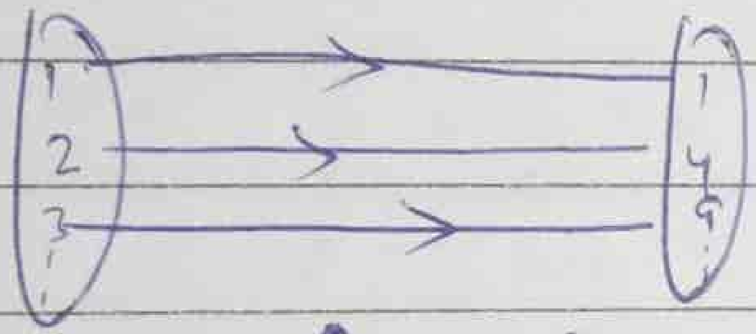
Mapping.

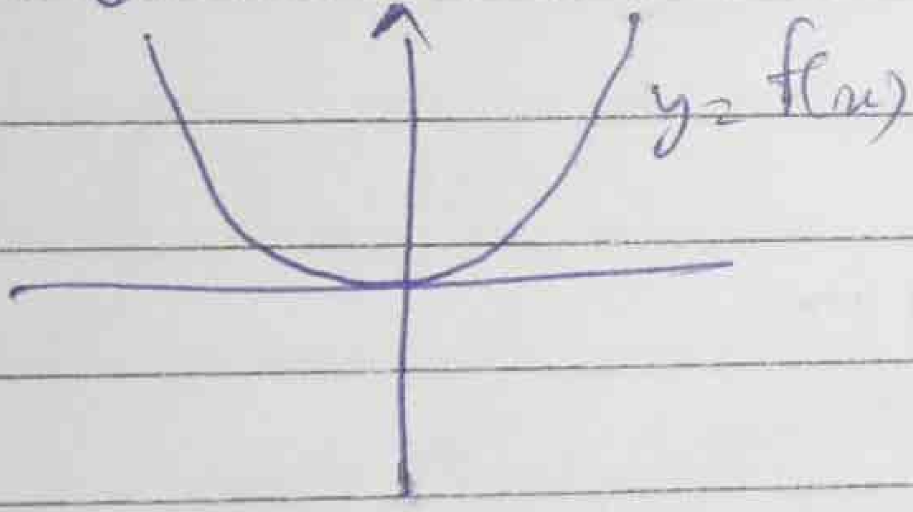
$f(x) = 2x + 3$

$f(x) = x^2$

$$\overbrace{f(x)=x^2}^{\text{fn is Mapping.}}$$

$$A \xrightarrow{f(x)=x^2} B$$





$z = 1+2i$

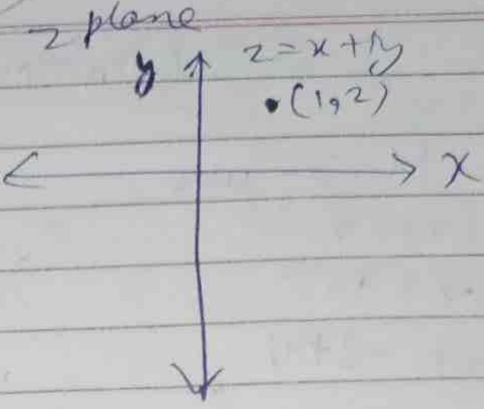
Date...............

