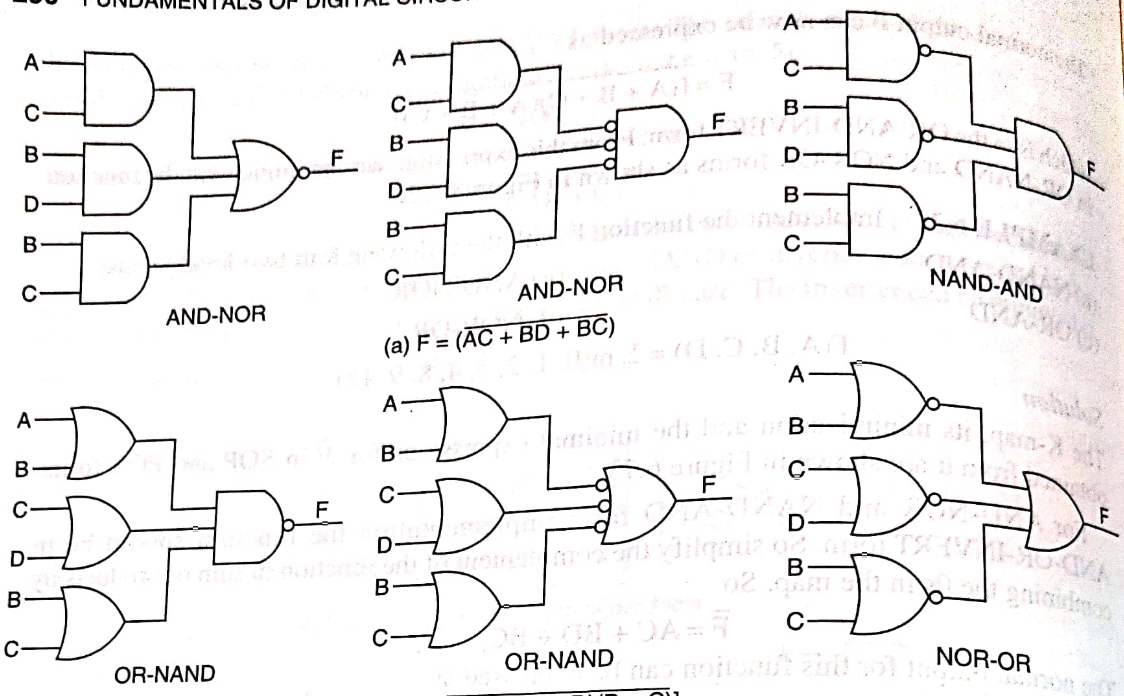

(a) $F = \overline{(AC + BD + BC)}$

(b) $F = \overline{[(A + B)(C + D)(B + C)]}$

Figure 6.74 AND-NOR and NAND-AND form, and OR-NAND and NOR-OR forms.

6.15 QUINE–McCLUSKEY METHOD

The minimization of Boolean expressions using K-maps is usually limited to a maximum of six variables. The Quine–McClusky method, also known as the *tabular method*, is a more systematic method of minimizing expressions of even larger number of variables. This is suitable for hand computation as well as computation by machines, i.e. it is programmable.

The fundamental idea on which this tabulation procedure is based is that, repeated application of the combining theorem $PA + P\bar{A} = P$ (where P is a set of literals) on all adjacent pairs of terms, yields the set of all prime implicants, from which a minimal sum may be selected.

Consider the minimization of the expression

$$\Sigma\, m(0, 1, 4, 5) = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + A\bar{B}\bar{C} + A\bar{B}C$$

The first two terms, and the third and fourth terms can be combined to yield

$$\bar{A}\bar{B}(C + \bar{C}) + A\bar{B}(C + \bar{C}) = \bar{A}\bar{B} + A\bar{B}$$

This expression can further be reduced to

$$\bar{B}(\bar{A} + A) = \bar{B}$$

In the first step, we combined two pairs of adjacent terms, each of three literals per term, into two terms each of two literals. In the second step, these two terms are combined again and reduced to one term of a single variable.

The same result can be obtained by combining m_0 and m_4, and m_1 and m_5 in the first step and the resulting terms in the second step. Minterms $m_0(\bar{A}\bar{B}\bar{C})$ and $m_1(\bar{A}\bar{B}C)$ are adjacent to each other because they differ in only literal C. Similarly, minterms $m_4(A\bar{B}\bar{C})$ and $m_5(A\bar{B}C)$ are adjacent to each other because they differ in only one literal C. Minterms $m_0(\bar{A}\bar{B}\bar{C})$ and $m_5(A\bar{B}C)$ or $m_1(\bar{A}\bar{B}C)$ and $m_4(A\bar{B}\bar{C})$ cannot be combined, being not adjacent to each other since they differ in more than one variable. If we consider the binary representation of minterms, $m_0(0\ 0\ 0)$ and $m_1(0\ 0\ 1)$, i.e. 0 0 0 and 0 0 1, they differ in only one position. When combined, they result in 0 0 –, i.e. variable C is absorbed. Similarly, $m_4(1\ 0\ 0)$ and $m_5(1\ 0\ 1)$, i.e. 1 0 0 and 1 0 1 differ in only one position. So, when combined, they result in 10 –. Now 0 0 – and 1 0 – can be combined because they differ in only one position. The result is a – 0 –.

