Population Dynamics

Rate of growth of population is proportional. For example, the bacteria population grows at a rate proportional to the population i.e., the growth rate dx/dt is proportional to x, where $x = x(t)$ denotes the number of bacteria present at time t. This fact can be represented by means of the differential equation:

$$\boxed{\frac{dx}{dt} = kx}$$

where k is a positive constant of proportionality.

Question 1:- A certain culture of bacteria grows at a rate that is proportional to the number present. It is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours?

Solution:- Let x denote the number of bacteria present at time t hours.

Then $\frac{dx}{dt} = kx$

$$\frac{dx}{x} = k\,dt$$

Integrating both sides, we get

$$\log x = kt + \log c$$

$$\log x = kt \log e + \log c$$

$$\log x = \log e^{kt} \cdot c$$

$$x = ce^{kt} \quad \text{--- (1)}$$

Assuming that $x = x_0$ at time $t = 0$

then $c = x_0$

$\therefore$ (1) can be written as $x = x_0 e^{kt}$ — (2)

We are given that if $t=4$, then $x=2x_0$. from(2)

$$2x_0 = x_0 e^{4t}$$

and so $e^{4t}=2$

If $t=12$, then by (1), $x = x_0 e^{12k}$

$$\therefore \quad x = x_0 (e^{4k})^3 = x_0 (2)^3 = 8x_0$$

Hence the growth of bacteria is 8 times the original number at the end of 12 hours.

Question 2:- The number of bacteria in a yeast culture grows at a rate which is proportional to the number present. If the population of a colony of yeast bacteria triples in 1 hour, find the number of bacteria which will be present at the end of 5 hours.

Solution:- Let x denote the number of bacteria in a yeast culture ~~gro~~ present at time t hours.

Then $$\frac{dx}{dt} = kx$$

$$\frac{dx}{x} = k\,dt$$

Integrating both sides, we get

$$\log x = kt + \log c$$

$$\log x = \log e^{kt}.c$$

$$x = c.e^{kt}$$

Assuming that $x = x_0$ at time $t=0$, then $c = x_0$

$$\therefore \quad x = x_0 e^{kt} \qquad \text{——(1)}$$

We are given that, $x = 3x_0$, when $t=1$.

$$\therefore \quad 3x_0 = x_0 e^{k}$$

$$e^{k} = 3$$

Now $t=5 \Rightarrow \quad x = x_0 e^{5k} = x_0 (e^{k})^5 = x_0 (3)^5.$

Hence at the end of five hours the number of bacteria is 243 times the number of bacteria present initially.

Question 3:- If the population of a country doubles in 50 years, in how many years will it triple under the assumption that the rate of increase is proportional to the number of inhabitants?

Solution:- Let x denote the population after t years, and x_0 the population at time $t=0$. Then

$$\frac{dx}{dt} = kx, \text{ } k \text{ being a constant of proportionality}$$

$$\Rightarrow \frac{dx}{dt} = kx.$$

$$\frac{dx}{x} = k\,dt$$

Integrating,

$$\log x = kt + \log c$$

$$x = c \cdot e^{kt}$$

Since $x = x_0$, when $t=0$

$\therefore c = x_0$

Thus, $x = x_0 e^{kt}$ ——— (1)

When $t = 50$, we are given that $x = 2x_0$

$$2x_0 = x_0 e^{50k}$$

$$e^{50k} = 2$$

When $x = 3x_0$, then (1) gives

$3 = e^{kt}$, and so

$$3^{50} = e^{50kt}$$

$$3^{50} = (e^{50k})^t$$

$3^{50} = 2^t$

taking log on both sides

$50 \log 3 = t \log 2$

Hence, $t = \frac{50 \log 3}{\log 2} = 79$ years (approx)

Question 4:- A colony of bacteria increases at a rate proportional to the amount present. If the number of bacteria doubles in 1 hour, how long does it take for the colony to attain four times its initial size?

Solution: The governing differential eq^n is

$$\frac{dx}{dt} = kx, \quad k > 0$$

Its solution is of the form

$$x = x_0 e^{kt} \qquad \text{—— (1)}$$

Here x_0 gives the initial size of the colony.

In 1 hour the size of the colony is $2x_0$, so

$$2x_0 = x_0 e^k$$

$$\Rightarrow e^k = 2$$

The time for the colony to attain four times its size is given by

$$4x_0 = x_0 e^{kt}$$

$$4 = e^{kt} = (e^k)^t = 2^t$$

Hence $t = \frac{\log 4}{\log 2} = \frac{2 \log 2}{\log 2} = 2$ hours.