UNIT 1: Function of Complex Variables - Complex Analysis I

- Introduction :

→ A number whose square is -1 is called an imaginary or a complex quantity and is denoted by i (iota).

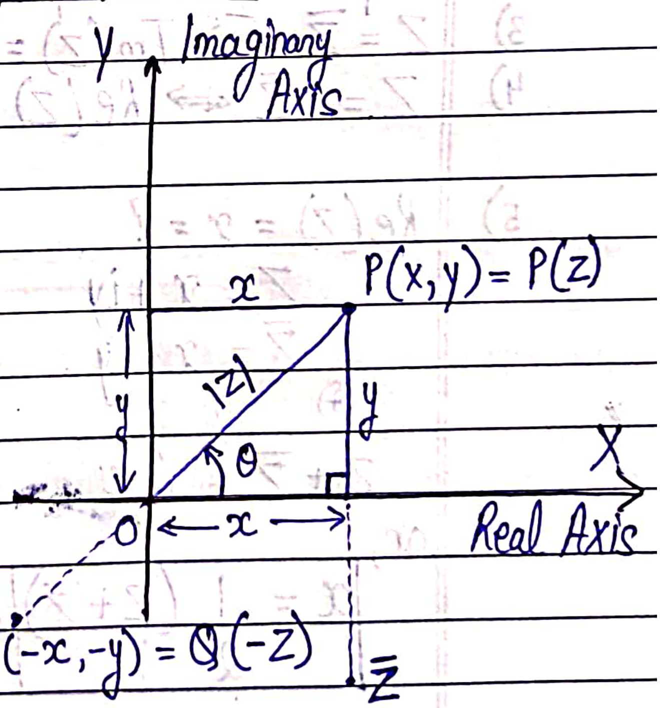
$$i = \sqrt{-1}$$
$$\boxed{i^2 = -1}$$

→ A complex number is written as,
$$\boxed{Z = x + iy} \qquad (x, y \in R).$$
where,
x is real part of Z , $Re(Z) = x$
y is imaginary part of Z ; $Im(Z) = y$

1) a) Geometric Representation of Complex Numbers :-

$Z = x + iy$
and,
conjugate of $Z = \bar{Z} = x - iy$.

$P(Z) = x + iy$
and,
$Q(-Z) = -x - iy$.



b) Modulus of a Complex Number :-

$$|Z| = \sqrt{x^2 + y^2}.$$

c) **Argument of a Complex Number :-**

- 'θ' is called argument or amplitude of Z.
- If argument lies in the range of,
$$-\pi < \theta \leq \pi$$
then, "θ" is called principal argument.
- General Argument is written as,
$$\text{General Argument} = \text{Principal Argument} + 2K\pi \quad (K \in I)$$

d) **Few Properties :-**

1) $\overline{(\bar{Z})} = Z$
2) $Z_1 = Z_2 \Leftrightarrow \bar{Z}_1 = \bar{Z}_2$
3) $Z = \bar{Z} \Leftrightarrow Im(Z) = 0$ (Purely Real)
4) $Z = -\bar{Z} \Leftrightarrow Re(Z) = 0$ (Purely Imaginary)

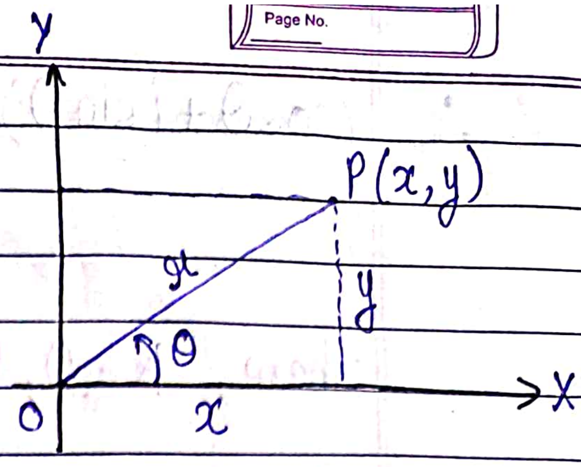
5) $Re(Z) = x = ?$
$$Z = x + iy$$
$$\bar{Z} = x - iy$$
$$\oplus \quad \oplus$$
$$Z + \bar{Z} = 2x$$
or,
$$\boxed{x = \frac{1}{2}(Z + \bar{Z})}$$

6) $Im(Z) = y = ?$
$$Z = x + iy$$
$$\bar{Z} = x - iy$$
$$\ominus \quad \ominus$$
$$Z - \bar{Z} = 2iy$$
or,
$$\boxed{y = \frac{1}{2i}(Z - \bar{Z})}$$

c) **Polar Form of a Complex Number :-**

Cartesian, $P(x, y) \longrightarrow$ Polar form, $Z(r, \theta)$.

