Variable: Is a parameter which is changing with outcome for and observed phenomena.

Data : The particular value of the variable is data-value.

Population: The collection of all data is called population.

Sample: A subset of population is called sample.

e.g. 1; Suppose a dice is thrown 45 times and outcomes are

6,1,6,2,1,6,6,4,1,5,5,5,1,2,6,6,4,2,3,2,4,1,4,4,6,4,3,6,4,3,2,6,4,3,4,3, 3,5,4,6,3,3,3,6,5. ...**(A)**

It can be arranged as **(A1)**

$x = number$	1	2	3	4	5	6
frequency	5	5	9	10	5	11
Cumulative freq. (≤)	5	10	19	29	34	45
Cumulative freq. (≥)	45	40	35	26	16	11

(A) Scattered data , & (A1) grouped data yielding the table called frequency distribution

e.g, 2; Consider the temperature of a city we have

71.1, 71.4, 71.9, 72.8, 75.9, 76.6, 76.9, 78.6, 80.7, 81.6, 81.8, 83.0, 84.0, 86.2, 87.8, 87.9, 88.8, 88.9, 89.4, 91.9, 92.3, 94.1, 94.4, 94.4, 94.6, 94.7, 95.0, 96.0, 96.8, 99.2, 101.0, 101.7, 103.0, 106.0, 107.5 ...**(B)**

It can be represented in form of class say

70-75, 75-80, 80-85, 85-90, 90-95, 95-100, 100-105, 105-110.

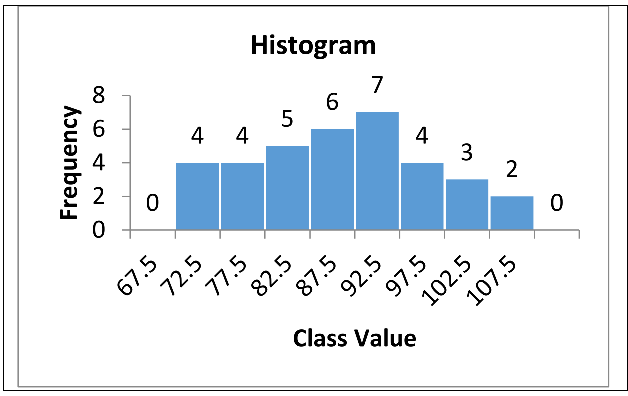
The mid value of an interval is taken as class value.

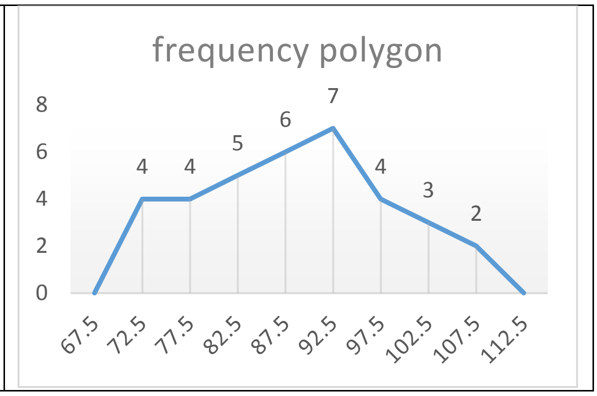
Any data falling on class boundaries to be assigned to higher class

Data can be represented as

Class bound	70-75	75-80	80-85	85-90	90-95	95-100	100-105	105-110
Class value	72.5	77.5	82.5	87.5	92.5	97.5	102.5	107.5
frequency	4	4	5	6	7	4	3	2
Cumulative freq.	4	8	13	19	26	30	33	35

Joining midpoint of the tips of the rectangle in the histogram. The polygon is closed on left and right by joining 67.5 112.5 and in this case sum of the area of rectangle equals to area bounded by frequency polygon and $x - axis$.





Mean

The mean (Arithmetic mean) of sample values $x_1, x_2, \dots x_n$ is given as

$$sample\ mean\ \ (\bar{x}) = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

or

$$= \frac{f_1 x_1 + f_2 x_2 + \cdots + f_k x_k}{f_1 + f_2 + \cdots + f_k} = \frac{\sum_{i=1}^{k} f_i x_i}{\sum_{i=1}^{k} f_i}$$

Mean is called average value.

e.g. 3; the mean of e.g. 1 is $\frac{5*1+5*2+9*3+10*4+5*5+11*6}{45} = \frac{173}{45} = 3.84$

Mid range

The mid range of a sorted sample $x_1, x_2, \dots x_n$ is the average of the smallest and the largest value ,

$$mid = \frac{x_1 + x_n}{2}$$

In case (A) $mid = \frac{1+6}{2} = 3.5$ & in case (B) $mid = \frac{72.5+107.5}{2} = 90.0$

Median ($\tilde{x}$)

If the sample data $x_1, x_2, \dots x_n$ is sorted in increasing order the median of the data is given as

$$\tilde{x} = \begin{cases} x_{k+1} & if\ n = 2k+1 \\ \frac{x_k + x_{k+1}}{2} & if\ \ n = 2k \end{cases}$$

e.g. 4; for 3,3,5,7,8 $\tilde{x} = 5$ & for 1,2,5,5,7,8,8,9 $\tilde{x} = \frac{5+7}{2} = 6$

In case of (A1) $\tilde{x} = 3$ (23rd term), and in case of (B) since data is grouped into the class, we can find median in two ways.

Since the number of data is 35 hence median is 18th term. Using the cumulative frequency it is the second term of class 85-90, thus

i. Simply $\tilde{x} = 87.5$ that is the class value.
ii. Linearly interpolate in the class $\tilde{x} = 85 + \frac{5}{6} * 5 = 89.166$

Clearly (ii) will be good approximation to the median.

In case of grouped frequency distribution median is obtained by the formula

$$\tilde{x} = l + \frac{h}{f}\left(\frac{N}{2} - C\right)$$

N: is population (or sample) size.

l: is lower limit of median class, i.e.; c.f. is just greater than N/2

f: is frequency of the median class,

h: width of the median class,

C: c.f. of the pre-median class.

<u>Mode</u>

It is the value which occurs with the greatest frequency (maybe more than one value). In case of grouped frequency it is given as

$$mode = l + \frac{h(f_m - f_1)}{2f_m - (f_1 + f_2)}$$

l: lower limit of modal class, i.e.; clas with max. frequency

$\boldsymbol{f_m}$: frequency of modal class

$\boldsymbol{f_1}$ & $\boldsymbol{f_2}$: Frequency of pre-modal class and post-model class

h: width of modal class

In case of (B) $mode = 90 + \frac{5(7-6)}{2*7-(6+4)} = 91.25$

<u>Variance and standard deviation</u>

Consider two samples

S1 : 7,9,9,10,10,11,14 & S2 : 7,7,8,10,11,13,14

For both the samples $\bar{x} = 10$ both has first and last term same, but in S1 values are closely clustered about the mean $\bar{x}$ then the values in S2.

Consider a sample of values $x_1, x_2, \ldots x_n$ and suppose $\bar{x}$ is the mean of the sample. The difference $(x_i - \bar{x})$ is called the deviation of the data value x_i from the mean $\bar{x}$. It is positive or negative accordingly as x_i is greater or less than $\bar{x}$.

The variance σ^2 is given as

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 \;, \; where \quad N = \sum_{i=1}^{n} f_i$$

$$= \frac{1}{N}\left\{\sum_{i=1}^{n} f_i(x_i^2 - 2x_i\bar{x} + \bar{x}^2)\right\}$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i\, x_i^2 - \frac{2\bar{x}}{N}\sum_{i=1}^{n} f_i x_i + \frac{\bar{x}^2}{N}\sum_{i=1}^{n} f_i$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i\, x_i^2 - \bar{x}^2 \qquad or \qquad = \frac{1}{N}\sum_{i=1}^{n} f_i\, x_i^2 - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2$$

And <u>*standard deviation (S.D.)*</u> is denoted by σ (non-negative square root of variance).

<u>Moments</u>

The r[th] moment of a variable x about any point $x = a$ denoted by μ_r' is given as

$$\mu_r' = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - a)^r$$

In case $a = \overline{x}$ then the moment is about mean, called central moment, denoted as μ_r is given as

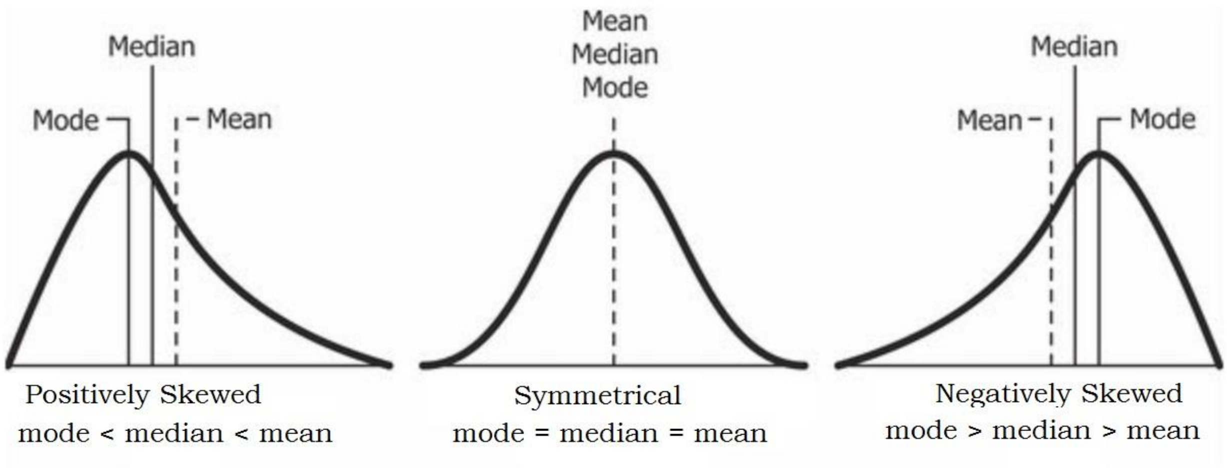
$$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^r$$

In particular we have $\mu_0 = 1$, $\mu_1 = 0$, $\mu_2 = \sigma^2$ & $\mu_0' = 1$, $\mu_1' = \bar{x} - a$

If $a = 0$, $\mu_1' = \overline{x}$, i.e.; first moment about the origin is the mean.

Skewness and kurtosis

Skewness is the measure of lack of symmetry. A distribution is symmetric or 'normal' when frequencies are symmetrically distributed about mean. If the frequency curve of distribution has a long tail to right of central maximum the distribution is said to be skewed right or positively skewed, and if reverse is true then said to be skewed to left or negatively skewed.



The expression

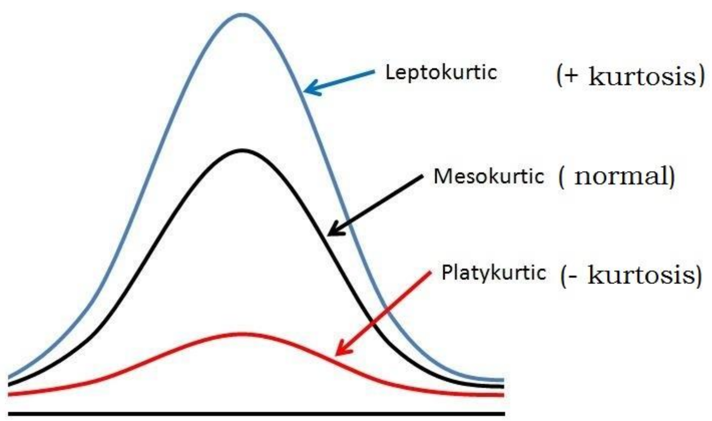
$$\frac{mean - mode}{S.D.}$$

is coefficient of skewness called Karl-Pearson's Coefficient of Skewness. It is a dimensionless number. For symmetric distribution, it is zero.
Thus skewness speaks about the manner in which the data is spread across the mean.

Kurtosis is the degree of peakedness or flatness of the frequency curve (usually taken relative to the normal distribution).

The data could be concentrated across the mean or dispersed equally across it which determines the peakedness of the curve.

If the data is more concentrated closer to the mean it is called as leptokurtosis. As the data gets more disperse, the peakedness reduces and the curve becomes Mesokurtic and if the data is much more dispersed, it is termed as platy kurtosis.



Probability

For any experiment the collection of all possible outcomes (events) is called sample space (S) and is exhaustive.
If no preference is given to any outcome, i.e., all may occur with equal opportunity then events are called equally likely.
If occurrence of one event precludes the occurrence of any other event of the sample space then such events are known as mutually exclusive.

Definition: (Classical)

If a random trial may result in 'n' exhaustive, mutually exclusive and equally likely outcomes, out of which 'm' are favouring to occurrence of any event E then the probability 'p' of that event E is given as

$$p = P(E) = \frac{no.\, of\ favourable\ cases}{total\ no.\, of\ cases} = \frac{m}{n}$$

From the definition it is clear that $0 \le P(E) \le 1$.

$P(E) = 0$ is impossible event.

$P(E) = 1$ is sure event.

Also the probability (q) of not occurrence of E , i.e., $\bar{E}\ or\ E^c$ will be,

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(E) = 1 - p$$

Also we have, $p + q = 1$ ie., $P(E) + P(\bar{E}) = 1$

Axiomatic approach

The concept of probability can be represented by set algebra. S (sample space) is the universal set. Outcomes are elements of S and any subset of S (proper or improper) is an event E then

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{no. of element in set } E}{\text{no. of element in set } S}$$

Definition

Let S be a sample space and an event $E \in S$, the probability of event E is $P(E)$ when the following axiom holds;

Axm.1. $0 \le P(E) \le 1 \quad for\ each \quad E \in S$

Axm.2. $P(S) = 1$

Axm.3. For any two disjoint events $E_1 \,\&\, E_2$ we have

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Using the Axm.3 we can extended it to any number of disjoint events

$P(E_1 \cup E_2 \cup E_2 \cup \ldots \cup\ E_n) = P(E_1) + P(\ E_2) + \cdots + P(E_n)$

Theorem

i. $P(\phi) = 0$
ii. $P(\bar{E}) = 1 - P(E)$
iii. $P(E_1) \le P(E_2)$, $if \quad E_1 \subseteq E_2$
iv. $P(E_2 \backslash E_1) = P(E_2) - P(E_1 \cap E_2)$

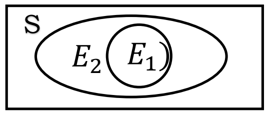
Proof:

i. $S \cup \phi = S \Longrightarrow P(S \cup \phi) = P(S)$

$P(S) + P(\phi) = P(S) \Longrightarrow P(\phi) = 0 \qquad \because \ S \ \& \ \phi \ \ are\ disjoint$

ii. S be sample space then $E \cup \bar{E} = S \Longrightarrow P(E) + P(\bar{E}) = P(S) = 1$
$P(\bar{E}) = 1 - P(E)$

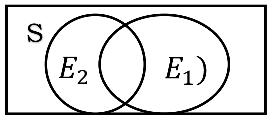
iii. $E_2 = E_1 \cup (E_2 \backslash E_1)$
$or \ \ P(E_2) = P(E_1) + P(E_2 \backslash E_1) \qquad \because \ \ E_1 \,\&\, E_1 \ are\ disjoint.$
$\Longrightarrow P(E_2) \ge P(E_1) \qquad \because P(E_2 \backslash E_1) \ge 0$



iv. $E_2 = (E_2 \backslash E_1) \cup (E_1 \cap E_2)$

$P(E_2) = P(E_2 \backslash E_1) \cup P(E_1 \cap E_2)$

$P(E_2 \backslash E_1) = P(E_2) - P(E_1 \cap E_2)$

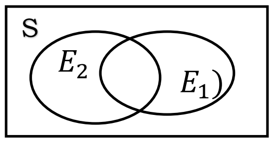


The sample space considered above is finite sample space that is number of outcome of a trial is finite for number of element in set S is finite. If all outcomes possess equal probabilities then it is called equiprobable space.

Addition law of probability

If $E_1 \,\&\, E_2$ are any two events of the sample space S , then

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$



From the figure we have

$E_1 \cup E_2 = (E_1 \backslash E_2) \cup E_2$, where $E_1 \backslash E_2 \ \& \ E_2$ are disjoint sets.

$P(E_1 \cup E_2) = P(E_1 \backslash E_2) + P(E_2) = P(E_1) - P(E_1 \cap E_2) + P(E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

For the events E_1 , E_2 , E_3

$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_2)$

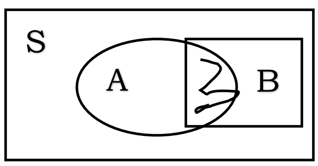
And similarly can be extended to any number of events.

