Interpolation With Equal Intervals

INTERPOLATION FOR EQUI-SPACED VALUES

Interpolation is often needed in engineering and Scientific and statistical problems. Here we discuss two important interpolation formulae employing the Forward and backward differences of a function.

1 Newton's Forward Interpolation formula

OR

Newton-Gregory Forward Differences Interpolation Formula

Let the function $y = f(x)$ take the values $y_0, y_1, y_2 \ldots$ corresponding to the values $x_0, x_0+h, x_0+2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number. Then

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{\lfloor 4} \Delta^4 y_0 + \cdots$$

is called Newton's forward Interpolation formula

Here $x = x_0 + ph$

Newton's Backward Interpolation formula

Let the function $y=f(x)$ take the values $y_0, y_1, y_2, \cdots$ corresponding to the values $x_0, x_0+h, x_0+2h, \cdots$ of x. suppose it is required to evaluate $f(x)$ for $x=x_0+ph$, where p is any real number. Thus we have

$$y_p = y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

$$\frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \cdots$$

Where $x = x_n + ph$

Find the cubic polynomial which takes the following values:

x :	0	1	2	3
$f(x)$:	1	2	1	10

Sol^n

Forward difference table

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
			-2	
1	2			12
		-1		
			10	
2	1			
		9		
3	10			

By Newton Forward Interpolation formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{\underline{|2}}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\underline{|3}}\Delta^3 y_0 + \cdots$$

$\because \quad x = x_0 + ph$

$x = 0 + p(1)$

$\therefore \quad p = x$

$$y_x = 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 1 + x - x(x-1) + 2(x)(x^2-3x+2)$$

$$= 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x$$

$$y_x = 2x^3 - 7x^2 + 6x + 1$$

$$\therefore \boxed{f(x) = 2x^3 - 7x^2 + 6x + 1} \text{ Ans}$$

Q. Find the value of $\sin 52^\circ$ from the given data

θ° :	45°	50°	55°	60°
$\sin\theta$:	.7071	.7660	.8192	.8660

Sol^n

Difference table is

$x=\theta$	$y=\sin\theta$	Δy	$\Delta^2 y$	$\Delta^3 y$
45	.7071			
		.0589		
			−.0057	
50	.7660			−.0007
		.0532		
			−.0064	
55	.8192			
		.0468		
60	.8660			

By Newton Interpolation forward formula, we know that

$$y_x = y_0 + P\Delta y_0 + \frac{P(P-1)}{\lfloor 2}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{\lfloor 3}\Delta^3 y_0 + \cdots$$

Here $X = x_0 + Ph$

$52 = 45 + P(5)$

$\frac{52-45}{5} = P \Rightarrow P = 1.4$

$$\therefore y_{52} = .7071 + 1.4(.0589) + \frac{1.4(.4)}{2}(-.0057) + \frac{1.4(.4)(-.6)}{6}(-.0007)$$

$$= .7071 + .08246 - .001596 + .0000392$$

$$y_{52} = .7880032$$

i.e $\sin 52^\circ = .7880$

Ans

From the following table, find the value of $e^{.24}$

x : .1 .2 .3 .4 .5

e^x : 1.10517 1.22140 1.34986 1.49182 1.64872

Sol^n

The difference Table is

x	$y = e^x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
.1	1.10517				
		.11623			
			.01223		
.2	1.22140			.00127	
		.12846			.00017
			.01350		
.3	1.34986			.00144	
		.14196			
			.01494		
.4	1.49182				
		.15690			
.5	1.64872				

By Newton Forward difference formula is

$$y_x = y_0 + P\Delta y_0 + \frac{P(P-1)}{\lfloor 2}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{\lfloor 3}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{\lfloor 4}\Delta^4 y_0$$

$\because$ $x = x_0 + Ph$

$.24 = .1 + P(.1) \Rightarrow \boxed{P = 1.4}$

$$y_{.24} = 1.10517 + 1.4\times.11623 + \frac{1.4\times.4}{2}(.01223) + \frac{1.4\times.4(-.6)}{6}(.00127) + \frac{1.4\times.4(-.6)(-1.6)}{24}(.00017)$$

$$= 1.10517 + .162722 + .0034244 - .00007112 + .000003808$$

$$= 1.27124$$

Ans

Q The following table gives the marks secured by 100 students in the numerical Analysis subject:

Range of marks :	30-40	40-50	50-60	60-70	70-80
No. of students :	25	35	22	11	7

Use Newton's Forward difference interpolation formula to find

(i) The Number of students who ~~secured~~ got more than 55 marks

(ii) The Number of students who secured marks in the range from 36 to 45.

