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• Convolution Theorem

If $L^{-1}(\bar{f}(s)) = f(t)$ and $L^{-1}(\bar{g}(s)) = g(t)$; then

$$L^{-1}(\bar{f}(s)\,\bar{g}(s)) = \int_0^t f(u)\, g(t-u)\,du$$

Ques. Solve: $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$ By apply convolution theorem.

Sol → $L^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right) = L^{-1}\left(\dfrac{s}{(s^2+a^2)} \cdot \dfrac{1}{(s^2+a^2)}\right)$

Let $\bar{f}(s) = \dfrac{s}{s^2+a^2}$ and $\bar{g}(s) = \dfrac{1}{s^2+a^2}$

$\Rightarrow L^{-1}(\bar{f}(s)) = L^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at = f(t)$

Also; $L^{-1}(\bar{g}(s)) = L^{-1}\left(\dfrac{1}{s^2+a^2}\right) = \dfrac{1}{a}\sin at = g(t)$

Using Convolution Theorem;

$$L^{-1}\left(\frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}\right) = \int_0^t \cos au \,\frac{1}{a}\sin a(t-u)\,du$$

[Use formula: $2\cos A \sin B = \sin(A+B) - \sin(A-B)$]

$$\Rightarrow \frac{1}{2a}\int_0^t 2\cos au \sin a(t-u)\,du$$

$$\Rightarrow \frac{1}{2a}\int_0^t \left(\sin a(u+t-u) - \sin a(u-t+u)\right) du$$

$$\Rightarrow \frac{1}{2a}\int_0^t \left(\sin at - \sin a(2u-t)\right) du$$

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$$\Rightarrow \frac{1}{2a}\left[(\sin at)[u]_0^t + \left[\frac{\cos a(2u-t)}{2a}\right]_0^t \right]$$

$$\Rightarrow \frac{1}{2a}\left(t \sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right)$$

$$\Rightarrow \frac{t \sin at}{2a} \quad \text{Ans}$$

Ques: Evaluate using convolution theorem

$$L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right)$$

Sol →

$$L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right) = L^{-1}\left(\frac{s}{(s^2+a^2)} \cdot \frac{s}{(s^2+b^2)}\right)$$

Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{s}{s^2+b^2}$

$$L^{-1}(\bar{f}(s)) = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at = f(t)$$

And $L^{-1}(\bar{g}(s)) = L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt = g(t)$

Using Convolution theorem;

$$L^{-1}\left(\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2}\right) = \int_0^t \cos au \cdot \cos b(t-u)\, du$$

Use formula:

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$\Rightarrow \frac{1}{2}\int_0^t 2\cos au \cos b(t-u)\,du$$

$$\Rightarrow \frac{1}{2}\int_0^t \{\cos(au + bt - bu) + \cos(au - bt + bu)\}\,du$$

$$\Rightarrow \frac{1}{2}\int_0^t [\cos((a-b)u + bt) + \cos\{(a+b)u - bt\}]\,du$$

$$\Rightarrow \cancel{\frac{1}{2}\int_0^t} \Rightarrow \frac{1}{2}\left[\frac{\sin((a-b)u + bt)}{a-b} + \frac{\sin((a+b)u - bt)}{a+b}\right]_0^t$$

$$\Rightarrow \frac{1}{2}\left[\frac{\sin(at - \cancel{bt} + \cancel{bt})}{a-b} - \frac{\sin bt}{a-b} + \frac{\sin(at + \cancel{bt} - \cancel{bt})}{a+b} - \frac{\sin(-bt)}{a+b}\right]$$

$$\Rightarrow \frac{1}{2}\left(\frac{\sin at - \sin bt}{a-b} + \frac{\sin(at) + \sin bt}{a+b}\right)$$

$$\Rightarrow \frac{1}{2(a^2-b^2)}(a\sin at + b\sin at - a\sin bt - b\sin bt + a\sin at + a\sin bt - b\sin at - b\sin bt)$$

$$\Rightarrow \frac{2(a\sin at - b\sin bt)}{2(a^2-b^2)} = \frac{a\sin at - b\sin bt}{a^2-b^2}$$ ~~Ans~~

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- Questions of Convolution Theorem

1. $L^{-1}\left(\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right)$

2. $L^{-1}\left(\dfrac{1}{s^3(s^2+1)}\right)$

3. $L^{-1}\left(\dfrac{1}{(s+1)(s^2+1)}\right)$

4. $L^{-1}\left(\dfrac{1}{(s+1)^2(s^2+4)}\right)$

5. $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$

Laplace Transform of a derivative : If $L(f(t)) = \bar{F}(s)$
then $L(f^n(t)) = s^n L(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - s^{n-4}f'''(0) - \_\_\_$

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• Application of Laplace Transform

Ques→ Solve:

$$(D^3 - 3D^2 + 3D - 1)\, y = t^2 e^t$$

given that : $y(0) = 1, y'(0) = 0, y''(0) = -2$.

