QUESTIONS :- (Applied Maths - II)

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(UNIT - 1) {Complex Analysis - I}

Q.1) Find $|z|$ and $\arg(z)$ when,
$$Z = \frac{(2-3i)(1+i)}{(2+i)}$$

Sol: $Z = \frac{-5i-1}{2+i} \times \frac{2-i}{2-i} = \frac{(1+5i)(i-2)}{5} = -\frac{7}{5} - \frac{9}{5}i$

So,
$$|z| = \sqrt{\left(\frac{-7}{5}\right)^2 + \left(\frac{-9}{5}\right)^2} = \frac{\sqrt{130}}{5}$$

and,
$$\arg(z) = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-9}{\cancel{5}} \times \frac{-\cancel{5}}{7}\right|$$

For 3rd Quadrant,
$$\Rightarrow \arg(z) = -(\pi - \theta)$$
$$\therefore \arg(z) = -\left(\pi - \tan^{-1}\left(\frac{9}{7}\right)\right).$$

Q.2) Find $\arg(z)$ if, $Z = \frac{(3+4i)(2-i)}{(2+3i)^2}$.

Sol: Concept same as above,
$$\Rightarrow Z = \frac{-5}{169}(-2+29i) = \frac{10}{169} - \frac{145}{169}i$$

$\because$ It lies in Quadrant IV,
$$\arg(z) = -\theta.$$

So,
$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-145}{169} \times \frac{169}{10}\right) = \tan^{-1}\left(\frac{-145}{10}\right).$$

Therefore, $\arg(z) = \tan^{-1}\left(\frac{145}{10}\right).$

Q.3) Proove that,

$$\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \sin 9\theta - i\cos 9\theta.$$

Sol: On LHS,

$$\Rightarrow \frac{(\cos\theta + i\sin\theta)^4}{[\cos(90-\theta) + i\sin(90-\theta)]^5}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^4 \cdot [\cos(90-\theta) + i\sin(90-\theta)]^{-5}$$

$$\Rightarrow (\cos 4\theta + i\sin 4\theta) \cdot [\cos(450-5\theta) - i\sin(450-5\theta)]$$

$$\Rightarrow (\cos 4\theta + i\sin 4\theta) \cdot (\sin 5\theta - i\cos 5\theta)$$

$$\Rightarrow \sin 5\theta \cdot \cos 4\theta + \sin 4\theta \cdot \cos 5\theta - \cos 4\theta \cdot \cos 5\theta\, i + \sin 4\theta \cdot \sin 5\theta\, i$$

$$\Rightarrow \sin(5\theta + 4\theta) + i[-(\cos 4\theta \cdot \cos 5\theta - \sin 4\theta \cdot \sin 5\theta)]$$

$$\Rightarrow \sin 9\theta - i \cdot \cos(4\theta + 5\theta)$$

$$\Rightarrow \sin 9\theta - i\cos 9\theta \quad = RHS.$$

Q.4) If $x + \frac{1}{x} = 2\cos\theta$, then proove that,

$$x^n + \frac{1}{x^n} = 2\cos n\theta.$$

Sol: $\Rightarrow x^2 + 1 = 2\cos\theta \cdot x$

$\Rightarrow x^2 - 2\cos\theta \cdot x + 1 = 0.$

$\Rightarrow x = \frac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2} = \cos\theta \pm i\sin\theta.$

$\Rightarrow x^n = (\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$.

So,

$\Rightarrow x^n + \frac{1}{x^n} = (\cos n\theta \pm i\sin\theta) + (\cos n\theta \mp i\sin n\theta)$

$= 2\cos n\theta \cancel{\pm i\sin\theta} \cancel{\mp i\sin n\theta}$

$\therefore x^n + x^{-n} = 2\cos n\theta = RHS$.

Q.5) Proove that,

$$(1+i)^n + (1-i)^n = 2^{\left(\frac{n}{2}+1\right)} \cdot \cos\frac{n\pi}{4}$$

Sol: (Use De-Moivre's Theorem on LHS...).

Q.6) If ω is a complex fourth root of unity, show that

$$1+\omega+\omega^2+\omega^3 = 0.$$

Sol: Let, $\Rightarrow x^4 = 1 \quad \Rightarrow x = (1)^{1/4} = (\cos 0 + i\sin 0)^{1/4}$.

Generalising, since x has four roots,

$x = \cos\frac{2K\pi}{4} + i\sin\frac{2K\pi}{4}$. $\quad (K \in I)$.

For $K=0$,

$\Rightarrow x_0 = \cos 0 + i\sin 0 = 1$.

For $K=1$,

$\Rightarrow x_1 = \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4} = \omega = 0 + i\times 1 = i$

For $K=2$,

$\Rightarrow x_2 = \cos 2\left(\frac{2\pi}{4}\right) + i\sin 2\left(\frac{2\pi}{4}\right) = \left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right)^2$

$\therefore x_2 = \omega^2 = (0+i)^2 = i^2 = -1$.

For $K=3$,

$\Rightarrow$ $x_3 = \cos 3\left(\frac{2\pi}{4}\right) + i\sin 3\left(\frac{2\pi}{4}\right)$

$= \left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right)^3 = \omega^3 = -i$

On LHS,

$\Rightarrow$ $1+\omega+\omega^2+\omega^3$

$= \cancel{1}+\cancel{i}-\cancel{1}-\cancel{i} = 0 = RHS$.

Q.7 Solve, $x^4+x^3+x^2+x+1=0$

Sol: * Multiply both sides by $(x-1)$,

$\Rightarrow$ $(x-1)(x^4+x^3+x^2+x+1)=0$.

$\Rightarrow$ $x^5+\cancel{x^4}+\cancel{x^3}+\cancel{x^2}+\cancel{x}-\cancel{x^4}-\cancel{x^3}-\cancel{x^2}-\cancel{x}-1=0$.

$\Rightarrow$ $x^5-1=0$.

$\Rightarrow$ $x^5=1$ $\Rightarrow x=(1)^{1/5}$

Or,

$\Rightarrow$ $x = \cos\frac{2K\pi}{5} + i\sin\frac{2K\pi}{5}$

For $K=0$,

$\Rightarrow$ $x_0 = \cos 0 + i\sin 0 = 1$

For $K=1$,

$\Rightarrow$ $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$

For $K=2$,

$\Rightarrow$ $x_2 = \cos 2\left(\frac{2\pi}{5}\right) + i\sin 2\left(\frac{2\pi}{5}\right) = \alpha^2$.

For. $K=3$,

$\Rightarrow x_3 = \cos 3\left(\frac{2\pi}{5}\right) + i \sin 3\left(\frac{2\pi}{5}\right) = \alpha^3$.

For $K=4$,

$\Rightarrow x_4 = \cos 4\left(\frac{2\pi}{5}\right) + i \sin 4\left(\frac{2\pi}{5}\right) = \alpha^4$.

Here,

$\Rightarrow \alpha = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = \frac{(-1+\sqrt{5})}{4} + i \frac{(10+2\sqrt{5})}{4}$

Similarly; α^2, α^3 and α^4 can be calculated;

$\therefore x = 1, \alpha, \alpha^2, \alpha^3, \alpha^4$.

Q.8) Solve, $z^6 + z^4 - z^2 - 1 = 0$.

Sol: $z^6 + z^4 - (z^2+1) = 0$.

$\Rightarrow z^4(z^2+1) - (z^2+1) = 0$

$\Rightarrow (z^2+1)(z^4-1) = 0$

i.e.,

$\Rightarrow z^2+1=0 \Rightarrow z = (-1)^{1/2} = [\cos\pi + i\sin\pi]^{+1/2}$.

$\Rightarrow z = \left(\cos\left(\frac{2K\pi+\pi}{2}\right) + i\sin\left(\frac{2K\pi+\pi}{2}\right)\right)$.

For $K=0$,

$\Rightarrow z_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0+i = i$.

For $K=1$,

$\Rightarrow z_2 = \cos\left(\pi+\frac{\pi}{2}\right) + i\sin\left(\pi+\frac{\pi}{2}\right) = -i$

And,

$\Rightarrow z^4 = 1 \Rightarrow z = (1)^{1/4} = (\cos 0 + i\sin 0)^{1/4}$.

$\Rightarrow Z = \cos\frac{2K\pi}{4} + i\sin\frac{2K\pi}{4}$ $(K \in I)$.

For $K = 0, 1, 2, 3$;

$\Rightarrow Z_3 = \cos 0 + i\sin 0 = 1$.

$\Rightarrow Z_4 = \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4} = i$

$\Rightarrow Z_5 = \left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right)^2 = i^2 = -1$.

$\Rightarrow Z_6 = \left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right)^3 = i^3 = -i$.

$\therefore$ All six roots of this equation are, (Distinct Roots)

$Z = 1, i, i, -i, -i, -1 \rightarrow \{1, -1, i, -i\}$.

Q.9) a) Represent $\frac{1+i}{1-i}$ in the form of $r(\cos\theta + i\sin\theta)$?

Sol: $Z = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+i^2+2i}{1-i^2} = \frac{2i}{2} = 0 + i$.

So,

$\Rightarrow r = \sqrt{(0)^2 + (1)^2} = 1$

$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$.

In Polar form,

$\Rightarrow Z = r(\cos\theta + i\sin\theta)$

$\therefore Z = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$.

b) If $Z = 1 - i$, Analytically find Z^8?

Sol: $r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$.

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left|\frac{-1}{1}\right| = \frac{\pi}{4}$.

$\because$ Z lies in Quadrant IV, then

$arg(Z) = -\theta = -\frac{\pi}{4}$.

