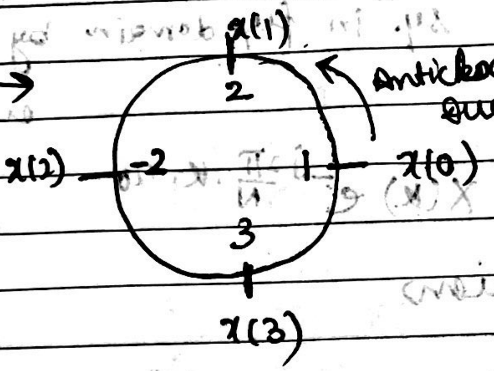
Date ....

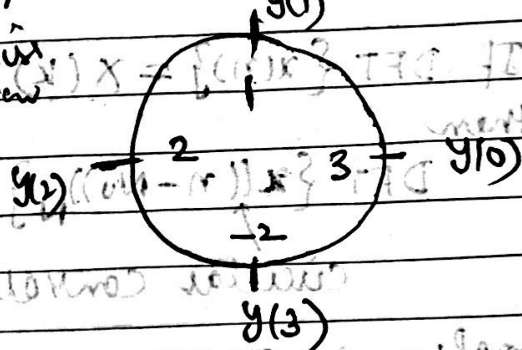
Concept of Circular Time Shift in DFT :→

First method

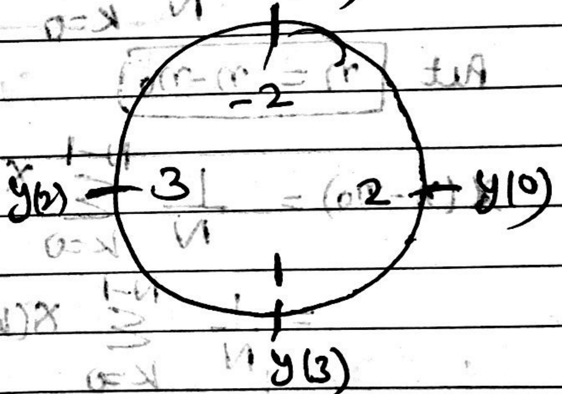
Q.19 $x(n) = \{1, 2, -2, 3\}$ ← Given Sequence.

If $y(n) = x((n-1))_4$ ← mode 4 ← To find





$\therefore y(n) = \{3, 1, 2, -2\}$



Q. $y(n) = x((n+4))_4$

$-ve \rightarrow$ Anticlockwise direction

$+ve \rightarrow$ clockwise direction

So $y(n) = \{2, -2, 3, 1\}$

Second method

$$y(n) = x((n-1))_4 = x(4+n-1)$$

for $n=0$ $\quad y(0) = x(4-1) = x(3) = +3$

out of range of given seq.

for $n=1$ $\quad y(1) = x(4+1-1) = x(4) = x(4-4) = x(0) = 1$

mode 4

for $n=2$ $\quad y(2) = x(6-1) = x(5) = x(5-4) = x(1) = 2$

for $n=3$ $\quad y(3) = x(7-1) = x(6) = x(6-4) = x(2) = -2$

So $y(n) = \{3, 1, 2, -2\}$

Spiral

Also known as Quadrature modulation Theorem

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(4) **Circular frequency shift:—** States that multiplication of seq. $x(n)$ by $e^{j\frac{2\pi}{N}nl}$ is equivalent to circular shift of DFT in time domain by l samples

if $DFT\{x(n)\} = X(k)$

then $DFT\{x(n)\, e^{j\frac{2\pi}{N}ln}\} = X((k-l))_N$

[freq domain signal will be circularly shifted

Proof:—

As $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}\cdot k\cdot n}$; $0 \le k \le N-1$

Putting $\boxed{k = k-l}$

$$X(k-l) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}(k-l)\cdot n}$$

$$= \underbrace{\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}k\cdot n}}\, e^{+j\frac{2\pi}{N}\cdot l\cdot n}$$

$$= X(k)\, e^{j\frac{2\pi}{N}\cdot l\cdot n}$$

$X(k-l) = X((k-l))_N = DFT\{x(n)\, e^{+j\frac{2\pi}{N}ln}\}$

As sequence is circular & DFT is periodic.

| or

$X(k+l) = X((k+l))_N = DFT\{x(n)\, e^{-j\frac{2\pi}{N}\cdot l\cdot n}\}$

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Q.20 A four point sequence $x(n) = \{1, 2, 3, 4\}$ has DFT $X(k)$, $0 \le k \le 3$. Without performing DFT or IDFT, determine the signal values which has DFT $X(k-1)$.

