VECTOR CALCULUS

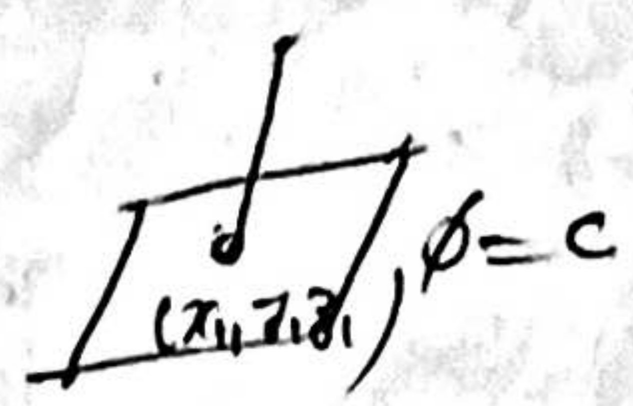
Scalar And Vector Fields

1) **Scalar Point fn.** :- Let R be a region of space at each point of which a scalar $\phi = \phi(x, y, z)$ is given, then ϕ is called a scalar fn. and 'R' the scalar field.

Eg. Temperature distribution in a medium etc.

2) **Vector Point fn.** :- Let R be a region of space at each point of which a vector $\vec{v} = \vec{v}(x, y, z)$ is given, then $\vec{v}$ is called a vector point fn. and R is called a vector field.

Gradient of a Scalar Field



$$\text{grad}\ \phi = \nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$\nabla\phi$ is a vector normal to the surface $\phi = c$ and has magnitude equal to the rate of change of ϕ along this normal. $\nabla \equiv \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

Divergence

The divergence of a ~~continuous~~ continuously differentiable vector point function $\vec{v}$ is denoted by div $\vec{v}$

$$\text{div}\ \vec{v} = \nabla \cdot \vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$= \cancel{\hat{i}\frac{\partial \vec{v}}{\partial x} + \hat{j}\frac{\partial \vec{v}}{\partial y} + \hat{k}\frac{\partial \vec{v}}{\partial z}} \left(\frac{\partial}{\partial x}v_1 + \frac{\partial}{\partial y}v_2 + \frac{\partial}{\partial z}v_3\right)$$

The divergence of a vector point function is a scalar quantity.

In particular, if $\nabla \cdot \vec{v} = 0$, the vector field is called Solenoidal field and $\vec{v}$ is called Solenoidal vector function.

Physical Interpretation

div $\vec{v}$ gives the rate of outflow per unit volume at a point of the fluid.

CURL OF VECTOR POINT FUNCTION

The curl (or rotation) of a differentiable vector point fn. $\vec{v}$ is denoted by curl $\vec{v}$ and is defined as

$$\text{curl } \vec{v} = \nabla \times \vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

Note:- If curl $\vec{v} = 0$
Field $\vec{v}$ is irrotational or conservative.

Curl of a vector point fn. is a vector quantity.

Physical Interpretation

The angular velocity at any point is equal to half the curl of linear velocity at that point of the body.

$$\vec{\omega} = \frac{1}{2} \text{ Curl } \vec{v}$$

Proof (From book).

Directional Derivative

The directional derivative of a scalar fn. ϕ at a point in the direction of a unit vector $\hat{a}$ is defined as component of $\nabla\phi$ in the direction of a.

It is given by $\nabla\phi \cdot \hat{a}$.

Physically, directional derivative of ϕ at a point P i.e. the rate of change of ϕ at P in the direction a.

Note: Directional derivative is maximum in its own direction.

Q1 Find grad ϕ, where $\phi(x,y,z) = x^2y + y^2x + z^2$, at point $(1,1,1)$

Sol
$$\phi(x,y,z) = x^2y + y^2x + z^2$$
$$\text{grad}\,\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$
$$= \hat{i}(2xy + y^2) + \hat{j}(x^2 + 2yx) + \hat{k}(2z)$$
$$\therefore \text{grad}\,\phi\Big]_{(1,1,1)} = \hat{i}(2+1) + \hat{j}(1+2) + \hat{k}(2)$$
$$= 3\hat{i} + 3\hat{j} + 2\hat{k}$$

Q2 Find the values of a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$.