$\therefore Z = \sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$.

$\Rightarrow Z^8 = (\sqrt{2})^8\left(\cos\frac{8\pi}{4} - i\sin\frac{8\pi}{4}\right) = 16(1-0) = \boxed{16}$.

c) Express $-1 + \sqrt{-3}$ in polar form?

Sol: $Z = -1 + \sqrt{-3} = -1 + i\sqrt{3}$

So,

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$arg(Z) = \pi - \theta = \pi - \tan^{-1}(\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Hence,

$\Rightarrow Z = 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$.

Q.10) Test the analyticity of function $w = \sin z$ and hence, derive that,

$$\frac{d}{dz}(\sin z) = \cos z.$$

Sol: Given function will be analytic if it satisfies C-R equation

$\Rightarrow$ $w = \sin z = \sin(x+iy)$
or,
$\Rightarrow$ $w = \sin x . \cos iy + \sin iy . \cos x$.

Note : [$\sin ix = i \sinh x$; $\cos ix = \cosh x$]

$\Rightarrow$ $w = \sin x (\cosh y) + (i \sinh y) . \cos x$.
or,
$\Rightarrow$ $u + iv = (\sin x . \cosh y) + i(\sinh y . \cos x)$
So,

$$u(x,y) = \sin x . \cosh y.$$
$$v(x,y) = \cos x . \sinh y.$$

Now,
$\Rightarrow$ $\frac{\partial u}{\partial x} = \cos x . \cosh y$, $\frac{\partial v}{\partial y} = \cos x . \cosh y$.
$\Rightarrow$ $\frac{\partial u}{\partial y} = \sin x . \sinh y$, $\frac{\partial v}{\partial x} = -\sin x . \sinh y$.
Since,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and, } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- Hence, function $f(z)$ is analytic.

$\therefore$ $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$.
$= \cos x . \cosh y - i \sin x . \sinh y$.
$= \cos x . \cos iy - \cancel{i} \sin x \frac{\sin iy}{\cancel{i}}$ (using above properties)
$= \cos x . \cos iy - \sin x . \sin iy$.
$= \cos(x+iy)$ (from trigonometry).
$\Rightarrow$ $f'(z) = \cos z$.

- Therefore, $\boxed{\frac{d}{dz}(\sin z) = \cos z}$. Hence prooved.

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Q.11) Show that the function $f(z) = z|z|$ is not analytic?

Sol: Given, $f(z) = z.|z|$

If, $z = x+iy$ then, $|z| = \sqrt{x^2+y^2}$.

$\therefore f(z) = w = (x+iy)\sqrt{x^2+y^2} = x\sqrt{x^2+y^2} + y\sqrt{x^2+y^2}\,i$

i.e.,

$u = x\sqrt{x^2+y^2}$, $v = y\sqrt{x^2+y^2}$.

For C-R equations,

$\Rightarrow \frac{\partial u}{\partial x} = \sqrt{x^2+y^2} + x\frac{1}{2\sqrt{x^2+y^2}}(2x) = \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}}$

$\Rightarrow \frac{\partial u}{\partial y} = x\frac{1}{2\sqrt{x^2+y^2}}(2y) = \frac{xy}{\sqrt{x^2+y^2}}$.

$\Rightarrow \frac{\partial v}{\partial x} = y\frac{1}{2\sqrt{x^2+y^2}}(2x) = \frac{xy}{\sqrt{x^2+y^2}}$.

$\Rightarrow \frac{\partial v}{\partial y} = \sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}}$.

$\because \left.\begin{matrix}\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \text{ and, } \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}\end{matrix}\right\}$ C-R conditions.

Therefore,

• given function $f(z) = z.|z|$ is not analytic.

Q.12) Find the value of C_1 and C_2 such that the function,

$$f(z) = x^2 + C_1 y^2 - 2xy + i(C_2 x^2 - y^2 + 2xy)$$

is analytic. Also find $f'(z)$?

Sol: Given,

$u = x^2 + C_1 y^2 - 2xy$; $v = C_2 x^2 - y^2 + 2xy$.

$\Rightarrow \dfrac{\partial u}{\partial x} = 2x - 2y$ $\Rightarrow \dfrac{\partial v}{\partial x} = 2C_2 x + 2y$

$\Rightarrow \dfrac{\partial u}{\partial y} = 2C_1 y - 2x$ $\Rightarrow \dfrac{\partial v}{\partial y} = -2y + 2x.$

From C-R equations,

$\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$\Rightarrow 2x - 2y = 2x - 2y$ (LHS = RHS)

and,

$\Rightarrow \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\Rightarrow 2C_1 y - 2x = -(2C_2 x + 2y)$

$\Rightarrow C_1 y - x = -(C_2 x + y) = -C_2 x - y.$

or,

$\Rightarrow C_1 y + C_2 x = x - y.$

$\Rightarrow C_1 y + y + C_2 x - x = 0.$

or,

$\Rightarrow x(C_2 - 1) + y(C_1 + 1) = 0 = x(0) + y(0).$

On comparing,

$\Rightarrow C_2 - 1 = 0 \quad \Rightarrow \boxed{C_2 = 1}$

$\Rightarrow C_1 + 1 = 0 \quad \Rightarrow \boxed{C_1 = -1}$

$\therefore f(z) = x^2 + y^2 - 2xy + i(2xy - x^2 - y^2)$

$\Rightarrow f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$= (2x - 2y) + i(2y - 2x)$

$= 2(x-y) - 2(x-y)i.$

or,

$$\boxed{f'(z) = 2(x-y)(1-i)}$$

Q.13) Show that the function $f(z) = \log z$ where, $z = x+iy$ and $z \neq 0$, is a analytic function? Also find $f'(z)$.

Sol: $z = x+iy$.

$\Rightarrow f(z) = \log z = \log(x+iy)$

or,

$\Rightarrow u+iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$ (Direct Formula)

On comparing,

$u = \frac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1}\left(\frac{y}{x}\right)$.

$\Rightarrow \frac{\partial u}{\partial x} = u_x = \frac{1}{2(x^2+y^2)} \times (2x) = \frac{x}{x^2+y^2}$

$\Rightarrow \frac{\partial u}{\partial y} = u_y = \frac{1}{2(x^2+y^2)} \times (2y) = \frac{y}{x^2+y^2}$

$\Rightarrow \frac{\partial v}{\partial x} = v_x = \frac{x^2}{x^2+y^2} \times y \times \frac{-1}{x^2} = \frac{-y}{x^2+y^2}$.

$\Rightarrow \frac{\partial v}{\partial y} = v_y = \frac{x^2}{x^2+y^2} \times \frac{1}{x} = \frac{x}{x^2+y^2}$

$\because$ From C-R equations,

$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and, $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$.

$\therefore$ Both equation's hold true for the function $f(z)$. So, $f(z)$ is a analytic function.

Now,

$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$\Rightarrow f'(z) = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right).$$

Therefore,

$$\Rightarrow \boxed{f'(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}}.$$

Q.14) Verify if the function $f(z)$ is analytic or not at $f(0)=0$ for $z \neq 0$,

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^4} \quad (z \neq 0).$$

[IMPORTANT]

Sol: $f(z) = \frac{xy^2(x+iy)}{x^2+y^4} = \frac{x^2y^2}{x^2+y^4} + i\frac{xy^3}{x^2+y^4}$

So,

$$u(x,y) = \frac{x^2y^2}{x^2+y^4}, \quad v(x,y) = \frac{xy^3}{x^2+y^4}$$

$\because$ Both functions $u(x,y)$ and $v(x,y)$ are implicit functions and, checking at origin $(0,0)$ first....

For,

$f(0)=0$; from first derivative principle,

$$\Rightarrow \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h}$$

$$\Rightarrow \left.\frac{\partial u}{\partial x}\right|_{x=0} = \lim_{h\to 0} \frac{u(0+h,y) - u(0,y)}{h}$$

Similarly at $(0,0) \rightarrow (x,y) = (0,0)$;

$$\Rightarrow \left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h}$$

$$\Rightarrow u(0+h,0) = \lim_{h\to 0} \frac{(0+h)^2 \times (0)^2}{(0+h)^2+(0)^4} = 0$$

also,

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$\Rightarrow u(0,0) = \frac{0^2 \times 0^2}{0^2+0^4} = 0.$

$\therefore \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{0-0}{h} = 0.$

Now,

$\Rightarrow \frac{\partial u}{\partial y} = \lim_{K\to 0} \frac{u(x,y+K)-u(0,0)}{K}.$

$= \lim_{K\to 0} \frac{u(0,0+K)-u(0,0)}{K}$ $(x,y = 0,0)$

$\therefore \frac{\partial u}{\partial y} = \frac{0-0}{K} = 0.$

Similarly, $\frac{\partial v}{\partial x} = 0$ and, $\frac{\partial v}{\partial y} = 0$

$\therefore$ Function $f(z)$ is analytic at $f(0) = 0$ and holds C-R equations at origin.

Now,

$\Rightarrow f'(z) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$

$\Rightarrow f'(z) = \lim_{z\to 0} \frac{\frac{xy^2(x+iy)}{x^2+y^4} - 0}{x+iy}.$ $(\because z = x+iy).$

*Concept used here: We will substitute y by (mx), i.e., as y = mx (straight line). If, f'(z) is independent of the term 'm' (m ↔ slope) then we can say f'(z) is independent of the path and has a one unique value resulting, f(z) being a ANALYTIC function.

But, if f'(z) is dependent of 'm' and is not independent then, f(z) is not a ANALYTIC function.

