11

STATISTICAL DISTRIBUTIONS ① To determine the most probable way in which a certain total ammount of energy E is distributed among to N particle of a System in thermal equilibrium at a aboslute temp. T.

② A Basic thing of statistical mechanics is that the greater the number W of different ways in which particles can be arranged among the available states, the most probable is the distribution. It is assumed that each State of a certain energy is equally likely to be occupied.

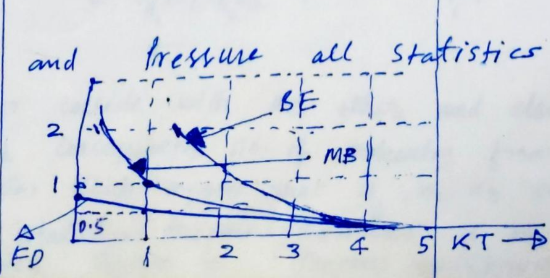
③ So Statistical mechanics begin with by finding a general formula for W for the Kind of particles being considered. Which corresponds to most propable distribution ie, System being in Thermal equilibrium

④ Condition is N remain constt. and E total energy remain constt)

COMPARISON OF The THREE STATISTICS :

M-B	F-D	B-E
① Particles : Are distinguishable Identical (but overlapping of wave functions is negli)	Indistinguishable, Identical (overlapping is in great extent)	Indistinguishable, Identical (overlapping is in great extent)
② only particles are taken into consideration	only Quantum states are taken into consideration.	only quantum states are taken into consideration.
③ No restriction of no. of particles in a state	③ Restriction on no. of particle in a state.	③ No restriction on no. of particle in a state (quantum)
④ Applicable to the gas molecules	⑤ Applicable to electrons of high concentrations like Fermi-gas ie electrons in the metal. spin : odd integral multiple of $\frac{1}{2}$. ex : electrons, protons, neutrons.	④ Applicable to photons and Symmetrical particles. (photon (1), graviton (2), He ~~atom~~ molecule (0), α particle (0), (Cooper pair) (0))
⑤ Particles are classical	⑤ quantum mechanical	⑥ quantum mechanical.
⑥ No wave function associated with particle	⑥ Antisymmetric wave function describe the system of two particle 1, 2 and state a, b. $\Psi_F = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)\right]$	⑥ Symmetric wavefunction $\Psi_B = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) + \psi_a(2)\psi_b(1)\right]$ System is described by above Symm function
⑦ CLASSICAL	⑦ Fermions	⑦ Bosons.
⑧	⑧ Fermions obey Pauli exclusion principle	⑧ Do not obey Pauli exclusion principle.
⑨ No. of distinguishable ways for arranging the particles $W = \prod_{i=1}^{N} \frac{g_i^{n_i}}{n_i!}$	$W = \prod_{i=} \frac{g_i!}{n!(g_i - n_i)!}$	$W = \prod_{i=1} \frac{[n_i + g_i - 1]!}{n_i!(g_i - 1)!}$
Correct ⑩ occupancy of state (Distribution function) $f(\epsilon) = \frac{n_i}{g_i} = \frac{1}{e^{\frac{\alpha + \beta\epsilon_i}{KT}}}$	$f(\epsilon) = \left(\frac{n_i}{g_i}\right) = \frac{1}{e^{\frac{\alpha+\beta\epsilon_i}{KT}} + 1} \leq 1$	$f(\epsilon) = \left(\frac{n_i}{g_i}\right) = \left(\frac{1}{e^{\frac{\alpha+\beta\epsilon_i}{KT}} - 1}\right) > 1$

⑪ At high temp and pressure all statistics gives the same result.



Molecular Energies in an Ideal Gas:

Consider MB distribution law.

$$n_i = g_i \, e^{-E_i/KT} \quad \text{—— (1)}$$

Consider an ideal gas that contains N molecules. Consider a continuous distribution of molecular energies instead of the discret set $E_1\ E_2\ E_3$ -----. Then (1) can be written as.

$$n(E)\,dE = g(E)\, e^{-(E/KT)}\, dE \quad \text{—— (2)}$$

Here $n(E)\,dE \longrightarrow$ No of molecules having energies between E and $E+dE$.

and $g(E) \longrightarrow$ represents the density of states having energies E and $E+dE$.

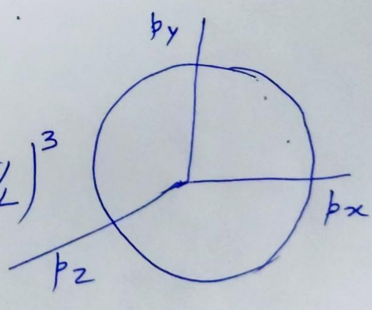
$p = \hbar K$

for $g(E)$: Consider momentum space

and compare with K space.

Let Volume $= \frac{4}{3}\pi p^3$

Momentum element Correspond to one allowed state $= \left(\hbar \pi/L\right)^3$



$$g(E)\,dE = C\sqrt{E}\, dE \quad \text{—— (3)}$$

$$\therefore \quad n(E)\,dE = C\sqrt{E}\, e^{-E/KT}\, dE \quad \text{—— (4)}$$

Apply normalization condition to find constant C.

$$\int_0^\infty n(E)\,dE = N = C\int_0^\infty \sqrt{E}\, e^{-E/KT}\, dE$$

$$N = C\,\frac{1}{2\left(\frac{1}{KT}\right)}\sqrt{\frac{\pi}{(1/KT)}}$$

$$N = \frac{C}{2}\,\frac{(\pi KT)^{3/2}}{\pi}$$

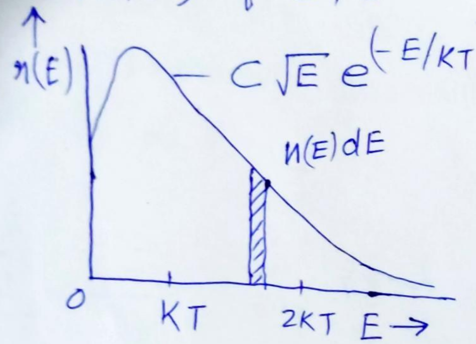
$$\int_0^\infty \sqrt{x}\, e^{-\alpha x}\, dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$\boxed{\alpha = \frac{1}{KT}}$$

$$\therefore\ C = \frac{2N\pi}{(\pi KT)^{3/2}}$$

$$\therefore\ \boxed{n(E)\,dE = \underbrace{\frac{2\pi N}{(\pi KT)^{3/2}}}\sqrt{E}\, e^{-E/KT}\, dE} \quad \text{—(5)}$$

Plotting of 5, Vs. KT.



* Total energy of the system :

$$E = \int_0^\infty E \cdot n(E)\,dE = \int_0^\infty \underbrace{C E^{3/2} e^{-E/KT}\, dE}$$

$$= \frac{3}{4\alpha^2}\sqrt{\frac{\pi}{\alpha}}$$

$$E = \frac{2\pi N}{(\pi KT)^{3/2}} \cdot \underbrace{\frac{3}{4}(KT)^2\sqrt{\pi KT}}_{\text{Value of definite integral}}$$

$$E = \frac{3}{2} NKT$$

Average energy of an ideal gas molecule $\overline{E} = \frac{3}{2} KT$

* Maxwell-Boltzmann Velocity distribution Law :

