$\therefore E(X) = \frac{1}{21}\{20\times3 + 30\times12 + 40\times6\}$

$= \frac{220}{7}$

$\simeq \boxed{31\,P}$

Practice Problems

Q1 Let $f(x) = Ke^{-ax}(1-e^{-ax})$, $x>0$, $a>0$.

(i) Find K such that $f(x)$ is a density function.

(ii) Find the corresponding cummulative distribution function.

(iii) Find $P(X>1)$.

Q2 A variate X has probability distribution

x	-3	6	9
$P(X=x)$	$1/6$	$1/2$	$1/3$

Find $E(X)$ & $E(X^2)$. Hence evaluate $E(2X+1)^2$.

Q3 A coin is tossed until a head appears. What is the expection of the number of tosses.

[Ans: 2]

Moment generating function

Mgf of the discrete probability distribution of the variate X about the value $x = a$ is defined by

$$M_a(t) = \sum_i p_i e^{t(x_i - a)}$$; which is a function of the parameter t only.

$$M_a(t) = E[e^{t(x-a)}].$$

$$M_a(t) = \sum_i p_i + t \sum_i p_i (x_i - a) + \frac{t^2}{2!} \sum p_i (x_i - a)^2 + \cdots$$

$$= 1 + t\mu_1'' + \frac{t^2}{2!}\mu_2'' + \cdots,$$

where

μ_r'' is the momement of order r about a.

Since $M_a(t)$ generates moments, it is called moment generating function.

$\therefore \mu_r'' =$ coefficient of $\frac{t^r}{r!}$ in the expansion of $M_a(t)$

Also, $\mu_r'' = \left.\frac{d^r}{dt^r}[M_a(t)]\right|_{t=0}$

We denote

μ_r' by moments about origin, that is,

$$\therefore \mu_r' = \frac{d^r}{dt^r}[M_0(t)]\Big|_{t=0}$$

and

μ_r by central moments, that is, moments about mean,

$$\mu_r = \frac{d^r}{dt^r}[M_\mu(t)]\Big|_{t=0}$$

$$\Rightarrow \mu_r = \sum_i (x_i-\mu)^r p_i$$

or

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu)^r f(x)\,dx.$$

<u>Ques</u> Find the mgf of $f(x) = \frac{1}{c} e^{-x/c}$, $0 \le x \le \infty$, $c>0$. Hence find its mean and standard deviation.

<u>Sol'n:</u> $M_0(t) = E(e^{tx})$

$$= \int_0^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \frac{1}{c} e^{-x/c}\, dx$$

$$= \frac{1}{c} \int_0^\infty e^{-\left(\frac{1}{c}-t\right)x}\, dx$$

$$= \frac{1}{c} \left[\frac{e^{-(1/c - t)x}}{-\left(\frac{1}{c}-t\right)} \right]\Bigg|_{x=0}^{\infty}$$

$$= \frac{1}{(ct-1)} [0-1]$$

$$= \frac{1}{1-ct} = (1-ct)^{-1} = 1 + ct + c^2t^2 + \dots$$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$= \mu_1'$ (moment of order 1 about origin)

$$= \frac{d}{dt} [M_0(t)] \Big|_{t=0}$$

$$= (c + 2c^2 t + 3c^3 t^2 + \dots) \Big|_{t=0}$$

$$= \boxed{c}$$

Moment of order 2 about origin, $\mu_2' = \int_{-\infty}^{\infty} x^2 f(x) dx$

$$= \frac{d^2}{dt^2} [M_0(t)] \Big|_{t=0} = \boxed{2c^2}$$

$$\therefore \sigma^2 = \mu_2' - (\mu_1')^2$$
$$= 2c^2 - c^2$$
$$= \boxed{c^2}$$

$$\Rightarrow \boxed{\sigma = c}$$

Binomial distribution

It is concerned with trials of a repetitive nature in which only the occurrence or non-occurrence, success or failure, acceptance or rejection of a particular even is of interest.

