MAL

Unit - III

1) Laplace Transform

Definition

Let $f(t)$ be a function of t defined for all positive values of t, then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ is defined by

$$\boxed{L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt}$$

provided the integral exist.

where; $s \rightarrow$ parameter which may be real or complex number.

Formulae List

1. $L(1) = 1/s$
2. $L(t^n) = \frac{\lfloor n}{s^{n+1}}$ (If n is integer)

   $L(t^n) = \frac{\Gamma n+1}{s^{n+1}}$ (If n is not integer)

   Gamma function
3. $L(e^{at}) = \frac{1}{s-a}$
4. $L(\sin at) = \frac{a}{s^2+a^2}$
5. $L(\cos at) = \frac{s}{s^2+a^2}$

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6. $L(\sinh at) = \dfrac{a}{s^2 - a^2}$

7. $L(\cosh at) = \dfrac{s}{s^2 - a^2}$

- Existence of Laplace Transform (Without Proof)

Theorem:

If $f(t)$ is a function which is piecewise continuous on every finite-interval in the range $t \geq 0$ & statisfies

$$|F(t)| \leq Me^{at} \cdot (\forall\, t \geq 0)$$

And for some constants a & M, then the Laplace transform of $f(t)$ exists $\forall\, p > a$.

Properties

1. Shifting Property

If $L(f(t)) = \bar{f}(s)$ then

$$L\left(e^{at} f(t)\right) = \bar{f}(s-a)$$

2. Division Property

If $L(f(t)) = \bar{f}(s)$

then $$L\left(\frac{1}{t} f(t)\right) = \int_s^\infty \bar{f}(s)\,ds$$

3. Transforms of integral

If $L(f(t)) = \bar{f}(s)$ then $L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}\bar{f}(s)$

4. Multiplication property

If $L(f(t)) = \bar{f}(s)$ then $L\left(t^n f(t)\right) = (-1)^n \frac{d^n}{ds^n}\left(\bar{f}(s)\right)$

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Problems on Laplace

Ques.

Find Laplace Transform of

$$f(t) = t e^{-4t} \sin 3t$$

Sol→

$$L(\sin 3t) = \frac{3}{s^2+9}$$

(Use multiplication property)

$$L(t \sin 3t) = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right) = (-3) \frac{d}{ds}(s^2+9)^{-1}$$

$$\Rightarrow (-3)(-1)(s^2+9)^{-2}(2s) \Rightarrow \frac{6s}{(s^2+9)^2}$$

(Use Shifting property)

$$L(e^{-4t}\, t \sin 3t) = \frac{6(s+4)}{((s+4)^2+9)^2}$$

$$\Rightarrow \frac{6(s+4)}{(s^2+8s+25)^2}$$ Ans

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Ques: find Laplace transform of

$$f(t) = e^{-t}\int_0^t \frac{\sin t}{t}\,dt$$

Soln $L(\sin t) = \frac{1}{s^2+1}$

(Use division property)

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}\,ds \Rightarrow \left[\tan^{-1} s\right]_s^\infty \Rightarrow \tan^{-1}\infty - \tan^{-1} s$$

$$\Rightarrow \pi/2 - \tan^{-1} s \Rightarrow \cot^{-1} s$$ (As; $\cot^{-1}\theta + \tan^{-1}\theta = \pi/2$)

(Use Transform of integral Property)

$$L\left(\int_0^t \frac{\sin t}{t}\,dt\right) = \frac{1}{s}\cot^{-1} s$$

(Use Shifting property)

$$L\left(e^{-t}\int_0^t \left(\frac{\sin t}{t}\right) dt\right) = \frac{1}{s+1}\cot^{-1}(s+1)$$ Ans

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Ques: Find Laplace transform of

(i) $e^{-3t}(2\cos 5t - 3\sin 5t)$

Sol→ $L(2\cos 5t - 3\sin 5t)$

$\Rightarrow 2L(\cos 5t) - 3L(\sin 5t)$

$\Rightarrow \frac{2s}{s^2+25} - \frac{3\times 5}{s^2+25} \Rightarrow \frac{2s-15}{s^2+25}$

(Use shifting property)

$L(e^{-3t}(2\cos 5t - 3\sin 5t)) = \frac{2(s+3)-15}{(s+3)^2+25}$

$\Rightarrow \frac{2s-9}{s^2+6s+34}$ Ans

(ii) $e^{2t}\cos^2 t$

Sol→ $f(t) = e^{2t}\cos^2 t$

$L(\cos^2 t) = \frac{1}{2}L(1+\cos 2t)$

(As; $\cos 2t = 2\cos^2 t - 1$)

$\frac{1}{2}\left(L(1) + L(\cos 2t)\right)$

$\Rightarrow \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$

(Use shifting Property)

$L(e^{2t}\cos^2 t) = \frac{1}{2}\left(\frac{1}{s-2} + \frac{(s-2)}{(s-2)^2+4}\right)$ Ans