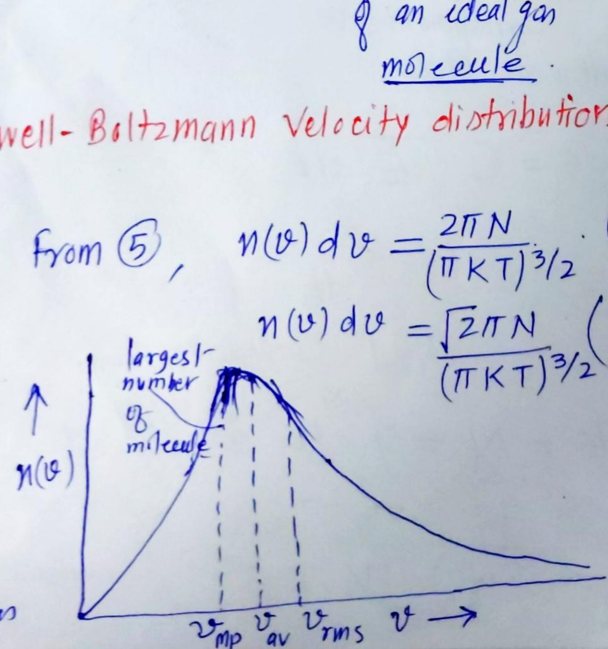
$$E = \frac{1}{2} m v^2$$

$$dE = mv\,dv$$

$\therefore$ from (5),

$$n(v)\,dv = \frac{2\pi N}{(\pi KT)^{3/2}} \cdot \left(\frac{1}{\sqrt{2}}\sqrt{m}\, v\right) \cdot e^{-\frac{mv^2}{2KT}} (mv\,dv)$$

$$n(v)\,dv = \frac{\sqrt{2}\pi N}{(\pi KT)^{3/2}} \left(m^{3/2} v^2\right) e^{-\frac{mv^2}{2KT}}\, dv$$



$$v_p = \sqrt{\frac{2KT}{m}} = 1.41\sqrt{\frac{KT}{m}}$$

$$\frac{n_1 v_1 + n_2 v_2 + n_3 v_3}{N} = \overline{v}_{av} = \sqrt{\frac{8KT}{\pi m}} = 1.6\sqrt{\frac{KT}{m}}$$

$$\int_0^\infty v^2 n(v)\,dv \leftarrow \overline{v^2}$$

$$v_{rms} = \sqrt{\frac{3KT}{m}} = 1.73\sqrt{\frac{KT}{m}}$$

$$\sqrt{\frac{n_1 v_1^2 + n_2 v_2^2 + n_3 v_3^2}{N}}$$

* $m \to$ mass of gas molecule.

27/01/15

Black body radiation: spectral distribution.

For any body $r + t + a = 1 \Rightarrow$ If $r = t = 0 \Rightarrow a = 1$ (100%)

Ex → black hole → neither particles nor radiation can escape from it

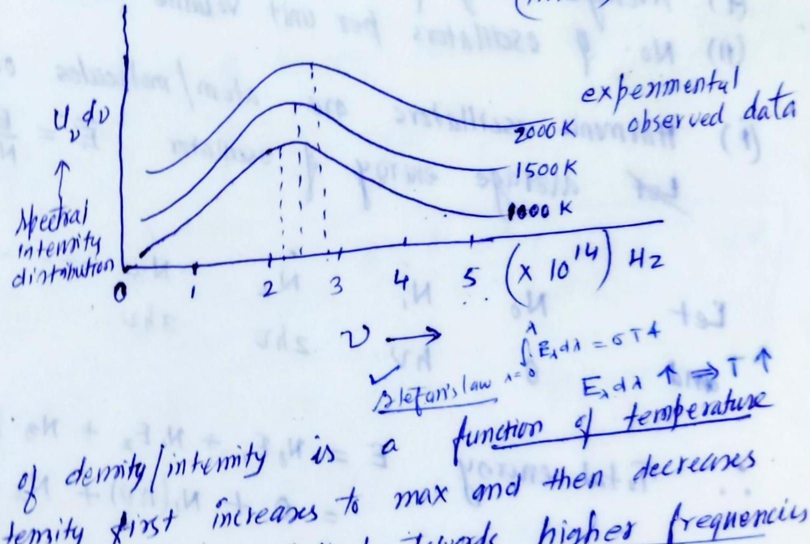
① Blackbody: A black body completely absorbs the radiations of all wavelengths incident (falling) on it.

② Blackbody radiations: When such a body is **heated**, it emits radiations which we call blackbody radiations.

③ Structure of blackbody:

(cu/fe) → A hollow container (cu/fe) with black lamp coating (inside)

④ Spectral distribution: (Fundamental laws of black body Radiation)



$\int_{\lambda=0}^{\lambda} E_\lambda d\lambda = \sigma T^4$ Stefan's law

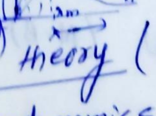
$E_\lambda d\lambda \uparrow \Rightarrow T \uparrow$

Graph shows the following: density/intensity is a function of temperature

(i) spectral distribution of density/intensity ⇒ intensity first increases to max and then decreases

(ii) As $\nu \uparrow$ ⇒ intensity peak shifted towards higher frequencies

(iii) position of maximum intensity peak shifted towards higher frequencies (lower wavelengths) with increase in temperature.



$\lambda_m T =$ Const (Wien's displacement law)

$K_B T$ ✓ Standing wave formation

⑤ The classical electromagnetic theory (based on wave nature of radiation) together with classical thermodynamics ($E = K_B T$) does not explain the characteristics of blackbody radiation

* Stefan's Law → Total intensity of radiation $\int_{\lambda=0}^{\infty} E_\lambda d\lambda = \sigma T^4$ → Stefan's constant.

* ~~Wien's and~~ Rayleigh Jean tried to explain by considering law of equipartition $(\frac{1}{2}K_BT)$ per dof energy of molecules

two SOP state of polarization due to E and B.

$\langle E \rangle = 2 \times \frac{1}{2} K_B T = K_B T$

⑥ Planck's Hypothesis : To explain blackbody radiation. Basic postulates
(1900) , got Nobel prize in 1918, (Germany)
for physics

of Planck's hypothesis —

(i) An oscillator (atom/molecule) can have only discrete energies (photon) i.e.

$$E = nh\nu \quad (n = 0, 1, 2, 3 \ldots) \quad (h = 6.67 \times 10^{-34}\ J.s)$$

(ii) Exchange of energies (emit/absorb) happens in the form of packets of energy ($h\nu$) but not continuous. i.e.

$$\Delta E = (n_2 - n_1) h\nu$$

Derivation : of Planck's Radiation formula : (for Radiation energy per unit volume)

(i) Average energy of an harmonic oscillator (
(ii) No. of oscillators per unit volume.

(i) Harmonic oscillators are atom/molecules on the wall of black body. Then

Let average energy of oscillator $\bar{E} = \frac{E}{N} = \frac{\text{Total energy of these oscillators}}{\text{Total no. of oscillators}}$

Let $N_0 \quad N_1 \quad N_2 \quad N_3 \ldots N_n \ldots$ be the no. of oscillators

and $0 \quad h\nu \quad 2h\nu \quad 3h\nu \ldots nh\nu \ldots$ be the corresponding energies

$\therefore$ Total energy $E = N_0 E_1 + N_1 E_2 + N_3 E_3 + \ldots + N_n E_n -$

$= 0 + N_1(h\nu) + N_2(2h\nu) + \ldots + N_n(nh\nu)$

where.

Total oscillators $N = N_0 + N_1 + \ldots + N_n + \ldots$

By Maxwell distribution - law (M-B) : $N_n = N_0 e^{-\frac{E_n}{KT}}$

$N_0 = N_0$

$N_1 = N_0 e^{-\frac{h\nu}{KT}}$

$N_2 = N_0 e^{-\frac{2h\nu}{KT}}$

$$\therefore E = 0 + \left(N_0 e^{-\frac{h\nu}{KT}}\right) h\nu + (2h\nu) N_0 e^{-\frac{2h\nu}{KT}} + \ldots + (nh\nu) N_0 e^{-\frac{nh\nu}{KT}}$$

$$= N_0 h\nu e^{-\frac{h\nu}{KT}} \left[1 + 2e^{-\frac{h\nu}{KT}} + 3e^{-\frac{2h\nu}{KT}} + \ldots + n e^{-\frac{(n-1)h\nu}{KT}} + \ldots \right]$$

$$= \frac{N_0 (h\nu) e^{-\frac{h\nu}{KT}} (1)}{(1 - e^{-h\nu/KT})^2} \quad \text{— (1)}$$

$$N = N_0 + N_0 e^{-\frac{h\nu}{KT}} + N_0 e^{-\frac{2h\nu}{KT}} + \cdots + N_0 e^{-\frac{nh\nu}{KT}} + \cdots$$

$$= N_0 \left[1 + e^{-\frac{h\nu}{KT}} + e^{-\frac{2h\nu}{KT}} + \cdots + e^{-\frac{nh\nu}{KT}}\right] + \cdots$$

$$= \frac{N_0 \cdot 1}{\left(1 - e^{-\frac{h\nu}{KT}}\right)}$$

$$\therefore \bar{E} = E/N = \frac{N_0 (h\nu)\, e^{-\frac{h\nu}{KT}}}{\left(1 - e^{-\frac{h\nu}{KT}}\right)^2} \cdot \frac{\left(1 - e^{-\frac{h\nu}{KT}}\right)}{N_0} = \frac{h\nu\, e^{-\frac{h\nu}{KT}}}{\left(1 - e^{-\frac{h\nu}{KT}}\right)}$$

$$= \left(\frac{h\nu}{e^{\frac{h\nu}{KT}} - 1}\right)$$

No of standing waves / volume due to reflections and incident on the cavity walls.