$w = f(z) = z^2$

$f(z) = (x+iy)^2$

$f(z) = x^2 + i^2y^2 + 2xy\,i$

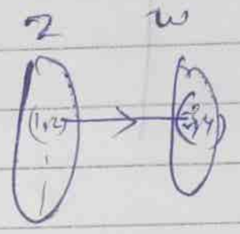
$f(z) = (x^2 - y^2) + i(2xy)$



$f(z) = u+iv$

$$\boxed{w = f(z) = u(x,y) + iv(x,y)}$$

Eg→ $z = 1 + 2i$

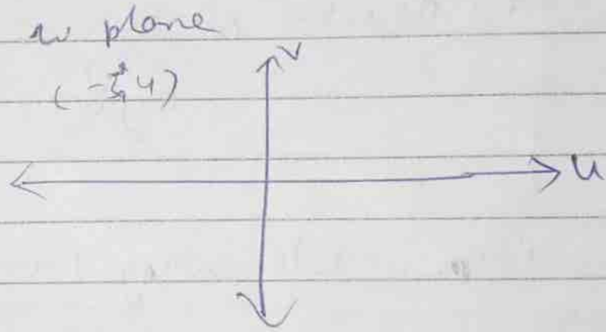


$w = (1+2i)^2$

$\Rightarrow 1 + (4i)^2 + 2(1)(2i)$

$\Rightarrow 1 - 4 + 4i$

$\Rightarrow -3 + 4i$



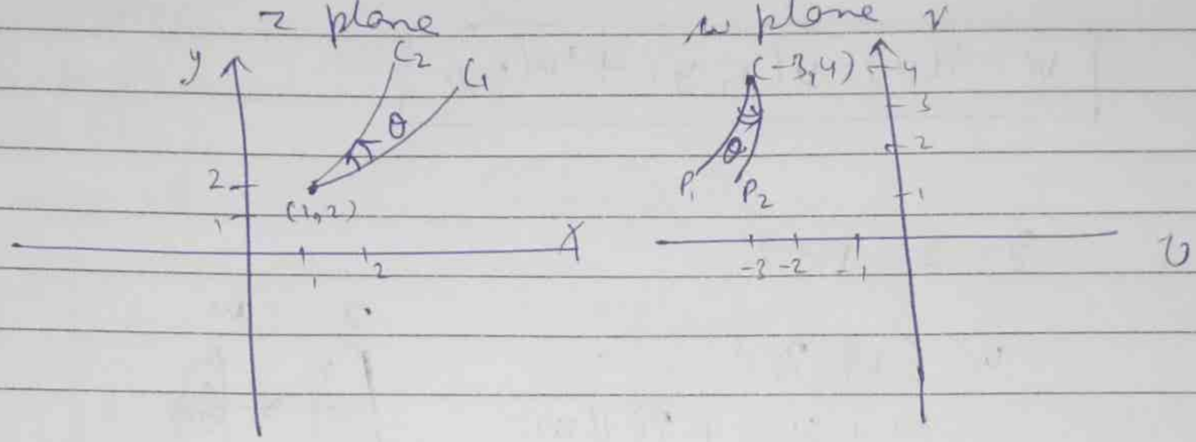
Conformal Mapping

$w = f(z) = u+iv$

$f(z) = z^2$

$z = 1+i2$

$f(z) = -3+4i$



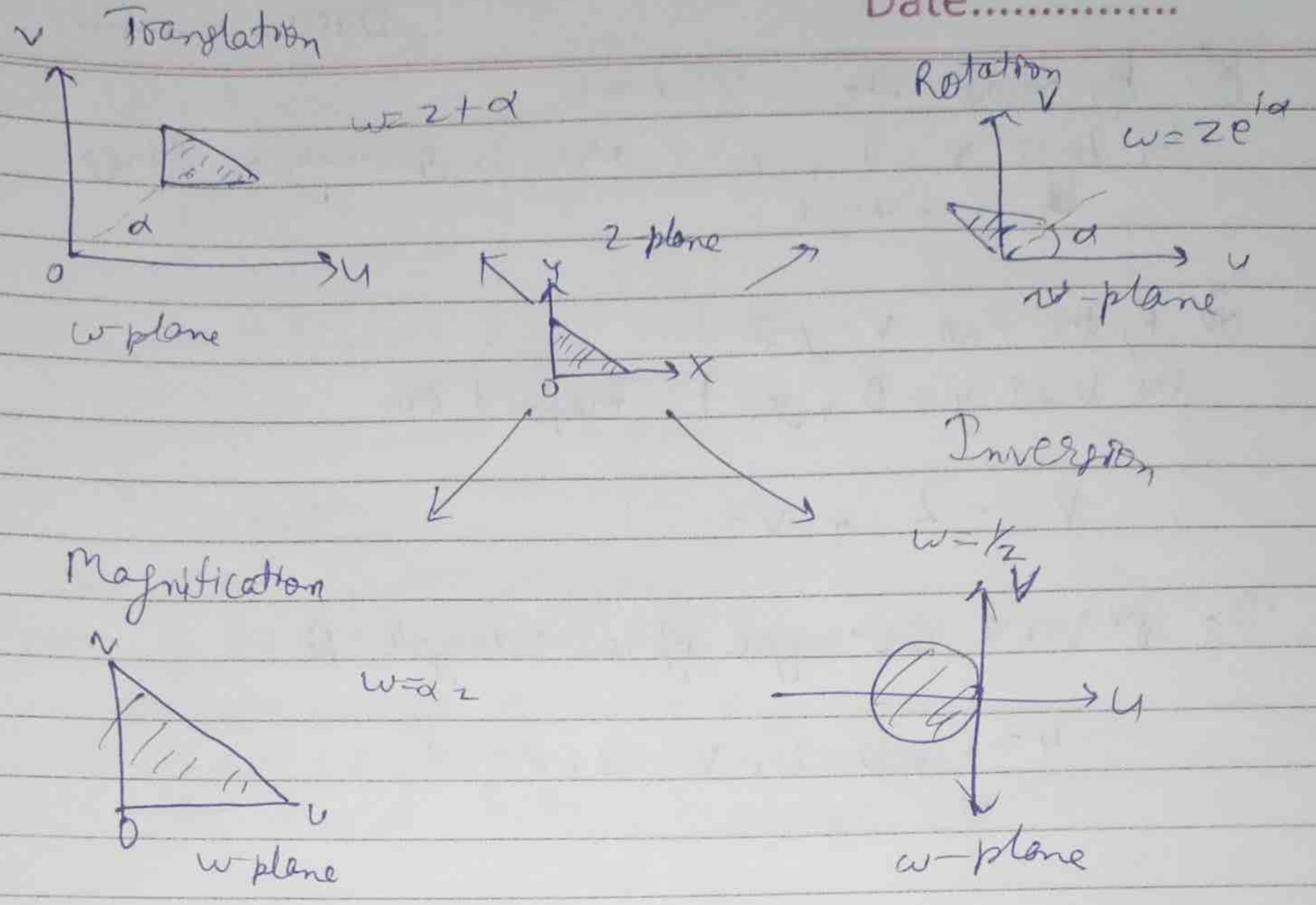
If angle b/w two curve in any mapping is same & if direction of curve is same. $(C_1 \to C_2)$. & $(P_1 \to P_2)$
(along angle) Anticlockwise direction

• Topic: Elementary Transformation

Elementary Transformation:

1. Translation → $w = z + \alpha$ where α is complex number
2. Rotation → $w = ze^{i\alpha}$ where α is real
3. Magnification → $w = \alpha z$ where α is complex numbers
4. Inversion → $w = 1/z$

Date...............



