Taylor Series Method

This is purely a numerical step by step method.

Consider the first order ordinary differential equation

$\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$,

Then by Taylor Series nth approximation is given as

$$\boxed{\begin{aligned} y_n &= y_{n-1} + h y'_{n-1} + \frac{h^2}{\lfloor 2} y''_{n-1} + \frac{h^3}{\lfloor 3} y'''_{n-1} + \cdots \\ y_1 &= y_0 + h y'_0 + \frac{h^2}{\lfloor 2} y''_0 + \frac{h^3}{\lfloor 3} y'''_0 + \cdots \end{aligned}}$$

Q.1 Solve $\frac{dy}{dx} = x - y^2$, given that $y(0) = 1$, for $y(0.1)$ correct to four places of decimal using Taylor series method.

Sol^n

Given diff. eq^n $\frac{dy}{dx} = x - y^2$, $y(0) = 1$

$f(x,y) = x - y^2$ i.e $x_0 = 0, y_0 = 1$

$\therefore y' = x - y^2$ i.e $y'_0 = x_0 - y_0^2 = 0 - (1)^2 = -1$

$y'' = 1 - 2yy'$ i.e $y''_0 = 1 - 2y(0)y'_0 = 1 - 2(1)(-1) = 1 + 2 = 3$

$y''' = 0 - 2[yy'' + y'y']$

$y'''(0) = -2[y(0)y''(0) + y'(0) \cdot y'(0)]$

$= -2[1(3) + (-1)(-1)] = -2[3+1] = -8$

By Taylor series

$$y_n = y_{n-1} + h y'_{n-1} + \frac{h^2}{\lfloor 2} y''_{n-1} + \frac{h^3}{\lfloor 3} y'''_{n-1} + \cdots$$

$$y_1 = y_0 + h y'_0 + \frac{h^2}{\lfloor 2} y''_0 + \frac{h^3}{\lfloor 3} y'''_0 + \cdots$$

$$= 1 + .1(-1) + \frac{(.1)^2}{2}(3) + \frac{(.1)^3}{6}(-8) + \cdots$$

$$= 1 - .1 + .015 - .00133$$

$y_1 = .91367$ Ans

$\because y_1 = x_0 + h$
$= 0 + .1$
$y_1 = .1$

$$\boxed{y(.1) = .9137}$$

Q.2. Apply Taylor's series method to find $y(.2)$, from $y' - 4y = 0$ given that $y(0) = 1$

Sol^n Given diff. Eq^n $y' = 4y$

$f(x, y) = 4y$ and $y(0) = 1$

$x_0 = 0, y_0 = 1$

$h = .2$

$y' = 4y \quad \therefore \quad y'(0) = 4y(0) = 4 \times 1 = 4$

$y'' = 4y' \quad \therefore \quad y''(0) = 4y'(0) = 4 \times 4 = 16$

$y''' = 4y'' \quad \therefore \quad y'''(0) = 4y''(0) = 4 \times 16 = 48.$

By Taylor's series method.

$$y_1 = y_0 + h y'_0 + \frac{h^2}{\lfloor 2} y''_0 + \frac{h^3}{\lfloor 3} y'''_0 + \cdots$$

$$= 1 + (.2)(4) + \frac{.04}{2}(16) + \frac{.008}{6} \times 48 + \cdots$$

$$= 1 + .8 + .32 + .064$$

$\because y_1 = x_0 + h$
$= 0 + .2$

y

$y_1 = 2.184$

$y(.2) = 2.184$ Ans.

Q.3 Using Taylor's series, find the solution of the differential equation $\frac{dy}{dx} = x+y$, Given $y(1)=0$, numerically upto $x = $ with $h = .1$. Compare the final result with the value of explicit solution.

Sol: Given diff. Eqn $\frac{dy}{dx} = x+y$, and $y(1)=0$

i.e $x_0 = 1, y_0 = 0$

$y' = x+y \Rightarrow y'(0) = x_0 + y_0 = 1+0 = 1$

$y'' = 1+y' \Rightarrow y''(0) = 1+y'(0) = 1+(1) = 2$

$y''' = y'' \Rightarrow y'''(0) = y''(0) = 2$

$y^{IV} = y''' \Rightarrow y^{IV}(0) = y'''(0) = 2$

By Taylor Series method

$$y_1 = y_0 + h y_0' + \frac{h^2}{\lfloor 2} y_0'' + \frac{h^3}{\lfloor 3} y_0''' + \frac{h^4}{\lfloor 4} y^{IV}_{(0)} e - - -$$

$$= 0 + (.1) + \frac{.01}{2} \times 2 + \frac{.001}{6} \times 2 + \frac{.0001}{24} \times 2$$

$$= .1 + .01 + .00033 + \cdots$$

$y_1 = .11033$ $\therefore x_1 = x_0 + h \Rightarrow x_1 = 1 + .1 = 1.1$

$$\therefore y_2 = y_1 + h y_1' + \frac{h^2}{\lfloor 2} y_1'' + \frac{h^3}{\lfloor 2} y_1''' e \cdots \longrightarrow (1)$$

$\therefore y' = x+y \Rightarrow y_1' = x_1 + y_1 = 1.1 + .1103$

$= 1.2103$

$y'' = 1+y' \Rightarrow y_1'' = 1 + y_1' = 1 + 1.2103$

$= 2.2103$

$y''' = y'' \Rightarrow y_1''' = y_1'' = 2.2103$

$$\therefore y_2 = .1103 + (.1)(1.2103) + \frac{.01 \times 2.2103}{2} \times 2.2103 + \frac{(.1)^3 \times 2.2103}{6} + \cdots$$

$$= .1103 + .12103 + .0110515 + .00036 \cdots$$

$$y_2 = .242$$

And $x_2 = x_1 + h = 1.1 + .1 = 1.2$

$$\therefore \boxed{y(1.2) = .242} \text{ Ans}$$

The Analytical solution

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} - y = x$$

$$I.F = e^{\int -dx} = e^{-x}$$

$$\therefore y \cdot e^{-x} = \int x \cdot e^{-x} dx + c$$

$$y e^{-x} = -x e^{-x} - e^{-x} + c$$

$$y = -x - 1 + c e^{x} \longrightarrow (1)$$

Putting $y(1) = 0$

$$0 = -1 - 1 + c e^{1} \implies c = 2e^{-1}$$

$$y = -x - 1 + 2e^{-1} \cdot e^{x}$$

$$= -x - 1 + 2e^{x-1}$$

Putting $x = 1.2$

$$y = -1.2 - 1 + 2e^{1.2-1}$$

$$y = -2.2 + 2e^{.2} = \underline{\underline{.242}}$$

Q4 Solve the differential equation $\frac{dy}{dx} = 1-y$, $y(0)=0$ in the range $0 \le x \le .2$ by Taking $h = .1$, Using Taylor's series method.