Planck's formula : Energy density of radiation (U_ν) in the frequency range ν to $\nu + d\nu$ depending upon the average energy of an harmonic oscillator.

$U_\nu\, d\nu = (\bar{E}) \times$ density of harmonic oscillators between ν and $\nu + d\nu$.

$$= \left(\frac{h\nu}{e^{\frac{h\nu}{KT}} - 1}\right) \times \frac{8\pi\nu^2}{c^3}\, d\nu$$

$$\boxed{U_\nu\, d\nu = \frac{8\pi\nu^2}{c^3}\left(\frac{h\nu}{e^{\frac{h\nu}{KT}} - 1}\right) = \frac{8\pi h\nu^3}{c^3}\left(\frac{1}{e^{\frac{h\nu}{KT}} - 1}\right) d\nu}$$

Planck's formula (law) in terms of frequency

In terms of wavelength $\Rightarrow$ $\nu = c/\lambda$

$$d\nu = -c/\lambda^2\, d\lambda$$

$$U_\lambda\, d\lambda = \frac{8\pi h}{\lambda^3}\left(\frac{c}{\lambda^2}\right)\frac{1}{\left(e^{\frac{hc}{\lambda KT}} - 1\right)}\, d\lambda$$

$$\boxed{U_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5}\left(\frac{1}{e^{\frac{hc}{\lambda KT}} - 1}\right) d\lambda}$$ In terms of wavelength

for lower wavelength (~~Rayleig~~ Weins law) $\rightarrow e^{\frac{hc}{\lambda KT}} >> 1$ So 1 can be neglected

$$\therefore \boxed{U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\left(e^{\frac{-hc}{\lambda KT}}\right) \cdot d\lambda}$$ Wein's law

for higher wavelength (Rayleigh - Jeans law), $e^{\frac{hc}{\lambda KT}} \neq (1 + \frac{hc}{\lambda KT} - 1)$

$$\therefore U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\left(\frac{\lambda KT}{hc}\right) \cdot d\lambda$$

$$\therefore \boxed{U_\lambda d\lambda = \frac{8\pi KT}{\lambda^4} \cdot d\lambda}$$

hold good for longer wavelength but fails at shorter (lower) wavelength known as (ultraviolet catastrophe) $\Rightarrow$ Total energy $= \int_{\lambda=0} U_\lambda d\lambda$, $E_\lambda = \infty$

$\Rightarrow E_\lambda \rightarrow \infty$ for $\lambda = 0$ (ultraviolet)
but experiment shows $E_\lambda \rightarrow 0$

* calculate wavelength at which human body radiates maximum energy

$T = 37 + 273 = 310$ K

$\lambda_m T = 2.9 \times 10^{-3}$ m K

$\lambda_m = 9 \times 10^{-6}$ m (IR region)

Wein displacement's law.

(ex) Mammals with temp 300 K, emits radiation about 10 μm in far Infrared (IR)

F-D Distribution Law: For a system consisting of fermions in thermal or statistical equilibrium.

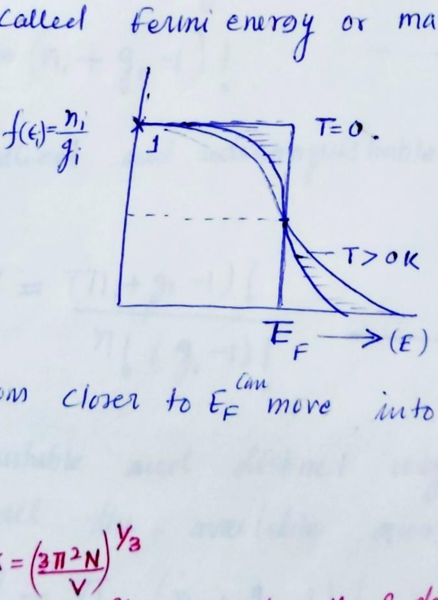
$$\beta = \frac{1}{KT} \quad \text{and} \quad \alpha = -\frac{E_F}{KT}$$

$$\therefore \quad n_i = \frac{g_i}{e^{\frac{E_i - E_F}{KT}} + 1}$$

The value of E_F is positive and is independent of temp.

for $T = 0$. $\begin{cases} \text{if } E_i < E_F \Rightarrow \text{All States are fully occupied. } \boxed{(n_i = g_i)} \\ \text{if } E_i > E_F \Rightarrow \text{All States are empty. } \boxed{n_i = 0} \end{cases}$

$\Rightarrow$ In the F-D Statistics, accumulation of particles at ground states is not allowed and at $T = 0\,K$, the particles occupied the lowest energy levels upto E_F. E_F called fermi energy or maximum energy of Fermions



At $T > 0K$, Fermions closer to E_F can move into the higher unoccupied energy states.

* $E_F = \frac{\hbar^2}{2m} K^2$ where $K = \left(\frac{3\pi^2 N}{V}\right)^{1/3}$

$K^2 = (3\pi^2 n)^{2/3}$; $n = N/V = \frac{\text{No. of electrons}}{\text{volume (m3)}}$

Fermi energy: Max. KE of electron at 0K.

($E < E_F$) below fermi-energy, energy states are filled.

($E > E_F$) Above - - - - - - - , - - - - - - - are vacant.

No of fermions density with energy between E and E+dE,

$$n(\epsilon)d\epsilon = g(\epsilon) f(\epsilon)\, d\epsilon$$

$$= \frac{8\sqrt{2}\pi}{h^3} m^{3/2} \sqrt{\epsilon} \frac{1}{\left(e^{\frac{\epsilon-\epsilon_F}{KT}}+1\right)} d\epsilon$$

(Total fermions density below ϵ_F at 0K)

$$n \doteq \frac{8\sqrt{2}\pi}{h^3} m^{3/2} \int_0^{\epsilon_F(0)} \left(\frac{\epsilon^{1/2}}{e^{\frac{\epsilon-\epsilon_F}{KT}}+1}\right) d\epsilon$$

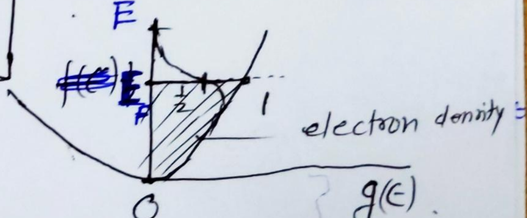
$$n = \frac{8\sqrt{2}\pi}{h^3} m^{3/2} \int_0^{\epsilon_F(0)} \epsilon^{1/2} d\epsilon$$

at obs. 0°K $e^{\frac{\epsilon-\epsilon_F}{KT}} \rightarrow 0$ for $\epsilon < \epsilon_F$

$$n = \frac{8\sqrt{2}\pi}{h^3} m^{3/2} \cdot \frac{3\,\epsilon_F^{3/2}(0)}{2}$$

$$\boxed{\epsilon_F(0) = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}}$$

where n is electron density



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$$n = \frac{8\sqrt{2}\pi}{h^3} m^{3/2} \int_0^{\epsilon_F(0)} \left(\frac{\epsilon^{1/2}}{e^{\frac{\epsilon-\epsilon_F}{KT}}+1}\right) d\epsilon$$

