Exact Differential Equation

Any first order differential equation $\frac{dy}{dx} = f(x,y)$ can be written in the form

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} \qquad \text{—— (1)}$$

(1) is said to be an exact differential equation if $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

Then the solution of (1) is given by

$$\boxed{\int M\,dx + \int (N \text{ w/o } x)\,dy = C}$$

i.e. $\int$(all terms in M treating y as constant)dx

+ $\int$(only those terms in N not involving x) $dy = C$

Question 1: Solve $(y\sin xy + xy^2\cos xy)\,dx + (x\sin xy + x^2y\cos xy)\,dy = 0$.

Solution: Here, $M = y\sin xy + xy^2\cos xy$ &

$N = x\sin xy + x^2y\cos xy$

$$\frac{\partial M}{\partial y} = \sin xy + xy\cos xy + 2xy\cos xy - x^2y^2\sin xy$$

$$\frac{\partial N}{\partial x} = xy\cos xy + \sin xy + 2xy\cos xy - x^2y^2\sin xy$$

Hence, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$ Given eq^n is an exact differential eq^n.

The general solution is

$$\int (y\sin xy + xy^2\cos xy)\,dx + \int 0\,dy = C$$

$$\frac{-y\cos xy}{y} + y^2\int x\cos xy\,dx = c$$

$$-\cos xy + y^2\left[\frac{x\sin xy}{y} - \int 1\cdot\frac{\sin xy}{y}dx\right] = c$$

$$-\cos xy + xy\sin xy + y^2\frac{\cos xy}{y^2} = c$$

Hence, $\boxed{xy\sin xy = c}$ is the required solⁿ.

Question 2:- Solve $y\sin 2x\,dx - (y^2+\cos^2 x)dy = 0$

Solution: Here, $M = y\sin 2x$ & $N = -y^2-\cos^2 x$

$$\frac{\partial M}{\partial y} = \sin 2x \quad \& \quad \frac{\partial N}{\partial y} = 2\cos x\sin x = \sin 2x$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

Hence the give equation is exact and its solution is

$$\int M dx + \int (N \text{ w/o } x) dy = 0$$

$$\int y\sin 2x\,dx + \int -y^2 dy = c$$

$$-\frac{y\cos 2x}{2} - \frac{y^3}{3} = c$$

$\boxed{3y\cos 2x + 2y^3 = C}$ is the required solution.

Question 3:- Solve $\left[y\left(1+\frac{1}{x}\right)\cos y\right]dx + (x+\log x)$
$[\cos y - y\sin y]dy = 0$

Solution:- Here, $M = y\left(1+\frac{1}{x}\right)\cos y$

& $N = (x+\log x)(\cos y - y\sin y)$

$$\frac{\partial M}{\partial y} = \left(1+\frac{1}{x}\right)(\cos y - y\sin y)$$

$$\frac{\partial N}{\partial x} = \left(1+\frac{1}{x}\right)(\cos y - y\sin y)$$

So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus, the given eq^n is exact and its sol^n is

$$\int M\,dx + \int (N \text{ w/o } x)\,dy = c$$

$$\int \left(1+\frac{1}{x}\right) y\cos y\,dx + \int 0\,dy = c$$

$\boxed{(x+\log x)\, y\cos y = C}$ is the required Sol^n.

Question 4:- Solve $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Solution: Here, $M = (e^y+1)\cos x$ & $N = e^y \sin x$

$$\frac{\partial M}{\partial y} = e^y \cos x$$

$$\frac{\partial N}{\partial x} = e^y \cos x$$

So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus, the given eq^n is exact and its sol^n is

$$\int M\,dx + \int (N \text{ w/o } x)\,dy = c$$

$$\int (e^y+1)\cos x\,dx + \int 0.\,dy = c$$

$\boxed{(e^y+1)\sin x = C}$ is the required solution.

Question 5:- Solve $(a^2 - 2xy - y^2)\,dx - (x+y)^2\,dy = 0$

Solution:- Here $M = a^2 - 2xy - y^2$ & $N = -(x+y)^2$

$$\frac{\partial M}{\partial y} = -2x - 2y \quad \& \quad \frac{\partial N}{\partial x} = -2(x+y)$$

So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus, the given equation is exact and its solution is

$$\int M dx + \int (N \sim x) dy = C$$

$$\int (a^2 - 2xy - y^2)\, dx + \int -y^2 dy = C$$

$$\boxed{a^2 x - x^2 y - xy^2 - \frac{y^3}{3} = C}$$ is the required sol^n.