Sol Let
$$\phi_1 = ax^2 - byz - (a+2)x = 0 \quad \text{--- (1)}$$
$$\phi_2 = 4x^2y + z^3 - 4 = 0 \quad \text{--- (2)}$$

Then
$$[\nabla\phi_1] = \hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}$$
$$= \hat{i}(2ax - 2a - 2) + \hat{j}(-bz) + \hat{k}(-by)$$
$$\nabla\phi_1\Big]_{(1,-1,2)} = (a-2)\hat{i} - 2b\hat{j} + b\hat{k}.$$

$$[\nabla\phi_2] = \hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z}$$
$$= \hat{i}(8xy) + \hat{j}(4x^2) + \hat{k}(3z^2)$$
$$\nabla\phi_2\Big]_{(1,-1,2)} = -8\hat{i} + 4\hat{j} + 12\hat{k}.$$

The surfaces ϕ_1 and ϕ_2 will cut orthogonally if

$\nabla\phi_1 \cdot \nabla\phi_2 = 0$

$\Rightarrow -8(a-2) - 8b + 12b = 0$

$\Rightarrow -8a + 4b = -16 \Rightarrow 2a - b = +4$ — (3)

Also since $(1,-1,2)$ lies on (1) and (2)

$\Rightarrow a + 2b - (a+2) = 0 \Rightarrow \underline{b=1}$ using in (3)

$a = \frac{5}{2}$.

Q Find the directional derivative of the fn. $\phi = 2xy + z^2$ at the point $(1,-1,3)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Sol $\phi = 2xy + z^2$

$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$= 2y\hat{i} + 2x\hat{j} + 2z\hat{k}$

$\therefore \nabla\phi\Big]_{(1,-1,3)} = -2\hat{i} + 2\hat{j} + 6\hat{k}$

Let $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$

$\Rightarrow \hat{a} = \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

$\therefore$ Directional derivative of ϕ in the direction of $\vec{a}$ is

$\nabla\phi \cdot \hat{a} = (-2\hat{i} + 2\hat{j} + 6\hat{k}) \cdot \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

$= \frac{1}{3}[-2(1) + 2(2) + 6(2)] = \frac{14}{3}$.

Q. Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \text{grad}\,(x^3+y^3+z^3-3xyz)$

Sol Let $\phi = x^3+y^3+z^3-3xyz$

Then $\vec{F} = \text{grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$

$$= (3x^2-3yz)\hat{i} + (3y^2-3zx)\hat{j} + (3z^2-3xy)\hat{k}$$

$\therefore$ div $\vec{F} = \nabla\cdot\vec{F}$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3zx) + \frac{\partial}{\partial z}(3z^2-3xy).$$

$$= 6x+6y+6z = 6(x+y+z)$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3zx & 3z^2-3xy \end{vmatrix}$$

$$= \hat{i}(-3x+3x) + \hat{j}(-3y+3y) + \hat{k}(-3z+3z) = \vec{0}.$$

Properties of Divergence And curl.

Eg Curl (grad ϕ) $= \vec{0}$

sol curl (grad ϕ) $= \nabla\times\nabla\phi$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right] + \hat{j}[0] + \hat{k}[0]$$

Etc.

$$= \vec{0}.$$

Q For a solenoidal $\vec{F}$, prove that curl curl curl curl $\vec{F} = \nabla^4 \vec{F}$.