$\underline{\underline{Sol^n}}$

Given $\frac{dy}{dx} = 1-y$

$f(x,y) = 1-y$, And $y=0$ when $x=0$

$y' = 1-y \Rightarrow y_0' = 1-y_0 \Rightarrow y_0' = 1-0 = 1$

$y'' = 0-y' \Rightarrow y_0'' = 0-y_0' = 0-1 = -1$

$y''' = 0-y'' \Rightarrow y_0''' = -y_0'' = 1$

$y^{iv} = -y''' \Rightarrow y_0^{iv} \Rightarrow -1 = -1$

By Taylor's series First approximation

$$y_1 = y_0 + h y_0' + \frac{h^2}{\lfloor 2} y_0'' + \frac{h^3}{\lfloor 3} y_0''' + \frac{h^4}{\lfloor 4} y_0^{iv} + \cdots$$

$$= 0 + .1(1) + \frac{(.1)^2}{2}(-1) + \frac{(.1)^3}{6}(1) + \frac{(.1)^4}{24}(-1) + \cdots$$

$$= .1 - .005 + .0001666 - .00000416$$

$$y_1 = .095$$

i.e $y = .095$ when $x = .1$ $\quad (\because x_1 = x_0 + h = 0 + .1 = .1)$

Second Approximation

$$y_2 = y_1 + h y_1' + \frac{h^2}{\lfloor 2} y_1'' + \frac{h^3}{\lfloor 3} y_1''' + \frac{h^4}{\lfloor 4} y_1^{iv} + \cdots$$

$\because y' = 1-y \Rightarrow y_1' = 1-y_1 = 1-.095 = .905$

$y'' = -y' \Rightarrow y_1'' = -y_1' = 0-.905 = -.905$

$y''' = -y'' \Rightarrow y_1''' = -y_1'' = +.905 = .905$

$y^{iv} = -y''' \Rightarrow y_1^{iv} = -y_1''' \Rightarrow y_1^{iv} = -.905$

$$y_2 = .095 + (.1)(.905) + \frac{(.1)^2}{2}(-.905) + \frac{(.1)^3}{6}(.905) + \frac{(.1)^4}{24}(-.905)$$

$$= .095 + .0905 - .0004525 + .00015083 - .00000377 = .1851$$

$x_2 = x_1 + h = .1 + .1 = .2$ Ans. Ans

Using Taylor Series method, Find correct to Four decimal Places, the value of $y(.1)$, given $\frac{dy}{dx} = x^2+y^2$ and $y(0)=1$

Given $\frac{dy}{dx} = x^2+y^2$

$y' = x^2+y^2$ and $y=1$ when $x=0$

$y'' = 2x+2yy'$

$y''' = 2+2[yy''+y'y']$

$y''' = 2+2yy''+2(y')^2$

$y^{iv} = 2[yy'''+y'y'']+4y'y''$

$= 2yy'''+2y'y''+4y'y''$

$y^{iv} = 2yy'''+6y'y''$

Here $y'_0 = x_0^2+y_0^2 = 0+(1)^2 = 1$

$y''_0 = 2x_0+2y(0)\cdot y'(0)$

$= 0+2(1)\cdot(1) = 2$

$y'''_0 = 2+2y(0)\cdot y''_0+2[y'(0)]^2$

$= 2+2(1)(2)+2[1]^2$

$= 2+4+2 = 8$

$y^{iv}_0 = 2y_0 y'''_0+6y'_0\cdot y''_0$

$= 2(1)(8)+6(1)(2)$

$= 16+12 = 28$

By Taylor series method.

$y_1 = y_0 + h y'_0 + \frac{h^2}{\underline{|2}} y''_0 + \frac{h^3}{\underline{|3}} y'''_0 + \frac{h^4}{\underline{|4}} y^{iv}_0$

$= 1+(.1)(1)+\frac{(.1)^2}{2}(2)+\frac{(.1)^3}{6}(8)+\frac{(.1)^4}{24}(28)$

$= 1+.1+.01+.001333+.00011666$

$y_1 = 1.11145$

i.e $y = 1.11145$ when $x = .1$

$y(.1) = 1.11145$ Ans

Q. Using Taylor method, compute $y(.2)$ and $y(.4)$ correct to 4 decimal places given $\frac{dy}{dx} = 1-2xy$ and $y(0)=0$

Sol: Given $\frac{dy}{dx} = 1-2xy$

$$y' = 1-2xy$$
$$y'' = -2[xy'+y]$$
$$y'' = -2xy'-2y$$
$$y''' = -2[xy''+y'] - 2[y']$$
$$= -2xy''-2y'-2y'$$
$$y''' = -2xy''-4y'$$
$$y^{iv} = -2[xy'''+y''] - 4y''$$
$$= -2xy'''-2y''-4y''$$
$$y^{iv} = -2xy'''-6y''$$

$\because$ $y(0)=0$

$\therefore$ $y'_0 = 1-0 = 1$

$$y''_0 = -2x_0y'_0 - 2y_0 = -0-2(0) = 0$$
$$y'''_0 = -2x_0y''_0 - 4y'_0$$
$$= -2(0)0 - 4(1) = -4$$
$$y^{iv}_0 = -2x_0y'''_0 - 6y''_0 = 0$$

By Taylor Series

$$y_1 = y_0 + hy'_0 + \frac{h^2}{\underline{|2}}y''_0 + \frac{h^3}{\underline{|3}}y'''_0 + \frac{h^4}{\underline{|4}}y^{iv}(0) + \cdots$$
$$= 0 + (.2)(1) + 0 + \frac{(.2)^3}{6}(-4) + 0$$
$$y_1 = .2 - .005333 + 0 = .19466 = .1947$$

i.e $y = .1947$ when $x = .2$

Again For finding $y(.4)$

We again starting with $x_0 = .2$, $y_0 = .1947$, $h = .2$

$y_0' = 1 - 2x_0y_0 = 1 - 2(.2)(.1947) = .92212$

$y_0'' = -2x_0y_0' - 2y_0$

$= -2(.2)(.92212) - 2(.1947)$

$y_0'' = -.7583$

$y_0''' = -2x_0y_0'' - 4y_0'$

$= -2(.2)(-.7583) - 4.(.92212)$

$y_0''' = -3.3851$

$y_0^{iv} = 5.9041$

Again By Taylor series method.

$$y = y_0 + hy_0' + \frac{h^2}{\underline{|2}}y_0'' + \frac{h^3}{\underline{|3}}y_0''' + \frac{h^4}{\underline{|4}}y_0^{iv}$$

$$= .1947 + \frac{(.2)}{2}(-.7583) + \frac{(.2)^3}{6}(-3.3851) + \frac{(.2)^4}{24}(5.9041)$$

$y. = .35988$

i.e $y = .35988$ at $x = .4$ Ans.

Picard's Method of Successive Approximation

Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

$$dy = f(x, y)dx \quad , \text{with } y(x_0) = y_0 \longrightarrow (1)$$

Integrating it on both sides between the corresponding limits for x and y, we get.

