UNIT II LAPLACE TRANSFORM

Fourier Series :- Any Non Sinusoidal Signal Can be expressed in terms of number of Sinusoidal Signals of different frequency (Frequency is multiple of fundamental frequency) and different amplitude.

Fourier Series is applicable to periodic ~~Signals~~ and Stationary Signals.

Note : Stationary Signals are those signals in which frequency components does not change with time. In other words, frequency of harmonic component does not change with time.

Harmonic components are sine wave whose frequency is integral multiple of fundamental frequency.

Any periodic function $f(t)$ can be represented in Fourier Series as

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

where $a_0 = \frac{1}{T}\int_0^T f(t)\, dt$; $a_n = \frac{2}{T}\int_0^T f(t) \cos n\omega t \, dt$

$b_n = \frac{2}{T}\int_0^T f(t) \sin n\omega t$. ; T is time period.

Fourier Series is very important in electrical engineering as it tells which frequency components are present in the Signal or waveform (For ex. Current waveform).

It is a discrete Spectrum. It gives the amplitude of discrete harmonic components.

Fourier Transform : Non periodic waveform is represented by Fourier Transform. A non periodic waveform is a single waveform of infinite time period.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \quad \rightarrow \text{Fourier Transform.}$$

$F(\omega)$ is fourier transform of $f(t)$.
$F(\omega)$ is the frequency domain representation of time domain signal $f(t)$. $F(\omega)$ is continuous frequency spectrum.

<u>Application of Fourier Transform.</u>

(i) It is used widely to study communication system.
(ii) Fourier Transform is used as mathematical transformation to solve differential equation.
This tool convert integro-differential equation into an algebraic equation in 'ω' domain, which can be solve easily.

<u>Limitation in Fourier Transform</u>

$f(t)$ is fourier transformable if the following condition is satisfied $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$

Function such as ramp, parabolic are not fourier transformable because integral $\int_{-\infty}^{\infty} |f(t)|\, dt$ does not converge.
In order to handle such functions, we modify our transformation by introducing a convergence factor e^{-at} where a is real number

$$\text{New transformation} = \int_{0}^{\infty} f(t)\, e^{-at}\, e^{-j\omega t}\, dt$$

The lower limit of integration is forced to be taken as zero rather than $-\infty$, since for $a > 0$, convergence factor e^{-at} will diverge when $t \rightarrow -\infty$.

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This does not put any serious limitation upon the new transformable since in the usual studies, time reference is normally chosen at the instant $t=0$.

This new transformation is Laplace transformation.

Laplace Transformation

- $F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$ → Laplace Transform

where $s = a + j\omega$ → ~~st~~ s is Complex Variable.

- Condition for Laplace transform to exist is

$$\int_{-\infty}^{\infty} |f(t)\, e^{-st}|\, dt < \infty$$

- **Property of Laplace Transform**

Let $F(s)$ be the Laplace transform of $f(t)$.

1. **Linear Property**

$$\mathcal{L}[k f(t)] \longrightarrow k F(s)$$

$$\mathcal{L}[f_1(t) + f_2(t)] \longrightarrow F_1(s) + F_2(s)$$

2. **Time shifting Property**

$$\mathcal{L}[f(t-a).u(t-a)] \longrightarrow e^{-as} F(s)$$

3. **Frequency shifting Property**

$$\mathcal{L}[e^{-at} f(t)] \longrightarrow F(s+a)$$

$$\mathcal{L}[e^{at} f(t)] \longrightarrow F(s-a)$$

4. **Scaling Property.**

$$\mathcal{L}[f(at)] \longrightarrow \frac{1}{a} F\left(\frac{s}{a}\right)$$

5. Laplace of Derivative of $f(t)$

$$\mathcal{L}\left[\frac{d f(t)}{dt}\right] \longrightarrow s F(s) - f(0^+)$$

$$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] \longrightarrow s^2 F(s) - s f(0^+) - f'(0^+)$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] \longrightarrow s^n F(s) - s^{n-1} f(0^+) - s^{n-2} f'(0^+) - s^{n-3} f''(0^+) - \dots$$

