Bessel's Differential Equation

Consider a linear second order homogeneous differential eq^n

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \quad \text{—— (1)}$$

Eq^n(1) is known as Bessel's Differential Eq

The complete solution may be expressed in the form

$$y = A J_n(x) + B J_{-n}(x)$$

$J_n(x)$ is called Bessel's fun^c of first kind of order 'n'.

$J_{-n}(x)$ is called Bessel's fun^c of first kind of order '$-n$'.

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{r!\,\Gamma(-n+r+1)}$$

Recurrence Formulae for $J_n(x)$

1. $\dfrac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

2. $\dfrac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

3. $J_n(x) = \dfrac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$

4. $J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

5. $J_n'(x) = \dfrac{n}{x} J_n(x) - J_{n+1}(x)$

6. $J_{n+1}(x) = \dfrac{2n}{x} J_n(x) - J_{n-1}(x)$

① Prove that

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

Proof :- We know,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{r!\,\Gamma(n+r+1)}$$

$$x^n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2n+2r}}{2^{n+2r}} \cdot \frac{1}{r!\,\Gamma(n+r+1)}$$

Diff. wrt x

$$\frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty} (-1)^r \frac{2(n+r)}{2^{n+2r}} \cdot \frac{x^{2n+2r-1}}{r!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} (-1)^r (n+r) \cdot x^n \frac{x^{n+2r-1}}{2^{n+2r-1}} \cdot \frac{1}{r!\,\Gamma(n+r)\,(n+r)}$$

$$= x^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r} \frac{1}{r!\,\Gamma((n-1)+r+1)} \quad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= x^n J_{n-1}(x)$$

H.P.

② Prove that

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

Proof :- We have

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{r!\,\Gamma(n+r+1)}$$

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{n+2r}} \cdot \frac{1}{r!\,\Gamma(n+r+1)}$$

Diff wrt x

$$\frac{d}{dx}[x^{-n} J_n(x)] = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2r \cdot x^{2r-1}}{2^{n+2r}} \cdot \frac{1}{r!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{r \cdot x^{2r-1}}{2^{n+2r-1}} \cdot \frac{1}{r(r-1)!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} (-1)^r x^{-n} . x^n . \frac{x^{2r-1}}{2^{n+2r-1}} . \frac{1}{(r-1)!\,\Gamma(n+r+1)}$$

$$= x^{-n} \sum_{r=0}^{\infty} (-1)^r . \frac{x^{n+2r-1}}{2^{n+2r-1}} . \frac{1}{(r-1)!\,\Gamma(n+r+1)}$$

$$= -x^{-n} \sum_{r=0}^{\infty} (-1)^{r-1} \left(\frac{x}{2}\right)^{n+2r-1+1-1} \frac{1}{(r-1)!\,\Gamma(n+r+1+1-1)}$$

$$= -x^{-n} \sum_{r=0}^{\infty} (-1)^{r-1} \left(\frac{x}{2}\right)^{(n+1)+2(r-1)} \frac{1}{(r-1)!\,\Gamma((n+1)+(r-1)+1)}$$

$$= -x^{-n} J_{n+1}(x)$$

H.P.

3) Prove that

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

Proof:- From 1^{st} Recurrence formula

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$$n x^{n-1} J_n(x) + x^n J_n'(x) = x^n J_{n-1}(x)$$

$$\frac{n}{x} J_n(x) + J_n'(x) = J_{n-1}(x) \quad \text{--- (1)}$$

From 2^{nd} Recurrence formula,

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

$$-n x^{-n-1} J_n(x) + x^{-n} J_n'(x) = -x^{-n} J_{n+1}(x)$$

$$-\frac{n}{x} J_n(x) + J_n'(x) = -J_{n+1}(x) \quad \text{--- (2)}$$

$Eq^n(1) - Eq^n(2)$, gives

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

H.P.

(4) Prove that

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

Proof :- We have,

$$\frac{n}{x} J_n(x) + J_n'(x) = J_{n-1}(x) \quad \text{--- (1)}$$

Also $-\frac{n}{x} J_n(x) + J_n'(x) = -J_{n+1}(x)$ — (2)

from (3) Result

$Eq^n(1) + Eq^n(2)$

$$2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

H.P.

