# Partial Differentiation

$\partial \rightarrow$ curly
$\partial \rightarrow$ del

$$\frac{\partial}{\partial x}(x^2) = 2x$$

$$\frac{\partial}{\partial x}(xy) = y(1) + x(0)$$
$$\Rightarrow y$$

# Differentiation

① $d$

② $\frac{d}{dx}(x^2) = 2x$

③ $\frac{d}{dx}(xy) = x\left(\frac{dy}{dx}\right) + y(1)$

$$\Rightarrow x\frac{dy}{dx} + y$$

2 variables
$x, y$

assume; both are independent

| | $x$ | $y$ |
|---|---|---|
| $\frac{\partial}{\partial x}$ | 1 | 0 → constant |
| $\frac{\partial}{\partial y}$ | 0 → constant | 1 |

# Partial derivative of two variable

Q. $\frac{\partial}{\partial x}(x^2 + y^2)$

$(2x + 0)$

Q. $\frac{\partial}{\partial y}(x^2+y^2)$

Sol. $(2y + 0)$

Q. $\frac{\partial}{\partial x}(xy)$

Sol. $y$

Q. $\frac{\partial}{\partial x}(x^2y^2)$

Sol. $2xy^2$

Q. $\frac{\partial(\sin xy)}{\partial x}$

Sol. $y \cos xy$ Ans

Q. $\frac{\partial}{\partial y}[\log(x^2+y^2)]$

Sol. $\frac{2y}{x^2+y^2}$ Ans

Q. $\frac{\partial}{\partial x}\left(\sin\left(\frac{x}{y}\right)\right)$

Sol. $\frac{\cos\left(\frac{x}{y}\right)}{y}$ Ans

## 2 variables

8 ↳ $x, y$

9 both are independent

10 11 12

| | $x$ | $y$ |
|---|---|---|
| $\partial/\partial x$ | 1 | 0 |
| $\partial/\partial y$ | 0 | 1 |

1

2 ## 3 variables

3 ↳ $\underbrace{x, y}_{\text{independent}}, z$ ↳ dependent

4 independent dependent

5 $z = f(x, y)$

6

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $\partial/\partial x$ | 1 | 0 | $\frac{\partial z}{\partial x}$ |
| $\partial/\partial y$ | 0 | 1 | $\frac{\partial z}{\partial y}$ |

## 4 variables

↳ $\underbrace{x, y, z}_{\text{independent}}, \underbrace{u}_{\text{dependent}}$

$u = f(x, y, z)$

| | $x$ | $y$ | $z$ | $u$ |
|---|---|---|---|---|
| $\partial/\partial x$ | 1 | 0 | 0 | $\partial u/\partial x$ |
| $\partial/\partial y$ | 0 | 1 | 0 | $\partial u/\partial y$ |
| $\partial/\partial z$ | 0 | 0 | 1 | $\partial u/\partial z$ |

$z = f(x, y)$

| $z$ | $f$ | $f$ | $z$ |
|---|---|---|---|
| $\partial z/\partial x$ | $\partial f/\partial x$ | $f_x$ | $p$ |
| $\partial z/\partial y$ | $\partial f/\partial y$ | $f_y$ | $q$ |
| $\frac{\partial^2 z}{\partial x^2}$ | $\frac{\partial^2 f}{\partial x^2}$ | $f_{xx}$ | $r$ |
| $\frac{\partial^2 z}{\partial y^2}$ | $\frac{\partial^2 f}{\partial y^2}$ | $f_{yy}$ | $t$ |
| $\frac{\partial^2 z}{\partial x \partial y}$ | $\frac{\partial^2 f}{\partial x \partial y}$ | $f_{xy}$ | $s$ |

Q. If $z(x+y) = (x^2+y^2)$ then prove that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Sol → $z(x+y) = x^2+y^2$

$$z = \frac{x^2+y^2}{x+y} \quad \text{—} \quad (1)$$

Partially Diff. eq (1) w.r.t $x$

$$\frac{\partial z}{\partial x} = \frac{(x+y)(2x) - (x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x^2-y^2+2xy}{(x+y)^2} \quad \text{—} \quad (\text{ii})$$

Partially diff. eq (1) w.r.t. $y$

$$\frac{\partial z}{\partial y} = \frac{2(x+y)y - (x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} = \frac{y^2-x^2+2xy}{(x+y)^2}$$

L.H.S.

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$$

$$\left(\frac{x^2-y^2+2xy}{(x+y)^2} - \frac{y^2+x^2-2xy}{(x+y)^2}\right)^2$$

$$\left(\frac{2x^2-2y^2}{(x+y)^2}\right)^2$$

$$\left(\frac{2\cancel{(x+y)}(x-y)}{\cancel{(x+y)}(x+y)}\right)^2$$

$$\boxed{4\frac{(x-y)^2}{(x+y)^2}} \text{ — } (iv)$$

R.H.S.

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$$4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$$

$$\Rightarrow 4\left(1-\frac{x^2+y^2-2xy}{(x+y)^2} - \frac{y^2+x^2-2xy}{(x+y)^2}\right)$$

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$$\Rightarrow 4\left(\frac{(x+y)^2 - x^2 + y^2 - 2xy - y^2 + x^2 - 2xy}{(x+y)^2}\right)$$

$$\Rightarrow 4\left(\frac{x^2+y^2+2xy - \cancel{x^2} + \cancel{y^2} - 4xy - y^2 + x^2}{(x+y)^2}\right)$$

$$\Rightarrow 4\left(\frac{x^2+y^2-2xy}{(x+y)^2}\right)$$

$$\Rightarrow 4\left(\frac{(x-y)^2}{(x+y)^2}\right) \quad IV$$

L.H.S. = R.H.S.
III   IV

Hence proved.

Q. If $u = \log(x^3+y^3+z^3-3xyz)$ then prove that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Sol. The given eqn is

$$u = \log(x^3+y^3+z^3-3xyz) \quad -(1)$$

Since there are 4 variables $x, y, z, u$. So; $x, y$ & $z$ are independent variables & $u$ is dependent variable.

L.H.S.

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]\cdot u$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{\partial u}{\partial x}+\left(\frac{\partial u}{\partial y}\right)+\frac{\partial u}{\partial z}\right] \quad -(11)$$

Now Partially differentiate (1) wrt $x$.

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz} \; ; \; \frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

Put the value in expression (11)

$$\Rightarrow \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3x^2-\cancel{3yz}+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}\right)$$

$$\Rightarrow 3\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{x^2+y^2+z^2-xy-yz-zx}{x^3+y^3+z^3-3xyz}\right)$$

$$\Rightarrow 3\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{x^2+y^2+z^2-xy-yz-zx}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}\right)$$

$$\left[\because a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\right]$$

$$\Rightarrow 3\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{1}{x+y+z}\right)$$

$$\Rightarrow 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$\Rightarrow 3\left[-1(x+y+z)^{-2}-1(x+y+z)^{-2}-1(x+y+z)^{-2}\right]$$

$$\Rightarrow -3\left[\frac{1+1+1}{(x+y+z)^2}\right]$$

$$\left[\frac{-9}{(x+y+z)^2}\right] = \underline{R.H.S}$$

Hence, proved

# Lec 2

* Q. State & Prove Euler's Theorem for Homogeneous Functions.

Sol →

Statement :- If $z$ is a homogenous function of degree $n$ in $x$ & $y$, then

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz} \rightarrow \text{always holds}$$

Proof → ~~Sinse~~ Since $z$ is a homogenous function of degree $n$ in $x$ & $y$.

So, $\boxed{z = x^n f\left(\frac{y}{x}\right)}$ — (1)

Now partially differentiate eq (1) w.r.t. $x$ & $y$, we get

$$\frac{\partial z}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2}\right] + f\left(\frac{y}{x}\right)\left(n x^{n-1}\right)$$

$$\frac{\partial z}{\partial x} = -x^{n-2} y f'(y/x) + nx^{n-1} f(y/x)$$

$$\frac{\partial z}{\partial y} = x^n f'(y/x)\frac{1}{x} = x^{n-1} f'(y/x)$$

Now, LHS

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$\Rightarrow x\left[-x^{n-2} y f'(y/x) + nx^{n-1} f(y/x)\right] + y\left[x^{n-1} f'(y/x)\right]$$



$$\Rightarrow -x^{n-1} y f'(y/x) + x^{n-1} y f'(y/x) + n x^n f(y/x)$$

$\Rightarrow nz = R.H.S.$

Hence proved

$\Rightarrow$ Euler's Theorem of 2 variables

Homogenous fn $z$ $\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = nz$

$\downarrow$ Degree

Q, $z = x^3 + y^3 + 3x^2y + 3xy^2$

find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$.

Sol: By Euler's Theorem

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

$n = 3$

$3(z)$ Ans

Q) Using Euler's Theorem on function

$$z = x^2 + y^2 + 2xy$$

Prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z$

Sol) The given fn is

$$z = x^2 + y^2 + 2xy$$

$$\Rightarrow x^2\left[1 + \frac{y^2}{x^2} + \frac{2xy}{x^2}\right]$$

$$\Rightarrow x^2\left[1 + \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)\right]$$

$$\Rightarrow z = x^2 f(y/x) \quad \text{--- (1)}$$

$\Rightarrow$ Now, by comparing eq: (1) with standard form of Homogeneous fn i.e. $z = x^n f(y/x)$ we get

$$\boxed{n = 2}$$

26 SUNDAY $\rightarrow$ Now as we know that by Euler's Theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$\Rightarrow$ Here, $n = 2$

2023 $\Rightarrow$ So, $\boxed{x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z}$ Hence, Proved.

Q. If $z = xy f\left(\frac{y}{x}\right)$ then prove that

$$x \frac{dz}{dx} + y \frac{dz}{dy} = 2z.$$

Sol → The given eq. is

$$z = xy f\left(\frac{y}{x}\right)$$

$$= x^2 \left[\frac{xy}{x^2} f\left(\frac{y}{x}\right)\right]$$

$$\Rightarrow x^2 \left[\frac{y}{x} f\left(\frac{y}{x}\right)\right]$$

$$\Rightarrow z = x^2 g\left(\frac{y}{x}\right) \quad \text{—} (1)$$

⇒ Now, by comparing eq. (1) with standard form of homogenous fn

i.e. $z = x^n g(y/x)$

we get

$$n = 2$$

⇒ Now, as we know that by Euler's Theorem

$$x \frac{dz}{dx} + y\, dz/dy = nz$$

⇒ Here, $n = 2$

⇒ So, $x \frac{dz}{dx} + y \frac{dz}{dy} = 2z$

⇒ Hence, Proved

Q. If $f(x,y) = \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{x^3+y^3}$ then

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is

(a) 0
(b) 9
(c) 1
(d) $-3f$

Sol. $f(x,y) = \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{x^3+y^3}$

$$\Rightarrow \frac{1}{x^3(1)} + \frac{1}{x^3(y/x)^3} + \frac{1}{x^3(1+(y/x)^3)}$$

$$\Rightarrow x^{-3}\left[1 + \frac{1}{(y/x)^3} + \frac{1}{1+(y/x)^3}\right]$$

$$f(x,y) = x^{-3}\, g(y/x) \quad \text{--- (1)}$$

→ Now by comparing eq (1) with standard form $f = x^n g(y/x)$, we get $n = -3$

Now, as we know that by Euler's Theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nf$$

Here, $n = -3$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -3f$$

(d)

Q. If $u = f(y/x)$ then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Sol. The given eq is

$$u = f(y/x)$$

$$u = x^0 f(y/x) \quad -(1)$$

by comparing eq (1) with standard form of homogeneous fn.

$$u = x^n f(y/x)$$

we get $n = 0$

As we know that by Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$\Rightarrow$ Here $n = 0$

So, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0(u)$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ Ans

Q. If $u = y^x \log(x/y)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is

(a) $u$
(b) $u^2$
(c) $2u$
(d) $3u$

Sol. The given eq is
$u = y^2 \log(x/y)$
$\Rightarrow y^2 f(x/y)$

$\rightarrow$ Here, $n = 2$
So, by Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

(c)

Q. If $u = \tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$ then

prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Sol → The given equation is

$$u = \tan^{-1}\underbrace{\left[\frac{x^3+y^3}{x-y}\right]}_{\text{Homogeneous}} \rightarrow \text{Non-Homogeneous}$$

$$\Rightarrow \overbrace{\tan u}^{\text{Let } z} = \frac{x^3+y^3}{x-y} \quad \text{— (1)}$$

⇒ Now eq (1) reduces to Homogeneous fn

$$z = \frac{x^3+y^3}{x-y}$$

$$= \frac{x^3}{x}\left(\frac{1+(y/x)^3}{1-(y/x)}\right)$$

$$\Rightarrow \boxed{z = x^2 f(y/x)} \quad \text{— (2)}$$

⇒ Now by comparing eq (2) with standard form of Homogeneous function

$$z = x^n f(y/x)$$

we get $\boxed{n = 2}$.

⇒ Now, as we know that by Euler's Theorem

$$x\frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz$$

⇒ Put $n = 2$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \quad -(3)$$

⇒ Now ; $z = \tan u$

$$\frac{\partial z}{\partial x} = \cancel{\sec^2 u \left(\frac{\partial u}{\partial x}\right)} \sec^2 u \left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial z}{\partial y} = \sec^2 u \left(\frac{\partial u}{\partial y}\right)$$

⇒ Put the values of $z$, $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ in eq (3), we get



$$\Rightarrow x\left(\sec^2 u \frac{\partial u}{\partial x}\right) + y\left(\sec^2 u \frac{\partial u}{\partial y}\right) = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u \cos^2 u$$



$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u}{\cos u} \times \cos^2 u$$

| D | | | | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| E | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | |
| | 25 | 26 | 27 | 28 | 29 | 30 | 31 | | | | | | | | |
| C | M | T | W | T | F | S | S | M | T | W | T | F | S | S | |

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$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u \cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

$\Rightarrow$ Hence, Proved.

# Implicit function

A function in which we can't represent $y = f(x)$; is called implicit function. Suppose $f(x, y) = \text{constant}$ is a implicit function.

$$\boxed{f(x, y) = c} \quad \text{—} \quad (1)$$

By ~~differenti~~ definition total differential coefficient, differentiate (1) wrt $x$

$$\frac{\partial f}{\partial x} \times \frac{\partial x}{\partial x} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial x} = \cancel{\frac{\partial f}{\partial}}\ 0$$

$$\frac{\partial f}{\partial y} \times \frac{\partial y}{\partial x} = -\frac{\partial f}{\partial x}$$

$$\partial y/\partial x = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$\boxed{\partial y/\partial x = -\frac{f_x}{f_y}}$$

BONUS

$f \equiv z < \begin{matrix} x \\ y \end{matrix}$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \quad ; \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$$

Q. Find $\partial y/\partial x$ ; if $x^y + y^x = c$ → Implicit function

Sol. Let $f = x^y + y^x - c = 0$

$$\frac{\partial f}{\partial x} = y x^{y-1} + y^x \log_e y \quad (y \text{ as constant}) \qquad \left(\because \frac{d a^x}{dx} = a^x \log_e a\right)$$

$$\frac{\partial f}{\partial y} = x^y \log_e x + x y^{x-1} \qquad \left(\because \frac{d}{dx} a^x = a^x \log_e a\right)$$

$$\because \frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-(y x^{y-1} + y^x \log_e y)}{(x^y \log_e x + x y^{x-1})} \quad \text{Ans}$$

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Q. If $x^4+y^4+4a^2xy=0$; find $dy/dx$.

Sol. formula of ~~sok~~ $dy/dx$ if implicit function is there:

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

$\Rightarrow f = x^4+y^4+4a^2xy$

$\Rightarrow f_x = \frac{\partial f}{\partial x} = (4x^3+4a^2y)$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = (4y^3+4a^2x)$

$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(4x^3+4a^2y)}{(4y^3+4a^2x)}$ Ans

Q. If $u = x^2y$ & $x^2+xy+y^2=1$. Find $\frac{du}{dx}$

Sol→ $x^2+xy+y^2-1=0$

$f(x,y)=0$

$\frac{du}{dx} = (2x)y + x^2\frac{dy}{dx}$

$\Rightarrow 2xy + x^2\frac{dy}{dx}$ —(1)

Formula of $dy/dx$ when there is implicit function.

$dy/dx = -\frac{f_x}{f_y}$

$f_x = \frac{\partial f}{\partial x} = 2x+y$

$f_y = \frac{\partial f}{\partial y} = x+2y$

$\frac{dy}{dx} = -\frac{f_x}{f_y} = \left[-\frac{(2x+y)}{(x+2y)}\right]$

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Now put value of $\frac{dy}{dx}$ in ①

$$\frac{du}{dx} = 2xy - x^2 \frac{(2x+y)}{(x+2y)} \text{ Ans}$$

Imp Q. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$

Prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)^2$$

Sol→ $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} - 1 = 0$

⇒ Differentiate both sides wrt $x$.

$$\Rightarrow \frac{(a^2+u)2x - x^2 U_x}{(a^2+u)^2} - \frac{(y^2)U_x}{(b^2+u)^2} - \frac{(z^2)U_x}{(c^2+u)^2} = 0$$

$$\Rightarrow \frac{2x}{a^2+u} - U_x\left(\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right) = 0$$

$$\left[\text{Let } F = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$\Rightarrow \frac{2x}{a^2u} - F U_x = 0$$

$$\Rightarrow U_x = \frac{2x}{F(a^2+u)}$$

Similarly, $U_y = \frac{2y}{F(b^2+u)}$ & $U_z = \frac{2z}{(c^2+u)F}$

$\therefore$ $\underline{L.H.S}$

$\Downarrow$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$$

$$\Rightarrow \frac{4x^2}{(a^2+u)^2F^2} + \frac{4y^2}{(b^2+u)^2F^2} + \frac{4z^2}{(c^2+u)^2F^2}$$

$$\Rightarrow \frac{4}{F^2}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$\Rightarrow \frac{4}{F^2} \times F \Rightarrow \frac{4}{F}$$

$\underline{RHS} \Rightarrow 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$

$$\Rightarrow 2\left[\frac{2x^2}{(a^2+u)F} + \frac{2y^2}{(b^2+u)F} + \frac{2z^2}{(c^2+u)F}\right]$$

$$\Rightarrow \frac{4}{F}\left[\underbrace{\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}}_{}\right]$$

$\rightarrow$ (1) $\rightarrow$ Means value is 1

$$\Rightarrow \frac{4}{F}(1) \quad \frac{4}{F}(1) = \frac{4}{F}$$

$\Rightarrow$ As ; LHS = R.H.S $= 4/F$

$\rightarrow$ Hence, proved

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Q. If $x^x y^y z^z = c$. Show that at $x = y = z$;

$$\frac{\partial^2 z}{\partial x \partial y} = -[x \log_e x]^{-1}$$

Sol → $x^x y^y z^z = c$

⇒ Taking log both sides.

$$\Rightarrow \log(x^x y^y z^z) = \log c$$

$$\Rightarrow \log x^x + \log y^y + \log z^z = \log c \quad [As;\ \log(xy) = \log x + \log y]$$

$$\Rightarrow x \log x + y \log y + z \log z = \log c \quad [As;\ \log m^n = n \log m]$$

Differentiate both sides wrt x

$$\Rightarrow x \times \frac{1}{x} + \log x + z \frac{1}{z}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)\log z = 0$$

$$\Rightarrow (1 + \log x) + (1 + \log z)\frac{\partial z}{\partial x} = 0 \quad — (1)$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-(\log x + 1)}{(\log z + 1)}$$

Similarly

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-(\log y + 1)}{(\log z + 1)}$$

⇒ Differentiate (1) partially ~~log y~~ wrt y.

$$\Rightarrow (1 + \log z)\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x} \cdot \frac{1}{z}\frac{\partial z}{\partial y}$$

$$\Rightarrow (1 + \log z)\frac{\partial^2 z}{\partial y \partial x} + \left[-\frac{(1 + \log x)}{(1 + \log z)}\right]\frac{1}{z}\left[\frac{-(1 + \log y)}{(1 + \log z)}\right] = 0$$

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$$\Rightarrow (1+\log z)\frac{\partial^2 z}{\partial y \partial x} = -\frac{(1+\log x)(1+\log y)}{z(1+\log z)^2}$$

$$\left[ (x=y=z) \text{ (Given)} \right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial y \partial x} = -\frac{\cancel{(1+\log x)}\,\cancel{(1+\log x)}}{x\,\cancel{(1+\log x)^2}\,(1+\log x)}$$

$$\Rightarrow \frac{\partial^2 z}{\partial y \partial x} = -\frac{1}{x(\log e + \log x)}$$ (As; $\log_e e = 1$)

$$\Rightarrow \frac{\partial^2 z}{\partial y \partial x} = -\frac{1}{x(\log ex)}$$ (As; $\log(xy) = \log x + \log y$)

$$\Rightarrow \frac{\partial^2 z}{\partial y \partial x} = -[x(\log ex)]^{-1}$$

Hence, Proved.

