Orthogonal Matrix

Definition :- A square matrix A is said to be orthogonal if $A'A = I$

If A is an orthogonal matrix, then $A'A = I$

$\Rightarrow |A'A| = |I|$

$\Rightarrow |A'||A| = 1$

$\Rightarrow |A|.|A| = 1$

$\Rightarrow |A|^2 = 1$

$\Rightarrow |A| = \pm 1$

$\Rightarrow |A| \neq 0$

$\Rightarrow$ A is invertible.

Also, then $A'A = I$

$\Rightarrow A' = A^{-1}$

$\Rightarrow AA' = I$

Thus A is an orthogonal matrix if and only if $\boxed{A'A = I = AA'}$

Theorem :- If A, B be n-rowed orthogonal matrices, AB & BA are also orthogonal matrices.

Proof :- Since A and B are both n-rowed square matrices, therefore AB is also an n-rowed square matrix

Since $|AB| = |A||B|$ & $|A| \neq 0$, also $|B| \neq 0$, therefore $|AB| \neq 0$. Hence AB is a non-singular matrix.

Now $(AB)' = B'A'$

$$\therefore (AB)'(AB) = (B'A')(AB)$$
$$= B'(A'A)B \qquad (\because A'A = I)$$
$$= B'B = I \qquad (\because B'B = I)$$

$\therefore$ AB is orthogonal. Similarly we can prove that BA is also orthogonal.

Problem 1:- Show that $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution:- Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$A' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$AA' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$\Rightarrow AA' = I$

$\Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix.

Problem 2:- Prove that $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is orthogonal.

Solution:- Let $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$AA' = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since $AA' = I$

A is an orthogonal matrix

Problem 3:- If (l_r, m_r, n_r), $r = 1, 2, 3$ be the direction cosines of three mutually perpendicular lines referred to an orthogonal cartesian co-ordinate system, then prove that

$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$ is an orthogonal matrix.

Solution:- Let $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$

Then $A' = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

We have $AA' = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

$$= \begin{bmatrix} l_1^3 + m_1^3 + n_1^3 & l_1 l_2 + m_1 m_2 + n_1 n_2 & l_1 l_3 + m_1 m_3 + n_1 n_3 \\ l_2 l_1 + m_2 m_1 + n_2 n_1 & l_2^2 + m_2^2 + n_2^2 & l_2 l_3 + m_2 m_3 + n_2 n_3 \\ l_3 l_1 + m_3 m_1 + n_3 n_1 & l_3 l_2 + m_3 m_2 + n_3 n_2 & l_3^2 + m_3^2 + n_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \qquad \left[\because l_1^2 + m_1^2 + n_1^2 = 1 \;\&\; l_1 l_2 + m_1 m_2 + n_1 n_2 = 0\right]$$

Hence the matrix A is orthogonal.

Problem 4:- Show that every 2-rowed real orthogonal matrix is of any one of the forms

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ or } \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Solution:- Let $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ be any 2-rowed real orthogonal matrix.

We have $A' = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

Therefore $A'A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

$$= \begin{bmatrix} a_1^2 + a_2^2 & a_1 b_1 + a_2 b_2 \\ a_1 b_1 + a_2 b_2 & b_1^2 + b_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Comparing these, we get

$$a_1^2 + a_2^2 = 1, \quad b_1^2 + b_2^2 = 1, \quad a_1 b_1 + a_2 b_2 = 0 \quad \text{—(1)}$$

Since a_1, a_2, b_1, b_2 are to be all real, therefore the numerical value of each of them cannot exceed unity. Hence there exist real angle θ and ϕ such that

$$\left.\begin{aligned} & a_1 = \cos\theta, \quad b_1 = \cos\phi \\ \& \quad & a_2 = \pm\sin\theta, \quad b_2 = \pm\sin\phi \end{aligned}\right\} \text{—— (2)}$$

From (1), $a_1 b_1 + a_2 b_2 = 0$, gives

$$\cos(\phi - \theta) = 0 \quad \text{or} \quad \cos(\phi + \theta) = 0$$

according as we take the same or different signs in (2). Considering all the possibilities for the values of a_1, a_2, b_1, b_2 we obtain the following four possible orthogonal matrices:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Changing θ to $-\theta$, we see that the first & second matrices respectively coincide with the fourth & third so that we have only two families of orthogonal matrices of order 2 given by

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$; θ being the parameter.

Problem 5:- Show that if A is an orthogonal matrix, then A' & A^{-1} are also orthogonal.

Solution:- A is orthogonal

$\Rightarrow A'A = I$

$\Rightarrow (A'A)' = I'$

$\Rightarrow A'(A')' = I$

$\Rightarrow A'$ is orthogonal.

Again A is orthogonal

$\Rightarrow (A'A) = I$

$\Rightarrow (A'A)^{-1} = I^{-1}$

$\Rightarrow A^{-1}(A')^{-1} = I$

$\Rightarrow A^{-1}(A^{-1})' = I$

$\Rightarrow A^{-1}$ is orthogonal.

Exercise

1) Verify that the following matrices are orthogonal

(a) $\begin{bmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

(c) $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

(d) $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$

2) Find the values of a, b, c if $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal

(Sol^n $a = \pm\frac{1}{\sqrt{2}}, b = \pm\frac{1}{\sqrt{6}}, c = \pm\frac{1}{\sqrt{3}}$)

3) Prove that $\begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix}$ is orthogonal when $l = \frac{2}{7}$, $m = \frac{3}{7}$, $n = \frac{6}{7}$.