Soln

According to the circular frequency shifting property of DFT, we have

$$DFT\left\{x(n) \cdot e^{-j2\frac{\pi}{N} \cdot ln}\right\} = X((k+l))_N = X(k-l)$$

As $l = 1$

Let the signal whose DFT is $X(k-1)$ be denoted by $x_1(n)$.

So

$$X(k-1) \cdot x_1(n) = x(n)\, e^{-j\frac{2\pi}{4} \cdot 1 \cdot n}$$

$$x_1(n) = x(n)\, e^{-j\frac{\pi}{2} n}$$

for $n=0$

$$x_1(0) = x(0)\, e^{0} = 1 \cdot 1 = 1$$

$n=1$

$$x_1(1) = x(1)\, e^{-j\frac{\pi}{2}} = 2e^{-j\frac{\pi}{2}} = -2j \qquad \left\{e^{-j\frac{\pi}{2}} = -j\right\}$$

$$x_1(2) = x(2)\, e^{-j\frac{\pi}{2} \cdot 2} = 3 \cdot e^{-j\pi} = -3 \qquad \left[e^{-j\pi} = -1\right]$$

$$x_1(3) = x(3)\, e^{-j\frac{\pi}{2} \cdot 3} = 4e^{-j\frac{3\pi}{2}} = 4j \qquad \left\{e^{-j\frac{3\pi}{2}} = +j\right\}$$

So $x_1(n) = \{1, -2j, -3, 4j\}$ Ans

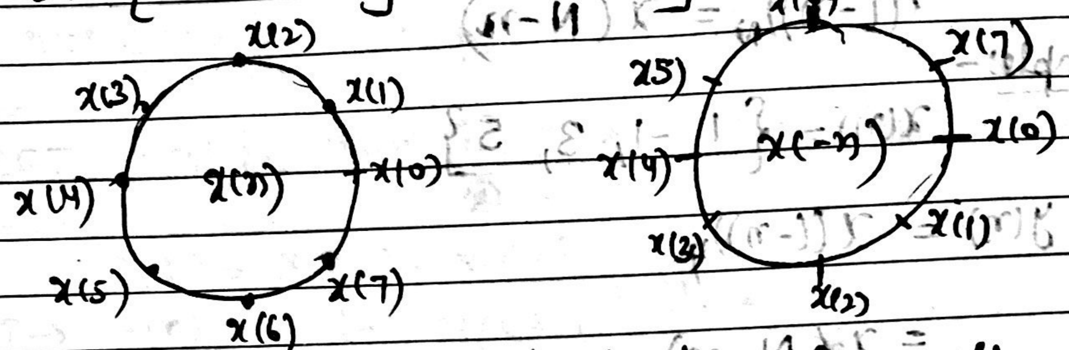
States that if a sequence is circularly folded, its DFT is also circularly folded.

Date ....

(5) Time Reversal :-

If $DFT\{x(n)\} = X(k)$

then $DFT\{x((-n))_N\} = X(N-n)\} = X((-k))_N = X(N-K)$



Time Reversal of a Sequence

Proof. $DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{-j\frac{2\pi}{N}\cdot k\cdot n}$

Putting $m = N-n$, for $n=0$, $\boxed{m=N}$ ← lower limit

so $\Rightarrow n = N-m$, for $N-1$, $m = \cancel{N} - \cancel{N} + 1$

$\boxed{m=1}$ ← upper limit

DFT $= \sum_{m=N}^{1} x(m)\, e^{-j\frac{2\pi}{N}k(N-m)}$

$= \sum_{m=1}^{N} x(m)\, e^{-j\frac{2\pi}{\cancel{N}}\cdot k\cancel{N}} \cdot e^{+j\frac{2\pi}{N}km}$

As $\boxed{e^{-j2\pi k} = 1}$

$= \sum_{m=1}^{N} x(m)\, e^{+j\frac{2\pi}{N}km} \cdot 1$

$= \sum_{m=1}^{N} x(m)\, e^{+j\frac{2\pi}{N}km} \cdot e^{-j\frac{2\pi}{N}\cdot N\cdot m}$

$= \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}(N-k)m}$

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$$DFT\{x(N-n)\} = X(N-k) = X((-k))_N$$

Hence Proved.