6. Laplace of integral of $f(t)$

$$\mathcal{L}\left[\int_0^t f(t)\, dt\right] \longrightarrow \frac{F(s)}{s}$$

7. $$\mathcal{L}\left[t^n f(t)\right] \longrightarrow (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\mathcal{L}\left[t f(t)\right] \longrightarrow -\frac{d}{ds} F(s)$$

8. $$\mathcal{L}\left[\frac{f(t)}{t}\right] \longrightarrow \int_s^\infty F(s)\, ds$$

9. Laplace of Periodic function.

Laplace of Periodic function $\longrightarrow \dfrac{1}{1 - e^{-Ts}} \times$ Laplace of First cycle

$f(t)$ is periodic with time period T

$$\mathcal{L}(f(t)) \longrightarrow \frac{1}{1 - e^{-Ts}} F_1(s) \quad \text{where } F_1(s) = \int_0^T f(t)\, e^{-st}\, dt$$

10. $$\mathcal{L}\left[f(t).u(t)\right] \longrightarrow F(s)$$

Initial Value Theorem

$$f(0^+) = \lim_{s \to \infty} [s F(s)]$$

If $F(s)$ is laplace of $f(t)$, then initial value of $f(t)$ (i.e $f(0^+)$) is given by above formula.

Proof: $$\int_0^\infty \frac{d f(t)}{dt} e^{-st} dt = s F(s) - f(0^+)$$

Let $s \to \infty$

$$\lim_{s \to \infty} \int_0^\infty \frac{d f(t)}{dt} e^{-st} dt = \lim_{s \to \infty} [s F(s) - f(0^+)]$$

since s is not a function of t. Therefore, it is allowable to let $s \to \infty$ before integrating. Then left hand side vanishes.

$$\therefore \quad 0 = \lim_{s \to \infty} [s F(s) - f(0^+)]$$

$$f(0^+) = \lim_{s \to \infty} [s F(s)]$$

Final Value Theorem

If $F(s)$ is laplace of $f(t)$, then final value of $f(t)$ is given by

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} [s F(s)]$$

Proof: $$\int_0^\infty \frac{d f(t)}{dt} e^{-st} dt = s F(s) - f(0^+)$$

Let $s \to 0$

$$\lim_{s \to 0} \int_0^\infty \frac{d f(t)}{dt} e^{-st} dt = \lim_{s \to 0} [s F(s) - f(0^+)]$$

$$\int_0^\infty d\, f(t) = \lim_{s \to 0} [s\, F(s) - f(0^+)]$$

$$f(\infty) - \cancel{f(0^+)} = \lim_{s \to 0} [s\, F(s)] - \cancel{f(0^+)}$$

$$f(\infty) = \lim_{s \to 0} [s\, F(s)]$$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} [s\, F(s)]$$

Laplace transform of some common function

	$f(t)$		$F(s)$
1.	1	$\longrightarrow$	$\frac{1}{s}$
2.	K	$\longrightarrow$	$\frac{K}{s}$
3.	$u(t)$	$\longrightarrow$	$\frac{1}{s}$
4.	t	$\longrightarrow$	$\frac{1}{s^2}$
5.	t^m	$\longrightarrow$	$\frac{m!}{s^{m+1}}$
6.	$\delta(t)$	$\longrightarrow$	1
7.	e^{at}	$\longrightarrow$	$\frac{1}{s-a}$
8.	e^{-at}	$\longrightarrow$	$\frac{1}{s+a}$
9.	$\sin \omega t$	$\longrightarrow$	$\frac{\omega}{s^2+\omega^2}$
10	$\cos \omega t$	$\longrightarrow$	$\frac{s}{s^2+\omega^2}$