Sol. For a solenoidal $\vec{F}$,

$$\nabla \cdot \vec{F} = 0$$

Now curl $\vec{F} = (\nabla \times \vec{F})$

Curl Curl $\vec{F} = \nabla \times (\nabla \times \vec{F})$ $\quad [\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}]$

$$= (\nabla \cdot \vec{F})\nabla - (\nabla \cdot \nabla)\vec{F}$$

$$= (0)\nabla - \nabla^2 \vec{F}$$

$$= -\nabla^2 \vec{F}$$

Again curl curl Curl Curl $\vec{F}$

$$= \nabla \times [\nabla \times (-\nabla^2 \vec{F})]$$

$$= [\nabla \cdot (-\nabla^2 \vec{F})]\nabla - [\nabla^2](-\nabla^2 \vec{F})$$

$$= 0 + \nabla^4 \vec{F}$$

$$= \nabla^4 \vec{F}.$$

Integration Of Vectors

① **Work:-** If $\vec{F}$ represents the variable force acting on a particle along ~~an~~ arc AB, then

total work done $= \int_A^B \vec{F}\cdot d\vec{r}$

* When the path of integration is closed curve, the notation is $\oint$ in place of $\int$.

* if $\oint \vec{F}\cdot d\vec{r} = 0$, the field $\vec{F}$ is called conservative i.e. no work is done and the energy is conserved.

Th If $F = \nabla\phi$ i.e $\vec{F}$ is conservative Show that the work done in moving a particle in the force field $\vec{F}$ from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$ is independent of the path joining the 2 points.

Proof Work done $= \int_A^B F\cdot dr = \int_A^B \nabla\phi\cdot dr$

$$= \int_A^B \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \int_A^B d\phi = [\phi]_A^B = \phi(B) - \phi(A)$$

Thus in a conservative (irrotational field), the work done depends on the end pts A and B and not on the path joining the points.

Note: $\boxed{F \text{ is conservative i.e. } \operatorname{curl} \vec{F} = 0 \iff \vec{F} = \nabla \phi}$

where ϕ is the scalar potential of the field.

Q Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3z^2x\hat{k}$ is a conservative field. Find its scalar potential and also work done in moving a particle from $(1,-2,1)$ to $(3,1,4)$

sol

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3z^2x \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x) = 0$$

$\therefore F$ is conservative.

$$\therefore \vec{F} = \nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$\Rightarrow (2xy+z^3)\hat{i} + (x^2)\hat{j} + (3z^2x)\hat{k} = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 2xy + z^3, \quad \frac{\partial \phi}{\partial y} = x^2, \quad \frac{\partial \phi}{\partial z} = 3z^2x$$

Also $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$$= (2xy + z^3)dx + x^2 dy + 3z^2x\, dz$$

$$\Rightarrow d\phi = (2xy\,dx + x^2 dy) + (z^3 dx + 3z^2x\, dz)$$

$$= d(x^2y) + d(z^3x)$$

$$= d(x^2y + z^3x)$$

$\Rightarrow \phi = x^2y + z^3x + c$

$\therefore$ which is the scalar potential of the vector field $\vec{F}$.

