Prerequisite: $(1+x)^{-1} = 1-x+x^2-x^3+\dots$
$(1-x)^{-1} = 1+x+x^2+x^3\dots$

Date 8/5/23

MAL

Unit - II

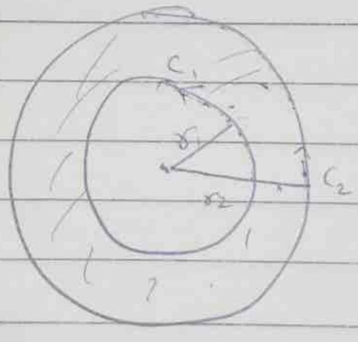
• Topic → Laurent's Series
Laurent's Theorem

If $f(z)$ is Analytic on C_1 & C_2 & the annular region formed by the two concentric circles of Radii r_1 & r_2, center 'a' then for all z in R.

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2}$$

where $a_n = \frac{1}{2\pi i}\int \frac{f(w)}{(w-a)^{n+1}}\,dw$

$b_n = \frac{1}{2\pi i}\int \frac{f(w)\,dw}{(w-a)^{-n+1}}$



Q. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's series valid from.

~~(i) $1 < |z| < 3$~~

~~Sol~~ Sol → $\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$

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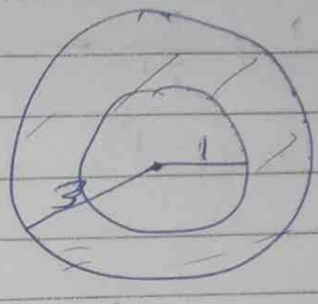
(i) $1 < |z| < 3$

Sol → $\frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$

Take bigger term as common



$\Rightarrow \frac{1}{2}\left[\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{3\left(1+\frac{z}{3}\right)}\right]$

$\Rightarrow \frac{1}{2}\left[\frac{1}{z}\left[1+\frac{1}{z}\right]^{-1} - \frac{1}{3}\left[1+\frac{z}{3}\right]^{-1}\right]$

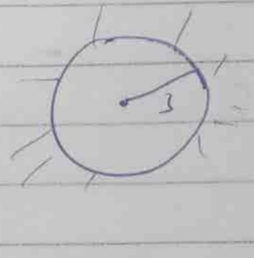
$\rightarrow \frac{1}{2}\left[\frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3}\right)\right]$

Ans → $\left(\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} \cdots\right) - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} \cdots$

(ii) $|z| > 3$

Sol → $f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$

$= \frac{1}{2}\left(\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{z\left(1+3/z\right)}\right)$



$\Rightarrow \frac{1}{2}\left(\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1+3/z\right)^{-1}\right)$

$\Rightarrow \frac{1}{2z}\left(1 - 1/z + 1/z^2 - \frac{1}{z^3} + \cdots\right) - \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right)$

Ans → $\left(\frac{1}{2z} - \frac{1}{2z}\right) + \left(\frac{-1}{2z^2} + \frac{3}{2z^2}\right) + \left(-\frac{1}{2z^3} + \frac{9}{2z^3}\right) + \left(\frac{-1}{2z^4} + \frac{27}{2z^4}\right)$ #

Ans → $\boxed{\frac{1}{z^2} + \frac{4}{z^3} + \frac{13}{z^4} + \cdots}$

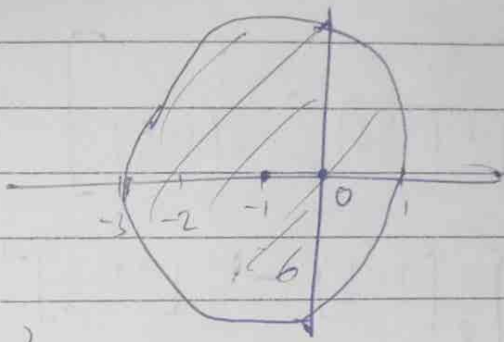
(iii) $0 < |z+1| < 2$

Sol → $f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right)$

~~(Put z+1~~

$\Rightarrow \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+1+2}\right)$

(Put $z+1 = t$)



Take

$\Rightarrow \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t+2}\right)$

(→ Take bigger term as common)

$\cancel{\Rightarrow \frac{1}{2}\left((1+z)^{-1} - \frac{1}{2(1+z+)}\right)}$

$\Rightarrow \frac{1}{2}\left(t^{-1} - \frac{1}{2\left(1+\frac{t}{2}\right)}\right)$

$\Rightarrow \frac{1}{2t} - \frac{1}{4}\left(1 - \frac{t}{2} + \frac{t^2}{4} \cdots\cdots\right)$

$\Rightarrow \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \cdots\cdots$ Ans

