CHAPTER 1

Partial Differentiation, Euler's Theorem, Change of Variables, Expansion of Functions, Approximation, Jacobians, Maxima-Minima

Partial Derivative of First and Second Order, Euler's Theorem for Homogenous Functions, Derivatives of Implicit Functions, Total Derivatives, Change of Variables, Applications of Partial Differentiation: Taylor's theorem for Functions of two Variables, Error and Approximation, Jacobian's, Maxima-minima, Lagrange's Method of Undetermined Multipliers

FUNCTIONS OF SEVERAL VARIABLES

In Engineering problems one frequently comes across a variable quantity which depends for its values on two or more independent variables. For instance, the area of a rectangle is a function of two independent variables, its length and breadth.

We use the notation $f(x, y)$, to denote the value of the function at (x, y) and write $z = f(x, y)$. Also, we may write $z = z(x, y)$. The concept can be easily extended to the functions of three or more variables. Thus, $\omega = f(x, y, z)$ denotes the value of the function f a point at (x, y, z), in three dimensional space.

PARTIAL DERIVATIVES:

Let us extend the concept of ordinary derivative of the function of one variable to the derivative of a function f of two independent variables x and y. Now, question arises, should we differentiate f with respect to x or y? The answer is simple : treat y as constant while differentiating f with respect to x and treat x as constant while differentiating f with respect to y. This way we define two different derivatives and call them partial derivatives to distinguish them from the ordinary derivative of a function of a single variable.

We denote the partial derivative of f with respect to x by $\dfrac{\partial f}{\partial x}$ or f_x

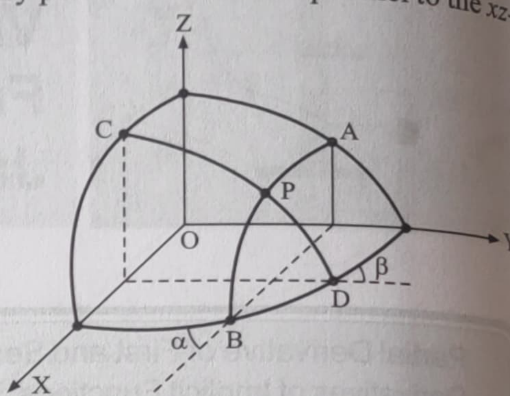
and the partial derivative of f w.r.t. y by $\dfrac{\partial f}{\partial y}$ or f_y

Thus, $\dfrac{\partial f}{\partial x} = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x, y) - f(x, y)}{\delta x} = \left(\dfrac{\partial f}{\partial x}\right)_y$ = value of $\dfrac{\partial f}{\partial x}$ when y is kept constant.

and $\dfrac{\partial f}{\partial y} = \lim\limits_{\delta y \to 0} \dfrac{f(x, y+\delta y) - f(x, y)}{\delta y} = \left(\dfrac{\partial f}{\partial y}\right)_x$ = value of $\dfrac{\partial f}{\partial y}$ when x is kept constant.

GEOMETRICAL INTERPRETATION OF PARTIAL DERIVATIVES

Partial derivative of a function of two variables $z = f(x, y)$ represents the equation of a surface in x, y, z coordinates system. Let APB be the curve cut by a plane through any point on the surface parallel to the xz-plane. As point P moves along this curve APB, its coordinates z and x vary while y remains constant. The slope of the tangent line at P to the curve APB represents the rate at which z changes w.r.t. x.



Thus $\dfrac{\partial z}{\partial x} = \tan\alpha =$ slope of the curve APB at the point P

and $\dfrac{\partial z}{\partial y} = \tan\beta =$ slope of the curve CPD at the point P as shown in the adjoining figure.

In general, $f_x = \left(\dfrac{\partial f}{\partial x}\right)$ and $f_y = \left(\dfrac{\partial f}{\partial y}\right)$ are functions of both x and y and therefore, we can also obtain the second order partial derivatives of $f(x, y)$ as

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}, \quad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy},$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}, \text{ etc.}$$

In $\dfrac{\partial^2 f}{\partial x\,\partial y}$ we first differentiate f partially with respect to y and then with respect to x. It should be noted that $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$, in general, meaning thereby, that the order of differentiation is immaterial.

EXAMPLE 1.1. (*a*) If $z = f(x + ay) + \phi(x - ay)$, then show that $z_{yy} = a^2 z_{xx}$. [*AKTU 2011*]

(*b*) If $u = e^{xyz}$, find $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$ [*GGSIPU 2013*]

SOLUTION: (*a*) From $z = f(x + ay) + \phi(x - ay)$, we get

$$z_x = \frac{\partial z}{\partial x} = f'(x + ay)\frac{\partial}{\partial x}(x + ay) + \phi'(x - ay)\frac{\partial}{\partial x}(x - ay) = f'(x + ay) + \phi'(x - ay)$$

where f' and ϕ' mean ordinary derivatives of f and ϕ with respect to $x + ay$ and $x - ay$ respectively.

Similarly, $$z_y = \frac{\partial z}{\partial y} = f'(x + ay)\frac{\partial}{\partial y}(x + ay) + \phi'(x - ay)\frac{\partial}{\partial y}(x - ay) = a f'(x + ay) - a\,\phi'(x - ay).$$

Now, differentiating z_x and z_y again partially with respect to x and y respectively, gives

$$z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x} f'(x + ay) + \frac{\partial}{\partial x}\phi'(x - ay)$$

$$= f''\frac{\partial}{\partial x}(x + ay) + \phi''\frac{\partial}{\partial x}(x - ay) = f''(x + ay) + \phi''(x - ay).$$

and $$z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}[af'(x+ay)] - \frac{\partial}{\partial y}[a\,\phi'(x-ay)]$$

$$= af''\frac{\partial}{\partial y}(x+ay) - a\,\phi''\frac{\partial}{\partial y}(x-ay) = a^2 f''(x+ay) + (-a)^2\phi''(x-ay)$$

which clearly gives $z_{yy} = a^2 z_{xx}$. **Hence Proved.**

(*b*) $u = e^{xyz}$ hence $\frac{\partial u}{\partial x} = e^{xyz}(yz) = yzu.$ Similarly $\frac{\partial u}{\partial y} = xz\,u$ and $\frac{\partial u}{\partial z} = xy\,u.$

Therefore, $$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial}{\partial y}(xyu) = x\left(y\frac{\partial u}{\partial y} + u\right) = x[y(xzu) + u] = u(x + x^2yz)$$

Further $$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{\partial}{\partial x}[u(x+x^2yz)] = (x+x^2yz)\frac{\partial u}{\partial x} + u(1+2xyz)$$

$$= uyz\,(x+x^2yz) + u(1+2xyz) = u[xy\,z(1+xy\,z) + 1 + 2xy\,z]$$

$$= u\,[1 + 3xy\,z + x^2y^2z^2] = (1 + 3xyz + x^2y^2z^2)\,e^{xyz}.$$ **Ans.**

EXAMPLE 1.2. Show that, at a point on the surface $x^x\, y^y\, z^z = c$ where $x = y = z$, we have $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{-1}{x\,\log(ex)}$. [*GGSIPU 2010*]

SOLUTION: Since $x^x y^y z^z = c$ $\quad x\log x + y\log y + z\log z = \log c.$ ...(1)

Here, we can take x as a function of y and z, or y as a function of z and x, or z as a function of x and y. Since we are to calculate $\frac{\partial^2 z}{\partial x \partial y}$, take z as a function of x and y.

