

UNIT-2.

1.

LAPLACE TRANSFORMATION

The knowledge of 'Integral Transforms' provide an easy and effective means for the solutions of many problems arising in science and engineering.

For example, The Laplace transformation replaces a given function  $F(t)$  by another function  $f(s)$ . Then Laplace transformation converts an ordinary differential equation with some given initial conditions into an algebraic equation in terms of  $f(s)$ . Finally, using inverse Laplace transformation we recover the original function  $F(t)$ .

LAPLACE TRANSFORM

Definition Given a function  $F(t)$  defined for all real  $t \geq 0$ , the Laplace transform of  $F(t)$  is a function of a new variable  $s$  is given by

$$L(F(t)) = f(s) = \bar{F}(s) = \int_0^{\infty} e^{-st} F(t) dt$$

we can also write  $L(f(t)) = \bar{f}(s)$ .

NOTE A function  $F(t)$  is called piecewise continuous or sectionally continuous in the closed interval  $[a, b]$  if there is a finite number of points  $t_1, t_2, \dots, t_n$  ( $a = t_1 < t_2 < t_3 < \dots < t_n = b$ ) such that  $F(t)$  is continuous in each of open subintervals  $(t_i, t_{i+1})$  and has finite right limit  $(F(t_{i+1}^+))$  and finite left limit  $(F(t_i^-))$ .

## EXISTENCE OF LAPLACE TRANSFORM.

(2)

Suppose a function  $f$  is piecewise continuous over  $\mathbb{R}^+$  and there are numbers  $M$  and  $K$  such that  $|f(t)| \leq M e^{Kt}$  for  $t \geq 0$ , then  $\int_0^\infty e^{-st} f(t) dt$  converges for  $s > K$  that is,  $F(s)$  exists for  $s > K$ .

Further we also define inverse Laplace transform.

$$\text{if } f(t) = L^{-1}(F(s)).$$

Linearity Theorem :- If  $L(f_1(t)) = \bar{f}_1(s)$  and  $L(f_2(t)) = \bar{f}_2(s)$

then for any constants  $c_1$  and  $c_2$

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \bar{f}_1(s) + c_2 \bar{f}_2(s).$$

First Shifting Theorem :- If  $L(f(t)) = \bar{f}(s)$  then  $L(e^{at} f(t)) = \bar{f}(s-a)$  ( $s > a$ )

$$\text{if } L(f(t)) = \bar{f}(s) \text{ then } L(e^{at} f(t)) = \bar{f}(s-a)$$

$$\begin{aligned} \text{Proof } L(e^{at} f(t)) &= \int_0^\infty e^{-st} (e^{at} f(t)) dt \\ &= \int_0^\infty e^{-(s-a)t} f(t) dt \end{aligned}$$

$$\text{let } p = s-a = + (s-a)$$

$$= \int_0^\infty e^{-pt} f(t) dt$$

$$= \bar{f}(p) = \bar{f}(s-a).$$

## Laplace Transform of some elementary functions

①  $f(t) = e^{at}$ ,  $a > 0$

$$\begin{aligned} L(e^{at}) &= \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{-(s-a)t} dt \quad (s > a) \\ &= \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = \frac{-1}{s-a} [0 - 1] \\ &= \frac{1}{s-a} \end{aligned}$$

Thus

$$L(e^{at}) = \frac{1}{s-a} \quad \text{where } s > a.$$

②  $f(t) = \sin at$  or  $\cos at$

We know  $e^{ait} = \cos at + i \sin at$

$$\Rightarrow L(e^{ait}) = L(\cos at) + i L(\sin at)$$

Now  $L(e^{ait}) = \frac{1}{s - ai} = \frac{s + ai}{s^2 + a^2}$

$$\therefore L(\cos at) + i L(\sin at) = \frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2}$$

$$\therefore L(\cos at) = \frac{s}{s^2 + a^2}$$

$$L(\sin at) = \frac{a}{s^2 + a^2}$$

(4)

$$\textcircled{3} \quad f(t) = \sinhat \quad \text{or} \quad \coshat$$

$$L(\sinhat) = L\left(\frac{e^{at} - \bar{e}^{-at}}{2}\right) = \frac{1}{2} L(e^{at}) - \frac{1}{2} L(\bar{e}^{-at}) \\ = \frac{1}{2(s-a)} - \frac{1}{2(s+a)}$$

$$\boxed{L(\sinhat) = \frac{a}{s^2 - a^2}}$$

Hence

$$\boxed{L(\coshat) = \frac{s}{s^2 - a^2}}$$

$$\textcircled{4} \quad f(t) = t^n$$

$$L(t^n) = \int_0^\infty e^{-st} \cdot t^n dt \quad \text{Putting } st = x \\ = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$$

$$\boxed{L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}} = \frac{n!}{s^{n+1}}$$

$$\textcircled{5} \quad L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}$$

$$\textcircled{6} \quad L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2} \quad \& \quad L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$$

$$\textcircled{7} \quad L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2} \quad \& \quad L(e^{ab} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$$

