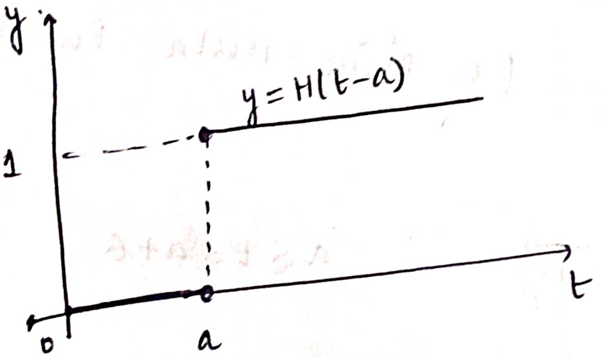
16.

LAPLACE TRANSFORM OF SOME SPECIAL FUNCTIONS.

(1) Heaviside Unit Step function.

An important discontinuous function that finds important application in connection with Laplace transform is Unit Step Function $H(t-a)$ or $U(t-a)$ with $a \geq 0$

It is defined as
$$H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$



(2) Unit Pulse Function

$y = H(t-a) - H(t-b)$ is defined as

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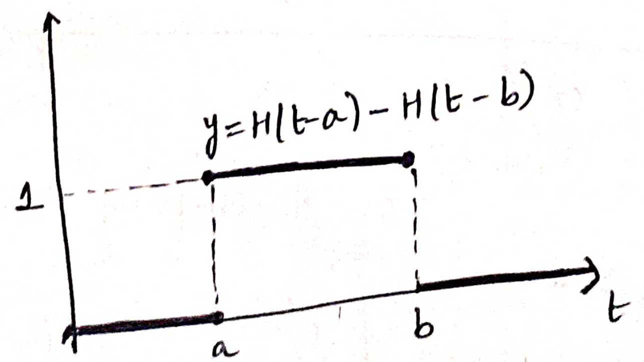
$$H(t-a) - H(t-b) = \begin{cases} 0 & t < a \\ 1 & a < t < b \\ 0 & t > b \end{cases} \qquad (b \geq a)$$



Laplace Transform

① $L(H(t-a)) = \int_a^\infty e^{-st}\,dt = \frac{e^{-as}}{s}$ for $s > a \geq 0$

② $L\left(H(t-a) - H(t-b)\right) = \left(\frac{e^{-as} - e^{-bs}}{s}\right)$

③ Second Shifting Theorem, $L[f(t-a)\,U(t-a)] = e^{-as} L[f(t)]$

or

If $L(f(t)) = \bar{f}(s)$ then $L(f(t-a)\,H(t-a)) = e^{-as}\bar{f}(s)$

$= e^{-as} L(f(t))$

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(3) Unit Impulse Function (or Dirac-Delta Function)

It is defined as

$$\delta(t-a) = \begin{cases} \frac{1}{\epsilon} & a \leq t \leq a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

④ $L(\delta(t-a)) = e^{-as}$

In particular for $a = 0$ $\quad L(\delta(t)) = 1$.

⑤ $L(f(t)\,\delta(t-a)) = e^{-as} f(a)$

Example/ To find $L^{-1}\left(\frac{s e^{-3s}}{s^2+4}\right)$.

We know $L^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t$

By II shifting theorem,

$L^{-1}\left(\frac{e^{-3s}\, s}{s^2+4}\right) = \cos 2(t-3)\,H(t-3)$

Ex 1: Find the Laplace transform of the function

$$f(t) = \begin{cases} t-1 & , \quad 1<t<2 \\ 3-t & , \quad 2<t<3 \end{cases}$$

Solution

$$f(t) = (t-1)\left[U(t-1) - U(t-2)\right] + (3-t)\left[U(t-2) - U(t-3)\right]$$

$$= (t-1)U(t-1) - 2(t-2)U(t-2) + (t-3)U(t-3)$$

$$\therefore \quad L[f(t)] = L\left((t-1)U(t-1)\right) - 2L\left((t-2)U(t-2)\right) + L\left((t-3)U(t-3)\right)$$

$$= e^{-s}L(t) - 2e^{-2s}L(t) + e^{-3s}L(t)$$

$$= \left(e^{-s} - 2e^{-2s} + e^{-3s}\right)L(t)$$

$$= \left(e^{-s} - 2e^{-2s} + e^{-3s}\right) \cdot \frac{1!}{s^2}$$

$$= \frac{e^{-s}}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right) = \frac{e^{-s}}{s^2}\left(1 - e^{-s}\right)^2$$

Ex 2: Express the function $f(t) = \begin{cases} 2t & 0<t<5 \\ 10 & t>5 \end{cases}$

in terms of unit step function and find its Laplace transform.