Sol→ Given: $(D^3 - 3D^2 + 3D - 1)\, y = t^2 e^t$

$\Rightarrow y''' - 3y'' + 3y' - y = t^2 e^t$

(Taking Laplace Transform on both sides)

$$L(y''') - 3L(y'') + 3L(y') - L(y) = L(t^2 e^t)$$

$$\Rightarrow (s^3 L(y) - s^2 y(0) - sy'(0) - y''(0)) - 3(s^2 L(y) - s y(0) - y'(0)) + 3(sL(y) - y(0)) - L(y) = \frac{2}{(s-1)^3}$$

$$\Rightarrow L(y)(s^3 - 3s^2 + 3s - 1) - s^2 - (-2) + 3s - 3 = \frac{2}{(s-1)^3}$$

$$\Rightarrow L(y)(s-1)^3 = s^2 - 3s + 1 + \frac{2}{(s-1)^3}$$

$$\Rightarrow L(y) = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\Rightarrow \frac{s^2 - 2s + 1 - s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\Rightarrow \frac{(s-1)^2 - s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\Rightarrow L(y) = \frac{1}{(s-1)} - \frac{s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\Rightarrow \frac{1}{s-1} - \frac{(s-1+1)}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\Rightarrow L(y) = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\Rightarrow y = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{(s-1)^2}\right) - L^{-1}\left(\frac{1}{(s-1)^3}\right) + 2L^{-1}\left(\frac{1}{(s-1)^6}\right)$$

(By Shifting Property)

$$y = e^t L^{-1}\left(\frac{1}{s}\right) - e^t L^{-1}\left(\frac{1}{s^2}\right) - e^t L^{-1}\left(\frac{1}{s^3}\right) + 2e^t L^{-1}\left(\frac{1}{s^6}\right)$$

$$\Rightarrow e^t(1) - \frac{e^t t^1}{\lfloor 1} - \frac{e^t t^2}{\lfloor 2} + \frac{2e^t t^5}{\lfloor 5}$$

$$\Rightarrow y = e^t\left(1 - t - t^2/2 + t^5/60\right)$$

which is required solution Ans

Ques: Solve : $ty'' + 2y' + ty = \cos t$; given $y(0) = 1$.

Sol → Taking LT on both sides

$$L(ty'') + 2L(y') + L(ty) = L(\cos t)$$

$$\Rightarrow -\frac{d}{ds} L(y'') + 2 \cdot L(y') - \frac{d}{ds} L(y) = \frac{s}{s^2+1}$$

$$\Rightarrow -\frac{d}{ds}\left(s^2 L(y) - sy(0) - y'(0)\right) + 2\left(sL(y) - y(0)\right) - \frac{d}{ds} L(y) = \frac{s}{s^2+1}$$

$$\Rightarrow -\left(s^2 \frac{d}{ds} L(y) + L(y)(2s) - (1)(1) - 0\right) + 2sL(y) - 2$$

$$-\frac{d}{ds} L(y) = \frac{s}{s^2+1}$$

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$\Rightarrow -s^2 \frac{d}{ds} L(y) - 2sL(y) + 1 + 2sL(y) - 2 - \frac{d}{ds} L(y) = \frac{s}{s^2+1}$

$\Rightarrow (-s^2-1) \frac{d}{ds} L(y) - 1 = \frac{s}{s^2+1}$

$\Rightarrow -(s^2+1) \frac{d}{ds} L(y) = 1 + \frac{s}{s^2+1}$

$\Rightarrow -\frac{d}{ds} L(y) = \frac{1}{s^2+1} + \frac{s}{(s^2+1)^2}$

Take ILT on both sides

$L^{-1}\left(-\frac{d}{ds} L(y)\right) = L^{-1}\left(\frac{1}{s^2+1}\right) + L^{-1}\left(\frac{s}{(s^2+1)^2}\right)$

