(Please write your Enrolment No. immediately) Enrolment No...........................

Class Test

Semester: II June, 2023

PAPER CODE: BS:112 Subject: Applied Mathematics-II

Time: $1\frac{1}{2}$ Hrs Max. Marks: 30

Note: attempt Q. No. 1 which is compulsory and any two more from remaining.

Q.1.

(a) State Convolution theorem for inverse Laplace transform. 2 CO 3

(b) If $f(t)=\begin{cases}1 & 1<t<2\\ 3-t & 2<t<3\end{cases}$ Find Laplace transform by using Unit Step Function 2 CO 3

(c) Find $L^{-1}\dfrac{1}{2s(s-1)}$ 2 CO 3

(d) Classify the Partial Differential Equation $2\dfrac{\partial^2 u}{\partial x^2}+4\dfrac{\partial^2 u}{\partial x\partial y}+3\dfrac{\partial^2 u}{\partial y^2}=0$ 2 CO4

(e) Write the Partial Differential Equation of one-dimensional Wave equation . 2 CO4

Q.2. (a) Find the Fourier Series expansion of function $f(x)=x^2$, $-\pi<x<\pi$

And also prove that $\dfrac{\pi^2}{12}=\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+.........$ 5 CO3

(b) Find Laplace transform of $\dfrac{\cos at-\cos bt}{t}$ 5 CO3

Q.3.(a) Using the method of Separation of Variable , Solve $\dfrac{\partial u}{\partial x}=2\dfrac{\partial u}{\partial t}+u$ Where $u(x,0)=6e^{-3x}$. 5 CO4

(b) A string is stretched and fastened to two points l apart. Motion is started by displacing the string In the form $y=a\sin\dfrac{\pi x}{l}$ from which it is released at a time t=0. Show that the displacement of any Point at a distance x from one end at time t is give by $y(x,t)=a\sin(\dfrac{\pi x}{l})\cos(\dfrac{\pi ct}{l})$ 5 CO 4

Q.4.(a) Determine the solution of one-Dimensional heat equation

$\dfrac{\partial u}{\partial t}=c^2\dfrac{\partial^2 u}{\partial x^2}$, subject to the boundary condition $u(0,t)=0$, $u(l,t)=0$ $u(x,0)=l$

. l being the length of the bar . 5 CO 4

(b) Using Laplace transforms, find the solution of initial value problem $y''+9y'=6\cos 3t$

Given $y(0)=2$, $y'(0)=0$. 5 CO3

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Solution:

1(a) If $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$, then.

$$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(u)\cdot G(t-u)\,du.$$

(b)

$$F(t) = 1[U(t-1) - U(t-2)] + (3-t)[U(t-2) - U(t-3)]$$

$$= U(t-1) + U(t-2)[3-t-1] + (3-t)U(t-3)$$

$$= U(t-1) + U(t-2)[2-t] + (t-3)U(t-3)$$

$$= U(t-1) - (t-2)U(t-2) + (t-3)U(t-3)$$

$$\therefore L[F(t)] = L[U(t-1) - (t-2)U(t-2) + (t-3)U(t-3)]$$

$$= L[U(t-1)] - \frac{e^{-2p}}{p^2} + \frac{e^{-3p}}{p^2}$$

$$= \frac{e^{-ap}}{p} - \frac{e^{-2p}}{p^2} + \frac{e^{-3p}}{p^2}$$ Ans.

(c) $$L^{-1}\left[\frac{1}{2s(s-1)}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s-1} - \frac{1}{s}\right] = \frac{1}{2}(e^t - 1)$$

(d) Compare with $AU_{xx} + BU_{xy} + CU_{yy} + f(x, y, z, p, q) = 0$

$A = 2, \quad B = 4, \quad C = 3$

$\therefore B^2 - 4AC = 16 - 4\times2\times3 = -8 < 0$

$\Rightarrow$ Elliptic equation.

