Continuous Random Variable

A continuous random variable can take any value in an interval, open or closed, so it has innumerable values

Examples: the height or weight of a chair

For such a variable X, the probability assigned to an exact value $P(X = a)$ is always 0, though the probability for it to fall into interval $[a, b]$, that is,

$P(a \leq X \leq b)$, can be a positive number

Probability Distributions and Probability Density Functions

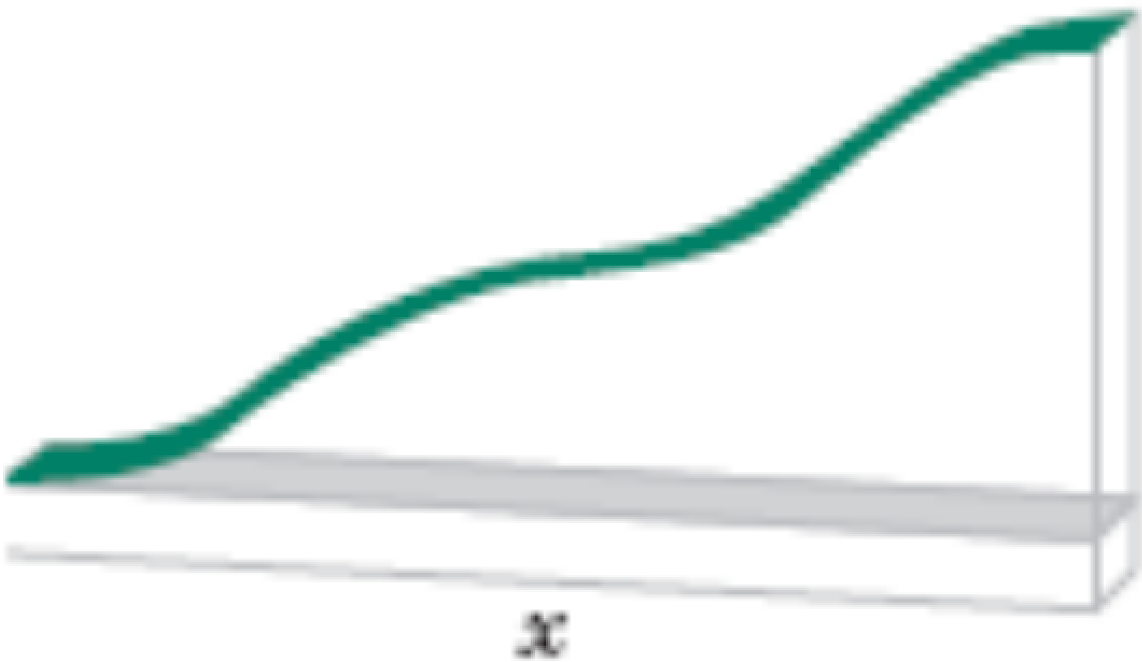
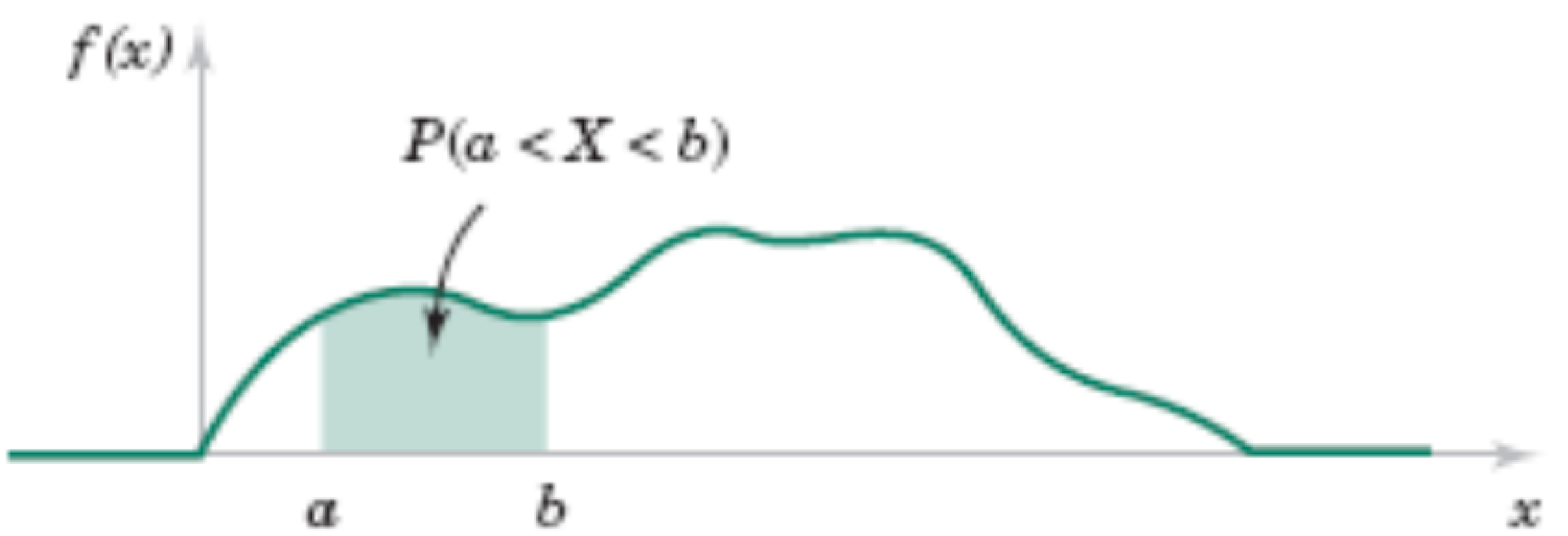


Figure 4-1 Density function of a loading on a long, thin beam.

Probability Distributions and Probability Density Functions



Probability determined from the area under $f(x)$.

4.1 Probability Distributions Probability Density Functions

Definition

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx =$ area under $f(x)$ from a to b

for any a and b (4-1)

Probability Distributions and Probability Density Functions

If X is a **continuous random variable**, for any x_1 and x_2,

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2) \quad (4\text{-}2)$$

Probability Distributions and Probability Density Functions

Example:

What proportion of parts is between 12.5 and 12.6 millimeters? Now,

$$P(12.5 < X < 12.6) = \int_{12.5}^{12.6} f(x)\,dx = -e^{-20(x-12.5)}\Big|_{12.5}^{12.6} = 0.865$$

Because the total area under $f(x)$ equals 1, we can also calculate $P(12.5 < X < 12.6) = 1 - P(X > 12.6) = 1 - 0.135 = 0.865$.

Cumulative Distribution Functions

Definition

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du \qquad (4\text{-}3)$$

for $-\infty < x < \infty$.

Mean and Variance of a Continuous Random Variable

Definition

Suppose X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \tag{4-4}$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

4.2 Mean and Variance of a Continuous Random Variable

Expected Value of a Function of a Continuous Random Variable

If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\, dx \qquad (4\text{-}5)$$

4.3 Continuous Uniform Random Variable

Definition

A continuous random variable X with probability density function

$$f(x) = 1/(b - a), \qquad a \le x \le b \tag{4-6}$$

is a **continuous uniform random variable.**

Continuous Uniform Random Variable

Continuous uniform probability density function.

Continuous Uniform Random Variable

Mean and Variance

If X is a continuous uniform random variable over $a \le x \le b$,

$$\mu = E(X) = \frac{(a+b)}{2} \quad \text{and} \quad \sigma^2 = V(X) = \frac{(b-a)^2}{12} \tag{4-7}$$

Continuous Uniform Random Variable

Example

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0, 20 mA], and assume that the probability density function of X is $f(x) = 0.05, 0 \leq x \leq 20$.

What is the probability that a measurement of current is between 5 and 10 milliamperes? The requested probability is shown as the shaded area in Fig. 4-9.

$$P(5 < X < 10) = \int_{5}^{10} f(x)\,dx$$

$$= 5(0.05) = 0.25$$

The mean and variance formulas can be applied with $a = 0$ and $b = 20$. Therefore,

$$E(X) = 10 \text{ mA} \quad \text{and} \quad V(X) = 20^2/12 = 33.33 \text{ mA}^2$$

Consequently, the standard deviation of X is 5.77 mA.