→ <u>Translation</u>

$w = z + \alpha$; where α is complex no.

Note: By this transformation, every point in z-plane is translated in direction of α.

Q: Let a rectangular domain R be bounded by $x=0$, $y=0$, $x=2$, $y=1$. Determine region R' of wplane into which R is mapped under transformation $w = z + (1-2i)$.

Sol: Given, $w = z + (1-2i)$

$u + iv = (x + iy) + (1 - 2i)$

$\Rightarrow u + iv = (x+1) + i(y-2)$

$\boxed{u = x+1}$ $\boxed{v = y-2}$

(*) By map the $u = x + 1$
the lines $x = 0$, $x = 2$ are respectively on lines
$u = 1$, $u = 3$

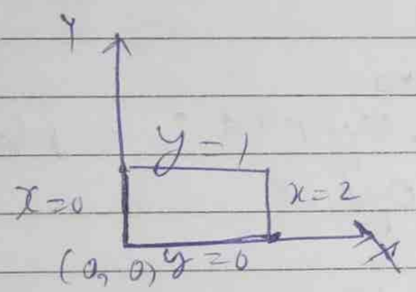
(*) By the map $v = y - 2$
the lines $y = 0$, $y = 1$ mapped on

$v = -2$, $v = -1$

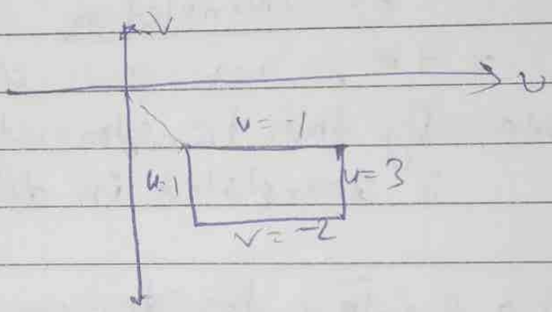
The required image is rectangle

$u = 1$, $u = 3$, $v = -2$, $v = -1$

z - plane



w plane



Q: Find image of $2x + y - 3 = 0$ under transformation

$w = z + 2i$. (AKTU)

Sol → Given: $w = z + 2i$

$\Rightarrow u + iv = (x + iy) + 2i$

$\Rightarrow u + iv = x + i(y + 2)$

$\Rightarrow u = x$ & $v = y + 2$

$\Rightarrow x = u$ & $y = v - 2$ — (1)

Date...............

Using these in

$\Rightarrow 2x + y - 3 = 0$

$\Rightarrow 2u + (v-2) - 3 = 0$

$\Rightarrow \boxed{2u + v = 5}$

which is required image of line $\boxed{2x + y - 3 = 0}$

• Rotation

$w = z e^{i\alpha}$, where α is real

Note: By this transformation, figure is z - plane rotates through an angle α in w plane.

i) If $\alpha > 0$; the rotation is anticlockwise

ii) If $\alpha < 0$; the rotation is clockwise

Q. Consider transformation $w = z e^{i\pi/4}$ & determine the region R' in w plane corresponding to triangular region R bounded by lines $x = 0, y = 0$ & $x + y = 1$ in z plane.

Sol→ Given: $w = z e^{i\pi/4}$

$\Rightarrow f(z) = (x+iy)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\Rightarrow u + iv = (x+iy)\left(\frac{1+i}{\sqrt{2}}\right)$

$\Rightarrow u + iv = \frac{1}{\sqrt{2}}(x-y) + \frac{i}{\sqrt{2}}(x+y)$ — 1

$\therefore u = \frac{1}{\sqrt{2}}(x-y)$; $v = \frac{1}{\sqrt{2}}(x+y)$ — 2

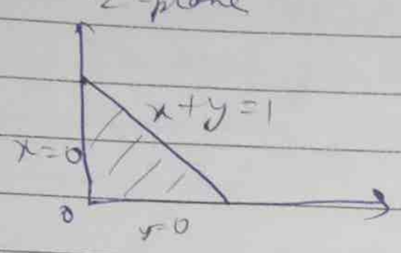
Put $x = 0$; we get $u = -\frac{y}{\sqrt{2}}$, $v = \frac{y}{\sqrt{2}}$ $\Rightarrow \boxed{v = -u}$

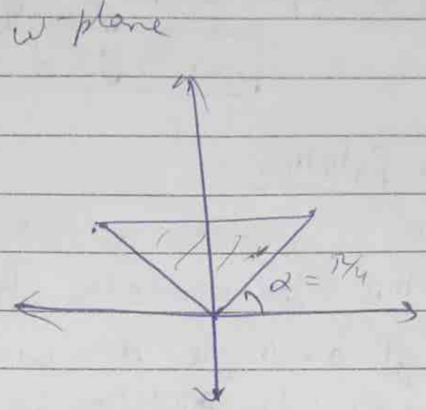
Put $y = 0$; we get $u = \frac{x}{\sqrt{2}}$, $v = \frac{x}{\sqrt{2}}$ $\Rightarrow \boxed{u = v}$

Date..............

