# 9 EVALUATION OF REAL DEFINITE INTEGRALS

## I. Integrals of type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$.

In this type we consider $z = e^{i\theta}$, that $z$ is a unit circle.

Putting, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\Rightarrow \cos\theta = \dfrac{z + z^{-1}}{2}$ and $\sin\theta = \dfrac{z - z^{-1}}{2i}$

and $dz = ie^{i\theta}\, d\theta$

Also as $\theta$ goes from 0 to $2\pi$ means $z$ is moving one round along the circle $C : |z| = 1$.

Ex Evaluate (a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}$, $a^2 < 1$

(b) $\displaystyle\int_0^{\pi} \frac{d\theta}{(2 + \cos\theta)^2}$.

(c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$.

(d) $\displaystyle\int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta}$ where $|a| < 1$.

(e). $$\int_0^{2\pi} \frac{2\,d\theta}{2+\cos\theta}.$$

(f) $$\int_0^{2\pi} \frac{d\theta}{2-\sin\theta}.$$

(g) $$\int_0^{2\pi} \frac{d\theta}{5-4\sin\theta}$$

(h) $$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}.$$

(i) $$\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$$

<u>Solution</u> (a) $$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}, \quad a^2<1.$$

Putting $z = e^{i\theta}$, $c : |z|=1$, $dz = ie^{i\theta}d\theta$, $\Rightarrow dz = iz\,d\theta$

$\therefore$ The given integral becomes

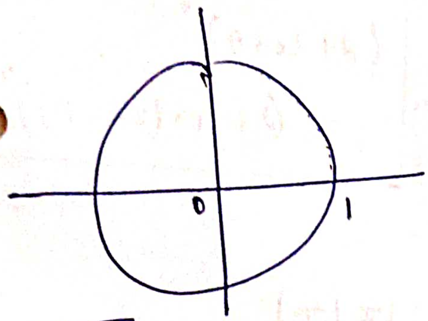
$$\int_C \frac{\frac{dz}{iz}}{1-2a\left(\frac{z+z^{-1}}{2}\right)+a^2.} = \int_C \frac{dz}{iz\left(1-a(z+z^{-1})+a^2\right)}$$

$$= \int \frac{dz}{i\left(z-a(z^2+1)+a^2z\right)}$$

$$= \int_C \frac{dz}{i\left((1+a^2)z - az^2 - a\right)}$$

$$= \int_C \frac{dz}{i\left[(z-a) + a^2 z - az^2\right]} = \int_C \frac{dz}{i\left[(z-a) + az(a-z)\right]}$$

$$= \int_C \frac{dz}{i\left[(z-a) - az(z-a)\right]} = \int \frac{dz}{i(z-a)(1-az)}$$




$\therefore$ $z = a$ is the pole lying inside the circle $|z| = 1$.

$$\boxed{\begin{array}{l} a^2 < 1 \\ \Rightarrow a < 1 \\ \text{and } \frac{1}{a} > 1 \end{array}}$$

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to a} \frac{1}{i(1-az)}$$

$$= \frac{1}{i(1-a^2)}.$$

$\therefore$ By Residue theorem,

$$\int_C \frac{dz}{i(z-a)(1-az)} = 2\pi i \cdot \frac{1}{i(1-a^2)}$$

$$= \frac{2\pi}{1-a^2}.$$

(b). $\displaystyle\int_0^{\pi} \frac{d\theta}{(2+\cos\theta)^2}$

Applying the property of definite integrations

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x). \end{cases}$$

$$\therefore \int_0^{\pi} \frac{d\theta}{(2+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$$

$\boxed{\text{Since } (2+\cos\theta)^2 = (2+\cos(2\pi-\theta))^2}$

Putting $z = e^{i\theta}$

$dz = iz\,d\theta$, $\quad C : |z| = 1$

$$= \frac{1}{2}\int_C \frac{dz/iz}{\left(2+\left(\frac{z+z^{-1}}{2}\right)\right)^2}$$

$$= \frac{1}{2}\int_C \frac{\frac{dz}{iz}}{\left[2+\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^2}.$$

$$= \frac{1}{2}\int \frac{\frac{dz}{iz}}{\left[\frac{4z + z^2 + 1}{2z}\right]^2} = \frac{1}{2}\int \frac{\frac{dz}{iz}}{\frac{(z^2+4z+1)^2}{4z^2}}$$

$$= \frac{1}{2}\int \frac{dz \cdot 4z^2}{iz(z^2+4z+1)^2} = \frac{2}{i}\int \frac{z\,dz}{(z^2+4z+1)^2}$$

