE}{dt} = \frac{1}{2}\epsilon\frac{dE^2}{dt} \quad \mu\vec{H}\frac{dH}{dt} = \frac{1}{2}\mu\frac{dH^2}{dt} \right\}$$

$$\Rightarrow -\vec{\nabla}\cdot\vec{S} = \vec{J}\cdot\vec{E} + \frac{1}{2}\frac{d}{dt}(\epsilon E^2 + \mu H^2)$$

or
$$\vec{J}\cdot\vec{E} + \frac{d}{dt}\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right) + \vec{\nabla}\cdot\vec{S} = 0$$

Taking volume Integral over the volume V enclosed by the surface S,

$$\int_V \vec{\nabla}\cdot\vec{S}\,dv + \frac{d}{dt}\int_V\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv = -\int_V \vec{J}\cdot\vec{E}\,dv$$

using Divergence Theorm

$$\boxed{\int_S \vec{S}\cdot d\vec{S} + \frac{d}{dt}\int_V\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv = -\int_V \vec{J}\cdot\vec{E}\,dv}$$

→ $\int_S \vec{S}\cdot d\vec{S}$ → Represent rate of flow of Energy or Power flux.

→ $\frac{d}{dt}\int_V\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv$ = Rate of change of total Energy

→ $\frac{\epsilon E^2}{2}$ & $\frac{\mu H^2}{2}$ represent Energy stored in Electric & Magnetic Field.

$\int_V \vec{J}\cdot\vec{E}\,dv$ = Rate of work done by the Electromagnetic Field in displacing the charge within the volume V.

Hence

$$\vec{\nabla}\cdot\vec{S} + \vec{J}\cdot\vec{E} + \frac{d}{dt}\left(\frac{1}{2}\mu H^2\right) + \frac{d}{dt}\left(\frac{1}{2}\epsilon E^2\right) = 0$$

OR

$$\int_V \vec{\nabla}\cdot\vec{S}\,dv + \int_V \vec{J}\cdot\vec{E}\,dv + \frac{d}{dt}\int_V \left(\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2}\right) dv = 0$$

Above Equation is known as Poynting Theorm or work- Energy Theorm.

According to Poynting theorm the power transformed into Electromagnetic field is Equal to the Sum of the time rate of change of EM energy. within certain volume and the time rate of Energy flowing out through the boundary surface. This is also known as Energy conservation. law in Electromagnetism.

Example 3.11 *The electromagnetic wave intensity received on the surface of the earth from the sun is found to be 1.33 kW/m^2. Find the amplitude of electric field vector associated with sunlight as received on earth surface. Assume Sun's light to be monochromatic ($\lambda = 6000$ Å).*

[GGSIPU, Feb. 2012 (5 marks) ; Feb. 2008 (3 marks)]

Solution. The energy transported by an electromagnetic wave per unit area per second during propagation is represented by Poynting vector **S** as

$$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$$

The energy flux per unit area per second is

$$|\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = EH \sin 90^\circ = EH$$

The energy flux per unit area per second at the earth surface.

$$|\mathbf{S}| = 1.33 \text{ kW/m}^2 = 1.33 \times 10^3 \text{Jm}^{-2}\text{s}^{-1}$$

$$\therefore \quad |\mathbf{S}| = 1330 \text{ Jm}^{-2}\text{s}^{-1} \qquad \text{...(i)}$$

We know that $$Z_0 = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi\times10^{-7}\ \mathrm{Wb/Am}}{8.854\times10^{-12}\ \mathrm{C^2/Nm^2}}} = 376.72\ \Omega$$

$$\frac{E}{H} = 376.72\ \Omega \qquad ...(ii)$$

Multiplying Eq. (*i*) and Eq. (*ii*), we get

$$EH\times\frac{E}{H} = 1330\times376.72$$

or $$E^2 = 501037.6, \quad E = 707.8\ \mathrm{V/m}$$

Substituting this value in Eq. (*ii*)

$$H = \frac{707.8}{376.72} = 1.879\ \mathrm{A/m}.$$

Therefore, the amplitudes of electric and magnetic fields of radiation are

$$E_0 = E\sqrt{2} = 707.8\sqrt{2} = 1000.8292 = 1000\ \mathrm{V/m}$$

and $$H_0 = H\sqrt{2} = 1.879\sqrt{2} = 2.657\ \mathrm{A.turn/m}.$$

Example 3.12 If the earth receives 2 Cal min^{-1}cm^{-2} solar energy, what are the amplitudes of electric and magnetic field of radiation ? **[GGSIPU., June 2015 (5 Marks), May 2016 (4 marks)]**

Solution. As Poynting vector,

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = EH\sin 90° = EH$$

$$\text{Solar energy} = 2\ \text{Cal min}^{-1}\text{cm}^{-2}$$

$$= \frac{2\times4.18\times10^4}{60}\ \mathrm{Jm^{-2}s^{-1}}$$

Both are energy flux per unit area per second

Hence $$EH = \frac{2\times4.18\times10^4}{60} \approx 1400$$

But $$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\ \pi = 377$$

$\therefore$ $$EH\times\frac{E}{H} = 1400\times377$$

$$E = \sqrt{1400\times377} = 726.5\ \mathrm{V/m}$$

Now, $$H = \frac{E}{377} = 1.927\ \mathrm{A/m}$$

Amplitudes of electric and magnetic field of radiation are

$$E_0 = E\sqrt{2} = 1024.3\ \mathrm{V/m}$$
$$H_0 = H\sqrt{2} = 2.717\ \mathrm{A/m}$$

Example 3.13 *Calculate the Poynting vector at the surface of the sun. Given that the energy radiated per second is* 3.8×10^{26} *J and the radius of the sun is* 0.7×10^{9} *m. Also calculate the amplitudes of electric and magnetic field vectors on the surface of the earth. The distance of the earth from the sun is* 0.15×10^{12} *m.*

[GGSIPU., May 2014 (6 Marks)]

Solution. **First case.** The Poynting vector at the surface of sun

$$|\mathbf{S}| = |\mathbf{E}\times\mathbf{H}| = EH\sin 90^\circ$$

$$= \frac{P_0}{4\pi r^2} = \frac{3.8\times10^{26}\ \text{J/s}}{4\pi\times(0.7\times10^{9}\,\text{m})^2}$$

$$= 6.17\times10^{7}\ \text{Jm}^{-2}\text{s}^{-1}$$

Second case. The Poynting vector at the surface of earth

$$EH = \frac{P_0}{4\pi r_1^2}$$

$$= \frac{3.8\times10^{26}\ \text{J/s}}{4\pi\times(0.15\times10^{12}\,\text{m})^2} = 1344.656\ \text{Jm}^{-2}\text{s}^{-1} \quad ...(i)$$

We also know

$$\frac{E}{H} = 120\pi\Omega = 377\ \Omega \quad ...(ii)$$

From Eqs. (*i*) and (*ii*)

$$\frac{E}{H}\times EH = 1344.656\times377$$

and

$$E^2 = \sqrt{1344.656\times377} = 711.99\ \text{V/m}$$

and from Eq. (*ii*)

$$\frac{E}{H} = 377\ \Omega$$

$$H = \frac{E}{377\Omega} = \frac{711.9938}{377}\ \text{A/m} = 1.888\ \text{A/m}$$

Amplitude of electric and magnetic field of radiation are

$$E_0 = E\sqrt{2} = 711.99\times1.414 = 1006.75\ \text{V/m}$$

and

$$H_0 = H\sqrt{2} = 1.888\times1.414 = 2.67\ \text{A/m}$$