Put $x+y=1$ in v of eqn (2)

$$\boxed{v = \frac{1}{\sqrt{2}}}$$





• Magnification

$w = \alpha z$; where α is complex no.

Notes: (1) The figure in w-plane is magnified α - times the size of the figure in z-plane.

(2) Magnification → stretching

Q. Consider transformation $w = 2z$ & determine the region R' of w-plane into which triangular region R enclosed by lines $x=0$, $y=0$, $x+y=1$ in the z plane is mapped under map.

Sol: Given: $w = 2z$

$\Rightarrow u + iv = 2(x+iy)$

$\Rightarrow u = 2x$; $v = 2y$

Put $x = 0$, $u = 0$

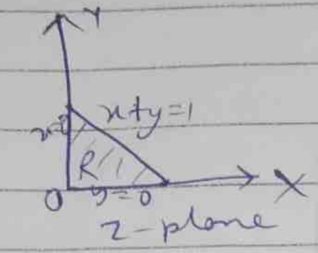
$y = 0$, $v = 0$

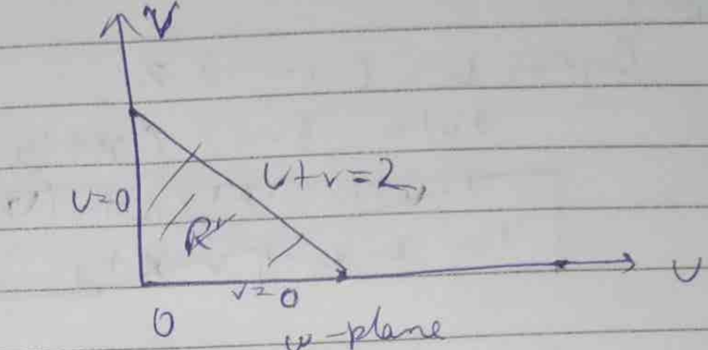
& $x+y=1$; $u + v = 2(x+y) = 2 \cdot 1 = 2$

$$\boxed{u + v = 2}$$

Date...............

Hence region R' is a triangular formed by lines $u = 0, v = 0, u + v = 2$.





Q: Find image of circle $|z| = 3$ under transformation $w = 3z$.

Sol→ Given: $w = 3z$

$\Rightarrow u + iv = 3(x + iy)$

$\Rightarrow u = 3x \; ; \; v = 3y$

$x = \frac{u}{3} \; ; \; y = \frac{v}{3}$ } - ①

Now ; $|z| = 3$

$\Rightarrow |z|^2 = 9$

$\Rightarrow |x + iy|^2 = 9$

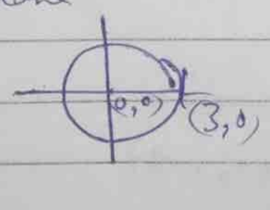
$\Rightarrow x^2 + y^2 = 9$

$\Rightarrow \frac{u^2}{3^2} + \frac{v^2}{3^2} = 9$

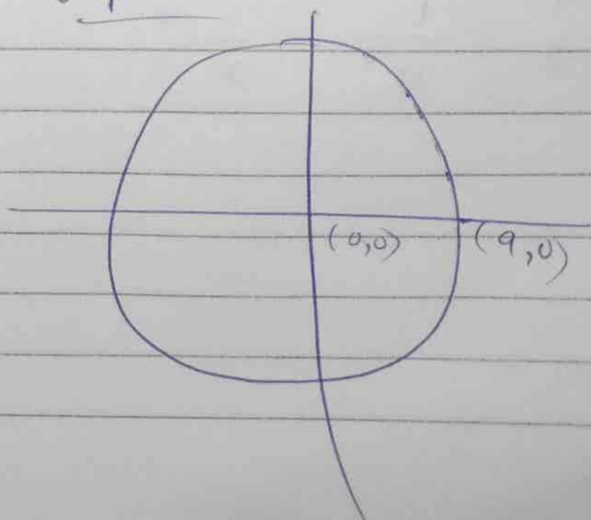
$\Rightarrow u^2 + v^2 = 81$

$\Rightarrow \boxed{u^2 + v^2 = 9^2}$

z plane



w plane



Date...............

Q. Find image of region $y > 1$ under transformation $w = (1-i)z$

Sol →

Given: $w = (1-i)z$

$\Rightarrow u + iv = (1-i)(x+iy)$

$\boxed{\begin{array}{l}\cancel{\Rightarrow u + iv = (x+y) + i(x+y)} \\ \cancel{\Rightarrow u = x-y \; ; \; v = x+y}\end{array}}$

$\Rightarrow u + iv = (x+y) + i(y-x)$

$\Rightarrow u = x+y \; ; \; v = y-x$

$\Rightarrow u + v = x+y+y-x$

$\Rightarrow u + v = 2y$

$\Rightarrow \boxed{y = \frac{u+v}{2}}$

The image of region $y > 1$ is

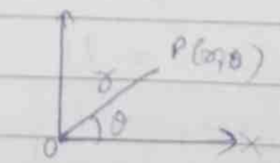
$\Rightarrow \frac{u+v}{2} > 1$

$\Rightarrow \boxed{u + v > 2}$

• Inversion

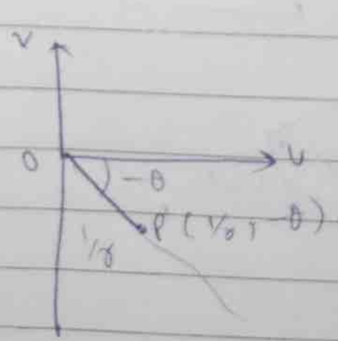
$w = 1/z$

Let z-plane; $z = re^{i\theta}$



for w-plane

$w = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$



Date...............