Now, poles of $f(z) = \frac{z}{(z^2+4z+1)^2}$

are at $z = -2 \pm \sqrt{3} = -2 \pm 1.732$. $\Rightarrow$ Only $z = -2+\sqrt{3}$ is lying inside the circle $c : |z| = 1$. and it is a double pole.

$$\therefore \operatorname{Res}_{z = -2+\sqrt{3}} f(z) = \lim_{z \to -2+\sqrt{3}} \frac{1}{1!} \frac{d}{dz}\left[\frac{z.(z+2-\sqrt{3})^2}{(z+2-\sqrt{3})^2(z+2+\sqrt{3})^2}\right]$$

$$= \lim_{z \to -2+\sqrt{3}} \frac{d}{dz}\left[\frac{z}{(z+2+\sqrt{3})^2}\right].$$

$$= \lim_{z \to -2+\sqrt{3}} \left[\frac{(z+2+\sqrt{3})^{\cancel{2}}(1) - z \cdot 2\,\cancel{(z+2+\sqrt{3})}}{(z+2+\sqrt{3})^{\cancel{4}\,3}}\right].$$

$$= \lim_{z \to -2+\sqrt{3}} \left[ \frac{(z+2+\sqrt{3}) - 2z}{(z+2+\sqrt{3})^3} \right]$$

$$= \frac{(-\cancel{2}+\sqrt{3}+\cancel{2}+\sqrt{3}) - 2(-2+\sqrt{3})}{(-2+\sqrt{3}+2+\sqrt{3})^3} = \frac{2\sqrt{3}+4-2\sqrt{3}}{(2\sqrt{3})^3}$$

$$= \frac{4}{8 \times 3\sqrt{3}} = \frac{1}{6\sqrt{3}}.$$

$\therefore$ By Residue theorem,

$$\frac{2}{i}\int \frac{z\,dz}{(z^2+4z+1)^2} = \frac{2}{i} * 2\pi i * \frac{1}{6\sqrt{3}}$$

$$= \frac{2\pi}{3\sqrt{3}}.$$

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(c). $$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, \quad a > |b|.$$

$$= \int_C \frac{dz}{iz\left[a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right]} = \frac{2}{i}\int_C \frac{dz}{bz^2+2az+b}$$

Singularities are roots of $bz^2+2az+b=0$.

$$\Rightarrow z = \frac{-a \pm \sqrt{4a^2-4b^2}}{2b}.$$

$$\Rightarrow z = -a \pm \sqrt{a^2-b^2}$$

Out of these two only $\frac{-a+\sqrt{a^2-b^2}}{b}$

lie inside $C$.

because if $a > |b|$
$\Rightarrow a > b$ and $a > -b$
$\Downarrow$
$-a < b$
$-\frac{a}{b} < 1$

$$\operatorname*{Res}_{z=\frac{-a+\sqrt{a^2-b^2}}{b}} f(z) = \lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \frac{1}{\left(z + \frac{a+\sqrt{a^2-b^2}}{b}\right)}$$

$$= \frac{1}{\frac{-a+\sqrt{a^2-b^2}}{b} + \frac{a+\sqrt{a^2-b^2}}{b}} = \frac{b}{2\sqrt{a^2-b^2}}.$$

$\therefore$ By Residue theorem

$$\frac{2}{i}\int \frac{dz}{bz^2+2az+b} = \frac{2}{i} * \frac{2\pi i}{b} * \frac{b}{2\sqrt{a^2-b^2}}$$

$$= \frac{2\pi b}{2b\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

(d) $\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$ , $|a|<1$

$$= \frac{1}{2}\int_0^{2\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$$

Putting $z=e^{i\theta}$ , $dz = iz\,d\theta$ , we get $c : |z|=1$.

$$= \frac{1}{2}\int_c \frac{a\frac{dz}{iz}}{a^2+\left(\frac{z-\frac{1}{z}}{2i}\right)^2} = \frac{1}{2}\int \frac{\frac{a\,dz}{iz}}{\frac{-4a^2z^2+(z^2-1)^2}{-4z^2}}$$

$$= \frac{a}{2i}\int \frac{dz\,(-4z^2)}{z\,(z^4+1-2z^2-4a^2z^2)}$$

$$= -\frac{2a}{i}\int \frac{z\,dz}{z^4+1-4a^2z^2-2z^2}$$

$z^2 = \frac{2+4a^2 \pm \sqrt{(4+16a^4-32a^2)-4}}{2}$

$\Rightarrow (z^4-2z^2+1)-(2az)^2$
$= (z^2-1)^2-(2az)^2$.