The Binomial probability distribution of the variate x is given by

$$P(x) = {}^nC_x \, p^x \, q^{n-x} \quad , \quad x = 0, 1, ---, n$$

$p \rightarrow$ probability of success

$q \rightarrow$ probability of failure

$$\therefore p + q = 1$$

Moment generating function about origin in Binomial distribution

$$M_0(t) = E(e^{tx}) = \sum_{x=0}^{n} P(x)\, e^{tx}$$

$$= \sum_{x=0}^{n} {}^{n}C_x\, p^x q^{n-x} e^{tx}$$

$$= \sum_{x=0}^{n} {}^{n}C_x\, (pe^t)^x q^{n-x}$$

$$= (q+pe^t)^n$$

Since mean $= \dfrac{d}{dt}[M_0(t)]\Big|_{t=0}$

$$= n(q+pe^t)^{n-1}(pe^t)\Big|_{t=0}$$

$$= n(p+q)^{n-1}(p)$$

$$= \boxed{np} \qquad (\because p+q=1)$$

$$\mu_2' = \frac{d^2}{dt^2}[M_0(t)]\Big|_{t=0}$$

$$= \Big\{ n\big[(n-1)(q+pe^t)^{n-2}(pe^t)^2\big] + n(q+pe^t)^{n-1}(pe^t) \Big\}\Big|_{t=0}$$

$$= \boxed{n(n-1)p^2 + np}$$

Similarly, $\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

Since $M_a(t) = e^{-at} M_0(t)$.

$\therefore$ mgf of the binomial distribution about its mean $(\mu) = np$ is given by

$$M_\mu(t) = e^{-npt}(q+pe^t)^n$$

$$= (qe^{-pt} + pe^{t-pt})^n$$

$$= (qe^{-pt} + pe^{qt})^n$$

$$= \left[1 + pq\frac{t^2}{2!} + pq(q^2-p^2)\frac{t^3}{3!} + pq(q^3+p^3)\frac{t^4}{4!} + \dots\right]^n$$

$$\Rightarrow 1 + \mu_1 t + \mu_2\frac{t^2}{2!} + \mu_3\frac{t^3}{3!} + \dots =$$

$$1 + npq\frac{t^2}{2!} + npq(q-p)\frac{t^3}{3!} + \dots$$

$(\because p+q=1)$

comparing the coeff. of like powers of t, we get

$\mu_1 = 0$, $\mu_2 = npq$, $\mu_3 = npq(q-p)$, ---

∴ variance = second central moment

$= \mu_2$

$= npq$

∴ standard deviation $= \sqrt{npq}$

Skewness and kurtosis

Skewness and kurtosis provide quantative measures of deviation from a theoretical distribution. Skewness is the degree of distortion from the symmetrical bell curve. Coefficient of skewness based on third moment is given by

$$\gamma_1 = \sqrt{\beta_1}, \text{ where } \beta_1 = \frac{\mu_3^2}{\mu_2^3}.$$

Kurtosis measures the degree of peakedness of a distribution and is given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$\gamma_2 = \beta_2 - 3$ gives the excess of kurtosis.

The curves with $\beta_2 > 3$ are called leptokurtic and those with $\beta_2 < 3$ as platykurtic.

normal curve for which $\beta_2 = 3$ is called Mesokurtic

In Binomial distribution,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(npq)^2 (q-p)^2}{(npq)^3} = \frac{(q-p)^2}{npq}$$

$$\Rightarrow \quad \beta_1 = \frac{(1-2p)^2}{npq}$$

$$\therefore \text{ skewness } = \frac{(1-2p)}{\sqrt{npq}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq\,[1+3(n-2)pq]}{(npq)^2}$$

$$= \frac{1+3(n-2)pq}{npq}$$

$$\therefore \text{ Kurtosis}, \ \beta_2 = 3 + \frac{1-6npq}{npq}$$

Ques 1 : A die is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean & variance of the number of successes.

Sol^n: Here, $n = 3$, $p = \frac{2}{6} = \frac{1}{3}$, $q = \frac{2}{3}$

$$\text{mean} = np = 3 \times \frac{1}{3} = \boxed{1}$$

$$\text{variance} = npq = 3 \times \frac{1}{3} \times \frac{2}{3} = \boxed{\frac{2}{3}}$$

Ques 2 : 10 % of screws produced in a certain turn out to be defective. Find the probability that in a sample of 10 screws chosen at random exactly 2 will be defective.

Soln: Here, $n = 10$, $p = \frac{1}{10}$, $q = \frac{9}{10}$, $r = 2$

$$P(r) = {}^nC_r \, p^r q^{n-r}$$

$$\therefore \boxed{P(2) = 0.1937}$$

Binomial frequency distribution :

