Power Series Method

Consider a homogeneous linear second order diff. eq^n with variable co-eff.

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \quad —(1)$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0$$

Let $P(x) = \frac{P_1(x)}{P_0(x)}$ & $Q(x) = \frac{P_2(x)}{P_0(x)}$

then $\boxed{\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0}$ — (2)

- eq^n (2) is called the normal/standard /canonical form of eq^n (1).

The power series sol^n about a point $x = x_0$ depends on following definitions:

(1) Ordinary point / regular point :- A point $x = x_0$ is called an ordinary pt. of eq^n (2), if $P(x)$ & $Q(x)$ are both analytic (i.e., differentiable) at x_0.

i.e, If $P_0(x) \neq 0$ at $x = x_0$, then x_0 is an ordinary pt.

(2) Singular point :— A point $x = x_0$ is called a singular point of eq^n (2) if either $P(x)$ or $Q(x)$ or both are not analytic at x_0.

i.e, If $P_0(x) = 0$ at $x = x_0$, then x_0 is a singular point.

Regular Singular Pt. :- A singular point is called a regular singular point of eqⁿ ② if $(x-x_0)P(x)$ & $(x-x_0)^2 Q(x)$ both are analytic (or differentiable) at $x=x_0$.

$$\lim_{x \to x_0} (x-x_0)P(x) = \text{finite value}$$

$$\lim_{x \to x_0} (x-x_0)^2 Q(x) = \text{finite value}$$

Irregular Singular point :- A singular point is called an irregular singular point of eqⁿ ② if either $(x-x_0)P(x)$ or $(x-x_0)^2 Q(x)$ or both are not analytic (or differentiable) at $x=x_0$

Q:- Find the singular point & ordinary point of the equation

$$(1-x^2)y'' - 6xy' - 4y = 0$$

Solution :- Given eqⁿ is

$$(1-x^2)y'' - 6xy' - 4y = 0 \quad \text{——} \quad (1)$$

Here $P_0(x) = 1-x^2$

At singular point

$$P_0(x) = 0$$

$$1-x^2 = 0$$

$x = \pm 1$ are singular points.

Eqⁿ ① can be written as (in normal form)

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' - \frac{6x}{1-x^2}y' - \frac{4}{1-x^2}y = 0 \quad \text{——} \quad (2)$$

$$P(x) = \frac{-6x}{1-x^2} \quad \& \quad Q(x) = \frac{-4}{1-x^2}$$

$$\Rightarrow P(x) = \frac{6x}{(x+1)(x-1)} \quad \& \quad Q(x) = \frac{4}{(x+1)(x-1)}$$

For $x = 1$,

$$\lim_{x \to 1} (x-1) \cdot \frac{6x}{(x+1)(x-1)} = \frac{6}{2} = 3$$

$$\lim_{x \to 1} (x-1)^2 \frac{4}{(x+1)(x-1)} = 0$$

$\Rightarrow x = 1$ is a regular singular point.

- For $x = -1$,

$$\lim_{x \to -1} (x+1) \frac{6x}{(x+1)(x-1)} = \frac{-6}{-2} = 3$$

$$\lim_{x \to -1} (x+1)^2 \frac{6x}{(x+1)(x-1)} = \frac{-6 \times 0}{-2} = 0$$

$\Rightarrow x = -1$ is also a regular singular point.

- All the values of $x \neq \pm 1$ are ordinary regular pt.

Series Solution about an ordinary pt.

Let the power series solution of eq^n

$$P_0(x) y'' + P_1(x) y' + P_2(x) y = 0 \quad \text{——} \ (1)$$

about an ordinary point x_0 be given as

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$= a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots \quad \text{——} \ (2)$$

The co-eff $a_1, a_2, a_3, \ldots$ are obtained as follows:

1) Let $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ be series solution of given eq^n

2) Differentiate y wrt x twice to get y, y', y'' then substitute y, y', y'' in eq^n (1)

3) Shift the summation index to obtain a common power of x in each term.

4) Equate the co-eff of various powers of x to zero to obtain $a_1, a_2, \ldots, a_n$ in terms of a_0.

5) Substitute $a_1, a_2, \ldots$ in eq^n (2) to obtain the required sol^n of given eq^n (1).

Question 1:- Find the power series solution of the given diff eq^n

$$(x^2+1)y'' + xy' - xy = 0 \text{ about } x=0.$$

Solution:- Given eq^n

$$(x^2+1)y'' + xy' - xy = 0 \quad \text{—— (1)}$$

Here $P_0(x) = 1 + x^2$

At $x=0$, $P_0(x) = 1 \neq 0$

$\Rightarrow x=0$ is an ordinary pt.

Let the series solution of eq^n (1) be

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

Also, $y'' = \sum_{n=0}^{\infty} a_n \cdot n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substituting y, y' & y'' in eq^n ①,

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n(x^n + x^{n-2}) + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

To obtain a common power of x in each term, put $n-2=m$ in the ~~a~~ first term & $n+1=t$ in the fourth term,

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n$$

$$- \sum_{t=1}^{\infty} a_{t-1} x^t = 0$$

Since m & t are dummy variable, replacing m & t by n.

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n$$

$$-\sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow \left[2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n\right] + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \left[a_1 x + \sum_{n=2}^{\infty} n a_n x^n\right] -$$

$$\left[a_0 x + \sum_{n=2}^{\infty} a_{n-1}\cdot x^n\right] = 0$$

$$\Rightarrow 2a_2 + (6a_3 + a_1 - a_0)x + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} + (n-1)na_n + na_n - a_{n-1}\right]x^n = 0$$

$$\Rightarrow 2a_2 + (6a_3 - a_1 - a_0)x + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} + n^2 a_n - a_{n-1}\right]x^n = 0$$

Equating the constant term, the co-effs of x & x^n to zero, we get

$$2a_2 = 0 \Rightarrow a_2 = 0$$

$$6a_3 + a_1 - a_0 = 0 \Rightarrow a_3 = \frac{1}{6}(a_0 - a_1)$$

$$(n+1)(n+2)a_{n+2} + n^2 a_n - a_{n-1} = 0 \quad (n \geq 2)$$

$$\Rightarrow a_{n+2} = \frac{a_{n-1} - n^2 a_n}{(n+1)(n+2)}$$

Put $n = 2, 3, \ldots$

$$a_4 = \frac{a_1 - 4a_2}{12} = \frac{1}{12}a_1$$

$$a_5 = \frac{a_2 - 9a_3}{20} = \frac{-9a_3}{20} = \frac{-9}{20} \times \frac{1}{6}(a_0 - a_1)$$

$$= \frac{3}{40}(a_1 - a_0)$$

and so on

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_1 x + 0 \cdot x^2 + \cdots$$

$$= \frac{1}{6}(a_0 - a_1)x^3 + \frac{1}{12}a_1 x^4 + \frac{3}{40}(a_1 - a_0) \cdot x^5 + \cdots$$

$$= a_0\left(1 + \frac{x^3}{6} - \frac{3}{40}x^5 + \cdots\right) + a_1\left(x - \frac{x^3}{6} + \frac{x^4}{12} + \frac{3}{40}x^5 + \cdots\right)$$