$$= 2ai\int \frac{z\,dz}{(z^2-2az-1)(z^2+2az-1)}$$

$$= 2ai\int \frac{z\,dz}{[(z-a)^2-(a^2+1)][(z+a)^2-(a^2+1)]}$$

$$= 2ai\int \frac{z\,dz}{(z-a-\sqrt{a^2+1})\underbrace{(z-a+\sqrt{a^2+1})}\underbrace{(z+a-\sqrt{a^2+1})}(z+a+\sqrt{a^2+1})}$$

Out of these $z_1 = a + \sqrt{a^2+1}$ and $z_2 = -a - \sqrt{a^2+1}$

are lying outside the circle C and

$z_3 = -a + \sqrt{a^2+1}$ and $z_4 = a - \sqrt{a^2+1}$ lie inside C.

$$\operatorname*{Res}_{z\to z_3} f(z) = \lim_{z\to z_3} \frac{z}{(z-a+\sqrt{a^2+1})(z-a-\sqrt{a^2+1})(z+a+\sqrt{a^2+1})}$$

$$= \lim_{z\to z_3} \frac{z}{\left((z-a)^2-(a^2+1)\right)\left(z+a+\sqrt{a^2+1}\right)}$$

$$= \frac{-a+\sqrt{a^2+1}}{\left((-2a+\sqrt{a^2+1})^2-(a^2+1)\right)\left(-\cancel{a}+\sqrt{a^2+1}+\cancel{a}+\sqrt{a^2+1}\right)}$$

$$= \frac{-a+\sqrt{a^2+1}}{\left(4a^2+\cancel{a^2}+\cancel{1}-4a\sqrt{a^2+1}-\cancel{a^2}-\cancel{1}\right)\left(2\sqrt{a^2+1}\right)}$$

$$= \frac{\cancel{-a+\sqrt{a^2+1}}\ \ -1}{4a\cancel{(a-\sqrt{a^2+1})}\,2\sqrt{a^2+1}}$$

$$= \frac{-1}{8a\sqrt{a^2+1}}$$

lly, $\operatorname*{Res}_{z\to z_4} f(z) = \dfrac{-1}{8a\sqrt{a^2+1}}$

$\therefore$ Value of given integral

$$= 2ai * 2\pi i \left[ \frac{-2}{8a\sqrt{a^2+1}} \right]$$

$$= \frac{\pi a}{a\sqrt{a^2+1}} = \frac{\pi}{\sqrt{a^2+1}}.$$

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(c). $$\int_0^{2\pi} \frac{2\,d\theta}{2+\cos\theta}$$

Putting $z = e^{i\theta}$ $\Rightarrow dz = iz\,d\theta$.

$$\Rightarrow \int_0^{2\pi} \frac{2\,d\theta}{2+\cos\theta} = \int_C \frac{2 * \frac{dz}{iz}}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

$$= \int_C \frac{4\,dz}{iz\left[4+\left(\frac{z^2+1}{z}\right)\right]}$$

$$= \int_C \frac{4\,dz}{i\cancel{z}\left[\frac{4z+z^2+1}{\cancel{z}}\right]}$$

$$= \int_C \frac{4\,dz}{i\left[z^2+4z+1\right]}.$$

Now $z^2+4z+1=0$

$$\Rightarrow z = \frac{-4 \pm \sqrt{16-4\times 1}}{2} = \frac{-4\pm\sqrt{12}}{2}$$

$$= -2 \pm \sqrt{3}.$$

Out of these two only $-2+\sqrt{3}$ lies inside c.

$$\therefore \int \frac{4\,dz}{i(z+2+\sqrt{3})(z+2-\sqrt{3})} = \frac{4}{i} \times 2\pi i \times \left(\begin{array}{c}\text{Res } f(z) \\ \text{at } z=-2+\sqrt{3}\end{array}\right).$$

$$\therefore \underset{z=-2+\sqrt{3}}{\text{Res}\, f(z)} = \lim_{z\to -2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}} = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

$$\therefore \int \frac{4\,dz}{i(z^2+4z+1)} = \frac{4}{i} \times 2\pi i \times \frac{1}{2\sqrt{3}} = \frac{4\pi}{\sqrt{3}}.$$