$\Rightarrow \quad Z = r\left[\cos\theta + i\sin\theta\right]$



where,

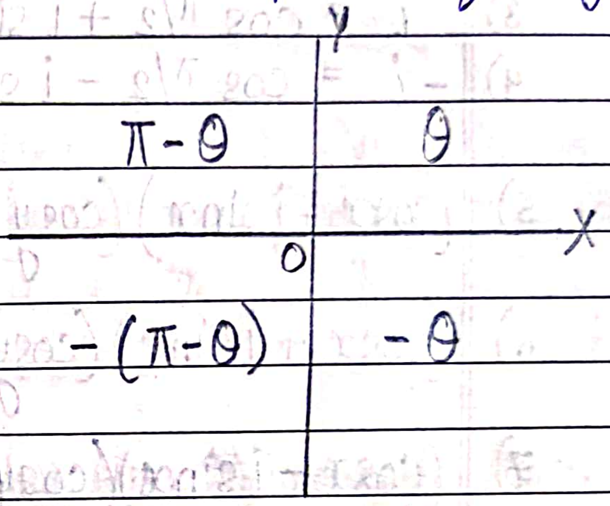
- $r = \sqrt{x^2+y^2}$.

and,

- $\theta = \tan^{-1}\left|\dfrac{y}{x}\right|$.

(NOTE: 'θ' will depend on coordinate axes / values of x & y)

Note: (TRICK to determine 'θ'):-



	Quadrant		Principal Argument
1)	I	→	θ
2)	II	→	$\pi-\theta$
3)	III	→	$-(\pi-\theta)$
4)	IV	→	$-\theta$

2) De-Moivre's Theorem :-

Let, $Z = r(\cos\theta + i\sin\theta)$

Then,

$$\boxed{Z^n = r^n(\cos n\theta + i\sin n\theta)}$$

where,

n is any positive or negative integer or a real rational number,

$$n \in I \text{ and } n \in Q.$$

2.1) Roots of a Complex Number :-

In general,

$$\cos\theta = \cos(2k\pi+\theta), \text{ where } k \in I$$
$$\sin\theta = \sin(2k\pi+\theta), \text{ where } k \in I.$$

$$\therefore (\cos\theta + i\sin\theta)^{1/n} = [\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)]^{1/n}$$

$$= \cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)$$

where, $K = 0, 1, 2, 3, \ldots\ldots, (n-1)$.

Note:

1) $1 = \cos 0 + i\sin 0$
2) $-1 = \cos\pi + i\sin\pi$
3) $i = \cos\pi/2 + i\sin\pi/2$
4) $-i = \cos\pi/2 - i\sin\pi/2$

(Standard numbers).

5) $(\cos x + i\sin x)(\cos y + i\sin y) = \cos(x+y) + i\sin(x+y)$.
6) $(\cos x + i\sin x)(\cos y - i\sin y) = \cos(x-y) + i\sin(x-y)$.
7) $(\cos x - i\sin x)(\cos y + i\sin y) = \cos(x-y) - i\sin(x-y)$
8) $(\cos x - i\sin x)(\cos y - i\sin y) = \cos(x+y) - i\sin(x+y)$

3)(i) Function of Complex Variables :-

$\Rightarrow f(Z) = u + iv = \underbrace{u(x,y)}_{\text{(Real part)}} + i\underbrace{v(x,y)}_{\text{(Imaginary part)}}$.

or,

$$\boxed{W = u + iv}$$

Here,
f(Z) is a function of complex variable and is denoted by w

(ii) Continuity of function f(Z) :-

$\Rightarrow$ A function $w = f(Z)$ is said to be continous at $Z = Z_0$ if,

$$\boxed{\lim_{Z\to Z_0} f(Z) = f(Z_0)}$$

(iii) Differentiability of function $f(z)$:-

⇒ Let, $f(z)$ be a singled value function of the variable z, then

$$\boxed{f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}}$$

- Provided that the limit exists and is independent of path along $\Delta z \to 0$.