# Chain Rule of Differentiation

$$z = f(x, y)$$

$$x = f(t) \; ; \; y(t)$$

z is a composite fn of 't'
(indirect relation)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$z = f(x, y)$$

$$x = h(r, \theta) \; ; \; y = g(r, \theta)$$

z is a composite fn of r & θ

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

Q. If $u = f(y-z, z-x, x-y)$ then show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Sol → The given ~~eqn is~~ expression is

$$u = f(\underbrace{x-y}_{r}, \underbrace{y-z}_{s}, \underbrace{z-x}_{t}) \quad \text{— ①}$$

Let $r = x-y$; $s = y-z$, $t = z-x$
Equation ① reduces to $u = f(r, s, t)$ — ②

Now;

$$u \to r, s, t \to x, y, z$$

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Differentiate equation (2) wrt $x$; we get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1)$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}}$$

⇒ Differentiate eq (2) again wrt $y$; we get

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$\boxed{\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}}$$

Similarly

$$\boxed{\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}}$$

⇒ Now; LHS

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\right) + \left(-\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\right) + \left(-\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}\right)$$

$= 0$

$=$ R.H.S

⇒ So, L.H.S = R.H.S

⇒ Hence, proved

Q. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Sol: The given equation is

$$u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right) \quad \text{--- (1)}$$

Let

$$r = \frac{x}{y}, \quad s = \frac{y}{z}, \quad t = \frac{z}{x}$$

So, eq (1) reduces to

$$u = f(r, s, t) \quad \text{--- (2)}$$

Now;

$$\begin{tikzcd} & r \arrow[dr] & \\ u \arrow[ur] \arrow[r] \arrow[dr] & s \arrow[r] & x, y, z \\ & t \arrow[ur] & \end{tikzcd}$$

Differentiate eq (2) wrt $x$, we get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(\frac{-z}{x^2}\right)$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) - \frac{z}{x^2}\left(\frac{\partial u}{\partial t}\right)}$$

Differentiate eq (2) wrt $y$, we get

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right) + \left(\frac{\partial u}{\partial s}\right)\frac{1}{z} + \frac{\partial u}{\partial t}(0)$$

$$\boxed{\frac{\partial u}{\partial y} = -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}}$$

Similarly;

$$\boxed{\frac{\partial u}{\partial z} = -\frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t}}$$

⇒ Now, LHS

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= x\left[\frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t}\right] + y\left[-\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}\right] +$$

$$z\left[-\frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t}\right]$$

$= 0$

$=$ R.H.S.

⇒ So, as L.H.S. = R.H.S

⇒ Hence, Proved

Q. If $u = f(2x-3y, 3y-4z, 4z-2x)$

Prove that

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$$

Sol. The given eq is

$u = f(2x-3y, 3y-4z, 4z-2x)$ — ①

Let $r = 2x-3y$, $s = 3y-4z$, $t = 4z-2x$

⇒ So, Eq ① reduces to

⇒ $u = f(r, s, t)$ — ②

⇒ Now;

$$u \to \begin{cases} r \\ s \\ t \end{cases} \to x, y, z$$

⇒ Partially differentiate eq (2) wrt $x$, we get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(2) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-2)$$

$$\boxed{\frac{\partial u}{\partial x} = \left(\frac{2\,\partial u}{\partial r} - 2\frac{\partial u}{\partial t}\right)}$$

→ Similarly

$$\boxed{\frac{\partial u}{\partial y} = -3\frac{\partial u}{\partial r} + 3\frac{\partial u}{\partial s}}$$

$$\boxed{\frac{\partial u}{\partial z} = -4\frac{\partial u}{\partial s} + 4\frac{\partial u}{\partial t}}$$

⇒ Now;

$$= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$$

$$= \frac{1}{2}\left(2\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial t}\right) + \frac{1}{3}\left(-3\frac{\partial u}{\partial r} + 3\frac{\partial u}{\partial s}\right) + \frac{1}{4}\left(-4\frac{\partial u}{\partial s} + 4\right.$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$$

$$= 0$$

= R.H.S.

⇒ So, as; LHS = RHS

⇒ Hence, Proved.

## Jacobian

* **Definition** ⇒ If u & v are the functions of two independent variables x & y ; then Jaco[bian] of u & v wrt x & y is defined as

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)}$$

$$= \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \longrightarrow \text{Determinant}$$

Similarly; $J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)}$

$$= \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

**Applications** ÷

→ To find functional relationship between two vari[ables]

Eg → 
$$\begin{tikzcd} u \arrow[r] \arrow[d] & y \\ x & \end{tikzcd} \qquad \begin{tikzcd} v \arrow[r] \arrow[dr] & y \\ & x \end{tikzcd}$$

• **Properties**

① If u & v are the function of x & y ; then $u = u($
$v = v(x,y)$

$$J = J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)}$$

$$J' = J\left(\frac{x,y}{u,v}\right) = \frac{\partial(x,y)}{\partial(u,v)}$$

$$\Rightarrow \boxed{JJ' = 1}$$

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(2) Chain Rule :- If u & v are the functions of x & y & x and y are the function of r & s then

$$J\left(\frac{u, v}{r, s}\right) = J\left(\frac{u, v}{x, y}\right) \cdot J\left(\frac{x, y}{r, s}\right)$$

$$\boxed{\frac{\partial(u, v)}{\partial(r, s)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(r, s)}}$$

- Types of functions

(1) Explicit functions

If $x = r\cos\theta$ ; $y = r\sin\theta$

$$J\left(\frac{x, y}{r, \theta}\right) = \frac{\partial(x, y)}{\partial(r, \theta)}$$

(2) Composite functions

Apply chain rule

$$J\left(\frac{u, v}{r, s}\right) = J\left(\frac{u, v}{x, y}\right) \cdot J\left(\frac{x, y}{r, s}\right)$$

(3) Implicit functions

The variables can be separated

$x = r\cos\theta$ ; $y = r\sin\theta$

After separating find :

$$\Rightarrow J\left(\frac{r, \theta}{x, y}\right) = \frac{\partial(r, \theta)}{\partial(x, y)}$$

$$= \begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix}$$

Variables can't be separated

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x, y)} \div \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

V imp Q. If $x = r\cos\theta$ ; $y = r\sin\theta$ then show that

$$J\left(\frac{x, y}{r, \theta}\right) = r$$

Sol→ Acc. to question:

$x = r\cos\theta$ — (1) ; $y = r\sin\theta$ — (2) ] Explicit function

Now; $J\left(\frac{x, y}{r, \theta}\right) = \frac{\partial(x, y)}{\partial(r, \theta)}$

$$= \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} \quad \text{— (3)}$$

Now, partially differentiate eqn (1) & (2) wrt $r$ & $\theta$ we get

$\partial x/\partial r = (1)\cos\theta$ $\qquad$ $\partial y/\partial r = (1)\sin\theta$

$\partial x/\partial\theta = -r\sin\theta$ $\qquad$ $\partial y/\partial\theta = r\cos\theta$

Put the values in eq (3), we get

L.H.S.

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= (\cos\theta)\, r\cos\theta + (\sin\theta)\, r\sin\theta$

$= r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta)$

$= r = \text{R.H.S.}$

Hence, proved

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Q. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$;

then find $\frac{\partial(u, v, w)}{\partial(x, y, z)} = ?$

Sol → The given equation is

$$u = \frac{yz}{x} \; ; \; v = \frac{zx}{y} \; ; \; w = \frac{xy}{z}$$ Explicit function

Partially differentiate given eqn wrt $x, y$ & $z$; we get

$\frac{\partial u}{\partial x} = -\frac{yz}{x^2}$ | $\frac{\partial v}{\partial x} = z/y$ | $\frac{\partial w}{\partial x} = y/z$

$\frac{\partial u}{\partial y} = \frac{z}{x}$ | $\frac{\partial v}{\partial y} = -2x/y^2$ | $\frac{\partial w}{\partial y} = x/z$

$\frac{\partial u}{\partial z} = y/x$ | $\frac{\partial v}{\partial z} = x/y$ | $\frac{\partial w}{\partial z} = -xy/z^2$

→ As; we know that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} -yz/x^2 & z/x & y/x \\ z/y & -2x/y^2 & x/y \\ y/z & x/z & -xy/z^2 \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left[\left(\frac{-2x}{y^2}\right)\left(\frac{-xy}{z^2}\right) - \left(\frac{x}{z}\right)\left(\frac{x}{y}\right)\right] - \frac{z}{x}\left[\left(\frac{z}{y}\right)\left(\frac{-xy}{z^2}\right) - \left(\frac{y}{z}\right)\left(\frac{x}{y}\right)\right] +$$
$$\frac{y}{x}\left[\left(\frac{z}{y}\right)\left(\frac{x}{z}\right) - \left(\frac{y}{z}\right)\left(-\frac{2x}{y^2}\right)\right]$$

$$= -\frac{yz}{x^2}\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[-\frac{x}{z} - \frac{x}{z}\right] + \frac{y}{x}\left[\frac{x}{y} + \frac{x}{y}\right]$$

$$= -\frac{yz}{x^2}(0) - \frac{z}{x}\left[-2\frac{x}{z}\right] + \frac{y}{x}\left[2\frac{x}{y}\right]$$

$= 0 + 2 + 2$

$= 4$ Ans

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Q. If $x = r\cos\theta$ ; $y = r\sin\theta$ then evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$ & $\frac{\partial(r,\theta)}{\partial(x,y)}$ = ? & prove that

$$\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1.$$

Sol:- The given eqn are

$x = r\cos\theta$ ; $y = r\sin\theta$

$\frac{\partial(x,y)}{\partial(r,\theta)} = r$ (check previous lectu

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix}$$

$x = r\cos\theta$ $y = r\sin\theta$

(1) $dx/dy = \frac{1}{(dy/dx)}$

(2) $\frac{\partial r}{\partial x} \neq \frac{1}{(\partial x/\partial r)}$

(3) Numerator variable must alone

(4) Numerator variable must be function of denominator variable & its groups
(Means; $x = f(r,\theta)$
$y = f(r,\theta)$
$r = f(x,y)$
$r = f(y,x)$)

→ Now, $\frac{\partial r}{\partial x} \Rightarrow r = \underline{f(x,y)}$
↳ Required $r$, $x$ & $y$
↳ Not Required $\theta$
↳ Eliminate $\theta$ by Hook or Crook

→ Now from eq (1)
$x = r\cos\theta$
$x/r = \cos\theta$

Now from eq (2)
$y = r\sin\theta$
$y/r = \sin\theta$

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Now as we know that

$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$x^2 + y^2 = r^2$$

$$\boxed{r = \sqrt{x^2+y^2}}$$

$\Rightarrow$ Now,

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}\, 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}(2y)$$

$$= \frac{y}{\sqrt{x^2+y^2}}$$

$\Rightarrow$ Now; $\frac{\partial\theta}{\partial x} \Rightarrow \theta = f(x, y)$

$\frac{\partial\theta}{\partial y}$

Required $\theta$, $x$ & $y$

Not required

by dividing eq (2) by eq (1); we get

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$

$$\Rightarrow \frac{y}{x} = \tan\theta$$

$$\Rightarrow \theta = \tan^{-1}(y/x)$$

Now; $$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2}$$

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$$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

Now

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$\Rightarrow \left(\frac{x}{\sqrt{x^2+y^2}}\right)\left(\frac{x}{x^2+y^2}\right) - \left(\frac{-y}{x^2+y^2}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\Rightarrow \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\Rightarrow \frac{(x^2+y^2)^1}{(x^2+y^2)^{3/2}}$$

$$\Rightarrow \frac{1}{(x^2+y^2)^{1/2}} \Rightarrow \frac{1}{(r^2)^{1/2}} \Rightarrow \frac{1}{r}$$

$\Rightarrow$ Now

LHS

$$\Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)}$$

$$\Rightarrow r \times \frac{1}{r} \Rightarrow 1$$

R.H.S

$\Rightarrow$ Hence, Proved.

# ODE

## Exact Differentials



**Definition** ⇒
A first order ODE is called exact if it can be derived from its premitive (solution).

**Necessary & sufficient condition** ⇒ A differential equation of the form $Mdx + Ndy = 0$ ; where $M = M(x, y)$, $N = N(x, y)$ is called exact DE if & only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

**Complete Solution** : The complete solution of exact differential equation $Mdx + Ndy = 0$ is given by :

$$\underset{(y\text{-as constant})}{\int M\,dx} + \underset{(\text{free from } x \text{ term})}{\int N\,dy} = C$$

**Q.1**→ Solve: $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Sol→ $(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$

$M = y\cos x + \sin y + y$ ; $N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$ ; $\frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int M\,dx + \underset{(\text{without } x)}{\int N\,dy} = C$$

$$\int (y\cos x + \sin y + y)\,dx + \int 0\,dy = c$$

$\boxed{y\sin x + x\sin y + xy = C}$ Ans

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Q.2 → Solve: $x\,dx + y\,dy = a^2\left(\dfrac{x\,dy - y\,dx}{x^2+y^2}\right)$

Sol →

$$x\,dx + y\,dy = \frac{a^2 x}{x^2+y^2}\,dy - \frac{a^2 y}{x^2+y^2}\,dx$$

$$\left(x + \frac{a^2 y}{x^2+y^2}\right)dx + \left(y - \frac{a^2 x}{x^2+y^2}\right)dy = 0$$

$$M = x + \frac{a^2 y}{x^2+y^2} \;;\; N = y - \frac{a^2 x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = a^2\left[\frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}\right] = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = -a^2\left[\frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2}\right]$$

$$= a^2\frac{(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int\left(x + \frac{a^2 y}{x^2+y^2}\right)dx + \int y\,dy = C$$

$$\frac{x^2}{2} + a^2 y\left[\frac{1}{y}\tan^{-1}\frac{x}{y}\right] + \frac{y^2}{2} = C$$

$$\frac{x^2}{2} + a^2\tan^{-1}\frac{x}{y} + \frac{y^2}{2} = C \quad \text{Ans}$$

Q.3 → Solve: $(2x - y + 1)\,dx + (2y - x - 1)\,dy = 0$

Sol → $M = 2x - y + 1 \;;\; N = (2y - x - 1)$

$$\frac{\partial M}{\partial y} = -1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial N}{\partial x} = 1$$

$$\int(2x - y + 1)\,dx + \int(2y - 1)\,dy = C$$

$$x^2 - xy + x + y^2 - y = C \quad \text{Ans}$$

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Q.4 Solve:

$$\left\{ y\left(1+\frac{1}{x}\right) + \cos y \right\} dx + (x + \log x - x\sin y)\, dy = 0$$

Sol. $M = y\left(1+\frac{1}{x}\right) + \cos y$

$N = x + \log x - x\sin y$

$\left(\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y\right)$ ; $\left(\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y\right)$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$\int M\, dx + \underset{(y \text{ wala})}{\int N\, dy} \Rightarrow \int \left(y\left(1+\frac{1}{x}\right) + \cos y\right) dx + \int 0\, dy$

$\underline{y[x + \log x] + x\cos y = C}$ Ans

Q5 $(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$

Sol. $M = 1 + e^{x/y}$ ; $N = e^{x/y}\left(1 - \frac{x}{y}\right)$

$\frac{\partial M}{\partial y} = -\frac{x}{y^2} e^{x/y}$ ; $\frac{\partial N}{\partial x} = -\frac{x}{y} e^{x/y}$

$$\left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right]$$

$\int M\, dx + \underset{(y \text{ wala term})}{\int N\, dy} \Rightarrow \int (1 + e^{x/y})\, dx + \int 0\, dy = C$

$\Rightarrow \boxed{x + ye^{x/y} = C}$ Ans

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## Remark :

If DE $Mdx + Ndy = 0$ is not exact then by multiplying a function $f(x, y)$ which called Integrating factor. The given DE can be ~~in~~ reduced into exact DE.

Some Rules for Integrating factor:

**Rule 1** By inspection method

1) $x\,dy + y\,dx = d(xy)$

2) $x\,dy - y\,dx \Rightarrow IF = \frac{1}{x^2}$

3) $y\,dx - x\,dy \Rightarrow IF = \frac{1}{y^2}$

**Rule 2** In Differential equation;
$Mdx + Ndy = 0$; $Mx + Ny \neq 0$ & M and N are both homogeneous then

$$\boxed{IF = \frac{1}{Mx + Ny}}$$

Remark: If $Mx + Ny = 0$

$\Rightarrow Mx = -Ny$

$\Rightarrow \frac{M}{N} = -\frac{y}{x}$

$\Rightarrow Mdx + Ndy = 0$

$\Rightarrow \frac{M}{N}dx + dy = 0$

$\Rightarrow -\frac{y}{x}dx + dy = 0$

$\Rightarrow -\frac{dx}{x} + \frac{dy}{y} = 0$

Integrating

$-\log x + \log y = \log c$

$\log\left(\frac{y}{x}\right) = \log c$

$$\boxed{y = cx}$$

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Q.1 → Find the Integrating ~~Factors~~ Factor of

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$$ & solve it.