Ex Find the Laplace transform of.

(5)

(1)  $\sin^2 t$

(2)  $\sin 2t \cos 3t$

(3)  $\cos^3 t$

(4)  $\sin t \cos t$

(5)  $e^{-2t} (3 \cos 4t - 2 \sin 5t)$

(6)  $\sin \hat{t} \cos \hat{t}$

Solution (1)  $L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}L(1) - \frac{1}{2}L(\cos 2t)$   
 $= \frac{1}{2s} - \frac{1}{2}\left(\frac{s}{s^2+4}\right)$   
 $= \frac{2}{s(s^2+4)}$

(2)  $L(\sin 2t \cos 3t) = L\left(\frac{1}{2}(\sin 5t - \sin t)\right)$   
 $= \frac{1}{2}L(\sin 5t) - \frac{1}{2}L(\sin t)$   
 $= \frac{1}{2}\left(\frac{s}{s^2+25}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right)$   
 $= \frac{2(s^2-5)}{(s^2+1)(s^2+25)}$

(3)  $L(\cos^3 t) = L\left(\frac{3 \cos t + \cos 3t}{4}\right)$

$$= \frac{3}{4}L(\cos t) + \frac{1}{4}L(\cos 3t)$$

(6)

$$= \frac{3}{4} \left( \frac{s}{s^2+1} \right) + \frac{1}{4} \left( \frac{s}{s^2+9} \right)$$

$$= \frac{s(s^2+9)}{(s^2+1)(s^2+9)}$$

$$(4) L(\sin t \cos t) = L\left(\frac{1}{2} \sin 2t\right) = \frac{1}{2} L(\sin 2t) \\ = \frac{1}{2} \left( \frac{2}{s^2+4} \right) = \frac{1}{s^2+4}$$

$$(5) L\left(e^{-2t}(3\cos 4t - 2\sin 5t)\right)$$

$$= 3L\left(e^{-2t}\cos 4t\right) - 2L\left(e^{-2t}\sin 5t\right)$$

$$= \frac{3 \cdot (s+2)}{(s+2)^2 + 16} - 2 \cdot \frac{5}{(s+2)^2 + 25}$$

$$= \frac{3(s+2)}{(s+2)^2 + 16} - \frac{10}{(s+2)^2 + 25}$$

$$(6) L(\sinh at \cos at) = L\left(\left(\frac{e^{at} - e^{-at}}{2}\right) \cos at\right)$$

$$= \frac{1}{2} L\left(e^{at} \cos at\right) - \frac{1}{2} L\left(e^{-at} \cos at\right)$$

$$= \frac{1}{2} \left[ \frac{s-a}{(s-a)^2 + a^2} - \frac{(s+a)}{(s+a)^2 + a^2} \right]$$

x

Ex 1. Let  $f(t) = \begin{cases} t & \text{for } 0 < t < a \\ 1 & \text{for } t > a \end{cases}$

Find the Laplace transform of  $f(t)$

Ex 2. Find the Laplace transform of the function

$$(1) f(t) = \begin{cases} 2 & : 0 < t < \pi \\ 0 & : \pi < t < 2\pi \\ \sin t & : 2\pi < t \end{cases}$$

$$(2) f(t) = \begin{cases} 2+t^2 & : 0 < t < 2 \\ 6 & : 2 < t < 3 \\ 2t-5 & : 3 < t < \infty \end{cases}$$

$$(3) f(t) = |t-1| + |t+1|, \quad t \geq 0.$$

Solution. .  $L(f(t)) = \int_0^\infty e^{-st} f(t) dt = \int_0^\infty e^{-st} \cdot \frac{t}{a} dt + \int_a^\infty e^{-st} \cdot 1 dt$