$\Rightarrow t \cdot y = \frac{1}{1} \sin t + L^{-1}\left(\frac{s}{(s^2+1)^2}\right)$ — (i)

Solve by convolution theorem

$L^{-1}\left(\frac{s}{(s^2+1)^2}\right) = L^{-1}\left(\frac{s}{s^2+1} \cdot \frac{1}{s^2+1}\right)$

$\Rightarrow$ Let $\bar{f}(s) = \frac{s}{s^2+1}$ and $\bar{g}(s) = \frac{1}{s^2+1}$

then $L^{-1}(\bar{f}(s)) = L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t = f(t)$

Also ; $L^{-1}(\bar{g}(s)) = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = g(t)$

Using Convolution Theorem

$L^{-1}\left(\frac{s}{s^2+1} \cdot \frac{1}{s^2+1}\right) = \int_0^t \cos u \sin(t-u)\, du$

(Use formula: $2\cos A \sin B = \sin(A+B) - \sin(A-B)$)

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$$\Rightarrow \frac{1}{2}\int_0^t 2\cos u \cdot \sin(t-u)\, du$$

$$\Rightarrow \frac{1}{2}\int_0^t (\sin(u+t-u) - \sin(u-t+u))\, du$$

$$\Rightarrow \frac{1}{2}\int_0^t (\sin t - \sin(2u-t))\, du$$

$$\Rightarrow \frac{1}{2}\left[(\sin t)(u)_0^t + \left[\frac{\cos(2u-t)}{2}\right]_0^t\right]$$

$$\Rightarrow \frac{1}{2}\left((\sin t)(t-0) + \tfrac{1}{2}(\cos t - \cos(-t))\right)$$

$$\Rightarrow \frac{t\sin t}{2}$$

from (i)

$$\Rightarrow ty = \sin t + L^{-1}\left(\frac{s}{(s^2+1)^2}\right)$$

$$\Rightarrow ty = \sin t + \frac{t\sin t}{2}$$

$$\Rightarrow ty = \sin t\left(1 + \frac{t}{2}\right) \quad \text{Ans}$$

which is required solution.

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Ques: Solve:

$$\frac{d^2x}{dt^2} + 9x = \cos t \quad ; \text{ Given}: x(0) = 1$$
$$\text{And } x(\pi/2) = -1$$

Sol→

$$\frac{d^2x}{dt^2} + 9x = \cos 2t$$

$$\Rightarrow (D^2 + 9)x = \cos 2t$$
$$\Rightarrow x'' + 9x = \cos 2t$$

Take LT on both sides

$$L(x'') + 9L(x) = L(\cos 2t)$$

$$\Rightarrow (s^2 L(x) - sX(0) - x'(0)) + 9(L(x)) = \frac{s}{s^2+4}$$

(Assume $x'(0) = A$)

$$\Rightarrow (s^2 L(x) - s(1) - A) + 9L(x) = \frac{s}{s^2+4}$$

$$\Rightarrow L(x)(s^2+9) = s + A + \frac{s}{s^2+4}$$

$$\Rightarrow L(x) = \frac{s}{s^2+9} + \frac{A}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

$$\Rightarrow \frac{s}{s^2+9} + \frac{A}{s^2+9} + \frac{1}{5}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right)$$

$$\Rightarrow L(x) = \frac{4}{5}\left(\frac{s}{s^2+9}\right) + \frac{A}{s^2+9} + \frac{1}{5}\left(\frac{s}{s^2+4}\right)$$

$$\Rightarrow x = \frac{4}{5}L^{-1}\left(\frac{s}{s^2+3^2}\right) + AL^{-1}\left(\frac{1}{s^2+3^2}\right) + \frac{1}{5}L^{-1}\left(\frac{s}{s^2+2^2}\right)$$

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$$\Rightarrow x = \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t + \frac{1}{5}\cos 2t$$

(Put $x = -1$ when $t = \pi/2$)

$$\Rightarrow -1 = \frac{4}{5}\cos\frac{3\pi}{2} + \frac{A}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\cos\frac{2\pi}{2}$$

$$\Rightarrow -1 = 0 - \frac{A}{3} - \frac{1}{5} \Rightarrow A = \frac{12}{5}$$

$$\therefore x = \frac{4}{5}\cos 3t + \frac{12}{15}\sin 3t + \frac{1}{5}\cos 2t$$

$$\boxed{x = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t}$$ Ans

which is required solution

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• Second Shifting Theorem

Unit Step function

A function of type $\begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$ is called unit step function and it is denoted by $H(t-a)$ i.e. $H(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$

Second Shifting Theorem

Let $f(t)$ be a function of t $L(f(t)) = f(s)$ then
$L\{f(t-a) H(t-a)\} = e^{-as} f(s)$

For inverse Laplace Theorem

Let $f(s)$ be a inverse Laplace transform of $f(t)$ so that

$L^{-1}(f(s)) = f(t)$
then $L^{-1}(e^{-as} f(s)) = f(t-a) H(t-a)$
where $H(t-a)$ is unit step function.

