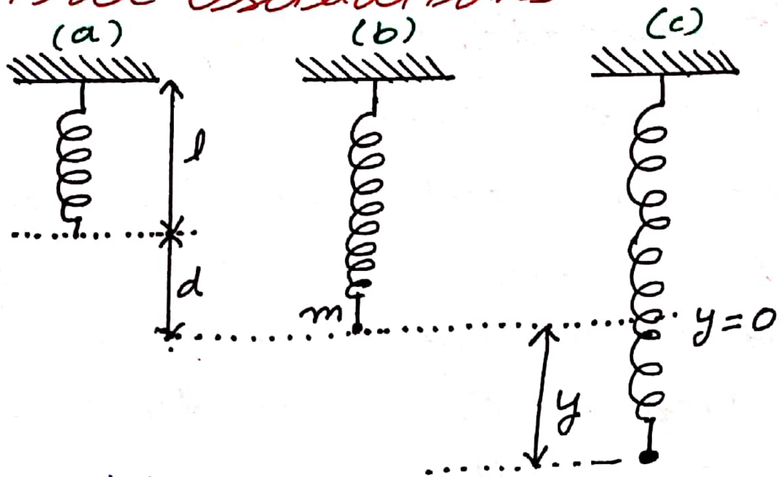
Modelling Mass-Spring System – Free Oscillations



(a) Unstretched
(b) Static equilibrium
(c) System in motion

Consider a spring with unstretched length l and spring modulus k, a measure of the stiffness of the spring. The spring is suspended vertically from a fixed support. Let a body of mass m is attached at the lower end of the spring, (assuming m to be large so that the mass of the spring is neglected), which stretches the spring by d units over its natural length before coming to rest in its equilibrium position. Next, let us suppose that the body is then displaced vertically (up, or down) by distance y_0 units and is released possibly with an initial velocity. We want to study the motion of this spring-mass system, refer to above figure.

The spring equation

Let $y(t)$ be the displacement of the object at time t from the equilibrium position, say $y=0$ and select the downward direction to be positive. Consider the forces acting on the body of mass m at time t.

The force due to gravity which pulls it downward is of magnitude mg.

The force the spring exerts on the body at time t, due to Hooke's law, is of magnitude ky. At equilibrium position the force of the spring is of magnitude $-kd$, sign is negative since the force acts upward. If the body is displaced downward by a distance y from the equilibrium position, then an additional force $-ky$ is exerted on it at time t.

Thus, the total force on the body due to gravity and the spring is

$$mg - kd - ky.$$

At equilibrium position $(y=0)$ this force is zero, and hence $mg = kd$.

Thus the net force acting on the object is therefore just $F_1 = -ky$, an upward force. It is a restoring force and has the tendency to restore the system, i.e., pull the body back to the equilibrium position $y=0$.

Next, every system is subjected to some damping or retarding forces, which may be air resistance or the viscosity of the medium if the body is suspended in some fluid such as oil etc. In case these forces are not negligible then we need to take the corresponding damping into account. Experiments show that the magnitude of the damping forces at any instant t is proportional to the velocity $y'(t)$ and direction is opposite to the instantaneous motion. Thus, the dumping force is $F_2 = -cy'$, where $c > 0$ is some constant, called the damping constant.

Also there may be a ~~de~~driving force of magnitude $F_3 = f(t)$ acting on the body, and then the total external force on the body has the magnitude

$$F_1 + F_2 + F_3 = -ky - cy' + f(t) \quad \text{——} \quad (1)$$

Using Newton's second law of motion, the equation of the mass-spring system is,

$$my'' = -ky - cy' + f(t)$$

or

$$my'' + ky + cy' = f(t) \quad \text{——} \quad (2)$$

This is called the spring equation.

In the absence of the external force, that is $f(t) = 0$, the eq^n (2) becomes homogeneous one given by $my'' + cy' + ky = 0$, and the motion of the mass-spring system are called the free motions/oscillations.

In the presence of the external force the motions of the spring-mass system are called the forced motions.

Mass - Spring System: Free Motions

In case of free motions the deriving force $f(t)=0$, hence the spring eq^n is

$$my'' + cy' + ky = 0 \qquad \text{—— (3)}$$

Now we study the following cases

A) Undamped system

In case the damping forces are negligible, then $c=0$, the eq^n (3) ~~beco~~ becomes

$$my'' + ky = 0 \qquad \text{—— (4)}$$

The general solution of (4)

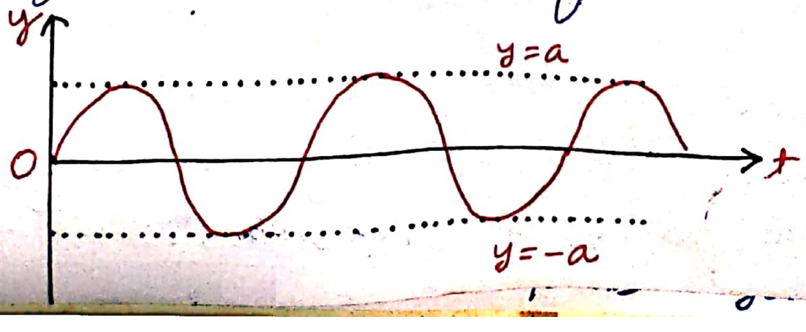
$$\boxed{y(t) = c_1 \cos \omega t + c_2 \sin \omega t}, \qquad \omega = \sqrt{\frac{k}{m}} \quad \text{— (5)}$$

In case the body is first pulled to a point at a distance a units from the position of static equilibrium and is released, then $y(0)=a$ and $y'(0)=0$.