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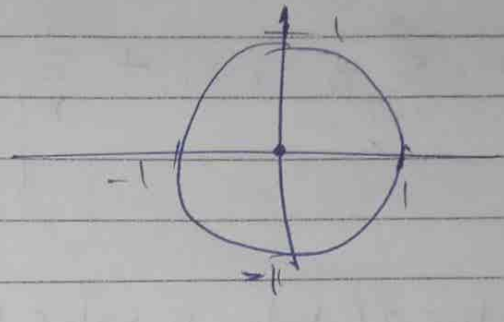
(iv) $|z| < 1$

Sol→ $f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$



$$\Rightarrow \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{3\left(1+\frac{z}{3}\right)}\right]$$

$$\Rightarrow \frac{1}{2}\left[(1+z)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$\Rightarrow \frac{1}{2}\left(1 - z + z^2 - z^3 + \ldots\ldots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \ldots\right)$$

$$\Rightarrow \left(\frac{1}{2} - \frac{1}{6}\right) + \left(-\frac{z}{2} + \frac{z}{18}\right) + \left(\frac{z^2}{2} - \frac{z^2}{54}\right) + \left(-\frac{z^3}{2} + \frac{z^3}{162}\right) + \ldots$$

$$\Rightarrow \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^3 - \frac{40}{81}z^3 + \ldots\ldots$$ Ans

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• Topic ⇒ Residue of Complex function

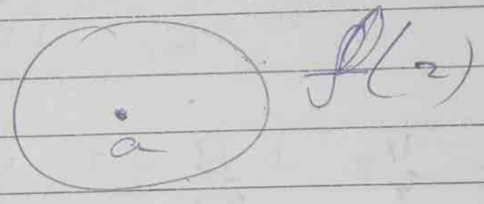
$f(z) = a$ be a pole of order 'm'

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a) + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

(By Laurent's series)

Residue at pole is value of b_1

★ Residue (at $z=a$) $= \lim_{z \to a} (z-a) f(z)$
(at simple pole)



★ Residue of Pole of order 'm' $= \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \{(z-z_0)^m f(z)\}$

Q: Find order of each pole & Residue at it of

$$\frac{1-2z}{z(z-1)(z-2)}$$

Sol → Poles:
Put $z(z-1)(z-2) = 0$
$z=0 \; ; \; z=1 \; ; \; z=2$

$$f(z) = \frac{1-2z}{z(z-1)(z-2)}$$

Residue of $f(z)$ at $(z=0)$ = $\lim_{z\to 0} (z-0) f(z)$

$$\Rightarrow \lim_{z\to 0} (z) \frac{1-2z}{(z-1)(z-2)(z)}$$

$$\Rightarrow \frac{1}{2}$$

Residue of $f(z)$ at $(z=1)$ = $\lim_{z\to 1} (z-1) f(z)$

$$\Rightarrow \lim_{z\to 1} \frac{(z-1)(1-2z)}{(z-1)(z-2)z} = 1$$

Residue of $f(z)$ at $(z=2)$ = $\lim_{z\to 2} (z-2) f(z)$

$$= \lim_{z\to 2} (z-2) \frac{(1-2z)}{z(z-1)(z-2)}$$

$$= -\frac{3}{2}$$

Q. Find Residue of fn $f(z) = \frac{z^2}{(z+1)^2(z-2)}$ at its Double Pole.

Sol: Poles: $(z+1)^2 (z-2) = 0$

$z = -1, -1, 2$

Residue at $(z=-1)$ = $\frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z)\right)$

$$= \frac{1}{(2-1)!} \lim_{z\to -1} \frac{d}{dz}\left(\frac{(z+1)^2 \cdot z^2}{(z+1)^2(z-2)}\right)$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(\frac{z^2}{z-2}\right)$$

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$$\Rightarrow \left[\frac{(z-2)(2z) - z^2 \cdot 1}{(z-2)^2}\right]_{z=-1}$$

$$\cancel{\Rightarrow (-1)^2 - 4}$$

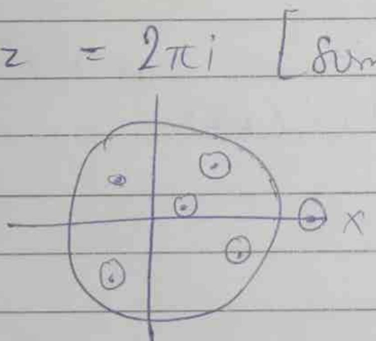
$$\Rightarrow \frac{(6 - (-1)^2)}{(-3)^2} \Rightarrow \frac{6-1}{9} \Rightarrow \frac{5}{9} \text{ Ans}$$