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(f) $$\int_0^{2\pi} \frac{d\theta}{2-\sin\theta}.$$

$$= \int_c \frac{\frac{dz}{iz}}{2-\left(\frac{z-z^{-1}}{2i}\right)} = \int_c \frac{2i\,dz}{iz\left(4i-\left(z-\frac{1}{z}\right)\right)}$$

$$= \int \frac{2\,dz}{z\left(4i-\left(\frac{z^2-1}{z}\right)\right)}$$

$$= \int \frac{2\,dz}{\cancel{z}\frac{(4iz-z^2+1)}{\cancel{z}}}$$

$$= \int_C \frac{-2\,dz}{z^2 - 4iz - 1}$$

Now $z^2 - 4iz - 1 = 0$

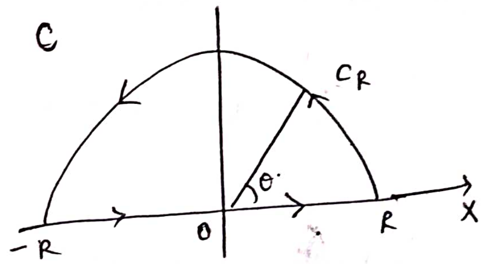
$$\Rightarrow z = \frac{4i \pm \sqrt{16i^2 - 4\times1\times(-1)}}{2} = \frac{4i \pm \sqrt{-12}}{2}$$

$$= 2i \pm \cancel{4}\sqrt{3}\,i$$

## II Integrals of type $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}$ where $f(x)$ and $F(x)$ are polynomials in $x$ such that $\frac{x f(x)}{F(x)} \to 0$ as $x \to \infty$ and $F(x)$ has no zero on the real axis.

To evaluate $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx$ we consider $\int_C \frac{f(z)dz}{F(z)}$

where $C$ is the closed contour consisting of the real axis from $-R$ to $R$. and the semicircle $C_R$ of radius $R$ in the upper half-plane.




Now,
$$\int_C \frac{f(z)\,dz}{F(z)} = \int_{-R}^{R} \frac{f(z)\,dz}{F(z)} + \int_{C_R} \frac{f(z)\,dz}{F(z)}$$

$$= \int_{-R}^{R} \frac{f(x)\,dx}{F(x)} + \int_{C_R} \frac{f(z)\,dz}{F(z)} \quad \boxed{\text{as } z = Re^{i\theta} \text{ on } C_R}$$

$$= \int_{-R}^{R} \frac{f(x)\,dx}{F(x)} + \int_{0}^{\pi} \frac{f(Re^{i\theta})\, Ri e^{i\theta}\, d\theta}{F(Re^{i\theta})}.$$

For large $R$, $\left| \int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})} Re^{i\theta} i\, d\theta \right|$ is of the order

$\frac{Rf(R)}{F(R)}$ which is given to tend to zero as $R \to \infty$.

$\therefore$ We are left with $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx$.

**Example :-** $\int_0^{\infty} \frac{\cos ax}{x^2+1} dx$.

**Solution** Since $\cos ax = \mathrm{Re}\left(e^{iax}\right)$

$\therefore$ We consider the function $f(z) = \frac{e^{aiz}}{z^2+1}$

which has poles at $z = i, -i$, out of which

only $z = i$ lies inside the semicircle contour.

$\therefore$ Residue at $z = i$ $= \lim_{z \to i} (z-i) f(z)$

$= \lim_{z \to i} \frac{e^{aiz}}{(z+i)} = \frac{e^{-a}}{2i}$.

$\therefore \int_C \frac{e^{aiz}}{z^2+1} dz = 2\pi i (\text{Residue}) = 2\pi i \left(\frac{e^{-a}}{2i}\right) = \pi e^{-a}$.

Now, $$\int_C f(z)\,dz = \underbrace{\int_{C_R} \frac{e^{aiz}}{z^2+1^2}\,dz}_{(=0)} + \int_{-R}^{R} \frac{e^{aix}}{x^2+1^2}\,dx.$$

As $R \to \infty$,

$$\therefore \int_C f(z)\,dz = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{aix}}{x^2+1^2}\,dx\right)$$

$$\Rightarrow \pi e^{-a} = \mathrm{Re}\left(\int_{-\infty}^{\infty}\left(\frac{\cos ax + i\sin ax}{x^2+1}\right)dx\right)$$

$$\Rightarrow \pi e^{-a} = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx$$

Thus $$\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi e^{-a}}{2}$$

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