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx$$

$$[y]_{y_0}^{y} = \int_{x_0}^{x} f(x, y)\,dx$$

$$y - y_0 = \int_{x_0}^{x} f(x, y)\,dx$$

$$\boxed{y = y_0 + \int_{x_0}^{x} f(x, y)\,dx} \longrightarrow (2)$$

Equations of the type (2) are called Integral equations and will be solved by successive approximations. We get first approximation for y, as $y^{(1)}$, by substituting y_0 for y in the integrand $f(x, y)$

Thus $$\boxed{y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx} \longrightarrow (3)$$

The integral ~~of~~ on right hand side of (3) can be evaluated and the resulting $y^{(1)}$ is substituted for y_0 in the integrand of (2) to obtai the second approximation $y^{(2)}$, given by

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

The process is successively continued till desired accur
is achieved. The nth approximation $y^{(n)}$ is given by

$$\boxed{y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx}$$

Q.5 Use Picards method to obtain y for $x = \cdot 2$, Given $\frac{dy}{dx} = x - y$, with initial condition $y = 1$ when $x = 0$

Sol^n Here $\frac{dy}{dx} = x - y$

$$f(x, y) = x - y \quad , \quad x_0 = 0, \ y_0 = 1$$

First approximation

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_0^x (x - 1)\,dx$$

$$= 1 + \left[\frac{x^2}{2} - x\right]_0^x$$

$$y^{(1)} = 1 + \frac{x^2}{2} - x$$

$$y^{(1)}_{at\ x = \cdot 2} = 1 + \frac{(\cdot 2)^2}{2} - (\cdot 2) = 1 + \frac{\cdot 04}{2} - \cdot 2 = \cdot 82$$

Second Approximation

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

$$= 1 + \int_0^x \left[x - \left(1 + \frac{x^2}{2} - x\right)\right]dx$$

$$= 1 + \int_0^x x - 1 - \frac{x^2}{2} + x\,]\,dx$$

$$= 1 + \int_0^x 2x - 1 - \frac{x^2}{2}\,]\,dx$$

$$= 1 + \left[x^2 - x - \frac{x^3}{6}\right]_0^x$$

$$y^{(2)} = 1 + x^2 - x - \frac{x^3}{6}$$

$$y^{(2)}_{at\ x = \cdot 2} = 1 + (\cdot 2)^2 - (\cdot 2) - \frac{(\cdot 2)^3}{6} = 1 + \cdot 04 - \cdot 2 - \cdot 00133$$

$$= \cdot 83867$$

Third Approximation

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\,dx$$

$$= 1 + \int_0^x x - 1 + x - x^2 + \frac{x^3}{6}]\,dx$$

$$= 1 + \int_0^x (2x - x^2 + \frac{x^3}{6} - 1)\,dx$$

$$y^{(3)} = 1 + x^2 - \frac{x^3}{3} + \frac{x^4}{24} - x$$

$$y^{(3)}_{at\ x=.2} = 1 - .2 + (.2)^2 - \frac{(.2)^3}{3} + \frac{(.2)^4}{24}$$

$$= 1 - .2 + .04 - .002666 + .000006$$

$$y^{(3)}_{at\ x=.2} = .8374$$

Fourth Approximation

$$y^{(4)} = y_0 + \int_{x_0}^{x} f(x, y^{(3)})\,dx$$

$$= 1 + \int_0^x x - [1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}]\,dx$$

$$= 1 + \int_0^x 2x - x^2 + \frac{x^3}{3} - \frac{x^4}{24} - 1]\,dx$$

$$= 1 + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120} - x \Big]_0^x$$

$$y^{(4)} = 1 + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120} - x$$

$$y^{(4)}_{at\ x=.2} = 1 + (.2)^2 - \frac{(.2)^3}{3} + \frac{(.2)^4}{12} - \frac{(.2)^5}{120} - .2$$

$$= 1 + .04 - .002666 + .0001333 - .00000266 - .2$$

$$= .83746$$

$$y^{(4)} = y^{(3)} = .8374 \text{ at } x = .2$$

<u>Ans</u>

Using Picard's process of successive approximation ~~about~~ obtain a solution upto the fifth approximation of equation $\frac{dy}{dx} = y + x$, such that $y = 1$ when $x = 0$. check your answer by finding the exact particular solution.

Given $\quad \frac{dy}{dx} = y + x \quad$ and $\quad y(0) = 1 \quad$ i.e $y_0 = 1$ at $x = 0$

$$f(x, y) = y + x$$

By Picard method

First approximation

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y^{(1)} = 1 + \int_0^x (1 + x)\, dx = 1 + x + \frac{x^2}{2}$$

Second approximation

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx$$

$$= 1 + \int_0^x \left(1 + x + \frac{x^2}{2} + x\right) dx$$

$$= 1 + \int_0^x \left(1 + 2x + \frac{x^2}{2}\right) dx$$

$$y^{(2)} = 1 + \left[x + x^2 + \frac{x^3}{6}\right]_0^x = 1 + x + x^2 + \frac{x^3}{6}$$

Third Approximation

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\, dx$$

$$= 1 + \int_0^x 1 + x + x^2 + \frac{x^3}{6} + x\,]\, dx$$

$$= 1 + \int_0^x 1 + 2x + x^2 + \frac{x^3}{6}\,]\, dx$$

$$y^{(3)} = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Fourth approximation

$$y^{(4)} = y_0 + \int_{x_0}^{x} f(x, y^{(3)})\,dx$$

$$= 1 + \int_0^x x + y^{(3)}]\,dx$$

$$= 1 + \int_0^x x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}]\,dx$$

$$= 1 + \int_0^x 1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}]\,dx$$

$$y^{(4)} = 1 + \left[x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120} \right]$$

Fifth approximation

$$y^{(5)} = y_0 + \int_{x_0}^{x} f(x, y^{(4)})\,dx$$

$$= 1 + \int_0^x x + y^{(4)}]\,dx$$

$$= 1 + \int_0^x x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}]\,dx$$

$$= 1 + \int_0^x 1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}]\,dx$$

$$y^{(5)} = 1 + [x + x^2 + x^3 + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}]$$ Ans

For Exact solution

Given $\frac{dy}{dx} = y + x$

$$\frac{dy}{dx} - y = x$$

$$\therefore I.F = e^{\int -1\,dx} = e^{-x}$$

$$\therefore y \cdot I.F = \int Q \cdot I.F\,dx + c$$

$$y \cdot e^{-x} = \int x \cdot e^{-x}\,dx + c$$

$$= x(-e^{-x}) - (1)(+e^{-x}) + 0 + c$$

$$y e^{-x} = -x e^{-x} - e^{-x} + c$$

$$y = -x - 1 + ce^{x}$$

$y = ce^{x} - x - 1$

Given $y=1$ when $x=0$

$1 = ce^{0} - 0 - 1$

$\cancel{1} = c - \cancel{1} \Rightarrow \boxed{c=2}$

$$\boxed{y = 2e^{x} - x - 1}$$

$$= 2\left[1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \cdots \infty\right] - x - 1$$

$$= 1 + x + \frac{2x^2}{\lfloor 2} + \frac{2x^3}{\lfloor 3} + \frac{2x^4}{\lfloor 4} + \frac{2x^5}{\lfloor 5}$$

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \cdots$$

Hence $y^{(5)}$ and y are same

Ans.

