Orthogonal Trajectories

(i) Trajectory :- A curve which cuts every member of a given family of curves according to some definite law is called a trajectory of the family.

(ii) Orthogonal Trajectory :- A curve which cuts every member of a given family of curves at right angles is called an orthogonal trajectory of the family.

(iii) Orthogonal Trajectories :- Two families of curves are said to be orthogonal trajectories if every member of either family cuts each member of the other family at right angles.

For example, parallels and meridians on a globe are orthogonal, as are equipotential and electric lines of force in an electric field, also the lines along which heat flows in a body are orthogonal to the isothermal surfaces. As a simple geometric example, the family consisting of circles about the origin, $x^2+y^2=c^2$; and the family of straight lines through the origin

$y = kx$ form orthogonal trajectories.

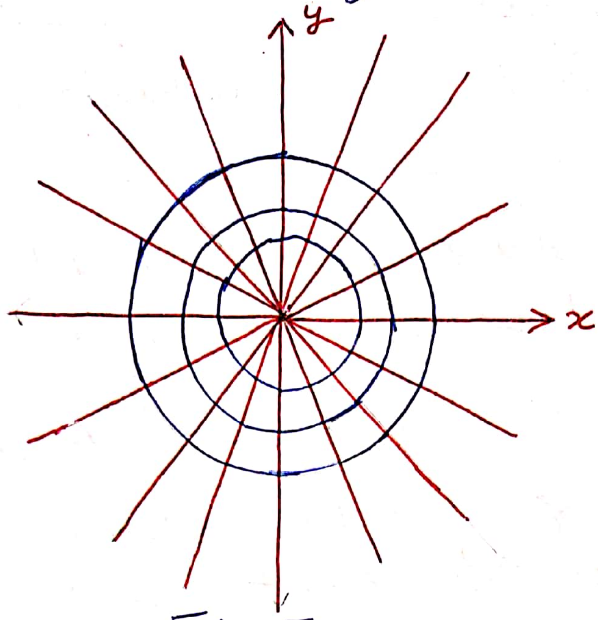

Fig. I

In general, two families of curves such that every member of either family cuts each member of the other family at a constant angle, say α are called isogonal trajectories of each other. For, orthogonal trajectories $\alpha = 90^\circ$.

Working Rule to find the eq^n of orthogonal trajectories.

(a) Cartesian curves $f(x, y, c) = 0$ —— (1)

(i) Differentiate (1) and eliminate the arbitrary constant c between (1) and the resulting eq^n. That gives diff. eq^n of the family (1).

Let it be $F\left(x, y, \frac{dy}{dx}\right) = 0$ —— (2)

(ii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

The diff. eq^n. of the orthogonal trajectory is $F\left(x, y, -\frac{dx}{dy}\right) = 0$ ——— (3)

(iii) Solve (3) to get the eq^n of the required orthogonal trajectory.

(b) Polar curves $f(r, \theta, c) = 0$ ——— (a)

(i) Differentiate (a) and eliminate the arbitrary constant c between (a) and the resulting eq^n. That gives the diff. eq^n of the family (a).

Let it be $F\left(r, \theta, \frac{dr}{d\theta}\right) = 0$ ——— (b)

(ii) Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$.

The differential eq^n of the orthogonal trajectory is $F\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0$ — (c)

(iii) Integrate (c) to get the eq^n of the required orthogonal trajectory.

Question 1:- Find the family of orthogonal trajectories to the family of parabolas $y = kx^2$.

Solution:- The given family of curves is

$$y = kx^2 \qquad \text{— (1)}$$

$$\Rightarrow \quad y - kx^2 = 0$$

which gives

$$\frac{dy}{dx} - 2kx = 0 \qquad \text{— (2)}$$

From ① & ②, we get

$$\frac{dy}{dx} = \frac{2y}{x} \qquad \text{——— (3)}$$

This is the differential eq^n of (1). The diff. eq^n of the family of orthogonal trajectories is obtained by replacing $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ in (3).