Solution

The given data is re-arranged as follows

Marks obtained	No. of students
Less than 40	25
Less than 50	25+35 = 60
Less than 60	60+22 = 82
Less than 70	82+11 = 93
Less than 80	93+7 = 100

(i) First we find the number of students who got less than 55 marks

Difference Table

Marks obtained Less than (x)	No. of students = y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	25				
		35			
50	60		-13		
		22		2	
60	82		-11		5
		11		7	
70	93		-4		
		7			
80	100				

Newton Forward difference formula

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{\lfloor 4}\Delta^4 y_0$$

$$\because x = x_0 + ph$$
$$55 = 40 + p(10)$$
$$15 = p(10) \Rightarrow p = 1.5$$

$$\therefore y_{55} = 25 + 1.5(35) + \frac{1.5(.5)}{2}(-13) + \frac{(1.5)(.5)(-.5)}{6}(2) + \frac{(1.5)(.5)(-.5)(-1.5)}{24}(5)$$

$$= 25 + 52.5 - 4.875 - \overset{-.125}{\cancel{.375}} + .11718$$

$$y_{55} = 72.6170 \approx 73$$

There are 73 students who got less than 55 marks

$\therefore$ Number of students who got more than 55 marks $= 100 - 73 = 27$ Ans

(ii) To calculate the number of students securing marks between 36 and 45, take the difference of y_{45} and y_{36}

First we find out y_{36}

$$x = x_0 + ph$$
$$36 = 40 + p(10) \Rightarrow p = -.4$$

$$\therefore y_{36} = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{\lfloor 4}\Delta^4 y_0$$

$$= 25 + (-.4)(35) + \frac{(-.4)(-.4-1)}{2}(-13) + \frac{(-.4)(-.4-1)(-.4-2)}{6}(2) + \frac{(-.4)(-.4-1)(-.4-2)(-.4-3)}{24}(5)$$

$$= 25 - 14 - 3.64 - .448 + .952$$

$$y_{36} = 7.864 \approx 8$$

Also

$$y_{45} = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-}{\lfloor 4}$$

Here $x = x_0 + ph$

$$45 = 40 + p(10) \Rightarrow p = .5$$

$$y_{45} = 25 + (.5)(35) + \frac{(.5)(.5-1)}{2}(-13) + \frac{.5(.5-1)(.5-2)}{6}(2) + \frac{(.5)(.5-1)(.5-2)(.5-3)}{24}(5)$$

$$= 25 + 17.5 + 1.625 + .125 - .1953125$$

$$y_{45} = 44.0546 \simeq 44$$

Hence the number of students who secured marks in the range from 36 to 45 is

$$y_{45} - y_{36} = 44 - 8 = 36 \quad \text{Ans}$$

Q1 The Population of city in the ~~decimal~~ decennial census is given in the following table :

Year x	1891	1901	1911	1921	1931
Population y (in thousand)	46	66	81	93	101

Estimate the population of city in year 1895 and 1925

Solⁿ First write the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

To compute the population in 1895 we use Newton forward differences formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{\lfloor 4}\Delta^4 y_0$$

Here $x = x_0 + ph$

$1895 = 1891 + p(10)$

$p = \frac{1895-1891}{10} = .4$

$$y_{1895} = 46 + .4(20) + \frac{.4(.4-1)}{2}(-5) + \frac{.4(.4-1)(.4-2)}{6}(2) + \frac{.4(.4-1)(.4-2)(.4-3)}{24}(-3)$$

$$= 46 + 8 + .6 + .128 + .1248 = 54.852, \text{ Thousand.}$$

Ans

To compute the population in 1925, we use the Newton backward difference formula.