(5) Prove that

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

Proof :- We have,

$$-\frac{n}{x} J_n(x) + J_n'(x) = -J_{n+1}(x) \quad [\text{from (3) result}]$$

$$\Rightarrow \quad J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

H.P.

(6) Prove that

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Proof :- From 4^{th} recurrence formula

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

Also, from 5^{th} recurrence formula

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

equating both $eq^n s$

$$\frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$$\left(1-\frac{1}{2}\right) J_{n+1}(x) = \frac{n}{x} J_n(x) - \frac{1}{2} J_{n-1}(x)$$

$$\frac{1}{2} J_{n+1}(x) = \frac{n}{x} J_n(x) - \frac{1}{2} J_{n-1}(x)$$

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

H.P.

Question 1:- Find $J_{\frac{1}{2}}(x)$

Solution:- $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

$$= \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^{n+2} \frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^{n+4} \frac{1}{2\,\Gamma(n+3)} - \cdots$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4 \frac{1}{2\,\Gamma(n+3)} - \cdots\right]$$

Put $n = \frac{1}{2}$

$$J_{\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{1/2} \left[\frac{1}{\Gamma\frac{3}{2}} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma\frac{5}{2}} + \left(\frac{x}{2}\right)^4 \frac{1}{2\,\Gamma\frac{7}{2}} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}} \left[\frac{1}{\frac{1}{2}\times\sqrt{\pi}} - \frac{x^2}{2^2} \cdot \frac{1}{1.\frac{3}{2}.\frac{1}{2}\times\sqrt{\pi}} + \frac{x^4}{2^4} \frac{1}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\times\sqrt{\pi}} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\sqrt{\pi}} \left[2 - \frac{x^2}{3} + \frac{x^4}{5.4.3.1} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\sqrt{\pi}} \left[\frac{2}{1} - \frac{2x^2}{3.2.1} + \frac{2x^4}{5.4.3.2.1} - \cdots\right] \times \frac{x}{x}$$

$$= \frac{\sqrt{x}}{\sqrt{2}\sqrt{\pi}} \cdot \frac{2}{x} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

Question 2:- Find $J_{-\frac{1}{2}}(x)$

Solution:- $J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{r! \, \Gamma{-n+r+1}}$

$$= \left(\frac{x}{2}\right)^{-n} \frac{1}{\Gamma{-n+1}} - \left(\frac{x}{2}\right)^{-n+2} \frac{1}{\Gamma{-n+2}} + \left(\frac{x}{2}\right)^{-n+4} \frac{1}{2\,\Gamma{-n+3}} - \cdots$$

$$= \left(\frac{x}{2}\right)^{-n} \left[\frac{1}{\Gamma{-n+1}} - \left(\frac{x}{2}\right)^{2} \frac{1}{\Gamma{-n+2}} + \left(\frac{x}{2}\right)^{4} \frac{1}{2\,\Gamma{-n+3}} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}} \left[\frac{1}{\Gamma\frac{1}{2}} - \frac{x^2}{2^2} \frac{1}{\Gamma\frac{3}{2}} + \frac{x^4}{2^4} \frac{1}{2\,\Gamma\frac{5}{2}} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}} \left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{2^2} \cdot \frac{1}{\frac{1}{2}\sqrt{\pi}} + \frac{x^4}{2^4} \frac{1}{2 \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}\,\sqrt{\pi}} \left[1 - \frac{x^2}{2} + \frac{x^4}{4 \cdot 3 \cdot 2 \cdot 1} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}\,\sqrt{\pi}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \cos x$$

Question 3:- Express $J_4(x)$ in terms of $J_0(x)$ & $J_1(x)$.