Continuous Uniform Random Variable

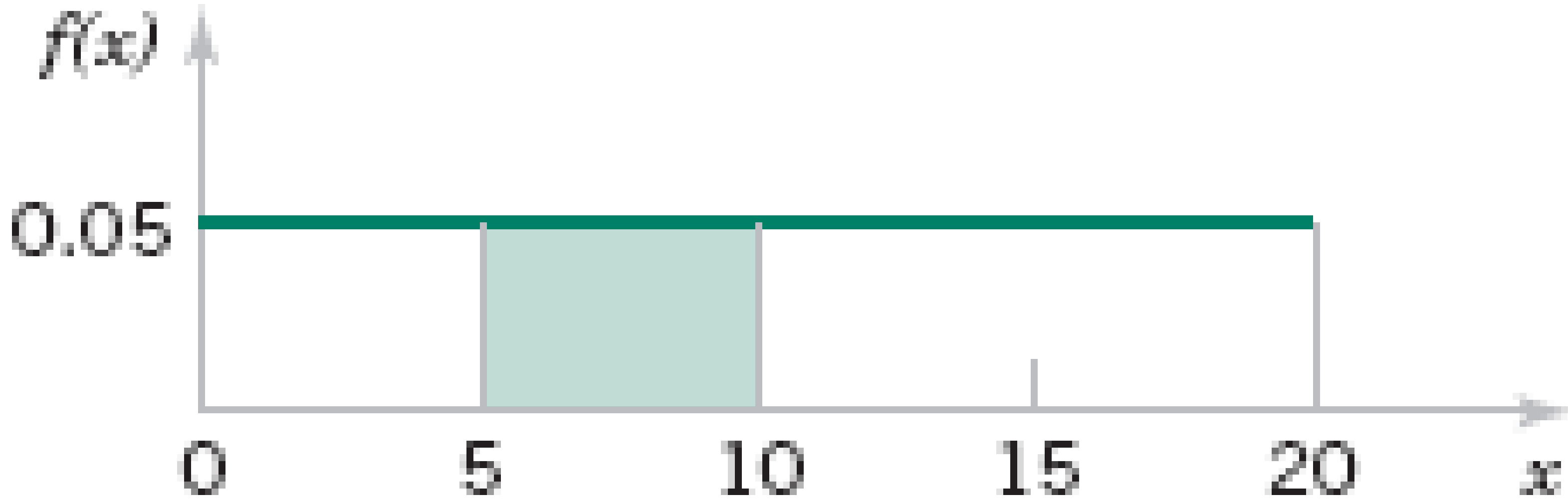


Figure 4-9 Probability for Example 4-9.

4.4Continuous Uniform Random Variable

The cumulative distribution function of a continuous uniform random variable is obtained by integration. If $a < x < b$,

$$F(x) = \int_a^x 1/(b-a)\,du = x/(b-a) - a/(b-a)$$

Therefore, the complete description of the cumulative distribution function of a continuous uniform random variable is

$$F(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x < b \\ 1 & b \le x \end{cases}$$

4.5 Normal Distribution

Definition

A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty \tag{4-8}$$

is a **normal random variable** with parameters μ, where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2 \tag{4-9}$$

and the notation $N(\mu, \sigma^2)$ is used to denote the distribution. The mean and variance of X are shown to equal μ and σ^2, respectively, at the end of this Section 5-6.

Normal Distribution

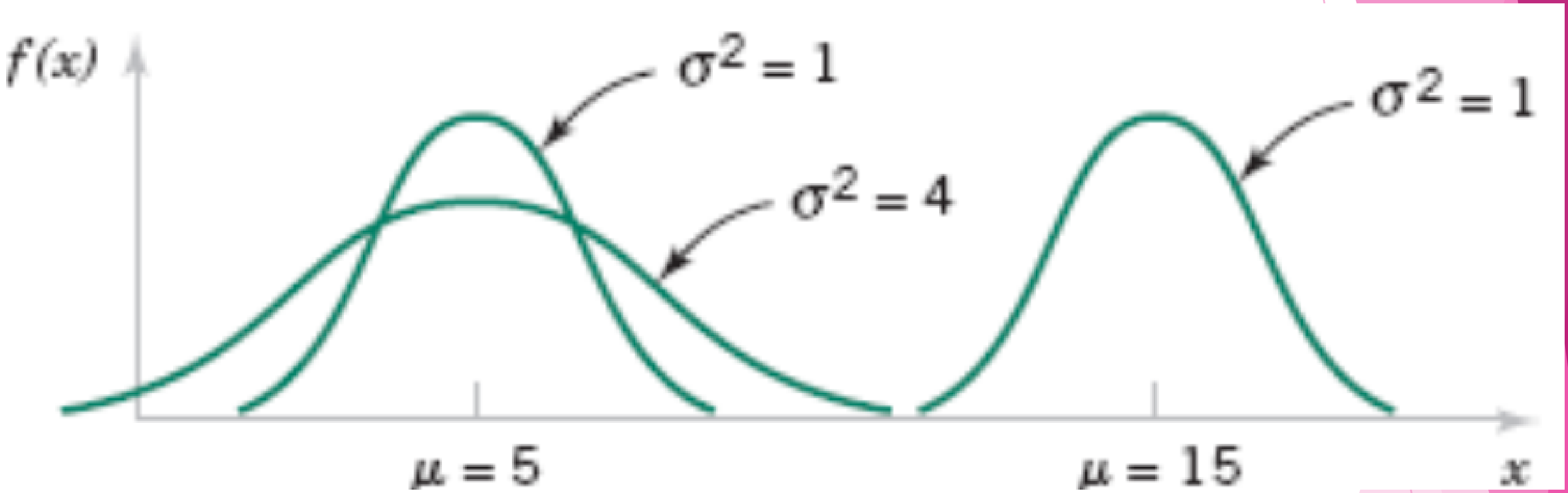


Figure : Normal probability density functions for selected values of the parameters μ and σ^2.

Normal Distribution

Some useful results concerning the normal distribution

For any normal random variable,

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

4.6 Normal Distribution

Definition : Standard Normal

A normal random variable with

$$\mu = 0 \quad \text{and} \quad \sigma^2 = 1$$

is called a **standard normal random variable** and is denoted as Z.

The cumulative distribution function of a standard normal random variable is denoted as

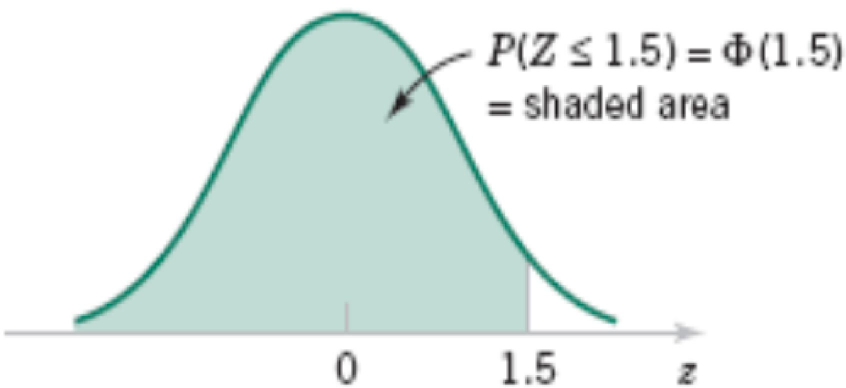
$$\Phi(z) = P(Z \leq z)$$

Normal Distribution

Example:

Assume Z is a standard normal random variable. Appendix Table II provides probabilities of the form $P(Z \leq z)$. The use of Table II to find $P(Z \leq 1.5)$ is illustrated in Fig. 4-13. Read down the z column to the row that equals 1.5. The probability is read from the adjacent column, labeled 0.00, to be 0.93319.

The column headings refer to the hundredth's digit of the value of z in $P(Z \leq z)$. For example, $P(Z \leq 1.53)$ is found by reading down the z column to the row 1.5 and then selecting the probability from the column labeled 0.03 to be 0.93699.



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

Figure 4-13 Standard normal probability density function.

Normal Distribution

Standardizing

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma} \tag{4-10}$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a standard normal random variable.

Normal Distribution

Example:

Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 $(\text{milliamperes})^2$. What is the probability that a measurement will exceed 13 milliamperes?

