Time Domain Analysis - Unit-2

Introduction ch-2 (Book + Vid-23)

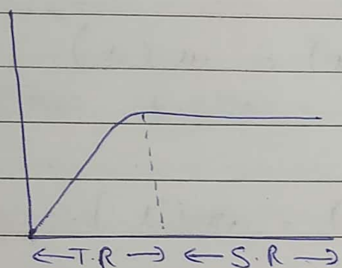
- Time Required to change from one State to another State is transient time
- Values of Current & voltages during this period is called transient response.

Also

- Output $c(t)$ with respect to time is time response analysis

Types of Response

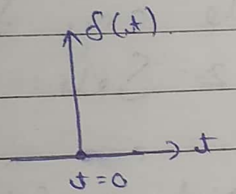
- Transient response (Non-Uniform)
- Steady-State response (Constant response)



Test Signals :

1) Impulse: ($\delta(t)$) — Graph — S-Domain

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

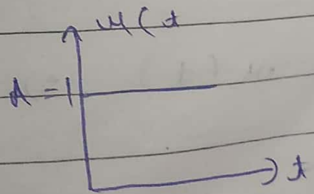


1

2) Step Signal: ($u(t)$)

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$A = 1$ (Unit Step

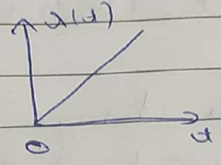


$\frac{A}{S}$, $\frac{1}{S}$ ← Unit Step

Spiral

3) Ramp Signal : ($r(t)$)   Graph   S - Domain

$$x(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

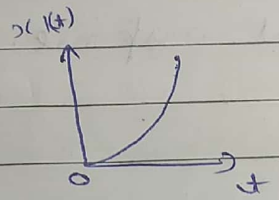


unit ramp

$\frac{A}{S^2} \rightarrow \frac{1}{S^2}$

4) Parabolic Signal : $x(t)$

$$x(t) = \begin{cases} At^2/2 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



unit Parabolic

$\frac{A}{S^3} , \frac{1}{S^3}$

Relations btw Signals :-

$\int \delta(t) = u(t)$ $\frac{d}{dt} u(t) = \delta(t)$

or

$\int u(t) = r(t)$ $\frac{d}{dt} r(t) = u(t)$

$\int r(t) = x(t)$ $\frac{d}{dt} x(t) = r(t)$

We can relate any signal with any other signal with these 2 - Sets.

$\delta(t)$ w.r.t $u(t)$

$\delta(t) = \frac{d}{dt} u(t) \Rightarrow \delta(t) = \frac{d}{dt} \frac{d}{dt} r(t) = \frac{d^2}{dt^2} r(t)$

teleco Dt.: Pg.:

Order of a System :

Let $T(s) = K \frac{P(s)}{Q(s)}$

K = Constant

$P(s)$ = Numerator Polynomial
$Q(s)$ = Denominator Polynomial

→ Order of the System will be determined by the power of S present in the $Q(s)$.

$$\boxed{Q(s) = a_0 S^n + a_1 S^{n-1} + a_2 S^{n-2} + \cdots a_{n-1} S + a_n}$$

If

→ $n = 0$ → Zero order system.

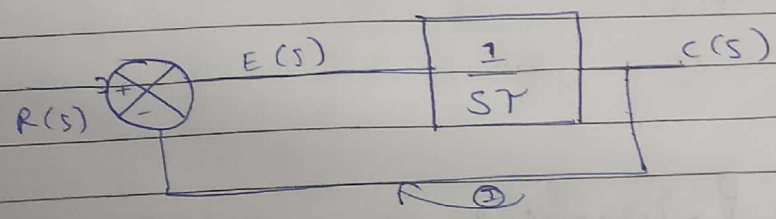
→ $n = 1$ → 1st order System.

→ $n = 2$ → 2nd order System.