Q Find laplace of $U(t-2)$

Sol. $\int_0^\infty U(t-2)\, e^{-st}\, dt = \int_0^2 0\, e^{-st}\, dt + \int_2^\infty 1\, e^{-st}\, dt$

$= \int_2^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_2^\infty = \left[0 - \left(\frac{e^{-2s}}{-s}\right)\right]$

$= \frac{e^{-2s}}{s}$

$$\boxed{\mathcal{L}[U(t-a)] \longrightarrow \frac{e^{-as}}{s}}$$

Q Find laplace of
(i) $f(t)$
(ii) $f(t-a)$
(iii) $f(t-a).U(t-a)$

Let $f(t) = t$

Sol (i) $\mathcal{L}[f(t)] = \mathcal{L}[t] = \frac{1}{s^2}$

(ii) $\mathcal{L}[f(t-a)] = \mathcal{L}[t-a]$

$= \int_0^\infty (t-a)\, e^{-st}\, dt = \int_0^\infty t\, e^{-st}\, dt - \int_0^\infty a\, e^{-st}\, dt$

$= \frac{1}{s^2} - \frac{a}{s}$

(iii) $\mathcal{L}[f(t-a).U(t-a)] = \mathcal{L}[(t-a).U(t-a)]$

$= \frac{e^{-as}}{s^2}$ (using Property)

Q Find laplace of
(i) $r(t)$
(ii) $r(t-a)$

Sol. (i) $\mathcal{L}[r(t)] = \mathcal{L}[t \cdot u(t)]$

$$= \int_0^\infty t\, e^{-st}\, dt = \frac{1}{s^2}$$

(ii) $\mathcal{L}[r(t-a)] = \mathcal{L}[(t-a) \cdot u(t-a)]$

$$= \frac{e^{-as}}{s^2}$$

- $\mathcal{L}[r(t)] \longrightarrow \frac{1}{s^2}$ (Laplace of ramp)
- $\mathcal{L}[r(t-a)] \longrightarrow \frac{e^{-as}}{s^2}$ (Laplace of delayed ramp)

Q Find Laplace of :-

(i) $\sin\omega(t-t_0) \cdot u(t)$

(ii) $\sin\omega t \cdot u(t-t_0)$

(iii) $\sin\omega(t-t_0) \cdot u(t-t_0)$

Sol. (i) $\mathcal{L}[\sin\omega(t-t_0) \cdot u(t)] = \int_0^\infty \sin\omega(t-t_0)\, e^{-st}\, dt$

$$= \int_0^\infty (\sin\omega t \cos\omega t_0 - \cos\omega t \sin\omega t_0)\, e^{-st}\, dt$$

$$= \cos\omega t_0 \int_0^\infty \sin\omega t\, e^{-st}\, dt - \sin\omega t_0 \int_0^\infty \cos\omega t\, e^{-st}\, dt$$

$$= \cos\omega t_0 \left[\frac{\omega}{s^2+\omega^2}\right] - \sin\omega t_0 \left[\frac{s}{s^2+\omega^2}\right]$$

$$= \frac{\omega \cos\omega t_0 - s \sin\omega t_0}{s^2+\omega^2}$$

(ii) $\mathcal{L}[\sin \omega t \cdot u(t-t_0)]$

$$= \int_{t_0}^{\infty} \sin \omega t \, e^{-st} \, dt = \int_{t_0}^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st} \, dt$$

$$= \frac{1}{2j} \int_{t_0}^{\infty} \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t} \right) dt$$

$$= \frac{1}{2j} \left[\frac{e^{-(s-j\omega)t}}{-(s-j\omega)} - \frac{e^{-(s+j\omega)t}}{-(s+j\omega)} \right]_{t_0}^{\infty}$$

$$= \frac{1}{2j} \left[\frac{e^{-(s-j\omega)t_0}}{s-j\omega} - \frac{e^{-(s+j\omega)t_0}}{s+j\omega} \right]$$