Exercise

Solve the following differential equations:

1) $(x^2 - ay)dx + (y^2 - ax)dy = 0$
2) $\left[y\left(1 + \frac{1}{x}\right) + \sin y\right] dx + (x + \log x + x\cos y)\, dy = 0$
3) $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$
4) $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$
5) $(2x - y + 1)dx + (2y - x - 1)dy = 0$
6) $(x^4 - 2xy^2 + y^4)dx = (2x^2 y - 4xy^3 + \sin y)\, dy$
7) $(ax + hy + g)dx + (hx + by + f)dy = 0$
8) $(xy^2 + x)dx + (yx^2 + y)dy = 0$
9) $(2ax + by + g)dx + (2cy + bx + e)dy = 0$
10) $(x^2 + y^2 - a^2)x\, dx + (x^2 - y^2 - b^2)y\, dy = 0$

Reducible to Exact Differential Equation

Sometimes a differential equation which is not exact, can be converted to exact differential equation by multiplying it by some suitable factor called Integrating Factor (I.F.)

In what follows, we list some rules for finding the integrating factors for the equation

$M\,dx + N\,dy = 0$, which is not exact.

Rule 1:- If $M\,dx + N\,dy = 0$ is a homogeneous equation in x and y then $\frac{1}{Mx+Ny}$ is the I.F. provided $Mx+Ny \neq 0$.

Rule 2:- If $M\,dx + N\,dy = 0$ is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$ then $\frac{1}{Mx-Ny}$ is the I.F. provided $Mx \neq Ny$.

Rule 3:- In the equation $M\,dx + N\,dy = 0$, if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x only $= f(x)$, say, then I.F. is $e^{\int f(x)dx}$.

Rule 4:- In the equation $M\,dx + N\,dy = 0$, if

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y only, $= f(y)$ say, then I.F. is $e^{\int f(y)\,dy}$.

Rule 5:- I.F. for the equations of the type $x^a y^b (py\,dx + qx\,dy) + x^c y^d (ry\,dx + sx\,dy) = 0$ is $x^h y^k$. The constants h and k are so chosen that the new equation (after multiplying by $x^h y^k$) becomes exact.

Rule 6:- Transformation to Polar co-ordinates: Sometimes, conversion from cartesian to polar co-ordinates helps solve the differential equation easily.

Question 1:- Solve $(y^3 - 3xy^2)dx + (2x^2y - xy^2)dy = 0$

Solution:- The given equation $Mdx + Ndy = 0$ is homogeneous where

$M = y^3 - 3xy^2$ & $N = 2x^2y - xy^2$

$$\frac{\partial M}{\partial y} = 3y^2 - 6xy \quad \& \quad \frac{\partial N}{\partial x} = 4xy - y^2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the equation is not exact but homogeneous

$$\therefore \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{-x^2y^2}$$

Multiplying the given eq^n by $\frac{-1}{x^2y^2}$, gives

$$\left(\frac{3}{x}-\frac{y}{x^2}\right)dx+\left(\frac{1}{x}-\frac{2}{y}\right)dy=0 \quad \text{—①}$$

Here, $M'=\frac{3}{x}-\frac{y}{x^2}$ & $N'=\frac{1}{x}-\frac{2}{y}$

$$\frac{\partial M'}{\partial y}=\frac{-1}{x^2} \quad \& \quad \frac{\partial N'}{\partial x}=\frac{-1}{x^2}$$

As $\frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$, so ① is an exact eq^n

and its solution is

$$\int M'dx+\int(N'-x)dy=c$$

$$\int\left(\frac{3}{x}-\frac{y}{x^2}\right)dx+\int\left(-\frac{2}{y}\right)dy=c$$

$$3\log x+\frac{y}{x}-2\log y=c$$

$$\frac{x^3}{y^2}=c_1e^{-y/x}$$

Question 2: Solve $(xy^2\sin xy+y\cos xy)dx+(x^2y\sin xy-x\cos xy)dy=0$

Solution :- The given eq^n is not exact but can be written as

$$(xy\sin xy+\cos xy)\,y\,dx+(xy\sin xy-\cos xy)\,x\,dy=0$$

which is of the form $f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0$

$$\therefore \text{I.F.}=\frac{1}{Mx-Ny}=\frac{1}{2xy\cos xy}$$

Multiplying the given eq^n by $\frac{1}{xy\cos xy}$, gives

$$\left(y\tan xy+\frac{1}{x}\right)dx+\left(x\tan xy-\frac{1}{y}\right)dy=0 \quad \text{—①}$$

which can be verified to be exact.