Spiral

Time Response of a 1st order System :-

Unit Step Signal :-



Basic 1st order $T(s) = \frac{1}{1+s\tau}$

$$T(s) = \frac{G}{1+GH}$$

$$= \frac{\frac{1}{s\tau}}{1+\frac{1}{s\tau}} = \frac{1}{1+s\tau}$$

we know

$r(t) = u(t) = 1 \qquad t \geq 0$

↓
Input

$$R(s) = L.T[r(t)] = L.T[1] = \frac{1}{s} \qquad t \geq 0$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{1}{1+s\tau}$$

$$C(s) = R(s) \cdot \frac{1}{1+s\tau}$$

Telco Dt.: Pg.:

$$= \frac{1}{S} \cdot \frac{1}{1+S\tau}$$

$$= \frac{1}{S} \cdot \frac{1}{\tau\left(\frac{1}{\tau}+S\right)} = \frac{1}{S} \cdot \frac{1/\tau}{\left(\frac{1}{\tau}+S\right)}$$

$$C(s) = \frac{1}{S} - \frac{1}{\left(S+\frac{1}{\tau}\right)}$$

Now taking Inverse Laplace transform

$$C(t) = L^{-1}\left[\frac{1}{S} - \frac{1}{\left(S+\frac{1}{\tau}\right)}\right]$$

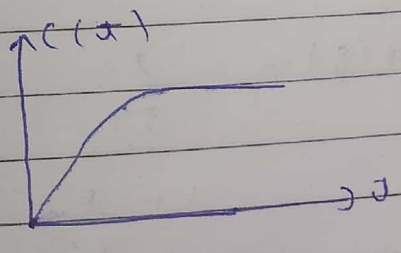
I.L.T of a) $\frac{1}{S} = 1$ $t \geq 0$

b) $S + \frac{1}{\tau} = e^{-at}$

$$\boxed{C(t) = 1 - e^{-\frac{1}{\tau}t}}$$ $t \geq 0$

τ = Time Const

Response of 1st order:



$t = \tau$

$C(t) = 1 - e^{-1} = 0.632$ or 63.2%

Spiral

Error $(e(t))$

$$e(t) = r(t) - c(t)$$

$r(t)$ ↓ input $c(t)$ ↓ output

$$e(t) = 1 - \left(1 - e^{-1/\tau\, t}\right)$$

$$\boxed{e(t) = e^{-\frac{1}{\tau} t}} \Rightarrow \text{Error}$$

② Time Respons of 1st Order System:

<u>Unit Impulse</u>: $(\delta(t))$

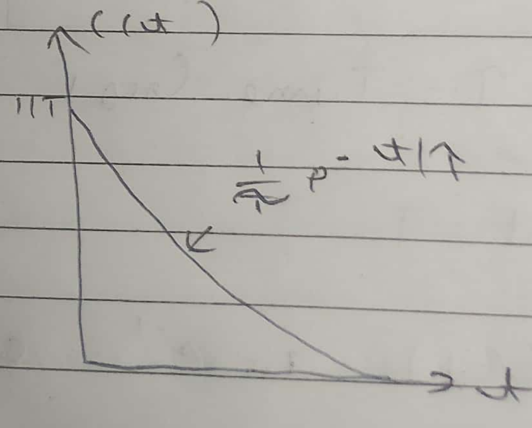
$$T(s) = \frac{C(s)}{R(s)} = \frac{1}{1 + s\tau} \quad \text{— ①}$$

$$T(s) = \frac{G}{1 + GH}$$

$$G = \frac{1}{s\tau}$$

$$T(s) = \frac{\frac{1}{s\tau}}{1 + \frac{1}{s\tau}} = \frac{1}{1 + s\tau}$$

Response:



telco Dt.:
Pg.:

$r(t) = \delta(t) = 1$

$\Downarrow$

Input

Inverse . L.T of $\delta(t)$

$R(s) = L.T[r(t)] = L.T[\delta(t)] = 1$

$$\frac{C(s)}{R(s)} = \frac{1}{1+sT}$$

$$C(s) = R(s) \cdot \frac{1}{1+sT} = \frac{1}{1+sT}$$

$$C(s) = \frac{1/T}{\left(\frac{1}{T} + s\right)}$$

Inverse . L.T. :

