(Please write your Enrolment No. immediately) Enrolment No. __________

MID TERM EXAMINATION

B.TECH PROGRAMMES (UNDER THE AEGIS OF USICT)

FIRST Semester, January, 2023

Paper Code: BS-111 **Subject: Applied Mathematics**

Time: 1½Hrs. **Max. Marks: 30**

Note: Attempt Q.No.1 which is compulsory and any two more questions from remaining.

Q.No.	Question	Max. Marks	CO(s)
1 (a)	Find the derivative with respect to x of the integral $\int_{x}^{x^2} \frac{\sin xt}{t} dt$.	2.5	CO1
1 (b)	Determine the differential $\left[\frac{x}{(x^2+y^2)}\right]dy = \left[\frac{y}{(x^2+y^2)}\right]dx$ is exact or not.	2.5	CO2
1 (c)	Express $5x^3+x^2-2x+1$ in terms of Legendre polynomial.	2.5	CO2
1(d)	Find the orthogonal trajectories of family of curves $y=cx^2$, c being a parameter.	2.5	CO2
2 (a)	If $u=\sin^{-1}(x-y), x=3t$ and $y=4t^3$, find $\frac{du}{dt}$.	4	CO1
2(b)	If $x=\rho\cos\phi, y=\rho\sin\phi$, transform the equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ into one in ρ and ϕ.	6	CO1
3 (a)	The temperature T at any point (x,y,z) of space is given by $T=400xyz^2$. Find the highest temperature at the surface of the sphere $x^2+y^2+z^2=1$.	5	CO1
3 (b)	Prove that $J_{3/2}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)$	5	CO2
4 (a)	Solve by the method of variation of parameters: $\frac{d^2y}{dx^2}+a^2y=\sec ax$	5	CO2
4 (b)	Solve $(1+y^2)dx=(\tan^{-1}y-x)dy$.	5	CO2

Solution of Mid-Term Exam 2023

Applied Mathematics-I

First Semester

Solution 1(a) Let $F(x) = \int_{x}^{x^2} \frac{\sin xt}{t}\, dt$

By using Leibnitz's Rule, we get

$$F'(x) = \int_{x}^{x^2} \frac{\partial}{\partial x}\left(\frac{\sin xt}{t}\right) dt + \frac{d(x^2)}{dx}\,\frac{\sin x(x^2)}{x^2} - \frac{d(x)}{dx}\,\frac{\sin x(x)}{x}$$

$$= \int_{x}^{x^2} \frac{t\cos xt}{t}\, dt + 2x\,\frac{\sin x^3}{x^2} - \frac{\sin x^2}{x}$$

$$= \int_{x}^{x^2} \cos xt\, dt + \frac{2\sin x^3}{x} - \frac{\sin x^2}{x}$$

$$= \left.\frac{\sin xt}{x}\right|_{x}^{x^2} + \frac{2\sin x^3}{x} - \frac{\sin x^2}{x}$$

$$= \frac{\sin x^3}{x} - \frac{\sin x^2}{x} + \frac{2\sin x^3}{x} - \frac{\sin x^2}{x}$$

$$F'(x) = \frac{1}{x}\left[3\sin x^3 - 2\sin x^2\right]$$

Solution 1(b):- Compare the given eq^n with $M\,dx + N\,dy = 0$, we get

$$M = \frac{-y}{x^2+y^2} \quad \& \quad N = \frac{x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = -\left[\frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}\right] = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then given eq^n is exact