Solution:- We have $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

$$\Rightarrow \frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$\Rightarrow J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Put $n=1$,

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

Put $n=2$,

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$= \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$$

$$= \frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)$$

$$J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

Put $n=3$,

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$$

$$= \frac{6}{x}\left[\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$= \left[\frac{8 \times 6}{x^3} - \frac{6}{x} - \frac{2}{x}\right] J_1(x) - \frac{24}{x^2} J_0(x) + J_0(x)$$

$$= \frac{8}{x}\left(\frac{6}{x^2} - 1\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

Question 4 :- Prove that

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3 - x^2}{x^2} . \sin x - \frac{3}{x} \cos x\right]$$

Solution :- $\because$ We have

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Put $n = \frac{1}{2}$

$$J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$= \sqrt{\frac{2}{\pi x}} \frac{\sin x}{x} - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

Put $n = \frac{3}{2}$

$$J_{5/2}(x) = \frac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= \frac{3}{x}\left[\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)\right] - \sqrt{\frac{2}{\pi x}} \sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3 \sin x}{x^2} - \frac{3 \cos x}{x} - \sin x\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2}-1\right)\sin x - \frac{3\cos x}{x}\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$$

H.P.

Question 5:- Prove that

$$\int J_3(x)\,dx = C - J_2(x) - \frac{2}{x}J_1(x)$$

Soln:-

From second recurrence formula

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow \int d\left[x^{-n}J_n(x)\right] = -\int x^{-n}J_{n+1}(x)\,dx$$

$$\Rightarrow -x^{-n}J_n(x) = \int x^{-n}J_{n+1}(x)\,dx$$

L.H.S. $\int J_3(x)\,dx = \int x^2 \cdot x^{-2}J_3(x)\,dx$

$$= x^2\int x^{-2}J_3(x)\,dx - \int \frac{d}{dx}x^2\int x^{-2}J_3(x)\,dx\Big]\,dx + C$$

$$= x^2\left(-x^{-2}J_2(x)\right) - \int 2x\left(-x^{-2}J_2(x)\right)dx + C$$

$$= -J_2(x) + 2\int x^{-1}J_2(x)\,dx + C$$

$$= -J_2(x) - 2x^{-1}J_1(x) + C$$

$$= C - J_2(x) - \frac{2}{x}J_1(x)$$

= R.H.S.

H.P.

Question 6:- Prove that

$$J_n''(x) = \frac{1}{4}\left[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\right]$$

Solution :- From fourth recurrence formula

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \quad \text{—— (I)}$$

Differentiating both sides w.r.t. x

$$J_n''(x) = \frac{1}{2}\left[J_{n-1}'(x) - J_{n+1}'(x)\right] \quad \text{—— (II)}$$

Changing n to $(n-1)$ in (I),

$$J_{n-1}'(x) = \frac{1}{2}\left[J_{n-2}(x) - J_n(x)\right]$$

Changing n to $n+1$ in

$$J_{n+1}'(x) = \frac{1}{2}\left[J_n(x) - J_{n+2}(x)\right]$$

Eq (II) becomes

$$J_n''(x) = \frac{1}{2}\left[\frac{1}{2}\left\{J_{n-2}(x) - J_n(x)\right\} - \frac{1}{2}\left\{J_n(x) - J_{n+2}(x)\right\}\right]$$

$$= \frac{1}{4}\left[J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)\right]$$

$$= \frac{1}{4}\left[J_{n-2} - 2J_n(x) + J_{n+2}(x)\right]$$

H. P.

Question 7 :- Prove that

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

Proof :- $\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right]$

$$= x J_n(x) J_{n+1}'(x) + J_{n+1}(x)\left[x J_n'(x) + J_n(x)\right]$$

$$= x J_n(x) J_{n+1}'(x) + x J_n'(x) J_{n+1}(x) + J_n(x) J_{n+1}(x)$$

From 5^{th} recurrence formula,

$$\boxed{J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)}$$

Also, we have,

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)$$

Changing n to $n+1$,

$$J_{n+1}'(x) = J_n(x) - \left(\frac{n+1}{x}\right) J_{n+1}(x)$$

From ③ recurrence formula

$$= x J_n(x) \left[J_n(x) - \left(\frac{n+1}{x}\right) J_{n+1}(x)\right] +$$

$$x\left[\frac{n}{x} J_n(x) - J_{n+1}(x)\right] J_{n+1}(x) + J_n(x) J_{n+1}(x)$$

$$= x J_n^2(x) - (n+1) J_n(x) J_{n+1}(x) + n J_n(x) J_{n+1}(x)$$
$$- x J_{n+1}^2(x) + J_n(x) J_{n+1}(x)$$

$$= x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

H. P.

Question 8 :- Prove that

$$\frac{d}{dx}\left[J_n^2(x)\right] = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