⇒ For Laplace Transform

Q: Find $L\{f(t)\}$
$f(t) = \begin{cases} \sin(t - \pi/3) & t \geq \pi/3 \\ 0 & t < \pi/3 \end{cases}$

Sol: $\sin(t - \pi/3) \begin{cases} 1 & t \geq \pi/3 \\ 0 & t < \pi/3 \end{cases}$

$f(t) = \sin(t - \pi/3)\, H(t - \pi/3)$

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$L(\sin t) = \frac{1}{s^2+1}$

$L(f(t)) = L(\sin(t - \pi/3) H(t - \pi/3)) = e^{-\pi/3 s} \frac{1}{s^2+1}$

Q. find $L\{f(t)\}$

$$f(t) = \begin{cases} e^{t-a} & t > a \\ 0 & t < a \end{cases}$$

Sol→

$f(t) = e^{t-a} \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$

$f(t) = e^{t-a} H(t-a)$

$L(e^t) = \frac{1}{s-1}$.

$L(e^{t-a} \cdot H(t-a)) = e^{-as}\left(\frac{1}{s-1}\right)$ Ans

Q. find $L(f(t))$

$$f(t) = \begin{cases} \cos(t - 2\pi/3) & t \geq 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$$

Sol→ $f(t) = \cos(t - 2\pi/3) \begin{cases} 1 & t \geq 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$

$f(t) \Rightarrow \cos(t - 2\pi/3) H(t - 2\pi/3)$

$L(\cos t) = \frac{s}{s^2+1}$

$L(\cos(t - 2\pi/3) H(t - 2\pi/3)) = e^{-2\pi/3\, s} \frac{s}{s^2+1}$ Ans

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Inverse Laplace Transform

Q. $L^{-1}\left(e^{-\pi/3 s}\dfrac{1}{s^2+1}\right)$

Soln

$$L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

$$L^{-1}\left(e^{-\pi/3 s}\frac{1}{s^2+1}\right) = \sin\left(t-\pi/3\right) H\left(t-\pi/3\right)$$

$$\Rightarrow \sin\left(t-\pi/3\right)\cdot\begin{cases}1 & t \geq \pi/3\\ 0 & t < \pi/3\end{cases}$$

$$\Rightarrow \begin{cases}\boxed{\sin\left(t-\pi/3\right)} & t \geq \pi/3\\ 0 & t < \pi/3\end{cases}$$ Ans

Q. $L^{-1}\left(e^{-as}\dfrac{1}{s-1}\right)$

Soln $L^{-1}\left(\dfrac{1}{s-1}\right) = e^{t}$

$$L^{-1}\left(e^{-as}\frac{1}{s-1}\right) = e^{t-a}H(t-a)$$

$$\Rightarrow \begin{cases}e^{t-a} & t \geq a\\ 0 & t < a\end{cases}$$ Ans

Q. $L^{-1}\left(e^{-2\pi/3 s}\left(\dfrac{s}{s^2+1}\right)\right)$

Soln

$$L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$$

$$\Rightarrow L^{-1}\left(e^{-2\pi/3 s}\cdot\frac{s}{s^2+1}\right) = \cos\left(t-2\pi/3\right)H\left(t-2\pi/3\right)$$

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$$\Rightarrow \begin{cases} \cos\left(t - 2\pi/3\right) & t \geq 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$$ Ans

Q $L^{-1}\left(\frac{1}{s^2} e^{-2s}\right)$

Sol → $L^{-1}\left(\frac{1}{s^2}\right) = t$

$\Rightarrow L^{-1}\left(e^{-2s} \cdot \frac{1}{s^2}\right) = (t-2)\, u(t-2)$.