Thus we obtain, $-\frac{dx}{dy} = \frac{2y}{x}$

$$\int 2y\,dy = \int -x\,dx$$

Integrating, we have

$$y^2 = -\frac{x^2}{2} + C$$

$$\Rightarrow \quad \frac{x^2}{2} + y^2 = C$$

which is a family of ellipse with parameter C as shown in Fig. II

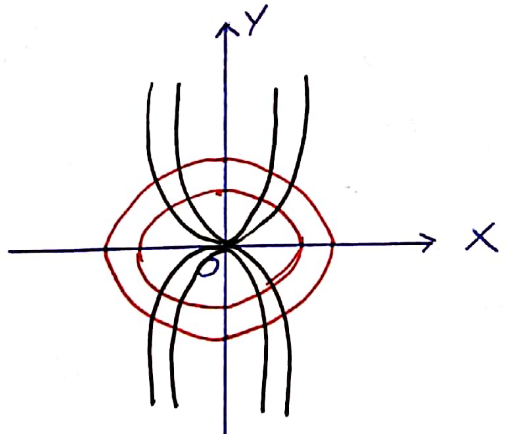

Fig. II

Question 2:- Find the orthogonal trajectories of the circles $r = k\sin\theta$, where k is a parameter.

Solution:- The given family of circles is

$$r = k\sin\theta \qquad \text{——— (1)}$$

Diff. (1) wrt θ, we get

$$\frac{dr}{d\theta} = k\cos\theta \quad \text{———} \quad (2)$$

eliminating k from (1) and (2), we obtain

$$\frac{dr}{d\theta} = r\tan\theta \quad \text{———} \quad (3)$$

which is the differential eqⁿ for the family of circles (1)

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (3), we get

$$-r^2\frac{d\theta}{dr} = r\tan\theta$$

$$\Rightarrow \quad -r\frac{d\theta}{dr} = \tan\theta$$

$$\int\frac{dr}{r} = \int -\cot\theta \, d\theta$$

Integrating, we obtain the circles, $r = c\cos\theta$, with c as parameter as the orthoganal trajectories to (1), as shown in Fig. III

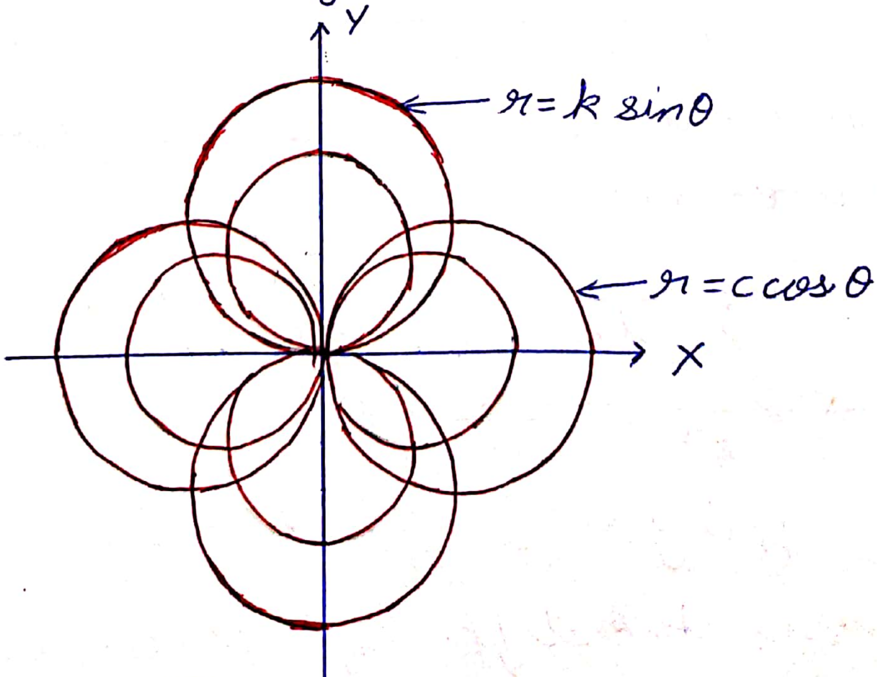

Fig. III

Question 3:- Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$, where λ is a parameter.