For the binary representation of two minterms to be different in just one position, it is necessary (but not sufficient) that the number of 1s in those two minterms differs exactly by one. Consequently, to facilitate the combination process, the minterms are arranged in groups according to the number of 1s in their binary representation.

The procedure for the minimization of a Boolean expression by the tabular method may, therefore, be described as follows.

Step 1. List all the minterms.

Step 2. Arrange all minterms in groups of the same number of 1s in their binary representation in column 1. Start with the least number of 1s group and continue with groups of increasing number of 1s. The number of 1s in a term is called the *index* of the term. The number of 1s in the binary form of a minterm is also called its *weight*.

Step 3. Compare each term of the lowest index group with every term in the succeeding group. Whenever possible, combine the two terms being compared by means of the combining theorem. Two terms from adjacent groups are combinable, if their binary representations differ by just a single digit in the same position; the combined terms consist of the original fixed representation with the differing one replaced by a dash (–). Place a check mark (✓) next to every term, which has been combined with at least one term (each term may be combined with several terms, but only a single check is required) and write the combined terms in column 2. Repeat this by comparing each term in a group of index i with every term in the group of index $i + 1$, until all possible applications of the combining theorem have been exhausted.

Step 4. Compare the terms generated in step 2 in the same fashion; combine two terms which differ by only a single 1 and whose dashes are in the same position to generate a new term. Two terms with dashes in different positions cannot be combined. Write the new terms in column 3 and put a check mark next to each term which has been combined in column 2. Continue the process with terms in columns 3, 4 etc. until no further combinations are possible. The remaining unchecked terms constitute the set of prime implicants of the expression. They are called *prime implicants* because they are not covered by any other term with fewer literals.

Step 5. List all the prime implicants and draw the *prime implicant chart*. (The don't cares if any should not appear in the prime implicant chart).

Step 6. Obtain the *essential prime implicants* and write the minimal expression.

6.15.1 The Decimal Representation

The tabulation procedure can be further simplified by adopting the decimal code for the minterms rather than the binary representation. Two minterms can be combined, only if they differ by a power of 2; that is, the difference between their decimal codes is 2^i, namely 1, 2, 4, 8, and so on. The combined term consists of the same literals as the minterms with the exception of the variable whose weight is 2^i, being deleted. For example, minterms m_0 (0000) and m_8 (1000) differ by $8 = 2^3$. So, they can be combined and the combined term can be written as 0, 8 (8), instead of – 0 0 0. The 8 in the parentheses indicates that the variable A whose weight is 8 can be deleted.

The condition that the decimal codes of two combinable terms must differ by a power of 2 is necessary but not sufficient. The terms whose codes differ by a power of 2, but which have the same index cannot be combined, since they differ by more than one variable. For example minterms 1 and 2, 2 and 4, 4 and 8, 10 and 12, etc. cannot be combined. Similarly, if a term with a smaller index has a higher decimal value than another term whose index is higher, then the two terms cannot be combined although they may differ by a power of 2; for example, minterms 9 and 7, 17 and 13, 20 and 19, 25 and 23, etc. cannot be combined although they differ by a power of 2. Except for the above phenomenon, the tabulation procedure using decimal representation is completely analogous to that using binary representation. In practice, only the decimal representations are used.

6.15.2 Don't Cares

When don't care terms are present in an expression, during the process of generating the set of prime implicants, don't care combinations are regarded as true combinations, i.e. combinations for which the expression assumes a 1. This, in effect, increases to the maximum, the number of possible prime implicants. The don't care terms are, however, not considered in the next step of selecting a minimum set of prime implicants, i.e. don't care minterms are not listed as column headings in the prime implicant chart since they do not have to be covered by the minimal expression. By not listing them, we actually leave the specification of don't care terms open. The prime implicant chart, thus, yields a minimal of an expression, which covers all the specified minterms.

6.15.3 The Prime Implicant Chart

The prime implicant chart is a pictorial representation of the relationships between the prime implicants and the minterms of the expression. It consists of an array of u rows and v columns where u and v designate the number of prime implicants and the number of minterms for which the expression takes on the value of 1 respectively. The entries of the ith row consist of ×s placed at the intersections with the columns corresponding to minterms covered by the ith prime implicant. The don't cares are not entered in the prime implicant chart.

6.15.4 Essential Prime Implicants

Essential prime implicants are the implicants which will definitely occur in the final expression. Any row in a prime implicant chart which has at least one minterm that is not present in any other row is called the *essential row* and the corresponding prime implicant is called the *essential prime implicant*. In other words, if any column contains a single ×, the prime implicant corresponding to the row in which this × appears is the essential prime implicant.

Once an essential prime implicant has been selected, all the minterms it covers are checked off. After all the essential prime implicants and their corresponding columns have been checked off, if all the minterms are covered, the union of all the essential prime implicants yields the minimal expression. If this is not the case, additional prime implicants are necessary. We have to draw a reduced PI chart and find the minimal set of PIs from that.

6.15.5 Dominating Rows and Columns

Two rows (or columns) I and J of a prime implicant chart, which have ×s in exactly the same columns (or rows) are said to be equal (written I = J).

A column I in a prime implicant chart is said to dominate another column J of that chart, if column I has a × in every row in which column J has a ×. Any minimal expression derived from a chart which contains both columns I and J can be obtained from a chart containing the dominated column. Hence, if column I dominates column J, then column I can be deleted from the chart without affecting the search for a minimal expression.

