Wronskian and Its Properties

We shall discuss some fundamental properties of a second order linear differential equation of the form

$$\left.\begin{aligned} a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \\ \text{or} \\ a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \end{aligned}\right\} \quad \text{---}(A)$$

We shall assume that $a_0(x)$, $a_1(x)$ & $a_2(x)$ are continuous on an interval (a, b) and $a_0(x) \neq 0$ for each $x \in (a, b)$.

Some important definitions :-

Definition 1 :- A real-valued function $u(x)$ is said to be identically zero on an interval (a, b), written as $u(x) \equiv 0$, if $u(x) = 0$, $\forall\ x \in (a, b)$.

Definition 2 :- A function $u(x)$ is said to be a solution of the equation (A) if

$$a_0(x)u''(x) + a_1(x)u'(x) + a_2(x)u(x) = 0 \ \forall x \in (a, b).$$

Definition 3 :- Two solutions $u_1(x)$ and $u_2(x)$ of (A) are said to be linearly dependent if there exist constants c_1 & c_2 not both zero, such that

$$c_1u_1(x) + c_2u_2(x) = 0 \quad \forall\ x \in (a, b).$$

Definition 4 :- If two solutions $u_1(x)$ & $u_2(x)$ are not linearly dependent, they are said to be linearly independent.

In other words, two solutions $u_1(x)$ and $u_2(x)$ are said to be linearly independent if

$c_1u_1(x) + c_2u_2(x) = 0$ implies $c_1 = c_2 = 0$,

$x \in (a, b)$

Definition 5:- The wronskian of two solutions $u_1(x)$ & $u_2(x)$ of (A) is defined as the determinant

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix}$$

$$= u_1(x)u_2'(x) - u_2(x)u_1'(x)$$

Definition 6:- The wronskian of three solutions y_1, y_2, y_3 of a third order differential equation is defined as

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Remark:- The wronskian determines whether the solutions are linearly independent or linearly dependent.

> Two solutions $y_1(x)$ and $y_2(x)$ are linearly independent if and only if their wronskian is not zero at some point $x_0 \in (a,b)$.

Question 1:- Show that $y = x + x\log x - 1$ is the unique solution of $xy'' - 1 = 0$ satisfying $y(1) = 0$ and $y'(1) = 2$.

Solution:- $xy'' - 1 = 0$

$$\frac{d^2y}{dx^2} = \frac{1}{x}$$

Integrating with respect to x

$$\frac{dy}{dx} = \log x + c_1 \quad — ①$$

Integrating again

$$y = x\log x - x + c_1 x + c_2 \quad — ②$$

Now $y(1) = 0 \Rightarrow (1\log 1 - 1) + c_1(1) + c_2 = 0$

$\Rightarrow c_1 + c_2 = 1. \quad — ③$

Now $y'(1) = 2 \Rightarrow \log 1 + c_1 = 2 \quad — ④$

$\Rightarrow c_1 = 2$

by using ③ & ④, we get

$c_1 = 2$ & $c_2 = -1$

putting $c_1 = 2$ & $c_2 = -1$ in (2), we get

$y = x + x\log x - 1$ is the unique solution

Fundamental Existence Theorem

Let $a_0(x), a_1(x), a_2(x)$ and $R(x)$ be continuous functions on an interval (a,b) and $a_0(x) \neq 0$ for each $x \in (a,b)$.

If c_0 & c_1 are arbitrary real numbers and $x_0 \in (a,b)$, then there exists a unique solution $y(x)$ of

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = R(x)$$

satisfying $y(x_0) = c_0$, $y'(x_0) = c_1$.

Further this solution is defined over the interval (a,b).

Question 2:- Show that $y = 3e^{2x} + e^{-2x} - 3x$ is the unique solution of the initial value problem $y'' - 4y = 12x$, where $y(0) = 4$, $y'(0) = 1$.

Solution:- We have $y = 3e^{2x} + e^{-2x} - 3x$.

$$\therefore \quad y' = 6e^{2x} - 2e^{-2x} - 3$$

$$y'' = 12e^{2x} + 4e^{-2x}$$

$$y'' - 4y = 12e^{2x} + 4e^{-2x} - 4(3e^{2x} + e^{-2x} - 3x) = 12x$$

Thus, $y = 3e^{2x} + e^{-2x} - 3x$ is a solution of

$y'' - 4y = 12x$, where

$y(0) = 3 + 1 - 0 = 4$, $y'(0) = 6 - 2 - 3 = 1$.

