Waves and Oscillations

Periodic Motion:- A motion which repeats itself after equal intervals of time is called "Periodic Motion" OR Harmonic motion

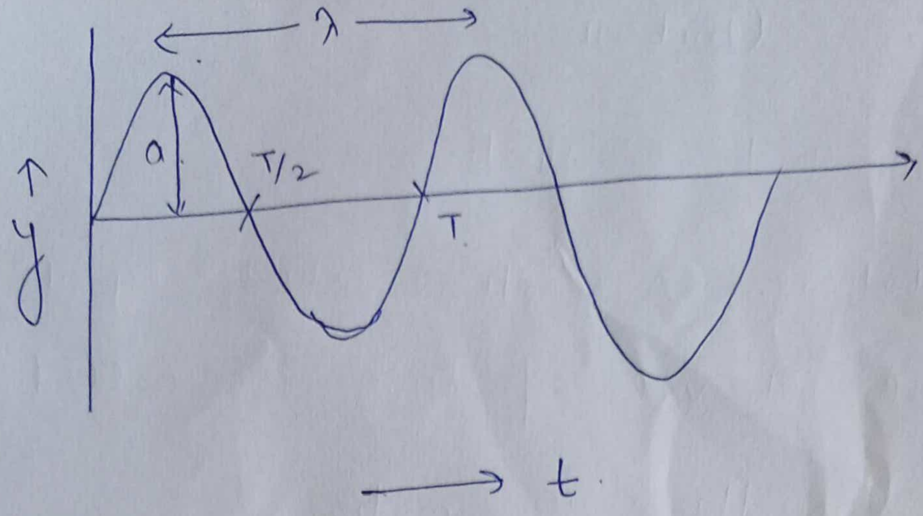
Ex. The spin of Earth, The motion of satellite around a planet, vibration of atoms in molecules etc.

→ **Periodic Time (T):-** The time after, which the motion is repeated OR the time taken for one oscillation is called periodic time.

→ **frequency:- (ν)** No. of oscillations in one second $\nu = \frac{1}{T}$

→ **Amplitude:- (a)** The maximum displacement or distance between the equilibrium position.

→ **Phase :-** The phase of an oscillatory particle is any instant defines the states of the particle as regards its position and direction of motion.



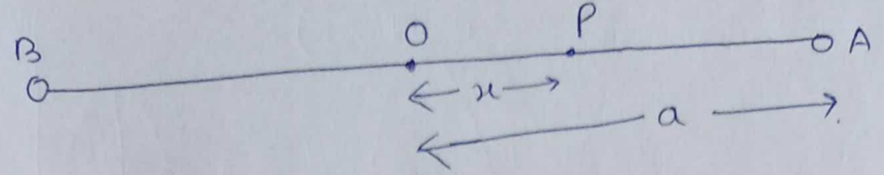
Simple Harmonic Motion and Harmonic Oscillator

It is a particular type of periodic motion and is very common in nature, in which body moves again and again over the same path about a fixed point (equilibrium position).

The acceleration of the particle is always directed towards a fixed point on the line and its magnitude is $\propto$ the displacement of the particle from this point. This fixed point is called the centre of oscillation.

A system executing SHM is called simple harmonic motion oscillator.

Let us consider a particle of mass (m) execute a SHM.



If the displacement of the particle at any instant t be x, then its acceleration be $\frac{d^2x}{dt^2}$

for SHM $\quad m\frac{d^2x}{dt^2} \propto -x.$

$$m\frac{d^2x}{dt^2} = -Cx.$$

Where C is a constant.

$-ve$ sign indicates that the force on the particle is directed opposite to x.

$$\frac{d^2x}{dt^2} + \frac{C}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{— (1)} \qquad \text{Where } \omega^2 = \frac{C}{m}$$

This is differential equation of motion of SHM

Now multiplied by $2\frac{dx}{dt}$ in equ. (1)

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + \omega^2 2x\frac{dx}{dt} = 0$$

on integrating it

$$\left(\frac{dx}{dt}\right)^2 + \omega^2x^2 = A \quad \text{— (11)}$$

When $x = a$, $\frac{dx}{dt} = 0$

ie displacement is maximum, velocity is zero.

Now from equ. (II)

$$0 + \omega^2 a^2 = A$$

$$A = \omega^2 a^2$$

Substitute the value of A in equ. (II)

$$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 = a^2 \omega^2$$

$$\frac{dx}{dt} = \sqrt{a^2 \omega^2 - \omega^2 x^2}$$

$$\boxed{\frac{dx}{dt} = \omega \sqrt{a^2 - x^2}} \quad \text{— (III)}$$

This equation gives the velocity of the particle at any time t.

$$\frac{dx}{\sqrt{a^2 - x^2}} = \omega\, dt$$

Integrate further -

$$\sin^{-1}\left(\frac{x}{a}\right) = \omega t + \phi.$$

$$\frac{x}{a} = \sin(\omega t + \phi)$$

$$\boxed{x = a \sin(\omega t + \phi)} \quad \text{— (IV)}$$

Where $a \rightarrow$ maximum value of displacement is called amplitude of oscillation

$\phi \rightarrow$ a constant, known as initial phase or phase constant.

$$\boxed{v = \frac{dx}{dt} = a\omega \cos(\omega t + \phi)} \quad \text{— (V)}$$

Energy of Harmonic Oscillator:-

In general, a SHM possesses two types of energy.

(I) Potential Energy, which is due to its displacement from the mean position

(II) Kinetic Energy, which is due to its velocity.

$\therefore$ At any instant the total energy

$$E = K + U \quad \text{——— (I)}$$

$$\because \frac{dx}{dt} = \omega\sqrt{a^2 - x^2}$$

$$= v \quad \text{— (II)}$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$= a \quad \text{— (III)}$$

$$\because f = ma$$

$$= -m\omega^2 x$$

$\because x = a\sin(\omega t + \phi)$

$$\frac{dx}{dt} = a\omega\cos(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -a\omega^2 \sin(\omega t + \phi)$$

$$= -\omega^2 x$$

$$f = -Cx \quad \text{— (IV)}$$

If the oscillator is displaced through dx distance, then work done on the oscillator is

$$dW = f\,dx$$
$$= Cx\,dx$$

If it is displaced $x = 0$ to $x = x$, then work done

$$W = \int_0^x Cx\,dx$$
$$= \frac{1}{2}Cx^2 \quad \text{— (V)}$$

This work done on the oscillator becomes its potential energy U

$$\therefore \boxed{U = \frac{1}{2}Cx^2} \quad \text{— (VI)}$$

K.E. of the oscillator at the displacement x is given by

$$K = \frac{1}{2}mv^2$$
$$= \frac{1}{2}m\omega^2(a^2 - x^2)$$
$$\boxed{K = \frac{1}{2}C(a^2 - x^2)} \quad \text{— (VII)}$$

