Aa

CM

Taylor's Theorem :- If $(a+h)$ be a function of the variable h s.t it can be expanded in ascending powers of h & this expansion be differentiable any no. of times, then

$$\boxed{f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots\cdots \infty}$$

Let $a+h = x \Rightarrow$

$$\boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots\cdots \infty}$$

Now if we will put $a=0$, this will give us $\Rightarrow$

$$\boxed{f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots \infty}$$

Mac Laurin's Series

Q. Expand $\tan^{-1}x$ in power of $\underbrace{\left(x-\frac{\pi}{4}\right)}$ by Taylor's Theorem.

Jo bhi '−' ya '+' hoga usey hm 'a' le lenge

$$\Rightarrow f(x) = f\Big(\underbrace{\tfrac{\pi}{4}}_{a} + \underbrace{\left(x-\tfrac{\pi}{4}\right)}_{h}\Big) = f\left(\tfrac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)}{1!} f'\left(\tfrac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\tfrac{\pi}{4}\right) + \cdots \infty$$

Here, $f(x) = \tan^{-1}x$

$\Rightarrow f'(x) = \dfrac{1}{1+x^2}$

$\Rightarrow f''(x) = \dfrac{-2x}{(1+x^2)^2}$

$f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$

$f'\left(\frac{\pi}{4}\right) = \dfrac{1}{1+\frac{\pi^2}{16}} = \dfrac{16}{16+\pi^2}$

$f''\left(\frac{\pi}{4}\right) = \dfrac{-\frac{\pi}{2}}{\left(1+\frac{\pi^2}{16}\right)^2} = \dfrac{-128\pi}{(16+\pi^2)}$

$$\therefore \tan^{-1}x = 1 + \frac{\left(x-\frac{\pi}{4}\right)}{1!}\left(\frac{16}{16+\pi^2}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\frac{(-128\pi)}{(16+\pi^2)} + \cdots\cdots \infty$$

Taylor's Theorem for function of two variable

If $f(x,y)$ & all it's partial derivatives upto n^{th} order are finite & continuous for all points (x,y) then,

$$\boxed{f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b) + \cdots\cdots}$$

Now in questions, we are given the two pt's & we put them in a & b

Now in questions, we are given the two pt's & we put them in a & b but if no pt. is given then we take a = 0 , b = 0. This is called Mac Laurin's Series.

Rolle's Theorem

If a function $f:[a,b] \to R$ is continuous - $[a,b]$ & differentiable - (a,b) and $f(a) = f(b)$ then there will be some $c \in (a,b)$ s.t $\boxed{f'(c) = 0}$

slope of tangent of the curve

Mean Value Theorem (extension of Rolle's theorem)

If a function $f:[a,b] \to R$ s.t f is continuous - $[a,b]$ & differentiable - (a,b) & $f(a) \neq f(b)$, so there will be a some $c \in (a,b)$;

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

UNIT-2

NUMERICAL ANALYSIS

Interpolation :- The technique or method of estimating unknown values from given set of observation is known as interpolation

Assumptions -
1. No sudden jumps
2. Sufficient number of obsevations
3. Absence of external forces
4. No continuous missing values
5. Stable relationship

Equal Interval

Newton Forward :- (↓)

Gauss Forward / Gauss Backward / Striling / Bessel's formula

Newton Backward (↑)

x	$f(x)$
1971	1000
1981	1025
1991	1080
2001	1120
2011	1200

1973

2008

Unequal Interval

Lagrange's Interpolation

Newton divided difference

x	$f(x)$
75	670
80	685
87	750
90	800
94	915

77

93

EQUAL INTERVAL

EQUAL INTERVAL

Q. Estimate the population in 1895 & 1925 from following :-

Year	1891	1901	1911	1921	1931
Population	46 K	66 K	81 K	93 K	101 K

A.

del operator

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1891	46 K				
		20 K			
1901	66 K		− 5 K		
		15 K		$-3-(-5)=2$	
1911	81 K		− 3 K		−3
		12 K		$-4-(-3)=-1$	
1921	93 K		− 4 K		
		8 K			
1931	101 K				

1895

1925

FORWARD

BACKWARD

Newton Forward :-

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) \text{ - - - - - -}$$

a → base (starting pt.)

h → difference b/w x

$f(1895)$

$a+hu = 1895$

$\Rightarrow a = 1891$

$h = 10$

$\Rightarrow 1891 + 10u = 1895$

$\Rightarrow u = 0.4$

$$= 46 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(-5) + \frac{0.4(0.4-1)(0.4-2)}{3!}(2) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(-3)$$

