Differentiation Under Integral Sign

The value of a definite integral $\int_a^b f(x,\alpha)\,dx$ is a function of α (parameter), $F(\alpha)$ say. To find $F'(\alpha)$, first we have to evaluate the integral $\int_a^b f(x,\alpha)\,dx$ & the differentiate $F(\alpha)$ w.r.t. α. However, it is not always possible to evaluate the integral and then to find its derivative. Such problems are solved by reversing the order of the integration & differentiation i.e., first differentiate $f(x,\alpha)$ partially w.r.t. α and then integrate it.

Leibnitz's Rule

If $f(x,\alpha)$ and $\frac{\partial f(x,\alpha)}{\partial \alpha}$ be continuous functions of x and α, then

$$\frac{d}{dx}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx.$$

Proof:- Let $F(\alpha) = \int_a^b f(x,\alpha)\,dx \qquad -①$

then $F(\alpha+\delta\alpha) = \int_a^b f(x,\alpha+\delta\alpha)\,dx \qquad -②$

Subtracting ① from ②, we get

$$F(\alpha+\delta\alpha) - F(\alpha) = \int_a^b f(x,\alpha+\delta\alpha)\,dx - \int_a^b f(x,\alpha)\,dx$$

$$= \int_a^b [f(x,\alpha+\delta\alpha) - f(x,\alpha)]\,dx$$

$$\frac{F(\alpha+\delta\alpha)-F(\alpha)}{\delta\alpha} = \int_a^b \frac{f(x,\alpha+\delta\alpha)-f(x,\alpha)}{\delta\alpha}\,dx$$

Taking limits of both sides as $\delta\alpha \to 0$, we have

$$\frac{dF}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\, dx$$

The above formula is useful for evaluating definite integrals (Improper integrals) which are otherwise impossible to evaluate.

Note:- If $f(x,\alpha)$, $\frac{\partial f(x,\alpha)}{\partial \alpha}$ be continuous functions of x and α, then

$$\frac{d}{d\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha}\, dx + \frac{d\psi(\alpha)}{d\alpha} f[\psi(\alpha),\alpha] - \frac{d\phi(\alpha)}{d\alpha} f[\phi(\alpha),\alpha]$$

Example 1. $\int_0^1 \frac{x^a - 1}{\log x}\, dx$

Solution: Let $F(a) = \int_0^1 \frac{x^a - 1}{\log x}\, dx$ — (1)

diff. both sides w.r.t. a

$$F'(a) = \int_0^1 \frac{\partial}{\partial a}\left[\frac{x^a - 1}{\log x}\right] dx$$

$$= \int_0^1 \frac{x^a \log x}{\log x}\, dx = \int_0^1 x^a\, dx$$

$$= \left[\frac{x^{a+1}}{a+1}\right]_0^1 \qquad \text{for } a > -1$$

$$F'(a) = \frac{1}{a+1}$$

integrating both sides w.r.t. a,

$$F(a) = \log(1+a) + C \qquad \text{— (2)}$$

From (1), when $a = 0$, $F(0) = \int_0^1 \frac{1-1}{\log x}\, dx = 0$

From (II), $0 = \log 1 + C \Rightarrow C = 0$

$F(a) = \log(1+a)$ where $a > -1$

Hence, $\int_0^1 \frac{x^a - 1}{\log x}\, dx = \log(1+a)$

Example 2. Evaluate $\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2}$

Solution: Let $I = \int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$ — (i)