Solution:- The eq^n of the family of given curves is $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$ —— (1)

Diff (1) wrt x

$$\frac{2x}{a^2}+\frac{2y}{b^2+\lambda}\frac{dy}{dx}=0$$

or

$$\frac{x}{a^2}+\frac{y}{b^2+\lambda}\frac{dy}{dx}=0 \quad \text{—— (2)}$$

To eliminate the parameter λ, we equate the value of $b^2+\lambda$ from (1) & (2).

From (1) $\quad b^2+\lambda = \frac{a^2y^2}{a^2-x^2}$

From (2) $\quad b^2+\lambda = -\frac{a^2y}{x}\frac{dy}{dx}$

$$\Rightarrow \frac{a^2y^2}{a^2-x^2} = -\frac{a^2y}{x}\frac{dy}{dx}$$

or

$$\frac{xy}{a^2-x^2}+\frac{dy}{dx}=0 \quad \text{—— (3)}$$

which is the diff. eq^n of the given family of curves (1)

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we get

$$\frac{xy}{a^2-x^2}-\frac{dx}{dy}=0$$

$$y\,dy-\left(\frac{a^2-x^2}{x}\right)dx=0 \quad \text{—— (4)}$$

which is the diff. eq^n of the orthogonal trajectories.

$$\frac{dr}{d\theta} = k\cos\theta \quad \text{——— (2)}$$

eliminating k from (1) and (2), we obtain

$$\frac{dr}{d\theta} = r\tan\theta \quad \text{——— (3)}$$

which is the differential eq^n for the family of circles (1)

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (3), we get

$$-r^2\frac{d\theta}{dr} = r\tan\theta$$

$$\Rightarrow \quad -r\frac{d\theta}{dr} = \tan\theta$$

$$\int\frac{dr}{r} = \int -\cot\theta \, d\theta$$

Integrating, we obtain the circles, $r = c\cos\theta$, with c as parameter as the orthoganal trajectories to (1), as shown in Fig. III