[Singled Value Function : Any function that gives only one output value for one input value is a singled value function. Examples are, $f(x) = x$. For $x = n$, $n \in R$, there will be always one value of $f(x)$...]

4) Analytic Function : (Regular Functions OR Hylomorphic Functions)

⇒ A singled valued function $f(z)$ which is differentiable at $z = z_0$ is said to be analytic function at the point $z = z_0$ (Point z_0 is said non-singular point or regular).

- Singular Point → The point at which the function is not differentiable is called a singular point of the function.

⇒ The Necessary Condition of $f(z)$ to be analytic : (at all points in a region R).

① $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ → Provided $\partial u/\partial x$, $\partial v/\partial y$, $\partial u/\partial y$, $\partial v/\partial x$ exist

② $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ → C-R Equations. (Cauchy - Reimann Equations)

★ Sufficient Condition for $f(z)$ to be Analytic,

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \longrightarrow (1)$$

OR

$$\boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}} \longrightarrow (2)$$

★ • C-R Equations in Polar Form :-

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \text{ and, } \boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

• Derivative of $\omega = f(z)$ in Polar form :-

i) $$\boxed{\frac{\partial w}{\partial z} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}} \text{ —— } ①$$

ii) $$\boxed{\frac{\partial w}{\partial z} = \frac{-i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}} \text{ —— } ②$$

4.1) Harmonic Functions :-

• Any function of x and y (of first and second order) if satisfies Laplace equation is called Harmonic function.

★ For example,

$$\left.\begin{aligned} &(\text{For 'u'}) \quad \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \\ &(\text{For 'v'}) \quad \boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0} \end{aligned}\right\} \text{Laplace Equation.}$$

★ Laplace Equations and Conjugates :-

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \qquad \boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

- How to find 'v' if 'u' is given →

If,

u(x,y) is given. Then,

$$\boxed{du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy}$$

and, f(z) is analytic i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

So,

$$\boxed{du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy}$$ (exact differential form) (For 'u(x,y)')

Similarly,

$$\boxed{dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy}.$$ (For 'v(x,y)')

4.2) Milne's Thomson Method →

If u(x,y) is given,

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)] . dz$$

Here,

$$\frac{\partial u}{\partial x} = \phi_1(x,y), \quad \frac{\partial u}{\partial y} = \phi_2(x,y)$$

If v(x,y) is given,

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)] dz$$

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Here,

$$\frac{\partial v}{\partial y} = \psi_1(x,y) \quad , \quad \frac{\partial v}{\partial x} = \psi_2(x,y)$$

5) (i) Exponential Function of Complex Variable →

⇒ Exponential function of complex variable $Z = x+iy$ is written as, e^Z or $\exp Z$ and, defined as,

$$\Rightarrow e^Z = 1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + \ldots\ldots + \frac{Z^n}{n!} + \ldots \infty$$

⇒ PROPERTIES →

(1) ⇒ Exponential form of $Z = re^{i\theta}$.

⇒ $Z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

(2) → e^Z is periodic function having imaginary period $2\pi i$

(3) → e^Z is not zero for any value of Z.

(ii) Euler's formula →

If θ be real or complex,

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \quad \text{OR} \quad \boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$$

→ Adding ① and ②,

$$\Rightarrow e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

or,

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

→ Similarly, subtracting eq-② from eq-①,

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} + e^{-i\theta}\right)$$

or,

$$\boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} + e^{-i\theta}\right)}$$

And,

$$\boxed{\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i\left(e^{i\theta} + e^{-i\theta}\right)}}$$

(iii) Hyperbolic Functions → For values of x (real or complex), the quantity, $\frac{e^{x}+e^{-x}}{2}$ is called hyperbolic cosine of x and is written as $\cosh x$.

And,

$$\sinh x = \frac{e^{x} - e^{-x}}{2}.$$

★ Relation between hyperbolic and circular functions :-

We know that,

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Putting $\theta = ix$,

$$\Rightarrow \sin ix = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} \times \frac{i}{i}$$

$$= \frac{-i^2(e^{x} - e^{-x})}{2i} = i\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$= i \sinh x.$$

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$$\boxed{\sin ix = i\sinh x}$$

Similarly,

$$\boxed{\cos ix = \cosh x}$$

$$\boxed{\tan ix = i\tanh x}$$

- To seperate the real and imaginary part of $\sin(x+iy)$.