If n independent trials constitute one experiment and this experiment is repeated N times, then the frequency of r successes is

$$F(r) = N \, {}^nC_r \, p^r q^{n-r}.$$

Ques Fit a binomial distribution to the following data :

x :	0	1	2	3	4	5	6	7	8	9	10
f :	6	20	28	12	8	6	0	0	0	0	0

Soln: Here $n = 10$, $N = \Sigma f_i = 80$

$$\text{mean} = \frac{\Sigma x_i f_i}{\Sigma f_i} = \boxed{2.175}$$

mean of binomial distribution = np

$= 10p = 2.175$

$\Rightarrow p = 0.2175$

$\therefore q = 1-p = 0.7825$

$\therefore F(r) = N\,{}^nC_r\, p^r q^{n-r}, \quad 0 \le r \le 10$

$\therefore$ Binomial distribution is

x:	0	1	2	3	4	5	6	7	8	9	10
f:	6.9	19.1	24.0	17.8	8.6	2.9	0.7	0.1	0	0	0

Practice Problems :

Q1 Fit a binomial distribution to the following frequency table :

x	0	1	3	4
f	28	62	10	4

Q2 Find binomial distribution if the sum and variance and the product of the mean and variance are $\frac{25}{3}$ and $\frac{50}{3}$ respectively :

Q3 Mean of a Binomial distribution is 4 a
its third moment about mean is 1.
Find other constants of the distribution.

Q4 Find the Binomial distribution whose
mean is 5 and variance is $\frac{10}{3}$.

Poisson distribution

The Poisson probability distribution of the variate x is

$$P(x) = e^{-m} \cdot \frac{m^x}{x!}, \quad x = 0, 1, \ldots$$

This distribution can be derived as a limiting case of the binomial distribution by making n very large and p very small, keeping np fixed ($= m$, say).

It is a distribution related to the probabilities of events which are extremely rare, but which have a large number of independent opportunities for occurrence.

For example, the number of hurricanes in a given season.

Constants of Poisson distribution

These constants can be derived from the corresponding constants of the binomial distribution by simply making $n \to \infty$, $p \to 0$, $q \to 1$ and taking $np = m$.

$$\text{Mean} = \lim_{\substack{n \to \infty \\ p \to 0 \\ q \to 1 \\ np \to m}} (np) = \boxed{m}$$

Variance $= \lim (npq)$

$= m \lim (q)$

$= \boxed{m} \longrightarrow \mu_2$

standard deviation $= \sqrt{m}$

$\mu_3 = \lim [npq(q-p)]$

$= m \lim q(q-p)$

$= \boxed{m}$

$\mu_4 = \lim [npq\{1+3(n-2)pq\}]$

$= m \lim [q\{1+3(mq-2pq)\}]$

$= \boxed{m(1+3m)}$

$\therefore$ skewness $= \sqrt{\beta_1}$, where

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{1}{m}$$

$\therefore$ $\boxed{\text{skewness} = \frac{1}{\sqrt{m}}}$

Kurtosis, $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{m(1+3m)}{m^2} = \dfrac{1+3m}{m}$

$\therefore$ $\boxed{\text{Kurtosis} = 3 + \frac{1}{m}}$

Poisson frequency distribution :

$$F(x) = N \frac{e^{-m} m^x}{x!}, \quad x = 0, 1, \cdots ;$$

where N is the total frequency.

Ques 1 : Fit a Poisson distribution to the set of observations:

x	0	1	2	3	4
f	122	60	15	2	1

soln: mean $= \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \boxed{0.5}$

Since mean of Poisson distribution $= m = 0.5$

$\therefore$ Poisson frequency distribution $= N \dfrac{e^{-m} m^x}{x!}$

$x = 0, 1, 2, 3, 4.$

$$= \frac{200\, e^{-0.5} (0.5)^x}{x!}, \quad x = 0, 1, \cdots, 4$$

$\therefore$ Poisson frequency distribution is

x :	0	1	2	3	4
f :	121	61	15	2	0

Que2: If the probability of a bad reaction from certain injection is 0.001, determine the chance out of 2000 individuals more than two will get bad reaction.

Soln:

$p = 0.001$

$n = 2000$

$\therefore$ mean $= m = np = \boxed{2}$

$$P(x>2) = 1 - P(x \le 2)$$
$$= 1 - e^{-m}\left[1 + m + \frac{m^2}{2}\right]$$
$$= 1 - \frac{1}{e^2}[5]$$
$$= \boxed{0.32}$$

Practice Problems :

Q1: In a Poisson distribution, $P(x)$ for $x = 0$ is 10%. Find the mean. [Ans: $m = 2.3026$]

Q2: If x is a Poisson variate with mean m, find MGF of $z = \frac{x - m}{\sqrt{m}}$.