Let X denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$. Let $Z = (X - 10)/2$. The relationship between the several values of X and the transformed values of Z are shown in Fig. 4-15. We note that $X > 13$ corresponds to $Z > 1.5$. Therefore, from Appendix Table II,

$$P(X > 13) = P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.93319 = 0.06681$$

Rather than using Fig. 4-15, the probability can be found from the inequality $X > 13$. That is,

$$P(X > 13) = P\left(\frac{(X - 10)}{2} > \frac{(13 - 10)}{2}\right) = P(Z > 1.5) = 0.06681$$

Normal Distribution

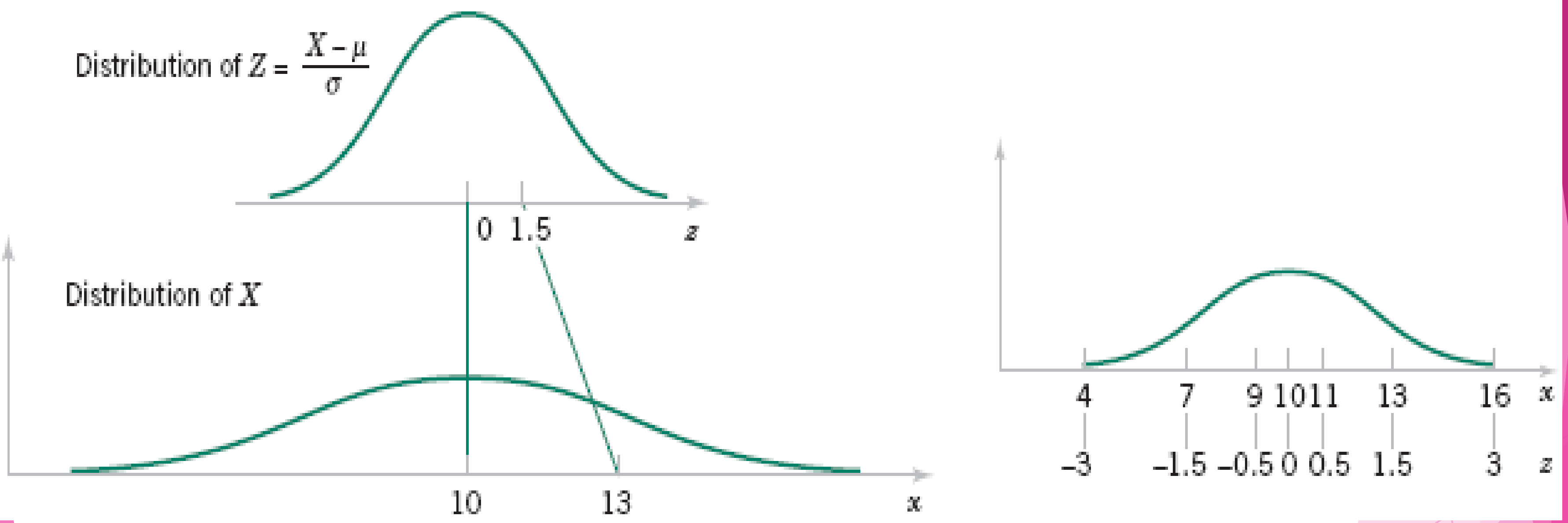


Figure 4-15 Standardizing a normal random variable.

Normal Distribution

To Calculate Probability

Suppose X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P(Z \leq z) \tag{4-11}$$

where Z is a **standard normal random variable**, and $z = \frac{(x-\mu)}{\sigma}$ is the z-value obtained by **standardizing** X.

The probability is obtained by entering Appendix Table II with $z = (x - \mu)/\sigma$.

Normal Distribution

Example:

Continuing the previous example, what is the probability that a current measurement is between 9 and 11 milliamperes? From Fig. 4-15, or by proceeding algebraically, we have

$$
\begin{aligned}
P(9 < X < 11) &= P((9 - 10)/2 < (X - 10)/2 < (11 - 10)/2) \\
&= P(-0.5 < Z < 0.5) = P(Z < 0.5) - P(Z < -0.5) \\
&= 0.69146 - 0.30854 = 0.38292
\end{aligned}
$$

Normal Distribution

Example (When prob. is given)

Determine the value for which the probability that a current measurement is below this value is 0.98. The requested value is shown graphically in Fig. 4-16. We need the value of x such that $P(X < x) = 0.98$. By standardizing, this probability expression can be written as

$$\begin{aligned} P(X < x) &= P((X - 10)/2 < (x - 10)/2) \\ &= P(Z < (x - 10)/2) \\ &= 0.98 \end{aligned}$$

Appendix Table II is used to find the z-value such that $P(Z < z) = 0.98$. The nearest probability from Table II results in

$$P(Z < 2.05) = 0.97982$$

Therefore, $(x - 10)/2 = 2.05$, and the standardizing transformation is used in reverse to solve for x. The result is

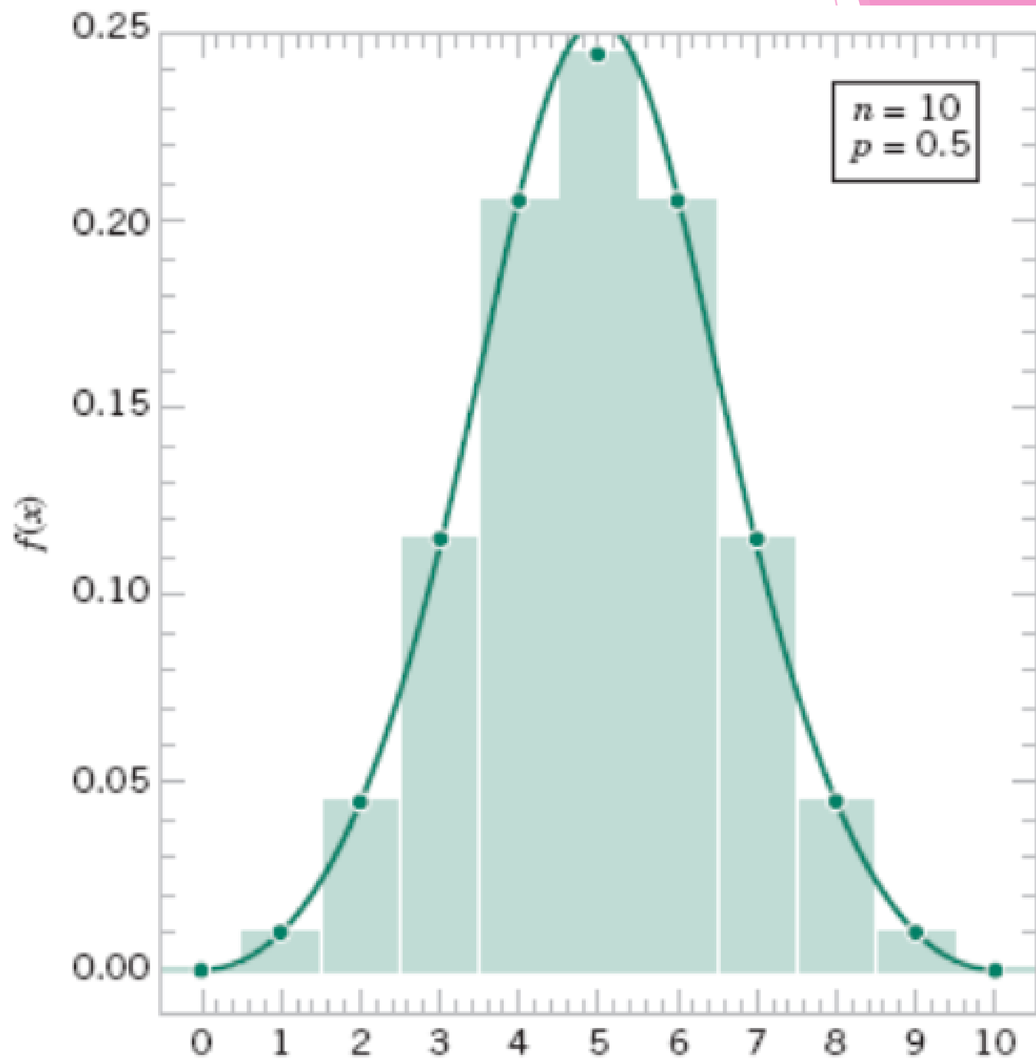
$$x = 2(2.05) + 10 = 14.1 \text{ milliamperes}$$

4-7 Normal Approximation to the Binomial and Poisson Distributions

- Under certain conditions, the normal distribution can be used to approximate the binomial distribution and the Poisson distribution.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Figure 4-19 Normal approximation to the binomial.



Normal Approximation to the Binomial and Poisson Distributions

Example (Normal Distri. can be used to compute the longer calculations)

In a digital communication channel, assume that the number of bits received in error can be modeled by a binomial random variable, and assume that the probability that a bit is received in error is 1×10^{-5}. If 16 million bits are transmitted, what is the probability that more than 150 errors occur?

Let the random variable X denote the number of errors. Then X is a binomial random variable and

$$P(X > 150) = 1 - P(x \le 150) = 1 - \sum_{x=0}^{150} \binom{16{,}000{,}000}{x} (10^{-5})^x (1 - 10^{-5})^{16{,}000{,}000-x}$$

Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Binomial Distribution

If X is a binomial random variable, with parameters n and p

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \tag{4-12}$$

is approximately a standard normal random variable. To approximate a binomial probability with a normal distribution a **continuity correction** is applied as follows

$$P(X \le x) = P(X \le x + 0.5) \cong P\left(Z \le \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

and

$$P(x \le X) = P(x - 0.5 \le X) \cong P\left(\frac{x - 0.5 - np}{\sqrt{np(1-p)}} \le Z\right)$$

The approximation is good for $np > 5$ and $n(1-p) > 5$.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4-18

The digital communication problem in the previous example is solved as follows:

$$P(X > 150) = P\left(\frac{X - 160}{\sqrt{160(1 - 10^{-5})}} > \frac{150 - 160}{\sqrt{160(1 - 10^{-5})}}\right)$$

$$= P(Z > -0.79) = P(Z < 0.79) = 0.785$$

Because $np = (16 \times 10^6)(1 \times 10^{-5}) = 160$ and $n(1 - p)$ is much larger, the approximation is expected to work well in this case.