$$\Rightarrow \begin{cases} (t-2) & t \geq 2 \\ 0 & t < 2 \end{cases}$$ Ans

Q. $L^{-1}\left(\frac{s}{s^2-w^2} e^{-as}\right)$

Sol →

$L^{-1}\left(\frac{s}{s^2-w^2}\right) = \cosh wt$

$\Rightarrow L^{-1}\left(e^{-as} \frac{s}{s^2-w^2}\right) = \cosh w(t-a)\, u(t-a)$

$$\Rightarrow \begin{cases} \cosh w(t-a) & t \geq a \\ 0 & t < a \end{cases}$$ Ans

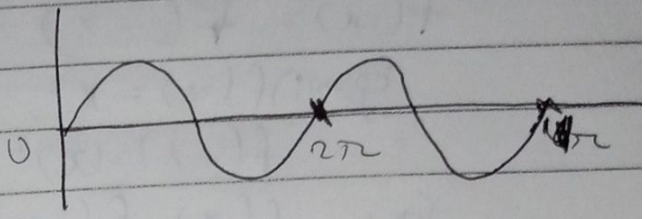
Date..............

• Fourier Series

→ Periodic :

$f(x) = \sin x$

$f(x) = \cos x$



→ Dirichlet's Condition :

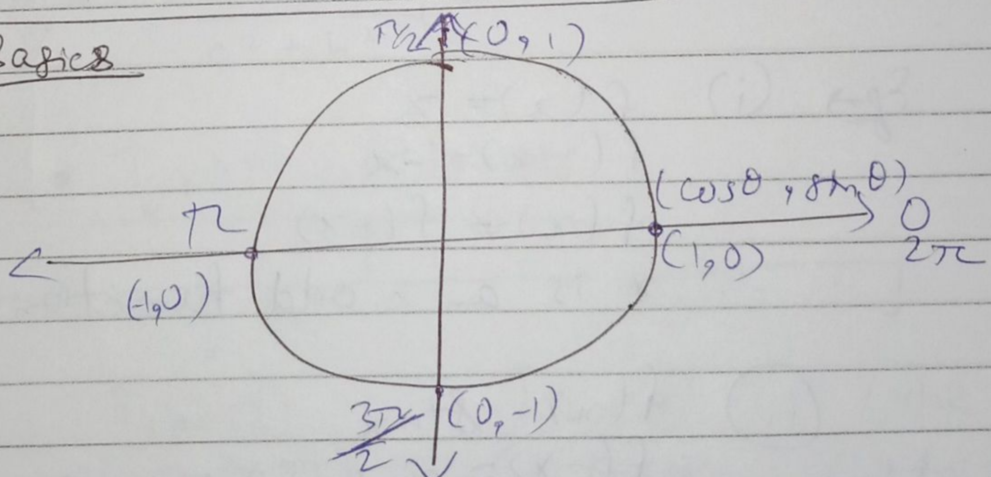
0 to 2π

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=0}^{\infty} b_n \sin nx$$

$$= \left[(a_0 \cos 0) + (a_1 \cos x) + (a_2 \cos 2x) + \dots\right] + \left[(\cancel{b_0 \sin 0}) + (b_1 \sin x) + (b_2 \sin 2x) + \dots\right]$$

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx}$$

• Basics



$z = x + iy$

$\Rightarrow \cos\theta + i\sin\theta$

① $\sin n\pi = 0$

② $\cos(\text{even})\pi = 1$

$\boxed{\cos(2n)\pi = 1}$

③ $\cos(\text{odd})\pi = -1$

$\cos(2n+1)\pi = \cos(2n-1)\pi$

$= -1$

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⇒ Even function & Odd function

① Even function

$f(x) = f(-x)$

Eg → (i) $f(x) = x^2$

$f(-x) = (-x)^2 = x^2$

So; $f(x) = f(-x)$

⇒ x^2 is an even function

(ii) $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x$

⇒ So; $f(x) = f(-x)$

⇒ $\cos x$ is an even function

② Odd function

$f(x) = -f(-x)$

Means $f(x) \neq f(-x)$

Eg → (i) $f(x) = x$

$f(-x) = -x$

$f(x) \neq f(-x)$

$\therefore$ x is an odd function

(ii) $f(x) = \sin x$

$f(-x) = \sin(-x) = -\sin x$

⇒ So; $f(x) \neq f(-x)$

⇒ So, $\sin x$ is an odd function

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⇒ Integration u.v

$$\int uv\,dx = \overbrace{u\int v\,dx}^{\text{ILATE}} - \int\left(\frac{du}{dx}\int v\,dx\right)dx + C$$

