NEW TOPICS ADDED FROM ACADEMIC SESSION [2022-23]
THIRD SEMESTER [B.TECH]
DISCRETE MATHEMATICS (CIC-205)

⇒ **Functions, Growth of functions, Permutation functions, Partially ordered sets**

[UNIT-2]

Q.1 Prove a function f: R → R f: R → R defined by f(x)= 2x – 3f(x) = 2x – 3 is a bijective function.

Ans. We have to prove this function is both injective and surjective.

If $f(x_1) = f(x_2)$

then $2x_1 - 3 = 2x_2 - 3$ and it implies that $x_1 = x_2$.

Hence, f is injective.

Here, $2x - 3 = y$

So, $x = (y + 5)/3$ which belongs to R and $f(x)=y$.

Hence, f is surjective.

Since f is both surjective and injective, we can say f is bijective.

Q.2 Let $f(x) = x + 2$ and $g(x) = 2x + 1$, find (fog) (x) and (gof) (x).

Ans. $(fog)(x) = f(g(x)) = f(2x + 1) = 2x + 1 + 2 = 2x + 3$

$(gof)(x) = g(f(x)) = g(x + 2) = 2(x + 2) + 1 = 2x + 5$

Hence, $(fog)(x) \neq (gof)(x)$

Q.3 Show that the function f: R → R, given by f(x) = 2x, is one-one and onto.

Ans. For one-one:

Let a, b ∈ R such that f(a) = f(b) then,

f(a) = f(b)

⇒ = 2a = 2b

⇒ a = b

Therefore, f : R → R is one-one.

For onto:

Let p be any real number in R (co-domain).

$$f(x) = p$$
$$\Rightarrow \quad 2x = p$$
$$\Rightarrow \quad x = p/2$$

p/2 ∈ R for p ∈ R such that f(p/2) = 2(p/2)= p

For each p ∈ R (codomain) there exists x = p/2 ∈ R (domain) such that $f(x) = y$

For each element in codomain has its pre-image in domain.

So, f: R → R is onto.

Since f: R → R is both one-one and onto.

f: R → R is one-one correspondent (bijective function).

Q.4 If f : R → R : $f(x) = 3x + 5$ is a bijective function then, find the inverse of f.

Ans. Let x ∈ R (domain), y ∈ R (codomain) such that f(a) = b

$f(x) = y$

$\Rightarrow 3x + 5 = y$

$\Rightarrow x = (y - 5)/3$

$\Rightarrow f^{-1}(y) = (y - 5)/3$

Thus, $f^{-1:R} \to R$ is defined as $f^{-1}(x) = (x-5)/3$ for all $x \in R$.

Q.5 Let f : R → R ; f(x) = cos x and g : R → R; g(x) = x^3. Find fog and gof.

Ans. Since the range of f is a subset of the domain of g and the range of g is a subset of the domain of f. So, fog and gof both exist.

$gof(x) = g(f(x)) = g(\cos x) = (\cos x)^3 = \cos^3 x$

$fog(x) = f(g(x)) = f(x^3) = \cos x^3$

Q.6 If f : Q → Q is given by f(x) = x^2, then find $f^{-1}(25)$.

Ans. Let $f^{-1}(25) = x$

$f(x) = 25$

$x^2 = 25$

$x = \pm 5$

$f^{-1}(25) = \{-5, 5\}$

Q.7 Consider the function f: {1, 2, 3, 4} → {1, 2, 3, 4} given by

$$f(n) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 4 \end{pmatrix}$$

(a) Find .f(1).

(b) Find an element n in the domain such that .f(n) = 1.

(c) Find an element n of the domain such that .f(n) = n.

(d) Find an element of the codomain that is not in the range.

Ans. f(1) = 4, Since 4 is the number below 1 in the two-line notation.

Such an n is, n = 2, since. f(2) = 1. Note that 2 is above a 1 in the notation.

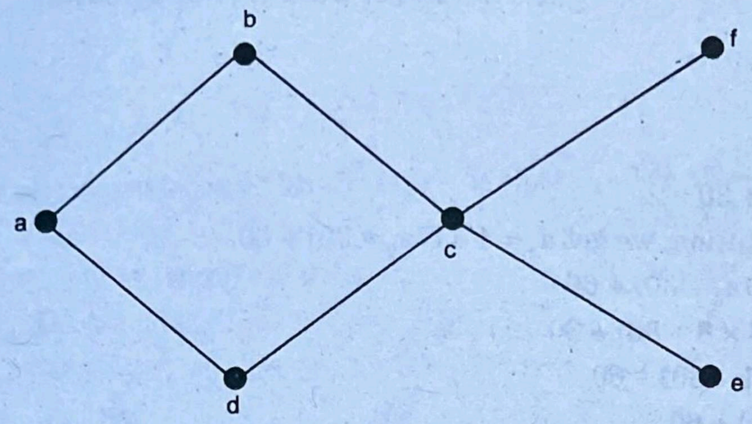
n=4 has this property. We say that 4 is a fixed point of .f Not all function have such a point.

Such an element is 2 (in fact, that is the only element in the codomain that is not in the range). In other words, 2 is not the image of any element under; f; nothing is sent to 2

Q.8 The coach of a basketball team is picking among 11 players for the 5 different positions in h is starting lineups. How many different lineups can he pick?

Ans. $P(11, 5) = \dfrac{11!}{(11-5)!} = \dfrac{11!}{6!} = 55440$ different lineups. {The permutation function is defined as : $P(n,k) = \dfrac{n!}{(n-k)!}$

Q.9 Determine all the maximal and minimal elements of the poset whose Hasse diagram is shown in fig.



Ans. The minimal elements of the Poset are d and c

The maximal elements of Poset are b and f.

Q.10 Consider A = {1, 2, 3, 5, 6, 10, 15, 30} is ordered by divisibility. Determine all the comparable and non-comparable pairs of of elements of A.

Ans. The comparable pairs of elements of A are:

{1, 2}, {1, 3}, {1, 5}, {1, 6}, {1, 10}, {1, 15}, {1, 30}

{2, 6}, {2, 10}, {2, 30}

{3, 6}, {3, 15}, {3, 30}

{5, 10}, {5, 15}, {5, 30}

{6, 30}, {10, 30}, {15, 30}

The non-comparable pair of elements of A are:

{2, 3}, {2, 5}, {2, 15}

{3, 5}, {3, 10}, {5, 6}, {6, 10}, {6, 15}, {10, 15}

Note: In an ordered set A. Two elements a and b of set A are called comparable if $a \le b$ or $b \le a$ in a relation. Whereas two elements a and b of set A are called noncomparable if neither $a \le b$ nor $b \le a$.

RECURRENCE RELATION [UNIT-2]

Q.11 Consider the recurrence relation $a_1 = 4$, $a_n = 5n + a_{n-1}$. What is the value of a_{64}?

Ans. $a_n = 5n + a_{n-1}$

$= 5n + 5(n - 1) + ... + a_{n-2}$

$= 5n + 5(n-1) + 5(n - 2) + ... + a_1$

$= 5n + 5(n-1) + 5(n - 2) + ...+ 4$ [since, $a_1 = 4$]

$= 5n + 5(n-1) + 5(n - 2) + ...+ 5.1 - 1$

$= 5 (n + (n - 1) + ... + 2 + 1) - 1$

$= 5 \times n(n + 1)/2-1$

$a_n = 5 \times n(n+1)/2-1$

Now, n = 64 so the answer is $a_{64} = 10399$.

Q.12 Determine the value of a_3 for the recurrence relation $a_n - 17a_{n-1} + 30n$ with $a_0 = 3$.

Ans. For n = 1,

$a_1 = 17a_0 + 30(1)$ ...(1)

$a_1 = 17(3) + 30$ since $a_0 = 3$

$a_1 = 51 + 30$

$a_1 = 81$

For n = 2,

$a_2 = 17a_1 + 30 \times 2$ ...(2)

By substitution, we get $a_2 = 17(17a_0 + 30) + 60.$ [from 2]

$a_2 = 17(17a_0 + 30) + 60$

$a_2 = 17(17 \times 3 + 30) + 60$

$a_2 = 17(51 + 30) + 60$

$a_2 = 17(81) + 60$

$a_2 = 1437$ ...(3)

For n = 3

$a_3 = 17a_2 + 30 \times 3$

$a_3 = 17(1437) + 30 \times 3$ {from 3}

$a_3 = 24519$

Then regrouping the terms, we get $a_3 = 24519$, where $a_0 = 3$, $a_1 = 81$, $a_2 = 1437$.

⇒ **GCD, LCM, Prime numbers [UNIT - 2]**

Q.13 Evaluate the expression: (X + Z) (X + XZ′) + XY + Y.

Ans. (X + Z) (X + XZ′) + XY + Y [Given]

= (X + Z) X(1 + Z′) + XY + Y [Using Distributive]

= (X + Z) X + XY + Y [Using Complement, Identity]

= (X + Z) X + Y (X + 1) ⇒ (X + Z) X + Y [Using Idempotent]

= XX + ZX + Y [Using Distributive]

= X + XZ + Y [Using Identity]

= X(1 + Z) + Y

= X + Y [Idempotent]

Q.14 What are the canonical forms of Boolean Expressions?

Ans. There are two kinds of canonical forms for a Boolean expression:

- sum of minterms (SOM) form
- Product of maxterms (SOM) form.

Q.15 Find the prime factorization of each of these integers. (a) 88 (b) 126 (c) 729 (Unit 2: GCD, LCM, Prime No.)

Ans. (a) $\sqrt{88} \approx 9.38$

88/2 = 44

44/2 = 22

22/2 = 11 Therefore $88 = 2^3 \cdot (11)$

(b) $\sqrt{126} \approx 11.22$

126/2 = 63

63/3 = 21

21/3 = 7 Therefore $63 = 2 \cdot 3^2 \cdot 7$

(c) $\sqrt{729} = 27$

729/3 = 243

$243/3 = 81$

$81/3 = 27$

$27/3 = 9$

$9/3 = 9$ Therefore $729 = 3^6$

Q.16 What are the greatest common divisors of these pairs of integers?

(a) $3^7 \cdot 5^3 \cdot 7^3, 2^{11} \cdot 3^5 \cdot 5^9$

Ans. $2^{\min(0,11)} \cdot 3^{\min(7,5)} \cdot 5^{\min(3,9)} \cdot 7^{\min(3,0)} = 3^5 \cdot 5^3$

(b) $11 \cdot 13 \cdot 17, 2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3$

Ans. $2^{\min(9,0)}, 3^{\min(7,0)}\ 5^{\min(5,0)}\ 7^{\min(7,0)}, 11^{\min(1,0)}, 13^{\min(1,0)}, 17^{\min(1,0)}$

(c) $41 \cdot 43 \cdot 53, 41 \cdot 43 \cdot 53$

Ans. $41 \cdot 43 \cdot 53$

(d) $23^{31} . 23^{17}$

Ans. $23^{\min(31,17)} = 23^{17}$

Q.17 What is the least common multiple of these pairs of integers?

(a) $3^7 \cdot 5^3 - 7^3, 2^{11} \cdot 3^5 \cdot 5^9$

Ans. $2^{\max(0,11)} \cdot 3^{\max(7,5)} \cdot 5^{\max(3,9)} \cdot 7^{\max(3,0)} = 2^{11} \cdot 3^7 \cdot 5^9 \cdot 7^3$

(b) $11 \cdot 13 \cdot 17, 2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3$

Ans. $11 \cdot 13 \cdot 17 \cdot 2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3$

(c) $23^{31}, 23^{17}$

Ans. $23^{\max(31,17)} = 23^{31}$

(d) $41 \cdot 43 \cdot 53, 41 \cdot 43 \cdot 53$

Ans. $41 \cdot 43 \cdot 53$

Q.18 Find gcd(92928, 123552) and 1cm(92928, 123552) and verify that gcd(92928, 123552) · 1cm(92928, 123552) = 92928 · 123552. [Hint: First find the prime factorizations of 92928 and 123552].

Ans. $92928 = 2^8 \cdot 3 \cdot 11^2$

$123552 = 2^5 \cdot 3^3 \cdot 11 \cdot 13$

$\gcd(92928, 123552) = 2^5 \cdot 3 \cdot 11$

$1\text{cm}(92928, 123552) = 2^8 \cdot 3^3 \cdot 11^2 \cdot 13$

$\gcd(92928, 123552).\ 1\text{ cm}(92928, 123552) = 92928 \cdot 123552$

$(2^5 \cdot 3 \cdot 11) \cdot (2^8 \cdot 3^3 \cdot 11^2 \cdot 13) = (2^8 \cdot 3 \cdot 11^2) \cdot (2^5 \cdot 3^3 \cdot 11 \cdot 13)$

$2^{13} \cdot 3^4 \cdot 11^3 \cdot 13 = 2^{13} \cdot 3^4 \cdot 11^3 \cdot 13$

Solution method for divide and conquer recurrence relation, Masters theorem (with proof)

[UNIT - 2]

Q.19 Suppose that $f(n) = f(n/5) + 3n^2$ when n is a positive integer divisible by 5, and $f(1) = 3$. Find (a) $f(5)$, (b) $f(3125)$.

Ans. (a) $f(5) = f(5/5) + 3(5)^2$

$= f(1) + 3(25)$ since $f(1) = 3$

$= 3 + 75 = 78$

(b) f (3125)

$f(625) = f(625/5) + 3(625)^2$

$= f(125) + 3(390625)$

$= 48829 + 1171875$

$= 1220704$

$f(3125) = f(3125/5) + 3(3125)^2$

$= f(625) + 3(9765625)$

$= 1220704 + 29296875 = 30517579$

{Note: Master Theorem: Let f be an increasing function that satisfies the recurrence relation

$f(n) = af(n/b) + cn^d$ whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1 and c and d are real numbers with c positive and d non-negative. Then

$$f(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

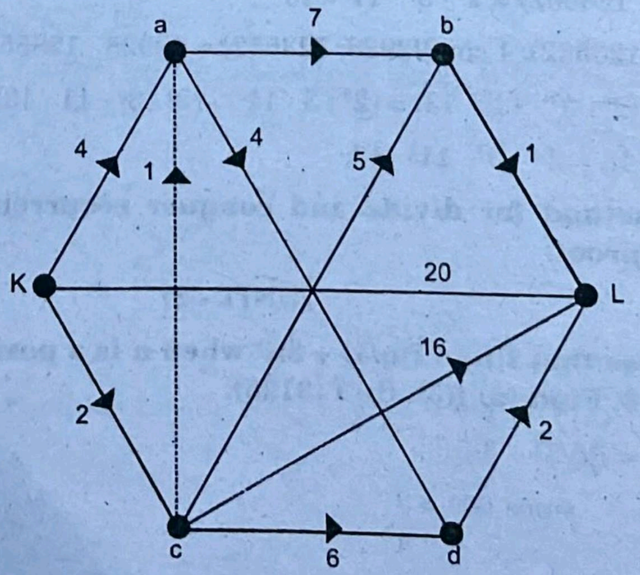
Q.20 Given a big-O estimable for the function f(n) = 2f(n)/3 + 4 if f is an increasing function and $n = 3^k$.

Ans. Use Master Theorem with a = 2, b = 3, c = 4, d = 0. Since $a > b^d$, we know that f(n) is $O(n^{\log b\ a}) = O(n^{\log 3\ 2})$.

⇒ **Shortest path and Minimum spanning trees and algorithms of the graph MST.**

[UNIT-4]

Q.21 Find the shortest paths between K and L in the graph shown in fig using Dijkstra's Algorithm.



Ans. To find the Shortest paths between the graph follow steps:

(a) Remove all the loops.

(b) Remove the parallel edges keeping only the edge carrying least weight.

(i) Determine all the direct paths from K to all other vertices without going through any other vertex.

Vertices	K	a	b	c	D	L
K	0	4	∞	2	∞	20
a	0	0	7	1	4	∞
b	0	0	0	0	0	1
c	0	1	5	0	6	16
d	0	0	0	0	0	2
L	0	0	0	0	0	0

(ii) Determine shortest paths to all vertices through this vertex and update the values. The closest vertex is c.

Vertices	K	a	b	c	D	L
K	0	3(K,c)	7(K,c)	2K	8(K,c)	18(K,c)

(iii) The vertex which is 2nd nearest to K is 9 which is used to determine the shortest paths and update the values.

Vertices	K	a	b	c	D	L
K	0	3(K,c)	7(K,c)	2K	7(K,c,a)	18(K,c)

(iv) The vertex which is 3rd nearest to K is b, which is used to determine the shortest paths and update the values.

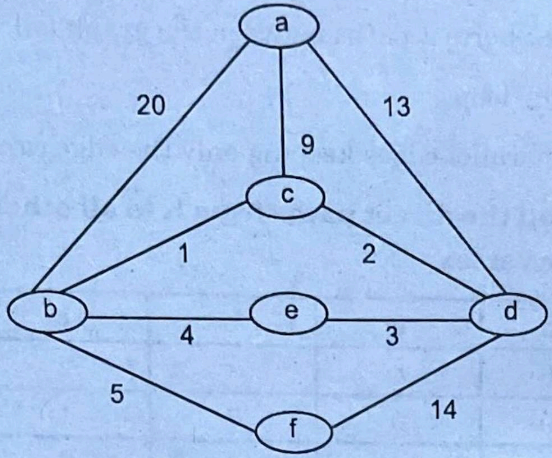
Vertices	K	a	b	c	D	L
K	0	3(K,c)	7(K,c)	2K	7(K,c,a)	18(K,c,b)

(v) The vertex which is 3rd nearest to K is d, which is used to determine the shortest paths and update the values.

Vertices	K	A	b	c	D	L
K(c,a,b,d)	0	3(K,c)	7(K,c)	2K	7(K,c,a)	18(K,c,b)

Thus, the shortest distance between K and L is 8 and the shortest path is K,c,b,L.

Q.22 Use Prim's algorithms to find minimum spanning tree for the following graph G



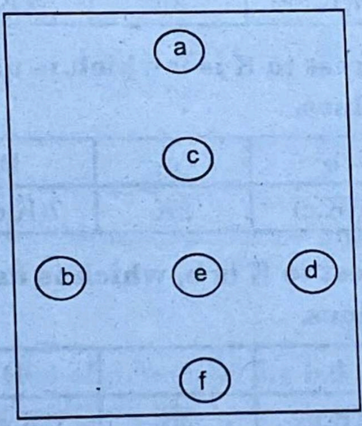
Ans. Prims Algorithm

(a) Initialize the minimal spanning tree with a single vertex, randomly chosen from the graph.

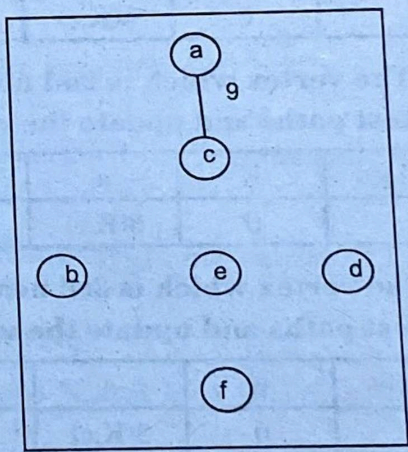
(b) Repeat steps 3 and 4 untill all the vertices are included in the tree.

(c) Select an edge that connects the tree with a vertex not yet in the tree, so that the weight of the edge is minimal and inclusion of the edge does not form a cycle.

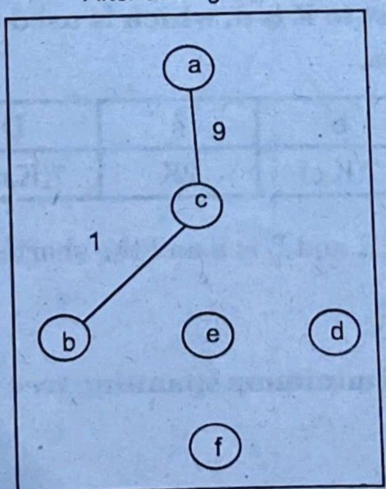
(d) Add the selected edge and the vertex that is connects to the tree.



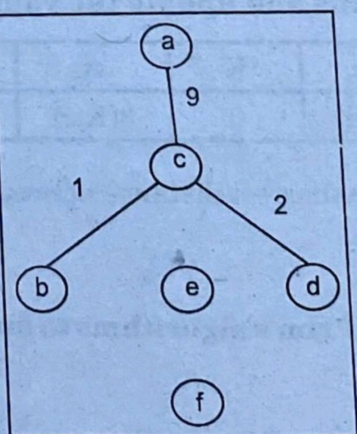
After adding vertex 'a'



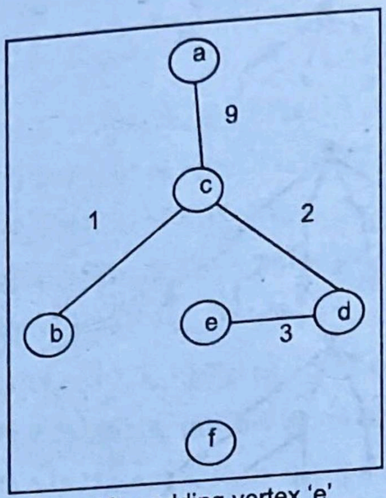
After adding vertex 'c'



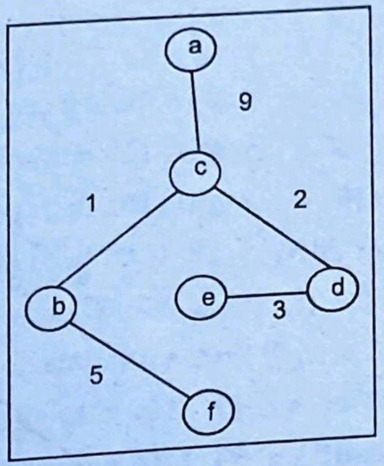
After adding vertex 'b'



After adding vertex 'd'



After adding vertex 'e'



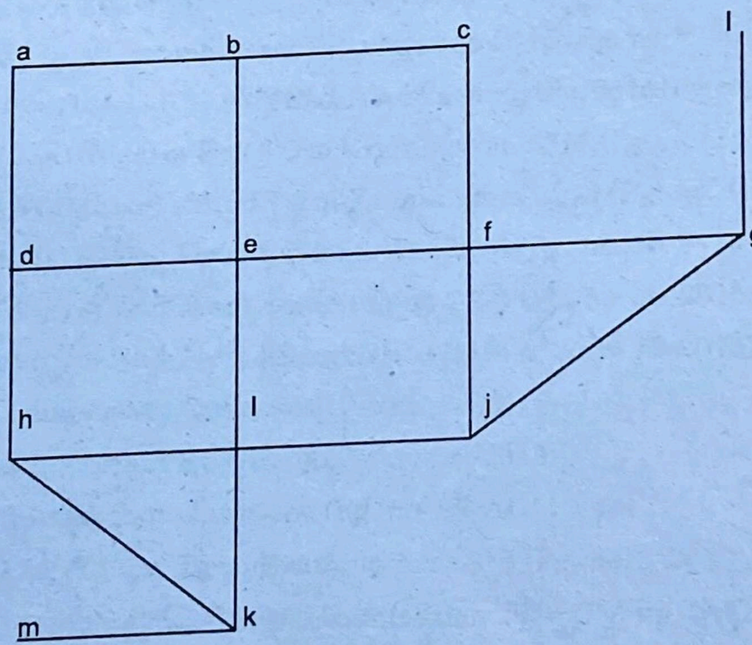
After adding vertex 'f'

The minimum spanning tree is formed carrying the weight as:

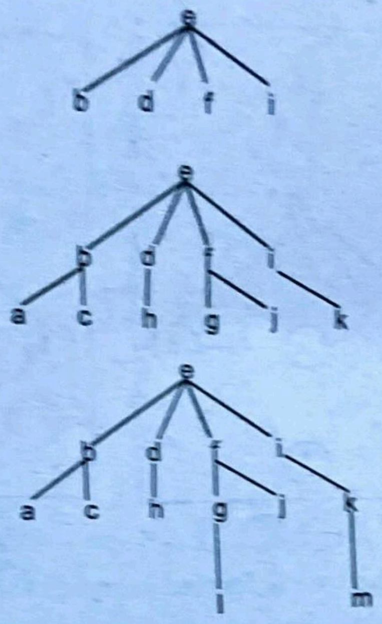
$1 + 2 + 3 + 5 + 9 = 20$

Breadth First search and Depth First Search

Q.23 Use BFS to find a spanning tree for graph given below.



Ans. Choose root vertex = e


e
b d f i
e
b d f i
a c h g j k
e
b d f i
a c h g j k
l m

UNIT - I

Q.1. Refer to Q.1 (a),(c),(d),(e) of First Term Examination 2014 (Pg. no. 1-2014)

Q.2. Refer to Q.2 (a),(b) of First Term Examination 2014 (Pg. no. 2-2014)

Q.3. Refer to Q.3 (a),(b) of First Term Examination 2014 (Pg. no. 2-2014)

Q.4. Refer to Q.4 (a),(b) of First Term Examination 2014 (Pg. no. 3-2014)

Q.5. Refer to Q.3 (a) of Second Term Examination 2014 (Pg. no. 8-2014)

Q.6. Refer to Q.1 (a),(b),(c),(d),(e) of End Term Examination 2014 (Pg. no. 11,12-2014)

Q.7. Refer to Q.2 (a),(b),(c) of End Term Examination 2014 (Pg. no. 14,15-2014)

Q.8. Refer to Q.3 (a),(b),(c) of End Term Examination 2014 (Pg. no. 16-2014)

Q.9. Refer to Q.4 (b),(c) of End Term Examination 2014 (Pg. no. 17,18-2014)

Q.10. Refer to Q.5 (a),(c) of End Term Examination 2014 (Pg. no. 18,19-2014)

Q.11. Refer to Q.6 (b) of End Term Examination 2014 (Pg. no. 19-2014)

Q.12. Refer to Q.1 (a),(b),(c),(d),(e) of First Term Examination 2015 (Pg. no. 1-2015)

Q.13. Refer to Q.2 (a),(b) of First Term Examination 2015 (Pg. no. 2-2015)

Q.14. Refer to Q.3 (a),(b) of First Term Examination 2015 (Pg. no. 2,3-2015)

Q.15. Refer to Q.1 (a) of Second Term Examination 2015 (Pg. no. 5-2015)

Q.16. Refer to Q.1 (a),(b),(c),(d),(e),(f) of End Term Examination 2015 (Pg. no. 12,13-2015)

Q.17. Refer to Q.2 (a),(b),(c) of End Term Examination 2015 (Pg. no. 14,15,16-2015)

Q.18. Refer to Q.3 (a),(b),(c) of End Term Examination 2015 (Pg. no. 16,17-2015)

Q.19. Refer to Q.4 (c) of End Term Examination 2015 (Pg. no. 18-2015)

Q.20. Refer to Q.5 (a) of End Term Examination 2015 (Pg. no. 18-2015)

Q.21. Refer to Q.6 (b) of End Term Examination 2015 (Pg. no. 19-2015)

Q.22. Refer to Q.1 Important Questions (Pg. no. 25-2015)

Q.23. Refer to Q.2 Important Questions (Pg. no. 25-2015)

Q.24. Refer to Q.3 Important Questions (Pg. no. 25-2015)

Q.25. Refer to Q.1 (c) of First Term Examination 2016 (Pg. no. 1-2016)

Q.26. Refer to Q.2 (a),(b) of First Term Examination 2016 (Pg. no. 2-2016)

Q.27. Refer to Q.3 (a) of First Term Examination 2016 (Pg. no. 2-2016)

Q.28. Refer to Q.4 (a),(b) of First Term Examination 2016 (Pg. no. 3,4-2016)

Q.29. Refer to Q.1 (a),(b),(d) of End Term Examination 2016 (Pg. no. 5,6-2016)

Q.30. Refer to Q.2 (b),(c) of End Term Examination 2016 (Pg. no. 6,7-2016)

Q.31. Refer to Q.3 (c) of End Term Examination 2016 (Pg. no. 8-2016)

Q.32. Refer to Q.6 (c) of End Term Examination 2016 (Pg. no. 11-2016)

Q.33. Refer to Q.1 (c),(d) of First Term Examination 2017 (Pg. no. 1-2017)

Q.34. Refer to Q.2 (a),(b) of First Term Examination 2017 (Pg. no. 2-2017)

Q.35. Refer to Q.3 (a),(b) of First Term Examination 2017 (Pg. no. 2,3-2017)

Q.36. Refer to Q.4 (a),(b) of First Term Examination 2017 (Pg. no. 3,4-2017)

Q.37. Refer to Q.1 (a),(c),(e),(f),(i) of End Term Examination 2017 (Pg. no. 5,6,7-2017)

Q.38. Refer to Q.2 (a),(b) of End Term Examination 2017 (Pg. no. 7-2017)

Q.39. Refer to Q.3 (a),(b),(c) of End Term Examination 2017 (Pg. no. 7,8-2017)

Q.40. Refer to Q.4 (a),(b) of End Term Examination 2017 (Pg. no. 8,9-2017)

Q.41. Refer to Q.5 (a),(b),(c) of End Term Examination 2017 (Pg. no. 9,10-2017)

Q.42. Refer to Q.1 (a),(b),(d),(e) of First Term Examination 2018 (Pg. no. 1,2-2018)

Q.43. Refer to Q.2 (a),(b),(c) of First Term Examination 2018 (Pg. no. 2-2018)

Q.44. Refer to Q.3 (a),(b) of First Term Examination 2018 (Pg. no. 2,3-2018)

Q.45. Refer to Q.4 (a),(b) of First Term Examination 2018 (Pg. no. 4-2018)

Q.46. Refer to Q.1 (a),(b),(e),(f),(h) of End Term Examination 2018 (Pg. no. 5,6,7-2018)

Q.47. Refer to Q.2 (a),(b),(c) of End Term Examination 2018 (Pg. no. 7,8-2018)

Q.48. Refer to Q.3 (a),(b),(c) of End Term Examination 2018 (Pg. no. 8,9-2018)

Q.49. Refer to Q.4 (a) of End Term Examination 2018 (Pg. no. 9-2018)

Q.50. Refer to Q.9 Important Questions (Pg. no. 28-2015)

Q.51. Refer to Q.1 (a) of First Term Examination 2016 (Pg. no. 1-2016)

Q.52. Refer to Q.3 (a) of End Term Examination 2016 (Pg. no. 7-2016)

Q.53. Refer to Q.1 (a) of First Term Examination 2017 (Pg. no. 1-2017)

UNIT - II

Q.1. Refer to Q.1 (c) of Second Term Examination 2014 (Pg. no. 5-2014)

Q.2. Refer to Q.2 (b) of Second Term Examination 2014 (Pg. no. 8-2014)

Q.3. Refer to Q.4 (a) of Second Term Examination 2014 (Pg. no. 10-2014)

Q.4. Refer to Q.1 (h) of End Term Examination 2014 (Pg. no. 13-2014)

Q.5. Refer to Q.4 (a) of End Term Examination 2014 (Pg. no. 17-2014)

Q.6. Refer to Q.5 (b) of End Term Examination 2014 (Pg. no. 18-2014)

Q.7. Refer to Q.6 (a),(c) of End Term Examination 2014 (Pg. no. 19,20-2014)

Q.8. Refer to Q.9 (a),(b),(c) of End Term Examination 2014 (Pg. no. 23,24-2014)

Q.9. Refer to Q.4 (b) of First Term Examination 2015 (Pg. no. 4-2015)

Q.10. Refer to Q.1 (b),(c) of Second Term Examination 2015 (Pg. no. 5-2015)

Q.11. Refer to Q.2 (a),(b) of Second Term Examination 2015 (Pg. no. 7,8-2015)

Q.12. Refer to Q.4 (a) of End Term Examination 2015 (Pg. no. 17-2015)

Q.13. Refer to Q.5 (b),(c) of End Term Examination 2015 (Pg. no. 18,19-2015)

Q.14. Refer to Q.6 (a),(c) of End Term Examination 2015 (Pg. no. 19,20-2015)

Q.15. Refer to Q.9 (a) of End Term Examination 2015 (Pg. no. 23-2015)

Q.16. Refer to Q.6 Important Question (Pg. no. 27-2015)

Q.17. Refer to Q.1 (b) of First Term Examination 2016 (Pg. no. 1-2016)

Q.18. Refer to Q.1 (e) of End Term Examination 2016 (Pg. no. 2-2016)

Q.19. Refer to Q.3 (b) of End Term Examination 2016 (Pg. no. 7-2017)

Q.20. Refer to Q.6 (b) of End Term Examination 2016 (Pg. no. 10-2016)

Q.21. Refer to Q.1 (b),(e) of First Term Examination 2017 (Pg. no. 1-2017)

Q.22. Refer to Q.1 (d),(g),(h) of First Term Examination 2017 (Pg. no. 5,6-2017)

Q.23. Refer to Q.6 (a),(b),(c) of End Term Examination 2017 (Pg. no. 10,11-2017)

Q.24. Refer to Q.1 (j) of First Term Examination 2018 (Pg. no. 7-2018)

Q.25. Refer to Q.4 (c) of End Term Examination 2018 (Pg. no. 9-2018)

Q.26. Refer to Q.5 (b) of End Term Examination 2018 (Pg. no. 10-2018)

Q.27. Refer to Q.6 (a) of End Term Examination 2018 (Pg. no. 10-2018)

Q.28. Refer to Q.9 (c) of End Term Examination 2018 (Pg. no. 16-2018)

Q.29. Refer to Q.1 (c) of End Term Examination 2016 (Pg. no. 5-2016)

Q.30. Refer to Q.1 (e) of End Term Examination 2016 (Pg. no. 5-2016)

UNIT - III

Q.1. Refer to Q.1 (b) of Second Term Examination 2014 (Pg. no. 5-2014)

Q.2. Refer to Q.3 (b) of Second Term Examination 2014 (Pg. no. 9-2014)

Q.3. Refer to Q.1 (g) of End Term Examination 2014 (Pg. no. 13-2014)

Q.4. Refer to Q.8 (a),(b),(c) of End Term Examination 2014 (Pg. no. 21,22-2014)

Q.5. Refer to Q.4 (a) of First Term Examination 2015 (Pg. no. 3-2015)

Q.6. Refer to Q.1 (e) of Second Term Examination 2015 (Pg. no. 7-2015)

Q.7. Refer to Q.3 (a),(b),(c) of Second Term Examination 2015 (Pg. no. 8,9,10-2015)

Q.8. Refer to Q.4 (a),(b) of Second Term Examination 2015 (Pg. no. 11-2015)

Q.9. Refer to Q.8 (a),(b) of End Term Examination 2015 (Pg. no. 22-2015)

Q.10. Refer to Q.9 (c) of End Term Examination 2015 (Pg. no. 24-2015)

Q.11. Refer to Q.4 Important Questions (Pg. no. 25-2015)

Q.12. Refer to Q.5 (a),(b),(c) of End Term Examination 2016 (Pg. no. 9,10-2016)

Q.13. Refer to Q.7 (b) of End Term Examination 2016 (Pg. no. 12-2016)

Q.14. Refer to Q.7 (a),(b) of End Term Examination 2017 (Pg. no. 11,12-2017)

Q.15. Refer to Q.8 (b) of End Term Examination 2017 (Pg. no. 15-2017)

Q.16. Refer to Q.9 (a),(b) of End Term Examination 2017 (Pg. no. 15,16-2017)

Q.17. Refer to Q.1 (c),(d) of First Term Examination 2018 (Pg. no. 5,6-2018)

Q.18. Refer to Q.4 (b) of End Term Examination 2018 (Pg. no. 9-2018)

Q.19. Refer to Q.7 (a) of End Term Examination 2018 (Pg. no. 11-2018)

Q.20. Refer to Q.8 (a),(b) of End Term Examination 2018 (Pg. no. 13-2018)

Q.21. Refer to Q.9 (a) of End Term Examination 2018 (Pg. no. 14-2018)

UNIT - IV

Q.1. Refer to Q.1 (a) of Second Term Examination 2014 (Pg. no. 5-2014)

Q.2. Refer to Q.1 (d),(e) of Second Term Examination 2014 (Pg. no. 5,6-2014)

Q.3. Refer to Q.2 (a) of Second Term Examination 2014 (Pg. no. 6-2014)

Q.4. Refer to Q.4 (b) of Second Term Examination 2014 (Pg. no. 10-2014)

Q.5. Refer to Q.1 (f) of End Term Examination 2014 (Pg. no. 13-2014)

Q.6. Refer to Q.7 (a),(d) of End Term Examination 2014 (Pg. no. 20,21-2014)

Q.7. Refer to Q.1 (d) of Second Term Examination 2015 (Pg. no. 6-2015)

Q.8. Refer to Q.7 (a),(b),(c),(d) of End Term Examination 2015 (Pg. no. 20,21-2015)

Q.9. Refer to Q.5 Important Questions (Pg. no. 26-2015)

Q.10. Refer to Q.7 Important Questions (Pg. no. 27-2015)

Q.11. Refer to Q.8 Important Questions (Pg. no. 28-2015)

Q.12. Refer to Q.3 (b) of First Term Examination 2016 (Pg. no. 3-2016)

Q.13. Refer to Q.4 (a),(b) of End Term Examination 2016 (Pg. no. 8,9-2016)

Q.14. Refer to Q.8 (b),(c) of End Term Examination 2016 (Pg. no. 12-2016)

Q.15. Refer to Q.7 (c) of End Term Examination 2017 (Pg. no. 12-2017)

Q.16. Refer to Q.8 (a),(c) of End Term Examination 2017 (Pg. no. 14,15-2017)

Q.17. Refer to Q.9 (c) of First Term Examination 2017 (Pg. no. 16-2017)

Q.18. Refer to Q.1 (g),(i) of End Term Examination 2018 (Pg. no. 7-2018)

Q.19. Refer to Q.6 (b) of End Term Examination 2018 (Pg. no. 11-2018)

Q.20. Refer to Q.7 (b),(c) of End Term Examination 2018 (Pg. no. 12-2018)

Q.21. Refer to Q.9 (b) of End Term Examination 2018 (Pg. no. 14-2018)

FIRST TERM EXAMINATION
THIRD SEMESTER (B.TECH) [ETCS-203]
FOUNDATION OF COMPUTER SCIENCE–SEPT. 2014

M.M. : 30

Time : 1.30 hrs.

