Numerical Integral

The Process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called numerical integration. This process when applied to a function of a single variable, is known as quadrature.

Newton-Cote's Quadrature Formula

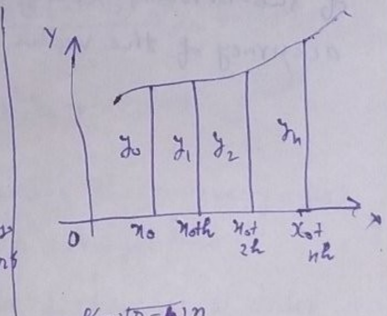
Let $I = \int_a^b f(x)\,dx$ or $\int_a^b y\,dx$, $\quad \because \boxed{y = f(x)}$

Where y takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x = x_0, x_1, x_2, \ldots, x_n$.

Let the interval of integration (a, b) be divided into n equal sub-intervals, each of width $h = \frac{b-a}{n}$ so that

$x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh = b$

$$\therefore\ I = \int_{x_0}^{x_0+nh} f(x)\,dx$$



Since any x is given by $x = x_0 + rh$

$dx = 0 + h\,dr$

Limit change: $x_0 = x_0 + rh$, $r = 0$; $x_0 + nh = x_0 + rh \Rightarrow \boxed{r = n}$

$$\therefore\ I = \int_0^n f(x_0 + rh)\cdot h\,dr$$

$$= h\int_0^n f(x_0 + rh)\,dr$$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{\lfloor 2}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{\lfloor 3}\Delta^3 y_0 + \cdots\right]dr$$

When $r = \frac{x - x_0}{h}$ or $\boxed{x = x_0 + rh}$

[by Newton Forward Interpolation Formula]

$$= h\left[y_0 r + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 + \cdots\right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \cdots\right.$$

$$= hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n}{12}(2n-3)\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right. \longrightarrow (1)$$

This is a general quadrature formula and is known as Newton-Cote's quadrature formula.

Note: If we put $n = 1, 2, 3$, we get Trapezoidal rule, Simpson's one third and Simpson three eight rule respectively.

TRAPEZOIDAL RULE ($n=1$)

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots+y_{n-1})\right]}$$

Which is known as Trapezoidal rule. By Increasing the number of subintervals, thereby making h very small, we can improve the accuracy of the value of the given interval.

Evaluate $\int_{.6}^{2} y\,dx$, Where y is given by the following table:

x :	.6	.8	1.0	1.2	1.4	1.6	1.8	2.0
y :	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45

by using Trapezoidal rule.

Solⁿ By Trapezoidal rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$$\therefore \int_{.6}^{2} y\,dx = \frac{.2}{2}\left[(y_0+y_7)+2(y_1+y_2+y_3+y_4+y_5+y_6)\right]$$

$$= .1\left[(1.23+12.45)+2(1.58+2.03+4.32+6.25+8.36+10.23)\right]$$

$$= .1\left[13.68+65.54\right] = 7.922 \text{ Ans.}$$

Q.2. Find from the following table, the area bounded by the curve and the x-axis from $x = 7.47$ to 7.52.

x :	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$:	1.93	1.95	1.98	2.01	2.03	2.06

Solⁿ By Trapezoidal Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$$\int_{7.47}^{7.52} f(x)\,dx = \frac{.01}{2}\left[(1.93+2.06)+2(1.95+1.98+2.01+2.03)\right]$$

$$= .09965 \text{ Ans}$$

Q. Use Trapezoidal rule to evaluate $\int_1^2 \frac{dx}{x^2}$ corresponding to 5 interval.

Sol.^{n} $h = \frac{2-1}{5} = \frac{1}{5}$

x	1	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	$9/5$	$\frac{10}{5}=2$
y	1	$\frac{25}{36}$	$\frac{25}{49}$	$\frac{25}{64}$	$\frac{25}{81}$	$\frac{25}{100}$
	y_0	y_1	y_2	y_3	y_4	y_n

By Trapezoidal rule

$$\int_1^2 \frac{dx}{x^2} = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$= \frac{1}{10}\left[\left(1+\frac{25}{100}\right) + 2\left(\frac{25}{36}+\frac{25}{49}+\frac{25}{64}+\frac{25}{81}\right)\right]$$

$$= \frac{1}{10}\left[1.25 + 2(.694 + .5102 + .3906 + .3086)\right] = \underline{\underline{.5056}}\ \text{A}$$

Q. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule taking $h = \frac{1}{4}$

Soln. Here $f(x) = \frac{1}{1+x^2}$, $h = \frac{1}{4}$

$\therefore x$:	0	$1/4$	$\frac{1}{2}$	$\frac{3}{4}$	1
$y = f(x)$:	1	$\frac{16}{17}$	$.8$	$.64$	$.5$
	y_0	y_1	y_2	y_3	y_n

By Trapezoidal rule

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3)\right]$$

$$= \frac{1}{4\times 2}\left[(1+.5) + 2\left(\frac{16}{17} + .8 + .64\right)\right]$$

$$= \frac{1}{8}\left[1.5 + 4.7623\right] = \underline{.7827}$$

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by Using Trapezoidal rule.

Solⁿ Divide the Interval $(0,6)$ into six parts each of width $h=1$

Let $f(x) = \frac{1}{1+x^2}$ $\therefore h = \frac{6-0}{6} = 1$

x :	0	1	2	3	4	5	6
$y=f(x)$:	1	.5	.2	.1	.0588	.0385	.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

We know that by Trapezoidal rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+y_4+\cdots+y_{n-1})\right]$$

$$= \frac{1}{2}\left[(1+.027) + 2(.5+.2+.1+.0588+.0385)\right]$$

$$= \frac{1}{2}\left[1.027 + 1.7946\right] = 1.4108 \text{ Ans}$$

Q. Use the Trapezoidal rule to estimate the integral $\int_0^2 e^{x^2}\,dx$, taking the number 10 intervals.

$$h = \frac{2-0}{10} = \frac{2}{10} = .2$$

x :	0	.2	.4	.6	.8	1	1.2	1.4	1.6
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
y :	1	1.0	1.1735	1.4333	1.8964	2.1782	4.2206	7.0993	12.9358

x :	1.8	2.0
	y_9	y_{10}
y :	25.5337	54.5981

$$\therefore \int_0^2 e^{x^2}\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$$= \frac{0.2}{2}\Big[(1+54.5981)+2(1.0408+1.1735+1.4333)+1.8964+2.71$$
$$+4.2246+7.0993+12.9358+25.5337)\Big]$$

$$= 17.0621$$ Ans