(e) one diamentional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

2 (a) $f(x) = x^2 \qquad -\pi < x < \pi$

Here $f(x)$ is an even function $\Rightarrow b_n = 0$

Fourier Series is.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad b_n = 0$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \int 2x\frac{\sin nx}{n}.dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{n}\left\{-x\frac{\cos nx}{n} + \frac{1}{n}\int \cos nx\,dx\right\}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n}\left[-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^2}\left[-\pi\cos n\pi\right]\right] = \frac{+4}{n^2}(-1)^n.$$

Fourier series is

$$F(x) = \frac{1}{2}\frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right]$$

putting $x=0$, we get

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

or $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

(b) $$L\left[\frac{\cos at - \cos bt}{t}\right]$$

Let $f(t) = \cos at - \cos bt$

$$L[f(t)] = L[\cos at - \cos bt)$$

$$= \frac{p}{p^2+a^2} - \frac{p}{p^2+b^2} = f(p)$$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \int_p^{\infty} f(x)\,dx$$

$$= \int_p^{\infty}\left[\frac{x}{x^2+a^2} - \frac{x}{x^2+b^2}\right]dx = \cancel{\frac{1}{2}\left[\log(x^2\right.}$$

$$= \frac{1}{2}\left[\log(x^2+a^2) - \log(x^2+b^2)\right]_p^{\infty}$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{p^2+a^2}{p^2+b^2}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{p^2+b^2}{p^2+a^2}\right). \quad \text{Ans.}$$

3(a) $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$, given $U(x,0) = 6e^{-3x}$

Let $U(x,t) = X(x)\cdot T(t)$ be the solution.

$\frac{\partial U}{\partial x} = T\frac{dX}{dx}$, $\frac{\partial U}{\partial t} = X\cdot\frac{dT}{dt}$

put these values in given eq. we get

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

or $T\left[\frac{dX}{dx} - X\right] = 2X\frac{dT}{dt}$ ~~or $\frac{1}{2}$~~

$$\frac{1}{2X}\left[\frac{dX}{dx} - X\right] = \frac{1}{T}\cdot\frac{dT}{dt} = K \text{ (say.)}$$

$\therefore$ $\frac{dX}{dx} - X = 2KX$ | $\frac{1}{T}\frac{dT}{dt} = K$

or $\frac{dX}{X} = (2K+1)\,dx$ | $\frac{dT}{T} = K\,dt$

$X = C_1\cdot e^{(2K+1)x}$ | $T = C_2\cdot e^{Kt}$

put these values in given equation, we get

$$U(x,t) = C_1\cdot C_2\, e^{(2K+1)x + Kt}$$

$$U(x,t) = C\, e^{(2K+1)x + Kt} \quad \because C_1C_2 = C$$

using initial condition, we get

$$U(x,0) = 6e^{-3x} = C\,e^{(2K+1)x}$$

$\Rightarrow$ $C = 6$, $2K+1 = -3$ $\Rightarrow$ $K = -2$

$$\therefore U(x,t) = 6\,e^{-(3x+2t)}$$

Ans.

3(b) wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —(1)

with $y(0,t) = 0 = u(l,t)$

$y(x,0) = a \sin \frac{\pi x}{l}$ at $t=0$

Solution of one dimentional wave equation is.

$$y(x,t) = \sum_{n=1}^{\infty} E_n \cdot \cos \frac{n\pi c t}{l} \cdot \sin \frac{\pi n x}{l} \quad —(2)$$

put $t=0$, we get

$$y(x,0) = \sum_{n=1}^{\infty} E_n \cdot \sin \frac{n\pi x}{l} \quad —(3)$$

where

$$E_n = \frac{2}{l} \int_0^l y(x,0) \cdot \sin \frac{n\pi x}{l} \quad —(4)$$

Solution is

$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

using $y(0,t) = 0 \Rightarrow c_1 = 0$

$y(x,t) = c_2 \sin px \,[c_3 \cos cpt + c_4 \sin cpt]$.

using $y(l,t) = 0 \Rightarrow p = \frac{n\pi}{l}$

$y(x,t) = c_2 \sin px \,[c_3 \cos cpt + c_4 \sin cpt]$

where $p = \frac{n\pi}{l}$

$\left(\frac{\partial y}{\partial t}\right) = c_2 \sin px \,\{ c_3 (-\sin cpt) cp + c_4 \cos cpt$

$c_2 c_3 = A$

$0 = \left(\frac{\partial y}{\partial t}\right)_{t=0} = c_2 c_4 \sin px \; cp$

$\frac{\pi x}{l} = \frac{n\pi x}{l} \Rightarrow n$

$y(x,t) = c_2 \sin px \; c_3 \cos cpt \rightarrow A \sin \frac{\pi x}{l} = c_2 c_3 \sin$

$= y(x,0) = c_2 \sin px \; c_3 \cos cpt \Rightarrow A \sin \frac{\pi x}{l} = c_2 c_3 \sin$