Solution $f(t) = 2t\left[U(t) - U(t-5)\right] + 10\left[U(t-5)\right]$

$= 2tU(t) - 2(t-5)U(t-5)$

$\therefore\ L(f(t)) = 2L(tU(t)) - 2L((t-5)U(t-5))$

$= 2e^{-0s}L(t) - 2e^{-5s}L(t)$

$= 2(1-e^{-5s})L(t) = \dfrac{2(1-e^{-5s})}{s^2}$.

Ex 3 :- Find the Laplace transform of

(1) $t^2U(t-3)$

$t^2 = [t-3+3]^2 = (t-3)^2 + 9 + 6(t-3)$

(2) $e^{-3t}U(t-2)$

[we have to write t^2 in terms of $(t-3)$].

Solution (1) $f(t) = t^2U(t-3)$

$= ((t-3)^2 + 6t - 9)U(t-3)$

$= ((t-3)^2 + 6(t-3) + 9)U(t-3)$

$\therefore\ L(f(t)) = L((t-3)^2U(t-3)) + 6L((t-3)U(t-3)) + 9L(U(t-3))$

$= e^{-3s}L(t^2) + 6e^{-3s}L(t) + \dfrac{9e^{-3s}}{s}$

$= e^{-3s}\cdot\dfrac{2!}{s^3} + 6e^{-3s}\dfrac{1!}{s^2} + \dfrac{9e^{-3s}}{s}$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right].$$

(2) $f(t) = e^{-3t}\, U(t-2)$ (writing e^{-3t} in term of $t-2$)

$$= \left(e^{-3(t-2)} * e^{-6}\right) U(t-2)$$

$$= e^{-6}\left(e^{-3(t-2)}\, U(t-2)\right)$$

$$\therefore\ L(f(t)) = e^{-6}\, L\left(e^{-3(t-2)}\, U(t-2)\right)$$

$$= e^{-6}\, e^{-2s} \cdot L\left(e^{-3t}\right)$$

$$= \frac{e^{-6-2s}}{s+3} = \frac{e^{-2(s+3)}}{s+3}.$$

Ex 4 Find the inverse Laplace Transform of

(1) $\dfrac{e^{-\pi s}}{s^2+4}$ (2) $\dfrac{(3s+1)}{s^2(s^2+4)}\, e^{-3s}$

Solution (1) We know $L^{-1}\left(\dfrac{1}{s^2+4}\right) = \dfrac{1}{2}\sin 2t$

Therefore, $L^{-1}\left(\dfrac{e^{-\pi s}}{s^2+4}\right) = \dfrac{1}{2}\sin 2(t-\pi)\, U(t-\pi)$

(Second shifting theorem)

$$= \begin{cases} 0 & 0 < t < \pi \\ \frac{1}{2}\sin 2t & t > \pi \end{cases}$$

(2) let $\bar{f}(s) = \dfrac{3s+1}{s^2(s^2+4)} = \dfrac{3s+1}{4}\left[\dfrac{1}{s^2} - \dfrac{1}{s^2+4}\right]$ 6

$$= \frac{3s+1}{4s^2} - \frac{3s+1}{4(s^2+4)}$$

$$\therefore \quad \bar{f}(s) = \frac{3}{4s} + \frac{1}{4s^2} - \frac{3s}{4(s^2+4)} - \frac{1}{4(s^2+4)}$$

Then $f(t) = L^{-1}(\bar{f}(s)) = \frac{3}{4}L^{-1}\left(\frac{1}{s}\right) + \frac{1}{4}L^{-1}\left(\frac{1}{s^2}\right)$

$$-\frac{3}{4}L^{-1}\left(\frac{s}{s^2+4}\right) - \frac{1}{4}L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$f(t) = \frac{3}{4}(1) + \frac{1}{4}t - \frac{3}{4}\cos 2t - \frac{\sin 2t}{8}$$

Now, $L(f(t-a)U(t-a)) = e^{-as}\bar{f}(s) = e^{-as}L(f(t))$

$$\therefore \quad L^{-1}\left(\frac{(3s+1)\cdot e^{-3s}}{s^2(s^2+4)}\right) = f(t-a)U(t-a) \qquad (a=3)$$

$$= \left(\frac{3+t-3}{4} - \frac{3}{4}\cos 2(t-3) - \frac{\sin 2(t-3)}{8}\right) U(t-3)$$

$$= \frac{1}{8}\left[2t - \sin 2(t-3) - 6\cos 2(t-3)\right]U(t-3).$$

GS Solve $\frac{d^2y}{dx^2} + 4y = U(x-2)$, $y(0)=0$, $y'(0)=1$.