Solution :- $\frac{d}{dx}\left[J_n^2(x)\right] = 2 J_n(x) J_n'(x)$

From 3rd recurrence formula,

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

From 4th recurrence formula,

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$\frac{d}{dx}\left[J_n^2(x)\right] = 2 J_n(x) J_n'(x)$$

$$= 2 \cdot \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \cdot \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

H. P.

Question for Practice

Question 1:- Prove that $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$

Question 2:- Prove that $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$.

Question 3:- Prove that $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2\left[\frac{n}{x}J_n^2(x) - \frac{n+1}{x}J_{n+1}^2(x)\right]$

Question 4:- Prove that $\int J_3(x) = -J_2(x) - \frac{2}{x}J_2(x)$

Question 5:- Prove that $\int x J_0^2 dx = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right] + C$

Question 6:- Show that $J_0'(x) = -J_1(x)$

Question 7:- Express $J_{3/2}$, $J_{-3/2}$, $J_{5/2}$ & $J_{-5/2}$ in terms of sine and cosine functions

Question 8:- Evaluate $\int x^{-1} J_4(x)\, dx$

Question 9:- Show that $J_3(x) = \left(\frac{8}{x^2} - 1\right)J_1 - \frac{4}{x}J_0$

Question 10:- Show that $J_3(x) + 3J_0'(x) + 4J_0'''(x) = 0$.

Question 11:- Express $\int x^{-2} J_2(x)\, dx$ in terms of Bessel functions.

Question 12:- Show that $\left(\frac{1}{x}\frac{d}{dx}\right)^m\left[x^{-n}J_n(x)\right] = \frac{(-1)^m}{x^{n+m}}J_{n+m}(x)$.

Generating Function for $J_n(x)$

Bessel function of first kind $J_n(x)$ can also be obtained by expanding the exponential function $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$.

This function is called generating function for $J_n(x)$, where 'n' is an integer.

$$e^{\frac{x}{2}(t-t^{-1})} = e^{\frac{xt}{2}} \cdot e^{\frac{-x}{2t}} = e^{\frac{xt}{2}} \cdot e^{(-1)\frac{x}{2t}}$$

$$= \left[1+\frac{xt}{2}+\frac{1}{2!}\left(\frac{xt}{2}\right)^2+\frac{1}{3!}\left(\frac{xt}{3}\right)^3+\cdots+\frac{1}{n!}\left(\frac{xt}{2}\right)^n+\cdots\right]$$

$$\times\left[1+\frac{(-1)x}{2t}+\frac{(-1)^2}{2!}\left(\frac{x}{2t}\right)^2+\frac{(-1)^3}{3!}\left(\frac{x}{2t}\right)^3+\cdots+\frac{(-1)^n}{n!}\left(\frac{x}{2t}\right)^n+\cdots\right]$$

Co-eff of $t^n = \frac{1}{n!}\left(\frac{x}{2}\right)^n - \left(\frac{1}{(n+1)!}\right)\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r} = J_n(x)$$

Co-eff of $t^{-n} = \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^{n+2}}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} \cdots$

$$= \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n - \frac{(-1)^n}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^n}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r(-1)^n}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r} = (-1)^n J_n(x) = J_{-n}(x)$$

$$\therefore e^{\frac{x}{2}(t-t^{-1})} = \left[J_0(x)+tJ_1(x)+t^2J_2(x)+\cdots\right] \times \left[J_0(x)+\frac{J_{-1}}{t}(x)+\frac{J_{-2}}{t^2}(x)+\cdots\right]$$

$$= \left[J_0(x)+tJ_1(x)+t^2J_2(x)+\cdots\right]\left[J_0(x)-\frac{J_1}{t}(x)+\frac{J_2(x)}{t^2}+\cdots\right]$$