$$= \frac{1}{a} \int_0^a t \cdot e^{-st} dt + \int_a^\infty e^{-st} dt$$

$$= \frac{1}{a} \left[ t \frac{e^{-st}}{-s} - \int_0^\infty \frac{1}{-s} e^{-st} dt \right]_0^\infty + \left[ \frac{e^{-st}}{-s} \right]_a^\infty$$

$$= \frac{1}{a} \left[ \frac{-a}{s} e^{-sa} + \frac{-st}{-s^2} \Big|_0^a \right] + \left[ \frac{-e^{-st}}{s} \right]_a^\infty$$

$$= -\frac{1}{s} e^{-as} - \frac{-sa}{as^2} + \frac{1}{as^2} + \frac{-sa}{s}$$

$$= -\frac{e^{-as}}{as^2} + \frac{1}{as^2}$$

Solution (2) (1)  $L(f(t)) = \int_0^\infty e^{-st} f(t) dt$

$$= \int_0^\pi 2e^{-st} dt + \int_\pi^{2\pi} 0 \cdot e^{-st} dt + \int_{2\pi}^\infty \sin t \cdot e^{-st} dt$$

$$= \left[ \frac{2e^{-st}}{-s} \right]_0^\pi + \left[ \frac{\sin t e^{-st}}{-s} \right]_\pi^{2\pi} - \left[ \frac{\cos t e^{-st}}{-s} \right]_{2\pi}^\infty$$

$$= \left[ \frac{2e^{-st}}{-s} \right]_0^\pi + 0 + \left[ \frac{-st}{1+s^2} (-s \sin t - \cos t) \right]_{2\pi}^\infty$$

$$= \frac{2}{s} \left[ 1 - e^{-s\pi} \right] + \frac{-2\pi s}{1+s^2} (s \sin 2\pi + \cos 2\pi)$$

$$= \frac{2}{s} \left( 1 - e^{-s\pi} \right) + \frac{e^{-s\pi}}{1+s^2}$$

① Using  $\lim_{t \rightarrow \infty} \frac{e^{-st}}{1+s^2} (-s \sin t) = 0$

$\sin t$  and  $\cos t$  are bounded functions.

②  $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$

and  $(\lim_{t \rightarrow \infty} -\text{zero}) \times \text{bdd func.} = 0$

(9)

$$(2) \cdot L(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^2 (2+t^2) e^{-st} dt + \int_2^3 6 e^{-st} dt + \int_3^\infty (2t-5) e^{-st} dt$$

$$= \left[ \frac{(2+t^2) e^{-st}}{-s} \right]_0^2 - \int_0^2 \frac{2t e^{-st}}{-s} dt + \left[ \frac{6 e^{-st}}{-s} \right]_2^3 + \left[ \frac{(2t-5) e^{-st}}{-s} \right]_3^\infty$$

$$- 2 \int_3^\infty \frac{e^{-st}}{-s} dt$$

$$= -\frac{6}{s} e^{-2s} + \frac{2}{s} + \frac{2}{s} \int_0^2 t e^{-st} dt - \frac{6}{s} (e^{-3s} - e^{-2s}) + \frac{e^{-3s}}{s}$$

$$+ \frac{2}{s} \left[ \frac{e^{-st}}{-s} \right]_3^\infty$$

$$= \frac{2}{s} + \frac{2}{s} \left[ \frac{t e^{-st}}{-s} - \int_0^2 \frac{e^{-st}}{-s} dt \right]_0^2 - \frac{6}{s} e^{-3s} + \frac{e^{-3s}}{s} + \frac{2}{s^2} e^{-3s}$$

$$= \frac{2}{s} + \frac{2}{s^2} \left[ -2e^{-2s} + 0 \right] + \frac{2}{s^2} \left[ \frac{e^{-st}}{-s} \right]_0^2 - \frac{5}{s} e^{-3s} + \frac{2}{s^2} e^{-3s}$$

$$= \frac{2}{s} - \frac{4e^{-2s}}{s^2} - \frac{2}{s^3} \left[ e^{-2s} - 1 \right] - \frac{5}{s} e^{-3s} + \frac{2}{s^2} e^{-3s}$$

$$= \frac{2}{s} + \frac{2}{s^3} - e^{-2s} \left( \frac{4}{s^2} + \frac{2}{s^3} \right) - e^{-3s} \left( \frac{5}{s} - \frac{2}{s^2} \right)$$