Let us first evaluate $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Dividing the numerator & denominator by $\cos^2 x$

$$\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\, dx$$

Put $t = \tan x$

$dt = \sec^2 x\, dx$

when $x = 0$, $t = 0$; when $x = \frac{\pi}{2}$, $t \to \infty$

$$\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} = \frac{1}{b^2}\int_0^{\infty} \frac{dt}{t^2 + \frac{a^2}{b^2}}$$

$$= \frac{1}{b^2} \cdot \frac{1}{\frac{a}{b}} \left[\tan^{-1} \frac{t}{\frac{a}{b}}\right]_0^{\infty}$$

$$= \frac{1}{ab}\left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$= \frac{\pi}{2ab} \quad \text{— (ii)}$$

diff both sides partially w.r.t. a

$$\int_0^{\pi/2} \frac{\partial}{\partial a}\left[\frac{1}{a^2\cos^2 x + b^2\sin^2 x}\right] dx = \frac{\partial}{\partial a}\left[\frac{\pi}{2ab}\right]$$

$$\int_0^{\pi/2} \frac{-2a\cos^2 x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\, dx = \frac{-\pi}{2a^2 b}$$

dividing both sides by $-2a$, we get

$$\int_0^{\pi/2} \frac{\cos^2 x}{(a^2\cos^2 x + b^2 \sin^2 x)^2}\, dx = \frac{\pi}{4a^3 b} \quad \text{— (iii)}$$

Similarly diff (ii), partially w.r.t. b

$$\int_0^{\pi/2} \frac{\sin^2 x}{(a^2\cos^2 x + b^2 \sin^2 x)^2}\, dx = \frac{\pi}{4ab^3} \quad \text{— (iv)}$$

Adding (iii) & (iv)

$$\int_0^{\pi/2} \frac{\cos^2 x + \sin^2 x}{(a^2\cos^2 x + b^2 \sin^2 x)^2}\, dx = \frac{\pi}{4a^3 b} + \frac{\pi}{4ab^3}$$

$$\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi(a^2+b^2)}{4a^3 b^3}$$

Hence $\quad I = \dfrac{\pi(a^2+b^2)}{4a^3b^3}$

Example 3: Evaluate $\displaystyle\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)}\, dx \quad (a \geq 0)$

Solution: Let $F(a) = \displaystyle\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)}\, dx$

diff both sides w.r.t. a

$$F'(a) = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{\tan^{-1} ax}{x(1+x^2)}\right] dx \quad \text{— (1)}$$

$$= \int_0^{\infty} \frac{1}{x(1+x^2)} \cdot \frac{1}{1+a^2x^2} \cdot x\, dx$$

$$F'(a) = \int_0^{\infty} \frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$= \frac{1}{1-a^2}\int_0^{\infty} \left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right] dx \quad \text{(by using Partial Fractions)}$$

$$= \frac{1}{1-a^2}\left[\tan^{-1} x\right]_0^{\infty} - \frac{a^2}{1-a^2} \cdot \frac{1}{a^2} \int_0^{\infty} \frac{dx}{x^2+\frac{1}{a^2}}$$

$$F'(a) = \frac{1}{1-a^2}\left[\tan^{-1}\infty - \tan^{-1}0\right] - \frac{a}{1-a^2}\left[\tan^{-1}ax\right]_0^\infty$$

$$= \frac{1}{1-a^2}\cdot\frac{\pi}{2} - \frac{a}{1-a^2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{2}\left[\frac{1-a}{1-a^2}\right] = \frac{\pi}{2(1+a)}$$

$$F'(a) = \frac{\pi}{2(1+a)}$$

integrating both sides w.r.t. a, we get

$$F(a) = \frac{\pi}{2}\log(1+a) + C$$

From (i), when $a=0$, $F(0)=0$

From (ii), $0 = \frac{\pi}{2}\log 1 + C$

$$\Rightarrow C = 0$$

$$F(a) = \frac{\pi}{2}\log(1+a)$$

Hence, $\displaystyle\int_0^\infty \frac{\tan^{-1}ax}{x(1+x^2)}\,dx = \frac{\pi}{2}\log(1+a)$

<u>Example 4</u>:- If $|a|<1$, prove that

$$\int_0^\pi \log(1+a\cos x)\,dx = \pi\log\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-a^2}\right]$$