Differentiating (1) partially with respect to y, gives

$$0 + y\cdot\frac{1}{y} + 1.\log y + z\cdot\frac{1}{z}\frac{\partial z}{\partial y} + \log z\cdot\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z}. \quad ...(2)$$

Similarly, differentiating (1) partially w.r.t. x, gives $\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{1+\log z}.$ ...(3)

Next, differentiating (2) partially w.r.t. x, we get

$$\frac{\partial^2 z}{\partial x\,\partial y} = -(1+\log y)\frac{\partial}{\partial x}\frac{1}{(1+\log z)} = -(1+\log y)\frac{(-1)}{(1+\log z)^2}\frac{\partial}{\partial x}(1+\log z) = \frac{1+\log y}{(1+\log z)^2}\frac{1}{z}\frac{\partial z}{\partial x}$$

Using (3) here, gives $$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{1+\log y}{(1+\log z)^2}\,\frac{(-1)(1+\log x)}{z\,(1+\log z)} = \frac{-(1+\log x)(1+\log y)}{z(1+\log z)^3}.$$

At the point $x = y = z$, we have $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{-1}{x(1+\log x)} = \frac{-1}{x\log(ex)}.$ **Hence Proved.**

EXAMPLE 1.3. (*a*) If $z\,(x+y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

(*b*) If $z = e^{ax+by} f(ax - by)$ show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2\,abz.$ [*GGSIPU 2005*]

(*c*) If $e^{-z/(x^2 - y^2)} = x - y$ show that $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x^2 - y^2.$ [*AKTU 2017*]

SOLUTION: (*a*) Differentiating $z(x+y)=x^2+y^2$, partially w.r.t. x and y separately, we get

$$\frac{\partial z}{\partial x}(x+y)+z=2x \quad \text{and} \quad \frac{\partial z}{\partial y}(x+y)+z=2y$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}=\frac{2(x-y)}{x+y} \quad \text{and} \quad \left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)=\frac{2(x+y)-2z}{(x+y)}.$$

$$\therefore \quad \left[1-\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)\right]=1-\left[\frac{2(x+y-z)}{x+y}\right]=\frac{-(x+y-2z)}{x+y}=-1+\frac{2z}{x+y}$$

$$=-1+\frac{2(x^2+y^2)}{(x+y)^2}=\left(\frac{x-y}{x+y}\right)^2=\frac{1}{4}\left[\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]^2.$$

Hence Proved.

(*b*) Given that $z=e^{ax+by}f(ax-by)$, we have

$$\frac{\partial z}{\partial x}=ae^{ax+by}f(ax-by)+ae^{ax+by}f'(ax-by)$$

and $$\frac{\partial z}{\partial y}=be^{ax+by}f(ax-by)-be^{ax+by}f'(ax-by)$$

Therefore $$b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=abe^{ax+by}[2f(ax-by)]=2abz.$$

Hence the result.

(*c*) $$e^{-z/(x^2-y^2)}=x-y \quad \text{hence} \quad z=(y^2-x^2)\log(x-y).$$

$$\therefore \quad \frac{\partial z}{\partial x}=\frac{y^2-x^2}{x-y}-2x\log(x-y) \quad \text{and} \quad \frac{\partial z}{\partial y}=-\left(\frac{y^2-x^2}{x-y}\right)+2y\log(x-y)$$

Hence $$y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=x^2-y^2.$$

Hence the result.

EXAMPLE 1.4. (*a*) If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, show that $u_x^2+u_y^2+u_z^2=2(xu_x+yu_y+zu_z)$.

(*b*) If $u=\log(x^3+y^3-x^2y-y^2x)$, show that $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}\right)^2 u=-\dfrac{4}{(x+y)^2}$. [*GGSIPU 2013*]

SOLUTION: (*a*) Presence of u_x, u_y and u_z indicates that we have to take u as a function of x, y and z.

Differentiating the given relation partially *w.r.t.* x, we get

$$\frac{2x}{a^2+u}-\frac{x^2u_x}{(a^2+u)^2}-\frac{y^2u_x}{(b^2+u)^2}-\frac{z^2u_x}{(c^2+u)^2}=0$$

or $$u_x=\frac{2x}{K(a^2+u)} \quad \text{where} \quad K=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}.$$

Similarly, differentiating the given relation partially *w.r.t.* y and z separately, gives

$$u_y=\frac{2y}{K(b^2+u)} \quad \text{and} \quad u_z=\frac{2z}{K(c^2+u)}.$$

Therefore, $$u_x^2+u_y^2+u_z^2=\frac{4}{K^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=\frac{4}{K^2}.K=\frac{4}{K}.$$

And $$xu_x+yu_y+zu_z=\frac{2}{K}\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]=\frac{2}{K}\cdot 1 \quad \text{(using the given relation.)}$$

This establishes that $u_x^2+u_y^2+u_z^2=2(xu_x+yu_y+zu_z)$.

Hence Proved.

(*b*) Differentiating the given relation *w.r.t.* x and y separately, we get

$$\frac{\partial u}{\partial x}=\frac{3x^2-2xy-y^2}{x^3+y^3-x^2y-y^2x} \quad \text{and} \quad \frac{\partial u}{\partial y}=\frac{3y^2-x^2-2xy}{x^3+y^3-x^2y-y^2x}$$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\frac{2x^2+2y^2-4xy}{x^3+y^3-xy\,(x+y)}=\frac{2\,(x-y)^2}{(x+y)\,[x^2+y^2-xy-xy]}=\frac{2}{x+y}.$$

Next, $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\frac{2}{x+y}=\frac{-4}{(x+y)^2}.$$ **Hence Proved.**

EXAMPLE 1.5. (*a*) If $\theta=t^n e^{-\frac{r^2}{4t}}$ find the value of n for which $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)=\dfrac{\partial\theta}{\partial t}$.

(*b*) If $V=(x^2+y^2+z^2)^{m/2}$ find the value of $m(\neq 0)$ which will make $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}+\dfrac{\partial^2 V}{\partial z^2}=0$ [GGSIPU 2006]

SOLUTION: (*a*) From the relation $\theta=t^n e^{-\frac{r^2}{4t}}$, it is clear that θ is a function of r and t.

Taking logarithm on the both sides, we get $$\log\theta=n\log t-\frac{r^2}{4t} \qquad \text{... (1)}$$

Differentiating (1) partially *w.r.t.* r, gives $\dfrac{1}{\theta}\dfrac{\partial\theta}{\partial r}=0-\dfrac{2r}{4t}$ hence $r^2\dfrac{\partial\theta}{\partial r}=-\dfrac{r^3\theta}{2t}$. ... (2)

Differentiating (2) partially *w.r.t.* r, gives

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{3r^2\theta}{2t}-\frac{r^3}{2t}\frac{\partial\theta}{\partial r}=-\frac{3r^2\theta}{2t}+\frac{r^4\theta}{4t^2} \quad \text{or} \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\left(-\frac{3}{2t}+\frac{r^2}{4t^2}\right)\theta. \qquad \text{...(3)}$$

Next, differentiating (1) partially *w.r.t.* t, we get $\dfrac{1}{\theta}\dfrac{\partial\theta}{\partial t}=\dfrac{n}{t}+\dfrac{r^2}{4t^2}$. ...(4)

Since $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)=\dfrac{\partial\theta}{\partial t}$, (given), we get $n=-3/2$. **Ans.**

(*b*) Given $V=(x^2+y^2+z^2)^{m/2}$ hence $$\frac{\partial V}{\partial x}=\frac{m}{2}(x^2+y^2+z^2)^{\frac{m}{2}-1}\cdot 2x$$

and $$\frac{\partial^2 V}{\partial x^2}=mx\left(\frac{m}{2}-1\right)(x^2+y^2+z^2)^{\frac{m}{2}-2}\cdot 2x+m(x^2+y^2+z^2)^{\frac{m}{2}-1}$$

$$=m(x^2+y^2+z^2)^{\frac{m}{2}-2}\,[(m-2)\,x^2+(x^2+y^2+z^2)].$$ Similarly for $\dfrac{\partial^2 V}{\partial y^2}$ and $\dfrac{\partial^2 V}{\partial z^2}$.