Total energy $E = U + K$

$$= \frac{1}{2}\cancel{Cx^2} + \frac{1}{2}Ca^2 - \cancel{\frac{1}{2}Cx^2}$$

$$\boxed{E = \frac{1}{2}Ca^2} \quad \text{— (VIII)}$$

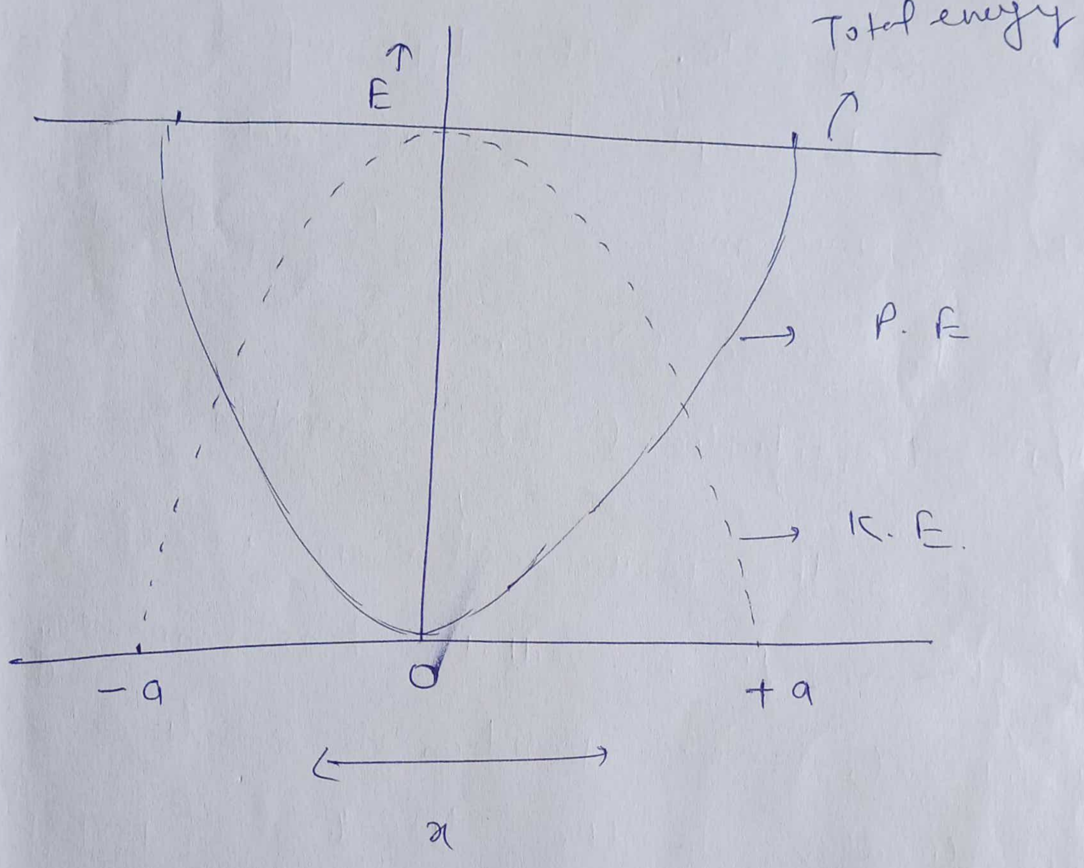
$\Rightarrow$ Total energy $\propto$ square of amplitude

$$E = \frac{1}{2} m \omega^2 a^2$$

$$= \frac{1}{2} m \left(\frac{2\pi}{T}\right)^2 a^2$$

$$= \frac{2\pi^2 m a^2}{T^2}$$

$$\boxed{E = 2\pi m a^2 \nu^2}$$



K.E. at $x = 0$ $\quad K = \frac{1}{2} C a^2$

$x = +a$ or $-a$ $\quad K = \frac{1}{2} C (a^2 - a^2)$

$= 0$

P.E at $x = 0$, $U = 0$

$x = +a/-a$ $\quad U = \frac{1}{2} C a^2$

Examples of SHM -
1. Simple pendulum
2. Compound pendulum
3. LC ckt

Waves. Wave motion is a form of disturbance in medium and due to the repeated periodic movements of the particles of the medium about their mean positions.

It is one of the most important modes of the transference of energy from one place to another, such as sound energy, e.m energy etc. This type of movement is called simple harmonic motion

The wave motion may to of two types—

(i) Transverse Wave motion

(ii) Longitudinal Wave motion

(i) It is the wave motion in which particles of the medium oscillate harmonically about its mean position in a direction perpendicular to the direction of propagation of wave.

Ex. E.M. waves.

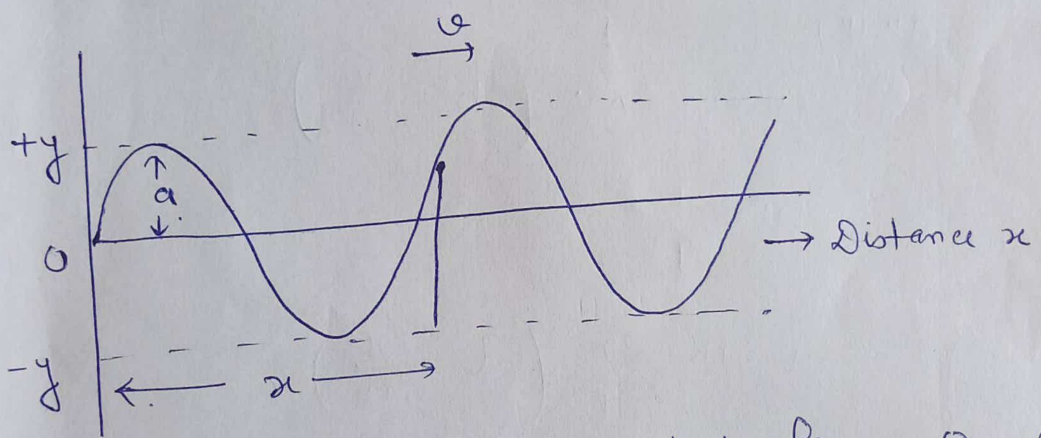
(ii) It is wave motion in which particles of the medium vibrates harmonically about its mean position in a direction parallel to the direction of propagation of the wave.

Longitudinal waves travels in the form of compressions and rarefactions.

During compression the particles of the medium come nearer than their normal separation.

Plane Progressive Harmonic Wave :-

The vibrating bodies which execute simple harmonic motion produces plane progressive harmonic waves. The amptitude of this wave remains constant. This wave may be longitudinal or transverse both. The phase difference between the two successive particles of the medium remains constant in this wave.



Let the wave is originated from O, and travels along +ve x axis.

The displacement of the particle at O at any instant 't' is given by

$$y = a \sin \omega t \quad \text{——— (1)}$$

$a \rightarrow$ amplitude and $\omega \rightarrow$ angular velocity

frequency $\nu = \cancel{f} \frac{\omega}{2\pi}$

If $v \rightarrow$ is the wave velocity, then disturbance produced at O will ~~be~~ reach at point P in $\frac{x}{v}$ seconds, where x is the distance between the origin and point P.