X

$$(3) f(t) = |t-1| + |t+1|, \quad t \geq 0$$

$$= \begin{cases} 1-t+1+t = 2 & 0 < t < 1 \\ 1-t+1+t = 2 & t \geq 1 \end{cases}$$

$$\therefore L(f(t)) = \int_0^\infty e^{-st} f(t) dt = \int_0^1 2e^{-st} dt + \int_1^\infty 2t e^{-st} dt$$

$$= \left[ \frac{2e^{-st}}{-s} \right]_0^1 + 2 \left[ \frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_1^\infty$$

$$= \frac{-2}{s} \left[ \frac{e^{-s}}{s} - 1 \right] + 2 \left[ \frac{\frac{-s}{s}}{-s} - \frac{e^{-s}}{s^2} \right]$$

$$= \frac{2}{s} + \frac{2e^{-s}}{s^2}$$

# PROPERTIES OF LAPLACE TRANSFORM

(11)

(1) Change of scale

$$L(f(at)) = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$$

Proof  $L(f(nt)) = \int_0^\infty e^{-st} f(nt) dt$  Putting  $at = x$

$$= \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x) d\left(\frac{x}{a}\right)$$

$$= \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x) dx = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right).$$

(2)

$$L(t f(t)) = -\frac{d}{ds} \bar{f}(s)$$

Proof we have  $\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt$   
Differentiating both sides w.r.t  $s$ .

$$\frac{d}{ds}(\bar{f}(s)) = \int_0^\infty \frac{\partial}{\partial s} \left( e^{-st} f(t) \right) dt$$

$$= \int_0^\infty -t e^{-st} f(t) dt = -L(t \cdot f(t)).$$

(3) In general

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} \bar{f}(s)$$

$n = 1, 2, 3 \dots$

(4)

$$L\left(\frac{1}{t} f(t)\right) = \int_s^\infty \bar{f}(s) ds$$

Proof.  $\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt$ .

Integrating both sides w.r.t.  $s$  under the limits  $s \rightarrow \infty$ , we get

$$\int_s^\infty \bar{f}(s) ds = \int_s^\infty \left[ \int_0^\infty e^{-st} f(t) dt \right] ds$$

changing the order of integration

$$= \int_0^\infty f(t) \left( \int_s^\infty e^{-st} ds \right) dt$$

$$= \int_0^\infty f(t) \left[ \frac{-e^{-st}}{-t} \right]_s^\infty dt = \int_0^\infty f(t) \left( 0 - \frac{-e^{-st}}{-t} \right) dt$$

$$= \int_0^\infty \frac{e^{-st} f(t)}{t} dt$$

$$= L\left(\frac{1}{t} f(t)\right)$$

(5)

$$L(f'(t)) = s\bar{f}(s) - f(0)$$

Proof  $L(f'(t)) = \int_0^\infty e^{-st} f'(t) dt$ .

Integration by parts:

$$= \left[ -e^{-st} f(t) \right]_0^\infty - \int_0^\infty -s e^{-st} f(t) dt$$

$$= \lim_{t \rightarrow \infty} t e^{-st} f(t) - f(0) + s \bar{f}(s),$$

(13)

Since  $f(t)$  is a continuous and bounded function

$$\therefore \lim_{t \rightarrow \infty} t e^{-st} f(t) = 0.$$

Thus, we have  $L(f'(t)) = s \bar{f}(s) - f(0)$

(5) If  $L(f(t)) = \bar{f}(s)$  then

$$L\left(\int_0^t f(u) du\right) = \frac{1}{s} \bar{f}(s)$$

$$(6) L(f''(t)) = s^2 \bar{f}(s) - sf(0) - f'(0)$$

Ex. ① Show that

$$L(t \sin at) = \frac{2as}{(s^2 + a^2)^2}$$

$$L(t \cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

② Find  $L(\sin^3 2t)$

③ Find  $L(e^{2t} e^{-2t})$

④ Find  $L\left(\frac{e^t \sin t}{t}\right)$

⑤ Find  $L(t e^t \sin 3t)$ .