$$e^{\frac{x}{2}(t-t^{-1})} = J_0(x) + \left(t - \frac{1}{t}\right) J_1(x) + \left(t^2 + \frac{1}{t^2}\right) J_2(x) + \left(t^3 - \frac{1}{t^3}\right) J_3(x) + \cdots$$

$$\boxed{e^{\frac{x}{2}(t-t^{-1})} = J_0(x) + \sum_{n=1}^{\infty} J_n(x)\left[t^n + (-1)^n t^{-n}\right]}$$

Question 1:- Show that

$$\cos(x\sin\theta) = J_0(x) + (2\cos 2\theta) J_2(x) + (2\cos 4\theta) J_4(x) + \cdots$$

Solution:- We know, the generating fun' of $J_n(x)$ is

$$e^{\frac{x}{2}(t-t^{-1})} = J_0(x) + \left(t - \frac{1}{t}\right) J_1(x) + \left(t^2 + \frac{1}{t^2}\right) J_2(x) + \left(t^3 - \frac{1}{t^3}\right) J_3(x) + \cdots$$

Put $t = e^{i\theta}$,

$$e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})} = J_0(x) + (e^{i\theta} - e^{-i\theta}) J_1(x) + (e^{2i\theta} + e^{-2i\theta}) J_2(x) + (e^{3i\theta} - e^{-3i\theta}) J_3(x) + \cdots$$

$$\Rightarrow e^{\frac{x}{2}(2i\sin\theta)} = J_0(x) + 2i\sin\theta\, J_1(x) + 2\cos 2\theta\, J_2(x) + 2i\sin 3\theta\, J_3(x) + \cdots$$

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = \left[J_0(x) + (2\cos 2\theta) J_2(x) + (2\cos 4\theta) J_4(x) + \cdots\right] + i\left[(2\sin\theta) J_1(x) + (2\sin 3\theta) J_3(x) + (2\sin 5\theta) J_5(x) + \cdots\right]$$

Comparing the real part on both sides

$$\cos(x\sin\theta) = J_0(x) + 2\cos 2\theta\, J_2(x) + 2\cos 4\theta \cdot J_4(x)$$

H.P.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Question 2: Show that

$$\sin(x\sin\theta) = 2\sin\theta\, J_1(x) + 2\sin 3\theta\, J_3(x) + 2\sin 5\theta\, J_5(x) + \cdots$$

~~Bessel's~~: Equation Reducible to Bessel Eqⁿ

The differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda^2x^2-n^2)y=0 \quad \text{— (I)}$$

the solution of eqⁿ (I) is

$$y=AJ_n(\lambda x)+BJ_{-n}(\lambda x)$$

But eqⁿ (I) can be reduced to Bessel's diff. eqⁿ of order n by putting $\lambda x = t$

$$\Rightarrow \lambda\frac{dx}{dy}=\frac{dt}{dy} \qquad (\lambda \text{ is any parameter})$$

$$\Rightarrow \frac{1}{\lambda}\frac{dy}{dx}=\frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dx}=\lambda\frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\lambda\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}=\lambda^2\frac{d^2y}{dt^2}$$

∴ eqⁿ (I) becomes, $\quad x=\frac{t}{\lambda}$

$$\left(\frac{t}{\lambda}\right)^2\cdot\lambda^2\frac{d^2y}{dt^2}+\frac{t}{\lambda}\cdot\lambda\frac{dy}{dt}+(t^2-n^2)y=0$$