<u>Solution</u>: Let $F(a) = \displaystyle\int_0^\pi \log(1+a\cos x)\,dx$ —(i)

diff w.r.t. a,

$$F'(a) = \int_0^\pi \frac{\partial}{\partial a}\left[\log(1+a\cos x)\right]dx = \int_0^\pi \frac{\cos x}{1+a\cos x}\,dx$$

$$= \frac{1}{a}\int_0^\pi \frac{a\cos x}{1+a\cos x}\,dx = \frac{1}{a}\int_0^\pi \frac{1+a\cos x-1}{1+a\cos x}\,dx$$

$$= \frac{1}{a}\int_0^\pi \left[1-\frac{1}{1+a\cos x}\right]dx$$

$$= \frac{1}{a}\left[x\right]_0^{\pi} - \frac{1}{a}\int_0^{\pi} \frac{dx}{1 + a\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$= \frac{\pi}{a} - \frac{1}{a}\int_0^{\pi} \frac{\left(1+\tan^2\frac{x}{2}\right)\,dx}{\left(1+\tan^2\frac{x}{2}\right) + a\left(1-\tan^2\frac{x}{2}\right)}$$

$$= \frac{\pi}{a} - \frac{1}{a}\int_0^{\pi} \frac{\sec^2\frac{x}{2}\,dx}{(1+a) + (1-a)\tan^2\frac{x}{2}}$$

Let $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

when $x = 0$, $t = 0$

when $x = \pi$, $t = \infty$

$$= \frac{\pi}{a} - \frac{1}{a}\int_0^{\infty} \frac{2\,dt}{(1+a) + (1-a)t^2}$$

$$= \frac{\pi}{a} - \frac{2}{a(1-a)}\int_0^{\infty} \frac{dt}{\frac{1+a}{1-a} + t^2}$$

$$= \frac{\pi}{a} - \frac{2}{a(1-a)}\sqrt{\frac{1-a}{1+a}}\left[\tan^{-1}\sqrt{\frac{1-a}{1+a}}\,t\right]_0^{\infty}$$

$$= \frac{\pi}{a} - \frac{2}{a\sqrt{1-a^2}}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{\pi}{a} - \frac{2}{a\sqrt{1-a^2}}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{a} - \frac{\pi}{a\sqrt{1-a^2}}$$

Integrating w.r.t. a

$$F(a) = \int \frac{\pi}{a}\,da - \pi\int \frac{da}{a\sqrt{1-a^2}} + C$$

$$= \pi\log a - \pi\int \frac{da}{a\sqrt{1-a^2}} + C$$

Put $a = \frac{1}{y}$

$$= \pi \log a - \pi \int \frac{-\frac{1}{y^2}\,dy}{\frac{1}{y}\sqrt{1-\frac{1}{y^2}}} + c$$

$$= \pi \log a + \pi \int \frac{dy}{\sqrt{y^2-1}} + c$$

$$= \pi \log a + \pi \log\left(y + \sqrt{y^2-1}\right) + c$$

$$= \pi \left[\log a + \log\left(\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1}\right)\right] + c$$

$$= \pi \log \left[a\left(\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1}\right)\right] + c$$

$$= \pi \log\left(1 + \sqrt{1-a^2}\right) + c \qquad \text{—(ii)}$$

From (i), when $a=0$, $F(0) = 0$

From (ii), $0 = \pi \log 2 + c$

$$c = -\pi \log 2$$

$$F(a) = \pi \log\left(1+\sqrt{1-a^2}\right) - \pi \log 2$$

$$F(a) = \pi \log \frac{1+\sqrt{1-a^2}}{2}$$

Hence $\displaystyle\int_0^{\pi} \log(1 + a\cos x)\,dx = \pi \log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-a^2}\right)$

<u>Example 5</u>: Evaluate $\displaystyle\int_0^a \frac{\log(1+ax)}{1+x^2}\,dx$ and hence show that $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2.$