$\therefore$ displacement $y = a \sin \omega \left(t - \frac{x}{v}\right)$ — (II)

If the wave is travelling along the $-ve$ x axis

then $y = a \sin \omega \left(t + \frac{x}{v}\right)$ — (III)

$\because \omega = 2\pi\nu$

from equ. (II)

$\therefore y = a \sin 2\pi\nu \left(t - \frac{x}{v}\right)$

$\nu = \frac{1}{T}$

$y = a \sin \frac{2\pi}{T}\left(t - \frac{x}{v}\right)$

$y = a \sin 2\pi \left[\frac{t}{T} - \frac{x}{vT}\right]$

we know that: $\because v = \nu\lambda$

$v = \frac{\lambda}{T}$

$vT = \lambda$

$\therefore y = a \sin 2\pi \left[\frac{t}{T} - \frac{x}{\lambda}\right]$

If $T = \frac{\lambda}{v}$

$$\therefore y = a \sin 2\pi \left[\frac{vt}{\lambda} - \frac{x}{\lambda}\right]$$

$$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

$$y = a \sin \frac{2\pi v}{\lambda}\left(t - \frac{x}{v}\right)$$

$\frac{2\pi}{\lambda} = K$ is called the propagation constant

$$\therefore y = a \sin Kv\left(t - \frac{x}{v}\right)$$

$$y = a \sin K(vt - x) \quad \text{——— (IV)}$$

$$\therefore \frac{2\pi}{\lambda} vt = Kvt$$

$$\frac{2\pi}{\lambda}(\nu\lambda)t = Kvt$$

$$2\pi\nu t = Kvt$$

$$\omega t = Kvt$$

from equ. (IV)

$$\therefore \boxed{y = a \sin(\omega t - Kx)} \quad \text{——— (V)}$$

Similare equation could be obtained for the wave travelling along -ve x-axis.

It is supposed in these equations that the particle at O just passes through its mean position in the +ve direction at $t = 0$

$$\therefore y = 0, \text{ at } t = 0$$

If the particle does not hold the above condition then it will have an initial phase ϕ, then above equation becomes

$$\boxed{y = a \sin(\omega t - kx - \phi)}$$

$\phi \rightarrow$ phase Constant.

Superposition of Progressive Waves:-

Stationary wave

Principle. When two or more waves are simultaneously present at a single point in space, the displacement of the medium at that point is sum of the displacement due to each individual wave.

$$y_1 = a \sin \frac{2\pi}{\lambda} (vt - x)$$

$$y_2 = a \sin \frac{2\pi}{\lambda} (vt + x)$$

Consider two plane waves one travelling along +ve x axis and other along -ve X-axis with an amplitude a, wave velocity v and wavelength λ.

Then by principle of superposition

$$Y = y_1 + y_2$$

$$= a \sin \frac{2\pi}{\lambda} (vt - x) + a \sin \frac{2\pi}{\lambda} (vt + x)$$

$$= 2a \cos \frac{2\pi}{\lambda} vt \cos \frac{2\pi}{\lambda} x$$

$$Y = A \cos^2 \frac{2\pi v}{\lambda} t$$

Electromagnetic Theory:-

There are four basic laws in electromagnetism. Such as

1. Gauss's law in electrostatics
2. Gauss's law in magnetism
3. faraday's law
4. Ampere's circuital law.

The entire theory of electromagnetic field is condensed into these four laws. These laws govern the interaction of bodies which are magnetic or electrically charged or both. A complete set of relations giving the connection between the charge at rest (electrostatics), charge in motion (current electricity) electric and magnetic fields (electromagnetism). Maxwell summarized these basic laws in four equations called Maxwell's equations. Maxwell's equations are back bone of electrodynamics. Maxwell's equations are used in integral as well as differential form depending upon the problems to be solved. But Maxwell's equations have certain limitations. They most successfully explain electromagnetic interaction between large aggregates of charges such as radiation antennas, electric circuits etc, but the electromagnetic interactions between fundamental particles could not be correctly explained on the basis of results derived from the Maxwell's equations. They are treated according to the laws of quantum mechanics by a technique called quantum electrodynamics

$$\vec{\nabla} = \hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z$$ (del operator).
OR
(Nabla ∇)

It is a differential operator.

<u>Gradient</u> :- The gradient of a scalar function (ϕ) is a vector whose magnitude at any point is equal to the maximum rate of increase of ϕ at the point and its direction is along the normal to the surface, i.e. in the direction of maximum rate of increase.

$$\text{grad}\ \phi = \frac{\partial \phi}{\partial n}\ \hat{n}$$

where $\hat{n}$ is the unit normal vector at the point.

<u>Ex.</u> If ϕ denotes the potential $\underline{V}$ in an electric field, the the intensity $\vec{E}$ of the field at any point is equal to the rate of change of potential and in the direction of the greatest rate of fall of potential $\qquad \vec{E} = -\frac{\partial V}{\partial r}$.

$$\vec{E} = -\frac{\partial \phi}{\partial n}\ \hat{n}$$

$$\vec{E} = -\ \text{grad}\ V = -\nabla V$$

$-$ve sign is used because the direction of field intensity is opposite to the direction of the increase of potential

Mathematically, the gradient of a scalar function is

$$\text{grad}\ \phi = \nabla \phi$$

$$= \left[\hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z\right] \phi$$

$$= \left[\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right]$$

<u>Note</u> :- A vector field which can be expressed as gradient of a scalar field is called non-curl or Lamellar field.

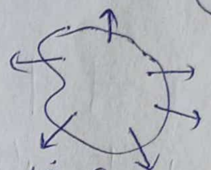
Divergence:- ~~of~~ The divergence of a vector field at a point is defined as the amount of flux per unit volume diverging from that point. The divergence is the amount of flux, is a scalar, and it is given by

$$\operatorname{div} \vec{A} = \vec{\nabla} \cdot \vec{A}$$

$$= \left[\hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z\right] \cdot \left[\hat{i} A_x + \hat{j} A_y + \hat{k} A_z\right]$$

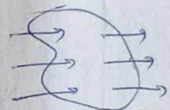
$$= \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]$$

- If divergence is +ve then something is diverging or coming out from a small volume.



- If divergence is -ve then something is converging or coming into a small volume.

- If the flux entering any element of space is exactly balanced by leaving it, then $\operatorname{div} \vec{A}$ is zero.



* If $\operatorname{div} \vec{A} = 0$, then vector is solenoidal.

Curl:- If a vector is derived as the gradient of a scalar field, then line integral of the vector round any closed path in the field is zero. Such a field is called lameller or non-curl field, however vector fields for which the line integral round any closed path is not zero and have a finite value. Such vector fields cannot be derived as gradient of any scalar field and they show the property of curl.

" Curl of a vector field may be defined as the maximum line integral of a vector computed per unit area along the boundary "

Mathematically, it is the cross product of operator ∇ and vector $\vec{A}$.

$$\text{Curl } \vec{A} = \nabla \times \vec{A}$$

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \times (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

The curl of a vector at any point is vector

Ex. If current is flowing in a conductor, then the curl of the magnetic field produced at a point is equal to the current flowing per unit area at that point which is called current density $\vec{J}$.

$$\text{Curl } \vec{H} = \vec{\nabla} \times \vec{H} = \vec{J}$$

If curl $\vec{A} = 0$ the vector $\vec{A}$ is called irrotational vector.

* **Gauss Divergence Theorem:-** This is used to transform a volume integral into a surface integral.

$$\int_V \vec{\nabla}.\vec{A}\, dv = \int_S \vec{A}.\vec{ds}$$

⇒ The volume integral of the divergence of the vector field $\vec{A}$ over the volume is equal to the surface integral of a vector field $\vec{A}$ over the closed surface.