A row I in a prime implicant chart is said to dominate another row J, if row I has a × in every column in which row J has a ×. Any minimal expression derived from a chart which contains both rows I and J can be derived from a chart which contains only the dominating row. Hence, if row I dominates row J, then row J can be deleted from the chart without affecting the search for a minimal expression.

6.15.6 Determination of Minimal Expressions in Complex Cases

In simple cases, we can determine the minimal expression by simply inspecting the prime implicant chart. In more complex cases, however, the inspection method becomes prohibitive. Here, the procedure is:

Step 1. Determine the essential prime implicants from the prime implicant chart.

Step 2. Form a reduced prime implicant chart by removing all essential prime implicants and the columns covered by them. Although, none of the rows in the reduced chart is essential, only some of them may be removed.

Step 3. Remove all the dominating columns and the dominated rows of this reduced chart and form a new reduced chart.

Step 4. Look for the secondary essential prime implicants in the new reduced chart, and form another chart by removing the secondary essential prime implicants and the columns covered by them and write the minimal expression in SOP form. Continue the process, if required.

6.15.7 The Branching Method

If the prime implicant chart has no essential prime implicants, dominated rows and dominated columns, the minimal expression can be obtained by a different approach called the *branching method*. Here, we consider any column and note the rows which cover that column. Make an arbitrary selection of one of those rows and apply the normal reduction procedure for the prime implicant chart without this row and the selected column and the columns covered by this row. The entire procedure is repeated for each row. Take the minimal of all such expressions obtained.

EXAMPLE 6.39 Obtain the set of prime implicants for the Boolean expression $f = \Sigma\, m(0, 1, 6, 7, 8, 9, 13, 14, 15)$ using the tabular method.

Solution

Group the minterms in terms of the number of 1s present in them and write their binary designations. The procedure to obtain the prime implicants is shown in Table 6.4.

Table 6.4 Example 6.39

Column 1			Column 2		Column 3	
Index	**Minterm**	**Binary designation**	**Pairs**	**A B C D**	**Quads**	**A B C D**
Index 0	0	0 0 0 0 ✓	0, 1 (1)	0 0 0 – ✓	0, 1, 8, 9 (1, 8)	– 0 0 – Q
Index 1	1	0 0 0 1 ✓	0, 8 (8)	– 0 0 0 ✓	… … … …	
	8	1 0 0 0 ✓	1, 9 (8)	– 0 0 1 ✓	6, 7, 14, 15 (1, 8)	– 1 1 – P
Index 2	6	0 1 1 0 ✓	8, 9 (1)	1 0 0 – ✓		
	9	1 0 0 1 ✓	6, 7 (1)	0 1 1 – ✓		
Index 3	7	0 1 1 1 ✓	6, 14 (8)	– 1 1 0 ✓		
	13	1 1 0 1 ✓	9, 13 (4)	1 – 0 1 S		
	14	1 1 1 0 ✓	7, 15 (8)	– 1 1 1 ✓		
Index 4	15	1 1 1 1 ✓	13, 15 (2)	1 1 – 1 R		
			14, 15 (1)	1 1 1 – ✓		

Comparing the terms of index 0 with the terms of index 1 of column 1, m_0(0000) is combined with m_1(0001) to yield 0, 1 (1), i.e. 000 –. This is recorded in column 2 and 0000 and 0001 are checked off in column 1. m_0(0000) is combined with m_8(1000) to yield 0, 8 (8), i.e. – 000. This is recorded in column 2 and 1000 is checked off in column 1. Note that 0000 of column 1 has already been checked off. No more combinations of terms of index 0 and index 1 are possible. So, draw a line below the last combination of these groups, i.e. below 0, 8 (8), – 000 in column 2. Now 0, 1 (1), i.e. 000 – and 0, 8 (8), i.e. – 000 are the terms in the first group of column 2.

Comparing the terms of index 1 with the terms of index 2 in column 1, m_1(0001) is combined with m_9(1001) to yield 1, 9 (8), i.e. – 001. This is recorded in column 2 and 1001 is checked off in column 1 because 0001 has already been checked off. m_8(1000) is combined with m_9(1001) to yield 8, 9 (1), i.e. 100 –. This is recorded in column 2. 1000 and 1001 of column 1 have already been checked off. So, no need to check them off again. No more combinations of terms of index 1 and index 2 are possible. So, draw a line below the last combination of these groups, i.e. 8, 9 (1), –– 001 in column 2. Now 1, 9 (8), i.e. – 001 and 8, 9 (1), i.e. 100– are the terms in the second group of column 2.

Similarly, comparing the terms of index 2 with the terms of index 3 in column 1,

m_6(0110) and m_7(0111) yield 6, 7 (1), i.e. 011–. Record it in column 2 and check off 6(0110) and 7(0111).

m_6(0110) and m_{14}(1110) yield 6, 14 (8), i.e. –110. Record it in column 2 and check off 6(0110) and 14(1110).

m_9(1001) and m_{13}(1101) yield 9, 13 (4), i.e. 1–01. Record it in column 2 and check off 9(1001) and 13(1101).

So, 6, 7 (1), i.e. 011–, and 6, 14 (8), i.e. –110 and 9, 13 (4), i.e. 1–01 are the terms in group 3 of column 2. Draw a line at the end of 9, 13 (4), i.e. 1–01.