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(iii) $e^{4t} \sin 2t \cos t$

Sol→ $L(\sin 2t \cos t) = \frac{1}{2} L(2 \sin 2t \cos t)$

$\Rightarrow \frac{1}{2} L(\sin 3t + \sin t) = \frac{1}{2}\left(\frac{3}{s^2+9} + \frac{1}{s^2+1}\right)$

(Use shifting Property)

$\Rightarrow L(e^{4t} \sin 2t \cos t) = \frac{1}{2}\left(\frac{3}{(s-4)^2+9} + \frac{1}{(s-4)^2+1}\right)$ Ans

Ques: Find Laplace Transform of

i) $t^3 e^{-3t}$

Sol→ $f(t) = t^3 e^{-3t}$

$L(e^{-3t}) = \frac{1}{s+3}$

(Use multiplication Property)

$L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) \Rightarrow -\frac{d^3}{ds^3}(s+3)^{-1}$

$\Rightarrow \frac{d^2}{ds^2}(s+3)^{-2} \Rightarrow (-2)\frac{d}{ds}(s+3)^{-3}$

$\Rightarrow (-2)(-3)(s+3)^{-4} \Rightarrow 6(s+3)^{-4} \Rightarrow \frac{6}{(s+3)^4}$ Ans

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(ii) $te^{-t}\sin 3t$

Sol: $f(t) = te^{-t}\sin 3t$

$$\boxed{L(\sin 3t) = \frac{3}{s^2+9}}$$

$\Rightarrow$ (Use multiplication property)

$$L(t\sin 3t) = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$\Rightarrow -3\frac{d(s^2+9)^{-1}}{ds}$$

$$\Rightarrow 3(s^2+9)^{-2}(2s) \Rightarrow \frac{6s}{(s^2+9)^2}$$

(Use shifting property)

$$L(\underbrace{e^{-t}}\ \underbrace{t\sin 3t}) = \frac{6(s+1)}{((s+1)^2+9)^2} \Rightarrow \frac{6(s+1)}{(s^2+2s+10)^2} \text{ Ans}$$

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Ques : Find Laplace Transform of

i) $\dfrac{\cos at - \cos bt}{t}$

Sol $\Rightarrow$ $L(\cos at - \cos bt) = \left(\dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}\right)$

(use division Property)

$L\left(\dfrac{\cos at - \cos bt}{t}\right) = \int_s^\infty \left(\dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}\right) ds$

$\Rightarrow \dfrac{1}{2}\int_s^\infty \left(\dfrac{2s}{s^2+a^2} - \dfrac{2s}{s^2+b^2}\right) ds$

$\Rightarrow \dfrac{1}{2}\left(\log(s^2+a^2) - \log(s^2+b^2)\right)_s^\infty$

$\Rightarrow \dfrac{1}{2}\ \cancel{\log(s^2+a^2)}$

$\Rightarrow \dfrac{1}{2}\left(\log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)\right)_s^\infty$

$\Rightarrow \dfrac{1}{2}\left(\lim_{s\to\infty} \log\left(\dfrac{s^2+a^2}{s^2+b^2}\right) - \log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)\right)$

$\Rightarrow \dfrac{1}{2}\left(\lim_{s\to\infty} \log\left(\dfrac{1+a^2/s^2}{1+b^2/s^2}\right) - \log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)\right)$

$\Rightarrow \dfrac{1}{2}\left(\log(1) - \log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)\right) \Rightarrow \dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$

$\Rightarrow \log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)^{1/2}$ Ans

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(ii) $(\sin t - \cos t)^2$

Sol $f(t) = (\sin t - \cos t)^2$

$= \sin^2 t + \cos^2 t - 2\sin t \cos t$

$\Rightarrow 1 - 2\sin t \cos t$

$\Rightarrow 1 - \sin 2t$

$L(f(t)) = L(1 - \sin 2t)$

$\Rightarrow L(1) - L(\sin 2t) = \frac{1}{s} - \frac{2}{(s^2+4)}$ Ans

Ques: Find Laplace Transform of

(i) $f(t) = 1 + 2\sqrt{t} + 3/\sqrt{t}$

Sol $\rightarrow L(f(t)) = L(1) + 2L(t^{1/2}) + 3L(t^{-1/2})$

$\Rightarrow \frac{1}{s} + \frac{2\,\lfloor 1/2}{s^{1/2+1}} + \frac{3\,\lfloor -1/2}{s^{-1/2+1}}$

$\lfloor n = \Gamma(n+1) = n\Gamma n$

$\Gamma 1/2 = \sqrt{\pi}$

$\Gamma 3/2 = 1/2 \times \sqrt{\pi}$

$\Rightarrow \frac{1}{s} + \frac{2\,\Gamma(1/2+1)}{s^{3/2}} + \frac{3\,\Gamma(-1/2+1)}{s^{1/2}}$