Sol → $M = x^2y - 2xy^2$
$N = -(x^3 - 3x^2y)$

$$\left(\frac{\partial M}{\partial y} = x^2 - 4xy\right) \neq \left(\frac{\partial N}{\partial x} = -(+3x^2 - 6xy)\right)$$

$$IF = \frac{1}{Mx + Ny}$$

$Mx + Ny = \cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x$
$\Rightarrow x^2y^2$

$$\boxed{IF = \frac{1}{x^2y^2}}$$

$\Rightarrow IF(Mdx + Ndy) = 0 \times IF$

$\Rightarrow \frac{1}{x^2y^2}\left[(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy\right] = 0$

$\Rightarrow \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$

$\Rightarrow M' = \frac{1}{y} - \frac{2}{x}$ ; $N' = -\left(\frac{x}{y^2} - \frac{3}{y}\right)$

$\Rightarrow \left(\frac{\partial M'}{\partial y} = \cancel{x} - \frac{1}{y^2}\right) = \left(\frac{\partial N'}{\partial x} = -\frac{1}{y^2}\right)$

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = 0$$

$$\boxed{\frac{x}{y} - 2\log x + 3\log y = C}$$

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Rule 3 → If the equation $M dx + N dy = 0$ has the form $f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$ and ~~Mx+~~ $Mx - Ny \neq 0$ then the I.F. $= \dfrac{1}{Mx - Ny}$

<u>Remark</u>: If $Mx - Ny = 0 \Rightarrow \dfrac{M}{N} = y/x$

$\Rightarrow M dx + N dy = 0$

$\Rightarrow \dfrac{M}{N} dx + dy = 0$

$\Rightarrow \dfrac{y}{x} dx + dy = 0$

$\Rightarrow \dfrac{dx}{x} + \dfrac{dy}{y} = 0$

(integrating)

$\Rightarrow \log x + \log y = \log C$

$\Rightarrow \log(xy) = \log C$

$\Rightarrow \boxed{xy = C}$

Q.2 → $(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0$

Sol → $M = y + xy^2$; $N = x - x^2 y$

$\Rightarrow \left(\dfrac{\partial M}{\partial y} = 1 + 2xy\right) \neq \left(\dfrac{\partial N}{\partial x} = 1 - 2xy\right)$

$\Rightarrow Mx - Ny = xy + x^2y^2 - (xy - x^2y^2)$

$\Rightarrow \cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2$

$\Rightarrow 2x^2y^2$

$\Rightarrow$ I.F. $= \dfrac{1}{Mx - Ny} = \dfrac{1}{2x^2y^2}$

$\dfrac{1}{2x^2y^2}\left[(y + xy^2)dx + (x - x^2y)dy\right] = 0$

$\left(\dfrac{1}{x^2y} + \dfrac{1}{x}\right)dx + \left(\dfrac{1}{xy^2} - \dfrac{1}{y}\right)dy = 0$

$\left[\dfrac{\partial M'}{\partial y} = \dfrac{\partial N'}{\partial x} = -\dfrac{1}{x^2y^2}\right]$

$\int \left(\dfrac{1}{x^2y} + \dfrac{1}{x}\right)dx + \int -\dfrac{1}{y}\,dy = C$

$\Rightarrow \dfrac{1}{y}\left(\dfrac{x^{-2+1}}{-2+1}\right) + \log x - \log y = c$

$\Rightarrow \boxed{-\dfrac{1}{xy} + \log x - \log y = c}$

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Q.3→ Solve:

$$(xy\sin xy + \cos xy)\, y\, dx + (xy\sin xy - \cos xy)\, x\, dy = 0$$

Sol→ $M = xy^2\sin xy + y\cos xy$ ; $N = x^2y\sin xy - x\cos xy$

$$\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right]$$

$$IF = \frac{1}{Mx - Ny}$$

$$Mx - Ny = \cancel{x^2y^2\sin xy} + xy\cos xy - \cancel{x^2y^2\sin xy} + xy\cos xy$$

$$\Rightarrow 2xy\cos xy$$

$$I.F. = \frac{1}{2xy\cos xy}$$

$$\frac{1}{2xy\cos xy}\left[(xy^2\sin xy + y\cos xy)\, dx + (x^2y^2\sin xy - x\cos xy)\, dy\right] =$$

$$\left(y\tan xy + \frac{1}{x}\right)dx + \left(x\tan xy - \frac{1}{y}\right)dy = 0$$

$$\frac{\partial M'}{\partial y} = \cancel{\tan xy + y\sec^2 xy\, x} \quad \tan xy + y\,x\sec^2 xy = \frac{\partial N'}{\partial x}$$

$$\int\left(y\tan xy + \frac{1}{x}\right)dx + \int -\frac{1}{y}\, dy = C$$

$$y\,\frac{\log\sec xy}{y} + \log x - \log y = C$$

$$\boxed{\log\sec xy + \log x - \log y = C}$$

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ule 4 → If in $Mdx + Ndy = 0$

$$\underbrace{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \underline{f(x)}}$$ [function of x only]

then $I.F. = e^{\int f(x)dx}$

Rule - 5 → If in DE $Mdx + Ndy = 0$

$$\underbrace{\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \underline{f(y)}}$$ [function of y only]

then

$\Rightarrow I.F. = e^{\int f(y)dy}$

Q.4 → Solve:

$(x^2+y^2+x)dx + xy\,dy = 0$

Sol → $M = x^2+y^2+x$ ; $N = xy$

$\left(\frac{\partial M}{\partial y} = 2y\right) \neq \left(\frac{\partial N}{\partial x} = y\right)$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y - y) = \frac{y}{xy} = \frac{1}{x}$ → $f(x)$

$I.F. = e^{\int \frac{1}{x}dx} = e^{\log_e x} = x$

$\Rightarrow x\left[(x^2+y^2+x)dx + xy\,dy\right] = 0$

$\Rightarrow (x^3+xy^2+x^2)dx + x^2y\,dy = 0$

$\frac{\partial M'}{\partial y} = 2xy = \frac{\partial N'}{\partial x}$

$\Rightarrow \int(x^3+xy^2+x^2)dx + \int 0\,dy = C$

$$\Rightarrow \boxed{\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C}$$

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Q.5 Solve:

$$(xy^2 - e^{1/x^3})dx - x^2 y\,dy = 0$$

Sol. $M = xy^2 - e^{1/x^3}$ ; $N = -x^2 y$

$$\left(\frac{\partial M}{\partial y} = 2xy\right) \neq \left(\frac{\partial N}{\partial x} = -2xy\right)$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2xy + 2xy}{-x^2 y} = \frac{4xy}{-x\,xy} \Rightarrow -4/x$$

$$I.F. = e^{-\int 4/x\,dx} = e^{-4\log_e x} \Rightarrow \frac{1}{x^4}$$

$$\Rightarrow \frac{1}{x^4}\left[(xy^2 - e^{1/x^3})dx - x^2 y\,dy\right] = 0$$

$$\Rightarrow \left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx - y/x^2\,dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{2y}{x^3} = \frac{\partial N'}{\partial x}$$

$$\int\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx + \int 0\,dy = C$$

$$\frac{1}{x^3} = t$$

$$\frac{-3}{x^4}dx = dt$$

$$\frac{1}{x^4}dx = \frac{-dt}{3}$$

$$\Rightarrow y^2\left(\frac{x^{-2}}{-2}\right) + \int \frac{e^t\,dt}{3} = C$$

$$\Rightarrow \frac{-y^2}{2x^2} + \frac{e^{1/x^3}}{3} = C$$ Ans

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Q.6- Solve:

$(2xy^2-2y)dx + (3x^2y-4x)dy=0$

Sol: $M = 2xy^2 - 2y$ ; $N = 3x^2y - 4x$

$\left(\frac{\partial M}{\partial y} = 4xy-2\right) \neq \left(\frac{\partial N}{\partial x} = 6xy-4\right)$

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{\cancel{2xy-2}}{y(\cancel{2xy-2})} = \frac{1}{y}$

$\Rightarrow$ I.F. $= e^{\int \frac{1}{y}dy} = y$

$\Rightarrow y\left[(2xy^2-2y)\,dx + (3x^2y-4x)\,dy\right] = 0$

$\Rightarrow (2xy^3 - 2y^2)\,dx + (3x^2y^2 - 4xy)\,dy = 0$

$\Rightarrow \frac{\partial M'}{\partial y} = 6xy^2 - 4y = \frac{\partial N'}{\partial x}$

$\Rightarrow \int (2xy^3 - 2y^2)\,dx + \int 0\,dy = C$

$\Rightarrow \boxed{y^3x^2 - 2xy^2 = C}$ Ans

Special Question
Exact differential Eq

Concept: Integrating factor for the equation of the type: $x^a y^b(my\,dx + nx\,dy) + x^{a'}y^{b'}(m'y\,dx + n'x\,dy) = 0$
an integrating factor is $x^h y^k$
where

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

# Maxima & Minima in 2 variables

## Lec-1

⇒ Steps to be followed

$f(x, y) \Rightarrow$ given

⇓

$\frac{\partial f}{\partial x} = ?$ ; $\frac{\partial f}{\partial y} = ?$

⇓

Put $\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$ → VTU, vellore-2014

⇓ solve the eqn

$\boxed{\text{Values of } x \ \& \ y = ?}$ → · Points of maxima & minima · stationary values / points · Critical values / points · Points of inflexion · Extreme Values

difficult

⇓

$\frac{\partial^2 f}{\partial x^2} = r = ?$ ; $\frac{\partial^2 f}{\partial y^2} = t = ?$ ; $\frac{\partial^2 f}{\partial x \partial y} = s = ?$

⇓

find $rt - s^2 = ?$

⇓

Put the points $(x, y)$ in $rt - s^2$; one by by one

⇓

Value of $rt - s^2$

- −ve ($<0$) → Neither Maxima nor Minima (Saddle Point) ← VTU, vellore 2013
- $= 0$ → Further investigation Required ⇓ $f(a+h, b+k) - f(a, b) = ?$
- +ve ($>0$) → Put point in $r = ?$
  - −ve → Point of maxima ⇓ Maximum value from $f(x, y)$ ← vtu vellore 2012 2014
  - +ve → Point of minima ⇓ Minimum value from $f(x, y)$

Lec-2

Q. Find the maximum & minimum values of the function
$$x^3 + y^3 - 3axy ; a>0$$

√ Imp

Sol→ The given equation is
Let $\boxed{f(x,y) = x^3 + y^3 - 3axy}$ — (1)

Partially differentiate eq (1) wrt x & y ; we get

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay \quad \text{—— (2)}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax \quad \text{—— (3)}$$

⇒ For Stationary values

| | |
|---|---|
| Put $\frac{\partial f}{\partial x} = 0$ | $\frac{\partial f}{\partial y} = 0$ |
| $3x^2 - 3ay = 0$ | $3y^2 - 3ax = 0$ |
| $x^2 - ay = 0$ | $y^2 - ax = 0$ |
| $\boxed{x^2 = ay}$ — (4) | $\boxed{y^2 = ax}$ — (5) |

from eq. (4)
$$x^2 = ay$$
$$\boxed{y = x^2/a}$$

Put the value of y in eq (5) we get
$$\left(\frac{x^2}{a}\right)^2 = ax$$
$$\frac{x^4}{a^2} = ax$$
$$x^4 = a^3x$$
$$x^4 - a^3x = 0$$
$$x(x^3 - a^3) = 0$$

| | |
|---|---|
| $\boxed{x = 0}$ | $x^3 - a^3 = 0$ |
| | $x^3 = a^3$ |
| | $\boxed{x = a}$ |

GOOD WRITE

⇒ Put the values of $x$ in eq (4), we get

| $x = 0$ | $x = a$ |
|---|---|
| $(0)^2 = ay$ | $(a)^2 = ay$ |
| $0 = ay$ | $y = a$ |
| $y = 0$ | |
| $(0,0)$ | $(a,a)$ |

So, the stationary values are

$$\boxed{(0,0) \ \& \ (a, a)}$$

⇒ Now, partially differentiate equation (2) & (3) further

$$\frac{\partial^2 f}{\partial x^2} = 6x = r$$

$$\frac{\partial^2 f}{\partial y^2} = 6y = t$$

$$\frac{\partial^2 f}{\partial x \partial y} = -3a = s$$

⇒ Now, $rt - s^2$

$$\Rightarrow (6x)(6y) - (-3a)^2$$

$$rt - s^2 \Rightarrow 36xy - 9a^2$$

⇒ Now at point $(0,0)$

$$rt - s^2 = 36(0)(0) - 9(a^2)$$

$$\Rightarrow -9a^2 < 0$$

Neither maxima nor minima
↳ saddle point

So, $(0,0)$ is a saddle point.

$\Rightarrow$ Now at point $(a, a)$

$rt - s^2 = 36(a)(a) - 9a^2 \Rightarrow 27a^2 > 0$

$\Rightarrow$ Value of $r = ?$

$r = 6x$

$r = 6(a) > 0$

(given $a > 0$)

So, Point of minima is $(a, a)$

Minimum value

Put value of minima in $f(x, y)$

$f(a, a) = (a)^3 + (a)^3 - 3a(a)(a)$

$\Rightarrow a^3 + a^3 - 3a^3 \Rightarrow -a^3$ Ans

▫ Lec – 3

(Discuss further investigation wala locha)

M.M.Imp.

Q. Discuss the maxima and minima of $f(x, y) = x^3y^2(1-x-y)$.

Sol→ The given Equation is :-

$f(x, y) = x^3y^2(1-x-y)$

$f(x, y) = x^3y^2 - x^4y^2 - x^3y^3$ — (1)

Partially differentiate eq (1) wrt $x$ & $y$, we get

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ — (2)

$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$ — (3)

$$\frac{\partial f}{\partial x} = x^2y^2(3-4x-3y)$$

$$\frac{\partial f}{\partial y} = x^3y(2-2x-3y)$$

Now, for critical value

$\Rightarrow$ Put $\frac{\partial f}{\partial x} = 0$

$\Rightarrow x^2y^2(3-4x-3y) = 0$

$\Rightarrow x=0, y=0, 3-4x-3y=0$

$\Rightarrow \boxed{x=0, y=0, 4x+3y=3}$

$\frac{\partial f}{\partial y} = 0$

$\Rightarrow x^3y(2-2x-3y) = 0$

$\Rightarrow x=0, y=0, 2-2x-3y=0$

$\Rightarrow \boxed{x=0, y=0, 2x+3y=2}$

Now on solving linear equation

$$\begin{array}{r} 4x+3y=3 \\ -\;2x+3y=2 \\ (-)\;(-)\;\;(-) \\ \hline 2x=1 \end{array}$$

$$\boxed{x = 1/2}$$

$\Rightarrow$ Put the value of $x$ in any linear equation we get

$$4(1/2)+3y=3$$

$$2+3y=3$$

$$\boxed{y = 1/3}$$

$\Rightarrow$ So, critical values are $(0,0)$ & $\left(\frac{1}{2}, \frac{1}{3}\right)$.

$\Rightarrow$ Now, partially differentiate (2) & (3) wrt $x$ & $y$, we get

$\Rightarrow \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$\boxed{r = 6xy^2(1 - 2x - y)}$

$\Rightarrow \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3 y$

$\boxed{t = 2x^3(1 - x - 3y)}$

$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = 6x^2 y - 8x^3 y - 9x^2 y^2$

$\Rightarrow \boxed{s = x^2 y(6 - 8x - 9y)}$

$\Rightarrow$ Now at point $(1/2, 1/3)$

$\Rightarrow r = 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2\left[1 - 2\left(\frac{1}{2}\right) - \frac{1}{3}\right]$

$\Rightarrow \cancel{3}^1 \times \frac{1}{\cancel{9}_3} \times \left(1 - 1 - \frac{1}{3}\right) \Rightarrow -1/9$

$\Rightarrow \boxed{r = -1/9}$

$\Rightarrow t = 2 \times \left(\frac{1}{2}\right)^3\left[1 - \frac{1}{2} - \cancel{3}^1\left(\frac{1}{\cancel{3}_1}\right)\right]$

$\Rightarrow \cancel{2}^1 \times \frac{1}{\cancel{8}_4}\left(-\frac{1}{2}\right) \Rightarrow -\frac{1}{8}$

$\Rightarrow \boxed{t = -1/8}$

$\Rightarrow s = \left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)\left[6 - 8\left(\frac{1}{2}\right) - 9\left(\frac{1}{3}\right)\right]$

$\Rightarrow \frac{1}{4} \times \frac{1}{3}[6 - 4 - 3]$

$\Rightarrow -\frac{1}{12} \Rightarrow \boxed{s = -1/12}$

Now
$\Rightarrow rt - s^2 = \left(\frac{-1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2$

$\Rightarrow \frac{1}{72} - \frac{1}{144} \Rightarrow \frac{1}{144} > 0$

So, as $rt - s^2 > 0$
& $r = -\frac{1}{9}$ which is $< 0$

$\Rightarrow$ So, $\left(\frac{1}{2}, \frac{1}{3}\right)$ is point of maxima.

$\Rightarrow$ Now at point $(0, 0)$

$$\boxed{\begin{matrix} r = 0 \\ t = 0 \\ s = 0 \end{matrix}}$$

$\Rightarrow rt - s^2 = 0 - 0 = 0$
$\Rightarrow$ So, $\boxed{rt - s^2 = 0}$; that's why further investigation required.

$\Rightarrow$ Now,
$f(a+h, b+k) - f(a, b)$ — (4)

where $h$ & $k$ are very small quantities.
& $(a, b)$ is the point where further investigation is required.

$\Rightarrow$ Here; $(a, b) = (0, 0)$
$\Rightarrow$ Put in eq (4); we get
$\Rightarrow f(0+h, 0+k) - f(0, 0)$
$\Rightarrow f(h, k) - f(0, 0)$
$\Rightarrow [h^3k^2(1-h-k)] - 0$
$\Rightarrow \boxed{h^3k^2 - h^4k^2 - h^3k^3}$

$\Rightarrow h^3k^2$ (ignoring high powers)

⇒ Now on changing sign of h the value of $f(a+h, b+k) - f(a, b)$ changes

Concept: $h^2 k^2 > 0$ ⇒ Point of minima
$-h^2 k^2 < 0$ ⇒ Point of maxima

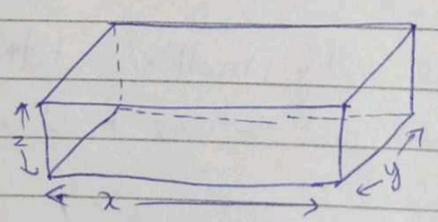
$h^3 k^2 > 0$ & $< 0$
↳ Saddle Point (Neither maxima nor minima)

⇒ So; $(0,0)$ is saddle point.

## Lec-4

Q. A Rectangular box opens at top is to have volume of 32 cubic meter. Find the dimensions of box that require the least material for the construction.

Sol→ Let the dimensions length, breadth & height of rectangular box be $x, y$ & $z$.




⇒ As, we know that;

Volume of cuboid $= lbh$

$32 = xyz$ (Given)

So, $xyz = 32$ —①
(Given)

⇒ Total surface area of cuboid (open at top) $= lb + 2bh + 2hl$

$$\boxed{S = xy + 2yz + 2zx} \quad -(2)$$

$\Rightarrow$ Now, from eq (1)

$\boxed{z = \frac{32}{xy}}$ Put the value of $z$ in eq (2)

we get

$$S = xy + 2y\left(\frac{32}{xy}\right) + 2\left(\frac{32}{xy}\right)x$$

$$\boxed{S = xy + \frac{64}{x} + \frac{64}{y}} \quad -(3)$$

$\Rightarrow$ Now partially differentiate eq (3) wrt $x$ & $y$, we get.

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} + 0 \qquad \partial S/\partial y = x + 0 - 64/y^2$$

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} \quad -(6) \qquad \partial S/\partial y = x - 64/y^2 \quad -(7)$$

$\Rightarrow$ for Critical values

Put

$\partial S/\partial x = 0$

$y - \frac{64}{x^2} = 0$

$y = 64/x^2$

$\boxed{x^2 y = 64}$ $-(4)$

$\partial S/\partial y = 0$

$x - 64/y^2 = 0$

$\boxed{xy^2 = 64}$ $-(5)$

from eq (4) $y = 64/x^2$

Put in eq (5), we get

$$x\left(\frac{64}{x^2}\right)^2 = 64$$

$$\Rightarrow x\left(\frac{64\times 64}{x^4}\right) = 64$$

$$64x = x^4$$

$$x^4 - 64x = 0$$

$$x(x^3-64) = 0$$

$\boxed{x = 0}$ (not Possible)

$$x^3 - 64 = 0$$

$$x^3 = 4^3$$

$$\boxed{x = 4}$$

⇒ Put the value of $x$ in eq (4)

$$(4)^2 y = 64$$

$$16y = 64$$

$$\underline{y = 4}$$

⇒ So, critical value is $(4, 4)$.

⇒ Now partially differentiate eq (3) & (7) wrt $x$ & $y$, we get

$$\frac{d^2S}{dx^2} = 0 - 64\left(\frac{-2}{x^3}\right) \Rightarrow \boxed{\frac{128}{x^3} = r}$$

$$\frac{d^2S}{dy^2} = 0 - 64\left(\frac{-2}{y^3}\right) \Rightarrow \boxed{\frac{128}{y^3} = t}$$

$$\frac{d^2S}{dxdy} = 1 - 0 \Rightarrow \boxed{1 = s}$$

⇒ Now; $rt - s^2$

$$\left(\frac{128}{x^3}\right)\left(\frac{128}{y^3}\right) - (1)^2$$

$$\Rightarrow \boxed{rt - s^2 = \frac{128\times 128}{x^3y^3} - 1}$$

$\Rightarrow$ at point $(4, 4)$

$$rt - s^2 = \frac{128 \times 128}{4^3 \times 4^3} - 1$$

$$\Rightarrow \frac{128 \times 128}{64 \times 64} - 1$$

$$\Rightarrow 4 - 1 \Rightarrow 3$$

$\Rightarrow rt - s^2 = 3$

$\Rightarrow$ So, $rt - s^2 > 0$

& $r = \frac{128}{x^3} \Rightarrow \frac{128}{4^3} \Rightarrow \frac{128}{64} \Rightarrow 2$

So, $r = 2$ which is $> 0$

$\hookrightarrow$ So, $(4, 4)$ is the point of minima.

$\Rightarrow$ Now from eq (1)

$$z = \frac{32}{xy} = \frac{32}{4 \times 4} \Rightarrow 2$$

$\boxed{z = 2}$

$\Rightarrow$ So, dimensions of rectangular box ~~for least~~ ~~meters~~

length $= x = 4m$
breadth $= y = 4m$
height $= z = 2m$ } Ans

Topic :
Lagrange's Method of undetermined multipliers
or
Lagrange's method of maxima & minima
or
Lagrange's method
or
Lagrange's multipliers method

Lec - 1

Definition

Mainly used to find maximum & minimum value (optimize) of a function of several (2 or more) variables, in which all the variables are not independent but connected by some given relation(s).