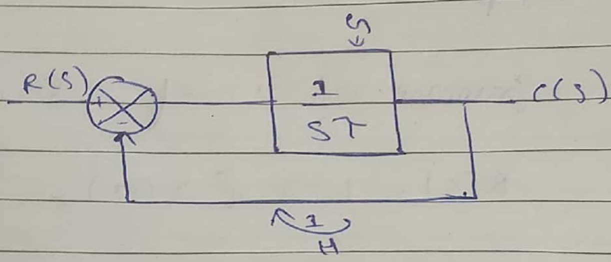
$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\left(\frac{1}{T}\right) \cdot \frac{1}{\left(s+\frac{1}{T}\right)}\right]$$

$$\boxed{c(t) = \frac{1}{T} e^{-at} \quad t \geq 0}$$

3 Time Response of 1st order System:

Unit Ramp Signal,

$$T(s) = \frac{C(S)}{R(S)} = \frac{1}{1+S\tau}$$



$$T(s) = \frac{G}{1+GH} = \frac{1/S\tau}{1+\frac{1}{S\tau}\times 1}$$

$$\boxed{T(s) = \frac{1}{1+S\tau}}$$

$$\frac{C(s)}{R(s)} = \frac{1}{1+S\tau} \quad -1$$

we know

$$R(t) = r(t) = t \quad , t \geq 0$$

g.L.T

$$R(s) = L.T\,[r(t)] = \frac{1}{S^2}$$

$$C(S) = R(S) \times \frac{1}{1+S\tau}$$ from 1, we get

$$C(s) = \frac{1}{s^2} \cdot \frac{1}{1+sT} \quad -② $$

Making eqⁿ ② easily inverse transable, we get

$$= \frac{1-sT}{s^2} + \frac{T^2}{1+sT}$$

$$= \frac{1}{s^2} - \frac{T}{s} + T \times \frac{1}{\left(s+\frac{1}{T}\right)}$$

I.L.T on both sides

I.L.T of a) $\frac{1}{s^2} = t$

b) $\frac{1}{s} = 1$

c) $\frac{1}{s+\frac{1}{T}}$ or $\frac{1}{s+a} = e^{-at}$

we get,

$$c(t) = t - T \cdot 1 + T e^{-\frac{1}{T}t}$$

$$\boxed{c(t) = t - T + T e^{-\frac{1}{T}t}}$$

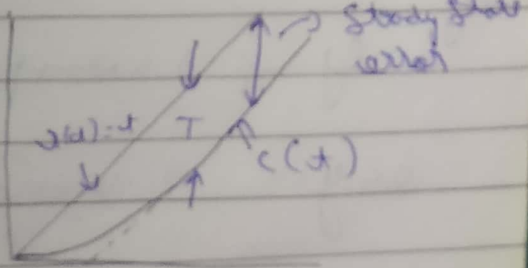
Error

$$e(t) = u(t) - c(t)$$

$$= \cancel{t} - \left(\cancel{t} - T + T e^{-\frac{1}{T}t}\right)$$

Spiral $= T - T e^{-\frac{1}{T}t}$

Unit Ramp response

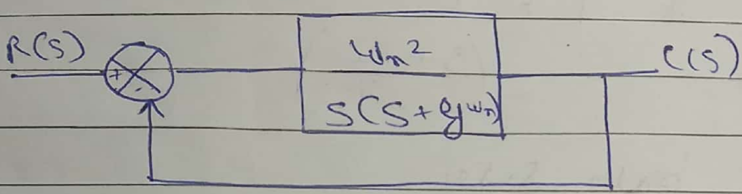


Steady State error:

$$= \lim_{t \to \infty} e(t) = t (T - Te^{-\frac{1}{T}t})$$

$$\boxed{S.S.E = T}$$

Second order System.



$$T(S) = \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

where

ω_n = undamped natural freq.