Also, comparing the terms of index 3 with the terms of index 4 in column 1,

m_7(0111) and m_{15}(1111) yield 7, 15 (8), i.e. –111. Record it in column 2 and check off 7(0111) and 15(1111).

m_{13}(1101) and m_{15}(1111) yield 13, 15 (2), i.e. 11–1. Record it in column 2 and check off 13 and 15.

m_{14}(1110) and m_{15}(1111) yield 14, 15 (1), i.e. 111–. Record it in column 2 and check off 14 and 15.

So, 7, 15 (8), i.e. –111, and 13, 15 (2), i.e. 11–1 and 14, 15 (1), i.e. 111– are the terms in group 4 of column 2. Column 2 is completed now.

Comparing the terms of group 1 with the terms of group 2 in column 2, the terms 0, 1 (1), i.e. 000– and 8, 9 (1), i.e. 100– are combined to form 0, 1, 8, 9 (1, 8), i.e. –00–. Record it in group 1 of column 3 and check off 0, 1 (1), i.e. 000–, and 8, 9 (1), i.e. 100– of column 2. The terms 0, 8 (8), i.e. –000 and 1, 9 (8), i.e. –001 are combined to form 0, 1, 8, 9 (1, 8), i.e. –00–. This has already been recorded in column 3. So, no need to record again. Check off 0, 8 (8), i.e. –000 and 1, 9 (8), i.e. –001 of column 2. Draw a line below 0, 1, 8, 9 (1, 8), i.e. –00–. This is the only term in group 1 of column 3. No term of group 2 of column 2 can be combined with any term of group 3 of column 2. So, no entries are made in group 2 of column 2.

Comparing the terms of group 3 of column 2 with the terms of group 4 of column 2, the terms 6, 7 (1), i.e. 011–, and 14, 15 (1), i.e. 111– are combined to form 6, 7, 14, 15 (1, 8), i.e. –11–. Record it in group 3 of column 3 and check off 6, 7 (1), i.e. 011– and 14, 15 (1), i.e. 111– of column 2. The terms 6, 14 (8), i.e. –110 and 7, 15 (8), i.e. –111 are combined to form 6, 7, 14, 15 (1, 8), i.e. –11–. This has already been recorded in column 3; so, check off 6, 14 (8), i.e. –110 and 7, 15 (8), i.e. –111 of column 2.

Observe that the terms 9, 13 (4), i.e. 1–01 and 13, 15 (2), i.e. 11–1 cannot be combined with any other terms. Similarly in column 3, the terms 0, 1, 8, 9 (1, 8), i.e. –00– and 6, 7, 14, 15 (1, 8), i.e. –11– cannot also be combined with any other terms. So, these 4 terms are the prime implicants. The terms, which cannot be combined further, are labelled as P, Q, R, and S. These form the set of prime implicants.

EXAMPLE 6.40 Obtain the minimal expression for f = Σ m(1, 2, 3, 5, 6, 7, 8, 9, 12, 13, 15) using the tabular method.

Solution

The procedure to obtain the set of prime implicants is illustrated in Table 6.5.

Table 6.5 Example 6.40

Column 1		Column 2	Column 3	
Index	**Minterm**	**Pairs**	**Quads**	
Index 1	1 ✓	1, 3 (2) ✓	1, 3, 5, 7 (2, 4)	T
	2 ✓	1, 5 (4) ✓	1, 5, 9, 13 (4, 8)	S
	8 ✓	1, 9 (8) ✓	2, 3, 6, 7 (1, 4)	R
Index 2	3 ✓	2, 3 (1) ✓	8, 9, 12, 13 (1, 4)	Q
	5 ✓	2, 6 (4) ✓	5, 7, 13, 15 (2, 8)	P
	6 ✓	8, 9 (1) ✓		
	9 ✓	8, 12 (4) ✓		
	12 ✓	3, 7 (4) ✓		
Index 3	7 ✓	5, 7 (2) ✓		
	13 ✓	5, 13 (8) ✓		
Index 4	15 ✓	6, 7 (1) ✓		
		9, 13 (4) ✓		
		12, 13 (1) ✓		
		7, 15 (8) ✓		
		13, 15 (2) ✓		

The non-combinable terms P, Q, R, S and T are recorded as prime implicants.

$$P \rightarrow 5, 7, 13, 15\ (2, 8) = X\,1\,X\,1 = BD$$

(Literals with weights 2 and 8, i.e. C and A are deleted. The lowest minterm is $m_5 (5 = 4 + 1)$. So, literals with weights 4 and 1, i.e. B and D are present in non-complemented form. So, read it as BD.)

$$Q \rightarrow 8, 9, 12, 13\ (1, 4) = 1\,X\,0\,X = A\bar{C}$$

(Literals with weights 1 and 4, i.e. D and B are deleted. The lowest minterm is m_8. So, literal with weight 8 is present in non-complemented form and literal with weight 2 is present in complemented form. So, read it as $A\bar{C}$.)

$$R \rightarrow 2, 3, 6, 7\ (1, 4) = 0\,X\,1\,X = \bar{A}C$$

(Literals with weights 1 and 4, i.e. D and B are deleted. The lowest minterm is m_2. So, literal with weight 2 is present in non-complemented form and literal with weight 8 is present in complemented form. So, read it as $\bar{A}C$.)

$$S \rightarrow 1, 5, 9, 13\ (4, 8) = X\,X\,0\,1 = \bar{C}D$$

(Literals with weights 4 and 8, i.e. B and A are deleted. The lowest minterm is m_1. So, literal with weight 1 is present in non-complemented form and literal with weight 2 is present in complemented form. So, read it as $\bar{C}D$.)

$$T \rightarrow 1, 3, 5, 7\ (2, 4) = 0\,X\,X\,1 = \bar{A}D$$

(Literals with weights 2 and 4, i.e. C and B are deleted. The lowest minterm is 1. So, literal with weight 1 is present in non-complemented form and literal with weight 8 is present in complemented form. So, read it as $\bar{A}D$.)