* **Stoke's Theorem:-** This is used to change the line integral into surface integral.

$$\oint_C \vec{A}.\vec{dl} = \int_S (\vec{\nabla} \times \vec{A}).\vec{ds}$$

⇒ The line integral of a vector field $\vec{A}$ around any closed curved C is equal to the surface integral of the curl $\vec{A}$ over the surface S bounded by C

$$\int_V \left(\quad \frac{\partial}{\partial t} \right)$$

$$\boxed{\operatorname{div} \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

This is the equation of continuity.

Physical significance:- Current flowing out of a given volume must be equal to the rate of decrease of charge within the surface.

$$\therefore \operatorname{div} \vec{J} = -\frac{\partial \rho}{\partial t}.$$

Maxwell's Equations

A theory describing the interaction between charges, currents, electric and magnetic fields was developed by James Clerk Maxwell in the form of four fundamental equations called Maxwell's equations or electro-magnetic field equations

First equation (Gauss law in electrostatics)

According to Gauss law the electric flux through any closed surface is equal to $1/\varepsilon_0$ times the total charge enclosed by the surface.

ie $$\int_S \vec{E}\cdot\vec{ds} = q/\varepsilon_0$$

If the surface encloses a continuous charge distribution, then

$$q = \int_V \rho\, dv$$ where ρ is the volume charge density

$$\therefore \int_S \vec{E}\cdot\vec{ds} = \frac{1}{\varepsilon_0}\int_V \rho\, dV$$

Using Gauss divergence theorem

$$\int_S \vec{E}\cdot\vec{ds} = \int_V \operatorname{div}\vec{E}\, dV$$

$$\therefore \int_V \operatorname{div}\vec{E}\, dV = \frac{1}{\varepsilon_0}\int_V \rho\, dV$$

$$\int_V \left(\operatorname{div}\vec{E} - \rho/\varepsilon_0\right) dV = 0$$

This is true for all volumes, hence the integrand, $\operatorname{div}\vec{E} - \rho/\varepsilon_0$ must vanish

ie $$\operatorname{div}\vec{E} - \rho/\varepsilon_0 = 0$$

$$\operatorname{div}\vec{E} = \rho/\varepsilon_0$$

or $$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

Physical significance

Maxwell's first equation states that the electric flux through any closed surface is $1/\varepsilon_0$ times the total charge enclosed by the surface.

2) Second equation (Gauss law in magnetostatics)

The net magnetic flux through any closed Gaussian surface is always zero.

$$\phi_B = \int_S \vec{B} \cdot \vec{ds} = 0$$

Using Gauss - divergence theorem

$$\int_S \vec{B} \cdot \vec{ds} = \int_V \text{div}\, \vec{B}\, dV$$

$$\therefore \int_V \text{div}\, \vec{B}\, dV = 0$$

The above equation is true for all volumes, so the integrand must vanish.

$$\text{ie} \quad \text{div}\, \vec{B} = 0 \quad \text{or} \quad \vec{\nabla} \cdot \vec{B} = 0$$

Physical significance

The net magnetic flux through any closed surface is zero. Since magnetic monopole does not exist, any closed volume always contain equal and opposite magnetic poles. ie the number of magnetic lines of flux entering a closed region is equal to that leaving it. ie the magnetic flux lines are continuous.

3) Third equation (Faraday's law)

According to Faraday's law, induced emf around a closed circuit is equal to the negative time-rate of change of magnetic flux linked with the circuit.

$$e = -\frac{\partial \phi_B}{\partial t} \quad \text{—— ①}$$

Suppose a magnetic field is produced by a current carrying coil enclosing a surface S. The magnetic flux linked with a small area $\vec{ds}$ is $d\phi = \vec{B} \cdot \vec{ds}$

The total flux linked with the coil is

$$\phi_B = \int_S \vec{B} \cdot \vec{ds} \quad \text{—— ②}$$

The changing magnetic flux induces an electric field, then

$$e = \oint_C \vec{E} \cdot \vec{dl} \quad \text{—— ③}$$

Substituting 2 & 3 in eqn. ①

$$\oint_C \vec{E} \cdot \vec{dl} = -\int_S \frac{\partial \vec{B}}{\partial t} \vec{ds}$$

Using Stokes theorem

$$\oint \vec{E} \cdot \vec{dl} = \int_S \text{Curl } \vec{E} \, ds$$

$$\therefore \int \text{Curl } \vec{E} \, ds = -\int \frac{\partial \vec{B}}{\partial t} \, ds$$

ie $$\text{Curl } \vec{E} = -\partial \vec{B} / \partial t$$

or $$\nabla \times \vec{E} = -\partial \vec{B} / \partial t$$

<u>Physical significance</u> – An electric field can be produced by changing magnetic flux.

(iv) Ampere's Law:-

The line integral of magnetic field vector $\vec{B}$ around a closed path is equal to μ_0 times the current flowing through the path.

$$\oint_c \vec{B} \cdot d\vec{l} = \mu_0 I$$

$\because \vec{B} = \mu_0 \vec{H}$

$\vec{H}$ = magnetic field intensity

or $$\oint_c \vec{H} \cdot d\vec{l} = I \quad \text{——— (1)}$$

$\because I = \int_s \vec{J} \cdot d\vec{s}$ $\vec{J}$ — Current density

$$\int_c \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot d\vec{s}$$

using Stokes's Theorem.

$$\int_s \text{curl } \vec{H} \cdot d\vec{s} = \int_s \vec{J} \cdot d\vec{s}$$

$$\int_s (\text{curl } \vec{H} - \vec{J})\, d\vec{s} = 0$$

$$\Rightarrow \boxed{\text{Curl } \vec{H} = \vec{J}} \quad \text{——— (ii)}$$ As the surface is arbitrary.

Let us examine the validity of this equation for time varying field.

Take divergence of this equ.

$$\text{div. Curl } \vec{H} = \text{div } \vec{J}$$

$\because \text{div Curl } \vec{H} = 0$

$$\therefore \text{div } \vec{J} = 0 \quad \text{——— (iii)}$$

but from equation of continuity

$$\text{div } \vec{J} = -\frac{\partial \rho}{\partial t}$$

$\therefore$ div $\vec{J} = 0$ only when $\frac{\partial \rho}{\partial t} = 0$ i.e charge density static. Thus we have to conclude that Ampere's equ. (1) is valid only for steady-state condition and it is insufficient

(27) for time varying field. Hence, Ampere's law must be modified

Maxwell investigated how one could alter the Ampere's equation, so to make it consistent with equation equation of continuity some thing say $\vec{J_d}$ must be added.

then $\quad \text{curl}\,\vec{H} = \vec{J} + \vec{J_d}$ ——— (iv)

Now take the div of above equation

$$\text{div}\,(\text{curl}\,\vec{H}) = \text{div}\,\vec{J} + \text{div}\,J_d$$

$$\Rightarrow \text{div}\,\vec{J} = -\,\text{div}\,J_d$$

$$\therefore \text{div}\,\vec{J_d} = \frac{\partial \rho}{\partial t} \quad \text{——— (v)}$$

But from Gauss's law in differential form.

$$\text{div}\,\vec{E} = \rho/\epsilon_0 \quad \text{or} \quad \text{div}\,\vec{D} = \rho$$

$$\because \vec{D} = \epsilon_0 \vec{E}$$

$$\therefore \text{div}\,J_d = \frac{\partial}{\partial t}(\text{div}\,\vec{D})$$

$$J_d = \frac{\partial \vec{D}}{\partial t} = \frac{\epsilon \partial \vec{E}}{\partial t}$$

$\therefore$ Hence Modified Ampere's law.