Modified

$$\int uv\,dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + u^{iv} v_5 - \cdots$$

⇒ Extra formulaes on integration by parts

(i) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

(ii) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx - b\sin bx\right]$

• Types of limits

- Type ① (0 to 2π)
- Type ② ($-\pi$ to π) → even/odd
- Type ③ (0 to $2l$) / (0 to $2a$)
- Type ④ ($-l$ to l) / ($-a$ to a)

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- Type 1

Limits (0 to 2π)

* Fourier coefficients

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Take integral on both sides, with limits c to $c + 2\pi$;

$$\int_c^{c+2\pi} f(x)\,dx = \int_c^{c+2\pi} a_0\,dx + \int_c^{c+2\pi} \sum_{n=1}^{\infty} a_n \cos nx\,dx + \int_c^{c+2\pi} \sum_{n=1}^{\infty} b_n \sin nx\,dx$$

(suppose $c = 0$)

$$\Rightarrow a_0 [x]_c^{c+2\pi} + 0 + 0$$

$$\Rightarrow \int_c^{c+2\pi} f(x)\,dx = a_0 [c + 2\pi - c]$$

$$\Rightarrow \boxed{a_0 = \frac{1}{2\pi} \int_c^{c+2\pi} f(x)\,dx}$$

$$\Rightarrow a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\,dx$$

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Type 1: Limits 0 to 2π

Q. Find the fourier series of $f(x) = x^2$ in $(0, 2\pi)$.

Sol → Let $x^2 = a_0 + \cancel{\sum_{n=1}^{\infty} a_n \sin nx} \sum_{n=1}^{\infty} a_n \cos nx$
$+ \sum_{n=1}^{\infty} b_n \sin nx$

① $a_0 = \frac{1}{2\pi} \int_0^{2\pi} x^2 \, dx = \frac{1}{2\pi} \left[\frac{x^3}{3} \right]_0^{2\pi}$

$\Rightarrow \frac{1}{6\pi} \left[(2\pi)^3 - 0 \right] \Rightarrow \frac{8\pi^3}{6\pi} \Rightarrow \frac{4\pi^2}{3}$

② $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$

$\Rightarrow \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx \, dx$

$\left(\int uv \, dx = uv_1 - u' v_2 + u'' v_3 - \ldots \right)$

$\Rightarrow \frac{1}{\pi} \left[x^2 \cancel{\left(\frac{\sin nx}{n} \right)} + 2x \left(\frac{\cos nx}{n^2} \right) + 2 \cancel{\left(\frac{-\sin nx}{n^3} \right)} \right]_0^{2\pi}$

$\Rightarrow \frac{2}{n^2 \pi} \left[x \cos nx \right]_0^{2\pi} \Rightarrow \frac{2}{n^2 \pi} \left[(2\pi \cos n 2\pi) - 0 \right]$

$\Rightarrow \frac{2}{n^2 \pi} \left[2\pi (\cos 2n\pi) \right] \Rightarrow \frac{2}{n^2 \pi} (2\pi) \Rightarrow \frac{4}{n^2}$

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$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$\Rightarrow \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \, dx$$

$$\Rightarrow \frac{1}{\pi}\left[-x^2\frac{\cos nx}{n} + \cancel{2x\frac{\sin nx}{n^2}}^{\,0} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$\Rightarrow \frac{1}{\pi}\left[\left[(2\pi)^2\left(-\frac{\cos n2\pi}{n}\right) + 2\left(\frac{\cos 2n\pi}{n^3}\right)\right] - \left(0 + \frac{2\cos 0}{n^3}\right)\right]$$

$$\Rightarrow \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^2} - \frac{2}{n^3}\right]$$

$$\Rightarrow -\frac{4\pi}{n}$$

• fourier series is

$$x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Ans $\Rightarrow$ $x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx + \sum_{n=1}^{\infty}\left(-\frac{4\pi}{n}\right)\sin nx$

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Q Obtain fourier expansion of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 \le x \le 2\pi$.