$\Rightarrow \frac{1}{s} + \frac{2 \times 1/2 \times \Gamma 1/2}{s^{3/2}} + \frac{3\,\Gamma 1/2}{s^{1/2}}$

$\Rightarrow \frac{1}{s} + \frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}}$ Ans

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(ii) $f(t) = \cos^3 2t$

Sol→ $\left(\begin{array}{l} \text{As; } \cos 3t = 4\cos^3 t - 3\cos t \\ \cos^3 t = \dfrac{\cos 3t + 3\cos t}{4} \end{array} \right)$

$\left(\text{As; } \cos^3 2t = \frac{1}{4}(\cos 6t + 3\cos 2t) \right)$

$L(\cos^3 2t) = \frac{1}{4} L(\cos 6t) + \frac{3}{4} L(\cos 2t)$

$\Rightarrow \frac{1}{4}\left(\frac{s}{s^2+36}\right) + \frac{3}{4}\left(\frac{s}{s^2+4}\right)$

$\Rightarrow \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right]$ Ans

Ques: Find Laplace Transform of

(i) $f(t) = (\sqrt{t} - 1/\sqrt{t})^3$

Sol→ $((a-b)^3 = a^3 - b^3 - 3ab(a-b))$

$f(t) = (t^{1/2} - t^{-1/2})^3 \Rightarrow t^{3/2} - t^{-3/2} - 3\, t^{1/2} t^{-1/2}(t^{1/2} - t^{-1/2})$

$\Rightarrow t^{3/2} - t^{-3/2} - 3t^{1/2} + 3t^{-1/2}$

$L(f(t)) = L(t^{3/2}) - L(t^{-3/2}) - 3L(t^{1/2}) + 3L(t^{-1/2})$

$\Rightarrow \frac{\Gamma(3/2+1)}{s^{3/2+1}} - \frac{\Gamma(-3/2+1)}{s^{-3/2+1}} - 3\frac{\Gamma(1/2+1)}{s^{1/2+1}} + 3\frac{\Gamma(-1/2+1)}{s^{-1/2+1}}$

$\left(\text{As } \Gamma(n+1) = n\Gamma(n) \right)$

$\Rightarrow \frac{3/2\, \Gamma(3/2)}{s^{5/2}} - \frac{\Gamma(-1/2)}{s^{-1/2}} - \frac{3 \times 1/2 \times \Gamma(1/2)}{s^{3/2}} + 3\frac{\Gamma(1/2)}{s^{1/2}}$

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$\Rightarrow \dfrac{\frac{3}{2}\,\Gamma\left(\frac{1}{2}+1\right)}{s^{5/2}} - \dfrac{\left(-2\sqrt{\pi}\right)}{s^{-1/2}} - \dfrac{3\times\frac{1}{2}\times\sqrt{\pi}}{s^{3/2}} + \dfrac{3\sqrt{\pi}}{s^{1/2}}$

$\Rightarrow \dfrac{\frac{3}{2}\times\frac{1}{2}\times\Gamma\frac{1}{2}}{s^{5/2}} + \dfrac{2\sqrt{\pi}}{s^{-1/2}} - \dfrac{3}{2}\dfrac{\sqrt{\pi}}{s^{3/2}} + \dfrac{3}{5}\dfrac{\sqrt{\pi}}{s^{1/2}}$

$\Rightarrow \dfrac{3}{4}\dfrac{\sqrt{\pi}}{s^{5/2}} + \dfrac{2\sqrt{\pi}}{s^{-1/2}} - \dfrac{3\sqrt{\pi}}{2s^{3/2}} + \dfrac{3\sqrt{\pi}}{5s^{1/2}}$ Ans

ii) $f(t) = \sinh^2 3t$

Sol→ $\left(2\sinh^2 t + 1 = \cosh 2t\right)$

So; $f(t) = \sinh^2 3t = \dfrac{\cosh 6t - 1}{2}$

$\Rightarrow L(f(t)) = \frac{1}{2}\, L(\cosh 6t - 1)$

$\Rightarrow \cancel{\frac{1}{2}\left(\frac{s}{s^2-36} - \frac{1}{s}\right)}$ ~~Ans~~

$\Rightarrow \dfrac{1}{2}\left(\dfrac{s}{s^2-36} - \dfrac{1}{s}\right)$ Ans

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Ques:

Find Laplace Transform of $\left(\frac{1-\cos t}{t^2}\right)$

Sol→ $L(1-\cos t) = 1/s - \frac{s}{s^2+1}$

(Use Division Property)

$L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds \Rightarrow \int_s^\infty \left(\frac{1}{s} - \frac{2s}{2(s^2+1)}\right) ds$

$\Rightarrow \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty \Rightarrow \left[\log s - \log(s^2+1)^{1/2}\right]_s^\infty$

