

FOURIER SERIES

PERIODIC FUNCTION :- A function $f(x)$ is said to be periodic if there exists a positive number T such that $f(x+T) = f(x)$ for all x belonging to the domain of definition of the function.

The least number possessing this property is called the **period** of the function $f(x)$ and we often say that $f(x)$ is T -periodic.

For example, $\sin x$ and $\cos x$ are 2π -periodic, since

$$\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \dots$$

$$\cos x = \cos(x+2\pi) = \cos(x+4\pi) = \dots$$

Also, $\sin nx$ and $\cos nx$ are $\frac{2\pi}{n}$ -periodic, since

$$\sin n\left(x + \frac{2\pi}{n}\right) = \sin(nx + 2\pi) = \sin nx$$

$$\cos n\left(x + \frac{2\pi}{n}\right) = \cos(nx + 2\pi) = \cos nx.$$

A 2π -periodic function $f(x)$ can be expressed as a trigonometric series

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

where $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ are some constants, provided that

$f(x)$ satisfies the following condition, called **DIRICHLET'S COND'**

(i) $f(x)$ is periodic

(ii) $f(x)$ and its integrals are finite and single valued

(iii) $f(x)$ is bounded and can have a finite number of discontinuities.

(iv) $f(x)$ has a finite number of maxima and minima over the range of time period (x).

Note :- (i) If $f(x)$ is continuous in the range of definition, series converges to $f(x)$.

(ii) If $f(x)$ has ordinary discontinuity at ' x ', the series converges to $\frac{1}{2} [f(x^+) + f(x^-)]$ for every x in the range of definition of $f(x)$.

This series (ii) is called Fourier series of $f(x)$.

Example :- The function $f(x) = \frac{1}{x-3}$ does not satisfy the

Dirichlet's condition in the interval $(0, 2\pi)$ as it is not defined at $x=3$.

$\therefore f(x) = \frac{1}{x-3}$ cannot possess Fourier series in $(0, 2\pi)$.

series in $(0, 2\pi)$.

To evaluate the unknowns, $a_0, a_1, a_2, \dots, b_1, b_2, \dots$
 we need to the following orthogonality properties :-

The trigonometric functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$
 are also orthogonal on the interval $(-\pi, \pi)$ and hence
 on $(c, c+2\pi)$. By orthogonality, we mean that the
 integral of the product of any two different such functions
 over that interval is zero.

$$(1) \int_c^{c+2\pi} \sin nx dx = \left[\frac{-1}{n} \cos nx \right]_c^{c+2\pi} = 0 ; n \neq 0$$

$$(2) \int_c^{c+2\pi} \cos nx dx = \left[\frac{1}{n} \sin nx \right]_c^{c+2\pi} = 0 ; n \neq 0$$

$$(3) \int_c^{c+2\pi} \sin mx \sin nx dx = \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_c^{c+2\pi} = 0 ; m \neq n$$

$$(4) \int_c^{c+2\pi} \cos mx \cos nx dx = \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n} \right]_c^{c+2\pi} = 0 ; m \neq n$$

$$(5) \int_c^{c+2\pi} \sin mx \cos nx dx = -\frac{1}{2} \left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n} \right]_c^{c+2\pi} = 0 .$$

Also,

some more results are :-

$$(1) \sin n\pi = 0$$

$$(2) \cos n\pi = (-1)^n$$

$$(3) \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

$$(4) \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

Evaluation of Fourier Coefficients

let the function $f(x)$ be represented in the interval

$c \leq x \leq c+2\pi$ by the Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx dx \quad n = 1, 2, \dots$$

Corollary :- ① If $c=0$, then the above integration

becomes for $0 \leq x \leq 2\pi$, the Fourier coefficients are,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx.$$

Corollary :- ② If $c = -\pi$, then the interval of integration

becomes for $-\pi \leq x \leq \pi$, and Fourier coefficients are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

Corollary :- ③ If $f(x)$ is an even function, i.e. $f(-x) = f(x)$

in $(-\pi, \pi)$ then by property of definite integrals,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$b_n = 0$$

Corollary :- ④ If $f(x)$ is an odd function, i.e. $f(-x) = -f(x)$

in $(-\pi, \pi)$ then

$$a_n = 0$$

and

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

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Corollary 5 The Fourier expansion of the function $f(x)$ as $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

is in two parts. The part $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ is an even part function of x and the part $\sum_{n=1}^{\infty} b_n \sin nx$ is an odd function of x .