Q. Find image of infinite strip $\frac{1}{4} < y < \frac{1}{2}$ under transformation $w = \frac{1}{z}$. Also show the regions graphically.

Soln Given: $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$\Rightarrow x + iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$

$\Rightarrow x + iy = \frac{u-iv}{u^2+v^2}$

$\Rightarrow x = \frac{u}{u^2+v^2}$; $y = \frac{-v}{u^2+v^2}$ } - (1)

$\frac{1}{4} < y < \frac{1}{2}$

Now, $y < \frac{1}{2}$; $\Rightarrow \frac{-v}{u^2+v^2} < \frac{1}{2}$

$\Rightarrow -2v < u^2+v^2$

$\Rightarrow u^2+v^2+2v > 0$

$\Rightarrow u^2 + (v^2+2v+1) > 1$

$\Rightarrow u^2 + (v+1)^2 > 1$ - (2)

& $y > \frac{1}{4} \Rightarrow \frac{-v}{u^2+v^2} > \frac{1}{4}$

$-4v > u^2+v^2$

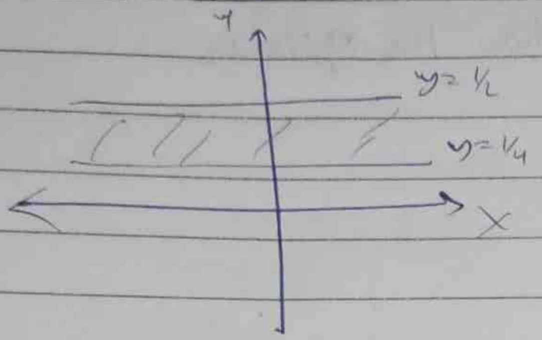
$\Rightarrow u^2+v^2+4v < 0$

$\Rightarrow u^2 + (v^2+4v+4) < 0\,4$

$\Rightarrow u^2 + (v+2)^2 > 4$ - (3)

Date..............

z - plane



$\frac{1}{4} < y < \frac{1}{2}$

w-plane

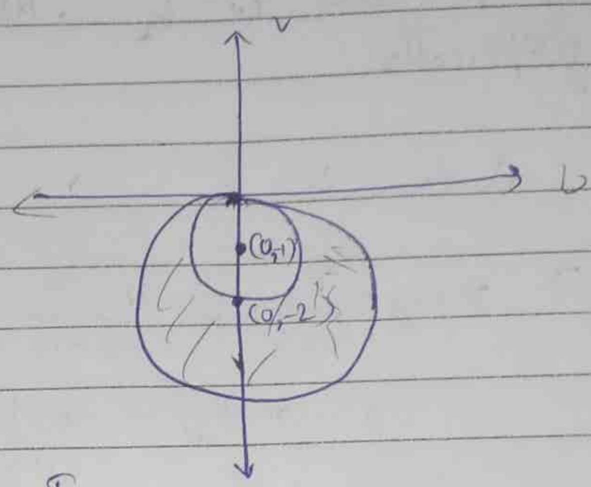


Image of strip.

Q. find image of $|z - 3i| = 3$ under mapping $w = \frac{1}{z}$.

Sol:- Given : $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$\Rightarrow x + iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$

$\Rightarrow x + iy = \frac{u-iv}{u^2+v^2}$

$\cancel{\Rightarrow x = \frac{u}{u^2+v^2}}$

$\Rightarrow x = \frac{u}{u^2+v^2}$ & $y^2 = \frac{-v}{u^2+v^2}$ —(1)

Given circle is

$|z - 3i| = 3$

$\Rightarrow |(x+iy) - 3i|^2 = 3^2$

$\Rightarrow |x + i(y-3)|^2 = 9$

$\Rightarrow x^2 + (y-3)^2 = 9$

Date...............

$$\Rightarrow \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2} - 3\right)^2 = 9$$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 9 + \frac{6v}{u^2+v^2} = 9$$

$$\Rightarrow \frac{u^2+v^2}{(u^2+v^2)^2} + \frac{6v}{(u^2+v^2)} = 0$$

$$\Rightarrow \frac{1}{u^2+v^2} + \frac{6v}{(u^2+v^2)} = 0$$

$$\Rightarrow \boxed{1 + 6v = 0}$$

*

Which is required image of circle under $w = \frac{1}{z}$.

• Linear Transformation

A combination of Translation, Rotation & magnification.
$w = z + \alpha$, $w = e^{i\alpha} z$, $w = az$

Q. Consider linear transformation $w = (1+i)z + (2-i)$ and determine the region in the w-plane into which the rectangular region bounded by lines $x = 0$, $y = 0$, $x = 1$ & $y = 2$ in the z-plane is mapped.