Sol→ As fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

① $a_0 = \frac{1}{2\pi} \int_c^{c+2\pi} f(x)\,dx$

$$\Rightarrow \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx = \frac{1}{8\pi} \int_0^{2\pi} (\pi - x)^2\,dx$$

$$a_0 = \frac{1}{8\pi} \left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = \frac{-1}{24\pi} \left[(\pi - 2\pi)^3 - (\pi - 0)^3\right]$$

$$\Rightarrow \frac{-1}{24\pi} \left[-\pi^3 - \pi^3\right]$$

$$\Rightarrow \frac{2\pi^3}{24\pi} \Rightarrow \frac{\pi^2}{12}$$

$$\Rightarrow a_n = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nx\,dx$$

$$\Rightarrow \frac{1}{4\pi} \left[\cancel{(\pi - x)^2 \frac{\sin nx}{n}} \to 0 - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + \cancel{2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)} \to 0 \right]_0^{2\pi}$$

$$\Rightarrow a_n = \frac{1}{4\pi}\left[-2(\pi - x)\left(\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$\Rightarrow \frac{-2}{4\pi}\left[(\pi - 2\pi)\left(\frac{\cos 2n\pi}{n^2}\right) - (\pi - 0)\left(\frac{\cos 0}{n^2}\right)\right]$$

$$\Rightarrow \frac{-1}{2\pi}\left[-\pi\left(\frac{1}{n^2}\right) - \pi\left(\frac{1}{n^2}\right)\right]$$

$$\Rightarrow \frac{-1}{2\pi}\left[-\frac{2\pi}{n^2}\right] = \frac{1}{n^2}$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nx \, dx$$

$$\Rightarrow \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2 \sin nx \, dx$$

$$\Rightarrow \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - \cancel{2(\pi - x)(-1)\left(\frac{-\sin nx}{n^2}\right)} \rightarrow 0\right.$$

$$\left. + 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$\Rightarrow \frac{1}{4\pi}\left[(\pi - 2\pi)^2\left(\frac{-\cos 2n\pi}{n}\right) + 2\left(\frac{\cos 2n\pi}{n^3}\right) - \right.$$

$$\left.(\pi - 0)^2\left(\frac{-\cos 0}{n}\right) + 2\left(\frac{\cos 0}{n^3}\right)\right]$$

$$\Rightarrow \frac{1}{4\pi}\left[\cancel{\frac{-\pi^2}{n}} + \cancel{\frac{2}{n^3}} + \cancel{\frac{\pi^2}{n}} - \cancel{\frac{2}{n^3}}\right] \Rightarrow 0$$

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$\Rightarrow$ fourier expansion is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow \left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx \quad \underline{\text{Ans}}$$

• Type 2 : $(-\pi$ to $\pi)$

Q Expand the fourier series for periodic function

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Sol $\rightarrow$ As fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\rightarrow a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$\rightarrow \frac{1}{2\pi} \left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx \right]$$

$$\Rightarrow \frac{1}{2\pi} \left[-\pi (x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_{0}^{\pi} \right]$$

$$\rightarrow \frac{1}{2\pi} \left(-\pi(\pi) + \frac{\pi^2}{2} \right) \Rightarrow \frac{-\pi}{4}$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$\Rightarrow \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$\Rightarrow \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right\}_{0}^{\pi}\right]$$

$$\Rightarrow \frac{1}{\pi}\left(-\pi(0) + \left\{\left(x(0) + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right\}\right)$$

$$\Rightarrow \frac{1}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)$$

$$\left(\begin{array}{l} \cos 2n\pi = 1 \\ \cos n\pi \begin{cases} \to \text{even} \to 1 \\ \to \text{odd} \to -1 \end{cases} \to \cos n\pi = (-1)^n \end{array}\right.$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$\Rightarrow a_n = \frac{(-1)^n - 1}{n^2\pi}$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$\Rightarrow \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

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$$\Rightarrow \frac{1}{\pi}\left[-\pi\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{-\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right\}_0^{\pi}\right]$$

$$\Rightarrow \frac{1}{\pi}\left[\frac{\pi(1-(-1)^n)}{n} + \left(-\frac{\pi}{n}(-1)^n\right)\right]$$

$$\Rightarrow \left(\frac{1-(-1)^n}{n} - \frac{(-1)^n}{n}\right)$$

$$\Rightarrow \left(\frac{1-2(-1)^n}{n}\right)$$

So, fourier series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Ans $f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi} \cos nx + \sum_{n=1}^{\infty}\left(\frac{1-2(-1)^n}{n}\right)\sin nx$

Even ka case

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

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Type 2 (Even & Odd function)

① $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{2\pi}\int_{0}^{\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$

② $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

Find ~~these~~ these two in even case

③ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ — find this in odd case.

