SOLVED EXAMPLES

Example 3.15. Three urns contain 6 red, 4 black; 4 red, 6 black, and 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the first urn.

Solution. Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = urn first is chosen, E_2 = urn second is chosen,

E_3 = urn third is chosen, and A = ball drawn is red.

Since there are three urns and one of the three urns is chosen at random, therefore

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}.$$

If E_1 has already occurred, then urn first has been chosen which contains 6 red and 4 black balls. The probability of drawing a red ball from it is 6/10.

So, $$P(A \mid E_1) = \frac{6}{10}.$$

Similarly $P(A \mid E_2) = \frac{4}{10}$ and $P(A \mid E_3) = \frac{5}{10}$

We are required to find $P(E_1 \mid A)$. *i.e.*, given that the ball drawn is red, what is the probability that it is drawn from the first urn.

By Baye's theorem, we have

$$P(E_1 \mid A) = \frac{P(E_1) \,.\, P(A \mid E_1)}{P(E_1) \,.\, P(A \mid E_1) + P(E_2) \,.\, P(A \mid E_2) + P(E_3) \,.\, P(A \mid E_3)}$$

$$= \frac{\frac{1}{3} \times \frac{6}{10}}{\frac{1}{3} \times \frac{6}{10} + \frac{1}{3} \times \frac{4}{10} + \frac{1}{3} \times \frac{5}{10}} = \frac{6}{15} = \frac{2}{5}.$$

Example 3.16. Urn A contains 2 white, 1 black and 3 red balls, urn B contains 3 white, 2 black and 4 red balls and urn C contains 4 white, 3 black and 2 red balls. One urn is chosen at random and 2 balls are drawn at random from the urn. If the chosen balls happen to be red and black, what is the probability that both balls come from urn B?

Solution. Let E_1, E_2, E_3 and A denote the following events.

E_1 = urn A is chosen, E_2 = urn B is chosen, E_3 = urn C is chosen, and A = two balls drawn at random are red and black. Since one of the urns is chosen at random, therefore

$$P(E_1) = \frac{1}{3} = P(E_2) = P(E_3)$$

If E_1 has already occurred, then urn A has been chosen. The urn A contains 2 white, 1 black and 3 red balls. Therefore the probability of drawing a red and a black ball is $\frac{{}^3C_1 \times {}^1C_1}{{}^6C_2}$.

So, $$P(A \mid E_1) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{3}{15} = \frac{1}{5}$$

Similarly, $$P(A \mid E_2) = \frac{{}^4C_1 \times {}^2C_1}{{}^9C_2} = \frac{2}{9}$$

and $$P(A \mid E_3) = \frac{{}^2C_1 \times {}^3C_1}{{}^9C_2} = \frac{1}{6}$$

We are required to find $P(E_2 \mid A)$. By Baye's theorem, we have

$$(E_2 \mid A) = \frac{P(E_2)\, P(A \mid E_2)}{P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2) + P(E_3)\, P(A \mid E_3)}$$

$$= \frac{\frac{1}{3} \times \frac{2}{9}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{2}{9} + \frac{1}{3} \times \frac{1}{6}} = \frac{\frac{2}{9}}{\frac{1}{5} + \frac{2}{9} + \frac{1}{6}} = \frac{20}{53}$$

Example 3.17. A factory has three machines, X, Y and Z, producing 1000, 2000 and 3000 bolts per day respectively. The machine X produces 1% defective bolts, Y produces 1.5% and Z produces 2% defective bolts. At the end of the day, a bolt is drawn at random and it is found to be defective. What is the probability that this defective bolt has been produced by the machine X?

Solution. Total number of bolts produced in a day

$$= (1000 + 200 + 3000) = 6000$$

Let E_1, E_2 and E_3 be the events of drawing a bolt produced by machine X, Y and Z respectively. Then,

$$P(E_1) = \frac{1000}{6000} = \frac{1}{6};\ P(E_2) = \frac{2000}{6000} = \frac{1}{3} \text{ and } P(E_3) = \frac{3000}{6000} = \frac{1}{2}$$

Let A be the event of drawing a defective bolt. Then,

$P(A / E_1)$ = Probability of drawing a defective bolt, given that it is produced by the machine X

$$= \frac{1}{100}$$

$P(A \mid E_2)$ = Probability of drawing a defective bolt, given that it is produced by the machine Y

$$= \frac{1.5}{100} = \frac{15}{1000} = \frac{3}{200}$$

$P(A / E_3)$ = Probability of drawing a defective bolt, given that it is produced by the machine Z

$$= \frac{2}{100} = \frac{1}{50}$$

Required probability = $P(E_1 \mid A)$

= Probability that the bolt drawn is produced by X, given that it is defective

$$= \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)+P(E_3).P(A/E_3)}$$

$$= \frac{\left(\frac{1}{6}\times\frac{1}{100}\right)}{\left(\frac{1}{6}\times\frac{1}{100}\right)+\left(\frac{1}{3}\times\frac{3}{200}\right)+\left(\frac{1}{2}\times\frac{1}{50}\right)}$$

$$= \left(\frac{1}{600}\times\frac{600}{10}\right) = \frac{1}{10} = 0.1$$

Hence, the required probability is 0.1.