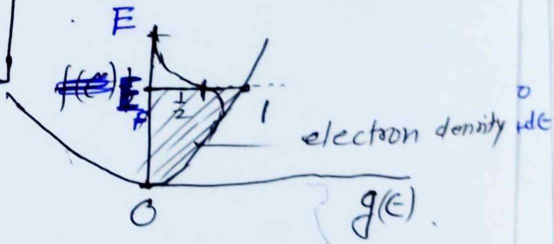
$$n = \frac{8\sqrt{2}\pi}{h^3} m^{3/2} \int_0^{\epsilon_F(0)} \epsilon^{1/2} d\epsilon$$

at obs. 0°K $e^{\frac{\epsilon-\epsilon_F}{KT}} \to 0$ for $\epsilon < \epsilon_F$

$$n = \frac{8\sqrt{2}\pi}{h^3} m^{3/2} \cdot \frac{3\,\epsilon_F^{3/2}(0)}{2}$$

$$\boxed{\epsilon_F(0) = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}}$$

where n is electron density



Average $\overline{E}_F = 3/5\, \epsilon_F(0)$

(Ex) Find the Fermi-energy in Cu on the assumption that each Cu atom contribute one free electron to the Fermi-gas.

$d = 8.94 \times 10^3\ kg/m^3$ and Atomic mass $= 63.5$ amu

Solution: $E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$; Here $n = \frac{\text{electron}}{m^3} = \frac{8.94\times10^3}{\text{Mass of one atom}}\left(\frac{kg}{m^3}\right) = \left(\text{atoms}/m^3\right)$

$= \text{electrons}/m^3$

$\simeq 1.13\times10^{-18}\ J$

$\simeq 7.04\ eV$

$n = \frac{8.94\times10^3}{63.5\times1.67\times10^{-27}} = 8.4\times10^{28}\ /m^3$

Speed corresponds to 6×10^6 m/s. at 0 K.

* But in ideal gas; energy of particle at 0K $= \underline{0}$ J. $\boxed{KT}$

* Find the dim of C in gaussian units.

7.3.4 Free Electron Gas

The behaviour of free electrons inside a metallic conductor can be studied by applying *Fermi-Dirac distribution law*. A metal can be considered to be composed of a system of fixed positive nuclei and a number of mobile electrons. These mobile electrons are assumed to move freely in the metal like the particles of a gas and collide continuously with fixed nuclei. Thus, they form a sort of gas known as an free *electron gas*. On this assumption the classical statistics could explain to a certain extent the various properties of metals depending upon the motion of free electrons in them such as electrical and thermal conductivities, thermoelectricity, thermionic emission, magnetic properties of metals and photo-electric effect etc. But in certain cases, like specific heat of metals, very serious difficulties were encountered in the use of classical statistics. That is why the theory of the electron gas was discredited to some extent.

Sommerfeld, in 1928, revised the electron theory of metals on the basis of the new quantum statistics. According to him, electrons in metals are not completely free but only partially free in the sense that though they are not bound to a particular atomic system, yet they are bound to the metal as a whole. Thus the interior of metal is considered as a region of uniform potential, which is positive relative to free electrons so that work is required to be done for extracting an electron from the metal. *Therefore electrons in metallic conductors cannot be compared with the free particles of a gas obeying the classical statistics*. Moreover, because of their small mass and dense packing, the electrons in metals should be considered as the particles of a gas under very high compression, hence a degenerate gas. Further since the electrons have half integral spin angular momentum and obey Pauli's exclusion principle, so they are Fermi-particle and should *obey the Fermi-Dirac statistics*. To a fist approximation, the mutual interaction of electrons can be neglected due to neutralizing effect of positive ions inside a metal.

To study the properties of electron gas at low temperatures in the region $T \to 0$, we shall make use of the last section.

Therefore from eqn. (7.32) we get *Fermi energy at 0K* as

$$\varepsilon_F(0) = \frac{h^2}{2m}\left[\frac{3n}{8\pi V}\right]^{2/3}$$

$$= \frac{h^2}{8m}\left(\frac{3n}{\pi V}\right)^{2/3} = \frac{h^2}{8m^{5/3}}\left(\frac{3nm}{\pi V}\right)^{2/3} = 0.625 \times 10^{-17}\,\rho^{2/3} \text{ joules}$$

$$\varepsilon_F(0) = 39\,\rho^{2/3}\,eV \qquad ...(7.37)$$

Substituting $m = 9.1 \times 10^{-31}$ kg, $h = 6.62 \times 10^{-34}$ Js, where $\rho = \left(\frac{mn}{V}\right)$ kg m^{-3} is the density of the electron gas. For conduction electrons in metals, $\rho \simeq 0.1$ kg m^{-3}.

The *Fermi temperature* T_F for electron gas is

$$T_F = \frac{\varepsilon_F(0)}{k} = \frac{h^2}{8mk}\left(\frac{3n}{\pi V}\right)^{2/3} = \frac{h^2}{8m^{5/3}k}\left(\frac{3mn}{\pi V}\right)^{2/3}$$

or

$$T_F = (4.52 \times 10^5\,\rho^{2/3})\text{K} \qquad ...(7.38)$$

For $\rho = 0.1$ kg m^{-3}, $T_F = 10^{+5}$ K. Thus electron gas below 10^5 K temperature is degenerate. The degeneracy factor of an electron gas is given by the expression,

$$D = \frac{1}{2}\frac{n}{V}\frac{h^3}{(2\pi mkT)^{3/2}}$$

As $T \to 0$, $D \to \infty$, therefore $\dfrac{1}{e^{\alpha + x} + 1} = \dfrac{1}{\frac{1}{D}e^x + 1} \to 1$

So for low temperatures

$$n = 2\frac{V}{h^3}(2\pi mkT)^{3/2}\frac{2}{\sqrt{\pi}}\int_0^\infty x^{1/2}dx$$

$$= \frac{4V}{\sqrt{\pi}}\frac{(2\pi mkT)^{3/2}}{h^3}\int_0^D x^{1/2}\,dx$$

Here we have replaced the upper limit by D at low temperatures near absolute zero as $D \to \infty$ when $T \to 0$.

or

$$n = \frac{hV}{\sqrt{\pi}}\frac{(2\pi mkT)^{3/2}}{h^3}\frac{D^{3/2}}{3/2}$$

or

$$D = \frac{h^2}{2mkT}\left(\frac{3n}{8\pi V}\right)^{2/3}$$

$$= \frac{h^2}{2m^{5/2}kT}\left(\frac{3mn}{8\pi V}\right)^{2/3}$$

or

$$D = \frac{h^2}{2m^{5/2}kT}\left(\frac{3\rho}{8\pi}\right)^{2/3} \qquad ...(7.39)$$

Now substituting $h = 6.6 \times 10^{-34}$ J s, $m = 9 \times 10^{-31}$ kg, $k = 1.38 \times 10^{-23}$ J/deg and $\rho = 0.1$ kg m^{-3}

$$D = \frac{4.66 \times 10^5}{T}$$

This means that at low temperature the electron gas is strongly degenerate. It shows clearly that for electron gas, the classical statistics is not valid and can be applied only at temperatures of the order of 10^5 (because only then $D \to$ unity). Therefore at low temperatures and other ordinary working temperatures it is necessary to use Fermi-Dirac statistics for studying electron gas in metals.