Using these initial conditions in eq^n (5) gives $c_1 = a$ and $c_2 = 0$, and hence eq^n (5)

$$y(t) = a \cos \omega t.$$

The motion is simple harmonic motion with period $\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$ with amplitude a as shown in below figure. The curve touches the line $y = \pm a$ when ωt is an integral multiple of π.



B) Damped system

If the damping forces are not negligible, then $c \neq 0$ and the spring eq^n is

$$my'' + cy' + ky = 0 \qquad \text{— (6)}$$

The corresponding characteristic eq^n is

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0, \text{ with roots}$$

$$\lambda_1 = -\alpha + \beta \text{ and } \lambda_2 = -\alpha - \beta$$

$$\text{where } \alpha = \frac{c}{2m} \;\&\; \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

As is evident, the form of the solution of (6) will depend on the mass m, the amount of damping and the stiffness of the spring. We have the following three cases:

Case I: $c^2 > 4mk$: Two distinct real roots λ_1, λ_2 (over damping)

Case II: $c^2 = 4mk$: Two equal and real roots (critical damping)

Case III: $c^2 < 4mk$: Complex conjugate roots (under damping)

Case I: Over damping

If the damping constant c is so large that $c^2 > 4mk$, then λ_1 & λ_2 are two real and distinct roots and the general solution of the equation (6) is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \qquad \text{——— (7)}$$

Obviously $\lambda_2 = -\alpha - \beta$ is negative, & also

$$\lambda_1 = -\alpha + \beta = -\frac{c}{2m} + \frac{\sqrt{c^2-4mk}}{2m} < -\frac{c}{2m} + \frac{c}{2m} = 0$$

Since both λ_1 & λ_2 are negative, therefore, the terms in (7) tend to be zero as t approaches infinity. Thus the body does not oscillate and after a sufficiently long time the mass will be at its static equilibrium position $y=0$, a case of over damping.

Case II : Critical damping

If $c^2 = 4mk$, then $\beta = 0$ and the two roots of the equation (6) are $\lambda_1 = \lambda_2 = -\frac{c}{2m} = -\alpha$. The general solution is

$$y(t) = (c_1 + c_1 t)e^{-\alpha t} \quad \text{—} \ (8)$$

Here also $y(t) \to 0$ as $t \to \infty$, as in the case of over damping. This case marks the boundary between the over damped ~~ing~~ behaviour discussed above and the oscillatory behaviour to be discussed next.

Case III : Under damping

If the damping coeff. is so small that $c^2 < 4mk$, then the roots of the eq^n (6) are complex conjugate, say $\lambda_1 = -\alpha + i\beta^*$ and $\lambda_2 = -\alpha - i\beta^*$, where $\alpha = \frac{c}{2m}$ and $\beta^* = \frac{1}{2m}\sqrt{4km - c^2}$. The general solution of eq^n (6) is

$$y(t) = e^{-\alpha t}(c_1 \cos\beta^* t + c_2 \sin\beta^* t) \quad \text{—} \ (9)$$

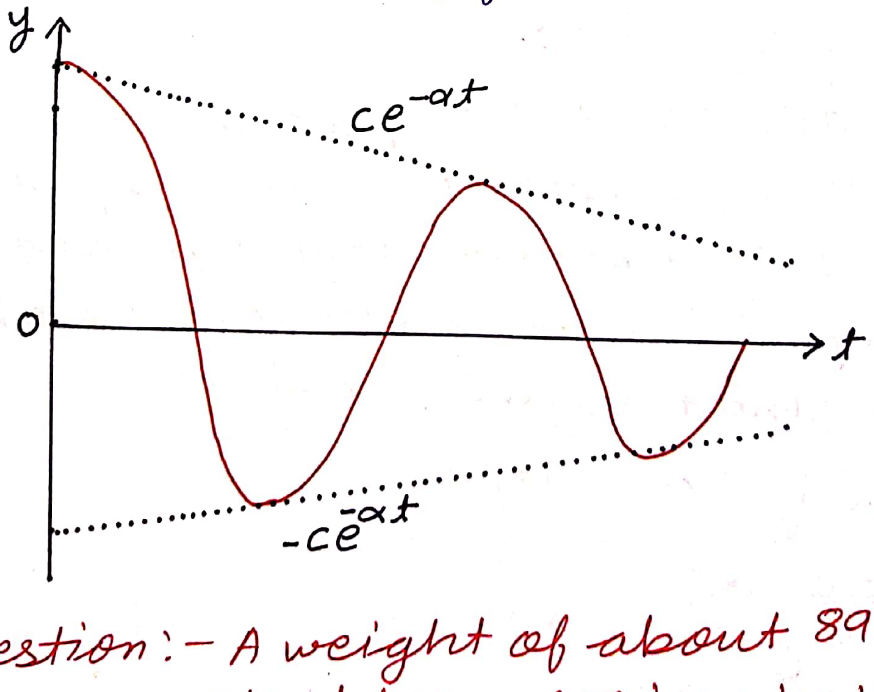
equation of motion ...