Solution 1(c):- As we know $P_0(x) = 1$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$\therefore\ x^2 = \frac{1}{3}(1+2P_2(x)) = \frac{1}{3}[P_0(x)+2P_2(x)]$$

$$x^3 = \frac{1}{5}[2P_3(x)+3P_1(x)]$$

$$5x^3+x^2-2x+1 = 5\times\frac{1}{5}(2P_3+3P_1)+\frac{1}{3}(P_0+2P_2) - 2P_1 + P_0$$

$$= 2P_3 + 3P_1 + \frac{P_0}{3} + \frac{2}{3}P_2 - 2P_1 + P_0$$

$$= 2P_3 + \frac{2}{3}P_2 + P_1 + \frac{4}{3}P_0$$

Solution 1(d):- The given family of curves is

$$y = c\cancel{x}^2 \quad \text{--- (1)}$$

$$\Rightarrow y - cx^2 = 0$$

which gives

$$\frac{dy}{dx} - 2cx = 0 \quad \text{--- (2)}$$

From (1) & (2), we get

$$\frac{dy}{dx} = \frac{2y}{x} \quad \text{--- } 3$$

This is the differential eq^n of (1). The differential eq^n of the family of orthogonal trajectories is obtained by replacing $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ in 3

Thus we obtain, $-\frac{dx}{dy} = \frac{2y}{x}$

$$\int 2y\,dy = \int -x\,dx$$

Integrating, we have

$$y^2 = -\frac{x^2}{2} + k$$

$$\Rightarrow \frac{x^2}{2} + y^2 = k$$

which is a family of ellipse with parameter k.

<u>Solution 2(a):-</u> $u = \sin^{-1}(x-y)$, $x = 3t$ & $y = 4t^3$

I^{st} Method, by chain rule, we get

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{3}{\sqrt{1-(x-y)^2}} + \frac{-12t^2}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

2^{nd} Method,

$$u = \sin^{-1}(x-y) = \sin^{-1}(3t-4t^3)$$

diff wrt t

$$\frac{du}{dt} = \frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

<u>Solution 2(b):-</u> Since $x = \rho\cos\phi$, $y = \rho\sin\phi$

we have $\rho^2 = x^2+y^2$ & $\phi = \tan^{-1}\frac{y}{x}$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho}\cdot\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi}\cdot\frac{\partial \phi}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \rho}\cdot\frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \phi}\cdot\frac{\partial \phi}{\partial y}$$

From $\rho^2 = x^2+y^2$

we have $2\rho \frac{\partial \rho}{\partial x} = 2x$ or $\frac{\partial \rho}{\partial x} = \frac{x}{\rho} = \cos\phi$ &

$$\frac{\partial \rho}{\partial y} = \frac{y}{\rho} = \sin\phi.$$

$\phi = \tan^{-1}\frac{y}{x}$ gives $\frac{\partial \phi}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-\sin\phi}{\rho}$

& $\frac{\partial \phi}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\phi}{\rho}$

Therefore $\frac{\partial f}{\partial x} = \cos\phi \frac{\partial f}{\partial \rho} - \frac{\sin\phi}{\rho} \cdot \frac{\partial f}{\partial \phi}$

& $\frac{\partial f}{\partial y} = \sin\phi \frac{\partial f}{\partial \rho} + \frac{\cos\phi}{\rho} \cdot \frac{\partial f}{\partial \phi}$

Thus, in terms of operators we can write

$$\frac{\partial}{\partial x} \equiv \cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho} \cdot \frac{\partial}{\partial \phi} \quad \& \quad \frac{\partial}{\partial y} \equiv \sin\phi \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho} \cdot \frac{\partial}{\partial \phi}$$

Therefore, $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left(\cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho} \frac{\partial}{\partial \phi}\right)\left(\cos\phi \frac{\partial f}{\partial \rho} - \frac{\sin\phi}{\rho} \frac{\partial f}{\partial \phi}\right)$

$$= \cos\phi \frac{\partial}{\partial \rho}\left(\cos\phi \frac{\partial f}{\partial \rho} - \frac{\sin\phi}{\rho} \cdot \frac{\partial f}{\partial \phi}\right) - \frac{\sin\phi}{\rho} \frac{\partial}{\partial \phi}\left(\cos\phi \frac{\partial f}{\partial \rho} - \frac{\sin\phi}{\rho} \frac{\partial f}{\partial \phi}\right)$$