→ Condition
→ Restriction
→ Constraint

---

Maxima & Minima of 2 variables

⇓

All stationary points
+
Given Constraints

⇓

Feasible Stationary Points

Constrained Optimization

## Working Rule

Let $f(x, y, z)$ be an objective function of 3 variables $x, y, z$ and variables are connected by the constraint $\phi(x, y, z) = 0$

max / min

**Step 1:** Identify the objective function $f(x, y, z)$

**Step 2:** Identify the constrained function $\phi(x, y, z) = 0$

**Step 3:** Find Lagrangian (Auxiliary) function (Expression)

[VTU, Vellore - 2009]

$$\boxed{F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)} \quad -①$$

Here:

$\lambda \rightarrow$ Non zero $\rightarrow$ Lagrange's Multiplier

**Step 4:** Partially differentiate eq ① wrt $x, y$ & $z$ and obtain the equations $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ & $\frac{\partial F}{\partial z} = 0$.

[Necessary Conditions (Lucknow university - 2011, 2012)]

**Step 5:** Solve the above 3 equations together with $\phi(x, y, z) = 0$ to find the stationary points.

* **Drawback:** We cannot determine the nature of stationary points by Lagrange's Method

(VTU, Vellore - 2011)

Step 6 :- Some alternate methods are found to find the nature of stationary point(s).

(1) By Physical considerations
(2) As discussed in lecture #4
(3) Bounded Hessian Matrix

$$1 \to x \quad 2 \to y \quad 3 \to z$$

Put
$$\begin{vmatrix} L_{11}-k & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22}-k & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33}-k & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix}_{4\times 4} = 0$$

Here :-

$L_{11} = \frac{\partial^2 f}{\partial x^2}$ ; $g_1 = \partial \phi / \partial x$

$L_{12} = \frac{\partial^2 F}{\partial x \partial y}$ ; $g_2 = \partial \phi / \partial y$

$L_{13} = \frac{\partial^2 f}{\partial x \partial z}$ ; $g_3 = \partial \phi / \partial z$

$\vdots \qquad \vdots \qquad \vdots$

and so on.

On solving the determinant or by using determinant property at stationary point :- if $k<0 \Rightarrow$ Point of maxima
$k > 0 \Rightarrow$ Point of minima

Step 7 :- Find maximum or Minimum value, put point of maxima or point of minima in function $f(x, y, z)$ respectively.

Rajesh sir explained

$$f(x,y,z)=0$$

subject to condt $\phi(x,y,z)=c$

find stationary at

$$\frac{\partial f}{\partial x}+\lambda\,\partial\phi/\partial x=0$$

$$\frac{\partial f}{\partial y}+\lambda\,\partial\phi/\partial y=0$$

$$\partial f/\partial z+\lambda\,\partial\phi/\partial z=0$$

Q→ Divide a number 'b' into 3 parts, such that their product will be maximum.

[LUCKNOW University 2015]

Sol→ Let the 3 parts be $x$, $y$ and $z$.

Now, according to the ques.

Given: $\boxed{x+y+z=b}$ — ①

$$x+y+z-b=0$$

& their product is given by :-

$\boxed{P=xyz}$ should be maximum (mentioned in ques)

Here → Objective function :-

$$f(x,y,z)=xyz$$

Constrained function

$$\boxed{\phi(x,y,z)=x+y+z-b}$$

Now; by Lagrange's Method of undetermined multipliers -

Lagrangian's function can be written as -

$$F(x,y,z) = f(x,y,z) + \lambda \, \phi(x,y,z)$$

$$\boxed{F(x,y,z) = xyz + \lambda\,(x+y+z-b)} \quad -(2)$$

$\Rightarrow$ Partially differentiate eq (2) wrt $x, y$ & $z$; we get :-

$$\frac{\partial F}{\partial x} = yz + \lambda$$

$$\frac{\partial F}{\partial y} = xz + \lambda$$

$$\frac{\partial F}{\partial z} = xy + \lambda$$

$\Rightarrow$ Put the terms $= 0$; we get

$$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda = 0 \Rightarrow \lambda = -yz \quad -(3)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow xz + \lambda = 0 \Rightarrow \lambda = -xz \quad -(4)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow xy + \lambda = 0 \Rightarrow \lambda = -xy \quad -(5)$$

From eq (3) & (4)

$$-yz = -xz \Rightarrow \boxed{x = y} \quad -(6)$$

From eq (4) & (5)

$$\cancel{-xy}\ -xz = -xy \Rightarrow \boxed{y = z} \quad -(7)$$

From eq (6) & (7); we get

$$\boxed{x = y = z}$$

Put $x = y = z$ in eq (1); we get

$$x + x + x = b$$

$$3x = b$$

$$x = b/3 = y = z$$

$\Rightarrow$ So, $\left(\frac{b}{3}, \frac{b}{3}, \frac{b}{3}\right)$ is the stationary point.

By direct method

So, $(b/3, b/3, b/3)$ is the point of maxima.

So, max value $\rightarrow$

$$f(b/3, b/3, b/3) = \frac{b}{3} \times \frac{b}{3} \times \frac{b}{3} = \frac{b^3}{27}$$

Max value

(Imp)

Q. Divide 24 into three parts such that the product of first, square of second & cube of third is maximum.

Sol→ Let 24 is divided into three parts x, y & z.

So; that

$$x+y+z = 24$$

$$\boxed{x+y+z-24=0}$$

So; Constraint $\boxed{\phi(x,y,z) = x+y+z-24}$ — ①

As per given in question

Product $= xy^2z^3$

So;

Objective function $\boxed{f(x,y,z) = xy^2z^3}$ — ②

Now; as per Lagrange's method of multipliers:-

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$$

Lagrangian fn — Objective fn — Lagrange's Multiplier — Constraint

$$\boxed{F = xy^2z^3 + \lambda(x+y+z-24)}$$ ③

Now; partially differentiate eq ③ wrt x, y & z respectively; we get:

$$\frac{\partial F}{\partial x} = y^2z^3 + \lambda(1+0+0-0) \Rightarrow y^2z^3+\lambda$$

$$\frac{\partial f}{\partial y} = 2xyz^3 + \lambda(0+1+0-0) \Rightarrow 2xyz^3+\lambda$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2 + \lambda(0+0+1-0) \Rightarrow 3xy^2z^2+\lambda$$

Put $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ & $\frac{\partial f}{\partial z} = 0$

we get:

$y^2z^3 + \lambda = 0$ — (4) $\Rightarrow \lambda = -y^2z^3$

$2xyz^3 + \lambda = 0$ — (5) $\Rightarrow \lambda = -2xyz^3$

$3xy^2z^2 + \lambda = 0$ — (6) $\Rightarrow \lambda = -3xy^2z^2$

Now, from eq (4) & (6)

$-y^2z^3 = -2xyz^3$

$y^2z^3 - 2xyz^3 = 0$

$yz^3(y-2x) = 0$

$\boxed{y=0}$ $\boxed{z=0}$ | $\boxed{y=2x}$ — (7)

Not possible

Now from eq (5) & (6)

$-2xyz^3 = -3xy^2z^2$

$2xyz^3 - 3xy^2z^2 = 0$

$xyz^2(2z-3y) = 0$

$x=0$ | $y=0$ | $z=0$ | $2z-3y=0$

Not Possible $\boxed{2z=3y}$ — (8)

Now put the values of x & z from eq (7) & (8) in eq (1)

$x+y+z = 24$

$\frac{y}{2} + y + \frac{3}{2}y = 24$

$\Rightarrow 3y = 24$
$\Rightarrow \boxed{y = 8}$

$\Rightarrow$ Now from eq (7) | from eq (8)

$8 = 2x$ | $2z = 3(8)$

$\boxed{x = 4}$ | $z = 12$

$\Rightarrow$ So, $(4, 8, 12)$ is the stationary point.

By Direct Method

Point $(4, 8, 12)$ is the point of maxima.

So, Maximum value -
$$f(4, 8, 12) = (4)(8)^2(12)^3$$
$$\boxed{f(4, 8, 12) = 442368}$$

Maximum value.

So, first part $= 4$
second part $= 8$
third part $= 12$ } Ans

Imp

Q. Find the maximum and minimum distances of the point $(3, 4, 12)$ from the sphere $x^2+y^2+z^2=1$.

Sol. The given eqn of sphere is :

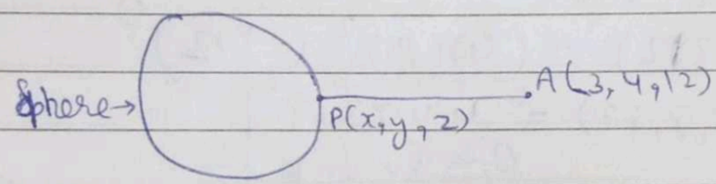
$$x^2+y^2+z^2=1 \quad -(1)$$

$$\boxed{x^2+y^2+z^2-1=0}$$

So, constraint is given as :

$$\boxed{\phi(x, y, z) = x^2+y^2+z^2-1} \quad -(2)$$

Now, as we know that :
Distance between Point $A(3, 4, 12)$ & point $P(x, y, z)$ On sphere is given :




By Distance formula :

$$AP = \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$$

$$\boxed{AP^2 = (x-3)^2+(y-4)^2+(z-12)^2}$$

So, now objective function is :

$$\boxed{f(x, y, z) = (x-3)^2+(y-4)^2+(z-12)^2}$$

Now by Lagrange's method of multipliers :

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

$$F = (x-3)^2+(y-4)^2+(z-12)^2+\lambda(x^2+y^2+z^2-1) \quad -(4)$$

GOOD WRITE

⇒ Now, partially differentiate eqn (4) wrt $x, y \& z$; we get:-

$$\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x$$

$$\frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y$$

$$\frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$$

Now; Put $\frac{\partial F}{\partial x} = 0$; $\frac{\partial F}{\partial y} = 0$ & $\frac{\partial F}{\partial z} = 0$

We get:-

$$2(x-3) + 2\lambda x = 0 \Rightarrow \boxed{x - 3 + \lambda x = 0} \quad -(5)$$

$$2(y-4) + 2\lambda y = 0 \Rightarrow \boxed{y - 4 + \lambda y = 0} \quad -(6)$$

$$2(z-12) + 2\lambda z = 0 \Rightarrow \boxed{z - 12 + \lambda z = 0} \quad -(7)$$

Now, from eq (5), (6) & (7), we get:-

$$\lambda = \frac{-(x-3)}{x} = \frac{-(y-4)}{y} = \frac{-(z-12)}{z}$$

$$= \pm \frac{\sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}}{\sqrt{x^2+y^2+z^2}} \quad \text{[Ratio's Property]}$$

So; $\lambda = \pm \frac{\sqrt{F}}{1} \Rightarrow \boxed{\lambda = \pm \sqrt{F}}$

from eq (1) & (3)

Put the value of $\lambda$ in eq (5) we get:-

$$x - 3 \pm \sqrt{F}\, x = 0$$

$$\Rightarrow x(1 \pm \sqrt{F}) = 3$$

$$\Rightarrow \boxed{x = \frac{3}{1 \pm \sqrt{F}}}$$

Similarly

$$y = \frac{4}{1 \pm \sqrt{f}}$$

$$z = \frac{12}{1 \pm \sqrt{f}}$$

Put the values of $x, y$ & $z$ in eqn (1) ; we get

$$\left(\frac{3}{1 \pm \sqrt{f}}\right)^2 + \left(\frac{4}{1 \pm \sqrt{f}}\right)^2 + \left(\frac{12}{1 \pm \sqrt{f}}\right)^2 = 1$$

$$\frac{9}{(1 \pm \sqrt{f})^2} + \frac{16}{(1 \pm \sqrt{f})^2} + \frac{144}{(1 \pm \sqrt{f})^2} = 1$$

$$9 + 16 + 144 = (1 \pm \sqrt{f})^2$$

$$\pm 13 = 1 \pm \sqrt{f}$$

$\Rightarrow$ Now ;

$13 = 1 + \sqrt{f} \Rightarrow \boxed{\sqrt{f} = 12} = AP$

$-13 = 1 + \sqrt{f} \Rightarrow \boxed{\sqrt{f} = -14}$ Not Possible — Because distance can't be -ive

$13 = 1 - \sqrt{f} \Rightarrow \boxed{\sqrt{f} = -12}$

$-13 = 1 - \sqrt{f} \Rightarrow \boxed{\sqrt{f} = 14} = AP$

So, $\sqrt{f} = 12$

$\boxed{AP = 12}$ Minimum distance

&

$\sqrt{f} = 14$

$\boxed{AP = 14}$ Maximum distance

Hence Proved

* We can also find Point of Maxima & Point of Minima as well.

# Leibnitz's Rule

(Differentiation under Integral sign)

Types of Ques
1. DUIS with constant limits & one parameter
2. DUIS with constant limits & two parameters
3. DUIS with parameter in limits
   - Put Leibnitz's Rule

* if $f(x, \alpha)$ is a function, having $\alpha$ as parameter & $f(x, \alpha)$ and $\frac{\partial}{\partial \alpha} f(x, \alpha)$ are continuous, then

$$\boxed{\frac{d}{d\alpha} \int_a^b f(x, \alpha)\,dx = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha)\,dx}$$

Here; limits a & b are constants.

⇒ Steps to be followed

1. Identify the parameter
   ⇓
2. Differentiate both sides wrt parameter
   ⇓
3. Apply Leibnitz's Rule
   ⇓
4. Simplify
   - Solve diffn. wrt Parameter
   - Integrate wrt the variable
   - Put the limits
   - Integrate wrt Parameter
   - Find the value of unknown

Most time consuming

GOOD WRITE

Q. By differentiation under Integral sign.
Evaluate: $\int_0^1 \frac{x^\alpha - 1}{\log x} dx$ given $\alpha \geq 0$.

Sol: The given Eq is $\int_0^1 \frac{x^\alpha - 1}{\log x} dx$

Here, $\alpha$ is the parameter.

Let

$$\boxed{F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x} dx} \quad -(1)$$

Differentiate both sides of eqn (1) wrt. parameter $\alpha$.

We get

$$\frac{d}{d\alpha} F(\alpha) = \frac{d}{d\alpha} \int_0^1 \frac{x^\alpha - 1}{\log x} dx$$

Now by Leibnitz's Rule

$$\frac{d}{d\alpha} F(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha}\left(\frac{x^\alpha - 1}{\log x}\right) dx$$

$$= \int_0^1 \frac{x^\alpha \log x - 0}{\log x} dx$$

$$\frac{d}{d\alpha} F(\alpha) = \int_0^1 x^\alpha dx$$

$$= \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 \Rightarrow \frac{1}{\alpha+1}\left[(1)^{\alpha+1} - (0)^{\alpha+1}\right]$$

$$\Rightarrow \frac{1}{\alpha+1}$$

$$\boxed{\frac{d}{d\alpha} F(\alpha) = \frac{1}{\alpha+1}} \quad -(2)$$

By integrating eqn (2) both side wrt parameter $\alpha$, we get

$$\int \frac{d}{d\alpha} F(\alpha) = \int \frac{1}{\alpha+1} d\alpha$$

$$F(\alpha) = \log|\alpha+1| + c \quad -(3)$$

Now, put $\alpha = 0$ in eq (3) we get

$$F(0) = \log(0+1) + c$$

$$\boxed{F(0) = c} \quad -(4)$$

Now put $\alpha = 0$ in eq (1) we get

$$F(0) = \int_0^1 \frac{x^0 - 1}{\log x} dx$$

$$F(0) = \int_0^1 \frac{1-1}{\log x} dx = 0$$

Put the value of $F(0)$ in eqn (4), we get

$$\boxed{0 = c}$$

Now put the value of $c$ in eqn (3); we get

$$F(\alpha) = \log(\alpha+1) + 0$$

$$\boxed{F(\alpha) = \log|\alpha+1|} \text{ Ans}$$

Q. Prove that
$$\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx = \tan^{-1}\left(\frac{1}{a}\right)$$
& hence show that
$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Sol: The given Eq is $\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx$

Here, a is the parameter

So, let $\boxed{f(a) = \int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx}$ — (1)

Differentiate eq (1) wrt parameter a ; we get
$$\frac{d}{da}F(a) = \frac{d}{da}\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx$$

By applying Leibnitz's Rule,
we get
$$\frac{d}{da}f(a) = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{e^{-ax}\sin x}{x}\right)dx$$
$$= \int_0^\infty \frac{\sin x}{x}\left[e^{-ax}(-x)\right]dx$$

$$\frac{d}{da}F(a) = -\int_0^\infty e^{-ax}\sin x\,dx$$
$$= -\left[\frac{e^{-ax}}{(-a)^2+(1)^2}(-a\sin x - \cos x)\right]_0^\infty$$
$$= \left[\frac{e^{-ax}}{a^2+1}(a\sin x + \cos x)\right]_0^\infty$$
$$= \left[\frac{e^{-\infty}}{a^2+1}(a\sin\infty + \cos\infty)\right] -$$

GOOD WRITE

$$\left[\frac{e^{0}}{a^2+1}(a\sin 0+\cos 0)\right]$$

$$= [0] - \left[\frac{1}{a^2+1}(0+1)\right]$$

$$\boxed{\frac{d}{da}F(a) = \frac{-1}{1+a^2}}$$

Integrating both sides wrt a we get

$$\int \frac{d}{da}F(a) = \int \frac{-1}{1+a^2}\,da$$

$$F(a) = \cot^{-1}(a) + c$$

$$F(a) = \tan^{-1}(1/a) + c \quad -(2)$$

$$\boxed{\cot^{-1} t = \tan^{-1}(1/t)}$$

$\Rightarrow$ Put $a=\infty$ in eq (2), we get

$$F(\infty) = \tan^{-1}(1/\infty) + c$$

$$F(\infty) = \tan^{-1}(0) + c$$

$$F(\infty) = \tan^{-1}(\tan 0) + c$$

$$\boxed{F(\infty) = c} \quad -(3)$$

Put $a=\infty$ in eq (1); we get

$$F(\infty) = \int_0^\infty \frac{e^{-\infty}\sin x}{x}\,dx$$

$$\boxed{F(\infty) = 0}$$

Now, put value of $F(\infty)$ in eq (3) we get

$$\boxed{0 = c}$$

$\Rightarrow$ Put the value of c in eq (3) ; we get

$$\boxed{F(a) = \tan^{-1}\left(\frac{1}{a}\right)}$$

~~Hence proved~~

So, $\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx = \tan^{-1}\left(\frac{1}{a}\right)$ — (4)

Hence, proved

Put $a = 0$; in eq (4), we get

$$\int_0^\infty \frac{e^0 \sin x}{x} = \tan^{-1}\left(\frac{1}{0}\right)$$

$$\int_0^\infty \frac{\sin x}{x} = \tan^{-1}(\infty)$$

$$\int_0^\infty \frac{\sin x}{x} = \tan^{-1}\left(\tan\frac{\pi}{2}\right)$$

So, $\boxed{\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}}$ Hence proved

⇒ Basic term in differential equation:

Let $y = f(x)$ be a function, then $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ ---

$\frac{d^n y}{dx^n}$ all are derivative of $y$.

Dependent & independent variables in DE

If $\frac{dy}{dx}$ is derivative in DE, then $y$ is called

dependent variable and $x$ is called independent variable.

⇒ Differential Equation

An equation involving dependent variables, independent variables and derivative of dependent variable is called an differential equation.

Eg 1 → $dy = (x + \sin x)\,dx$

Eg 2 → $\frac{d^3y}{dx^3} + \frac{dy}{dx} + y = e^x$

Eg 3 → $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

here equation (1) & (2) containing one dependent variable and one independent variable but equation (3) containing one dependent variable & 3 independent variable.

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## ⇒ Ordinary Differential Equation

A differential equation involving derivatives wrt a single independent variable is called an ordinary differential equation.
Example 1 & 2 are example of ODE.

## Partial Differential Equation

A differential equation involving partial derivatives wrt more than one independent variables is called PDE.
Example 3 is a partial differential equation

## ⇒ Order of Differential Equation

The order of the highest order derivative in a differential equation is called the order of the differential equation.