Using general curve,

$$y = mx \quad \text{(Straight line at origin)}$$

So,

$$z = x + iy = x + i(mx) = x(1 + im).$$

If,

$$\lim_{z \to 0} f(z) \Rightarrow z \to 0 \text{ then } x \to 0.$$

Therefore,

$$\Rightarrow f'(z) = \lim_{x \to 0} \frac{\frac{x(mx)^2(x + i(mx))}{x^2 + (mx)^4} - 0}{x + i(mx)}$$

$$\Rightarrow f'(z) = \lim_{x \to 0} \frac{\frac{x^3 m^2 \cdot \cancel{x(1+im)}}{x^2 + x^4 m^4}}{\cancel{x(1+im)}}$$

$$\Rightarrow f'(z) = \lim_{x \to 0} \frac{x^{\cancel{3}\,1} m^2}{\cancel{x^2}(1 + x^2 m^4)} = \lim_{x \to 0} \frac{x m^2}{1 + x^2 m^4}$$

$$\Rightarrow f'(z) = \frac{0 \times m^2}{1 + 0^2 \times m^4} = \frac{0}{1 + 0} = \underline{\underline{0}}.$$

$\because$ $f'(z)$ has a unique value $= 0$, i.e., it is independent of the term 'm'.

Therefore,

<u>$f(z)$ is A ANALYTIC function.</u>

Q.15) Show that the function defined by $f(z) = \sqrt{|xy|}$ is not regular at origin, although C-R Equations are satisfied?

Sol: Given function $f(z)$ is not a singled-value function which already fails the first condition of analyticity.

$\Rightarrow$ $f(z) = \sqrt{|xy|} \Rightarrow u + iv = \sqrt{|xy|} + i(0)$.

So,

$$u = \sqrt{|xy|} \quad \text{and,} \quad v = 0$$

At origin, C-R equations are satisfied.

But,

$\Rightarrow$ $f'(0) = \lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z}$ (from first principle)

$\Rightarrow$ $f'(0) = \lim\limits_{z \to 0} \dfrac{\sqrt{|xy|} - 0}{x + iy}$

For a general curve,

$$y = mx$$

$z \to 0$, $x \to 0$ for $z = x + iy = x(1 + im)$.

$\Rightarrow$ $f'(0) = \lim\limits_{x \to 0} \dfrac{\sqrt{|x \times mx|} - 0}{x + i(mx)}$

$\therefore$ $f'(0) = \lim\limits_{x \to 0} \dfrac{\cancel{x}\sqrt{|m|}}{\cancel{x}(1 + im)} = \dfrac{\sqrt{m}}{1 + im}$ $(m > 0)$

$\therefore$ Function $f(z)$ is NOT analytic, since $f'(0)$ is dependent on 'm'.

Q.16) Show that an analytic function with constant modulus is constant?

Sol: Let, $f(z) = u + iv$ be an analytic function.

$\Rightarrow$ $|f(z)| = |u + iv| = u^2 + v^2$.

or,

$\Rightarrow$ $C = u^2 + v^2$ —— ① (C: constant)

Given,

$f(z)$ is analytic, i.e., it satisfies C-R equations,

→ Given equation : $u^2+v^2=a^2$ —— ①

• Partially differentiating eq-① w.r.t x and y,

$\Rightarrow 2u.\frac{\partial u}{\partial x}+2v.\frac{\partial v}{\partial x}=0$

$\Rightarrow u.\frac{\partial u}{\partial x}+v.\frac{\partial v}{\partial x}=0$ —— ②

and,

$\Rightarrow u.\frac{\partial u}{\partial y}+v.\frac{\partial v}{\partial y}=0$ —— ③

→ From equation ②,

$\Rightarrow u.\frac{\partial u}{\partial x}+v\left(-\frac{\partial u}{\partial y}\right)=0$.

$\Rightarrow u.\frac{\partial u}{\partial x}-v.\frac{\partial u}{\partial y}=0$

→ From equation ③,

$\Rightarrow u.\frac{\partial u}{\partial y}+v.\frac{\partial u}{\partial x}=0$

• Eliminating $\partial u/\partial y$,

$\Rightarrow u.\frac{\partial u}{\partial x}-v\left(-\frac{v}{u}.\frac{\partial u}{\partial x}\right)=0$.

$\Rightarrow u.\frac{\partial u}{\partial x}+\frac{v^2}{u}.\frac{\partial u}{\partial x}=0$.

$\Rightarrow \left(\frac{u^2+v^2}{u}\right)\frac{\partial u}{\partial x}=0$

$\therefore \frac{\partial u}{\partial x}=0$

On integration, $\boxed{u=\text{Constant}}$

→ From equation ②,

$\Rightarrow u.\frac{\partial u}{\partial x} + v.\frac{\partial v}{\partial x} = 0.$

$\Rightarrow u.\frac{\partial v}{\partial y} + v.\frac{\partial v}{\partial x} = 0 \qquad \left(\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right).$

similarly,

$\Rightarrow u.\frac{\partial u}{\partial y} + v.\frac{\partial v}{\partial y} = 0 \qquad$ (equation-③)

$\Rightarrow u.\left(-\frac{\partial v}{\partial x}\right) + v.\frac{\partial v}{\partial y} = 0 \qquad \left(\because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right).$

$\Rightarrow -u.\frac{\partial v}{\partial x} + v.\frac{\partial v}{\partial y} = 0.$

• Eliminating $\partial v/\partial x$ from both equations,

$\Rightarrow u.\frac{\partial v}{\partial y} + v\left(\frac{v}{u}.\frac{\partial v}{\partial y}\right) = 0.$

$\Rightarrow u.\frac{\partial v}{\partial y} + \frac{v^2}{u}.\frac{\partial v}{\partial y} = 0.$

$\Rightarrow \left(\frac{u^2+v^2}{u}\right).\frac{\partial v}{\partial y} = 0$

$\Rightarrow \frac{\partial v}{\partial y} = 0$

On integration, $\boxed{v = \text{Constant}}$

$\therefore f(z) = u + iv = \text{Constant} + i(\text{Constant}) = K = \underline{\text{Constant}}.$

Q.17) Show that the function $u = \log\sqrt{x^2+y^2}$ is harmonic. Also, find its harmonic conjugate?

Sol: $u = \frac{1}{2}\log\left(\sqrt{x^2+y^2}\right)^2 = \frac{1}{2}\log(x^2+y^2)$

We know that,

a given function will be harmonic if it satisfies the Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

So,

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{+\cancel{y^2}-\cancel{x^2}+\cancel{x^2}-\cancel{y^2}}{(x^2+y^2)^2} = 0.$$

(Laplace equation is satisfied)

$\therefore$ Given function $u(x,y)$ is a harmonic function.

Here, $v = ?$

$$\Rightarrow dv = \frac{\partial v}{\partial x} . dx + \frac{\partial v}{\partial y} . dy.$$

Using C-R equations,

$$\Rightarrow dv = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy.$$

$$\Rightarrow dv = \left(-\frac{y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy$$

$\because$ It is a form of exact differential equation and is exact D.E. so,

$$\Rightarrow u(x,y) = \int M(x,y).dx, \text{ where } M = \frac{-y}{x^2+y^2}$$

$$\Rightarrow u(x,y) = \int \frac{-y}{x^2+y^2}\,dx = -y\int \frac{dx}{x^2+y^2}.$$

$$\Rightarrow u(x,y) = -\cancel{y} \times \frac{1}{\cancel{y}} \tan^{-1}\left(\frac{x}{y}\right) + C.$$

$$\therefore u(x,y) = -\tan^{-1}\left(\frac{x}{y}\right) + C.$$

also,

$\Rightarrow F(y) = \int N(y).dy$ [Here, $N(y)$ should be independent term of x]

$$\therefore F(y) = \int 0.\,dy = 0.$$

Hence,

$$\Rightarrow V = u(x,y) + F(y)$$

$$\Rightarrow V = -\tan^{-1}\left(\frac{x}{y}\right) + 0 + C$$

$$\therefore \boxed{V = -\tan^{-1}\left(\frac{x}{y}\right) + C}$$

$$\therefore f(z) = u + iv$$

$$\Rightarrow \boxed{f(z) = \frac{1}{2}\log(x^2+y^2) - i\tan^{-1}\left(\frac{x}{y}\right) + C}$$ Final Answer

Q.18) Show that the function $u(x,y) = x^3 - 3xy^2$ is a harmonic function. Find the corresponding analytic function $f(z)$. Also express $f(z)$ in terms of variable 'z' only ?

Sol: $u(x,y) = x^3 - 3xy^2.$

So,

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x.$$

$$\Rightarrow \frac{\partial u}{\partial y} = -6xy, \quad \frac{\partial^2 u}{\partial y^2} = -6x.$$

$\Rightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 6x + (-6x) = 0$

$\because$ $u(x,y)$ satisfies the Laplace equation, i.e., the function is harmonic.

Now, $v = ?$

$\Rightarrow dv = \dfrac{\partial v}{\partial x} \cdot dx + \dfrac{\partial v}{\partial y} \cdot dy$

$\Rightarrow dv = \left(-\dfrac{\partial u}{\partial y}\right) dx + \left(\dfrac{\partial u}{\partial x}\right) dy$

$\Rightarrow dv = 6xy \cdot dx + 3(x^2 - y^2)\, dy$. (exact D.E.)