Also, we can write $f(x) = \frac{1}{2} [f(x) + f(-x)] + \frac{1}{2} [f(x) - f(-x)]$

out of which $\frac{1}{2} [f(x) + f(-x)]$ is even function

and $\frac{1}{2} [f(x) - f(-x)]$ is odd function.

\therefore we have,

$$\boxed{\frac{1}{2} [f(x) + f(-x)] = \frac{1}{2} a_0 + \sum_n a_n \cos nx}$$

and

$$\boxed{\frac{1}{2} [f(x) - f(-x)] = \frac{a_0}{2} + \sum_n b_n \sin nx}$$

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Ex. Find the Fourier series for the function
 $f(x) = e^{ax}$ in the interval $(-\pi, \pi)$.

Solution Let the Fourier series for $f(x)$ be

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{Hence, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} dx = \left[\frac{e^{ax}}{a\pi} \right]_{-\pi}^{\pi} = \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2 \sinh(a\pi)}{a\pi}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos nx dx = \left[\frac{e^{ax}}{\pi(a^2+n^2)} (\cos nx + n \sin nx) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi(a^2+n^2)} \left[e^{a\pi} (\cos a\pi + \pi \sin a\pi) - e^{-a\pi} (\cos a\pi - \pi \sin a\pi) \right] \\ &= \frac{\cos a\pi (e^{a\pi} - e^{-a\pi})}{\pi(a^2+n^2)} = \frac{2a(-1)^n \sinh(a\pi)}{\pi(a^2+n^2)}. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \sin nx dx = \left[\frac{e^{ax}}{\pi(a^2+n^2)} (-\sin nx - n \cos nx) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi(a^2+n^2)} \left[e^{a\pi} (-\sin a\pi - \pi \cos a\pi) - e^{-a\pi} (-\sin a\pi - \pi \cos a\pi) \right] \end{aligned}$$

$$= \frac{1}{\pi(a^2+n^2)} \left[e^{a\pi} (\sin a\pi - \pi \cos a\pi) - e^{-a\pi} (-\sin a\pi - \pi \cos a\pi) \right]$$

$$= -\frac{n \cos n\pi}{\pi(a^2+n^2)} (e^{a\pi} - e^{-a\pi}) = -\frac{2n(-1)^n \sinh(a\pi)}{\pi(a^2+n^2)}$$

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Substituting these values of a_0, b_n, a_n , we get

$$f(x) = e^{ax} = \frac{1}{a\pi} \sinh a\pi + \sum_{n=1}^{\infty} \frac{2(-1)^n \sinh a\pi}{\pi(a^2+n^2)} (\cos nx - n \sin nx)$$

is the required Fourier series.

Ex Find a Fourier series to represent $x-x^2$

from $x=-\pi$ to $x=\pi$ and hence show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots = \frac{\pi^2}{12}$$

Solution let the Fourier series be given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x-x^2) dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{1}{\pi} \left[-\frac{\pi^3}{3} - \frac{\pi^3}{3} \right] = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x-x^2) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} n \cos nx dx - \frac{1}{\pi} \int_{-\pi}^{\pi} n^2 \cos nx dx \right]$$

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$$= 0 \quad (\text{odd}) - \frac{2}{\pi} \int_0^{\pi} n^2 \cos nx dx$$

$$= -\frac{2}{\pi} \left[\frac{n^2 \sin nx}{n} - \frac{(2n)(-\cos nx)}{m^2} + \frac{2(-\sin nx)}{m^3} \right]_0^{\pi}$$

$$= -\frac{2}{\pi} \left[\frac{2\pi \cos \pi}{n^2} - 0 \right] = -\frac{4\pi (-1)^n}{\pi n^2} = -\frac{4(-1)^n}{n^2}$$

$$\boxed{bn} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \sin nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx - \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx dx$$

↑ odd.

$$= \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$$

$$= \frac{2}{\pi} \left[\frac{x(-\cos nx)}{n} - \frac{(1)(-\sin nx)}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{\pi (-1)^n}{n} \right] = -\frac{2(-1)^n}{n}$$

$$\therefore f(x) = -\frac{\pi^2}{3} + \sum_{m=1}^{\infty} \left[\frac{-4(-1)^n \cos nx}{n^2} - \frac{2(-1)^n \sin nx}{n} \right]$$

$$\Rightarrow f(x) = -\frac{\pi^2}{3} + 4 \left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right] + 2 \left[\frac{\sin x}{1^2} - \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} - \dots \right]$$

Putting $x=0$ on both sides, we get

$$f(0) = 0 = -\frac{\pi^2}{3} + 4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$$

Example 3: Find the Fourier series to represent the function $f(x) = x \sin x$, $-\pi < x < \pi$.