Eg-1 $$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 = e^x$$

Eg-2 $$\left(\frac{dy}{dx}\right)^3 + y = e^{2x}$$

Here eg-1 is of third order and eg-2 is of first order.

## ⇒ Degree of differential Equation

The degree of a differential equation is the degree of the highest derivative is called degree of differential equation.

In above eg-1 & eg-2 ; degree of differential equation is 1 & 3.

- **Result**: A differential equation contain following types of terms like $\log\left(\frac{dy}{dx}\right)$, $e^{(dy/dx)}$, $\sin\left(\frac{dy}{dx}\right)$, $\cos\left(\frac{dy}{dx}\right)$ then degree of this differential equation doesn't exist.

Eg-1 $\left(\frac{d^2y}{dx^2}\right)^2 + \sin\left(\frac{dy}{dx}\right) = 0$

then order $= 2$, but degree doesnot exist.

Eg-2 $\left(\frac{d^3y}{dx^3}\right) + \left(\frac{d^2y}{dx^2}\right)^2 + \log\left(\frac{dy}{dx}\right) = 0$

then order $= 3$, but degree doesn't exist.

Q.1 → The order & degree of differential equation $y = x\left(\frac{dy}{dx}\right) + a\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2}$

(A) 1 & 2 (C) 1 & 1
(B) 2 & 1 (D) 2 & 2

$\boxed{\text{Ans : A}}$

Q.2 → The order & degree of Differential Equation
$$\left\{y + x\left(\frac{dy}{dx}\right)^2\right\}^{7/3} = x\left(\frac{d^2y}{dx^2}\right)$$

(A) 3 & 2 (C) 2 & 2
(B) 2 & 3 (D) 3 & 3

$\boxed{\text{Ans: B}}$

# Formation of Differential Equation

Suppose we have family of curves containing $n$ arbitrary constants, then we can obtain an $n^{th}$ order differential equation whose soln is the given family.

Example:
A family of curves $y = e^{mx}$ containing one arbitrary constant.

- Working Rule

1) Write the equation of family of curves
2) Differential of the equation of (1).
3) Eliminate arbitrary constant from getting equation.
   Thus, we get the required DE.

Example 1

Let $y = e^{mx}$ — (1)

Diff. (1) $\frac{dy}{dx} = me^{mx}$ — (2)

Using (1) & (2)

$$\frac{dy}{dx} = my \Rightarrow m = \frac{1}{y}\left(\frac{dy}{dx}\right) \quad \text{— (3)}$$

Again from (1); $mx = \log y$

$$m = \frac{\log y}{x} \quad \text{— (4)}$$

Equ from (3) & (4)

$$\frac{dy}{dx} = \frac{y}{x}\log y \Rightarrow \boxed{\frac{dy}{dx} - \frac{y\log y}{x} = 0}$$

## Example -2

Find the differential equation which has $y = a\cos(mx+b)$ for its integral, where $a$ & $b$ are two arbitrary constant.

Sol→ Given that $y = a\cos(mx+b)$ — ①

$$\frac{dy}{dx} = -am\sin(mx+b) \quad \text{— ②}$$

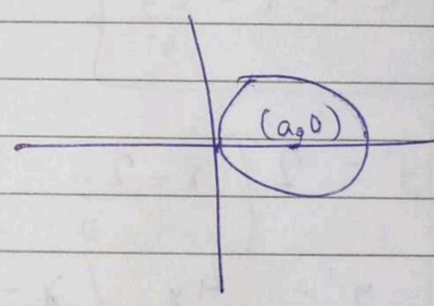
$$\frac{d^2y}{dx^2} = -am^2\cos(mx+b) \quad \text{— ③}$$

Using ① in ③; $\frac{d^2y}{dx^2} = -m^2 y$

$\Rightarrow \frac{d^2y}{dx^2} + m^2 y = 0$ ; which is required solution.

Q.1→ Find the differential eqn of all circles which passes through the origin and whose centres are on the x-axis.

Sol→ $(x-a)^2 + (y-0)^2 = a^2$




$x^2 - 2ax + a^2 + y^2 = a^2$

$x^2 - 2ax + y^2 = 0$

$$\boxed{2x - 2a + 2y\frac{dy}{dx} = 0}$$

$$\frac{x^2+y^2}{x} = 2a$$

$$2x - \left(\frac{x^2+y^2}{x}\right) + 2y\frac{dy}{dx} = 0$$

$$\frac{2x^2 - x^2 - y^2}{x} + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} + \frac{x^2-y^2}{x} = 0 \Rightarrow \boxed{\frac{dy}{dx} + \frac{x^2-y^2}{2xy} = 0}$$

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Q.2 → The differential eqn corresponding to the family of curves $y = c(x-c)^2$, where $c$ is an arbitrary constant.

Sol →
$$y = c(x-c)^2$$
$$\frac{dy}{dx} = 2c(x-c)$$
$$\frac{dy}{dx} \Big/ y = \frac{2\cancel{c}\cancel{(x-c)}}{\cancel{c}(x-c)^{\cancel{2}}}$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x-c}$$
$$\frac{y'}{y} = \frac{2}{x-c}$$
$$x - c = \frac{2y}{y'}$$
$$\boxed{c = x - \frac{2y}{y'}}$$

$$y' = 2\left(x - \frac{2y}{y'}\right)\left(x - x + \frac{2y}{y'}\right)$$
$$y' = \frac{4y}{y'}\left(x - \frac{2y}{y'}\right)$$
$$y' = \frac{4y}{y'}\left(\frac{xy' - 2y}{y'}\right)$$
$$\boxed{(y')^3 = 4y(xy' - 2y)}$$

# Variable Separable Method

- Working Rule to solve

STEP-1 : Let $\frac{dy}{dx} = f_1(x) \cdot f_2(y)$ —①

be given equation. $f_1(x)$ is a function of x-alone and $f_2(y)$ is ~~on~~ a function of y-alone

STEP-2 : from ①

Separating variables $\left[\frac{1}{f_2(y)}\right] dy = f_1(x)dx$

STEP-3 : Integration both side and we get required solution.

Example-1→ Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Sol→ $\frac{dy}{dx} = e^x e^{-y} + x^2 e^{-y}$

$\frac{dy}{dx} = e^{-y}(e^x + x^2)$

$\frac{dy}{e^{-y}} = (e^x + x^2)\, dx$

$\int e^y dy = \int (e^x + x^2) dx$

$\boxed{e^y = e^x + \frac{x^3}{3} + C}$ Ans

Good Write

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Example-2 ]

Solve $e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$

Sol → $\frac{e^x}{1-e^x}\tan y\, dx + \sec^2 y\, dy = 0$

$$\int \frac{e^x}{1-e^x}\, dx + \int \frac{\sec^2 y}{\tan y}\, dy = 0$$

$$-\int \frac{dt}{t} + \log(\tan y) = \log c$$

$$-\log(1-e^x) + \log(\tan y) = \log c$$

$$\log\left(\frac{\tan y}{1-e^x}\right) = \log c$$

$$\boxed{\tan y = c(1-e^x)} \text{ Ans}$$

Q.1 → Solve ~~ye~~ $y - x\left(\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$

(A) $(x+a)(1+ay) = cy$

(B) $(x+a)(1-ay) = cy$

(C) $(x-a)(1-ay) = cy$

(d) $(x-a)(1-ay) = cy$

Sol → $y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$

$$\Rightarrow (a+x)\frac{dy}{dx} = y(1-ay)$$

$$\Rightarrow \int \frac{dy}{y(1-ay)} = \int \frac{dx}{a+x}$$

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(By Partial fraction;
$\frac{1}{y(1-ay)} = \frac{A}{y} + \frac{B}{1-ay} = \frac{1}{y} + \frac{a}{1-ay}$ )

$$\int \left(\frac{1}{y} + \frac{a}{1-ay}\right) dy = \int \frac{dx}{a+x}$$

$$\log y + \left(\frac{a}{-a}\right) \log(1-ay) = \log(a+x) + \log C$$

$$\log\left(\frac{y}{1-ay}\right) = \log C(a+x)$$

$$\frac{y}{1-ay} = C(a+x)$$

$$\frac{1}{C} y = (1-ay)(a+x)$$

$$\boxed{cy = (x+a)(1-ay)} \qquad \boxed{\frac{1}{C} = c}$$

(B) $(x+a)(1-ay) = cy$

~~Reduction~~.

## Reducible into separation of variable

Let $\frac{dy}{dx} = f(ax+by+c)$ or $\frac{dy}{dx} = f(ax+by)$

### · Working process.

1) Let $ax+by+c = v$ or $ax+by = v$ after differentiation put in DE.
2) Separating variables
3) Integration & get solution

**Q.1→** Solve $(x+y)^2\left(\frac{dy}{dx}\right) = a^2$

**Sol→** Let $x+y = t$.

$$1+\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$t^2\left(\frac{dt}{dx} - 1\right) = a^2$$

$$t^2\frac{dt}{dx} - t^2 = a^2$$

$$t^2\frac{dt}{dx} = a^2 + t^2$$

$$\int \frac{t^2}{a^2+t^2}\,dt = \int dx$$

Good Write

$$\int \frac{t^2+a^2-a^2}{t^2+a^2}\,dt = x+c$$

$$\int\left(1-\frac{a^2}{t^2+a^2}\right)dt = x+c$$

$$t - a^2\left(\frac{1}{a}\tan^{-1}\frac{t}{a}\right) = x+c$$

$$t - a\tan^{-1}\frac{t}{a} = x+c$$

$$\boxed{(x+y) - a\tan^{-1}\left(\frac{x+y}{a}\right) = x+c}$$

$$\boxed{y - a\tan^{-1}\left(\frac{x+y}{a}\right) = c}$$

Q.2 → Solve $(x+y+1)\frac{dy}{dx} = 1$

Sol → Let $x+y+1 = t$

$$1+\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$t\left(\frac{dt}{dx}-1\right) = 1$$

$$t\frac{dt}{dx} - t = 1$$

$$t\frac{dt}{dx} = 1+t$$

$$\int \frac{1+t-1}{1+t} = \cancel{dt}\int dx$$

$$\int\left(1-\frac{1}{1+t}\right)dt = x+c$$

$$t - \log(1+t) = x+c$$

$$x+y+1 - \log(2+x+y) = x+c$$

$$\boxed{y+1-\log(2+x+y) = c}$$

Good Write

# Homogeneous Differential Equation

A differential equation of first order and first degree is said to be homogeneous, if it can be put in the form $\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)} = f\left(\frac{x}{y}\right)$

• **Working rule for solving homogeneous equation**

Step-1: Let $y = vx$ then differentiate wrt $x$
Step-2: Put both value in given differential equation
Step-3: Separating variable $x$ & $v$, and integrates this.
Step-4: After integration, replace $v$ by $y/x$.

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$x = vy$
in $f(x/y)$ case

Q.1 Solve $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)\,dy = 0$

Sol: $\frac{dy}{dx} = \frac{-(x^3 + 3xy^2)}{y^3 + 3x^2y}$

$\frac{dy}{dx} = \frac{-(x^3(1 + 3(y/x)^2))}{x^3((y/x)^3 + 3(y/x))}$

$\frac{dy}{dx} = \frac{-(1 + 3(y/x)^2)}{(y/x)^3 + 3(y/x)}$

Let $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{-(1 + 3v^2)}{v^3 + 3v}$

$x\frac{dv}{dx} = \frac{-(1 + 3v^2)}{v^3 + 3v} - v$

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$$\Rightarrow x\frac{dv}{dx} = \frac{-1-3v^2-v^4-3v^2}{v^3+3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-(v^4+6v^2+1)}{v^3+3v}$$

$$\Rightarrow \int \frac{v^3+3v}{v^4+6v^2+1}\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow v^4+6v^2+1 = t$$

$$4v^3+12v^1 = \frac{dt}{dv}$$

$$(v^3+3v)\,dv = \frac{dt}{4}$$

$$\Rightarrow \frac{1}{4}\int \frac{dt}{t} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{\log t}{4} = -\log x + \log c$$

$$\Rightarrow \log t = -4\log x + \log c$$

$$\Rightarrow \log t = -\log x^4 + \log c$$

$$\Rightarrow \log t = \log \frac{c}{x^4}$$

$$\Rightarrow t = \frac{c}{x^4}$$

$$\Rightarrow tx^4 = c$$

$$\Rightarrow (v^4+6v^2+1)\,x^4 = c$$

$$\Rightarrow \left(\frac{y^4}{x^4}+\frac{6y^2}{x^2}+1\right)x^4 = c$$

$$\Rightarrow \frac{(y^4+6x^2y^2+x^4)}{x^4}\times x^4 = c$$

$$\Rightarrow \boxed{y^4+6x^2y^2+x^4=c}$$ Ans

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Q.2 → Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Sol → Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = v + \tan v$

$\Rightarrow \int \cot v \, dv = \int \frac{dx}{x}$

$\Rightarrow \log \sin v = \log x + \log c$

$\Rightarrow \log \sin v = \log cx$

$\Rightarrow \sin v = cx$

$\Rightarrow \boxed{\frac{y}{x} = \sin^{-1}(cx)}$

$\Rightarrow \boxed{y = x \sin^{-1}(xc)}$

Q.3 → Solve: $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

Sol → $x\frac{dy}{dx} - y = \sqrt{x^2+y^2}$

$\Rightarrow x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$

$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2(1+(y/x)^2)}}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{x}{x}\sqrt{1+(y/x)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{1+(y/x)^2}$

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$\Rightarrow$ Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\Rightarrow \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \log\left|v + \sqrt{1+v^2}\right| = \log x + \log c$$

$$\Rightarrow \log\left|v + \sqrt{1+v^2}\right| = \log xc$$

$$\Rightarrow v + \sqrt{1+v^2} = xc$$

$$\Rightarrow \boxed{\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = xc}$$

# Reducible into Homogeneous Equation

## Non-Homogeneous Differential Equation

A differential equation of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$

is called non-homogeneous differential Equation.

There are two types

Type-1 : If $\frac{a}{a'} \neq \frac{b}{b'}$ — (1)

In this we major discuss it.

Working Rule

1) Take $x = X+h$ & $y = Y+k$ — (2)

$dx = dX$ and $dy = dY$

So, that $\frac{dy}{dx} = \frac{dY}{dX}$ — (3)

Type-2 : If $\frac{a}{a'} = \frac{b}{b'}$

Then it can be solved by reducible variable separable form.

Q.1→ Solve:

$$\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$$

Sol→ ~~x =~~ Let ; $x = X+h$

$y = Y+k$

$dx = dX$

$dy = dY$

So, $\frac{dY}{dX} = \frac{(Y+k)+(X+h)-2}{(Y+k)-(X+h)-4}$

Good Write

$$\Rightarrow \frac{dY}{dX} = \frac{Y + X + k + h - 2}{Y - X + k - h - 4}$$

Assume; $k + h - 2 = 0$

$$\begin{aligned} k - h - 4 &= 0 \\ \hline 2h + 2 &= 0 \\ \hline h &= -1 \\ k &= 3 \end{aligned}$$

$$\frac{dY}{dX} = \frac{Y + X}{Y - X}$$

$$\frac{dY}{dX} = \frac{X\left(\frac{Y}{X} + 1\right)}{X\left(\frac{Y}{X} - 1\right)}$$

Let $\boxed{Y = vX}$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$v + X\frac{dv}{dX} = \frac{v + 1}{v - 1}$$

$$X\frac{dv}{dX} = \frac{v + 1 - v^2 + v}{v - 1}$$

$$X\frac{dv}{dX} = \frac{1 + 2v - v^2}{v - 1}$$

$$\int \frac{v - 1}{1 + 2v - v^2}\, dv = \int \frac{dX}{X}$$

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Let $1+2v-v^2 = t$

$$(2-2v)\,dv = dt$$

$$(1-v)\,dv = \frac{dt}{2}$$

$$(v-1)\,dv = -\frac{dt}{2}$$

$$-\frac{1}{2}\int \frac{dt}{t} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\log t = \log x + \log C'$$

$$\log t = -2\log x + \log c$$

$$\log t = \log \frac{c}{x^2}$$

$$t = \frac{c}{x^2}$$

$$\boxed{c = x^2 t}$$

$$x^2(1+2v-v^2) = c$$

$$x^2\left(1+\frac{2y}{x}-\frac{y^2}{x^2}\right) = c$$

$$x^2\left(\frac{x^2+2xy-y^2}{x^2}\right) = c$$

$$\boxed{x^2+2xy-y^2 = c}$$

$$(x-h)^2+2(x-h)(y-k)-(y-k)^2 = c$$

$(h=-1 \,\&\, k=3)$

$$\boxed{(x+1)^2+2(x+1)(y-3)-(y-3)^2 = c}$$ Ans

Good Write

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Q.2 → Solve: $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Sol → $x = X + h$ $\quad\left(\begin{matrix} X = x-h \\ Y = y-k \end{matrix}\right)$

$y = Y + k$

$$\frac{dY}{dX} = \frac{X + 2Y + h + 2k - 3}{2X + Y + 2h + k - 3}$$

$$h + 2k - 3 = 0$$
$$2h + k - 3 = 0$$
$$h = 1, \quad k = 1$$

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} \Rightarrow \frac{dY}{dX} = \frac{1 + 2(Y/X)}{2 + (Y/X)}$$

( Let $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$v + X\frac{dv}{dX} = \cancel{X + 2} \; \frac{1+2v}{2+v}$$

$$X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$$

$$X\frac{dv}{dX} = \frac{1 + 2v - 2v - v^2}{2+v}$$

$$X\frac{dv}{dX} = \frac{1-v^2}{2+v}$$

$$\int \frac{2+v}{1-v^2}\, dv = \int \frac{dX}{X} \cancel{+ \log C}$$

$$\int \frac{2}{1-v^2}\, dv + \int \frac{v}{1-v^2}\, dv = \log X + \log C$$

$$2\left(\frac{1}{2}\log\frac{1+v}{1-v}\right) + \int \frac{dt}{-2t} = \log XC$$

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$$\log\left(\frac{1+v}{1-v}\right) - \frac{1}{2}\log t = \log xc$$

$$\log\left(\frac{1+v}{1-v}\right) - \frac{1}{2}\log(1-v^2) = \log xc$$

$$2\log\left(\frac{1+v}{1-v}\right) - \log(1-v^2) = 2\log xc$$

$$\log\left(\frac{1+v}{1-v}\right)^2 - \log(1-v^2) = \log(xc)^2$$

$$\log\left[\frac{(1+v)^2}{(1-v)^2} \times \frac{1}{(1-v)(1+v)}\right] = \log(xc)^2$$

$$\frac{(1+v)}{(1-v)^3} = x^2c^2$$

$$1+v = x^2c^2(1-v)^3$$

$$1+\frac{y}{x} = x^2c^2(1-y/x)^3$$

$$\frac{x+y}{x} = x^2c^2\frac{(x-y)^3}{x^3}$$

$$\boxed{x+y = \cancel{x^2}c^2(x-y)^3}$$

$$(x-h)+(y-k) = c^2(x-h-(y-k))^3$$

$$(x-1)+(y-1) = c^2(x-1-(y-1))^3$$

$$\boxed{x+y-2 = c^2(x-y)^3}$$ Ans

# Linear Differential Equation

There are two form:

Form 1 → $\frac{dy}{dx} + Py = Q$

, where P & Q are function of x.

$IF = e^{\int P dx}$

Then soln

$y \times IF = \int Q(IF)\, dx + C$

Q.1 → Solve:

$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

Sol → $\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$

$P = -\frac{1}{x+1}$ ; $Q = e^{3x}(x+1)$

$IF = e^{\int P dx} = e^{-\int \frac{1}{x+1} dx} = e^{\log_e (x+1)^{-1}}$

$\boxed{IF = \frac{1}{x+1}}$

$y \times IF = \int Q \times (IF)\, dx$

$y \times \frac{1}{x+1} = \int \frac{e^{3x}\cancel{(x+1)}}{\cancel{(x+1)}} dx$

$\frac{y}{x+1} = \int e^{3x} dx$

Good Write

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$$\boxed{y = \left(\frac{e^{3x}}{3} + c\right)(1+x)}$$

Q.2 → Solve: $x\cos x\left(\frac{dy}{dx}\right) + y(x\sin x + \cos x) = 1$

Sol → $\frac{dy}{dx} + y\left(\frac{x\sin x + \cos x}{x\cos x}\right) = \frac{1}{x\cos x}$

$P = \frac{x\sin x}{x\cos x} + \frac{\cos x}{x\cos x} = \tan x + \frac{1}{x}$

$Q = \frac{\sec x}{x}$

$IF = e^{\int P dx} = e^{\int(\tan x + \frac{1}{x})dx} \Rightarrow e^{\log\sec x + \log x} \Rightarrow e^{\log(x\sec x)}$

$$\boxed{IF = x\sec x}$$

$y \times IF = \int Q \times (IF)\,dx + c$

$x y\sec x = \int x\sec x \times \frac{\sec x}{x}\,dx + c$

$x y\sec x = \int \sec^2 x\,dx + c$

$$\boxed{x y\sec x = \tan x + c}$$

Form 2 →

$\frac{dx}{dy} + Px = Q$

, where $P$, $Q$ are the function of $y$.