Note: *Q.1. is compulsory and answer any 2 more questions.*

Q.1.(a) Show that the proposition $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent? **(2)**

Ans.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

$\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$(\neg p \vee \neg q)$

Above there two tables has the same meaning.

Q.1. (*b*) Determine the contrapositive of the statement " If John is a poet, then he is poor".

Ans. Refer Q.4. (*c*) of First Term Exam 2016.

Q.1.(c) Show that $n[p[p[p[\phi]]]] = 4$.

Ans. As we know that in given equation P mean power set. In fact, the number of elements in Power (S) is 2 raised to the cardinality of S; that is

$$n\,(\text{Power}\,(S)) = 2^{n(S)}$$

So, L.H.S.:

$$p[\phi] = \{\phi\} = 2^0 = 1$$

$$p[p[\phi]] = \{\phi, \{\phi\}\} = 2^1 = 2$$

$$p[p]p[\phi]]] = \{\phi, \{\phi\}, [\{\phi\}\}, \{\phi, \{\phi\}\}\} = 4$$

So,

$$n[p[p[p[\phi]]]] = 4. \text{ Hence proved.}$$

Q.1.(*d*) Explain pigeohole principle? **(2)**

Ans. If n pigeonholes are occupied by $n + 1$ or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.

This principle can be applied to many problems where we want to show that a given situation can occur.

For example: Suppose a department contains 13 professors. Then two of the professors (pigeons) wee born in the same month (pigeon holes)

Q.1.(c) Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of the following statements. **(2)**

(*i*) $(\exists x \in A)\ (x + 3 = 10)$

(*ii*) $(\forall x \in A)\ (x + 3 < 10)$

Ans. (*i*) False. For no number in A is a solution to $x + 3 = 10$.

(*ii*) True. For every number in A satisfies $x + 3 < 10$.

Q.2.(a) Given that

C1 : P→S

C2 : S→U

C3 : P

C4 : U

Show that C4 is a logical consequence of C1, C2 and C3?

Ans. Logical consequence means prove that logical argument or valid argument is formalized:

$(P \to S) \wedge (S \to U) \wedge P \vdash U$

P	S	U	P→S	S→U	(P→S) ^ (S→U) ^ P
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	F
F	T	F	T	F	F
F	F	T	T	T	F
F	F	F	T	T	F

Now P→S, S→U, P are true simultaneously only in the first row of the table, where U is also true. Hence, the argument is valid.

Q.2.(b) Use mathematical induction to prove that $1 + 2 + 3 + 4 \ldots + n = n(n+1)/2$ for any integer $n \geq 1$.

Ans. The proposition holds for $n = 1$ since

$$P(1) : 1 = \frac{1}{2}(1)(1+1)$$

Assuming $P(n)$ is true, we add $(n + 1)$ to both sides of $p(n)$, obtaining

$$1 + 2 + 3 + \ldots + n + (n+1) = \frac{1}{2}n(n+1) + (n+1)$$

$$= \frac{1}{2}\left[n(n+1) + 2(n+1)\right]$$

$$= \frac{1}{2}\left[(n+1)(n+2)\right]$$

Which is $p(n + 1)$. That is, $p(n + 1)$ is true whenever $p(n)$ is true. By the principle of induction, p is true for all n.

Q.3.(a) Test the validity of the following argument: If two sides of a triangle are equal, then opposite angles are equal.

Two sides of a triangle are not equal. Therefore, the opposite angles are not equal.

Ans. Now, Let p : Two sides of a triangle are equal.

q : The opposite angles are equal.

First translate the argument into the symbolic form:

$$\begin{array}{l} p \to q \\ \neg p \\ \hline \neg q \end{array}$$

Therefore, $p \to q, \neg p \vdash \neg q$

p	q	$p \to q$	$\neg p$	$(p \to q) \wedge n \neg p$	$\neg q$	$[(p \to q) \wedge p] \to \neg q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Since the proposition $[(p \to q) \wedge \neg p] \to q$ is not a tautology, the argument is a fallacy. Equivalently, the argument is a fallacy since in third line of truth table $p \to q$ and $\neg p$ are true but $\neg q$ is false.

Q.3.(*b*) Use the method of proof by contrapositive to show that $\sqrt{2}$ is an irrational number.

Ans. Here $p : \sqrt{2}$ is an irrational number. We assume that $\sim p$ is true, that is, $\sqrt{2}$ is not an irrational number. This implies that $\sqrt{2}$ is a rational number. We know that every rational number can be expressed in the form of $\frac{p}{q}(q \neq 0)$, where p and q have no common factor (assuming these are the lowest terms).

Let $\sqrt{2} = \frac{p}{q}$ such that p and q have no common factor

$\Rightarrow \qquad \sqrt{2}q = p$

$\Rightarrow \qquad 2q^2 = p^2$

$\Rightarrow p^2$ is an even number.

$\Rightarrow p$ is an even number (since if p^2 is even, p must be even).

$\Rightarrow p = 2k$ for some integer k.

$\Rightarrow p^2 = 4k^2$

$\Rightarrow q^2 = 2k^2$ (on substituting the value of p^2 in $2q^2 = p^2$).

$\Rightarrow q^2$ is an even number.

$\Rightarrow q$ is an even number.

$\Rightarrow$ 2 is the common factor of a and b.

This is a contradiction that p and q have no common factor. Thus, the assumption '$\sim p$ is true" that is, '$\sqrt{2}$ is not an irrational no.' is false. Hence, $\sqrt{2}$ is an irrational number.

Q.4.(*a*) Give examples of relations R on $A = \{1, 2, 3\}$ having the stated properly.

(*a*) R is both symmetric and antisymmetric.

(*b*) R is neither symmetric nor antisymmetric.

Ans. There are several possible example for each answer. One possible set of examples follows:

(*a*) $R = \{(1, 1), (2, 2)\}$

(*b*) $R = \{(1, 2), (2, 1), (2, 3)\}$

Q.4.(*b*) If R is an equivalence relation on a set X, then prove that R^{-1} is also an equivalence relation. **(6)**

Ans. A relation R on a set A is called an equivalene relation if R is:

- Reflexive
- Symmetric
- Transitive

We can prove given Question statement with the help of example such as:

Example: Consider the following relation on the set $A = \{1, 2, 3, 4\}$. $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$

The relation R is equivalence relation because R is reflexive, symmetric and transitive.

In this case, $R^{-1} = \{(1, 1), (2, 1), (1, 2), (2, 2), (4, 3), (3, 4), (3, 3), (4, 4)\}$.

If you can see that Relation R and Relation R^{-1} than ordering is different but number of element in an pair is same. If we can say that R is an equivalence relation than R^{-1} is also an equivalence relation on a set A.

SECOND TERM EXAMINATION
THIRD SEMESTER (B.TECH) [ETCS-203]
FOUNDATION OF COMPUTER SCIENCE–NOV. 2014

Time : 1.30 hrs. **M.M. : 30**

Note: *Q.1. is compulsory and answer any 2 more questions.*

Q.1.(*a*) Suppose that a connected planner simple graph has 20 vertices, each of degree 3 into how many regions does a representation of this planner graph split the plane? **(2)**

Ans. This graph has 20 vertices, each of degree 3, so $v = 20$. Because the sum of degrees of the vertices, $3v = 60$ is equal to twice the number of edges, $2e$. We have $2e = 60$, or $e = 30$.

Consequently, from Euler's formula, the number of regions is

$$r = e - v + 2 = 30 - 20 + 2 = 12$$

Q.1.(*b*) Define Normal Subgroup and give an example?

Ans. Normal Subgroup: A subgroup H of a group G is called a Normal subgroup of G if $Ha = aH$ for all $a \in G$. Clearly, G and e are normal subgroups of G and referred to as the trivial normal subgroups.

Example: Show that every subgroup of an abelian group is normal.

Solution: Let G be an abelian group and H a subgroup of G. Let $x \in G$ and $h \in H$. Then,

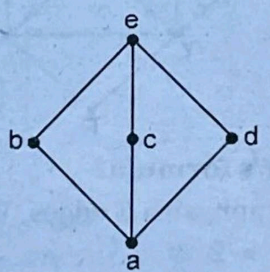
$$\begin{aligned} xhx^{-1} &= xx^{-1}h \text{ (since } G \text{ is and thus } hx^{-1} = x^{-1}h) \\ &= eh = h \in H. \end{aligned}$$

Thus, $x \in G, h \in H \Rightarrow xhx^{-1} \in H$. Hence H is normal in G.

Q.1.(*c*) Define Lattice and give an example? **(2)**

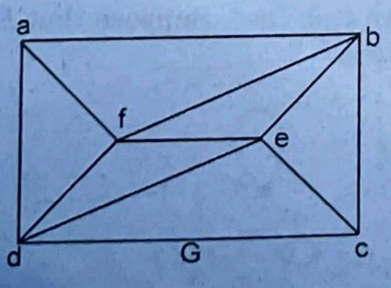
Ans. Lattice: A POSET (L, α) is called a Lattice if every pair of elements in L has an LUB and GLB. The GLB of x and y is called the meet of x and y, and it is denoted by $x \wedge y$. The LUB of x and y is called the join of x and y, and it is denoted by xvy.

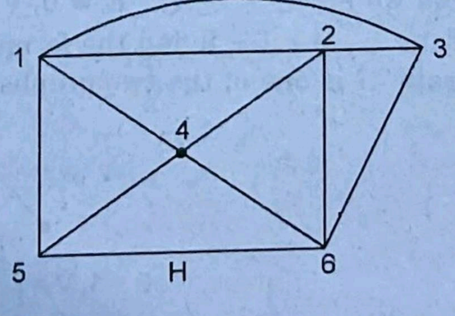
Example:



Posets represent Lattice.

Q.1.(*d*) Define Isomorphic graphs. Determine whether the given pair of graphs is isomorphic.

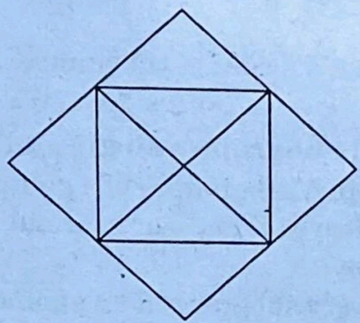




Ans. Isomorphic Graph: The simple graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function f from v_1 to v_2 with the property that a and b are adjacent in G, if and only if $f(a)$ and $f(b)$ are adjacent in G_2 for all a and b in v_1. Such a function f is called an isomorphism.

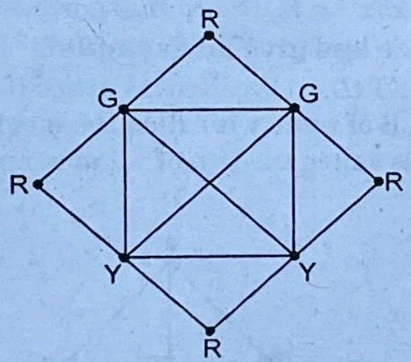
Both G and H have six vertices and Eleven edges. Both have four vertices of degrees four and two vertices of degree three. It is also to see that the subgraphs of G and H consisting of all vertices of degree four and the edges connecting them are isomorphic. Because G and H agree with respect to these invariants, it is reasonable to try to find an isomorphism f.

Q.1.(e) Define Chromatic number of graph. Find the chromatic number of the given graph.



Ans. Chromatic Number: The chromatic nuber of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a Graph G is denoted by $X(G)$. (Here x is the Greek Letter Chi).

⇒ Given graph, the chromatic number is equal to 3.



Q.2.(a) Prove that Euler's formula? **(6)**

Ans. If G is a connected graph with E edges, V verticles and R region, then

$$V - E + R = 2$$

Proof: We shall use induction on the number of edges. Suppose that $E = 0$. Then the graph G consists of a single vertex, say P. Thus, G is as show below:

$$\bullet P$$

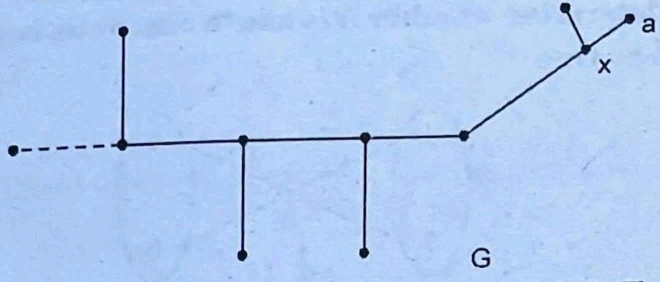
and we have $$E = 0, V = 1, R = 1$$

Thus $1 - 0 + 1 = 2$ and the formula holds in this case. Suppose that $E = 1$. Then the graph G is one of the two graphs shown below:

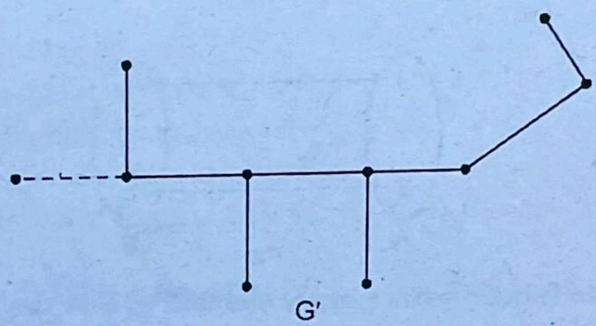
E = 1, V = 2, R = 1 E = 1, V = 1, R = 2

we see that, in either case, the formula holds.

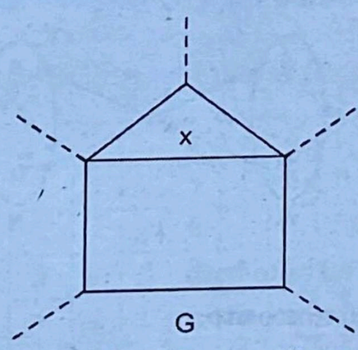
Suppose that the formula holds for connected planar graph with n edges. We shall prove that this holds for graph with $n + 1$ edges. So, Let G be the graph with $n + 1$ edges. Suppose first that G contains no cycles. Choose a vertex V_1 and trace a path starting at V_1. Ultimately, we will reach a vertex a with degree 1, that we cannot Leave.



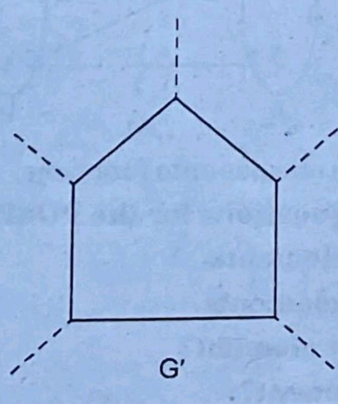
We delete "a" and the edge x incident on "a" from the graph G. The resulting graph G' has n edges and so by induction hypothesis, the formula holds for G'. Since G has one more edge than G', one more vertex than G' and the same number of faces as G', it follows the formula $V - E + R = 2$ holds also for G.



Now suppose that G contains a cycle. Let x be an edge in a cycle as shown below.



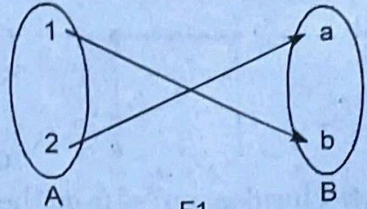
Now the edge x is part of a boundary for two faces. We delete the edges x but no vertices to obtain the graph G' as shown in below:



Thus G' has n edges and so by induction hypothesis the formula holds. Since G has one more faces (region) than G', one more edge than G' and the same number of vertices as G' it follows that the formula $V - E + R = 2$ also holds for G. Hence, by mathematical induction, the theorem is true.

Q.2.(b) Let $A = \{1, 2\}$ and $B = \{a, b\}$. Find all functions $f : A \to B$ and for each such function, determine whether it is one to one, onto, both or neither. (4)

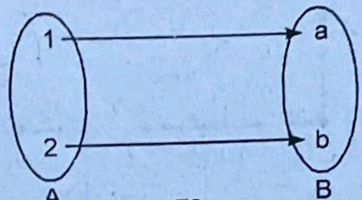
Ans. (i) One-to-One



F1

Here, F1 is one to one since no element of B is the image of more than one element of A.

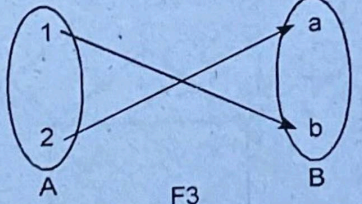
(ii) Onto



F2

Here, F2 is onto function since every element of B is the image under f_2 of some element of A.

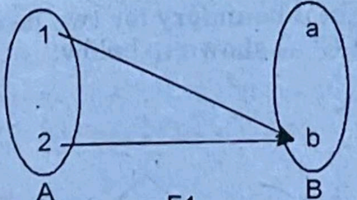
(iii) Both (one-to-one and onto):



F3

Here, F3 is one-to-one and onto both.

(iv) Neither one to one nor onto:



F4

Here F4 is neither one to one nor onto functions.

Q.3.(a) Answer these questions for the POSET ({3, 5, 9, 15, 24, 45}, 1)

(i) Find the maximal elements.

(ii) Find the minimal elements.

(iii) Is there a greatest element?

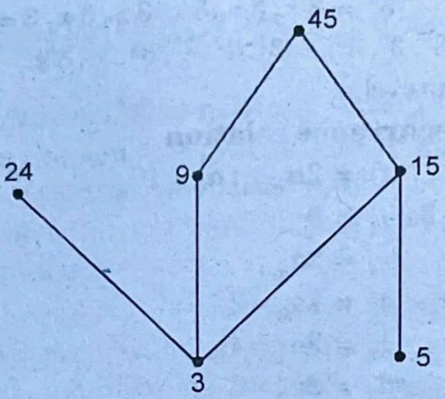
(iv) Is there a least element?

Ans. $S = \{3, 5, 9, 15, 24, 45\}$

$R = \{(3, 9), (3, 15), (3, 24), (3, 45), (5, 15), (5, 45), (9, 45), (15, 45)\}$.

To draw Hasse diagram use the following rules:

- Omit all edges implied by Reflexive property such as (a, a)
- Neglect all edges implied by transitive property such as (3, 9) and (9, 45) then (3, 45) neglected and (5, 15) and (15, 45) then (5, 45) neglected.



(*i*) Maximal element is 45.

(*ii*) Minimal elements are 3 and 5.

(*iii*) The least element does not exist.

(*iv*) The greatest element does not exist.

Q.3.(*b*) Consider the group $G = \{1, 2, 3, 4, 5, 6\}$ under multiplication modules 7 ?

(*i*) Find the multiplication table of G.

(*ii*) Find 2^{-1}, 3^{-1}.

(*iii*) Find the orders and subgroups generated by 2.

(*iv*) Is *G* cyclic?

Ans. (*i*) Let us form the multiplication table of *G*:

x_7	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(*ii*) **For Find Inverse:**

2^{-1}: $2 \, x_7 \, 4 = 1$

So, $\boxed{2^{-1} = 4}$

3^{-1}: $3 \, x_7 \, 5 = 1$

So, $\boxed{3^{-1} = 5}$

(*iii*) **Order of 2:**

$$(2)^1 = 2$$

$$(2)^2 = 2 \, x_7 \, 2 = 4$$

$$(2)^3 = 2\, x_7\, 2\, x_7\, 2 = 1 = \text{identify of } G.$$

$\boxed{0(2)=3}$ and subgroup generated by $2 = \{1, 2, 4\}$.

(*iv*) Let $G = \{1, 2, 3, 4, 5, 6\}$, we can write:

$$3^1 = 3, 3^2 = 3\, x_7\, 3 = 2, 3^3 = 3\, x_7\, 3\, x_7\, 3 = 6$$
$$3^4 = 3\, x_7\, 3\, x_7\, 3\, x_7\, 3 = 4, 3^5 = 3\, x_7\, 3\, x_7\, 3\, x_7\, 3\, x_7\, 3 = 5$$
$$3^6 = 3\, x_7\, 3\, x_7\, 3\, x_7\, 3\, x_7\, 3\, x_7\, 3 = 1$$

Thus, $G = \{3^6, 3^2, 3^1, 3^4, 3^5, 3^3\} = \{3^1, 3^2, 3^3, 3^4, 3^5, 3^6\}$

$\Rightarrow G = <3>$ and so G is cyclic.

Q.4.(*a*) Solve the recurrence relation

$$\boldsymbol{a_n = 2a_{n-1}; a_0 = 1}$$

Ans. $a_n - 2a_{n-1} = 0$

$\Rightarrow$ $a_n = 2a_{n-1}$

$\Rightarrow$ $a_1 = 2a_0 = 2$

$\Rightarrow$ $a_2 = 2a_1 = 4$

$\Rightarrow$ $a_3 = 2a_2 = 8$

Thus, we can write $a_n = 2^n$ for $n \geq 1$.

Verification: For $n = 0, a_0 = 2^0 = 1$ and thus, it is true for $n = 0$.

Let $a_n = 2^n$ be true for $n = k$; that is, $a_k = 2^k$.

Now $a_{k+1} = 2a_k$ (using the recurrence relation)

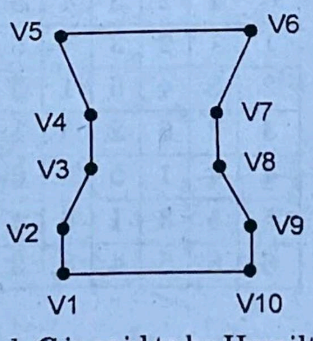
$$= 2.2^k = 2^k + 1$$

Hence, $a_n = 2^n$ is true for all non-negative integers.

Q.4.(*b*) Define Hamiltonian Circuit. Give an example. (5)

Ans. Hamiltonian Circuit is a circuit drawn form G that contains each vertex of G exactly once except the beginning and the ending vertex. Since no vertex except the start vertex is repeated, no edge will repeat.

For example, Figure represent Hamiltonian the circuit $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_1\}$



A circuit in a connected graph G is said to be Hamiltonian if it includes every vertex of G. Hence a Hamiltonian circuit in a graph of n vertices consists of exactly n edges.

If we remove one of the edge from the given circuit say the edge (v_1, v_{10}) then we are left with a path $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ in which we travel along all the vertices exactly once is the Hamiltonian path. Since a Hamiltonian path is a sub–graph of Hamiltonian Circuit which we get by removing an edge from the circuit. Therefore every graph that has a Hamiltonian circuit also has a Hamiltonian path.

Therefore, the length of a Hamiltonian path (if exists) in a connected graph of n vertices and $(n - 1)$ edges.

END TERM EXAMINATION
THIRD SEMESTER (B.TECH) [ETCS-203]
FOUNDATION OF COMPUTER SCIENCE–DEC. 2014

Time : 3.00 hrs. **M.M. : 75**

Note: *Attempt any five questions including Q.No. 1 which is compulsory. Internal Choice is indicated.*

Q.1.(a) Define Predicates and Quantifiers. Give an example for each? **(3)**

Ans. Predicates: The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part the predicate, "is greater than 3" by $p(x)$, where P denotes the predicate "is greater than 3" and x is the variable.

Example: $p(x)$: x is a student.

Quantifiers: When the variables in propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. However, there is another important way, called quantification, to create a proposition from a propositional function. Quantification expresses the extent to which a predicate is true over a range of elements.

For example: $\forall x\, p(x)$: All students are clever.

Q.1.(b) Prove by Contradiction that at least four of any 22 days must fall on the same day of the world.

Ans. Let p be the proposition 'At least 4 of 22 chosen days fall on the same day of the week'. Suppose that $\neg p$ is true. This means that at most of 3 of the 22 days fall on the same day of the week.

Because there are 7 days in a week, this implies that at most 21 days could be chosen because for each of the day of the week, at most 3 of the chosen days could fall on that day. This contradicts that we have 22 days under consideration.

That is, if R is the statement that "22 days are chosen", then we have that $\neg P \rightarrow (R \wedge \neg R)$. So we know P is true.

Q.1.(c) Explain Principle of Inclusion and Exclusion with an example? **(3)**

Ans. Suppose two tasks A and B can occur in n_1 and n_2 ways, where some of the n_1 and n_2 ways may be the same. In this situation, we cannot apply the sum rule, because the same number of ways will be counted twice. In such situations, we apply the inclusion—exclusion principle, which has already been discussed. According to this principle, if A and B are two sets, then the no. of elements in the set AUB is given by

$$n\,(AUB) = n(A) + n(B) - n(A \cap B)$$

This principle holds for any no. of sets. For three sets, it can be stated as follows:

$$n\,(AUBUC) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + (A \cap B \cap C)$$

Example: In how many ways can we select an ace or a heart from a pack of cards?

Ans. There are 4 aces and 13 cards of heart in a pack of cards, and 1 card is common to both. If E_1 is the event of getting an ace and E_2 is the event of getting a heart, then $n(E_1) = 4$

$$n(E_2) = 13 \text{ and } n\,(E_1 \cap E_2) = 1.$$

Thus, the no. of ways in which we can select an ace or heart from a pack of cards is

$$n\,(E_1 \cup E_2) = n\,(E_1) + n\,(E_2) - n\,(E_1 \cap E_2)$$
$$= 4 + 13 - 1 = 16$$

Q.1.(*d*) Give an example for the following:

(*i*) Representing Relations using Matrices.

(*ii*) Representing Relations using Digraphs.

Ans. (*i*) Representing Relations using Matrices: A relation between finite sets can be represented using a zero-one matrix.

$$m_{ij} = \begin{bmatrix} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{bmatrix}$$

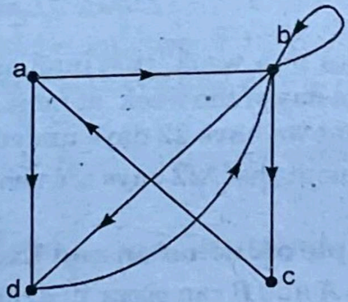
For example: $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1*s* in MR show that the pairs (2, 1), (3, 1) and (3, 2) belong to R. The O_S show that no other pairs belong to R.

(*ii*) Representing Relations Using Digraphs: A directed graph or digraph, consist of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b) and the vertex b is called the terminal vertex of this edge.

Example: The directed graph with vertices a, b, c and d and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b) and (d, b) is displayed in figure below:



Q.1.(*e*) Define Principle of Mathematical Induction? **(3)**

Ans. PRINCIPAL OF MATHEMATICAL INDUCTION: A proof by mathematical induction that $p(n)$ is true for every positive integer n consists of two steps:

1. **Basic step:** The proposition $p(1)$ is shown to be true.
2. **Inductive step:** The implication $p(n) \to p(n + 1)$ is shown to be true for every positive integer n.

   When we complete both steps of a proof by mathematical induction, we have proved that $p(n)$ is true for all positive integers, that is, we have shown than $\forall n\, p(n)$ is true. It is to be noticed that in the proof by mathematical induction it is not assumed that $p(n)$ is true for all positive integers. It is only shown that if it assumed that $p(n)$ is true then $p(n + 1)$ is also true.

For example: To prove that the sum of the first n odd positive integers is n^2.

Let $P(n)$ is the proposition that sum of the first n odd positive integers is n^2.

Step 1: $p(1)$ says, the sum of the first one odd integer is $(1)^2$ which is automatically true as the first odd positive integer is 1.

Step 2: Let us assume that $p(n)$ is true *i.e.*, sum of first n positive odd integer is n^2.

i.e., $1 + 3 + 5 + ... + (2n - 1) = n^2$

Consider $p(n + 1)$, the sum of first $(n + 1)$ positive odd integers:

$1 + 3 + 5 + ... + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$

This implies that $p(n + 1)$ is true.

Therefore, by the principle of mathematical induction the result is true.

Q.1.(*f*) Give the proof for Five Color theorem? **(3)**

Ans. Proof: assume planer g then show colourable 5g

proof (induct rule: graph-measure-induct)

fix g :: α graph

assume IH : ^g′ :: α grpah.size g′ < size $g \Rightarrow$ planer $g' \Rightarrow$ colourable $5g'$.

assume planer g then obtain t where triangulation t and $g < t$ then obtain v where v ε Vt and d : degree $tv \leq 5$ have size $(t \odot v) <$ size t .. also have size t = size g ... also have planer $(t \odot v)$..

ultimately obtain colourable 5 $(t \odot v)$ by (rules dest : IH) from d have colourable $5t$

proof cases

assume degree $tv < 5$ show colourable $5t$ by

next

assume degree $tv = 5$ show colourable $5t$ by

qed

then show colourable 5g ...

qed

qed.

Q.1.(*g*) Mention the axioms to be satisfied in a ring *R*. **(4)**

Ans. Axioms for Ring:

(*i*) Addition is closed.

(*ii*) Addition is Associative

(*iii*) Existence of Additive Identity

(*iv*) Existence of Additive Inverse

(*v*) Addition is Commutative.

(*vi*) Multiplication is closed.

(*vii*) Multiplication is Associative.

(*viii*) Multiplication is distributive over Addition.

First five properties of above definition of a ring says that every ring R is an abelian group under addition or additive abelian group.

Q.1.(*h*) Define Automorphism. Give an example for illustration? **(3)**

Ans. Automorphism: A mapping $f : G \to G$ is called an automorphism if f is homomorphism, one-one and onto.

In other words, a mapping $f : G \to G$ where G is a group under the binary operation '*' is called an automorphism if:

(*i*) f is a homomorphism *i.e.* $f(x + y) = f(x) * f(y)$ for all $x, y \in G$.

(*ii*) f is one-one.

(*iii*) f is onto.

Example: Let $I: G \to G$ be the identity function on G *i.e.* $I(x) = x$ for all $x \in G$. Prove that I is an automorphism.

Solution: (*i*) ***I* is homomorphism:** Let $x, y \in G$ be two arbitrary elements. Then

$$I(x, y) = xy = I(x) I(y)$$

$\therefore$ I is homomorphism.

(*ii*) ***I* is one-one:** Let $x, y \in G$ be any two elements such that

$$I(x) = I(y) \Rightarrow x = y$$

$\therefore$ I is one-one.

(*iii*) **I is onto:** Let $x \in G$ be any element, then $I(x) = x$. So x itself is the pre-image of x under I

$\therefore$ I is onto.

Hence I is an automorphism of G.

Q.2.(*a*) Give an example to illustrate proofs by contraposition and contradiction methods. **(6)**

Ans. Example of Proof by Contraposition: Prove that if n^2 is odd, then n is odd.

If we consider this example, then using direct method of proof it is tedious to prove the statement.

In this case, proof by contrapositive is quite easy.

Here, $P : n^2$ is odd and $Q : n$ is odd.

Thus $\sim P : n^2$ is even and $\sim Q : n$ is even.

To prove the statement $P \to Q$ using the method of contrapositive, we shall take $\sim Q$ as premise.

Let $\sim Q$ is true, that is, n is even.

n is even $\Rightarrow n = 2k$ for some integer p_2.

$\Rightarrow$ $$n^2 = 4k^2 = 2(2k^2)$$

$\Rightarrow n^2$ is even

This shows that $\sim Q \to \sim P$, hence the equivalent statement of this is $P \to Q$, that is if n^2 is odd then n is odd.

Example of proof by Contradiction

$\Rightarrow$ Prove that for all non-negative real numbers x, y and z if $x^2 + y^2 = z^2$, then $x + y \geq z$.

Here, $P : x^2 + y^2 = z^2$ and $Q : x + y \geq z$.

We shall assume that P is true and υQ is true.

Thus $$x^2 + y^2 = z^2 \text{ and } x + y < z.$$

$$x + y < z \Rightarrow (x + y)^2 < z^2$$

$\Rightarrow$ $$x^2 + y^2 + 2xy < z^2$$

$\Rightarrow$ $$x^2 + y^2 < z^2 \text{ (Since } 2xy \text{ is also non-negative real no.)}$$

This is a contradiction to the assumption $x^2 + y^2 = z^2$ thus $x + y < 2$ is not true, that is $x + y \geq z$. This proves that for all non-negative real numbers x, y and z if $x^2 + y^2 = z^2$, then $x + y \geq z$.

Q.2.(b) Mention the Rules of Inference for proportional logic. (2)

Ans.

Rule of Inference	Tautology	Name
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus Ponen
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to p$	Modus tollens
$p \to q$ $z \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical Syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \wedge q) \wedge p] \to q$	Disjunctive Syllogism
p $\therefore pvq$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplication
p q $\therefore p \wedge q$	$[(p) \wedge (q)) \to (p \wedge q)$	Conjunction
pvq $\neg pvr$ $\therefore qvr$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

Q.2.(c) Let m, n be two positive integers. Prove that if m, n are perfect squares, then the product $m * n$ is also a perfect square? (2.5)

Ans. To produce a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely we assume that m and n are both perfect squares.

By definition of a perfect square, it follows that there are integers s and t such that $m = s^2$ and $n = t^2$. The goal of the proof is to show that mn must also be a perfect square when m and n are: Looking ahead we see how we can show this by multiplying the two equations $m = s^2$ and $n = t^2$ together. This shows that $mn = s^2t^2$, which implies that $mn = (st)^2$ (using commutativity and associativity of multiplication). By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of st, which is an integer. We have proved that if m and n are both perfect squares, then mn is also a perfect square.

Q.3.(*a*) Provide a proof by contradiction for the following for every integer n if n^2 is odd, than n is odd. (5)

Ans. This theorem has the form "P if and only if q", where p is "n is odd" and q is "n^2 is odd". To prove this theorem, we need to show that $p \to q$ and $q \to p$ are true.

Sometimes a theorem states that several propositions are equivalent. Such a theorem states that propositions $p_1, p_2, p_3 \cdots p_n$ are equivalent. This can be written as

$$p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n.$$

which states that all n propositions have the same truth values, and consequently that for all i and j with $i \le i < n$ and $1 < j < n, p_i$ and p_j are equivalent. One way to prove these mutually equivalent is to use the tautology.

$$[p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n] \leftrightarrow [(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \ldots \wedge (p_n \to p_1)]$$

This shows that if the conditional statements $p_1 \to p_2, p_2 \to p_3 \cdots p_n \to p_1$ can be shown to be true, then the propositions $p_1, p_2 \ldots, p_n$ are all equivalent. This is much more efficient than proving that

$p_i \to p_j$ for all $i \ne j$ with $1 \le i \le n$ & $1 \le j \le n$.