ζ = damping ratio = $\frac{\text{actual damping}}{\text{critical damping}}$

if

$\zeta = 0 \Rightarrow$ undamped sys

$0 < \zeta < 1 \Rightarrow$ underdamped sys

$\zeta = 1 \Rightarrow$ critically damped system

$\zeta > 1 \Rightarrow$ over damped System.

telco Dt.: Pg.:

$$\underbrace{s^2}_{a} + \underbrace{2 \xi \omega_n}_{b} s + \underbrace{\omega_n^2}_{c} = 0$$

$$s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - (2\omega_n)^2}}{2}$$

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$\xi = 0 \rightarrow s_1, s_2 = \pm j\omega_n$, Purely imaginary

$\xi = 1 \rightarrow s_1, s_2 = -\omega_n$, Purely real

$0 < \xi < 1 \rightarrow s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

$\xi > 1 \rightarrow s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

Spiral

Response of Undamped 2nd order System for Unit Step i/p:

$$T(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

for undamped $\xi = 0$, $T(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}$

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

& $r(t) = u(t) = 1$

$R(s) = L.T[u(t)] = \frac{1}{s}$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + \omega_n^2} \quad -(1)$$

$$= \frac{A(s^2 + \omega_n^2) + s(Bs + C)}{s(s^2 + \omega_n^2)}$$

$A = 1$

$A + B = 0$

$1 + B = 0$, $B = -1$, $C = 0$

from (1)

$$= \frac{1}{s} - \frac{s}{s^2 + \omega_n^2} \quad -(2)$$

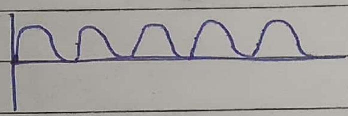
L.T of ②

$$c(t) = L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right]$$

$$\boxed{c(t) = 1 - \cos\omega_n t} \quad , \quad t \geq 0$$

L.T of $\dfrac{s}{s^2+\omega_n^2} = \cos\omega_n t$

Response



→ For Under damped system :-

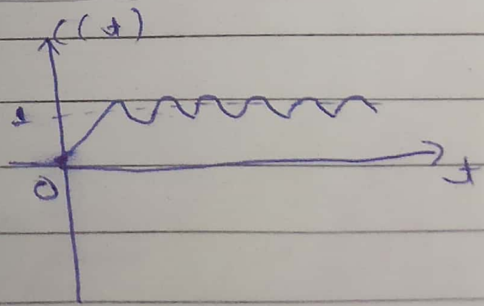
$0 < \zeta < 1$ $\qquad r(t) = u(t)$

$\qquad R(s) = 1/s$

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad - ①$$

Put value of ζ betw 0 & 1 in eqn - ① & then solving it by inverse L.T, we get

$$\boxed{c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)}$$



$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

* Critically damped System: $(\zeta = 1)$

$$\boxed{c(t) = 1 - e^{-\omega_n t}\left(1 + \omega_n t\right)}$$

* Overdamped System:

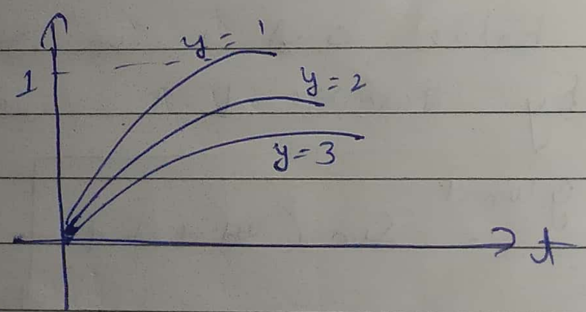
$$\zeta > 1$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right]$$

where

$$\left.\begin{aligned} s_1 &= \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \\ s_2 &= \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \end{aligned}\right\} \Rightarrow \text{roots}$$

Response:



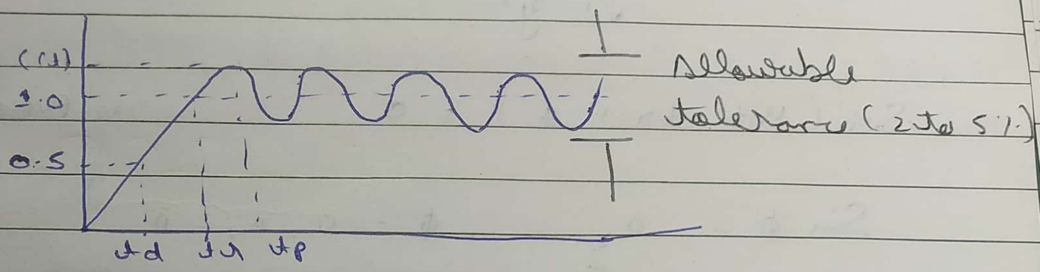
Telco Dt.: Pg.:

Time Domain Specifications :.