= solve & get answer

nebla $\Rightarrow$ $\nabla f(x) = f(x) - f(x-h)$

Newton Backward :-

$$f(a+hu) = f(a) + \frac{u}{1!}\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)(u+3)}{3!}\nabla^3 f(a) \text{ - - - - - -}$$

CENTRAL DIFFERENCE INTERPOLATION

Q. Find u_9 if $u_0 = 14$, $u_4 = 24$, $u_8 = 32$, $u_{12} = 35$, $u_{16} = 40$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	14 y_{-2}				
		10 y_{-2}			
4	24 y_{-1}		−2 y_{-2}		
		8 y_{-1}		−3 y_{-2}	
8	32 y_0		−5		10 y_{-2}
		3			

(9)

Q. Find u_9 if $u_0 = 14$, $u_4 = 24$, $u_8 = 32$, $u_{12} = 35$, $u_{16} = 40$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	14 y_{-2}				
		10 y_{-2}			
4	24 y_{-1}		-2 y_{-2}		
		8 y_{-1}		-3 y_{-2}	
8 (9)	32 y_0		-5 y_{-1}		10 y_{-2}
		3 y_0		7 y_{-1}	
12	35 y_1		2 y_0		
		5 y_1			
16	40 y_2				

Gauss Forward :-

$$f(a+hu) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} \text{ -----}$$

Gauss Backward :-

$$f(a+hu) = y_0 + \frac{u}{1!}\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u-1)u(u+1)}{3!}\Delta^2 y_{-2} + \frac{(u-1)u(u+1)(u+2)}{4!}\Delta^4 y_{-2} \text{ ----}$$

Stirling Formula :- $\dfrac{\text{Gauss Forward} + \text{Gauss Backward}}{2}$

$$f(a+hu) = y_0 + \frac{u}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} \text{ ----}$$

Bessel's Formula :-

$$f(a+hu) = \frac{1}{2}(y_0+y_1) + \frac{\left(u-\frac{1}{2}\right)}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right) + \frac{\left(u-\frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] \text{ -----}$$

UNEQUAL INTERVAL

Lagrange's Interpolation :-

Q. Find value of y when $x = 10$ by Lagrange's Interpolation formula

x	5	6	9	11

Lagrange's Interpolation :-

Q. Find value of y when $x = 10$ by Lagrange's Interpolation formula

x	5	6	9	11
$f(x) = y$	12	13	14	16

$$f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13) + \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14)$$

$$+ \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

Now put $x = 10$ & solve !!

NEWTON's DIVIDED DIFFERENCE :-

solving above same question !!

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0 5	12			
		$\frac{13-12}{6-5} = 1$		
x_1 6	13		$\frac{\frac{1}{3}-1}{9-5} = -\frac{1}{6}$	
		$\frac{14-13}{9-6} = \frac{1}{3}$		$\frac{\frac{2}{15}+\frac{1}{6}}{11-5} = \frac{1}{20}$
x_2 9	14			
		$\frac{16-14}{11-9} = 1$	$\frac{1-\frac{1}{3}}{11-6} = \frac{2}{15}$	
x_3 11	16			

$$\boxed{f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) - - - - -}$$

Errors in polynomial Interpolation

Let $f(x)$ have $(n+1)$ continuous derivatives on $[a,b]$ & let $x_0, x_1, \ldots x_n$ be distinct nodes in $[a,b]$, $x_0 < x_1 < x_2 - - - - < x_n$

then, error in polynomial is

derivative

$$\boxed{f(x) - P_n(x) = \frac{(x-x_0)(x-x_1)(x-x_2) - - - - (x-x_n)}{(n+1)!} f^{(n+1)}(c_x) \quad \text{where } x_0 < c_x < x_n}$$

Proof :- Let $P_n(x)$ be the n^{th} degree interpolating polynomial,

$P_n(x) = y_i$, $i = 0, 1, 2, \ldots n$

Proof :- Let $P_n(x)$ be the n^{th} degree interpolating polynomial,

$P_n(x) = y_i$, $i = 0, 1, 2, \ldots\ldots n$

$f(x) - P_n(x) = L(x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_n)$

Let $\pi(x) = (x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_n)$

$\Rightarrow L = \dfrac{f(x) - P_n(x)}{\pi(x)}$

Let $F(x) = f(x) - P_n(x) - L(x-x_0)(x-x_1)\ldots\ldots(x-x_n)$

For any $x_0 \le x' \le x_n$, $F'(x) = 0$

$F(x)$ has $(n+1)$ zeroes, $x_0, x_1, x_2 \ldots\ldots x_n, x'$

min. degree of $F(x)$ is $(n+2)$

$\Rightarrow F^{(n+1)}(x) = f^{(n+1)}(x) - 0 - L(n+1)!$

and $F^{(n+1)}(x)$ is linear

$\Rightarrow c_x \in [a, b]$ s.t $F^{(n+1)}(c_x) = 0 \Rightarrow f^{(n+1)}(c_x) - L(n+1)! = 0$