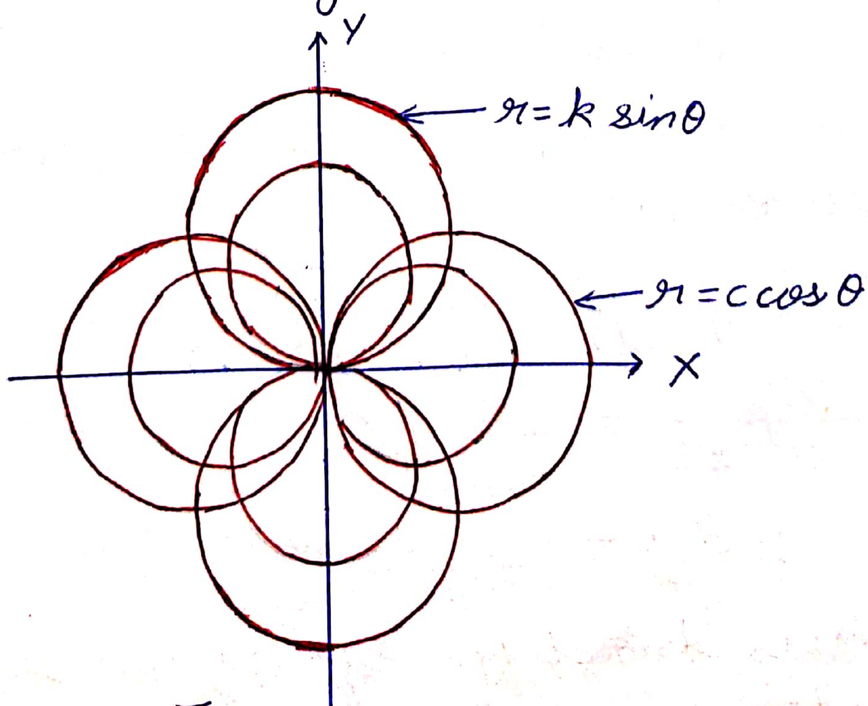

Fig. III

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Integrating (4), we get

$$\int y\,dy - \int\left(\frac{a^2}{x} - x\right)dx = C$$

$$\frac{y^2}{2} - a^2 \log x + \frac{x^2}{2} = C$$

$$x^2 + y^2 = 2a^2 \log x + C$$

which is the eqn of the required orthogonal trajectories of (1).

Question 4 :- Prove that the system of confocal and coaxial parabolas $y^2 = 4a(x+a)$ is self orthogonal.

Solution :- The eqn of the family of given parabolas is $y^2 = 4a(x+a)$ — (1)

Diff. (1) w.r.t x

$$2y\frac{dy}{dx} = 4a \quad \text{or} \quad y\frac{dy}{dx} = 2a \quad \text{— (2)}$$

Eliminating a between (1) & (2), we have

$$y^2 = 4 \cdot \frac{y}{2}\frac{dy}{dx}\left[x + \frac{y}{2}\cdot\frac{dy}{dx}\right]$$

or $$y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

or $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \quad \text{— (3)}$$

which is the differential eqn of the given family (1).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we get

$$y\left(\frac{dx}{dy}\right)^2 - 2x\left(\frac{dx}{dy}\right) - y = 0$$

or $$y - 2x\frac{dy}{dx} - y\left(\frac{dy}{dx}\right)^2 = 0$$

$$y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}-y=0 \quad \text{———— (4)}$$

which is the differential eq^n of the orthogonal trajectories.

Since (4) is the same as (3), the system of confocal and coaxial parabolas is self orthogonal.

H.P.

Question 5:- Find the orthogonal trajectories of the cardioids $r = a(1-\cos\theta)$.

Solution:- The eq^n of the family of given cardioids is $r = a(1-\cos\theta)$ ———— (1)

Diff. (1) w.r.t θ

$$\frac{dr}{d\theta} = a\sin\theta \quad \text{———— (2)}$$

From (1) & (2), we get

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\frac{\theta}{2} \quad \text{———— (3)}$$

which is the diff. eq^n of (1)

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (3), we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot\frac{\theta}{2}$$, which gives

$$\frac{dr}{r} + \tan\frac{\theta}{2}\,d\theta = 0 \quad \text{—— (4)}$$

which is the diff. eq^n of the family of orthogonal trajectories.

Integrating (4), we get

$$\log r - 2\log\cos\frac{\theta}{2} = \log c$$

$\Rightarrow \quad \log r = \log c + \log \cos^2\frac{\theta}{2}$

$\Rightarrow \quad r = c\cos^2\frac{\theta}{2}$

$\Rightarrow \quad r = \frac{c}{2}(1+\cos\theta)$

$\Rightarrow \quad r = C(1+\cos\theta) \qquad$ where $C = \frac{c}{2}$

which is the orthogonal trajectories of (1)

Exercise

Question:- Find the orthogonal trajectories of the family of following curves:-

(1) parabolas $y^2 = 4ax$

(2) parabolas $y = ax^2$

(3) semi-cubical parabolas $ay^2 = x^3$

(4) $r = a(1+\cos\theta)$

(5) $r^n \sin n\theta = a^n$

(6) $r^2 = a^2\cos 2\theta$

(7) $r^n = a^n \cos n\theta$

(8) $r = \frac{2a}{1+\cos\theta}$

(9) $xy = c$

(10) $r^n = a^n \sin n\theta$