The prime implicant chart of the expression

$$f = \Sigma\, m(1, 2, 3, 5, 6, 7, 8, 9, 12, 13, 15)$$

is as shown in Table 6.6. It consists of 11 columns corresponding to the number of minterms and 5 rows corresponding to the prime implicants P, Q, R, S, and T generated. Row R contains four ×s at the intersections with columns 2, 3, 6, and 7, because these minterms are covered by the prime implicant R. A row is said to cover the columns in which it has ×s. The problem now is to select a minimal subset of prime implicants, such that each column contains at least one × in the rows corresponding to the selected subset and the total number of literals in the prime implicants selected is as small as possible. These requirements guarantee that the number of unions of the selected prime implicants is equal to the original number of minterms and that, no other expression containing fewer literals can be found.

Table 6.6 Example 6.40: Prime implicant chart

		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
PIs/Minterms	1	2	3	5	6	7	8	9	12	13	15
*P → 5, 7, 13, 15 (2, 8)				×		×				×	×
*Q → 8, 9, 12, 13 (1, 4)							×	×	×	×	
*R → 2, 3, 6, 7 (1, 4)		×	×		×	×					
S → 1, 5, 9, 13 (4, 8)	×			×				×		×	
T → 1, 3, 5, 7 (2, 4)	×		×	×		×					

In the prime implicant chart of Table 6.6, m_2 and m_6 are covered by R only. So, R is an essential prime implicant. So, check off all the minterms covered by it, i.e. m_2, m_3, m_6, and m_7. Q is also an essential prime implicant because only Q covers m_8 and m_{12}. Check off all the minterms covered by it, i.e. m_8, m_9, m_{12}, and m_{13}. P is also an essential prime implicant, because m_{15} is covered only by P. So check off m_{15}, m_5, m_7, and m_{13} covered by it. Thus, only minterm 1 is not covered. Either row S or row T can cover it and both have the same number of literals. Thus, two minimal expressions are possible.

$$P + Q + R + S = BD + A\bar{C} + \bar{A}C + \bar{C}D$$

or

$$P + Q + R + T = BD + A\bar{C} + \bar{A}C + \bar{A}D$$

EXAMPLE 6.41 Using the tabular method, obtain the minimal expression for

$$f = \Sigma\, m(6, 7, 8, 9) + d(10, 11, 12, 13, 14, 15)$$

Solution

In the given Boolean expression there are don't cares. Treat the don't cares as minterms and apply the usual procedure to obtain the set of prime implicants as shown in Table 6.7.

Table 6.7 Example 6.41

Column 1		Column 2	Column 3	Column 4
Index	**Minterm**	**Pairs**	**Quads**	**Octets**
Index 1	8 ✓	8, 9 (1) ✓	8, 9, 10, 11 (1, 2) ✓	8, 9, 10, 11, 12, 13, 14, 15 (1, 2, 4) P
	6 ✓	8, 10 (2) ✓	8, 9, 12, 13 (1, 4) ✓	
	9 ✓	8, 12 (4) ✓	8, 10, 12, 14 (2, 4) ✓	
Index 2	10 ✓	6, 7 (1) ✓	6, 7, 14, 15 (1, 8) Q	
	12 ✓	6, 14 (8) ✓	9, 11, 13, 15, (2, 4) ✓	
	7 ✓	9, 11 (2) ✓	10, 11, 14, 15 (1, 4) ✓	
	11 ✓	9, 13 (4) ✓	12, 13, 14, 15 (1, 2) ✓	
Index 3	13 ✓	10, 11 (1) ✓		
	14 ✓	10, 14 (4) ✓		
Index 4	15 ✓	12, 13 (1) ✓		
		12, 14 (2) ✓		
		7, 15 (8) ✓		
		11, 15 (4) ✓		
		13, 15 (2) ✓		
		14, 15 (1) ✓		

From this table, we see that the prime implicants are P $\rightarrow$ 8, 9, 10, 11, 12, 13, 14, 15 (1, 2, 4) and Q $\rightarrow$ 6, 7, 14, 15 (1, 8). The term 6, 7, 14, 15 (1, 8) means that literals with weights 1 and 8, i.e. D and A are deleted and the lowest designated minterm is $m_6(4 + 2)$, i.e. literals with weights 4 and 2 are present in non-complemented form. So it is read as BC. The term 8, 9, 10, 11, 12, 13, 14, 15 (1, 2, 4) means that literals with weights 1, 2, and 4, i.e. D, C, and B are deleted. The lowest designated minterm is, therefore, m_8. So, literal with weight 8 is present in non-complemented form. So, it is read as A.

In the prime implicant chart of Σ (6, 7, 8, 9) + d(10, 11, 12, 13, 14, 15) shown in Table 6.8, all the don't care minterms are omitted.