Conditional probability

The probability for the event A to occur while it is known that the event B has already occurred is called conditional Probability and defined as following

$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad \ldots(1)$

$P(A \cap B) = \frac{n(A \cap B)}{n(S)}$, $\qquad P(B) = \frac{n(B)}{n(S)} \quad \Longrightarrow P(A|B) = \frac{n(A \cap B)}{n(B)}$



Multiplication law of probability

From equation (1) we have

$P(A \cap B) = P(B)P(A|B) \quad or \quad P(A)P(B|A)$

In case of two Independents events A & B with $P(A) \neq 0$ & $P(B) \neq 0$ it becomes

$P(A \cap B) = P(A)P(B)$ ...(2)

(as occurrence of one does not influence occurrence of other)

The relation given in equation (2) is the condition of being two events independent.

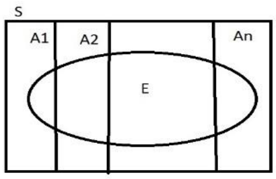
Its extension for three events A, B *and* C will be,

i. $P(A \cap B \cap C) = P(A)P(B)P(C)$
ii. $A, B, \& C$ are pair wise independent

Both the condition holds simultaneously.

Partition , Total probability, and Bay's formula

Let set S is the union of mutually disjoint subsets $A_1, A_2, \dots, A_n$ then $A_1, A_2, \dots, A_n$ forms the partition of the set S. Let E be any event of S then



$E = E \cap S = E \cap (A_1 \cup A_2 \cup \dots \cup A_n)$
$\quad = (E \cap A_1) \cup (E \cap A_2) \cup \dots \cup (E \cap A_n)$

Moreover the $'n'$ subsets in the right side are mutually disjoint and form partition of E .

$P(E) = P(E \cap S) = P(E \cap A_1) + P(E \cap A_2) + \dots + P(E \cap A_n)$ ($\because$ mutually disjoint sets)

$\quad = P(A_1)P(E|A_1) + P(A_2)P(E|A_2) + +P(A_n)P(E|A_n)$ ...(3)

Relation (3) is called total probability.

Bay's formula

Let E be an event in the sample space S and let $A_1, A_2, \dots, A_n$ are disjoint event whose union is S , i.e.;

$S = \bigcup_{i=1}^{n} A_i$, then for $i = 1,2, \dots n$

$$P(A_i|E) = \frac{P(A_i \cap E)}{P(E)} = \frac{P(A_i \cap E)}{P(A_1)P(E|A_1) + P(A_2)P(E|A_2) + +P(A_n)P(E|A_n)}$$

Independent events

If the occurrence of an event does not get influenced by occurrence of other events is called independent.

If the occurrence of A does not affect the occurrence of B , then A & B are called independent events, i.e.; $P(A|B)$ *or* $P(B|A)$ is same as $P(A)$ *or* $P(B)$,

Hence, $P(A \cap B) = P(B)P(A|B)$ becomes $P(A \cap B) = P(A)P(B)$.

(Mutually exclusive are different than independent)

UNIT-I

- If an event can happen in 'm' ways, other event can happen in 'n' ways then they happen together in 'mn' ways.
- Permutation — $^nP_r = \frac{n!}{(n-r)!}$
- Combination — $^nC_r = \frac{n!}{r!(n-r)!}$
- Exhaustive events - The events which can be occurred i.e. all possible outcomes.
- Mutually exclusive events - The events whose intersection is null i.e. $A \cap B = \phi$.
- Equally likely events - Same chances of happening.
- Odds in favour of an event - $\frac{m}{n}$ m → Chances of happening event, n → Chances of not happening.
- Probability — $\frac{\text{Favourable Outcomes}}{\text{Total Outcomes}}$
- Random Experiment - Events which are happening randomly and results are not predictable before happening of event.
  Ex :- Tossing of Coin.
- Sample Space - Set of all outcomes is called sample space.
- Event - Outcome of a random experiment.

SOME RESULTS :-

a) $P(\phi) = 0$

b) $P(S) = 1$ → S → Sample Space

c) $0 \leq P(A) \leq 1$

d) $P(A \cap B)$ or $P(AB)$ → Probability of A and B.

e) $P(A \cup B)$ or $P(A+B)$ → Probability of A or B.

- Addition Law of Probability :-

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A)+P(B)+P(C)-P(A \cap B)-P(B \cap C)-P(C \cap A)+P(A \cap B \cap C)$.

Q Find Probability of getting an ace or a spade out of deck of cards.

→ Let A be event of getting an ace

$P(A) = \frac{4}{52} = \frac{1}{13}$

B be event of getting a spade

$P(B) = \frac{13}{52} = \frac{1}{4}$

$A \cap B = 1$, $P(A \cap B) = \frac{1}{52}$

Probability of A or B

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{16}{52} = \frac{8}{26} = \boxed{\frac{4}{13}}$ Answer

- Independent events – The events in which happening of an event doesn't effect the happening of other event.

- Dependent events – The events in which happening of an event effects the happening of other event as well.

- If A and B are dependent events, Probability of B is A is already happened is $P(B/A)$ then probability of happening of both A and B $P(AB) = P(A) \cdot P(B/A)$
or $P(A \cap B) = P(A) \cdot P(B/A)$.
||ly $P(AB) = P(B) \cdot P(A/B)$.

If A and B are independent events, $P(B/A) = P(B)$ and $P(AB) = P(A) \cdot P(B)$.

Q If we toss a fair coin three times
i) What is probability of three heads
ii) What is probability of exactly one head.
iii) Given that we have observed atleast one head. What is probability that we observed atleast ~~three~~ two heads.

→ i) $\frac{1}{8}$ (HHH) ii) $\frac{3}{8}$ (HTT, THT, TTH)

iii) Let A_1 be event that we observed atleast one head and A_2 be event that we observed atleast two heads.

$P(A_1) = \frac{7}{8}$ $[S - \{TTT\}]$.

$P(A_2) = \frac{4}{8}$ $\{HHH, HHT, HTH, THH\}$, $\boxed{A_1 \cap A_2 = A_2}$

$P(A_2/A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \left(\frac{4}{8}\right) / \left(\frac{7}{8}\right) = \boxed{\frac{4}{7}}$ Answer

Q What is probability that total of two dice will be greater than 9. Given that first dice is a 5

→ $P(A_1) = \frac{1}{6}$ → Event of getting 5.

$P(A_2) = \frac{1}{6}$ → Event of getting sum greater than 9
$\{(5,5), (5,6), (4,6), (6,4), (6,6), (6,5)$

$P(A_1 \cap A_2) =$ (5,6), (5,5) $= \frac{2}{36} = \frac{1}{18}$

$P(A_2/A_1) = \left(\frac{2}{36}\right) / \left(\frac{6}{36}\right) = \boxed{\frac{1}{3}}$ Answer.

Q Suppose a family has two children and one of child is a boy. What is probability of both are boys.

→ Let A be event of getting a boy

$P(A) = \frac{3}{4}$ (BG, BB, GB)

Let B be event both are boys

$P(B) = \frac{1}{4}$ (BB)

$P(A \cap B) = \frac{1}{4}$

$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{(1/4)}{(3/4)} = \boxed{\frac{1}{3}}$

Q Two cards are drawn one by one from deck of 52 cards. Find probability that first is King and second is a queen if first card is
i) Replaced ii) Not replaced.

→ 52 cards 4-King 4-Queen

i) A - event of getting K with replacement

$P(A) = \frac{4}{52} = \frac{1}{13}$

$P(B) = \frac{4}{52} = \frac{1}{13}$ B - getting queen after replacing card, ~~$P(A \cap B) = \frac{4}{52} = \frac{1}{13}$~~

~~$P(B/A) = \frac{1}{169}$~~ $P(AB) = \boxed{\frac{1}{169}} = P(A) \cdot P(B)$

ii) A - event of getting K

$P(A) = \frac{4}{52} = \frac{1}{13}$

B - event of getting Q after without replacing first card

$P(B) = \frac{4}{51}$

$P(A \cap B) = \frac{1}{13} \times \frac{4}{51} = \boxed{\frac{4}{663}}$ Answer

Q A boxA has 2 white & 4 black balls other box B contain 5 white & 7 black balls. A Ball is transfered from A to B, ball is drawn from box B find probability that it is white.

→

A	B
2-W	5-W
4-B	7-B

Case I - if transfered is W

$P(A_1) = \frac{2}{6} = \frac{1}{3}$ - Transfering White balls

$P(A_2) = \frac{6}{13}$ - Drawing ~~$(A_1 \cap A_2)$~~ white ball after transfering white ball

~~$P(A_1/A_2) =$~~

Probability when white transfered and drawing white ball is $6/39$

Case II - if Black is transfered

$P(A_1') = \frac{4}{6} = \frac{2}{3}$ - Transfering black ball

$P(A_2') = \frac{5}{13}$ - Drawing white from box B after transfering black ball

Probability when black transfered and drawing white is $\frac{10}{39}$

Total probability of getting white ball drawn from box B $= \frac{6}{39} + \frac{10}{39} = \boxed{\frac{16}{39}}$ Answer

• Baye's Theorem:-

An event A corresponds to a number of exhaustive events $B_1, B_2 \ldots B_n$. if $P(B_i)$ and $P(A/B_i)$ are given then $P(B_i/A) = \dfrac{P(B_i)\,P(A/B_i)}{\sum P(B_i)\,P(A/B_i)}$

Q The Probability of X, Y, Z of becoming managers are 4/9, 2/9, 1/3 respectively. The Probability that Bonus scheme will be introduced if X, Y and Z becomes managers are 3/10, 1/2, 4/5 respectively.

i) Find probability that bonus scheme introduced

ii) If Bonus scheme has been introduced find probability that manager was X.

iii) If Bonus scheme has been introduced find probability the manager was X or Y.

→ Let B_1, B_2, B_3 be the events that X, Y and Z become manager and A be the event that bonus scheme is introduced.

$P(B_1) = \frac{4}{9}$ $\quad P(B_2) = \frac{2}{9}$ $\quad P(B_3) = \frac{1}{3}$

$P(A/B_1) = \frac{3}{10}$ $\quad P(A/B_2) = \frac{1}{2}$ $\quad P(A/B_3) = \frac{4}{5}$

ii) $P(B_1/A) = \dfrac{P(B_1)\,P(A/B_1)}{\sum_{i=1}^{3} P(B_i)\,P(A/B_i)} = \dfrac{\frac{4}{9}\times\frac{3}{10}}{\frac{4}{9}\times\frac{3}{10} + \frac{2}{9}\times\frac{1}{2} + \frac{4}{5}\times\frac{1}{3}} = \dfrac{\frac{2}{15}}{\frac{23}{45}} = \boxed{\frac{6}{23}}$

Answer

$\frac{2}{15} + \frac{1}{9} + \frac{4}{15} = \frac{6+5+12}{45} = \frac{23}{45}$

(i) $P(E) = \frac{4}{9}\times\frac{3}{10} + \frac{2}{9}\times\frac{1}{2} + \frac{4}{5}\times\frac{1}{3} = \frac{23}{45}$

iii) $P(B_2/A) = \dfrac{\frac{2}{9}\times\frac{1}{2}}{\frac{23}{45}} = \dfrac{\frac{1}{9}}{\frac{23}{45}} = \frac{1}{9}\times\frac{45}{23} = \frac{5}{23}$

$P(B_1/A) + P(B_2/A) = P(E)$

$E \rightarrow$ Event of introducing bonus scheme when X & Y are managers

$P(E) = \frac{5}{23} + \frac{6}{23} = \frac{11}{23}$

Q Three machines M_1, M_2, M_3 produce identical articles. Of respective output 5%, 4%, 3% of articles are defective on a certain day M_1 has produced 25%, $M_2 \rightarrow 30\%$ & $M_3 \rightarrow 45\%$ of total o/p. An article selected at random is found to be defective. What are chances that it was produced by machine with highest o/p (M_3).

$\rightarrow$ Let A be event of getting a defective articles and B_1, B_2, B_3 are o/p from M_1, M_2, M_3 respectively.

$P(B_1) = 0.25$, $P(B_2) = 0.30$, $P(B_3) = 0.45$, $P(A/B_1) = 0.05$

$P(B_3) = 45\% = \frac{45}{100}$, $P(A/B_2) = 0.04$, $P(A/B_3) = 0.03$

$P(A/B_3) = \frac{3}{100}$

$E \rightarrow$ event of getting defective article produced from M_3.

$$P(E) = P(B_3/A) = \frac{\frac{45}{100} \times \frac{3}{100}}{\frac{5}{100} \times \frac{25}{100} + \frac{4}{100} \times \frac{30}{100} + \frac{3}{100} \times \frac{45}{100}} = \frac{135}{125 + 120 + 135}$$

$$= \frac{135}{380} = 0.35526$$

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Random Variable:- If a variable 'X' be associated with the outcome of a random experiment then since the value which 'X' takes depend on the chance it is called a random variable.

Ex:- If we toss two coins the random variable 'X' which is no. of heads are given by

Outcomes — HH, HT, TH, TT

X 2 1 1 0

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Ex:- If we toss a pair dice the sum X can have the value

$X - 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12$

- **One - Dimensional Random Variable :-**

Let S be sample space associated with a random experiment a real valued function defined on S and taking values in $R\,(-\infty, \infty)$ is called a one-dimensional random variable.

- **Discrete Random Variable :-**

If a random variable takes a finite set of values it is called a discrete random variable.

- Ex :- Tossing two coins

- **Continuous Random Variable :-**

If a random variable contains an infinite number of uncountable values it is called a continuous Random Variable.

- Ex :- Height, Weight.

Probability - Mass Function (for Discrete Random Variables) :-

- If X is a random variable with distinct values $x_1, x_2, \ldots x_n$ then

$$P_X(x) = \begin{cases} P(X = x_i) = p_i, & x = x_i \\ 0 & x \neq x_i \end{cases}$$

$i = 1, 2, \ldots n.$

then function $P_X(x)$ is called Probability - Mass function of random variable 'X'.

Remark :- • The number $P(x_i)$ must satisfy following condition

a) $0 \leq P(x_i) \leq 1$

b) $\sum_{i=1}^{n} P(x_i) = 1$

Probability distribution of Random Variable X :-

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- The set of ordered pairs $(x_1, p_1), (x_2, p_2) \dots (x_n, p_n)$ specifies the probability distribution of random variable X.
- **Distribution function:-**

Let 'X' be random variable the function 'F' defined for all real x $F(x) = P(X \le x)$ is called distribution function of random variable X. It is called "cummulated distribution fn" That is for discrete f(n) $\boxed{F(x) = \sum_{i : x_i \le x} p_i}$.

Properties:-
(a) $0 \le F(x) \le 1$
(b) If $x < y \Rightarrow F(x) \le F(y)$ ∴ Distribution function is monotonically non-decreasing and lies b/w '0 and 1'.
(c) $P(a < x \le b) = F(b) - F(a)$.

Q: A random variable X has following probability Distribution

x	0	1	2	3	4	5	6	7
$p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

$9k + 10k^2 = 1$
$k(9+10k) = 1$
$0 - \frac{8}{10}$

i) Find k
ii) Find $P(X<6)$, $P(X \ge 6)$, $P(0<X<5)$.
iii) Find distribution function of X.

→ i) $\sum P(x) = 1$

$8k + 10k^2 + k = 1$

$10k^2 + 9k = 1$

$k(10k+9) = 1$

~~$k = 1$~~ ~~$10k+9 = 1$~~

$10k^2 + 9k - 1 = 0$

$10k(k-1)+1(k-1)$

$(k-1)\ldots$ $10k^2 + 10k - k - 1 = 0$

$(k+1)(10k-1) = 0$

$k = -1, \boxed{+1/10}$ Answer ~~neglected~~

$\boxed{k = 1/10}$.

ii) a) $P(x<6) \rightarrow P_1 + P_2 + P_3 + P_4 + P_5 = 8k + k^2$

$= 8\left(\frac{1}{10}\right) + \frac{1}{100} = \frac{81}{100} = 0.81$

b) $P(x \geq 6) = P_6 + P_7 = 9k^2 + k = 0.19$.

c) $P(0 < x < 5) = 8k - 0 = P_1 + P_2 + P_3 + P_4 = 0.8$,

iii) $F_X(x) = P(X \leq x)$

$$= \begin{cases} 0 & x = 0 \\ 1/10 & x = 1 \\ 3/10 & x = 2 \\ 5/10 & x = 3 \\ 8/10 = 4/5 & x = 4 \\ 81/100 & x = 5 \\ 83/100 & x = 6 \\ 1 & x = 7 \end{cases}$$

Q. $P(x) = \begin{cases} \frac{x}{15} & , x = 1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases}$

Find i) $P(X = 1 \text{ or } 2)$ ii) $P(1/2 < X < \frac{5}{2} / x > 1)$.