Thus, $$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=m(x^2+y^2+z^2)^{\frac{m}{2}-2}\,[(m-2)\,(x^2+y^2+z^2)+3(x^2+y^2+z^2)]$$

$$=m(m+1)\,(x^2+y^2+z^2)^{\frac{m}{2}-1}=0 \quad \text{when } m=0 \text{ or } m=-1$$

$\therefore$ non-zero value of m is -1. **Ans.**

Example 1.6. (a) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

[GGSIPU 2012; 2013; 2017; AKTU 2004]

(b) If $x = \frac{\cos\theta}{u}$, $y = \frac{\sin\theta}{u}$ show that $\left(\frac{\partial x}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial y}\right)_x = 1$

[GGSIPU 2003, 2006]

Solution: (a) Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$, we have

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}, \quad \frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}, \quad \frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\therefore \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} = \frac{3}{x+y+z}.$$

Next $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \frac{3}{(x+y+z)} = \frac{-9}{(x+y+z)^2}.$$ **Hence proved.**

(b) From $x = \frac{\cos\theta}{u}$, $y = \frac{\sin\theta}{u}$, we can get $\left(\frac{\partial x}{\partial u}\right)_\theta = -\frac{\cos\theta}{u^2}$ and $\left(\frac{\partial y}{\partial u}\right)_\theta = -\frac{\sin\theta}{u^2}$.

Next, to find $\left(\frac{\partial u}{\partial x}\right)_y$ and $\left(\frac{\partial u}{\partial y}\right)_x$ we express u as a function of x and y.

Eliminating θ in the given relations, we get $x^2u^2 + y^2u^2 = 1$ or $u^2 = \frac{1}{x^2 + y^2}$

Therefore, $2u\left(\frac{\partial u}{\partial x}\right)_y = \frac{-2x}{(x^2+y^2)^2} = -2xu^4$ or $\left(\frac{\partial u}{\partial x}\right)_y = -xu^3 = -u^2\cos\theta$.

Similarly we can get $\left(\frac{\partial u}{\partial y}\right)_x = -yu^3 = -u^2\sin\theta$.

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial y}\right)_x = \frac{-\cos\theta}{u^2}(-u^2\cos\theta) - \frac{\sin\theta}{u^2}(-u^2\sin\theta) = 1.$$ **Hence Proved.**

Example 1.7. If $u = f(r)$ where $r^2 = x^2 + y^2 + z^2$, show that $u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r}f'(r)$.

Solution: From the relation $r^2 = x^2 + y^2 + z^2$, we get $2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Since $u = f(r)$ we have $u_x = \frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$.

Differentiating again w.r.t x partially, we get

$$u_{xx} = \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] = f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r}f''(r)\frac{\partial r}{\partial x}$$

$$= f'(r)\left(\frac{1}{r}-\frac{x}{r^2}\cdot\frac{\partial r}{\partial x}\right)+\frac{x^2}{r^2}f''(r) = \frac{f'(r)}{r}+x^2\left(\frac{1}{r^2}f''(r)-\frac{f'(r)}{r^3}\right).$$

Similarly, $u_{yy}=\frac{f'(r)}{r}+y^2\left(\frac{1}{r^2}f''(r)-\frac{f'(r)}{r^3}\right)$ and $u_{zz}=\frac{f'(r)}{r}+z^2\left(\frac{1}{r^2}f''(r)-\frac{f'(r)}{r^3}\right)$

$\therefore$ $u_{xx}+u_{yy}+u_{zz} = \frac{3f'(r)}{r}+(x^2+y^2+z^2)\left\{\frac{1}{r^2}f''(r)-\frac{f'(r)}{r^3}\right\} = f''(r)+\frac{2}{r}f'(r)$

Hence Proved.

Homogeneous Functions:

An expression of the form $a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+...+a_ny^n$ in which all the terms are of same degree, is called homogeneous function in x and y of degree n. The definition can be extended to functions of three or more variables. The above expression can also be written as

$$x^n\,[a_0+a_1\,(y/x)+a_2\,(y/x)^2+...+a_n\,(y/x)^n] = x^n f(y/x)$$

where $f(y/x)$ is a n^{th} degree polynomial in (y/x). Thus, $x^n f(y/x)$ defines a homogeneous function in x and y, of degree n.

For example, $x^4\tan(y/x)$ is homogeneous in x and y of degree 4 while, the expression $\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$ is a homogeneous function of degree $-\frac{1}{6}\left(=\frac{1}{3}-\frac{1}{2}\right)$. [AKTU 2009; 2013, 2015]

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS:

If u is a homogeneous function in x and y, of degree n, then $\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,u.}$

Proof : We can write $u=x^n f(y/x)$ or $u=x^n f(t)$ where $t=y/x$.

Then $\frac{\partial u}{\partial x} = nx^{n-1}f(t)+x^n f'(t)\frac{\partial t}{\partial x} = nx^{n-1}f(t)+x^n f'(t)(-y/x^2) = nx^{n-1}f(t)-x^{n-2}yf'(t)$,

and $\frac{\partial u}{\partial y} = x^n f'(t)\frac{\partial t}{\partial y} = x^n f'(t)\frac{1}{x} = x^{n-1}f'(t)$

Therefore, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nx^n f(t)-x^{n-1}yf'(t)+x^{n-1}yf'(t) = nu.$

Some Useful Deductions of Euler's Theorem:

I. If u is a homogeneous function in x, y, z, of degree n, we can write $u=x^n f(y/x, z/x)$ then

$$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.}$$

II. If u is a homogeneous function in x and y of degree n, then

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)\,u.}$$

Proof : We already have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$ ...(1)

Differentiating (1) partially *w.r.t.* x and y separately, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}.1+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \quad \text{...(2)}$$

and $$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+1.\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y} \quad \text{...(3)}$$

Multiplying (2) by x and (3) by y and adding these, gives

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=nx\frac{\partial u}{\partial x}+ny\frac{\partial u}{\partial y}$$

Using (1) here and the fact that $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\,x\frac{\partial u}{\partial x}+(n-1)\,y\frac{\partial u}{\partial y}=n(n-1)u$$

III. If z is homogeneous in x and y of degree n, and z is a function of u as $z=f(u)$, **then**

$$\boxed{(i)\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{n\,f(u)}{f'(u)}}$$

$$\boxed{(ii)\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\,[g'(u)-1]\quad \text{where}\quad g(u)=\frac{n\,f(u)}{f'(u)}.}$$

PROOF: (*i*) Since z is homogenous in x and y of degree n, we have $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$ ...(4)

Next, since $z=f(u)$, we have $\frac{\partial z}{\partial x}=f'(u)\frac{\partial u}{\partial x}$, $\quad \frac{\partial z}{\partial y}=f'(u)\frac{\partial u}{\partial y}$.

Therefore (4) becomes $\quad x\,f'(u)\frac{\partial u}{\partial x}+y\,f'(u)\frac{\partial u}{\partial y}=n f(u)$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{n\,f(u)}{f'(u)}=g(u)\quad \text{where}\quad \boxed{g(u)=\frac{n\,f(u)}{f'(u)}}. \quad \text{...(5)}$$

(*ii*) Differentiating (5) partially *w.r.t.* x and y separately, gives

$$x\frac{\partial^2 u}{\partial x^2}+1.\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=g'(u)\frac{\partial u}{\partial x} \quad \text{...(6)}$$

and $$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+1.\frac{\partial u}{\partial y}=g'(u)\frac{\partial u}{\partial y} \quad \text{...(7)}$$

Multiplying (6) by x and (7) by y and than adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=g'(u)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)(g'(u)-1)=g(u)\,(g'(u)-1).$$

EXAMPLE 1.8. (*a*) If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, then evaluate $x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy}$.

[GGSIPU 2011]

(*b*) If $z = x^4 y^2 \sin^{-1}\frac{x}{y} + \log x - \log y$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 6x^4 y^2 \sin^{-1}\frac{x}{y}$.

SOLUTION: (a) Here $\tan^{-1}(y/x)$ and $\tan^{-1}(x/y)$ are both of degree zero and hence behave like constants. Therefore z is homogeneous function in x and y of degree two. Applying Euler's theorem, we get

$$x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = 2(2-1)\, z = 2z.$$ **Ans.**

(b) Let us write $z = u + v$ where $u = x^4 y^2 \sin^{-1}\frac{x}{y}$ and $v = \log x - \log y = \log\left(\frac{x}{y}\right)$.