$M(x,y) = 6xy \quad ; \quad N(y) = -3y^2$

$\Rightarrow U(x,y) = \int M(x,y) \cdot dx$

$\Rightarrow U(x,y) = \int 6xy \cdot dx = 6y \int x \cdot dx = 3yx^2 + C$

$\Rightarrow F(y) = \int N(y) \cdot dy = -3\int y^2 \cdot dy = -y^3$

Hence,

$\Rightarrow U(x,y) + F(y) = v(x,y)$.

$\therefore \boxed{v(x,y) = 3yx^2 - y^3 + C}$

$\therefore f(z) = u + iv$

$\boxed{f(z) = x^3 - 3xy^2 + i(3x^2y - y^3) + C}$

(Analytic function)

From direct identity,

$\Rightarrow f(z) = (x+iy)^3 = \boxed{z^3}$ $\quad \left((a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2\right)$

Q.19) Construct the analytic function $f(z)$ of which the real part is $e^x \cdot \cos y$?

Sol: Here, $u = e^x \cos y$.

So,

$\Rightarrow \frac{\partial u}{\partial x} = e^x \cos y \Rightarrow \phi_1(x,y) = e^x \cos y$

$\Rightarrow \phi_1(z,0) = e^z \cdot \cos 0 = e^z$.

and,

$\Rightarrow \frac{\partial u}{\partial y} = -e^x \sin y \Rightarrow \phi_2(x,y) = -e^x \cdot \sin y$

$\Rightarrow \phi_2(z,0) = -e^z \cdot \sin 0 = 0$.

$\because$ $u(x,y)$ is given,

$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + C$.

$f(z) = \int (e^z - 0) \cdot dz + C$

$\therefore$ $\boxed{f(z) = e^z + C}$. (C : constant).

Q.20) Construct the analytic function whose imaginary part is,

$$v = \log(x^2+y^2) + x - 2y.$$

Sol: Here, $v = \log(x^2+y^2) + x - 2y$

So,

$\Rightarrow \psi_1 = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\{\log(x^2+y^2) + x - 2y\}$

$\Rightarrow \psi_1(x,y) = \frac{2y}{x^2+y^2} - 2$

$\Rightarrow \psi_1(z,0) = 0 - 2 = -2$.

$$\Rightarrow \psi_2 = \frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1.$$

$$\Rightarrow \psi_2(x,y) = \frac{2x}{x^2+y^2} + 1.$$

$$\Rightarrow \psi_2(z,0) = \frac{2z}{z^2+0} + 1 = \frac{2z}{z^2} + 1 = \frac{2}{z} + 1$$

$$\therefore f(z) = \int \left[\psi_1(z,0) + i\,\psi_2(z,0) \right] dz + C.$$

$$= \int \left(-2 + i\left(\frac{2}{z} + 1 \right) \right) dz + C$$

$$\Rightarrow \boxed{f(z) = -2z + i\,(2\log z + z) + C}$$

Q.21) If $f(z) = u + iv$ is analytic function of z in any domain, then show that,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] |f(z)|^2 = 4\,|f'(z)|^2.$$

Sol: Here, $f(z) = u + iv$

$$\Rightarrow |f(z)| = (u^2+v^2)^{1/2}$$

or,

$$\Rightarrow |f(z)|^2 = u^2+v^2 = \phi$$

On LHS,

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

here,

$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(u^2+v^2)$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 2u.\frac{\partial u}{\partial x} + 2v.\frac{\partial v}{\partial x}.$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = 2\left[\left(u.\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right) + v.\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

Again,

$$\Rightarrow \frac{\partial^2\phi}{\partial y^2} = 2\left[u.\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v.\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right].$$

Now LHS,

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

$$= 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right].$$

$\because$ $f(z)$ is Analytic, so $f(z)$ is also harmonic and it satisfies the Laplace equation. So,

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$\left(\because \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right).$$

- By C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and,} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right].$$

$$= 2\left[2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right)\right] = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= 4.\left|f'(z)\right|^2 = RHS.$$

On RHS,

$$\Rightarrow f(z) = u + iv$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

$$\Rightarrow |f'(z)| = \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right)^{1/2}$$

So,

$$\Rightarrow |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2.$$

or,

$$\Rightarrow 4|f'(z)|^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \times 4 = RHS.$$

Q.22) If $f(z)$ is a harmonic function of z, show that

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2.$$

Sol: Let, $f(z) = u + iv$.

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

So,

$$\Rightarrow |f(z)| = (u^2+v^2)^{1/2}$$

and,

$$\Rightarrow |f'(z)| = \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right)^{1/2}.$$

or,

$$\Rightarrow |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = RHS.$$

Now on LHS,

$$\Rightarrow \frac{\partial}{\partial x}\left[(u^2+v^2)^{1/2}\right] = \frac{\partial}{\partial x}(\sqrt{\phi}) \qquad [\phi = u^2+v^2].$$

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$$= \frac{1}{2\sqrt{\phi}} \cdot \frac{\partial \phi}{\partial x} = \frac{1}{2\sqrt{\phi}}\left[2u.\frac{\partial u}{\partial x} + 2v.\frac{\partial v}{\partial x}\right].$$

$$= \frac{1}{\sqrt{\phi}}\left(u.\frac{\partial u}{\partial x} + v.\frac{\partial v}{\partial x}\right) \text{ --- } ①.$$

and,

$$= \frac{\partial}{\partial y}(\sqrt{\phi}) = \frac{1}{\sqrt{\phi}}\left(u.\frac{\partial u}{\partial y} + v.\frac{\partial v}{\partial y}\right) \text{ --- } ②,$$

→ From ① and ②,

$$= \frac{1}{\phi}\left\{\left[u.\frac{\partial u}{\partial x} + v.\frac{\partial v}{\partial x}\right]^2 + \left[u.\frac{\partial u}{\partial y} + v.\frac{\partial v}{\partial y}\right]^2\right\}.$$

→ From C-R equations,

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and, } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

$$= \frac{1}{\phi}\left[\left[u.\frac{\partial u}{\partial x} + v.\frac{\partial v}{\partial x}\right]^2 + \left[-u.\frac{\partial v}{\partial x} + v.\frac{\partial u}{\partial x}\right]^2\right].$$

$$= \frac{1}{\phi}\left\{u^2.\left(\frac{\partial u}{\partial x}\right)^2 + v^2.\left(\frac{\partial v}{\partial x}\right)^2 + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2\right\}.$$

$$= \frac{1}{\phi}\left[(u^2+v^2)\left(\frac{\partial u}{\partial x}\right)^2 + (u^2+v^2)\left(\frac{\partial v}{\partial x}\right)^2\right].$$

$$= \frac{1}{\cancel{u^2+v^2}} \times \cancel{(u^2+v^2)}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right).$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = LHS.$$

$\therefore$ LHS = RHS (hence prooved).

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Q23) Find the value of Z for which e^Z is real?

Sol: We know that

$$\Rightarrow e^{z} = e^{x+iy}$$

$$\Rightarrow e^{z} = e^{x} \cdot e^{iy}$$

$$\Rightarrow e^{z} = e^{x} \cdot \cancel{e}[\cos y + i\sin y]$$

$$\Rightarrow e^{z} = e^{x}(\cos y + i\sin y)$$

If e^z is real,

$$\Rightarrow e^{x}\sin y = 0$$

$$\Rightarrow \sin y = 0$$

$$\Rightarrow \boxed{y = n\pi} \quad , n \in N, \forall n \in I^{-}$$

$$\therefore Z = x + i(n\pi) = \boxed{x + n\pi i}$$

Q24) Show that, $\exp\left(\pm i\frac{\pi}{2}\right) = \pm i$

Sol: We know that,

$$\exp\left(\pm i\frac{\pi}{2}\right) = \cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}$$

$$= 0 \pm i(1) = \pm i$$

Q25) Find all values of Z which satisfies $e^z = 1+i$.

Sol: $e^z = 1+i$

$\Rightarrow \exp(x+iy) = 1+i$

$\Rightarrow \exp(x) \cdot \exp(iy) = 1+i$

$\Rightarrow e^{x}[\cos y + i\sin y] = 1+i$

$\Rightarrow e^{x}\cos y + ie^{x}\sin y = 1+i$

So,

$$e^{x}\cos y = 1 \quad \text{—①}, \quad e^{x}\sin y = 1 \quad \text{—②}$$

→ On squaring ① and ② and adding,

$$\Rightarrow e^{2x}\left[\cos^2 y + \sin^2 y\right] = 2$$

$$\Rightarrow e^{2x} = 2 \Rightarrow 2x = \ln 2$$

$$\Rightarrow \boxed{x = \frac{1}{2}\ln 2}$$

→ Divide ①/②,

$$\Rightarrow \frac{e^x \sin y}{e^x \cos y} = 1 \Rightarrow \tan y = 1 \Rightarrow \boxed{y = \frac{\pi}{4} + 2n\pi}$$

$$\therefore Z = x + iy = \boxed{\frac{1}{2}\ln 2 \pm i\left(\frac{\pi}{4} + 2n\pi\right)}$$

Q 26) To seperate the real and imaginary part of $\sinh(x+iy)$?

Sol: We know that,

$$\Rightarrow \sin ix = i \sinh x$$

$$\Rightarrow \sinh x = \frac{1}{i}\sin ix.$$

$$\Rightarrow \sinh(x+iy) = \frac{1}{i}\sin i(x+iy) = \frac{1}{i}.\sin(ix-y)$$

$$= \frac{1}{i}\times\frac{+i}{i}\left[\sin ix \cos y - \cos ix . \sin y\right]$$

$$= \frac{i}{i^2}\left[i \sinh x \cos y - \cosh x \sin y\right]$$

$$= -\left(-\sinh x \cos y - i\cosh x . \sin y\right)$$

$$= \underbrace{\sinh x \cos y}_{\text{Real part}} + i\,\underbrace{\cosh x . \sin y}_{\text{Imaginary part}}$$

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Q27) If $\tan(\theta + i\phi) = \sin(x+iy)$, then proove that $\coth y \cdot \sinh 2\phi = \cot x \cdot \sin 2\theta$.