$$= \frac{e^{-st_0}}{2j} \left[\frac{(s+j\omega) e^{j\omega t_0} - e^{-j\omega t_0}(s-j\omega)}{(s^2+\omega^2)} \right]$$

$$= \frac{e^{-st_0}}{2j} \left[\frac{s(e^{j\omega t_0} - e^{-j\omega t_0}) + j\omega(e^{j\omega t_0} + e^{-j\omega t_0})}{s^2+\omega^2} \right]$$

$$= \frac{e^{-st_0}}{s^2+\omega^2} \left[s \left(\frac{e^{j\omega t_0} - e^{-j\omega t_0}}{2j} \right) + j\omega \left(\frac{e^{j\omega t_0} + e^{-j\omega t_0}}{2j} \right) \right]$$

$$= e^{-st_0} \left[\frac{s \sin \omega t_0 + \omega \cos \omega t_0}{s^2+\omega^2} \right]$$

(iii) $\mathcal{L}[\sin \omega(t-t_0) \cdot u(t-t_0)] = e^{-st_0} \mathcal{L}[\sin \omega t]$

$$= e^{-st_0} \frac{\omega}{s^2+\omega^2}$$

Q If $F(s) = \dfrac{5s+3}{s(s+1)}$. Find $f(0^+)$ & $f(\infty)$

Soln. $f(0^+) = \lim_{s \to \infty} [s\,F(s)]$

$$= \lim_{s\to\infty}\left[\cancel{s} \times \frac{5s+3}{\cancel{s}(s+1)}\right]$$

$$= \lim_{s\to\infty}\left[\frac{\frac{5s}{s}+\frac{3}{s}}{\frac{s}{s}+\frac{1}{s}}\right] = \frac{5+0}{1+0} = 5$$

$$f(\infty) = \lim_{s\to 0}[s\,F(s)]$$

$$= \lim_{s\to 0}\left[\cancel{s} \times \frac{5s+3}{\cancel{s}(s+1)}\right]$$

$$= \frac{0+3}{0+1} = 3$$

Note: Final value theorem is not valid if the denominator of $[s\,F(s)]$ contains any zero whose real part is zero or positive

Q If $F(s) = \dfrac{\omega}{s^2+\omega^2}$, Find $f(\infty)$

Sol. $f(\infty) = \lim_{s\to 0}[s\,F(s)]$

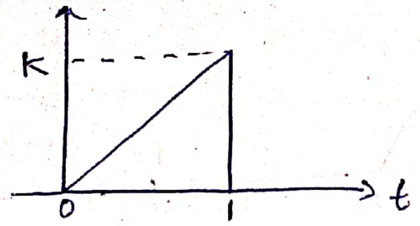
$$= \lim_{s\to 0}\left[s\,\frac{\omega}{s^2+\omega^2}\right] = 0 \text{ (Incorrect)}$$

since ~~denominator of~~ $s\,F(s) = \dfrac{s\,\omega}{(s+j\omega)(s-j\omega)}$

Since denominator of $s\,F(s)$ contain two zeros whose real part is zero. Therefore final value theorem cannot be applied.

Hence $f(\infty) = 0$ is incorrect solution.

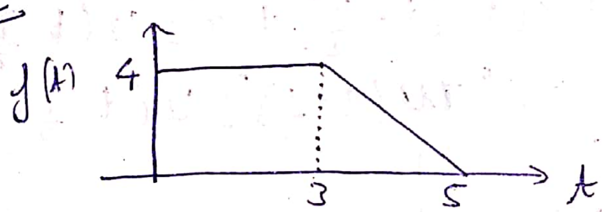
Q Find the laplace transform of given waveform.