Therefore, the solution of ① is

$$\int \left(y\tan xy + \frac{1}{x}\right)dx + \int \frac{-1}{y}\,dy = c$$

$$y \log\sec(xy)\cdot\frac{1}{y} + \log x - \log y = c$$

$$\log\sec(xy) + \log\frac{x}{y} = c$$

$x\sec xy = c'y$ is the required solⁿ.

Question 3:- solve $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$

Solution:- Here $M = xy^2 - e^{\frac{1}{x^3}}$, $N = -x^2y$

$$\frac{\partial M}{\partial y} = 2xy \quad \& \quad \frac{\partial N}{\partial x} = -2xy$$

hence not exact.

However, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4xy}{-x^2y} = -\frac{4}{x}$, a function of x alone.

$\therefore$ I.F. $= e^{\int -\frac{4}{x}dx} = \frac{1}{x^4}$

Multiplying the given eqⁿ throughout by $\frac{1}{x^4}$, gives

$$\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}\,dy = 0$$ which can be verified to be exact.

Hence the general solution is

$$\int\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx + \int 0\,dy = c$$

$$-\frac{y^2}{2x^2} + \frac{1}{3}e^{\frac{1}{x^3}} = c$$ is the required solⁿ.

Question 4 :- Solve $(x+2y^3)\frac{dy}{dx} = y + 2x^3y^2$

Solution :- Here $M = y+2x^3y^2$ & $N = -x-2y^3$

$$\frac{\partial M}{\partial y} = 1+4x^3y \quad \& \quad \frac{\partial N}{\partial x} = -1$$

As $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, the given eq^n is not exact.

Now $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{2}{y}$, a function of y alone.

$$\therefore \quad I.F. = e^{\int \frac{-2}{y} dy} = \frac{1}{y^2}$$

Now the eq^n is multiplying by $\frac{1}{y^2}$, we get

$$\left(\frac{1}{y} + 2x^3\right)dx + \left(-\frac{x}{y^2} - 2y\right)dy = 0$$

which can be easily verified to be exact.

The general solution is

$$\int\left(\frac{1}{y} + 2x^3\right)dx + \int -2y\,dy = C$$

$$\frac{x}{y} + \frac{2x^4}{4} - y^2 = C$$

$2x + x^4y - 2y^3 = Cy$ is the required solution.

Question 5 :- Solve $(3xy + 8y^5)dx + (2x^2 + 24xy^4)dy = 0$

Solution :- The given eq^n can be written as

$$x(3y\,dx + 2x\,dy) + 8y^4(y\,dx + 3x\,dy) = 0 \quad -(1)$$

It is of the form mentioned in the rule 5,

above, therefore we take the I.F. as $x^h y^k$.
Multiplying the given equation throughout by $x^h y^k$, we get

$$(3x^{h+1}y^{k+1} + 8x^h y^{k+5})dx + (2x^{h+2}y^k + 24x^{h+1}y^{k+4})dy = 0 \quad (2)$$

Here, $M = 3x^{h+1}y^{k+1} + 8x^h y^{k+5}$

& $N = 2x^{h+2}y^k + 24x^{h+1}y^{k+1}$

$$\frac{\partial M}{\partial y} = 3(k+1)x^{h+1}y^k + 8(k+5)x^h y^{k+4}$$

$$\frac{\partial N}{\partial x} = 2(h+2)x^{h+1}y^k + 24(h+1)x^h y^{k+4}$$

For the eq^n (2) to be exact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore, equating the coefficients of like terms, gives.

$$3(k+1) = 2(h+2) \;\&\; 8(k+5) = 24(h+1)$$

which on solving for h & k, yield $h=1$, $k=1$, then (2) becomes

$$(3x^2y^2 + 8xy^6)dx + (2x^3y + 24x^2y^5)dy = 0$$

which is exact.

Its solution is

$$\int (3x^2y^2 + 8xy^6)dx + \int 0\,dy = C$$

$x^3y^2 + 4x^2y^6 = C$ is the required solution.

Question 6:- Solve $\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$

Solution:- Conversion to polar coordinates, converts the given equation to

$$\frac{r\,dr}{r^2 d\theta} = \sqrt{\frac{a^2-r^2}{r^2}} \quad \text{or} \quad \frac{dr}{d\theta} = \sqrt{a^2-r^2}$$

$$\frac{dr}{\sqrt{a^2-r^2}} = d\theta$$

Integrating both sides, gives

$$\sin^{-1}\frac{r}{a} = \theta + c$$

$$\sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{a}\right) = \tan^{-1}\left(\frac{y}{x}\right) + C$$ is the required solution.