Sol → $w = (1+i)z + (2-i)$

$(u+iv) = (1+i)(x+iy) + (2-i)$

$(u+iv) = x + iy + ix - y + 2 - i$

$u + iv = (x - y + 2) + i(x + y - 1)$

Date..............

Compare ;

$v = x + y - 1 \quad - (2)$

$u = x - y + 2 \quad - (1)$

$1+2 \Rightarrow u + v = 2x + 1 \quad - (3)$

$1-2 \Rightarrow u - v = -2y + 3 \quad - (4)$

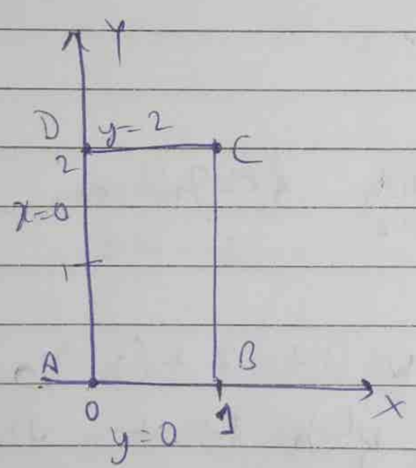
(a) $x = 0 \Rightarrow u + v = 1$

(b) $y = 0 \Rightarrow u - v = 3$

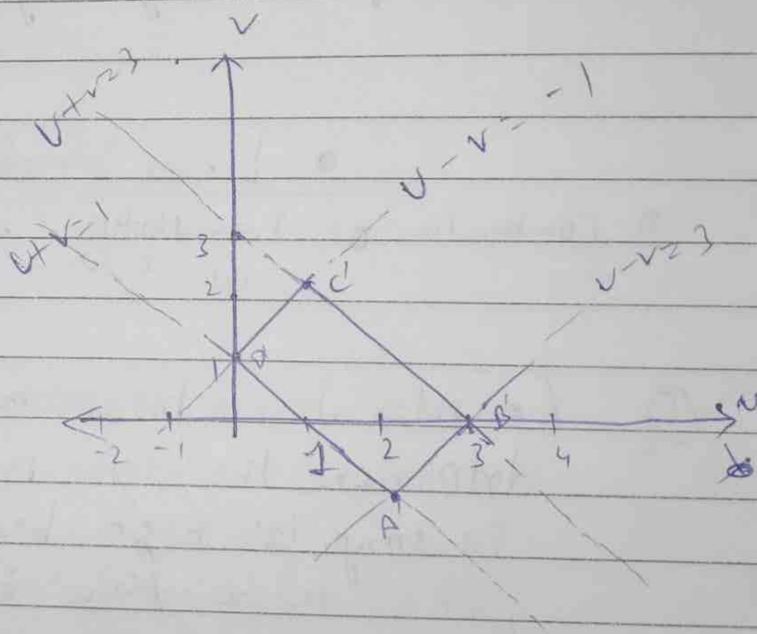
(c) $x = 1 \Rightarrow u + v = 3$

(d) $y = 2 \Rightarrow u - v = -1$

z-plane.



w-plane



Date..............

Q. Determine the linear transformation that maps the rectangle $z_1 = 0$, $z_2 = \frac{1}{2}$, $z_3 = i$ & $z_4 = \frac{1}{2} + i$ onto rectangle $w_1 = i$, $w_2 = 2i$, $w_3 = -2+i$ & $w_4 = -2+2i$.