When we prove that a group of statements are equivalent, we can establish any chain of conditional statements we choose as long as it is possible to work through the chain to go from any one of these statements to any other statement. For example, we can show that p_1, p_2 and p_3 are equivalent by showing that $p_1 \to p_3, p_3 \to p_2$ and $p_2 \to p_1$.

Q.3.(*b*) Let $A = \{1, 2, \ldots, 9, 10\}$. Consider each of the following sentences. If it is a statement, then determine its truth value. If it is a propositional function, determine its truth set.

(*i*) $(\forall x \in A)\ (\exists y \in A)\ (x + y < 14)$

(*ii*) $(\forall x \in A)\ (\forall y \in A)\ (x + y < 14)$

(*iii*) $(\forall x \in A)\ (x + y < 14)$

(*iv*) $(\exists y \in A)\ (x + y < 14)$

Ans. (*i*) The open sentence in two variables is preceded by two quantifiers; hence it is a statement. Moreover he statement is **true**.

(*ii*) It is a statement and it is **false**: if $x_0 = 8$ and $y_0 = 9$, then $x = + y_0 < 14$ is **not true**.

(*iii*) The open sentence is preceded by one quantifier hence it is a propositional function of the other variable. Note that for every $x \in A, x_0 + y_0 < 14$ if and only if $y_0 = 1$, 2, or 3. Hence the truth set is $\{1, 2, 3\}$.

(*iv*) It is an open sentence in x. The truth set is A itself.

Q.3.(*c*) Define Demorgan's Laws. Find the negation of $P \leftrightarrow Q$. (2.5)

Ans. DeMorgan's Laws: It tell us how to negate conjunctions and how to negate disjunctions. In particular, the equivalence $\neg(p \wedge q) = \neg p \wedge \neg q$ tell p as that the negation of a disjunction is formed by taking the conjunction of the negations of the component propositions.

Negation of $P \leftrightarrow Q$

Note that $\sim [P \leftrightarrow Q]$ has exactly same truth values as $\sim [(p \to q) \wedge (q \to p)]$.

where $p \to q = \neg p \vee q$

So, $\sim [(\sim p \vee q) \wedge (\sim q \wedge p)]$

According to Demorgan's Law

$$\sim (p \wedge q) = \sim p \vee \sim q$$

$$\sim (p \vee q) = \sim p \wedge \sim q$$

$[(\sim\sim p \wedge \sim q) \vee (\sim\sim q \wedge \sim p)]$ $[\because \sim\sim p = p]$

$[(p \wedge \sim p) \vee (q \wedge \sim p)]$

Q.4.(*a*) Draw the Hasse diagram representing the partial ordering {(*a*, *b*)/a divides *b*} on (1, 2, 3, 4, 6, 8, 12}.

Ans. First of all, make a relation depends on given statement:

$$R = \{(1, 2), (1, 3), (1, 4), (1, 6), (1, 8), (1, 12), (2, 4), (2, 6), (2, 8), (2, 12), (3, 6), (3, 12), (4, 8), (4, 12), (6, 12)\}$$

Remove transitive property such as $(a, b) \in R$, $(b, c) \in R$ than we can remove $(a, c) \in R$.

So, $R = \{(1, 2), (1, 3), (2, 4), (2, 6), (3, 6), (4, 8), (4, 12), (6, 12)\}$

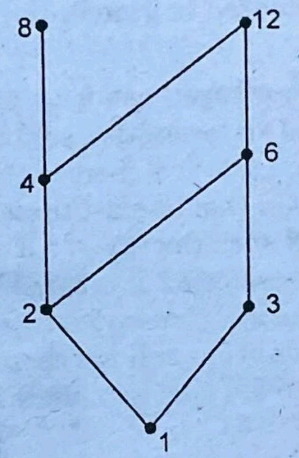


Fig. Hasse Diagram of Partial Ordering

Q.4.(*b*) Let *A*, *B* and *C* be any three subsets of the Universal set *U*. Then prove that:

(*i*) $A - (BUC) = (A - B) \cap (A - C)$

Ans. (*i*) We will prove this identity by showing that each side is a subset of the other side.

Suppose that $x \in A - (BUC)$. Then $x \in A$ and $x \notin (BUC)$. By the definition of union, it follows that $x \in A$ and $x \notin B$ or $x \notin c$. Consequently, we know that ($x \in A$ and $x \notin B$) and that $x \in$ A and $x \notin C$). By the definition of difference, it follows that $x \in A - B$ and $x \in A - C$. Using the definition of Intersection, we conclude that $x \in (A - B) \cap (A - C)$. We conclude that $A - (BUC) \subseteq (A - B) \cap (A - C)$.

Now suppose that $x \in (A - B) \cap (A - C)$. Then, by the definition of Intersection, $x \in (A - B)$ and $x \in (A - C)$. By the definition of Difference, it follows that $x \in$ A and $x \notin B$ and that $x \in$ A and $x \notin C$. From this we see that $x \in$ A and $x \notin B$ or $x \notin C$. Consequently, by the definition of union we se that $x \in A$ and $x \notin B \cup C$. Furthermore by the definition of difference, it follows that $x \in A - (BUC)$. We conclude that $(A - B) \cap (A - C) \subseteq A - (BUC)$. This complete the proof of the identity.

(*ii*) $(A \cap B) - C = A \cap (B - C)$

Ans. Suppose that $x \in (A \cap B) - C$. Then $x \in (A \cap B)$ and $x \notin C$. By the definition of Intersection, it follows that $x \in A$ and $x \in B$ and $x \in C$. Consequently, we know that $x \in A$

and that $x \in B$ and $x \notin C$. By the definition of difference, it follows that $x \in B - C$. Using the definition of Intersection, we conclude that $x \in A \cap (B - C)$. We conclude that $(A \cap B) - C \subseteq A \cap (B - C)$.

Now, suppose that $x \in A \cap (B - C)$. Then, by the definition of Intersection, $x \in A$ and $x \in B - C$. By the definition of difference, it follows that $x \in B$ and $x \notin C$. Consequently, by the definition of Intersection, we see that $x \in A$ and $x \in B$. Furthermore, by the definition of difference, it follows that x $(A \cap B) - C$. We conclude that $A \cap (B - C) \subseteq (A \cap B) - C$. This complete the proof of the identity.

Q.4.(c) What is the power set of the set {0, 1, 2}? **(2.5)**

Ans. The power set $P(\{1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$P(\{0, 1, 2\}) = \{\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$

Q.5.(a) Using Pigeonhole principle calculate the following:

(i) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

(ii) How many must be selected to guarantee that at least three hearts are selected? **(5)**

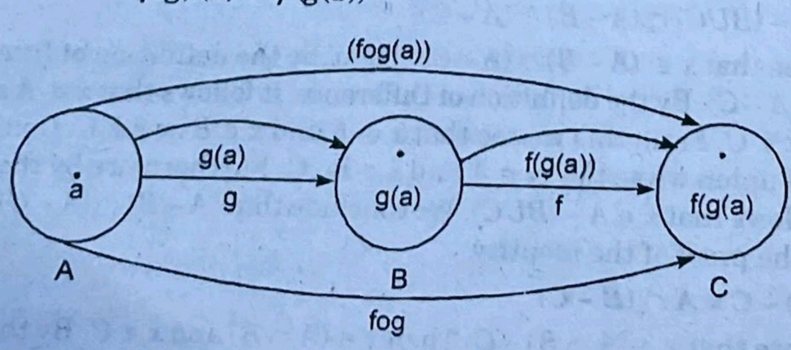
Ans. (i) Suppose there are four boxes, one for each suit, and as cards are selected they are placed in the box reserved for card of that suit. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing at least $[N/4]$ cards. Consequently we know that at least three cards of one suit are selected if $(N/4) \geq 3$, The smallest integer N such that $(N/4) \geq 3$ is $N = 2.4 + 1 = 9$, so nine cards suffice. Note that if eight cards are selected, it is possible to have two cards of each suit, so more than eight cards are needed. Consequently, nine cards must be selected to guarantee that at least three cards of one suit are chosen. One good way to think about this is to note that after the eighth card is chosen, there is no way to avoid having a third card of some suit.

(ii) We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three heards, not just three cards of one suit. Note that in the worst case, we can select all the clubs, diamonds and spades, 3g cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three heards.

Q.5.(b) Define Composition of Functions. Prove that Composition of functions is not commutative. **(5)**

Ans. Composition of Functions: Let g be a function from the set A to the set B and set f be a function from the set B to the set C. The composition of functions of and g, denoted by $f \circ g$, is defined by

$$(fog)(a) = f(g(a))$$



Composition of the Functions f and g

Composition of Functions is not commutative: Let us consider f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

$$(fog)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$
$$(gof)(x) = g(f(x)) = g(2x+3) = 3(2x + 3) + 2 = 6x + 11$$

In above case fog and gof are not equal. In other words, the commutative law does not hold for the composition of functions.

Q.5.(c) Is the POSET $\{Z^+, 1\}$ a Lattice? **(2.5)**

Ans. Yes, POSET $(Z^+, 1)$ is a lattice. A POSET (S, α) is a Lattice if for any items x and y there is a unique LUB and a unique GLB.

In this case Z^+ defines set of positive integers which can start from $\{1, 2, 3...\}$. So, when there exist LUB and GLB than POSET is a Lattice.

Q.6.(a) Develop a general explicit formula for a Homogeneous recurrence relation of the form $a_n = ra_{n-1} + sa_{n-2}$ where r and s are constant?

Ans. A homogeneous recurrence relation

$$a_n = ra_{n-1} + sa_{n-2}$$

can be written in the form

$$a_n - ra_{n-1} - sa_{n-2} = 0 \quad ...(1)$$

We associate the quadratic polynomial $x^2 - rs - s$ with it. The polynomial $x^2 - rs - s$ is the characteristics polynomial of the recurrence relation. For example, $x^2 - 3x + 8$ is characteristics polynomial of recurrence relation $a_n = 3a_{n-1} + 8a_{n-2}$. Here the basic approach for solving Linear Homogeneous recurrence relation is to compute solutions of the form $a_n = r^n$. Here $a_n = r^n$ is a solution of recurrence relation.

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + ... + C_k a_{n-k} \quad ...(2)$$

if and only if

$$r^n = C_1 r^{n-1} + C_2 r^{n-2} + ... + C_k r^{n-k} \quad ...(3)$$

Where r is constant.

Let us divide both sides of this equation (3) by r^{n-k} and the right hand side is subtracted from the left, we get

$$r^k - C_1 r^{k-1} + C_2 r^{k-2} - ... - Ck_{-1} r - C_k = 0 \quad ...(4)$$

Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution of the above Eq. (4). The Eq. (4) is called the characteristic Eq. of the recurrence relation (Eq. (2). The solutions of this equation are called characteristic roots of recurrence relation. For recurrence relation of order two. Let x_1 and x_2 be roots of the polynomial $x^2 - rx - s$. Then the solution of recurrence relation.

$$a_n = ra_{n-1} + sa_{n-2}, n \geq 2$$
$$a_n = C_1 x_1^n + C_2 x_2^n \text{ if } x_1 \neq x_2$$
$$= C_1 x^n + C_2 n x^n \text{ if } x_1 = x_2 = x$$

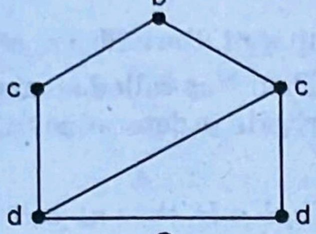
Q.6.(b)(i) Prove by Mathematical Induction. For every positive integer n, the expression $2^{n+2} + 3^{2n+1}$ is divisible by 7. **(5)**

Ans. Step 1: Prove for $n = 1$

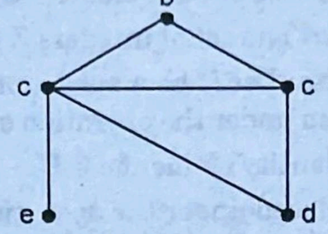
$2^{n+2} + 3^{2n+1} = 2^{1+2} + 3^{2(1)+1} = 2^3 + 3^3 = 8 + 27 = 35$

which is divisible by 7.

Q.7.(c) Show that the graphs displayed in the following figures are not isomorphic. **(5)**



G



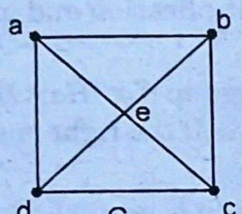
H

Ans. Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic.

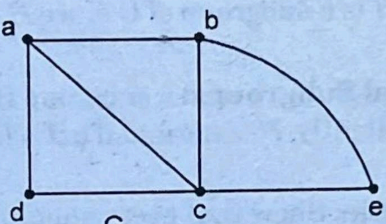
Q.7.(d) Define Euler and Hamiltonian paths in a graph. **(2.5)**

Ans. Euler Path: An Euler Path in G is a simple path containing every edge of G.

For example:



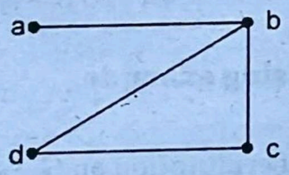
G_2



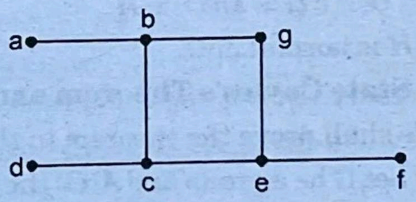
G_1

G_2 does not have an Euler path. However, G_1 has an Euler path, namely, a, c, d, e, b, d, a, b.

Hamilton Path: A simple path in a Graph G that passes through every vertex exactly once is called a Hamilton path.



G1



G2

G_1 has a Hamilton path, namely, a, b, c, d. G_2 has not a Hamilton path.

Q.8.(a) If f is a homomorphism from a commutative semigroup $\{S, *\}$ onto a semigroup $\{T, *\}$, then $\{T, *\}$ is also commutative. **(2.5)**

Ans. Consider two semigroups $(s, *)$ and $(s', *')$. A function $f: s \to s'$ is called a semigroup homomorphism or simply a homomorphism if

$$f(a * b) = f(a) *' f(b) \text{ or simply } f(ab) = f(a) f(b)$$

Suppose f is also one-to-one and onto. Then f is called an isomorphism between S and S' and S and S' are said to be isomorphic semigroups, written $S = S'$.

Q.8.(b) Define Groups, sub-groups and Normal Sub-groups. Give an example for each.

Ans. Group: A set G together with a binary operation $*$ is called a group if it satisfies the following properties:

1. G is closed w.r.t the binary operation $*$.

2. G is associative w.r.t. the binary operation *.
3. There exists an identity element in G w.r.t. the binary operation *.
4. The inverse of each element $a \in G$ exists in G.

Example: The set of integers Z form an Group w.r.t. the addition of integers.

Subgroup: Let H be a subset of a group G. Then H is called a subgroup of G if H itself is a group under the operation of G. Simple criteria to determine subgroups follow.

(*i*) The identity element $e \in H$.

(*ii*) H is closed under the operation of G, *i.e.* if $a, b \in H$, then $ab \in H$ and

(*iii*) H is closed under inverses, *i.e.*, if $a \in H$, then $a^{-1} \in H$.

Example: Consider the group G of 2×2 matrices with rational entries and non-zero determined. Let H be the subset of G consisting of matrices whose upper-right entry is zero; *i.e.* matrices of the form

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$$

Then H is a subgroup of G since H is closed under multiplication and inverses and $I \in H$.

Normal Subgroups: A subgroup H of G is normal subgroup if $a^{-1}Ha \leq H$ for every $a \leq G$. Equivalently, H is normal if $aH = Ha$ for every $a \in G$, *i.e.*, if the right and left cosets coincide.

Example: Show that every subgroup of an abelian group is normal.

Sol. Let G be an abelian group and H a subgroup of G. Let $x \in G$ and $h \in H$. Then,

$$\begin{aligned} xhx^{-1} &= xx^{-1}h \qquad \text{(since } G \text{ is and thus } hx^{-1} = x^{-1}h) \\ &= eh = h \in H \end{aligned}$$

Thus $x \in G, h \varepsilon H \Rightarrow xhx^{-1} \in H$.

Hence, H is normal in G.

Q.8.(*c*) State Cayles's Theorem and Explain using example. (4)

Ans. We shall prove the theorem in three steps.

Step 1: Let G be a group and $A(G)$ the group of all permutation on G. *i.e.* $A(G)$ is the set of all one-one onto mapping of G to G. For each $A \in G$, we define the mapping.

$fa: G \rightarrow G$ by setting $fa(x) = ax \ \forall x \varepsilon G$

We claim that fa is a permutation on G.

***(i) fa* is one-one:** Let $x, y \in G$ be any two element wrt.

$$fa(x) = fa(y) \Rightarrow ax = ay \Rightarrow x = y \ \forall \ x, y \in G$$

***(ii) fa* is onto:** For any $g \in G$ we have

$$g = e.g. = a(a^{-1}g) = ag_1 \text{ where } g_1 = a^{-1}g \in G$$

Using (1), we have $fa(g_1) = g \Rightarrow g_1$ is a preimage of g under $fa \Rightarrow fa$ is onto.

Since fa is one-to-one and onto fa is permutation of G. Thus $fa \in A(G) \ \forall a \in G$.

Step II: Let we claim that G' is a subgroup of $A(G)$ let $f_{a,} f_b \in G'$ be any two elements. To prove that G' is a subgroup of $A(G)$, it is sufficient to prove that $f_a.f_b \in G'$ and $(f_a)^{-1} \in G'$.

For any $x \in G$ we have

$$(f_a.f_b)(x) = f_a[f_b(x)] = f_a(bx)$$

$$= a(bx)$$
$$= (ab)\,x = f_{ab}(x)$$

$\Rightarrow$ $f_a.f_b = f_{ab}.$

since $ab \in G, f_{ab} \in G'$ *i.e.* $f_a.f_b \in G'$

Now, we show that $(f_a)^{-1} = fa^{-1}$ for all $fa \in G'$

Using $b = a^{-1}$ in (2) we have

$$fa.fa^{-1} = faa^{-1} = fe = I$$

Where $fe(x) = ex = x \forall x \in G$ *i.e.* $fe = I$ = identity function similarly $fa^{-1}.f_a = I$.

Thus $(fa)^{-1} = fa^{-1}$. Since $a^{-1} \in G, fa^{-1} \in G' \Rightarrow (fa)^{-1} \in G'$

Hence G' is a subgroup of $A(G)$.

Step III: Finally we prove that $G \cong G'$.

Define a mapping $\varphi : G \to G'$ by setting $j(a) = f_a \; \forall \; a \in g$

(*i*) φ is homomorphism: let $a, b \in G$ be arbitrary elements, then we have

$$\varphi\,(ab) = f_{ab} = f_a.f_b \qquad \text{[using (2)]}$$
$$= \varphi(a).\varphi(b)$$

(*ii*) φ is one to one: let $a, b \in G$ be any two element such that

$$\varphi\,(a) = j(b)$$
$$\Rightarrow \quad f_a = f_b$$
$$\Rightarrow \quad fa(x) = fb(x) \; \forall \; x \in G$$
$$\Rightarrow \quad ax = bx \qquad \text{[using (1)]}$$
$$\Rightarrow \quad a = b \qquad \text{[By cancelation law in } G]$$

(*iii*) φ is onto: Let $fa \in G'$ be any element, then $a \in G$ and $\varphi(a) = fa$

$\Rightarrow a$ is a pre image of f_a

$\Rightarrow \varphi$ is onto.

Thus $\varphi : G \to G'$ is isomorphism and therefore $G \cong G'$.

Since G' consists of permutations group and so we can say that G is isomorphism to a permutation group.

Q.9.(*a*) Give an example to represent and minimize the Boolean function.

Ans. Let take an example

$$E = x'z' + xy + x'y' + yz'$$

Step I: Expres each prime implicant of E as a complete sum of product to obtain.

$$x'z' = x'z'(y + y') = x'yz' + x'y'z'$$
$$xy = xy(z + z') = xyz + xyz'$$
$$x'y' = x'y'(z + z') = x'y'z + x'y'z'$$
$$yz' = yz'\,(x + x') = xyz' + x'yz'$$

Step 2: The summands of $x'z'$ are $x'yz + x'y'z'$ which appear among the other commands. Thus delete $x'z'$ to obtain.

$$E = xy + x'y' + yz'$$

The summands of no other prime implicant appear among the summands of the remaining prime implicants, and hence this is a minimal sum of products form for E. In other words, none of the remaining prime implicants is superflows, that is, none can be deleted without changing E.

Q.9(b) Prove langrange's theorem? (5)

Ans. Langrange's theorem: The order of each subgroup of a finite group is a divisor of the order of the group.

Proof: Let G be a finite group of order n. Let H be a subgroup of G and let $O(H) = m$. Let $H = \{h_1, h_2 \dots h_m\}$ for $a \in G$, Ha is the right coset of H in G, given by $Ha = \{h, a, h_2a, \dots hma\}$ clearly Ha has m distinct members, since if $h_{ia} = h_{ja}$

$\Rightarrow$ $h_i = h_j$ not possible.

Hence each rigid coset of H in G has m distinct members. Any two distinct right coset of H in G are disjoint. *i.e.* they have no element in common. Now let H has k distinct right coset in G namely $H_{a1}, H_{a2} \dots H_{ak}$. are disjoint and their union is G so, we must have

Number of elements in G = Number of element in Ha_1 + Number of element in Ha_2 ... element in H_{ak}.

$$\Rightarrow \quad n = m + m + \dots + m \text{ (} k \text{ times)}$$

$$\Rightarrow \quad n = m.k$$

$$\Rightarrow \quad m \text{ divides } n.$$

$$\Rightarrow \quad O(H) \text{ divides } O(G).$$

Q.9.(c) Show that in a ring R: (i) $\{-a\}\{-b\} = ab$

(ii) $\{-1\}\{-1\} = 1$

if R has an identity element 1.

Ans. (*i*) Using $b + (-b) = (-b) + b = 0$, we have

$$ab + a(-b) = a(b + (-b)) = a.0 = 0$$

$$a(-b) + ab = a(-b) + b) = a.0 = 0$$

Hence $(-a)(-b)$ is the negative of $(-ab)$ that is $(-a)(-b) = ab$.

(*ii*) We have

$$a + (-1)a = 1.a + (-1)a = (1 + (-1))a = 0.a = 0$$

$$(-1)a + a = (-1)a + 1.a = ((-1) + 1)a = 0.a = 0$$

Hence $(-1)(-1)$ is the negative of (-1); that is, $(-1)(-1) = 1$.

FIRST TERM EXAMINATION [SEPT. 2015]
THIRD SEMESTER [B.TECH]
FOUNDATION OF COMPUTER SYSTEM
[ETCS–203]

Time. 1.30 Hours **M.M. : 30**

Note: Q. No.1 is compulsory. Attempt any two from the rest of the questions.

Q.1. (a) What is partition of a set. Explain with examples.

Ans. A partition of a set A is a collection of disjoint an empty subject have A as their union. st. $Ai \subseteq A\ i \in I$.

$$\left.\begin{array}{l}(i) A :\notin \forall i \in I \\ (ii) Ai \cap Aj \neq \phi; i \neq j \\ (iii) U_{i \in I} A\ i \neq A\end{array}\right\} \text{Properties}$$

e.g: {1,2,3} has 5 partitions

(1) {{1,2}, {3}; (2) {{1}, {2}, {3}}; (3) {{1,3} {2}}; (4) {{1}, {2,3}}; (5) {{1,2,3}}

Q.1. (b) Bracket the formulas to correctly interpret.

(i) $p \to q \to \sim p \vee q$

(1) Bracketting: implies sign

(2) Then negation

(3) Then negation with OR

(4) To get the final result propositional logic.

Ans. $(r \to q) \leftrightarrow ((\sim p) \vee q)$

(ii) $p \vee q \wedge r \sim \to \sim p \vee q \to P \wedge T$

(iii) Solving equivalence

Now first solve If- then

so first evaluate OR alongwith A with

$\Rightarrow ((p \vee q) \wedge r) \to (\sim p \vee q) \leftrightarrow (p, r)$

Q.1. (c) Prove sum of 2 odd intgers is even.

Ans. Let the two odd intergs be x and y

$$\text{sum}(s) = x + y = x' + 1 + y' + 1$$

where $x' + 1 = x$ and $y' + 1 = y \Rightarrow x'\ y' \in I$

$\Rightarrow$ $s = x' + y' + 2$ **s is a even number**

Hence proved.

Q.1. (d) How many no's between 4000 and 9000 can be formed using digits 2,4,7,9, if each digit may be repeated.

Ans. We have only 3 choices for 1st place.

Place $\overline{1}\ \overline{2}\ \overline{3}\ \overline{4}$

for 2nd place any numbers among 4 can occurs choice = 4

for 3rd place any numbers among 4 can occurs choice = 4

For 4th place nay numbers among 4 occurs choice = 4

($\because$ Repetition of digits are allowed)

$\Rightarrow$ total options possibilities $= 3 \times 4 \times 4 \times 4 = 192$

Q.1. (e) Give matrix representation of relation R on set A = {a, b, c, d} and B = (1,2,3) R = {(a, 1), (a, 3), (b, 2) (b, 3), (c, 1), (d, 1) (d, 2), (d, 3)}.

Ans. $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3) (c, 1), (c, 2), (c, 3), (d, 1) (d, 2), (d, 3)\}$

$\because R = \{(a,1), (a, 3), (b, 2), (b, 3), (c, 1), (d,1), (d, 2), (d, 3), (d, 4)\}$

$\therefore$ Using matrix representation for the above

$$R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Q.2. (a) What is PNCF and PDNF. Derive PDNF for $(\neg p \vee \sim q) \rightarrow (p \leftrightarrow \sim q)$ without constructing truth table.

Ans. A product of the variables and their negations in formula is called an elementary product A sum of elementary product. A sum of elementary products is called PDNF.

$$(p \rightarrow q) \wedge \sim q = (\sim p \wedge \sim q) \vee (q \wedge \sim q) \text{ is PDNF}$$

PCNF:

A formula which consists of a product of elementary sums is called PCNF.

Eg. $(p \rightarrow q) \wedge \sim q$

$$\equiv (\sim p \vee q) \wedge \sim q \text{ is in PCNF}$$

$$(\sim p \vee \sim q) \rightarrow (p \leftrightarrow \sim q)$$

$$= \sim (\sim p \vee \sim q) \vee (p \leftrightarrow \sim q) \text{ (solving If then)}$$

$$= (p \wedge q) \sim [(\sim p \wedge q) \vee (p \wedge \sim q)] \text{ (solving Iff)}$$

$$= q \vee (p \wedge \sim q) \text{ (Using And and OR simplification)}$$

$$= q \vee p \text{ (Idempotent law)}$$

Q.2. (b) Using rule of inference prove that s is a valid conclusion from premises $p \rightarrow q, p \rightarrow r, \sim (q \wedge r), (s \vee p)$

$p \rightarrow q$ (Given)

$p \rightarrow q$

$p \rightarrow r$

$\sim (q \wedge r) = (\sim q \vee \sim r)$ (De Margan's Law)

$(s \vee p)$

s (To Prove)

$p \rightarrow q$

$\sim q$

$\sim p$ (Modus Tollens)

$s \vee p \Rightarrow s$ or p (property)

$\Rightarrow$ $s \rightarrow r$

s

s (Modus Pones)

Hence proved.

Q.3. (a) Let $\sqrt[3]{3}$ is rational by indirect proof of contradiction.

Ans. $$\sqrt[3]{3} = \frac{a}{b} \Rightarrow 3b^3 = a^3$$

$$\Rightarrow \quad b^3 = \frac{a^3}{3} \Rightarrow a^3 \text{ is divisible by 3}$$

$\Rightarrow$ 3 divides $a^3 \Rightarrow$ 3 divides a.

$$\Rightarrow \qquad a = 3k \text{ (say)}$$
$$3b^3 = (3k)^3$$
$$b^3 = 9k^3 \Rightarrow 3 \text{ divides } b$$

Contradition occurs here. $3\sqrt{3}$ is rational numbers Hence proved.

Q.3. (*b*) (*i*) What is pigeon-hole principle? Give its proof.

Ans. Theorem: If m pigeons are put into m pigeon holes these is an empty hole iff there is a hole with more than 1 pigon in it.

Proof: Let $n < m$

E a function (f) ; $N_m = \{1, 2, 3, ..., m\}$ to $N_n = \{1, x, ... n\}$

$$\forall\, k \in N_m$$
$$f(k) \in N_n$$

Let no element of N_n is associated with >1 element of Nm

$$\therefore \qquad i\, j \in N_m \text{ and } i \neq j \Rightarrow f(i) \neq f(j)$$
$$f(N_m) \subseteq N_n$$

This contradicts that $n < m$

Hence proved

Q.3. (*b*) (*ii*) How many permutations can be made with letters of word CONSTITUTION when consonants and vowels accur alternately?

Ans.

CONSTITUTION (12 letters)

5 vowels 7 consonants

Consider the sequence of letters in the bag as a single set and arrange it.

No. of ways of doing this = 4. Coming to the no. different sets of vowels arrangements possible

$$0 - 0 - I - I - U$$

$\Rightarrow$ No of ways in which vowels can be arranged $\frac{\lfloor 5}{\lfloor 2 \lfloor 2}$.

Coming to the no of ways for consonants

$$\boxed{- - o - o - I - I - U -}$$

$$= \frac{\lfloor 7}{\lfloor 3 \lfloor 2}$$

Combining all the above we get

$$= 4 \times \frac{\lfloor 5}{\lfloor 3 \lfloor 2} \times \frac{\lfloor 7}{\lfloor 3 \lfloor 2} = \boxed{10 \lfloor 7}$$

Q.4. (*a*) Give the hasse diagram of D_{12} if $D_n = \{x : x/n$ such that $x \in N\}$.

Ans. Consider $D_{12} = (p, \leq)$

where
$$P = \{c : c \text{ divides } 12\}$$
$$P = \{1, 2, 3, 4, 6.12\}$$

$x \le y$ in p if x divides y

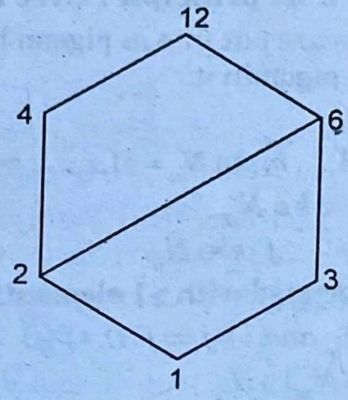
$1 < 2, 3, 4, 6, 12$

$2 < 4, 6, 12$

$3 < 6, 12$

$4 < 12$

$6 < 12$



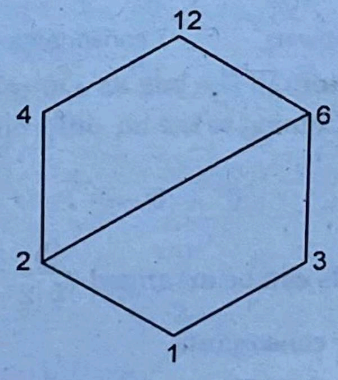
Hasse Diagram

Q.4. (b) What is Lattice? Explain, least upper bound and Greatest lower band?

Ans. A lattice consists of a partially ordered set POSET in which every 2 elements have a unique supremum and a unique infimum.

LUB: Least Upper Bound (LUB) is defined as the least element that is present in the upper bound set. It should be connected to all elements below it.

In D_{12} Hasse Diagram.



6 is LUB, so LUB = {6}

GLB: Greatest lower Bound is the greatest level element that is present in LB set. GLB = {1} .

SECOND TERM EXAMINATION [NOV. 2015]
THIRD SEMESTER [B.TECH]
FOUNADTION OF COMPUTER SYSTEM
[ETCS-203]

Time. 1.30 Hours M.M. : 30

Note: Q. No.1 is compulsory. Attempt any two the rest of the questions.

Q.1. (a) Prove by mathematical induction: for all $n \geq 1$, $n^3 + 2n$ is a multiple of 3.

Ans.

When
$$n = 1$$
$$s_1 : (1^3 + 2) = 3 \text{ which is divisible by } 3$$

Hence true for $n = 1$

Let
$$s_n = n^3 + 2^n \text{ is divisible by 3 therefore 3k} \quad ...(1)$$
$$s_{n+1} = (n+1)^3 + 2(n+1)$$
$$= n^3 + 2n + (3n^2 + 3n + 3)$$
$$= 3k + 3(n^2 + n + 1) \qquad \text{Using...1}$$
$$s_{n+1} = 3(k + n^2 + n + 1)$$

Hence $(n + 1)^3 + 2(n + 1)$ is divisible by 3

True for s_{n+1}

i.e. the inductive step is true

$\therefore s_n$ is true for $n \geq 1$.

Q.1. (b) Identify homogeneous and non-homogenous recurrence relation:

(i) $a_n - \sqrt{a_{n-1}} + (a_{n-1})^2 = 0$

Ans. It is non-linear homogeneous recurrence relation since it can be expresses as $f(n) = 0$

(ii) $a_n - 5a_{n-1} + n(a_{n-2}) = 0$

Ans. It is homogeneous recurrence relation but not with constant coefficient

(iii) $a_n = \sin a_{n-1} + \cos a_{n-2} + \sin a_{n-3} + \cos a_{n-4} + --- + e^n$

Ans. It is non-homogeneous recurrence relation since it is not of the form $f(n) = 0$ rather.

$a_n - \sin a_{n-1} - \cos a_{n-2} - \sin a_{n-3} - \cos a_{n-4} \text{ -------- } = e^n$

(iv) $a_n = a_{n-1} + a_{n-2} + a_{n-3} + ...a_0$

Ans. $a_n - a_{n-1} - a_{n-2} - a_{n-3} \text{ ------} a_0 = 0$

It is homogeneous recurrence relation since it is of the form $f(n) = 0$

Q.1. (c) What is Generating function? Explain with example.

Ans. The generating function of a sequence $a_0, a_1 a_2$ ...... is the expression

$$G(x) = a_0 + a_1x + a_2x^2 + ...\infty = \sum_{n=0}^{\infty} a_n x^n$$

For example:

(*i*) The generating function for the sequence 1, 1, 1, 1... is given by

$$G(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(*ii*) The generating function for the sequence 1, 2, 3, 4... is given by

$$G(x) = \sum_{n=0}^{\infty}(n+1)x^n = 1+2x+3x^2+\ldots.. = \frac{1}{(1-x)^2}$$

(*iii*) The generating function for the sequence $1, a, a^2, a^3$ .... is given by

$$G(x) = 1+ax+a^2x^2\ldots\infty\frac{1}{1-ax} \text{ for } |ax|<|$$

To solve a recurrence relation (both homogeneous and non homegeneous) with given initial conditions, we shall multiply the relation by an appropriate power of x and sum up suitably so as to get an explicit formula for the associated generating function. The solution of the recurrence relation a_n is then obtained as the coefficient of x^n in the expansion of the generating function. The procedure is explained clearly in example below

To solve $a_n = 3a_{n-1}+1\,;\, n \geq 1$ given $a_0 = 1$

Let the generating function of $\{a_n\}$ be $G(x) = \sum_{n=0}^{\infty} a_n x^n$

The given R.R. is $a_n = 3a_{n-1}+1$

$$\therefore \qquad \sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} x^n$$

on multiplying both sides of (1) by x^n and summing up

i.e.
$$G(x)-a_0 = 3xG(x)+\frac{x}{1-x}$$

i.e.
$$(1-3x)\,G(n) = 1+\frac{x}{1-x} \quad (\because a_0 = 1)$$

$$\therefore \qquad G(x) = \frac{1}{(1-x)(1-3x)} = \frac{-\frac{1}{2}}{1-x}+\frac{\frac{3}{2}}{1-3x}$$

i.e.
$$G(x) = \frac{-1}{2}(1-x)^{-1}+\frac{3}{2}(1-3x)^{-1}$$

i.e.
$$\sum_{n=0}^{\infty} a_n x^n = \frac{-1}{2}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} 3^n x^n$$

$$a_n = \text{coefficient of } x^n \text{ in } G(x)$$

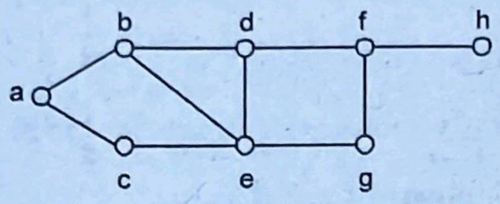
$$= \frac{1}{2}(3^{n+1}-1)$$

Q.1. (*d*) What is difference between cut set and cut edge? Explain with example.