Solution since product of two odd functions is an even function. $\therefore f(x)$ is an even function of x .

Hence, $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$\begin{aligned} \text{Now } a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x dx \\ &= \frac{1}{\pi} \left[x(-\cos x) - (-1)(\sin x) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[-\pi \cos \pi - \sin \pi + (-\pi) \cos(-\pi) + \sin(-\pi) \right] \\ &= \frac{1}{\pi} [\pi + \pi] = 2. \end{aligned}$$

$$\text{Now } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \cos nx dx$$

$x \sin x \rightarrow \text{odd} * \text{odd} = \text{even}$
 $\cos nx \rightarrow \text{even}$
 $\therefore x \sin x \cos nx \rightarrow \text{even}$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \left[\sin((n+1)x) - \sin((n-1)x) \right] dx$$

$$= \frac{1}{\pi} \left[\pi \left\{ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right\} - (-1) \left\{ -\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^\pi$$

$$= \frac{1}{\pi} \left[\pi \left\{ -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} \right\} + \boxed{0} \right] + 0 \quad , \quad n \neq 1$$

$$= (-1)^n \left[\frac{1}{n+1} - \frac{1}{n-1} \right] = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1} \quad , \quad n \neq 1$$

For $n=1$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{2\pi} \int_0^\pi 2n \sin x \cos x dx$$

$$= \frac{1}{\pi} \int_0^\pi x \sin x dx = \frac{1}{\pi} \left[-x \frac{\cos 2x}{2} + (-1)(\sin 2x) \right]_0^\pi$$

$$= \frac{1}{\pi} \left[-\frac{\pi}{2} + 0 \right] = -\frac{1}{2}$$

$$\therefore f(x) = 1 - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cos nx.$$

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Example 4 :- The function $f(x)$ is given by

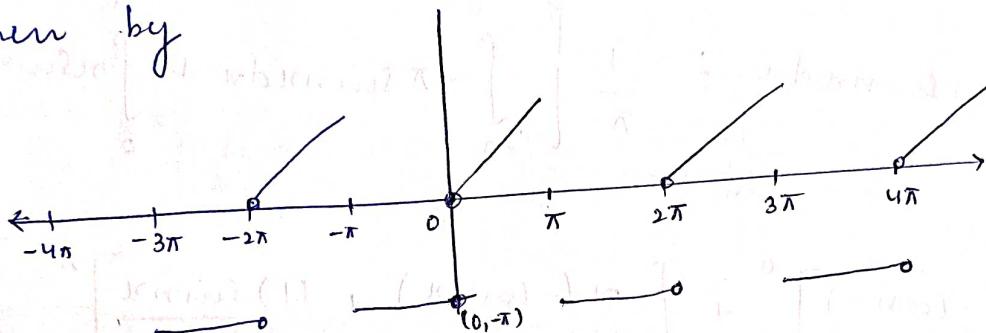
$$f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Draw its graph and find its Fourier series

and hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{8}$.

Solution. The graph of the 2π -periodic function is

given by



Let the Fourier series of the given function be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{Then } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-\pi) dx + \int_0^{\pi} x dx \right]$$

$$= \frac{1}{\pi} \left[[-\pi x]_{-\pi}^0 + \left[\frac{x^2}{2} \right]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[0 + (-\pi)^2 + \frac{\pi^2}{2} \right] = -\frac{\pi^2}{2\pi} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \cos nx dx + \int_0^\pi x \cos nx dx \right]$$

$$= - \left[\frac{\sin nx}{n} \right]_{-\pi}^0 + \frac{1}{\pi} \left[x \frac{\sin nx}{n} + (-1) \frac{\cos nx}{n^2} \right]_0^\pi$$

$$= 0 + \frac{1}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2} \right] = \frac{1}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right]$$