$$= \cos\phi \left[\cos\phi \frac{\partial^2 f}{\partial \rho^2} + \frac{\sin\phi}{\rho^2} \cdot \frac{\partial f}{\partial \phi} - \frac{\sin\phi}{\rho} \frac{\partial^2 f}{\partial \rho \partial \phi}\right]$$

$$- \frac{\sin\phi}{\rho}\left[-\sin\phi \frac{\partial f}{\partial \rho} + \cos\phi \frac{\partial^2 f}{\partial \phi \partial \rho} - \frac{\cos\phi}{\rho} \cdot \frac{\partial f}{\partial \phi} - \frac{\sin\phi}{\rho} \frac{\partial^2 f}{\partial \phi^2}\right]$$

$$= \cos^2\phi \frac{\partial^2 f}{\partial \rho^2} - \frac{2\sin\phi\cos\phi}{\rho} \frac{\partial^2 f}{\partial \phi \partial \rho} + \frac{\sin^2\phi}{\rho^2} \frac{\partial^2 f}{\partial \phi^2}$$

$$+ \frac{2\sin\phi\cos\phi}{\rho^2} \frac{\partial f}{\partial \phi} + \frac{\sin^2\phi}{\rho} \cdot \frac{\partial f}{\partial \rho}$$

Similarly, $\frac{\partial^2 f}{\partial y^2} = \sin^2\phi \frac{\partial^2 f}{\partial \rho^2} + \frac{2\sin\phi\cos\phi}{\rho} \frac{\partial^2 f}{\partial \rho \partial \phi} + \frac{\cos^2\phi}{\rho^2} \frac{\partial^2 f}{\partial \phi^2}$

$$- \frac{2\sin\phi\cos\phi}{\rho^2} \frac{\partial f}{\partial \phi} + \frac{\cos^2\phi}{\rho} \frac{\partial f}{\partial \rho}$$

Therefore, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \boxed{\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} = 0}$

Solution 3(a):– $T = 400\,xyz^2$ where $x^2+y^2+z^2=1$.

Let $T' = xyz^2 = xy(1-x^2-y^2)$

hence $\dfrac{\partial T'}{\partial x} = y(1-x^2-y^2) + xy(-2x) = -y(3x^2+y^2-1)$

& $\dfrac{\partial T'}{\partial y} = x(1-x^2-y^2) + xy(-2y) = -x(3y^2+x^2-1)$

For T' to be maximum or minimum

$$\frac{\partial T'}{\partial x} = \frac{\partial T'}{\partial y} = 0$$

$$y(3x^2+y^2-1)=0 \ \& \ x(3y^2+x^2-1)=0$$

The above relations when solved for x & y, gives $x=0, y=0$ & $x^2=\frac{1}{4}, y^2=\frac{1}{4}$.

Thus, the critical points are $(0,0), (\frac{1}{2},-\frac{1}{2}), (-\frac{1}{2},\frac{1}{2}), (-\frac{1}{2},-\frac{1}{2})$.

Next, $r = \dfrac{\partial^2 T'}{\partial x^2} = -6xy$

$t = \dfrac{\partial^2 T'}{\partial y^2} = -6xy$

$s = \dfrac{\partial^2 T'}{\partial x \partial y} = -3x^2-3y^2+1$

At $(0,0)$, $r = t = 0$ & $s = 1$ so $rt < s^2$. Hence $(0,0)$ is a saddle point.

At $(\frac{1}{2},\frac{1}{2})$ & $(-\frac{1}{2},-\frac{1}{2})$, $r = t = -\frac{3}{4}$ & $s = -\frac{1}{2}$

$\therefore\ rt - s^2 = \frac{5}{16} > 0$ & $r < 0$

& at $(\frac{1}{2},-\frac{1}{2})$ & $(-\frac{1}{2},\frac{1}{2})$, $r = t = \frac{3}{4}$ & $s = -\frac{1}{2}$

$\therefore\ rt - s^2 = \frac{5}{16} > 0$ & $r > 0$

$\therefore (\frac{1}{2},\frac{1}{2})$ & $(-\frac{1}{2},-\frac{1}{2})$ are the point of maxima & $(\frac{1}{2},-\frac{1}{2})$ & $(-\frac{1}{2},\frac{1}{2})$ are the point of minima. 5