Example 3.18. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter driver, car driver and truck is 0.01, 0.03 and 0.15 respectively. One of the insured person meets with an accident. What is the probability that he is a scooter driver?

Solution. Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = person chosen is a scooter driver,

E_2 = person chosen is a car driver,

E_3 = person chosen is a truck driver, and

A = person meets with an accident.

Since there are 12000 persons, therefore

$$P(E_1) = \frac{2000}{12000} = \frac{1}{6}$$

and

$$P(E_2) = \frac{4000}{12000} = \frac{1}{3}$$

and

$$P(E_3) = \frac{6000}{12000} = \frac{1}{2}$$

It is given that

$P(A/E_1)$ = Probability that a person meets with an accident given that he is a scooter driver = 0.01.

Similarly, $P(A/E_2) = 0.03$ and $P(A/E_3) = 0.15$

We are required to find $P(E_1/A)$, *i.e.*, given that the person meets with an accident, what is the probability that he was a scooter driver.

By Baye's rule, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)+P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{6}\times 0.01}{\frac{1}{6}\times 0.01+\frac{1}{3}\times 0.03+\frac{1}{2}\times 0.15}$$

$$= \frac{1}{1+6+45} = \frac{1}{52}.$$

Example 3.19. A company has two plants to manufacture scooters. Plant I manufactures 70% of the scooters and Plant II manufactures 30%. At Plant I, 80% of the scooters are rated as of standard quality and at Plant II, 90% of the scooters are rated as of standard quality. A scooter is chosen at random and is found to be of standard quality. What is the probability that it has come from Plant II?

Solution. Let E_1, E_2 and A be the following events.

E_1 = Plant I is chosen, E_2 = Plant II is chosen, and A = Scooter is of standard quality

Then,
$$P(E_1) = \frac{70}{100},\ P(E_2) = \frac{30}{100},$$

$$P(A/E_1) = \frac{80}{100} \text{ and } P(A/E_2) = \frac{90}{100}$$

We are required to find $P(E_2/A)$. By Baye's theorem, we have

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{30}{100}\times\frac{90}{100}}{\frac{70}{100}\times\frac{80}{100}+\frac{30}{100}\times\frac{90}{100}} = \frac{27}{56+27} = \frac{27}{83}$$

Example 3.20. In a test, an examine either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct, given that he copies it, is 1/8. Find the probability that he knew the answer to the question, given that he correctly answered it.

Solution. Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = the examinee guesses the answer, E_2 = the examinee copies the answer, E_3 = the examinee knows the answer, and A = the examinee answers correctly.

We have $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{6}$. Since E_1, E_2, E_3 are mutually exclusive and exhaustive events, therefore

$$\Rightarrow \quad P(E_1) + P(E_2) + P(E_3) = 1$$

$$P(E_3) = 1 - (P(E_1) + P(E_2))$$

$$= 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}.$$

If E_1 has already occurred, then the examinee guesses. Since there are four choices out of which one is correct, therefore, the probability that he answers correctly given that he has made a guess is $\frac{1}{4}$, i.e., $P(A/E_1) = 1/4$. It is given that $P(A/E_2) = 1/8$, and

$P(A/E_3) =$ Probability that he answers correctly given that he knew the answer

$= 1.$

By Baye's theorem, we have

$$\text{Required probability} = P(E_3/A)$$

$$= \frac{P(E_3)\,P(A/E_3)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1} = \frac{24}{29}.$$

Example 3.21. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probability that he will be late are $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

Solution. Let E_1, E_2, E_3, E_4 be the events that the doctor comes by train, bus, scooter and other means of transport respectively. Then,

$$P(E_1) = \frac{3}{10},\ P(E_2) = \frac{1}{5},\ P(E_3) = \frac{1}{10} \text{ and } P(E_4) = \frac{2}{5}$$

Let A be the event that the doctor visits the patient late. Then,

$P(A/E_1) =$ Probability that the doctor will be late if he comes by train

$= \frac{1}{4}$

$P(A/E_2) =$ Probability that the doctor will be late if he comes by bus

$= \frac{1}{3}$

$P(A/E_3) =$ Probability that the doctor will be late if he comes by scooter

$= \frac{1}{12}$

$P(A/E_4) =$ Probability that the doctor will be late if he comes by other means of transport

$= 0$

We have to find $P(E_1/A)$

By Baye's theorem, we have (Giv

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3) + P(E_4)\,P(A/E_4)}$$

$$= \frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0} = \frac{3}{40}\times\frac{120}{18} = \frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$.