Sol: Given,

$$\sin(x+iy) = \tan(\theta+i\phi).$$

$\Rightarrow \sin x \cdot \cos iy + \cos x \cdot \sin iy = \dfrac{2\sin(\theta+i\phi)}{2\cos(\theta+i\phi)} \times \dfrac{\cos(\theta-i\phi)}{\cos(\theta-i\phi)}$

$\Rightarrow \sin x \cdot \cosh y + \cos x \cdot i\sinh y = \dfrac{\sin 2\theta + \sin 2i\phi}{\cos 2\theta + \cos 2i\phi}$.

$\Rightarrow \underbrace{\sin x \cdot \cosh y}_{R.P} + i\underbrace{\cos x \sinh y}_{I.P.} = \underbrace{\dfrac{\sin 2\theta}{\cos 2\theta + \cos 2i\phi}}_{R.P} + \underbrace{\dfrac{\sin 2i\phi}{\cos 2\theta + \cos 2i\phi}}_{I.M.}$

→ On comparing real and imaginary parts.

$\Rightarrow \sin x \cdot \cosh y = \dfrac{\sin 2\theta}{\cos 2\theta + \cos 2i\phi}$ —— ①

$\Rightarrow \cos x \sinh y = \dfrac{\sin 2i\phi}{\cos 2\theta + \cos 2i\phi}$ —— ②.

$\Rightarrow$ Divide eq-② by eq-①,

$\Rightarrow \tan x \cdot \coth y = \sin 2\theta \cdot \operatorname{cosec} 2i\phi$

$\Rightarrow \sin 2i\phi \cdot \coth y = \cot x \cdot \sin 2\theta$.

or,

$\Rightarrow \boxed{\coth y \cdot \sinh 2\phi = \cot x \cdot \sin 2\theta}$

Hence prooved.

Q28) Seperate into real and imaginary parts : $\text{Log}(3+4i)$

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Sol: We know that,

$$\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left[\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi\right].$$

$$= \frac{1}{2}\log(3^2+4^2) + i\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$$

$$\therefore \text{Log}(x+iy) = \log 5 + i\left[\tan^{-1}\left(\frac{4}{3}\right) + 2n\pi\right]$$

- Real part $= \log 5$, Imaginary part $= \tan^{-1}\left(\frac{4}{3}\right) + 2n\pi$.

Q29) Show that,

$$\log(-\log i) = \log\frac{\pi}{2} - i\frac{\pi}{2}.$$

Sol: On LHS,

$$\log i = \log(0+i) = \frac{1}{2}\log(0^2+1^2) + i\frac{\pi}{2} = \frac{i\pi}{2}$$

now,

$$\log\left(-\frac{i\pi}{2}\right) = \log\left(0 - i\frac{\pi}{2}\right)$$

$$= \frac{1}{2}\log\left[0^2 + \frac{\pi^2}{4}\right] + i\left(\frac{-\pi}{2}\right).$$

$$= \boxed{\log\frac{\pi}{2} - \frac{i\pi}{2}} = RHS.$$

Q30) Seperate real and imaginary part of i^i?

Sol:

$$i^i = e^{\log i^i}$$

$(0+i)$

$$= e^{i\log i} = e^{i\left[0 + i\frac{\pi}{2}\right]} = e^{-\pi/2}.$$

Hence, i^i is a real number.

Sol: We know that,

$$\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left[\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi\right].$$

$$= \frac{1}{2}\log(3^2+4^2) + i\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$$

$$\therefore \text{Log}(x+iy) = \log 5 + i\left[\tan^{-1}\left(\frac{4}{3}\right) + 2n\pi\right]$$

- Real part $= \log 5$, Imaginary part $= \tan^{-1}\left(\frac{4}{3}\right) + 2n\pi$.

Q29) Show that,

$$\log(-\log i) = \log\frac{\pi}{2} - \frac{i\pi}{2}.$$

Sol: On LHS,

$$\log i = \log(0+i) = \frac{1}{2}\log(0^2+1^2) + \frac{i\pi}{2} = \frac{i\pi}{2}$$

now,

$$\log\left(-\frac{i\pi}{2}\right) = \log\left(0 - \frac{i\pi}{2}\right)$$

$$= \frac{1}{2}\log\left[0^2 + \frac{\pi^2}{4}\right] + i\left(\frac{-\pi}{2}\right).$$

$$= \boxed{\log\frac{\pi}{2} - \frac{i\pi}{2}} = \text{RHS}.$$

Q30) Seperate real and imaginary part of i^i?

Sol:

$$i^i = e^{\log i^i} \quad (0+i)$$

$$= e^{i\log i} = e^{i\left[0 + \frac{i\pi}{2}\right]} = e^{-\pi/2}.$$

Hence, i^i is a real number.

Q31) Seperate real and imaginary parts of :-

$$\log[(1+i)\log i]$$

Sol: $\log\left[(1+i)\frac{i\pi}{2}\right] = \log\left(\frac{i\pi}{2} - \frac{\pi}{2}\right)$ $\pi - \frac{\pi}{4}$

or,

$\Rightarrow \log\left[-\frac{\pi}{2} + i\frac{\pi}{2}\right]$ $\theta = \pi - \theta$

$= \frac{1}{2}\log\left(\frac{\pi^2}{4}\times\frac{2}{2}\right) + i\tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right]$ $\pi - \frac{\pi}{4}$

$= \boxed{\log\left(\frac{\pi}{\sqrt{2}}\right) + i\frac{3\pi}{4}}$ $\frac{3\pi}{4}$

Q32) Show that, $\log(1+e^{i\theta}) = \log\left(2\cos\frac{\theta}{2}\right) + \frac{1}{2}i\theta$

if $\theta \in (-\pi, \pi)$?

$\cos 2\theta = c^2 - s^2$
$= 1 - 2s^2$
$= 2c^2 - 1$

Sol: $\log(1+e^{i\theta}) = \log(1+\cos\theta + i\sin\theta)$

$= \frac{1}{2}\log\left[1+\cos^2\theta + 2\cos\theta + \sin^2\theta\right] + i\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$

$= \frac{1}{2}\log\left(2(1+\cos\theta)\right) + i\tan^{-1}\left(\frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}\right)$

$= \frac{1}{2}\log\left(2\times 2\cos^2\frac{\theta}{2}\right) + i\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$= \frac{1}{2}\log\left[2\cos\frac{\theta}{2}\right]^2 + i\tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$= \log\left(2\cos\frac{\theta}{2}\right) + i\frac{\theta}{2}$

Hence prooved

Q33) Proove that, $\sin(\log i^i) = -1$.

Sol: $\sin\left[\log\left(e^{-\pi/2}\right)\right]$ $\quad\left(\because i^i = e^{-\pi/2}\right)$

$= \sin\left[\frac{-\pi}{2} \times \log_e e\right]$

$= \sin\left(\frac{-\pi}{2}\right) = -\sin\frac{\pi}{2} = -1 = RHS$

Q34) Evaluate the integral, $I = \int_C Re(z^2).dz$ from $z=0$ to $2+4i$ along the parabola $y = x^2$?

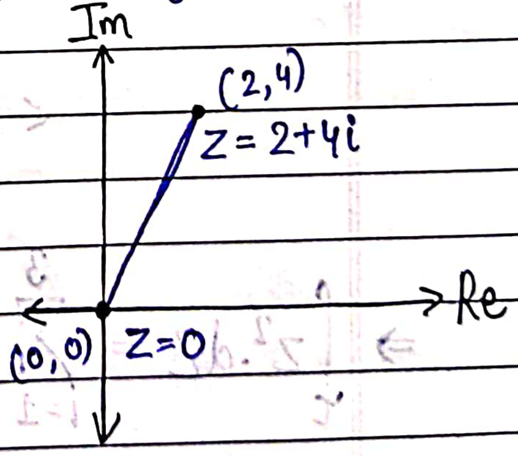
Sol: $z = x+iy \Rightarrow z^2 = (x+iy)^2$

So,

$\Rightarrow z^2 = (x^2-y^2) + 2xyi$

$\Rightarrow Re(z^2) = x^2 - y^2$.