$\Rightarrow$ $\sin(x+iy) = \sin x.\cos iy + \cos x.\sin iy$

$$= \underbrace{\sin x.\cosh y}_{\text{Real part}} + i\,\underbrace{\cos x.\sinh y}_{\text{Imaginary part}}.$$

- To seperate the real and imaginary part of $\tan(x+iy)$

$\Rightarrow$ $\tan(x+iy) = \dfrac{\sin(x+iy)}{\cos(x+iy)} \times \dfrac{\cos(x-iy)}{\cos(x-iy)} \times \dfrac{2}{2}$

$$= \frac{2\sin(x+iy).\cos(x-iy)}{2\cos(x+iy).\cos(x-iy)}$$

$$= \frac{\sin(x+\cancel{iy}+x-\cancel{iy}) + \sin(\cancel{x}+iy-\cancel{x}+iy)}{\cos(x+\cancel{iy}+x-\cancel{iy}) + \cos(\cancel{x}+iy-\cancel{x}+iy)}$$

$$= \frac{\sin 2x + \sin(2iy)}{\cos 2x + \cos(2iy)}$$

$\Rightarrow$ $\tan(x+iy) = \dfrac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$

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→ Real part = $\frac{\sin 2x}{\cos 2x + \cosh(2y)}$

→ Imaginary part = $i \cdot \frac{\sinh 2y}{\cos 2x + \cosh 2y}$.

(iv) Logarithm of Complex Numbers :

$\Rightarrow Z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\Rightarrow x + iy = re^{i\theta}$

or,

$\Rightarrow \log(x+iy) = \log(re^{i\theta}) = \log r + i\theta$.

$\because r = \sqrt{x^2+y^2}$ and, $\tan^{-1}\left(\frac{y}{x}\right) = \theta$

$$\Rightarrow \boxed{\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)}$$

→ $\log(x+iy) \longrightarrow$ Principal value.

→ $\mathrm{Log}(x+iy) \longrightarrow$ General value. $(n \in N)$

$\therefore \mathrm{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left[\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi\right]$.

6) Line Integral in the Complex Plane :-

$\Rightarrow \int_C f(Z)\, dZ = \int_C (u+iv)(dx + i\,dy)$ (C → closed integral)

Here,

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$\Rightarrow Z = x + iy$

$\Rightarrow f(Z) = u + iv = u(x,y) + iv(x,y)$

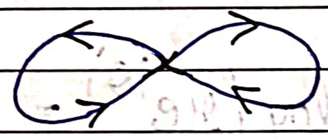
and,

$\Rightarrow dZ = dx + i\,dy$

7) Cauchy's Integral Theorem and Cauchy's Integral Formula :-

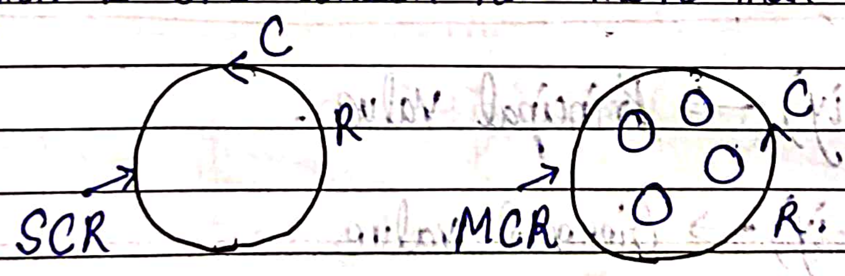
$\Rightarrow$ A simple closed curve is a curve that does not cross itself.

(a) Multiple curve: A curve which crosses itself is called a multiple cure.



(b) Simply connected-region: A simply connected region is one which has no pole (holes).

(c) Multiple or Multiply connected region: A multiple connected region is one which has more than one poles (holes).



Note: We can convert multiply connected region into simply connected region with the help of cross-cuts.