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J_d}$$

$\vec{D}$ = Electric displacement vector.

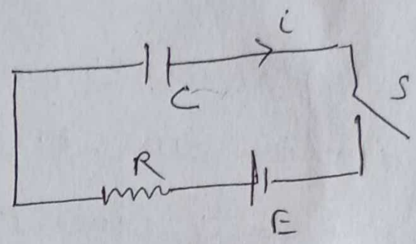
$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

$\vec{J_d}$ – Displacement current density

Modified Ampere's Law :- The idea of displacement current originating from the study of discharge of a condenser leads to a modification in Ampere's law.

Consider the process of charging of a || plate capacitor through a series circuit containing a cell, R, & C and a switch S.

→ When S is pressed, ckt is closed and a current flows and charge starts accumulating gradually on the plates of the capacitor. The current in the ckt decreases as the charge grows on the plates. When the capacitor is fully charged to the e.m.f of the cell, the current stops.



There is no actual flow of charge b/w the plates during charging. If we place a compass needle in the space between the plates, the needle deflects. This indicates that there must be some other source of magnetic field in the gap. The other source is nothing but the changing electric field between the plates.

Displacement current :- Suppose q be the charge collecting on the capacitor plates at any instant t. Then electric field b/w the plates

$$E = \frac{q}{\epsilon_0 A}$$

where A → surface area of each plate

$$E = \frac{\sigma}{\epsilon}$$

$$\frac{dE}{dt} = \frac{1}{\epsilon_0 A}\frac{dq}{dt}$$

$$\frac{dE}{dt} = \frac{1}{\epsilon_0 A}\cdot I$$

$$I_c = \epsilon_0 A \frac{dE}{dt} \qquad \text{———} \quad (1) \Rightarrow$$

by definition $I_d = \epsilon_0 \frac{d\phi_E}{dt}$

$\because \vec{J_d} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\frac{I_d}{A} = \epsilon_0 \frac{\partial \vec{E}}{dt}$

$\phi_E = EA$

$i_d = \frac{d}{dt} \phi . E$

$= \frac{d}{dt} (\vec{E} . \vec{A})$

$= \frac{d\vec{E}}{dt} . \vec{A}$

$I_d = \epsilon_0 A \frac{dE}{dt}$ — (ii)

$J_d = \frac{d\vec{E}}{dt}$

from (i) & (ii) $\boxed{I = I_d}$

⇒ displacement current in the gap is identical with the conduction current in the connecting wires.

⇒ Displacement current does not mean an actual flow current through the capacitor. It is simply an apparent current which represents the rate at which charge flow from one electrode to another in the external ckt.

Characteristics of displacement Current:

→ Displacement current is a current in the sense that it produces a magnetic field. As it is not linked with the motion of charge, displacement current has no other properties of current. It has a finite value even in perfect vacuum, where there is no charge at all.

I	I_d
follows ohm's law	does not
depends on V, R $J_c = \sigma E$ $I = V/R$	depends on Electric field $J_d = \frac{\partial D}{\partial t} = \epsilon \frac{\partial E}{\partial t}$
If $E = 0$ $I \neq 0$	$E = 0$ $I_d = 0$

→ Displacement current exists whenever there is a change of electric flux. Unlike conduction current it does not exist under steady condition.

→ It is not a current. It is only adds to current density in Ampere's law. It produces magnetic field, so called current.

→ The magnitude of i_d = rate of displacement of

→ Together conduction current, displacement current satisfy the property of continuity.

by definition $I_d = \epsilon_0 \frac{d\phi_E}{dt}$

$\because \vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\frac{I_d}{A} = \epsilon_0 \frac{\partial \vec{E}}{dt}$

$\phi_E = EA$

$i_d = \frac{d}{dt}\phi_E$

$= \frac{d}{dt}(\vec{E}\cdot\vec{A})$

$= \frac{d\vec{E}}{dt}\cdot\vec{A}$

$I_d = \epsilon_0 A \frac{dE}{dt}$ — (ii)

$J_d = \frac{d\vec{E}}{dt}$

from (i) & (ii) $\boxed{I = I_d}$

⇒ displacement current in the gap is identical with the conduction current in the connecting wires.

⇒ Displacement current does not mean an actual flow current through the capacitor. It is simply an apparent current which represents the rate at which charge flow from one electrode to another in the external ckt.

Maxwell's Equations in Integral form

(I) $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

Taking volume integral over a volume v enclosed by a closed surface s.

$$\int_v \vec{\nabla}\cdot\vec{E}\, dv = \frac{1}{\epsilon_0}\int_v \rho\, dv$$

use Gauss-divergence theorem in L.H.S.

$$\boxed{\int_s \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0}\int_v \rho\, dv}$$

(II) $\vec{\nabla}\cdot\vec{B} = 0$

$$\int_v \vec{\nabla}\cdot\vec{B}\, dv = 0$$

$$\int_s \vec{B}\cdot d\vec{s} = 0$$

(III) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$$\int_s (\vec{\nabla}\times\vec{E})\, d\vec{s} = \frac{-\partial}{\partial t}\int_s \vec{B}\cdot d\vec{s}$$

$$\oint_c \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_s \cancel{\phi_B}\ \cancel{\partial B}\ \vec{B}\cdot d\vec{s}.$$

(IV) $$\int_s (\vec{\nabla}\times\vec{H})\, ds = \int_s (\vec{J}+\vec{J}_d)\, d\vec{s}$$

$$\int_c \vec{H}\cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) d\vec{s}$$

Poynting Theorem:- The net power flowing out of a given volume V is equal to the time rate of decrease in energy stored within V minus the ohmic losses.

i.e. $\oint \text{div}(\vec{E}\times\vec{H})\,dv = -\int_V \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right]dv - \int_V \sigma E^2 dv$

Proof:- ∴ Electrostatic energy $u_e = \frac{1}{2}\int_V \epsilon E^2\,dv$ — (i)

magnetostatic energy $u_m = \frac{1}{2}\int_V \mu H^2\,dv$ — (ii)

or $u_m = \frac{1}{2}\int_V \frac{B^2}{\mu}\,dv$ — (iii)

Maxwell's equations.

