Quadratic Forms

Definition:- A homogeneous polynomial of second degree in any number of variables is called a quadratic form. For example,

(i) $ax^2 + 2hxy + by^2$

(ii) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$

(iii) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2lxw + 2myw + 2nzw$

are quadratic forms in two, three and four variables.

In n-variables $x_1, x_2, \ldots, x_n$, the general quadratic form is $\boxed{\sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij} x_i x_j}$

In the expansion, the co-efficient of $x_i x_j = (b_{ij} + b_{ji})$.

Suppose $2a_{ij} = b_{ij} + b_{ji}$ where $a_{ij} = a_{ji}$ & $a_{ii} = b_{ii}$

$$\therefore \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij} x_i x_j = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j \text{ where } a_{ij} = \frac{1}{2}(b_{ij} + b_{ji}).$$

Hence every quadratic form can be written as $\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j = X'AX$, so that the matrix A is always symmetric, where $A = [a_{ij}]$ & $X' = [x_1, x_2, \ldots, x_n]$.

Rule to write the matrix A of quadratic form

$$A = \begin{bmatrix} \text{co-eff. of } x_1^2 & \frac{1}{2}\text{co-eff. of } x_1x_2 & \frac{1}{2}\text{co-eff. of } x_1x_3 & \cdots \\ \frac{1}{2}\text{co-eff. of } x_2x_1 & \text{co-eff. of } x_2^2 & \frac{1}{2}\text{co-eff. of } x_2x_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Now writing the above said examples of quadratic forms in matrix form, we get

(i) $ax^2+2hxy+by^2 = [x\ y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(ii) $ax^2+by^2+cz^2+2hxy+2gyz+2fzx$

$$= [x \quad y \quad z]\begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(iii) $ax^2+by^2+cz^2+dw^2+2hxy+2gyz+2fzx+$ $2lxw+2myw+2nzw$

$$= [x \quad y \quad z \quad w]\begin{bmatrix} a & h & f & l \\ h & b & g & m \\ f & g & c & n \\ l & m & n & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Linear Transformation of a quadratic form

Let $X'AX$ be a quadratic form in n-variables and let $X=PY$ —— (1)

where P is a non-singular matrix, be the non-singular transformation.

From (1), $X'=(PY)'=Y'P'$ and hence

$$X'AX = Y'P'APY = Y'(P'AP)Y = Y'BY \quad \text{—(2)}$$

where $B=P'AP$. Therefore $Y'BY$ is also a quadratic form in n-variables. Hence it is a linear transformation of the quadratic form $X'AX$ under the linear transformation $X=PY$ and $B=P'AP$.

Canonical Form

If a real quadratic form be expressed as a sum or difference of the squares of new variables by means of any real non-singular linear transformation, then the latter quadratic expression is called a canonical form of the given quadratic form.

i.e., if the quadratic form $X'AX = \sum_{j=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ can be reduced to the quadratic form $Y'BY = \sum_{j=1}^{n} \lambda_i y_i^2$ by a non-singular linear transformation $X = PY$ then $Y'BY$ is called the canonical form of the given one.

$\therefore$ If $B = P'AP = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ then

$$X'AX = Y'BY = \sum_{j=1}^{n} \lambda_i y_i^2 .$$

Note 1. Here some of λ_i (eigenvalues) may be positive or negative or zero.

Note 2. A quadratic form is said to be real if the elements of the symmetric matrix are real.

Note 3. If $\rho(A) = r$, then the quadratic form $X'AX$ will contain only r terms.

Index and Signature of the Quadratic form

The number p of positive terms in the canonical form is called the index of the quadratic form.

(The no. of positive terms) − (The no. of negative terms) i.e. $p-(r-p) = 2p-r$, is called the signature of the quadratic form, where $\rho(A) = r$.

Definite, Semi-definite and Indefinite Real Quadratic Forms

Let $X'AX$ be a real quadratic form in n-variables $x_1, x_2, \cdots, x_n$ with rank r and index p. Then we say that the quadratic form is

(i) positive definite if $r = n$, $p = r$.

(ii) negative definite if $r = n$, $p = 0$

(iii) positive semi-definite if $r < n$, $p = r$

(iv) negative semi-definite if $r < n$, $p = 0$

If the canonical form has both positive and negative terms, the quadratic form is said to be indefinite.

Note:- If $X'AX$ is positive definite then $|A| > 0$

Problem 1:- Reduce $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into canonical form.

Solution:- The given quadratic form can be written as $X'AX$ where $X' = [x \; y \; z]$ & the symmetric matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}.$$

Let us reduce A into diagonal matrix.

We know that $A = IAI$

$$\begin{bmatrix}3&2&4\\2&0&4\\4&4&3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Operating $R_2 \to R_2-\frac{2}{3}R_1$, $R_3\to R_3-\frac{4}{3}R_1$ (for A on LHS and pre-factor on RHS), we get

$$\begin{bmatrix}3&2&4\\0&-\frac{4}{3}&\frac{4}{3}\\0&\frac{4}{3}&-\frac{7}{3}\end{bmatrix}=\begin{bmatrix}1&0&0\\-\frac{2}{3}&1&0\\-\frac{4}{3}&0&1\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Operating $C_2\to C_2-\frac{2}{3}C_1$, $C_3\to C_3-\frac{4}{3}C_1$ (for A on LHS and post-factor on RHS), we get

$$\begin{bmatrix}3&0&0\\0&-4/3&4/3\\0&4/3&-7/3\end{bmatrix}=\begin{bmatrix}1&0&0\\-2/3&1&0\\-4/3&0&1\end{bmatrix}A\begin{bmatrix}1&-2/3&-4/3\\0&1&0\\0&0&1\end{bmatrix}$$

Operating $R_3\to R_3+R_2$, we get

$$\begin{bmatrix}3&0&0\\0&-4/3&4/3\\0&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\-2/3&1&0\\-2&1&1\end{bmatrix}A\begin{bmatrix}1&-2/3&-4/3\\0&1&0\\0&0&1\end{bmatrix}$$

Operating $C_3\to C_3+C_2$, we get

$$\begin{bmatrix}3&0&0\\0&-4/3&0\\0&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\-2/3&1&0\\-2&1&1\end{bmatrix}A\begin{bmatrix}1&-2/3&-2\\0&1&1\\0&0&1\end{bmatrix}$$

or $\text{Diag}\left(3,-\frac{4}{3},-1\right)=P'AP$

$\therefore$ The canonical form of the given quadratic form is

$$Y'(P'AP)Y=\begin{bmatrix}y_1&y_2&y_3\end{bmatrix}\begin{bmatrix}3&0&0\\0&-4/3&0\\0&0&-1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=3y_1^2-\frac{4}{3}y_2^2-y_3^2.$$

Here $\rho(A) = 3$

index $= 1$

Signature $= 1 - 2 = -1$

Note. In this problem the non-singular transformation which reduces the given quadratic form into the canonical form is $X = PY$

i.e. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

i.e., $x = y_1 - \frac{2}{3} y_2 - 2y_3$

$y = y_2 + y_3$

$z = y_3$

Note. The above example can also be questioned as 'Diagonalise' the quadratic form $3x^2 + 3z^2 + 4xy + 8xyz + 8yz$ by linear transformation and write the linear transformation.

Or

Reduce the quadratic form $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into the sum of squares.