→ i) $P(X = 1 \text{ or } 2)$

$P(1) = \frac{1}{15}$, $P(2) = \frac{2}{15}$

$P(X = 1 \text{ or } 2) = \frac{3}{15} = 0.2$

ii) $P\left(1/2 < x < 5/2 \,/\, x > 1\right) = \frac{P\left(\left(\frac{1}{2} < x < \frac{5}{2}\right) \cap x > 1\right)}{P(x > 1)}$

$= \frac{P(X=2)}{P(X>1)} = \frac{P(X=2)}{P(X=2)+P(X=3)+P(X=4)+}$

$= \frac{2/15}{14/15} = \boxed{\frac{1}{7}}$. Answer

Q Two dices are rolled let X be random variable which counts total of pts. on the dices. Construct table for Probability mass function also find distribution function of X.

→

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Probability Mass function

x	0	2	3	4	5	6	7	8	9	10	11	12
F(x)	0	1/36	3/36	6/36	10/36	15/36	21/36	26/36	30/36	33/36	35/36	1

Distribution function

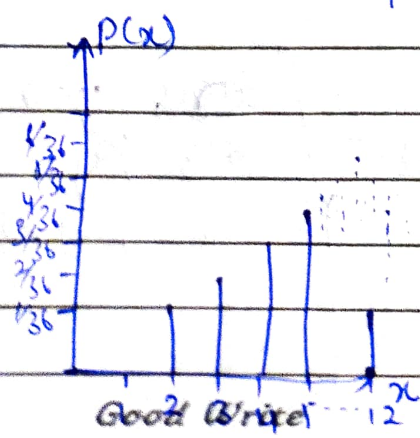
This implies

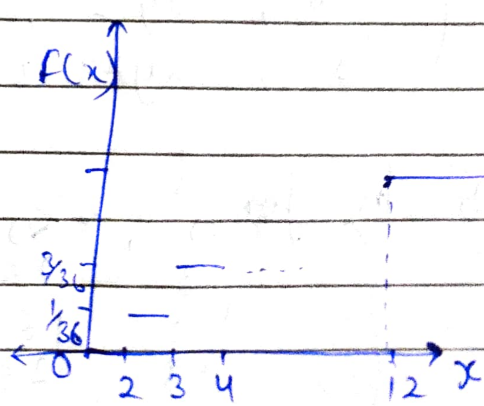
$$F(x) = \sum_{i: x_i \le x} p_i$$

$$F(x) = \begin{cases} 0 & x < 2 \\ 1/36 & 2 \le x < 3 \\ 3/36 & 3 \le x < 4 \\ 6/36 & 4 \le x < 5 \\ \vdots & \\ 1 & x \ge 12 \end{cases}$$

In case of discrete random variable the graph of distribution function will be a step function

In above exmple draw graph of P(x) and F(x)





Q An exp. consist of three independent tosses of a fair coin. Let X = the no. of heads, Y = the no. of head runs which is defined as consecutive occurence of ~~exactly~~ atleast two heads. Z = the length of head runs which is defined as no. of heads occurring together. Find $P(x)$ of

i) X ii) Y iii) Z iv) $X+Y$ v) XY

Also distribution for X & Y.

→

S. No	1	2	3	4	5	6	7	8
Events	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
X	3	2	2	2	1	1	1	0
Y	1	1	0	1	0	0	0	0
Z	3	2	0	2	0	0	0	0
X+Y	4	3	2	3	1	1	1	0
XY	3	2	0	2	0	0	0	0
P(x)								

$$P_X(x) = \begin{cases} 1/8, & x=0 \\ 3/8, & x=1 \\ 3/8, & x=2 \\ 1/8, & x=3 \end{cases} \qquad P_Y(y) = \begin{cases} 5/8, & y=0 \\ 3/8, & y=1 \end{cases}$$

$$P_Z(z) = \begin{cases} 5/8, & z=0 \\ 0, & z=1 \\ 2/8, & z=2 \\ 1/8, & z=3 \end{cases}$$

P(x)	f(x)		
5/8	0	$z=0$	$z<0$
0	5/8	$z=1$	$0\le z<1$
2/8	5/8	$z=2$	$1\le z<2$
1/8	7/8	$z=3$	$2\le z<3$
	1		$z\ge 3$

$$P_{X+Y}(x+y) = \begin{cases} 1/8, & x+y=0 \\ 3/8, & x+y=1 \\ 1/8, & x+y=2 \\ 2/8, & x+y=3 \\ 1/8, & x+y=4 \end{cases}$$

$F_{X+Y}(x+y)$: 0, $(x+y<0)$; $1/8$, $0\le x+y<1$; $4/8$, $1\le x+y<2$; $5/8$, $2\le x+y<3$; $7/8$, $3\le x+y<4$; 1, $x+y\ge 3$

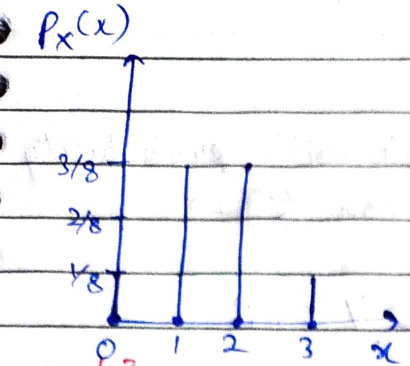
$$P_{XY}(xy) = \begin{cases} 5/8, & xy=0 \\ 0, & xy=1 \\ 2/8, & xy=2 \\ 1/8, & xy=3 \end{cases}$$

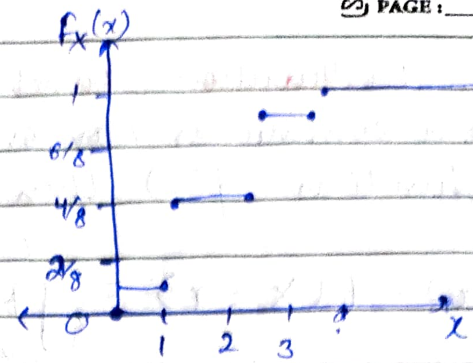
F_{XY}: 0, $xy<0$; $5/8$, $0\le xy<1$; $5/8$, $1\le xy<2$; $7/8$, $2\le xy<3$; 1, $xy\ge 3$

$$F_Y(y) = \begin{cases} 0, & y<0 \\ 5/8, & 0\le y<1 \\ 1, & y\ge 1 \end{cases}$$

Distributionⁿ fn for X

$$F_X(x) = \begin{cases} 0, & x<0 \\ 1/8, & 0\le x<1 \\ 4/8, & 1\le x<2 \\ 7/8, & 2\le x<3 \\ 1, & x\ge 3 \end{cases}$$





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Continuous Random Variable :-

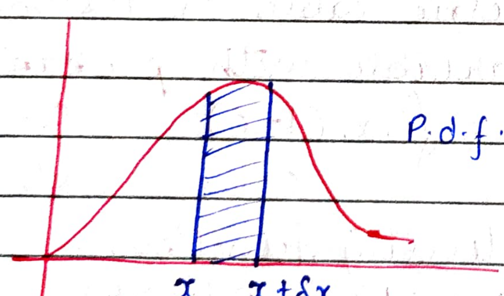
• Probability Density :-

$$f_X(x) = \lim_{\delta x \to 0} \frac{P(x \le X \le x + \delta x)}{\delta x} \quad \forall x \in R$$

* Remark :- Let $f(x)$ be any continuous function of x consider the small interval $(x, x+\delta x)$ of length δx then $f(x).\delta x$ represents probability that random variable x fall in interval $(x, x+\delta x)$ i.e. probability will be

$P(x \le X \le x + \delta x) = f(x) \cdot \delta x$,

i.e. area bounded by the curve $y = f(x)$, x-axis and the ordinates x and $x + \delta x$.



P.d.f. $y = f(x)$

* Remark :- i) $P(a \le x \le b) = P(a < x \le b) = P(a < x < b) = P(a \le x < b) = \int_a^b f(x)dx$.

ii) In case of continuous random variable probability $P(x = a) = 0$, $f(x) \ge 0$,

$\int_{-\infty}^{\infty} f(x)dx = 1$, $P(E) = \int_E f(x)dx$

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• Continuous Distribution function :-

If 'x' is a continuous random variable with probability density function $f(x)$ then the function $F(x)$ =

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \quad \forall x \in R$$

is called distribution function or cummulated distribution function of random variable x.

= Properties :-

i) $0 \leq F_X(x) \leq 1$ (ii) $F'(x) = f(x) \geq 0 \Rightarrow F_X$ is non-decreasing function.

(iii) $F(-\infty) = \int_{-\infty}^{-\infty} f(x)\,dx = 0$ (iv) $F(\infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1$.

(v) $F(x)$ is continuous function of 'x' on right.

(vi) The discontinuity of $F(x)$ at the most countable.

Q The diameter of an electric cable is assumed to be continuous random variable with probability density function $f(x) = \begin{cases} 6x(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

i) Check $f(x)$ is probability density function.

ii) Find no. 'b' such that $P(x<b) = P(x>b)$

→ i) To Prove

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{1} (6x - 6x^2)\,dx = \left[3x^2 - 2x^3\right]_0^1$$

$$= [3-2] - 0$$

$$= \boxed{1} \text{ Answer}$$

Hence, $f(x)$ is a probability density function

Good Write

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$$P(x<b) = P(b \le x)$$

$$\int_{-\infty}^{b} f(x)\,dx = \int_{b}^{\infty} f(x)\,dx$$

$$\int_{0}^{b} f(x)\,dx = \int_{b}^{1} f(x)\,dx$$

$$\int_{0}^{b} 6x^2 - 6x^3$$

$$3b^2 - 2b^3 = 1 - [3b^2 - 2b^3]$$

$$\cancel{3b^2 - 2b + 3b^2 - 2b = 1} \qquad 3b^2 - 2b^3 = 1 - 3b^2 + 2b^3$$

$$\cancel{6b^2 - 4b - 1 = 0} \qquad 6b^2 - 4b^3 - 1 = 0$$

$$\cancel{3b^2 - 2b - \tfrac{1}{2} = 0} \qquad 4b^3 - 6b^2 + 1 = 0$$

$$\cancel{b = 2 \pm \sqrt{\ }} \qquad \boxed{b = \frac{1}{2}, \frac{1 \pm \sqrt{3}}{2}}$$

Hence, $b = \frac{1}{2}$ is valid no.

$\boxed{b = \frac{1}{2}}$ Answer

Q The amount of bread (x) in a hundreds at pounds is able to sell in a day is found to be a random phenomena with probability density f^n given by

$$f(x) = \begin{cases} kx, & 0 \le x \le 5 \\ k(10-x), & 5 < x \le 10 \\ 0, & \text{otherwise} \end{cases}$$

i) Find 'k'.

ii) What is Prob. that no. of pounds of bread that will be sold tomorrow is

a) more than 500 pounds.

b) less than 500 ''

c) 250 & 750 ''

(iii) Denoting by a, b, c (A B C) events that pounds of bread sold are as in b(i), b(ii), b(iii) respectively. find Probability P(A/B) & P(A/C) are A and B independent events, are A and C independent events.

(i) $\int_0^5 kx + \int_5^{10} k(10-x) = 1$

$\left.\frac{kx^2}{2}\right]_0^5 + \left.10kx - \frac{kx^2}{2}\right]_5^{10} = 1$

$\frac{25k}{2} + 100k - \frac{100k}{2} - 50k + \frac{25k}{2} = 1$

$-\frac{100k}{2} + 75k = 1$

$-100k + 150k = 2$

$-100k + 150k - 2 = 0$

$-50k + 75k - 2 = 0$

~~$50k^2 - 75k + 2 = 0$~~

~~$50k^2 =$~~

$25k = 1$

$\boxed{k = \frac{1}{25}}$

(ii) a) $P(X>5) = \int_5^{10} \frac{1}{25}(10-x)dx$

$= \frac{1}{25}\left[10x - \frac{x^2}{2}\right]_5^{10} = \frac{1}{25}\left(100 - \frac{100}{2} - 50 + \frac{25}{2}\right)$

$= \frac{1}{25}\left(\frac{25}{2}\right) = \frac{1}{2} = \boxed{0.5}$ Answer.

b) $P(X<5) = 1 - P(X>5) = 1 - \frac{1}{2} = \boxed{0.5}$ Answer

c) $\int_{2.5}^{5} \frac{1}{25}(x) + \int_5^{7.5} \frac{1}{25}(10-x)$

Good Write

$$\frac{1}{25}\left[\frac{x^2}{2}\right]_{2.5}^{5} + \frac{1}{25}\left[10x - \frac{x^2}{2}\right]_{5}^{7.5}$$

$$\frac{1}{25}\left(\left[\frac{25}{2} - \frac{6.25}{2}\right] + \left(75 - \frac{56.25}{2} - 50 + \frac{25}{2}\right)\right)$$

$$\frac{1}{25}\left(\frac{25}{2} - \frac{6.25}{2} + 75 - \frac{56.25}{2} - 50 + \frac{25}{2}\right)$$

$$\frac{1}{25}\left[\frac{25+75-62.50}{2} \cancel{+25} - 50\right]$$

$$\frac{1}{25}\left[50 - \cancel{\frac{62.50}{2}}^{31.25}\right]$$

$$\frac{1}{\cancel{25}_{5}}\left[\cancel{18.75}^{3.75}\right] = \boxed{0.75} \text{ Answer}$$

(iii) let A, B and C given by

$A: 5 \le x \le 10$, $B: 0 \le x \le 5$; $C: 2.5 \le x \le 7.5$

$P(A) = 0.5$, $P(B) = 0.5$, $P(C) = 0.75$

$A \cap B = \phi \Rightarrow P(A \cap B) = 0 \ne P(A).P(B)$

$\Rightarrow$ A & B are not independent.

$A \cap C : 5 \le x \le 7.5$, $P(A \cap C)$

$$= \int_{5}^{7.5} \frac{1}{25}(10-x)\,dx = \frac{3}{8} = P(A).P(C)$$

$\Rightarrow$ A & C are independent.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = 0$$

$$P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{3/8}{3/4} = \boxed{\frac{1}{2}} = \boxed{0.5} \text{ Answer.}$$

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Q Verify that following is a distribution f^n.

$$F(x) = \begin{cases} 0 & x < -a \\ \frac{1}{2}\left(\frac{x}{a}+1\right) & -a \le x < a \\ 1 & x > a \end{cases}$$

→ i) $0 \le F(x) \le 1$

ii) $F'(x) \ge 0$, $F'(x) = \frac{1}{2a}$

iii) $F(-\infty) = 0$ (iv) $F(\infty) = 1$

(v) $F(-a^+) = 0$.

(vi) Discontinuous at a & $-a$.

$F(x)$ is distributive func^n if $f(x)$

$$f(x) = \begin{cases} \frac{1}{2a} & , -a \le x \le a \\ 0 & , \text{otherwise} \end{cases}$$

$f(x) \ge 0$

$$\int_{-\infty}^{\infty} \frac{1}{2a}\, dx = \left.\frac{x}{2a}\right]_{-a}^{a}$$

$$= \frac{1}{2} + \frac{1}{2} = \boxed{1}$$ Answer.

⇒ $f(x)$ is probability density f^n.

$F(x)$ ⟶ distribution f^n.

Q The diameter x of an electric cable has probability density function $f(x) = 6x(1-x)$ $0 \le x \le 1$.

(iii) Find expression for distribution function.

→

$$F(x) = \begin{cases} 0 & x < 0 \\ 3x^2 - 2x^3 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Q $$f(x) = \begin{cases} kx, & 0 \le x < 1 \\ k, & 1 \le x < 2 \\ -kx + 3k, & 2 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$$

i) find k

ii) find CDF.