4-7 Normal Approximation to the Binomial and Poisson Distributions

hypergometric distribution	$\approx$ $\frac{n}{N} < 0.1$	binomial distribution	$\approx$ $np > 5$ $n(1-p) > 5$	normal distribution

Figure 4-21 Conditions for approximating hypergeometric and binomial probabilities.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Poisson Distribution

If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \tag{4-13}$$

is approximately a standard normal random variable. The approximation is good for

$$\lambda > 5$$

4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4-20

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that less than 950 particles are found?

This probability can be expressed exactly as

$$P(X \leq 950) = \sum_{x=0}^{950} \frac{e^{-1000} x^{1000}}{x!}$$

The computational difficulty is clear. The probability can be approximated as

$$P(X \leq x) = P\left(Z \leq \frac{950 - 1000}{\sqrt{1000}}\right) = P(Z \leq -1.58) = 0.057$$

4-8 Exponential Distribution

Definition

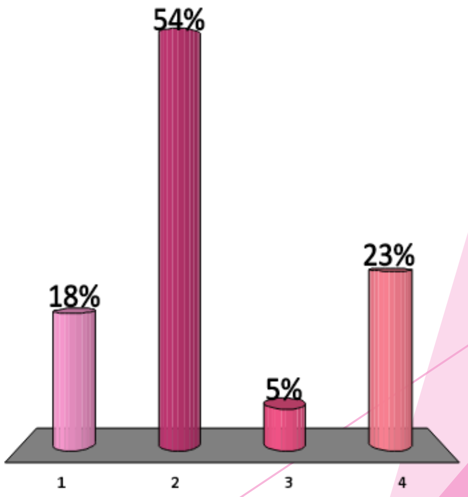
The random variable X that equals the distance between successive events of a Poisson process with mean $\lambda > 0$ is an **exponential random variable** with parameter λ. The probability density function of X is

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad 0 \leq x < \infty \tag{4-14}$$

Poisson or not?

Which of the following is most likely to be well modelled by a Poisson distribution?

1. Number of trains arriving at Falmer every hour
2. Number of lottery winners each year that live in Brighton
3. Number of days between solar eclipses
4. Number of days until a component fails



Are they Poisson?
***Answers*:**

1. Number of trains arriving at Falmer every hour

   NO, (supposed to) arrive *regularly* on a timetable not at random

2. Number of lottery winners each year that live in Brighton

   Yes, is number of random events in fixed interval

3. Number of days between solar eclipses

   NO, solar eclipses are not random events and this is a time *between* random events, not the number in some fixed interval

4. Number of days until a component fails
   NO, random events, but this is time until a random event, not the *number* of random events

Time between random events / time till first random event ?

If a Poisson process has constant average rate ν, the mean after a time t is

What is the probability distribution for the time to the first event?

$\Rightarrow$ **Exponential distribution**

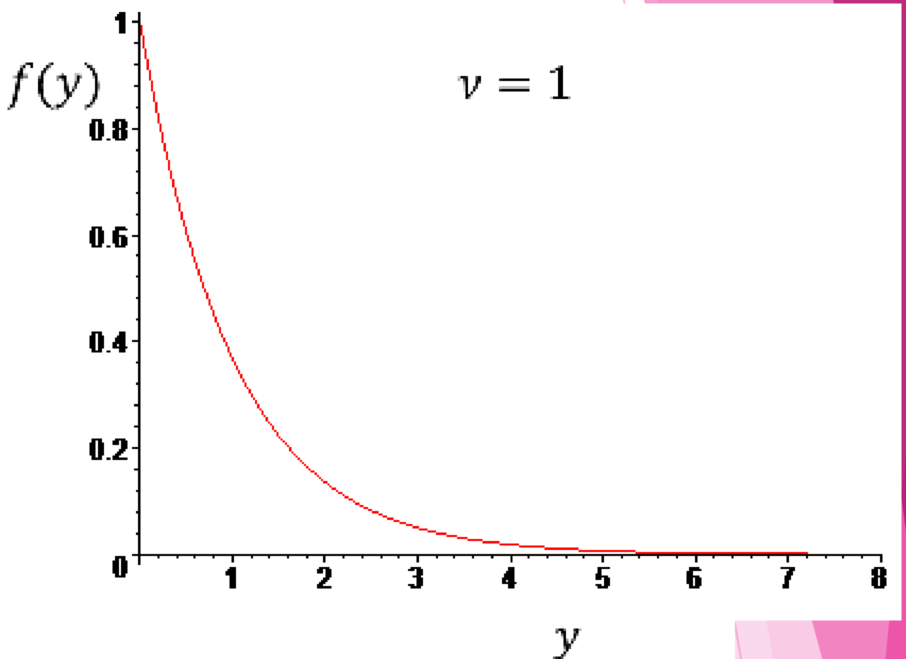
Poisson - *Discrete* distribution: P(number of events)

Exponential - *Continuous* distribution: P(time till first event)

Exponential distribution

The continuous random variable Y has the Exponential distribution, with ***constant*** rate parameter ν if:

$$f(y) = \begin{cases} \nu e^{-\nu y}, & y > 0 \\ 0, & y < 0 \end{cases}$$



Occurrence

1) Time until the failure of a part.

2) Separation between randomly happening events

- Assuming the probability of the events is constant in time: $\nu =$

Relation to Poisson distribution

If a Poisson process has constant average rate ν, the mean after a time t is

The probability of no-occurrences in time t is

$$P(k=0) = \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} = e^{-\nu t}.$$

If $f(t)$ is the pdf for the first occurrence, then the probability of no occurrences

$$P(\text{no occurrence by } t) = 1 - P(\text{first occurrence has happened}$$

$$= 1 - \int_0^t f(t)dt$$

$$\Rightarrow 1 - \int_0^t f(t)dt = e^{-\nu t} \qquad \Rightarrow \int_0^t f(t)\,dt = 1 - e^{-\nu t}$$

Solve by differentiating both sides respect to t assuming constant ν,

$$\frac{d}{dt}\int_0^t f(t)\,dt = \frac{d}{dt}(1 - e^{-\nu t})$$

$$\Rightarrow f(t) = \nu e^{-\nu t}$$

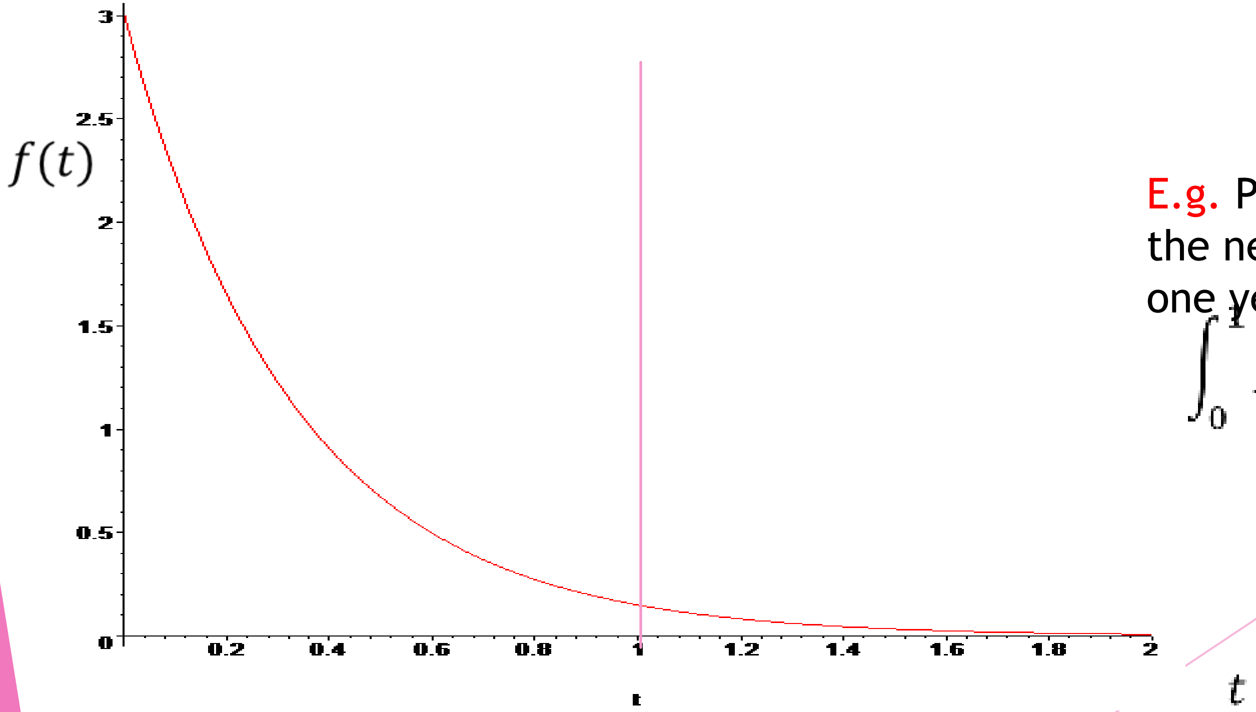
The time until the first occurrence (and between subsequent occurrences) has the Exponential distribution, parameter

Example

On average lightening kills three people each year in the UK, $\lambda = 3$. So the rate is $\nu = 3/\text{year}$.