Zero Point Energy of Electron Gas (Energy of electron gas at 0 K)

We have seen that at temperature $T = 0$ K, all the electrons have energies less than or equal to. So from eqn. (7.33), we have

$$n(\varepsilon)\, d\varepsilon = dn(\varepsilon) = \frac{3}{2} n \varepsilon_F^{-3/2}(0)\, \varepsilon^{1/2}\, d\varepsilon$$

$\therefore$ Energy at 0 K, is

$$U_0 = \int_0^{\varepsilon_F(0)} \varepsilon\, n(\varepsilon)\, d\varepsilon$$

$$= \frac{3}{2} n\, \varepsilon_F^{-3/2}(0) \int_0^{\varepsilon_F(0)} \varepsilon^{3/2}\, d\varepsilon$$

$$= \frac{3}{2} n\, \varepsilon_F^{-3/2}(0) \left[\frac{2}{5} \varepsilon^{5/2}\right]_0^{\varepsilon_F(0)}$$

$$U_0 = \frac{3}{2} n\, \varepsilon_F^{-3/2}(0)\, \frac{2}{5}\, \varepsilon_F^{5/2}(0) = \frac{3}{5} n \varepsilon_F(0) \qquad \text{...(7.40)}$$

7.3.5 Electronic heat Capacity

In the previous section, we have applied Fermi-Dirac statistics to Fermi electron gas in which mutual interactions of the eletrons in neglected due to neutralizing effect of the positive ions inside a metal. Drude in 1900 suggested that electrical and thermal behaviour of metals can be correlated by assuming that free electrons exist in thermal equilibrium with the atoms of the metals. Loroutz in 1905 explained that the electrons in metals obey MB statistics.

If this is so, then the free electron should contribute an amount 3/2 k per free (valence) electron to the heat capacity, in addition to the contribution form the atomic vibrations. At room temperature the latter leads to Dulong and Petit'law which contributed to the heat capacity, as

$$C_V = \left(\frac{\partial U}{\partial t}\right)_V = 3R = 249 \text{ Jmol}^{-1} \text{ K}^{-1}$$

The correct explanation of heat capacity was given assuming that assembly of electrons in metal (free electron gas) obey *Fermi-Dirac statistics*.

We have seen that the FD distribution depends only slightly on temperature. As the temperature is raised from 0K to TK, each free electron does not gain energy by an amount kT because most of them are occupying states of energy less $\varepsilon_F(0)$, represents the energy of the highest level occupied at 0K $\varepsilon_F(0) = 39\,\rho^{2/3}$ eV of electrons. In accordance with the Pauli principle they cannot be excited to the states as they are already fully occupied. It is only a small fraction of electrons with energy close to $\varepsilon_F(0)$ that can be excited to empty states lying in the range kT. This number of excited electrons N is given by

$$N_{\text{exc}} = \frac{3N}{\varepsilon_F(0)}\, kT = \frac{3}{2}\, N \frac{T}{T_F}$$

Clearly, only a small fraction ($\sim 3T/2T_F$) of the conduction electrons are excited. For $T_F \sim 10^4$ K, $T \sim$ 300, only a few percent are excited. The electron energy is obtained as

$$U(T) = N_{\text{exc}} kT = \frac{3}{2}\, Nk \frac{T^2}{T_F}$$

and the electronic heat capacity (specific heat) is

$$(C_V)_{el} = \left(\frac{\partial U}{\partial T}\right)_V = 3Nk\,\frac{T}{T_F} = aT \quad ...(7.41)$$

which is close to the correct result due to Sommerfeld and a is called Sommerfeld constant

Thus, we see that at room temperature the electronic heat capacity per electron, $3k(T/T_F)$, calculated according to *FD statistics* is very small compared to the atomic specific beat of about $3k$ per atom, in agreement with experimental results.

The total heat capacity of a metal in made up of two parts. Electronic and lattice heat capacity (due to photons). At very low temperatures, the lattice heat capacity is proportional to T^3, or

$$(C_V)_{ph} = bT^3$$

where $b = \dfrac{16\pi^5 k^4}{15h^2 u_1^3}$, $u_1 = 22.6 \times 10^3$ cm/s. We have seen that electronic heat capacity varies linearly with T *i.e.*

$$(C_V)_{el} \propto T$$

For very low temperatures, $(C_V)_{ph}$ decreases very rapidly and $(C_V)_{el}$ begins to dominate. This is in good agreement with experimental results. Accordingly, the total that capacity of a metal is expressed as

$$(C_V)_{\text{total}} = aT + bT^3. \quad ...(7.41)$$

7.4 COMPARISON OF M-B, B-E AND F-D STATISTICS

Maxwell-Boltzmann	Bose-Einstein	Fermi-Dirac
1. Particles are considered as distinguishable.	1. Particles are considered as indistinguishable.	1. Particles are considered as indistinguishable.
2. Only particles are taken into consideration.	2. Only quantum states are taken into consideration.	2. Only quantum states are taken into consideration.
3. No restriction on number of particles in a particular state.	3. No restriction on number of particles in a particular quantum state.	3. There is restriction on number of particles in a particular quantum state.

4. It is applicable to gas molecules.	4. It is applicable to photons.	4. It is applicable to electrons.
5. Volume of a cell in phase space is unknown.	5. Volume of a cell in phase space is given by $V = h^4$.	5. Volume of a cell in phase space is given by $V = h^4$.
6. No. of distinguishable ways are given by $W = n_i \prod \frac{(g_i)^{n_i}}{n_i!}$	6. No. of distinguishable ways are given by $W = \prod \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}$	6. No. of distinguishable ways are given by $W = \prod \frac{g_i}{n_i!(g_i - n_i)!}$
7. Maximum probability distribution is proportional to $\frac{1}{e^{\alpha} e^{\beta u_i}}$	7. Maximum probability distribution is proportional to $\frac{1}{e^{\alpha} e^{\beta u_i} + 1}$	7. Maximum probability distribution is proportional to $\frac{1}{e^{\alpha} e^{\beta u_i} + 1}$
8. —	8. At high temperature B-E distribution approaches to M-B distribution.	8. At high temperature F-D distribution approaches to M-B distribution.
9. At absolute zero, energy of molecules is zero.	9. The energy is zero at absolute zero.	9. The energy is not zero at absolute zero.
10. Distribution law of energy state is given by $n_i = \frac{g_i}{e^{\alpha} e^{\beta u_i}}$	Distribution law of energy states is given by $n_i = \frac{g_i}{e^{\alpha} e^{\beta u_i} - 1}$	Distribution law of energy states is given by $n_i = \frac{g_i}{e^{\alpha} e^{\beta u_i} + 1}$
11. Occupation index is given by $\frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\beta u_i}}$	11. Occupation index is given by $\frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\beta u_i} - 1}$ When $\frac{n_i}{g_i} << 1$, then the value of denominator in R.H.S. is very large and the expression, $e^{\alpha} e^{\beta u_i} = e^{\alpha} e^{u_i/kT}$ is very large than 1. Therefore, when $u_i >> kT$ then $e^{\alpha}\ e^{\beta u_i} >> 1$ and $\frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\beta u_i}}$ which gives the same result as given by M-B statistics. Again when $u_i << kT$, $\frac{n_i}{g_i}$ gives much higher value as compared to M-B statistics due to -1 term in denominator.	11. Occupation index is given by $\frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\beta u_i} + 1}$ When $\frac{n_i}{g_i} << 1$, then the value of denominator in R.H.S. is very large and the expression, $e^{\alpha} e^{\beta u_i} = e^{\alpha} e^{u_i/kT}$ is very large than 1. Therefore, when $u_i >> kT$ then $e^{\alpha} e^{\beta u_i} >> 1$ and $\frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\beta u_i}}$ which gives the same result as given by M-B statistics. Again when $u_i << kT$, $\frac{n_i}{g_i}$ gives much lower value as compared to M-B statistics due to $+1$ term in denominator.

From the above table (last point), we find that for very small values of occupation index, the two quantum statistics yield M-B statistics. Also, it is clear that for energy states with energy << kT, the three distribution greatly differ from one another but for energy >> kT, three statistics give similar energy distribution.

Thus the classical M.B. statistics is a special case of the quantum statistics when $(n_i/g_i) << 1$, it means that only a very small fraction of the total number of phase space cells are occupied by a particle while the rest are empty. In such a state Pauli's exclusion principle does not play any role because the probability of two particles simultaneously sharing a cell is very small. Further in this case we can identify the particles on the basis of the cells occupied by them in phase space. Thus even though the particles are identical, they can be treated as distinguishable from one another. This is the reason why the quantum statistics gives, in this case the same result as that of M.B. statistics.