Q.3. Given differential equation $\frac{dy}{dx} = \frac{x^2}{1+y^2}$ with the initial condition $y=0$ when $x=0$. Use Picards method to obtain y for $x = .25$, $.5$ correct to three decimal places.

Soln Given $\frac{dy}{dx} = \frac{x^2}{1+y^2}$

$$f(x,y) = \frac{x^2}{1+y^2}, \quad x_0 = 0, \; y_0 = 0$$

First approximation $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$

$$= 0 + \int_0^x \frac{x^2}{1+y_0^2}\,dx$$

$$= \int_0^x \frac{x^2}{1+0}\,dx = \left(\frac{x^3}{3}\right)_0^x$$

$$y^{(1)} = \frac{x^3}{3} \longrightarrow (1)$$

$$y^{(1)}_{\text{at } x=.25} = \frac{(.25)^3}{3} = .0052$$

$$y^{(1)}_{\text{at } x=.5} = \frac{(.5)^3}{3} = .04166$$

For Second Approximation

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

$$= 0 + \int_0^x \frac{x^2}{1+\{y^{(1)}\}^2}\, dx$$

$$= \int_0^x \frac{x^2}{1+\left(\frac{x^3}{3}\right)^2}\, dx = \int_0^x \frac{x^2}{1+\frac{x^6}{9}}\, dx$$

$$= \int_0^x x^2 \left(1+\frac{x^6}{9}\right)^{-1} dx$$

$$= \int_0^x x^2 \left(1-\frac{x^6}{9} \cdots\right) dx$$

$$= \int_0^x x^2 - \frac{x^8}{9} \cdots \Big] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^9}{81} \cdots\right]_0^x$$

$$y^{(2)}_{\text{at } x=.25} = \frac{(.25)^3}{3} - \frac{(.25)^9}{81} = .0052 - .00000047$$

$$= .0052$$

$$y^{(2)}_{\text{at } x=.5} = \frac{(.5)^3}{3} - \frac{(.5)^9}{81} = .04166 - .0000241$$

$$= .0416$$

Hence

y at $x = .25 = .005$

y at $x = .5 = .041$

correct to three decimal places.

Find $y(0.1)$ by solving the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0)=1$ Using Picard's method with one iteration.

Given $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$

$$\therefore f(x,y) = \frac{y-x}{y+x}$$

We know that by Picard's method

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx \longrightarrow (1)$$

$$= 1 + \int_0^x \frac{y^{(0)} - x}{y^{(0)} + x}\, dx$$

$$= 1 + \int_0^x \frac{1-x}{1+x}\, dx$$

$$= 1 + \int_0^x \frac{1-x-1+1}{1+x}\, dx$$

$$= 1 + \int_0^x \frac{2}{1+x} - 1\,]dx$$

$$= 1 + \left[2\,Log(1+x) - x \right]_0^x$$

$$y^{(1)} = 1 - x + 2\,Log(1+x) \longrightarrow (2)$$

$$y(.1) = 1 - .1 + 2\,Log(1+.1)$$

$$= .9 + 2\,Log(1.1)$$

$$= .9 + .19062$$

$$\boxed{y(.1) = 1.09062}$$ Ans

Q. Use Picard's method with 3 iterations to solve for $y(.1)$ and $y(.2)$, the equation $\frac{dy}{dx} = x^2+y^2$, $y(0)=0$

Sol^n Given $\frac{dy}{dx} = x^2+y^2$

$f(x,y) = x^2+y^2$, $y(0)=0$

As we know that by Picard's method

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\,dx \longrightarrow ①$$

$\because f(x,y) = x^2+y^2$

$f(x,y^0) = x^2+y_0^2 =$

$$= 0 + \int_0^x x^2\,dx$$

$$y^{(1)} = \frac{x^3}{3} \longrightarrow (2)$$

$f(x,y^{(1)}) = x^2+y_1^2$

$= x^2+\left(\frac{x^3}{3}\right)^2$

$= x^2+\frac{x^6}{9}$

Then $y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\,dx$

$$y^{(2)} = 0 + \int_0^x \left(x^2+\frac{x^6}{9}\right)dx$$

$$y^{(2)} = \frac{x^3}{3}+\frac{x^7}{63} \longrightarrow (3)$$

$$y^{(3)} = y_0 + \int_0^x f(x, y^{(2)})\,dx$$

$f(x,y^{(2)}) = x^2+y_2^2$

$= x^2+\left(\frac{x^3}{3}+\frac{x^7}{63}\right)^2$

$$= 0 + \int_0^x x^2+\left(\frac{x^3}{3}+\frac{x^7}{63}\right)^2\Big]dx$$

$$= \int_0^x x^2+\frac{x^6}{9}+\frac{x^{14}}{3969}+\frac{2x^{10}}{189}\Big]dx$$

$$y^{(3)} = \frac{x^3}{3}+\frac{x^7}{63}+\frac{x^{15}}{2079}+\frac{x^{15}}{59535}+\cdots \quad ③$$

$\therefore y(.1) = .00033$

$y(.2) = .00267$ Ans

Euler's Method

Let $x_0, x_1, x_2, \ldots$ be equi-spaced values of x at a spacing h and let $y_0, y_1, y_2, \ldots$ be the corresponding values of y on a curve satisfying the differential equation

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \longrightarrow (1)$$

Suppose the solution of (1) is $y = \phi(x)$

For the first step we find the value of y at $x = x_0 + h$ and call it y_1, then by (1)

$$y_1 = \phi(x_1) = \phi_1(x_0 + h)$$

$$= \phi(x_0) + h\,\phi'(x_0) + \frac{h^2}{\lfloor 2}\phi''(x_0) + \cdots$$

$$= y_0 + h f(x_0, y_0) + \frac{h^2}{\lfloor 2} f'(x_0, y_0) + \cdots$$

Hence, if h is small, neglecting terms involving h^2 and higher powers of h

$$\therefore \boxed{y_1 = y_0 + h f(x_0, y_0)}$$

And $y_2 = y_1 + h f(x_1, y_1)$

Continuing the process, we have in general

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

Note Let the differential equation of first order and first degree be, with initial condition $y(x_0) = y_0$,

$$\frac{dy}{dx} = f(x,y),$$

Then Euler Formula is given as

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

Q.1 Apply Euler's method to solve

$$\frac{dy}{dx} = x+y, \quad y(0) = 0$$

Choosing $h = .2$, and compute $y(.4)$ and $y(.6)$

Sol^n Given equation $\frac{dy}{dx} = x+y$

$\therefore f(x,y) = x+y$

And $x_0 = 0,\ y_0 = 0,\ h = .2$

Approximate value of y at $x = .2$ is given by

$y_1 = y_0 + h f(x_0, y_0)$

$y_1 = 0 + .2[x_0 + y_0] = .2[0+0] = 0$ $\quad \boxed{y(.2) = 0}$

Approximate value of y at $x = .4$

$y_2 = y_1 + h f(x_1, y_1)$

$= 0 + .2[x_1 + y_1] = .2[.2+0] = .04$ $\quad \boxed{y(.4) = .04}$

Approximate value of y at $x = .6$

$y_3 = y_2 + h f(x_2, y_2)$

$= .04 + .2[x_2 + y_2] =$

$= .04 + .2[.4 + .04]$

$= .04 + .2(.44) = .04 + .088 = .128$

$\boxed{y(.6) = .128}$

x	0	.2	.4	.6
y	0	0	.04	.128

Ans

Q.2 Given $\frac{dy}{dx} = \frac{y-x}{y+x}$, with the initial condition $y=1$ at $x=0$ find y for $x = \cdot1$ and $x = \cdot2$, Using Euler's method.