$\Rightarrow I = \int_C Re(z^2).dz = \int_C (x^2-y^2).(dx+idy)$



$\Rightarrow I = \int_{x=0}^{x=2} (x^2-x^4)(dx + i2x\,dx)$

$(\because y = x^2 \Rightarrow dy = 2x.dx)$

$\Rightarrow I = \int_0^2 (x^2-x^4)(1+2xi)\,dx$

$\Rightarrow I = \int_0^2 (x^2 - 2x^3 i - x^4 - 2x^5 i)\,dx$.

i.e.,

$$\Rightarrow I = \left[\frac{x^3}{3} + \frac{2ix^4}{4} - \frac{x^5}{5} - \frac{2ix^6}{6}\right]_0^2$$

$96 - 40 = 56$

$64 - 24 = 40$

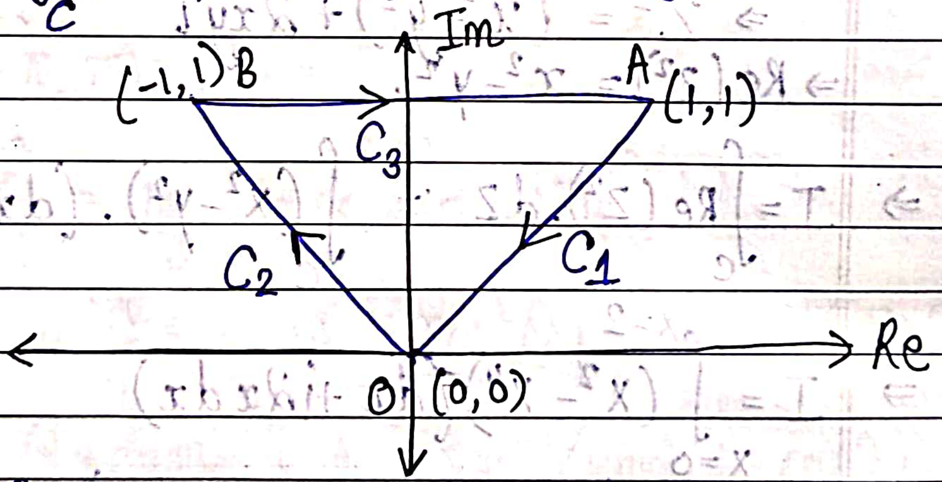
$$\Rightarrow I = \frac{8}{3} + i(8) - \frac{1}{5}(32) - \frac{i}{3}(64).$$

$$\therefore I = \left(\frac{8}{3} - \frac{32}{5}\right) + i\left(8 - \frac{64}{3}\right)$$

$$\therefore \boxed{I = \frac{-56}{15} + \frac{-40}{3} i}.$$

Q35) Evaluate the line integral $\int_C z^2 . dz$ where C is the boundary of a triangle with vertices $0, 1+i, -1+i$ clockwise.

Sol: $$\int_C z^2 . dz = \int_C (x^2 - y^2 + 2ixy)(dx + idy)$$



$$\Rightarrow \int_C z^2 . dz = \sum_{i=1}^{3} \int_{C_i} (x^2 - y^2 + 2ixy)(dx + idy)$$

$$\Rightarrow \int_{C_1} (x^2 - y^2 + 2ixy)(dx + idy)$$

$$= \int_{x=1}^{0} (\cancel{x^2} - \cancel{x^2} + 2x^2 i)(dx + idx)$$

For OA,

O $(0,0)$ ← A $(1,1)$

$$y = \frac{1-0}{1-0}x + 0$$

$$\boxed{y = x}$$

$$= \int_1^0 (2x^2 i)(1+i)\,dx = \int_1^0 (2x^2 i - 2x^2)\,dx$$

$$= 2\int_1^0 x^2 i\,dx - 2\int_1^0 x^2\,dx$$

$$= \frac{2}{3}\left[x^3\right]_1^0 i - \frac{2}{3}\left[x^3\right]_1^0 = \frac{i2}{3}(-1) + \frac{2}{3} = -\frac{2}{3}i + \frac{2}{3}$$

$$\Rightarrow \int_{C_2} (x^2 - y^2 + 2ixy)(dx + idy)$$

For OB, $\underset{O}{(0,0)} \rightarrow \underset{B}{(-1,1)}$.

$y = -x$.

$$\Rightarrow \int_0^{-1} (\cancel{x^2} - \cancel{x^2} - 2x^2 i)(1+(-i))\,dx$$

$$= \int_0^{-1} (-2x^2 i)(1-i)\,dx$$

$$= 2\left[\int_0^1 (x^2 i)(i-1)\,dx\right] = 2\left[\int_0^1 -x^2\,dx - \int_0^1 x^2 i\,dx\right]$$

$$= 2\left[-\frac{1}{3}\left[x^3\right]_0^1 - \frac{1}{3}\left[x^3\right]_0^1 i\right]$$

$$= -\frac{2}{3} - \frac{2}{3}i$$

$$\Rightarrow \int_{C_3} (x^2 - y^2 + 2ixy)(dx + idy)$$

"here, 'y' is constant so 'y' doesn't change $\Rightarrow y = 1$"

$$\Rightarrow \int_{-1}^{1} (x^2 - 1 + 2xi)\,dx = \left[\frac{x^3}{3} - x + x^2\right]_{-1}^{1}$$ $2 - \frac{1}{3}$

$$= \frac{1}{3} - \cancel{1} + \cancel{1} - \left(-\frac{1}{3} + 1 + 1\right) = \frac{1}{3} - \frac{5}{3} = \boxed{-\frac{4}{3}}$$

$\therefore$ On adding all values,

$$= \cancel{\frac{2}{3}} - \cancel{\frac{2}{3}}i + \left(\cancel{\frac{+2}{3}}\right) + \cancel{\frac{2}{3}}i - \cancel{\frac{4}{3}} = 0$$

Q36) Evaluate $\int_C (z-a)^n \, dz$, where C is the circle with centre 'a' and radius r. Also discuss the case when $n = -1$?

Sol: We know that,
the equation of circle is $\Rightarrow |z-a| = r$, where a is centre and r is radius of circle.

$$\Rightarrow |z-a| = r$$

$$\Rightarrow \boxed{z-a = e^{i\theta} r}$$

$$\Rightarrow dz = e^{i\theta}(ir)\,d\theta = ire^{i\theta}.d\theta$$

$$\Rightarrow \int_C (z-a)^n dz = \int_C r^n . e^{in\theta} . ire^{i\theta} . d\theta$$

$$= \int_0^{2\pi} r^{n+1} . e^{in\theta + i\theta} . i \, d\theta$$

$$= \int_0^{2\pi} (ir^{n+1}) . e^{i\theta(n+1)} . d\theta$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} . d\theta$$

$$= ir^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}\left[\cancelto{1}{\cos(n+1)2\pi} + \cancelto{0}{i\sin(n+1)2\pi} - 1\right]$$

$$= \frac{r^{n+1}}{n+1}\left(\cancel{1} + 0 - \cancel{1}\right)$$

$$= \underline{\underline{0}}$$

$$\boxed{\begin{aligned}\cos n\pi &= (-1)^n \\ \sin n\pi &= 0\end{aligned}}$$

Q37) Find $\int_C \frac{1}{(z-a)}\,dz$, where C is simple closed curve and the point $z=a$ is inside.

Sol: $\because \int_C \frac{f(z)}{(z-a)}\,dz = 2\pi i\, f(a)$.

Put denominator $= 0$
$\Rightarrow z-a=0$
$\Rightarrow \boxed{z=a}$

Here a is lies outside,

$$\Rightarrow \int_C \frac{1}{z-a}\,dz = 2\pi i\, f(a)$$

$$= 2\pi i\left[f(z)\right]_{z=a}$$

$$= \boxed{2\pi i}. \qquad (f(z)=1).$$

Q38) Solve the integral, $\int_C \frac{\cos \pi z}{z-1}\,dz$, where C is the circle $|z| = 3$?

$|z-a| = r$
$a \rightarrow$ centre
$r \rightarrow$ radius.

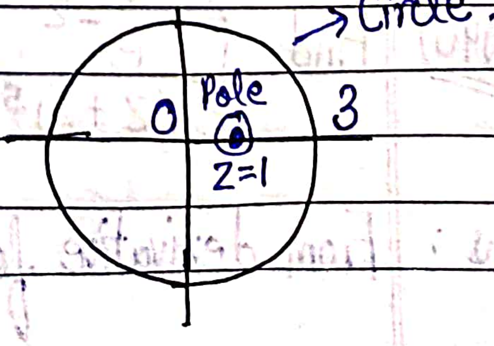
Sol: For pole,

$\Rightarrow z-1=0$

$\Rightarrow \boxed{z=1}$.

Given,
$|z-0| = r = 3$
Centre $= 0$
Radius $= 3$

i.e.,
pole lies inside C.



$\Rightarrow$ By Cauchy integral formula,

$$\Rightarrow f(a).\,2\pi i = \int_C \frac{f(z)}{z-a}\,dz$$

$$\Rightarrow \int_C \frac{\cos \pi z}{z-1}\, dz = 2\pi i \left[f(z) \right]_{z=a}.$$

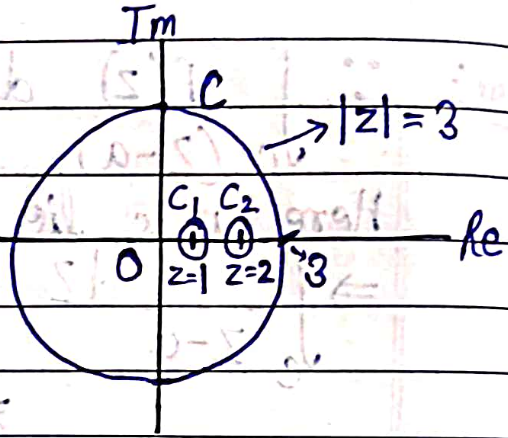
$$= 2\pi i \left[\cos \pi z \right]_{z=1}$$

$$= 2\pi i \times \cos \pi (1)$$

$$= \boxed{-2\pi i}.$$

Q39) Find $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz$, where C is $|z| = 3$?