$$\boxed{b_n} = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin nx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \sin nx dx + \int_0^\pi x \sin nx dx \right]$$

$$= \frac{1}{\pi} \left[\left[-\pi \left(\frac{-\cos nx}{n} \right) \right]_{-\pi}^0 + \left[x \left(\frac{-\cos nx}{n} \right) + (-1) \frac{\sin nx}{n^2} \right]_0^\pi \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi (1 - (-1)^n)}{n} + \frac{\pi ((-1)^{n+1} - 1)}{n} + 0 \right]$$

$$= \frac{1 - 2(-1)^n}{n}$$

\therefore The Fourier series for $f(x)$ is

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\left(\frac{(-1)^n - 1}{n^2} \right) \cos nx \right] + \sum_{n=1}^{\infty} \left(\frac{1 - 2(-1)^n}{n} \right) \sin nx$$

Now, we need to prove the equality :-

The expansion of $f(x)$ at $x=0$, which is a point of

discontinuity of $f(x)$. The value of $f(x)$ at 115
 $x=0$ is given by the mean of its left
 hand and right hand limits, equal to

$$\frac{f(0^-) + f(0^+)}{2} = \frac{-\pi + 0}{2} = -\frac{\pi}{2}$$

Thus, we have (Putting $n=0$) in ④
 $\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} + 0$

$$-\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi} \left[\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \right] + 0.$$

$$\Rightarrow \pi^2 \left(\frac{-1}{2} + \frac{1}{n^2} \right) = \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}$$

$$\Rightarrow -\frac{2\pi^2}{8} = \frac{(-2)}{1^2} + \frac{(-2)}{3^2} + \frac{(-2)}{5^2} + \dots$$

∴ we have

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

Example 5, obtain the Fourier series for the function

$f(x)$ given by

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi} & 0 \leq x \leq \pi \end{cases}$$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.

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Solution Clearly $f(-x) = f(x)$ and hence

$f(x)$ is even in $(-\pi, \pi)$ and will have only cosine series. Also the continuous at $x=0$.

$$\text{let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{Here } a_0 = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx = \frac{2}{\pi} \left[x - \frac{x^2}{\pi} \right]_0^{\pi} = 0$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx dx \\ &= \frac{2}{\pi} \left[\left(1 - \frac{2x}{\pi}\right) \left(\frac{\sin nx}{n}\right) - \left(\frac{-2}{\pi}\right) \left(-\frac{\cos nx}{n^2}\right) \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{-2}{\pi n^2} \cos nx \right]_0^{\pi} = \frac{-4}{\pi^2 n^2} (-1)^n - 1 \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{\pi^2 n^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

$$\text{Thus, } f(x) = \sum_{n \text{ is odd}} \frac{8}{\pi^2 n^2} \cos nx = \frac{8}{\pi^2} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \dots \right]$$

$$\text{Putting } n=0, \quad f(0) = \frac{8}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots \right]$$

$$\therefore \frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots \right].$$

Example: Find the Fourier series to

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represent the function $f(x) = \begin{cases} x & 0 \leq x \leq \pi \\ 2\pi - x & \pi \leq x \leq 2\pi \end{cases}$

and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

Solution

$$[a_0] = \pi$$

$$[a_n] = \frac{2}{\pi n^2} [(-1)^n - 1] = \begin{cases} 0 & n \text{ even} \\ -\frac{4}{\pi n^2} & n \text{ odd} \end{cases}$$

$$[b_n] = 0.$$

$$\therefore f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=\text{odd}} \frac{\cos nx}{n^2}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

Putting $x = \pi$ on both sides,

$$\pi = \frac{\pi}{2} + \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

Die \mathbb{R} -Vektorraumstruktur auf $\mathcal{A} = \{g_A\}_{A \in \mathcal{A}}$ ist definiert durch

$\alpha \cdot g_A = g_{\alpha A}$ für $\alpha \in \mathbb{R}$ und $b \in \mathcal{B}$

$(g_A + g_B) = g_{A+B}$ für $A, B \in \mathcal{A}$

$\left(\sum_{i=1}^n \alpha_i g_{A_i} \right) = \sum_{i=1}^n \alpha_i g_{A_i}$ für $\alpha_i \in \mathbb{R}$ und $A_i \in \mathcal{A}$

\mathcal{A} ist ein \mathbb{R} -Vektorraum mit der Dimension n .