7.5 BIRTH AND DEATH OF STARS

Like living things, star eventually reach the end of its life several billion years after its life starts. How the star dies, depends upon the type, mass and size of the star. The most massive stars die by exploding as supernovae.

Similar to number of things in the universe, stars begin very small, mere particles in vast clouds of dust and gas. For from active stars, these nebulae remain cold for ages. Then like some sleepy little town, everything stirs up when a new comer speeds through. This disturbance might take the form of a streaking comet or the shockwave from a distant supernova. As the resulting force moves through the cloud, particles collide and begin to form clumps. Individually, a clump attains more mass and therefore a stronger gravitational pull, attracting even more particles from the surrounding cloud.

As more matter falls into the clump, its center grows denser and hotter. Over the course of a million years, the clump grows into a small, dense body called a protostar. It continues to draw in even more gas and grows even more hotter. When the protostar becomes hot enough (7 million Kelvin), its hydrogen atoms begin to fuse, producing helium and an out flow of energy in the process. This atomic reaction is just like nuclear fusion. The out ward push of its fusion energy is still weaker than the inward pull of gravity at this point in the star's life.

Material continues to flow into the protostar, providing increased mass and heat. Finally after millions of years, some of these stars reach the tipping point. If enough mass (0.1 solar mass) collapses into the protostar, a bipolar flow occurs. Two massive gas jet erupt from the protostar and blast the remaining gas and dust clears away from its surface. At this point, the young star stabilizes and it reaches the point where its output exceeds its intake. The outward pressure from hydrogen fusion now counteracts gravity's inward pull. It is now a *main sequence star* and will remain so until it burns through all its fuel.

Star like the Sun: A star of the size of the sun takes roughly 50 million years to reach main sequence and maintains that level for approximately 10 billion years.

As stated above, when the core runs out of hydrogen fuel, it will contract under the weight of gravity. However, some hydrogen fusion will occur in the upper layers. As the core contracts, it heats up. This heats the upper layers, causing them to expand. As the outer layers expand, the radius of the star will increase and it will become a *red giant*. The radius of the red giant sun will be just beyond earth's orbit. At some point after this, the core will become hot enough to cause the helium to fuse into carbon. When the helium fuel runsout, the core will expand and cool. The upper layers will expand and eject materials that will collect around the dying star to form a planetry nebula. Finally the core will cool into a white dwarf and then eventually into a *black dwarf*. This entire process will take billion years.

Stars more massive then the Sun: When the core runs out of hydrogen, these stars fuse helium into carbon just like the sun. However, after the helium is gone, their mass is enough to fuse carbon into heavier elements such as oxygen, neon, silicon, magnesium, sulfur and iron. Once the core has turned into iron, it can no longer burn. The star collapses by its own gravity and the iron core heats up. The core becomes so tightly packed that protons and electrons merge to form *neutrons*.

In around 1930, this was explained by S. Chandrasekhar that when star's core mass reaches about 1.4 solar masses, the electron degeneracy can no longer support the star's core. The whole mess comes crashing down, as everything (including the iron in the core) turns back into *neutrons*. This is expected to happen when the star's initial mass exceeds about eight suns.

In less than a second, the iron core, which is about the size of the earth, shrinks to a neutron core with a radius of about 10 kilometers. The outer layers of the star fall inwards on the neutron core, thereby crushing it further. The core heats to billions of degrees and explodes (supernova), thereby releasing large amounts of energy and materials into space. The shockwave from the supernova can initiate star formation in other intersteller clouds. The remains of the core can form a *neutron star* or a *black hole* depending upon the mass of the original star.

SOLVED EXAMPLES

Example 1. *Calculate the Fermi energy in electron volts for sodium assuming that it has one free electron per atom. Given density of sodium = 0.97 g cm^{-3}, atomic weight of sodium = 24.*

Solution: Fermi energy is given by

$$\varepsilon_F(0) = \frac{h^2}{8m}\left(\frac{3n}{\pi V}\right)^{2/3}$$

Assuming one electron per sodium atom, electron density $\frac{n}{V}$ is given by

$$\frac{n}{V} = \frac{N_0\rho}{W}$$

where $N_0 = 6 \times 10^{26}$ atoms/kg mole, Avogadro's number

$\rho = 0.97$ g cm^{-3} = 0.97×10^3 kg m^{-3}, density of sodium, $W = 23$, atomic weight of sodium

$h = 6.62 \times 10^{-34}$ Js, Planck's constant, $m = 9.1 \times 10^{-31}$ kg, mass of electron

$$\therefore \text{ Electron density, } \frac{n}{V} = \frac{(6\times10^{26}\,\text{atom/kg male})(0.97\times10^{3}\,\text{kg m}^{-3})}{23} = 2.52\times10^{28}\ \text{electron/m}^3$$

$\therefore$ Fermi energy $\varepsilon_F(0)$ is given by

$$\varepsilon_F(0) = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{n}{V}\right)^{2/3}$$

$$= \left[\frac{(6.62\times10^{-34}\,\text{Js})^2}{8\times(9\times10^{-31}\,\text{kg})}\times\frac{3}{3.14}(2.52\times10^{23})\text{electron/m}^3\right]^{2/3} = 5.032\times10^{-19}\ \text{J}$$

$$= \frac{5.032\times10^{-19}}{1.6\times10^{-19}} = 4.145\ \text{eV}.$$

Example 2. *The metal mass of lithium is 0.00694 and its density is 0.53×10^3 kg m^{-4}. Calculate the Fermi energy and Fermi temperature of the electrons.*

Solution: The Fermi energy is given by

$$\varepsilon_F(0) = \frac{h^2}{8m}\left(\frac{3n}{\pi V}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 n}{V}\right)^{2/3} \quad \text{where } \hbar = \frac{h}{2\pi}$$

The unit volume contains the atoms and as many conduction electrons as

$$\frac{0.53 \times 10^3}{6.94 \times 10^{-3}} \times 6.02 \times 10^{23} = 0.4597 \times 10^{32}$$

Thus $$\frac{n}{V} = 0.4597 \times 10^{32}, \frac{h}{2\pi} = \frac{6.6 \times 10^{-34}}{2 \times 3.14} = 1.05 \times 10^{-34} \text{ Js}$$

So $$\varepsilon_F(0) = \frac{(1.05 \times 10^{-34})^2}{2 \times (9.1 \times 10^{-31})}[3\pi^2 \times 0.4597 \times 10^{32}]^{2/3} = 7.4 \times 10^{-17} \text{ J}$$

$$T_F = \frac{\varepsilon_F(0)}{k} = \frac{7.4 \times 10^{-17}}{1.38 \times 10^{-23}} = 5.32 \times 10^6 \text{ K}$$

Example 3. *Calculate the free electron density and the Fermi energy in electron volts for sodium assuming that it has one free electron per atom. Given density of sodium = 0.97 gm/cm³, atomic weight of sodium = 23.*

Solution: The Fermi-energy is given by eqn.

$$\varepsilon_F(0) = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 n}{V}\right)^{2/3} = \frac{h^2}{2m}\left(\frac{3n}{8\pi V}\right)^{2/3}$$

Assuming, one electron per sodium atom, the electron density $\frac{n}{V}$ is given by

$$\frac{n}{V} = \frac{N\rho}{W}$$

where Avogadro's number $N = 6 \times 10^{26}$ atoms/kg. mole. Density of sodium $\rho = 0.97$ gm/cm³ $= 0.97 \times 10^3$ kg/m³. Atomic weight $W = 23$

∴ Free electron density $$\frac{n}{V} = \frac{6 \times 10^{26} \times 0.97 \times 10^3}{23} = 2.53 \times 10^{28} \text{ electrons/m}^3$$

Now Planck's constant $h = 6.62 \times 10^{-34}$ J sec., Mass of electron $m = 9.1 \times 10^{-31}$ kg