→ i) $$\int_{-\infty}^{\infty} f(x) = 1$$

$$\int_{-\infty}^{0} \cancel{0\,dx} + \int_0^1 kx\,dx + \int_1^2 k\,dx + \int_2^3 (-kx + 3k)\,dx + \int_3^{\infty} \cancel{0\,dx}$$

$$\left.\frac{kx^2}{2}\right]_0^1 + \left.kx\right]_1^2 + \left.-\frac{kx^2}{2} + 3kx\right]_2^3$$

$$\frac{k}{2} + 2k - k + \left(-\frac{9k}{2}\right) + 9k + \frac{4k}{2} - 6k = 1$$

$$-4k + k + 9k - 4k = 1$$

$$2k = 1$$

$$\boxed{k = \frac{1}{2}}.$$

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ii) for $x < 0$

$$F(x) = \int_{-\infty}^{x} 0\,dx = 0.$$

for $0 \le x \le 1$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} \frac{x}{2}\,dx = \frac{x^2}{4}$$

for $1 \le x \le 2$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{x} \frac{1}{2}\,dx$$

$$= 0 + \frac{x^2}{4} + \left[\frac{x}{2} - \frac{1}{2}\right] = \frac{x}{2} - \frac{1}{4} = \frac{2x-1}{4}$$

for $2 \le x < 3$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{x} \left(\frac{-x}{2} + \frac{3}{2}\right)dx$$

$$= -\frac{x^2}{4} + \frac{6x}{4} - \frac{5}{4}$$

for $x \ge 3$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{3} \left(\frac{-x}{2} + \frac{3}{2}\right)dx$$

$$+ \int_{3}^{x} 0\,dx = 1$$

$$F(x) = \begin{cases} 0 & , x < 0 \\ x^2/4 & , 0 \le x < 1 \\ 2x - 1/4 & , 1 \le x < 2 \\ -\frac{x^2}{4} + \frac{3x}{2} - \frac{5}{4} & , 2 \le x < 3 \\ 1 & , x \ge 3. \end{cases}$$

Q $f(x) = \begin{cases} \frac{100+x}{10,000}, & -100 \le x < 0 \\ \frac{100-x}{10,000}, & 0 \le x < 100 \\ 0, & \text{otherwise} \end{cases}$, find C.D.F.

for $x \geq 100$

$\rightarrow F(x) = \int_{-\infty}^{x} 0\,dx = 0$

for $-100 \le x < 0$

$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{-100} 0\,dx + \int_{-100}^{x} \frac{100+x}{10,000}\,dx$

$= \frac{100x}{10,000} + \frac{x^2}{20,000} + \frac{100(100)}{10,000} - \frac{10000}{20000}$

$F(x) = \frac{x^2}{20,000} + \frac{x}{100} + \frac{1}{2}$

for $0 \le x < 100$

$F(x) = \int_{-\infty}^{100} f(x)\,dx = \int_{-\infty}^{-100} 0\,dx + \int_{-100}^{0} \frac{100+x}{10,000}\,dx + \int_{0}^{x} \frac{100-x}{10,000}\,dx$

$= \frac{-x^2}{20000} + \frac{x}{100} + \frac{1}{2}$

for $x \geq 100$

$= 1$ Answer

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Various measures of Central Tendency:-

Let $f(x)$ ($P(x)$) be the p.d.f. (p.m.f.) of a random variable X, let $f(x)$ be defined in $[a,b]$, then

i) Mean $(\mu) = \int_a^b x f(x) dx$

$\Sigma x P(x)$

ii)a) Variance $\sigma^2 = \int_a^b (x-\mu)^2 f(x) dx$.

$\Sigma (x-\mu)^2 P(x)$.

ii)b) Standard Deviation $= \sigma$

iii) r^{th} moment about $x = A$

$$\mu_r' = \int_a^b (x-A)^r f(x) dx.$$

$$\Sigma (x-A)^r P(x).$$

r^{th} moment about mean

$$\mu_r = \int_a^b (x-\mu)^r f(x) dx \; ; \; \Sigma (x-\mu)^r P(x)$$

r^{th} moment about origin

$$\mu_r' = \int_a^b x^r f(x) dx \; ; \; \Sigma (x)^r P(x)$$

μ_1' (about origin) $= \mu$

μ_2' (about origin) $= \int_a^b x^2 f(x) dx$

$\Rightarrow \mu_2 = \mu_2' - (\mu_1')^2$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$.

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1' - 3(\mu_1')^4$.

iv) Mean deviation :-

$$\int_a^b |x - mean|\, f(x)\, dx$$

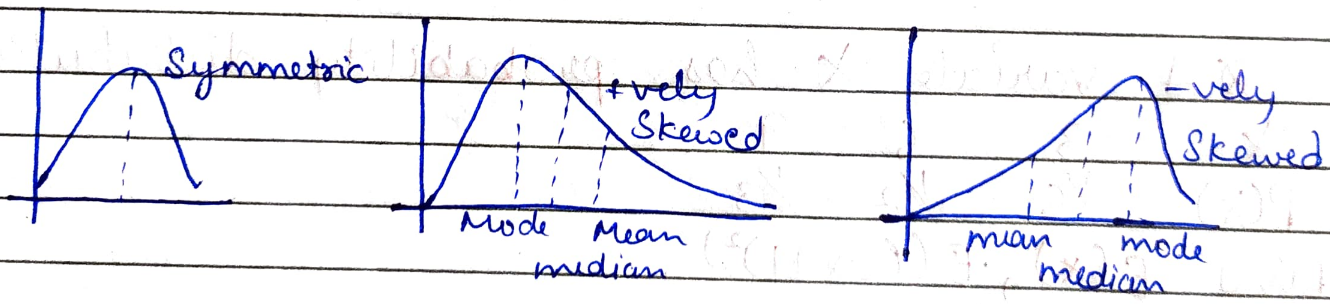
v) Mode :- ⊙ It is value of X for which $f(x)$ is maximum i.e $f'(x)=0$ and $f''(x)<0$.

vi) Median :- ⊙ It is pt. which divides entire distribution ~~interval~~ into two equal parts i.e. $\int_a^m f(x)dx = \int_m^b f(x)\cdot dx = \frac{1}{2}$.

vii) Skewness :- It measures degree of symmetry

· Coefficient of Skewness $= \sqrt{\beta_1}$, $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$

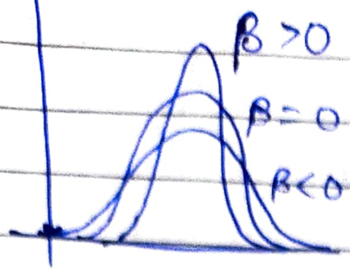
· When skewness = 0 then mean = median = mode.



viii) Kurtosis :- It measures degree of peakness.

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2}$$

- If $\beta_2 > 0$, curve is called Leptokurtic
- If $\beta_2 < 0$, curve is called Platykurtic
- If $\beta_2 = 0$, curve is called Mesokurtic



ix) Expectation:- The mean value μ of prohibited distribution is known as its expectation and it is denoted by '$E(x)$' if $f(x)$ is p.d.f. of r.v. 'x' then $E(x) = \int_{-\infty}^{\infty} x f(x) dx$; $\Sigma x P(x)$

• In general, expectation of any function $\phi(x)$

$$E(\phi(x)) = \int_{-\infty}^{\infty} \phi(x) f(x) dx.$$

$$= \Sigma \phi(x) P(x).$$

Properties:-

a) $E(\phi(x) \pm \psi(x)) = E(\phi(x)) \pm E(\psi(x))$

b) $E(c\phi(x)) = cE(\phi(x))$

c) $E(k) = k$

Ex:- A variate X has probability distribution

X	-3	6	9
$P(X)$	$1/6$	$1/2$	$1/3$

Find $E(x)$, $E((2x+1)^2)$

→ $E(x) = \Sigma x P(x)$

$$= (-3)\left(\frac{1}{6}\right) + (6)\left(\frac{1}{2}\right) + 9\left(\frac{1}{3}\right)$$

$$= \frac{-1}{2} + 3 + 3 = \frac{11}{2}.$$

$E((2x+1)^2) = \Sigma(4x^2 + 1 + 4x)$

$$= 4E(x^2) + E(1) + 4E(x)$$

$E(X^2) = \sum x^2 P(x)$

$= \cancel{9}^{3} \times \frac{1}{\cancel{6}\,2} + \cancel{36}^{18} \times \frac{1}{\cancel{2}} + \cancel{81}^{27} \times \frac{1}{\cancel{3}}$

$= \frac{3}{2} + 18 + 27 = \frac{93}{2}$

$E((2x+1)^2) = 4 \times \frac{93}{2} + 1 + 4 \times \frac{11}{2} = 209$

Q A die is tossed, success of getting 1 or 6 find mode, mean, variance, S.D.

→ $X = 0, 1, 2, 3$

Prob of success $= \frac{2}{6}$

" " failure $= \frac{4}{6}$

$P(X=0) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

$P(X=1) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} \times {}^3C_1 = \frac{4}{9}$

$P(X=2) = \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times {}^3C_2 = \frac{2}{9}$

$P(X=3) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \cancel{{}^3C_3}^{\,1} = \frac{1}{27}$

Prob mass fun^c

X	0	1	2	3
P(x)	8/27	4/9	2/9	1/27

Mean $= \sum X P(x)$

$(\mu) = 1$

Variance $= \sum x^2 p(x)$

$= 0 + \frac{4}{9} + \frac{8}{9} + \frac{\cancel{9}\,3}{\cancel{27}\,9} - \cancel{1} = \frac{15}{9} = \frac{5}{3} - 1 = \frac{2}{3}$

Mode $= (1 = x)$.

Q In continuous distribution

$f(x) = \frac{3}{4}x(2-x)$, $0 \le x \le 2$

$\mu, \sigma^2, \beta_1, \beta_2$

→ $f(x) = \frac{3(2x)}{4} - \frac{3x^2}{4} = \frac{3x}{2} - \frac{3x^2}{4}$

$\mu_r' = \int_0^2 x^r f(x)\,dx$

$= \int_0^2 x^r \cdot \frac{3}{4}x(2-x)\,dx$

$=$

Moment Generating Function :-

1) MGF about pt $x = a$ is :-

$M_a(t) = \int e^{t(x-a)} f(t)\,dt$

$\sum p_i e^{t(x_i-a)}$

2) MGF about origin

$M_a(t) = \int e^{tx} f(t)\,dt = \sum p_i e^{tx_i}$

$\Rightarrow M_a(t) = e^{at} M_0(t)$

3) Expansion of $M_a(t)$

$M_a(t) = E(e^{t(x-a)})$

$= E\left(1 + t(x-a) + \frac{t^2}{2!}(x-a)^2 + \cdots\right)$

$= 1 + t\,(E(x-a)) + \frac{t^2}{2!} E((x-a)^2) + \cdots$

$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots$

Coeff of $\frac{t^r}{r!} = \mu_r'$.

4) Differentiating the M.G.F. :-

$$\left[\frac{d^r}{dt^r}(M_a(t))\right]_{t=0} = \mu_r'$$

Some Discrete Distributions :-

(I) Discrete Uniform Distribution :-

This random variable assumes only a finite no. of possible values each with equal probability.

Ex:- A dice experiment

Probability mass function

A random variable X set as a discrete uniform distribution over range (1-n) of probability mass function is

$$f(x) = \begin{cases} 1/n, & x = 1, 2, 3 \cdots n \\ 0, & \text{otherwise} \end{cases}$$

'n' - is called parameter of distribution.

Mean & Variance

Let 'X' be a discrete uniform distribution random variable on consecutive integers, a, a+1, a+2, b, a≤b

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$a, a+1, a+2, \ldots\ldots b \; ; \; a \leq b$.

Mean $\mu = E(x) = \frac{a+b}{2}$.

Variance $\sigma^2 \; E(x^2) - (E(x))^2 \quad \frac{(b-a+1)^2 - 1}{12}$

Q Let the random variable 'x' denote no of 48 ~~boys~~ voice lines that are in use at a particular time. Find probability mass function, mean, variance. Also draw graph of probability mass function :-

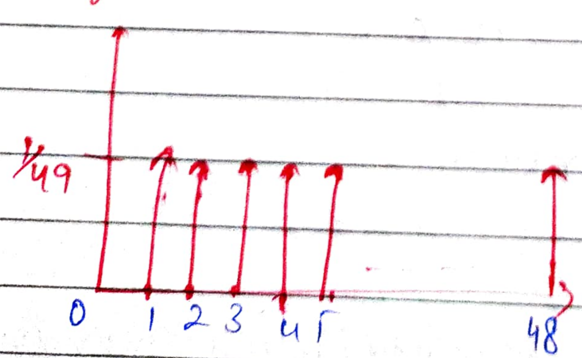
→ Pmf: $f(x) = \begin{cases} \frac{1}{49} & , x = 0, 1, 2 \ldots 48. \\ 0 & , \text{otherwise} \end{cases}$

$\text{Mean}(\mu) = \frac{a+b}{2} = \frac{0+48}{2} = 24$.

$S.D. = \sqrt{\text{Variance}} = \sqrt{\frac{(b-a+1)^2 - 1}{12}}$

$= \sqrt{\frac{(48-0+1)^2 - 1}{12}} = 14.14$.

Pmf graph



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Physical Interpretation:
The average number of lines used in 24 but standard deviation is large.
∴ At many times ~~far~~ far more or fewer than 24 lines are in use.

Q If $y = cx$, then Mean of $Y = c \cdot$ Mean of X

Variance of $Y = c^2 \cdot$ Variance of X

→ Let Y denote proportion of 48 voice lines that are in use at a particular time, then $Y = \frac{X}{48}$

$$\Rightarrow \text{Mean of } Y = \frac{1}{48} \cdot \text{Mean of}(X) = \frac{1}{48} \cdot 24 = 0.5$$

$$\text{Variance of } Y = \left(\frac{1}{48}\right)^2 \cdot \text{Variance of } X = \frac{1}{48^2} \cdot \frac{\cancel{(\ )}^3 \times 10}{12}$$

$$= 0.08678.$$

Bernoulli's Distribution :-

~~p.m.f.~~

A random variable X is said to have bernoullis distribution with parameter p if its probability mass function is $p(x)$

$$f(x) = \begin{cases} p^x (1-p)^{1-x}, & x = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

Ex:- In toss of dice let getting multiple of 3 be a success therefore $p = 2/6 = 1/3 \Rightarrow f(x) = \begin{cases} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{1-x}, & x = 0, 1 \\ 0, & \text{otherwise} \end{cases}$

Binomial Distribution

It consist of n-Bernoulli's trial such that

i) Trials are independent

ii) Each trial results in only two possible outcomes labelled as success and failure.

iii) The probability of success in each trial is P(constant)

A random variable 'X' is said to follow binomial distribution if it assumes only non-empty values and its probability mass function is given by

$$f(x) = \begin{cases} {}^{n}C_{x}\, p^{x} q^{n-x} & , x = 0,1,2 \ldots n; \; q = 1-p \\ 0 & , \text{otherwise} \end{cases}$$

Remark:-

i) $\sum\limits_{x=0}^{n} f(x) = 1$

ii) If n trial constitute an exp and this exp is repeated 'N' times the frequency of binomial distribution is given by
$Y(x) = N f(x) = N\, {}^{n}C_{x}\, p^{x} q^{n-x}$
$, x = 0,1,2 \ldots n.$

iii) # Mean $(\mu) = E(x) = \sum x f(x) = np.$

iv) Variance $(\sigma^2) = E(x^2) - (E(x))^2 = np(1-p).$

Q The chance that a bit transmitted through a channel is receive in error is 0.1. Let x be the no of bits in error in next four bits transmitted. Find

i) $P(X=2)$

→ $n=4,\ x=2,\ p=0.1$

$P(x) = {}^nC_x (0.1)^x (9)^{n-x}$

${}^4C_2 (0.1)^2 (0.9)^2$

$= 0.0486$

Q A coffee founder distinguish b/w a cup of A coffee & fircalata coffee 75% of time. It is agreed that his claim will be accepted if he correctly identify atleast 5 of 6 cups. Find chance of having claim
i) accepted ii) rejected

→ $p = 0.75$, $n = 6$,

$$f(x) = \begin{cases} {}^6C_x (0.75)^x (0.25)^{6-x}, & x = 0,1,2 \ldots 6 \\ 0, & \text{otherwise} \end{cases}$$

i) $P(X \geq 5) = P(X=5) + P(X=6)$

$= {}^6C_5 (0.75)^5 (0.25) + {}^6C_6 (0.75)^6$

$= \frac{6!}{5!} (0.75)^5 (0.25) + (0.75)^6$

$= 0.534$

ii) $1 - 0.534 = 0.466$

$P(X < 5) = 0.466$

Q Mean and variance of binomial distribution are find prob. mass function

→ Mean $= np = 4$ $n = \frac{4 \times 3}{2} = \boxed{6}$

Variance $= np(1-p) = \frac{4}{3}$

$4(1-p) = \frac{4}{3}$

$1 - p = \frac{1}{3}$ $1 - \frac{1}{3} = p = \boxed{\frac{2}{3}}$

$q = 1/3$

$$f(x) = \begin{cases} {}^6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x} & , x = 0,1,2,3,4,5,6 \\ 0 & , \text{otherwise} \end{cases}$$

Q : 7 coins are tossed and no of heads noted. The exp is repeated for 128 times and following distribuⁿ is obtained

No. of heads	0	1	2	3	4	5	6	7	Total
Frequency	7	6	19	35	30	23	7	1	128

Fit binomial distrⁿ assuming i) Coin is unbiased ii) Nature of coin is not known

→ $n = 7, \quad N = 128$

i) Coin is unbiased

$p = \frac{1}{2}, \quad q = \frac{1}{2}$

$\frac{7 \times 6}{2}$ $\quad \frac{7 \times 6 \times 5}{3 \times 2}$ $\quad \frac{7 \times 6 \times 5 \times 4}{}$

pmf

$$f(x) = \begin{cases} {}^7C_x \left(\frac{1}{2}\right)^7 & , x = 0,1,2 \text{---} 7 \\ 0 & , \text{otherwise} \end{cases}$$

frequency $= N f(x)$ $\quad x = 0,1,2, \text{---} 7$

The expected frequency

x	$f(x)$	frequency
0	$\frac{1}{128}$	1
1	$\frac{7}{128}$	7
2	$\frac{21}{128}$	21
3	$\frac{35}{128}$	35
4	$\frac{35}{128}$	35
5	$\frac{21}{128}$	21
6	$\frac{7}{128}$	7
7	$\frac{1}{128}$	1

ii) When nature of coin is unknown.