$\vec{\nabla}\cdot\vec{D} = \rho$ — (iv) $\quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

$\vec{\nabla}\cdot\vec{B} = 0$ — (v)

$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ — (vi)

$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ — (vii) $\quad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\epsilon\,\partial\vec{E}}{\partial t}$

Now taking the dot product of (vi) & (vii) with $\vec{H}$ & $\vec{E}$

$\vec{H}\cdot(\vec{\nabla}\times\vec{E}) = -\vec{H}\frac{\partial\vec{B}}{\partial t}$ — (viii)

$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{E}\cdot\vec{J} + E\cdot\frac{\partial\vec{D}}{\partial t}$ — (ix)

(viii) − (ix)

$$\underbrace{\vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})} = -\vec{H}\frac{\partial\vec{B}}{\partial t} - \vec{E}\frac{\partial\vec{D}}{\partial t} - \vec{E}\cdot\vec{J}$$

$$\text{div}(\vec{E}\times\vec{H}) = -\left[\vec{H}\frac{\partial\vec{B}}{\partial t} + \vec{E}\frac{\partial\vec{D}}{\partial t}\right] - \vec{E}\cdot\vec{J}$$

$$= -\left[\frac{1}{2}\cdot 2\,\vec{H}\frac{\partial}{\partial t}\mu_0\vec{H} + \frac{1}{2}\cdot 2\,\vec{E}\cdot\frac{\partial}{\partial t}\epsilon_0\vec{E}\right] - \vec{E}\cdot\vec{J}$$

$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} - (\vec{a}\times\vec{c})\cdot\vec{b}$

$a = \nabla$, $b = E$, $c = H$

$$= -\frac{\partial}{\partial t}\Big[\underbrace{\frac{1}{2}\mu_0 H^2}_{\text{Magnetostatic energy per unit volume}} + \underbrace{\frac{1}{2}\epsilon_0 E^2}_{\text{Electrostatic energy per unit/vol.}}\Big] - \vec{E}\cdot\vec{J}$$

...magnetic wave in free space:- In free space, $q = 0$

Taking integration over a volume V bounded by surface S

$$\int_V \text{div}(\vec{E} \times \vec{H})\, dv = -\int_V \left\{ \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 \right) \right\} dv - \int_V \vec{J} \cdot \vec{E}\, dv$$

using Gauss divergence theorem, the L.H.S. —

$$\boxed{\oint_S (\vec{E} \times \vec{H})\, d\vec{s} = -\frac{d}{dt} \int_V \left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 \right) dv - \int_V \vec{J} \cdot \vec{E}\, dv}$$

Total Power leaving the volume = Rate of decrease in energy stored in electric and magnetic fields − ohmic power dissipated

$\int_V \vec{J} \cdot \vec{E}\, dv = \int_V \sigma E^2\, dv$ represents rate of energy transformed into electromagnetic field through the motion of free charge in volume V.

$\vec{J} = \sigma \vec{E}$ ohm's law

$V = IR$, $R = \rho l/A$, $V = \rho I l / A$, $\frac{1}{\rho} \frac{V}{l} = \frac{I}{A}$, $J = \sigma E$

If $\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu H^2 = u$ (Total E.M. energy)

$$\oint_S (\vec{E} \times \vec{H})\, d\vec{s} = -\frac{d}{dt} \int_V u\, dv - \int_V (\vec{J} \cdot \vec{E})\, dv$$

$$= -\int_V \frac{\partial u}{\partial t}\, dv - \int_V (\vec{J} \cdot \vec{E})\, dv$$

Energy in Electromagnetic waves - Poynting vector - when e.m. waves travel from one point to another in space, it transport electric and magnetic energies from point to point. The rate of flow of this energy in a plane e.m. wave can be described by a vector per unit area $\vec{S} = \vec{E} \times \vec{H}$, watt/m^2

$\vec{S}$ called Poynting vector direction in the flow in which energy moves.

energy / Time × area = J/sec × m^2

Electromagnetic wave in free space:- In free space, $q = 0$, $I = 0$

So, $\rho = 0$, $\vec{J} = 0$, then Maxwell's equation can be written as- (34)

(i) $\vec{\nabla} \cdot \vec{E} = 0$ ——— (i)

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ ——— (ii)

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ——— (iii)

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ ——— (iv)

$\cancel{\bar{a} \times (\bar{b} \times \bar{c})}$

Taking curl of equ. (iii), $\text{Curl}(\vec{\nabla} \times \vec{E}) = -\text{curl} \frac{\partial \vec{B}}{\partial t}$

$$= -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c}) b - (\bar{a} \cdot b)\bar{c}$

$$= -\frac{\partial}{\partial t}\left[\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$$

$$\vec{\nabla}(\cancel{\vec{\nabla} \cdot \vec{E}})^{0} - \nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\cancel{-} \nabla^2 \vec{E} = \cancel{-} \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \text{ ——— (v)}$$

Similarly, $\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$ ——— (vi)

On comparing these equ. with a standard classical wave equation propagating with velocity v.

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t}$$

$$\therefore v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{\frac{4\pi}{4\pi \epsilon_0 \mu_0}} = \sqrt{\frac{\cancel{4\pi} \times 9 \times 10^9}{\cancel{4\pi} \times 10^{-7}}}$$

$$= 3 \times 10^8$$

$$= c = \text{speed of light}$$

Hence, we see that field vector $\vec{E}$ and $\vec{B}$ are propagating in ~~field~~ free space as waves at the ~~so~~ speed of light.

(35) $\therefore \vec{\nabla}\cdot\vec{E} = 0$? Plane wave

Transverse Nature of E-M waves:-

The equation of plane e.m. waves (vi & (vi) are given as

$$\vec{E}(\vec{r},t) = \vec{E_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} \quad \text{——} \quad (vii)$$

$$\vec{B}(\vec{r},t) = \vec{B_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} \quad \text{——} \quad (viii)$$

The no. of complete wave cycles of an e.m field exist in one meter of linear dist.

Where $\vec{E_0}$ and $\vec{B_0}$ are amplitudes of the electric and magnetic field, and $\vec{K}$ is the propagation const given by

$$\vec{K} = K\hat{n} = \frac{2\pi}{\lambda}\cdot\hat{n} = \frac{2\pi\nu}{c}\hat{n} = \frac{\omega}{c}\hat{n} \qquad \lambda = \frac{c}{\nu}$$

Frequency of a wave either in cycles per second distance of radians per ... no. of waves

Now, apply the condition of free space

$$\vec{\nabla}\cdot\vec{E} = 0$$

L.H.S. $\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{E_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)}$ —— (ix)

$$\vec{K}\cdot\vec{r} = (\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= k_x x + k_y y + k_z z \quad , \quad \vec{E_0} = (\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})$$

$$\vec{\nabla}\cdot\vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\cdot e^{i(k_x x + k_y y + k_z z) - \omega t}\right]$$

$$= \frac{\partial}{\partial x}\left[E_{0x}\, e^{i(k_x x + k_y y + k_z z) - i\omega t}\right]$$
$$+ \frac{\partial}{\partial y}\left[E_{0y}\, e^{i(k_x x + k_y y + k_z z) - i\omega t}\right]$$
$$+ \frac{\partial}{\partial z}\left[E_{0z}\, e^{i(k_x x + k_y y + k_z z) - i\omega t}\right]$$

$$= (ik_x)E_{0x}\, e^{i(k_x x + k_y y + k_z z - \omega t)} + (ik_y)\left[E_{0y}\, e^{i(\;) - i\omega t}\right]$$
$$+ ik_z E_{0z}\left[e^{i(\;) - i\omega t}\right]$$

$$= (E_{0x}\, ik_x + E_{0y}\, ik_y + E_{0z}\, ik_z)\left[e^{i(k_x x + k_y y + k_z z) - i\omega t}\right]$$

$$= i(k_x E_{0x} + k_y E_{0y} + k_z E_{0z})\, e^{i\vec{K}\cdot\vec{r} - i\omega t}$$

$$= i\,\vec{K}\cdot\vec{E_0}\, e^{i\vec{K}\cdot\vec{r} - i\omega t}$$

$$= i\,\vec{K}\cdot\vec{E}$$

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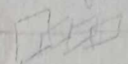
$\because \vec{\nabla}\cdot\vec{E} = 0$

$\therefore i\vec{K}\cdot\vec{E} = 0$

or $\vec{K}\cdot\vec{E} = 0$ ———— (X)

Similarly, $\vec{K}\cdot\vec{B} = 0$ ———— (XI)

Plane wave – whose wavefronts (surfaces of Constt phase) are infinite ∥ planes



from equ. (X) & (XI), it is clear that "electromagnetic field vector $\vec{E}$ and $\vec{B}$ are both ⊥ to the direction of propagation vector $\vec{K}$. This shows that electromagnetic waves are transverse in character.