Solution

①  $L(\sin at) = \frac{a}{s^2 + a^2}$

$$\Rightarrow L(t \sin at) = -\frac{d}{ds} \left( \frac{a}{s^2 + a^2} \right)$$

$$= -a \frac{(2s)(-1)}{(s^2 + a^2)^2}$$

$$= \frac{2as}{(s^2 + a^2)^2}$$

②  $L(\sin^3 2t)$

Now  $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ .

$$\Rightarrow \sin^3(2t) = \frac{1}{4}(3 \sin 2t - \sin 6t)$$

(15)

$$\therefore L(\sin^3 2t) = \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t)$$

$$= \frac{3}{4} \cdot \frac{2}{s^2 + 2^2} - \frac{1}{4} \cdot \frac{6}{s^2 + 6^2}$$

$$= \frac{3}{2} \left[ \frac{1}{s^2 + 4} - \frac{1}{s^2 + 36} \right].$$

③ we know if  $L(f(t)) = \bar{f}(s)$

$$\text{then } L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} \bar{f}(s).$$

$$\text{Now } L(\bar{e}^{2t}) = \frac{1}{s+2}$$

$$\Rightarrow L(t^2 \bar{e}^{2t}) = (-1)^2 \frac{d^2}{ds^2} \left( \frac{1}{s+2} \right)$$

$$= \frac{d}{ds} \left( \frac{(-1)}{(s+2)^2} \right) = \frac{(-1)(-2)}{(s+2)^3}.$$

$$= \frac{2}{(s+2)^3}$$

$$\text{④ } L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}$$

$$\therefore L(\bar{e}^t \sin t) = \frac{b=1}{(s+1)^2 + 1}$$

$$\text{Also, } L\left(\frac{f(t)}{t}\right) = \int_s^{\infty} f(s) ds$$

$$\begin{aligned} \therefore L\left(\frac{1}{t}(\bar{e}^{ts} \sin t)\right) &= \int_s^{\infty} \frac{ds}{(s+1)^2 + 1} = [\tan^{-1}(s+1)]_s^{\infty} \\ &= \frac{\pi}{2} - \tan^{-1}(s+1) \\ &= \cot^{-1}(s+1). \end{aligned}$$

Ex Find (a)  $L(2e^t \sin 4t \cos 2t)$

(b)  $L\left(\frac{\sin at}{t}\right)$

(c) Using Laplace transform, show that

$$L \int_0^t \frac{\cos at - \cos bt}{t} dt = \frac{1}{2s} \log \frac{s^2 + b^2}{s^2 + a^2}.$$

(d) Show that  $L \int_0^t \frac{e^t \sin t}{t} dt = \frac{1}{s} \cot^{-1}(s-1)$ .

Solution (a)  $L(2 \sin 4t \cos 2t) = L(\sin 6t + \sin 2t)$

$$= \frac{6}{s^2 + 36} + \frac{2}{s^2 + 4}$$

By first shifting theorem

$$\begin{aligned} L(e^t \sin 6t + e^t \sin 2t) &= L(e^t \sin 6t) + L(e^t \sin 2t) \\ &= \frac{s}{(s-1)^2 + 36} + \frac{2}{(s-1)^2 + 4} \end{aligned}$$

(17)

$$(b) L(\sin at) = \frac{a}{s^2 + a^2}$$

$$\therefore L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2 + a^2} ds = \left[ -\tan^{-1}\left(\frac{s}{a}\right) \right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$\int \frac{dx}{1+x^2} = \tan^{-1} x$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

$$(c) L(\cos at - \cos bt) = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}$$

$$\therefore L\left(\frac{1}{t}(\cos at - \cos bt)\right) = \int_s^\infty \left( \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right) ds$$

$$= \left[ \frac{1}{2} \log(s^2 + a^2) - \frac{1}{2} \log(s^2 + b^2) \right]_s^\infty$$

$$= 0 - \frac{1}{2} \log \left( \frac{s^2 + a^2}{s^2 + b^2} \right) = \frac{1}{2} \log \left( \frac{s^2 + b^2}{s^2 + a^2} \right)$$

because  $\lim_{s \rightarrow \infty} \log \left( \frac{s^2 + a^2}{s^2 + b^2} \right) = \log 1 = 0$ .

Now, Using  $L\left(\int_0^t f(t) dt\right) = \frac{1}{s} \bar{f}(s)$

we have  $L\left[\int_0^t \frac{\cos at - \cos bt}{t} dt\right] = \frac{1}{2s} \log \left( \frac{s^2 + b^2}{s^2 + a^2} \right)$

$$(d) \quad L(e^t \sin t) = \frac{1}{(s-1)^2 + 1}$$

Hence  $L\left(\frac{e^t \sin t}{t}\right) = \int_s^\infty \frac{ds}{(s-1)^2 + 1} = \tan^{-1}(s-1)$

$$= \frac{\pi}{2} - \tan^{-1}(s-1) = \cot^{-1}(s-1)$$

$$\therefore L\left(\int_0^t \frac{e^s \sin s}{s} ds\right) = \frac{1}{s} f(s) = \frac{1}{s} \cot^{-1}(s-1)$$

Ex Evaluate  $\int_0^\infty t^3 e^{-t} \sin t dt$ .