$\Rightarrow \left[\log \frac{s}{(s^2+1)^{1/2}}\right]_s^\infty \Rightarrow \lim_{s\to\infty} \log \frac{s}{s(1+1/s^2)^{1/2}} - \log \frac{s}{(s^2+1)^{1/2}}$

$\Rightarrow \log 1 - \log \frac{s}{(s^2+1)^{1/2}} \Rightarrow \log \frac{(s^2+1)^{1/2}}{s} \Rightarrow \log\left(\frac{s^2+1}{s^2}\right)^{1/2}$

$\Rightarrow \frac{1}{2}\log\left(1+\frac{1}{s^2}\right) \Rightarrow \frac{1}{2}(\log(1+s^{-2}))$

Again use division Property;

$L\left(\frac{1-\cos t}{t^2}\right) = \frac{1}{2}\int_s^\infty \underset{\mathrm{I}}{\log(1+s^{-2})}\, \underset{\mathrm{II}}{ds}$

$\Rightarrow \frac{1}{2}\left[\log(1+s^{-2})\int ds - \int\left(\frac{d}{ds}\log(1+s^{-2})\int ds\right) ds\right]_s^\infty$

$\Rightarrow \frac{1}{2}\left[s\log(1+s^{-2}) - \int\left(\frac{1}{1+s^{-2}}(-2)s^{-3}\cdot s\right) ds\right]_s^\infty$

$\Rightarrow \frac{1}{2}\left[s\log(1+s^{-2}) + 2\int \frac{s^{-2}}{1+s^{-2}}\, ds\right]_s^\infty$

$$\Rightarrow \frac{1}{2}\left[s\log(1+s^{-2}) + 2\int \frac{s^{-2}}{s^{-2}\left(\frac{1}{s^{-2}}+1\right)}\,ds\right]_s^{\infty}$$

$$\Rightarrow \frac{1}{2}\left(s\log(1+s^{-2}) + 2\int \frac{ds}{1+s^2}\right)_s^{\infty}$$

$$\Rightarrow \frac{1}{2}\left(s\log(1+s^{-2}) + 2\tan^{-1}s\right)_s^{\infty}$$

$$\Rightarrow \frac{1}{2}\left(0 + 2\tan^{-1}\infty - s\log(1+s^{-2}) - 2\tan^{-1}s\right)$$

$$\Rightarrow \frac{1}{2}\left(\left(2\times\frac{\pi}{2}\right) - 2\tan^{-1}s - s\log(1+s^{-2})\right)$$

$$\Rightarrow \frac{1}{2}\left(2\left(\frac{\pi}{2} - \tan^{-1}s\right) - s\log(1+s^{-2})\right)$$

$$\Rightarrow \frac{1}{2}\left(2\cot^{-1}s - s\log(1+s^{-2})\right)$$

$$\Rightarrow \boxed{\cot^{-1}s - \frac{1}{2}s\log\left(1+\frac{1}{s^2}\right)}$$ Ans

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Ques → Find Laplace Transform of

$$f(t) = \begin{cases} \sin t & ; 0 < t < \pi \\ 0 & ; t > \pi \end{cases}$$

Sol → By definition of Laplace Transform;

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\Rightarrow \int_0^\pi e^{-st} \sin t\, dt + \cancel{\int_\pi^\infty e^{-st}\, 0\, dt} \to 0$$

(Use formula: $\int e^{at} \sin bt\, dt = \frac{e^{at}}{a^2+b^2}(a \sin bt - b\cos bt)$).

$$\Rightarrow \left[\frac{e^{-st}}{s^2+1}(-s \sin t - \cos t)\right]_0^\pi$$

$$\Rightarrow \left(\frac{e^{-s\pi}}{s^2+1}(-s.\cancel{\sin \pi} - \cos \pi)\right) - \left(\frac{e^0}{s^2+1}(\cancel{-s\sin 0} - \cos 0)\right)$$

$$\Rightarrow \frac{1}{s^2+1}\left[e^{-s\pi}((-1)(-1)) + 1\right] = \frac{1 + e^{-s\pi}}{s^2+1} \quad \text{Ans}$$

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Evaluation of integrals by Laplace Transform

Ques: Evaluate

$$\int_0^\infty t e^{-3t} \sin t \, dt$$

Sol → $L(\sin t) = \dfrac{1}{s^2+1}$

(Use multiplication property)

$$L(t \sin t) = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$\Rightarrow -\frac{d}{ds}(s^2+1)^{-1} \Rightarrow -(-1)(s^2+1)^{-2}(2s) \Rightarrow \frac{2s}{(s^2+1)^2}$$

(Use Shifting property)

$$L(e^{-3t} t \sin t) = \frac{2(s+3)}{((s+3)^2+1)^2}$$

Now by definition of Laplace Transform

$$L(f(t)) = \int_0^\infty e^{-st} f(t) \, dt$$

$$L[e^{-3t} t \sin t] = \int_0^\infty e^{-st} t e^{-3t} \sin t \, dt = \frac{2(s+3)}{((s+3)^2+1)^2}$$