4.(a) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$

It's soln is given as

$u(x,t) = (C_1 \cos px + C_2 \sin px)\, C_3 e^{-c^2 p^2 t} \longrightarrow (2)$

And condition

$u(0,t) = 0 \longrightarrow (i)$

$u(l,t) = 0 \longrightarrow (ii)$

$u(x,0) = l \longrightarrow (iii)$

Using (i) condition i.e $u(0,t) = 0$

$0 = (C_1 \cos 0 + 0) \Rightarrow C_1 = 0$ putting in (2)

$u(x,t) = (C_2 \sin px)\, C_3 e^{-c^2 p^2 t} \longrightarrow (3)$

Now using (ii) condition $u(l,t) = 0$

$0 = (C_2 \sin pl)$ i.e $\sin pl = 0$

$C_2 \neq 0$ $\qquad pl = n\pi$

$p = \frac{n\pi}{l}$

Putting in (3)

$u(x,t) = C_2 \sin \frac{n\pi x}{l} \cdot C_3 e^{-c^2 \left(\frac{n^2\pi^2}{l^2}\right) t}$

$= C_2 C_3 \sin \frac{n\pi x}{l} \cdot e^{-c^2 \frac{n^2\pi^2}{l^2} t}$

$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2 n^2 \pi^2}{l^2} t} \longrightarrow (4)$

Using (4) condition i.e $u(x,0) = l$

$l = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^{-0}$

$l = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$

Now we find out by Half Range Sin Series

$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

$= \frac{2}{l} \int_0^l l \cdot \sin \frac{n\pi x}{l}\, dx$

$$= \frac{2}{l}\left[\left\{(l)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^l - \left\{(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^l + 0\right.$$

$$= \frac{2}{l}\left[-l\cdot\frac{l}{n\pi}\left\{\cos n\pi - \cos 0\right\} - 0\right.$$

$$B_n = -\frac{2l}{n\pi}\left\{\cos n\pi - 1\right\}$$

$$= -\frac{2l}{n\pi}\left\{-1-1\right\} \text{ if } n \text{ is odd}$$

$$B_n = \frac{4l}{n\pi} \text{ if } n \text{ is odd}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot e^{-\frac{c^2n^2\pi^2}{l^2}t}$$

when $B_n = \frac{4l}{n\pi}$ if n is odd.

4-(b) $y'' + 9y = 6\cos 3t$

Taking L.T both side, we get.

$$L(y'') + 9L(y) = 6L[\cos 3t]$$

$$p^2 L(y) - p\,y(0) - y'(0) + 9L(y) = \frac{6\cdot p}{p^2+3^2}$$

$$(p^2+9)L(y) - 2p = \frac{6p}{p^2+3^2}$$

$$L(y) = \frac{2p}{p^2+9} + \frac{6p}{(p^2+3^2)(p^2+9)}$$

$$y = 2L^{-1}\left[\frac{p}{p^2+3^2}\right] + 6L^{-1}\left[\frac{p}{(p^2+3^2)(p^2+3^2)}\right]$$

$$= 2\cos 3t + 6\left[-\frac{t\cos 3t}{6}\right] = 2\cos 3t - t\cos 3t.$$

For $L^{-1}\left[\frac{p}{p^2+3^2}\right] = \cos 3t = f(t)$ say.

$$L^{-1}\left[\frac{1}{p^2+3^2}\right] = \frac{1}{3}\cdot\sin 3t = g(t)$$

$$\therefore L^{-1}\left[\frac{p}{p^2+3^2}\cdot\frac{1}{p^2+3^2}\right] = \int_0^t \frac{1}{3}\cdot\cos 3u\cdot\sin 3(t-u)\cdot du$$

$$= \frac{1}{6}\int_0^t 2\sin 3(t-u)\cdot\cos 3u\, du$$

$$= \frac{1}{6}\int_0^t \sin[3t) + \sin(3t-6u)]\, du.$$

$$= \frac{1}{6}\cdot\left[-\frac{\cos 3t}{3}u + \frac{\cos(3t-6u)}{6}\right]_0^t$$

$$= \frac{1}{6}\left[-\frac{t\cos 3t}{3} + \frac{1}{3}\frac{\cos 3t}{6} - \frac{\cos 3t}{6}\right]$$

$$= -t\cos 3t/6$$