Assuming strikes occur randomly at any time during the year so ν is constant, time from today until the next fatality has pdf (using t in year

$$f(t) = \nu e^{-\nu t} = 3\, e^{-3t}$$



E.g. Probability the time till the next death is less than one year?

$$\int_0^1 f(t)dt = \int_0^1 3\, e^{-3t} dt$$

$$= \left[\frac{3e^{-3t}}{-3}\right]_0^1$$

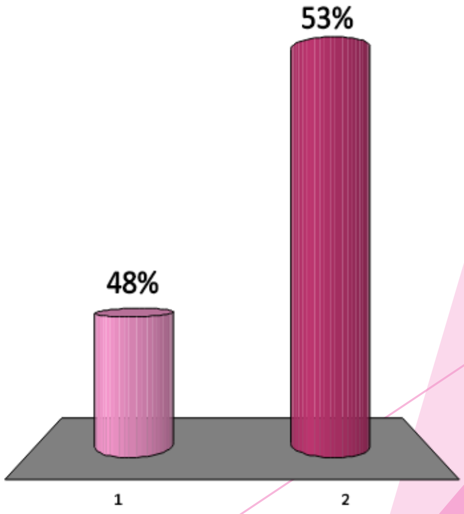
$$= -e^{-3} + 1 \approx 0.95$$

Exponential distribution

A certain type of component can be purchased new or used. 50% of all new components last more than five years, but only 30% of used components last more than five years. Is it possible that the lifetimes of new components are exponentially distributed?

Question from Derek Bruff

1. YES
2. NO

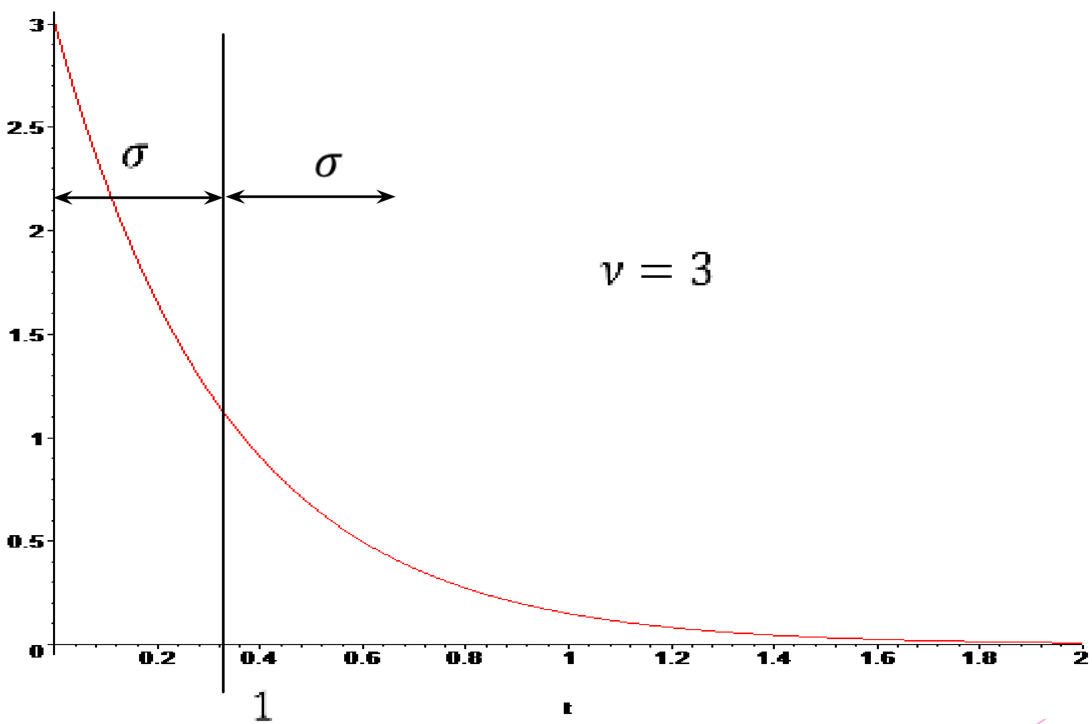


Mean and variance of exponential distribution

$$\mu = \int_{-\infty}^{\infty} y\, f(y)dy \; = \int_0^{\infty} y\nu e^{-\nu y} dy = [-ye^{-\nu y}]_0^{\infty} + \int_0^{\infty} e^{-\nu y} dy \; = \left[-\frac{e^{-\nu y}}{\nu}\right]_0^{\infty} = \frac{1}{\nu}$$

$$\sigma^2 = \int_{-\infty}^{\infty} y^2 f(y)dy \; - \mu^2 = \int_0^{\infty} y^2 \nu e^{-\nu y} dy - \frac{1}{\nu^2}$$

$$= [-y^2 e^{-\nu y}]_0^{\infty} + 2\int_0^{\infty} y\, e^{-\nu y} dy - \frac{1}{\nu^2} = 0 + 2\frac{\mu}{\nu} - \frac{1}{\nu^2} = \frac{1}{\nu^2}$$



$$\mu = \frac{1}{3}$$

4-8 Exponential Distribution

Mean and Variance

If the random variable X has an exponential distribution with parameter λ,

$$\mu = E(X) = \frac{1}{\lambda} \quad \text{and} \quad \sigma^2 = V(X) = \frac{1}{\lambda^2} \tag{4-15}$$

It is important to **use consistent units** in the calculation of probabilities, means, and variances involving exponential random variables. The following example illustrates unit conversions.

4-8 Exponential Distribution

Example 4-21

In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour. What is the probability that there are no log-ons in an interval of 6 minutes?

Let X denote the time in hours from the start of the interval until the first log-on. Then, X has an exponential distribution with $\lambda = 25$ log-ons per hour. We are interested in the probability that X exceeds 6 minutes. Because λ is given in log-ons per hour, we express all time units in hours. That is, 6 minutes $= 0.1$ hour. The probability requested is shown as the shaded area under the probability density function in Fig. 4-23. Therefore,

$$P(X > 0.1) = \int_{0.1}^{\infty} 25e^{-25x}\,dx = e^{-25(0.1)} = 0.082$$

4-8 Exponential Distribution

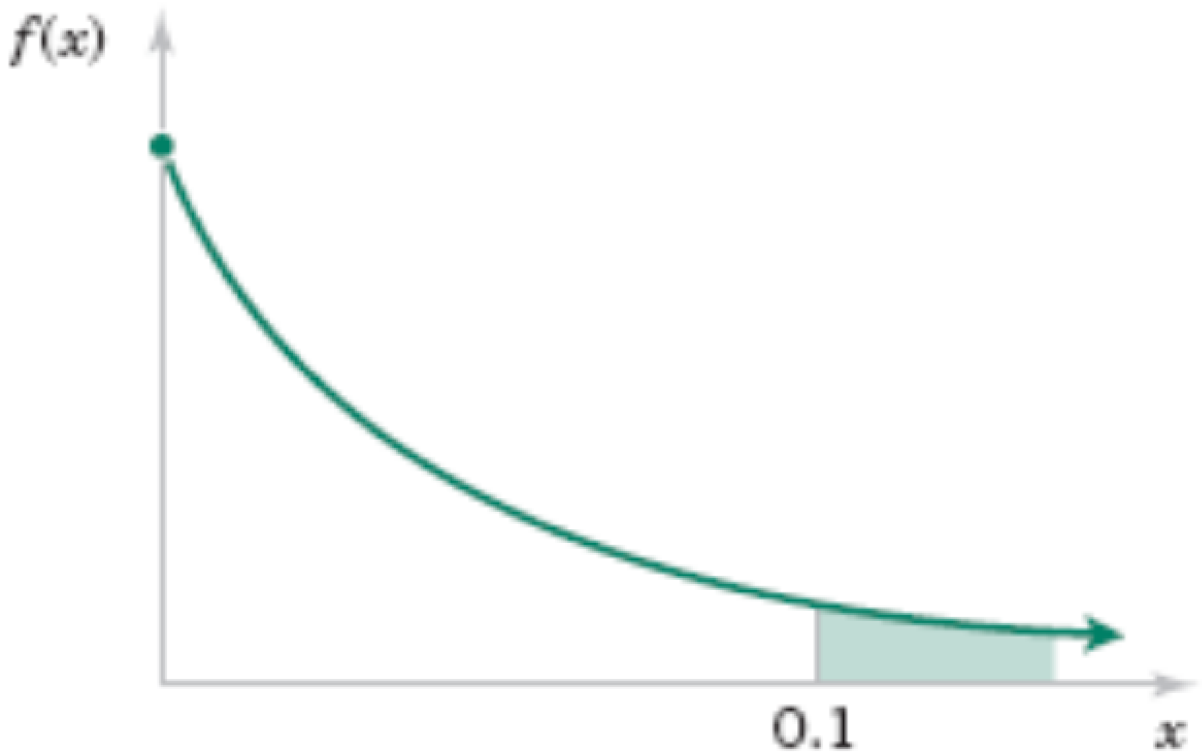


Figure 4-23 Probability for the exponential distribution in Example 4-21.