Observe here that $\frac{x}{y}$ is homogenous of degree 0, so is $\sin^{-1}\frac{x}{y}$ and, in turn, u is homogenous in x, y of degree $6(= 4 + 2)$. Also $\log\left(\frac{x}{y}\right)$ is homogenous in x and y of degree 0, therefore by Euler's theorem, we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6u$ and $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0$.

Adding these, gives $x\frac{\partial}{\partial x}(u+v) + y\frac{\partial}{\partial y}(u+v) = 6u + 0$

or $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 6x^4 y^2 \sin^{-1}\frac{x}{y}$. **Hence Proved.**

EXAMPLE 1.9. If $u = \sin^{-1}\frac{x^2 + y^2}{x + y}$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$. [GGSIPU 2007]

SOLUTION: Given $u = \sin^{-1}\frac{x^2 + y^2}{x + y}$ hence $\sin u = \frac{x^2 + y^2}{x + y}$.

Clearly $\sin u$ is a homogeneous function in x and y of degree one, hence by Euler's theorem

$$x\frac{\partial}{\partial x}\sin u + y\frac{\partial}{\partial y}\sin u = 1 \sin u \quad \text{or} \quad x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$. **Hence the result.**

EXAMPLE 1.10. (*a*) If $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$.

Also find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y}\, y^2\frac{\partial^2 u}{\partial y^2}$. [GGSIPU 2010]

(*b*) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u = \sin 2u(1 - 4\sin^2 u).$$ [GGSIPU 2011]

SOLUTION: (*a*) $u = \log\left(\dfrac{x^2+y^2}{x+y}\right) \quad \therefore \quad \dfrac{x^2+y^2}{x+y} = e^u = f(u)$, say.

Here $f(u)$ is a homogeneous function in x and y of degree 1, hence

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = 1\cdot\frac{e^u}{e^u} = 1.$$

Also we know that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y}\, y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u)-1]$

where $\quad g(u) = \dfrac{nf(u)}{f'(u)}$ which is equal to 1 as shown above and $g'(u) = 0$.

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 1(0-1) = -1.$$

Ans.

(*b*) Give that $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ hence $\tan u = \dfrac{x^3+y^3}{x-y} = f(u)$, say.

Clearly $\tan u$ is a homogeneous function in x and y of degree 2, hence by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = \frac{2\tan u}{\sec^2 u} = \sin 2u = g(u), \text{ say.}$$

Also, by Euler's theorem (Deduction III), we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[.g'(u)-1] = \sin 2u\,[2\cos 2u - 1] = \sin 4u - \sin 2u.$$

Hence the result.

EXAMPLE 1.11. (*a*) If $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0$.

[*AKTU 2019*]

(*b*) If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\sin u \cos 2u}{4\cos^3 u}$

[*GGSIPU 2009; 2006; 2012*]

SOLUTION: (*a*) $\cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = z$, say where z is homogenous factor in x and y of degrees $\dfrac{1}{2}$

$\therefore$ By Euler's theorem $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{1}{2}z$ or $x\dfrac{\partial}{\partial x}\cos u + y\dfrac{\partial}{\partial y}\cos u = \dfrac{1}{2}\cos u$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

(*b*) Since $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ we have $\sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = f(u)$, say.

Clearly, $\sin u$ is homogeneous in x and y of degree 1/2, hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u = g(u), \text{ say.}$$

Also by Euler's theorem (Deduction III), we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u)-1] = \frac{1}{2}\tan u\left[\frac{1}{2}\sec^2 u-1\right] = \frac{1}{4}\tan u\,(\sec^2 u-2)$$

$$= \frac{1}{4}\frac{\sin u}{\cos u}\left[\frac{1}{\cos^2 u}-2\right] = \frac{-1}{4}\frac{\sin u\cos 2u}{\cos^3 u}.$$ **Hence the result.**

EXAMPLE 1.12. (*a*) If $u = \sin^{-1}(x^2+y^2)^{1/5}$ prove that

$$x^2 u_{xx}+2xy\,u_{xy}+y^2 u_{yy} = \frac{2}{25}\tan u\,(2\tan^2 u-3).$$ [*GGSIPU 2003*]

(*b*) If $u = \dfrac{(x^2+y^2)^n}{2n(2n-1)}+x\,f\left(\dfrac{y}{x}\right)+\phi\left(\dfrac{x}{y}\right)$, evaluate $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}$.

SOLUTION: (a) Since $u = \sin^{-1}(x^2+y^2)^{1/5}$ we have $\sin u = (x^2+y^2)^{1/5} = f(u)$, Say.

Since $\sin u$ is homogeneous in x and y of degree 2/5, we have, by Euler's theorem

$$xu_x+yu_y = \frac{nf(u)}{f'(u)} = \frac{2}{5}\tan u = g(u).$$

Also, by Euler's theorem (Deduction III), we have

$$x^2 u_{xx}+2xy\,u_{xy}+y^2 u_{yy} = g(u)\,[g'(u)-1] = \frac{2}{5}\tan u\left[\frac{2}{5}\sec^2 u-1\right] = \frac{2}{25}\tan u\,[2\tan^2 u-3].$$

(*b*) Let $u = u_1+u_2+u_3$ where $u_1 = \dfrac{(x^2+y^2)^n}{2n(2n-1)}$, $u_2 = x\,f\left(\dfrac{y}{x}\right)$, $u_3 = \phi\left(\dfrac{x}{y}\right)$. Clearly u_1 is homogeneous in x and y of degree $2n$, u_2 is homogeneous in x and y of degree 1 and u_3 is homogenous in x and y of degree 0. Thus, applying Euler's theorem to u_1, u_2, u_3, separately we get

$$x^2\frac{\partial^2 u_1}{\partial x^2}+2xy\frac{\partial^2 u_1}{\partial x\partial y}+y^2\frac{\partial^2 u_1}{\partial y^2} = 2n(2n-1)\,u_1$$

$$x^2\frac{\partial^2 u_2}{\partial x^2}+2xy\frac{\partial^2 u_2}{\partial x\partial y}+y^2\frac{\partial^2 u_2}{\partial y^2} = 1\,(1-1)\,u_2 = 0$$

$$x^2\frac{\partial^2 u_3}{\partial x^2}+2xy\frac{\partial^2 u_3}{\partial x\partial y}+y^2\frac{\partial^2 u_3}{\partial y^2} = 0\,(0-1)\,u_3 = 0$$

Adding these, gives $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2} = 2n(2n-1)\,u_1+0+0 = (x^2+y^2)^n$. **Ans.**

EXAMPLE 1.13. (*a*) If $u = x^n f\left(\frac{y}{x}\right) + y^{-n} \phi\left(\frac{x}{y}\right)$, show that

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + xu_x + yu_y = n^2 u.$$

(*b*) If $u = \sin^{-1}\left(\frac{x^3 + y^3 + z^3}{ax + by + cz}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2 \tan u.$

[*GGSIPU 2015; AKTU 2018*]

SOLUTION: (a) Let $u = u_1 + u_2$ where $u_1 = x^n f\left(\frac{y}{x}\right)$ and $u_2 = y^{-n} \phi\left(\frac{x}{y}\right)$.