Put $s=0$

$$\int_0^\infty t e^{-3t} \sin t \, dt = \frac{2(3)}{((3)^2+1)^2} \Rightarrow \frac{6}{(10)^2} \Rightarrow \frac{6}{100} \Rightarrow \frac{3}{50} \text{ Ans}$$

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Ques: Evaluate:

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt$$

or Prove that

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \log b/a$$

Sol: $L(e^{-at} - e^{-bt}) = \frac{1}{s+a} - \frac{1}{s+b}$

(Use division property)

$$L\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{(s+b)}\right) ds$$

$$\Rightarrow \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$\Rightarrow \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty \Rightarrow \lim_{s\to\infty} \log\left(\frac{s+a}{s+b}\right) - \log\left(\frac{s+a}{s+b}\right)$$

$$\Rightarrow \lim_{s\to\infty} \log \frac{s(1+a/s)}{s(1+b/s)} - \log\left(\frac{s+a}{s+b}\right)$$

$$\Rightarrow \log 1 + \log\left(\frac{s+b}{s+a}\right) \Rightarrow \log\left(\frac{s+b}{s+a}\right)$$

By definition of Laplace Transform

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \int_0^\infty e^{-st}\left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = \log\left(\frac{s+b}{s+a}\right)$$

Put $s = 0$

$$\Rightarrow \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} = \log\frac{b}{a}$$

Ans

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Inverse Laplace Transform

⇒ Formulae

1. $L^{-1}\left(\frac{1}{s}\right) = 1$

2. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

3. $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\lfloor n-1}$

4. $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

5. $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

6. $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

7. $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

⇒ Properties

If $L^{-1}(\bar{f}(s)) = f(t)$

1. $L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t)$ ⟶ Shifting Property

2. $L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\} = \frac{f(t)}{t}$ ⟶ Integral Property

3. $L^{-1}\left(\frac{\bar{f}(s)}{s}\right) = \int_0^t f(t)\,dt$ ⟶ Division Property

4. $L^{-1}\left(\frac{d}{ds}\bar{f}(s)\right) = -t\,f(t)$ ⟶ Derivative Property.

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Problems on Inverse Laplace Transform

Ques: Find I L T:

$$L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right)$$

Sol→ $\frac{1}{s(s+1)(s+2)} = \frac{1}{s} \cdot \frac{1}{(s+1)(s+2)}$

(Use Partial fraction)

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2)+B(s+1)}{(s+1)(s+2)}$$

$\Rightarrow 1 = A(s+2) + B(s+1)$

$\Rightarrow$ Put $s=-1 \Rightarrow 1 = A + 0 \Rightarrow A = 1$

$\Rightarrow$ Put $s=-2 \Rightarrow 1 = 0 + B(-1) \Rightarrow B = -1$

$$\frac{1}{(s+1)(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

(Take ILT on both sides)

$$L^{-1}\left(\frac{1}{(s+1)(s+2)}\right) = L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right)$$

$$\Rightarrow \underbrace{e^{-t} - e^{-2t}}_{f(t)}$$

(Use division Rule)

$$L^{-1}\left(\frac{1}{s}\left(\frac{1}{(s+1)(s+2)}\right)\right) = \int_0^t (e^{-t} - e^{-2t})\,dt$$

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$$\Rightarrow \left[\frac{e^{-t}}{-1} + \frac{e^{-2t}}{2}\right]_0^t$$

$$\Rightarrow -e^{-t} + 1 + \frac{e^{-2t}}{2} - \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} \quad \text{Ans}$$

Ques: Evaluate

$$L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$$

Sol → Given:

$$f(s) = \frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s+2)}$$

$$\Rightarrow \frac{4s+5}{(s-1)^2(s+2)} = \frac{A(s-1)(s+2) + B(s+2) + C(s-1)^2}{(s-1)^2(s+2)}$$

$$\Rightarrow 4s+5 = A(s^2+2s-s-2) + B(s+2) + C(s^2-2s+1)$$

$$\Rightarrow 0s^2 + 4s + 5 = A(s^2+s-2) + B(s+2) + C(s^2-2s+1)$$

On comparing coefficients of

(i) s^2 ; $0 = A + C \Rightarrow A = -C$

(ii) s ; $4 = A + B - 2C \Rightarrow 4 = -C + B - 2C \Rightarrow 4 = B - 3C$ --- (i)

(iii) constants; $5 = -2A + 2B + C = 2C + 2B + C$

$$\Rightarrow 5 = 2B + 3C \quad -(ii)$$

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from (i) & (ii)

$$\begin{aligned} B - 3C &= 4 \\ 2B + 3C &= 5 \\ \hline 3B &= 9 \end{aligned}$$

$$B = 3$$

from (i); $B - 3C = 4 \Rightarrow B - 4 = 3C \Rightarrow C = -1/3$

$$\Rightarrow A = 1/3$$

$$\bar{f}(s) = \frac{4s+5}{(s-1)^2(s+2)} = \frac{1}{3}\left(\frac{1}{s-1}\right) + 3\left(\frac{1}{s-1}\right)^2 - \frac{1}{3}\left(\frac{1}{s+2}\right)$$