Table 6.8 Example 6.41: Prime implicant chart

	✓	✓	✓	✓
PIs/Minterms	6	7	8	9
*P $\rightarrow$ 8, 9, 10, 11, 12, 13, 14, 15 (1, 2, 4)			×	×
*Q $\rightarrow$ 6, 7, 14, 15 (1, 8)	×	×		

As seen from the table, P and Q are both essential prime implicants. So, the minimal expression is A + BC.

EXAMPLE 6.42 Using the Quine–McCluskey method of tabular reduction minimize the given combinational single output function f(W, X, Y, Z) = Σ m(0, 1, 5, 7, 8, 10, 14, 15).

Solution

Minimization using the tabular method is as shown in Table 6.9.

Table 6.9 Example 6.42

Column 1		Column 2	
Index	**Minterm**	**Pairs**	
Index 0	0 ✓	0, 1 (1)	A
Index 1	1 ✓	0, 8 (8)	B
	8 ✓	1, 5 (4)	C
Index 2	5 ✓	8, 10 (2)	D
	10 ✓	5, 7 (2)	E
Index 3	7 ✓	10, 14 (4)	F
	14 ✓	7, 15 (8)	G
Index 4	15 ✓	14, 15 (1)	H

None of the terms in any group of step 2 can be combined with any other term in the next group. So all of them are prime implicants. The PI chart is shown in Table 6.10.

Table 6.10 Example 6.42: Prime implicant chart

PIs/Minterms	0	1	5	7	8	10	14	15
A → 0, 1 (1)	×	×						
B → 0, 8 (8)	×				×			
C → 1, 5 (4)		×	×					
D → 8, 10 (2)					×	×		
E → 5, 7 (2)			×	×				
F → 10, 14 (4)						×	×	
G → 7, 15 (8)				×				×
H → 14, 15 (1)							×	×

We can see from the PI chart that there are no essential prime implicants and one possible minimal combination of PIs that can cover all minterms is A, E, D, H. Therefore,

$$f_{min} = A + E + D + H = 000- + 01-1 + 10-0 + 111- = \bar{W}\bar{X}\bar{Y} + \bar{W}XZ + W\bar{X}\bar{Z} + WXY$$

$$= \overline{\overline{(\bar{W}\bar{X}\bar{Y})}\;\overline{(\bar{W}XZ)}\;\overline{(W\bar{X}\bar{Z})}\;\overline{(WXY)}}$$

EXAMPLE 6.43 Minimize the following expression:

$$f = \Sigma\, m(0, 1, 2, 8, 9, 15, 17, 21, 24, 25, 27, 31)$$

Solution

Table 6.11 shows the procedure for obtaining all the prime implicants.

Table 6.11 Example 6.43

Column 1		Column 2	Column 3
Index	**Minterm**	**Pairs**	**Quads**
Index 0	0 ✓	0, 1 (1) ✓	0, 1, 8, 9 (1, 8) R
	1 ✓	0, 2 (2) W	1, 9, 17, 25 (8, 16) Q
Index 1	2 ✓	0, 8 (8) ✓	8, 9, 24, 25 (1, 16) P
	8 ✓	1, 9 (8) ✓	
	9 ✓	1, 17 (16) ✓	
Index 2	17 ✓	8, 9 (1) ✓	
	24 ✓	8, 24 (16) ✓	
Index 3	21 ✓	9, 25 (16) ✓	
	25 ✓	17, 21 (4) V	
Index 4	15 ✓	17, 25 (8) ✓	
	27 ✓	24, 25 (1) ✓	
Index 5	31 ✓	25, 27 (2) U	
		15, 31 (16) T	
		27, 31 (4) S	

From Table 6.11 we see that the prime implicants are P, Q, R, S, T, U, V, and W. The prime implicant chart is shown in Table 6.12.

Table 6.12 Example 6.43: Prime implicant chart

0, 9, 24, 25, 15, 21,

	✓		✓	✓	✓	✓	✓	✓	✓	✓		✓
PIs/Minterms	0	1	2	8	9	15	17	21	24	25	27	31
*P → 8, 9, 24, 25				×	×				×	×		
Q → 1, 9, 17, 25		×			×		×			×		
R → 0, 1, 8, 9	×	×		×	×							
S → 27, 31											×	×
*T → 15, 31						×						×
U → 25, 27										×	×	
*V → 17, 21							×	×				
*W → 0, 2	×		×									

From the PI chart we observe that P, T, V, and W are the essential prime implicants because m_{24} is covered by P only, m_{15} is covered by T only, m_{21} is covered by V only and m_2 is covered by

W only. Delete the essential prime implicants and the columns covered by them and form the reduced prime implicant chart as shown in Table 6.13.

Table 6.13 Example 6.43: Reduced prime implicant chart

PIs/Minterms	1	27
Q	×	
R	×	
S		×
U		×

m_1 and m_{27} can be covered by Q + S or Q + U or R + S or R + U. So there are 4 minimal SOP expressions as shown below.

$$P+T+V+W+Q+S = B\bar{C}\bar{D} + BCDE + A\bar{B}\bar{D}E + \bar{A}\bar{B}\bar{C}\bar{E} + \bar{C}\bar{D}E + ABDE$$

$$P+T+V+W+Q+U = B\bar{C}\bar{D} + BCDE + A\bar{B}\bar{D}E + \bar{A}\bar{B}\bar{C}\bar{E} + \bar{C}\bar{D}E + ABC\bar{E}$$

$$P+T+V+W+R+S = B\bar{C}\bar{D} + BCDE + A\bar{B}\bar{D}E + \bar{A}\bar{B}\bar{C}\bar{E} + \bar{A}\bar{C}\bar{D} + ABDE$$

$$P+T+V+W+R+U = B\bar{C}\bar{D} + BCDE + A\bar{B}\bar{D}E + \bar{A}\bar{B}\bar{C}\bar{E} + \bar{A}\bar{C}\bar{D} + ABC\bar{E}$$

EXAMPLE 6.44 Find the minimal expression for $f = \Pi M(2, 3, 8, 12, 13) \cdot d(10, 14)$.