∴ Fermi energy $$\varepsilon_F(0) = \frac{h^2}{2m}\left(\frac{3n}{8\pi V}\right)^{2/3}$$

$$= \frac{(6.62 \times 10^{-34})^2}{2 \times 9.1 \times 10^{-31}}\left[\frac{3 \times 7 \times 2.53 \times 10^{28}}{8 \times 22}\right]^{2/3} = 5.032 \times 10^{-19} \text{ J}$$

$$= \frac{5.032 \times 10^{-19}}{1.6 \times 10^{-19}} \text{ eV} = 3.145 \text{ eV}.$$

Example 4. *The Fermi level in potassium is 2.1 eV at a particular temperature. Calculate the number of free electrons per unit volume in potassium at the same temperature.*

Solution: The Fermi energy is given as

$$\varepsilon_F(0) = \frac{h^2}{8m}\left(\frac{3n}{\pi V}\right)^{2/3},$$

where $\varepsilon_F(0) = 2.1$ eV (given) $= 2.1 \times 1.6 \times 10^{-19}$ J
where n/V = umber of free electrons perunit volume

$$\frac{n}{V} = \left(\frac{8m}{h^2}\varepsilon_F(0)\right)^{3/2}\frac{\pi}{3}$$

$$= \left(\frac{8 \times 9.1 \times 10^{-31} \times 2.1 \times 1.6 \times 10^{-19}}{(6.625 \times 10^{-34})^2}\right)^{3/2} \times \frac{3.14}{3}$$

or $$\frac{n}{V} = 1.379 \times 10^{28} \text{ electrons/m}^3$$

Example 5. *The density of zinc is 7.13 u 10^3 kg/m^3 and its atonic weight is 65.4 calculate the Fermi energy and the mean energy at 0K.,*

Solution: The expression for the Fermi energy is

$$\varepsilon_F(0) = \frac{h^2}{8m}\left(\frac{3n}{\pi V}\right)^{2/3}$$

given $$m = 9.1 \times 10^{-31} \text{ kg}, h = 6.625 \times 10^{-34} \text{ J-s, we get}$$

$$\varepsilon_F(0) = 3.65 \times 10^{-19} (n/V)^{2/3} \text{ eV}$$

and $$n/V = \frac{2\rho N_A}{M} = \frac{2 \times \text{density} \times \text{Avogadro's number}}{\text{molecular weight}}$$

here $$\rho = 7.13 \times 10^3 \text{ kg/m}^3, \text{ molecular weight} = 65.4$$

$$N_A = 6.023 \times 10^{26} \text{ kg/mol}^3$$

Thus $$N/V = 1313 \times 10^{26}$$

$\therefore$ $$\varepsilon_F(0) = 3.65 \times 10^{-19} \times (1313 \times 10^{26})^{-2/3} = 11.1 \text{ eV}$$

Mean energy $$\bar{\varepsilon} = \frac{3}{5}\varepsilon_F(0) = \frac{3}{5} \times 11.1 = 6.66 \text{ eV}$$

Example 6. *Calculate the Fermi temperature (T_F) for liquid helium using experimental data.*

Solution: The expression for Fermi temperature is

$$T_F = \frac{\varepsilon_F(0)}{k} = \frac{h^2}{2mk}\left(\frac{3n}{8\pi V}\right)^{2/3}$$

For liquid Ferium, $$\frac{V}{n} = 63 \text{ Å/ atom} = 6.3 \times 10^{-24} \text{ cm}^{-3}\text{/atom}$$

or $$\frac{n}{V} = \frac{10^{24}}{63} \text{ atom cm}^{-3}$$

Thus $$T_F = 4.9 \text{ K}$$

Example 7. *The number of conduction electrons per cm³ in beryllium is 2.42 × 10²³ and in cerium 0.91 × 10². If the Fermi energy of conduction electrons in Be is 14.14 eV, calculate its value in case of Cs.*

***Solution*:** Fermi energy of electron in metal is given as

$$\varepsilon_F(0) = \frac{h^2}{8m}\left(\frac{3n}{\pi V}\right)^{2/3}$$

In case of beryllium, $n/V = 2.42 \times 10^{23}$, Fermi energy is given as

$$(\varepsilon_F(0))_{Be} = \frac{h^2}{8m}\left(\frac{3}{\pi} \times 2.42 \times 10^{23}\right)^{2/3}$$

In case of Cerium, $n/V = 0.91 \times 10^2$, Fermi energy is given as

$$(\varepsilon_F(0)_{Ce} = \frac{h^2}{8m}\left(\frac{3 \times 0.91 \times 10^{22}}{\pi}\right)^{2/3}$$

Dividing eqn. (*B*) by (*A*), we get

$$\frac{(\varepsilon_F \rightarrow \varepsilon_F(0))_{Ce}}{(\varepsilon_F \rightarrow \varepsilon_F(0))_{Be}} = \left(\frac{0.9 \times 10^{22}}{2.42 \times 10^{23}}\right)^{2/3}$$

But $$(\varepsilon_F \rightarrow \varepsilon_F(0))_{Be} = 14.14 \text{ eV (given)}$$

$\therefore$ $$(\varepsilon_F \rightarrow \varepsilon_F(0))_{Ce} = 14.14\left(\frac{0.91}{24.2}\right)^{2/3} \text{ eV}$$

or $$(\varepsilon_F \rightarrow \varepsilon_F(0))_{Ce} = 1.6 \text{ eV}$$

Example 8. *Calculate the Fermi energy of free electrons in copper from the given data: Atomic weight of copper = 63.5, Density of copper = 8.94 gm/cm³, Planck constant = 6.62 × 10⁻²⁷, mass of the electron = 9.1 × 10⁻²⁸ gm, Avogadro's number = 6.02 × 10²³.*

***Solution*:** Concentration of free electrons in

$$Cu = \frac{\text{Avogadro's number} \times \text{density}}{\text{Atonic weight}}$$

$$= \frac{6.02 \times 10^{23} \times 8.94}{63.5} = 8.48 \times 10^{22} \text{ edlectrons/cm}^3$$

Fermi energy of electrons in $$Cu = \frac{h^2}{8m}\left(\frac{3n}{\pi V}\right)^{2/3} = \frac{(6.62 \times 10^{-27})^2}{8 \times 9.1 \times 10^{-28}}\left(\frac{3 \times 7 \times 8.48 \times 10^{22}}{22}\right.$$

or $$\varepsilon_F(0) = 0.602\,(80.94)^{2/3} \times 10^{-12}$$

or $$\varepsilon_F(0) = 1.126 \times 10^{-11} \text{ ergs}$$

or $$\varepsilon_F(0) = 7.04 \text{ eV}$$

(as 1 eV = 1.6×10^{-12} erg)

SHORT ANSWER QUESTIONS

Q1. *Distinguish between classical and quantum statistics?*

Ans. (*i*) Classical statistics applies to particles which obey the laws of classical mechanics whereas quantum statistics is applied to particles obeying quantum mechanical laws.

(*ii*) The state of the particle in classical statistics is specified by giving its position and momentum whereas the state of a particle in quantum statistics is described by the wave function of particle.

Q2. *What was the need to introduce quantum statistics?*

Ans. Classical statistics successfully explained the energy and velocity distribution of molecules of an ideal gas but failed to explain several experimentally observed phenomenon such as energy distribution of electrons in metals, black body radiations and photo-electric effect etc. Since the behaviour of particles constituting these systems is governed by the laws of quantum mechanics, a need was felt to develop quantum statistics.

Q3. *Mention briefly the assumptions of B.E. and F.D. statistics.*

Ans. Assumptions of B.E. statistics

(*i*) The particles of the system are identical and indistinguishable.

(*ii*) Any no. of particles can occupy a single cell in phase space.

(*iii*) The size of the cell cannot be less than h^3 where h is Planck's constant.

(*iv*) B.E. statistics is applicable to particles known as bosons.

Assumptions of F.D. statistics

(*i*) the particles of the system are identical and indistinguishable.

(*ii*) There cannot be more than one particle in a single cell in phase space.