(Taking ILT on both sides)

$$L^{-1}(\bar{f}(s)) = \frac{1}{3} L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{(s-1)^2}\right) - \frac{1}{3} L^{-1}\left(\frac{1}{s+2}\right)$$

$$\Rightarrow \frac{e^t}{3} + 3e^t L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{3} e^{-2t}$$

$$\Rightarrow \frac{e^t}{3} + 3e^t \cdot \frac{t^1}{\lfloor 1} - \frac{1}{3} e^{-2t}$$

$$\Rightarrow \frac{e^t}{3} + 3te^t - \frac{e^{-2t}}{3} \quad \text{Ans}$$

Date..............

Ques: Evaluate

$$L^{-1}\left\{\frac{s+2}{s^2(s+1)(s-2)}\right\}$$

Sol → Solving $\frac{s+2}{(s+1)(s-2)}$ by partial fraction

$$\frac{s+2}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} \Rightarrow \frac{A(s-2)+B(s+1)}{(s+1)(s-2)}$$

$\Rightarrow s+2 = A(s-2) + B(s+1)$

$\Rightarrow$ Put $s=2$; $B = 4/3$

Put $s=-1$; $A = -1/3$

$$\frac{s+2}{(s+1)(s-2)} = \frac{-1}{3}\left(\frac{1}{s+1}\right) + \frac{4}{3}\left(\frac{1}{s-2}\right)$$

Take ILT on both sides

$$L^{-1}\left(\frac{s+2}{(s+1)(s-2)}\right) = \frac{-1}{3}L^{-1}\left(\frac{1}{s+1}\right) + \frac{4}{3}L^{-1}\left[\frac{1}{s-2}\right]$$

$$\Rightarrow \frac{4}{3}e^{2t} - \frac{e^{-t}}{3}$$

Now; $L^{-1}\left(\frac{1}{s}\cdot\frac{s+2}{(s+1)(s-2)}\right) = \int_0^t \left(\frac{4}{3}e^{2t} - \frac{1}{3}e^{-t}\right)dt$

(Division Property)

$$\rightarrow \left[\frac{4}{3}\frac{e^{2t}}{2} + \frac{e^{-t}}{3}\right]_0^t \Rightarrow \left[\frac{2}{3}e^{2t} + \frac{e^{-t}}{3}\right]_0^t$$

$$\Rightarrow \frac{2}{3}e^{2t} + \frac{e^{-t}}{3} - \frac{2}{3} - \frac{1}{3} \Rightarrow \frac{2}{3}e^{2t} + \frac{e^{-t}}{3} - 1$$

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Again using division property;

$$L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s}\cdot\frac{s+2}{(s-2)(s+1)}\right) = \int_0^t\left(\frac{2}{3}e^{2t}+\frac{e^{-t}}{3}-1\right)dt$$

$$\Rightarrow \left[\frac{2}{3}\frac{e^{2t}}{2}+\frac{1}{3}\frac{e^{-t}}{(-1)}-t\right]_0^t$$

$$\Rightarrow \left[\frac{e^{2t}}{3}-\frac{e^{-t}}{3}-t\right]_0^t$$

$$\Rightarrow \frac{e^{2t}}{3}-\frac{e^{-t}}{3}-t-\frac{e^0}{3}+\frac{e^0}{3}+0 \Rightarrow \left(\frac{e^{2t}}{3}-\frac{e^{-t}}{3}-t\right) \text{ Ans}$$

Q. Evaluate

$$L^{-1}\left(\frac{s}{(s+3)^2+4}\right)$$

Sol → $L^{-1}\left(\frac{s+3-3}{(s+3)^2+4}\right)$

$$\Rightarrow L^{-1}\left(\frac{s+3}{(s+3)^2+4}\right) - 3L^{-1}\left(\frac{1}{(s+3)^2+4}\right)$$

$$\Rightarrow e^{-3t}L^{-1}\left(\frac{s}{s^2+2^2}\right) - 3e^{-3t}L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

(Used shifting Property)

$$\Rightarrow e^{-3t}\cos 2t - 3e^{-3t}\left(\frac{1}{2}\sin 2t\right)$$

$$\Rightarrow e^{-3t}\left(\cos 2t - \frac{3}{2}\sin 2t\right) \text{ Ans}$$

Date...............