• Topic : Residue Integration Method

Cauchy's Residue Theorem

If $f(z)$ is Analytic in a closed curve 'C', Except a finite no. of poles within C, then

$$\int_C f(z)\,dz = 2\pi i \text{ [Sum of all Residues at the poles within C]}$$



$f(z)$ - Analytic
finite → Poles

Q. Evaluate following integral using Residue Theorem.

$$\int_C \frac{1+z}{z(2-z)}\,dz \quad \text{where C is circle } |z|=1.$$

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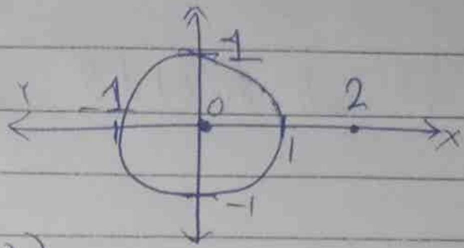
Sol: Poles :

$z(2-z)=0$

$z=0,\ 2$



$$\oint f(z)\,dz = 2\pi i\,(\text{Residue at } z=0)$$

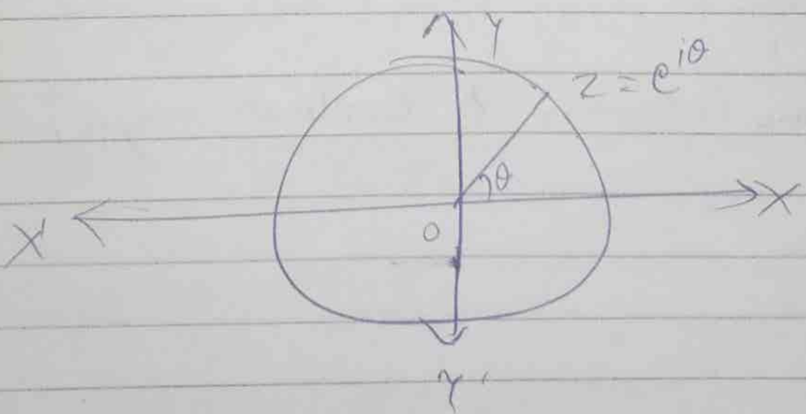
Residue at $z=0$ $= \lim_{z\to z_0} (z-z_0) f(z)$

$$= \lim_{z\to 0} (z-0)\frac{(1+z)}{z(2-z)}$$

$$\Rightarrow \left[\frac{(1+z)}{2-z}\right]_{z=0} \Rightarrow \frac{1}{2}$$

$$\int_C \frac{1+z}{z(2-z)}\,dz = 2\pi i\left(\frac{1}{2}\right) = \pi i \text{ Ans}$$

[• Topic : Integration Round a Unit Circle]



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Also known as Real Definite Integrals

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

Steps

1. Convert $\sin\theta$ & $\cos\theta$ into z

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ (as $z = re^{i\theta}$ $= e^{i\theta}$ $(r=1)$ unit circle)

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \Rightarrow \frac{1}{2}\left(z + \frac{1}{z}\right)$

2. $\int$ Convert into f^n of z.

$z = e^{i\theta}$

$dz = e^{i\theta}\, i\, d\theta$

$\boxed{d\theta = \frac{dz}{iz}}$

3. Apply Cauchy Residue Theorem to Evaluate integral.

Q: Evaluate.

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$$

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Ans

$$\int_0^{2\pi} \frac{d\theta}{5-3\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)}$$ $\left(\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)\right)$ $\left(z = re^{i\theta} = e^{i\theta} (r=1)\right)$

$$= \int_C \frac{dz}{iz\left(5-3\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)\right)}$$

$$\Rightarrow \int_C \left(\frac{1\times 2}{10-3z-\frac{3}{z}}\right)\frac{dz}{iz}$$

$$\Rightarrow \frac{1}{i}\int_C \frac{2\,dz}{10z-3z^2-3}$$

$$\cancel{\Rightarrow -\frac{1}{i}\int_C \frac{2\,dz}{3z^2-10z+3}}$$

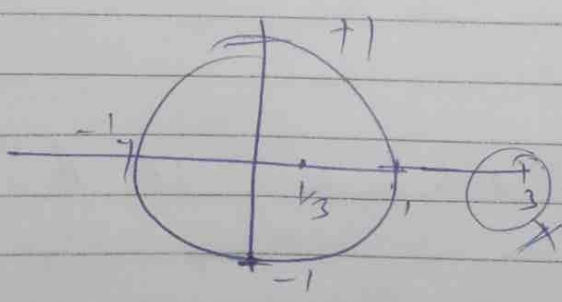
$$-\frac{1}{i}\times\frac{-1}{-i} = \frac{i^2}{-i} = -i$$

$$\Rightarrow i\int_C \frac{2\,dz}{-10z+3z^2+3}$$

$$-\frac{1}{i}\times\frac{-1}{-1} = \frac{i^2}{1} \Rightarrow i$$

$$\Rightarrow 2i\int_C \frac{dz}{(3z-1)(z-3)}$$

Poles $\Rightarrow z = 1/3, 3$



(By Cauchy Residue)