Sol.

$$f(t) = K\,t\,[u(t) - u(t-1)]$$
$$= K[t\,u(t) - t\,u(t-1)]$$
$$= K[t\,u(t) - (t-1)u(t-1) - u(t-1)]$$
$$F(s) = K\left[\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}\right]$$
$$= K\left[\frac{1 - e^{-s} - s\,e^{-s}}{s^2}\right]$$

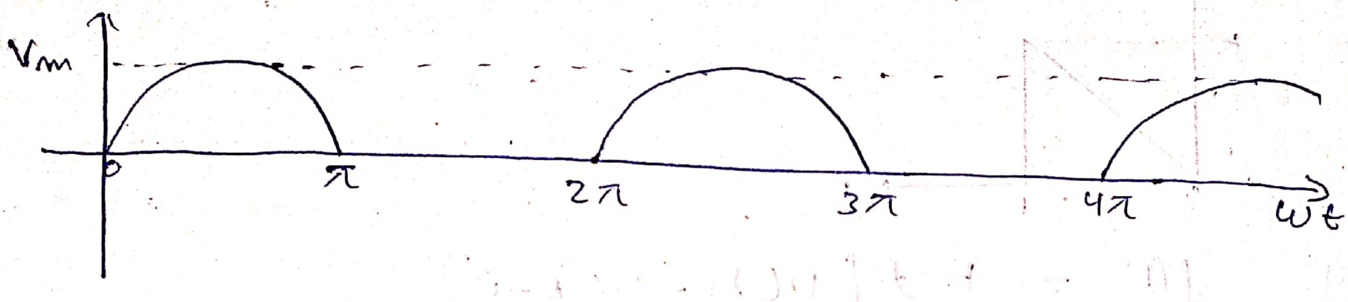
Q Find the laplace transform of given waveform.



Sol.

$$f(t) = 4[u(t) - u(t-3)] + [-2(t-5)[u(t-3) - u(t-5)]$$
$$= 4u(t) - 4u(t-3) - 2(t-5)u(t-3) + 2(t-5)u(t-5)$$
$$= 4u(t) - 4u(t-3) - 2(t-3-2)u(t-3) + 2(t-5)u(t-5)$$
$$= 4u(t) \cancel{- 4u(t-3)} - 2(t-3)u(t-3) \cancel{+ 4u(t-3)} + 2(t-5)u(t-5)$$
$$= 4u(t) - 2(t-3)u(t-3) + 2(t-5)u(t-5)$$
$$F(s) = 4\left(\frac{1}{s}\right) - 2\frac{e^{-3s}}{s^2} + 2\frac{e^{-5s}}{s^2}$$
$$= \frac{4s - 2e^{-3s} + 2e^{-5s}}{s^2}$$

Q. Find the laplace transform of given waveform.



Sol. Since it is periodic waveform with $T = 2\pi$

$\therefore$ Laplace of $f(t) = \dfrac{1}{1-e^{-Ts}} \times$ Laplace of First cycle

First cycle, $f_1(t) = V_m \sin\omega t \left[U(t) - U\left(t - \frac{\pi}{\omega}\right)\right]$

$$f_1(t) = V_m \sin\omega t\, U(t) - V_m \sin\omega t\, U\left(t - \frac{T}{2}\right)$$

$$= V_m \sin\omega t\, U(t) - V_m \sin\left[\omega\left(t - \frac{T}{2}\right) + \omega\frac{T}{2}\right] U\left(t - \frac{T}{2}\right)$$

$$= V_m \sin\omega t\, U(t) - V_m \sin\left[\omega\left(t - \frac{T}{2}\right) + \pi\right] U\left(t - \frac{T}{2}\right)$$

$$= V_m \sin\omega t\, U(t) + V_m \sin\left[\omega\left(t - \frac{T}{2}\right)\right] U\left(t - \frac{T}{2}\right)$$

$$F_1(s) = V_m\left[\frac{\omega}{s^2+\omega^2} + e^{-\frac{T}{2}s}\frac{\omega}{s^2+\omega^2}\right]$$