$np = 3.3828$

$p = 0.4832$

pmf

$$f(x) = \begin{cases} {}^7C_x (0.483)^x (0.5167)^{7-x}, & x = 0, 1, 2 \ldots 7 \\ 0, & \text{otherwise} \end{cases}$$

freq. $Nf(x)$

x	$f(x)$	frequency
0	0.009	1.085
1	0.0643	8.29
2	0.180	23
3	0.281	35
4	0.263	33
5	0.147	19
6	0.046	6
7		1

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Geometric Distribution.

Consider random experiment that assumes a series of Bernoulli trials but instead of fixed no. of trials. Trials are conducted until success is obtained. let random variable 'X' denot no of trials until first success then 'X' is a geometric random variable with parameter 'p' and its probability mass function.

$$f(x) = \begin{cases} (1-p)^{x-1} p, & x = 1, 2, 3, 4, \ldots \\ 0, & \text{otherwise} \end{cases}$$

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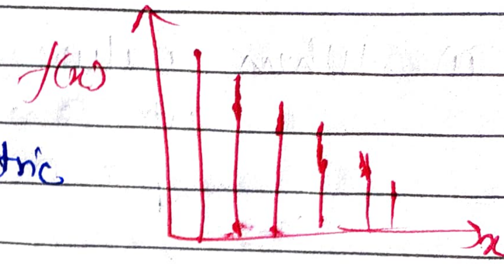
It is clear that height of line at x is $(1-p)$ times the height of line at $(x-1)$. this is called geometric distribution.



- Mean $(\mu) = E(x) = \sum x\, p(x) = \frac{1}{p}$.
- Variance $(\sigma^2) = E(x^2) - (E(x))^2 = \frac{1-p}{p^2}$.

- Lack of memory:- Let an event occur at one of the time $(t = 1, 2, 3 \ldots)$ and the waiting time 'x' has a geometric distribution with parameter 'p'.

Let 'E' has not occured before $t = k$
$\therefore$ $X >= k$. Let $Y = X - k$, Y is the amount of additional time needed for 'E' to occur.

$\therefore$ $P(Y = t \mid X \geq k) = pq^{t-1}$.

Additional time required for occurrence has same distribution as the initial time of occurrence, which is all equal to.

$$P(Y = t \mid X \geq k) = P(x = t).$$

Distribution doesn't depend upon 'k' in a sense how much we have shifted the time horizon. If 'B' were waiting for the event 'E' and is relieved by 'C' immediately before time 'k' then waiting time distribution is same as that of 'B'.

Q The probability that a waffer contains large particle of contamination is 0.01. It is assumed that the waffers are independent. What is probability that exactly '125' waffers need to be analysed before a large particle is detected.

→ $X = 125$, $p = 0.01$

$p(x=125) = (1-0.01)^{125-1} \cdot (0.01) \approx 0.0029.$

Negative Bernoulli's Distribution :-

In this no. of Bernoulli's trials are required to obtain 'r' successes result in the negative binomial Distribution.

In a series of Bernoulli's trial let random variable 'X' denote no. of trials until R success occur, then 'X' is a -ve binomial random variable with parameter 'p' and 'r' where $r = 0, 1, 2 \ldots$ and probability mass function is

$$f(x) = \begin{cases} {}^{x-1}C_{r-1} (1-p)^{x-r} (p)^r , & x = r, r+1, r+2 \ldots \\ 0, & \text{otherwise} \end{cases}$$

Remark :-

i) At least 'r' trials are required to obtain 'r' successes.

∴ Range of x is from r to ∞.

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ii) If $r = 1$, -ve binomial distribution equals to geometric ~~bino~~ distribution.

iii) A binomial random variable is a count of no of successes in 'n'- bernoulli's trials.

∴ No. of trials is fixed and no. of successes is random.

A -ve binomial random variable is a count of no. of trials required to obtain 'r' success i.e. no. of success is fixed and no. of trials is random.

∴ -ve binomial random variable is opposite of binomial random variable.

iv) Let X denote the total no. of trials required to obtain 'r' successes. Let x_1 denote the no. of trials required to obtained first success, x_2 denote the no. of extra trials required to obtain second success, x_3 denote no of extra trials required to obtain third success and so on. Now each x_1, x_2, upto x_r has a geometric distribution with same probability 'p'.

∴ -ve binomial random variable can be taken as sum of 'r' geometric random variables.

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- Normal distribution:-

Let random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

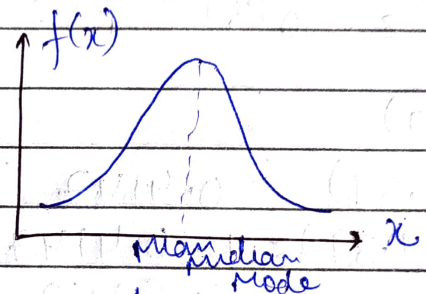
is a normal random variable with parameter μ

$-\infty < \mu < \infty$, $\sigma > 0$.

Mean $E(x) = \mu$

Variance $V(x) = \sigma^2$

$N(\mu, \sigma^2)$ is used to denote distribution



Remarks:-

- From symmetry we have

$$P(x > \mu) = P(x < \mu) = 0.5$$

- $P(\mu - \sigma < x < \mu + \sigma) = 0.6827$ 68%

$P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.9545$ 95%

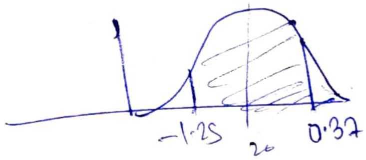
$P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$. 99.73%

Since, 0.9973 of the probability of a normal distribution is within the interval $(\mu - 3\sigma)$ to $(\mu + 3\sigma)$.

$\therefore$ 6σ is often called the width of normal distribution.

- The area under a normal probability density function beyond 3σ from mean is quite small

- A normal random variable with $(\mu = 0)$ & $\sigma^2 = 1$ is called standard normal random variable and it is denoted by 'Z'.



0·3962

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Q Solve probability 'Z' ~~P(Z) > 1·26~~

(I) $P(Z > 1.26)$

→ $P(Z < 1.26) = 0.8962$

$P(Z > 1.26) = 1 - 0.8962 = 0.1038$

$P(Z > 1.26) = 0.5 - 0.3962 = 0.1038$

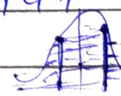
(II) $P(Z < -0.86)$

→ $P(Z < 0.86) = 0.8051$

$P(Z > 0.86) = 1 - 0.8051 = 0.1949$

By Symmetry

$P(Z < -0.86) = P(Z > 0.86) = 0.1949$ Answer

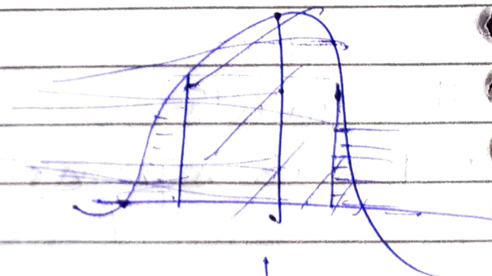


(III) $P(Z > 1.37)$

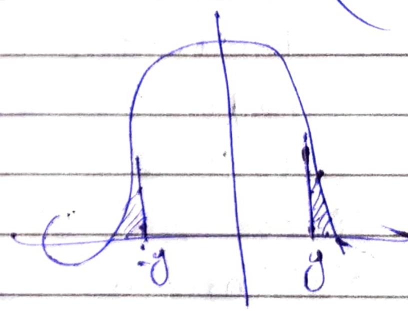
→ $P(Z > 1.37) = 0.4147$

$P(Z > -1.37) = 1 - 0.4147$

$= 0.5853$



(IV) $P(-1.25 < Z < 0.37)$



$Z > 1.25$

0·3944

$Z > -1.25$ 0·8944 − 0·6443

0·3944

(x > ∞ − y)

~~501~~

$P(Z < 0.37) - P(Z < -1.25)$

$P(Z < 0.37) = 0.6443$

$P(Z < -1.25) = 0.1057$

$P(-1.25 < Z < 0.37) = 0.5386$

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If X is a normal random variable $E(X) = \mu$ and $V(X) = \sigma^2$ then random variable $Z = \frac{X-\mu}{\sigma}$, is a standard normal variable with mean zero and variance 1

Q $P(9 < X < 11)$

$\rightarrow P(-0.5 < Z < 0.5) = P(Z < 0.5) - P(Z < 0.5)$

$0.6915 - .1915$

$= .5000$

Normal approximation to the binomial / poisson distribution:-

- If X is binomial random variable $(\mu = np, \sigma^2 = npq)$

$Z = \frac{X - np}{\sqrt{npq}}$ is approximately standard normal random variable. The approximation is good for $np > 5$ and nq or $n(1-p) > 5$

Q In a digital communication channel no of bits in an error can be modeled by a binomial random variable and assume that the prob that bit is recieved in error 1×10^{-5} if 16 million bits are transmitted what is prob that 150 error coming.

$\rightarrow$ $p = 1 \times 10^{-5}$ $n = 16$ million $= 16,000,000$

$P(X > 150)$

$1 - P(X \leq 150)$

$1 - \sum^{150} {}^nC_x \, p^x (1-p)^{n-x}$

Good Write

which is difficult to solve using binomial distribution we use normalization

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \quad X > 150$$

$$Z > \frac{150 - 160}{\sqrt{16000000 \times 10^{-5}(1-10^{-5})}}$$

$= -0.79$

$P(X > 150) = P(Z > -0.79)$

$= P(Z < 0.79)$

$= 0.7852$

If X is a poison random variable with mean 'λ' variance λ

$\therefore$ $Z = \frac{X - \lambda}{\sqrt{\lambda}}$ is approximately standard normal random variable ; for $\lambda > 5$ & approximation is good

NOTE:-

Hypergeometric distribution $\approx$ binomial distribution $\approx$ normal distribution

$\frac{n}{N} < 0.1$ $\qquad$ $np > 5$

$n(p-1) > 5$

Exponential Distribution :

Let random variable 'n' no of flaws in 'x' millimeter of wire. If mean no of flaws is 'λ' per mm N has poison distribution with mean 'λx'. We assume that wire is longer than 'x'. Now

$$P(X > x) = P(N=0) = \frac{e^{-\lambda x}(\lambda x)^0}{(0)!} = e^{-\lambda x}.$$

$$\Rightarrow F(x) = P(X \le x) = 1 - P(X > x) = 1 - e^{-\lambda x},$$

If cummulative distribution function is $F(x)$.

$\Rightarrow$ probability density function

$$f(x) = \lambda e^{-\lambda x}. \quad , x \geq 0.$$

The random variable X that equals the distance b/w successive counts of a poison process with mean '$\lambda > 0$' is an exponential random variable with parameter 'λ' and p.d.f. $f(x) = \lambda e^{-\lambda x}$

Mean of this distribution is $\boxed{\frac{1}{\lambda}}$ and variance is $\boxed{\frac{1}{\lambda^2}}$

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$$= \int_{0.1}^{\infty} 25e^{-25x}\,dx = 25\int_{0.1}^{\infty} e^{-25x}\,dx = 0.082$$

Remark :- The probability that there are no logons in 6 min interval is 0.082 regardless of starting time of interval if there is no ~~plus~~ clustering of the event

However if there are high use period during day, followed by a period of low use. Then we can model each of high and low use period by a separate poison process, then an exponential distribution with corresponding 'λ' can be used to calculate logon probabilities for high and low use periods.

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Lack of memory property :-

For an exponential distribution we have $P(X < t_1 + t_2 / X > t_1)$
$= P(X < t_2)$

Ex:- let 'x' denote time b/w detection of a particle with a counter and assume that x has an exponential distribution with $\lambda = 1.4$ min the probability we detect a particle with in 30 seconds of starting the counter is probability.

→ $$P(X < 0.5) = \int_{-\infty}^{0.5} f(x)\,dx = 0.30$$

Suppose we turn on counter & wait 3 min without detecting particle then prob. that particle is detected next 30sec is $P(X < 3.5 / X > 3)$

$$= \frac{P(3 < X < 3.5)}{P(X > 3)} = 0.30$$

Erlang Distribution :-

The random variable x that equals interval length until hour counts occur in poison process with mean 'λ' > 0' has an erlang random variable with parameters λ and r. The probability density function of 'x' is.

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$$

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Since, erlang R.V. can be represented as sum of 'r' exponential random variable.

$$\text{Mean} = \frac{r}{\lambda}, \quad \sigma^2 = \frac{r}{\lambda^2} \text{ (Variance)}$$

Remark :- If $r=1$, an Erlang variable is an exponential random variable.

Q. The of large computer system are often modelled as a poison process. Assume that mean number of failures per hour is 0.0001. Let x denote time until '4' failures occur in an system. Let n-occurs 40,000 hours

→ $\lambda = 0.0001$, $r = 4$

$$P(X>40,000) = \int_{40,000}^{\infty} \frac{(0.0001)^4 x^3 e^{-(0.0001)x}}{\ldots}$$

$= 0.433.$

Γ-Distribution :- The random variable x in pdf $f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}$

~~If r an integer~~

β- Distribution :-

β-function :-

1) $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m>0, n>0$$

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2) $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$, $m>0, n>0$

3) $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$.

β-distribution :- A random variable X is said to have β-distribution of first with parameter $m>0$ & $n>0$. If its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} & , 0<x<1 \\ 0 & , \text{otherwise} \end{cases}$$

Mean $= \frac{m}{m+n}$ $\qquad \sigma^2 = \frac{mn}{(m+n)^2(m+n+1)}$

If $m=n=1$, $f(x) = \frac{1}{\beta(1,1)} = 1$

Q Find c, $f(x) = cx^3(1-x)^6$ is a β-distribution of first kind find mean & variance

→ Since,

$f(x)$ is probability density function

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 cx^3(1-x)^6 dx = 1$$

$$c\int_0^1 x^3(1-x)^6 dx = 1$$

$$c\,\beta(4,7) = 1$$

$$c = \frac{1}{\beta(4,7)} \quad , \quad f(x) = \frac{x^3(1-x)^6}{\beta(4,7)} \quad , \quad 0<x<1$$

Mean $= \frac{m}{m+n} = \frac{4}{11} = 0.363$ $\sigma^2 = \frac{mn}{(m+n)^2(m+n+1)} = 0.0192$

β-distribution Second Kind :-

A random variable X said to have β- distribution of second kind with parameter $m>0, n>0$. If its probability density function

$$f(x) = \begin{cases} \frac{x^{m-1}}{\beta(m,n)(1+x)^{m+n}} , & 0<x<1 \\ 0 , & \text{otherwise} \end{cases}$$

Mean $= \frac{m}{n-1}, n>1$; $\sigma^2 = \frac{m(m+n-1)}{(n-1)^2 (n-2)}, n>2.$

Q Find K such that $f(x) = \frac{Kx^3}{(1+x)^7}$; $0<x<\infty$ is probability density function of second kind. Also find mean and variance.