Solution

The tabulation method for the POS form is exactly the same as that for the SOP form. Treat the maxterms as if they are the minterms and complete the process. While writing the minimal expression in the POS form, treat the non-complemented variable as a 0 and the complemented variable as a 1 and write the terms in sum form. The prime implicants are obtained as shown in Table 6.14.

Table 6.14 Example 6.44

Column 1		Column 2	Column 3
Index	**Minterm**	**Pairs**	**Quads**
Index 1	2 ✓	2, 3 (1) S	8, 10, 12, 14 (2, 4) P
	8 ✓	2, 10 (8) R	
	3 ✓	8, 10 (2) ✓	
Index 2	10 ✓	8, 12 (4) ✓	
	12 ✓	10, 14 (4) ✓	
Index 3	13 ✓	12, 13 (1) Q	
	14 ✓	12, 14 (2) ✓	

There are four prime implicants P, Q, R, and S. Draw the prime implicant chart as shown in Table 6.15.

Table 6.15 Example 6.44: Prime implicant chart

	✓	✓	✓	✓	✓
PIs/Minterms	2	3	8	12	13
*P → 8, 10, 12, 14 (2, 4)			×	×	
*Q → 12, 13 (1)				×	×
R → 2, 10 (8)	×				
*S → 2, 3 (1)	×	×			

In Table 6.15 M_8 is covered by P only. Further, M_{13} is covered by Q only and M_3 is covered by S only. Therefore, P, Q, and S are the essential prime implicants. Check them out and also check the Maxterms covered by them. They cover all the Maxterms. So, the final expression is given by

$$f_{min} = PQS = (\bar{A} + D)(\bar{A} + \bar{B} + C)(A + B + \bar{C}) \quad \text{(11 gate inputs)}$$

EXAMPLE 6.45 Obtain the minimal POS expression for the following:

$$f = \Pi\, M(0, 1, 4, 5, 9, 11, 13, 15, 16, 17, 25, 27, 28, 29, 31) \cdot d(20, 21, 22, 30)$$

Solution

The prime implicants are obtained as shown in Table 6.16.

Table 6.16 Example 6.45

Column 1	Column 2	Column 3	Column 4
Minterms	Pairs	Quads	Octets
0 ✓	0, 1 (1) ✓	0, 1, 4, 5 (1, 4) ✓	0, 1, 4, 5, 16, 17, 20, 21 (1, 4, 16) R
1 ✓	0, 4 (4) ✓	0, 1, 16, 17 (1, 16) ✓	1, 5, 9, 13, 17, 21, 25, 29 (4, 8, 16) Q
4 ✓	0, 16 (16) ✓	0, 4, 16, 20 (4, 16) ✓	9, 11, 13, 15, 25, 27, 29, 31 (2,4,16) P
16 ✓	1, 5 (4) ✓	1, 5, 9, 13 (4, 8) ✓	
5 ✓	1, 9 (8) ✓	1, 5, 17, 21 (4, 16) ✓	
9 ✓	1, 17 (16) ✓	1, 9, 17, 25 (8, 16) ✓	
17 ✓	4, 5 (1) ✓	4, 5, 20, 21 (1, 16) ✓	
20 ✓	4, 20 (16) ✓	16, 17, 20, 21 (1, 4) ✓	
11 ✓	16, 17 (1) ✓	5, 13, 21, 29 (8, 16) ✓	
13 ✓	16, 20 (4) ✓	9, 11, 13, 15 (2, 4) ✓	
21 ✓	5, 13 (8) ✓	9, 11, 25, 27 (2, 16) ✓	
22 ✓	5, 21 (16) ✓	9, 13, 25, 29 (4, 16) ✓	
25 ✓	9, 11 (2) ✓	17, 21, 25, 29 (4, 8) ✓	
28 ✓	9, 13 (4) ✓	20, 21, 28, 29 (1, 8) U	
15 ✓	9, 25 (16) ✓	20, 22, 28, 30 (2, 8) T	
27 ✓	17, 21 (4) ✓	11, 15, 27, 31 (4, 16) ✓	
29 ✓	17, 25 (8) ✓	13, 15, 29, 31 (2, 16) ✓	

(Contd.)

Table 6.16 Example 6.45 *(Contd.)*

Column 1 Minterms	Column 2 Pairs	Column 3 Quads	Column 4 Octets
30 ✓	20, 21 (1) ✓	25, 27, 29, 31 (2, 4) ✓	
31 ✓	20, 22 (2) ✓	28, 29, 30, 31 (1, 2) S	
	20, 28 (8) ✓		
	11, 15 (4) ✓		
	11, 27 (16) ✓		
	13, 15 (2) ✓		
	13, 29 (16) ✓		
	21, 29 (8) ✓		
	22, 30 (8) ✓		
	25, 27 (2) ✓		
	25, 29 (4) ✓		
	28, 29 (1) ✓		
	28, 30 (2) ✓		
	15, 31 (16) ✓		
	27, 31 (4) ✓		
	29, 31 (2) ✓		
	30, 31 (1) ✓		

From Table 6.16 we see that there are six prime implicants P, Q, R, S, T, and U. The prime implicant chart is shown in Table 6.17.