The desired performance char of Control System are Specified interms of Time domain Specification :.

1) Delay time (t_d) 2) Rise time (t_r) 3) Peak time (t_p)

4) Maximum overshoot (M_p) 5) Settling time (t_s)



a) Delay time (t_d) = Time req for the response to reach 50% of the final value in 1st time

b) Rise time (t_r) = Response to reach 0% to 100% (underdamped Sy) but for overdamped it is from (10 to 90%)

c) Peak time (t_p) = Response to reach ~~to~~ the first peak

d) Maximum overshoot (M_p) = Ratio of Difference of maximum value $c(t_p)$ & final value => $c(\infty)$

So $$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

e) Settling time (t_s) = Time req for the response to reach & stay within the Specified range (2% to 5%) of its final value

Spiral

Rise time (t_r) : Underdamped System (0% to 100%)

$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \phi)$$

$c(t) = c(t_r) = 1$

$$\cancel{1} = \cancel{1} - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \phi)$$

$\sin(\omega_d t_r + \phi) = 0$, we know $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

$\sin \phi = 0$, $\phi = 0, \pi, 2\pi \ldots$

$\sin \pi = 0$

So

$\sin(\omega_d t_r + \phi) = \sin \pi$

$\therefore \omega_d t_r + \phi = \pi$

$$t_r = \frac{\pi - \phi}{\omega_d t_r}$$

We know :

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$\therefore \boxed{t_r = \frac{\pi - \phi}{\omega_n \sqrt{1-\zeta^2}}} , \quad \phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Telco Dt.: Pg.:

Peak time (t_p) :- Underdamped System

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \times \sin(\omega_d t_p + \phi)$$

$$\frac{d}{dt} c(t) \Big|_{t = t_p} = 0$$

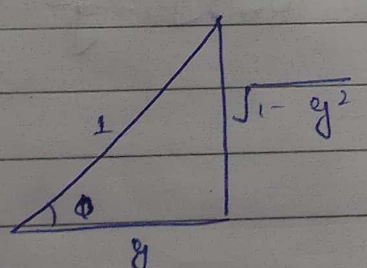
$$0 - \frac{1}{\sqrt{1-\xi^2}} \left[e^{-\xi \omega_n t} (-\xi \omega_n) \sin(\omega_d t_p + \phi) + e^{-\xi \omega_n t_p} \omega_d \cos(\omega_d t_p + \phi) \right] = 0$$

$$= \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} = \text{Common}$$, So remaining

$$\xi \omega_n \sin(\omega_d t_p + \phi) - \omega_d \cos(\omega_d t_p + \phi) = 0$$

$$= \xi \omega_n \sin(\omega_d t_p + \phi) - \omega_n \sqrt{1-\xi^2} \cos(\omega_d t_p + \phi) = 0$$

$$= \xi \sin(\omega_d t_p + \phi) - \sqrt{1-\xi^2} \cos(\omega_d t_p + \phi) = 0$$



$$\sin \phi = \frac{\sqrt{1-\xi^2}}{1}$$

$$\cos \phi = \xi$$

Replacing ξ & $\sqrt{1-\xi^2}$ with $\sin \phi$ & $\cos \phi$

Spiral

$\cos\phi \sin(\omega_d t_p + \phi) - \sin\phi \cos(\omega_d t_p + \phi) = 0$

$\sin A \cos B - \cos A \sin B = \sin A - B$

So

$\sin(\omega_d t_p + \phi - \phi) = 0$

$\sin(\omega_d t_p) = 0$

we know

$\sin\phi = 0$

$\phi = 0, \pi, 2\pi, 3\pi$

$\sin\pi = 0$

So, we can write

$\sin(\omega_d t_p) = \sin\pi$

$\omega_d t_p = \pi$

$\boxed{t_p = \frac{\pi}{\omega_d}}$ $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