Ans: Poles are $z = 1$ and $z = 2$, i.e., both poles lie inside C.



$$\Rightarrow \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} = \int_{C_1} + \int_{C_2}$$

$$= \int_{C_1} \frac{\left(\frac{\cos \pi z^2}{z-2}\right)}{(z-1)}\, dz + \int_{C_2} \frac{\left(\frac{\cos \pi z^2}{z-1}\right)}{(z-2)}\, dz.$$

$$= 2\pi i \left[\frac{\cos \pi z^2}{z-2} \right]_{z=1} + 2\pi i \left[\frac{\cos \pi z^2}{z-1} \right]_{z=2}.$$

$$= 2\pi i \left(\frac{\cos \pi}{-1} \right) + 2\pi i \left(\frac{\cos 4\pi}{1} \right) \quad \left(\begin{matrix} \cos \pi = -1 \\ \cos 4\pi = +1 \end{matrix} \right).$$

$$= 2\pi i + 2\pi i$$

$$= \boxed{4\pi i}$$

Q40) Find $\int_C \frac{e^{-z}}{(z+2)^5}\, dz$, where C is the circle $|z| = 3$?

Ans: From derivative formula,

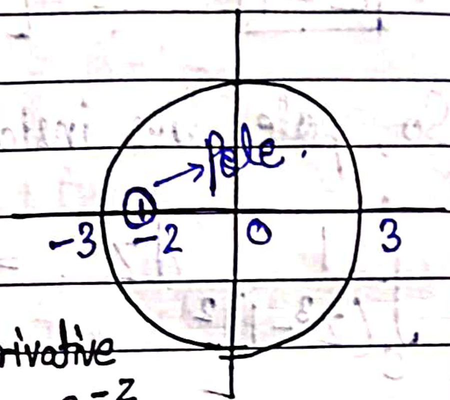
$$\Rightarrow f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz. \quad \text{(formula)}.$$

For $n=4$,

$$\Rightarrow f^4(a) = \frac{4!}{2\pi i}\int_C \frac{f(z)}{(z-a)^5}\,dz.$$

$\Rightarrow$ Here, pole $z=-2$ (order 5)

$\because$ Pole lies inside C,



$$\Rightarrow \int_C \frac{e^{-z}}{(z+2)^5}\,dz$$

$$= \frac{2\pi i}{4!} f^4(e^{-z})$$

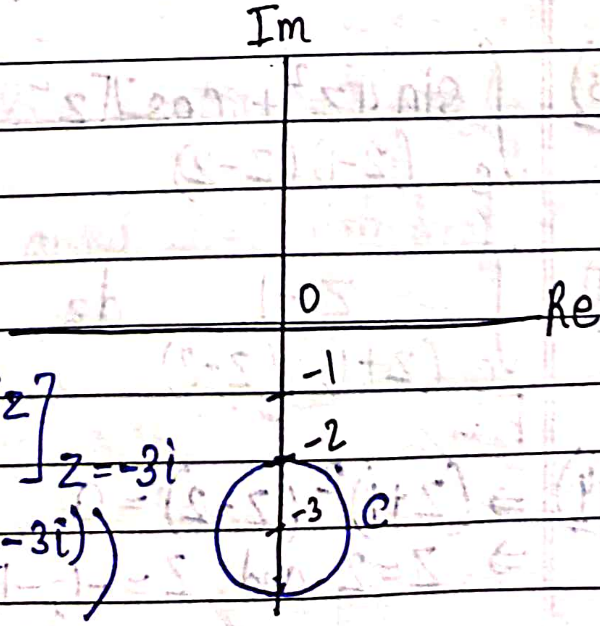
fourth derivative of $e^{-z} = e^{-z}$.

$$= \frac{2\pi i}{24} \times \left[e^{-z}\right]_{z=-2}$$

$$= \frac{2\pi i}{24} \times e^2$$

$$= \boxed{\frac{\pi i e^2}{12}}$$

Q41) Find $\int_C \frac{e^{iz}}{z+3i}\,dz$, where C is the circle $|z+3i| = 1$.



Ans: Pole, $z+3i=0$

$\Rightarrow z = -3i$

i.e., pole lies inside C.

$$\Rightarrow \int_C \frac{e^{iz}}{z+3i}\,dz = 2\pi i\left[e^{iz}\right]_{z=-3i}$$

$$= 2\pi i\left(e^{i(-3i)}\right)$$

$$= \boxed{2\pi i e^3}.$$

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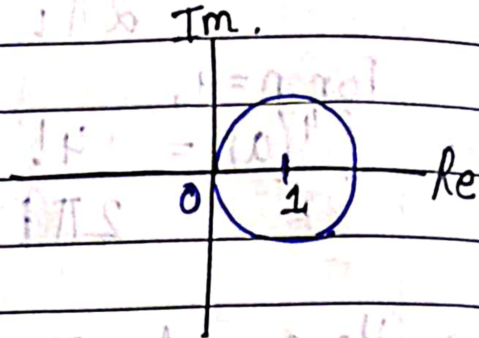
$\left(-\frac{4}{9}\pi i\right)$

Q42) Integrate $(z^3-1)^{-2}$ around the circle, $|z-1| = 1$.

Sol: To find, $\int_c \frac{dz}{(z^3-1)^2}$



Pole is at,

$\Rightarrow z^3-1=0$

$\Rightarrow (z-1)(z^2+1+z)=0$

$\Rightarrow \boxed{z=1}$ $\qquad [z^2+z+1 \neq 0]$

So, pole lies inside C,

$$\Rightarrow \int_C \frac{1}{(z^3-1)^2}\,dz = \int_C \frac{\frac{1}{(z^2+z+1)}}{(z-1)}\,dz.$$

Using cauchy's integral formula,

$$\Rightarrow \int_C \frac{\frac{1}{(z^2+z+1)}}{(z-1)}\,dz = 2\pi i\left[(z^2+z+1)^{-1}\right]_{z=1}.$$

$$= 2\pi i\left[\frac{1}{1+1+1}\right]$$

$$= \boxed{\frac{2\pi i}{3}}.$$

Q43) $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$, where C is $|z|=3$.

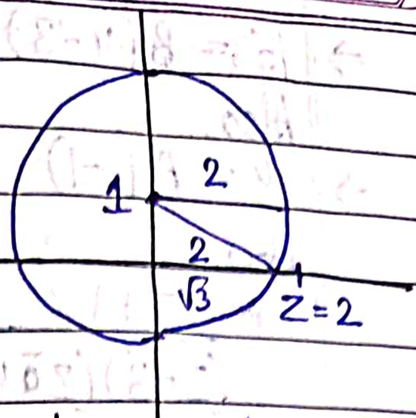
Q44) $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$, where C is $|z-i|=2$.

Ans. 44) $\Rightarrow (z+1)^2(z-2)=0$

$\Rightarrow z=2$ and $z=-1,-1$

$= -1$ (order 2)

Here, $Z=2$ pole lies outside C,
and $Z=-1$ pole lies inside C.



Answer $= -\frac{2\pi i}{9}$.

Q45) $\int_C \frac{z}{z^2+1}\,dz$, where C is $\left|z+\frac{1}{z}\right| = 2$! (Assig.)

Ans: $\Rightarrow \frac{|z^2+1|}{|z|} = 2$

$\Rightarrow |x^2-y^2+2xyi+1| = 2|x+iy|$.

or, $\Rightarrow \sqrt{(x^2-y^2+1)^2+(2xy)^2} = 2\sqrt{x^2+y^2}$.

$\Rightarrow (x^2-y^2+1)^2+(2xy)^2 = 4(x^2+y^2)$ ... Solve yourself.

Q46) Expand e^z about a?

Ans: From taylor series,

$\Rightarrow f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \dots + \frac{(z-a)^n}{n!}f^n(a)$

$\Rightarrow e^z = e^a + (z-a)e^z + \frac{(z-a)^2}{2}e^z + \dots + \frac{(z-a)^n}{n!}(e^z)^n$

Q.47) Expand $f(z) = \frac{z+1}{(z-3)(z-4)}$ about $z=2$ upto first 4 terms?

Ans: By partial fractions,

$\Rightarrow \frac{z+1}{(z-3)(z-4)} = \frac{A}{z-3} + \frac{B}{z-4}$

$\Rightarrow z+1 = A(z-4) + B(z-3)$.

So,

$\Rightarrow 5 = B(4-3) \Rightarrow \boxed{B=5}$

Also,

$\Rightarrow 4 = A(-1) \Rightarrow \boxed{A=-4}$

Now,

$$\Rightarrow \frac{Z+1}{(Z-3)(Z-4)} = \frac{-4}{Z-3} + \frac{5}{Z-4}$$

Now,

$$\Rightarrow f(Z) = \frac{-4}{(Z-3)} + \frac{5}{(Z-4)}$$

$$\Rightarrow f'(Z) = \frac{-4}{(Z-3)^2}(-1) + \frac{5}{(Z-4)^2}(-1)$$

$$\Rightarrow f'(Z) = \frac{4}{(Z-3)^2} - \frac{5}{(Z-4)^2}$$

and,

$$\Rightarrow f''(Z) = 4\left[-2(Z-3)^{-3}\right] - 5\left[-2(Z-4)^{-3}\right]$$

$$\Rightarrow f''(Z) = \frac{-8}{(Z-3)^3} + \frac{10}{(Z-4)^3}$$

$$\Rightarrow f'''(Z) = -8(-3)(Z-3)^{-4} + 10(-3)(Z-4)^{-4}$$

$$\Rightarrow f'''(Z) = \frac{24}{(Z-3)^4} - \frac{30}{(Z-4)^4}$$

$$\Rightarrow f''''(Z) = \frac{24(-4)}{(Z-3)^5} - \frac{30(-4)}{(Z-4)^5} = \frac{-96}{(Z-3)^5} + \frac{120}{(Z-4)^5}$$

$\Rightarrow$ At $Z=2$,

$$\Rightarrow f(2) = \frac{2+1}{(-1)(-2)} = \frac{3}{2}.$$

$$\Rightarrow f'(2) = \frac{4}{(-1)^2} - \frac{5}{(-2)^2} = 4 - \frac{5}{4} = \frac{11}{4},$$

$\Rightarrow f''(2) = \frac{-8}{(-1)^3} + \frac{10}{(-2)^3} = 8 - \frac{\cancel{10}^{5}}{\cancel{8}_{4}} = \frac{27}{4}$.