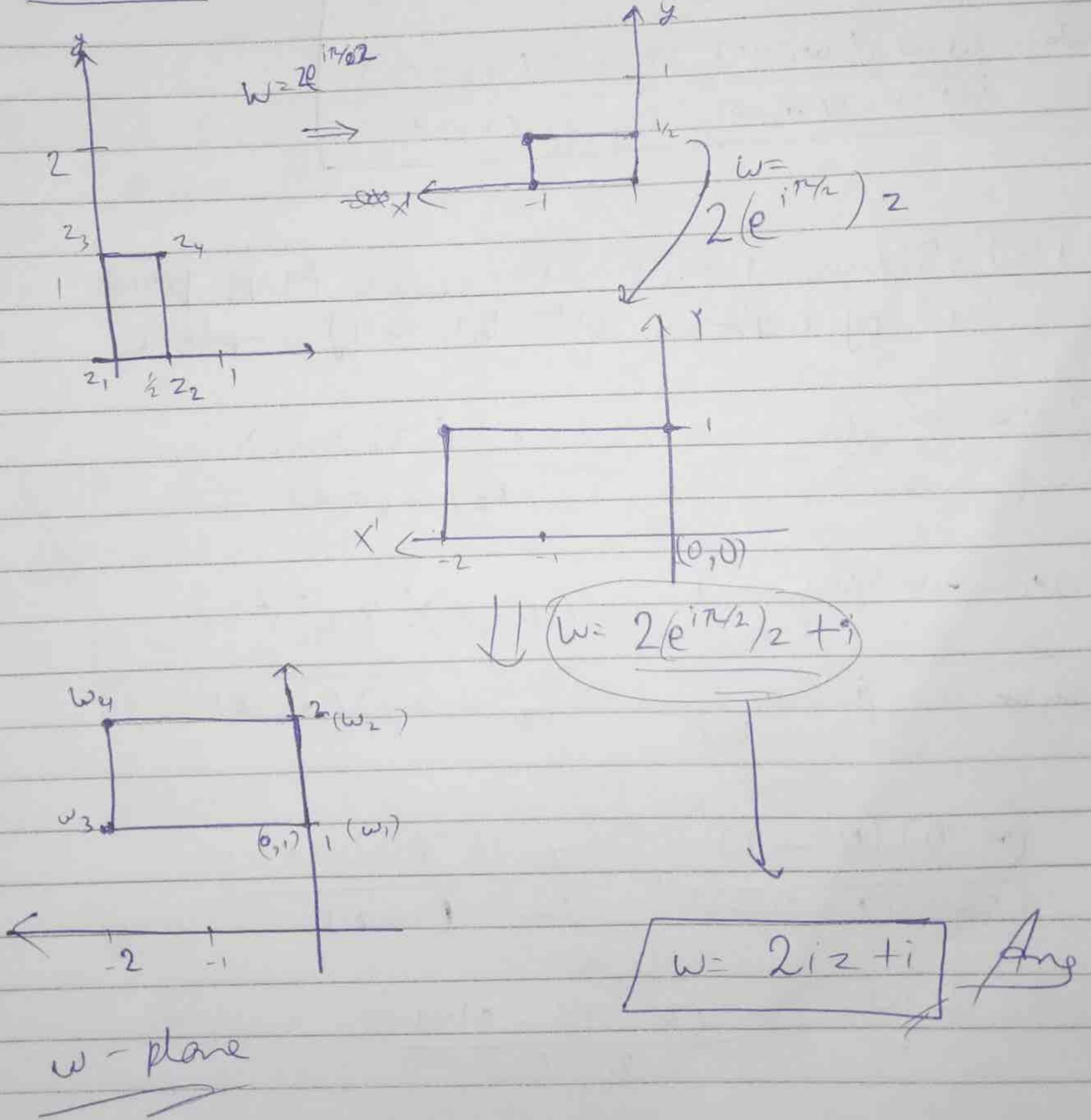
for → $z_1 \to 0 + i0 \to (0, 0)$ $\quad z_2 \to \frac{1}{2} + i0 \to (\frac{1}{2}, 0)$

$z_3 \to 0 + i \to (0, 1)$ $\quad z_4 \to \frac{1}{2} + i \to (\frac{1}{2}, 1)$

$w_1 = 0 + i \to (0, 1)$ $\quad w_2 = 0 + 2i \to (0, 2)$

$w_3 = (-2, 1)$ $\quad w_4 = (-2, 2)$

z - plane



$\boxed{w = 2iz + i}$ Ans

w - plane

Date..............

• Topic: Bilinear Transformation

A transformation of type $w = \frac{az+b}{cz+d}$ where a, b, c & d are real or complex constant so that $ad - bc \neq 0$ is called Bilinear Transformation or Mobious Transformation.

Determining Bilinear Mapping

z_1, z_2, z_3 in z plane $\quad w = \frac{az+b}{cz+d}$

w_1, w_2, w_3 in w plane

Cross Ratio formula: $\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

Q. Find Bilinear Transformation which Maps points $i, -i, 1$ of z-plane into $0, 1, \infty$ of w-plane.

Sol → $\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$\Rightarrow \frac{\cancel{w_3}(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\cancel{w_3}(w_1-w_2)\left(1-\frac{w}{w_3}\right)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-i)(-i-1)}{(i+i)(1-z)}$

$\frac{-w}{-1} = \frac{(z-i)(-i-1)}{2i\ (1-z)}$

Date...............

$\Rightarrow w = \frac{-1(1+i)}{+1(z-1)2i}(z-i)$

$\Rightarrow w = \frac{(z-i)}{(z-1)} \frac{(1+i)}{2i}$

$\Rightarrow w = -i\frac{(1+i)}{2}\frac{(z-i)}{(z-1)}$

$= \frac{(-i+1)}{2}\frac{(z-i)}{(z-1)}$

$\Rightarrow w = \frac{(1-i)}{2}\left(\frac{z-i}{z-1}\right)$ Ans

$\frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} \Rightarrow \frac{-i}{-i \times i}$

$\Rightarrow -i$

Q. Find BLT that maps $z = 1, i, -1$ into $w = i, 0, -i$.
Hence, find image of $|z| < 1$

Soln $z_1 = 1$ $\quad w_1 = i$

$z_2 = i$ $\quad w_2 = 0$

$z_3 = -1$ $\quad w_3 = -i$

$\Rightarrow \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$

$\Rightarrow \frac{(z-1)(i+1)}{(1-i)(-1-z)} = \frac{(w-i)(0+i)}{(i-0)(-i-w)}$

$\Rightarrow \frac{(z-1)}{(z+1)}\frac{(1+i)}{(1-i)}\frac{(1+i)}{(1+i)} = \frac{w-i}{w+i}$

$\Rightarrow \frac{(z-1)}{(z+1)}\frac{(1+i)^2}{1^2-i^2} = \frac{w-i}{w+i}$

Date...............