Table 6.17 Example 6.45: Prime implicant chart

	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓
	0	1	4	5	9	11	13	15	16	17	25	27	28	29	31
*P					×	×	×	×			×	×		×	×
Q		×		×	×		×			×	×			×	
*R	×	×	×	×					×	×					
S													×	×	×
T													×		
U													×	×	

Here M_{11}, M_{15}, and M_{27} are covered by P only. So P is an essential prime implicant. Also M_0, M_4, and M_{16} are covered by R only. So, R is an essential prime implicant. The only Maxterm left uncovered is M_{28}. It can be covered by S or T or U. So the minimum POS expression is given by PRS or PRT or PRU and each one requires ten gate inputs.

$$f_{min} = PRS = (\bar{B} + \bar{E})(B + D)(\bar{A} + \bar{B} + \bar{C})$$
$$f_{min} = PRT = (\bar{B} + \bar{E})(B + D)(\bar{A} + \bar{C} + E)$$
$$f_{min} = PRU = (\bar{B} + \bar{E})(B + D)(\bar{A} + \bar{C} + D)$$

EXAMPLE 6.46 Use the tabular procedure to simplify the given expression

$$F(V, W, X, Y, Z) = \Sigma\, m(0, 4, 12, 16, 19, 24, 27, 28, 29, 31)$$

in SOP form and draw the circuit using only NAND gates.

Solution

The tabular procedure is shown in Table 6.18.

Table 6.18 Example 6.46

Column 1		Column 2	
Index	**Minterms**	**Pairs**	
Index 0	0 ✓	0, 4 (4)	A
Index 1	4 ✓	0, 16 (16)	B
	16 ✓	4, 12 (8)	C
Index 2	12 ✓	16, 24 (8)	D
	24 ✓	12, 28 (16)	E
Index 3	19 ✓	24, 28 (4)	F
	28 ✓	19, 27 (8)	G
Index 4	27 ✓	28, 29 (1)	H
	29 ✓	27, 31 (4)	I
Index 5	31 ✓	29, 31 (2)	J

No term in any group of step 2 can be combined with any other term in the next group. So all the terms in step 2 are prime implicants. The prime implicant chart is shown in Table 6.19.

Table 6.19 Example 6.46: Prime implicant chart

					✓		✓			
PIs/Minterms	0	4	12	16	19	24	27	28	29	31
A → 0, 4 (4)	×	×								
B → 0, 16 (16)	×			×						
C → 4, 12 (8)		×	×							
D → 16, 24 (8)				×		×				
E → 12, 28 (16)			×					×		
F → 24, 28 (4)						×		×		
G* → 19, 27 (8)					×		×			
H → 28, 29 (1)								×	×	
I → 27, 31 (4)							×			×
J → 29, 31 (2)									×	×

In the PI chart m_{19} can be covered only by the prime implicant G. So G is an essential PI. It also covers m_{27}. From the PI chart we can observe that the remaining minterms are covered by the minimal set of prime implicants A, D, E, J. Therefore,

$$f_{min} = G + A + D + E + J = 1\text{–}011 + 00\text{–}00 + 1\text{–}000 + \text{–}1100 + 111\text{–}1$$
$$= V\bar{X}YZ + \bar{V}\bar{W}\bar{Y}\bar{Z} + V\bar{X}\bar{Y}\bar{Z} + WX\bar{Y}\bar{Z} + VWXZ$$
$$= \overline{\overline{(V\bar{X}YZ)}\,\overline{(\bar{V}\bar{W}\bar{Y}\bar{Z})}\,\overline{(V\bar{X}\bar{Y}\bar{Z})}\,\overline{(WX\bar{Y}\bar{Z})}\,\overline{(VWXZ)}}$$

The above functions can be realized using AOI logic and NAND logic.

EXAMPLE 6.47 Apply branching method to simplify the following function:

$$f(A, B, C, D) = \Sigma\, m(2, 3, 4, 6, 9, 11, 12, 13)$$

Solution

The branching method is applied if the prime implicant chart contains no essential prime implicants, dominated rows, and dominating columns. The PI chart is obtained using the tabular method.

Table 6.20 Example 6.47

Column 1		Column 2
Index	**Minterm**	**Pairs**
Index 1	2	2, 3 (1) W
	4	2, 6 (4) V
Index 2	3	4, 6 (2) U
	6	4, 12 (8) T
	9	3, 11 (8) S
	12	9, 11 (2) R
Index 3	11	9, 13 (4) Q
	13	12, 13 (1) P

There are eight prime implicants of equal size. The prime implicant chart is shown in Table 6.21.

Table 6.21 Example 6.47: Prime implicant chart

PIs/Minterms	2	3	4	6	9	11	12	13
W → 2,3 (1)	×	×						
V → 2,6 (4)	×			×				
U → 4,6 (2)			×	×				
T → 4,12 (8)			×				×	
S → 3,11(8)		×				×		
R → 9,11 (2)					×	×		
Q → 9,13 (4)					×			×
P → 12,13(1)							×	×