Cramer's Rule

Let A be a non-singular matrix, then by Cramer's rule the solution of $AX = B$, is given by $$x_i = \frac{|A_i|}{|A|}, \quad i = 1, 2, \cdots, n$$
where $|A_i|$ is the determinant of the matrix A_i which is obtained by replacing the i^{th} column of A by the right hand side column vector B.

Following cases arise here:

Case I :- When $|A| \neq 0$ the system of equations is consistent and the solution obtained is unique.

Case II :- When $|A| = 0$ and one or more of $|A_i|$, $i = 1, 2, \cdots, n$ are not zero, then the system of equations has no solution and the system is inconsistent.

Case III :- When $|A| = 0$ & all $|A_i| = 0$, $i = 1, 2, \cdots, n$, then the system of equations is consistent and has infinite number of solution

Problem 1:- Solve the following system of equations

$x+2y+3z=0$, $2x+3y-2z=0$, $4x+7y+4z=0$

Solution:- The system of eq^ns can be written as $AX=0$

where $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -2 \\ 4 & 7 & 4 \end{bmatrix}$ & $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Here $|A|=0$, hence A is a singular matrix.

& $|A_1| = |A_2| = |A_3| = 0$

Therefore, the system bears infinite no. of solutions.

Taking $z=t$, the first two equations give

$x+2y+3t=0$ & $2x+3y-2t=0$ by taking $z=t$

$\Rightarrow x=13t$ & $y=-8t$, $z=t$, where t is arbitrary

Also, this solution satisfies the third equation as well.

Problem 2:- Solve the following system of equations

$4x+9y+3z=6$, $2x+3y+z=2$, $2x+6y+2z=7$

Solution:- The system of equations is $AX=B$

where $A=\begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B=\begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}$

Here $|A|=0$

$|A_1| = \begin{vmatrix} 6 & 9 & 3 \\ 2 & 3 & 1 \\ 7 & 6 & 2 \end{vmatrix} = 0$, $|A_2| = \begin{vmatrix} 4 & 6 & 3 \\ 2 & 2 & 1 \\ 2 & 7 & 2 \end{vmatrix} = 6$

$$|A_3| = \begin{vmatrix} 4 & 9 & 6 \\ 2 & 3 & 2 \\ 2 & 6 & 7 \end{vmatrix} = -18$$

Since $|A| = 0$ & $|A_2| \neq 0$ the system of eq^{n}s is inconsistent.

Problem 3:- Solve the following system of eq^{n}s

$x - y + z = 4$, $2x + y - 3z = 0$, $x + y + z = 2$

Solution:- The given system of equations is $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \ \& \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$|A| = 10$$

$$|A_1| = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 20$$

$$|A_2| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix} = -10$$

$$|A_3| = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 10$$

$$\therefore x = \frac{|A_1|}{|A|} = 2, \quad y = \frac{|A_2|}{|A|} = \frac{-10}{10} = -1$$

$$z = \frac{|A_3|}{|A|} = \frac{10}{10} = 1$$

Problem 4:- Solve the following system of equations

$x - y + 3z = 3$, $2x + 3y + z = 2$, $3x + 2y + 4z = 5$

Solution :- The system of equations is

$AX = B$ where

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; \& \; B = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Here $|A| = 0$

$$|A_1| = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix} = 0, \quad |A_2| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 0$$

$$\& \quad |A_3| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 0$$

This means that the system of equations has infinite no. of solutions. From the first two equations, taking $z = t$, we have

$x - y = 3 - 3t$ & $2x + 3y = 2 - t$

These, on solving for x & y, give

$x = \dfrac{11 - 10t}{5}$ & $y = \dfrac{5t - 4}{5}$, $z = t$ where t is arbitrary.

Taking various values of t we can have various values of x & y.

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