Here u_1 is homogeneous in x and y of degree n while u_2 is homogenous in x and y of degree $-n$. Making use of Euler's theorem to u_1 and u_2 separately, we get

$$x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} = nu_1, \quad \ldots(1) \qquad x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} = -nu_2. \quad \ldots(2)$$

Also, $$x^2\frac{\partial^2 u_1}{\partial x^2} + 2xy\frac{\partial^2 u_1}{\partial x \partial y} + y^2\frac{\partial^2 u_1}{\partial y^2} = n(n-1)\, u_1 \quad \ldots(3)$$

and $$x^2\frac{\partial^2 u_2}{\partial x^2} + 2xy\frac{\partial^2 u_2}{\partial x \partial y} + y^2\frac{\partial^2 u_2}{\partial y^2} = -n(-n-1)u_2. \quad \ldots(4)$$

Adding (1), (2), (3) and (4), gives

$$x^2\frac{\partial^2}{\partial x^2}(u_1 + u_2) + 2xy\frac{\partial^2}{\partial x \partial y}(u_1 + u_2) + y^2\frac{\partial^2}{\partial y^2}(u_1 + u_2) + x\frac{\partial}{\partial x}(u_1 + u_2) + y\frac{\partial}{\partial y}(u_1 + u_2)$$
$$= n(n-1)\, u_1 + (-n)(-n-1)\, u_2 + nu_1 - nu_2$$

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u_1(n^2 - n + n) + u_2(n^2 + n - n) = n^2 u.$$

Hence Proved.

(**b**) $f(u) = \sin u = \frac{x^3 + y^3 + z^3}{ax + by + cz}$ is homogeneous in x, y, z of degree 2.

Now using Euler's theorem deduction I, we get

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{2 f(u)}{f'(u)} = \frac{2 \sin u}{\cos u} = 2 \tan u.$$

Hence Proved.

EXERCISE 1A

1. If $u = \dfrac{e^{x+y+z}}{e^x + e^y + e^z}$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 2u.$

2. If $V = \log(\tan x + \tan y + \tan z)$ find the value of
$$\sin 2x \frac{\partial V}{\partial x} + \sin 2y \frac{\partial V}{\partial y} + \sin 2z \frac{\partial V}{\partial z}.$$
[*GGSIPU 2015; AKTU 2012*]

3. Given that $z = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, find the value of $\dfrac{\partial^2 z}{\partial y \, \partial x}$ [*AKTU 2018*]

4. If $u^2 = x^2 + y^2 + z^2$ find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}.$

5. If $u(x, t) = ae^{-gx} \sin(nt - gx)$ where a, g and n are constants, and $\dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}$, show that $g = \dfrac{1}{a}\sqrt{\dfrac{n}{2}}$.

6. If $u = (1 - 2xy + y^2)^{-1/2}$ prove that $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = y^2 u^3.$
And also evaluate $\dfrac{\partial}{\partial x}\left\{(1 - x^2)\dfrac{\partial u}{\partial x}\right\} + \dfrac{\partial}{\partial y}\left\{y^2 \dfrac{\partial u}{\partial y}\right\}.$

7. If $u = e^{r\cos\theta} \cos(r \sin\theta)$ and $v = e^{r\cos\theta} \sin(r\sin\theta)$, show that
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

8. If $u = r^n$ and $r^2 = x^2 + y^2 + z^2$, prove that $u_{xx} + u_{yy} + u_{zz} = n(n+1) r^{n-2}.$

9. Given that $u = x\log(x + r) - r$ where $r^2 = x^2 + y^2$, evaluate and simplify $u_{xx} + u_{yy}$.

10. Let $u = lx + my$ and $v = mx - ly$ where l and m are constants, show that
$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2} = \left(\frac{\partial y}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x$$

11. If $u = \log(\tan x + \tan y)$, evaluate $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y}.$

12. Let $xu + yv = 0$ and $\dfrac{u}{x} + \dfrac{v}{y} = 1$, then show that $\left(\dfrac{\partial u}{\partial x}\right)_y \left(\dfrac{\partial v}{\partial y}\right)_x = \dfrac{y^2 + x^2}{y^2 - x^2}.$

13. If $u = e^{xyz} f\left(\dfrac{xz}{y}\right)$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2xyzu = y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}.$
and hence deduce that $x\dfrac{\partial^2 u}{\partial x \, \partial z} = y\dfrac{\partial^2 u}{\partial y \, \partial z}.$

14. Given $x = \cos\theta - r\sin\theta$ and $y = \sin\theta + r\cos\theta$, show that
(*i*) $\dfrac{\partial \theta}{\partial x} = \dfrac{-\cos\theta}{r}$, $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ (*ii*) $\dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{\cos\theta}{r^3}(\cos\theta - 2r\sin\theta)$

15. If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r).$
[*GGSIPU 2010; AKTU 2016*]

16. If $u=\dfrac{x^3+y^3}{y\sqrt{x}}+\dfrac{1}{x^7}\sin^{-1}\left(\dfrac{x^2+y^2}{x^2+2xy}\right)$ find the value of $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}+xu_x+yu_y$ at the point (1, 2).

17. If $u=\sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ show that $\dfrac{\partial u}{\partial x}=\dfrac{-y}{x}\dfrac{\partial u}{\partial y}$.

18. If $z=x^3e^{-x/y}$ prove that $x^2z_{xx}+2xy\,z_{xy}+y^2z_{yy}=6z$.

19. If $u=\sin^{-1}\left(\dfrac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right)$, evaluate $x^2\dfrac{\partial^2u}{\partial x^2}+2xy\dfrac{\partial^2u}{\partial x\,\partial y}+y^2\dfrac{\partial^2u}{\partial y^2}$. [AKTU 2017]

20. If $y=x\sin u$, find the value of $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}$.

21. If $f(x,y)=\dfrac{1}{x^2}+\dfrac{1}{xy}+\dfrac{\log x-\log y}{x^2+y^2}$, show that $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}+2f=0$.

22. Determine the value of $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}$ if $u=\sin^{-1}(x^3+y^3)^{2/5}$.

23. (a) If $u=\sec^{-1}\left(\dfrac{x^3-y^3}{x-y}\right)$, then evaluate $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}$. [AKTU 2018]

(b) If $u=\operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ evaluate $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}$

24. If $u=\dfrac{x^2y^2z^2}{x^2+y^2+z^2}+\cos\dfrac{2xy+yz+zx}{x^2+y^2+z^2}$ evaluate $xu_x+yu_y+zu_z$.

25. If $u=\dfrac{1}{r}f(\theta)$ where $x=r\cos\theta$, $y=r\sin\theta$, then evaluate $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$. [AKTU 2019]

26. (a) If $u=\left(\sqrt{x}+\sqrt{y}\right)^5$, find the value of $x^2\dfrac{\partial^2u}{\partial x^2}+2xy\dfrac{\partial^2u}{\partial x\,\partial y}+y^2\dfrac{\partial^2u}{\partial y^2}$.

(b) If $u=\log\sqrt{x^2+y^2}$, evaluate $x^2\dfrac{\partial^2u}{\partial x^2}+2xy\dfrac{\partial^2u}{\partial x\,\partial y}+y^2\dfrac{\partial^2u}{\partial y^2}$.

27. If $x=e^u\tan v$, $y=e^u\sec v$ find the value of $\left(x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}\right)\left(x\dfrac{\partial v}{\partial x}+y\dfrac{\partial v}{\partial y}\right)$.

28. If $u=\log\dfrac{x^4+y^4}{x+y}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=3$. [GGSIPU 2012]

29. If $u=\sin^{-1}\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=-3\tan u$. [GGSIPU 2014]

30. Prove that $xu_x+yu_y=\dfrac{5}{2}\tan u$ where $u=\sin^{-1}\left(\dfrac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$. [AKTU 2015]

TOTAL DERIVATIVE:

If $u=f(x,y)$ is a function of two independent variables x and y, and x and y are separately functions of a single independent variable t, then u can be expressed as a function of t alone and then we can find the ordinary derivative $\frac{du}{dt}$ which is called the *total derivative* of u w.r.t. t.