Weibull Distribution :- This distribution is used to model the time until failure of many different physical system.

The random variable X with probability density function is $f(x) = \begin{cases} \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} e^{-\left(\frac{x}{\delta}\right)^{\beta}} , & x>0 \end{cases}$

is a Weibull random variable with parameter $\delta>0$, $\beta>0$.

Remark :-

i) If $\beta=1$ it is exponential distribution

ii) Mean $(\mu) = \delta \Gamma\left(1+\frac{1}{\beta}\right)$ $\sigma^2 = \delta^2 \Gamma\left(1+\frac{2}{\beta}\right) - \delta^2 \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2$

iii) The cummulated distribution function of x is
$$F(x) = 1 - e^{-\left(\frac{x}{\delta}\right)}$$

Q Time of failure of bearing in mechanical shaft is modelled as a weibull random variable with $\beta = \frac{1}{2}$, $\delta = 5000$ hrs.

i) Find mean time until failure.
ii) Find probability that bearing lasts at least 6000 hrs.

→ i) Mean $= \delta \Gamma\left(1 + \frac{1}{\beta}\right)$

$$= 5000\, \Gamma\left(1 + \frac{1}{1/2}\right) = 5000\, \Gamma 3 =$$

$$= 5000\, \lfloor 2 = 10{,}000$$

ii) $P(X > 6000) = 1 - P(X \leq 6000) = 1 - F(X = 6000)$

$$= 1 - e^{-\left(\frac{6000}{5000}\right)^{\beta}}$$

$$= 0.33439$$

Lognormal Distribution :- Let w have a normal distribution with mean (θ), variance (ω^2) then $X = e^{\omega}$ is a lognormal random variable with probability density function

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \theta)^2}{2\omega^2}\right), \quad 0 < x < \infty$$

(μ) Mean $= e^{\theta + \omega^2/2}$ $\qquad \sigma^2 = e^{2\theta + \omega^2}\left(e^{\omega^2} - 1\right)$

Q The lifetime of semiconductor laser has a lognormal distribution $\theta = 10$hrs, $\omega = 1.5$ hrs. What is probability that lifetime exceeds 10,000hrs.

$\rightarrow$ $P(X > 10000) = 1 - P(X \leq 10000) = 1 - (e^{w} \leq 10000)$

$= 1 - P(w \leq \ln(10,000))$

$$Z = \frac{X - \mu}{\sigma}$$

$$Z = \frac{w - \theta}{\omega} = \frac{\ln(10,000) - 10}{1.5}$$

$$Z = \frac{9.2 - 10}{1.5} = -0.526 = -0.52$$

$= 1 - P(Z \leq -0.52)$

$= 1 - 0.3 = \boxed{0.7}$ Answer.

$$\mu = e^{\theta + \omega^2/2} = 67846.3$$

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Covariance and Corelation :-

$$E(h(x,y)) = \begin{cases} \iint_R h(x,y) \, f_{xy}(x,y) \, dy \, dx & \text{(continuous)} \\ \sum\sum_R h(x,y) \, f_{xy}(x,y) . & \text{(Discrete)} \end{cases}$$

The covariance between two random variables x & y is denoted by $Cov(X,Y)$ or σ_{XY} and it is given by

$$\sigma_{XY} = E((X - \mu_X)(Y - \mu_Y)) = E(XY) - \mu_X \mu_Y$$

Remark :-

- $E((X - \mu_X)(Y - \mu_Y)) = \iint_R (X - \mu_X)(Y - \mu_Y) f_{XY} \, dx \, dy$

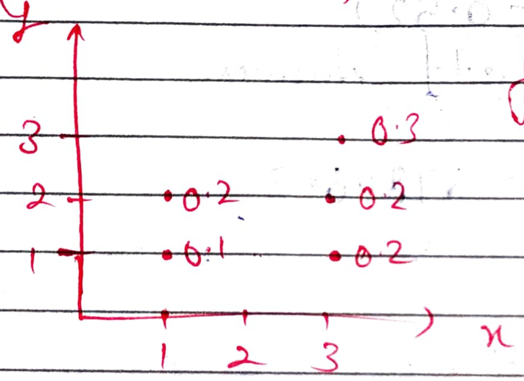
$$= \iint_R XY f_{xy}\, dy\, dy - \mu_x \iint_R Y f_{xy}\, dx\, dy - \mu_y \iint_R X f_{xy}\, dx\, dy$$

$$+ \mu_x \mu_y \iint_R f_{xy}\, dx\, dy$$

$$= E(XY) - \mu_x\mu_y - \mu_y\mu_x + \mu_x\mu_y$$

$$= E(XY) - \mu_x \mu_y .$$

Q The join probabitily distrib^n of x & y is shown as



find is Covariance of xy

$\sigma_y = \sqrt{V(y)}$
$= 0.1 + 0$
$+ 0.2 + 0$
$+ 0.3$
$= \sqrt{0.6}$
$= 0.7745$

$$\sigma_x = \sqrt{V(x)} = \sum\sum (x-\mu_x)^2 f_{xy}$$

$$= 0.196 + 0.392 + 0.072$$
$$+ 0.072 + 0.108$$
$$= \sqrt{0.84} = 0.9165$$

$$E(XY) = \sum_R \sum xy\, f_{XY}(x,y).$$

$$= (1)(1)(0.1) + (1)(2)(0.2) + (1)(3)(0.2) + 3(2)(0.2)$$
$$+ 3(3)(0.3)$$
$$= 0.1 + 0.4 + 0.6 + 1.2 + 2.7$$
$$= 5.0.$$

$$\rho = \frac{0.2}{0.7745 \times 0.9165}$$
$$= .$$

$$\mu_x = 0.1 + 0.2 + 1.2 + 0.9$$
$$= 2.4$$

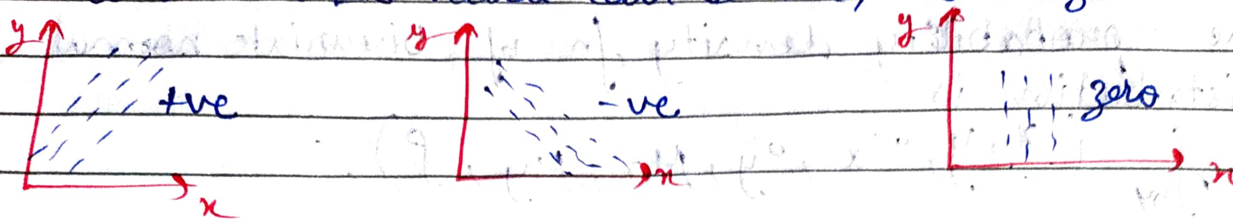
$$\mu_y = 0.1 + 0.4 + 0.2 + 0.4 + 0.9$$
$$= 2.0$$

$$= E(XY) - \mu_x\mu_y$$

$$\text{Cov} = 5 - (2.4)2 = 0.2$$

Remark :-

1. Covariance is a measure of linear relationship b/w random variable. Its value can be +ve, −ve and zero



Q Let random variable x & y are no of acceptable & suspect bits among 4 bits received during communication. Is C.V. b/w x & y +ve or −ve.

→ −ve

The covariance by relation is

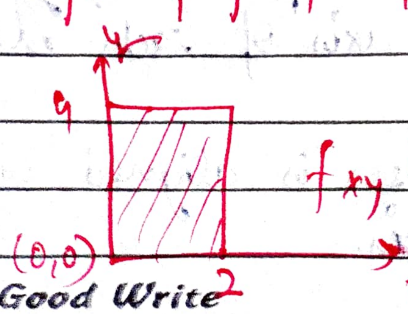
Correlation

$$\rho_{xy} = \frac{Cov(x,y)}{\sqrt{V(x)\,V(y)}} = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y}$$

Remark

1. Since $\sigma_x > 0$ & $\sigma_y > 0$. the correlation is +ve, −ve or zero if covariance +ve, −ve or zero respectively.
2. The correlation is a dimensionless quantity.
3. $-1 \leq \rho_{xy} \leq 1$
4. Correlation is also a linear relationship b/w random varia
5. If x & y are independent random variable then $\sigma_{xy} = \rho_{xy} = 0$.

Q The joint pdf of x & y is given as



$$f_{xy}(x,y) = \frac{1}{16}xy$$

find corelation, C.V.

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Bivariate Normal Distribution:-

The probability density fn of bivariate normal distribution is

$$f_{XY}(x, y, \sigma_x, \sigma_y, \mu_x, \mu_y, \rho)$$

$$= \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}}$$

$$\exp\left\{ \frac{-1}{2(1-\rho^2)} \left[\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right] \right\}$$

where $-\infty < x < \infty$, $-\infty < y < \infty$ will parameter

$\sigma_x > 0$, $\sigma_y > 0$, $-\infty < \mu_x < \infty$, $-\infty < \mu_y < \infty$, $-1 < \rho < 1$

Remarks:-

① $f_{xy} \geq 0$

② $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{xy}\, dx dy = 1$

③ Each curve is set of points for which pdf is constant. As we can see from graph bivariate normal distribution is constant on elipses in x-y plane. The centre of ellipse is at pt (μ_x, μ_y)

④ If $\rho > 0$ $(\rho < 0)$ the major axis of each ellips has +ve (-ve) slope respectively

⑤ If $\rho = 0$ the major axis of ellipses is aligned with either x or y coordinate axes.

- **Marginal distribution of bivariate normal random variable:-**

→ If x & y have bivariate normal random variable with join probability density function f_{xy} then marginal probability distribution of x & y are normal with means μ_x, μ_y and standard deviations σ_x, σ_y respectively.

ρ is correlation b/w x & y.

Ex:- Let x & y have bivariate normal distribution with $\sigma_x = 0.04$, $\sigma_y = 0.08$, $\mu_x = 3$, $\mu_y = 7.70$, $\rho = 0$. Find probability i) $(2.95 < x < 3.05), (7.60 < y < 7.80)$

ii) $\rho = 0.8$

→ $P(2.95 < X < 3.05) . P(7.60 < Y < 7.80)$

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

$$P\left(\frac{2.95-3}{0.04} < Z < \frac{3.05-3}{0.04}\right) \quad P\left(\frac{7.60-7.70}{0.08} < Z < \frac{7.80-7.70}{0.08}\right)$$

$$P\left(\frac{-0.05}{0.04} < Z < \frac{0.05}{0.04}\right) \quad P\left(\frac{-0.10}{0.08} < Z < \frac{0.10}{0.08}\right)$$

$$P(-1.25 < Z < 1.25) \quad P(-1.25 < Z < 1.25)$$

$$(2 \times 0.3944) \quad (2 \times 0.3944)$$

Linear combinations of random variables:-

If $x_1, x_2, x_3 \ldots x_p$ are given random variables and $c_1, c_2, c_3 \ldots c_p$ are constants then y is equal to $y = c_1x_1 + c_2x_2 + \ldots c_px_p$ is a linear combination of $(x_1, x_2 \ldots x_p)$.

• Mean =

$$E(Y) = \int y f_y(y)dy$$

$$= \int \ldots \iint (c_1x_1 + c_2x_2 + \ldots c_px_p) f_{x_1x_2x_3\ldots x_p}(x_1 x_2 x \ldots x_p) dx_1 dx_2 \ldots dx_p$$

$$= c_1 \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 f_{x_1} f_2 \ldots x_p + c_p \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_p f_{x_1 \ldots x_p} dx_1 \ldots dx_p$$

$$= c_1 E(x_1) + c_2 E(x_2) + \ldots + c_p E(x_p).$$

$$\Rightarrow \mu_y = c_1 \mu_{x_1} + c_2 \mu_{x_2} + \ldots + c_p \mu_{x_p}$$

• Variance =

$$V(y) = c_1^2 V(x_1) + c_2^2 V(x_2) + c_3^2 V(x_3) + \ldots + c_p^2 V(x_p) + 2 \sum_{i<j}\sum c_i c_j \, \mathrm{cov}(x_i, x_j)$$

If $x_1, x_2, \ldots x_p$ are independent then covariance is 0.

$$V(Y) = c_1^2 V(x_1) + c_2^2 V(x_2) + \ldots + c_p^2 V(x_p).$$

Ex:- If Y is -ve binomial random variable parameter p & r $\therefore$ $y = x_1 + x_2 + \ldots + x_r$ where each x_i is geometric r.v. parameter p and are independent.

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$Y = X_1 + X_2 + X_3 - - - - - X_r$

Each X_i is geometric random variable with parameter p

Mean $= \frac{r}{p}$, Variance $= \frac{r(1-p)}{p^2}$.

Q Let x_1 & x_2 denote length & width of path assume that expected value of X_1 is 2cm $E(X_1) = 2cm$, $\sigma(X_1) = 0.1cm$ $E(X_2) = 5cm$, $\sigma(X_2) = 0.2cm$ Also assume that covariance equal to -0.005 cm^2, then $y = 2x_1 + 2x_2$ is a random variable that represent perimeter of path

→ $E(Y) = 2E(X_1) + 2E(X_2)$

$E(Y) = 14cm$

$V(Y) = 2^2 V(X_1) + 2^2 V(X_2) + 2\,Cov(X_1, X_2)$

$= 4(0.01) + 4(0.04) + 8(-0.005)$

$= 0.04 + 0.16 - 0.04$

$= 0.16$

Mean and Variance :-

If $\bar{X} = \frac{X_1 + X_2 + - - - + X_p}{p}$ $E(\bar{X}) = \frac{E(X_1) + E(X_2) + - - + E(X_p)}{p}$

If $E(X_i) = \mu$

$E(\bar{X}) = \frac{p\mu}{p} = \mu$

$V(\bar{X}) = \frac{V(X_1) + V(X_2) + - - - + V(X_p)}{p^2}$

$X_1, X_2, X_3 - - - - X_p$ are independent.

If $V(X_i) = \sigma^2 \quad \forall \; i = 1, 2, - - - p.$

$$\boxed{V(\bar{X}) = \frac{\sigma^2}{p}}$$

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In above examples X_1, X_2 are independent variables and they are normal r.v find prods that parameter exceeds by 14.5cm ; Cov = 0

→ $E(Y) = 14cm$

$V(Y) = 0.2cm^2$

$P(Y > 14.5) = P\left(Z > \frac{14.5 - 14}{0.447}\right)$

$= P(Z > 1.12)$

$= 0.3686$ (0.5)

$= 0.5 - 0.3686$

$= 0.1314$

Q Soft Drinks are filled by an automated filling machine. The mean fill volume is 12.1 fluid ounces and standard deviation 0.1 fluid ounces. Assume that filled volume of can are independent normal values. What is probability that avg volume of 10 cans selected from this process is less than 12 fluid ounces

→ $\bar{X} = X_1 + X_2 + \cdots + X_{10}$

$E(\bar{X}) = \mu = E(X_i) = 12.1$ fluid ounces

$V(\bar{X}) = \frac{(0.1)^2}{10} = 0.001$

$P(\bar{X} < 12) = P\left(Z < \frac{12 - 12.1}{0.1}\right) = \cancel{3.16}\ P(Z < -3.16)$

$= 0.5 - 0.4992$

$= 0.0008$

Q Suppose X is discrete r.v. with probability density function.

$Y = h(X)$ a one to one transformation b/w values of x and y so that equation

$y = h(x)$

can be solved uniquely for x in terms of y

Let this solution be $x = \mu(y)$

then probability density function of random variable y.

$$f_Y(y) = f_X(\mu(y))$$

Q Let x be a geometric r.v. with probability distribution $f_X(x) = p(1-p)^{x-1}$, $x = 1, 2, \ldots$

Find probability distribution of $Y = X^2$

→ Since, $X \geq 0$ the transformation is one to one $y = x^2 \Rightarrow x = \sqrt{y}$. Now this implies

$$f_Y(y) = f_X(\sqrt{y}) = p(1-p)^{\sqrt{y}-1}$$

$y = 1, 4, 9, 16, \ldots$

Suppose x_1 & x_2 are discrete random variables with joint probability distribution $f_{X_1 X_2}(x_1, x_2)$. Let $y_1 = h_1 f(x_1, x_2)$

$y_2 = h_2 f(x_1, x_2)$

Define one to one transformation b/w pts (x_1, x_2) & (y_1, y_2) So, that equations $y_1 = h_1(x_1, x_2)$, $y_2 = h_2(x_1, x_2)$ can be solved uniquely for (x_1, x_2) in terms of y_1 & y_2 ~~that~~.