Q3: If x is a Poisson variate with mean m, find

(i) $E(e^{-kx})$

(ii) $E(x e^{-kx})$

Moment generating function of Poisson distribution

$$M_0(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-m} \cdot m^x}{x!}$$

$$= e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!} = e^{-m} \cdot e^{me^t}$$

$$= \boxed{e^{m(e^t-1)}}$$

$$\therefore M_\mu(t) = e^{-\mu t} M_0(t)$$

$$= e^{-mt} [e^{m(e^t-1)}]$$

$$= \boxed{e^{m(e^t-1-t)}}$$

Normal distibution

The normal distribution of the continuous variate x with mean μ and standard deviation σ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

Mean deviation about mean for a normal variate with mean μ and s.d. σ.

Mean deviation $= E(|x-\mu|)$

$$= \int_{-\infty}^{\infty} |x-\mu| \, f(x) \, dx$$

$$= \int_{-\infty}^{\infty} |x-\mu| \, \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

Put $z = \frac{x-\mu}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|\frac{x-\mu}{\sigma}\right| e^{-\frac{1}{2}z^2} \, dz$$

$$\left(\because \frac{dx}{\sigma} = dz\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| \, e^{-\frac{1}{2}z^2} \, dz$$

Put $\frac{1}{2}z^2 = t \Rightarrow \frac{2z}{2} dz = dt$

$\Rightarrow z \, dz = dt$

$$\therefore = \frac{2\sigma}{\sqrt{2\pi}} \left[\int_0^{\infty} e^{-t} \, dt \right]$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \left(-e^{-t}\right)\Big|_0^{\infty}$$

$$= \sigma\sqrt{\frac{2}{\pi}} = 0.7979\,\sigma$$

$$\simeq \frac{4}{5}\,\sigma$$

...ents about mean

...dd order moment

$$\mu_{2n+1} = E\left[(x-\mu)^{2n+1}\right]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow dz = \frac{dx}{\sigma}$

$$\Rightarrow dx = \sigma\, dz$$

$$= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} \cdot e^{-\frac{z^2}{2}} dz$$

$= 0$, since it is an odd function.

$\therefore$ all odd order moments about mean vanish.

even order moment

$$\mu_{2n} = E\left[(x-\mu)^{2n}\right]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{z^{2n-1}}_{I} \cdot \underbrace{\left(z e^{-\frac{z^2}{2}}\right)}_{II} dz$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}}\left[-z^{2n-1} \cdot e^{-z^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}(2n-1)\, z^{2n-2} \cdot e^{-z^2/2}\, dz\right]$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}}\left[(0-0) + (2n-1)\int_{-\infty}^{\infty} z^{2n-2} \cdot e^{-\frac{z^2}{2}}\, dz\right]$$

$$= \sigma^2 (2n-1)\, \mu_{2n-2}$$

$$\therefore \boxed{\mu_{2n} = \sigma^2 (2n-1)\, \mu_{2n-2}}$$

Since $\mu_2 = \sigma^2$

for $n=2$ $\quad \therefore \mu_4 = \sigma^2 (3)\, \mu_2$

$$\Rightarrow \boxed{\mu_4 = 3\sigma^4}$$

$$\therefore \beta_1 = \frac{\mu_3^4}{\mu_2^3} = 0 \qquad (\because \mu_3 = 0)$$

Also,

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\sigma^4}{\sigma^4} = \boxed{3}$$

$$\therefore \beta_2 = 3.$$

Since $\gamma_1 = \sqrt{\beta_1} = 0$, i.e. coefficient of skewness is zero, the curve is symmetrical. Moreover, the kurtosis is 3.

Ques1: For a certain normal distribution the first moment about 10 is 40 and that the 4th moment about 50 is 48; what is mean and standard deviation of the distribution?

soln: let μ be the mean and σ be the standard deviation.

first moment about 10, $E(X-10) = 40$.

$$\Rightarrow E(X) - 10 = 40$$

$$\Rightarrow \boxed{\mu = 50}$$

Moreover, fourth moment about 50,

$$\mu_4 = 3\sigma^4$$
$$= 48$$

$$\Rightarrow \sigma^4 = 16$$

$$\Rightarrow \boxed{\sigma = 2}$$

Ques 2: If X is a normal variate with mea[illegible] and SD 5, find the probabilities that

(i) $26 \le X \le 40$

(ii) $|X-30| > 5$.