Ans. A vertex cut-set of a connected graph G is a set S of vertices with the following properties:

- The removal of all the vertices in S disconnects G.
- The removal of some (but not all) of vertices in S does not disconnects G

consider the following graph.

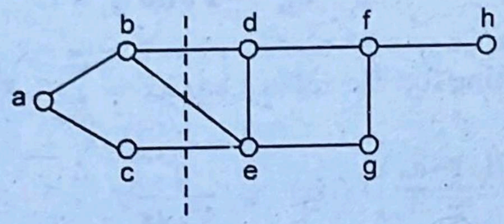


We can disconnect the graph by removing the two vertices b and e, but we can not disconnect it by removing just one of these vertices. The vertex cut set of G is $\{b, e\}$

A cut edge of a connected graph G is a set S of edges with the following properties:

- The removal of all edges in S disconnects G.
- The removal of some (but not all) of edges in S does not desconnects G.

Consider the following graph



We can disconnect G by removing the three edges bd, bc and ce, but we cannot disconnect it by removing just two of these edges. A cut edges is a set of edges in which no edge is redundant.

Q.1. (e) What is an Abelian group?

Ans. If G is a non empty set and * is a binary operation of G, then the algebraic system, $\{G, *\}$ is called a group if the following conditions are satisfied.

1. For all $a, b, c, \in$ G

$$(a*b)*c = a*(b*c) \text{ (Associativity)}$$

2. There existing an element $e \in G$ such that, for any $a \in G$.

$$a*e = e*a = a \text{ (Existence of Identity)}$$

3. For every $a \in G$ there exists an element $a^{-1} \in G$ such that

$$a*a^{-1} = a^{-1}*a = e \text{ (Existence of Inverse)}$$

A group $\{G, *\}$ in which the binary operation * is commutative is called an abelian group.

Q.2. (a) Find solution of non-homogeneous recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = 2^n$.

Ans. The characteristic equation of the RR is

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0 \; i.e. \; r = 2, 2$$

$$a_n^{(H)} = (c_1 + c_2 n).\, 2^n$$

since $R.S.$ of the $R.R.$ is 2^n where 2 is the double root of the character equation we assume the particular solution of $R.R.$ is

$$a_n^{(p)} = c_3 n^2 2^n$$

Using this in the $R.R.$ we have

$$c_3 n^2 2^n - 4c_3 (n-1)^2 2^{n-1} + 4c_3 (n-2)^2 2^{n-2} = 2^n$$

$$4c_3 n^2 - 8c_3 (n-1)^2 + 4c_3 (n-2)^2 = 4$$

$$C_3 \cancel{n}^2 - 2\cancel{c}_3 n^2 - 2c_3 + 4\cancel{n}c_3 + \cancel{c_3 n^2} + 4c_3 - 4\cancel{n}c_3 = 1$$

$$2c_3 = 1$$

$$c_3 = \frac{1}{2}$$

$$\therefore \qquad a_n{}^{(p)} = \frac{1}{2}n^2 2^n = n^2 2^{n-1}$$

Hence, the general solution of the $R.R.$ is

$$a^n = a_n{}^{(h)} + a_n\,(p) = \left[c_1 + c_2 n + \frac{n^2}{2}\right]2^n$$

Q.2. (*b*) Solve the recurrence relation $a_n - 2a_{n-1} + a_{n-2} = 2^n$ by generating functions with initial conditions

$$a_0 = 2 \text{ and } a_1 = 1.$$

Ans. Let the generating function of $\{a_n\}$ be $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\frac{G(x) - a_o - a_1 n}{x^2} - 2\left(\frac{G(x) - a_0}{x}\right) + G(x) = \frac{1}{1-2x}$$

Now, put $a_0 = 2$ and $a_1 = 1$ in above equation and after simplification, we get

$$G(x) - 2 - x - 2x\,G(x) - 2x + x^2\,G(x) = \frac{x^2}{1-2x}$$

$$\Rightarrow \qquad (1-x)^2\,G(x) = 2 + 3x + \frac{x^2}{1-2x}$$

$$\Rightarrow \qquad G(x) = \frac{2}{(1-x)^2} + \frac{3x}{(1-x)^2} + \frac{x^2}{(1-2x)(1-x)^2}$$

By partial fraction, we get

$$\frac{x^2}{(1-2x)(1-x)^2} = \frac{1}{(1-2x)} - \frac{1}{(1-x)^2}$$

Hence,

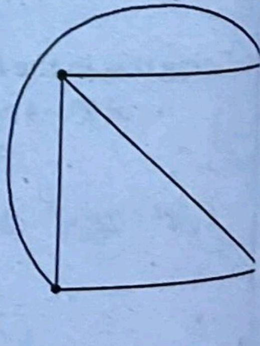
$$G(x) = \frac{1}{(1-x)^2} + \frac{3x}{(1-x)^2} + \frac{1}{(1-2x)}$$

Thus, $\qquad a_n = (n+1) + 3n + 2^n,$

i.e. $\qquad \boxed{a_n = 1 + 4n + 2^n}$

Q.3. (a) Define planar graph. Give the proof of Euler's formul a for connected planar graph.

Ans. A graph or multigraph which can be drawn in the plane so that its edges do not cross is said to be plannar.



Example:

The complete graph with four vertices k_4 is a planar graph.

Euler gave a formula which connects the number V of vertices, the number E of edges and the number R of regions of any connected map

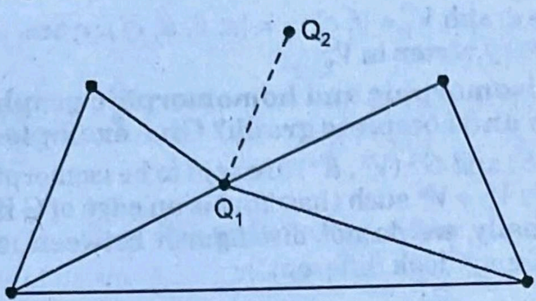
$$V - E + R = 2$$

Proof: suppose the connected map M consists of a single vertex P as

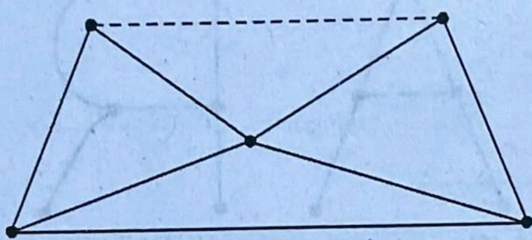
• P

Then $V = 1$, $E = 0$ and $R = 1$. Hence $V - E + R = 2$. Otherwise M can built up from a single vertex by the following two constructions:

1. Add a new vertex Q_2 and connect it to an existing vertex Q_1 by an edge which does not cross any existing edge as

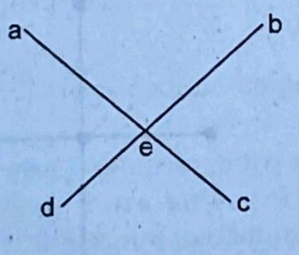


2. Connect two existing vertices Q_1 and Q_2 by an edge e which does not cross any existing edge as

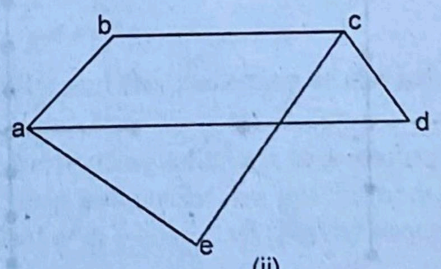


Neither operation changes the value of $V - E + R$. Hence M has the same value of $V - E + R$ as the map consisting of a single vertex, that is $V - E + R = 2$. Thus the theorem is proved.

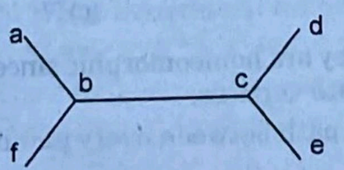
Q.3. (b) What is a bipartite graph? Determine whether following are bipartite with reason.



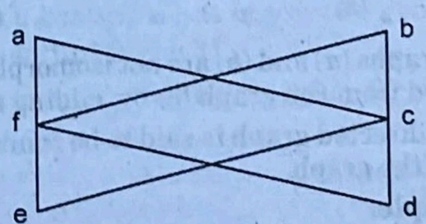
(i)



(ii)



(iii)

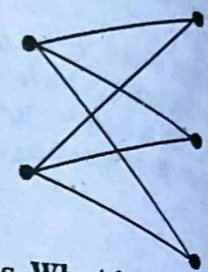


(iv)

Ans. A graph G is said to be bipartite if its vertices V can be partitioned into two subsets M and N such that each edge of G connects a vertex of M to a vertex N. By a complete bipartite graph, we mean that each vertex of M is connected to each vertex of N; this graph is denoted by $k_{m,n}$ where m is the number of vertices in M and n is the number of vertices in N, and, for standardization we will assume $m \leq n$

Example:

(*i*) It is bipartite graph $V_1 = \{e\}$ $V_2 = \{a, b, c, d\}$ since e is conncected to each vertex in V_2.


$K_{2,3}$

(*ii*) It is not bipartite graph since it cannot be partitioned as $V = \{V_1\ V_2\}$ such that every vertex in V_1 is adjacent to every vertex in V_2

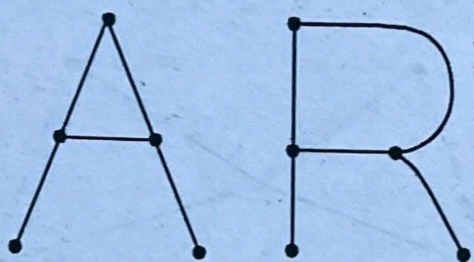
(*iii*) It is not bipartite graph since it cannot be partitioned as $V = \{V_1\ V_2\}$ such that every vertex in V_1 is adjacent to every vertex in V_2.

(*iv*) It is a bipartite graph $V_1 = \{f, c\}$ $v_2 = \{a, b, d, e\}$ f, c are connected *n* each and every vertex in V_2.

Q.3. (c) What are isomorphic and homomorphic graphs. What is connected graph, regular graph and complete graph? Give examples.

Ans. Graphs $G\,(V, E)$ and $G^*\,(V^*, E^*)$ are said to be isomorphic if there exists a one-to-one correspondence $f : V \to V^*$ such that $\{u, v\}$ is an edge of G if and only if $\{f(u)\}, f(v)\}$ is an edge of G^*. Normally, we do not distinguish between isomorphic graph (even through their diagrams may "look different").

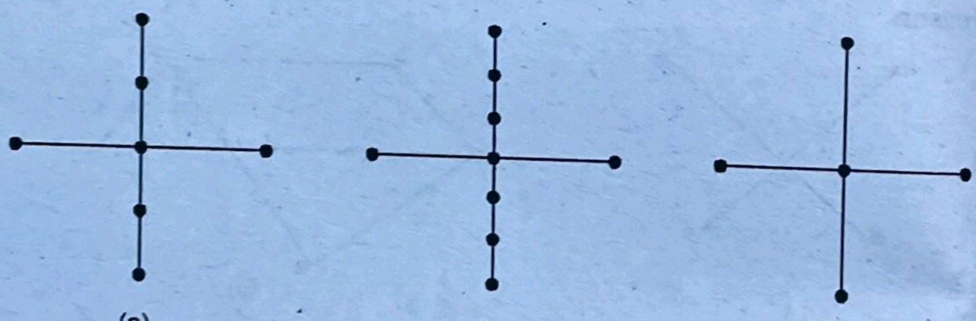
Example:



These both are isomorphic graphs

Two graphs G and G^* are said to be homeomorphic if they can be obtained from the same graph or isomorphic graphs by diverding an edge of G with additional vertices.

Example:

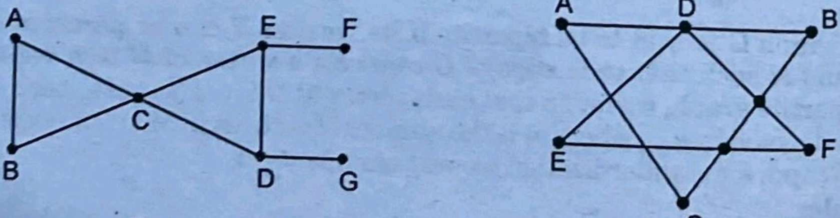


(a) (b) (c)

The graphs (*a*) and (*b*) are not isomorphic but they are homeomorphic since they can be obtained from the graph (*c*) by adding appropriate vertices.

An undirected graph is said to be connected if a path between every pair of distinct vertices of the graph.

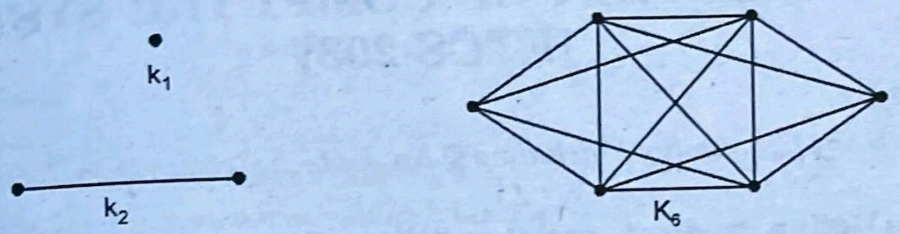
Example:



G_1 G_1 and G_2 are connected graphs G_2

A graph G is said to be complete if every vertex in G is connected to every other vertex in G. Thus a complete graph G must be connected. The complete graph with n vertices is denoted by k_n

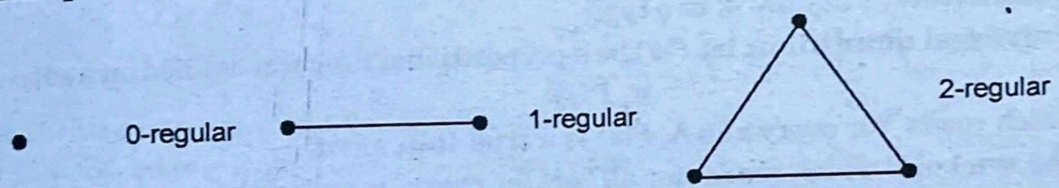
Example:



They are complete graphs

A graph G is regular of degree K or K- regular if every vertex has degree K. In other words, a graph is regular if every vertex has the same degree.

Example:



Q.4. (a) Let Q be a set of positive rational numbers which can be expressed in form $2^a\, 3^b$ where a and b are integers. Prove that {Q,.} is a group where. is multiplication operator.

Ans. The requirements on a group are strong enough to introduce the idea of cancellation. In a Group G, if $a * b = a*c$, then $b = c$ (this is Left cancellation). To see this let a^{-1} be the inverse of a in G. Then

$$a^{-1} * (a*b) = a^{-1} * (a*c)$$

from which it is immediate using associativity and the operation of the identity that $b = c$.

Under group requirements, we can also verify that solutions to linear equations of the form $a*x = b$ are unique. Using the group properties we get immediately that $x = a^{-1} b$. If x, and x_2 are two solutions, such that $a*x_1 = b = a * x_2$, then by cancellation we get immediately that $x_1 = x_2$.

Q.4. (b) What is order of an element in a group? What is a cyclic group? If in a group G, $x^5 = e$, $xyx^{-1} = y^2$ for $x, y \in$ G show that $0(y) = 31$

Ans. The smallest positive intger m such that a^m = e is called the order of element a where e is the indentity element.

A group $\{G, *\}$ is said to be cyclic, if there exists an element $a \in G$ such that every element x of G can be expressed as $x = a^n$ for some integer n.

In such a case, the cyclic group is said to be generated by a or a is a generator of G. G is also denoted by $\{a\}$

For example: if G = $\{1, -1, i, -i\}$ then $\{G, x\}$ is a cyclic group with the generator i, *for* $I = i^4, -1 = i^2, i = i^1$ and $-i = i^3$

For this cyclic group, $-i$ is also a generator.

END TERM EXAMINATION [JAN. 2015]
THIRD SEMESTER [B.TECH]
FOUNDATION OF COMPUTER SYSTEM
[ETCS-203]

Time. 3 Hours **M.M. : 75**

Note: *Attemp any five questions including Q.No. 1 which is compulsory. Internal choice is indicated.*

Q.1. (*a*) Define Predicate and Quantifiers. Give an example for each.

Ans. Predicate: We express statement as predicate of x/ function *i.e* $P(x)$ where domain is set of all inputs to the function.

e.g. $P(x) : x$ is mertal

Here $P(x)$ is predicate for the given statement

Quantifiers: They are of two types

Universal quantifier: let $P(x)$ be a propositional function defined on a set A. Then

$$\forall_x P(x)$$

which reads "for every x in A, $P(x)$ is a true statement"

The symbol $\forall$ which reads "for all" or" for every" is called the universal quantifier.

Existential quantifier: Let $P(x)$ be a propositional function defined on a set A. Then

$$\exists x \; P(x)$$

Which reads "For some x, $P(x)$"

The symbol $\exists$ which reads "there exists" or "for some" is called existential quantifier.

Q.1. (*b*) Prove by contradiction that at least four of any 22 days must fall on the same day of the week.

Ans.

Let p : At least four of 22 chosen days fall on the same day of the week

q : 22 days are chosen

p	q	$\neg p$	$\neg q$	$q \wedge \neg q$	$\neg p \to (q \wedge \neg q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	F
F	F	T	T	T	F

Explanation:

Let p be the proposition 'At least 4 of 22 chosen days fall on the same day of the week' suppose that $\neg$p is true. This means that at most of 3 of the 22 days fall on the same day of the week. Because there are 7 days in a week, this implies that atmost 21 days could be chosen because for each of the days of the week, at most 3 of the chosen days could fall on that day.This contradicts that we have 22 days under consideration. That is if q is the statement that, 22 days are chosen then we have that $\neg p \to (q \wedge \neg q)$. so we know p is true.

Q.1. (*c*) Explain principle of inclusion and exclusion with an example

Ans. Let A and B be any finite set, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

In other words, to find the number of $n(A \cup B)$ of elements in the union $A \cup B$, we add $n(A)$ and $n(B)$ and then subtract $n(A \cap B)$; that is "include" $n(A)$ and $n(B)$ and we "exclude" $n(A \cap B)$. This follows from the fact that, when we add $n(A)$ and $n(B)$ we have counted the elements of $A \cap B$ twice. This principle holds for any number of sets.

Example:

	Students
Let $n(F) = 65$	who study French
$n(G) = 45$	who study German
$n(R) = 42$	who study Russian
$n(F \cap G) = 20$	who study French and German
$n(G \cap R) = 15$	who study German and Russian
$n(F \cap R)$. 25	who study French and Russian
$n(F \cap G \cap R) = 8$	who study all three.

Then number of students taking at least one of the languages

$$\begin{aligned} n(F \cup G \cup R) &= n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) \\ &\quad - n(G \cap R) + n(F \cap G \cap R) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

Q.1.(*d*) Give an example for the following:

(*i*) Representing Relations Using Matrices

Ans. Let

$$A = \{1, 2, 3\}$$

$$R = \{(1, 1), (1, 2), (2,1)\}$$

$$\therefore \quad m_{ij} = \begin{cases} 0 \text{ if } (a_i, b_j) & \in R \\ 1 \text{ if } (a_i, b_j) & \notin R \end{cases}$$

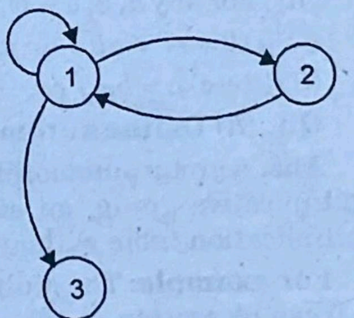
$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(*ii*) Representing Relations using Digraphs

Let

$$A = \{1, 2, 3\}$$

$$R = \{(1, 1), (1, 2)\ (2,1), (1,3)\}$$



Q.1. (*e*) Define principle of Mathematical Induction.

Ans. Let S be a set of positive integers with the following two properties:

(*i*) 1 belongs to s.

(*ii*) If k belongs to S, then k + 1 belongs to S Then S is a set of all positive integers it can be also stated as

Let P be a proposition defined on the integers $n \geq 1$ such that

(*i*) $P(1)$ is true

(*ii*) $P(n + 1)$ is true whenever $P(n)$ is true.

Then $P(n)$ is true for every integer $n \geq 1$.

Q.1. (*f*) Give the proof for five colour theorem.

Ans. The proof is by induction on the number P of vertices of G. If $P \leq 5$, then the theorem obviously holds suppose $P > 5$, and the theorem holds for graphs with less than P vertices. Let G has a vertex V such that deg $(V) \leq 5$. By induction, the subgraph G–V is 5- colorable. Assume one such coloring. If the vertices adjacent to V use less than the five colors, then we simply point V with one of the remaining colours and obtain *a*

5-colouring of G. We are still left with the case that V is adjacent to five vertices which are painted different colours say the vertices, moving counter clockwise about V, are $V_1 \ldots V_5$, and are painted respectively by the colours $c_1 \ldots c_5$ consider now the subgraph H of G generated by the vertices painted c_1 and c_3. Note H includes V_1 and V_3. If V_1 and V_3 belong to different components of H, then we can interchange the colours and c_1 and c_3 in the component containing V_1 without destroying the colouring of G–V. Then V_1 and V_3 are painted by c_3 c_1 can be chosen to paint V and we have a 5- colouring of G. On the other hand suppose v_1 and v_3 are in the same component of H. Then there is a path P from V_1 to V_3 whose vertices are painted with either c_1 or c_3. The path p together with the edges $\{v, v\}$ and $\{v, v_3\}$ form a cycle c which encloses either v_2 or v_4. Consider now the subgraph K generated by the vertices painted c_3 or c_4. Since C encloses V_2 or V_4 but not both, the vertices V_2 and V_4 belong to different components of K. Thus we can interchange the colours c_2 and c_4 in the component containing v_2 without destroying the colouring of G-V. Then v_2 and v_4 are painted by c_4, and we can choose c_2 to paint V and obtain a 5-colouring of g. Thus G is 5- colourable and the theorem is proved.

Q.1. (*g*) Mention the axioms to be satified in a ring R.

Ans. The axioms are if R be a non empty set with two binary operations addition and multiplication.

[R_1] For any $a, b, c, \in R$ we have $(a + b) + c = a + (b + c)$

[R_2] There exists an element $0 \in R$, called the zero element, such that $a + 0 = 0 + a = a$ for every $a \in R$.

[R_3] For each $a \in R$ there exists an element $-a \in R$, called the negative of a, such that $a + (-a) = (-a) + a = 0$.

[R_4] For any $a, b, \in R$, we have $a + b = b + a$.

[R_5] For any $a, b, c \in R$, we have $(ab)\,c = a\,(b\,c)$

[R_6] For any $a, b, c \in R$, we have

(*i*) $a\,(b + c) = ab + ac$ and

(*ii*) $(b + c)\,a = ba + ca$

Q.1. (*h*) Define automorphism. Give an example for illustration.

Ans. A group automorphism is an isomorphism from a group to itself. If G is a finite multiplicative group, an automorphism of G can be described as a rewriting of its multiplication table without altering its pattern of repeated elements.

For example: The multiplication table of the group of 4th roots of unity $G = \{1, -1, i, -i\}$ can be written as shown below.

	1	–1	i	$-i$
1	1	–1	i	$-i$
–1	–1	1	$-i$	i
i	i	$-i$	–1	1
$-i$	$-i$	i	1	–1

	1	–1	$-i$	i
1	1	–1	$-i$	i
–1	–1	1	i	$-i$
$-i$	$-i$	i	–1	1
i	i	$-i$	1	–1

Which means that the map defined by

$1 \to 1 \quad -1 \to -1 \quad i \to -i \quad -i \to i$ is an automorphism of G.

In general the automorphisiom group of an algebraic object 0, like a ring or field is the set of isomorphisms of o and is denoted Aut (0).

Q.2. (*a*) Give an example to illustrate proofs by Contraposition and Contradiction methods.

Ans. Proof by Contraposition: To prove if a product of two positive real numbers is greater than 100, then at least one of the number is greater than 10 *i.e.* $\forall$ positive real number r and s, if $(r.s.) > 100$, then $r > 10$ or $5 > 10$

Proof:

Contrapositive of the given statement is

$\forall$ positive real number r and s if $r \le 10$ and $s \le 10$ then $(r.s.) \le 100$

Now suppose r and s are positive real numbers and $r \le 10$ and $s \le 10$ then

since $r \le 10$ Multiply both side by s

we get $r.s. \le 10.5$ ...(1)

Since $s \le 10$ Multiply both side by 10

we get $10.5 \le 10.10$

$10.5 \le 100$ ...(2)

Since '$\le$' holds transitivity property therefore by (1) and (2)

we get $r.s \le 100$

This completes the proof.

Q.2. (*b*) Mention the Rules of Inference for Propostional logic

Ans. They are:

(*a*) Modus Pones or Rule of Detachment:

$$\begin{array}{l} p \Rightarrow q \\ \underline{\text{p}} \\ \therefore\ q \end{array}$$

Here the premises are

$P \to q$ (" p implies q")

p ("p is assumed to be true")

Conclusion: q ("so q is true")

(*b*) Law of contrapostion (or Modus Tollens)

$$\begin{array}{l} p \to q \\ \underline{\sim p} \\ \therefore\ \underline{\sim p} \end{array}$$

Here the premises are

$p \to q$ ("p implies q")

$\sim q$ ("q is assumed to be false")

conclusion: $\sim p$ ("so p is false")

(c) Disjunctive syllogism

$$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ \therefore\ \underline{q} \end{array}$$

Here the premises are

$p \vee q$ (" p or q")

$\sim p$ (" p is assumed to be false")

Conclusion: q ("so q is true")

Proof by Contradiction:

To prove the negative of any irrational number is irrational

i.e. $\forall$ real numbers x, if x is irrational then $-x$ is irrational

Suppose not [will take the negation of the given statement and suppose it be true.]

Assume, to the contrary that

$\exists$ irrational number x such that $-x$ is rational

By defination of rational we have

$-x = a/b$ for some integers a and b with $b \neq 0$

Multiply both sides by –1 gives

$$x = -(a/b) = -a/b$$

But $-a$ and b are integers [since a and b are integers] and $b \neq 0$ [by zero product property]. Thus x is a ratio of the two integres $-a$ and b with $b \neq 0$. Hence by definition of ration x is rational, which is contradiction [This contradiction shows that the supposition is false and so the given statement is true]

This competes the proof

(*d*) Hypothetical Syllogism

$$\begin{array}{l} P \to q \\ \underline{q \to r} \\ \therefore\ \underline{p \to r} \end{array}$$

Here premises are

$p \to q$ ("p implies q")

$q \to r$ ("q implies r")

Conclusion: $p \to r$ (so p imples r")

Q.2. (c) Let m, *n* be two positive integers prove that if *m, n* are perfect squares, then the product *m*n* is also a perfect square.

Ans. By definition of "perfect square" we know that $m = k^2$ and $n = j^2$, for some integrs k and j

so then
$$m.n = k^2 j^2$$
$$m.n = (k.j)^2$$

Since k and j are integers so is $k.j$. Since mn is the square of the integer $(k.j)$, mn is a perfect sqaure.

Q.3. (*a*) Provide a proof by contradiction for the following. For every interger *n* if n^2 is odd, then *n* is odd.

Ans. Suppose not. [we take the negation of the given statement and suppose it to be true]

Assume to the contrary that $\exists$ an integer n such that n^2 is odd and n is even. By definition of even we have $n = 2k$ for some integer k.

so by substitution we have

$$n.n = (2K).(2K) = 2(2.K.K)$$

Now (2.k.k) is an integer because products of integers are integer and 2 and k are integers

Hence

$$n.n = 2\,(\text{some integer})$$

or

$$n^2 = 2\,(\text{some integer})$$

and so by definition of n^2 even, is even so the conclusion is since n is even, n^2 which is the product of n with itself is also even. This contradicts the supposition that n^2 is odd.

Hence, the supposition is false and the proposition is true.

Q.3. (*b*) Let A {1, 2...9,10}. Consider each of the following sentences. If it is a statement, then determine its truth value. If it is a propositional function, determine its truth set

(*i*) $(\forall x \in A)\ (\exists y \in A)\ (x + y < 14)$

Ans. True for each x let $y = 1$ then $x + 1 < 14$ is a true statment

(*ii*) $(\forall x \in A)\ (\forall y \in A)\ (x + y < 14)$

Ans. False for if $x = 10$ and $y = 6$ then $x + y < 14$ is not a true statement *(iii)* $(\forall x \in A)\ (x + y < 14)$

Ans. The open sentence is preceded by one quantifier hence it is a propositional function of the other variable. For every $x \in A\ x + y < 14$ if and only if $y = 1, 2$, or 3. Hence truth set is {1, 2, 3}

(iv) $(\exists\ y \in A)\ (x + y < 14)$

Ans. It is an open sentence in x. The truth set is A itself.

Q.3. (c) Define De Morgans's Laws. Find the negation of $P \to Q$

Ans. De Morgan's Laws are

(a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

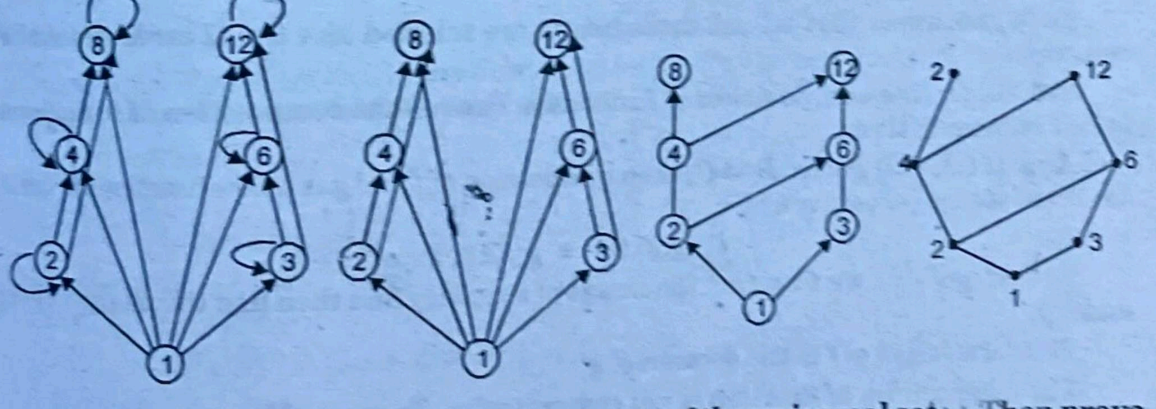
(b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

$$\begin{aligned}
P \to Q &\equiv (p \to Q) \wedge (Q \to p)\\
&\equiv [(\sim p \vee Q) \wedge (\sim Q \vee p)]\\
\sim(p \to Q) &\equiv \sim[(\sim p \vee Q) \wedge (\sim Q \vee P)]\\
&\equiv [\sim(\sim p \vee Q) \vee \sim(\sim Q \vee P)]\\
&\equiv (p \wedge \sim Q) \vee (Q \wedge \sim P)\\
&\equiv [(p \wedge \sim Q) \vee Q] \wedge [(p \wedge \sim Q) \vee \sim p]\\
&\equiv [Q \vee (p \wedge \sim Q) \wedge [\sim p \vee (P \wedge \sim Q)]\\
&\equiv [(Q \vee p) \wedge (Q \vee \sim Q)] \wedge [\sim P \vee P) \wedge (\sim P \vee \sim Q)]\\
&\equiv [(Q \vee p) \wedge T] \wedge [T \wedge (\sim p \vee \sim Q)] \text{ Using Identity law}\\
&\equiv [(Q \vee p) \wedge (\sim P \vee \sim Q)]\\
&\equiv [(\sim P \vee \sim Q) \wedge (Q \vee P)]\\
&\equiv [(P \to \sim Q) \wedge (\sim Q \to P)]\\
\sim(p \leftrightarrow Q) &\equiv p \leftrightarrow \sim Q
\end{aligned}$$

Q.4. (a) Draw the Hasse diagram representing the partial ordering {(a, b)}1 a divides b} on {1, 2, 3, 4, 6, 8, 12}.

Ans.

R = { (1,1) (1, 2) (1, 3) (1, 4) (1, 6) (1, 8) (1,12)
(2,4) (2,6) (2,8) (2,12) (2,2)
(3,6) (3,12) (3,3)
(4,8) (4,12), (6,12) (4,4) (6,6) (8,8) (12,12)}



Q.4. (b) Let A,B and C be any three subsets of the universal set ∪. Then prove that:

(i) $A - (B \cup C) = (A - B) \cap (A - C)$

(ii) $(A \cap B) - C = A \cap (B - C)$

Ans. *(i)* $A - (B \cup C) = \{x \mid x \in A \text{ and } x \notin B \cup C\}$

$= \{x \mid x \in A \text{ and } x \notin B \text{ and } x \notin C\}$

$$= \{x \mid x \in A \text{ and } x \notin B \text{ and } x \in A \text{ and } x \notin C\}$$
$$= \{x \mid x \in (A-B) \text{ and } x \in (A-C)\}$$
$$= \{x \mid x \in (A-B) \cap (A-C)\}$$
$$= (A-B) \cap (A-C)$$

(ii)
$$(A \cap B) - C = \{x \mid x \in (A \cap B) \text{ and } x \notin C\}$$
$$= \{x \mid x \in A \text{ and } x \in B \text{ and } x \notin C\}$$
$$= \{x \mid x \in A \text{ and } x \in B \text{ and } x \in \bar{C}\}$$
$$= \{x \mid x \in (A \cap B \cap \bar{C})\}$$
$$A \cap (-B-C) = \{x \mid x \in A \text{ and } x \in (B-C)\}$$
$$= \{x \mid x \in A \text{ and } (x \in B \text{ and } x \notin C)\}$$
$$= \{x \mid x \in A \text{ and } x \in B \text{ and } x \in \bar{C}\}$$
$$= \{x \mid x \in (A \cap B \cap \bar{C})\}$$

since LHS = RHS Hence the result.

Q.4. (c) What is the power set of the set {0, 1, 2}?

Ans. Let
$$A = \{0, 1, 2\}$$
$$\text{Power set of } A = P(A)$$
$$P(A) = \{\phi, \{0\}, \{1\}, \{2\}, \{0, 1\} \{1,2\}, \{0,2\}, \{0,1,2\}$$

A has 3 elements. Then $P(A)$ has $2^3 = 8$ elements

Q.5. (a) Using Pigeonhole principle calculate the following.

(i) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Ans. Suppose for each suit we have a box that contain cards of that suit. Here $n = 4$ suits are the Pigeonholes and $k + 1 = 3$ so $k = 2$.

Thus according to Pigeonhole principle among any $k\, n + 1 = 2 \times 4 + 1 = 9$ cards, 3 of them belong to the same suit.

(ii) How many must be selected to guarantee that atleast three hearts are selected

Ans. The worst case, we may select all the clubs, diamonds, spades (39 cards) before any hearts.

So to guarantee that atleast three hearts are selected 39 + 3 = 42 cards should be selected.

Q.5. (b) Define composition of functions. Prove that composition of functions is not commutative.

Ans. If $f: A \to B$ and $g: B \to C$, then compostion of f and g is a new functions from A to C denoted by gof is given by

$$(gof)(x) = g\{f(x)\}, \text{ for all } x \in A$$

To find $(gof)(x)$, we first find the image of x under f and then find the image of $f(x)$ under g.

Thus the range of f is the domain of g.

The composition of function is not commutative. To prove this

Let $f(x) = x + 1$ and $g(x) = x^2$, which are both real valued functions with domain R

$$(fog)(1) = f(g(1)) = f(1) = 2 \text{ but}$$
$$(gof)(1) = g(f(1) = g(2) = 4 \neq 2$$

Since the two composite functions have different values at the element 1 of the domain, they are not the same function. Thus, function composition of F is not commutative.