Q. Find the fourier series of $f(x) = x^2$; $-\pi \le x \le \pi$.

Ans. As $f(x) = x^2$ & $f(-x) = (-x)^2 = x^2$

$f(x) = f(-x)$; Given function is an even function

fourier series will be

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

① $a_0 = \frac{1}{2\pi}\int_{c}^{c+2\pi} f(x)dx$

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$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx \Rightarrow \frac{2}{2\pi}\int_{0}^{\pi} x^2 dx$$

$$\therefore a_0 = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{1}{3\pi}\left[\pi^3 - 0\right] \Rightarrow \frac{\pi^2}{3}$$

$$(2)\ a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \underset{u}{x^2}\, \underset{v}{\cos nx}\, dx$$

$$\Rightarrow \frac{2}{\pi}\left[x^2\left(\cancel{\frac{\sin nx}{n}}\right)^{0} - 2x\left(-\left(\frac{\cos nx}{n^2}\right)\right) + 2\left(\cancel{\frac{-\sin nx}{n^3}}\right)^{0}\right]_0^{\pi}$$

$$\Rightarrow \frac{2\times 2}{\pi}\left[\frac{\pi\cos n\pi}{n^2} - 0\right] \Rightarrow \frac{4}{\pi}\left(\frac{\pi(-1)^n}{n^2}\right)$$

$$\Rightarrow \frac{4(-1)^n}{n^2}$$

$\therefore$ Fourier Expanstion is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx \quad \text{Ans}$$

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Q Obtain fourier series for

$f(x) = x + \pi/2 \quad ; -\pi < x < 0$

$\quad = \pi/2 - x \quad ; 0 < x < \pi$

Sol→ $f(-x) = -x + \pi/2 \quad ; \pi < -x < 0$

$\quad = \pi/2 + x \quad ; 0 < -x < \pi$

$\Rightarrow f(-x) = \pi/2 - x \quad ; \pi > x > 0$

$\Rightarrow \pi/2 + x \quad ; 0 > x > \pi$

$\therefore f(x) = f(-x)$ (Given function is Even function)

Fourier Expansion will be

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

① $$a_0 = \frac{1}{2\pi}\int_{c}^{c+2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(x+\pi/2)\,dx + \int_{0}^{\pi}(\pi/2 - x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{x^2}{2} + \frac{\pi}{2}x\right)_{-\pi}^{0} + \left(\frac{\pi}{2}(x) - \frac{x^2}{2}\right)_{0}^{\pi}\right]$$

$$\Rightarrow \frac{1}{2\pi}\left[0 - \left(\frac{(-\pi)^2}{2} + \frac{(\pi^2)}{2}\right) + \left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) - 0\right]$$

$\Rightarrow a_0 = 0$

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(2) $a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cos nx \, dx$

$\Rightarrow \frac{1}{\pi} \left[\int_{-\pi}^{0} \underbrace{\left(x + \frac{\pi}{2}\right)}_{u} \underbrace{\cos nx \, dx}_{v} + \int_{0}^{\pi} \underbrace{\left(\frac{\pi}{2} - x\right)}_{u} \underbrace{\sin nx \, dx}_{v} \right]$

$\boxed{\text{OR}}$

$a_n \Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$\Rightarrow \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx$

$\Rightarrow \frac{2}{\pi} \int_{0}^{\pi} \left(\frac{\pi}{2} - x\right) \cos nx \, dx$

$\Rightarrow \frac{2}{\pi} \left[\left(\frac{\pi}{2} - x\right) \left(\frac{\sin nx}{n}\right) - (-1) \left(-\frac{\cos nx}{n^2}\right) \right]_{0}^{\pi}$

$\Rightarrow \frac{2}{\pi} \left[-\frac{\cos n\pi}{n^2} + \frac{\cos 0}{n^2} \right]$

$\Rightarrow \frac{2}{\pi} \left[-\frac{(-1)^n}{n^2} - \frac{(-1)}{n^2} \right]$

$\Rightarrow \frac{2}{\pi} \left(\frac{-(-1)^n + 1}{n^2} \right) = \frac{2(1 - (-1)^n)}{n^2 \pi}$

$\Rightarrow$ Now; $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$\Rightarrow f(x) = 0 + \sum_{n=1}^{\infty} \left(\frac{2(1 - (-1)^n)}{n^2 \pi} \right) \cos nx$

Ans