Mid term Examination

Discrete Mathematics

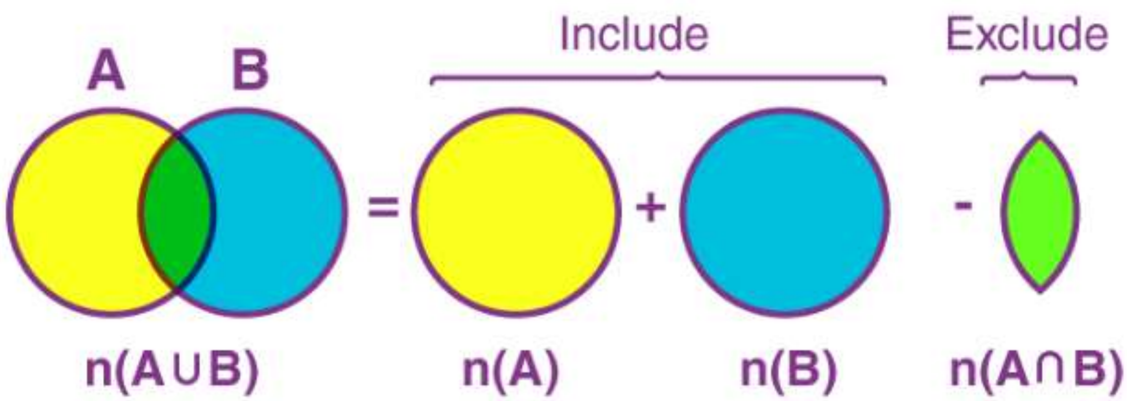
Q1) a)

Ans: The **principle of inclusion and exclusion (PIE)** is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

Consider two finite sets, A and B. We can denote the Principle of Inclusion and Exclusion formula as follows:

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Here, $n(A)$ denotes the cardinality of set A, $n(B)$ denotes the cardinality of set B and $n(A \cap B)$ denotes the cardinality of $(A \cap B)$. We have included A and B and excluded their common elements.



Q1 b))

Ans: A typical Divide and Conquer technique solves a problem using the following three steps.

1. **Divide**: Break the given problem into subproblems of same type. This step involves breaking the problem into smaller sub-problems. Sub-problems should represent a part of the original problem. This step generally takes a recursive approach to divide the problem until no sub-problem is further divisible. At this stage, sub-problems become atomic in nature but still represent some part of the actual problem.

2. **Conquer**: Recursively solve these sub-problems. This step receives a lot of smaller sub-problems to be solved. Generally, at this level, the problems are considered 'solved' on their own.
3. **Combine**: Appropriately combine the answers. When the smaller sub-problems are solved, this stage recursively combines them until they formulate a solution of the original problem. This algorithmic approach works recursively and conquer & merge steps works so close that they appear as one.

Binary Search is a searching algorithm. In each step, the algorithm compares the input element (x) with the value of the middle element in array. If the values match, return the index of middle. Otherwise, if x is less than the middle element, then the algorithm recurs to the left side of the middle element, else it recurs to the right side of the middle element. Divide and Conquer technique solves the problem in `O(log(n)) time`.

Closest Pair of Points The problem is to find the closest pair of points in a set of points in x-y plane. The problem can be solved in `O(n^2)` time by calculating distances of every pair of points and comparing the distances to find the minimum. The Divide and Conquer technique solves the problem in `O(nLogn)` time.

1) c)

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1 (c)

If x is positive then $x \neq 0$;
$\underbrace{\text{If } x \text{ is positive}}_{p}$ then $\underbrace{x \neq 0}_{q}$

Converse: $q \rightarrow p$

If $x \neq 0$ then x is positive.

Contrapositive

$\sim q \rightarrow \sim p$

If $x = 0$ then x is not positive.

1 (d)

If $x > 5$ then $x^2 > 25$

Let $F(x)$ be "x is greater than 5"

and

$P(x)$ be The square of x is always greater than 25"

$\forall x \, (F(x) \rightarrow P(x))$

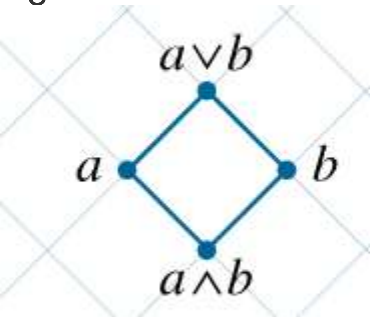
~~De Morgan (Negate)~~

$\sim (\forall x \, (F(x) \rightarrow P(x))$

$\exists x \, F(x) \wedge \sim P(x)$

It mean, Even for a single x value greater than 5, the square of x would never ~~less than 25.~~
or equal