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-n^2)y=0 \quad \text{— (II)}$$

the solution of eqⁿ (II) is

$$y=AJ_n(t)+BJ_{-n}(t)$$

Question 1:- Solve the following diff eqⁿ

$$y''+\frac{1}{x}y'+\left(8-\frac{1}{x^2}\right)y=0$$

Solution:- Given differential equation is

$$y'' + \frac{1}{x}y' + \left(8 - \frac{1}{x^2}\right)y = 0$$

$$\Rightarrow x^2y'' + xy' + (8x^2 - 1)\,y = 0 \quad -(1)$$

On comparing with eq^n

$$x^2y'' + xy' + (\lambda^2x^2 + n^2)\,y = 0 \quad -(2)$$

we get , $\lambda^2 = 8 \Rightarrow \lambda = 2\sqrt{2}$

& $n^2 = 1 \Rightarrow n = \pm 1$

We know the solution of eq^n (2) is

$y = A J_n(\lambda x) + B J_{-n}(\lambda x)$

For $n = 1$, $\lambda = 2\sqrt{2}$,

the sol^n of eq^n (1) is

$y = A J_1(2\sqrt{2}x) + B J_{-1}(2\sqrt{2}x)$

For $n = -1$, $\lambda = 2\sqrt{2}$, we get the sol^n of eq^n (1) is

$y = C J_{-1}(2\sqrt{2}x) + D J_1(2\sqrt{2}x)$

When n is not an integer, then sol^n of eq^n (III) is

$$y = x^{\alpha}\left[c_1 J_n(\beta x^{\gamma}) + c_2 J_{-n}(\beta x^{\gamma})\right]$$

where

$$x^2\frac{d^2y}{dx^2} + (1-2\alpha)x\frac{dy}{dx} + \left[\beta^2\gamma^2x^{2\gamma} + (\alpha^2 - n^2\gamma^2)\right]y = 0 \quad -(III)$$

Question 2:- Solve the following diff eq^{n}s in terms of Bessel fun^c

$$y'' - \frac{2}{x}y' + 4\left(x^2 - \frac{1}{x^2}\right)y = 0$$

Solution:- The given equation is $x^2y'' - 2xy' + (4x^4 - 4)y = 0$

Comparing with general form, we get

$1 - 2\alpha = -2$, $\beta^2\gamma^2 = 4$, $x^{2\gamma} = x^4$, $(\alpha^2 - n^2\gamma^2) = -4$

i.e. $\alpha = \frac{3}{2}$, $\beta = 1$, $\gamma = 2$, $n = \frac{5}{4}$, Here n is not integer

then the solution is

$$y = x^{3/2}\left[c_1 J_{5/4}(x^2) + c_2 J_{-5/4}(x^2)\right]$$

Orthogonality of Bessel's functions

Prove that

$$\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = \begin{cases} 0 & ; \text{if } \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2 & ; \text{if } \alpha = \beta \end{cases}$$

Proof:- We know, $J_n(\lambda x)$ is the solution of

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0 \quad -①$$

Let $u = J_n(\alpha x)$ & $v = J_n(\beta x)$ be the solution of eq^n ①, then

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (\alpha^2 x^2 - n^2)u = 0 \quad -②$$

$$\& \; x^2 \frac{d^2v}{dx^2} + x\frac{dv}{dx} + (\beta^2 x^2 - n^2)v = 0 \quad -③$$

Multiplying ② by $\frac{v}{x}$ & ③ by $\frac{u}{x}$ and subtracting

$$\frac{v}{x}\left[x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (\alpha^2x^2-n^2)u\right] - \frac{u}{x}\left[x^2\frac{d^2v}{dx^2} + x\frac{dv}{dx} + (\beta^2x^2-n^2)v\right] = 0$$

$$\Rightarrow x\left[v\frac{d^2u}{dx^2} - u\frac{d^2v}{dx^2}\right] + \left(v\frac{du}{dx} - u\frac{dv}{dx}\right) + \frac{uv}{x}(\alpha^2x^2 - n^2 - \beta^2x^2 + n^2) = 0$$

$$\Rightarrow \frac{d}{dx}\left[x\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\right] + \frac{uv}{x}(\alpha^2x^2 - \beta^2x^2) = 0$$

$$\Rightarrow (\alpha^2-\beta^2)\,uvx = -\frac{d}{dx}\left[x\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\right]$$