Sol^n Given $\frac{dy}{dx} = \frac{y-x}{y+x}$

$f(x,y) = \frac{y-x}{y+x}$ Given $x_0 = 0, y_0 = 1$

taking $h = \cdot1$

First approximation value of y at $x = \cdot1$ is given as

$y_1 = y_0 + h f(x_0, y_0)$

$= 1 + \cdot1 \left[\frac{y_0 - x_0}{y_0 + x_0}\right] = 1 + \cdot1 \left[\frac{1-0}{1+0}\right] = 1 + \cdot1 = 1\cdot1$

$\boxed{y(\cdot1) = 1\cdot1}$

Second approximation value of y at $x = \cdot2$ is given as

$y_2 = y_1 + h f(x_1, y_1)$

$= 1\cdot1 + \cdot1 \left[\frac{y_1 - x_1}{y_1 + x_1}\right] = 1\cdot1 + \cdot1 \left[\frac{1\cdot1 - \cdot1}{1\cdot1 + \cdot1}\right]$

$= 1\cdot1 + \cdot1 \left[\frac{1}{1\cdot2}\right]$

$= 1\cdot1 + \cdot1 [\cdot833]$

$= 1\cdot1 + \cdot0833 = 1\cdot1833$

Ans $\boxed{y(\cdot2) = 1\cdot1833}$

Q.3 Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 1$, compute $y(\cdot02)$ by Euler method, taking $h = \cdot01$

Sol^n Given $\frac{dy}{dx} = x^3 + y$

$\therefore f(x,y) = x^3 + y$ And $y_0 = 1, x_0 = 0, h = \cdot01$

First approximation of y at $x = \cdot01$

$y_1 = y_0 + h f(x_0, y_0) = 1 + \cdot01 [x_0^3 + y_0] = 1 + \cdot01 [0 + 1]$

$= 1 + \cdot01 = 1\cdot01$

$\boxed{y(\cdot01) = 1\cdot01}$

Second approximate Value of y at $x = .01 + .01 = .02$ is

$y_2 = y_1 + h\, f(x_1, y_1)$

$= 1.01 + .01 \left[x_1^3 + y_1 \right]$

$y_2 = 1.01 + .01 \left[(.01)^3 + 1.01 \right] = 1.0201$ $\boxed{y(.02) = 1.0201}$

x	0	.01	.02
y	1	1.01	1.0201

Q.4. Solve by Euler's method $\frac{dy}{dx} = \frac{x-y}{2}$, $y(0) = 1$ over $[0, 3]$ using step size $1/2$

Sol^n Given $f(x,y) = \frac{x-y}{2}$, $x_0 = 0$, $y_0 = 1$, $h = .5$

First approximate value of y when $x = .5$

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 1 + .5 \left[\frac{x_0 - y_0}{2} \right] = 1 + .5 \left[\frac{0-1}{2} \right] = .75$

$\boxed{y(.5) = .75}$

Second approximate value of y when $x = .5 + .5 = 1$ is

$y_2 = y_1 + h\, f(x_1, y_1)$

$= .75 + .5 \left[\frac{x_1 - y_1}{2} \right] = .75 + \frac{.5}{2}(.5 - .75) = .6875$

Third approximate value of y at $x = 1 + .5 = 1.5$ is $y(1) = .6875$

$y_3 = y_2 + h\, f(x_2, y_2) = .6875 + .5 \left[\frac{x_2 - y_2}{2} \right]$

$= .6875 + \frac{.5}{2} \left[1 - .6875 \right] = .7656$

$\boxed{y(1.5) = .7656}$

Fourth approximate value of y at $x = 1.5 + .5 = 2$ is

$$y_4 = y_3 + h f(x_3, y_3)$$

$$= .7656 + \frac{.5}{2}[1.5 - .7656] = .9492$$

$$\boxed{y(2) = .9492}$$

Fifth approximate value of y at $x = 2 + .5 = 2.5$ is

$$y_5 = y_4 + h f(x_4, y_4)$$

$$= .9492 + \frac{.5}{2}[x_4 - y_4]$$

$$= .9492 + \frac{.5}{2}[2 - .9492] = 1.2119$$

$$\boxed{y(2.5) = 1.2119}$$

Sixth approximate value of y at $x = 2.5 + .5 = 3$ is

$$y_6 = y_5 + h f(x_5, y_5)$$

$$= 1.2119 + \frac{.5}{2}[2.5 - 1.2119] = 1.533925$$

$$\boxed{y(3) = 1.5339}$$

x	0	.5	1	1.5	2	2.5	3
y	1	.75	.6875	.7656	.9492	1.2119	1.5339

Euler's Modified Method

Modified Euler's Method

A generalised form of Euler's modified formula is

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]; \quad n = 0, 1, 2, \ldots$$

Where $y_1^{(n)}$ is the nth approximation to y_1

Note: The above iteration formula can be started by choosing $y_1^{(1)}$ from Euler's formula $y_1^{(1)} = y_0 + h f(x_0, y_0)$

Since this formula attempts to correct the values of y_{n+1} using the predicted value of y_{n+1} (by Euler method), it is classified as one-step predictor-corrector method

Q1 Solve $\frac{dy}{dx} = 1-y$, $y(0)=0$ in the range $0 \le x \le$. by taking $h = .1$ using modified Euler method.

Sol^n:

$$\frac{dy}{dx} = 1-y$$

$$f(x,y) = 1-y, \quad x_0 = 0, \; y_0 = 0, \; h = .1$$

First we find out y at $x = .1$ by Euler method

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$y_1 = 0 + .1\,[1-y_0] = .1\,[1-0] = .1$$

Now we correct this value by Modified Euler method

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0,y_0) + f(x_1,y_1)] \qquad \because f(x,y) = 1-y$$

$$= 0 + \frac{.1}{2}\,[1-y_0 + 1-y_1]$$

$$y_1^{(1)} = \frac{.1}{2}\,[1-0+1-.1] = .095$$

It's second Approximation

$$y_1^{(2)} = y_0 + \frac{h}{2}\,[f(x_0,y_0) + f(x_1, y_1^{(1)})]$$

$$= 0 + \frac{.1}{2}\,[1-y_0 + 1-y_1^{(1)}]$$

$$= \frac{.1}{2}\,[1-0+1-.095] = .0952$$

It's third Approximation

$$y_1^{(3)} = y_0 + \frac{h}{2}\,[f(x_0,y_0) + f(x_1, y_1^{(2)})]$$

$$= 0 + \frac{.1}{2}\,[1-y_0 + 1-y_1^{(2)}]$$

$$= \frac{.1}{2}\,[1-0+1-.0952] = .0952$$

i.e $y_1^{(2)} = y_1^{(3)} = .0952$

nce $y_1 = .0952$ i.e $\boxed{y(.1) = .0952}$

ge $0 \le x \le .2$
method.

