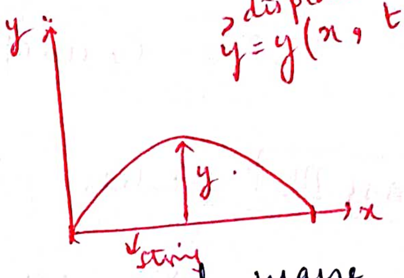
(4) WAVE ~~...~~ EQUATION

The equation of motion for the vibrations of a stretched string as a function of the distance x and the time t is given by

displacement $y = y(x, t)$:



$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$ ~~where~~

This is also called as the one dimensional wave equation.

The boundary conditions which the equation has to satisfy are :-

(1) $y = 0$ when $x = 0$

(2) $y = 0$ when $x = l$

If the string is made to vibrate by pulling it into a curve $y = f(x)$, then releasing it, the initial conditions are

(1) $y = f(x)$ $t = 0$

(2) $\frac{\partial y}{\partial t} = 0$ $t = 0$

SOLUTION OF THE WAVE EQUATION

The wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —— (1)

speed

Let $y = XT$ be the solution of (1)

Then $XT'' = c^2 X'' T$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = K \text{ (some constant)}$$

$$\Rightarrow X'' - KX = 0 \quad \text{and} \quad T'' - c^2 K T = 0$$

$\boxed{\text{Case I}}$ when K is positive and $= p^2$. as when $x = 0$ $y \neq 0$.

Then $X = c_1 e^{px} + c_2 e^{-px}$, $T = c_3 e^{cpt} + c_4 e^{-cpt}$.

This is not a feasible solution.

Case II when k is negative and $=-p^2$,

$$X = c_1 \cos px + c_2 \sin px$$
$$T = c_3 \cos cpt + c_4 \sin cpt$$

This is the only feasible solution which satisfies the boundary cond^ns.

Case III when $k=0$.

$X = c_1 x + c_2$ and $T = c_3 t + c_4$.

Since we are dealing with problems on vibrations, y must be a periodic function of x and t.

$\therefore$ $y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ is the only suitable solution of the wave equation and it corresponds to $k=-p^2$.

Now applying boundary conditions, we get $(y=0$ when $x=0)$ and $(y=0$ when $x=l,)$

$$0 = c_1 (c_3 \cos cpt + c_4 \sin cpt) \quad \text{——— (2)}$$

and $0 = (c_1 \cos pl + c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt)$ — (3)

From (2), we have, $c_1 = 0$.

$\therefore$ Equation (3) reduces to

$$0 = (c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt)$$

~~when~~ $\Rightarrow \sin pl = 0$

$\Rightarrow pl = n\pi \quad \Rightarrow \quad p = \frac{n\pi}{l}, \quad n = 1, 2, 3 \ldots$

$\therefore$ A solution of the wave equation satisfying the boundary conditions is

$$y = c_2\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

$$\boxed{y = \left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}}.$$

Adding up the solutions for different values of n,

$$\boxed{y = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}}.$$

is also a solution.

Applying the initial conditions.

$y = f(x)$ and $\dfrac{\partial y}{\partial t} = 0$ when $t = 0$, we have.

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}. \quad (4)$$

and $$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l}\, b_n \sin\frac{n\pi x}{l}. \quad (5)$$

Fourier series of $f(x)$ in interval $(c, c+2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_c^{c+2\pi} f(x)\,dx.$

$a_n = \dfrac{1}{\pi}\displaystyle\int_c^{c+2\pi} f(x)\cos nx\,dx$

$b_n = \dfrac{1}{\pi}\displaystyle\int_c^{c+2\pi} f(x)\sin nx\,dx.$ $n = 1, 2 \ldots$

Equation (4) represents Fourier series for $f(x)$.

we have, $$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \quad (6)$$

from (5) $b_n = 0 \quad \forall n.$

Hence y reduces to

$$y = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}.$$

where $a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx.$

Ex A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin\frac{\pi x}{l}$ from which it is released at time $t=0$. Show that the displacement of any point at a distance x from one end point at time t is given by

$$y(x,t) = a \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l}.$$

Solution Here the boundary conditions are

$y(0,t) = y(l,t) = 0$

and initial conditions are $y(x,0) = a \sin\frac{\pi x}{l}$

and $\frac{\partial y}{\partial t} = 0$, when $t = 0$

$\therefore$ we have.