Thus, maximum $T = 400\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\left(1-\frac{1}{4}-\frac{1}{4}\right) = 50$

Solution 3(b) :- Find $J_{\frac{1}{2}}(x)$ & $J_{-\frac{1}{2}}(x)$

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$$

$$= \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^{n+2}\frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^{n+4}\frac{1}{2\,\Gamma(n+3)} - \cdots$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^{2}\frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^{4}\frac{1}{2\,\Gamma(n+3)} - \cdots\right]$$

Put $n = \frac{1}{2}$

$$J_{\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\Gamma\frac{3}{2}} - \left(\frac{x}{2}\right)^2\frac{1}{\Gamma\frac{5}{2}} + \left(\frac{x}{2}\right)^4\frac{1}{2\,\Gamma\frac{7}{2}} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}}\left[\frac{1}{\frac{1}{2}\sqrt{\pi}} - \frac{x^2}{2^2}\cdot\frac{1}{1\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} + \frac{x^4}{2^4}\,\frac{1}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\sqrt{\pi}}\left[2 - \frac{x^2}{3} + \frac{x^4}{5\cdot4\cdot3\cdot1} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\sqrt{\pi}}\left[\frac{2}{1} - \frac{2x^2}{3\cdot2\cdot1} + \frac{2x^4}{5\cdot4\cdot3\cdot2\cdot1} - \cdots\right]\times\frac{x}{x}$$

$$= \frac{\sqrt{x}}{\sqrt{2}\sqrt{\pi}}\cdot\frac{2}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x \qquad \text{— (1)}$$

Similarly, for $n = -\frac{1}{2}$, we get

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\cos x \qquad \text{— (2)}$$

We have

$$J_{n+1}(x) = \frac{2n}{x}\,J_n(x) - J_{n-1}(x)$$

Put $n = \frac{1}{2}$

$$J_{\frac{3}{2}}(x) = \frac{1}{x}\,J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

H.P.

Solution 4(a):- $\frac{d^2y}{dx^2} + a^2y = \sec ax$ — ①

Eqn ① can be written as

$$(D^2+a^2)y = \sec ax \qquad D \equiv \frac{d}{dx}$$

A. E. is $\quad m^2+a^2=0$

$$m = \pm ai$$

$\therefore$ C.F. is $\quad y = C_1 \cos ax + C_2 \sin ax$

Here $y_1 = \cos ax$, $y_2 = \sin ax$ & $X = \sec ax$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0$$

$$P.I. = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -\cos ax \int \frac{\sin ax \cdot \sec ax}{a} dx + \sin ax \int \frac{\cos ax \cdot \sec ax}{a} dx$$

$$= \frac{-\cos ax}{a} \int \tan ax \, dx + \frac{\sin ax}{a} \int dx$$

$$= \frac{\cos ax}{a^2} \log \cos ax + \frac{x \sin ax}{a}$$

Hence the complete solution is

$$y = C.F. + P.I.$$

$$y = \cos ax + \sin ax + \frac{\cos ax}{a^2} \log \cos ax + \frac{x \sin ax}{a}$$

Solution 4(b):- $(1+y^2)dx = (\tan^{-1}y - x)dy$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \quad \text{— (1)}$$

Eqⁿ (1) is a linear differential eqⁿ, where

$$P = \frac{1}{1+y^2} \quad \& \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

$$I.F. = e^{\int P dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$$

The solution of eqⁿ (1) is

$$x(I.F.) = \int (I.F.) Q \, dy + C$$

$$x e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \frac{\tan^{-1}y}{1+y^2} dy + C$$

Let $\tan^{-1}y = t$

$$dt = \frac{dy}{1+y^2}$$

$$x e^{\tan^{-1}y} = \int e^t \cdot t \, dt + C$$

$$x e^{\tan^{-1}y} = e^t(t-1) + C$$

$$x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C$$