Now we find out y at $x = .2$ by Euler method

$y_2 = y_1 + h f(x_1, y_1)$

$= .0952 + .1 [1 - y_1]$

$y_2 = .0952 + .1 [1 - .0952] = .1857$

Now we correct this value by Modified Euler method.

$y_2^{(1)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2)]$ $\because f(x,y) = 1-y$

$= .0952 + \frac{.1}{2} [1 - y_1 + 1 - y_2]$

$= .0952 + \frac{.1}{2} [1 - .0952 + 1 - .1857] = .1812$

It's second Approximation

$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= .0952 + \frac{.1}{2} [1 - y_1 + 1 - y_2^{(1)}]$

$= .0952 + \frac{.1}{2} [1 - .0952 + 1 - .1812] = .1814$

$y_2^{(3)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= .0952 + \frac{.1}{2} [1 - y_1 + 1 - y_2^{(2)}]$

$= .0952 + \frac{.1}{2} [1 - .0952 + 1 - .1814] = .1814$

Hence $y_2^{(2)} = y_2^{(3)} = .1814$

$y_2 = .1814$ $\boxed{y(.2) = .1814}$

x :	0	.1	.2
y :	0	.0952	.1814

Ans

Q.2 Solve $\frac{dy}{dx} = x^2+y$, $y(0)=1$ to find y at $x=.1$ by taking $h=.05$

Euler modified method.

Sol^n

$$\frac{dy}{dx} = x^2+y$$

$$f(x,y) = x^2+y \quad , \quad x_0=0, \; y_0=1, \; h=.05$$

First we find out y at $x=.05$ by Euler method

$$\because f(x,y) = x^2+y$$

$$y_1 = y_0 + h \, f(x_0, y_0)$$

$$= 1 + .05\,[x_0^2 + y_0]$$

$$y_1 = 1+.05\,[0+1] = 1.05$$

Now we correct this value by Modified Euler method

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0,y_0) + f(x_1,y_1)]$$

$$f(x,y) = x^2+y$$

$$f(x_1,y_1) = x_1^2+y_1$$

$$= 1 + \frac{.05}{2}\,[x_0^2+y_0 + x_1^2 + \overset{y_1}{\cancel{1.05}}]$$

$$= 1 + \frac{.05}{2}\,[0+1+(.05)^2+1.05]$$

$$= 1.0513$$

It's second Approximation is

$$y_1^{(2)} = y_0 + \frac{h}{2}\,[f(x_0,y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{.05}{2}\,[x_0^2+y_0 + x_1^2 + 1.0513]$$

$$= 1 + \frac{.05}{2}\,[0+1+(.05)^2+1.0513]$$

$$y_1^{(2)} = 1.0513$$

Hence $y_1^{(1)} = y_1^{(2)} = 1.0513$

So $y = 1.0513$ at $x = .05$

Now we find out y at $x = .1$ by Euler method

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 1.0513 + .05 [x_1^2 + y_1]$$

$$= 1.0513 + .05 [(.05)^2 + 1.0513]$$

$$y_2 = 1.1039$$

Now correct this value by modified Euler method.

$$y_2^{(1)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2)]$$

$$= 1.0513 + \frac{.05}{2} [x_1^2 + y_1 + x_2^2 + y_2]$$

$$= 1.0513 + \frac{.05}{2} [(.05)^2 + 1.0513 + (.1)^2 + 1.1039]$$

$$y_2^{(1)} = 1.1054$$

It's second Approximation is

$$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1.0513 + \frac{.05}{2} [x_1^2 + y_1 + x_2^2 + y_2^{(1)}]$$

$$= 1.0513 + \frac{.05}{2} [(.05)^2 + 1.0513 + (.1)^2 + 1.1054]$$

$$= 1.0513 + \frac{.05}{2} [$$

$$y_2^{(2)} = 1.1055$$

Third Approximation is

$$y_2^{(3)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= y_1 + \frac{h}{2} [x_1^2 + y_1 + x_2^2 + y_2^{(2)}]$$

$$= 1.0513 + \frac{.05}{2} [x_1^2 + y_1 + x_2^2 + 1.1055] = 1.1055$$

$$y_2^{(3)} = 1.0513 + \frac{.05}{2} [(.05)^2 + 1.0513 + (.1)^2 + 1.1055]$$

i.e. $y_2^{(2)} = y_2^{(3)} = 1.1055$. Hence $y = 1.1055$ at $x = .1$ Ans

x :	0	.05	.1
y :	1	1.0513	1.1055

Ans

Q.3. Given $\frac{dy}{dx} = x - y^2$: $y(.2) = .02$, Find $y(.4)$ by Euler's method correct to 3 decimal places, taking

Given $\frac{dy}{dx} = x - y^2$

$f(x,y) = x - y^2$, $x_0 = .2$, $y_0 = .02$, $h = .2$

We find out y at $x = .4$ by Euler Method

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$= .02 + .2\,[x_0 - y_0^2] =$$

$$y_1 = .02 + .2\,[.2 - (.02)^2] = .060$$

Now correct this value by modified Euler method.

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1)]$$

$$= .02 + \frac{.2}{2}\,[x_0 - y_0^2 + x_1 - y_1^2]$$

$$y_1^{(1)} = .02 + \frac{.2}{2}\,[.2 - (.02)^2 + .4 - (.060)^2] = .080$$

It's second Approximation is

$$y_1^{(2)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= .02 + \frac{.2}{2}\,[.2 - (.02)^2 + .4 - (.080)^2]$$

$$y_1^{(2)} = .079$$

It's third Approximation is

$$y_1^{(3)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= .02 + \frac{.2}{2}\,[.2 - (.02)^2 + .4 - (.079)^2] = .079$$

Hence $y_1^{(2)} = y_1^{(3)} = .079$

$\therefore y = .079$ when $x = .4$

Ans.

x :	.2	.4
y :	.02	.079

Ans

Forth Order Runge-Kutta Method

Forth order Runge-kutta method is also termed as Runge-kutta Method.

Let $\frac{dy}{dx} = f(x,y)$, with the initial condition $y(x_0) = y_0$.