Also, when *gof* is defined *fog* need not be defined for when *gof* is defined $R_f = Dg$ where R_f is range of *f* and *Dg* is domain of g. This does not mean $R_g = Df$ which is the required condition for *fog* to exist

For example:

if
$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{1, 2, 3, 8, 9\} \quad f : A \to B, g : A \to A$$
$$f = \{(1, 8) (3, 9) (4, 3) (2, 1) (5, 2)\}$$
$$g = \{(1, 2) (3, 1) (2, 2) (4, 3) (5, 2)\}$$

Here
$$fog = \{(1, 1), (2, 1) (3, 8), (4, 9), (5, 1)\}$$

but
$$R_f \neq D_g$$

$\therefore$ *gof* is not defined

Q.5. (c) Is the poset (Z^+ 1) a lattice?

Ans. The poset (Z^+, 1) is a lattice for any pair of positive integers *a* and *b* that we take, the greatest lower bound *i.e. g/b* of $\{a, b\}$ is gcd (a,b) while the least upper bound of $\{a,b\}$ is the lcm (a,b) and any poset is a lattice if every power of elements has a *lub* and a *glb*.

Q.6. (a) Develop a general explicit formula for a non-homegeneous recurrence relation of the form $a_n = r\,a_{n-1} + s$ where *r* and *s* are constant

Ans. Give
$$a_n = ra_{n-1} + s \qquad \ldots(1)$$

We can substitute the value of a_{n-1} with the help of Equation. (1), we get
$$= r(ra_{n-2} + s) + s$$
$$= r^2 a_{n-2} + rs + s$$
$$\vdots \quad \vdots \quad \vdots$$
$$\vdots \quad \vdots \quad \vdots$$
$$\vdots \quad \vdots \quad \vdots$$
$$= r^{n-1} a_1 + r^{n-2} s + \ldots\ldots + s$$
$$\boxed{a_n = r^{n-1} a_1 + s.\frac{r^{n-1} - 1}{r - 1} (r \neq 1)}$$

If
$$r = 1$$

then,
$$\boxed{a_n = a_1 + (n-1)s}$$

Q.6. (b) Prove by mathematical induction. For every positive integer *n*, the expression $2^{n+2} + 3^{n+1}$ is divisible by 7.

Ans. When
$$n = 1$$
$$2^{n+2} + 3^{2n+1} = 2^{1+2} + 3^{2\times1+1} = 8 + 27 = 35$$

which is divisible by 7

Hence true for $n = 1$

Let it be true for n

Thus $2^{n+2} + 3^{n+1} = 7\,k$ *i.e.* divisible by 7 ...(1)

for $n + 1$
$$= 2^{(n+1)+2} + 3^{(2(n+1)+1)}$$
$$= 2^{n+2}.2 + 3^{2n+1}\,3^2$$

From (1) $2^{n+2} = 7k - 3^{2n+1}$ substituting it
$$= (7k - 3^{2n+1}).2 + 3^{2n+1}.3^2$$
$$= 14\,k - 2.3^{2n+1} + 3^2.3^{2n+1}$$
$$= 14k - 3^{2n+1}(2-9)$$

$$= 14k - 3^{2n+1}(-7) = 14k + 7.3^{2n+1}$$
$$= 7(2k + 3^{2n+1});$$

i.e. divisible by 7. Hence true for $n + 1$.

Q.6. (c) Define linear recurrence relations with constant coefficient. Give an example with illustration.

Ans. A recurrence relation of the form $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + ...+ c_k a_{n-k} = f(n)$ is called a linear recurrence relation of degree k with constant coefficients where $c_0, c_1 ....c_k$ are real numbers and $c_k \neq 0$. The recurrence relation is called linear because each a_r is raised to the power 1 and there are no products such as $a_r a_s$.

For example the Fibenacci sequence

0, 1, 1, 2, 3, 5, 8, 13....

which can be represented as

$$F_{n+2} = F_{n+1} + F_n \text{ where } n \geq 0 \text{ and } F_0 = 0\ F_1 = 1$$

it is linear recurrence relation with constant coefficient.

Q.7. (a) Show that the edge chromatic number of a graph must be atleast as large as the maximum degree of a vertex of the graph.

Ans. In graph theory, Vizing's theorem states that the edges of every simple undirected graph may be coloured using a number of colours that is at most one larger then the maximum degree Δ of the graph.

Proof: Let $G = (V, E)$ be a simple undirected graph. We proceed by induction on m, the number of edges. If the graph is empty, the theorem trivially holds. Let $m > 0$ and suppose a proper $(\Delta + 1)$ – edge – colouring exists for all $G - xy$ where $xy \in E$.

We say that colour $\alpha \in \{1, ...\Delta + 1\}$ is missing in $x \in V$. *w.r.t* proper $(\Delta + 1)$ edge-colouring c if $C(x,y) \neq \alpha$ for all of $y \in N(x)$

Also, let α/β – path from x denote the unique maximam path starting in x with α – coloured edge and alternating the colours of edges, its length can be 0. Note that if c is a proper $(\Delta + 1)$ - edge-colouring of G then every vertex has a missing colour with respect to c.

Let p be the α/β - path from y_k with respect to c_k. From (1) p has to end in x. But α is missing in x, so it has to end with an edge of colour β. Therefore, the last edge of p is $y_{i-1}x$. The edge leading to y_k clearily has colour. But β is missing in y_k which is a contradiction with (1) above.

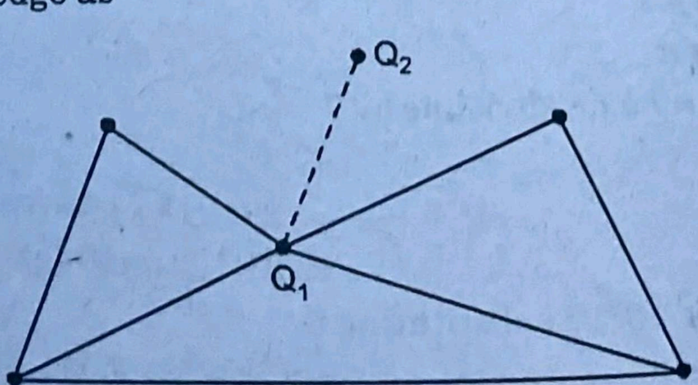
Q.7. (b) Prove the Euler Formula.

Ans. Suppose the connected map consists of a single vertex P as

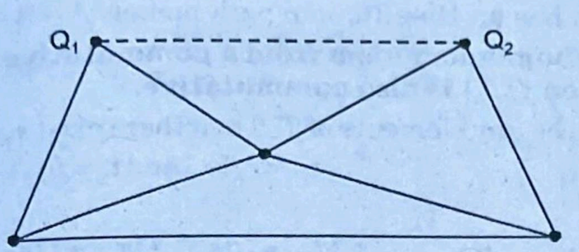
•P

Then $V = I, E = 0, R = 1$. Hence $V - E + R = 2$. Otherwise M can be built up from a single vertex by the following two constructions.

1. Add a new vertex Q_2 and connect it to an existing vertex Q_1 by an edge which does not cross any exisiting edge as



2. Connect two existing vertices Q_1 and Q_2 by an edge e which does not cross any existing edge as

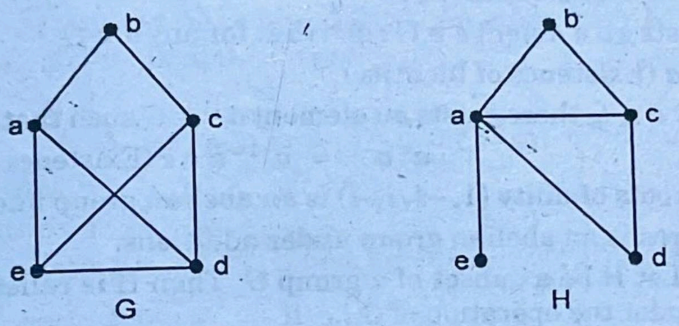


Neither operation changes the value of $V - E + R$. Hence M has the same value of $V - E + R$ as the map consisting of a single vertex that is $V - E + R = 2$. Thus the theorem is proved.

Thus in any connected map where v is the number of vertices, E is the nomber of edges, R is number of regions.

$$V - E + R = 2.$$

Q.7. (c) Show that the graph displayed in the following figures are not isomorphic.



Ans. In both the graphs G and H number of vertices = 5 and number of edges = 6

For graph G deg (a) = deg (b) = deg (d) = 2

deg (c) = deg (e) = 3

For graph H deg (b) = deg (d) = 2

deg (e) = 1; deg (a) = 4; deg (c) = 3

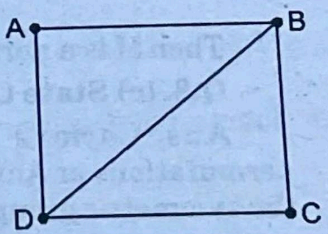
Thus for vertex e, and a of graph H there is no one-to-one correspondence in graph (G). Hence they are not isomorphic.

Q.7. (d) Define Euler and Hamiltonian paths in a graph.

Ans. A path of graph G is called an Eulerian path if it includes each edge of G exactly once

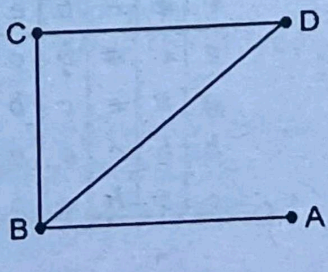
For example:

The given graph contains an Eulerian path between B and D namely, B – D – C– B–A–D, since it includes each of edges exactly once.



A path of a graph G is called a Hamiltonian path of it includes each vertex of G exactly once.

For example:



The given Graph has an Hamiltonian path namely A – B – C– D

Q.8. (a) If f is a homomorphism from a commutative semigroup {s*} into a semigroup {T,*'} then {T,*'} is also commutative.

Ans. Let t_1 and t_2 be any elements of T. Then there exist s_1 and s_2 in S with

$$t_1 = f(s_1) \text{ and } t_2 = f(s_2)$$

Therefore

$$t_1 * t_2 = f(s_1) *' f(s_2), = f(s_1 * s_2) = f(s_2 * s_1)$$
$$= f(s_2) *' f(s_1) = t_2 *' t_1$$

Hence (T, *') is also commutative.

Q.8. (b) Define groups, sub-groups and normal sub-groups. Give an example for each.

Ans. Group: Let G be non empty set and * is a binary operation of G, then the algebraic system {G, *} is called a group if the following conditions are satisfied

(*i*) For all $a, b, c \in G$

$(a*b)*c = a*(b*c)$ (Associativity)

(*ii*) There exists an element $e \in G$ such that for any $a \in G$

$a * e = e * a = a$ (Existence of Identity)

(*iii*) For every $a \in G$ there exists an element $a^{-1} \in G$ such that

$$a * a^{-1} = a^{-1} * a = e \text{ (Existence of Inverse)}$$

Example: 4 roots of unity $(1, -1, i, -i)$ is an abelian group under multiplication set of Z integres is an abelian group under additions.

Sub-group: Let H be a subset of a group G. Then H is called a sub-group of G if itself is a group under the operation of G *i.e.* if

(*i*) The identity element $e \in H$

(*ii*) H is closed under the operation of G *i.e.* if $a, b, \in H$ then $ab \in H$ and

(*iii*) H is closed under inverse, that is if $a \in H$ then $a^{-1} \in$ H

Example: Let $G = \{1, -1, i - i\}$ be a group under multiplication then its subset $H = \{1, -1\}$ is its sub-group.

Normal sub-groups: A sub-group H of G is a normal sub-group if $a^{-1} Ha \subseteq$ H for every $a \in G$. Equivalently H is normal if aH = Ha for every $a \in$ G *i.e.* if the rights and left cosets coincide.

Example: Consider the group Z of integers under addition. Let H denote the multiples of 5 that is

$$H = \{....., -10, -5, 0, 5, 10 .....\}$$

Then H is a normal sub group of Z.

Q.8. (c) State Cayles's Theorem and explain using an example.

Ans. Cayles's theorem states that every group is isomorphic to a group of permutations or Any group G is isomorphic to a subgroup of sym (G) where sym (G) is the symmetric group *i.e.* the group of all permutations on a set G

Consider the following Cayley table of a group G = {e, a, b, c}

v	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

We then have

$$\lambda_e = \begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}, \qquad \theta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)$$

$$\lambda_a = \begin{pmatrix} e & a & b & c \\ e & e & c & b \end{pmatrix}, \qquad \theta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34)$$

$$\lambda_b = \begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix}, \qquad \theta_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24)$$

$$\lambda_c = \begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix}, \qquad \theta_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$$

Hence G is isomorphic to the subgroup of s_4

$\{(1, (12)(34), (13)(24), (14)(23)\} = \{12)(34), (13)(24)\}$.

Q.9. (a) Give an example to represent and minimize the Boolean fuction.

Ans. If x_1, x_2 -----x_n are Boolean variables, a function from $B^n = \{x_1, x_2, ...x_n\}$ to $B = \{0,1\}$ is called a Boolean function of degree n.

For example:

$$f(a, b, c, d) = a'b'c'd' + a'bc'd + a'b'cd + a'b'cd' + a'bcd$$

It is a Boolean function.

Minimising it using k-map method

The given minterms in $f(a, b, c, d)$ correspond to the binary numbers 0000, 0101,0011, 0010 and 0111.

ab \ cd	00	01	11	10
00	1	0	1	1
01	0	1	1	0
11	0	0	0	0
10	0	0	0	0

The number 1 is entered in the cells correspoding to these numbers and the numbers 0 is centered in the remaining cells.

The minimum possible number of loops containing maximum possible number of 1's will are shown in the map.

The terms corresponding to the loops are $a'b'd', a'bd, a'cd$

Hence minimum $f(a, b, c, d) = a'b'd' + a'bd + a'cd$.

Q.9. (b) Prove Largrange's Theorem.

Ans. Lagrange's theorem states that the order of a subgroup of a finite group is a divisor of the order of the group.

Proof: Let aH and bH be two left cosets of the subgroup $\{H, *\}$ in the group $\{G,*\}$. Let the two cosets aH of the bH be not disjoint

Then let c be an element common to aH and bH i.e. $C \in aH \cap bH$

Since $C \in aH$, $C = a^*h$, for some $h_1 \in H$ ...(1)

Since $C \in bH$, $C = b^*h_2$ for some $h_2 \in H$ ...(2)

From (1) and (2) we have

$$a^*h_1 = b * h_2$$

$$\therefore \quad a = b^* h_2 {}^* h_1^{-1} \quad ...(3)$$

Let x be an element in aH

$$x = a^*h_3 \text{ for some } h_3 \in H$$

$$= b * h_2 * h_1^{-1} * h_3 \text{ using (3)}$$

since H is a subgroup, $h_2 * h_1^{-1} * h_3 \in H$

Here, (3) means $x \in bH$

Thus, any element in aH is also an element in bH.

$$\therefore \quad aH \subseteq bH$$

similarly we can prove that $bH \subseteq aH$

Hence $aH = bH$

Thus, if aH and bH are not disjoint, they are identical

$\therefore$ The two cosets aH and bH are disjoint or identical

Now every element $a \in G$ belongs to one and only one

Left coset of H in G for

$$a = ae \in aH, \textit{ since } e \in H$$

i.e. $$a \in aH$$

$a \notin bH$, since aH and bH are disjoint *i.e.* a belongs to one and only left coset of H in G *i.e.* aH from (4) and (5) we see that the set of left cosets of H in G from a partition of G.

Now let the order of H be m.

viz let $H = \{h_1, h_2 ....h_m\}$ where h_i's are distinct

Then $$aH = \{ah, ah_2....ah_m\}$$

The elements of aH are also distinct for $ah_i = ah_j$

$$\Rightarrow \quad h_i = h_j \text{ which is not true}$$

Thus H and aH have the same number of elemetns namely m.

In fact every coset of H in G has exactly m elements. Now Let the order of the group $(G, *)$ be n *i.e.* there are n elements in G.

Let the number of distinct left cosets of H in G be p [p is called the index of H in G]

$\therefore$ The total number of elements of all the left cosets $= pm =$ the total number of elements of G *i.e.* $n = p.m.$

i.e. m, the order of H is a divisor of n, the order of G.

Q.9. (c) Show that in ring R:

(i) $(-a)(-b) = ab$

Ans. We have

$$(-a).(-b) = -(a).b] \text{ by replacing } a \text{ by } -a \text{ in}$$

$$-(a.b) = a.(-b)$$

$$= -[-(a.b)] \text{ from } -(a.b) = (-a).b$$

$$(-a).(-b) = a.b \text{ by } -(-a) = a$$

(ii) $(-1)(-1) =$ (if R has identity element. 1)

$(-1)(1) = (-1) = -(1.1)$ $\quad \because -(a.b) = a.(-b)$

$(-1)(-1) = -(1.(-1))$ $\quad$ replacing b by $-a$

$(-1)(-1) = (-(1.1))$ $\quad \because -(a.b) = (-a).b$

$(-1)(-1) = (1.1) = 1$ $\quad \because -(-a) = a$

IMPORTANT QUESTIONS

Q.1. Explain the principal of mathematical Induction. **(2.5)**

Ans. PRINCIPAL OF MATHEMATICAL INDUCTION: A proof by mathematical induction that $p(n)$ is true for every positive integer n consists of two steps:

1. **Basic step:** The proposition $p(1)$ is shown to be true.
2. **Inductive step:** The implication $p(n) \to p(n+1)$ is shown to be true for every positive integer n.

When we complete both steps of a proof by mathematical induction, we have proved that $p(n)$ is true for all positive integers, that is, we have shown than $\forall n\, p(n)$ is true. It is to be noticed that in the proof by mathematical induction it is not assumed that $p(n)$ is true for all positive integers. It is only shown that if it assumed that $p(n)$ is true then $p(n+1)$ is also true.

For example: To prove that the sum of the first n odd positive integers is n^2.

Let $P(n)$ is the proposition that sum of the first n odd positive integers is n^2.

Step 1: $p(1)$ says, the sum of the first one odd integer is $(1)^2$ which is automatically true as the first odd positive integer is 1.

Step 2: Let us assume that $p(n)$ is true *i.e.,* sum of first n positive odd integer is n^2.

i.e., $1 + 3 + 5 + ... + (2n - 1) = n^2$

Consider $p(n + 1)$, the sum of first $(n + 1)$ positive odd integers:

$1 + 3 + 5 + ... + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$

This implies that $p(n + 1)$ is true.

Therefore, by the principle of mathematical induction the result is true.

Q.2. How many different 8 bit strings are there that end with 0111. **(2.5)**

Ans. The ways to construct a bit string of length 8-bit end with 4-bit such as 0111. There are $2^4 = 16$ ways to construct such a string.

Q.3. If Set x has 10 members then How many members in $P(x)$ and how many members of $P(x)$ are proper subset of x. **(2.5)**

Ans. If X has 10 members then $P(X)$ has 2^{10} members, out of which $(2^{10} - 1)$ are proper subset of X, as are of the member is set X itself.

For example:

$$X = \{1, 2\}$$

$$P(X) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$$

In above power set $\{1, 2\}$ is a element that contain in $P(X)$. So, $\{1, 2\}$ is not a proper subset of X. So, $(2^n - 1)$ are proper subset of X where n represent total number of members.

Q.4. Define the concept of Pascal's triangle. **(2.5)**

Ans. Pascal's Triangle: The co-efficient of the successive powers of $x + y$ can be arranged in a triangular array of numbers, called Pascal's Triangle.

The co-efficient in the expansion of $(x + y)^n$ as

$$(x + y)^n = \sum_{k=0}^{n} {}^nc_k\, x^{n-k} y^k$$

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

In other way, considering the coefficients we will have.

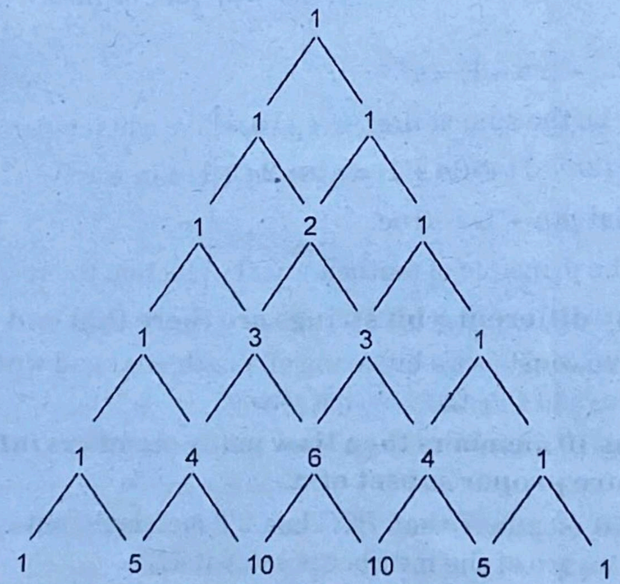
The number in Pascal's Triangle have the following intersecting properties:

1. The first number and the last number in each row is 1,
2. Every other number in the array can be obtained by adding the two numbers appearing directly above it.

For example, 4 = 1 + 3, 10 = 6 + 4, 15 = 5 + 10

Since the no's appearing in Pascal's Triangle are binomial co-efficient, property of pascal's Triangle come from the theorem given below:

$$c(n+1, r) = C(n, r-1) + c(n, r)$$



Q.5. List the differences between BFS and DFS. **(5)**

Ans. BFS: (*i*) Breadth-First Search (BFS) is a graph search algorithm that beings at the root node and explores all the neighbouring nodes.

(*ii*) Breadth-First search uses a queue as an auxillary data structure to store nodes for further processing.

(*iii*) It will gives the best in space complexity, completences and optimal for a graph that has edges of equal length.

DFS: (*i*) The depth-First Search algorithm progresses by expanding the starting node of G and thus going deeper and deeper untill a goal node is found or until a node that has no children is encountered.

(*ii*) When a dead-end is reached, the algorithm backtracks, returning to the most recend node that has not been completely explored.

(iii) It uses a stack as an data structure.

(iv) The space complexity of a DFS is lower than that of a breadth-first search.

Q.6. Solve the following recurrence relation? **(7.5)**

Ans. $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)\,2^n$ ...(1)

The homogenous solution of the difference equation is given by

$$a_{r(h)} = (C_1 + C_2 n)\,2^n \quad \text{...(2)}$$

because it has two real and equal roots i.e. 2 and 2.

To find the particular solution, let us assume the general form of the solution = $2^n(A_{1n} + A_0)$, but due to occurence of there terms in equation (2), we multiply this by suitable power of n so that none of the terms will occur in Eq. (2). Thus multiply by n^2.

Hence, the general form of the solution becomes = $2^n(A_1 n + A_0)n^2$

Putting this solution in L.H.S. of equation (1), we get

$= 2^n(A_1 n + A_0).\,n^2 - 4.2^{n-1}[A_1(n-2) + A_0].(n-1)^2 + 42^{n-2}[A_1(n-2) + A_0].\,(n-2)^2$

$= 2^n(A_1 n + A_0).n^2 - 2(n^2 + 1 - 2n)\,2^n(A_1 n - A_1 + A_0) + (n^2 + 4 - 4n)\,2^n.(A_1 n - 2A_1 + A_0)$

$= n.2^n(6A_1) + 2^n(-6A_1 + 2A_0)$ ...(3)

Equating Equation (3) with R.H.S. of Eq. (1), we get

$$6A_1 = 1 \quad \therefore \quad \boxed{A_1 = \frac{1}{6}}$$

$$-6A_1 + 2A_0 = 1 \quad \therefore \quad \boxed{A_0 = 1}$$

Therefore, the particular solution is

$$\boxed{n^2.2^n\left(\frac{n}{6} + 1\right)} \text{ Ans.}$$

Q.7. In the defination of Euber circuit discuss the requirements that the Euler circuit intersects with every vertex at least once. **(6)**

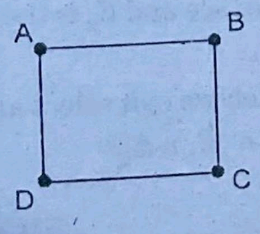
Ans. An Euler Circuit is a route through a connected graph such that

1. Each edge of the graph is traced exactly once.
2. The route starts and ends at the same vertex.

The requirement that the Euler Circuit:

(i) The given graph should be in the form of Hamiltonian Graph.

(ii) You only consider Euler Circuits in connected graphs.



Above graph meet with the requirement

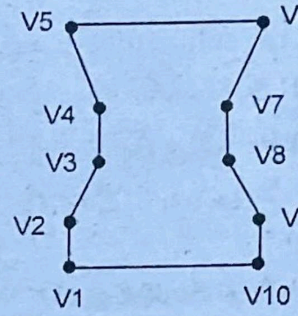
(i) The path exist $A - B - C - D$. This will shows that given graph is in Hamiltonian graph.

(ii) The path exist $A - B - C - D - A$. This is a connected graph is in form of Euler Circuits.

Q.8. Explain Hamiltonian circuit with example. (6)

Ans. Hamiltonian Circuit is a circuit drawn form G that contains each vertex of G exactly once except the beginning and the ending vertex. Since no vertex except the start vertex is repeated, no edge will repeat.

For example, Figure represent Hamiltonian the circuit $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$.



A circuit in a connected graph G is said to be Hamiltonian if it includes every vertex of G. Hence a Hamiltonian circuit in a graph of n vertices consists of exactly n edges.

If we remove one of the edge from the given circuit say the edge (v_1, v_{10}) then we are left with a path $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ in which we travel along all the vertices exactly once is the Hamiltonian path. Since a Hamiltonian path is a sub–graph of Hamiltonian Circuit which we get by removing an edge from the circuit. Therefore every graph that has a Hamiltonian circuit also has a Hamiltonian path.

Therefore, the length of a Hamiltonian path (if exists) in a connected graph of n vertices and $(n - 1)$ edges.

Q.9. Explain Principle of Inclusion and Exclusion with an example? (3)

Ans. Suppose two tasks A and B *can* occur in n_l and n_2 ways, where some of the n_l and n_2 ways may be the same. In this situation, we cannot apply the sum rule, because the same number of ways will be counted twice. In such situations, we apply the inclusion—exclusion principle, which has already been discussed. According to this principle, if A and B are two sets, then the no. of elements in the set AUB is given by $n(AUB) = n(A) + n(B) - n(A \text{ n } B)$

This principle holds for any no. of sets. For three sets, it can be stated as follows: n $iAUBUC) = n(A) + n(B) + n(C) - n(AnB) - n(BnC) - n(AnC) + (A \text{ nBn.C})$

Example: In how many ways can we select an ace or a heart from a pack of cards?

Ans. There are 4 aces and 13 cards of heart in a pack of cards, and 1 card is common to both. If E_l is the event of getting an ace and E_2 is the event of getting a heart, then «(#j) = 4 $n(E_2) = 13$ and $n\ (E_l \text{ n } E_2) = 1$.

Thus, the no. of ways in which we can select an ace or heart from a pack of cards is

$n\ (E_1 \text{ u } E_2) = n\ (EJ + n\ (E_2) - n\ (E_1\ n\ E_2)$

$= 4 + 13 - 1 = 16$

FIRST TERM EXAMINATION [SEPT. 2016]
THIRD SEMESTER [B.TECH]
FOUNDATION OF COMPUTER SCIENCE
[ETCS–203]

Time : 1.5 Hrs. M.M. : 30

Note: Attempt three questions in all. Question 1 is compulsory. All questions carry equal marks.

Q.1. (a) What is pigeonhole principle? Explain in brief.

Ans. In a more quantified version: For natural numbers k and m, if $n = \text{km} + 1$ objects are distributed among m sets, then the pigeonhole principle asserts that at least one of the sets will contain at least $k + 1$ objects. For arbitrary n and m this generalizes to $k + 1 = \lfloor (n-1)/m \rfloor + 1$, where $\lfloor ... \rfloor$ is the floor function.

Example: If there are n people who can shake hands with one another (where $n > 1$), the pigeonhole principle shows that there is always a pair of people who will shake hands with the same number of people. As the 'holes', or m, correspond to number of hands shaken, and each person can shake hands with anybody from 0 to $n - 1$ other people, this creates $n - 1$ possible holes. This is because either the '0' or the '$n - 1$' hole must be empty (if one person shakes hands with everybody, it's not possible to have another person who shakes hands with nobody; likewise, if one person shakes hands with no one there cannot be a person who shakes hands with everybody). This leaves n people to be placed in at most n – 1 non-empty holes, guaranteeing duplication.

Q.1.(b) Write the condition of the function to be surjective?

Ans. A surjective function is a function whose image is equal to its codomain. Equivalently, a function f with domain X and codomain Y is surjective if for every y in Y there exists at least one x in X with f (x) = y .

Symbolically,

If $f : X \to Y$ f is said to be surjective if

$\forall y \in Y, \exists x \in X, f(x) = y$

Q.1. (c) Compute truth table of $(p \leftrightarrow q) \vee (\sim q \leftrightarrow r)$.

Ans. p q r (p<->q) $\vee$ (~q <->r)

F F F T

F F T T

F T F T

F T T F

T F F F

T F T T

T T F T

T T T T

Q.1. (d)

Ans. p: if $x > 5$

q: $x^2 > 25$

predicate : $(\forall x \in R)(p \to q)$

negation: $(\exists x \in X)(p \wedge \sim q)$

Q.1. (e) Define lattices?

Ans. A lattice is an abstract structure studied in the mathematical subdisciplines of order theory and abstract algebra. It consists of a partially ordered set in which every two elements have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet). An example is given by the natural numbers, partially ordered by divisibility, for which the unique supremum is the least common multiple and the unique infimum is the greatest common divisor.

Q.2. (a) Show that $\sim p \wedge (\sim q \wedge r) \vee (q \vee r) \vee (p \wedge r) \equiv r$.

Ans.

p	q	r	~p	~q	~q^r	q^r	p^r	RESULT OF EXPRESSION
F	F	F	T	T	F	F	F	F
F	F	T	T	T	T	F	F	T
F	T	F	T	F	F	T	F	F
F	T	T	T	F	F	T	F	T
T	F	F	F	T	F	F	F	F
T	F	T	F	T	T	F	T	T
T	T	F	F	F	F	F	F	F
T	T	T	F	F	F	T	T	T

Q.2. (b) Prove the statement "if × is an integer and x^2 is even then × is also even.

Ans. We have the following hypothesis: An integer n is even iff n^2 is even

Proof must be in two parts:

Part I: If n is even, then n^2 is even.

Proof: If n is even then by definition $n = 2a$ for some integer a. Therefore $n^2 = n \times n =$ $(2a)(2a) = 2(2a^2)$ which is also even by definition. Therefore I have shown that if n is even then n^2 is also even.

Part 2: If n^2 is even, then n is even

Proof: For this proof I will use the contrapositive statement which is equivalent to the original statement. If n is odd, then n^2 is odd.

Therefore by definition of odd we have that $n = 2p + 1$ for some integer p. Now consider $n^2 = n \times n = (2p + 1)(2p + 1) = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1 = 2k + 1$, where k is the integer value $2p^2 + 2p$. So by the definition of odd we have that n^2 is odd. There the contrapositive is true and the original statement (which is equivalent) is also true.

Have successful proven both parts of my hypothesis, I now have that an integer n is even iff n^2 is even.

Q.3. (a) In how many ways can a team of 11 criketers be chosen from 6 bowlers, 4 wicket keepers and 11 batsmen to give a majority of batsmen if atleast 4 bowlers are to be included and there is one wicket keeper? (5)

Ans. 1 wicket keeper can be selected in C(4,1) ways

4 boulers chosen = C(6,4)

Remaining 6 batsmen = C(11,6)

Total possibilities = C(4,1) * C(6,4) * C(11,6) = 27720.

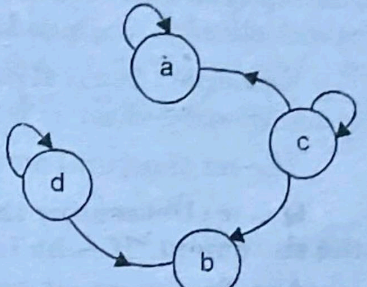
The batsmen has to be majority. So the split cannot be 1 WC, 5 Bowlers, 5 Batsmen. It can only be 1 WC, 4 bowlers and 6 batsmen.

Q.3. (b) Let A = {a, b, c, d} and R be the relation on set A that has the matrix representation given as. (5)

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Construct the diagraph of R and find the indegree and outdegree of all the vertices.

Ans.



INDEGREE OUTDEGREE

A 2 1

B 3 1

C 1 3

D 1 2

Q.4. (a) Check the validity of the argument. If the races are fixed or the casions are cooked, then the tourist trade will decrease, if the tourist trade decrease, then the police will be happy. The Police force is never happy therefore the races are not fixed.

Ans. A: $(p+q) \to r$

B: $r \to s$

C: $\sim s$

$\sim p$

p	q	r	s	p+q	(p+q)->r(1)	r->s(2)	1^2 (3)	~s(4)	3^4(5)	~p(6)	5->6
F	F	F	F	F	T	T	T	T	T	T	T
F	F	F	T	F	T	T	T	F	F	T	T
F	F	T	F	F	F	F	F	F	F	T	T
F	F	T	T	F	F	F	F	F	F	T	T
F	T	F	F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	F	F	T	T
F	T	T	F	T	T	F	F	T	F	T	T
F	T	T	T	T	T	T	T	F	F	T	T
T	F	F	F	T	T	T	T	T	T	F	F
T	F	F	T	T	T	T	T	F	F	F	T
T	F	T	F	T	T	F	F	T	F	F	T
T	F	T	T	T	T	T	T	F	F	F	T
T	T	F	F	T	T	T	T	T	T	F	F
T	T	F	T	T	T	T	T	F	F	F	T
T	T	T	F	T	T	F	F	T	F	F	T
T	T	T	T	T	T	T	T	F	F	F	T

Q.4. (b) What is the necessary condition for the relation to become poset? Explain with example.

Ans. In mathematics, especially order theory, a partially ordered set (also poset) formalizes and generalizes the intuitive concept of an ordering, sequencing, or arrangement of the elements of a set. A poset consists of a set together with a binary relation indicating that, for certain pairs of elements in the set, one of the elements precedes the other in the ordering. The word "partial" in the names "partial order" or "partially ordered set" is used as an indication that not every pair of elements need be comparable. That is, there may be pairs of elements for which neither element precedes the other in the poset. Partial orders thus generalize total orders, in which every pair is comparable.

To be a partial order, a binary relation must be reflexive (each element is comparable to itself), antisymmetric (no two different elements precede each other), and transitive (the start of a chain of precedence relations must precede the end of the chain).A poset can be visualized through its Hasse diagram, which depicts the ordering relation.

Example : The real numbers ordered by the standard less-than-or-equal relation ≤ (a totally ordered set as well).

The set of natural numbers equipped with the relation of divisibility.

Q.... (c) Determine the converse, inverse ana contrapositive statement for the statement "If John is a poet, then he is poor".

Ans. Statement : If John is a poet, then he is poor.

Converse : If he is poor then John is a poet.

Inverse : If John is not a poet then he is not poem.

Contrapositive : If he is not poor then John is not a poet.

END TERM EXAMINATION [DEC. 2016]
THIRD SEMESTER [B.TECH]
FOUNDATION OF COMPUTER SCIENCE
[ETCS–203]

Time : 3 Hrs. M.M. : 75

Note: Attempt three questions in all. Question 1 is compulsory. All questions carry equal marks.

Q.1.(a) Define the connectives conjuncton and disjunction and give the truth table for p V q.

Ans. In logic, a conjunction is a compound sentence formed by using the word and to join two simple sentences. The symbol for this is ^. (whenever you see ^ read 'and') When two simple sentences, p and q, are joined in a conjunction statement, the conjunction is expressed symbolically as p ^ q. In logic, a disjunction is a compound sentence formed by using the word or to join two simple sentences. The symbol for this is v. (whenever you see v read 'or') When two simple sentences, p and q, are joined in a disjunction statement, the disjunction is expressed symbolically as p ψ q.

TRUTH TABLE

p	q	p v q
T	T	T
T	F	T
F	T	T
F	F	F

Q.1.(b) Determine the contrapositive of the statement "If John is a poet, then he is poor".

Ans. If John is not poor, then he is not a poet.