$\therefore$ req. work done $= \int_A^B \vec{F}\cdot d\vec{r} = [\phi]_A^B$

$$= \left[x^2y + z^3x + c\right]_{(1,-2,1)}^{(3,1,4)}$$

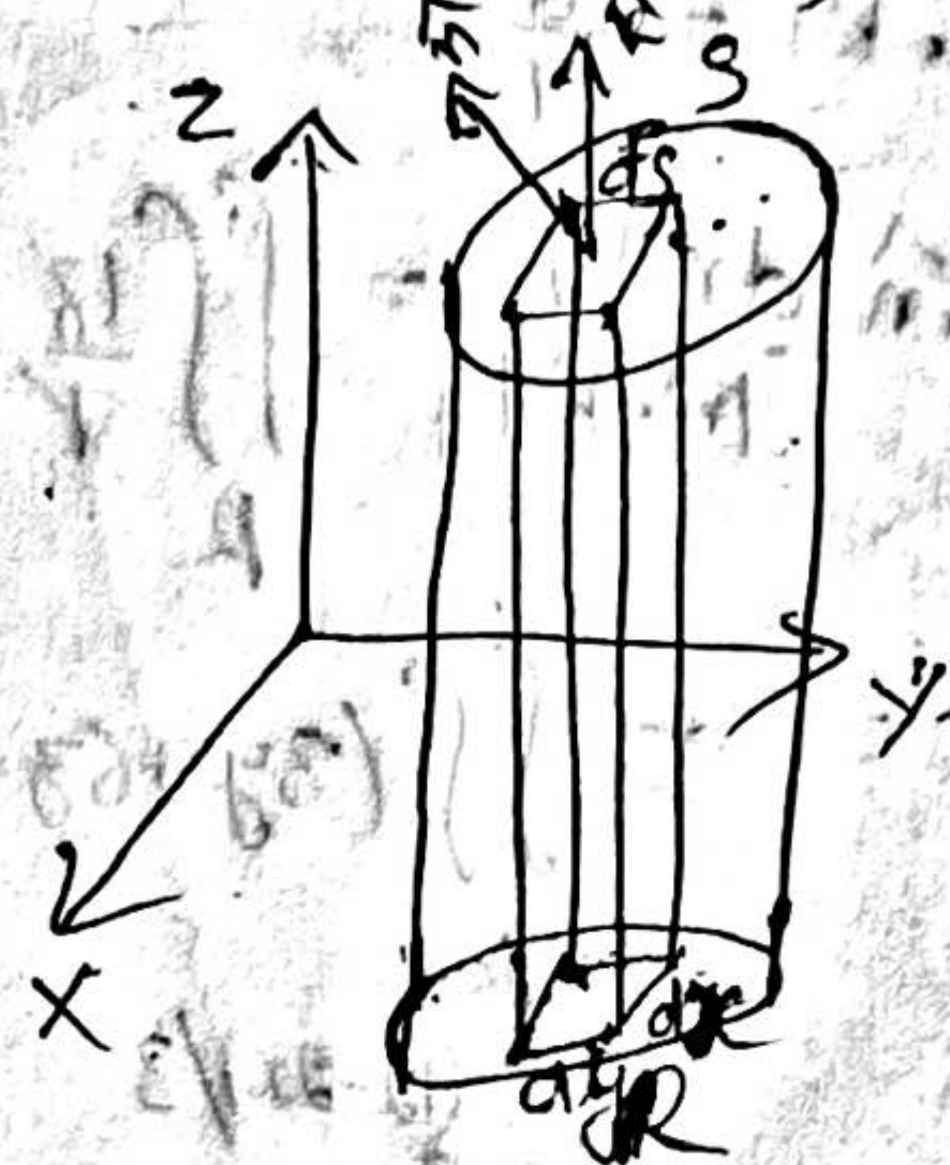
$$= (9+192) - (-2+1) = 202 \text{ units}$$

Surface Integral

Any integral which is to be evaluated over a surface, is called a surface integral.

1. If a given surface S is projected on the xy-plane, we call it's projection as R, then

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n}\, \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$



2. If R is projection of S on yz-plane

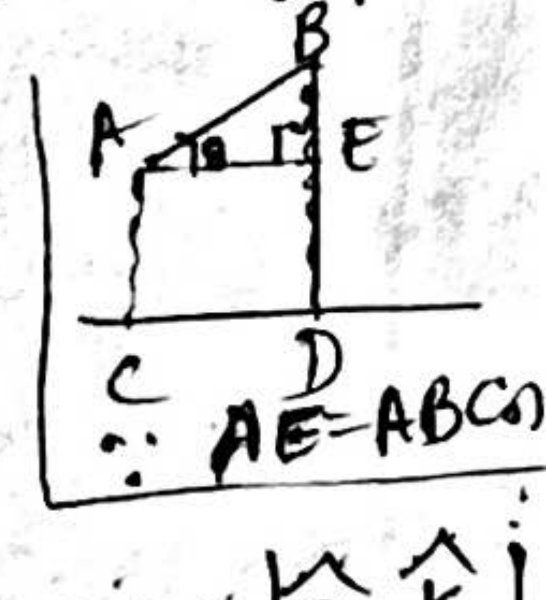
$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n}\, \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$dx\,dy$ = projection of dS on xy-plane

$= dS\cos\gamma$

$\Rightarrow dS = \dfrac{dx\,dy}{\cos\gamma}$

$\therefore = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$, $\because \cos\gamma = \dfrac{|\hat{n}\cdot\hat{k}|}{|1\cdot 1|}$



3. If R is projection of S on zx-plane

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n}\, \frac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$$