From $y'' - 4y = 12x$, we see that

$a_0(x) = 1$, $a_1(x) = 0$, $a_2(x) = -4$, $R(x) = 12x$

are continuous functions in $(-\infty, \infty)$ and $a_0(x) \neq 0$ for each x in $(-\infty, \infty)$. So by Fundamental

Existence Theorem, $y = 3e^{2x} + e^{-2x} - 3x$ is the unique solution of $y'' - 4y = 12x$ with $y(0) = 4$, $y'(0) = 1$.

Question 3:- Show that $y = \frac{1}{4}\sin 4x$ is a unique solution of the initial value problem $y'' + 16y = 0$ with $y(0) = 0$, $y'(0) = 1$.

Solution:- We have $y' = \cos 4x$

$$y'' = -4\sin 4x$$

Now, $y'' + 16y = -4\sin 4x + 16\left(\frac{1}{4}\sin 4x\right) = 0$.

Thus $y = \frac{1}{4}\sin 4x$ is a solution of $y'' + 16y = 0$, where $y(0) = 0$, $y'(0) = 1$.

Since $a_0(x) = 1 \neq 0$ for each x in $(-\infty, \infty)$, it follows from Fundamental Existence Theorem that, $y = \frac{1}{4}\sin 4x$ is a unique solution of $y'' + 16y = 0$ with $y(0) = 0$, $y'(0) = 1$

Theorem 1:- If $y_1(x)$ & $y_2(x)$ are any two solutions of $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ then the linear combination $c_1 y_1(x) + c_2 y_2(x)$, where c_1 & c_2 are constants is also a solution of the given equation.

Theorem 2:- There exist two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

such that its every solution $y(x)$ may be written as

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad x \in (a, b)$$

where c_1 and c_2 are suitably chosen constants.

Theorem 5:- Two solutions $y_1(x)$ & $y_2(x)$ of the equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ are linearly dependent if and only if their wronskian is identically zero.

Corollary:- Two solutions $y_1(x)$ & $y_2(x)$ are linearly independent if and only if their wronskian is not zero at some point $x_0 \in (a,b)$.

Question 4:- Show that $\sin x$ and $\cos x$ are linearly independent solutions of $y'' + y = 0$

Solution:- If $y = \sin x$

then $y' = \cos x$

$y'' = -\sin x$

Thus $y'' + y = 0$ and so $\sin x$ is a solution of $y'' + y = 0$

Similarly, $\cos x$ is a solution of $y'' + y = 0$

The wronskian of $y_1(x) = \sin x$ & $y_2(x) = \cos x$ is

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0$$

Hence, $y_1(x) = \sin x$ and $y_2(x) = \cos x$ are linearly independent solutions of $y'' + y = 0$.

Question 5:- Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are linearly independent solutions of $y'' + y = 0$. Determine constants c_1 and c_2 so that the solution

$$\sin x + 3\cos x \equiv c_1 y_1(x) + c_2 y_2(x).$$

Solution:- We have $y_2' = \cos x + \sin x$ &

$$y_2'' = -\sin x + \cos x$$

Thus, $y_2'' + y_2 = 0$ and so y_2 is a solution of $y'' + y = 0$.

Clearly $y_1 = \sin x$ is a solution of $y'' + y = 0$.

The wronskian of $y_1(x)$ & $y_2(x)$ is

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \sin x - \cos x \\ \cos x & \cos x + \sin x \end{vmatrix}$$

$$= (\sin x \cos x + \sin^2 x) - (\sin x \cos x - \cos^2 x)$$

$$= 1 \neq 0$$

Hence, y_1 & y_2 are linearly independent solutions of $y'' + y = 0$.

Consider the relation

$$\sin x + 3\cos x \equiv c_1 y_1(x) + c_2 y_2(x)$$

i.e., $\sin x + 3\cos x \equiv c_1 \sin x + c_2(\sin x - \cos x)$

Comparing the coefficients of $\sin x$ and $\cos x$ on both the sides, we obtain

$$c_1 + c_2 = 1 \quad \& \quad -c_2 = 3$$

Hence $\quad c_1 = 4$ & $c_2 = -3$

Exercise:

Q1) If $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$ are two solutions of $y'' + 9y = 0$, show that $y_1(x)$ & $y_2(x)$ are linearly independent solutions.

Q2) Show that linearly independent solutions of $y'' - 3y' + 2y = 0$ are e^x & e^{2x}.

Find the solution $y(x)$ with the property that $y(0) = 0$, $y'(0) = 1$.