(*iii*) The size of the cell cannot be less than h^3 where h is Planck's constant.

(*iv*) F.D. statistics is applicable to particles known as fermions.

Q4. *What are the constraints which are obeyed by a photon gas at a given temperature?*

Ans. The photon gas obeys only one constraint *i.e.* the total energy of the photon gas remains constant. But the number of photons is not constant.

Q5. *Write a short note on Bose-Einstein statistics?*

Ans. In Bose-Einstein statistics, particles are treated as *indistinguishable* and they don't obey Pauli's exclusion principle and so any number of particles can occupy a singie cell in phace space. It applies to particles having integral spin angular momentum (in units of $\hbar$) *i.e.* photon (spin 1), mesons, α–particles (spin zero) etc. and such particles are called bosons.

Q6. *What is free electrons Fermi gas?*

Ans. The gas of free electrons and non-interacting electrons subject to Pauli principle is called free electron Fermi gas.

Q7. *What is a photon gas? Are the number of photons constant?*

Ans. A system of black body radiations contained in an encloser is called photon gas. No, the number of photons in the encloser does not remain constant.

Q8. *What is the value of α for a photon gas?*

Ans. Value of α is zero for a photon gas. It is because the number of photons in the system is not constant.

Q9. *How does free electron gas differ from ordinary gas.*

Ans. Free electrons gas differs from ordinary gas in the following ways:

(*i*) Free electron gas is constituted by electrons which are charged particles while the atoms or molecules which constitute the ordinary gas are neutral.

(ii) The concentration of electrons in free electron gases is large ≈ 10^{29} per m^3 as compared to the concentration of atoms or molecules of ordinary gas (10^{25} per m^3).

Q10. *What do you understand by photon gas and electron gas? Which statistics is obeyed by them?*

Ans. In metallic conductors there are free electrons called *conduction electrons* and these electrons are free to move about inside the volume of the conductor like a gas and is called electron gas. It obeys F.D. statistics. A hollow enclosure, at a given constant temperature would be filled with radiations (called photons) which are characteristics of that temperature. These photons are supposed to form a photon gas. It obeys B.E. statistics.

Q11. *What is meant by Fermi energy of a metal?*

Ans. The energy of the highest filled level in metal at 0 K is known as Fermi energy. All the energy states above it are completely empty and all the energy states below it are completely filled.

Q12. *What is Fermi gas? Give an example.*

Ans. An assembly of indistinguishable elementary particles having half integral spin, called fermions confined in a volume is known as Fermi gas. The conduction electrons in a metal is an example of Fermi gas.

Q13. *Do electrons have zero energy at 0 K? Why Or What do you mean by zero point energy for Fermi gas?*

Ans. The electrons do not have zero energy at 0 K due to Pauli exclusion principle—rather electrons have definite energy at 0 K. This energy is given by $\bar{\varepsilon} = \dfrac{3n\,\varepsilon_F(0)}{5}$.

Q14. *Explain the conditions under which B.E. and F.D. statistics yield to classical statistics.*

Ans. Both these statistics yield to classical statistics at high temperatures and low concentrations.

Q15. *Explain why at occupation index << 1 B.E. and F.D. statistics give the same result as is given by M.B. statistics?*

Ans. The expression for the occupation index n_i/g_i for these statistics is

$$\frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{u_i/kT}} \quad \text{M.B. statistics}$$

$$\frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{u_i/kT} - 1} \quad \text{B.E. statistics}$$

$$\frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{u_i/kT} + 1} \quad \text{F.D. statistics}$$

If $(n_i/g_i) << 1$, then denominators of R.H.S. of equations (*ii*) and (*iii*) are very large. That is $e^{\alpha} e^{u_i/kT} >> 1$.

Then 1 can be neglected in comparison to $e^{a}\, e^{ui/kT}$. Therefore $(n_i/g_i) = \dfrac{1}{e^{\alpha} e^{u_i/kT}}$.

Q16. *Distinguish between boson and fermion.*

Ans. The particles which obey Pauli's exclusion principle are called fermions Electrons, positrons, protons and neutrons are examples.

The particles which have integral spin and do not obey Pauli's exclusion principle are called boson. Photons, π–mesons are examples.

Q17. *Distinguish the following particles as bosons or fermions (i) Hydrogen atom (ii) $^3He^+$ (iii) α-particle (iv) $^6Li^+$ ion (v) $^7Li^+$ ion (vi) hydrogen molecule.*

Ans. (*i*) Hydrogen atom contains 1 proton and 1 electron *i.e.* 2 fermions. Then it is a boson.

(*ii*) $^3He^+$ nucleus contains 3 fermions so it is a fermion.

(*iii*) ∝-particle is a nucleus of atom. It contains 2*n* and 2*p* *i.e.* 4 fermions. Thus it is a boson.

(*iv*) $^6L_i^+$ nucleus has 6 fermions, so it is boson.

(*v*) $^7L_i^+$ nucleus has 7 fermions, so it is fermion.

(*vi*) Hydrogen molecule contains 2*e* and 2*p* *i.e.* 4 fermion so it is boson.

Q18. *What is F.D. distribution law? explain.*

Ans. F.D. distribution law is

$$n_i = \frac{g_i}{e^{\alpha} e^{u_i/kT} + 1}$$

where k is Boltzmann constant and T is absolute temperature and $\alpha = -\mu/kT$ where μ is chemical potential per mole, n_i is number of particles in ith energy state and g_i is the degenracy of the ith energy level and u_i the energy of a particle associated with this level.

Q19. *How do you define Fermi energy and Fermi level at absolute zero?*

Ans. At absolute zero all the levels below a certain level will be filled with electrons and all the levels above it will be empty. The level which divides the filled and vacant levels is called Fermi level at absolute zero and the energy of this level is called Fermi energy at 0 K.

LONG QUESTIONS

1. State and explain clearly the basic differences between classical and quantum statistics.
2. Differentiate between distinguishable and indistinguishable particles. Explain with examples.
3. Derive an expression for the most probable distribution of particles for a system obeying Bose-Einstein statistics.
4. Starting from basic assumption of B.E. Statistics, derive the relation
$$n_i = \frac{g_i}{e^{\alpha} e^{u_i/kT} - 1}$$
where the symbols have their usual meanings.
5. Write down the postulates of B-E statistics. Derive the distribution of particles governed by B-E statistics.
6. Discuss the salient features of black body radiation.
7. Starting from basic assumption of F.D. statistics derive the relation
$$n_i = \frac{g_i}{e^{\alpha} e^{u_i/kT} + 1}$$
where the symbols have their usual meanings.
8. Apply the Fermi-Dirac distribution law to derive the energy distribution of free electrons in a conductor.
9. Using F-D distribution law, discuss thé behaviour of electrons in conductor at 0 K and at higher temperature.
10. (*a*) Analyse the difference between classical M.B. statistics and quantum B.E. and F.D. statistics. Show that in limiting case both B.E. and F.D. statistics are reduced to M.B. statistics.
(*b*) How does free electron gas differ from ordinary gas.
11. Give the significance of Fermi energy of a metal at absolute zero temperature. Show that at any normal temperature, half of the total number of particles of an electron gas are below the fermi level.
12. Starting from basic postulates, obtain F.D. distribution law.
13. What is the difference between a boson and fermion. Find an expression for the energy distribution for electron gas in a metal.

NUMERICAL PROBLEMS

1. Fermi energy of conduction electrons in silver is 5.48 eV. Calculate the concentration of thse electrons (e/V), 6.62×10^{-34} J-s, $m = 9.1 \times 10^{-31}$ kg.
[**Ans.** 5.88×10^{22}/CC]
2. Calculate the Fermi temperature of gold, having Fermi energy $\varepsilon_F(0) = 5.54$ eV. Given $k = 1.38 \times 10^{-23}$ J/K. and 1 eV = 1.6×10^{-19} J.
[**Ans.** 6.42×10^4 K]
3. Fermi energy for silver is 5.51 eV. What is the average energy of free electron at 0K.
[**Ans.** 3.306 eV]