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{\lfloor 4}\nabla^4 y_n$$

Here $x = x_n + ph$

$1925 = 1931 + p(10) \Rightarrow p = -.6$

$$y_{1925} = 101 + (-.6)(8) + \frac{(-.6)(-.6+1)}{2}(-4) + \frac{(-.6)(-.6+1)(-.6+2)}{6}(-1) + \frac{(-.6)(-.6+1)(-.6+2)(-.6+3)}{24}(-3)$$

$$= 101 - 4.8 + .48 + .056 + .1008 = 96.84 \text{ Thousand}$$

Ans

Q.2. Find the cubic polynomial which takes on the values $f(0) = 4$, $f(1) = 1$, $f(2) = 2$, $f(3) = 11$, $f(4) = 32$, $f(5) = 71$. Also find $f(6)$ and $f(2.5)$.

Soln According to given

x	0	1	2	3	4	5
$f(x) = y$	4	1	2	11	32	71

Difference Table

	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	4					
		-3				
1	1		4			
		1		4		
2	2		8		0	2
		9		4		
3	11		12		2	
		21		6		
4	32		18			
		39				
5	71					

By Newton Forward Interpolation formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{\underline{|2}}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\underline{|3}}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{\underline{|4}}\Delta^4 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)(p-4)}{\underline{|5}}\Delta^5 y_0$$

$$= 4 + p(-3) + \frac{p^2-p}{2}(4) + \frac{(p^2-p)(p-2)}{6}(4) + \frac{(p^2-p)(p-2)(p-3)}{24}(0)$$

$$+ \frac{(p^2-p)(p-2)(p-3)(p-4)}{120} \times 2$$

$$y_x = 4 - 3p + 2(p^2-p) + \frac{2}{3}[p^3-3p^2+2p] + 0 + \frac{1}{60}[(p^3-3p^2+2p)(p^2-7p+12)$$

$$= 4 - 3p + \cancel{2p^2} - 2p + \frac{2}{3}p^3 - \cancel{2p^2} + \frac{4}{3}p + \cdots$$

$$\boxed{y_x = 4 - \frac{11}{3}p + \frac{2}{3}p^3}$$

$$y_6 = 4 - \frac{11}{3}\times 6 + \frac{2}{3}\times 6^3 = 4 - 22 + \frac{2\times 6\times 6\times 6}{3} = 126 \text{ Ans}$$

$$y_{2.5} = 4 - \frac{11}{3}(2.5) + \frac{2}{3}(2.5)^3 = 4 - 9.166 + 10.416 = \underline{\underline{5.25}} \text{ Ans}$$

Q.3. Following table gives the grouped data for number of students lying in various weight groups:

Wt.	0-40	40-60	60-80	80-100	100-120
No. of students	250	120	100	70	50

Find the number of students having weight between 60 and 70.

Soln. We prepare cumulative frequency distribution and finite difference table.

x Weight	y No. of students	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
below 40	250				
		+120			
below 60	370		−20		
		+100		−10	
below 80	470		−30		20
		+70		10	
below 100	540		−20		
		+50			
below 120	590				

Now we calculate from the above table the number of students having weight less than 70 by using Newton Forward difference interpolation formula.

$\therefore \; x = x_0 + ph$

$70 = 40 + p(20)$

$\boxed{p = 1.5}$

$$y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{\lfloor 4}\Delta^4 y_0$$

$$y_0 = 250 + 1.5\times 120 + \frac{(1.5)(1.5)}{2}\times(-20) + \frac{(1.5)(.5)(-.5)}{6}(-10)$$

$$+ \frac{(1.5)(.5)(-.5)(-1.5)}{24}(20)$$

$$= 250 + 180 - 7.5 + .625 + .46875 = 423.59$$

$$= 424$$

Hence Number of students having weight ~~was~~ between 60 and 70 is

$424 - 370 = \underline{54}$

Q.4. From the following table estimate the number of students who obtained marks between 40 and 45.