4-8 Exponential Distribution

Example 4-21 (continued)

Also, the cumulative distribution function can be used to obtain the same result as follows:

$$P(X > 0.1) = 1 - F(0.1) = e^{-25(0.1)}$$

An identical answer is obtained by expressing the mean number of log-ons as 0.417 log-ons per minute and computing the probability that the time until the next log-on exceeds 6 minutes. Try it.

What is the probability that the time until the next log-on is between 2 and 3 minutes? Upon converting all units to hours,

$$P(0.033 < X < 0.05) = \int_{0.033}^{0.05} 25e^{-25x}\,dx = -e^{-25x}\Big|_{0.033}^{0.05} = 0.152$$

4-8 Exponential Distribution

Example 4-21 (continued)

An alternative solution is

$$P(0.033 < X < 0.05) = F(0.05) - F(0.033) = 0.152$$

Determine the interval of time such that the probability that no log-on occurs in the i
val is 0.90. The question asks for the length of time x such that $P(X > x) = 0.90$. Now,

$$P(X > x) = e^{-25x} = 0.90$$

Take the (natural) log of both sides to obtain $-25x = \ln(0.90) = -0.1054$. Therefore,

$$x = 0.00421 \text{ hour} = 0.25 \text{ minute}$$

4-8 Exponential Distribution

Example 4-21 (continued)

Furthermore, the mean time until the next log-on is

$$\mu = 1/25 = 0.04 \text{ hour} = 2.4 \text{ minutes}$$

The standard deviation of the time until the next log-on is

$$\sigma = 1/25 \text{ hours} = 2.4 \text{ minutes}$$

4-8 Exponential Distribution

Our starting point for observing the system does not matter.

- An even more interesting property of an exponential random variable is the **lack of memory property**.

In Example 4-21, suppose that there are no log-ons from 12:00 to 12:15; the probability that there are no log-ons from 12:15 to 12:21 is still 0.082. Because we have already been waiting for 15 minutes, we feel that we are "due." That is, the probability of a log-on in the next 6 minutes should be greater than 0.082. *However, for an exponential distribution this is not true.*

4-8 Exponential Distribution

Example 4-22

Let X denote the time between detections of a particle with a geiger counter and assume that X has an exponential distribution with $\lambda = 1.4$ minutes. The probability that we detect a particle within 30 seconds of starting the counter is

$$P(X < 0.5 \text{ minute}) = F(0.5) = 1 - e^{-0.5/1.4} = 0.30$$

In this calculation, all units are converted to minutes. Now, suppose we turn on the geiger counter and wait 3 minutes without detecting a particle. What is the probability that a particle is detected in the next 30 seconds?

4-8 Exponential Distribution

Example 4-22 (continued)

Because we have already been waiting for 3 minutes, we feel that we are "due." T is, the probability of a detection in the next 30 seconds should be greater than 0.3. Howe for an exponential distribution, this is not true. The requested probability can be expre as the conditional probability that $P(X < 3.5 \mid X > 3)$. From the definition of conditi probability,

$$P(X < 3.5 \mid X > 3) = P(3 < X < 3.5)/P(X > 3)$$

where

$$P(3 < X < 3.5) = F(3.5) - F(3) = [1 - e^{-3.5/1.4}] - [1 - e^{-3/1.4}] = 0.0035$$

and

$$P(X > 3) = 1 - F(3) = e^{-3/1.4} = 0.117$$

4-8 Exponential Distribution

Example 4-22 (continued)

Therefore,

$$P(X < 3.5 \mid X > 3) = 0.035/0.117 = 0.30$$

After waiting for 3 minutes without a detection, the probability of a detection in the next 30 seconds is the same as the probability of a detection in the 30 seconds immediately after starting the counter. The fact that you have waited 3 minutes without a detection does not change the probability of a detection in the next 30 seconds.

Example: *Reliability*

The time till failure of an electronic component has an Exponential distribution and it is known that 10% of components have failed by 1000 hours.

(a) What is the probability that a component is still working after 5000 hours?

(b) Find the mean and standard deviation of the time till failure.

Answer

Let Y = time till failure in hours; $f(y) =$

(a) First we need to find $P(Y \leq 1000) = \int_0^{1000} \nu e^{-\nu y}$

$$= [-e^{-\nu y}]_0^{1000} \quad = 1 - e^{-1000\nu}$$

$$P(Y \leq 1000) = 0.1 \Rightarrow \quad 1 - e^{-1000\nu} = 0.1$$

$$\Rightarrow e^{-1000\nu} = 0.9$$

$$\Rightarrow -1000\nu = \ln 0.9 = \; -0.10536 \quad \Rightarrow \nu \approx 1.05 \times 10^{-4}$$

If Y is the time till failure, the question asks for $P(Y > 5($

$$P(Y > 5000) = \int_{5000}^{\infty} \nu e^{-\nu y} dy$$

$$= [-e^{-\nu y}]_{5000}^{\infty} \qquad = e^{-5000\nu} \approx 0.59$$

(b) Find the mean and standard deviation of the time till failure.

Answer
:

Mean = $1/\nu$ = 9491 hours

Standard deviation = $\sqrt{\text{Variance}}$

$$= \sqrt{\frac{1}{\nu^2}} = 1/\nu \ = \ 9491 \text{ hours}$$

4-8 Exponential Distribution

Example 4-22 illustrates the **lack of memory property** of an exponential random variable and a general statement of the property follows. In fact, the exponential distribution is the only continuous distribution with this property.

Lack of Memory Property

For an exponential random variable X,

$$P(X < t_1 + t_2 \mid X > t_1) = P(X < t_2) \qquad (4\text{-}16)$$

4-8 Exponential Distribution

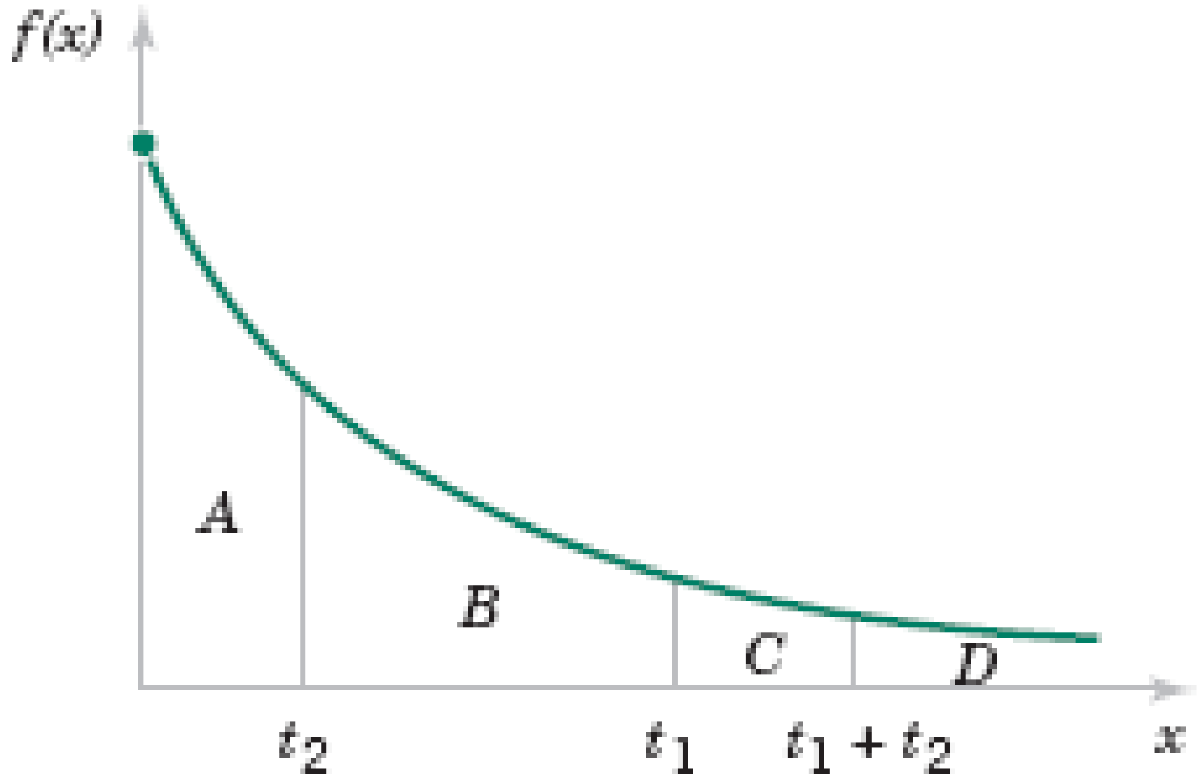


Figure 4-24 Lack of memory property of an Exponential distribution.