Let this solⁿ be $x_1 = \mu_1(y_1, y_2)$, $x_2 = u_2(y_1, y_2)$.

Then joint probability distribution of y_1, y_2 is

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(u_1, u_2).$$

Remark

- If want to find distribution of y_1 alone that will be given by marginal distribution of y_1

$$f_{Y_1}(y_1) = \sum_{y_2} f_{X_1,X_2}(u_1, u_2)$$

Some Discrete Probability Distributions:

Discrete Uniform Distribution:

If the discrete random variable X assumes the values x_1, x_2, …,x_k with equal probabilities, then X has the discrete uniform distribution given by:

$$f(x) = P(X = x) = f(x, k) = \begin{cases} \dfrac{1}{k} ; & x = x_1, x_2, \cdots, x_k \\ 0 ; & elsewhere \end{cases}$$

Note:

· $f(x)=f(x;k)=P(X=x)$

k is called the parameter of the distribution.

Example 1 :

Experiment: tossing a balanced die.

- Sample space: S={1,2,3,4,5,6}
- Each sample point of S occurs with the same probability 1/6.
- Let X= the number observed when tossing a balanced die.
- The probability distribution of X is,

$$f(x) = P(X = x) = f(x;6) = \begin{cases} \frac{1}{6}\ ; & x = 1, 2, \cdots, 6 \\ 0\ ; & elsewhere \end{cases}$$

Result:

If the discrete random variable X has a discrete uniform distribution with parameter k, then the mean and the variance of X are;

$$E(X) = \mu = \frac{\sum_{i=1}^{k} x_i}{k}$$

$$Var(X) = \sigma^2 = \frac{\sum_{i=1}^{k} (x_i - \mu)^2}{k}$$

Uniform Distribution

Also $E(X)= (k+1)/2$ (why?)

Hint: $(1/k).k(k+1)/2 = (k+1)/2$

Similarly, $Var(X)=(k+1)(k-1)/12$

Example 5.3:

Find E(X) and Var(X) in Example 1.

Solution:

$$E(X) = \mu = \frac{\sum_{i=1}^{k} x_i}{k} = \frac{1+2+3+4+5+6}{6} = 3.5$$

$$Var(X) = \sigma^2 = \frac{\sum_{i=1}^{k} (x_i - \mu)^2}{k} = \frac{\sum_{i=1}^{k} (x_i - 3.5)^2}{6}$$

$$= \frac{(1-3.5)^2 + (2-3.5)^2 + \cdots + (6-3.5)^2}{6} = \frac{35}{12}$$

Binomial Distribution:

Bernoulli Trial:

- Bernoulli trial is an experiment with only two possible outcomes.
- The two possible outcomes are labeled:
  success (s) and failure (f)
- The probability of success is P(s)=p and the probability of failure is P(f)= $q = 1-p$.
- Examples:

1. Tossing a coin (success=H, failure=T, and p=P(H))
2. Inspecting an item (success=defective, failure=non-defective, and p=P(defective))

Bernoulli Process:

Bernoulli process is an experiment that must satisfy the following properties:

1. The experiment consists of n repeated Bernoulli trials.
2. The probability of success, $P(s)=p$, remains constant from trial to trial.
3. The repeated trials are independent; that is the outcome of one trial has no effect on the outcome of any other trial.

Binomial Random Variable:

Consider the random variable :

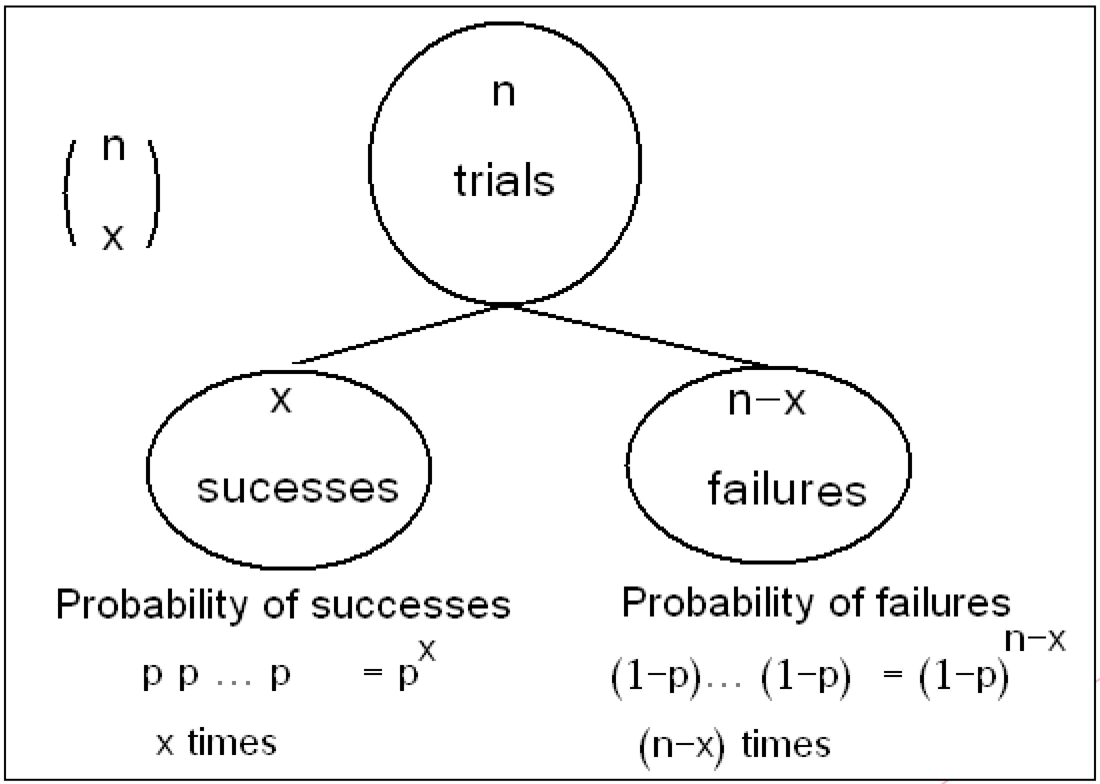
X = The number of successes in n trials in a Bernoulli process.

The random variable X has a binomial distribution with parameters n (number of trials) and p (probability of success), and we write:

$$X \sim \text{Binomial}(n,p) \quad \text{or} \quad X \sim b(x;n,p)$$

The probability distribution of X is given by:

$$f(x) = P(X = x) = b(x; n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}; & x = 0, 1, 2, \ldots, n \\ 0; & otherwise \end{cases}$$



We can write the probability distribution of X in table as follows.

x	$f(x)=P(X=x)=b(x;n,p)$
0	$\binom{n}{0} p^0 (1-p)^{n-0} = (1-p)^n$
1	$\binom{n}{1} p^1 (1-p)^{n-1}$
2	$\binom{n}{2} p^2 (1-p)^{n-2}$
$\vdots$	$\vdots$
$n-1$	$\binom{n}{n-1} p^{n-1} (1-p)^1$
n	$\binom{n}{n} p^n (1-p)^0 = p^n$
Total	1.00

Example 2:

Suppose that 25% of the products of a manufacturing process are defective. Three items are selected at random, inspected, and classified as defective (D) or non-defective (N). Find the probability distribution of the number of defective items.

Solution:

- Experiment: selecting 3 items at random, inspected, and classified as (D) or (N).
- The sample space is

S={DDD,DDN,DND,DNN,NDD,NDN,NND,NNN}

- Let X = the number of defective items in the sample
- We need to find the probability distribution of X.

(1) First Solution:

Outcome	Probability	X
NNN	$\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}=\frac{27}{64}$	0
NND	$\frac{3}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{9}{64}$	1
NDN	$\frac{3}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{9}{64}$	1
NDD	$\frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{3}{64}$	2
DNN	$\frac{1}{4}\times\frac{3}{4}\times\frac{3}{4}=\frac{9}{64}$	1
DND	$\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{3}{64}$	2
DDN	$\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{64}$	2
DDD	$\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{1}{64}$	3

The probability distribution of X is,

x	$f(x)=P(X=x)$
0	$\frac{27}{64}$
1	$\frac{9}{64}+\frac{9}{64}+\frac{9}{64}=\frac{27}{64}$
2	$\frac{3}{64}+\frac{3}{64}+\frac{3}{64}=\frac{9}{64}$
3	$\frac{1}{64}$

(2) Second Solution:

Bernoulli trial is the process of inspecting the item. The results are success=D or failure=N, with probability of success $P(s)=25/100=1/4=0.25$.

The experiments is a Bernoulli process with:

- number of trials: $n=3$
- Probability of success: $p=1/4=0.25$
- $X \sim$ Binomial(n,p)=Binomial(3,1/4)
- The probability distribution of X is given by:

$$f(x)=P(X=x)=b(x;3,\frac{1}{4})=\begin{cases}\binom{3}{x}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{3-x}; & x=0,1,2,3\\ 0\ ; & otherwise\end{cases}$$

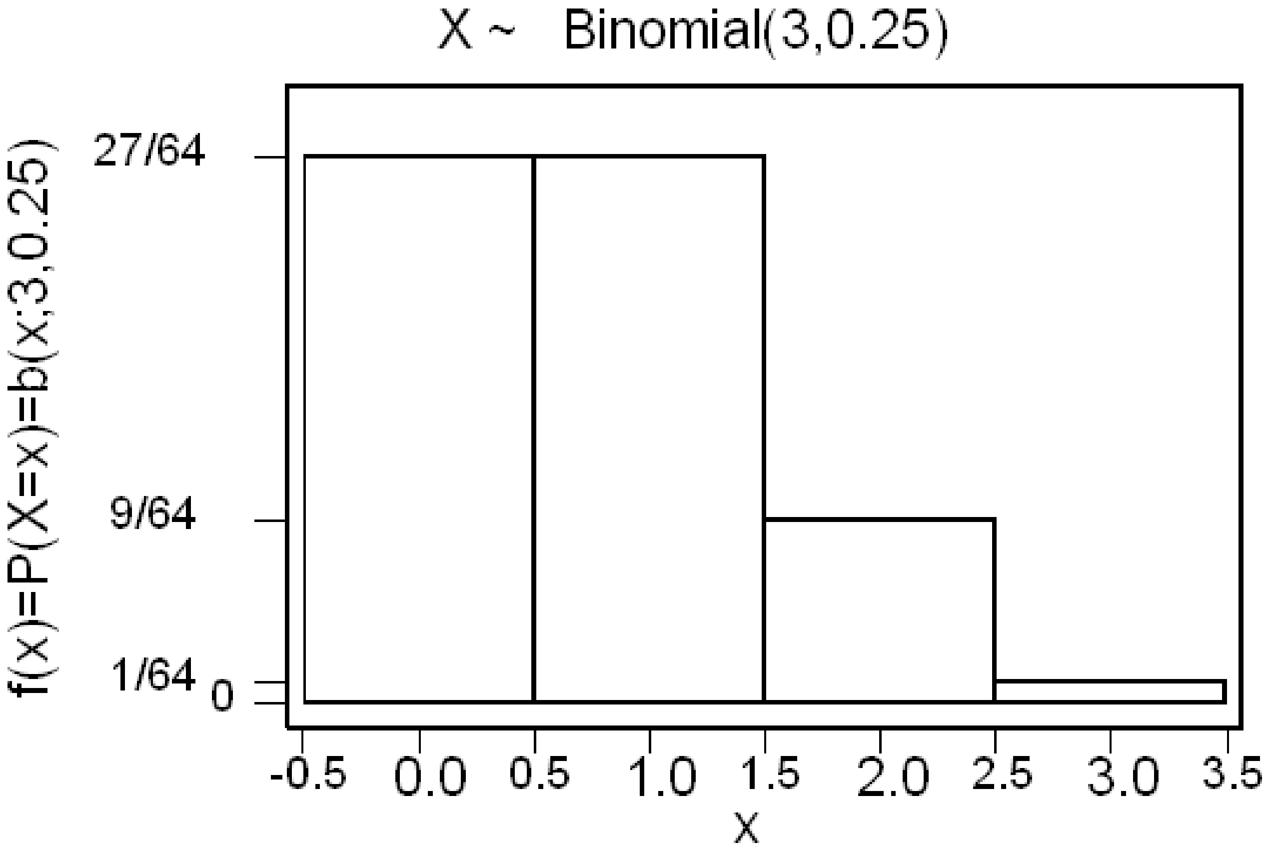
$$f(0)=P(X=0)=b(0;3,\frac{1}{4})=\binom{3}{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{3}=\frac{27}{64}$$

$$f(2)=P(X=2)=b(2;3,\frac{1}{4})=\binom{3}{2}\left(\frac{1}{4}\right)^{2}\left(\frac{3}{4}\right)^{1}=\frac{9}{64}$$

$$f(3)=P(X=3)=b(3;3,\frac{1}{4})=\binom{3}{3}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{0}=\frac{1}{64}$$

The probability distribution of X is,

x	f(x)=P(X=x) =b(x;3,1/4)
0	27/64
1	27/64
2	9/64
3	1/64



Result:

The mean and the variance of the binomial distribution b(x;*n*,*p*) are:

$$\mu = n\,p$$

$$\sigma^2 = n\,p\,(1-p)$$

Example:

In the previous example, find the expected value (mean) and the variance of the number of defective items.

Solution:

- X = number of defective items
- We need to find $E(X)=\mu$ and $Var(X)=\sigma^2$
- We found that $X \sim \text{Binomial}(n,p)=\text{Binomial}(3,1/4)$
- .$n=3$ and $p=1/4$

The expected number of defective items is

$$E(X)=\mu = n\,p = (3)\,(1/4) = 3/4 = 0.75$$

The variance of the number of defective items is

$$Var(X)=\sigma^2 = n\,p\,(1-p) = (3)\,(1/4)\,(3/4) = 9/16 = 0.5625$$

Example:

In the previous example, find the following probabilities:

(1) The probability of getting at least two defective items.

(2) The probability of getting at most two defective items.

Solution:

X ~ Binomial(3,1/4)

$$f(x) = P(X = x) = b(x;3,\frac{1}{4}) = \begin{cases} \binom{3}{x} (\frac{1}{4})^x (\frac{3}{4})^{3-x} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

x	.f(x)=P(X=x)=b(x;3,1/4)
0	27/64
1	27/64
2	9/64
3	1/64

(1) The probability of getting at least two defective items:

$$P(X\geq 2)=P(X=2)+P(X=3)= f(2)+f(3)= \frac{9}{64}+\frac{1}{64}=\frac{10}{64}$$

(2) The probability of getting at most two defective item:

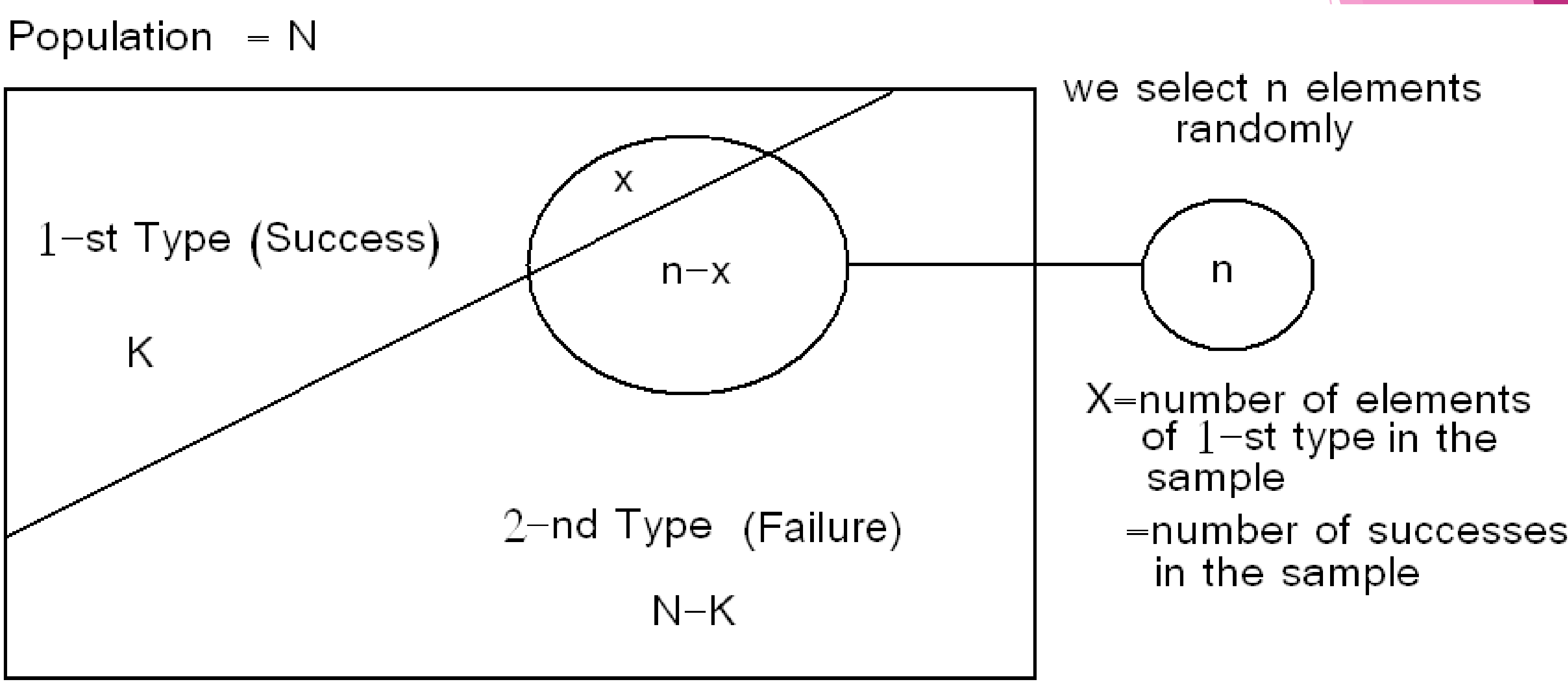
$$P(X\leq 2) = P(X=0)+P(X=1)+P(X=2)$$

$$= f(0)+f(1)+f(2) = \frac{27}{64}+\frac{27}{64}+\frac{9}{64}=\frac{63}{64}$$

or

$$P(X\leq 2)= 1-P(X>2) = 1-P(X=3) = 1- f(3) = 1- \frac{1}{64}=\frac{63}{64}$$

Hypergeometric Distribution :



- Suppose there is a population with 2 types of elements:

1-st Type = success

2-nd Type = failure

- N= population size
- K= number of elements of the 1-st type
- $N - K$ = number of elements of the 2-nd type

· We select a sample of n elements at random from the population.