$\Rightarrow L = \dfrac{f^{(n+1)}(c_x)}{(n+1)!}$

$$\Rightarrow \boxed{f(x) - P_n(x) = \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_n)}{(n+1)!} f^{(n+1)}(c_x)}$$

Hence, proved

Q. Arrive at the error in linear interpolation of $f(x) = e^x$ on $[0, 1]$

Let (x_0, y_0) & (x_1, y_1) be the nodal points.

$f(x) - P_1(x) = \dfrac{(x-x_0)(x-x_1)}{2} f''(c_x)$, $x_0 \le c_x \le x_1$

$= \dfrac{(x-x_0)(x-x_1)}{2} e^{c_x}$

$\because x_0 \le c_x \le x_1$

$\Rightarrow e^{x_0} \le c_x \le e^{x_1} \Rightarrow \dfrac{(x-x_0)(x_1-x)}{2} e^{x_0} \le \dfrac{(x-x_0)(x_1-x)}{2} e^{c_x} \le \dfrac{(x-x_0)(x_1-x)}{2} e^{x_1}$

$\Rightarrow \dfrac{(x-x_0)(x_1-x)}{2} e^{x_0} \le \left| f(x) - P_1(x) \right| \le \dfrac{(x-x_0)(x_1-x)}{2} e^{x_1}$

max. $[(x-x_0)(x_1-x)]$ occurs at $\dfrac{x_1+x_0}{2}$

mid pt.

max. $\left[(x-x_0)(x_1-x)\right]$ occurs at $\dfrac{x_1+x_0}{2}$

mid pt.

$$\frac{x-x_0}{2} = \frac{x-x_0}{2} = \frac{h}{2} \;;\; h = x_1 - x_0 \Rightarrow \frac{h^2}{8}e^{x_0} \leq \left|f(x)-P_1(x)\right| \leq \frac{h^2}{8}e^{x_1}$$

$\therefore$ Bound for error is $\left|f(x)-P_1(x)\right| \leq \dfrac{h^2}{8}e$

In particular if $h = 0.01$ then, $\left|f(x)-P_1(x)\right| \leq \dfrac{(0.01)^2}{8}e \sim 0.000034$

NUMERICAL INTEGRATION

As we know from class 12^{th}, the area bounded by the curve $f(x)$ & x-axis b/w limit a & b is denoted by $I = \int_a^b f(x)\,dx$ — ①

Now divide the interval (a,b) into n equal interval with length h (step size) ie

$(a,b) = (a = x_0, x_1, x_2, \ldots\ldots x_{n-1}, x_n = b)$

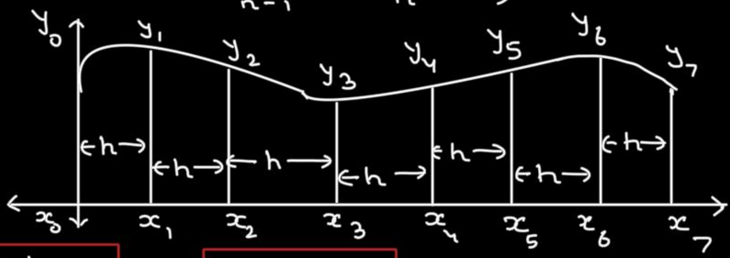
$a = x_0$

$x_1 = x_0 + h$

$x_2 = x_1 + h$

$\vdots$

$x_n = x_{n-1} + h$ where $\boxed{h = \dfrac{b-a}{h}}$ OR $\boxed{h = \dfrac{b-a}{n}}$



Newton - Cote's Quadrature Formula :-

$$\boxed{= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{1}{2}\Delta^2 y_0 + \frac{1}{3!}\left(\frac{u^4}{4} - \frac{3u^3}{3} + \frac{2u^2}{2}\right)\Delta^3 y_0 \ldots\right]}$$