<u>Solution</u>: Let $F(a) = \displaystyle\int_0^a \frac{\log(1+ax)}{1+x^2}\,dx \qquad$ —(i)

diff w.r.t. a

$$F'(a) = \int_0^a \frac{\partial}{\partial a}\left[\frac{\log(1+ax)}{1+x^2}\right]dx + \frac{\log(1+a^2)}{1+a^2}\cdot\frac{d(a)}{da} - \frac{\log(1+a0)}{1+0}\cdot\frac{d0}{da}$$

$$= \int_0^a \frac{1}{1+x^2} \cdot \frac{1}{1+ax} \cdot x \, dx + \frac{\log(1+a^2)}{1+a^2}$$

$$= \int_0^a \frac{x}{(1+ax)(1+x^2)} \, dx + \frac{\log(1+a^2)}{1+a^2} \quad \text{— (ii)}$$

To resolve the integrand into partial fractions

Let $\dfrac{x}{(1+ax)(1+x^2)} = \dfrac{A}{1+ax} + \dfrac{Bx+C}{1+x^2}$

$$x = A(1+x^2) + (Bx+C)(1+ax)$$

$$A = \frac{-a}{1+a^2}, \quad B = \frac{1}{1+a^2} \quad \& \quad C = \frac{a}{1+a^2}$$

$$\therefore \frac{x}{(1+ax)(1+x^2)} = \frac{1}{1+a^2}\left[\frac{-a}{1+ax} + \frac{x+a}{1+x^2}\right]$$

$$\Rightarrow \int_0^a \frac{x}{(1+ax)(1+x^2)} \, dx = \frac{1}{1+a^2}\left[-\log(1+ax) + \frac{1}{2}\log(1+x^2) + a\tan^{-1}x\right]_0^a$$

$$= \frac{1}{1+a^2}\left[-\log(1+a^2) + \frac{1}{2}\log(1+a^2) + a\tan^{-1}a\right]$$

$$= \frac{1}{1+a^2}\left[-\frac{1}{2}\log(1+a^2) + a\tan^{-1}a\right]$$

From (ii), we get

$$F'(a) = \frac{1}{1+a^2}\left[-\frac{1}{2}\log(1+a^2) + a\tan^{-1}a\right] + \frac{\log(1+a^2)}{1+a^2}$$

$$F'(a) = \frac{1}{1+a^2}\left[\frac{1}{2}\log(1+a^2) + a\tan^{-1}a\right]$$

Integrating w.r.t. a,

$$F(a) = \frac{1}{2}\int \log(1+a^2) \cdot \frac{1}{1+a^2} \, da + \int \frac{a\tan^{-1}a}{1+a^2} \, da + C$$

$$= \frac{1}{2}\left[\log(1+a^2)\tan^{-1}a - \int \frac{2a}{1+a^2}\tan^{-1}a \, da\right] + \int \frac{a\tan^{-1}a}{1+a^2} \, da + C$$

$$= \frac{1}{2}\log(1+a^2)\tan^{-1}a + C \quad \text{— (iii)}$$

Putting $a=0$ in (i), $F(0)=0$

Putting $a=0$ in (iii), $F(0)=C \Rightarrow C=0$

From (iii), we have

$$F(a) = \frac{1}{2}\log(1+a^2)\tan^{-1}a$$

Hence $$\int_0^a \frac{\log(1+ax)}{1+x^2}\,dx = \frac{1}{2}\log(1+a^2)\tan^{-1}a$$

Putting $a=1$, we get

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2.$$

Exercise

Question 1:- Show that $\int_0^{\pi} \frac{\log(1+\sin\alpha.\cos x)}{\cos x}\,dx = \pi\alpha$

Question 2:- Show that $\int_0^{\infty} \frac{e^{-x}\sin bx}{x}\,dx = \tan^{-1}b$

Question 3:- Show that

$$\int_0^{a^2} \tan^{-1}\left(\frac{x}{a}\right)dx = a^2\tan^{-1}a - \frac{1}{2}a\log(1+a^2)$$