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\mathcal{A} ist ein \mathbb{R} -Vektorraum mit der Dimension n .

Die \mathbb{R} -Vektorraumstruktur auf $\mathcal{A} = \{g_A\}_{A \in \mathcal{A}}$ ist definiert durch

$\alpha \cdot g_A = g_{\alpha A}$ für $\alpha \in \mathbb{R}$ und $A \in \mathcal{A}$

$(g_A + g_B) = g_{A+B}$ für $A, B \in \mathcal{A}$

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Example Obtain the Fourier series for

the function $f(x) = \left(\frac{\pi-x}{2}\right)^2$ in $[0, 2\pi]$

and hence show that

$$(1) \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

$$(2) \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

In this problem the function is defined in $[0, 2\pi]$. In this problem the function is defined in

Solution In this problem the function is defined in $[0, 2\pi]$. If we substitute $t = x - \pi$, then

$$f(x) = \phi(t) = \frac{t^2}{4} \text{ when } -\pi < t < \pi$$

Clearly $\phi(t) = \frac{t^2}{4}$ is an even function of t .

$$\text{Let } \phi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t) dt = \frac{2}{\pi} \int_0^{\pi} \frac{t^2}{4} dt$$

$$= \frac{2}{\pi} \left[\frac{t^3}{12} \right]_0^{\pi} = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t) \cos nt dt = \frac{2}{\pi} \int_0^{\pi} \frac{t^2}{4} \cos nt dt$$

$$= \frac{2}{\pi} \left[\frac{t^2}{4} \left(\frac{\sin nt}{n} \right) - \left(\frac{2t}{4} \right) \left(-\frac{\cos nt}{n^2} \right) + \frac{2}{4} \left(\frac{-\sin nt}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{2t}{4n^2} \cos nt \right]_0^\pi$$

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$$= \frac{1}{\pi} \left[\frac{\pi \cos n\pi}{n^2} - 0 \right] = \frac{(-1)^n}{n^2}$$

\therefore The Fourier series is given by

$$\phi(t) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt$$

Replacing t by $x-\pi$, we get

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n(x-\pi)$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} [\cos nx \cos n\pi + \sin nx \sin n\pi]$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n \cos nx}{n^2}$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$\Rightarrow \frac{(\pi-x)^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$\text{Putting } n=0, \text{ we get } \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\text{Putting } x=\pi, \text{ we get } \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$$

Example. Obtain the Fourier expansion

for $\sqrt{1-\cos x}$ in the interval $-\pi < x < \pi$

$$\text{Solution. } f(x) = \sqrt{1-\cos x} = \sqrt{2\sin^2 \frac{x}{2}} = \pm \sqrt{2} \sin \frac{x}{2}$$

$$f(x) = \begin{cases} \sqrt{2} \sin \frac{x}{2} & 0 < x < \pi \\ -\sqrt{2} \sin \frac{x}{2} & -\pi < x < 0 \end{cases}$$

is an even function of x .

Let its Fourier series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$\text{where } a_0 = \frac{2}{\pi} \int_0^{\pi} \sqrt{2} \sin \frac{x}{2} dx = \frac{2\sqrt{2}}{\pi} \left[\frac{-\cos \frac{x}{2}}{\frac{1}{2}} \right]_0^{\pi} = \frac{4\sqrt{2}}{\pi}.$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sqrt{2} \sin \frac{x}{2} \cos nx dx$$

$$= \frac{\sqrt{2}}{\pi} \int_0^{\pi} 2 \sin \frac{x}{2} \cos nx dx$$

$$= \frac{\sqrt{2}}{\pi} \int_0^{\pi} \left[\sin \left(n + \frac{1}{2} \right)x - \sin \left(n - \frac{1}{2} \right)x \right] dx$$

$$= \frac{\sqrt{2}}{\pi} \left[\frac{-\cos \left(n + \frac{1}{2} \right)x}{\left(n + \frac{1}{2} \right)} + \frac{\cos \left(n - \frac{1}{2} \right)x}{\left(n - \frac{1}{2} \right)} \right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi} \left[\frac{-1}{\left(n + \frac{1}{2} \right)} \cos \left(n\pi + \frac{\pi}{2} \right) + \frac{1}{\left(n - \frac{1}{2} \right)} \cos \left(n\pi - \frac{\pi}{2} \right) + \frac{1}{\left(n + \frac{1}{2} \right)} - \frac{1}{\left(n - \frac{1}{2} \right)} \right]$$