Q.1.(c) State and prove the De Morgan's law for a Boolean algebra.

Ans. The complement of the union of two sets is equal to the intersection of their complements and the complement of the intersection of two sets is equal to the union of their complements. These are called De Morgan's laws.

For any two finite sets A and B;

(i) $(A \cup B)' = A' \cap B'$ (which is a De Morgan's law of union).

(ii) $(A \cap B)' = A' \cup B'$ (which is a De Morgan's law of intersection).

Proof of De Morgan's law: $(A \cup B)' = A' \cap B'$

Let $P = (A \cup B)'$ and $Q = A' \cap B'$

Let x be an arbitrary element of P then $x \in P \Rightarrow x \in (A \cup B)'$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

$\Rightarrow x \in Q$ ...(i)

Therefore, $P \subset Q$

Again, let y be an arbitrary element of Q then $y \in Q \Rightarrow y \in A' \cap B'$

$\Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$

$\Rightarrow y \in P$

Therefore, $Q \subset P$

Now combine (*i*) and (*ii*) we get; $P = Q$ i.e. $(A \cup B)' = A' \cap B'$ ...(*ii*)

Proof of De Morgan's law: $(A \cap B)' = A' \cup B'$

Let $M = (A \cap B)'$ and $N = A' \cup B'$

Let x be an arbitrary element of M then $x \in M \Rightarrow x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

$\Rightarrow x \in N$

Therefore, $M \subset N$ ...(*i*)

Again, let *y* be an arbitrary element of N then $y \in N \Rightarrow y \in A' \cup B'$

$\Rightarrow y \in A'$ or $y \in B'$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin (A \cap B)$

$\Rightarrow y \in (A \cap B)'$

$\Rightarrow y \in M$

Therefore, $N \subset M$ ... (*ii*)

Now combine (*i*) and (*ii*) we get; $M = N$ i.e. $(A \cap B)' = A' \cup B'$

Q.1. (*d*) Find DNF for the fundtion $F(x, y, z) = (x + y)z'$.

Ans. DNF FORM : (x^z') ∨ (y^z')

Q.1. (*e*) Define function, domain, Co-domain and range of a tnction.

Ans. A function relates each element of a set with exactly one element of another set(possibly the same set).The "domain" of a function or relation is: theset of all values for which it can be evaluated the set of allowable "input" values. he "range" of a function or relation is: the set of all values that it can produce its "output" set of values. The "co-domain" of a function or relation is a set of values that inclues the Range as described above, but may also include additional values beyond thos in the range.

Q.2. Prove the following:

Q.2. (*a*) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Ans. For any sets A and B, prove that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

We have by venn diagram

A – B, B – A, $(A \cap B)$ are disjoint and their union is $(A \cup B)$.

$$\begin{aligned} n(A \cup B) &= n(A - B) + n(A \cap B) + n(B - A) \\ &= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B) \\ &= n(A) + n(B) - n(A \cap B) \end{aligned}$$

$$n(A - B) + n(A \cap B) = n(A) \text{ and } n(B - A) + n(A \cap B) = n(B)]$$

Hence $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Q.2. (*b*) $p \rightarrow (q \vee r)$, $(s \wedge t) \rightarrow q$, $(q \vee r) \rightarrow (s \wedge t)$ then $p \rightarrow q$. (5)

Ans. Consider the two propositions ∃x(P(x) ^ Q(x)) and ∃xP(x)^∃Q(x) where P(x) and Q(x) are propositional functions, and x belongs to some Universef Discourse.

∃x(P(x)^Q(x) ⇒ ∃xP(x)^∃xQ(x). If ∃x(P(x)^Q(x)) is true,then theris an element of the U of D, call it c such that P(c) ^ Q(c) is true. But then ∃xP(x) is trubecause we can choose x = c and ∃xQ(x) is true because we can choose x=c. Therefore ∃xP(x)^∃xQ(x) is true.

Q.2. (c) (∃x) (p(x) ∧ Q(x)) ⇒ (∃x) ∧ (p(x)) ∧ (∃x). (Q(x)).

Ans. consider

p -> (qvr)

and

(qvr) -> (s ^ t)

Therefore, p -> (s ^ t) [acc. To Hypothetical Syllogism]

Now,

(s ^ t) -> q

Therefore,

p -> q [acc. To Hypothetical Syllogism]

Q.3.(a) Expian the principle of mathematical induction. (4)

Ans. It is common that the validity of a statement must be proved for all natural numbers starting from a certain number. This starting number is usually 0 or 1. Assume, for now, that the starting number is 1.

Example: 'The sum of the natural numbers from 1 to n equals n(n+1)/2'.

This is a property which depends on n. We write this fact as E(n). E(3) means in our example :

'The sum of the natural numbers from 1 to 3 equals 3(3+1)/2'.

A proof by induction is a common method to prove such a property. We show how the method works and why it is correct.

Let V = { n | E(n) is true }. First we prove E(1). Then we know that 1 is in V.

Then we prove that: If E(k) is true, then E(k+1) is true.

If that is proved, we know that: if k is in V then (k+1) is in V.

But 1 is in V, so 2 is in V, so 3 is in V, etc...

Then V = {1, 2, 3, 4, ...}

We have shown that the property E(n) is true for all natural numbers starting with 1.

Q.3.(b) Explain the partial ordered relation with the help of suitable example. (4)

Ans. Let R be a binary relation on a set A.

R is antisymmetric if for all x, y A, if xRy and yRx, then x = y.

R is a partial order relation if R is reflexive, antisymmetric and transitive.

In terms of the digraph of a binary relation R, the antisymmetry is tantamount to saying there are no arrows in opposite directions joining a pair of (different) vertices.

Example: Let A = {0,1,2} and R = {(0,0),(0,1),(0,2),(1,1), (1,2), (2,2)} and S = {(0, 0),(1, 1),(2, 2)} be 2 relations on A. Show

(i) R is a partial order relation.

(ii) S is an equivalence relation.

Solution: We choose to use digraphs to make the explanations in this case.

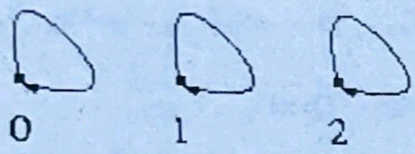
(i)

The digraph for R on the right implies

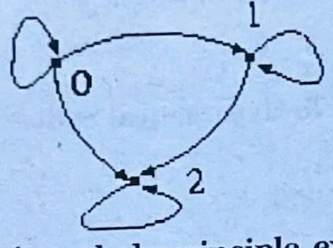
Reflexive: loops on every vertex.

Transitive: if you can travel from vertex v to vertex w along consecutive arrows of same direction, then there is also a single arrow pointing from v to w.

 Antisymmetric: no type of arrows.



(*ii*) The digraph for S on the right is reflexive due to loops on every vertex, and is symmetric and transitive because no no-loop arrows exist.



Q.3.(*c*) What is extended pigeonhole principle, explain with suitable example. (4.5)

Ans. The Extended PigeonHole Principle

If there are m pigeonholes and more than 2m pigeons , then three or more pigeons will have to be assigned to at least one pigeonhole.

If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain at least

$$[(n-1)/m] + 1 \text{ pigeons.}$$

Proof: If each pigeonhole contains no more than $[(n-1)/m]$ pigeons, then there are at least

m*[(n – 1) / m] <= m × (n – 1) / m = n – 1 pigeons in all.

The above contradicts our assumptions, so one of the pigeonholes must contain at least [(n – 1)/ m] + 1 pigeons.

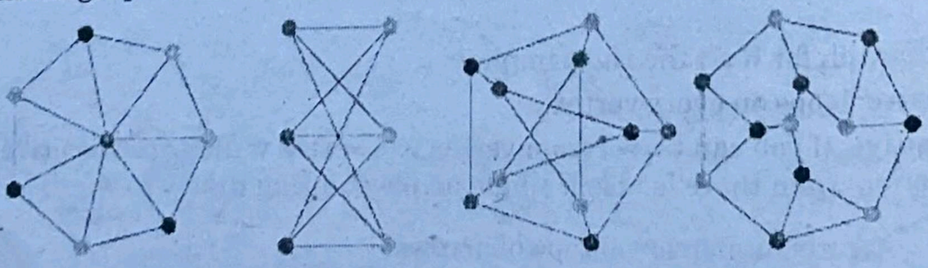
Example: If 71 letters are distributed to 70 mailboxes, then atleast one mailbox must contain at least 2 letters. If 71 letters are distributed to 10 mailboxes, then at least one mailbox must contain at least 8 letters.

[(n – 1) / m] + 1 = [(71 – 1)/70] + 1 = 1 (71 letters in 70 mailboxes)

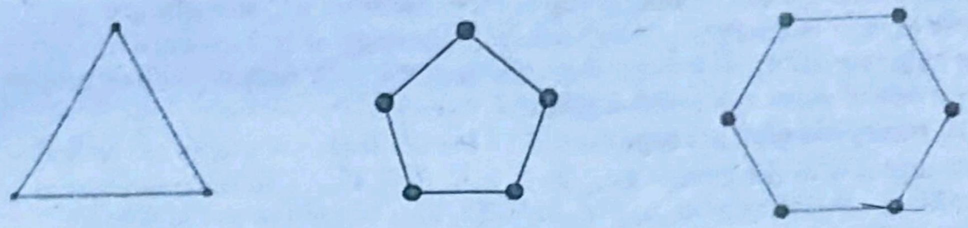
[(n – 1) / m] + 1 = [(71 – 1)/10] + 1 = 8 (71 letters in 10 mailboxes)

Q.4. (*a*) Explain Vertex coloring problem and chromatic number of graph using example. (6)

Ans. A vertex coloring is an assignment of labels or colors to each vertex of a graph such that no edge connects two identically colored vertices. The most common type of vertex coloring seeks to minimize the number of colors for a given graph. Such a coloring is known as a minimum vertex coloring, and the minimum number of colors which with the vertices of a graph G may be colored is called the chromatic number, denoted chi(G). A vertex coloring of a graph with k or fewer colors is known as a k-coloring. A graph having a k-coloring (and therefore chromatic number chi(G)<=k) is said to be a k-colorable graph, while a graph having chromatic number chi(G)=k is called a k-chromatic graph. The only one-colorable (and therefore one-chromatic) graphs are empty graphs, and two-colorable graphs are exactly the bipartite graphs. The four-color theorem establishes that all planar graphs are 4-colorable.



The smallest number of colors which are needed to color a graph is known as the chromatic number of the graph. Cycle Graph Chromatic Number: 2 colors if number of vertices is even



3 colors if number of vertices is odd

Q.4.(b) Show that minimum number of edges in a connected graph with n vertices is (n – 1).

Ans. Let G be a connected Graph : If G has no cycles then G is connected with no cycles =>G is a Tree. So G has n-1 edges.

If G has cycles : and G is connected then for every two vertices there is a path between them. Assuming that G have only one cycle: lets look at the path : v1,v2..vn,v1 we can remove the edge v1,v1 and we will get a connected sub Graph v1, v2 with no cycles and $E(H) + 1 = E(G)$ so $E(G) = n$.

And by induction you will get that for every number of cycles n $E(G) \geq n$.

So if G has cycles $E(G) = n - 1$ else $E(G) \geq n$.

Q.5. (a) Show that all proper subgroups of groups of order 8 must be abelian. **(4)**

Ans. Consider the following set of invertible 2×2 matrices with entries in the field of complex numbers.

Subgroup of quaternion group Q_8. are:

$$\{\{1\}, \{\pm 1\}, \{\pm 1, \pm i\},$$
$$\{\pm 1, \pm j\}, \{\pm 1, k\}, Q_8\}$$

Matrices with entries in the field of complex numbers.

$$\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \pm\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$i^2 = j^2 = k^2 = -1;$$
$$ij = k, jk = i, ki = j;$$
$$ji = -k, kj = -i, ik = -j.$$

In order 8 called the quaternion group, or group of quaternion.

These elements form a abelian group Q of order 8 called the quaternion group,

Q.5.(b) Define cyclic group with example. **(4)**

Ans. A cyclic group is a group that can be generated by a single element X (the group generator). Cyclic groups are Abelian.

A cyclic group of finite group order n is denoted C_n, Z_n, Z_n, or C_n and its generator X satisfies

$$X\hat{\ }n = I,$$

where I is the identity element.

The ring of integers Z form an infinite cyclic group under addition, and the integers 0, 1, 2, ..., n-1 (Z_n) form a cyclic group of order n under addition (mod n). In both cases, 0 is the identity element.

There exists a unique cyclic group of every order n> = 2, so cyclic groups of the same order are always isomorphic . Furthermore, subgroups of cyclic groups are cyclic, and all groups of prime group order are cyclic. In fact, the only simple Abelian groups are the cyclic groups of order n = 1 or n a prime .

The nth cyclic group is represented in the Wolfram Language as CyclicGroup[n].

Examples of cyclic groups include C_2, C_3, C_4, ..., and the modulo multiplication groups M_m such that m=2, 4, p^n, or 2p^n, for p an odd prime and n>=1

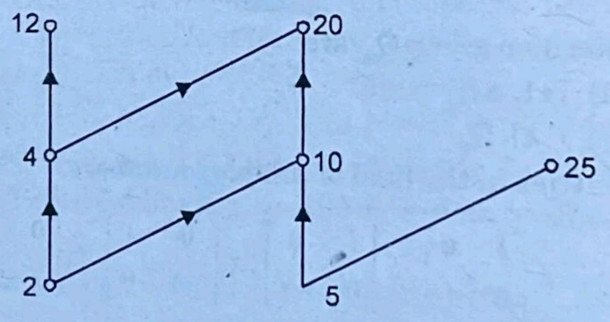
Q.5.(*c*) Prove that the group (G, +6) is a cyclic group where G = {0, 1, 2, 3, 4, 5}. **(4.5)**

Ans. A group G is called cyclic if there exists an element g in G such that G = {g} = {gn I n is an integer}. Since any group generated by an element in a group is a subgroup of that group, showing that the only subgroup of a group G that contains g is G itself suffices to show that G is cyclic.

For example, if G = { g0, g1, g2, g3, g4, g5 } is a group of order 6, then g6 = g0, and G is cyclic. In fact, G is essentially the same as (that is, isomorphic to) the set {0, 1, 2, 3, 4, 5 } with addition modulo 6. For example, $1 + 2 \equiv 3 \pmod 6$ corresponds to g1 · g2 = g3, and $2 + 5 \equiv 1 \pmod 6$ corresponds to g2 · g5 = g7 = g1, and so on. One can use the isomorphism χ defined by $\chi(gi) = i$.

Q.6. (*a*) Draw the hasse diagram for the divisibility for the divisibility relation on {2, 4, 5, 10, 12, 20, 25} starting from the digraph. **(4)**

Ans. Example: Draw the Hasse diagram represented the partial ordering {(a, b)/ a/b} on {2, 4, 5, 10, 12, 20, 25}.



Sol. Here the relation set = {(2, 4), (2, 12), (4, 12), (5, 10), (5, 20), (5, 25), (10, 20)}

Q.6.(*b*) Define lattice and give an example. **(4)**

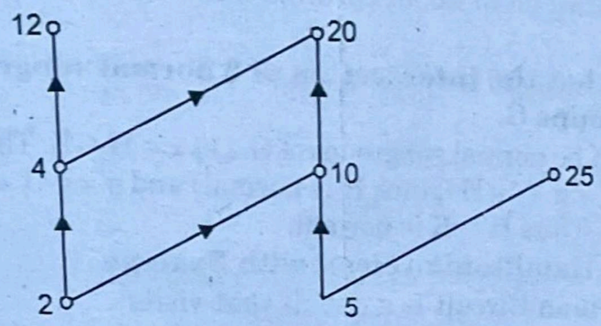
Ans. A lattice is an abstract structure studied in the mathematical subdisciplines of order theory and abstract algebra. It consists of a partially ordered set in which every two elements have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet). An example is given by the natural numbers, partially ordered by divisibility, for which the unique supremum is the least common multiple and the unique infimum is the greatest common divisor. If (L, ≤) is a partially ordered set (poset), and S ⊆ L is an arbitrary subset, then an element u ∈ L is said to be an upper bound of S if s ≤ u for each s ∈ S. A set may have many upper bounds, or none at all. An upper bound u of S is said to be its least upper bound, or join, or supremum, if u ≤ x for each upper bound x of S. A set need not have a least upper bound, but it cannot have more than one. Dually, l ∈ L is said to be a lower bound of S if l ≤ s for each s ∈ S. A lower bound l of S is said to be its greatest lower bound, or meet, or infimum, if x ≤ l for each lower bound x of S. A set may have many lower bounds, or none at all, but can have at most one greatest lower bound.

(*ii*) On similar conditions the lexicographic order is zoo ≤ zoology ≤ zoom.

Hasse Diagram: We can represent the poset as a directed graph. Every poset consisting of a set and a relation can be represented as a graph. We have to do minor modification in this relational graph. Since partial ordering relation is reflexive and transitive the relational graph will consist of self loops and edges corresponding to the transitive relations.

In the relational graph, the self loops and edges corresponding to the pairs (a, c) are removed whenever (a, b) and (b, c) are present. Finally, each edge is arranged so that its initial vertex is below its terminal vertex. All the arrows on the directed edges, are removed since all the edges point upward towards their terminal vertex.

Example: Draw the Hasse diagram represented the partial ordering {(a, b) / a/b} on {2, 4, 5, 10, 12, 20, 25}.



Sol. Here the relation set = {(2, 4), (2, 12), (4, 12), (5, 10), (5, 20), (5, 25), (10, 20)}

Q.6.(c) Explain principle of inclusion and exclusion with an example. (4.5)

Ans. In combinatorics (combinatorial mathematics), the inclusion–exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets; symbolically expressed as

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

where A and B are two finite sets and $|S|$ indicates the cardinality of a set S (which may be considered as the number of elements of the set, if the set is finite). The formula expresses the fact that the sum of the sizes of the two sets may be too large since some elements may be counted twice. The double-counted elements are those in the intersection of the two sets and the count is corrected by subtracting the size of the intersection.

Examples:

Counting integers

As a simple example of the use of the principle of inclusion–exclusion, consider the question:

How many integers in {1,...,100} are not divisible by 2, 3 or 5?

Let S = {1,...,100} and P1 the property that an integer is divisible by 2, P2 the property that an integer is divisible by 3 and P3 the property that an integer is divisible by 5. Letting Ai be the subset of S whose elements have property Pi we have by elementary counting: $|A1| = 50$, $|A2| = 33$, and $|A3| = 20$. There are 16 of these integers divisible by 6, 10 divisible by 10 and 6 divisible by 15. Finally, there are just 3 integers divisible by 30, so the number of integers not divisible by any of 2, 3 or 5 is given by:

$$100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

Q.7.(*a*) Explain lagrange's Theorem with proof. (8)

Ans. Lagrange's theorem, in the mathematics of group theory, states that for any finite group G, the order (number of elements) of every subgroup H of G divides the order of G.

This can be shown using the concept of left cosets of H in G. The left cosets are the equivalence classes of a certain equivalence relation on G and therefore form a partition of G. Specifically, x and y in G are related if and only if there exists h in H such that x = yh. If we can show that all cosets of H have the same number of elements, then each coset of H has precisely |H| elements. We are then done since the order of H times the number of cosets is equal to the number of elements in G, thereby proving that the order of H divides the order of G. Now, if aH and bH are two left cosets of H, we can define a map f : aH → bH by setting f(x) = ba – 1x. This map is bijective because its inverse is given by f – 1 (y) = a b – 1 y .

This proof also shows that the quotient of the orders |G| / |H| is equal to the index [G : H] (the number of left cosets of H in G). If we allow G and H to be infinite, and write this statement as

$$|G| = [G:H].|H|,$$

then, seen as a statement about cardinal numbers, it is equivalent to the axiom of choice.

Q.7. (b) Show that the intersection of 2 normal subgroups of a group G is also normal subgroups G.

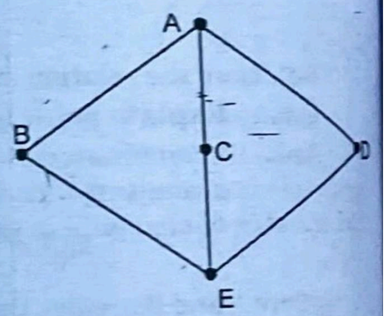
Ans. Let H and K be normal subgroups of G. Let x ∈ H ∩ K. Then x ∈ H and x ∈ K. For any element g ∈ G g × g – 1∈H (since H is normal) and g × g–1 ∈ K since K is normal). So g × g – 1 ∈ H ∩ K. Thus H ∩ K is normal.

Q.8. (a) Define Hamiltonian ciruit with Example. (2.5)

Ans. A Hamiltonian Circuit is a circuit that visits every vertex exactly once.

Do these graphs have a Hamiltonian circuit?

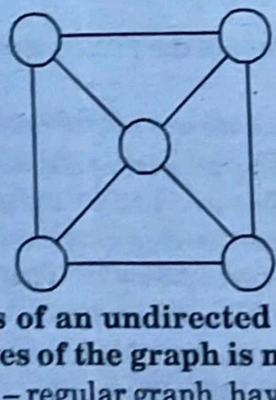
Example 1:



Q.8. (b) Give an example of graph which contains a Hamiltonian circuit, but not a Eulerian Circuit. (5)

Ans. Eulerian Trail: A connected graph G is Eulerian if there is a closed trail which includes every edge of G, such a trail is called an Eulerian trail.

Hamiltonian Cycle: A connected graph G is Hamiltonian if there is a cycle which includes every vertex of G; such a cycle is called a Hamiltonian cycle.



Q.8. (c) If all the vertices of an undirected graph are each odd degree show that the number of edges of the graph is multiple of k.

Ans. Let G be a graph i.e. k – regular graph, having n vertices, where k is add

Let $n = 2\sigma$

must be even

Now, sum of degree of all vertices in $6 = nk = 2\sigma k = 2e$

So $2\sigma k = 2e$ yields $\sigma k = e$

$= \text{multiple of } k$

Hence proved.

FIRST TERM EXAMINATION [SEPT. 2017]
THIRD SEMESTER [B.TECH]
FOUNDATION OF COMPUTER SCIENCE
[ETCS–203]

Time : 1.30 hrs. **M.M. : 30**

Note: *Attempt any three question including Q.1 is compulsory.*

Q.1. (a) What is principle of Inclusion and Exclusion? Explain in brief. (2)

Ans. In combinatorics (combinatorial mathematics), the **inclusion–exclusion principle** is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets. The double-counted elements are those in the intersection of the two sets and the count is corrected by subtracting the size of the intersection.

Examples

Counting integers

As a simple example of the use of the principle of inclusion–exclusion, consider the question

How many integers in {1,...,100} are not divisible by 2, 3 or 5?

Let $S = \{1,...,100\}$ and P_1 the property that an integer is divisible by 2, P_2 the property that an integer is divisible by 3 and $P_3$the property that an integer is divisible by 5. Letting A_i be the subset of S whose elements have property P_i we have by elementary counting: $|A_1| = 50$, $|A_2| = 33$, and $|A_3| = 20$. There are 16 of these integers divisible by 6, 10 divisible by 10 and 6 divisible by 15. Finally, there are just 3 integers divisible by 30, so the number of integers not divisible by any of 2, 3 or 5 is given by:

$100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$

Q. 1. (b) What is Function? Write the condition of the function to be injective? (2)

Ans. A **function** is a *rule* which relates the values of one variable quantity to the values of another variable quantity, and does so in such a way that the value of the second variable quantity is *uniquely determined by* (i.e. *is a function of*) the value of the first variable quantity.In mathematics, an injective function or injection or one-to-one function is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its codomain. In other words, every element of the function's codomain is the image of at most one element of its domain.

Q. 1. (c) Find the Converse and contrapositive of the statement "If x is positive then $x \neq 0$" (2)

Ans. Converse: if x not equal to 0 then x is positive

Contrapositive: if x is equal to 0 then x is negative

Q. 1. (d) Represent the statement using predicate and quantifier and negate it. For all the real number x if x > 5 then $x^2 > 25$ (2)

Ans. $\forall$ R(x):X>5 -> X²>25

Q. 1. (e) Define lattices? (2)

Ans. A **lattice** is an abstract structure studied in the mathematical subdisciplines of order theory and abstract algebra. It consists of a partially ordered set in which every two elements have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet). An example is given by the natural numbers, partially ordered by divisibility, for which the unique supremum is the least common multiple and the unique infimum is the greatest common divisor.

Q. 2. (a) Show that $((p \to q) \wedge (q \to r)) \to (p \to r)$ is tautology By Rules of preposition. (5)

Ans.

p	q	r	p→q	q→r	(p→q)^(q→r)	p→r	[(p→q) ^ (q→r)] → (p→r)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Q. 2. (b) Prove that if X,Y,∈Z (set of integer) such that xy is odd then both x and y are odd, by proving its contrapositive. (5)

Ans. Let x,yx,y be integers. "If xyxy is even, then xx is even or yy is even". Pattern matching that to what is above, then A = "xyxy is even" and B = " xx is even or yy is even." Hence the contrapositive would be:

If ¬(x¬(x is even or yy is even)) then ¬(xy¬(xy is even))

= If xx is not even and yy is not even, then xyxy is not even.

= If xx is odd and *yy* is odd then *xyxy* is odd.

At this point you can proceed as you correctly guessed, letting x = 2a + 1x = 2a +1 and y = 2b + 1y = 2b + 1 for some a, b ∈ Za, b ∈ Z. You'll find that xy = 2c + 1xy = 2c + 1 for some c ∈ Z c ∈ Z.

Q. 3. (a) In how many ways can a team of 11 cricketers be chosen from 6 bowlers, 4 wicket keepers and 11 batsmen to give a majority of bastmen if atleast 4 bowlers are to be included and there is one wicket keeper? (5)

Ans. 1 wicket keeper can be selected in C(4,1) ways

4 bowlers chosen = C(6,4)

Remaining 6 batsmen = C(11,6)

Total possibilities = C(4,1) * C(6,4) * C(11,6) = 27720.

Q. 3. (b) Let A = (1,2,3,4,6) and R is a Relation on the Set A such that aRb if a divides b (5)

Find (i) Relation R

(ii) Diagraph of R

(iii) Find Adjacency matrix of R

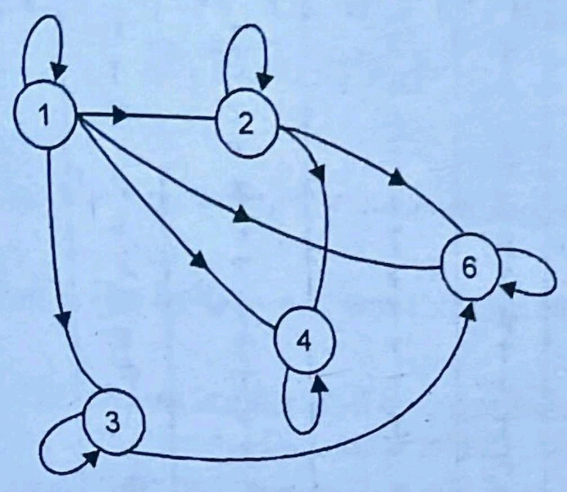
(iv) Indergree and outdegree of each node

(v) Find its Hasse diagram

Ans. A = {1, 2, 3, 4, 6} aRb of a divides b (*i*) Relation R

= {(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)}

(*ii*) Diagraph



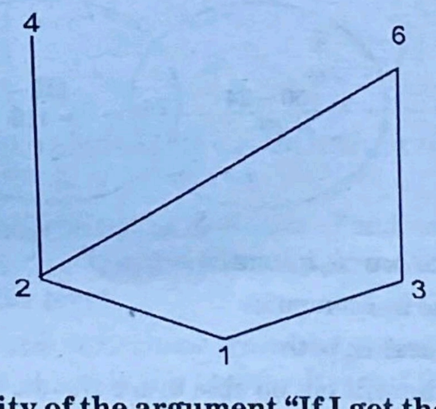
(*iii*) Adjacency matrix

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array}\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(*iv*)

vertices	1	2	3	4	6
Indegree	1	2	2	3	4
Outdegree	4	2	1	0	0

(*v*) Hasse diagram



Q. 4. (a) Prove the validity of the argument "If I get the job and work hard, then I will get promoted. If I get promoted, then I will be happy I will not be happy. Therefore, either I will not get the job or I will not work hard". (5)

Ans.

p = I get the job

r = I will get promoted

s = I will be happy

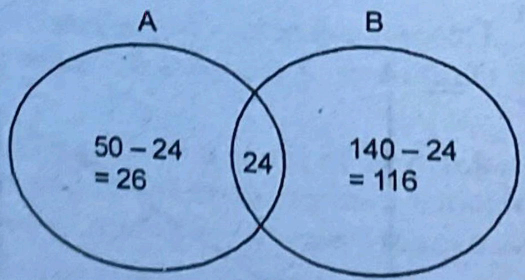
q = I will work hard

p	q	r	s	$(p \sim q) \to r$ ①	$r \to s$ ②	$\sim s$ ③	① ∧ ② ∧ ③ ⇒ $\tilde{p} \vee \tilde{q}$
T	T	T	T	T	T	F	T
T	T	T	F	T	F	T	T
T	T	F	T	F	T	F	T
T	T	F	F	F	T	T	T
T	F	T	T	T	T	F	T
T	F	T	F	T	F	T	T
T	F	F	T	T	T	F	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	T	F	T	T	F	T
F	T	F	T	T	T	T	T
F	T	F	F	T	T	F	T
F	F	T	T	T	T	T	T
F	F	T	F	T	F	F	T
F	F	F	T	T	T	T	T
F	F	F	F	T	T	F	T

Hence validity is proved

Q. 4. (b) Out of the 200 students, 50 of them take the course in mathematics 140 of them take the course economics and 24 of them take both the course Since both courses have schedule examinations for the following day, only those students who are not taken any of these courses will be able to go to see movie. How many students will be able to go to see movie. (5)

Ans.



A : 50 student take course in maths

B : 140 take course in economics

A ∪ B : 24 take course in both

Find : Students who will not be able to see the movie

$$\therefore \text{Total students studying} = \begin{array}{r} 26 \\ +\ 24 \\ +116 \\ \hline 166 \\ \hline \end{array}$$

No. of total students = 200

$$\therefore \text{Students left} = \begin{array}{r} 200 \\ -166 \\ \hline 34 \\ \hline \end{array}$$

END TERM EXAMINATION [DEC. 2017]
THIRD SEMESTER [B.TECH]
FOUNDATION OF COMPUTER SCIENCE [ETCS-203]

Time : 3 hrs. M.M. : 75

Note: *Attempt any five questions including Q.1 is compulsory. Select one question from each unit.*

Q.1. (a) What do you mean by Quantifiers? Explain nested quantifiers. (2.5)

Ans. The variable of predicates is quantified by quantifiers. There are two types of quantifier in predicate logic " Universal Quantifier and Existential Quantifier.

Universal Quantifier: Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol $\forall$.

$\forall xP(x) \forall\ xP(x)$ is read as for every value of x, P(x) is true.

Existential Quantifier: Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol $\exists$.

$\exists xP(x) \exists\ xP(x)$ is read as for some values of x, P(x) is true.

Two quantifiers are nested if one is within the scope of the other

Q. 1. (b) Show that logical expression $\neg(p \rightarrow q) \rightarrow p$ is a tautology.

Ans.

P	q	$(p\rightarrow q)$	$\sim(p\rightarrow q)$	$\sim(p\rightarrow q)\rightarrow p$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

Q. 1. (c) Prove by contradiction that "If n is an integer and 3n + 2 is odd, then n is odd". (2.5)

Ans. n = 2k +1 for some integer k by the definition of odd integers. 3n +2 = 3(2k+1) + 2 = 6k +3 +2 = 6k +4 +1 =2(3k+2) + 1, so 3n+2 is odd, but we assumed 3n+2 was even. The contradiction completes the proof.

Q. 1. (d) Let f and g be the functions from the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of fog and gof ? (2.5)

Ans. $f(x)= 2x+3$

$g(x) = 3x+2$

fog= $f(g(x))= 2(3x+2) +3 = 6x+7$

gof= $g(f(x))= 3(2x+3) +2 = 6x+ 11$

Q. 1. (e) Use mathematical induction to prove the inequality $n<2^n$. (2.5)

Ans. Step 1: prove for n = 1

$1 < 2$

Step 2:

$n + 1 < 2 . 2^n$

$n < 2 . 2^n - 1$

$n < 2^n + 2^n - 1$

The function $2^n + 2^n - 1$ is surely higher than $2^n - 1$ so if $n < 2^n$ is true (induction step), $n < 2^n + 2^n - 1$ has to be true as well.

Q. 1. (f) How many bit strings of length four do not have two consecutive 1s? (2.5)

Ans. Consider the following recurrence relation:

$$a_1 = 2$$
$$a_2 = 3$$
$$a_{n+1} = a_n + a_{n-1}, n \geq 2$$

We have 8 bit strings

Q. 1. (g) What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

Ans. c_1=1,c_2=2

Characteristic equation: $r^2 - r - 2 = 0$

Solutions: r = [–(–1) ±((–1)2 – 4·1·(–2))1/2]/2·1 =(1±91/2)/2 =(1±3)/2, so r = 2 or r = –1.

So $a_n = \alpha_1$ 2n + α_2 (–1)n.

To find α_1 and α_2, solve the equations for the initial conditions a_0 and a_1: $a_0 = 2 =$ α_1(0) + α_2 (–1)0 a_1 = 7 = α_1(2) (1) + α_2 (–1)1 Simplifying, we have the pair of equations: 2 = $\alpha_1 + \alpha_2$ 7 = $2\alpha_1 - \alpha_2$ which we can solve easily by substitution: $\alpha_2 = 2 - \alpha_1$; 7 = $2\alpha_1$ – (2– α_1) = $3\alpha_1$ – 2; 9 = $3\alpha_1$; α_1 = 3; α_2 = 1.

Final answer: a_n = 3·2n – (–1)n.

Q. 1. (h) Explain the Pascal's Identify and Triangle. (2.5)

Ans. The rows of Pascal's triangle are conventionally enumerated starting with row $n = 0$ at the top (the 0th row). The entries in each row are numbered from the left beginning with $k = 0$ and are usually staggered relative to the numbers in the adjacent rows. The triangle may be constructed in the following manner: In row 0 (the topmost row), there is a unique nonzero entry 1. Each entry of each subsequent row is constructed by adding the number above and to the left with the number above and to the right, treating blank entries as 0. For example, the initial number in the first (or any other) row is 1 (the sum of 0 and 1), whereas the numbers 1 and 3 in the third row are added to produce the number 4 in the fourth row.

The entry in the rth row and kth column of Pascal's triangle is denoted $\binom{n}{k}$. For example, the unique nonzero entry in the topmost row is $\binom{0}{0} = 1$. With this notation, the construction of the rervious paragraph may be written as follows:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

For any non-negative integer n and any integer k between 0 and n inclusive. This recurrence for the binomial coefficients is known as Pascal's rule.

Pascal's triangle has higher dimensional generalizations. The three-dimensional verison is called Pascal's tetrahedron, while the general versions are called Pascal's simplices.

Q. 1. (i) Give the formula for the number of elements in the union of 4 sets A_1, A_2, A_3 & A_4. **(2.5)**

Ans. P (A U B U C U D) = P(A) + P(B) + P(C) +P(D) - P(A ∩ B) - P(A ∩ C) - P(A ∩ D)- P(B ∩ C) - P(B ∩ D) - P(C ∩ D) + P(A ∩ B ∩ C) + P(A ∩ B ∩ D) + P(A ∩ C ∩ D) + P(B ∩ C ∩ D) - P(A ∩ B ∩ C ∩ D).

Q. 1. (j) Find the value of the Boolean Function represented by **(2.5)**
$F(x,y,z) = xy + \overline{z}$.

Ans.

$$F(x,y,z) = x\,y + \sim z$$
$$= (\sim z+x)(\sim z+y)$$

UNIT–I

Q. 2. (a) Explain the pigeonhole principle. How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from O to 100 points? **(4.5)**

Ans.In mathematics, the **pigeonhole principle** states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item. This theorem is exemplified in real life by truisms like "in any group of three gloves there must be at least two left gloves or two right gloves".