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l},$$ where $a_n = \frac{2}{l}\int_0^l y(x,0) \sin\frac{n\pi}{l} dx$

$n^2\pi^2 \Big[\quad n\pi \qquad l \int_0$

$= \begin{cases} 0 & n \text{ is even} \\ \frac{8\lambda l^2}{n^3\pi^3}. & n \text{ is odd} \end{cases}$

$\cos 2x = 1 - 2\sin^2 x$

$$= a_1 \cos\frac{\pi ct}{l} \sin\frac{\pi x}{l} + a_2 \cos\frac{2\pi ct}{l}\sin\frac{2\pi x}{l} + \cdots$$

$$a_1 = \frac{2}{l}\int_0^l y(x,0)\sin\frac{\pi x}{l}\,dx = \frac{2}{l}\int_0^l a\sin\frac{\pi x}{l}\cdot\sin\frac{\pi x}{l}\,dx$$

$$= \frac{2a}{l}\int_0^l \sin^2\frac{\pi x}{l}\,dx$$

$$= \frac{a}{l}\int_0^l \left(1-\cos\frac{2\pi x}{l}\right)dx = \frac{2a}{l}\left[x - \frac{l}{2\pi}\sin\frac{2\pi x}{l}\right]_0^l$$

$$= \frac{a}{l}\left[l - \frac{l}{2\pi}\times 0\right] = a$$

$$\frac{2}{l}\int_0^l a\sin\frac{\pi x}{l}\cdot\sin\frac{n\pi x}{l}$$

$$a_2 = \frac{2}{l}\int_0^l a\sin\frac{\pi x}{l}\sin\frac{2\pi x}{l}\,dx = \frac{2a}{l}\left[\frac{1}{2}\left\{\cos\left(\frac{n\pi-\pi}{l}\right)x - \cos\left(\frac{(n+1)\pi x}{l}\right)\right\}\right]$$

$$a_n = \frac{a}{l}\left[\frac{l}{(n-1)\pi}\sin\frac{(n-1)\pi x}{l} - \frac{l}{(n+1)\pi}\sin\frac{(n+1)\pi x}{l}\right]_0^l$$

$n > 1$

$$= \frac{a}{l}\left[\int_0^l \cos\frac{(\pi x - 2\pi x)}{l} - \cos\frac{(\pi x + 2\pi x)}{l}\right]dx$$

at $n = 1 \to \infty$. we calculate a_1 separately

$$= \frac{a}{l}\left[\int_0^l \cos\frac{\pi x}{l} - \cos\frac{3\pi x}{l}\,dx\right]$$

$$= \frac{a}{l}\left[\frac{\sin\frac{\pi x}{l}}{\frac{\pi}{l}} - \frac{\sin\frac{3\pi x}{l}}{\frac{3\pi}{l}}\right]_0^l = \frac{a}{l}[0]$$

||ly $a_3 = a_4 \cdots = 0$.

$\therefore \quad y(x,t) = a \cos\frac{\pi c t}{l} \sin\frac{\pi x}{l}$.

Ex. A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time t.

Solution Here the boundary conditions are $y(0,t) = y(l,t) = 0$

$y(x,0) = 0$, $y(0,t) = y(l,t) = 0$.

$\frac{\partial y}{\partial t} = \lambda x(l-x)$ at $t=0$

By wave equation

$$y(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi c t}{l} + b_n \sin\frac{n\pi c t}{l}\right)\sin\frac{n\pi x}{l}.$$

Since the string was at rest initially, $\boxed{y(x,0)=0}$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} = 0$$

$$\Rightarrow a_n = 0 \quad \forall n. \qquad (*)$$

$$\therefore \quad y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi c t}{l}\sin\frac{n\pi x}{l}$$

$\boxed{\frac{\partial y}{\partial t} = \lambda x(l-x) \text{ at } t=0}$

and
$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \cdot \frac{n\pi c}{l} \cos\frac{n\pi c t}{l}\sin\frac{n\pi x}{l}.$$

$$= \frac{\pi c}{l}\sum_{n=1}^{\infty} n b_n \cos\frac{n\pi c t}{l}\sin\frac{n\pi x}{l}$$

But $\frac{\partial y}{\partial t} = \lambda x(l-x)$ at $t=0$

$\therefore \lambda x(l-x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin\frac{n\pi x}{l}$

using Fourier half range series formula, $0 < x < l$.