$$= \frac{2\sqrt{2}}{\pi} \left[\frac{1}{(2n+1)} \sin n\pi + \frac{1}{(2n-1)} \sin n\pi + \frac{1}{2n+1} - \frac{1}{2n-1} \right]$$

$$= \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

$$\therefore f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{4n^2-1}$$

$$= \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{(2n+1)(2n-1)}$$

$$= \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi} \left[\frac{1}{2} \left(\frac{1}{1-x^2} + \frac{1}{1+x^2} \right) \right] = \boxed{2A}$$

$$= \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi} \left[\frac{1}{2} \left(\frac{1}{1-x^2} + \frac{1}{1+x^2} \right) \right] = \boxed{2A}$$

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$$= \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi} \left[\frac{1}{2} \left(\frac{1}{1-x^2} + \frac{1}{1+x^2} \right) \right] = \boxed{2A}$$

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$$= \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi} \left[\frac{1}{2} \left(\frac{1}{1-x^2} + \frac{1}{1+x^2} \right) \right] = \boxed{2A}$$

Example :- Expand $f(x) = |\cos x|$, as a Fourier series in $(-\pi, \pi)$. 123

[IPU 2010, 2014, 2016]

Solution :- $f(x) = |\cos x|$ is an even function of x in $(-\pi, \pi)$.

$$\text{let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{where } [a_0] = \frac{2}{\pi} \int_0^\pi f(x) dx = \frac{2}{\pi} \int_0^\pi |\cos x| dx = \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^\pi -\cos x dx \right]$$

$$= \frac{2}{\pi} \left[\sin x \Big|_0^{\pi/2} \right] - \frac{2}{\pi} \left[\sin x \Big|_{\pi/2}^\pi \right]$$

$$= \frac{2}{\pi} [1 - 0] - \frac{2}{\pi} [0 - 1] = \frac{4}{\pi}$$

$$[a_n] = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos nx dx + \int_{\pi/2}^\pi -\cos x \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi/2} \cos((n+1)x) + \cos((n-1)x) dx \right]$$

$$- \frac{1}{\pi} \left[\int_{\pi/2}^\pi \cos((n+1)x) + \cos((n-1)x) dx \right]$$

$$= \frac{1}{\pi} \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \Big|_0^{\pi/2} \right]$$

$$- \frac{1}{\pi} \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \Big|_{\pi/2}^\pi \right]$$

$$= \frac{2}{\pi} \left[\frac{\sin(n+1)\frac{\pi}{2}}{(n+1)} + \frac{\sin(n-1)\frac{\pi}{2}}{(n-1)} \right]; \quad (n \neq 1)$$

$$= \frac{2}{\pi(n^2-1)} \left[(n-1)\sin(n+1)\frac{\pi}{2} + (n+1)\sin(n-1)\frac{\pi}{2} \right]; \quad (n \neq 1)$$

$$= \frac{2}{\pi(n^2-1)} \left[(n-1)\cos n\frac{\pi}{2} + (n+1)(-\cos n\frac{\pi}{2}) \right]; \quad (n \neq 1)$$

$$= -\frac{4}{\pi(n^2-1)} \cos n\frac{\pi}{2}. \quad (n \neq 1)$$

Also, $a_1 = \frac{2}{\pi} \int_0^{\pi} f(x) \cos x dx = \frac{2}{\pi} \int_0^{\pi} \cos^2 x dx - \frac{2}{\pi} \int_{\pi/2}^{\pi} \cos^2 x dx$

$$= \frac{1}{\pi} \int_0^{\pi/2} (1+\cos 2x) dx - \frac{1}{\pi} \int_{\pi/2}^{\pi} (1+\cos 2x) dx$$

$$= \frac{1}{\pi} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2} - \frac{1}{\pi} \left[x + \frac{\sin 2x}{2} \right]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} + 0 \right] - \frac{1}{\pi} \left[\pi - \frac{\pi}{2} \right]$$

$$= \frac{1}{2} - \frac{1}{2} = 0.$$

$$\therefore f(x) = |\cos x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{\cos n\frac{\pi}{2} \cos nx}{(n^2-1)}$$