Peak Overshoot or Maximum Overshoot (M_p)
(Underdamped System)

$$C(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi)$$

We know.

$$M.P = \frac{C(t_p) - C(\infty)}{C(\infty)}$$

$C(t_p)$:.

$$C(t_p) = 1 - \frac{e^{-\zeta \omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \phi)$$

We know,

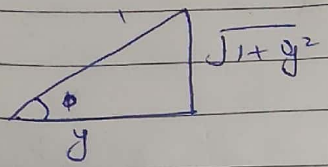
$$\boxed{t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}}$$

$$C(t_p) = 1 - \frac{e^{-\zeta \cancel{\omega_n} \frac{\pi}{\cancel{\omega_n}\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\left(\cancel{\omega_d} \frac{\pi}{\cancel{\omega_d}} + \phi\right)$$

$$C(t_p) = 1 - \frac{e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(\pi + \phi)$$

We know $\sin \pi + \phi = -\sin \phi$

$$C(t_p) = 1 + \frac{e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin \phi$$



$$\sin\phi = \sqrt{1-\zeta^2}$$

$$c(t_p) = 1 + \frac{e^{-\zeta\frac{\pi}{\sqrt{1-\zeta^2}}}}{\cancel{\sqrt{1-\zeta^2}}} \times \cancel{\sqrt{1-\zeta^2}}$$

$$\boxed{c(t_p) = 1 + e^{-\zeta\frac{\pi}{\sqrt{1-\zeta^2}}}}$$

$c(\infty)$:-

$$= 1 - \frac{e^{-\zeta w_n t}}{\sqrt{1-\zeta^2}} \sin(w_d t + \phi)$$

$$= 1 - \frac{e^{-\infty}}{\sqrt{1-\zeta^2}} \sin(w_d \times \infty + \phi)$$

$$= 1 - 0$$

$$c(\infty) = 1$$

. Now

$$M.P = \frac{\cancel{1} + e^{-\zeta\pi/\sqrt{1-\zeta^2}} - \cancel{1}}{1}$$

Dt.:
Pg.:

Settling time (t_s)

$$C(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi)$$

t_s = $\underbrace{2\%}_{\text{Taking this}}$ to 5% tolerance band

$$\frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} = 0.02$$

$$e^{-\zeta \omega_n t_s} = 0.02$$

$$-\zeta \omega_n t_s = \ln(0.02)$$

$$\cancel{-}\zeta \omega_n t_s = \cancel{-}4$$

$$\therefore \boxed{t_s = \frac{4}{\zeta \omega_n}} \rightarrow 4T$$

$T = \frac{1}{\zeta \omega_n}$ = Time const of the System

At 5% tolerance, we get

$$\boxed{t_s = \frac{3}{\zeta \omega_n}} \rightarrow 3T$$

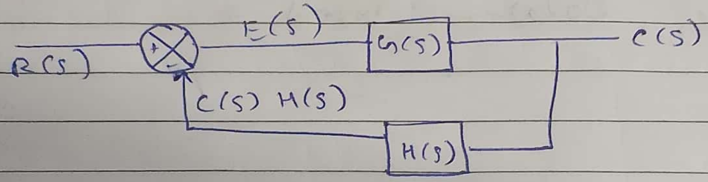
$$\boxed{T_1 = \frac{1}{\zeta \omega_n}}$$

Spiral

Steady State error : (Difference betw i/p & o/p of the Sys at Steady State).