Soln: Here $\mu = 30$, $\sigma = 5$. $\therefore Z = \frac{X-\mu}{\sigma} = \frac{X-30}{5}$

(i) if $X = 26$, $Z = -0.8$ & if $X = 40$, $Z = 2$

$$\therefore P(26 \le X \le 40) = P(-0.8 \le Z \le 2)$$
$$= P(-0.8 \le Z \le 0) + P(0 \le Z \le 2)$$
$$= P(0 \le Z \le 0.8) + P(0 \le Z \le 2)$$
$$= 0.2881 + 0.4772$$
$$= \boxed{0.7653}$$

(ii) $$P(|X-30| > 5) = 1 - P(|X-30| \le 5)$$
$$= 1 - P(25 \le X \le 35)$$
$$= 1 - P\left(-1 \le \frac{X-30}{5} \le 1\right)$$
$$= 1 - P(-1 \le Z \le 1)$$
$$= 1 - 2P(0 \le Z \le 1)$$
$$= 1 - 2(0.3413)$$
$$= 1 - 0.6826$$
$$= \boxed{0.3174}$$

Assignment Problems Probability 1

Unit-2

Ten coins are thrown simultaneously. Find the probability of getting at least seven ~~kato~~ heads.

[Ans: $\frac{15}{1024}$]

Q2 Let X & Y be discrete variates. Prove that

$E(X+K) = E(X) + K$

$E(KX) = K E(X)$

$E(X+Y) = E(X) + E(Y)$

$Var(X+K) = Var(X)$

$Var(KX) = K^2 Var(X)$

Q3 Suppose a continuous R.V. x has the probability density

$$f(x) = \begin{cases} K(1-x^2), & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

a) Find K

b) Find $P(0.1<x<0.2)$

c) Find $P(x>0.5)$

Using distribution function, determine the probabilities that

d) $x < 0.3$

e) $0.4 < x < 0.6$

f) Calculate mean & variance for the probability density f^n.

Q4 Find the mean & variance of PDF.

$$f(x) = \begin{cases} \frac{1}{4} e^{-x/4} & , x > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

[Ans: $\mu = 4$, $\sigma^2 = 80$]

Q5 A person wins Rs 80 if 3 heads occur, Rs 30 if 2 occur, Rs 10 if only one head occurs in a single toss of 3 fair coins. If the game is to be fair, how much should he lose if no heads occur?

[Let Rs x be the amount lost if no heads occur

Then

$$\text{Expectation} = 80 \cdot \frac{1}{8} + 30 \cdot \frac{3}{8} + 10 \cdot \frac{3}{8} - x \cdot \frac{1}{8}$$

$$= 0$$

$\Rightarrow \boxed{x = 200}$]

Q6 Two dice are thrown 120 times. Find the average number of times in which the number on first dice exceeds the number on the second dice.

[$E(X) = np = 120 \left(\frac{5}{12}\right) = 50$

possibilities are (2,1); (3,1), (4,1), (5,1), (6,1), (3,2), (4,2), (5,2), ---, (6,5) $= \frac{15}{36} = \frac{5}{12}$]

If X is a BD random variable with $E(X) = 2$ & $Var(X) = \frac{4}{3}$, find the distribution of X

Sol^n:

$$E(X) = np = 2, \quad Var(X) = npq = \frac{4}{3}$$

$$\Rightarrow q = \frac{2}{3}, \quad p = \frac{1}{3}$$

$$\therefore n\left(\frac{1}{3}\right) = 2 \Rightarrow \boxed{n = 6}$$

$$f(x) = {}^nC_x \, p^x q^{n-x}$$

x_i	0	1	2	3	4	5	6
$f(x_i)$	$\frac{64}{729}$	$\frac{192}{729}$	$\frac{240}{729}$	$\frac{160}{729}$	$\frac{60}{729}$	$\frac{12}{729}$	$\frac{1}{729}$

Q8. For a Poisson variate x with parameter λ, show that

$$\mu'_{r+1} = \lambda \mu'_r + \lambda \frac{d\mu'_r}{d\lambda}, \text{ where}$$

$$\mu'_r = E(x^r), \; r \geq 0 \text{ integer.}$$

$$\mu'_r = E(x^r) = \sum_{x=0}^{\infty} x^r \cdot e^{-\lambda} \cdot \frac{\lambda^x}{x!}$$

$$\Rightarrow \frac{d\mu'_r}{d\lambda} = -\sum_{x=0}^{\infty} x^r \cdot e^{-\lambda} \cdot \frac{\lambda^x}{x!} + \frac{1}{\lambda} \sum_{x=0}^{\infty} x^{r+1} \cdot e^{-\lambda} \cdot \frac{\lambda^x}{x!}$$

$$= -\mu'_r + \frac{1}{\lambda} \mu'_{r+1}$$

$$\Rightarrow \mu'_{r+1} = \lambda \mu'_r + \lambda \frac{d\mu'_r}{d\lambda}$$