I. Trapezoidal Rule :- It is applicable on any no. of interval (odd, even)

$$\boxed{\int_a^b f(x)\,dx = h\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1}\right)}$$

first term   last term

II. Simpson's 1/3 Rule :- It is applicable in even intervals

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4\underbrace{(y_1 + y_3 + y_5 + \ldots)}_{\text{odd terms}} + 2\underbrace{(y_2 + y_4 + y_6 \ldots)}_{\text{even terms}}\right]}$$

III. Simpson's 3/8 Rule :-

It is applicable when 3 ke multiple mein intervals ho.

$$\boxed{\int_a^b f(x)\,dx = \frac{3h}{8}\Big[(y_0+y_n) + 3\underbrace{(y_1+y_2+y_4+y_5----)}_{\text{other than 3 multiple terms}} + 2\underbrace{(y_3+y_6+y_9----)}_{\text{3 multiple terms}}\Big]}$$

Q. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (i) Trapezoidal Rule (ii) Simpson's $\frac{1}{3}$ Rule (iii) Simpson's $\frac{3}{8}$ Rule.

Also find the value of π in each case.

A. $h = \frac{b-a}{n} = \frac{1-0}{6}$ → coz we combine Simpson's $\frac{1}{3}$ & $\frac{3}{8}$ Rule then interval becomes 6.

$y = 1/(1+x^2)$

⇒ $x_0 = \frac{1}{6}$	$y_0 = \frac{1}{1+x_0^2} = 1$	And By Direct Integration,
$x_1 = \frac{1}{6}$	$y_1 = \frac{1}{1+x_1^2} = 36/37$	$\int_0^1 \frac{1}{1+x^2}dx = (\tan^{-1}x)_0^1$
$x_2 = 2/6$	$y_2 = \quad = 0.9$	
$x_3 = 3/6$	$y_3 = \quad = 0.8$	$= \tan^{-1}1 - \tan^{-1}0$
$x_4 = 4/6$	$y_4 = \quad = 9/13$	$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$ — (4)
$x_5 = 5/6$	$y_5 = \quad = 36/61$	
$x_6 = 6/6$	$y_6 = \quad = 0.5$	

(i) $$\int_0^1 \frac{1}{1+x^2}dx = h\left[\frac{y_0+y_6}{2} + y_1+y_2+y_3+y_4+y_5\right]$$

$$= \frac{1}{6}\left[\frac{1+0.5}{2} + \text{Put the values \& solve !!}\right] = \text{Ans} \_\_\_ -(1)$$

(ii) $$\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$ → remember to take terms till $(n-1)$.

Put the values & solve. = Ans ___ —(2)

(iii) $$\int_0^1 \frac{1}{1+x^2}dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 3(y_3)\right]$$

Put the values & solve. = Ans ___ —(3)

∴ value of π (using (1) & (4))

⇒ $\frac{\pi}{4}$ = Ans — (1) ⇒ π = Ans — (1) × 4 = ____

Value of π (using (2) & (4))

Value of π (using ② & ④)

$\Rightarrow \frac{\pi}{4} =$ Ans ____ ② $\Rightarrow \pi =$ Ans ____ ② $\times 4 =$ ________

value of π (using ③ & ④)

$\Rightarrow \frac{\pi}{4} =$ Ans ____ ③ $\Rightarrow \pi =$ Ans ____ ③ $\times 4 =$ ________

Errors in Quadrature Formula :- The error in quadrature formulae is :-

$$\boxed{E = \int_a^b y\,dx - \int_a^b Q(x)\,dx}$$

where $Q(x)$ is the polynomial of $y = f(x)$ in the interval $[a, b]$.

→ **Error in Trapezoidal Rule :-**

$$\boxed{E = -\frac{(b-a)}{12} h^2 \cdot y''(x)}$$

Error is in order h^2

→ **Error in Simpsons 1/3 Rule :-**

→ **Error in Simpsons 3/8 Rule :-**

ROMBERG INTEGRATION :-

$h \downarrow$

h	I_1	$I_1' = I_2 + (I_2 - I_1)\frac{1}{3}$	$I_1'' = I_2' + \frac{1}{3}(I_2' - I_1')$	$I_1''' = I_2'' + \frac{1}{3}(I_2'' - I_1'')$
$h/2$	I_2	$I_2' = I_3 + (I_3 - I_1)\frac{1}{3}$	$I_2'' = I_3' + \frac{1}{3}(I_3' - I_2')$	
$h/4$	I_3	$I_3' = I_4 + (I_4 - I_1)\frac{1}{3}$		
$h/8$	I_4			