Question 7:- Solve $(x^2+y^2+1)dx - 2xy\,dy = 0$

Solution:- Here $M = x^2+y^2+1$, $N = -2xy$

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = -2y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4y}{-2xy} = \frac{-2}{x}$$

$$I.F. = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$$

Multiplying the given eq^n by $\frac{1}{x^2}$

$$\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$M_1\,dx + N_1\,dy = 0$$

$$\frac{\partial M_1}{\partial y} = \frac{2y}{x^2} = \frac{\partial N_1}{\partial x}$$

The general solution is

$$\int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx + \int 0\,dy = c$$

$$x - \frac{y^2}{x} - \frac{1}{x} = c \Rightarrow x^2-y^2 = cx+1$$

Question 8:- Solve $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$

Solution:- Here $M = y^4+2y$, $N = xy^3+2y^4-4x$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{1}{M}\left(-\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right) = \frac{-3}{y}$$

$$I.F. = e^{-\int \frac{3}{y} dy} = \frac{1}{y^3}$$

Multiplying the given equation by $\frac{1}{y^3}$, we get

$$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0$$

The above is exact. so, the general soln is

$$\int\left(y+\frac{2}{y^2}\right) dx + \int 2y\,dy = C$$

$\left(y+\frac{2}{y^2}\right) x + y^2 = C$ is the required solution.

Question 9:- Solve $x^2y\,dx - (x^3+y^3)dy=0$

Solution:- The given eqn is homogeneous & $Mx+Ny \neq 0$

$$I.F. = \frac{1}{Mx+Ny} = -\frac{1}{y^4}$$

Multiplying the given eqn by $\frac{-1}{y^4}$, we get

$$\frac{-x^2}{y^3} dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

$$M_1 = \frac{-x^2}{y^3} , \quad N_1 = \frac{x^3}{y^4} + \frac{1}{y}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Then the general solution is

$$\int M_1\,dx + \int (N_1 \text{ not } x)\,dy = c$$

$$\frac{-x^3}{3y^3} + \log y = \log c$$

$$y = C e^{x^3/3y^3}$$

Question 10:- Solve $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Solution:- The given equation is of the form

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying the given equation by $\frac{1}{3x^3y^3}$, we get

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

which is exact.

The required solution is given by

$$\int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx - \int\frac{1}{3y}\,dy = c$$

$$x^2 = k y e^{\frac{1}{xy}}$$

Question 11:- Solve $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0$

Solution:- The given equation can be written as

$$(4xy + 3y^4)dx + (2x^2 + 5xy^3)dy = 0$$

Multiply the above equation by $x^\alpha y^\beta$ to obtain

$$(4x^{\alpha+1}y^{\beta+1} + 3x^{\alpha}y^{\beta+4})dx + (2x^{\alpha+2}y^{\beta} + 5x^{\alpha+1}y^{\beta+3})dy = 0 \quad -(1)$$

This is exact for values of α & β for which $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$4(\beta+1)x^{\alpha+1}y^{\beta} + 3(\beta+4)x^{\alpha}y^{\beta+3} = 2(\alpha+2)x^{\alpha+1}y^{\beta} + 5(\alpha+1)x^{\alpha}y^{\beta+3}$$

On comparing the co-efficient

$$4(\beta+1) = 2(\alpha+2)$$

$$3(\beta+4) = 5(\alpha+1)$$

by solving these eq^n, we get $\alpha=2$, $\beta=1$

Thus, (1) becomes

$$(4x^3y^2 + 3x^2y^5)dx + (2x^4y + 5x^3y^4)dy = 0$$

which is exact.

Then the solution is

$$x^4y^2 + x^3y^5 = C$$

Exercise

Solve the following differential equations

1) $xe^x(dx - dy) + e^x dx + ye^y dy = 0$
2) $(x^2+y^2+x)dx + xy\,dy = 0$
3) $(x^2y^3+2y)dx + (2x-2x^3y^2)dy = 0$
4) $(x^2+y^2+2x)dx + 2y\,dy = 0$
5) $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)dy = 0$
6) $(x^2y-2xy^2)dx - (x^3-3x^2y)dy = 0$
7) $y(x^2y^2+2)dx + x(2-2x^2y^2)dy = 0$
8) $(2x^2y^2+y)dx - (x^3y-3x)dy = 0$