Integrating Factor (I.F.) $= e^{\int P dy}$

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Then Solution

$$x(I.F.) = \int Q\,(I.F.)\,dy + c$$

Q.3→ Solve:

$$(x + 2y^3)\left(\frac{dy}{dx}\right) = y$$

Sol→ $(x + 2y^3) = y\frac{dx}{dy}$

$$y\frac{dx}{dy} - x = 2y^3$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2$$

$P = -1/y \;;\; Q = 2y^2$

$$IF = e^{\int P dy} = e^{-\int \frac{1}{y} dy} = e^{-\log y} \Rightarrow 1/y$$

$$x.(IF) = \int Q \times (IF)\, dy + C$$

$$x \times \frac{1}{y} = \int \frac{1}{y}\,(2y^2)\,dy + c$$

$$\frac{x}{y} = 2\int y\,dy + c$$

$$\frac{x}{y} = y^2 + c$$

$\boxed{x = y^3 + cy}$ Ans

# Reducible into Linear Form

An equation of the form $f'(y)\frac{dy}{dx} + P f(y) = Q$

where P & Q are function of $x$ alone or constant.

## Working Rule

(1) Putting $f(y) = v$; so that $f'(y)\frac{dy}{dx} = \frac{dv}{dx}$

(2) DE become $\frac{dv}{dx} + Pv = Q$

which is linear differential equation.

**Q.1** → Solve: $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Sol → $\frac{dy}{dx} + x 2\sin y\cos y = x^3\cos^2 y$

$\frac{1}{\cos^2 y}\frac{dy}{dx} + 2x\frac{\sin y}{\cos y} = x^3$

$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$

$\left(\text{Let } \tan y = v \quad \sec^2 y\frac{dy}{dx} = \frac{dv}{dx}\right)$

$\frac{dv}{dx} + 2vx = x^3$

$P = 2x \; ; \; Q = x^3$

$IF = e^{\int 2x\,dx} = e^{x^2}$

$v(IF) = \int Q(IF)\,dx + c$

$v(e^{x^2}) = \int e^{x^2} x^2 x\,dx + c$

$\left(\text{Let } x^2 = p \quad 2x\,dx = dp \quad x\,dx = \frac{dp}{2}\right)$

$ve^{x^2} = \frac{1}{2}\int(e^p p)\,dp + c$

$2ve^{x^2} = (pe^p - e^p) + 2c$

$2\tan y\, e^{x^2} = x^2 e^{x^2} - e^{x^2} + 2c$

Ans↑

Q.2 → Solve:

$$\frac{dy}{dx} = e^{x-y}(e^x - e^y)$$

Sol → $\frac{dy}{dx} = e^{x-y+x} - e^{x-y+y}$

$$\frac{dy}{dx} = e^{2x-y} - e^x$$

$$\frac{dy}{dx} + e^x = e^{2x} e^{-y}$$

$$e^y \frac{dy}{dx} + e^x e^y = e^{2x}$$

$$\left( \text{Let } e^y = v \quad e^y \frac{dy}{dx} = \frac{dv}{dx} \right)$$

$$\frac{dv}{dx} + ve^x = e^{2x}$$

$$P = e^x \; ; \; Q = e^{2x}$$

$$IF = e^{\int P dx} = e^{\int e^x dx} = e^{e^x}$$

$$ve^{e^x} = \int e^{e^x} e^x e^x dx + c$$

$$\left( \text{Let } e^x = p \Rightarrow e^x dx = dp \right)$$

$$ve^{e^x} = \int e^p p \, dp + c$$

$$e^y e^{e^x} = (pe^p - e^p) + c$$

$$e^y e^{e^x} = e^x e^{e^x} - e^{e^x} + c$$

$$\boxed{e^y = e^x - 1 + ce^{-e^x}}$$ Ans

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## • Bernoulli's Equation

An equation of the form $\frac{dy}{dx} + Py = Qy^n$

where P & Q are function of x alone or constant is called a Bernoulli's equation.

Working Rule :

1) Divide both side by $y^n$.
So, DE become $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$

2) Put $y^{1-n} = v$ & $y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$

Q.3 Solve :

$$x\left(\frac{dy}{dx}\right) + y = y^2 \log x$$

Sol → $\frac{dy}{dx} + \frac{y}{x} = y^2 \frac{\log x}{x}$

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{\log x}{x}$

$\left(\text{Let } \frac{1}{y} = v \right.$

$\left. \frac{-1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}\right)$

$-\frac{dv}{dx} + \frac{v}{x} = \frac{\log x}{x}$

$\frac{dv}{dx} - \frac{v}{x} = \frac{-\log x}{x}$

$P = -\frac{1}{x}$ ; $Q = -\frac{\log x}{x}$

$IF = e^{-\int \frac{1}{x} dx}$

$= e^{\log x^{-1}}$

$\Rightarrow 1/x$

$v \times \left(\frac{1}{x}\right) = \int -\frac{\log x}{x} \times \frac{1}{x}\, dx + c$

~~$\frac{v}{x} = -\int \frac{1}{}$~~

$\left(\text{Let } \log x = p \right.$

$\left. \frac{1}{x} dx = dp \right)$

$\frac{v}{x} = -\int \frac{1}{e^p} p\, dp + c$

$\frac{v}{x} = -\int e^{-p} p\, dp + c$

Good Write

$$\frac{v}{x} = -(-pe^{-p} - e^{-p}) + c$$

$$\frac{1}{xy} = \frac{\log x}{x} + \frac{1}{x} + c$$

$$\boxed{\frac{1}{y} = \log x + 1 + cx}$$ Ans

Q.4→ $(xy + x^3y^3)\frac{dy}{dx} = 1$

Sol→ $\frac{dx}{dy} - xy = x^3y^3$

$$\frac{1}{x^3}\frac{dx}{dy} - \frac{y}{x^2} = y^3$$

(Let $\frac{-1}{x^2} = v$

$\frac{2}{x^3}\frac{dx}{dy} = \frac{dv}{dy}$)

$$\frac{1}{2}\frac{dv}{dy} + yv = y^3$$

$$\frac{dv}{dy} + 2yv = 2y^3$$

$P = 2y$ & $Q = 2y^3$

$IF = e^{\int 2y\,dy} = e^{y^2}$

$ve^{y^2} = \int e^{y^2} \times 2y y^2\,dy + C$

(Let $y^2 = p$

$2y\,dy = dp$)

$ve^{y^2} = \int (e^p p)\,dp + C$

$ve^{y^2} = (pe^p - e^p) + C$

$$\frac{-e^{y^2}}{x^2} = y^2e^{y^2} \cancel{+} - e^{y^2} + C$$

$$\boxed{\frac{1}{x^2} = -y^2 + 1 \cancel{+} - ce^{-y^2}}$$

(Q. For Practice?

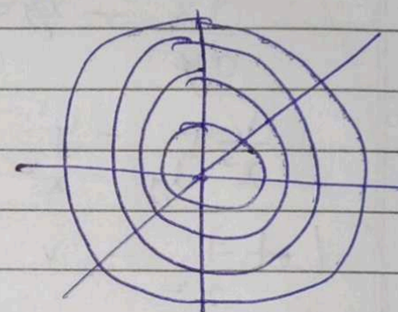
$x\left(\frac{dy}{dx}\right) + y = x^3y^6$)

# Orthogonal Trajectory

**Definition:**

A curve which cuts every member of a given family of curves is called trajectories.

Let $x^2+y^2=a^2$ be a family of circle, then $y=x$ cuts every member of $x^2+y^2=a^2$. So, $y=x$ is a trajectory of given family of circle.



## Orthogonal Trajectories

If curve cuts every member of given family of curves at right angles, then it is called an orthogonal trajectory.

**Working Rule for finding orthogonal trajectory**

1) Differentiate the given equation of family of curve & eliminate parameter.
2) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$
3) Solve this new DE & get orthogonal trajectory.

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Example 1 ]

Find orthogonal trajectories of $y = ax^2$.

Sol → Given that : $y = ax^2$

Differentiate: $\frac{dy}{dx} = 2ax$

$$\frac{dy}{dx} = 2\left(\frac{y}{x^2}\right)x = \frac{2y}{x}$$

Now; $-\frac{dx}{dy} = \frac{2y}{x}$

Solve this DE.

$$\int -x\,dx = \int 2y\,dy$$

$$\frac{-x^2}{2} = y^2 + b^2$$

$$\boxed{\frac{x^2}{2b^2} + \frac{y^2}{b^2} = 1}$$

This is orthogonal trajectories

Q.1 → Find the orthogonal trajectory of $x^2 + y^2 = 2ax$.

(A) $x^2 + y^2 = c_1 y$ (C) $x^2 - y^2 = -c_1 y$

(B) $x^2 + y^2 = e - c_1 y$ (D) $x^2 - y^2 = c_1 y$

Sol → $x^2 + y^2 = 2ax$

$$2x + 2y\frac{dy}{dx} = 2a$$

$$x^2 + y^2 = \left(2x + 2y\frac{dy}{dx}\right)x$$

$$x^2 + y^2 = 2x^2 + 2xy\frac{dy}{dx}$$

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$$2xy\frac{dy}{dx} = y^2 - x^2$$

$$-2xy\frac{dx}{dy} = y^2 - x^2$$

$$\frac{dy}{dx} = -\frac{2xy}{y^2-x^2}$$

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2} \Rightarrow \frac{dy}{dx} = \frac{2(y/x)}{(1-(y/x)^2)}$$

(Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx})$$

$$v + x\frac{dv}{dx} = \frac{2v}{1-v^2}$$

$$x\frac{dv}{dx} = \frac{2v}{1-v^2} - v$$

$$x\frac{dv}{dx} = \frac{2v - v + v^3}{1-v^2}$$

$$x\frac{dv}{dx} = \frac{v+v^3}{1-v^2}$$

$$\int\left(\frac{1-v^2}{v(1+v^2)}\right)dv = \int\frac{dx}{x}$$

(By using Partial fraction)

$$\int\left(\frac{1}{v} - \frac{2v}{1+v^2}\right)dv = \log x + \log c$$

$$\log v - \log(1+v^2) = \log xc$$

$$\log\left(\frac{v}{1+v^2}\right) = \log xc$$

Good Write

$$\log \frac{v}{1+v^2} = xc$$

$$v = xc(1+v^2)$$

$$\frac{y}{x} = xc\left(1+\frac{y^2}{x^2}\right)$$

$$y = c(x^2+y^2)$$

$$\boxed{x^2+y^2 = c_1 y}$$

(A) ✓

(self O.T. wala)

**Q. 2 →** Find orthogonal trajectory of family of parabola $y^2 = 4a(x+a)$.

**Sol →** $2y\frac{dy}{dx} = 4a$

$$y^2 = 2y\frac{dy}{dx}\left(x+\frac{y}{2}\frac{dy}{dx}\right)$$

$$\boxed{y^2 = 2yx\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2}$$

$$y^2 = -2xy\frac{dx}{dy} + y^2\left(\frac{dx}{dy}\right)^2$$

$$y^2 = \frac{-2xy}{(dy/dx)} + \frac{y^2}{(dy/dx)^2}$$

$$\left(\frac{dy}{dx}\right)^2 y^2 = -2xy\left(\frac{dy}{dx}\right) + y^2$$

$$\boxed{y^2 = 2xy\left(\frac{dy}{dx}\right) + y^2\left(\frac{dy}{dx}\right)^2}$$

both same

So, self orthogonal trajectory. Ans

So, after integrate & solving we get; $y^2 = 4a(x+a)$

# Orthogonal Trajectory (Polar Form)

- Orthogonal Trajectory in Polar coordinates

Working Rule :

1) Differentiate the given equation w.r.t. $\theta$ and eliminate the parameter.
2) Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$
3) Solve new DE and get required soln.

Example→ Find orthogonal trajectories of cardiods $r = a(1-\cos\theta)$, a parameters.

Sol→ Given that:

$$r = a(1-\cos\theta)$$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$r = \frac{dr}{d\theta \sin\theta}(1-\cos\theta)$$

$$r = -r^2 \frac{d\theta}{dr} \frac{2\sin^2\theta/2}{2\sin\theta/2 \cos\theta/2}$$

$$1 = -r\frac{d\theta}{dr} \tan\theta/2$$

$$\int \frac{dr}{r} = -\int \tan\theta/2 \, d\theta$$

$$\log r = 2\log\cos\theta/2 + \log c$$

$$\log r = \log \cos^2\theta/2 \times c$$

$$r = \cos^2\theta/2 \; c$$

$$r = \left(\frac{1+\cos\theta}{2}\right) c$$

Good Write

$$r = (1+\cos\theta)\, c_1$$ Ans

Q.1→ Find orthogonal trajectories of $a = r^2 \cos\theta$.

[A] $b = r\cos^2\theta$ [C] $b = -r\sin^2\theta$

[B] $b = r\sin^2\theta$ [D] $b = -r\cos^2\theta$

Sol→ $a = r^2\cos\theta$

$0 = 2r\frac{dr}{d\theta}\cos\theta + r^2(-\sin\theta)$

$2r\frac{dr}{d\theta}\cos\theta = r^2\sin\theta$

$-2r^2\frac{d\theta}{dr}\cos\theta = \cancel{r}\sin\theta$

$-2\int \cot\theta \, d\theta = \int \frac{dr}{r}$

$-2\log\sin\theta = \log r + \log c$

$\log \frac{1}{\sin^2\theta} = \log rc$

$rc = 1/\sin^2\theta$

$r\sin^2\theta = 1/c$

$\boxed{b = r\sin^2\theta}$

(B)

Q.2→ Find orthogonal trajectories of $r^2 = a^2\cos 2\theta$.

[A] $r^2 = b^2\sin 2\theta$ [C] $r^2 = -b^2\sin^2\theta$

[B] $r^2 = b^2\sin^2\theta$ [D] $r^2 = -b^2\cos^2\theta$

Sol→ $r^2 = a^2\cos 2\theta$

$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta$

$r\frac{dr}{d\theta} = -a^2\sin 2\theta$

$\frac{r\frac{dr}{d\theta}}{r^2} = \frac{-a^2\sin 2\theta}{a\cos 2\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = -\tan 2\theta$

$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan 2\theta$

$\int \cot 2\theta \, d\theta = \int \frac{dr}{r}$

$$\frac{1}{2}\log \sin 2\theta = \log r + \log c$$

$$\log (\sin 2\theta)^{1/2} = \log rc$$

$$r^2c^2 = \sin 2\theta$$

$$\boxed{r^2 = b^2 \sin 2\theta}$$

(A) ✓

## Linear ~~Differentiation~~ Differential Eqn with constant coefficient

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = f(x)$$

Higher Order → 2nd & more derivatives are allowed.

Linear differential → Power of derivatives ~~are~~ will be 1. (degree)
&
None of dependent variables & its derivatives will be multiplied.

$y\frac{dy}{dx}$ ×

$\frac{dy}{dx} \cdot \frac{d^2y}{dx^2}$ ×

with derivates

~~Eq~~ Equation → Equal sign will be there.

Constant Coefficients → $k_1, k_2 \dots k_n$ will be constants.

$f(x)$ → Function of $x$ (independent variable)

## Complete / General Solution

$$\boxed{y = C.F. + P.I.}$$

C.F. → Complementary function

P.I. → Particular Integral

$y$ → Dependent variable

C.F. + P.I. → in term of independent variables

| Quest | D.V. | I.V. | Solution |
|---|---|---|---|
| $dy/dx$ | $y$ | $x$ | $y = C.F. + P.I.$ |
| $dy/dt$ | $y$ | $t$ | $y = C.F. + P.I.$ |
| $dx/dt$ | $x$ | $t$ | $x = C.F. + P.I.$ |
| $dz/dy$ | $z$ | $y$ | $z = C.F. + P.I.$ |

# Linear differential Eqn with constant coefficient

## Complete / General Solution

$y = C.F. + P.I.$

C.F. = Complementary Function

P.I. = Particular Integral

$\frac{d}{dx} = D \; ; \; \frac{d^2}{dx^2} = D^2$ - - -

- Differential Operators
- Operator

$\Uparrow$

Used in symbolic form.

## Flow chart for finding C.F.

Linear Differential Equation

$\downarrow$

Symbolic form $(D, D^2, D^3 \ldots)$

$\downarrow$

$f(D) = ?$

$\downarrow$

Auxiliary Eqn (Put $f(m) = 0$)

$\downarrow$ Solve

Values of $m = ?$

$\downarrow$ ?

C.F.