7.1) Cauchy's Integral Theorem statement:

$\Rightarrow$ If a function $f(Z)$ is analytic and its derivative $f'(Z)$ continuous at all points inside and on a simple closed curve C, then;

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$$\boxed{\int_C f(z) \, dz = 0}$$

$$\Rightarrow \int_C F(z) \, dz = \int_C (u+iv)(dx+idy) \qquad \left[\begin{array}{l} z = x+iy. \\ \text{So,} \\ f(z) = u+iv \end{array}\right].$$

$$= \int_C u \, dx + iu \, dy + iv \, dx - v \, dy$$

$$= \int_C (u\,dx - v\,dy) + i(u\,dy + v\,dx)$$

$$= \int_C [(u\,dx - v\,dy) + i(v\,dx + u\,dy)].$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

Note: (Green's Theorem) →

$$\int_C \phi \, dx + \psi \, dy = \iint_R \left(\frac{\partial}{\partial x}\psi - \frac{\partial}{\partial y}\phi\right) dx \, dy.$$

$$= \iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx \, dy + i \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx \, dy.$$

$\because$ $f(z)$ is analytic, i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

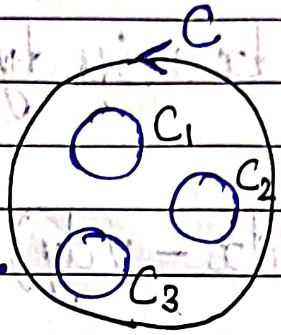
$$= \iint \left(\cancel{\frac{\partial u}{\partial y}} - \cancel{\frac{\partial u}{\partial y}}\right) dx \, dy + i \iint \left(\cancel{\frac{\partial v}{\partial y}} - \cancel{\frac{\partial v}{\partial y}}\right) dx \, dy$$

$$\Rightarrow \int_C F(z).\,dZ = 0 + i(0) = \underline{\underline{0}}.$$

(Hence proved).

7.2) Extension of Cauchy's theorem to multiple connected region

$$\Rightarrow \int_C f(z).\,dz = \int_{C_1} f(z).\,dZ + \int_{C_2} f(z).\,dZ + \int_{C_3} f(z).\,dZ \ldots$$



$$\int_C f(z).\,dz = 0$$

7.3) Cauchy's Integral formula :-

Statement: If $f(z)$ is analytic function within and on a closed curve C and if 'a' is any point within C, then

$$\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\,dZ}$$

7.4) Cauchy's Integral formula for the derivative of an analytic function :-

$\Rightarrow$ If a function $f(z)$ is analytic in a region R, then its derivative at any point $z=a$, if R is also analytic in R, and is, given as;

$$\boxed{f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

If $n = \underline{1}$;

$$f'(a) = \frac{1!}{2\pi i} \int_c \frac{f(z)}{(z-a)^2} dz .$$

If $n=2$,

$$f''(a) = \frac{2!}{2\pi i} \int_c \frac{f(z)}{(z-a)^3} dz .$$

8) Expansion of functions :-

① Taylor series
② Maclaurin series
③ Laurents series

8.1) Taylor Series: If a function $f(z)$ is analytic inside a circle C with centre at $z=a$ then it can be expanded in the series,

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots + \frac{(z-a)^n}{n!} \times f^n(a) + \dots$$

⇒ If putting $a=0$,

⇒ $f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \dots + \frac{z^n}{n!} f^n(0) + \dots$

This series is called Maclaurin series.

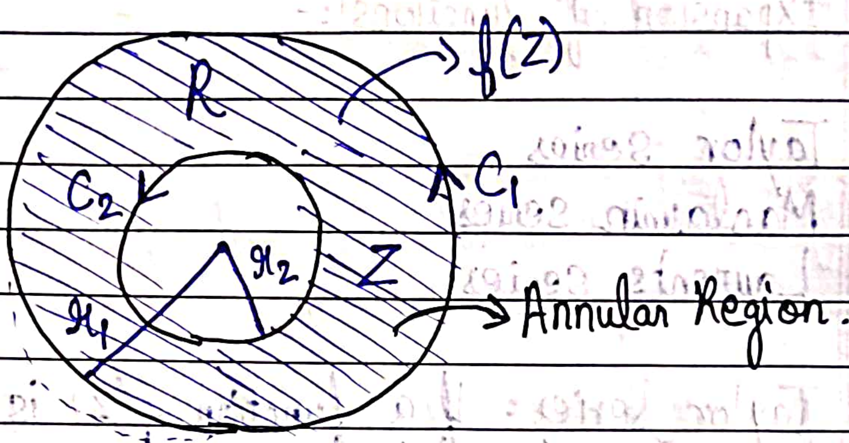
8.2) Laurent's Series :-

If $f(z)$ is analytic in annular (co-centric) region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 $(r_1 \geq r_2)$ and centre at 'a', then for all z in R, we have;