1) e) **lattice** is a partially ordered set (L, ≤) in which every subset {a, b} consisting of two elements has a least upper bound and a greatest lower bound.
We denote LUB({a, b}) by $a \vee b$ and call it **join** or **sum of** ***a*** **and** ***b***.
Similarly, we denote GLB ({a, b}) by $a \wedge b$ and call it **meet** or **product of** ***a*** **and** ***b***.
Eg:



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Q2) a)

$$
\begin{aligned}
& ((p \to q) \wedge (q \to r)) \to (p \to r) & \\
\equiv\ & \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r) & \text{decomposition of } \to \\
\equiv\ & \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee \neg p \vee r & \text{De Morgan} \\
\equiv\ & (\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg r) \vee \neg p \vee r & \text{De Morgan} \\
\equiv\ & \neg p \vee (\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg r) \vee r & \text{commutativity} \\
\equiv\ & ((\neg p \vee \neg\neg p) \wedge (\neg p \vee \neg q)) \vee ((\neg\neg q \vee r) \wedge (\neg r \vee r)) & \text{distributivity} \\
\equiv\ & (\top \wedge (\neg p \vee \neg q)) \vee ((\neg\neg q \vee r) \wedge \top) & \text{negation law} \\
\equiv\ & (\neg p \vee \neg q) \vee (\neg\neg q \vee r) & \text{identity law} \\
\equiv\ & \neg p \vee (\neg q \vee \neg\neg q) \vee r & \text{associativity} \\
\equiv\ & \neg p \vee \top \vee r & \text{negation law} \\
\equiv\ & \top & \text{domination law}
\end{aligned}
$$

Q2(b)

Date : ..../ ..../ ....

L If $x, y, z \in$ (Set of integers) such that xy is odd then both x and y are odd.

Ans

In Contrapositive

$\sim q \rightarrow \sim p$

$q \rightarrow x, y$ are odd

$\sim q \rightarrow x, y$ are both - even.

Let take

$x \rightarrow 2m$ { $m \in$ Set of integer)

as x is even

$y \rightarrow 2n$ { $n \in$ Set of Integer)

as y is even.

Let multibly

$x \cdot y = 2m \cdot 2n$

$= 2(2mn)$: $mn \in$ Set of Integer)

Which is again even form.

Hence We have proved

$\sim p$ i.e xy is even.

Hence by Contradiction We can say

It is always true.

Q3a)

Q3 (a). Date : ..../......../........

$f(x,y,z) = xyz + \bar{x}yz + x\bar{y}z + xy\bar{z} + \bar{x}\bar{y}\bar{z}$

x \ yz	$\bar{y}\bar{z}$ 00	$\bar{y}z$ 01	yz 11	$y\bar{z}$ 10
$\bar{x}$ 0	1		1	
x 1		1	1	1

$f(x,y,z) = yz + xz + xy + \bar{x}\bar{y}\bar{z}$

b)

Definition

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i < n$).

Example – Fibonacci series – $F_n = F_{n-1} + F_{n-2}$, Tower of Hanoi – $F_n = 2F_{n-1} + 1$

Linear Recurrence Relations

A linear recurrence equation of degree k or order k is a recurrence equation which is in the format $x_n = A_1 x_{n-1} + A_2 x_{n-1} + A_3 x_{n-1} + \dots A_k x_{n-k}$ (A_n is a constant and $A_k \neq 0$) on a sequence of numbers as a first-degree polynomial.

How to solve linear recurrence relation

Suppose, a two ordered linear recurrence relation is − $F_n = AF_{n-1} + BF_{n-2}$ where A and B are real numbers.

The characteristic equation for the above recurrence relation is −

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots −

Case 1 − If this equation factors as $(x - x_1)(x - x_1) = 0$ and it produces two distinct real roots x_1 and x_2 , then $F_n = ax_1^n + bx_2^n$ is the solution. [Here, a and b are constants]

Case 2 − If this equation factors as $(x - x_1)^2 = 0$ and it produces single real root x_1 , then $F_n = ax_1^n + bnx_1^n$ is the solution.

Case 3 − If the equation produces two distinct complex roots, x_1 and x_2 in polar form $x_1 = r\angle\theta$ and $x_2 = r\angle(-\theta)$, then $F_n = r^n(acos(n\theta) + bsin(n\theta))$ is the solution.