Example 3.22. By examining the chest X-ray, the probability that a person is diagnosed with TB when he is actually suffering from it, is 0.99. The probability that the doctor incorrectly diagnoses a person to be having TB, on the basis of X-ray reports, is 0.001. In a certain city, 1 in 1000 persons suffers from TB. A person is selected at random and is diagnosed to have TB. What is the chance that he actually has TB?

Solution. Let

$E_1 =$ event that the person selected in suffering from TB,

$E_2 =$ event that the person selected is not suffering from TB,

$A =$ event that the doctor diagnoses TB.

Then,

$$P(E_1) = \frac{1}{1000} \text{ and } P(E_2) = \left(1 - \frac{1}{1000}\right) = \frac{999}{1000}$$

$P(A/E_1) =$ probability that TB is diagnosed, when the person actually has TB

$= \frac{99}{100}$

$P(A/E_2) =$ probability that TB is diagnosed, when the person has no TB

$= \frac{1}{1000}$

Using Bayes' theorem, we have

$P(E_1/A) =$ probability of a person actually having TB, if it is known that he is diagnosed to have TB

$$= \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2)}$$

$$= \frac{\frac{1}{1000}\times\frac{99}{100}}{\frac{1}{1000}\times\frac{99}{100}+\frac{1}{1000}\times\frac{99}{100}} = \frac{110}{221}.$$

Hence, the required probability is $\frac{110}{221}$.

Example 3.23. **Suppose the 5% of men and 0.25% of women have a grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.**

Solution. Let us define the following events:

E_1 : a male is chosen

E_2 : a female is chosen

A : a grey haired person is chosen

Then,
$$P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}$$

$P(A/E_1)$ = Probability that a grey haired person is chosen, when it is known that a person is a male

$$= 5\%$$
$$= \frac{5}{100} = 0.05$$

$P(A/E_2)$ = Probability that a grey haired person is chosen, when it is known that a person is a female

$$= 0.25\%$$
$$= \frac{0.25}{100} = 0.0025$$

Hence, by Baye's theorem, we have

$P(E_1/A)$ = Probability that the person is a male when it is known that the person chosen is a grey haired

$$= \frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times 0.05}{\frac{1}{2}\times 0.05 + \frac{1}{2}\times 0.0025} = \frac{0.05}{0.05 + 0.0025}$$

$$= \frac{0.05}{0.0525} = \frac{5\times 100}{525} = \frac{20}{21}.$$

Example 3.24. **If a machine is correctly set up, it produces 90% acceptable items. If it is incorrectly set up, it produces only 40% acceptable items. Past experience shows that 80% of the set ups are correctly done. If after a certain set up, the machine produces 2 acceptable items, find the probability that the machine is correctly set up.**

Solution. Let us define the events as:

E_1 : the machine set up is correct

E_2 : the machine set up is incorrect

A : the machine produces 2 acceptable items

Then,

$P(E_1)$ = Probability that the machine set up is correct

$$= 80\% = \frac{80}{100} = 0.8$$

$P(E_2)$ = Probability that the machine set up is incorrect

$$= 20\% = \frac{20}{100} = 0.2$$

$P(A/E_1)$ = Probability that the machine produces 2 acceptable items given that the machine set up is correct

$$= \frac{90}{100}\times\frac{90}{100} = 0.81$$

$P(A/E_2)$ = Probability that the machine produces 2 acceptable items given that the machine set up is incorrect

$$= \frac{40}{100}\times\frac{40}{100} = 0.16$$

Then by Baye's theorem,

$P(E_1/A)$ = Probability that the machine is correctly set up given that the machine produces 2 acceptable items

$$= \frac{P(B_1)\,P(A/B_1)}{P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2)}$$

$$= \frac{0.8\times 0.81}{0.8\times 0.81 + 0.2\times 0.16}$$

$$= \frac{0.648}{0.648 + 0.032} = \frac{0.648}{0.680} = \frac{648}{680} = \frac{81}{85}$$

Example 1.25. **Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?**

Solution. Let us define the following events:

E_1 : Getting 5 or 6 in a single throw of a die

E_2 : Getting 1, 2, 3 or 4 in a single throw of a die

A : Getting exactly one head

Then,
$$P(E_1) = \frac{2}{6} = \frac{1}{3}, P(E_2) = \frac{4}{6} = \frac{2}{3}$$

$P(A/E_1)$ = Probability of getting exactly one head given that a coin is tossed three times

$$= {}^3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{3-1} = {}^3C_1\times\left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$P(A/E_2)$ = Probability of getting exactly one head given that a coin tossed once (whether a head or tail is obtained)

$$= \frac{1}{2}$$

Hence, by Baye's theorem,

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{2}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{3}\left(\frac{3}{8}+1\right)}$$

$$= \frac{8}{3+8} = \frac{8}{11}.$$

Example 3.27. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Solution. Let E_1, E_2 and A be the events defined as follows:

E_1 = six occurs, E_2 = six does not occur, and A = the man reports that it is a six

We have,

$$P(E_1) = \frac{1}{6},\ P(E_2) = \frac{5}{6}$$

Now,

$P(A/E_1)$ = Probability that the man reports that there is a six on the die given that six has occurred on the die

= Probability that the man speaks truth = $\frac{3}{4}$

and

$P(A/E_2)$ = Probability that the man reports that there is a six on the die given that six has not occurred on the die

= Probability that the man does not speak truth = $1-\frac{3}{4}=\frac{1}{4}$

We have to find $P(E_1/A)$ *i.e.*, the probability that there is six on the die given that the man reported that there is six. By Baye's theorem, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}} = \frac{3}{8}$$

Example 3.28. A letter is known to have come either from TATANAGAR or CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter has come from: (*i*) Calcutta, (*ii*) Tatanagar?

Solution. Let E_1 be the event that the letter came from Calcutta and E_2 be the event that the letter came from Tatanagar. Let A denote the event that two consecutive letters visible on the envelope are TA.

Since the letters have come either from Calcutta or Tatanagar, therefore,

$$P(E_1) = \frac{1}{2} = P(E_2)$$

If E_1, has occurred, then it means that the letter came from Calcutta. In the word CALCUTTA there are 8 letters in which TA occurs in the end. Considering TA as one letter there are seven letters out of which one can be in 7 ways. Therefore,

$$P(A/E_1) = \frac{1}{7}$$

If E_2 has occurred, then the letter came from Tatanagar. In the word TATANAGAR there are 9 letters in which TA occurs twice. Considering one of the two TA's as one letter there are 8 letters. Therefore,

$$P(A/E_2) = \frac{2}{8}$$

By Baye's Theorem, we have

(*i*)
$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{1}{7}}{\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{2}{8}} = \frac{4}{11}$$

(*ii*)
$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{2}{8}}{\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{2}{8}} = \frac{7}{11}$$

Example 3.29. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart.

Solution. Let E_1, E_2, E_3, E_4 and A be the events as defined below:

E_1 = the missing card is a heart card,

E_2 = the missing card is a spade card,

E_3 = the missing card is a club card,

E_4 = the missing card is a diamond card, and

A = Drawing two heart cards from the remaining cards.

Then,
$$P(E_1) = \frac{13}{52} = \frac{1}{4},$$
$$P(E_2) = \frac{13}{52} = \frac{1}{4},$$
$$P(E_3) = \frac{13}{52} = \frac{1}{4},$$
$$P(E_4) = \frac{13}{52} = \frac{1}{4},$$

$P(A / E_1) =$ Probability of drawing two heart cards given that one heart card is missing

$$= \frac{^{12}C_2}{^{51}C_2}$$

$P(A / E_2) =$ Probability of drawing two heart cards given that one spade card is missing

$$= \frac{^{12}C_2}{^{51}C_2}$$

Similarly, $P(A / E_3) = \dfrac{^{13}C_2}{^{51}C_2}$ and $P(A / E_4) = \dfrac{^{13}C_2}{^{51}C_2}$

By Baye's Theorem, we have

Required probability $= P(E_1 / A)$

$$= \frac{P(E_1)\, P(A / E_1)}{P(E_1)\, P(A / E_1) + P(E_2)\, P(A / E_2) + P(E_3)\, P(A / E_3) + P(E_4)\, P(A / E_4)}$$

$$= \frac{\frac{1}{4} \cdot \frac{^{12}C_2}{^{51}C_2}}{\frac{1}{4} \cdot \frac{^{12}C_2}{^{51}C_2} + \frac{1}{4} \cdot \frac{^{13}C_2}{^{51}C_2} + \frac{1}{4} \cdot \frac{^{13}C_2}{^{51}C_2} + \frac{1}{4} \cdot \frac{^{13}C_2}{^{51}C_2}}$$

$$= \frac{^{12}C_2}{^{12}C_2 + {^{13}C_2} + {^{13}C_2} + {^{13}C_2}} = \frac{66}{66 + 78 + 78 + 78} = \frac{11}{50}.$$