Let us be interested in finding $\frac{du}{dt}$ without actually substituting the values of x and y in terms of t, in $f(x,y)$. We derive below the relation between the total derivative $\frac{du}{dt}$ and the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

When t is given an increment δt, suppose x and y get increments δx and δy respectively and this, in turn, causes u to get an increment of δu. In other words, when t becomes $t+\delta t$, let x become $x+\delta x$ and y become $y+\delta y$ and consequently, u becomes $u+\delta u$. Thus, δx and δy both tend to 0 as δt tends to 0 and then,

$$\frac{du}{dt} = \lim_{\delta t\to 0}\frac{f(x+\delta x, y+\delta y)-f(x,y)}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{f(x+\delta x, y+\delta y)-f(x, y+\delta y)+f(x, y+\delta y)-f(x,y)}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{f(x+\delta x, y+\delta y)-f(x, y+\delta y)}{\delta x}\cdot\frac{\delta x}{\delta t}+\lim_{\delta t\to 0}\frac{f(x, y+\delta y)-f(x,y)}{\delta y}\cdot\frac{\delta y}{\delta t}$$

$$= \lim_{\delta x\to 0}\frac{f(x+\delta x, y+\delta y)-f(x, y+\delta y)}{\delta x}\cdot\left(\frac{dx}{dt}\right)+\lim_{\delta y\to 0}\frac{f(x, y+\delta y)-f(x,y)}{\delta y}\cdot\left(\frac{dy}{dt}\right)$$

$$=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt} \quad \text{since } u=f(x,y).$$

Thus, we have $$\boxed{\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}} \quad ...(1)$$

and in terms of differentials this result can be better written as

$$\boxed{du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy} \quad ...(2)$$

This du is called the total derivative of u.

On extending this result to functions of three variables, $u=u(x,y,z)$, we have

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt} \quad \text{or} \quad du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz.$$

Some Deductions

If $f(x,y)=0$ is an implicit relation between x and y, *i.e.*, if y is an implicit function of x, then

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0 \qquad \therefore \quad \boxed{\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}}. \quad ...(3)$$

Now we intend to calculate $\frac{d^2y}{dx^2}$ in terms of partial derivatives. But, before that, let us introduce the following conventional notations:

$$\boxed{\frac{\partial f}{\partial x}=p,\ \frac{\partial f}{\partial y}=q,\ \frac{\partial^2 f}{\partial x^2}=\frac{\partial p}{\partial x}=r,\ \frac{\partial^2 f}{\partial y^2}=\frac{\partial q}{\partial y}=t,\ \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}=s.}$$

Thus, the relation (3) can be written as $\boxed{\dfrac{dy}{dx} = \dfrac{-p}{q}}$.

Hence $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(-\dfrac{p}{q}\right) = -\dfrac{q\dfrac{dp}{dx} - p\dfrac{dq}{dx}}{q^2}$. Now, using (1) we can write

$$\frac{dp}{dx} = \frac{\partial p}{\partial x}.1 + \frac{\partial p}{\partial y}.\frac{dy}{dx} = r + s\left(\frac{-p}{q}\right) = \frac{qr - ps}{q}$$

and $$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\frac{dy}{dx} = s + t\left(\frac{-p}{q}\right) = \frac{qs - pt}{q}.$$

Thus $$\frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q.\frac{qr - ps}{q} - p.\frac{qs - pt}{q}\right] = -\frac{1}{q^3}\left[q^2r - 2pqs + p^2t\right]$$

Thus, we have $$\boxed{\frac{d^2y}{dx^2} = -\frac{(q^2r - 2pqs + p^2t)}{q^3}}.$$

CHANGE OF INDEPENDENT VARIABLES

Let $u = f(x, y)$ where $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$. It is frequently necessiated to change the expressions involving $u, x, y, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$ to the expressions involving u, t_1, t_2, $\dfrac{\partial u}{\partial t_1}, \dfrac{\partial u}{\partial t_2}$, etc.

In this constext, the earlier result $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt}$ can now be extended as follows:

$$\boxed{\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_1} \quad \text{and} \quad \frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}}.$$

Next, if we are given that $u = u(t_1, t_2)$, where $t_1 = \phi_1(x, y)$ and $t_2 = \phi_2(x, y)$
Then the equations of transformation will be

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}.\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}.\frac{\partial t_2}{\partial y}}.$$

Conversion from cartesian to polar coordinates and *vice versa*, are frequently encountered examples. Thus, if $u = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial \theta}$$

and then other way round, we have the formulae

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

EXAMPLE 1.14. Find $\dfrac{dy}{dx}$ if (i) $(\cos x)^y - (\sin y)^x = 0$. (ii) $y^x + x^y = (x + y)^{(x+y)}$

SOLUTION: (i) The given relation is $(\cos x)^y = (\sin y)^x$ or $y\log(\cos x) = x\log(\sin y)$
Now, let us write $f(x, y) = y\log(\cos x) - x\log(\sin y) = 0$

So y can be taken as an imlicit function of x or *vice versa.*

Here $\quad \dfrac{\partial f}{\partial x} = \dfrac{y(-\sin x)}{\cos x} - 1.\log(\sin y) = -(y\tan x + \log\sin y)$

and $\quad \dfrac{\partial f}{\partial y} = \log\cos x\,.\,1 - \dfrac{x\cos y}{\sin y} = \log\cos x - x\cot y.$

Therefore, $\quad \dfrac{dy}{dx} = -\dfrac{\partial f}{\partial x}\Big/\dfrac{\partial f}{\partial y} = \dfrac{(y\tan x + \log\sin y)}{\log\cos x - x\cot y}.$ **Ans.**

(ii) Let $f(x,y) = y^x + x^y - (x+y)^{(x+y)} = 0.$

$\therefore \quad \dfrac{\partial f}{\partial x} = y^x\log y + y\,.\,x^{y-1} - (x+y)^{x+y}[1+\log(x+y)]$

and $\quad \dfrac{\partial f}{\partial y} = x\,y^{x-1} + x^y\log x - (x+y)^{x+y}[1+\log(x+y)]$

Hence $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\left[\dfrac{y^x\log y + y\,x^{y-1} - (x+y)^{x+y}(1+\log(x+y))}{x\,y^{x-1} + x^y\log x - (x+y)^{x+y}(1+\log(x+y))}\right].$ **Ans.**

EXAMPLE 1.15. (*a*) If $f(x,y)=0$ and $\phi(y,z)=0$, show that $\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial\phi}{\partial z}\dfrac{dz}{dx} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial\phi}{\partial y}.$

(*b*) If $x^n + y^n = a^n$, find $\dfrac{d^2y}{dx^2}$.

SOLUTION: (*a*) Since $f(x,y)=0$, we have $\dfrac{dy}{dx} = -\dfrac{\partial f}{\partial x}\Big/\dfrac{\partial f}{\partial y}.$

And since $\phi(y,z)=0$, we have $\dfrac{dz}{dy} = -\dfrac{\partial\phi}{\partial y}\Big/\dfrac{\partial\phi}{\partial z}$

$$\therefore \quad \frac{dy}{dx}\cdot\frac{dz}{dy} = (-1)^2\,\frac{\dfrac{\partial f}{\partial x}\,\dfrac{\partial\phi}{\partial y}}{\dfrac{\partial f}{\partial y}\,\dfrac{\partial\phi}{\partial z}} \quad \text{or} \quad \frac{\partial f}{\partial y}\cdot\frac{\partial\phi}{\partial z}\frac{dz}{dx} = \frac{\partial f}{\partial x}\,\frac{\partial\phi}{\partial y}.$$ **Hence Proved.**

(b) Let $f(x,y) = x^n + y^n - a^n = 0$ then $p = \dfrac{\partial f}{\partial x} = n\,x^{n-1}$ and $q = \dfrac{\partial f}{\partial y} = n\,y^{n-1},$

and $\quad r = \dfrac{\partial^2 f}{\partial x^2} = n(n-1)\,x^{n-2},\ \ s = \dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{\partial p}{\partial y} = 0$ and $t = \dfrac{\partial^2 f}{\partial y^2} = n(n-1)\,y^{n-2}.$

Since $f(x,y)=0$ we can take y as an implicit function of x.