Integrating both sides w.r.t x from 0 to 1

$$(\alpha^2-\beta^2)\int_0^1 uvx\,dx = -\int_0^1 \frac{d}{dx}\left[x\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\right]$$

$$\Rightarrow \int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \frac{-1}{\alpha^2-\beta^2}\left[x\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\right]_0^1$$

$$= \frac{-1}{\alpha^2-\beta^2}\left[x\{J_n(\beta x)\alpha J_n'(\alpha x) - J_n(\alpha x)\cdot\beta J_n'(\beta x)\}\right]_0^1$$

$$\left[\begin{array}{lll} \because\ u = J_n(\alpha x) & \Rightarrow & \frac{du}{dx} = \alpha\ J_n'(\alpha x) \\ \&\ \ v = J_n(\beta x) & \Rightarrow & \frac{dv}{dx} = \beta\ J_n'(\beta x) \end{array}\right]$$

$$= \frac{-1}{\alpha^2-\beta^2}\left[1\{J_n(\beta)\alpha J_n'(\alpha) - J_n(\alpha)\cdot\beta J_n'(\beta) - 0\right]$$

$$= \frac{-1}{\alpha^2-\beta^2}\left[\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\right]$$

$$\left[\begin{array}{l} \because\ \alpha \ \&\ \beta \text{ be roots of } J_n(x)=0 \\ \Rightarrow J_n(\alpha)=0 \ \&\ J_n(\beta)=0 \end{array}\right]$$

- <u>Case I:</u> When $\alpha \neq \beta$

$$\int_0^1 x\, J_n(\alpha x).\, J_n(\beta x)\, dx = \frac{-1}{\alpha^2-\beta^2} \times 0 = 0$$

<u>Case II:</u> When $\alpha = \beta$

$$\int_0^1 x\, J_n(\alpha x)\cdot J_n(\beta x)\, dx = \frac{0}{0} \text{ form}$$

$$= \lim_{\alpha \to \beta} \frac{0 - \beta J_n'(\alpha) J_n'(\beta)}{-2\alpha}$$

- $$= \frac{1}{2}\left[J_n'(\alpha)\right]^2 \qquad [\because \alpha=\beta]$$

$$= \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2$$

$$\left[\begin{array}{l} \text{From } 5^{th} \text{ Recurrence formula} \\ J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \\ J_n'(\alpha) = \frac{n}{\alpha} J_n(\alpha) - J_{n+1}(\alpha) \\ \Rightarrow J_n'(\alpha) = -J_{n+1}(\alpha) \qquad \because J_n(\alpha)=0 \end{array}\right]$$

Questions for practice

Question 1:- Find the solutions of the following differential equations in terms of Bessel functions:

(i) $xy'' + y = 0$

(ii) $4y'' + 9xy = 0$

(iii) $xy'' - y' + 4x^2 y = 0$

(iv) $y'' + \frac{1}{x}y' + 4\left(1 - \frac{1}{x^2}\right)y = 0$

(v) $y'' + \frac{1}{x}y' + \left(3 - \frac{1}{4x^2}\right)y = 0$

(vi) $xy'' + y' + \frac{1}{4}y = 0$

(vii) $y'' + \left(9x - \frac{20}{x^2}\right)y = 0$

(viii) $xy'' - 3y' + xy = 0$

Solutions (i) $y = \sqrt{x}\left[C_1 J_1(2\sqrt{x}) + C_2 J_{-1}(2\sqrt{x})\right]$

(ii) $y = \sqrt{x}\left[C_1 J_{1/3}(x^{3/2}) + C_2 J_{-1/3}(x^{3/2})\right]$

(iii) $y = x\left[C_1 J_{1/2}(x^2) + C_2 J_{-1/2}(x^2)\right]$

(iv) $y = C_1 J_2(2x) + C_2 J_{-2}(2x)$

(v) $y = C_1 J_{1/2}(\sqrt{3}x) + C_2 J_{-1/2}(\sqrt{3}x)$

(vi) $y = C_1 J_0(\sqrt{x}) + C_2 Y_0(\sqrt{x})$

(vii) $y = \sqrt{x}\left[C_1 J_3(2x^{3/2}) + C_2 J_{-3}(2x^{3/2})\right]$

(viii) $y = x^2\left[C_1 J_2(x) + C_2 J_{-2}(x)\right]$