$\Rightarrow f'''(2) = \frac{24}{(-1)^4} - \frac{30}{(-2)^4} = 24 - \frac{\cancel{30}^{15}}{\cancel{16}_{8}} = \frac{177}{8}$.

$\therefore$ Series expansion,

$\Rightarrow f(z) = \frac{3}{2} + (z-2)\frac{11}{4} + \frac{(z-2)^2}{2} \times \frac{27}{4} + \frac{(z-2)^3}{3\times 2} \times \frac{177}{8}$

$\therefore f(z) = \frac{3}{2} + \frac{11(z-2)}{4} + \frac{27(z-2)^2}{8} + \frac{177(z-2)^3}{48}$.

Q48) Expand $f(z) = \frac{1}{z^2+4}$ about $z = -i$?

Ans: $f(z) = \frac{1}{z^2 - i^2(4)} = \frac{1}{(z+2i)(z-2i)} = \frac{A}{z+2i} + \frac{B}{z-2i}$

$\Rightarrow f(z) = \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right)$.

$\Rightarrow f'(z) = \frac{1}{4i}\left[\frac{-1}{(z-2i)^2} + \frac{1}{(z+2i)^2}\right]$.

$\Rightarrow f''(z) = \frac{1}{4i}\left[\frac{2}{(z-2i)^3} - \frac{2}{(z+2i)^3}\right]$.

$\Rightarrow f'''(z) = \frac{1}{4i}\left[\frac{-6}{(z-2i)^4} + \frac{6}{(z+2i)^4}\right]$

Now,

$\Rightarrow f(a) = f(-i) = \frac{1}{(-i)^2+4} = \frac{1}{3}$.

$\Rightarrow f'(-i) = \frac{1}{4i}\left[\frac{-1}{(-3i)^2} + \frac{1}{(i)^2}\right] = \frac{1}{4i}\left(\frac{-1}{-9} - 1\right)$

Now,

$$\Rightarrow f'(-i) = \frac{1}{4i}\left(\frac{1}{9} + 1\right) = \frac{\cancel{10}}{\cancel{36i}} = \frac{\cancel{5}}{\cancel{18i}} = \frac{-2}{9i}.$$

$$\Rightarrow f''(-i) = \frac{1}{4i}\left(\frac{2}{(-3i)^3} - \frac{2}{(i)^3}\right) = \frac{-28}{54}.$$

$$\therefore f(z) = \frac{1}{3} + (z+i)\frac{2}{9}i + \ldots\ldots n\text{-terms}.$$

Q49) Find the laurents expansion of $f(z) = \dfrac{1}{z(z^2-3z+2)}$ in the region $1 \leq |z| < 2$?

Ans: $$\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

On solving,

$$A = 1/2$$
$$B = -1$$
$$C = 1/2$$

$$\therefore \frac{1}{z(z-1)(z-2)} = \frac{1}{2(z)} + \frac{-1}{z-1} + \frac{1}{2(z-2)}$$

Note: ① $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots\infty$

② $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots\ldots\infty.$

$$= \frac{1}{2z} - \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2(-2)\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{2z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{4}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots\right) + \left(\frac{-1}{4}\right)$$

$$\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots\right) \quad \text{Ans.}$$

Q50) Find the Laurent's expansion of $f(z) = \frac{1}{(z+1)(z+3)}$

a) $1 < |z| < 3$

b) $|z| > 3$

c) $|z| < 1$

d) $0 < |z+1| < 2$

Sol: $f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right)$.

(a) $1 < |z| < 3$

$$\Rightarrow f(z) = \frac{1}{2}\left[\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{2}{2} \cdot \frac{1}{3\left(1+\frac{z}{3}\right)}\right]$$

$$\Rightarrow f(z) = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2} \cdot \frac{1}{3\left(1+\frac{z}{3}\right)}$$

(d) $0 < |z+1| < 2$.

$$\Rightarrow f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$= \frac{1}{2(z+1)} - \frac{1}{2} \cdot \frac{1}{(z+1)+2}$$

$$\Rightarrow f(z) = \frac{1}{2(z+1)} - \frac{1}{2} \cdot \frac{1}{\left[2\left(1+\left(\frac{z+1}{2}\right)\right)\right]}$$

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$$\Rightarrow f(z) = \frac{1}{2}(z+1)^{-1} - \frac{1}{4}\left[1+\left(\frac{z+1}{2}\right)\right]^{-1}$$

$$\therefore f(z) = \frac{1}{2}\left(1-z+z^2-z^3+z^4-\ldots\infty\right) - \frac{1}{4}\left(1 - \frac{z+1}{2} + \left(\frac{z+1}{2}\right)^2 - \left(\frac{z+1}{2}\right)^3 + \left(\frac{z+1}{2}\right)^4 - \ldots\infty\right),$$

Q51) Find the radius of convergence of the power series,

(a) $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Ans: Here, $a_n = \frac{1}{n!}$, $a_{n+1} = \frac{1}{(n+1)!}$

$$\Rightarrow \frac{1}{R} = \lim_{n\to\infty}\left(\frac{1}{(n+1)!} \Big/ \frac{1}{n!}\right)$$

$$\Rightarrow \frac{1}{R} = \lim_{n\to\infty}\left(\frac{\cancel{n!}}{\cancel{n!}(n+1)}\right) = \frac{1}{\infty+1} = 0$$

Or, $\boxed{R=\infty}$.

(b) $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^n}$ (HW)

(c) $f(z) = \sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$ (HW).

$\boxed{\int_C f(z).\,dz = 2\pi i \sum R^+}$

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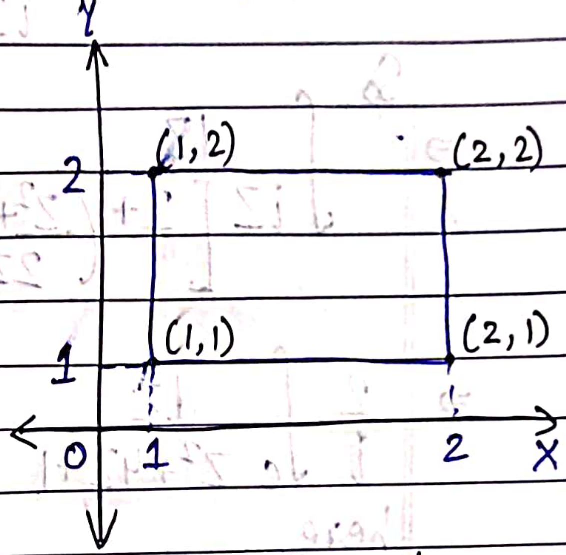
Q52) Evaluate $\int_C \frac{2z^2+5}{(z+2)^2(z^2+4)}\,dz$, where C is the square with the vertices at $1+i$, $2+i$, $2+2i$, $1+2i$. (Use Cauchy's residue theorem).

Ans: Given, $I = \int_C \frac{2z^2+5}{(z+2)^2(z^2+4)}\,dz$

Coordinates are: (1, 1), (2, 1), (2, 2) & (1, 2).

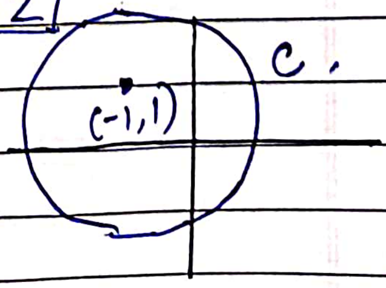
The pole will be,
$z = -2$,
Here, no point lies on the real or imaginary axis. So,
$\boxed{I = 0}$



Q53) Evaluate $\int_C \frac{z-3}{z^2+2z+5}\,dz$, where C is $|z+1-i| = 2$.

Ans: $\Rightarrow |x+iy+1-i| = 2$
$\Rightarrow |(x+1)+i(y-1)| = 2$
$\Rightarrow (x+1)^2 + (y-1)^2 = 4$.
$\Rightarrow x^2+1+2x+y^2+1-2y = 4$
$\Rightarrow x^2+y^2+2x-2y-2 = 0$.
So,
$C \equiv (-1, 1)$ and $r = \sqrt{g^2+f^2-c} = \sqrt{1+1-(-2)} = 2$.
Hence,
$C \equiv (-1, 1)$ and $\boxed{r = 2}$

Here,
pole lies outside the circle
i.e.,
$\boxed{I = 0}$.



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Q 54) Evaluate : $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ by contour integration ?

Ans : Putting $Z = e^{i\theta}$

$\Rightarrow dZ = e^{i\theta} . i d\theta .$

$\Rightarrow dZ = i Z . d\theta$

$\Rightarrow d\theta = \frac{dZ}{iZ}$

So,

$$\Rightarrow \int \frac{dZ}{iZ\left[2+\left(\frac{Z^2+1}{2Z}\right)\right]} = \int_c \frac{2}{i} \frac{dZ}{Z^2+4Z+1}$$

$$\Rightarrow \frac{2}{i}\int_c \frac{dZ}{Z^2+4Z+1}$$

Here, $C : |Z| = 1$.
centre = (0, 0) ; radius = 1.

here,
pole is = $(-2+\sqrt{3})$ and not $(-2-\sqrt{3})$.
The pole $(-2+\sqrt{3})$ lies inside the circle C. So,