Q. Evaluate

$$L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right]$$

Sol → $\bar{f}(s) = \frac{3(s^2-2)^2}{2s^5} = 3\,\frac{(s^4-4s^2+4)}{2s^5}$

$$\Rightarrow \frac{3}{2}\left[\frac{1}{s} - \frac{4}{s^3} + \frac{4}{s^5}\right]$$

Taking ILT on both sides

$$L^{-1}\{\bar{f}(s)\} = \frac{3}{2}\left[\left[L^{-1}(1/s)\right] - 4L^{-1}(1/s^3) + 4L^{-1}(1/s^5)\right]$$

$$\Rightarrow \frac{3}{2}\left[1 - \frac{4t^2}{\lfloor 2} + \frac{4t^4}{\lfloor 4}\right] \Rightarrow \frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right]$$

$$\Rightarrow 3/2 - 3t^2 + \frac{t^4}{4}$$ Ans

Q. Evaluate:

$$L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]$$

Sol → $\bar{f}(s) = \frac{s^2-3s+4}{s^3} \Rightarrow \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}$

Now; take ILT on both sides

$$L^{-1}\{\bar{f}(s)\} = L^{-1}(1/s) - 3L^{-1}(1/s^2) + 4L^{-1}(1/s^3)$$

$$\Rightarrow 1 - 3t/\lfloor 1 + \frac{4t^2}{\lfloor 2}$$

$$\Rightarrow 1 - 3t + 2t^2$$ Ans

Date..............

Q Evaluate:

$$L^{-1}\left(\frac{s+2}{s^2-4s+13}\right)$$

Sol→ Let $L^{-1}(\bar{f}(s)) = L^{-1}\left(\frac{s+2}{((s^2-4s+4)+9)}\right)$

$$\Rightarrow L^{-1}\left(\frac{s+2}{(s-2)^2+3^2}\right)$$

$$\Rightarrow L^{-1}\left(\frac{s-2+2+2}{(s-2)^2+3^2}\right) = L^{-1}\left(\frac{(s-2)+4}{(s-2)^2+3^2}\right)$$

$$\Rightarrow L^{-1}\left(\frac{s-2}{(s-2)^2+3^2}\right) + 4L^{-1}\left(\frac{1}{(s-2)^2+3^2}\right)$$

(Use shifting Property)

$$L^{-1}(\bar{f}(s)) = e^{2t} L^{-1}\left(\frac{s}{s^2+3^2}\right) + 4e^{2t} L^{-1}\left(\frac{1}{s^2+3^2}\right)$$

$$\Rightarrow e^{2t}\cos 3t + \frac{4e^{2t}}{3}\sin 3t$$

$$\Rightarrow e^{2t}\left(\cos 3t + \tfrac{4}{3}\sin 3t\right) \text{ Ans}$$

Q Evaluate:

$$L^{-1}\left(\frac{s+1}{s^2+s+1}\right)$$

$\hookrightarrow$

Date..............

Sol →

Here;

$$\bar{f}(s) = \frac{s+1}{s^2+s+1} \Rightarrow \frac{s+1}{(s^2+s+1/4)+(1-1/4)}$$

$$\Rightarrow \frac{s+1}{(s+1/2)^2+(\sqrt{3}/2)^2} = \frac{(s+1/2)+1/2}{(s+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Take ILT on both sides

$$L^{-1}[\bar{f}(s)] = L^{-1}\left(\frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{(s+1/2)^2+(\sqrt{3}/2)^2}\right)$$

(⇒ Use Shifting Property)

$$L^{-1}(\bar{f}(s)) = e^{-t/2} L^{-1}\left(\frac{s}{s^2+(\sqrt{3}/2)^2}\right) + \frac{e^{-t/2}}{2} L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$\Rightarrow e^{-t/2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}e^{-t/2}\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

$$\Rightarrow e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right) \text{ Ans}$$

Imp.

Ques: Prove that:

$$L^{-1}\left(\frac{s}{s^4+s^2+1}\right) = \frac{2}{\sqrt{3}}\sinh\frac{t}{2}\cdot\sin\frac{\sqrt{3}}{2}t$$

↳

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Sol> Let

$$\bar{f}(s) = \frac{s}{s^4+s^2+1} = \frac{s}{(s^4+2s^2+1)-s^2} = \frac{s}{(s^2+1)^2-s^2}$$

$$\Rightarrow \frac{s}{(s^2+s+1)(s^2-s+1)} = \frac{As+B}{(s^2+s+1)} + \frac{Cs+D}{(s^2-s+1)}$$

Using partial fraction technique

$$s = (As+B)(s^2-s+1) + (Cs+D)(s^2+s+1)$$

$$s = As^3 - As^2 + As + Bs^2 - Bs + B + Cs^3 + Cs^2 + Cs + Ds^2 + Ds + D$$

On Equating the coefficients of

s^3 ; $0 = A + C$

s^2 ; $0 = -A + B + C + D$

s ; $1 = A - B + C + D \Rightarrow -B + D = 1$

Constants; $0 = B + D \Rightarrow -A + C = 0$

Also; $B + D = 0$
$-B + D = 1$
$D = 1/2$
$B = -1/2$

$A + C = 0$
$-A + C = 0$
$C = 0$
$A = 0$

~~Also, B~~

Date..............