Solution Taking laplace transform on both sides

$$s^2\bar{y}(s) - s\underset{0}{\underset{\shortparallel}{y(0)}} - \underset{1}{\underset{\shortparallel}{y'(0)}} + 4\bar{y}(s) = \frac{e^{-2s}}{s}$$

$$\therefore \quad \bar{y}(s)(s^2+4) = \frac{e^{-2s}}{s} + 1$$

$$\Rightarrow \quad \bar{y}(s) = \frac{e^{-2s}}{s(s^2+4)} + \frac{1}{s^2+4}$$

Therefore $L(y(t)) = \frac{e^{-2s}}{s(s^2+4)} + \frac{1}{s^2+4}$

$$\Rightarrow \quad y(t) = L^{-1}\left(\frac{e^{-2s}}{s(s^2+4)}\right) + L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= L^{-1}\left(\frac{e^{-2s}}{s(s^2+4)}\right) + \frac{1}{2}\sin 2t$$

Now $L^{-1}\left(\frac{1}{s(s^2+4)}\right) = \int_0^t \frac{1}{2}\sin 2t\, dt = \left[-\frac{\cos 2t}{4}\right]_0^t$

$$= \frac{1-\cos 2t}{4} = \frac{\sin^2 t}{2}$$

Since $L\left(f(t-a)U(t-a)\right) = \bar{f}(s)e^{-as}$, we have

$$L^{-1}\left(\frac{e^{-2s}}{s(s^2+4)}\right) = \frac{1}{2}\sin^2(t-2)\,U(t-2)$$

Thus, $$y = \frac{1}{2}\sin 2t + \frac{1}{2}\sin^2(t-2)\,U(t-2).$$

Ex Solve the initial value problem

$$y''+3y'+2y = H(t-\pi)\sin 2t, \quad y(0)=1, \; y'(0)=0$$

Solution Taking laplace transform on both sides, we get

$$s^2\bar{y}(s) - s\,y(0) - y'(0) + 3\left(s\bar{y}(s) - y(0)\right) + 2\bar{y}(s)$$

$$= L\left(H(t-\pi)\sin 2t\right)$$

$$\Rightarrow (s^2+3s+2)\,\bar{y}(s) - s - 3 = \frac{2e^{-\pi s}}{s^2+4}$$

$$\Rightarrow \bar{y}(s) = \frac{s+3}{s^2+3s+2} + \frac{2e^{-\pi s}}{(s^2+3s+2)(s^2+4)}$$

$$= \frac{2}{s+1} - \frac{1}{s+2} + e^{-\pi s}\left[\frac{2}{5(s+1)} - \frac{1}{4(s+2)} - \frac{1}{20}\left(\frac{2}{s^2+4}\right) - \frac{3s}{20(s^2+4)}\right]$$

$$\Rightarrow \bar{y}(s) = 2L(e^{-t}) - L(e^{-2t}) + \frac{2}{5}\left(e^{-(t-\pi)}H(t-\pi) - \frac{1}{4}e^{-2(t-\pi)}H(t-\pi)\right)$$

$$-\frac{1}{20}\sin 2(t-\pi)H(t-\pi) - \frac{3}{20}\cos 2(t-\pi)H(t-\pi).$$

Ex. Find $L\left(tU(t-4) - t^3\delta(t-2)\right)$

Solution $L[tU(t-4)] = L[(t-4+4)U(t-4)]$

$$= L[(t-4)U(t-4)] + 4L[U(t-4)]$$

$$= e^{-4s}L(t) + 4e^{-4s}L(1)$$

$$= e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right).$$

Again, $L[t^3\delta(t-2)] = e^{-2s}(2)^3 = 8e^{-2s}.$

Therefore, $L[tU(t-4) - t^3\delta(t-2)] = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - 8e^{-2s}.$