The principle has several generalizations and can be stated in various ways. In a more quantified version: for natural numbers k and m, if $n = km + 1$ objects are distributed among m sets, then the pigeonhole principle asserts that at least one of the sets will contain at least $k + 1$ objects. For arbitrary n and m this generalizes to $k + 1 = [(n - 1)/m] + 1 = [n/m]$, where [...] is the floor function and [...] is the ceiling function.To use pigeonhole principle, first find boxes and objects. Suppose that for each score, we have a box that contains a student which got that score in the final exam. The number of boxes is 101, so by the pigeonhole principle, the number of students must be 102 or more.

Q. 2. (b) Find the solution to the recurrence relation- **(8)**

(i) $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

Ans. Let c1, c2, ..., ck be real numbers. Consider the linear homogeneous recurrence relation $a_n = c1a_n-1 + c2a_n-2 + ... + ck\ a_n-k$. Suppose the characteristic equation of the recurrence relation has k distinct characteristic roots r1,r2, ...,rk . Then $\{a_n\}$ is a solution of the recurrence relation if and only if $a_n = \alpha_1 r\ n\ 1 + \alpha_2 r\ n\ 2 + ... + \alpha_k\ r\ n\ k$ for n = 0, 1, 2, ..., where $\alpha_1, \alpha_2, ..., \alpha_k$ are constants.

CE is $r_3-6r_2+11r-6=0 \Rightarrow (r-1)(r-2)(r-3)=0$

Thus the soln is: $a_n=(\alpha_1 1n+ \alpha_2 2n+ \alpha_3 3n)$

(ii) (i) $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

If {a (p) n } is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients an = $c1a_n-1 + c2a_n-2 + ... + ck\ a_n-k + F(n)$, then every solution is of the form {a (p) n + a (h) n }, where {a (h) n } is a solution of the associated homogeneous recurrence relation $a_n = c1a_n-1 + c2a_n-2 + ... + ck\ a_n-k$.

Q. 3. (a) Let R be the relation on the set A = {0, 1, 2, 3} containing the ordered pairs (0, 1), (1, 1), (1, 2), (2, 0), (2, 2) and (3. 0). **(4.5)**

Find (i) Reflexive closure of R (ii) Symmetric closure of R.

Ans. (i) The reflexive closure of R is the relation containing the ordered pairs (0,0), (0,1), (1,1), (1,2), (2,0), (2,2), (3,0), (3,3)

(ii) The symmetric closure of R is the relation containing the ordered pairs (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (2,1), (2,2), (3,0)

Q. 3. (b) Let R be the relation on the set of real numbers such that aRb if and only if a-b an integer. Is R an equivalence relation? (4)

Ans. Reflexive: $\forall a \in R$, aRa $[(a - a) \in Q]$

$a - a = 0$ and $0 \in Q$

Symmetric: $\forall a, b \in R$, $aRb \rightarrow bRa$ $[(a - b) \in Q \rightarrow (b - a) \in Q] \forall$

Let $a - b = c$, where $c \in Q$, then $b - a = -c$, Well if $c \in Q$, then $(-c) \in Q$ as well.

So $(b - a) \in Q$ as well.

Transitive: $\forall a, b, c \in R$, $aRb \wedge bRc \rightarrow aRc$ $[((a - b) \in Q) \wedge ((b - c) \in Q) \rightarrow (a - c) \in Q]$

Let $a - b = d$, where $d \in Q$ and also let $b - c = e$, where $e \in Q$,

Consider $d + e$. Since rational numbers are closed under addition, then $(d + e) \in Q$

Which implies that $[(a - b) + (b - c)] \in Q \rightarrow (a - c) \in Q$

Since the relation is reflexive, symmetric, and transitive, then the relation is an equivalence relation.

Q. 3. (c) Show that $\neg (p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalence. (4)

Ans. $\neg(p (\vee \neg(p \wedge q))$

$\Leftrightarrow \neg p \wedge \neg(\neg(p \wedge q))$ De Morgan's Law

$\Leftrightarrow \neg p \wedge (p \wedge q)$ Double Negation Law

$\Leftrightarrow (\neg p \wedge p) \wedge q$ Associative Law

$\Leftrightarrow F \wedge q$ Contradiction

$\Rightarrow F$

Also for $\sim p \wedge \sim q$, we draw the truth table

p	q	~p	~q	~p ∧ ~q
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	F

So both are logically equivalent since False occurs

UNIT–II

Q. 4. (a) Explain principle of Inclusion-Exclusion. Find how many positive integers not exceeding 1000 are divisible by 7 or 11. (4.5)

Ans. In combinatorics (combinatorial mathematics), the **inclusion–exclusion principle** is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets; symbolically expressed as

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

where A and B are two finite sets and $|S|$ indicates the cardinality of a set S (which may be considered as the number of elements of the set, if the set is finite). The formula expresses the fact that the sum of the sizes of the two sets may be too large since some elements may be counted twice. The double-counted elements are those in the intersection of the two sets and the count is corrected by subtracting the size of the intersection.

The number of positive integers n less than 1000 that are divisible by 7: [1000/7] = 142

The number of positive integers divisible by both 7 and 11 are those that are divisible by 77: [1000/77] = 12

⇒ The number of positive integers divisible by 7 but not by 11 is 142 – 12 = 130.

The number of positive integers less than 1000 that are divisible by 11: [1000/11] = 90

⇒ the number of positive integers less than 1000 that are divisible by either 7 or 11 is 142 + 90 - 12 = 220.

⇒ the number of positive integers less than 1000 that are divisible by exactly by one of 7 and 11 is 220 - 12 = 208

Q. 4. (b) Obtain: **(8)**

(i) PDNF form of $\{(p \wedge q) \vee (\neg p \wedge r) \vee (p \wedge r)\}$

(ii) PCNF form of $[(p \vee q) \wedge (\neg p \rightarrow \neg q)]$.

Ans. (i) $[(p \wedge q) \vee (\sim p \wedge r) \vee (p \wedge r)$

PDNF: $[(p \wedge q) \vee F] \vee [(\sim p \wedge r) \vee F] \vee [p \wedge r) \vee F]$

$= [(p \wedge q) \vee [(r \wedge \sim r)] \vee [(\sim p \wedge r) \vee (q \wedge \sim q) \vee [(p \wedge r) \vee (q \wedge \sim q)]$

$= (p \wedge q \wedge r) \vee [(p \wedge q \wedge \sim r)] \vee [(\sim p \wedge r \wedge q) \vee (\sim p \wedge r \wedge \sim q) \vee (p \wedge \sim q \wedge r)$

(ii) $[(p \vee q) \wedge (\sim p \rightarrow \sim q)]$

PCNF: $(p \vee q) \wedge (p \vee \sim q)$

Q. 5. (a) Show that the hypothesis "It is not sunny this afternoon and it is colder than yesterday" We will go swimming only if it is sunny". "if we do not go swimming, then we will take a conoe trip", and "If we take a canoe trip then we will be home by sunset" lead to the conclusion "we will be home by sunset." **(5.5)**

Ans. p: it is not sunny this afternoon

q: it is colder than yesterday

r: we will go swimming

s: we will take a canoe trip

t: we will be home by sunset

$p \wedge q$

$r \rightarrow p$ (acc to implication rule)

$\sim r \rightarrow s$ (acc to association rule)

$s \rightarrow t$

Therefore: t is true.

Q. 5. (b) Let A = $\{x \mid 3x^2 - 7x - 6 = 0\}$ and B = $x \mid 6x^2 - 5x - 6 = 0\}$, then find $A \cap B$. **(3)**

Ans. $A = \{x \mid 3x^2 - 7x - 6 = 0\}$

$x = -2/3 = -0.667$

$x = 3$

$B = x \mid 6x^2 - 5x - 6 = 0\}$

$x = -2/3 = -0.667$

$x = 3/2 = 1.500$

so A INTERSECTION B : x = –0.667

Q. 5. (c) Prove that $\sqrt{2}$ is irrational by giving proof by Contradiction. (4)

Ans. Here $p : \sqrt{2}$ is an irrational number. We assume that $\sim p$ is true, that is, $\sqrt{2}$ is not an irrational number. This implies that $\sqrt{2}$ is a rational number. We know that every rational number can be expressed in the form of $\frac{p}{q}(q \neq 0)$, where p and q have no common factor (assuming these are the lowest terms).

Let $\sqrt{2} = \frac{p}{q}$ such that p and q have no common factor

$$\Rightarrow \quad \sqrt{2}\,q = p$$

$$\Rightarrow \quad 2q^2 = p^2$$

$\Rightarrow p^2$ is an even number.

$\Rightarrow p$ is an even number (since if p^2 is even, p must be even).

$\Rightarrow p = 2k$ for some integer k.

$\Rightarrow p^2 = 4k^2$

$\Rightarrow q^2 = 2k^2$ (on substituting the value of p^2 in $2q^2 = p^2$).

$\Rightarrow q^2$ is an even number.

$\Rightarrow q$ is an even number.

$\Rightarrow$ 2 is the common factor of a and b.

This is a contradiction that p and q have no common factor. Thus, the assumption '$\sim p$ is true" that is, '$\sqrt{2}$ is not an irrational no.' is false. Hence, $\sqrt{2}$ is an irrational number.

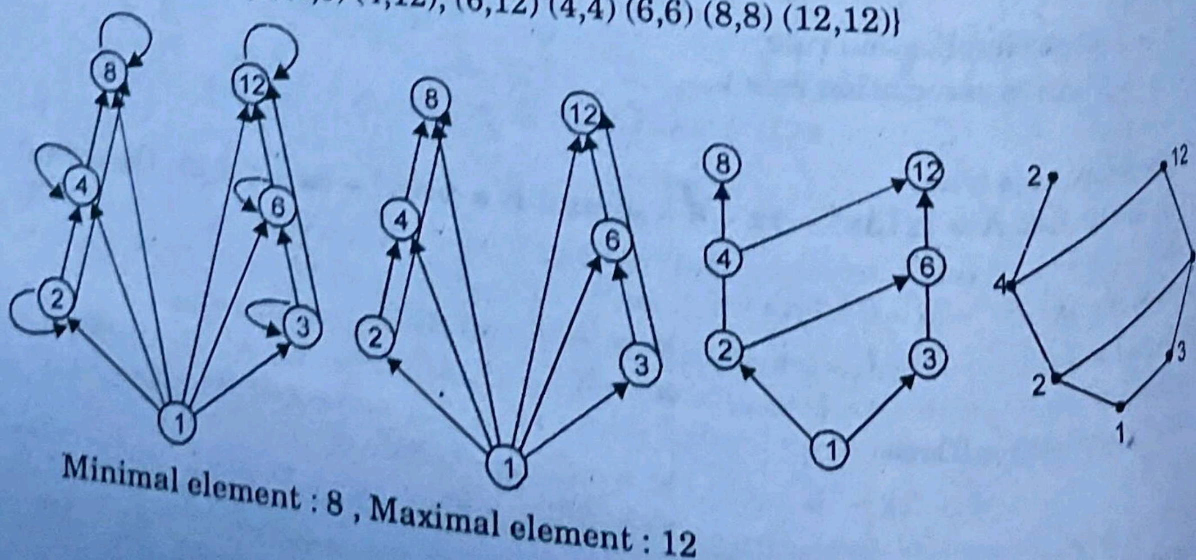
UNIT–III

Q. 6. (a) Is the poset (Z^+. 1) a lattice. (4)

Ans. The poset (Z^+ 1) is a lattice for any pair of positive integers a and b that we take, the greatest lower bound *i.e. g/b* of $\{a, b\}$ is gcd (a,b) while the least upper bound of $\{a,b\}$ is the lcm (a,b) and any poset is a lattice if every power of elements has a *lub* and a *glb*.

Q. 6. (b) Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid$ a divides b$\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$. Also find the minimal and maximal elements of the Hasse diagram. (4.5)

Ans. R = { (1,1) (1, 2) (1, 3) (1, 4) (1, 6) (1, 8) (1,12)
(2,4) (2,6) (2,8) (2,12) (2,2)
(3,6) (3,12) (3,3)(4,8) (4,12), (6,12) (4,4) (6,6) (8,8) (12,12)}



Minimal element : 8 , Maximal element : 12

Q. 6. (c) What is distributive lattice? Show that in any distributive lattice, the set of all complemented elements is a sublattice. **(4)**

Ans. In mathematics, a **distributive lattice** is a lattice in which the operations of join and meet distribute over each other. The prototypical examples of such structures are collections of sets for which the lattice operations can be given by set union and intersection. Indeed, these lattices of sets describe the scenery completely: every distributive lattice is—up to isomorphism—given as such a lattice of sets.

A complemented and distributive lattice is a boolean algebra, so we will use ++ and ... in place of ∨∨ and ∧∧ respectively. Now, of course, every element does have a complement (by definition); the real task is to show uniqueness.

Let xx be an arbitrary element, and let yy and zz be its complements. We want to show that y=zy=z. We start from

y = y. 1,y=y. 1,

and replace 1 by x+zx+z. Then applying distributivity and the fact that yx=0 we get

y=y(x+z)=yx+yz=0+yz=yz.(1)

Repeating this argument after switching yy and zz, we get

z=zy.(2)

Comparing (1) and (2), we are done.

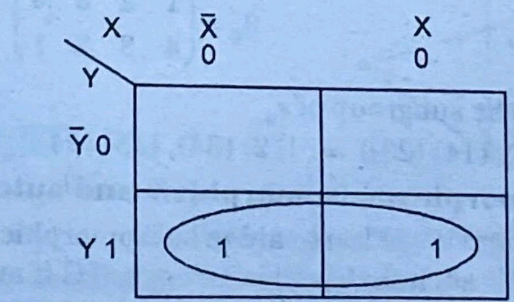
Q. 7. (a) Use the K-maps and simplify:

(i) $XY + \bar{X}Y$.

(ii) $X\bar{Y} + \bar{X}Y$

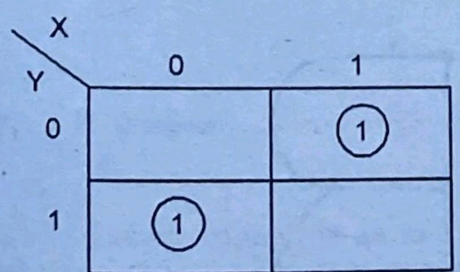
(iii) $X\bar{Y} + \bar{X}Y + \bar{X}\bar{Y}$

Ans. (i) $XY + \bar{X}Y$



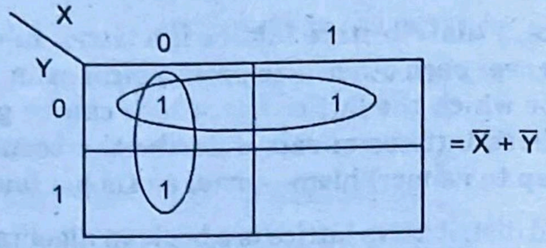
= Y

(ii) $X\bar{Y} + \bar{X}Y$



$= \bar{X}Y + X\bar{Y} \rightarrow$ Cannot we simplified further

(iii) $X\overline{Y} + \overline{X}Y + \overline{X}\overline{Y}$



Q. 7. (b) Explain Cayley's theorem by using an example. (3)

Ans. Cayles's theorem states that every group is isomorphic to a group of permutations or Any group G is isomorphic to a subgroup of sym (G) where sym (G) is the symmetric group *i.e.* the group of all permutations on a set G

Consider the following Cayley table of a group G = {*e*, *a*, *b*, *c*}

v	*e*	*a*	*b*	*c*
e	*e*	*a*	*b*	*c*
a	*a*	*e*	*c*	*b*
b	*b*	*c*	*e*	*a*
c	*c*	*b*	*a*	*e*

We then have

$$\lambda_e = \begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}, \qquad \theta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)$$

$$\lambda_a = \begin{pmatrix} e & a & b & c \\ e & e & c & b \end{pmatrix}, \qquad \theta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)\,(34)$$

$$\lambda_b = \begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix}, \qquad \theta_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)\,(24)$$

$$\lambda_c = \begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix}, \qquad \theta_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)\,(23)$$

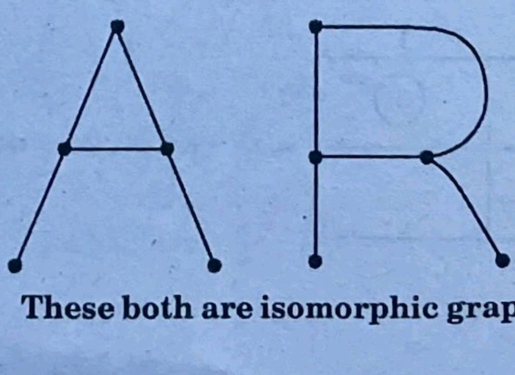
Hence G is isomorphic to the subgroup of s_4

{(1, (12) (34), (13) (24), (14) (23)} = {12) (34), (13) (24)}.

Q. 7. (c) Explain homomorphism, isomorphism and automorphism? (3.5)

Ans. Graphs $G\,(V, E)$ and $G^*\,(V^*, E^*)$ are said to be isomorphic if there exists a one-to-one correspondence $f : V \to V^*$ such that $\{u, v\}$ is an edge of G if and only if $\{f(u)\}, f(v)\}$ is an edge of G^*. Normally, we do not distinguish between isomorphic graph (even through their diagrams may "look different").

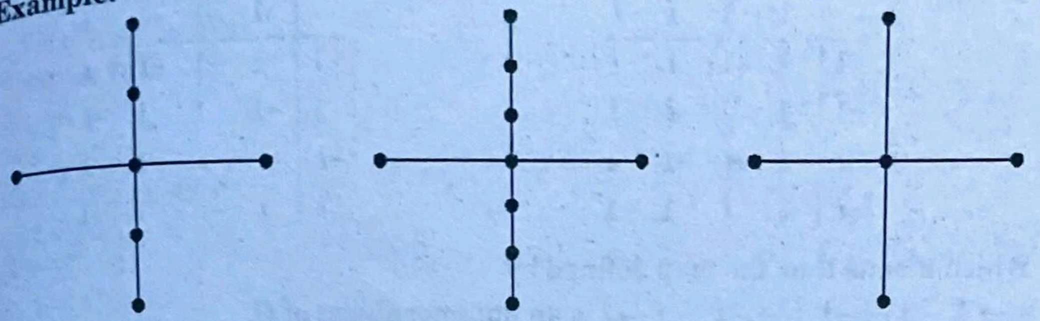
Example:



These both are isomorphic graphs

Two graphs G and G^* are said to be homeomorphic if they can be obtained from the same graph or isomorphic graphs by diverding an edge of G with additional vertices.

Example:

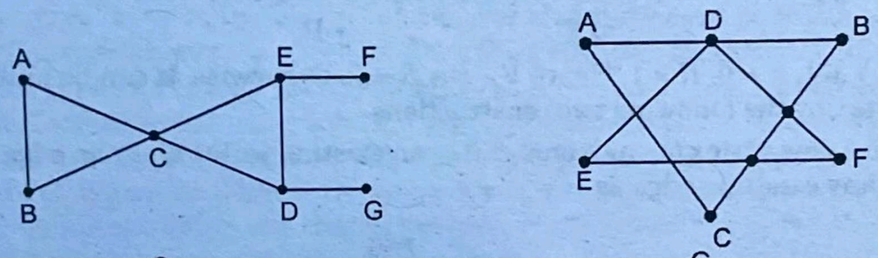


(a) (b) (c)

The graphs (a) and (b) are not isomorphic but they are homeomorphic since they can be obtained from the graph (c) by adding appropriate vertices.

An undirected graph is said to be connected if a path between every pair of distinct vertices of the graph.

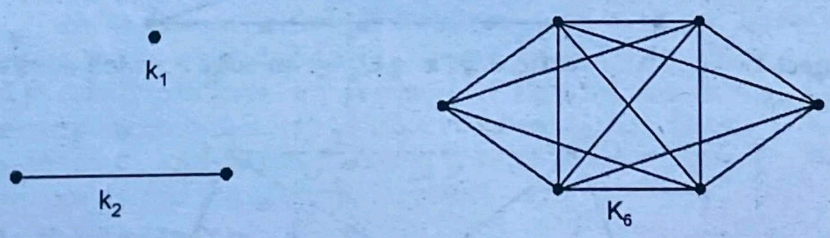
Example:



G_1 G_1 and G_2 are connected graphs G_2

A graph G is said to be complete if every vertex in G is connected to every other vertex in G. Thus a complete graph G must be connected. The complete graph with n vertices is denoted by k_n

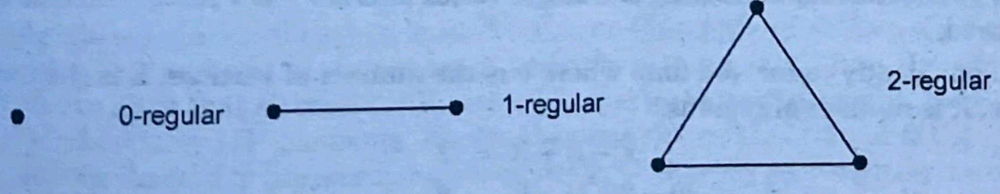
Example:



They are complete graphs

A graph G is regular of degree K or K- regular if every vertex has degree K. In other words, a graph is regular if every vertex has the same degree.

Example:



A group automorphism is an isomorphism from a group to itself. If G is a finite multiplicative group, an automorphism of G can be described as a rewriting of its multiplication table without altering its pattern of repeated elements.

For example: The multiplication table of the group of 4th roots of unity G = {1, –1, i, – i} can be written as shown below.

	1	–1	i	$-i$
1	1	–1	i	$-i$
–1	–1	1	$-i$	i
i	i	$-i$	–1	1
$-i$	$-i$	i	1	–1

.	1	–1	$-i$	i
1	1	–1	$-i$	i
–1	–1	1	i	$-i$
$-i$	$-i$	i	–1	1
i	i	$-i$	1	–1

Which means that the map defined by

$1 \to 1\ \ -1 \to -1\ \ i \to -i\ \ -i \to i$ is an automorphism of G.

In general the automorphisiom group of an algebraic object 0, like a ring or field is the set of isomorphisms of o and is denoted Aut (0).

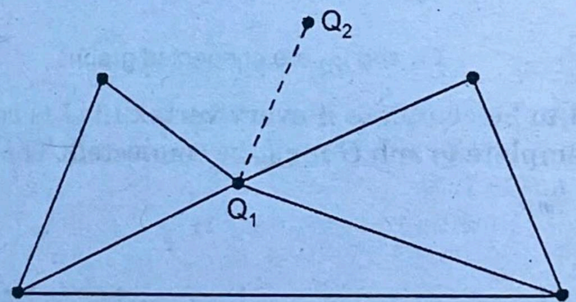
Q. 8. (a) Give the proof of Euler's formula. Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane. **(4.5)**

Ans. Suppose the connected map consists of a single vertex P as

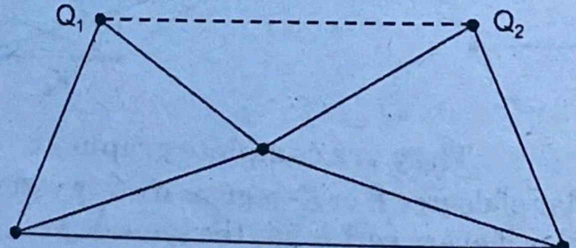
• P

Then $V = I, E = 0, R = 1$. Hence $V - E + R = 2$. Otherwise M can be built up from a single vertex by the following two constructions.

1. Add a new vertex Q_2 and connect it to an existing vertex Q_1 by an edge which does not cross any exisiting edge as



2. Connect two existing vertices Q_1 and Q_2 by an edge e which does not cross any existing edge as



Neither operation changes the value of $V - E + R$. Hence M has the same value of $V - E + R$ as the map consisting of a single vertex that is $V - E + R = 2$. Thus the theorem is proved.

Thus in any connected map where v is the number of vertices, E is the number of edges, R is number of regions.

$$\begin{aligned} V - E + R &= 2 \\ 20 - 20(3) + R &= 2 \\ 20 - 60 + R &= 2 \\ R &= 42 \end{aligned}$$

Q. 8. (b) Show that if $a^2 = e$ for all a in a group G(A, *), then G is commutative. **(4)**

Ans. Let b, c belongs to G

As closure holds in G

bc belongs to G

$(bc)^2 = e$

$(bc)(bc) = e$

Premultiplyng by b

bbcbc = be

ecbc= b

cbc= b

Postmultiplying by c

cbcc= bc

cbe= bc

cb= bc

G is commutative

Q. 8. (c) Explain the color theorem with suitable example. **(4)**

Ans. The proof is by induction on the number P of vertices of G. If $P \le 5$, then the theorem obviously holds suppose $P > 5$, and the theorem holds for graphs with less than P vertices. Let G has a vertex V such that deg $(V) \le 5$. By induction, the subgraph $G–V$ is 5- colorable. Assume one such coloring. If the vertices adjacent to V use less than the five colors, then we simply point V with one of the remaining colours and obtain a 5-colouring of G. We are still left with the case that V is adjacent to five vertices which are painted different colours say the vertices, moving counter clockwise about V, are $V_1 \ldots V_5$, and are painted respectively by the colours $c_1 \ldots c_5$ consider now the subgraph H of G generated by the vertices painted c_1 and c_3. Note H includes V_1 and V_3. If V_1 and V_3 belong to different components of H, then we can interchange the colours and c_1 and c_3 in the component containing V_1 without destroying the colouring of $G–V$. Then V_1 and V_3 are painted by c_3 c_1 can be chosen to paint V and we have a 5- colouring of G. On the other hand suppose v_1 and v_3 are in the same component of H. Then there is a path P from V_1 to V_3 whose vertices are painted with either c_1 or c_3. The path p together with the edges $\{v, v\}$ and $\{v, v_3\}$ form a cycle c which encloses either v_2 or v_4. Consider now the subgraph K generated by the vertices painted c_3 or c_4. Since C encloses V_2 or V_4 but not both, the vertices V_2 and V_4 belong to different components of K. Thus we can interchange the colours c_2 and c_4 in the component containing v_2 without destroying the colouring of G-V. Then v_2 and v_4 are painted by c_4, and we can choose c_2 to paint V and obtain a 5-colouring of g. Thus G is 5- colourable and the theorem is proved.

Q. 9. Write short notes on: **(4)**

(a) Lagrange's theorem

Ans. Lagrange's theorem, in the mathematics of group theory, states that for any finite group G, the order (number of elements) of every subgroup H of G divides the order of G.

This can be shown using the concept of left cosets of H in G. The left cosets are the equivalence classes of a certain equivalence relation on G and therefore form a partition of G. Specifically, x and y in G are related if and only if there exists h in H such that x = yh. If we can show that all cosets of H have the same number of elements, then each coset of H has precisely | H | elements. We are then done since the order of H times the number of cosets is equal to the number of elements in G, thereby proving that the order of H divides the order of G. Now, if aH and bH are two left cosets of H, we can define a map f : aH → bH by setting f(x) = ba – 1x. This map is bijective because its inverse is given by f – 1 (y) = a b – 1 y .

This proof also shows that the quotient of the orders |G| / |H| is equal to the index [G : H] (the number of left cosets of H in G). If we allow G and H to be infinite, and write this statement as

$$|G| = [G:H].|H|,$$

then, seen as a statement about cardinal numbers, it is equivalent to the axiom of choice.

Q. 9. (b) Normal Subgroups and Ring. (4)

Ans. In abstract algebra, a **normal subgroup** is a subgroup which is invariant under conjugation by members of the group of which it is a part. In other words, a subgroup *H* of a group *G* is normal in *G* if and only if $gH = Hg$ for all *g* in *G*. The definition of normal subgroup implies that the sets of left and right cosets coincide. In fact, a seemingly weaker condition that the sets of left and right cosets coincide also implies that the subgroup *H* of a group *G* is normal in *G*. Normal subgroups (and *only* normal subgroups) can be used to construct quotient groups from a given group.

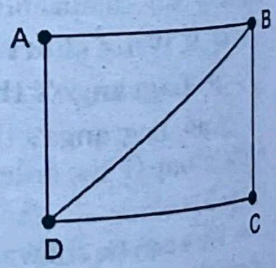
In mathematics, a ring is one of the fundamental algebraic structures used in abstract algebra. It consists of a set equipped with two binary operations that generalize the arithmetic operations of addition and multiplication. Through this generalization, theorems from arithmetic are extended to non-numerical objects such as polynomials, series, matrices and functions.

A ring is an abelian group with a second binary operation that is associative, is distributive over the abelian group operation, and has an identity element (this last property is not required by some authors, see § Notes on the definition). By extension from the integers, the abelian group operation is called *addition* and the second binary operation is called *multiplication*.

Whether a ring is commutative or not (i.e., whether the order in which two elements are multiplied changes the result or not) has profound implications on its behavior as an abstract object. As a result, commutative ring theory, commonly known as commutative algebra, is a key topic in ring theory. Its development has been greatly influenced by problems and ideas occurring naturally in algebraic number theory and algebraic geometry. Examples of commutative rings include the set of integers equipped with the addition and multiplication operations, the set of polynomials equipped with their addition and multiplication, the coordinate ring of an affine algebraic variety, and the ring of integers of a number field. Examples of noncommutative rings include the ring of $n \times n$ real square matrices with n e" 2, group rings in representation theory, operator algebras in functional analysis, rings of differential operators in the theory of differential operators, and the cohomology ring of a topological space in topology.

Q. 9. (c) Euler and Hamiltonian paths (4.5)

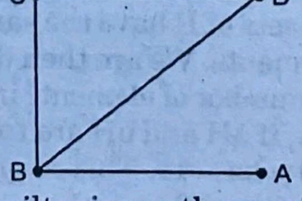
Ans. path of graph G is called an Eulerian path if it includes each edge of G exactly once



For example: The given graph contains an Eulerian path between B and D namely, B – D – C– B–A–D, since it includes each of edges exactly once.

A path of a graph G is called a Hamiltonian path of it includes each vertex of G exactly once.

For example:



The given Graph has an Hamiltonian path namely A – B – C– D

FIRST TERM EXAMINATION [SEPT. 2018]
THIRD SEMESTER [B.TECH]
FOUNDATION OF COMPUTER SCIENCE
[ETCS-203]

Time : 1.5 hrs. **M.M. : 30**

Note: Attempt any three questions including Q. No. 1 which is compulsory.

Q. 1. (a) Prove that $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$ is a tautology. **(2)**

Ans. Construct the truth table of $[((p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

p	q	r	$(p \vee q)$	$(p \rightarrow r)$	$(q \rightarrow r)$	$(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)$	$[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Since the truth value of $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$ is T for all values of p, q and the proposition is a tautology.

Q. 1. (b) Determine whether the following arguments are valid or invalid:

Premises:

If I read the newspaper in the kitchen, my glasses would be on the kitchen table

I did not read the newspaper in the kitchen.

Conclusion: My glasses are not on the kitchen table. **(2)**

Ans. Let p: I read newspaper in the kitchen

q: my glasses are on kitchen table

The argument in symbolic form is:

(i) $p \rightarrow q$

(ii) $\sim p$

(iii) $\sim q$

Construct the truth table for $[(p \rightarrow q) \wedge \sim p)] \rightarrow \sim q$.

p	q	$p \rightarrow q$	$\sim p$	$(p \rightarrow q) \wedge \sim p$	$\sim q$	$[(p \rightarrow q) \wedge \sim p] \rightarrow \sim q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Since the proposition $[(p \rightarrow q) \wedge \sim p)] \rightarrow \sim q$ is not a tautology, the argument is valid

Q. 1. (c) Given: A = {1, 2}. B = {x, y, z}, and C = {3, 4}. Find A × B × C. (2)

Ans. A = {1, 2}, B = {x, y, z} and C = {3, 4}

A × B × C = {(1, x, 3), (1, x, 4), (1, y, 3), (1, y, 4), (1, z, 3), (1, z, 4), (2, x, 3), (2, x, 4), (2, y, 3), (2, y, 4), (2, z, 3), (2, z, 4)}

Q. 1. (d) If we are on vacation then we go fishing. Find out the converse, inverse and contra positive of the statement. (2)

Ans. Let p: we are on vacation

q: we go fishing

In symbolic form, the given statement can be written as:

$$p \to q$$

Converse $q \to p$: If we go fishing we are on vocation.

inverse $\sim p \to \sim q$: If we are not on vacation, we don't go fishing.

Converse $\sim q \to \sim p$: If we don't go fishing, we are not on vacation.

Q. 1. (e) What is Pigeonhole principle, explain with suitable example. (2)

Ans. Pigeon hole Principle: It states that if there are more pigeons than pigeonhole then there must be sat least one pigeonhole, then there must be at least one pigeonhole with at least two pigeons in it. This principle is applicable to other objects besides pigeons and pigeonholes. Thus, it may be stated that if (n + 1) or more objects are placed into n boxes then there exists at least one box containing two or more objects.

For Example: Suppose the department of mathematics contains 13 professors. Then 2 of the professor (pigeons) were born in the same month (pigeoholes) out of 12 months.

- If n pigeons are assigned to m pigeonholes, then at least one pigeonholes contains two or more pigeons ($m < n$).

Q. 2. (a) Show that $\neg\ \forall x\ (P(x) \to Q\ (x))$ and $\exists x\ (P(x) \wedge \neg Q\ (x))$ are logically equivalent. (4)

Ans. $\sim \forall x(P(x) \to Q(x))$

$\equiv\ \sim \forall x[\sim P(x) \vee Q(x)]$ [using $p \to q \equiv\ \sim p \vee q$]

$\equiv\ \exists x \sim [\sim P(x) \vee Q(x))]$ [using $\sim \forall x \equiv \exists x$]

$\equiv\ \exists x[P(x) \wedge \sim Q(x)]$ [using De Morgan's law]

They are logically equivalent.

Q. 2. (b) Given a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (4)

Ans. Let $m = a^2$ as it is a perfect square

$n = b^2$ as it is also a perfect square

where a, b are integers

Then $nm = b^2a^2 = (ba)^2$

$\therefore$ nm is also a perfect square.

Q. 2. (c) Negate each of the following statements: (2)

(i) $\exists x\ \forall y,\ p(x, y)$; (ii) $\exists x\ \forall y,\ p(x, y)$; (iii) $\exists y\ \exists x\ \forall z,\ p(x, y, z)$.

Ans. Negation of following:

(i) $\sim \exists x \forall y\ \ p(x,y)\ =\ \forall x \exists y \sim p(x,y)$

(ii) $\sim \exists x\ \forall y\ \ p(x,y)\ =\ \forall x \exists y \sim p(x,y)$

(iii) $\sim \exists y \exists x\ \ \forall z p(x,y,z) = \forall y \forall x \exists z \sim p\ (x,y,z)$

Q. 3. (a) Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

"All hummingbirds are richly colored."

"No large birds live on honey."

"Birds that do not live on honey are dull in color."

"Hummingbirds are small."