$$= \frac{V_m\,\omega}{s^2+\omega^2}\left[1 + e^{-\frac{T}{2}s}\right]$$

$$F(s) = \frac{1}{1-e^{-Ts}} \times \frac{V_m\,\omega}{s^2+\omega^2}\left[1 + e^{-\frac{T}{2}s}\right]$$

$$= \frac{1}{1^2 - \left(e^{-\frac{T}{2}s}\right)^2} \times V_m\frac{\omega}{s^2+\omega^2}\left[1 + e^{-\frac{T}{2}s}\right]$$

$$= \frac{1}{\cancel{(1+e^{-T/2s})}(1-e^{-T/2s})}\,\frac{V_m\,\omega}{s^2+\omega^2}\,\cancel{(1+e^{-T/2s})}$$

$$F(s) = \frac{V_m\,\omega}{(s^2+\omega^2)}\cdot\frac{1}{(1-e^{-T/2s})}$$

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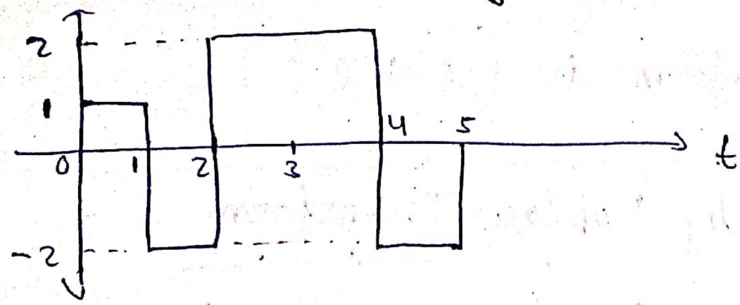
Q Find the laplace of the given waveform:



Sol $f(t) = [u(t) - u(t-1)] - 2[u(t-1) - u(t-2)]$
$+ 2[u(t-2) - u(t-4)] - 2[u(t-4) - u(t-5)]$

$f(t) = u(t) - 3u(t-1) + 4u(t-2) - 4u(t-4) + 2u(t-5)$

$$F(s) = \frac{1}{s} - 3\frac{e^{-s}}{s} + 4\frac{e^{-2s}}{s} - 4\frac{e^{-4s}}{s} + 2\frac{e^{-5s}}{s}$$

$$= \frac{1}{s}\left[1 - 3e^{-s} + 4e^{-2s} - 4e^{-4s} + 2e^{-5s}\right]$$

Q Find laplace of impulse function
$\delta(t^2 - 3t + 2)$

Sol. $\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$

$\delta(t^2 - 3t + 2) = \begin{cases} 1 & t^2 - 3t + 2 = 0 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} 1 & \cancel{t =} (t-2)(t-1) = 0 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} 1 & t = 1, 2 \\ 0 & \text{otherwise} \end{cases}$

$= \delta(t-1) + \delta(t-2)$

$$\mathcal{L}[\delta(t-1)+\delta(t-2)] = e^{-s} + e^{-2s}$$

Required Laplace transform is $(e^{-s} + e^{-2s})$

Circuit Analysis by Laplace Transform

- Any electrical circuit can be modelled in integro-differential equation (or simply in algebraic equation in case of pure resistive circuit).

Solution of differential equation can be found in 3 steps

(i) Find complementary function

(ii) Find Particular Integral

(iii) Find the value of arbitrary constants using initial conditions.

- Laplace transform can be used as a tool to solve the differential equation easily.

- Advantage of Laplace transform for solving differential eq.

(i) It transforms the time domain differential eq. into algebraic eq. in s domain, which can be solved easily

(ii) It gives the complete solution both complementary as well as particular integral in one operation.

(iii) Initial conditions are automatically considered in the transformation operation.

- In circuit analysis problems, the Laplace transform is preferred than the Fourier Transform, not only because a larger class of waveforms have Laplace transform, but also because it takes directly into account initial condition at $t=0$ due to the lower limit of integration.