Problem 1

Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$

Solution

The characteristic equation of the recurrence relation is –

$$x^2 - 5x + 6 = 0,$$

So, $(x-3)(x-2) = 0$

Hence, the roots are –

$x_1 = 3$ and $x_2 = 2$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is –

$$F_n = ax_1^n + bx_2^n$$

Here, $F_n = a3^n + b2^n \ (As\ x_1 = 3\ and\ x_2 = 2)$

Therefore,

$$1 = F_0 = a3^0 + b2^0 = a + b$$

$$4 = F_1 = a3^1 + b2^1 = 3a + 2b$$

Solving these two equations, we get $a = 2$ and $b = -1$

Hence, the final solution is –

$$F_n = 2.3^n + (-1).2^n = 2.3^n - 2^n$$

Non-Homogeneous Recurrence Relation and Particular Solutions

A recurrence relation is called non-homogeneous if it is in the form

$F_n = AF_{n-1} + BF_{n-2} + f(n)$ where $f(n) \neq 0$

Its associated homogeneous recurrence relation is $F_n = AF_{n-1} + BF_{n-2}$

The solution (a_n) of a non-homogeneous recurrence relation has two parts.

First part is the solution (a_h) of the associated homogeneous recurrence relation and the second part is the particular solution (a_t) .

$a_n = a_h + a_t$

Solution to the first part is done using the procedures discussed in the previous section.

To find the particular solution, we find an appropriate trial solution.

Let $f(n) = cx^n$; let $x^2 = Ax + B$ be the characteristic equation of the associated homogeneous recurrence relation and let x_1 and x_2 be its roots.

- If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
- If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
- If $x = x_1 = x_2$, then $a_t = An^2x^n$

Example

Let a non-homogeneous recurrence relation be $F_n = AF_{n-1} + BF_{n-2} + f(n)$ with characteristic roots $x_1 = 2$ and $x_2 = 5$. Trial solutions for different possible values of $f(n)$ are as follows –

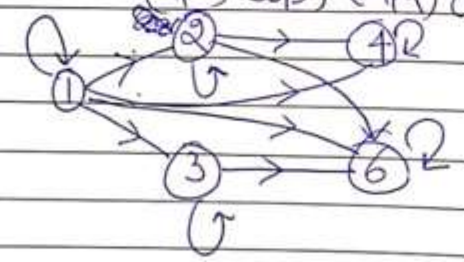
f(n)	Trial solutions
4	A
5.2^n	$An2^n$
8.5^n	$An5^n$
4^n	$A4^n$
$2n^2+3n+1$	An^2+Bn+C

Q4)

Q4(a) $A = \{1, 2, 3, 4, 6\}$

(i) $= R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,6)\ (2,4)\ (2,6),\ (3,6)\ (2,2)\ (3,3)\ (4,4)\ (6,6)\}$

(ii)



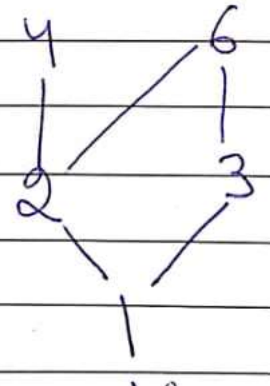
(iii)

a\b	1	2	3	4	6
1	1	1	1	1	1
2		1		1	1
3			1		1
4				1	
6					1

(iv)

	Indegree	Outdegree
1	1	4
2	2	2
3	1	2
4	3	1
6	4	1

(v)



Hass digram

Q4 b)

Master Theorem

The master method is a formula for solving recurrence relations of the form:

```
T(n) = aT(n/b) + f(n),
where,
n = size of input
a = number of subproblems in the recursion
n/b = size of each subproblem. All subproblems are assumed
     to have the same size.
f(n) = cost of the work done outside the recursive call,
      which includes the cost of dividing the problem and
      cost of merging the solutions

Here, a ≥ 1 and b > 1 are constants, and f(n) is an asymptotically positive function.
```

An asymptotically positive function means that for a sufficiently large value of `n`, we have `f(n) > 0`.

Master Theorem

If `a ≥ 1` and `b > 1` are constants and `f(n)` is an asymptotically positive function, then the time complexity of a recursive relation is given by

```
T(n) = aT(n/b) + f(n)

where, T(n) has the following asymptotic bounds:

    1. If f(n) = O(n^(log_b a-ϵ)), then T(n) = Θ(n^(log_b a)).

    2. If f(n) = Θ(n^(log_b a)), then T(n) = Θ(n^(log_b a) * log n).

    3. If f(n) = Ω(n^(log_b a+ϵ)), then T(n) = Θ(f(n)).

ϵ > 0 is a constant.
```