$x(n) \rightarrow N-$ Point sequence.

Time Reversal

$$x((-n))_N = x(N-n)$$

Q.21 Example:-

$$x(n) = \{1, -1, 3, 5\}$$

find $y(n) = x((-n))_4$

$$= x(N-n) = x(4-n)$$

for $n=0$

$y(n) = x(4-n)$

out of range of given sequence

$y(0) = x(4) = x(4-4) = x(0) = 1$ (mode)

$y(1) = x(3) = 5$

$y(2) = x(4-2) = x(2) = 3$

$y(3) = x(1) = -1$

So $y(n) = \{1, 5, 3, -1\}$ Ans

Q.22 $x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

find $y(n) = x((-n))_8 = x(8-n)$

$y(n) = x(8-n)$ mode

$n=0$

$y(0) = x(8) = x(8-8) = x(0) = 1$

$y(1) = x(8-1) = x(7) = 8$

$y(2) = x(8-2) = x(6) = 7$

$y(3) = x(8-3) = x(5) = 6$

$y(4) = x(8-4) = x(4) = 5$

$y(5) = x(8-5) = x(3) = 4$

$y(6) = x(8-6) = x(2) = 3$

$y(7) = x(8-7) = x(1) = 2$

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So $y(n) = \{1, 8, 7, 6, 5, 4, 3, 2\}$ Ans.

(6) Circular Convolution :-

If DFT $\{x(n)\} = X(k)$

then

$$\text{DFT}\{y(n)\} = x_1(n) \; (N) \; x_2(n) = Y(K) = X_1(K) \cdot X_2(K)$$

Proof :-

$$Y(K) = X_1(K) \cdot X_2(K)$$

$$\text{IDFT} \rightarrow y(n) = \frac{1}{N} \sum_{K=0}^{N-1} Y(K) \, e^{+j\frac{2\pi}{N} \cdot K \cdot n}$$

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} [X_1(K) \cdot X_2(K)] \, e^{+j\frac{2\pi}{N} \cdot K \cdot n} \quad \text{--- (1)}$$

As $X_1(K) = \sum_{m=0}^{N-1} x_1(m) \, e^{-j\frac{2\pi}{N} Km}$

$X_2(K) = \sum_{l=0}^{N-1} x_2(l) \, e^{-j\frac{2\pi}{N} Kl}$

Putting values of $X_1(K)$ & $X_2(K)$ in eq^n (1) :-

$$y(n) = \frac{1}{N} \sum_{K=0}^{N-1} \sum_{m=0}^{N-1} x_1(m) \, e^{-j\frac{2\pi}{N} Km} \sum_{l=0}^{N-1} x_2(l) \, e^{-j\frac{2\pi}{N} Kl} \cdot e^{j\frac{2\pi}{N} Kn} \quad \text{--- (2)}$$

Rearranging eq^n (2) :-

$$y(n) = \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{K=0}^{N-1} e^{j\frac{2\pi}{N}(n-m-l)K} \right] \quad \text{--- (3)}$$

we know that

$$\sum_{K=0}^{N-1} \alpha^K = \frac{1-\alpha^N}{1-\alpha} \quad ; \; \alpha \neq 1 \quad \text{--- (4)}$$

$$= N \quad ; \; \alpha = 1$$

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Date ....

$$\alpha = e^{j\frac{2\pi}{N}(n-m-l)}$$

$\alpha = 1$; for $l = n-m$

$\alpha \neq 1$; $l \neq n-m$

using this analysis in eq^n (4) :-

$$\frac{1 - e^{j\frac{2\pi}{N}(n-m-l) \cdot N}}{1 - e^{j\frac{2\pi}{N}(n-m-l)}} = 0 ; \quad l \neq n-m$$

$$N ; \quad l = n-m$$

So
$$y(n) = \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \boxed{\sum_{l=0}^{N-1}} x_2(l) \cdot N$$

we can ignore this summation because for $l = n-m$ only we get the non-zero value.

$$y(n) = \frac{1}{N} \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m) \cdot N$$

$$\boxed{y(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)}$$

$$y(n) = x_1(n) \; (N) \; x_2(n)$$

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Q.23 Compute the circular convolution of

$x_1(n) = \{2, 1, 2, 1\}$ & $x_2(n) = \{1, 2, 3, 4\}$ using DFT & IDFT.