$$\frac{\pi c}{l} n b_n^* = \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\int_0^l \underbrace{(lx-x^2)}_{1} \underbrace{\sin\frac{n\pi x}{l}}_{2}\,dx$$

$$= \frac{2\lambda}{l}\left[\underbrace{(lx-x^2)\frac{(-l)}{n\pi}\cos\frac{n\pi x}{l}}_{0} + \int_0^l (l-2x)\frac{l}{n\pi}\cos\frac{n\pi x}{l}\,dx\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\frac{l}{n\pi}\int_0^l \underbrace{(l-2x)}_{1}\underbrace{\cos\frac{n\pi x}{l}}_{2}\,dx\right]$$

$$= \frac{2\lambda}{n\pi}\left[\underbrace{(l-2x)\frac{l}{n\pi}\sin\frac{n\pi x}{l}}_{0} - \int_0^l \frac{(-2)l}{n\pi}\sin\frac{n\pi x}{l}\,dx\right]_0^l$$

$$= \frac{2\lambda}{n\pi}\left[\frac{2l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4\lambda l}{n^2\pi^2}\left[\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l = \frac{-4\lambda l^2}{n^3\pi^3}\left[\cos n\pi - \cos 0\right]$$

$$= \begin{cases} 0 & n \text{ is even} \\ \frac{8\lambda l^2}{n^3\pi^3} & n \text{ is odd} \end{cases}$$

$$\therefore \quad \frac{\pi c n b_n}{l} = \begin{cases} 0 & n \text{ is even} \\ \dfrac{8\lambda l^2}{n^3 \pi^3} & n \text{ is odd} \end{cases}$$

Replacing $n \to 2m-1$

$$\Rightarrow \quad \frac{\pi c (2m-1) b_m}{l} = \frac{8\lambda l^3}{(2m-1)^3 \pi^3}$$

$$\Rightarrow \quad \boxed{b_m = \frac{8\lambda l^4}{c(2m-1)^4 \pi^4}} \qquad m = 1, 2, 3 \ldots$$

Putting the value of b_m in ⊛

$$y = \sum_{m=1}^{\infty} b_m \sin \frac{(2m-1)\pi c t}{l} \cdot \sin \frac{(2m-1)\pi x}{l}$$

$$= \sum_{m=1}^{\infty} \frac{8\lambda l^4}{c(2m-1)^4 \pi^4} \cdot \sin \frac{(2m-1)\pi c t}{l} \cdot \sin \frac{(2m-1)\pi x}{l}.$$

X

Ex Solve the vibrating string problem with

(1) $u(0,t) = 0 = u(l,t)$

(2) $u(x,0) = \begin{cases} x & 0 < x < l/2 \\ l-x & l/2 < x < l \end{cases}$

(3) $u_t(x,0) = x(l-x) \quad 0 < x < l.$

Solution The vibrating string problem given by $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ has the solution

given by $u = \sum_{n=1}^{\infty} \left[A_n \cos \frac{an\pi t}{l} + B_n \sin \frac{an\pi t}{l} \right] \sin \frac{n\pi x}{l}$

From (1) $u(0,t) = 0 = u(l,t)$ is satisfied.

From(2). $u(x,0) = \begin{cases} x & 0 < x < l/2 \\ l-x & l/2 < x < l \end{cases}$

we get

$$u = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

for finding the value of A_n, using Fourier series

we get $A_n = \frac{2}{l} \int_0^l u \cdot \sin \frac{n\pi x}{l} \, dx$

$$= \frac{2}{l}\left[\int_0^{l/2} x \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[-\frac{xl}{n\pi}\cos\frac{n\pi x}{l} + \frac{l}{n\pi}\int_0^{l/2} 1.\cos\frac{n\pi x}{l}\,dx\right]_0^{l/2}$$

$$+ \frac{2}{l}\left[-\frac{(l-x)\,l}{n\pi}\cos\frac{n\pi x}{l} - \int_{l/2}^{l}(-1)\left(\frac{-l}{n\pi}\right)\cos\frac{n\pi x}{l}\,dx\right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[\cancel{\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2}} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$+ \frac{2}{l}\left[\cancel{+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}} - \int_{l/2}^{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2l}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2l}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

From (3) $\quad u_t(x,0) = x(l-x) \quad 0<x<l$

we get

$$u_t = \sum_{n=1}^{\infty}\left(-A_n\frac{an\pi}{l}\sin\frac{an\pi t}{l} + B_n\frac{an\pi}{l}\cos\frac{an\pi t}{l}\right)\sin\frac{n\pi x}{l}$$

$$u_t(x,0) = \sum_{n=1}^{\infty}\left(B_n\frac{an\pi}{l}\right)\sin\frac{n\pi x}{l}$$