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{1}{q^3}[q^2 r - 2pqs + p^2 t]$$

$$= -\frac{1}{n^3\,y^{3n-3}}[n^2y^{2n-2}\cdot n(n-1)\,x^{n-2} - 2\,n^2x^{n-1}y^{n-1}\cdot(0) + n^2x^{2n-2}\cdot n(n-1)\,y^{n-2}]$$

$$= -\frac{n^3(n-1)}{n^3\,y^{3n-3}}[x^{n-2}y^{2n-2} + y^{n-2}x^{2n-2}]$$

$$= -\frac{(n-1)}{y^{3n-3}}x^{n-2}y^{n-2}(y^n + x^n) = -\frac{(n-1)\,x^{n-2}\,.\,a^n}{y^{2n-1}}.$$ **Ans.**

EXAMPLE 1.16. (*a*) If $z=\sqrt{x^2+y^2}$ and $x^3+y^3+3axy=5a^3$, find the value of $\dfrac{dz}{dx}$ when $x=y=a$.

(*b*) If $z=xy\,f(y/x)$ and z is constant, show that $\dfrac{f'(y/x)}{f(y/x)}=\dfrac{x\left(y+x\dfrac{dy}{dx}\right)}{y\left(y-x\dfrac{dy}{dx}\right)}$.

[GGSIPU 2005] [GGSIPU 2010; 2016]

SOLUTION: (a) Since $z=\sqrt{x^2+y^2}$ we have, from total derivative concept,

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \quad \text{or} \quad \frac{dz}{dx}=\frac{\partial z}{\partial x}\cdot 1+\frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$\therefore \quad \frac{dz}{dx}=\frac{1}{2}(x^2+y^2)^{-1/2}2x\cdot 1+\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y\cdot\frac{dy}{dx}=\frac{1}{\sqrt{x^2+y^2}}\left[x+y\frac{dy}{dx}\right]$$

From the relation $x^3+y^3+3axy=5a^3$, we can have

$$\frac{dy}{dx}=\frac{-\dfrac{\partial}{\partial x}(x^3+y^3+3axy)}{\dfrac{\partial}{\partial y}(x^3+y^3+3axy)}=\frac{-(3x^2+3ay)}{3y^2+3ax}=-1 \text{ at } (a,a).$$

$$\therefore \quad \left(\frac{dz}{dx}\right)_{(a,a)}=\left[\frac{1}{\sqrt{x^2+y^2}}\{x+y(-1)\}\right]_{(a,a)}=0. \qquad \textbf{Ans.}$$

(b) Since z is constant we write $xy\,f(y/x)=z=$ constant

$$\therefore \quad \frac{dy}{dx}=\frac{-\dfrac{\partial}{\partial x}[xyf(y/x)]}{\dfrac{\partial}{\partial y}[xyf(y/x)]}=\frac{-[yf(y/x)+xyf'(y/x)(-y/x^2)]}{[xf(y/x)+xyf'(y/x)(1/x)]}$$

or

$$\frac{x\dfrac{dy}{dx}}{y}=\frac{\dfrac{y}{x}f'(y/x)-f(y/x)}{\dfrac{y}{x}f'(y/x)+f(y/x)}.$$ Applying componendo and dividendo here, we get

$$\frac{y+x\dfrac{dy}{dx}}{y-x\dfrac{dy}{dx}}=\frac{2\dfrac{y}{x}f'(y/x)}{2f(y/x)}=\frac{yf'(y/x)}{xf(y/x)} \quad \therefore \quad \frac{f'(y/x)}{f(y/x)}=\frac{x\left(y+x\dfrac{dy}{dx}\right)}{y\left(y-x\dfrac{dy}{dx}\right)}.$$ **Hence the result.**

EXAMPLE 1.17. (*a*) If $u=f(x-y,y-z,z-x)$ show that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$ [AKTU 2019]

(*b*) If $u=f(e^{x-y},e^{y-z},e^{z-x})$ find the value of $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}$. [GGSIPU 2011]

SOLUTION: (*a*) Let $x'=x-y,\ y'=y-z,\ z'=z-x$ then we have $u=f(x',y',z')$

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial u}{\partial y'}\cdot\frac{\partial y'}{\partial x}+\frac{\partial u}{\partial z'}\cdot\frac{\partial z'}{\partial x}=\frac{\partial u}{\partial x}\cdot 1+0+\frac{\partial u}{\partial z}(z)^{-3}$$

But $\dfrac{\partial x'}{\partial x}=1,\ \dfrac{\partial y'}{\partial x}=0$ and $\dfrac{\partial z'}{\partial x}=-1$, hence $\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial x'}-\dfrac{\partial u}{\partial z'}$

Similarly, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\cdot\frac{\partial y'}{\partial y} + \frac{\partial u}{\partial z'}\cdot\frac{\partial z'}{\partial y} = -\frac{\partial u}{\partial x'} + \frac{\partial u}{\partial y'} + 0$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial u}{\partial y'}\cdot\frac{\partial y'}{\partial z} + \frac{\partial u}{\partial z'}\cdot\frac{\partial z'}{\partial z} = 0 - \frac{\partial u}{\partial y'} + \frac{\partial u}{\partial z'}$$

Therefore, $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x'} - \frac{\partial u}{\partial z'} - \frac{\partial u}{\partial x'} + \frac{\partial u}{\partial y'} - \frac{\partial u}{\partial y'} + \frac{\partial u}{\partial z'} = 0.$$ **Hence Proved**

(b) Putting $X' = e^{x-y}$, $Y' = e^{y-z}$, $Z' = e^{z-x}$ we have $u = u(X', Y', Z')$

then $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X'}\cdot\frac{\partial X'}{\partial x} + \frac{\partial u}{\partial Y'}\frac{\partial Y'}{\partial x} + \frac{\partial u}{\partial Z'}\frac{\partial Z'}{\partial x} = e^{x-y}\cdot\frac{\partial u}{\partial X'} + 0\cdot\frac{\partial u}{\partial Y'} - e^{z-x}\cdot\frac{\partial u}{\partial Z'}$$

Similarly $$\frac{\partial u}{\partial y} = -e^{x-y}\cdot\frac{\partial u}{\partial X'} + e^{y-z}\cdot\frac{\partial u}{\partial Y'} + 0\cdot\frac{\partial u}{\partial Z'} \quad\text{and}\quad \frac{\partial u}{\partial z} = 0\cdot\frac{\partial u}{\partial X'} - e^{y-z}\cdot\frac{\partial u}{\partial Y'} + e^{z-x}\cdot\frac{\partial u}{\partial Z'}$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$ **Ans.**

EXAMPLE 1.18. (*a*) If $x = u + v + w$, $y = uv + vw + wu$, $z = uvw$ and F is a function of x, y, z, then show that $u\frac{\partial F}{\partial u} + u\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$.

[*GGSIPU 2014*]

(*b*) If $z = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$ [*GGSIPU 2009, 2011*]

SOLUTION: (a) By change of independent variables, we have

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u} = 1\cdot\frac{\partial F}{\partial x} + (v+w)\frac{\partial F}{\partial y} + vw\frac{\partial F}{\partial z},$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial v} = 1\cdot\frac{\partial F}{\partial x} + (w+u)\frac{\partial F}{\partial y} + wu\frac{\partial F}{\partial z}$$

and $$\frac{\partial F}{\partial w} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial w} = 1\cdot\frac{\partial F}{\partial x} + (u+v)\frac{\partial F}{\partial y} + uv\frac{\partial F}{\partial z}.$$

Therefore $$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = (u+v+w)\frac{\partial F}{\partial x} + 2(uv+vw+wu)\frac{\partial F}{\partial y} + 3uvw\frac{\partial F}{\partial z}$$

$$= x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$ **Hence Proved.**

(b) Here $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \qquad ...(1)$$

and $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

or $$\frac{1}{r}\frac{\partial z}{\partial \theta} = -\sin\theta\frac{\partial z}{\partial x} + \cos\theta\frac{\partial z}{\partial y}. \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta + 2\sin\theta\cos\theta\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$+\left(\frac{\partial z}{\partial x}\right)^2 \sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \cos^2\theta - 2\sin\theta\cos\theta\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$=\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

Hence Proved.