Now taking Maxwell's III equation $\quad \nabla\times E = \frac{\partial B}{\partial t}$, $\vec{B} = \mu_0 \vec{H}$

$$\vec{\nabla}\times\vec{E} = -\mu_0\frac{\partial\vec{H}}{\partial t} \quad ———— \quad (XII)$$

L.H.S.

$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(\vec{E}_0\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right)$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}\left[E_{oz}\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right] - \frac{\partial}{\partial z}\left[E_{oy}\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right]\right\}$$
$$+\hat{j}\left\{\frac{\partial}{\partial z}\left[E_{ox}\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right] - \frac{\partial}{\partial x}\left[E_{oz}\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right]\right\}$$
$$+\hat{k}\left\{\frac{\partial}{\partial x}\left[E_{oy}\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right] - \frac{\partial}{\partial y}\left[E_{ox}\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right]\right\}$$

On solving eqs.

$$= \hat{i}\left\{i E_{oz} K_y\, e^{i(\vec{K}\cdot\vec{r}-\omega t)} - i E_{oy} K_z\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right\} + \hat{j}\left\{i E_{ox} K_z\, e^{i(\vec{K}\cdot\vec{r}-\omega t)} - i E_{oy} K_z\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right\} + \hat{k}\left\{i E_{oy} K_x\, e^{i(\vec{K}\cdot\vec{r}-\omega t)} - i E_{ox} K_y\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right\}$$

$$= \hat{i}\left\{i\left[E_{oz}K_y - E_{oy}K_z\right] + \hat{j}\left[E_{ox}K_z - E_{oz}K_x\right] + \hat{k}\left[E_{oy}K_x - E_{ox}K_y\right]\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right\}$$

$\Downarrow$

$$= i\left\{\vec{K}\times\vec{E}_0\right\}\cdot e^{i(\vec{K}\cdot\vec{r}-\omega t)}$$
$$= i\left(\vec{K}\times\vec{E}\right)$$

$$\therefore\ i\left(\vec{K}\times\vec{E}\right) = -\mu_0\frac{\partial\vec{H}}{\partial t}$$

$$i(\vec K \times \vec E) = -\mu_0 \frac{\partial}{\partial t}\left(\vec H_0 e^{i(\vec K \cdot \vec r - \omega t)}\right)$$

$$= \mu_0 \, i\omega \vec H_0 e^{(i\vec K \cdot \vec r - i\omega t)}$$

$$= i\mu_0 \omega \vec H$$

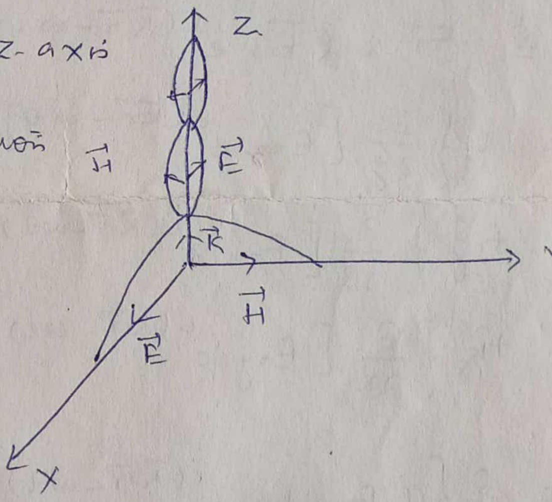
or $\vec K \times \vec E = \mu_0 \omega \vec H$ ——— (XIII)

Similarly $\vec K \times \vec H = -\epsilon_0 \omega \vec E$ ——— (XIV)

$\Rightarrow$ It is clear that $\vec H$ is $\perp$ to both $\vec K$ and $\vec E$ and $\vec E$ is $\perp$ to both $\vec H$ & $\vec K$.

$\Rightarrow$ the field vector $\vec E$ and $\vec H$ are mutually perpendicular and they are also $\perp$ to direction of propagation

If a wave is propaging along z-axis then we can taken the direction of $\vec H$ and $\vec E$ in either Y and X or X and Y directions as shown in fig.



Wave Impedance:-

Now from equ. (XIII)

$$\vec K \times \vec E = \mu_0 \omega \vec H$$

$$\vec H = \frac{1}{\mu_0 \omega}(\vec K \times \vec E)$$

$$= \frac{K}{\mu_0 \omega}(\hat n \times \vec E)$$

$\vec K = K\hat n$

$\frac{\omega}{K} = c$

$n = c/v$

$$\vec H = \frac{1}{\mu_0 c}(\hat n \times \vec E) \quad \text{——— (XV)}$$

$$|\vec H| = \frac{1}{\mu_0 c}|(\hat n \times \vec E)|$$

$$H = \frac{1}{\mu_0 c} E$$

$$\frac{E}{H} = \frac{E_0}{H_0} = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}} \qquad \because c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

The ratio of magnitude of $\vec{E}$ to the H is symbolized as Z_0

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} = 376.6 \text{ ohm}$$

$\because$ Z_0 is reffered as the "wave impedance" of free space or Intrinsic impendance.

$\because Z_0 = \frac{E}{H} = \frac{V/m}{amp/m} = \text{ohm}$

Energy flow in Plane electromagnetic wave:- (Poynting vector)

The amount of energy carried by plane e.m. wave in free space is given by

$$\vec{S} = \vec{E} \times \vec{H} \quad \text{——— (1)}$$

Substituting the value of $\vec{H}$

$$\vec{S} = \vec{E} \times \frac{1}{\mu_0 c} (\hat{n} \times \vec{E})$$

$$\because (\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \cdot \vec{B} - (\vec{A} \cdot \vec{B}) \cdot \vec{C}$$

$$= \frac{1}{\mu_0 c} \vec{E} \times (\hat{n} \times \vec{E})$$

$$= \frac{1}{\mu_0 c} \left[(\vec{E} \cdot \vec{E}) \cdot \hat{n} - (\vec{E} \cdot \hat{n}) \cdot \vec{E} \right]$$

$\because \vec{E} \cdot \hat{n} = 0$ because $\vec{E}$ is $\perp$ to the direction of propagation

$\therefore \theta = 90°$

$$\vec{S} = \frac{1}{\mu_0 c} E^2 \hat{n}$$

$$= \frac{E^2 \hat{n}}{Z_0} \qquad \because Z_0 = \mu_0 c.$$

The average value of $\vec{S}$ over a complete cycle is

$$\langle \vec{S} \rangle = \frac{1}{Z_0} \langle E^2 \rangle \hat{n}$$

$$= \frac{1}{Z_0} \langle E_0 e^{i(\vec{K} \cdot \vec{r} - \omega t)} \rangle^2_{Real} \hat{n}$$