4-9 Erlang and Gamma Distributions

Erlang Distribution

The random variable *X* that equals the interval length until *r* counts occur in a Poisson process with mean λ > 0 has and **Erlang random variable** with parameters λ and *r*. The probability density function of *X* is

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$$

for $x > 0$ and $r = 1, 2, 3, \ldots$

4-9 Erlang and Gamma Distributions

Gamma Distribution

The **gamma function** is

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx, \quad \text{for } r > 0 \qquad (4\text{-}17)$$

4-9 Erlang and Gamma Distributions

Gamma Distribution

The random variable X with probability density function

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, \quad \text{for } x > 0 \tag{4-18}$$

has a **gamma random variable** with parameters $\lambda > 0$ and $r > 0$. If r is an integer, X has an Erlang distribution.

4-9 Erlang and Gamma Distributions

Gamma Distribution

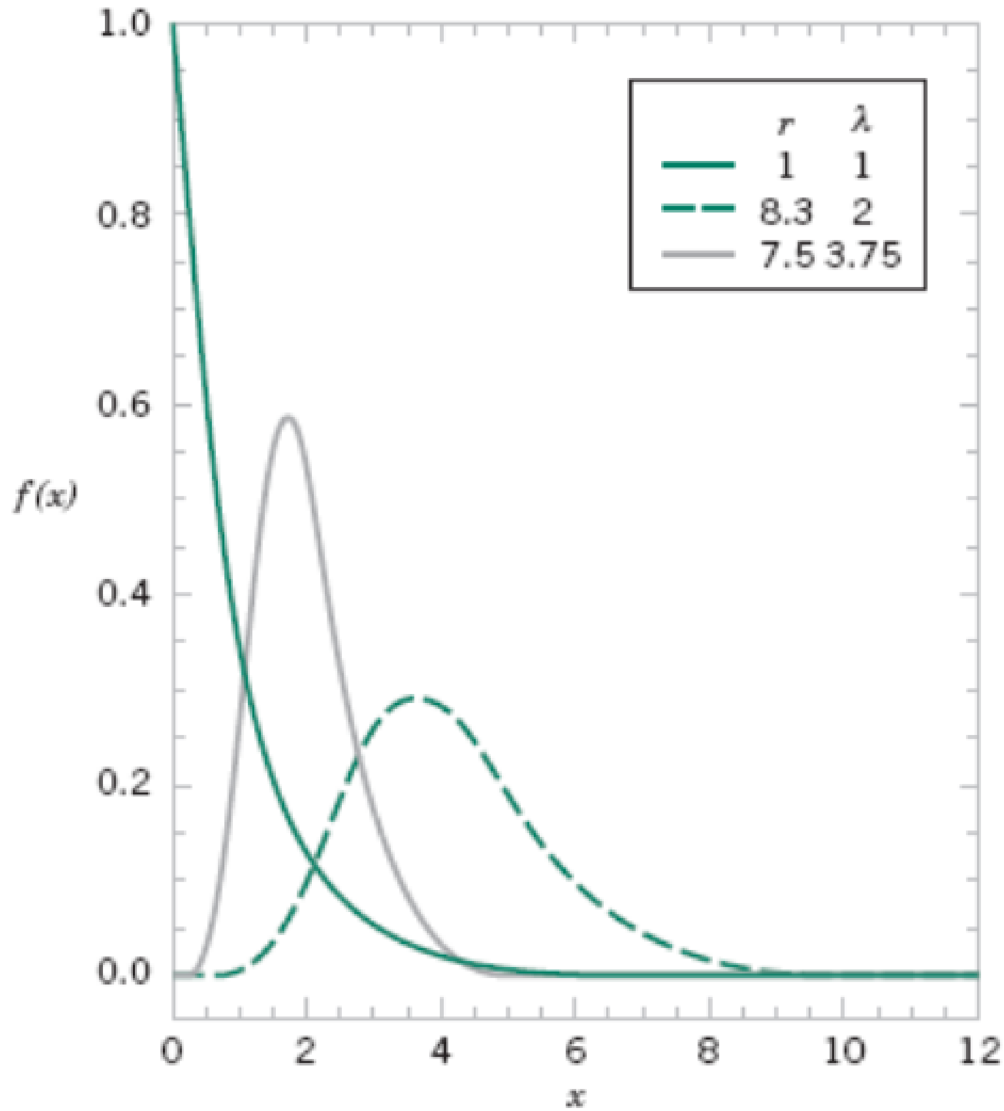


Figure 4-25 Gamma probability density functions for selected values of *r* and λ.

4-9 Erlang and Gamma Distributions

Gamma Distribution

If X is a **gamma random variable** with parameters λ and r,

$$\mu = E(X) = r/\lambda \quad \text{and} \quad \sigma^2 = V(X) = r/\lambda^2 \tag{4-19}$$

4-10 Weibull Distribution

Definition

The random variable X with probability density function

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right], \qquad \text{for } x > 0 \qquad (4\text{-}20)$$

is a **Weibull random variable** with scale parameter $\delta > 0$ and shape parameter $\beta > 0$.

4-10 Weibull Distribution

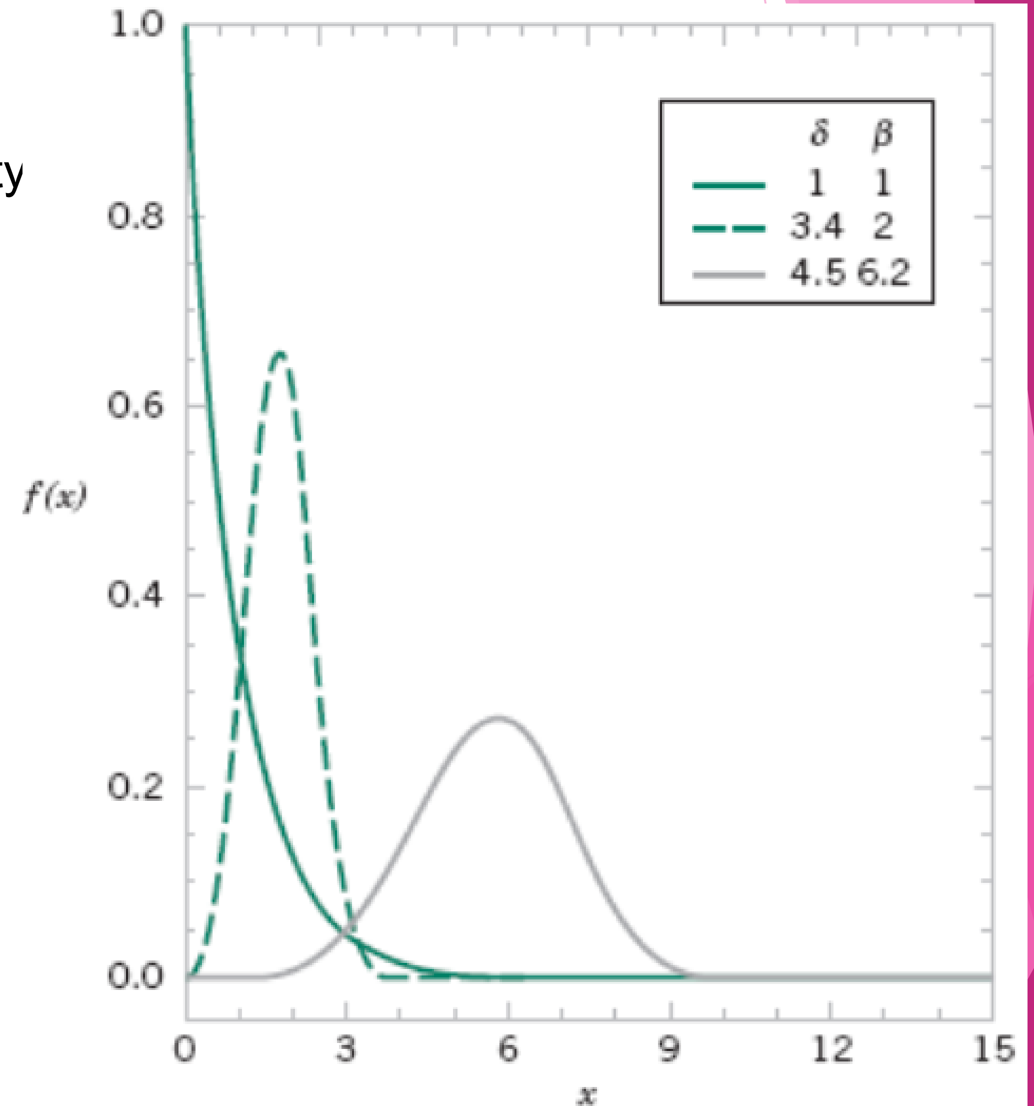


Figure 4-26 Weibull probability density functions for selected values of α and β.