## Writing of C.F. for values of m

| Values of m | C.F. |
| --- | --- |
| ① Real & Distinct | |
| $-2, 3, 5$ | $c_1 e^{-2x} + c_2 e^{3x} + c_3 e^{5x}$ |
| ② Real & Repeated | |
| $-2, 3, 3$ | $c_1 e^{-2x} + c_2 e^{3x} + c_3 x e^{3x}$ |
| ③ Imaginary / In pairs | |
| $\pm 3i$ | $c_1 \cos 3x + c_2 \sin 3x$ |
| ④ Complex / in pairs | |
| $2 \pm 3i$ | $e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$ |
| ⑤ Complex & Repeated | |
| $2 \pm 3i$, $2 \pm 3i$ | $e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + x e^{2x}(c_3 \cos 3x + c_4 \sin 3x)$ |
| ⑥ Mixing | |
| $2, \pm 3i$ | $c_1 e^{2x} + (c_2 \cos 3x + c_3 \sin 3x)$ |

# Linear Differential Equation with constant coefficient

Q. 1 → Solve

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + dy = 0$$

Sol → The given differential Equation is

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} + dy = 0 \quad -(1)$$

which is a linear differential equation

Now from eqn (1)

$$D^2y + 3Dy + 2y = 0$$

$$(D^2 + 3D + 2)\,y = 0 \quad -(2)$$

which is the required symbolic form

Now from eqn (2)

$$\boxed{f(D) = D^2 + 3D + 2}$$

Now for Auxiliary Equation

Put $f(m) = 0$; we get

$$m^2 + 3m + 2 = 0$$

$$m^2 + 2m + m + 2 = 0$$

$$m(m+2) + 1(m+2) = 0$$

$$(m+1)(m+2) = 0$$

$m = -1$ | $m = -2$

Since; m is real & distinct

So; $\boxed{C.F. = c_1 e^{-x} + c_2 e^{-2x}}$

Good Write

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Finding of P.I.

$$P.I. = \frac{1}{f(D)} \cdot RHS$$

$$= \frac{1}{D^2+3D+2} \cdot (0)$$

$$\boxed{P.I. = 0}$$

Now; complete solution of Linear Differential Equation is written as

$$y = C.F. + P.I.$$

$$y = (c_1 e^{-x} + c_2 e^{-2x}) + 0$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x}}$$ Ans

$y'' - 2y' + 10y = 0$ ; $\underbrace{y(0) = 4 , y'(0) = 1}$

↳ Boundary Conditions

The given differential equation is

$$y'' - 2y' + 10y = 0 \quad - (1)$$

which is a linear differential equation

Now, from eq (1)

$$D^2 y - 2Dy + 10y = 0$$

$$(D^2 - 2D + 10)y = 0 \quad - (2)$$

which is the required symbolic form;

Now, from eq (2)

$$\boxed{f(D) = D^2 - 2D + 10}$$

Now for Auxiliary Equation

Put $f(m) = 0$

we get

$$m^2 - 2m + 10 = 0$$

$$m = \frac{2 \pm \sqrt{4-40}}{2(1)} \qquad \left[ \because \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right]$$

$$= \frac{2 \pm \sqrt{-36}}{2}$$

$$= \frac{2 \pm 6i}{2}$$

$$\Rightarrow \boxed{m = 1 \pm 3i}$$

Good Write

Now; C.F. is written as

$$\boxed{C.F. = e^{x}(c_1 \cos 3x + c_2 \sin 3x)}$$

We know that

$$P.I. = \frac{1}{f(D)}(R.H.S.)$$

$$= \frac{1}{D^2 - 2D + 10}(0)$$

$$= 0$$

Now, as we know that
complete solution of Linear Differential Equation is written as

$$\boxed{y = C.F. + P.I.}$$

$$= e^{x}(c_1 \cos 3x + c_2 \sin 3x) + 0$$

$$\boxed{y = e^{x}(c_1 \cos 3x + c_2 \sin 3x)} \quad -(3)$$

Boundary conditions → to eliminate unknown

Now; $y(0) = 4$;
$y = 4$ at $x = 0$
Put in eqn (3), we get

$$4 = e^{0}(c_1 \cos(0) + c_2 \sin(0))$$

$$4 = 1(c_1(1) + 0) \Rightarrow \boxed{c_1 = 4} \quad -(4)$$

Now, $y'(0) = 1$
↳ $y' = 1$ at $x = 0$
from eq (3)

$$y = e^{x}(c_1 \cos 3x + c_2 \sin 3x)$$

$y' = e^x(-3c_1 \sin 3x + 3c_2 \cos 3x) + (c_1 \cos 3x + c_2 \sin 3x)e^x$

$y' = e^x(-3c_1 \sin 3x + 3c_2 \cos 3x + c_1 \cos 3x + c_2 \sin 3x)$

Put $y' = 1$ at $x = 0$

$1 = e^0[-3c_1 \sin(0) + 3c_2 \cos(0) + c_1 \cos(0) + c_2 \sin(0)]$

$1 = 1(0 + 3c_2 + c_1 + 0)$

$1 = 3c_2 + c_1$

$1 = 3c_2 + 4$ (from eq 4)

$\boxed{c_2 = -1}$

Now put the values of $c_1$ & $c_2$ in eq (3), we get

$\boxed{y = e^x(4\cos 3x - \sin 3x)}$ Ans

# Linear differential Eqn with constant coefficient

## (Particular Integral)

$$P.I. = \frac{1}{f(D)} . R.H.S.$$

Basic

1. zero → Homogeneous Linear Differential eqn → $\boxed{PI = 0}$
2. $e^{ax}$ → Put $D = a$
3. constant → Put $D = 0$
4. $\sinh bx / \cosh bx$ → convert into exponential form

$$\sinh bx = \frac{e^{bx} - e^{-bx}}{2} \qquad \cosh bx = \frac{e^{bx} + e^{-bx}}{2}$$

5. $\sin bx / \cos bx$ → Put $D^2 = -b^2$
6. $x, x^2, x^3$ → By expansion
7. $e^{ax} \sin bx$
8. $x^2 e^{ax}$
9. $x \sin bx$
10. $x e^{ax} \sin bx$

Linear Differential Equation with constant coefficient

Q→ Solve:

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 2y = e^x$$

Sol→ The given differential equation is

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 2y = e^x \quad -(1)$$

It is a linear differential equation.

Now;

$$D^3y - D^2y - 4Dy - 2y = e^x$$

$$(D^3 - D^2 - 4D - 2)y = e^x \quad -(2)$$

which is the required symbolic form.

Now; from eq (2)

$$f(D) = D^3 - D^2 - 4D - 2$$

Now for Auxiliary Equation

Put $f(m) = 0$

We get

$$m^3 - m^2 - 4m - 2 = 0 \quad -(3)$$

⇒ By hit & trial method

Put $m = 1$

$$1 - 1 - 4 - 2 \neq 0$$

Put $m = -1$

$$(-1)^3 - (-1)^2 - 4(-1) - 2$$

$$-1 - 1 + 4 - 2 = 0 = R.H.S.$$

Good Write

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$\Rightarrow$ So, $\boxed{m=-1}$ is one of the root of equation 3

Now by synthetic division method

$$\begin{array}{c|cccc} -1 & 1 & -1 & -4 & -2 \\ & & -1 & 2 & 2 \\ \hline & 1 & -2 & -2 & 0 \end{array}$$

So, now equation 3 can be written as:

$$(m+1)(m^2-2m-2)=0$$

$m+1=0$ | $m^2-2m-2=0$

$\boxed{m=-1}$ | $m = \frac{+2 \pm \sqrt{4-4.1.-2}}{2(1)}$

$$m = \frac{2 \pm \sqrt{12}}{2}$$

$$m = \frac{2 \pm 2\sqrt{3}}{2}$$

$$m = 1 \pm \sqrt{3}$$

So, $\boxed{m = -1, 1-\sqrt{3}, 1+\sqrt{3}}$

All values are real & Distinct.

So, $\left[C.F. = c_1 e^{-x} + c_2 e^{(1-\sqrt{3})x} + c_3 e^{(1+\sqrt{3})x}\right]$

Now, $P.I. = \frac{1}{f(D)} \cdot RHS$

$$= \frac{1}{D^3-D^2-4D-2}(e^x)$$

$\boxed{\text{Put } D=1}$

$$= \frac{1}{1-1-4-2}(e^x)$$

Good Write

$$\boxed{PI = -\frac{1}{6}e^{x}}$$

Now, as we know that in Linear Differential Equation complete / general solution ~~as~~ is given as :-

$$\boxed{y = C.F. + P.I.}$$

$$y = \left[c_1 e^{-x} + c_2 e^{(1-\sqrt{3})x} + c_3 e^{(1+\sqrt{3})x}\right] + \left[-\frac{1}{6}e^{x}\right]$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{(1-\sqrt{3})x} + c_3 e^{(1+\sqrt{3})x} - \frac{1}{6}e^{x}}$$ Ans

Q. Solve the differential equation:

$$\frac{d^2y}{dx^2} - \frac{6dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$$

Sol. The given differential equation is, exponential

$$\frac{d^2y}{dx^2} - \frac{6dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \underbrace{\log 2}_{\text{Constant}} \quad -(1)$$

$$D^2y - 6Dy + 9y = 6e^{3x} + 7e^{-2x} - \log 2$$

$$(D^2 - 6D + 9)\,y = 6e^{x} + 7e^{-2x} - \log 2 \quad -(2)$$

which is the required symbolic form

Now; $f(D) = D^2 - 6D + 9$

Now for Auxiliary Equation

Put $f(m) = 0$

$$m^2 - 6m + 9 = 0$$

$$m^2 - 3m - 3m + 9 = 0$$

$$m(m-3) - 3(m-3) = 0$$

$$(m-3)(m-3) = 0$$

| | |
|---|---|
| $m - 3 = 0$ | $m - 3 = 0$ |
| $m = 3$ | $m = 3$ |

Since m is real & repeated

So, $\boxed{CF \Rightarrow c_1 e^{3x} + c_2 x e^{3x}}$

Now, as we know that

$$PI = \frac{1}{f(D)} \times RHS$$

$$\Rightarrow \frac{1}{D^2-6D+9}\left(6e^{3x}+7e^{-2x}-\log 2\right)$$

$$\Rightarrow \frac{1}{D^2-6D+9}\left(6e^{3x}\right)+\frac{1}{D^2-6D+9}\left(7e^{-2x}\right)$$

$$-\frac{1}{D^2-6D+9}(\log 2)$$

$$\left(\because \frac{1}{f(D)}(u+v)=\frac{1}{f(D)}\cdot u+\frac{1}{f(D)}\cdot v\right)$$

$$\Rightarrow \frac{6e^{3x}}{9-18+9}+\frac{7e^{-2x}}{4+12+9}-\frac{1}{9}(\log 2)$$

$$\Rightarrow \frac{6e^{3x}}{0}+\frac{7e^{-2x}}{25}-\frac{\log 2}{9}$$

$\hookrightarrow \frac{1}{0}$ is the case of failure

[Note $\rightarrow$ Case of failure

$$\frac{1}{f(D)}\cdot RHS=\frac{x}{f'(D)}\cdot RHS=\frac{x^2}{f''(D)}\,RHS\ \ldots\ -$$

If again $\frac{1}{0}$ then ]

$$\Rightarrow \frac{x\,6e^{3x}}{2D-6}+\frac{7e^{-2x}}{25}-\frac{\log 2}{9}$$

$$\Rightarrow \frac{x\,6e^{3x}}{0}+\frac{7e^{-2x}}{25}-\frac{\log 2}{9}$$

(Case of failure)

$$\Rightarrow \frac{x^2 6e^{3x}}{2}+\frac{7e^{-2x}}{25}-\frac{\log 2}{9}$$

$$\boxed{P.I = 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{\log 2}{9}}$$

Now; complete solution is written as

$$y = C.F. + P.I.$$

$$\boxed{y = c_1 e^{3x} + c_2 x e^{3x} + 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{\log 2}{9}}$$ Ans

Q. Solve :
$(D^2 - 4)\, y = e^{2x}$

Sol → The given differential equation is

$$(D^2 - 4)\, y = e^{2x} \quad -(1)$$

which is already in symbolic form.

Now, $f(D) = D^2 - 4$

Now for auxiliary equation.

Put $f(m) = 0$

$$m^2 - 4 = 0$$

$$m^2 = 4$$

$$\boxed{m = \pm 2}$$

⇒ Now since ~~value~~ values of m are real & distinct

$C.F. = c_1 e^{-2x} + c_2 e^{2x}$

⇒ Now as we know that

$$P.I. = \frac{1}{f(D)} \times R.H.S.$$

$$\Rightarrow \frac{1}{D^2 - 4} (e^{2x})$$

$$\left[ \because \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)} \right]$$

$$\Rightarrow \frac{1}{2^2 - 4} (e^{2x})$$

$$\Rightarrow \frac{1}{4-4} (e^{2x}) \Rightarrow \frac{e^{2x}}{0}$$

(Case of failure)

Good Write

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In case of failure:

$$\boxed{\frac{1}{f(D)} \cdot RHS = \frac{x}{f'(D)}(RHS)}$$

$\Rightarrow \frac{x}{2D}(e^{2x})$

$$\boxed{\Rightarrow \frac{x\,e^{2x}}{4} = P.I.}$$

Now, complete solution is given as

$y = C.F. + P.I.$

$$\boxed{y = c_1 e^{-2x} + c_2 e^{2x} + \frac{xe^{2x}}{4}}$$ Ans

Q. Solve:
$(D^2 - 3D + 2)\,y = \cosh 4x$
↳ hyperbolic function

Sol. The given differential equation is
$(D^2 - 3D + 2)\,y = \cosh 4x$ —(1)
which is already in symbolic form.
Now, $f(D) = D^2 - 3D + 2$
Now, for auxiliary equation
Put $f(m) = 0$
$m^2 - 3m + 2 = 0$
$(m-1)(m-2) = 0$
$\boxed{m = 1, 2}$

Good Write

⇒ Now, since values of m are real & distinct

$$\boxed{C.F. = c_1 e^{x} + c_2 e^{2x}}$$

⇒ Now, as we know that

$$P.I. = \frac{1}{f(D)} \cdot (R.H.S.)$$

$$= \frac{1}{D^2 - 3D + 2} \cdot \cosh 4x$$

$$\left[ \because \cosh bx = \frac{e^{bx} + e^{-bx}}{2} \right]$$

$$= \frac{1}{D^2 - 3D + 2} \left( \frac{e^{4x} + e^{-4x}}{2} \right)$$

$$= \frac{1}{2} \left( \frac{1}{D^2 - 3D + 2} \cdot e^{4x} \right) + \frac{1}{2} \left( \frac{1}{D^2 - 3D + 2} e^{-4x} \right)$$

$$= \frac{1}{2} \left( \frac{e^{4x}}{16 - 12 + 2} \right) + \frac{1}{2} \left( \frac{e^{-4x}}{16 + 12 + 2} \right)$$

$$\Rightarrow \frac{e^{4x}}{12} + \frac{e^{-4x}}{60}$$

$$\Rightarrow \boxed{P.I. = \frac{5e^{4x} + e^{-4x}}{60}}$$

⇒ As; we know that complete solution is written as

$$y = C.F. + P.I.$$

$$y = \boxed{c_1 e^{x} + c_2 e^{2x} + \frac{5e^{4x} + e^{-4x}}{60}} \text{ Ans}$$

Q→ Solve the differential Equation:

$$y''' - y'' + 4y' - 4y = \sin 3x$$

Sol. → The given differential equation is

$$y''' - y'' + 4y' - 4y = \sin 3x \quad \text{—(1)}$$

Now;

$$D^3y - D^2y + 4Dy - 4y = \sin 3x$$

$$(D^3 - D^2 + 4D - 4)\,y = \sin 3x \quad \text{—(2)}$$

which is the required symbolic form

Now;

$$f(D) = D^3 - D^2 + 4D - 4$$

for Auxiliary Equation.

Put $f(m) = 0$

$$m^3 - m^2 + 4m - 4 = 0$$

$$m^2(m-1) + 4(m-1) = 0$$

$$(m^2+4)(m-1) = 0$$

| | |
|---|---|
| $m^2 + 4 = 0$ | $m - 1 = 0$ |
| $m^2 = -4$ | $m = 1$ |
| $m = \pm 2i$ | |

$$\boxed{m = 1, \pm 2i}$$

Since; m is real & imaginary.

$$C.F. = c_1 e^{x} + (c_2 \cos 2x + c_3 \sin 2x)$$

Now, as we know that

$$P.I. = \frac{1}{f(D)} \cdot (RHS)$$

Good Write

$$\Rightarrow \frac{1}{D^3 - D^2 + 4D - 4} \cdot \sin 3x$$

$$\left[ \frac{1}{f(D)} \cdot \sin ax \Rightarrow \text{Put } D^2 = -a^2 \right]$$

$\Rightarrow$ So, put $D^2 = -(3)^2$

$$\boxed{D^2 = -9}$$

$$\Rightarrow \frac{1}{(-9)D + 9 + 4D - 4} \cdot \sin 3x$$

$$\Rightarrow \frac{\sin 3x}{-5D + 5}$$

$$\Rightarrow -\frac{1}{5}\left( \frac{1}{D-1} \cdot \sin 3x \right)$$

$$\Rightarrow -\frac{1}{5}\left( \frac{D+1}{D^2-1} \cdot \sin 3x \right)$$

$$\Rightarrow \frac{D(\sin 3x) + \sin 3x}{-5(-9-1)} \Rightarrow \frac{D(\sin 3x) + \sin 3x}{50}$$

$$\Rightarrow \frac{3\cos 3x + \sin 3x}{50}$$

$\Rightarrow$ Now complete solution is written as

$$y = C.F. + P.I.$$

Ans $\rightarrow$ $y = c_1 e^x + (c_2 \cos 2x + c_3 \sin 2x) + \frac{(3\cos 3x + \sin 3x)}{50}$

Q. Solve:

$(D^3+1)y = 2\cos^2 x$

Sol. The given differential equation is

$(D^3+1)y = 2\cos^2 x$ —(1)

which is already in symbolic form.

Now, $f(D) = D^3+1$

for Auxiliary equation

Put $f(m) = 0$

$m^3 + 1 = 0$

$(m+1)(m^2+1-m) = 0$

$[\because a^3+b^3 = (a+b)(a^2+b^2-ab)]$

$m+1 = 0$

$\boxed{m = -1}$

$m^2 - m + 1 = 0$

$m = \dfrac{1 \pm \sqrt{1-4}}{2(1)}$

$m = \dfrac{1 \pm \sqrt{-3}}{2}$

$m = \dfrac{1 \pm \sqrt{3}\,i}{2}$

So; $m = -1, \ \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

So, C.F. $= c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$

Now, as we know that

$P.I. = \dfrac{1}{f(D)} \times R.H.S.$

$\Rightarrow \dfrac{1}{D^3+1} \cdot 2\cos^2 x$

Good Write

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$$\Rightarrow \frac{1}{D^3+1} \cdot 2\left(\frac{1+\cos 2x}{2}\right) \quad \left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$\Rightarrow \frac{1}{D^3+1} \cdot (1+\cos 2x)$$

$$\Rightarrow \frac{1}{D^3+1} \cdot (1) + \frac{1}{D^3+1}(\cos 2x)$$

Put $D = 0$ Put $D^2 = -4$

$$\Rightarrow \frac{(1)}{0+1} + \frac{1}{(-4)D+1}(\cos 2x)$$

$$\Rightarrow 1 + \frac{1}{1-4D}(\cos 2x)$$

$$\Rightarrow 1 + \left(\frac{1}{1-4D}\right)\left(\frac{1+4D}{1+4D}\right)\cos 2x$$

$$\Rightarrow 1 + \frac{(1+4D)}{1-16D^2}\cos 2x$$

$$\Rightarrow 1 + \frac{(1+4D)}{1+16(4)}(\cos 2x)$$

$$\Rightarrow 1 + \frac{1}{65}[\cos 2x + 4D\cos 2x]$$

$$\Rightarrow 1 + \frac{1}{65}[\cos 2x + (-8\sin 2x)]$$

$$\Rightarrow \left[P.I. = 1 + \frac{1}{65}[\cos 2x - 8\sin 2x]\right]$$

$\Rightarrow$ Now, as we know that complete soln is written as

$$y = C.F. + P.I.$$

Ans $\rightarrow$ $y = c_1 e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) + 1 + \frac{1}{65}(\cos 2x - 8\sin 2x)$

Q. Solve the differential equation

$(D^2+4)y = \cos 2x$

Sol→ The given differential equation is

$$(D^2+4)y = \cos 2x \quad \text{--- (1)}$$

It is already in symbolic form

Now; $f(D) = D^2+4$

For Auxiliary equation

Put $f(m)=0$

$$m^2+4=0$$

$$m^2=-4$$

$$m=\pm 2i$$

Since; m is imaginary, then

$$C.F. = c_1\cos 2x + c_2\sin 2x$$

Now; as we know that

$$PI = \frac{1}{f(D)} \times RHS$$

$$= \frac{1}{D^2+4} \times \cos 2x$$

Put $D^2 = -(4)$

$$= \frac{1}{-4+4} \times \cos 2x$$

$$= \underbrace{\left(\frac{1}{0}\right)}_{\text{case of failure}} \cos 2x$$

$$= \frac{x}{2D+0} \times \cos 2x$$

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$\Rightarrow \dfrac{x}{2}\left(\dfrac{1}{D}\cos 2x\right)$

$\Rightarrow \dfrac{x}{2}\,\dfrac{(\sin 2x)}{2}$

$\Rightarrow \boxed{\dfrac{x}{4}\sin 2x} = P\,I$

(Denote
$D \rightarrow$ Derivative
$\frac{1}{D} \Rightarrow$ Integration)

$\Rightarrow$ Now, as we know that complete solution is written as

$$y = C.F. + P.I.$$

$$\boxed{y = c_1\cos 2x + c_2\sin 2x + \frac{x}{4}\sin 2x}$$ Ans

Q. Solve & tell what is the solution of P.I.

$$(D^2+3D+2)y = x^2-6x$$

Sol → $P.I. = \frac{1}{f(D)} \times RHS$

$$\Rightarrow \frac{1}{D^2+3D+2}(x^2-6x)$$

$$\Rightarrow \frac{1}{2\left(\frac{D^2}{2}+\frac{3D}{2}+1\right)}(x^2-6x)$$

$$\Rightarrow \frac{1}{2}\left[1+\left(\frac{D^2}{2}+\frac{3D}{2}\right)\right]^{-1}(x^2-6x)$$

[Recall: Binomial Expansion
$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots$]

$$\Rightarrow \frac{1}{2}\left[1+(-1)\left\{\frac{D^2}{2}+\frac{3D}{2}\right\}+\frac{(-1)(-2)}{2!}\left(\frac{D^2}{2}+\frac{3D}{2}\right)^2+\ldots\right](x^2-6x)$$

$$\Rightarrow \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{D^4}{4}+\frac{9D^2}{4}+\frac{3D^3}{2}+\ldots\right](x^2-6x)$$

$$\Rightarrow \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{9D^2}{4}\right](x^2-6x) \quad \text{(ignoring higher power than } D^2\text{)}$$

$$\Rightarrow \frac{1}{2}\left[1-\frac{3D}{2}+\frac{7D^2}{4}\right](x^2-6x)$$

$$\Rightarrow \frac{1}{2}\left[(x^2-6x)-\frac{3}{2}D(x^2-6x)+\frac{7}{4}D^2(x^2-6x)\right]$$

Good Write

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$$\Rightarrow \frac{1}{2}\left[(x^2-6x)-\frac{3}{2}(2x-6)+\frac{7}{4}(2-0)\right]$$

$$\Rightarrow \frac{1}{2}(x^2-6x-3x+9+7/2)$$

$$\Rightarrow \frac{1}{2}\left(x^2-9x+\frac{25}{2}\right)$$

$$\boxed{P.I. = \frac{1}{4}(2x^2-18x+25)}$$ Ans

Good Write

Q. Solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \sin x$$

Sol→ The given differential equation is

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \sin x \quad \text{—(1)}$$

It is a linear differential equation

Now;

$$D^2y - 2Dy + y = e^x \sin x$$

$$\boxed{(D^2 - 2D + 1)y = e^x \sin x}$$

which is the required symbolic form

Here; $f(D) = D^2 - 2D + 1$

Now, for Auxiliary equation

Put $f(m) = 0$

$$m^2 - 2m + 1 = 0$$

$$(m-1)(m-1) = 0$$

$$\boxed{m = 1, 1}$$

Since values of m are real & repeated.