Marks x	30-40	40-50	50-60	60-70	70-80
No. of students y	31	42	51	35	31

Sol^n First we prepare a cumulative frequency and diff. table

Marks less than (x)	No. of student y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	191				

Now $x = x_0 + ph$

$45 = 40 + p(10)$

$\boxed{p = .5}$

By Newton Forward diff. formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{\lfloor 4}\Delta^4 y_0$$

$$= 31 + .5 \times 42 + \frac{(.5)(-.5)}{2}(9) + \frac{(.5)(-.5)(-1.5)}{6} \times (-25)$$

$$+ \frac{(.5)(-.5)(-1.5)(-2.5)}{24}(37)$$

$$= 31 + 21 - 1.125 \cancel{+} 1.5625 - 1.4453 = 47.9047$$

$$\approx 48$$

∴ Number of students getting marks between 40 and 45

$= 48 - 31 = 17$ Ans

3. Apply Newton's Backwards difference formula to find a Polynomial of degree three which includes the following x, y pairs.

x : 3 4 5 6

y : 6 24 60 120

Sol^n Difference Table is

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$
3	6			
		18		
4	24		18	
		36		6
5	60		24	
		60		
6	120			

$\because \quad X = x_n + ph$

$x = 6 + p(1) \Rightarrow \boxed{p = x - 6}$

By Newton Backward Interpolation formula

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{\underline{2}}\nabla^2 y_n + \frac{p(p+1)(p+2)}{\underline{3}}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{\underline{4}}\nabla^4 y_n$$

$$= 120 + (x-6)(60) + \frac{(x-6)(x-5)}{2}(24) + \frac{(x-6)(x-5)(x-4)}{6} \times 6$$

$$= x^3 - 27x^2 \quad \text{Ans}$$

Q. Evaluate $f(3.8)$ from following table

x : 0 1 2 3 4

$f(x)$: 1 1.5 2.2 3.1 4.6

Backward difference Table

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0	1				
		.5			
1	1.5		.2		
		.7		0	
2	2.2		.2		.4
		.9		.4	
3	3.1		.6		
		1.5			
4	4.6				

We know that $X = x_n + ph$

$$3.8 = 4.6 + P(1) \Rightarrow \boxed{P = -.2}$$

By Newton Backward Interpolation formula

$$y_x = y_n + P\nabla y_n + \frac{P(P+1)}{\lfloor 2}\nabla^2 y_n + \frac{P(P+1)(P+2)}{\lfloor 3}\nabla^3 y_n + \frac{P(P+1)(P+2)(P+3)}{\lfloor 4}\nabla^4 y_n$$

$$y_{3.8} = 4.6 + (-.2)(1.5) + \frac{(-.2)(-.2+1)}{2}\times .6 + \frac{(-.2)(-.2+1)(-.2+2)}{6}\times .4$$

$$+ \frac{(-.2)(-.2+1)(-.2+2)(-.2+3)}{24}\times .4$$

$$= 4.6 - .3 \cancel{+} .048 - .0192 - .01344 = 4.2193 \text{ Ans}$$

Q. From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policy maturing at the age of 63:

Age :	45	50	55	60	65
Premium : in Rupees	114.84	96.16	83.32	74.48	68.48

The difference Table

Age x	Premium (y)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.32		4		.68
		-8.84		-1.16	
60	74.48		2.84		
		-6			
65	68.48				

$\because \; x = x_n + ph \Rightarrow 63 = 65 + p(5)$

$p = -\frac{2}{5} = -.4$

By Newton Backward Interpolation formula

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2} \nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3} \nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{\lfloor 4} \nabla^4 y_n$$

$$y_{63} = 68.48 + (-.4)(-6) + \frac{(-.4)(-.4+1)}{2}(2.84) + \frac{(-.4)(-.4+1)(-.4+2)}{6} \times (-1.16)$$

$$+ \frac{(-.4)(-.4+1)(-.4+2)(-.4+3)}{24} \times .68$$

$$= 68.48 + 2.4 - .3408 + .07424 - .028288$$

$$= 70.585 \text{ Ans}$$