EXAMPLE 1.19. Let $z = f(x, y)$ where $x = e^u \cos v$ and $y = e^u \sin v$, then show that

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y} \quad \text{and} \quad \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right].$$

[GGSIPU 2017][AKTU 2018]

SOLUTION: We have $z = f(x, y)$ and x and y are functions of u and v,

Hence $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = e^u \cos v\frac{\partial z}{\partial x} + e^u \sin v\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \quad ...(1)$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -e^u \sin v\frac{\partial z}{\partial x} + e^u \cos v\frac{\partial z}{\partial y} = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} \quad ...(2)$$

$$\therefore\quad y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} - xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y}$$

$$= (x^2 + y^2)\frac{\partial z}{\partial y} = (e^{2u}\cos^2 v + e^{2u}\sin^2 v)\frac{\partial z}{\partial y} = e^{2u}\frac{\partial z}{\partial y}.$$

Next, squaring and adding (1) and (2), gives

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = x^2\left(\frac{\partial z}{\partial x}\right)^2 + 2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + y^2\left(\frac{\partial z}{\partial y}\right)^2 + y^2\left(\frac{\partial z}{\partial x}\right)^2 - 2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + x^2\left(\frac{\partial z}{\partial y}\right)^2$$

$$= (x^2 + y^2)\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right].$$

or $$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$ **Hence the result.**

EXAMPLE 1.20. By changing the independent variables u and v to x and y by means of the relations $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$. Also, show that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$ transforms into $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$.

[GGSIPU 2006]

SOLUTION: Since $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}.$$

$$\therefore \quad u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v} = (u\cos\alpha - v\sin\alpha)\frac{\partial z}{\partial x}+(u\sin\alpha+v\cos\alpha)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$$ **Hence proved**

Next
$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial x}\left(\cos\alpha\frac{\partial z}{\partial x}+\sin\alpha\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial u}+\frac{\partial}{\partial y}\left(\cos\alpha\frac{\partial z}{\partial x}+\sin\alpha\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial u}$$

$$= \left(\cos\alpha\frac{\partial^2 z}{\partial x^2}+\sin\alpha\frac{\partial^2 z}{\partial x\partial y}\right)\cos\alpha+\left(\cos\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin\alpha\frac{\partial^2 z}{\partial y^2}\right)\sin\alpha$$

$$= \cos^2\alpha\frac{\partial^2 z}{\partial x^2}+2\sin\alpha\cdot\cos\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2 z}{\partial y^2}.$$

and
$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial x}\left(-\sin\alpha\frac{\partial z}{\partial x}+\cos\alpha\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial v}+\frac{\partial}{\partial y}\left(-\sin\alpha\frac{\partial z}{\partial x}+\cos\alpha\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial v}$$

$$= \left(-\sin\alpha\frac{\partial^2 z}{\partial x^2}+\cos\alpha\frac{\partial^2 z}{\partial x\partial y}\right)(-\sin\alpha)+\left(-\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\cos\alpha\frac{\partial^2 z}{\partial y^2}\right)\cos\alpha$$

$$= \sin^2\alpha\frac{\partial^2 z}{\partial x^2}-2\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y}+\cos^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}.$$ **Hence the result.**

EXAMPLE 1.21. Transform the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ into polar co-ordinates r and θ.

[*GGSIPU 2009; 2013*]

SOLUTION: Since $x=r\cos\theta$, $y=r\sin\theta$ we have $r^2=x^2+y^2$ and $\theta=\tan^{-1}\frac{y}{x}$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}.$$

From $r^2=x^2+y^2$, we have $2r\frac{\partial r}{\partial x}=2x$ or $\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta$ and $\frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta$.

and $\theta=\tan^{-1}\frac{y}{x}$ gives $\frac{\partial \theta}{\partial x}=\frac{-y}{x^2+y^2}=\frac{-\sin\theta}{r}$ and $\frac{\partial \theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{\cos\theta}{r}$.

Therefore
$$\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial y}=\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}.$$

Thus, in terms of operators we can write

$$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial}{\partial y}\equiv\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}.$$

Therefore, $\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial u}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta}\right]$$

$$-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r}+\cos\theta\frac{\partial^2 u}{\partial\theta\,\partial r}-\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\,\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2}=\sin^2\theta\frac{\partial^2 u}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}$$

Therefore, $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}$$

Thus, the Laplace equation in cartesian form $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ gets transformed into polar form as $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0.$ **Ans.**

EXAMPLE 1.22. (a) Let f be a composite function of u and v and u and v be functions of x and y, given by $u=x^2-y^2$, $v=2xy$, then show that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4(x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right).$$

(b) If $z=f(x,y)$, $x^2=uv$, $y^2=u/v$ then change the independent variables x and y to u and v in the equation $x^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial xy}+y^2\frac{\partial^2 z}{\partial y^2}+2y\frac{\partial z}{\partial y}=0.$

[GGSIPU 2010]

SOLUTION: (a) Given that $f=f(u,v)$ and $u=x^2-y^2$, $v=2xy$, we have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial f}{\partial u}(2x)+\frac{\partial f}{\partial v}(2y)\Rightarrow\frac{\partial}{\partial x}\equiv 2\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)$$

and $$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial f}{\partial u}(-2y)+\frac{\partial f}{\partial v}(2x)\Rightarrow\frac{\partial}{\partial y}\equiv 2\left(-y\frac{\partial}{\partial u}+x\frac{\partial}{\partial v}\right)$$

$$\therefore\quad\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=4\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)$$

$$=4x\left(x\frac{\partial^2 f}{\partial u^2}+y\frac{\partial^2 f}{\partial u\,\partial v}\right)+4y\left(x\frac{\partial^2 f}{\partial u\,\partial v}+y\frac{\partial^2 f}{\partial v^2}\right)$$

$$=4x^2\frac{\partial^2 f}{\partial u^2}+8xy\frac{\partial^2 f}{\partial u\,\partial v}+4y^2\frac{\partial^2 f}{\partial v^2}.$$

Similarly, $$\frac{\partial^2 f}{\partial y^2}=4y^2\frac{\partial^2 f}{\partial u^2}-8xy\frac{\partial^2 f}{\partial u\,\partial v}+4x^2\frac{\partial^2 f}{\partial v^2}$$

$$\therefore\quad\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4(x^2+y^2)\frac{\partial^2 f}{\partial u^2}+4(y^2+x^2)\frac{\partial^2 f}{\partial v^2}=4(x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right).$$ **Hence Proved.**

(b) $x^2 = uv,\ y^2 = u/v \Rightarrow u^2 = x^2 y^2,\ v^2 = x^2/y^2,\quad \therefore\ u = xy,\ v = x/y.$

Therefore $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = y\dfrac{\partial z}{\partial u} + \dfrac{1}{y}\dfrac{\partial z}{\partial v}$ and $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = x\dfrac{\partial z}{\partial u} - \dfrac{x}{y^2}\dfrac{\partial z}{\partial v}$

$\Rightarrow\quad x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 2\dfrac{x}{y}\dfrac{\partial z}{\partial v} = 2v\dfrac{\partial z}{\partial v}$. Squaring the operators on both sides we get

$$\therefore\quad x^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 4v^2\frac{\partial^2 z}{\partial v^2}$$

Also, we have $\quad 2y\dfrac{\partial z}{\partial y} = 2u\dfrac{\partial z}{\partial u} - 2v\dfrac{\partial z}{\partial v}$.

Therefore the equation $\quad x^2\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x\,\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} + 2y\dfrac{\partial z}{\partial y} = 0$ becomes

$$4v^2\frac{\partial^2 z}{\partial v^2} + 2u\frac{\partial z}{\partial u} - 2v\frac{\partial z}{\partial v} = 0.$$ **Ans.**