Since $\alpha > 0$, thus $y(t) \to 0$ as $t \to \infty$. However, the motion is now oscillatory because of the sine and cosine terms in the solution. But it is not the periodic one because of the exponential factors which causes the amplitude of the oscillations to decay to zero as t becomes sufficiently large. The eq^n (9) can be expressed as

$$y(t) = ce^{-\alpha t}\cos(\beta^\circ t - \theta), \text{ where } c = \pm\sqrt{c_1^2 + c_2^2}$$

$$\& \ \theta = \tan^{-1}\frac{c_2}{c_1}$$

Thus the solution curve lies between $\pm ce^{-\alpha t}$, touching these curves when $(\beta^\circ t - \theta)$ is an integral multiple of π, as shown in below figure.



Question:- A weight of about 89.0 Newton stretches a spring by 10.0 cm. How many cycles per second will this mass-spring system execute? What will be its equation of motion in case the weight is pulled down by 15.00 cm from its position of static equilibrium and then released, ignore the damping forces. How does the motion will change if the system has damping given by

(i) $c = 200.0$ kg/sec

(ii) $c = 179.81$ kg/sec

(iii) $c = 100.0$ kg/sec ?

Solution:- If k is the coefficient of stiffness for the spring, then using Hooke's law, we have

$$89.0 = 0.1k \qquad (10.0 \text{ cm} = 0.1 \text{ m})$$

hence $k = 890$ N/m. Also mass,

$$m = \frac{w}{g} = \frac{89.0}{9.80} = 9.082 \text{ kg}.$$

Thus frequency, $\frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{890}{9.082}}$

$$\frac{\omega}{2\pi} = 1.576.$$

If $y(t)$ denotes the displacement of the body at any instant from its position of static equilibrium, then ignoring the damped forces, motion is described by the intial value problem

$$\left.\begin{aligned} & y'' + \omega^2 y = 0 \\ & y(0) = 0.15 \\ & y'(0) = 0 \\ & \omega = \sqrt{\frac{k}{m}} = 9.899 \end{aligned}\right\} \quad \text{--- } (1)$$

We can check that the solution of (1) is

$y(t) = 0.1500 \cos 9.899t.$

In case the system has damping, then the spring eq^n is $my'' + cy' + ky = 0.$

(i) When $c = 200.0$, the springequation for $m = 9.082$ and $k = 890$ becomes

$$9.082\, y'' + 200.0\, y' + 890.0\, y = 0 \quad — (2)$$

The characteristic eq^n has the roots

$\lambda = -11.01 \pm 4.82 = -6.190, -15.83.$

The general solution is

$$y(t) = c_1 e^{-6.19t} + c_2 e^{-15.83t} \quad — (3)$$

Using the initial conditions $y(0) = 0.15$, and $y'(0) = 0$. Eq^n (3) becomes

$$y(t) = 0.2463 e^{-6.19t} - 0.0963 e^{-15.83t}$$

Thus, $y(t)$ tends to zero as $t \to \infty$. It is the case of over damping.

(ii) When $c = 179.81$, then $c^2 = 4mk$ and hence the characteristic eq^n has the double root $\lambda = -9.899$. Therefore the sol^n is

$$y(t) = (c_1 + c_2 t) e^{-9.899t} \quad — (4)$$

Using the initial conditions $y(0) = 0.15$, and $y'(0) = 0$, Eq^n (4) becomes

$$y(t) = (0.150 + 1.48t) e^{-9.899t}$$

In this case also $y(t) \to 0$ as $t \to \infty$.

It is the case of critical damping.

(iii) When $c = 100$, the roots of the characteristic equation are $\lambda = -5.506 \pm 8.227i$, the complex conjugate. The general sol^n is

$$y(t) = e^{-5.506t}(c_1 \cos 8.227t + c_2 \sin 8.227t) \quad (5)$$

Using the initial conditions $y(0) = 0.15$ and $y'(0) = 0$, eq^n (5) becomes

$$y(t) = e^{-5.506t}(0.1500 \cos 8.227t + 0.1004 \sin 8.227t)$$

$$= 0.1805\, e^{-5.506t} \cos(8.227t - 0.981)$$

It is a case of damped oscillation with frequency, $\frac{\omega}{2\pi} = \frac{8.227}{2\pi} = 1.309$