$\Rightarrow$ Reside at pole $\left(z = \frac{1}{3}\right)$

$$= \lim_{z \to 1/3} \left((z - 1/3) \frac{1}{(3z-1)(z-3)} \right)$$

$$\Rightarrow \left(\frac{1}{3(z-3)} \right)_{z = 1/3}$$

$$\Rightarrow -\frac{1}{8}$$

$\Rightarrow$ So, $\displaystyle\int_C \frac{dz}{(3z-1)(z-3)} = 2\pi i \left[-1/8 \right] \Rightarrow -\frac{\pi i}{4}$

$\Rightarrow \displaystyle 2i \int_C \frac{dz}{(3z-1)(z-3)} = 2i \times -\frac{\pi i}{4} \Rightarrow \frac{\pi}{2}$ Ans

- ~~Residue Integral~~
- Topic: Integration Round the Semi Circle

Evaluation of $\displaystyle\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} = f(x)$

Eg $\rightarrow$ $\displaystyle\int \frac{x \sin \pi x}{x^2 + 2x + 5} \, dx$

1. $\displaystyle\int_C f(z) dz$

C is curve, upper half-semicircle $|z| = R$

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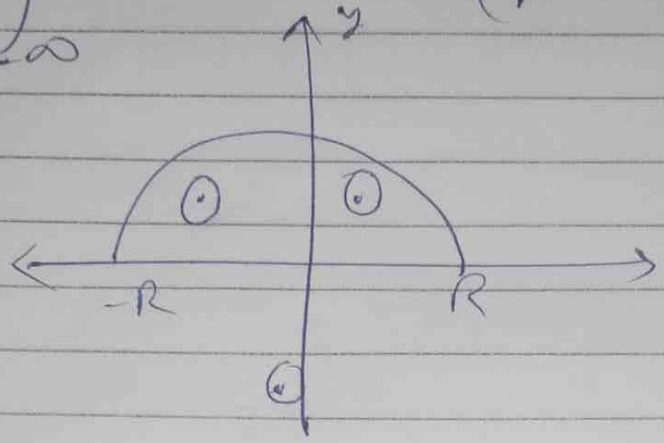
7. Cauchy Residue Theorem.

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \left(\text{Residue ka sum}\right)$$



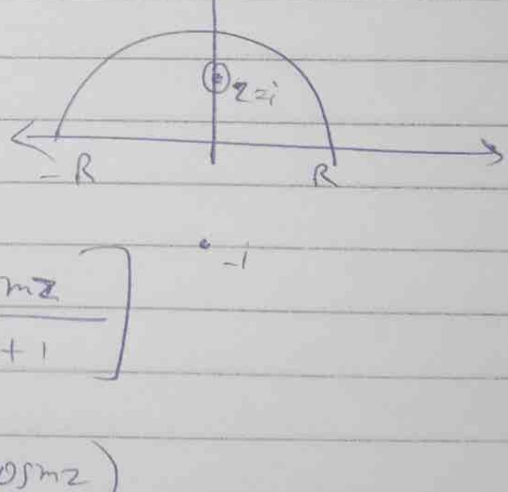
Q. Evaluate $\displaystyle\int_0^{\infty} \frac{\cos mx}{(x^2+1)}\,dx$

Sol →

$$\int_c f(z)\,dz = \int_c \frac{\cos mz}{(z^2+1)}\,dz$$

Poles: $z^2+1=0$

$z = \pm i$



Residue of $f(z)$ at $z=i$ $= \displaystyle\lim_{z\to i}\left[(z-i)\frac{\cos mz}{z^2+1}\right]$

$$\Rightarrow \lim_{z\to i}\left(\frac{(z-i)\cos mz}{(z-i)(z+i)}\right)$$

$$\Rightarrow \left[\frac{\cos mz}{(z+i)}\right]_{z=i} \Rightarrow \frac{\cos(im)}{2i}$$

$$\Rightarrow \int_c f(z)\,dz = 2\pi i\,(\text{Sum of Residue}) = 2\pi i \times \frac{\cos(im)}{2i} \Rightarrow \pi\cos(im)$$

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\,dx = \pi\cos im \;\Rightarrow\; \int_0^{\infty}\frac{\cos mx}{x^2+1}\,dx = \frac{\pi\cos im}{2} \quad \text{Ans}$$