Express the statements in the argument using quantifiers, assume that the domain consists of all birds. (5)

Ans. Let B(x) denote that x is a bird, H(x) denote that x is a hummingbird. R(x) denote that x is richly colored, L(x) denote that x lives on honey, and S(x) denote that x is small. Then the above statement (along with the additional assumptions) yields the following argument:

(1) $\forall x\ H(x) \rightarrow B(x)$

(2) $\forall x\ H(x) \rightarrow R(x)$

(3) $\neg\exists x\ B(x) \wedge \neg S(x) \wedge L(x)$

(4) $(B(x) \wedge \neg L(x)) \rightarrow \neg R(x)$

$\therefore\ \forall x\ H(x) \rightarrow S(x)$

The validity of the argument can be proved through following steps:-

(5) $\forall x\ \neg(B(x) \wedge \neg S(x) \wedge L(x) \Leftrightarrow \forall x\ \neg B(x) \vee S(x) \vee \neg L(x)$	logically equiv to (3)
(6) $\forall x\ B(x) \rightarrow (S(x) \vee \neg L(x))$	(5), implication
(7) $\forall x\ H(x) \rightarrow (S(x) \vee \neg L(x))$	(1), (6), hyp. syllogism
(8) $\forall x\ R(x) \rightarrow \neg(B(x) \wedge \neg L(x))$	contrapositive of (4)
(9) $\forall x\ H(x) \rightarrow \neg(B(x) \wedge \neg L(x))$	(2), (8), hyp. syllogism
(10) $\forall x\ H(x) \rightarrow (\neg B(x) \vee L(x))$	(9), DeMorgan
(11) $\forall x\ H(x) \rightarrow (B(x) \wedge (\neg B(x)) \vee L(x)))$	(1), (10), hyp. conjunction
(12) $\forall x\ H(x) \rightarrow (B(x) \wedge L(x))$	(11), distributive, domination
(13) $\forall x\ H(x) \rightarrow L(x)$	(12), hyp. simplification
(14) $\forall x\ H(x) \rightarrow (L(x) \wedge (S(x) \vee \neg L(x)))$	(7), (13). hyp. conjunction
(15) $\forall x\ H(x) \rightarrow (L(x) \wedge S(x))$	(14), distributive, domination
(16) $\forall x\ H(x) \rightarrow S(x)$	(15), hyp. simplification

Q. 3. (b) Find the PDNF and PCNF of $(p \vee q) \wedge (p \wedge r) \vee (q \rightarrow r)$. (5)

Ans. PDNF and PCNF of $(p \vee q) \wedge (p \wedge r) \vee (q \rightarrow r)$

Construct the truth table of the $(p \vee q) \wedge (p \wedge r) \vee (q \rightarrow r)$

p	q	r	$(p \vee q)$	$(p \wedge q)$	$(q \rightarrow r)$	$(p \vee q) \wedge (p \wedge r)$	$(p \vee q) \wedge (p \wedge r) \vee (q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	F	T	F	T
F	T	F	T	F	F	F	F
F	F	T	F	F	T	F	T
F	F	F	F	F	T	F	T

Here the rows of p, q, r in which truth value appears T forms the PDNF and rows of truth value F form PCNF. Therefore

PDNF:- $(p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

PCNF: - $(\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee r)$

Q. 4. (a) Consider the following five relations on the set A = {1, 2, 3, 4}:

R1 = {(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)}

R2 = {(1, 1) (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

R3 = {(1, 3), (2, 1)}

R4 = ϕ, the empty relation

R5 = A × A, the universal relation

Determine which of the relations are reflexive, symmetric, antisymmetric, and transitive. (7)

Ans. R_1 = {(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)

R_2 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)

R_3 = {(1, 3), (2, 1)}

$R_4 = \phi$

R_5 = A × A, universal relation.

Reflexive: Only R_2 and R_5 are reflexive as they satisfy the condition of reflexivity i.e. they contain (1, 1), (2, 2), (3, 3) (4, 4),

Symetric: Relation R_2, R_4,R_5 are symetric as R_2 contains (1, 2) and (2, 1); R_4 containts ϕ; R_5 is universal relation.

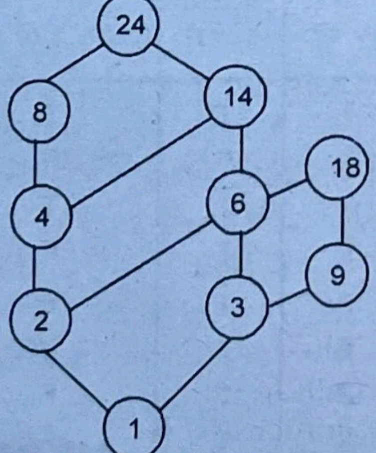
Antisymetric: R_2 is not antisymetric since (2, 1) and (1, 2) belong to R_2, but $1 \neq 2$. Similarly universal set R_5 is not antisymetric. All other are antisymetric.

Transitive: R_3 is not transitive since (2, 1), (1, 3) ∈ R but ∉ R. All other relations are transitive.

Q. 4. (b) Let A = {1, 2, 3, 4, 6, 8, 9, 12, 18, 24} be ordered by the relation "x divides y". (3)

Draw the Hasse diagram of A. Determine whether it is a lattice.

Ans. A = {1, 2, 3, 4, 6, 8, 9, 12, 18, 24}. Hasse diagram if A:



It is not a lattice.

END TERM EXAMINATION [NOV-DEC 2018]
THIRD SEMESTER [B.TECH]
FOUNDATION OF COMPUTER SCIENCE
[ETCS–203]

Time : 3 hrs. **M.M. : 75**

Note: *Attempt any five qustions in all including Q. No. 1 is compulsory. Select one question from each unit.*

Q. 1. Explain following in brief: **(2.5)**

(a) Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Ans. Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Construct the truth table of $(p \wedge q) \rightarrow (p \vee q)$

p	q	$(p \wedge q)$	$(p \vee q)$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

As all the values are T for any value of p'q hence it is a tautology.

Q. 1. (b) Show that $\exists y\, \forall x P(x, y) \Rightarrow \forall x\, \exists y\, P(x, y)$. **(2.5)**

Ans. $\exists x\, \forall y\; p(x,y)$

$\equiv \forall x\; p(x,b)$ [By existential instantiation]

$\equiv p(x, b)$ [By universal instantiation]

$\equiv \exists y\; p(x,y)$ [By existential generalization]

$\equiv \forall x\; \exists y\, p(x,y)$ [By universal generalization]

Q. 1. (c) Define groups, subgroups and normal subgroups with suitable example. **(2.5)**

Ans. Groups: Let G be non empty set and * is a binary operation of G, then the algebraic system {G, *} is called a group if the following conditions are satisfied

(*i*) For all $a,b,c \in G$

$(a*b)*c = a*(b*c)$ (Associativity)

(*ii*) There exists an element $e \in G$ such that for any $a \in G$

$a * e = e * a = a$ (Existence of Identity)

(*iii*) For every $a \in G$ there exists an element $a^{-1} \in G$ such that

$a * a^{-1} = a^{-1} * a = e$ (Existence of Inverse)

Example: 4 roots of unity $(1, -1, i, -i)$ is an abelian group under multiplication set of Z integres is an abelian group under additions.

Sub-groups: Let H be a subset of a group G. Then H is called a sub-group of G if H itself is a group under the operation of G *i.e.* if

(*i*) The identity element $e \in H$

(*ii*) H is closed under the operation of G *i.e.* if $a, b, \in H$ then $ab \in H$ and

(*iii*) H is closed under inverse, that is if $a \in H$ then $a^{-1} \in H$

Example: Let $G = \{1, -1, i - i\}$ be a group under multiplication then its subset $H = \{1, -1\}$ is its sub-group.

Normal sub-groups: A sub-group H of G is a normal sub-group if a^{-1} $Ha \subseteq$ H for every $a \in G$. Equivalently H is normal if aH = Ha for every $a \in$ G *i.e.* if the rights and left cosets coincide.

Example: Consider the group Z of integers under addition. Let H denote the multiples of 5 that is

$$H = \{.....,-10, -5, 0, 5, 10 .....\}$$

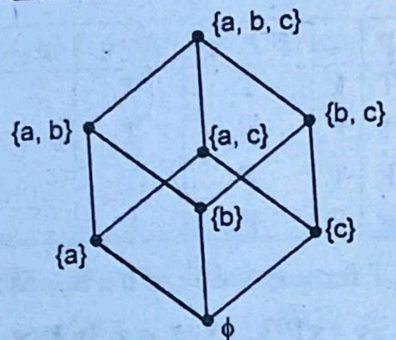
Then H is a normal sub group of Z.

Q. 1. (d) Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set P(S) where S = {a, b, c}. **(2.5)**

Ans. $\{(A,B) \mid A \subseteq B\}$ on P(S) where s = {a, b, c}

$$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$$

The Hasse diagram can be drawn as



Q. 1. (e) Define a Chain. Give an example of an infinite set which is a Chain? **(2.5)**

Ans. A partial ordering is a pair (P, <), where P is a nonempty set and < is a binary relation on P which is transitive and irreflexive. A subset C $\subseteq$ P is a chain if (C, <) is linearly ordered. A subset A $\subseteq$ P is an antichain if no two distinct elements of A are comparable under <. It follows from Ramsey's theorem for pairs that every infinite partial ordering has an infinite chain or an infinite antichain.

Q. 1. (f) A set A = {1, 2, 3, 4, 5, 6, 7}. Find the (2 4 6 7) o (1 3 5). That composition is even permutation or odd permutation? **(2.5)**

Ans. A = {1, 2, 3, 4, 5, 6, 7]

$$(2\,4\,6\,7) = \begin{pmatrix} 2 & 4 & 6 & 7 & 1 & 3 & 5 \\ 4 & 6 & 7 & 2 & 1 & 3 & 5 \end{pmatrix}$$

$$(1\,3\,5) = \begin{pmatrix} 1 & 3 & 5 & 2 & 4 & 6 & 7 \\ 3 & 5 & 1 & 2 & 4 & 6 & 7 \end{pmatrix}$$

$$(2\,4\,6\,7) \circ (1\,3\,5) = \begin{pmatrix} 2 & 4 & 6 & 7 & 1 & 3 & 5 \\ 4 & 6 & 7 & 2 & 1 & 3 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 3 & 5 & 2 & 4 & 6 & 7 \\ 3 & 5 & 1 & 2 & 4 & 6 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 & 6 & 7 & 1 & 3 & 5 \\ 4 & 6 & 7 & 2 & 3 & 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 6 & 7 & 1 & 3 & 5 \\ 4 & 6 & 7 & 2 & 3 & 5 & 1 \end{pmatrix} = \text{can be written as } (2\,4\,6\,7)\,(1\,3\,5)$$

(2 4 6 7) (1 3 5) = (2 4) (2 6) (2 7) (1 3) (1 5)

Since it is a product of odd number of transpositions, the permutation is an odd permutation.

Q. 1. (g) A connected plane graph has 10 vertices each of degree 3. Into how many region, does a representation of this planner graph split the plane. (2.5)

Ans. Number of regions = e– v + 2

No. of vertices = 10

degree of each vertex = 3

Sum of degree of edges = 10 * 3 = 30 = 2* no. of edges

2*e = 30

e = 15

Regions = 15 – 10 + 2 = 7

Q. 1. (h) Find the minimum number of students in a class to be sure that four out of them are born in the same month. (2.5)

Ans. We consider each month as a pigeonhole, m = 12 and we have to find the no. of students (pigeons) so that 4 out of them are born in same month.

$$\lfloor (n-1)/m \rfloor + 1 = 4$$

$$(n-1)/m = 3$$

$$(n-1)/12 = 3$$

n = 37, which is required no. of students.

Q. 1. (i) If H is a subgraph of G such that $x^2 \in H$ for every $x \in G$, then prove that H is a normal subgroup of G. (2.5)

Ans. For any $g \in G, h \in H$:

$$(gh)^2 \in H \text{ and } g^{-2} \in H$$

since H is a subgroup $h^{-1} g^{-2} \in H$ and so,

$$(gh)^2 h^{-1} g^{-2} \in H. \quad \text{This gives that}$$

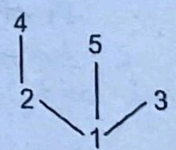
$$= gh\,gh\,h^{-1} g^{-2} \in H$$

$$= gh g^{-1} \in H$$

Hence H is a normal subgroup of G.

Q. 1. (j) Determine whether the poset ({1, 2, 3, 4, 5}, /)a lattice. (2.5)

Ans. ({1, 2, 3, 4, 5} |) has Hasse diagram:



The poset represented by Hasse diagram is lattice because in this poset their exist both lub and gub for each pair.

UNIT-I

Q. 2. (a) Prove that if n = ab where a and b are positive integer, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$. (4)

Ans. If n =ab, where a and b are positive integers, then

$b \le \sqrt{n}$ or $a \le \sqrt{n}$

Proof (by contraposition):

Assume $b > \sqrt{n}$ and $a > \sqrt{n}$

$ab > (\sqrt{n}).(\sqrt{n}) = n$

So, $n \ne ab$.

By contraposition, if n = ab, then $b \le \sqrt{n}$ or $a \le \sqrt{n}$

Q. 2. (b) Find the CNF of the function $f = [p \wedge (\neg q \wedge r)] \vee \neg r$ and then find its DNF from it. (5

Ans. $p \wedge (\sim q \wedge r)] \vee \sim r$

$\equiv (p \vee \sim r) \wedge [\sim r \vee (\sim q \wedge r)]$ [using distributive law]

$\equiv (p \vee \sim r) \wedge [(\sim r \vee \sim q) \wedge (\sim r \vee r)]$ [using distributive law]

$\equiv (p \vee \sim r) \wedge [(\sim r \vee \sim q) \wedge T]$

$\equiv (p \vee \sim r) \wedge (\sim r \vee \sim q)$

It is the required CNF.

$\equiv (p \wedge \sim q) \vee \sim r$ [Distributive law]

It is the required DNF.

Q. 2. (c) Write an equivalent expression for $(p \Rightarrow q \wedge r) \vee (r \Leftrightarrow s)$ which contains neither the bi-conditional nor the conditional. (3.5)

Ans. $(p \rightarrow q \wedge r) \vee (r \leftrightarrow s)$

$\equiv (p \rightarrow (q \wedge r)] \vee [(r \rightarrow s) \wedge (s \rightarrow r)]$

$\equiv [\sim p \vee (q \wedge r)] \vee [(\sim r \vee s) \wedge (\sim s \vee r)]$

$\equiv [(\sim p \vee q) (\sim p \vee r)] \vee [(\sim r \vee s) \wedge (\sim s \vee r)]$

Q. 3. (a) Give an example to illustrate proofs by contraposition and contradiction methods. (6)

Ans. Proof by contraposition (example)

Theorem:

If n is an integer and 3n+2 is odd, then n is odd.

Proof (by contraposition):

Assume n is even.

$\exists$ integer k, such that n = 2k

3n+2 = 3(2k)+2 = 2(3k+1)

Let m = 3k+1.

3n+2 = 2m

So, 3n+2 is even.

By contraposition, if 3n+2 is odd, then n is odd.

Proof by contradiction (example)

Prove if 3n+5 is even then n is odd.

Proof (proof by contradiction):

Assume 3n+5 is even and n is even.

n = 2k (k is some integer)

3n+5 = 3(2k) + 5 = 6k + 5 = 2(3k + 2) + 1

Assume m = 3k+2.

3n+5 = 2m + 1

So, 3n+5 is odd

Assume p is "3n+5 is even".

p ^ ¬ p is a contradiction.

By contradiction, if 3n+5 is even then n is odd.

Q. 3. (b) Prove that n(n + 1) (2n + 1) is divisible by 6 $\forall n \in N$ using mathematical induction. (4)

Ans. Let P(n): n(n + 1)(n + 2) is divisible by 6.

P(1): 1(1 +1)(1 + 2) = 6 which is divisible by 6. Thus P(n) is true for n = 1.

Let P(k) be true for some natural number k.

i.e. P(k): k(k + 1)(k + 2) is divisible by 6.

Now we prove that P(k + 1) is true whenever P(k) is true.

Now, (k + 1)(k + 2)(k + 3) = k(k + 1)(k + 2) + 3(k + 1)(k + 2)

Since, we have assumed that k(k + 1)(k + 2) is divisible by 6, also (k + 1)(k + 2) is divisible by 6 as either of (k +1) and (k + 2) has to be even number.

$\Rightarrow$ P(k + 1)is true.

Thus P(k +1) is true whenever P(k) is true.

$\therefore$ By principle mathematical induction $n(n + 1)(n + 2)$ is divisible by 6 for all n

Q. 3. (c) Write the symbolic form of the proposition: "This is a criminal who has committed every crime". Also derive its negation. **(2.5)**

Ans. "This is a criminal who has committed very crime."

Let A be set of criminals and B be set of crimes.

Then, $(\exists x \in A)(\forall x \in B)$ (C has committed x)

Negation of the proposition.

$\sim (\exists C \in A)(\forall x \in B)$ (C has committed x)

$\equiv (\forall C \in A)(\exists x \in B)$ (C has not committed x)

We can write it as "For every criminal, there is a crime that this person has not commited".

UNIT-II

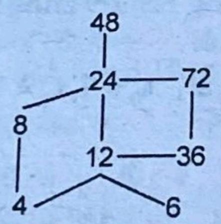
Q. 4. (a) If A = {4, 5, 7, 8, 10}, B = {4, 5, 9}, C = {1, 4, 6, 9}, then verify that (A ∩ B) ∪(A ∩ C). **(4)**

Ans. A = {4, 5, 7, 8, 10}, B = {4, 5, 9}, C = {1, 4, 6, 9}

(A ∩ B) ∪ (A ∩ C) = ({4, 5} ∪ {4} = {4, 5}

Q.4. (b) Let A = {4, 6, 8, 12, 24, 36, 48, 72} with the partial order of divisibility. Draw its Hasse diagram. **(5)**

Ans. A = {4, 6, 8, 12, 24, 36, 48, 72}. Hasse daigram is:



Q. 4. (c) Prove that s lattice (L, ≤) is modular if and only if $(a \vee b) \vee (a \vee c) = a \wedge (b \vee (a \wedge c))$ for all a, b, c ∈ L. **(3.5)**

Ans. A lattice is said to be modular if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a < c \; \forall a, b, c \in L$.

$a \wedge [b \vee (a \wedge c)]$

$\equiv a \wedge [(b \vee a) \wedge (b \vee c)]$

$\equiv a \wedge (b \vee a) \wedge (b \vee c)$

$\equiv [a \wedge (b \vee c)] \wedge (b \vee a)$

$\equiv [(a \wedge b) \vee (a \wedge c)] \wedge (b \vee a)$

$\therefore$ It is a modular lattice.

Q. 5. (a) Explain the pigeonhole principle with suitable examples. **(6)**

Ans. Refer Q. 4. (a) First Term Examination 2018.

Q. 5. (b) Using Binomial theorem prove that; **(6.5)**

$$3^n = \sum_{r=0}^{n} C = (n;r)2^r$$

Ans. First of all, examine every piece:

$$3^n = \sum_{r=0}^{n}\binom{n}{r}y^r$$

The first thing which strikes when looking at this is that we have binomial coefficients:

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

Comparing the statement of the bionomial theorem of our problem we notice that both have a summation with binomial coefficient equal to the power of a number.

$$3^n (x+y)^n$$

This means that we have only partially determined the solution be comparing the results of the summations.

$$\binom{n}{r}2^r \quad \binom{n}{r}x^{n-r}y^r$$

Examing the binomial theorem, we see that

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

$$(1+2)^n = \sum_{r=0}^{n}\binom{n}{r}(1)^{n-r}(2)^r$$

$$3^n = \sum_{r=0}^{n}\binom{n}{r}(2)^r$$

UNIT-III

Q. 6. (a) Solve the following recurrence relation. $t_n - 7t_{n-1} + 10t_{n-2} = 5.2^n$, with initial conditions $t_0 = 5, t_1 = 16$. **(6.5)**

Ans. $t_n - 7t_{n-1} + 10t_{n-2} = 5.2^n$, $t_0 = 5, t_1 = 16$

Rewite the recurrence,

$$t_n - 7t_{n-1} + 10t_{n-2} = 0$$

$$x^2 - 7x + 10 = 0$$

$$x = 2, 5$$

$$t_n = c_1 2^n + c_2 5^n$$

using initial conditions, we get

$$c_1 + c_2 = 5$$

$$2c_1 + 2c_2 = 16$$

Solving these two equation, we get $c_1 = 3, c_2 = 2$

Therefore homogeneous solution is $t_n = 3(2)^n + 2(5)^n$

General form of particular solution is np. 2^n.

put this in given relation we get.

$n^p 2^n - 7(n-1)p\, 2^{n-1} + 10(n-2)p\, 2^{n-2} = 5.2^n$

put n = 0, we get

$0 - 7 \times - p \times 2^{-1} + 10 \times (-2) \times p \times 2^{-2} = 5$

$$\frac{7}{2}p - \frac{20p}{4} = 5$$

$$\frac{14p - 20p}{4} = 5$$

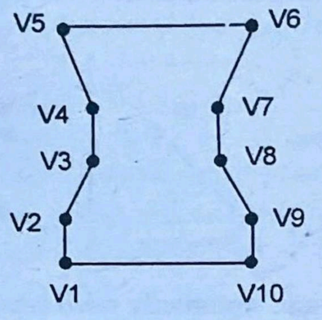
$$p = \frac{-20}{6} = \frac{-10}{3}$$

Hence total solution is $3(2^n) + 2(5^n) - \frac{10}{3}n.2^n$

Q. 6. (b) Explain Hamiltonian Circuit with suitable examples. **(6)**

Ans. Hamiltonian Circuit is a circuit drawn form G that contains each vertex of G exactly once except the beginning and the ending vertex. Since no vertex except the start vertex is repeated, no edge will repeat.

For example, Figure represent Hamiltonian the circuit $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$.



A circuit in a connected graph G is said to be Hamiltonian if it includes every vertex of G. Hence a Hamiltonian circuit in a graph of n vertices consists of exactly n edges.

If we remove one of the edge from the given circuit say the edge (v_1, v_{10}) then we are left with a path $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ in which we travel along all the vertices exactly once is the Hamiltonian path. Since a Hamiltonian path is a sub–graph of Hamiltonian Circuit which we get by removing an edge from the circuit. Therefore every graph that has a Hamiltonian circuit also has a Hamiltonian path.

Therefore, the length of a Hamiltonian path (if exists) in a connected graph of n vertices and $(n - 1)$ edges.

Q. 7. (a) State and prove 5 color theorem. **(6)**

Ans. The proof is by induction on the number P of vertices of G. If $P \leq 5$, then the theorem obviously holds suppose $P > 5$, and the theorem holds for graphs with less than P vertices. Let G has a vertex V such that deg $(V) \leq 5$. By induction, the subgraph $G–V$ is 5- colorable. Assume one such coloring. If the vertices adjacent to V use less than the five colors, then we simply point V with one of the remaining colours and obtain a 5-colouring of G. We are still left with the case that V is adjacent to five vertices which are painted different colours say the vertices, moving counter clockwise about V, are $V_1 ....V_5$, and are painted respectively by the colours $c_1 ...c_5$ consider now the subgraph H of G generated by the vertices painted c_1 and c_3. Note H includes V_1 and V_3. If V_1 and V_3 belong to different components of H, then we can interchange the colours and c_1 and c_3 in the component containing V_1 without destroying the colouring of $G–V$. Then V_1 and V_3 are painted by c_3 c_1 can be chosen to paint V and we have a 5- colouring of G. On the other hand suppose v_1 and v_3 are in the same component of H. Then there is a path P from V_1 to V_3 whose vertices are painted with either c_1 or c_3. The path p together with the edges $\{v, v\}$ and $\{v, v_3\}$ form a cycle c which encloses either v_2 or v_4. Consider now the subgraph

K generated by the vertices painted c_3 or c_4. Since C encloses V_2 or V_4 but not both, the vertices V_2 and V_4 belong to different components of K. Thus we can interchange the colours c_2 and c_4 in the component containing v_2 without destroying the colouring of G-V. Then v_2 and v_4 are painted by c_4, and we can choose c_2 to paint V and obtain a 5-colouring of g. Thus G is 5- colourable and the theorem is proved.

Q. 7. (b) Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane. (3)

Ans. Number of regions = e – v + 2

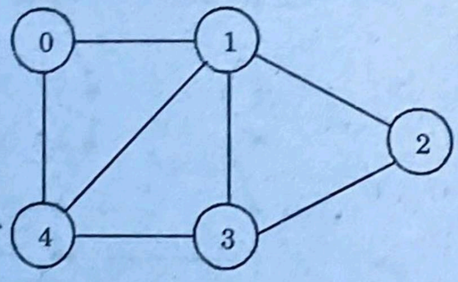
Sum of degrees of edges = 20 × 3 = 60 = 2*number of edges

$$2 * e = 60$$
$$e = 30$$

regions = 30 – 20 + 2 =12

Q. 7. (c) Illustrate the trade-off between adjacency lists and adjacency matrices. (3.5)

Ans. Following is an example of an undirected graph with 5 vertices.



Following two are the most commonly used representations of a graph.

1. Adjacency Matrix 2. Adjacency List

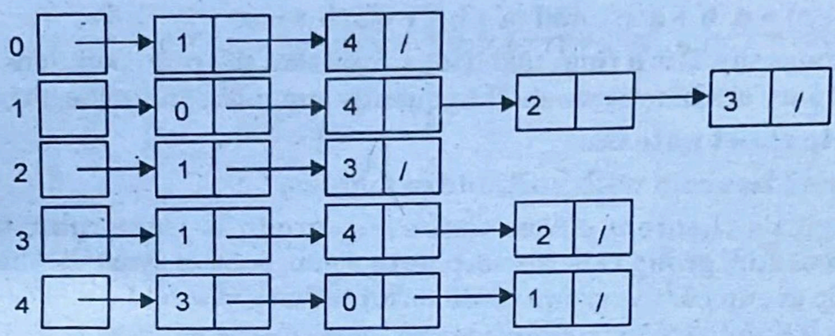
There are other representations also like, Incidence Matrix and Incidence List. The choice of the graph representation is situation specific. It totally depends on the type of operations to be performed and ease of use.

Adjacency Matrix: Adjacency Matrix is a 2D array of size V × V where V is the number of vertices in a graph. Let the 2D array be adj[] [], a slot adj[i][j] = 1 indicates that there is an edge from vertex i to vertex j. Adjacency matrix for undirected graph is always symmetric. Adjacency Matrix is also used to represent weighted graphs. If adj[i][j] = w, then there is an edge from vertex i to vertex j with weight w.

The adjacency matrix for the above example graph is

	0	1	2	3	4
0	0	1	0	0	1
1	1	0	1	1	1
2	0	1	0	1	0
3	0	1	1	0	1
4	1	1	0	1	0

An array of lists is used. Size of the array is equal to the number of vertices. Let the array be array []. An entry array[i] represents the list of vertices adjacent to the i^{th} vertex. This representation can also be used to represent a weighted graph. The weights of edges can be represented as lists of pairs. Following is adjacency list representation of the above graph.



UNIT-IV

Q. 8. (a) Define a cyclic group, show that the set {1, ω, ω²) is a cyclic group of order 3 with generators ω and ω² with respect to multiplication, ω being the cube root of unity. (6)

Ans. A cyclic group G is a group that can be generated by a single element a, so that every element in G has the form a^i for some integer i. We denote the cyclic group of order n by Z_n, since the additive group of Z_n is a cyclic group of order n.

Theorem: All subgroups of a cyclic group are cyclic. If G = <a> is cyclic, then for every divisor d of |G| there exists exactly one subgroup of order d which may be generated by $a^{|G|/d}$.

Show that G = $\{1, w, w^2\}$ where w is a cube root of unity is a cyclic group.

It suffices to show that the only subgroup H of G that contains w is G itself.

Let $H \le G, w \in H$. Now the identity element of the group is the identity element for every subgroup, so $1 \in H$.

H is a subgroup if and only if it is closed under multiplication. If H={1,w} then it is not closed, since w^2 is not in H. Thus H must be {1 ,w,w^2} which is G.

Then G is cyclic.

Another approach is to show that G is a group that is generated by w. w^0= 1 ,w^1 = w, w^2=w^2,w^3= 1

Q. 8. (b) In the ring (S , +,.), S is asset of 2 × 2 matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where a, b, c are even integers and + and are respectively matrix addition and multiplication. Show that (S, + ,.), is non commutative ring with no unity element. (6.5)

Ans. The definition of a ring. A structure (R, +, •) is a ring if R is a non-empty set and + and • are binary operations:

$$+ : R \times R \to R, \quad (a, b) \to a + b$$

$$\bullet : R \times R \to R, \quad (a, b) \to a.b$$

such that

Addition: (R, +) is an abelian group, that is,

(A1) associativity: for all a, –b, c ∈ R we have a + (b + c) = (a + b) + c

(A2) zero element: there exists 0 ∈ R such that for all a ∈ R we have a+0 = 0 + a = a

(A3) inverses: for any a ∈ R there exists – a ∈ R such that a + (– a) = (– a) + a = 0

(A4) commutativity: for all a, b∈ R we have a + b = b + a

Multiplication:

(M1) associativity: for all a, b, c ∈ R we have a . (b . c) = (a . b) .c

Addition and multiplication together

(D) for all a, b, c ∈ R,

a. (b + c) = a. b + a. c and (a + b) . c = a. b + b.c.

We sometimes say 'R is a ring', taken it as given that the ring operations are denoted + and. As in ordinary arithmetic we shall frequently suppress. and write ab instead of a . b.

Q. 9. Write short note on:

(a) Cayles Theorem with suitable examples. **(4)**

Ans. Cayles's theorem states that every group is isomorphic to a group of permutations or Any group G is isomorphic to a subgroup of sym (G) where sym (G) is the symmetric group *i.e.* the group of all permutations on a set G

Consider the following Cayley table of a group G = {e, a, b, c}

v	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

We then have

$$\lambda_e = \begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}, \qquad \theta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)$$

$$\lambda_a = \begin{pmatrix} e & a & b & c \\ e & e & c & b \end{pmatrix}, \qquad \theta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\ 2)(3\ 4)$$

$$\lambda_b = \begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix}, \qquad \theta_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1\ 3)(2\ 4)$$

$$\lambda_c = \begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix}, \qquad \theta_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (1\ 4)(2\ 3)$$

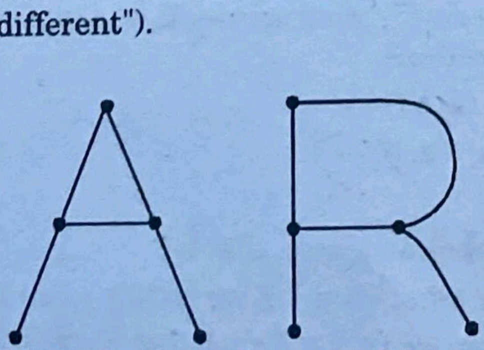
Hence G is isomorphic to the subgroup of s_4

{(1, (1 2) (3 4), (1 3) (2 4), (1 4) (2 3)} = {(1 2) (3 4), (1 3) (2 4)}.

Q. 9. (b) Homomorphism, isomorphism and automorphism with examples. **(4.5)**

Ans. Graphs $G\,(V, E)$ and $G^*\,(V^*, E^*)$ are said to be isomorphic if there exists a one-to-one correspondence $f : V \rightarrow V^*$ such that $\{u,v\}$ is an edge of G if and only if $\{f(u)\}, f(v)\}$ is an edge of G^*. Normally, we do not distinguish between isomorphic graph (even through their diagrams may "look different").

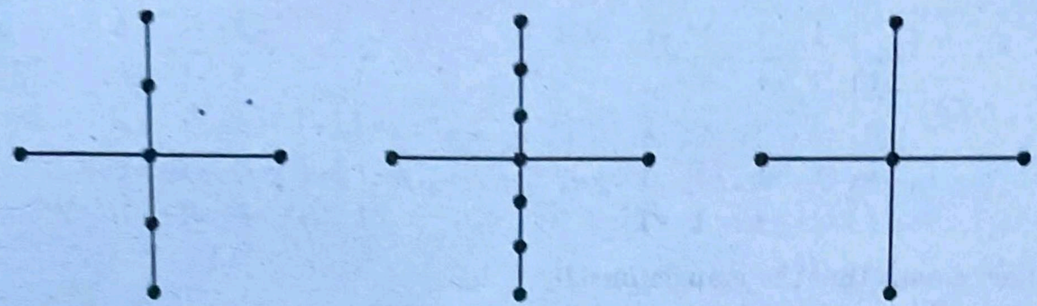
Example:



These both are isomorphic graphs

Two graphs G and G^* are said to be homeomorphic if they can be obtained from the me graph or isomorphic graphs by diverding an edge of G with additional vertices.

Example:

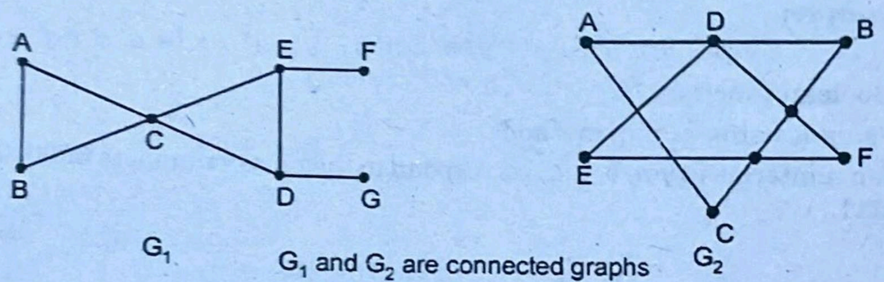


(a) (b) (c)

The graphs (a) and (b) are not isomorphic but they are homeomorphic since they can e obtained from the graph (c) by adding appropriate vertices.

An undirected graph is said to be connected if a path between every pair of distinct ertices of the graph.

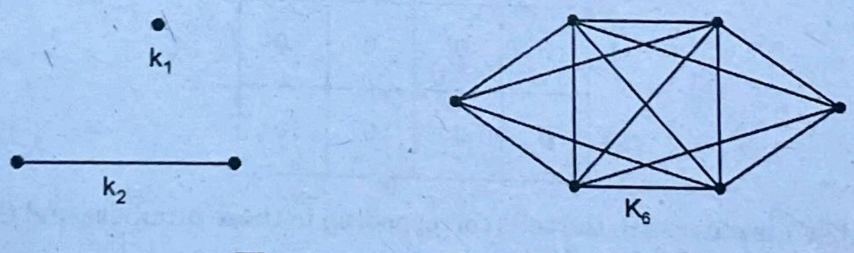
Example:



G_1 and G_2 are connected graphs

A graph G is said to be complete if every vertex in G is connected to every other vertex in G. Thus a complete graph G must be connected. The complete graph with n vertices is denoted by k_n

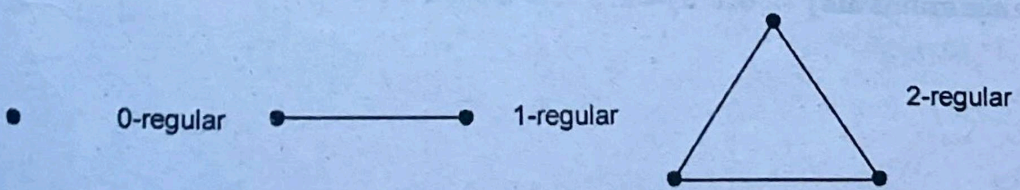
Example:



They are complete graphs

A graph G is regular of degree K or K- regular if every vertex has degree K. In other words, a graph is regular if every vertex has the same degree.

Example:



A group automorphism is an isomorphism from a group to itself. If G is a finite multiplicative group, an automorphism of G can be described as a rewriting of its, multiplication table without altering its pattern of repeated elements.

For example: The multiplication table of the group of 4th roots of unity $G = \{1, -1, i, -i\}$ can be written as shown below.

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

	1	-1	$-i$	i
1	1	-1	$-i$	i
-1	-1	1	i	$-i$
$-i$	$-i$	i	-1	1
i	i	$-i$	1	-1

Which means that the map defined by

$1 \to 1 \quad -1 \to -1 \quad i \to -i \quad -i \to i$ is an automorphism of G.

In general the automorphisiom group of an algebraic object 0, like a ring or field is the set of isomorphisms of o and is denoted Aut (0).

Q. 9. (c) Minimization of Boolean function. **(4)**

Ans. If x_1, x_2 -----x_n are Boolean variables, a function from $B^n = \{x_1, x_2, ...x_n\}$ to $B = \{0,1\}$ is called a Boolean function of degree n.

For example:

$$f(a, b, c, d) = a'b'c'd' + a'bc'd + a'b'cd + a'b'cd' + a'bcd$$

It is a Boolean function.

Minimising it using k-map method

The given minterms in $f(a, b, c, d)$ correspond to the binary numbers 0000, 0101,0011, 0010 and 0111.

ab \ cd	00	01	11	10
00	1	0	1	1
01	0	1	1	0
11	0	0	0	0
10	0	0	0	0

The number 1 is entered in the cells correspoding to these numbers and the numbers 0 is centered in the remaining cells.

The minimum possible number of loops containing maximum possible number of 1's will are shown in the map.

The terms corresponding to the loops are $a'b'd'$, $a'bd$, $a'cd$

Hence minimum $f(a, b, c, d) = a'b'd' + a'bd + a'cd$.