(39)

$\because e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore e^{-i(\omega t - \vec{K}\cdot\vec{r})} = \cos(\omega t - \vec{K}\cdot\vec{r}) - i\sin(\omega t - \vec{K}\cdot\vec{r})$

$$\therefore \langle \vec{S} \rangle = \frac{1}{Z_0} E_0^2 \langle \cos^2(\omega t - \vec{K}\cdot\vec{r}) \rangle \hat{n}$$

$$\because \langle \cos^2(\omega t - \vec{K}\cdot\vec{r}) \rangle = \frac{1}{2}$$

$$\therefore \langle \vec{S} \rangle = \frac{1}{Z_0} \frac{E_0^2}{2} \hat{n} \qquad \Big|\ \because E_{rms} = E_0/\sqrt{2}$$

$$\therefore \boxed{\langle \vec{S} \rangle = \frac{E^2_{rms}}{Z_0} \hat{n}}$$

Thus, the direction of Poynting vector is along the direction of propagation of plane e.m. wave in free space. Hence, the flow of energy in a plane e.m. wave in free space is along the direction of propagation of wave

<u>Energy Density in Plane Electromagnetic wave in free space</u>:-

The electric energy per unit volume or electric field energy density

$$u_E = \frac{1}{2} \epsilon_0 E^2 \quad \text{———} \quad (I)$$

The magnetic energy per unit volume or magnetic field energy density.

$$u_B = \frac{1}{2\mu_0} B^2 \quad \text{———} \quad (II)$$

The total e.m. field energy density (u)

$$u = u_E + u_B$$
$$= \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 \quad \text{———} \quad (III) \qquad \because \vec{B} = \mu_0 \vec{H}$$

$$\because \frac{E}{H} = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$\Rightarrow H = E \sqrt{\frac{\epsilon_0}{\mu_0}} \quad \text{———} \quad (IV)$$

(49) using equ. (iv) in equ. (iii) (5)

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\cancel{\mu_0}\frac{\epsilon_0}{\cancel{\mu_0}}E^2$$

$$\boxed{u = \epsilon_0 E^2} \quad \text{———} \quad (v)$$

Average energy density

$$\langle u \rangle = \langle \epsilon_0 E^2 \rangle$$

$$= \epsilon_0 E_0^2 \langle e^{i(\vec{k}\cdot\vec{r} - \omega t)} \rangle^2_{real} \qquad \because \langle \cos^2(\omega t - \vec{k}\cdot\vec{r}) \rangle = \frac{1}{2}$$

$$= \epsilon_0 E_0^2 \langle \cos^2(\omega t - \vec{k}\cdot\vec{r}) \rangle$$

$$= \frac{\epsilon_0 E_0^2}{2}$$

$$\langle u \rangle = \epsilon_0 E^2_{rms} \quad \text{———} \quad (vi)$$

Divide equ.

$$\frac{\langle \vec{S} \rangle}{\langle u \rangle} = \frac{\frac{\cancel{E^2_{rms}}}{Z_0}\hat{n}}{\epsilon_0 \cancel{E^2_{rms}}} = \frac{\hat{n}}{Z_0 \epsilon_0} = \frac{\hat{n}}{\epsilon_0\sqrt{\frac{\mu_0}{\epsilon_0}}}$$

$$= \frac{\hat{n}}{\sqrt{\mu_0 \epsilon_0}} = c\,\hat{n}$$

$$\langle \vec{S} \rangle = c\,\hat{n}\,\langle u \rangle$$

Energy flux = velocity of light × Energy density

⇒ The energy density associated with an e.m. wave in free space travels with a speed equal to the velocity of light with which the field vectors propagate.

(14)

The ratio of electric and magnetic energy densities

$$\frac{u_E}{u_B} = \frac{\frac{1}{2}\epsilon_0 E_0^2}{\frac{1}{2}\mu_0 H_0^2} = \frac{\epsilon_0 E^2}{\mu_0 H^2} = \frac{\epsilon_0}{\mu_0} \cdot \frac{\mu_0}{\epsilon_0}$$

$$= 1$$

$u_E = u_B$ energy.

$\Rightarrow$ The electric field energy density is equal to the magnetic field energy density.

---x---x---

Angular Momentum in Electromagnetic field

E.M. field also carries momentum. This can be derive as same manner as poynting vector

Momentum density of e.m. field is given by

$$= \frac{S}{c^2}$$

In dimensional way

$$\because S = \frac{\text{Energy}}{\text{area time}} = \frac{(\text{mass}) \times (\text{velo})^2}{(\text{length})^2 (\text{time})} = \frac{M \cancel{L^2} T^{-2}}{\cancel{L^2} \cdot T} = [M T^{-3}]$$

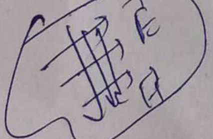
$$\therefore \text{Momentum density} = \frac{(\text{mass}) \times (\text{velo})}{(\text{length})^3}$$

$$= \frac{M \cancel{L} T^{-1}}{L^{\cancel{3}\,2}} = [M T^{-1} L^{-2}]$$

$$\frac{S}{c^2} = \cancel{\frac{M L^2 T^{-2}}{L^2\, T}} \cdot \cancel{\frac{T^2}{L^2}} = \frac{M}{\;}$$

$$= M T^{-3} \times L^{-2} T^{2}$$

$$= [M L^{-2} T^{-1}]$$



$$\therefore \text{Momentum density} = \frac{S}{c^2}$$

If there is same region has $\vec{E}$ (electric field vector) and $\vec{B}$ (magnetic field vector) then momentum contain in this -

Momentum density = $\frac{\vec{S}}{c^2} = \frac{1}{c^2} \frac{1}{\mu_0} (\vec{E} \times \vec{B})$

of momentum per unit vol

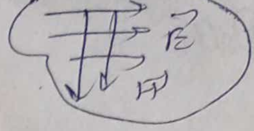
$\because \frac{1}{c^2} = \mu_0 \epsilon_0$

$$= \cancel{\mu_0} \epsilon_0 \frac{1}{\cancel{\mu_0}} (\vec{E} \times \vec{B})$$

$\therefore$ Momentum density $= \epsilon_0 (\vec{E} \times \vec{B})$.

Electromagnetic fields also carries angular momentum also

→ Angular momentum density of e.m. field about an arbitrary volume —



$$\therefore \quad \vec{L} = \vec{r} \times \vec{p}$$

$$= \vec{r} \times \frac{\vec{S}}{c^2}$$

p → linear

L = angular

$$= \vec{r} \times \frac{1}{c^2} (\vec{E} \times \vec{H})$$

$\therefore$ Angular momentum of the field

$$\boxed{\vec{L}_{em} = \frac{1}{c^2} \int_V \vec{r} \times (\vec{E} \times \vec{H}) \, dV}$$

or $$\boxed{\vec{L}_{em} = \epsilon_0 \int_V \left[\vec{r} \times (\vec{E} \times \vec{B}) \right] dV}$$