Let h be the interval between equidistant values of x then the first increment in y is computed from the furmula as

$$x_1 = x_0 + h \quad \text{And} \quad y_1 = y_0 + \Delta y$$

Here Δy find as.

$$\Delta y = \frac{1}{6}\left[K_1 + 2(K_2 + K_3) + K_4\right]$$

Where K_1, K_2, K_3 and K_4 given as

$$K_1 = h\,f(x_0, y_0)$$

$$K_2 = h\,f\left[x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right]$$

$$K_3 = h\,f\left[x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right]$$

$$K_4 = h\,f\left[x_0 + h,\ y_0 + K_3\right]$$

Q.1 Solve the equation $\frac{dy}{dx} = x+y$, with initial condition $y(0) = 1$ by Runge kutta method when $x = .1$, Given $h=$

$$\frac{dy}{dx} = x+y$$

$f(x,y) = x+y$, $x_0 = 0$, $y_0 = 1$, $h = .1$

Here $x_1 = x_0 + h = 0 + .1 = .1$

And $y_1 = y_0 + \Delta y \longrightarrow (1)$

Now $\Delta y = \frac{1}{6}[K_1 + 2(K_2+K_3) + K_4]$

Here $K_1 = h\, f(x_0, y_0)$

$K_2 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$K_3 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$

$K_4 = h\, f(x_0 + h, y_0 + K_3)$

$\therefore K_1 = .1[x_0 + y_0] = .1[0+1] = .1$ $\quad \because f(x,y) = x+y$

$K_2 = .1[x_0 + \frac{h}{2} + y_0 + \frac{K_1}{2}]$

$K_2 = .1[0 + \frac{.1}{2} + 1 + \frac{.1}{2}] = .11$

$K_3 = .1[x_0 + \frac{h}{2} + y_0 + \frac{K_2}{2}]$

$K_3 = .1[0 + \frac{.1}{2} + 1 + \frac{.11}{2}] = .1105$

$K_4 = .1[0 + .1 + 1 + .1105] = .12105$

$\therefore \Delta y = \frac{1}{6}[K_1 + 2(K_2+K_3) + K_4] = \frac{1}{6}[.1 + 2(.11 + .1105) + .12105$

$\Delta y = .11034$

$\therefore y_1 = y_0 + \Delta y = 1 + .11034 = 1.11034$

$\therefore$ Hence $y = 1.11034$ when $x = .1$ Ans

Q2. Apply Runge-Kutta method to find y for $x = \cdot 2$ in steps of $\cdot 1$ if $\frac{dy}{dx} = x + y^2$, given that $y = 1$, when $x = 0$.

Sol^n $\frac{dy}{dx} = x + y^2$

$f(x,y) = x + y^2$, $x_0 = 0$, $y_0 = 1$

Here $x_1 = x_0 + h = 0 + \cdot 1 = \cdot 1$

$y_1 = y + \Delta y$

$\Delta y = \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$

$k_1 = \cancel{h[f(x_0 + \frac{h}{2}, y_0 +}$ $h\, f(x_0, y_0)$

$= \cdot 1 [x_0 + y_0^2] = \cdot 1 [0 + 1^2] = \cdot 1$

$k_2 = h [f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})]$

$= \cdot 1 [x_0 + \frac{h}{2} + (y_0 + \frac{k_1}{2})^2]$

$= \cdot 1 [0 + \frac{\cdot 1}{2} + (1 + \frac{\cdot 1}{2})^2] = \cdot 1 [\frac{\cdot 1}{2} + 1.1025] = \cdot 1152$

$k_3 = h [f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})]$

$= \cdot 1 [x_0 + \frac{h}{2} + (y_0 + \frac{k_2}{2})^2]$

$= \cdot 1 [0 + \frac{\cdot 1}{2} + (1 + \frac{\cdot 1152}{2})^2] = \cdot 11685 \simeq \cdot 1168$

$k_4 = h [f(x_0 + h, y_0 + k_3)]$

$= \cdot 1 [x_0 + h + (y_0 + k_3)^2]$

$k_4 = \cdot 1 [0 + \cdot 1 + (1 + \cdot 1168)^2] = \cdot 1347$

$\therefore \Delta y = \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4] = \frac{1}{6}[\cdot 1 + 2(\cdot 1152 + \cdot 1168) + \cdot 1347]$

$\Delta y = \cdot 1165$

$\therefore y_1 = y_0 + \Delta y = 1 + \cdot 1165 = 1.1165$

i.e. $y = 1.1165$ when $x = \cdot 1$

In second step

$x_2 = x_1 + h = .1 + .1 = .2$ Here we use $x_0 = .1$, $y_0 = 1.1165$

$y_2 = y_1 + \Delta y$

$K_1 = h f[x_0, y_0]$

$K_1 = h f(x_0, y_0) = .1[x_0 + y_0^2]$

$= .1[.1 + (1.1165)^2] = .1347$

$K_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$= .1[x_0 + \frac{h}{2}, (y_0 + \frac{K_1}{2})^2]$

$= .1[.1 + \frac{.1}{2} + (1.1165 + \frac{.1347}{2})^2] = .1551$

$K_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$

$= .1[x_0 + \frac{h}{2}, (y_0 + \frac{K_2}{2})^2]$

$K_3 = .1[.1 + \frac{.1}{2} + (1.1165 + \frac{.1551}{2})^2] = .1576$

$K_4 = h f(x_0 + h, y_0 + K_3)$

$= .1[x_0 + .1 + (1.1165 + .1576)^2]$

$K_4 = .1[.1 + .1 + (1.1165 + .1576)^2] = .1823$

$\therefore \Delta y = \frac{1}{6}[K_1 + 2(K_2 + K_3) + K_4]$

$= \frac{1}{6}[.1347 + 2(.1551 + .1576) + .1823]$

$\Delta y = .1571$

$\therefore y_1 = y_0 + \Delta y$

$y = 1.1165 + .1571 = 1.2736$

$\therefore y = 1.2736$ at $x = .2$

x : 0	.1	.2
y : 1	1.1165	1.2736

Ans

Milne's Predictor - Corrector Method

Milne's Predictor formula is

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

This formula is used to Predict the value of y_4 when y_0, y_1, y_2, y_3 are known.

After calculation of y_4, we correct this value by corrector formula

$$y_4^{(1)(C)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

For second Approximation

$$y_4^{(2)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + \{y_4^{(1)}\}']$$

For third Approximation

$$y_4^{(3)} = y_3 + \frac{h}{3}[y_2' + 4y_3' + \{y_4^{(2)}\}']$$

1 Find $y(2)$ if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x+y)$
Where $y(0) = 2$, $y(.5) = 2.636$, $y(1) = 3.595$,
$y(1.5) = 4.968$ by Milne Predictor corrector method.