Exercise 16B

Ex Find the laplace transform of function of

$$e^{-t}(1 - U(t-2)).$$

Solution $L[e^{-t}(1-U(t-2))]$

$$= L(e^{-t}) - L(e^{-t}U(t-2))$$

$$= L(e^{-t}) - L\left(e^{-(t-2)} * e^{-4}U(t-2)\right)$$

$$= \frac{1}{s+1} - e^{-4}L\left(e^{-(t-2)}U(t-2)\right)$$

$$= \frac{1}{s+1} - e^{-4}\left\{ e^{-2s} \cdot L\left(e^{-t}\right)\right\}$$

$$= \frac{1}{s+1} - \frac{e^{-2s-4}}{s+1} = \frac{1-e^{-2(s+2)}}{s+1}$$

Ex 4 Find the inverse Laplace transform of

(1) $\dfrac{s e^{-s/2} + \pi e^{-s}}{s^2+\pi^2}$

Solution $L^{-1}\left(\dfrac{s e^{-s/2} + \pi e^{-s}}{s^2+\pi^2}\right)$

$$= L^{-1}\left(\frac{s e^{-s/2}}{s^2+\pi^2}\right) + \pi L^{-1}\left(\frac{e^{-s}}{s^2+\pi^2}\right)$$

We know, $L^{-1}\left(\dfrac{s}{s^2+\pi^2}\right) = \cos \pi t.$

By second shifting theorem,

$$L^{-1}\left(\frac{e^{-s/2} s}{s^2+\pi^2}\right) = \cos\left(\pi\left(t-\tfrac{1}{2}\right)\right) H\left(t-\tfrac{1}{2}\right)$$

$$= \cos\left(\pi t - \pi/2\right) H\left(t - 1/2\right)$$

$$= \cos\left(\pi/2 - \pi t\right) H\left(t-1/2\right)$$

$$= \sin \pi t \, H\left(t-1/2\right).$$

$$L^{-1}\left(\frac{e^{-s}}{s^2+\pi^2}\right) = \frac{1}{\pi}\sin \pi(t-1)H(t-1)$$

$$= \frac{1}{\pi}\sin(\pi t-\pi)\,H(t-1)$$

$$= \frac{-1}{\pi}\sin \pi t\, H(t-1).$$

$$\therefore\ L^{-1}\left(\frac{s e^{-s/2}}{s^2+\pi^2}\right) + \pi L^{-1}\left(\frac{e^{-s}}{s^2+\pi^2}\right)$$

$$= \sin \pi t\, H(t-1/2) - \frac{\pi}{\pi}\sin \pi t\, H(t-1)$$

$$= \sin \pi t\left(H(t-1/2) - H(t-1)\right).$$

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(2) $$L^{-1}\left(\frac{e^{-s} - 3e^{-3s}}{s^2}\right)$$

<u>Solution</u> $$L^{-1}\left(\frac{e^{-s}}{s^2}\right) - 3\,L^{-1}\left(\frac{e^{-3s}}{s^2}\right)$$

Now $L^{-1}\left(\frac{1}{s^2}\right) = t$ ~~and t~~

Using second shifting theorem,

$$L^{-1}\left(\frac{e^{-s}}{s^2}\right) = (t-1)H(t-1)$$

$$L^{-1}\left(\frac{e^{-3s}}{s^2}\right) = (t-3)\,H(t-3).$$

$$\therefore\ (t-1)H(t-1) - 3(t-3)H(t-3).$$

Ex Find the inverse Laplace transform of (12).

$$\left(\frac{1-\sqrt{s}}{s^{3/2}}\right)e^{-s}.$$

Solution $L^{-1}\left(\frac{e^{-s}}{s^{3/2}}\right) - L^{-1}\left(\frac{e^{-s}}{s}\right)$

Now $L^{-1}\left(\frac{1}{s^{3/2}}\right) = \frac{t^{1/2}}{\Gamma 3/2} = \frac{\sqrt{t}}{\sqrt{\pi}/2}$

$L^{-1}\left(\frac{1}{s}\right) = 1$

Using second shifting theorem

$L^{-1}\left(\frac{e^{-s}}{s^{3/2}}\right) = \sqrt{\frac{2}{\pi}}\cdot\sqrt{(t-1)}\, H(t-1)$

and $L^{-1}\left(\frac{e^{-s}}{s}\right) = H(t-1)$.

Thus, $L^{-1}\left(\left(\frac{1-\sqrt{s}}{s^{3/2}}\right)e^{-s}\right) = \left(\sqrt{\frac{2}{\pi}}\cdot\sqrt{t-1} - 1\right) H(t-1)$.