or

Value of error Signal when $t \to \infty$



$$E(s) = R(s) - C(s)\,H(s) \quad -(1)$$

Also

$$\therefore C(s) = E(s)\,G(s)$$

$$E(s) = R(s) - E(s)\,G(s)\,H(s)$$

$$E(s)\left[1 + G(s)\,H(s)\right] = R(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)\,H(s)}$$

$$e(t) = L^{-1}\left[E(s)\right] = L^{-1}\left[\frac{R(s)}{1 + G(s)\,H(s)}\right]$$

$$\boxed{e_{ss} = \underset{t \to \infty}{Lt}\ e(t)}$$

$$= e(\infty) = \underset{s \to 0}{Lt}\ s\,E(s)$$

$$\boxed{e_{ss} = \underset{s \to 0}{Lt}\ s\,E(s)}$$

teleco Dt.:
Pg.:

Static Error Coefficients or Constants :-

When a control System is excited with Std input Sig the Steady State error may be zero, const or infinite. Value of Steady State error depends upon the type no of i/p Sig.

3 - Types :

→ Positional error const (Unit Step i/p)

$$K_p = \lim_{s \to 0} G(s) H(s)$$

→ Velocity error const (Unit ramp i/p)

$$K_v = \lim_{s \to 0} s\, G(s) H(s)$$

→ Accⁿ error const (Parabolic i/p)

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

(K_p, K_v, K_a = Static error Const.)

→ Response with P, PI & PID Controllers :-

P - Control :-

→ Actuating Signal ∝ error Sig

$$E(s) = R(s) - B(s)$$

$e_a(t) \propto e(t)$

$e_a(t) = K_p\, e(t)$ — (1)

K_p = Proportional const (Taking Laplace both Side)

$E_a(s) = K_p\, E(s)$ ∴ $K_p = \frac{E_a(s)}{E(s)}$ ⇒ Actuating Sig ⇒ Error Sig

Spiral

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PI Controller :

It is made to reduce the Steady State error without disturbing the Stability.

→ PI Controller consist of Proportional & Integral Controller to make a PI Controller.

$u(t) \propto e(t) + \int e(t)$

$u(t) = K_P e(t) + K_I \int e(t)$

let L.T on B.S

$u(s) = K_P E(s) + K_I \frac{E(s)}{S}$

$u(s) = E(s)\left(K_P + \frac{K_I}{S}\right)$

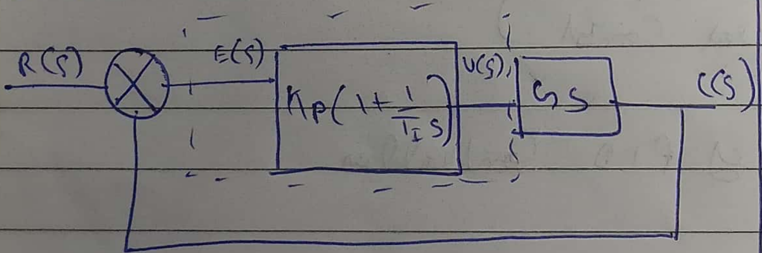
$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{S} \rightarrow K_P\left(1 + \frac{K_I}{K_P}\right)$

→ Transfer fn

$= K_P\left(1 + \frac{1}{T_I S}\right)$

$\therefore \boxed{T_I = \frac{K_P}{K_I}}$

B.D



PID Controller :

O/P is the Combination of Proportional + Integral + Derivative Controller :

Derivative C ⇒ Steady State prob

Integral C ⇒ Stability prob

PID

$u(t) \propto e(t) + \int e(t) + \frac{d}{dt} e(t)$

Note

Integral L.T = $\frac{E(S)}{S}$

Derivative L.T = $S\,E(S)$.

So $\frac{U(S)}{E(S)} = K_P + \frac{K_I}{S} + K_D S$ — (1)

$T_i = \frac{K_P}{K_I}$, $T_D = \frac{K_D}{K_P}$

$\frac{U(S)}{E(S)} = K_P\left[1 + \frac{1}{T_i S} + T_D S\right]$

Eqn (1) Can be written as

$= \frac{K_P S + K_I + K_D S^2}{S}$

We have two S in above eqn's num^r one S for Steady state error & one S for Stability & S in denominator Shows for horizon.

Spiral