Solution

$$L(t^3 \sin t) = (-1)^3 \frac{d^3}{ds^3} \left( \frac{1}{s^2 + 1} \right) = \frac{24s(s^2 - 1)}{(s^2 + 1)^4}$$

That is  $L(t^3 \sin t) = \int_0^\infty e^{-st} t^3 \sin t dt = \frac{24s(s^2 - 1)}{(s^2 + 1)^4}$

Putting  $s=1$

we get  $\int_0^\infty t^3 e^{-t} \sin t dt = 0$ .

Ex Evaluate  $L(\sinh^3 2t)$

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Solution

We know

$$\sinh 3t = 3\sinh t + 4\sinh^3 t$$

$$\Rightarrow \sinh^3 t = \frac{1}{4} [\sinh 3t - 3\sinh t].$$

$$\Rightarrow \sinh^3(2t) = \frac{1}{4} [\sinh 6t - 3\sinh 2t]$$

$$\therefore L(\sinh^3 2t) = \frac{1}{4} L(\sinh 6t) - \frac{3}{4} L(\sinh 2t)$$

$$= \frac{6}{4(s^2 - 6^2)} - \frac{3}{4} \cdot \frac{2}{(s^2 - 2^2)}$$

$$= \frac{48}{(s^2 - 4)(s^2 - 36)}.$$

Ex, If  $L(f(t)) = \bar{f}(s)$  and  $g(t) = \begin{cases} f(t-a) & t>a \\ 0 & 0 < t < a \end{cases}$

then prove that  $L(g(t)) = e^{-as}\bar{f}(s)$ .

$$\text{Solution } L(g(t)) = \int_0^\infty e^{-st} g(t) dt = \int_0^a e^{-st} g(t) dt + \int_a^\infty e^{-st} g(t) dt$$

$$= \int_0^a e^{-st} \cdot 0 dt + \int_a^\infty e^{-st} f(t-a) dt.$$

Putting  $t-a=u$ .

$$\Rightarrow dt = du.$$

$$= \int_0^\infty e^{-s(a+u)} f(u) du = -as \int_0^\infty e^{-su} f(u) du.$$

$$= -\frac{as}{e^s} \bar{f}(s).$$

Hence proved. Q.E.D.

Ex Evaluate  $L(\sin \sqrt{t})$

Solution  $\sin \sqrt{t} = (\sqrt{t}) - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \dots$

$$= t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \dots$$

$$L(\sin \sqrt{t}) = L(t^{1/2}) - \frac{1}{3!} L(t^{3/2}) + \frac{1}{5!} L(t^{5/2}) - \dots$$

$$= \frac{\sqrt{3/2}}{s^{3/2}} - \frac{1}{6} \frac{\sqrt{5/2}}{s^{5/2}} + \frac{1}{120} \frac{\sqrt{7/2}}{s^{7/2}}$$

$$\text{as } L(t^n) = \frac{\sqrt{n+1}}{s^{n+1}}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{s^{5/2}} + \frac{1}{120} \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{s^{7/2}} - \dots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \left[ 1 - \left(\frac{1}{4s}\right) + \frac{1}{2!} \left(\frac{1}{4s}\right)^2 - \frac{1}{3!} \left(\frac{1}{4s}\right)^3 - \dots \right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$$

(21)

Ex Find the laplace transform of the function  
f(t) given by

$$f(t) = \begin{cases} \sin(t-\alpha) & t > \alpha \\ 0 & t < \alpha \end{cases}$$