4-10 Weibull Distribution

If X has a Weibull distribution with parameters δ and β, then the cumulative distribution function of X is

$$F(x) = 1 - e^{-\left(\frac{x}{\delta}\right)^{\beta}} \tag{4-21}$$

If X has a Weibull distribution with parameters δ and β,

$$\mu = E(X) = \delta\Gamma\left(1 + \frac{1}{\beta}\right) \quad \text{and} \quad \sigma^2 = V(X) = \delta^2\Gamma\left(1 + \frac{2}{\beta}\right) - \delta^2\left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \tag{4-21}$$

4-10 Weibull Distribution

Example 4-25

The time to failure (in hours) of a bearing in a mechanical shaft is satisfactorily mode
Weibull random variable with $\beta = 1/2$, and $\delta = 5000$ hours. Determine the mean tin
failure.

From the expression for the mean,

$$E(X) = 5000\Gamma[1 + (1/0.5)] = 5000\Gamma[3] = 5000 \times 2! = 10{,}000 \text{ hours}$$

Determine the probability that a bearing lasts at least 6000 hours. Now

$$P(x > 6000) = 1 - F(6000) = \exp -\left[\left(\frac{6000}{5000}\right)^{1/2}\right] = e^{-1.095} = 0.334$$

Consequently, only 33.4% of all bearings last at least 6000 hours.

4-11 Lognormal Distribution

Let W have a normal distribution mean θ and variance ω^2; then $X = \exp(W)$ is a **lognormal random variable** with probability density function

$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} \exp\left[-\frac{(\ln x - \theta)^2}{2\omega^2}\right] \qquad 0 < x < \infty$$

The mean and variance of X are

$$E(X) = e^{\theta + \omega^2/2} \qquad \text{and} \qquad V(X) = e^{2\theta + \omega^2}\left(e^{\omega^2} - 1\right) \qquad (4\text{-}22)$$

The parameters of a lognormal distribution are θ and ω^2, but care is needed to interpret that these are the mean and variance of the normal random variable W. The mean and variance of X are the functions of these parameters shown in (4-22). Figure 4-27 illustrates lognormal distributions for selected values of the parameters.

4-11 Lognormal Distribution

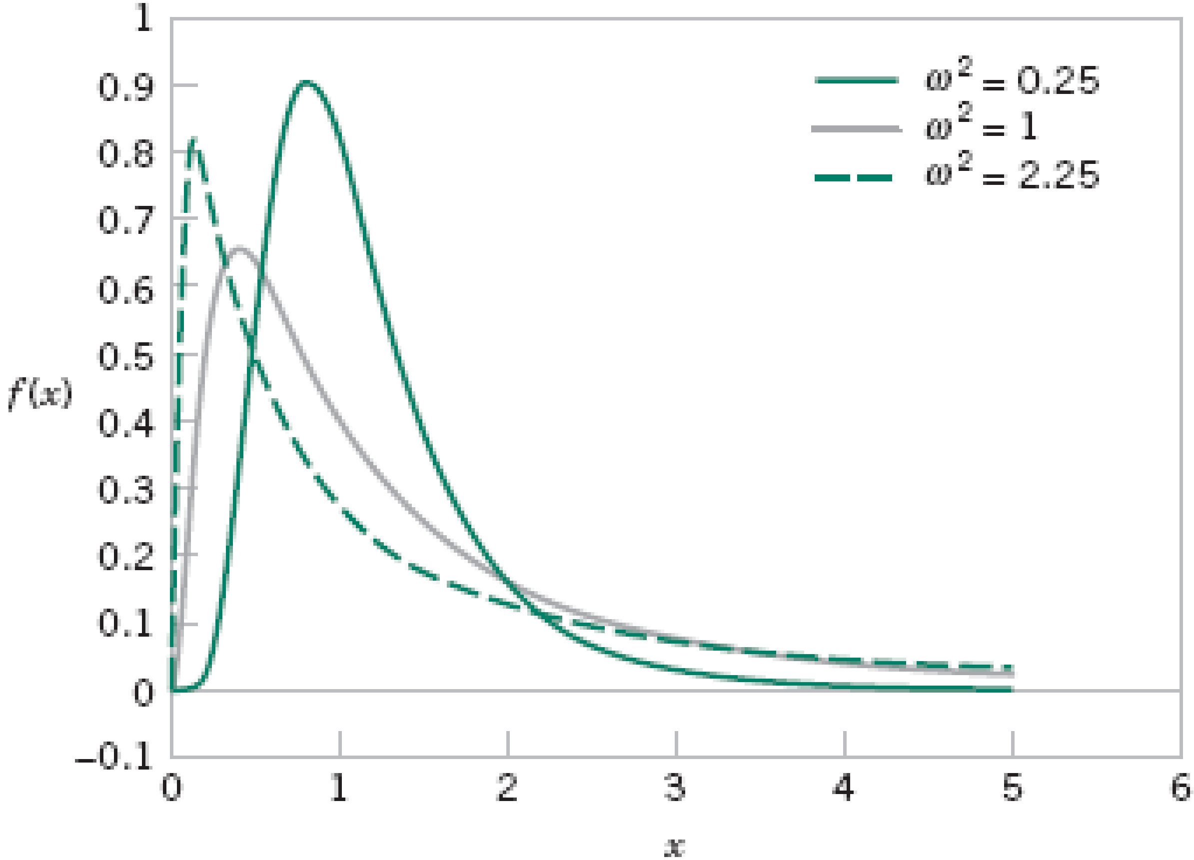


Figure 4-27 Lognormal probability density functions with $\theta = 0$ for selected values of ω^2.

4-11 Lognormal Distribution

Example 4-26

The lifetime of a semiconductor laser has a lognormal distribution with $\theta = 10$ hours and $\omega = 1.5$ hours. What is the probability the lifetime exceeds 10,000 hours?

From the cumulative distribution function for X

$$P(X > 10{,}000) = 1 - P[\exp(W) \le 10{,}000] = 1 - P[W \le \ln(10{,}000)]$$
$$= \Phi\left(\frac{\ln(10{,}000) - 10}{1.5}\right) = 1 - \Phi(-0.52) = 1 - 0.30 = 0.70$$

4-11 Lognormal Distribution

Example 4-26 (continued)

What lifetime is exceeded by 99% of lasers? The question is to determine x such that $P(X > x) = 0.99$. Therefore,

$$P(X > x) = P[\exp(W) > x] = P[W > \ln(x)] = 1 - \Phi\left(\frac{\ln(x) - 10}{1.5}\right) = 0.99$$

From Appendix Table II, $1 - \Phi(z) = 0.99$ when $z = -2.33$. Therefore,

$$\frac{\ln(x) - 10}{1.5} = -2.33 \quad \text{and} \quad x = \exp(6.505) = 668.48 \text{ hours.}$$

4-11 Lognormal Distribution

Example 4-26 (continued)

Determine the mean and standard deviation of lifetime. Now,

$$E(X) = e^{\theta + \omega^2/2} = \exp(10 + 1.125) = 67{,}846.3$$

$$V(X) = e^{2\theta + \omega^2}(e^{\omega^2} - 1) = \exp(20 + 2.25)[\exp(2.25) - 1] = 39{,}070{,}059{,}886.6$$

so the standard deviation of X is 197,661.5 hours. Notice that the standard deviation of lifetime is large relative to the mean.

IMPORTANT TERMS AND CONCEPTS

Chi-squared distribution
Continuous uniform distribution
Continuity correction
Cumulative probability distribution function-continuous random variable
Erlang distribution
Exponential distribution
Gamma distribution
Lack of memory property-continuous random variable
Lognormal distribution
Mean-continuous random variable
Mean-function of a continuous random variable
Normal approximation to binomial and Poisson probabilities
Normal distribution
Probability density function
Probability distribution-continuous random variable
Standard deviation-continuous random variable
Standard normal distribution
Standardizing
Variance-continuous random variable
Weibull distribution