· Let X = number of elements of 1-st type (number of successes) in the sample.

· We need to find the probability distribution of X.

There are to two methods of selection:

1. selection with replacement
2. selection without replacement

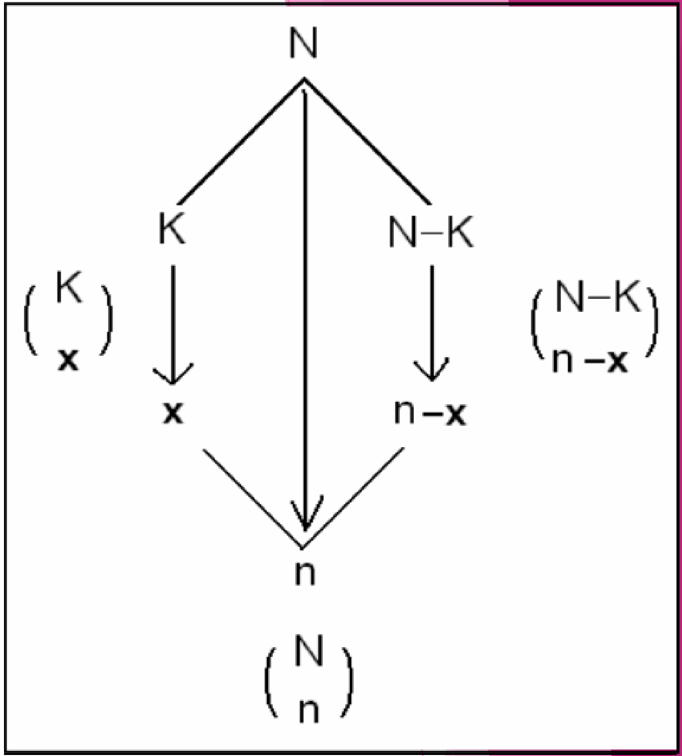
(1) If we select the elements of the sample at random and with replacement, then

$X \sim$ Binomial(n,p); where $p = \frac{K}{N}$

(2) Now, suppose we select the elements of the sample at random and without replacement. When the selection is made without replacement, the random variable X has a hyper geometric distribution with parameters N, n, and K. and we write X~h(x;N,n,K).

$$f(x) = P(X = x) = h(x; N, n, K)$$

$$= \begin{cases} \dfrac{\binom{K}{x} \times \binom{N-K}{n-x}}{\binom{N}{n}}; & x = 0, 1, 2, \cdots, n \\ 0; \ otherwise & \end{cases}$$



Note that the values of X must satisfy:

$0 \le x \le K$ and $0 \le n - x \le N - K$

$\Leftrightarrow$

$0 \le x \le K$ and $n - N + K \le x \le n$

Hypergeometric Distribution

Also $E(X)= (nK)/N$

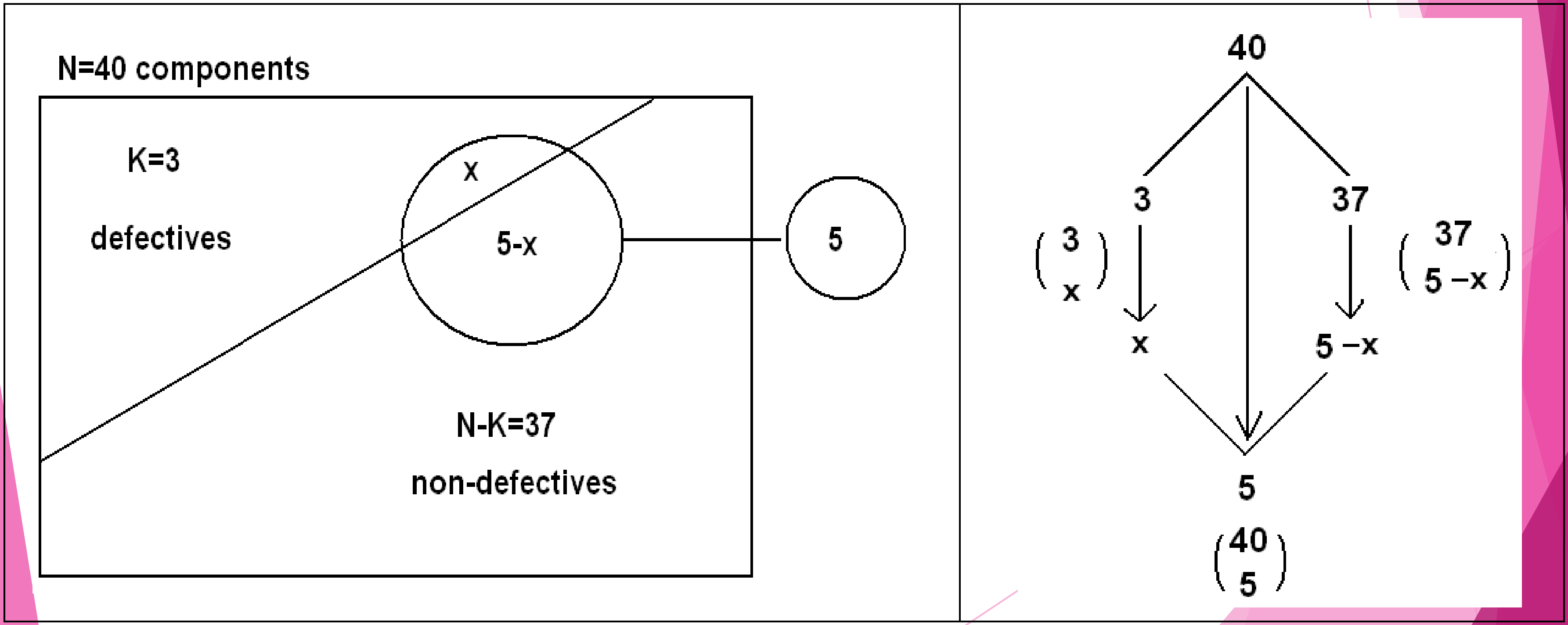
Hint: $(1/k).k(k+1)/2 = (k+1)/2$

Similarly, $Var (X)=NK(N-K)(N-n)/N^2(N-1)$

Example:

Lots of 40 components each are called acceptable if they contain no more than 3 defectives. The procedure for sampling the lot is to select 5 components at random (without replacement) and to reject the lot if a defective is found. What is the probability that exactly one defective is found in the sample if there are 3 defectives in the entire lot.

Solution:



- Let X= number of defectives in the sample
- N=40, K=3, and n=5
- X has a hypergeometric distribution with parameters N=40, n=5, and K=3.
- X~h(x;N,n,K)=h(x;40,5,3).
- The probability distribution of X is given by:

$$f(x) = P(X = x) = h(x;40,5,3) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{37}{5-x}}{\binom{40}{5}}; & x = 0, 1, 2, \cdots, 5 \\ 0; & otherwise \end{cases}$$

But the values of X must satisfy:

$0 \le x \le K$ and $n-N+K \le x \le n \Leftrightarrow 0 \le x \le 3$ and $-42 \le x \le 5$

Therefore, the probability distribution of X is given by:

$$f(x) = P(X = x) = h(x;40,5,3) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{37}{5-x}}{\binom{40}{5}}; & x = 0, 1, 2, 3 \\ 0; & otherwise \end{cases}$$

Now, the probability that exactly one defective is found in the sample is

$$.f(1)=P(X=1)=h(1;40,5,3)= \frac{\binom{3}{1} \times \binom{37}{5-1}}{\binom{40}{5}} = \frac{\binom{3}{1} \times \binom{37}{4}}{\binom{40}{5}} = 0.3011$$

Question?

- A bag contains 50 light bulbs of which 5 are defective and 45 are not. A Quality Control Inspector randomly samples 4 bulbs without replacement. Let X = the number of defective bulbs selected. Find the probability mass function, $f(x)$, of the discrete random variable X.

Result.

- Note: One of the key features of the hypergeometric distribution is that it is associated with sampling without replacement. When the samples are drawn with replacement, the discrete random variable follows what is called the **binomial distribution**.
- Note: In cases where the sample size is relatively large compared to the population, a discrete distribution called **hypergeometric** may be useful.

Question 1. A lake contains 600 fish, eighty (80) of which have been tagged by scientists. A researcher randomly catches 15 fish from the lake. Find a formula for the probability mass function of the number of fish in the researcher's sample which are tagged.

Question 2. Let the random variable X denote the number of aces in a five-card hand dealt from a standard 52-card deck. Find a formula for the probability mass function of X.

Question 3: Determine whether the given scenario describes a binomial setting. Justify your answer.
(a) Genetics says that the genes children receive from their parents are independent from one child to another. Each child of a particular set of parents has probability 0.25 of having type O blood. Suppose these parents have 5 children. Count the number of children with type O blood.

(b). Shuffle a standard deck of 52 playing cards. Turn over the first 10 cards, one at a time. Record the number of aces you observe.

Question 4: Each child of a particular set of parents has probability 0.25 of having type O blood. Suppose these parents have 5 children. What's the probability that exactly one of the five children has type O blood?

Solution 3 (a)

- **Binary?** "Success" 5 has type O blood. "Failure" 5 doesn't have type O blood.
- **Independent?** Knowing one child's blood type tells you nothing about another child's because they inherit genes independently from their parents.
- **Number?** n = 5
- **Same probability?** p = 0.25

Solution 3 (b)

- **Binary?** “Success” get an ace. “Failure” don’t get an ace.
- **Independent?** No. If the first card you turn over is an ace, then the next card is less likely to be an ace because you’re not replacing the top card in the deck. If the first card isn’t an ace, the second card is more likely to be an ace.

- This is not a binomial setting because the independent condition is not met.

Poisson Distribution:

- It is discrete distribution.
- The discrete r. v. X is said to have a Poisson distribution with parameter (average) λ if the probability distribution of X is given by

$$P(X=x)=\begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & ; \quad for \quad x=0,1,2,3,\ldots \\ 0 & ; \quad otherwise \end{cases}$$

where e = 2.71828.

We write:

$X \sim$ Poisson (λ)

- The mean (average) of Poisson (λ) is $\mu = \lambda$ $\quad \sigma^2 = \lambda$
- The variance is:

- The Poisson distribution is used to model a discrete r. v. which is a <u>count</u> of how many times a specified random event occurred in an interval of time or space.

Example:

- No. of patients in a waiting room in an hour.
- No. of serious injuries in a particular factory in a month.
- No. of calls received by a telephone operator in a day.
- No. of rates in each house in a particular city.

Note:

λ Is the average (mean) of the distribution.

If X = The number of calls received in a month and

$X \sim$ Poisson (λ)

then:

(i) Y = The no. calls received in a year.

$Y \sim$ Poisson (λ^*), where $\lambda^* = 12\lambda$

$Y \sim$ Poisson (12λ)

(ii) W = The no. calls received in a day.

$W \sim$ Poisson (λ^*), where $\lambda^* = \lambda/30$

$W \sim$ Poisson $(\lambda/30)$

Example

Suppose that the number of snake bites cases seen in a year has a Poisson distribution with average 6 bite cases.

1. What is the probability that in a year:

(i) The no. of snake bite cases will be 7?

(ii) The no. of snake bite cases will be less than 2?

2- What is the probability that in 2 years there will be 10 bite cases?

3- What is the probability that in a month there will be no snake bite cases?

Solution:

(1) X = no. of snake bite cases in a year.

X ~ Poisson (6) (λ=6)

$$P(X = x) = \frac{e^{-6}6^x}{x!}; \quad x = 0, 1, 2, \ldots$$

(i) $$P(X = 7)) = \frac{e^{-6}6^7}{7!} = 0.13768$$

(ii) $$P(X < 2) = P(X = 0) + P(X = 1)$$

$$= \frac{e^{-6}6^0}{0!} + \frac{e^{-6}6^1}{1!} = 0.01735$$

Y = no of snake bite cases in 2 years

$Y \sim$ Poisson(12) $\qquad (\lambda^* = 2\lambda = (2)(6) = 12)$

$$P(Y = y) = \frac{e^{-12}12^y}{y!} : \qquad y = 0, 1, 2\ldots$$

$$\therefore \; P(Y = 10) = \frac{e^{-12}12^{10}}{10} = 0.1048$$

3- W = no. of snake bite cases in a month.

$W \sim$ Poisson (0.5) $\qquad \lambda^{**} = \frac{\lambda}{12} = \frac{6}{12} = 0.5$

$$P(W = w) = \frac{e^{-.5}.5^w}{w!} : \qquad w = 0, 1, 2\ldots$$

$$P(W = 0) = \frac{e^{-0.5}(0.5)^0}{0!} = 0.6065$$

Note: Poisson distribution is for counts—if events happen at a constant rate over time, the Poisson distribution gives the probability of X number of events occurring in time T.

$$Result:\ p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}\ (\text{for } k \in \mathbb{N})$$

$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} e^{\lambda} = e^0 = 1$$

(Why? Taylor Series for e^x

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x)$$

$X \sim \mathrm{Unif}(a,b)$

$$p_X(k) = \frac{1}{b-a+1}$$

$$\mathbb{E}[X] = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)(b-a+2)}{12}$$

$X \sim \mathrm{Ber}(p)$

$$p_X(0) = 1-p;\quad p_X(1) = p$$

$$\mathbb{E}[X] = p$$

$$\mathrm{Var}(X) = p(1-p)$$

$X \sim \mathrm{Bin}(n,p)$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathbb{E}[X] = np$$

$$\mathrm{Var}(X) = np(1-p)$$

$X \sim \mathrm{Geo}(p)$

$$p_X(k) = (1-p)^{k-1} p$$

$$\mathbb{E}[X] = \frac{1}{p}$$

$$\mathrm{Var}(X) = \frac{1-p}{p^2}$$

$X \sim \mathrm{NegBin}(r,p)$

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

$$\mathbb{E}[X] = \frac{r}{p}$$

$$\mathrm{Var}(X) = \frac{r(1-p)}{p^2}$$

$X \sim \mathrm{HypGeo}(N,K,n)$

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$\mathbb{E}[X] = n\frac{K}{N}$$

$$\mathrm{Var}(X) = \frac{K(N-K)(N-n)}{N^2(N-1)}$$

$X \sim \mathrm{Poi}(\lambda)$

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\mathbb{E}[X] = \lambda$$

$$\mathrm{Var}(X) = \lambda$$