Solution

$$L(f(t)) = \int_0^\infty e^{-st} f(t) dt = \int_0^\alpha e^{-st} \cdot 0 dt + \int_\alpha^\infty e^{-st} \sin(t-\alpha) dt$$

$$\text{Put } t-\alpha = u$$

$$= \int_0^\infty e^{-s(u+\alpha)} \sin u du$$

$$= \frac{-e^{-\alpha s}}{s} \int_0^\infty e^{-su} \sin u du = \frac{-e^{-\alpha s}}{s} L(\sin u)$$

$$= \frac{-e^{-\alpha s}}{s^2+1}$$

$$\text{Ex} \quad \text{Given that } L\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{1}{s^{3/2}}$$

$$\text{Show that } L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$$

$$\text{Solution} \quad \text{let } f(t) = 2\sqrt{\frac{t}{\pi}}$$

$$\text{Now } L(f(t)) = \frac{1}{s^{3/2}} = \bar{f}(s)$$

Using property :-

$$L(f'(t)) = s\bar{f}(s) - f(0) = s \cdot \frac{1}{s^{3/2}} - 0 = \frac{1}{\sqrt{s}}$$

$$\therefore L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$$

Ex Using Laplace transformation

Show that  $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

Solution We know that  $L(\sin at) = \frac{a}{s^2 + a^2}$

$$\begin{aligned}\therefore L\left(\frac{\sin at}{t}\right) &= \int_s^\infty \frac{\frac{a}{s^2 + a^2}}{s} ds \\ &= \left[ a \cdot \frac{1}{a} \tan^{-1}\left(\frac{s}{a}\right) \right]_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)\end{aligned}$$

Thus, by def<sup>n</sup>,

$$\int_0^\infty e^{-st} L(\sin at) dt = \cot^{-1}\left(\frac{s}{a}\right)$$

Taking  $a=1$  and  $s=0$  in above relation,

$$\int_0^\infty \frac{\sin t}{t} dt = \cot^{-1}(0) = \frac{\pi}{2}$$

Eii Evaluate  $\int_0^\infty t e^{-2t} \cos t dt$

Solution we know  $L(t \cos t) = -\frac{d}{ds} \left( \frac{s}{s^2+1} \right)$

$$= - \left[ \frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2} \right]$$

$$= \frac{s^2-1}{(s^2+1)^2}$$

By def<sup>n</sup>,

$$\int_0^\infty e^{-st} t \cos t dt = \frac{s^2-1}{(s^2+1)^2}$$

Taking  $s=2$ ,

$$\int_0^\infty e^{-2t} t \cos t dt = \frac{3}{25}$$

## EXTRA QUESTIONS

(24)

Evaluate

$$(1) \quad L\left(\frac{\cos 2t - \cos 3t}{t}\right)$$

$$\frac{1}{2} \log \frac{s^2 + 9}{s^2 + 4}$$

$$(2) \quad L\left(\frac{1 - \cos t}{t^2}\right)$$

$$\cot^{-1}s - \frac{s}{2} \log(1 + s^2)$$

$$(3) \quad L(\sinh at \sin at)$$

$$\frac{2a^2s}{s^4 + 4a^4}$$

$$L(\cos 2t) = \frac{s^2 + 4}{s^2 + 1}$$

Ex If  $L(\cos^2 at) = \frac{s^2 + 2}{s(s^2 + 4)}$ , find  $L(\cos^2 at)$ .

Solution.

(2).

$$\text{Let } f(t) = \frac{1 - \cos t}{t^2}$$

$$\text{We let, } g(t) = t^2 f(t) = 1 - \cos t$$

$$L(g(t)) = L(t^2 f(t)) = L(1 - \cos t)$$

$$\Rightarrow (-1)^2 \frac{d^2}{ds^2} \bar{f}(s) = \frac{1}{s} - \frac{s}{s^2 + 1}$$

$$\Rightarrow \bar{f}(s) = \int \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) ds$$

$$= \int \left( \log s - \frac{1}{2} \log(s^2 + 1) \right) ds$$

$$= \left( \log s \cdot (s) - \int \frac{1}{s} (s) ds \right) - \frac{1}{2} \left( \left( s \log(s^2+1) - \int \left( \frac{2s^2}{s^2+1} \right) ds \right) \right)$$

$$= -s + s \log s - \frac{s \log(s^2+1)}{2} + \int \left( \frac{s^2}{s^2+1} \right) ds$$

$$= -s + s \log s - \frac{s \log(s^2+1)}{2} + \int \left( 1 - \left( \frac{1}{s^2+1} \right) \right) ds$$

$$= -s + s \log s - \frac{s \log(s^2+1)}{2} + s + \cot^{-1}s$$

$$= s \log \left( \frac{s}{\sqrt{s^2+1}} \right) + \cot^{-1}s$$