$$\therefore \frac{s}{s^4+s^2+1} = \frac{-1/2}{s^2+s+1} + \frac{1/2}{s^2-s+1}$$

Take ILT on both sides

L.H.S. =

$$L^{-1}(\bar{f}(s)) = -\frac{1}{2} L^{-1}\left(\frac{1}{s^2+s+1}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s^2+s+1}\right)$$

$$\Rightarrow -\frac{1}{2} L^{-1}\left(\frac{1}{(s^2+s+1/4)+(1-1/4)}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{(s^2-s+1/4)+(1-1/4)}\right)$$

$$\Rightarrow -\frac{1}{2} L^{-1}\left(\frac{1}{(s+1/2)^2+(\sqrt{3}/2)^2}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{(s-1/2)^2+(\sqrt{3}/2)^2}\right)$$

$$\cancel{\Rightarrow -\frac{1}{2} L^{-1}\left(\frac{1}{\quad}\right)}$$

Use shifting Property

$$\Rightarrow -\frac{1}{2} e^{-t/2} L^{-1}\left(\frac{1}{s^2+(\sqrt{3}/2)^2}\right) + \frac{1}{2} e^{t/2} L^{-1}\left(\frac{1}{s^2+(\sqrt{3}/2)^2}\right)$$

$$\Rightarrow -\frac{e^{-t/2}}{2} \times \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t + \frac{e^{t/2}}{2} \times \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t$$

$$\Rightarrow \frac{e^{t/2}}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t - \frac{e^{-t/2}}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t$$

$$\Rightarrow \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \left(e^{t/2} - e^{-t/2}\right) \times \frac{2}{2} \qquad \left(\frac{e^{t}-e^{-t}}{2} = \sinh t\right)$$

$$\Rightarrow \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \left(\frac{e^{t/2}-e^{-t/2}}{2}\right)$$

$$\Rightarrow \frac{2}{\sqrt{3}} \left(\sin\left(\frac{\sqrt{3}}{2} t\right)\right) \sinh t/2 = \text{R.H.S.}$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

$\Rightarrow$ Hence, Proved.

Date..............

Q Evaluate

$$L^{-1}\left(\log\left(\frac{s+3}{s+2}\right)\right)$$

Soln Let $\bar{f}(s) = \log\left(\frac{s+3}{s+2}\right) = \log(s+3) - \log(s+2)$.

Differentiating wrt s; we get

$$\frac{d}{ds}(\bar{f}(s)) = \frac{1}{s+3} - \frac{1}{s+2}$$

Take I LT on both sides

$$L^{-1}\left(\frac{d}{ds}\bar{f}(s)\right) = L^{-1}\left(\frac{1}{s+3}\right) - L^{-1}\left(\frac{1}{s+2}\right)$$

$$\Rightarrow -t f(t) = e^{-3t} - e^{-2t}$$

$$\Rightarrow f(t) = \frac{e^{-2t} - e^{-3t}}{t} \quad \text{Ans}$$

Q Evaluate:

$$L^{-1}\left(\log\left(\frac{s^2+1}{s(s+1)}\right)\right)$$

Soln Let $\bar{f}(s) = \log\left(\frac{s^2+1}{s(s+1)}\right) = \log(s^2+1) - \log s - \log(s+1)$

Different wrt s; we get

$$\frac{d}{ds}\bar{f}(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

Take ILT on both sides

$$L^{-1}\left[\frac{d}{ds}\bar{f}(s)\right] = 2L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$\Rightarrow -t f(t) = 2\cos t - 1 - e^{-t}$

$\Rightarrow t f(t) = 1 + e^{-t} - 2\cos t$

$\Rightarrow f(t) = \dfrac{1+e^{-t}-2\cos t}{t}$ Ans

Ques: find inverse laplace transform of $\cot^{-1}(s/2)$.

Sol → Let $\bar{f}(s) = \cot^{-1}(s/2) = \tan^{-1}\left(\frac{1}{s/2}\right) = \tan^{-1}\left(\frac{2}{s}\right)$

$\boxed{\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)}$

Differentiate both sides wrt s;

$$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\tan^{-1}\left(\frac{2}{s}\right) = \frac{1}{1+(2/s)^2}\left(2\cdot\left(\frac{-1}{s^2}\right)\right) = \frac{-2}{s^2\left(\frac{s^2+2^2}{s^2}\right)}$$

$$\Rightarrow \frac{-2}{s^2+2^2}$$

Take ILT on both sides

$$L^{-1}\left(\frac{d}{ds}\bar{f}(s)\right) = -2L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$\Rightarrow -t f(t) = -\frac{2}{2}\sin 2t$

$\Rightarrow f(t) = \dfrac{\sin 2t}{t}$ Ans