Solⁿ

Given $f(x,y) = \frac{1}{2}(x+y)$

$y(0) = 2$, $y(.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$

Here $x_0 = 0$, $x_1 = .5$, $x_2 = 1$, $x_3 = 1.5$

$y_0 = 2$, $y_1 = 2.636$, $y_2 = 3.595$, $y_3 = 4.968$

By Predictor Formula

$$y_4 = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$\because y' = \frac{1}{2}(x+y) \Rightarrow y_1' = \frac{1}{2}(x_1+y_1) = \frac{1}{2}(.5+2.636) = 1.568$

$\cancel{y'} = \cancel{\frac{1}{2}(x+y)} \Rightarrow \cancel{y'}\ y_2' = \frac{1}{2}(x_2+y_2) = \frac{1}{2}(1+3.595) = 2.2975$

$y_3' = \frac{1}{2}(x_3+y_3) = \frac{1}{2}(1.5+\cancel{3.595}\ 4.968) = \cancel{2.5475}\ 3.234$

$\therefore y_4 = 2 + \frac{4}{3}\times.5[2\times1.568 - 2.295 + 2\times3.234] = 6.872$

Now corrected this Value

$\because y' = \frac{1}{2}(x+y)$

$y_4' = \frac{1}{2}(x_4+y_4)$

$= \cancel{\frac{1}{2}(2+1.568)}$

$= \frac{1}{2}(2+6.872)$

$= 4.436$

$$y_4^{(1)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 3.595 + \frac{.5}{3}[2.2975 + 4(3.234) + 4.436]$$

$$y_4^{(1)} = 6.8732$$

$\{y_4^{(1)}\}' = \frac{1}{2}(x_4+y_4^{(1)})$

$= \frac{1}{2}(2+6.8732$

$$y_4^{(2)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + \{y_4^{(1)}\}']$$

$$= 3.595 + \frac{.5}{3}[2.2975 + 4(3.234) + 4.4366] = 6.8732$$

Hence $y_4^{(1)} = y_4^{(2)} = 6.8732$

$\therefore y(2) = 6.8732$ Ans

Q.2 Using Milne's method, solve $y' = 1+y^2$ with $y(0) = 0$, y
$y(.4) = .4228$, $y(.6) = .6841$, obtain $y(.8)$ and $y(1)$

Sol^n Given $y' = 1+y^2$

$f(x,y) = 1+y^2$

Given $x_0 = 0$, $x_1 = .2$, $x_2 = .4$, $x_3 = .6$

$y_0 = 0$, $y_1 = .2027$, $y_2 = .4228$, $y_3 = .6841$

By Milne's Predictor method

$$y_4^p = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

Here $y_1' = 1+y_1^2 = 1+(.2027)^2 = 1.0411$

$y_2' = 1+y_2^2 = 1+(.4228)^2 = 1.1788$

$y_3' = 1+y_3^2 = 1+(.6841)^2 = 1.4680$

$$\therefore y_4^p = 0 + \frac{4}{3}(.2)[2\times 1.0411 - (1.1788) + 2(1.4680)]$$

$$y_4^{(p)} = 1.0238$$

Now correct this value by Corrector Formula

$$y_4^{(1)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$\because y_4' = 1+y_4^2 = 1+(1.0238)^2$
$= 2.04816$

$$= .4288 + \frac{.2}{3}[1.1788 + 4(1.4680) + 2.04816]$$

$$= .4288 + \frac{.2}{3}[1.1788 + 5.872 + 2.0482] = 1.035$$

$$y_4^{(2)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + \{y_4^{(1)}\}']$$

$$= .4288 + \frac{.2}{3}[1.1788 + 4(1.4680) + \overset{2.071}{\cancel{2.035}}]$$

$$= 1.036$$

$\{y_4^{(1)}\}' = 1+(y_4^{(1)})^2$

$= 1+(1.035)^2$

$= 2.071$

$\therefore y_4 = 1.036$

$y = 1.036$ at $x = .8$

Ans

ADAMS - BASHFORTH FORMULA

Adams - Bashforth Predictor Corrector formula is denoted generally as

$$y_4^p = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

And

$$y_4^c = y_3 + \frac{h}{24}\left[9y_4'^p + 19y_3' - 5y_2' + y_1'\right]$$

Q.1 Using Adam's-Bashforth method to find $y(1.4)$ given
$\frac{dy}{dx} = x^2(1+y)$, $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$
And $y(1.3) = 1.979$

Given $\frac{dy}{dx} = x^2(1+y)$

$y' = x^2(1+y)$, $h = .1$

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$

$y_0 = 1$, $y_1 = 1.233$, $y_2 = 1.548$, $y_3 = 1.979$

By Adams-Bashforth Predictor formula is

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right] \longrightarrow ①$$

Here $y' = x^2(1+y)$

$y_0' = x_0^2(1+y_0) = 1(1+1) = 2$

$y_1' = x_1^2(1+y_1) = (1.1)^2(1+1.233) = 2.70193$

$y_2' = x_2^2(1+y_2) = (1.2)^2(1+1.548) = 3.66912$

$y_3' = x_3^2(1+y_3) = (1.3)^2(1+1.979) = 5.03451$

Putting these value in ①

$$y_4 = 1.979 + \frac{.1}{24}\left[55(5.03451) - 59(3.66912) + 37(2.70193) - 9(2)\right] = 2.5722974$$

$\therefore$ By corrector formula

$$y_4^{(C)} = y_3 + \frac{h}{24}\left[9y_4'^{P} + 19y_3' - 5y_2' + y_1'\right]$$

Now $y_4' = x_4^2(1+y_4) = (1.4)^2(1+2.5722974) = 7.0017029$

$y_4^{(c)} = 1.979 + \frac{.1}{24}[9(7.0017029) + 19(5.03451) - 5(3.66912) + 2.70193]$

$y_4^{(c)} = 2.5749$

$\therefore \boxed{y(.4) = 2.5749}$ Ans

Q.2 Solve and get $y(2)$, Given $\frac{dy}{dx} = \frac{1}{2}(x+y)$, $y(0) = 2$, $y(.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$. by Adam's method.

Sol^n Given $\frac{dy}{dx} = \frac{1}{2}(x+y)$

$f(x,y) = \frac{1}{2}(x+y)$, $h = .5$

$x_0 = 0$, $x_1 = .5$, $x_2 = 1$, $x_3 = 1.5$

$y_0 = 2$, $y_1 = 2.636$, $y_2 = 3.595$, $y_3 = 4.968$

By Adams predictor formula is

$$y_4^{p} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0'] \longrightarrow (1)$$

Here $y' = \frac{1}{2}(x+y)$

$y_0' = \frac{1}{2}(x_0+y_0) = \frac{1}{2}(0+2) = 1$

$y_1' = \frac{1}{2}(x_1+y_1) = \frac{1}{2}(.5+2.636) = 1.568$

$y_2' = \frac{1}{2}(x_2+y_2) = \frac{1}{2}(1+3.595) = 2.2975$

$y_3' = \frac{1}{2}(x_3+y_3) = \frac{1}{2}(1.5+4.968) = 3.234$

Putting in (1)

$$y_4^{(p)} = 4.968 + \frac{.5}{24}[55 \times 3.234 - 59(2.2975) + 37(1.568) - 9 \times 1]$$

$$y_4^{(p)} = 6.8708$$

By corrector method

$$y_4^{(c)} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1'] \longrightarrow (2)$$

$$\therefore y_4' = \frac{1}{2}(x_4 + y_4) = \frac{1}{2}(2 + 6.8708) = 4.4354$$

Putting in (2)

$$y_4^{(c)} = 4.968 + \frac{.5}{24}[9(4.4354) + 19(3.234) - 5(2.2975) + 1.5680]$$

$$y_4^{(c)} = 6.8731$$

$\therefore y = 6.8731$ At $x = 2$ Ans