So;

$$\boxed{CF = c_1e^x + c_2xe^x}$$

Now;

$$\boxed{P.I. = \frac{1}{f(D)} \cdot RHS}$$

$$= \frac{1}{D^2 - 2D + 1} \cdot e^x \sin x$$

Good Write

$\Rightarrow$ Now,

$$\boxed{\frac{1}{f(D)}\underset{L}{} \left(e^{ax}\underset{R}{\sin bx}\right) = \underset{L}{e^{ax}} \cdot \frac{1}{f(D+a)} \underset{R}{\sin bx}}$$

$$\Rightarrow e^{x} \cdot \frac{1}{(D+1)^2 - 2(D+1) + 1} \cdot \sin x$$

$$\Rightarrow e^{x} \cdot \frac{1}{D^2 + \cancel{1} + \cancel{2D} - \cancel{2D} - 2 + \cancel{1}} \cdot \sin x$$

$$\Rightarrow e^{x} \cdot \frac{1}{D^2} \cdot \sin x$$

$$\left(\Rightarrow \text{Put } D^2 = -(1)^2\right)$$

$$\Rightarrow e^{x} \cdot \frac{1}{-1} \cdot \sin x$$

$$\boxed{P.I. = -e^{x}\sin x}$$

$\Rightarrow$ Now complete solution is given by

$$y = C.F. + P.I.$$

$$\boxed{y = c_1 e^{x} + c_2 x e^{x} - e^{x}\sin x}$$ Ans

Q. Solve:

$(D^2 - 5D + 6) y = x^2 e^{3x}$

Sol → The given differential equation is

$(D^2 - 5D + 6) y = x^2 e^{3x}$ — (1)

it is already in symbolic form.

Here; $f(D) = D^2 - 5D + 6$

Now, for auxiliary equation

Put $f(m) = 0$

$m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$\boxed{m = 2, 3}$

Since values of m are real & distinct.

$C.F. = c_1 e^{2x} + c_2 e^{3x}$

Now; $\boxed{PI = \frac{1}{f(D)} . RHS}$

$= \frac{1}{D^2 - 5D + 6} . x^2 e^{3x}$

As we know that

$\frac{1}{f(D)} . e^{ax} x^2 = e^{ax} . \frac{1}{f(D+a)} . x^2$

$= e^{3x} . \frac{1}{(D+3)^2 - 5(D+3) + 6} . x^2$

$= e^{3x} . \frac{1}{(D^2 + 9 + 6D) - 5D - 15 + 6} . x^2$

$= e^{3x} . \frac{1}{D^2 + D} . x^2$

Good Write

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$= e^{3x} \cdot \frac{1}{D(D+1)} x^2$

$= e^{3x} \cdot \frac{1}{D}(1+D)^{-1} \cdot x^2$

(Now by Binomial expansion)

$= e^{3x} \cdot \frac{1}{D}\left[1+(-1)D+\frac{(-1)(-2)}{2!}D^2+\frac{(-1)(-2)(-3)}{3!}D^3+\ldots\right]x^2$

{ignoring higher power than $D^3$}

$= e^{3x} \cdot \left[\frac{1}{D}-1+D-D^2\right]x^2$

$= e^{3x} \cdot \left[\frac{1}{D}(x^2)-x^2+D(x^2)-D^2(x^2)\right]$

$= e^{3x} \cdot \left[\frac{x^3}{3}-x^2+2x-2\right]$

$\boxed{PI = \left(\frac{x^3-3x^2+6x-6}{3}\right)e^{3x}}$

Now complete solution is given as

$y = CF + PI$

Ans $\rightarrow$ $y = c_1 e^{2x} + c_2 e^{3x} + \left(\frac{x^3-3x^2+6x-6}{3}\right)e^{3x}$

$$\frac{d^2y}{dx^2} + 4y = x\sin 2x$$

Sol → The given differential equation

$$\frac{d^2y}{dx^2} + 4y = x\sin 2x \quad \text{(1)}$$

It is a linear differential equation.

$$D^2y + 4y = x\sin 2x$$

$$(D^2+4)y = x\sin 2x$$

which is the required symbolic form.

Here, $f(D) = D^2+4$

For auxiliary equation

Put $f(m)=0$

$$(m^2+4) = 0$$

$$m^2 = -4$$

$$\boxed{m = \pm 2i}$$

values of m are imaginary

$$C.F. = c_1\cos 2x + c_2\sin 2x$$

Now, $P.I. = \frac{1}{f(D)} \cdot RHS$

$$= \frac{1}{D^2+4} \cdot x\sin 2x$$

Now, as we know that $e^{i\theta} = \underset{RP}{\cos\theta} + i\underset{IP}{\sin\theta}$

$\cos\theta$ = Real part of $(e^{i\theta})$

$\sin\theta$ = ~~Real~~ Img part of $(e^{i\theta})$

Good Write

$$\Rightarrow \frac{1}{D^2+4}\, x \left[\text{Img part of } e^{i(2x)}\right]$$

$$\Rightarrow \text{I.P. of} \left[\frac{1}{D^2+4}\, x e^{2ix}\right]$$

$$\Rightarrow \text{I.P. of} \left[e^{2ix} \cdot \frac{1}{(D+2i)^2+4} \cdot x\right]$$

$$\left[\because \frac{1}{f(D)} e^{ax} x = e^{ax} \frac{1}{f(D+a)} x\right]$$

$$\Rightarrow \text{I.P. of} \left[e^{2ix} \frac{1}{D^2+4i^2+4Di+4} \cdot x\right]$$

$$\Rightarrow \text{I.P. of} \left[e^{2ix} \frac{1}{D^2-4+4Di+4} \cdot x\right]$$

$$\Rightarrow \text{I.P. of} \left[e^{2ix} \frac{1}{D^2+4Di}\, x\right]$$

$$= \text{I.P. of} \left[e^{2ix} \frac{1}{4Di\left(\frac{D}{4Di}+1\right)}\, x\right]$$

$$\Rightarrow \text{I.P. of} \left[e^{2ix} \cdot \frac{1}{4Di} \left(1+\frac{D}{4Di}\right)^{-1} x\right]$$

(By Binomial Expansion)

$$\Rightarrow \text{I.P. of} \left[e^{2ix} \cdot \frac{1}{4Di} \left\{1+(-1)\left(\frac{D}{4Di}\right) + \frac{(-1)(-2)}{2!}\left(\frac{D}{4Di}\right)^2 + \cdots\right\} x\right]$$

$$\Rightarrow \text{I.P. of} \left[e^{2ix} \frac{1}{4Di\left(\frac{D}{4i}+1\right)}\, x\right]$$

$$\Rightarrow \text{I.P. of} \left[e^{2ix} \frac{1}{4Di} \left(1+\frac{D}{4i}\right)^{-1} x\right]$$

$$\Rightarrow \text{I.P of } \left[ e^{2ix} \frac{1}{4Di} \left\{ 1 + (-1)\left(\frac{D}{4i}\right) + \frac{(-1)(-2)}{2i} \left(\frac{D}{4i}\right)^2 - - - \right\} x \right]$$

(By Binomial Expansion)

$$\Rightarrow \text{I.P. of } \left[ e^{2ix} \frac{1}{4Di} \left( 1 - \frac{D}{4i} - \frac{D^2}{16} \right) x \right]$$

(ignoring higher Power than $D^2$)

$$\Rightarrow \text{I.P. of } \left[ e^{2ix} \left( \frac{1}{4Di} + \frac{1}{16} - \frac{D}{64i} \right) x \right]$$

$$\Rightarrow \text{I.P. of } \left[ e^{2ix} \left( \frac{-i}{4D} + \frac{1}{16} + \frac{Di}{64} \right) x \right]$$

$$\left( \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = -i \right)$$

$$\Rightarrow \text{I.P. of } \left[ e^{2ix} \left( -\frac{ix}{4D} + \frac{x}{16} + \frac{iD(x)}{64} \right) \right]$$

$$\Rightarrow \text{I.P. of } \left[ e^{2ix} \left( -\frac{i}{4} \cdot \frac{x^2}{2} + \frac{x}{16} + \frac{i}{64} \right) \right]$$

$$\Rightarrow \text{I.P. of } \left[ e^{2ix} \left( \frac{-x^2}{8} i + \frac{x}{16} + \frac{i}{64} \right) \right]$$

$$\Rightarrow \text{I.P. of } \left[ (\cos 2x + i \sin 2x) \left( \frac{-x^2}{8} i + x/16 + i/64 \right) \right]$$

$$\Rightarrow \text{I.P. of } \left[ -\frac{x^2}{8} (\cos 2x) i + \frac{x^2}{8} \sin 2x + \frac{x}{16} \cos 2x + \frac{x}{16} (\sin 2x) i + \frac{(\cos 2x)}{64} i - \frac{(\sin 2x)}{64} \right]$$

$$\Rightarrow \boxed{\text{P.I.} = -\frac{x^2}{8} \cos 2x + \frac{x}{16} \sin 2x + \frac{1}{64} (\cos 2x)}$$

Now complete soln is given as

$y = \text{C.F.} + \text{P.I.}$

Ans $\rightarrow$ $y = c_1 \cos 2x + c_2 \sin 2x - \frac{x^2}{8} \cos 2x + \frac{x}{16} \sin 2x + \frac{\cos 2x}{64}$

# Method of Variation of Parameters (MOVOP)

Linear Differential Equation with constant coefficient

⇓

Complete / General soln

⇓

$y = C.F. + \underline{P.I.}$

↓

Symbolic form

⇓

$f(D) = ?$

⇓

Auxiliary Eqn

⇓

values of $m = ?$

⇓

C.F.

→ Differential operator method / Operator method

$P.I. = \frac{1}{f(D)} \underline{RHS}$

↓ Total 11 types of RHS

↳ Handle differently each RHS

→ Method of variate of Parameter

$$\boxed{P.I. = -y_1 \int \frac{y_2 \cdot X}{W} dx + y_2 \int \frac{y_1 \cdot X}{W} dx}$$

→ same way of handling each type of RHS

★ → Only applicable to 2nd order Differential Eqn

→ Integeration in each P.I.

→ Undetermined Coefficient method

P.I. = Trial Solution

→ Different as per RHS

→ Easy to solve algebric RHS

## MO VOP

**Method of Variation of Parameters** → To get P.I.

$$P.I. = -y_1 \int \frac{y_2 \cdot X}{W}\, dx + y_2 \int \frac{y_1 \cdot X}{W}\, dx$$

$y_1$ = coefficient of $c_1$ in C.F.

$y_2$ = coefficient of $c_2$ in C.F.

$X$ = R.H.S.

$W$ = Wronskian's coefficient = $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

**(*) Advantage**

→ Handle all 11 types of RHS with simple way.

**(*) Disadvantages**

→ Only applicable to 2nd order differential equation.

→ 2 Integrations are there in P.I.

**Q.** Using method of variation of parameter, solve:

$$y'' - 6y' + 9y = \frac{e^{3x}}{x^2} \qquad \text{(IPU - 2013, 2014)}$$

**Sol.** The given differential equation is

$$y'' - 6y' + 9y = \frac{e^{3x}}{x^2} \quad \text{— (i)}$$

it is a linear differential equation.

$$D^2 y - 6Dy + 9y = e^{3x}/x^2$$

$$(D^2 - 6D + 9)y = e^{3x}/x^2$$

which is required symbolic form.

Here; $\boxed{f(D) = D^2 - 6D + 9}$

Good Write

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Now for Auxiliary equation
Put $f(m) = 0$

$$m^2 - 6m + 9 = 0$$
$$(m-3)(m-3) = 0$$
$$\boxed{m = 3, 3}$$

Since values of m are real & ~~rest~~ repeated, so
$$\boxed{C.F. = c_1 e^{3x} + c_2 x e^{3x}}$$

Now by M.O.V.O.P.

$$\boxed{P.I. = -y_1 \int \frac{y_2 \cdot X}{w} dx + y_2 \int \frac{y_1 \cdot X}{w} dx} \quad (2)$$

Here, $y_1$ = coefficient of $c_1$ in C.F. $= e^{3x}$
$y_2$ = " " $c_2$ in C.F. $= x e^{3x}$
$X$ = R.H.S. $= e^{3x}/x^2$
$W$ = wronskian coefficient $= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & e^{3x} + 3x e^{3x} \end{vmatrix}$$

$$w = e^{3x}(3x e^{3x} + e^{3x}) - 3x e^{6x}$$
$$= \cancel{3e^{6x} x} + e^{6x} - \cancel{3x e^{6x}}$$
$$\boxed{w = e^{6x}}$$

Put the values in eq. (2), we get

$$P.I. = -e^{3x} \int \frac{x e^{3x}}{e^{6x}} \left(\frac{e^{3x}}{x^2}\right) dx + x e^{3x} \int \frac{e^{3x}}{e^{6x}} \left(\frac{e^{3x}}{x^2}\right) dx$$

$$= -e^{3x} \int \frac{1}{x} dx + x e^{3x} \int x^{-2} dx$$

$$= -e^{3x} \log|x| + x e^{3x} \left(\frac{x^{-1}}{-1}\right)$$

$$P.I. = -e^{3x}\log x - e^{3x}$$

$$\boxed{P.I. = -e^{3x}(1+\log x)}$$

Now complete solution is

$$y = C.F. + P.I.$$

$$\boxed{y = c_1 e^{3x} + c_2 x e^{3x} - e^{3x}(1+\log x)}$$ Ans

SPP D P

Q. Using M.O.V.O.P., solve:

(IPU 2013)

$$\frac{d^2y}{dx^2} + 4y = \tan 2x$$

Sol→ The given differential equation is

$$\frac{d^2y}{dx^2} + 4y = \tan 2x \quad \text{--- (i)}$$

it is a linear differential equation.

Now;

$D^2 y + 4y = \tan 2x$

$(D^2+4) y = \tan 2x$

which is the required symbolic form.

Here; $\boxed{f(D) = D^2+4}$

For auxiliary equation

Put $f(m)=0$

$m^2+4=0$

$m^2 = -4$

$\boxed{m = \pm 2i}$

Good Write

→ Since values of m are imaginary so

$$\boxed{C.F. = c_1 \cos 2x + c_2 \sin 2x}$$

Now by using Method of variation of parameter

$$\boxed{P.I. = -y_1 \int \frac{y_2 \cdot X}{W} dx + y_2 \int \frac{y_1 X}{W} dx} \quad (2)$$

$y_1$ = coefficient of $c_1$ in C.F. = $\cos 2x$

$y_2$ = " " $c_2$ " " = $\sin 2x$

$X$ = R.H.S. = $\tan 2x$

$W$ = wronskian's coefficient = $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\sin^2 2x + \cos^2 2x)$$

$$= 2 \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$\boxed{W = 2}$$

Put all values in eq (2), we get

$$P.I. = -\cos 2x \int \frac{(\sin 2x)(\tan 2x)}{2} dx + \sin 2x \int \frac{(\cos 2x)(\tan 2x)}{2} dx$$

$$= -\frac{\cos 2x}{2} \int \frac{\sin^2 2x}{\cos 2x} dx + \frac{\sin 2x}{2} \int \sin 2x \, dx$$

$$= -\frac{\cos 2x}{2} \int \frac{1 - \cos^2 2x}{\cos 2x} dx + \frac{\sin 2x}{2} \left( -\frac{\cos 2x}{2} \right)$$

$$= -\frac{\cos 2x}{2} \int (\sec 2x - \cos 2x) dx + \left( \frac{-\sin 2x \cos 2x}{4} \right)$$

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$$= \frac{-\cos 2x}{2}\left[\frac{\log|\sec 2x+\tan 2x|}{2}-\frac{\sin 2x}{2}\right]-\frac{\sin 2x\cos 2x}{4}$$

$$\left[\because \int \sec\theta\, d\theta = \log|\sec\theta+\tan\theta|\right]$$

$$= \frac{-\cos 2x}{4}\log|\sec 2x+\tan 2x| + \cancel{\frac{\cos 2x\sin 2x}{4}} - \cancel{\frac{\sin 2x\cos 2x}{4}}$$

$$\boxed{P.I. = -\frac{\cos 2x}{4}\log|\sec 2x+\tan 2x|}$$

~~Now,~~

As, we know that complete soln is written as

$y = C.F. + P.I.$

Ans $y = c_1\cos 2x + c_2\sin 2x - \frac{\cos 2x}{4}(\log|\sec 2x+\tan 2x|)$

# Gamma Function

The Gamma Function is denoted by '$\Gamma n$' (read as gamma 'n') where $n>0$, and denoted by a definite integral.

$$\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx} \quad (n>0)$$

Eg → $\Gamma 5 = \int_0^\infty e^{-x} x^{5-1} dx$

□ Properties of Gamma Function

1) $\Gamma 1 = 1$
2) $\Gamma n+1 = n\Gamma n = \lfloor n$
3) $\Gamma n = z^n \int_0^\infty e^{-zx} x^{n-1} dx \quad (n, z > 0)$
4) $\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$
5) $\Gamma n+1 = \int_0^\infty e^{-y^{(1/n)}} dy$
6) $\Gamma 1/2 = \sqrt{\pi}$

Q. Prove that:

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$$

Sol→ $\boxed{L.H.S.} = \int_0^\infty e^{-x^2}\,dx$

Let $x^2 = t$ (means $x = \sqrt{t}$)

$2x\,dx = dt$

$dx = \frac{dt}{2x}$

$\boxed{dx = \frac{dt}{2\sqrt{t}}}$ (No change in limits)

$\Rightarrow \int_0^\infty e^{-t}\frac{dt}{2\sqrt{t}}$

$\Rightarrow \frac{1}{2}\int_0^\infty e^{-t}\,t^{-1/2}\,dt$ (Gamma function $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$)

$\Rightarrow \frac{1}{2}\Gamma\frac{1}{2} \Rightarrow \frac{1}{2}\sqrt{\pi} = R.H.S.$

$(\Gamma\frac{1}{2} = \sqrt{\pi})$

L.H.S. = R.H.S.

Hence Proved.

Q. $\int_0^\infty \sqrt{x}\,e^{-3\sqrt{x}}\,dx$ . Find the answer.

Sol. Let $3\sqrt{x} = t$ (means $\sqrt{x} = t/3$)

$9x = t^2$

$9dx = 2t\,dt$

$\boxed{dx = \frac{2t\,dt}{9}}$ (No change in limit)

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$$\Rightarrow \int_0^\infty \left(\frac{t}{3}\right) e^{-t} \frac{2t\,dt}{9}$$

$$\Rightarrow \frac{2}{27}\int_0^\infty e^{-t} t^2\,dt \qquad \text{(Gamma function)}$$

$$\Rightarrow \frac{2}{27}\Gamma 3 \Rightarrow \frac{2}{27}\times(2\times 1) \Rightarrow \frac{4}{27} \text{ Ans}$$

$(\Gamma(n+1) = n\Gamma n)$