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{General term}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-a)^{-n}}_{\text{Principal term}} .$$

where,
$$a_n = \frac{1}{2\pi i}\int_c \frac{f(t)}{(t-a)^{n+1}}\,dt \quad , n=0,1,2,3,\ldots$$

$$b_n = \frac{1}{2\pi i}\int_c \frac{f(t)}{(t-a)^{-n+1}}\,dt \quad , n=1,2,3,\ldots$$



9) Power Series: A series in powers of $(z-z_0)$ is called power series:

$$\sum_{n=0}^{\infty} a(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots\ldots + a_n(z-z_0)^n$$

Note: If centre of series, i.e., $z_0 = 0$, then;
$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots$$

9.1) Radius of Convergence of Power Series :-

⇒ Let the power series, $\sum a_n z^n$, by cauchy theorem on limits.

⇒ Radius of convergence (R) is given by,

$$\boxed{\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$$

10) **Zero of Analytic Function :-**

→ A zero of analytic function $f(z)$ is the value of z for which $f(z) = 0$, i.e.,

$$f(z) = z - 2$$

So for $z = 2$, $f(z) = 0$.

11) • **Singular Point :-**

⇒ A point at which a function $f(z)$ is not analytic is known a singular point or singularity of the function.

Let, $f(z) = \dfrac{1}{z-2}$ has a singular point at $z = 2$.

• **Isolated Singular Point :-**

⇒ If $z = a$ is a singularity of $f(z)$ and if there is no other singularity with in a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of function $f(z)$, otherwise it is called non-isolated singular point.

Example: ① $f(z) = \dfrac{1}{(z-1)(z-3)}$

Here, $z = 1, 3$ are two isolated singular point.

② $f(z) = \dfrac{1}{\sin\left(\frac{\pi}{z}\right)}$

here, $\sin \pi/z = 0$

$\Rightarrow \pi/z = n\pi$.

$\Rightarrow z = 1/n$.

Here, $z = \dfrac{1}{n}$ $(n = 0, 1, 2, 3, \ldots \infty)$ $\therefore z = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $(n \in N)$

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- Pole of order 'm' :-

→ By Laurent's series,

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$$\Rightarrow f(z) = a_0 + a_1(z-z_0) + \ldots\ldots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \ldots\ldots +$$

$$(1) \longrightarrow \frac{b_m}{(z-z_0)^m} + \ldots + \frac{b_{m+1}}{(z-z_0)^{m+1}} + \ldots$$

⇒ If the number of terms of negative powers in (1) is infinite then, $z = z_0$ is called an essential singularity point of $f(z)$.

- Removable Singularity :-

Note → If Laurent's series does not contain negative power of $(z-z_0)$ then, $z = a$ is called a removable singularity.

Example: $f(z) = \frac{\sin(z-a)}{(z-a)}$

$$\Rightarrow f(z) = \frac{1}{(z-a)} \Bigg[\underbrace{(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \ldots\ldots \infty}_{\text{Expansion of } \sin x} \Bigg]$$

$$\Rightarrow f(z) = 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - \ldots\ldots \infty$$

→ Now, there is no singularity.

Note: ① If $\lim_{z \to a} f(z)$ = Exists and finite, then $z = a$ is a removable singular point.

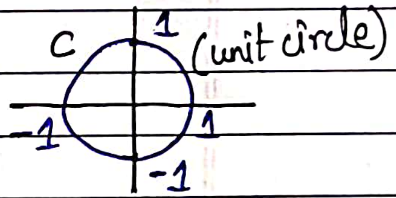
② If $\lim_{z \to a} f(z)$ does not exists, then $z = a$ is an essential singular point.

③ If $\lim_{z \to a} f(z)$ is infinite then $f(z)$ has pole at $z = a$

12) Evaluation of real definite integrals by Contour Integration

1) Integration round the unit circle of the type,

$$\int_0^{2\pi} f(\cos\theta, \sin\theta) \, . \, d\theta \, .$$



here, $\int_C \rightarrow$ Unit circle.

To change $\theta \rightarrow$ 'z'. we have to substitute $z = e^{i\theta}$.

$\Rightarrow z = e^{i\theta}$

$\Rightarrow dz = i e^{i\theta} . d\theta$.

$\Rightarrow dz = iz . d\theta \qquad \Rightarrow d\theta = \dfrac{dz}{iz}$.

$$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\Rightarrow \cos\theta = \frac{z + z^{-1}}{2} = \frac{z^2 + 1}{2z}$$

and,

$$\Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2 - 1}{2iz}.$$