$$\Rightarrow \frac{(z-1)}{(z+1)} \frac{2i}{2} = \frac{w-i}{w+i}$$

$$\Rightarrow (zi-i)(w+i) = (w-i)(z+1) = 0$$

$$\Rightarrow z[iw - (w-i)] = i(w+i) + (w-i)$$

$$\Rightarrow z((i-1)w + (i-1)) = (i+1)w - (i+1)$$

$$\Rightarrow z = \frac{(i+1)}{(i+1)} \frac{(i+1)}{(i-1)} \frac{(w-1)}{(w+1)} \Rightarrow z = \frac{2i}{-2}\left(\frac{w-1}{w+1}\right)$$

$$\Rightarrow \boxed{z = i\left(\frac{1-w}{1+w}\right)}$$

$$|z| < 1 \qquad \Rightarrow |z|^2 < 1$$

$$z\bar{z} < 1$$

$$i\left(\frac{1-w}{1+w}\right)\left(-i\left(\frac{1-\bar{w}}{1+\bar{w}}\right)\right) < 1$$

$$(1-w)(1-\bar{w}) < (1+w)(1+\bar{w})$$

$$1 - (w+\bar{w}) + w\bar{w} < 1 + (w+\bar{w}) + w\bar{w}$$

$$0 < 2(w+\bar{w})$$

$$2(w+\bar{w}) > 0$$

$w = u + iv$

$\bar{w} = u - iv$

$2u = w + \bar{w}$

$$2(2u) > 0$$

$$\boxed{4u > 0}$$

Date...............

Q. Show that transformation $w = \frac{2z+3}{z-4}$

maps the circle $x^2 + y^2 - 4x = 0$ into straight line $4u + 3 = 0$.

Sol: Eqn of circle : $x^2 + y^2 - 4x = 0$

$|(z-2)| = 2$ — ①

Given : $w = \frac{2z+3}{z-4}$

$wz - 4w - 2z - 3 = 0$

$(w-2)z = 4w + 3$

$$\boxed{z = \frac{4w+3}{w-2}}$$

$\Rightarrow |z-2| = 2$

$\Rightarrow \left| \frac{4w+3}{w-2} - 2 \right| = 2$

$\Rightarrow \left| \frac{4w+3-2w+4}{w-2} \right| = 2$

$\Rightarrow \left| \frac{2w+7}{w-2} \right| = 2$

$\Rightarrow |2w+7| = 2|w-2|$

$\Rightarrow |2w+7|^2 = 4|w-2|^2$

$\Rightarrow (2w+7)(2\bar{w}+7) = 4(w-2)(\bar{w}-2)$

$\Rightarrow 4w\bar{w} + 14(w+\bar{w}) + 49 = 4(\cancel{w\bar{w}} - 2(w+\bar{w}) + 4)$

$\Rightarrow 22(w+\bar{w}) + 33 = 0$

$\Rightarrow 2(w+\bar{w}) + 3 = 0$

$\Rightarrow 2(2u) + 3 = 0$

$\Rightarrow \boxed{4u + 3 = 0}$

$w = u + iv$
$\bar{w} = u - iv$
$w + \bar{w} = 2u$

Q: Show that transformation $w = i\left(\frac{1-z}{1+z}\right)$ transform the circle $|z| = 1$ into real axis at w-plane & the interior of circle $|z| < 1$ into upper half of w-plane.

Sol: $|z| < 1$ & $|z| = 1$

$|z| \leq 1$

$$w = \frac{i(1-z)}{(1+z)}$$

$w + wz - i + iz = 0$

$(w+i)z = -(w-i)$

$$\boxed{z = -\left(\frac{w-i}{w+i}\right)}$$

$|z|^2 \leq 1$

$z\bar{z} \leq 1$

$$\left(\frac{w-i}{w+i}\right)\left(\frac{\bar{w}+i}{\bar{w}-i}\right) \leq 1$$

$\Rightarrow (w-i)(w+i) \le (w+i)(\bar{w}-i)$

$\Rightarrow \cancel{w\bar{w}} + i(w-\bar{w}) + 1 \le w\bar{w} + (-i(w-\bar{w})) + 1$

$\Rightarrow 2i(w-\bar{w}) \le 0$

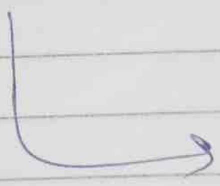
$\Rightarrow 2i(2iv) \le 0$

$\Rightarrow -4v \le 0$

$\Rightarrow 4v \ge 0$

$|z| = 1 \rightarrow 4v = 0$ i.e $v = 0$ (Real axis of w plane)

$|z| < 1 \rightarrow 4v > 0$ i.e. upper half of w-plane.



Date...............

Fixed Point or Invariant Point

A point $z = z_0$ is called fixed point or invariant point of Bilinear Mapping if

$$w(z_0) = z_0$$

$$w = \frac{az+b}{cz+d} = z$$

Q Find fixed points for (i) $w = \frac{3iz+1}{z+i}$

Sol → $w = \frac{3iz+1}{z+i} = z$

$\Rightarrow z^2 + iz = 3iz + 1$

$\Rightarrow z^2 + iz - 3iz - 1 = 0$

$\Rightarrow z^2 - 2iz - 1 = 0$

$\Rightarrow (z-i)^2 = 0$

$\Rightarrow z = i$ Ans

(ii) $w = \frac{z}{z-2}$

Sol → $w = \frac{z}{z-2} = z$

$\Rightarrow z^2 - 2z = z \Rightarrow z^2 - 3z = 0$

$\Rightarrow z(z-3) = 0$

$\boxed{z = 0, 3}$ Ans