Using Fourier series, we find B_n

$$\frac{an\pi B_n}{\cancel{l}} = \frac{2}{\cancel{l}} \int_0^l u_t(x,0) \cdot \sin\frac{n\pi x}{l}\, dx$$

$$\Rightarrow \quad B_n = \frac{2}{an\pi} \left[\int_0^l \underbrace{x(l-x)}_{\text{I}} \sin\frac{n\pi x}{l}\, dx \right]$$

$$= \frac{2}{an\pi} \left[\underbrace{(lx - x^2)\left(\frac{-l}{n\pi}\right)\cos\frac{n\pi x}{l}}_{0} - \int (l-2x)\left(\frac{-l}{n\pi}\right)\cos\frac{n\pi x}{l} \right]_0^l$$

$$= \frac{2l}{an^2\pi^2} \int_0^l (l-2x) \cos\frac{n\pi x}{l}\, dx$$

$$= \frac{2l}{an^2\pi^2} \left[\underbrace{(l-2x)\frac{l}{n\pi}\sin\frac{n\pi x}{l}}_{0} - \int_0^l (-2)\frac{l}{n\pi}\sin\frac{n\pi x}{l}\, dx \right]_0^l$$

$$= \frac{4l^2}{an^3\pi^3} \int_0^l \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{4l^2}{an^3\pi^3} \left(\frac{-l}{n\pi}\right) \left[\cos\frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-4l^3}{an^4\pi^4} \left[\cos n\pi - \cos 0 \right] = \frac{-4l^3}{an^4\pi^4} \left[(-1)^n - 1 \right]$$

$$= \begin{cases} \dfrac{8l^3}{an^4\pi^4} & n \text{ is odd} \\ 0 & n \text{ is even.} \end{cases}$$

$\therefore$ The desired solution is

$$u = \frac{4l}{\pi^2}\left[\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2} \cos\frac{an\pi t}{l} \sin\frac{n\pi x}{l}\right]$$

$$+ \frac{8l^3}{a\pi^4}\left[\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)a\pi t}{l} \sin\frac{(2n-1)\pi x}{l}\right]$$

Ex. The vibrations of an elastic string is governed by the partial differential equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$. The length of the string is π and the ends are fixed. The initial velocity is zero and the initial deflection is $u(x,0) = 2(\sin x + \sin 3x)$. Find the deflection $u(x,t)$ of the vibrating string for $t>0$.

Solution. The solution of the equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ is given by.

$$u = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mt + c_4 \sin mt)$$

The initial conditions are

(1) $u(0,t) = 0$

(2) $u(\pi,t) = 0$

(3) $\frac{\partial u}{\partial t}(x,0) = 0$

(4) $u(x,0) = 2(\sin x + \sin 3x)$

From (1) $u(0,t)=0$, we get

$$0 = c_1(c_3 \cos mt + c_4 \sin mt)$$

$$\Rightarrow c_1 = 0.$$

Thus, $u = (c_2 \sin mx)(c_3 \cos mt + c_4 \sin mt)$

From (2) $u(\pi,t)=0$, we get

$$0 = (c_2 \sin m\pi)(c_3 \cos mt + c_4 \sin mt)$$

Since $c_2 \neq 0$, $\sin m\pi = 0$

$$m\pi = n\pi$$

$$\Rightarrow m = n.$$

Thus, $u = \sin nx (c_3' \cos nt + c_4' \sin nt)$

From (3) $\frac{\partial u}{\partial t}(x,0) = 0$ we get

$$\Rightarrow \frac{\partial u}{\partial t} = \sin nx(-nc_3' \sin nt + c_4' n \cos nt)$$

$$\Rightarrow \frac{\partial u}{\partial t}(x,0) = \sin nx \cdot c_4' n$$

$$0 = \sin nx \, c_4' n$$

$$\Rightarrow c_4' = 0.$$

$\therefore$ $u = \sin nx \, c_3' \cos nt$

Hence the general solution will be given by

$$u = \sum_{n=1}^{\infty} b_n \sin nx \cos nt .$$

from $u(x,0) = 2(\sin x) + \sin 3x$ we get

$$2\sin x + \sin 3x = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$b_1 = 2$

$b_2 = 0$

$b_3 = 1$

$b_4 = b_5 = \cdots = 0$.

$\therefore u = 2\sin x \cos t + \sin 3x \cos 3t .$

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