Solⁿ:-

As we know

$$y(n) = x_1(n) \; (N) \; x_2(n) \longleftrightarrow X_1(K) \cdot X_2(K)$$

Using matrix method:-

$$X_N = [W_N] \cdot x_N$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

i) $DFT\{x_1(n)\} = X_1(k)$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1+2+1 \\ \cancel{2}-\cancel{j}-\cancel{2}+\cancel{j} \\ 2-\cancel{1}+\cancel{2}-\cancel{1} \\ \cancel{2}+\cancel{j}-\cancel{2}-\cancel{j} \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

ii) $DFT\{x_2(n)\} = X_2(k)$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

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$$X_1(k)\cdot X_2(k) = \begin{bmatrix} 6\times(10) \\ 0\times(-2+2j) \\ 2\times(-2) \\ 0\times(-2-2j) \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

ii) Taking IDFT $\{X_1(k)\cdot X_2(k)\}$

$$x_N = \frac{1}{N}\left[W_N^*\right] X_N$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 56 \\ 64 \\ 56 \\ 64 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

So, $y(n) = \{14, 16, 14, 16\}$ Ans

Date ....

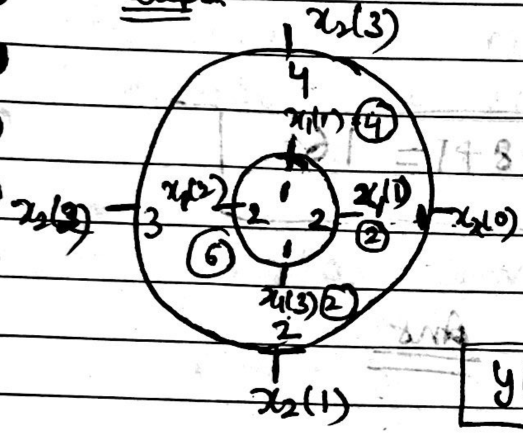
Q. Find the circular convolution of
$x_1(n) = \{2, 1, 2, 1\}$ & $x_2(n) = \{1, 2, 3, 4\}$

Sol^n General formula for circular convolution :-

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N$$

Method 1 :- [Stockham's method]

Step 1 :-

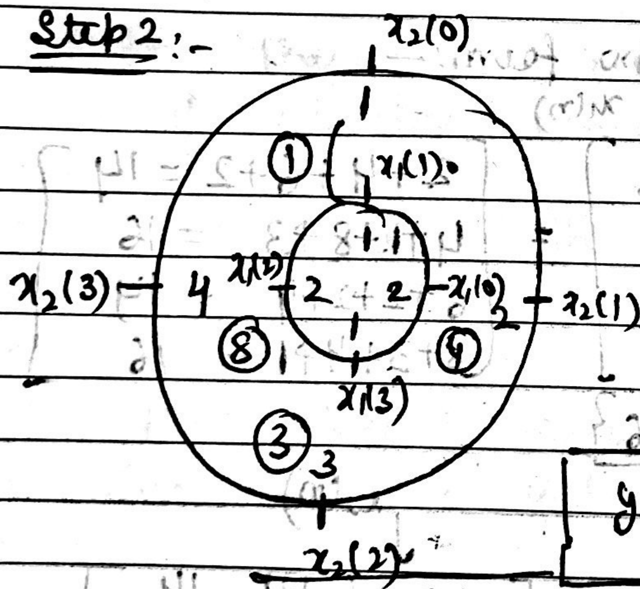


Inner circle → $x_1(n)$ [Anti-Clockwise Direction]

Outer circle → $x_2(n)$ [Clockwise Direction]

$\boxed{y(0) = 2 + 4 + 6 + 2 = 14}$

Step 2 :-

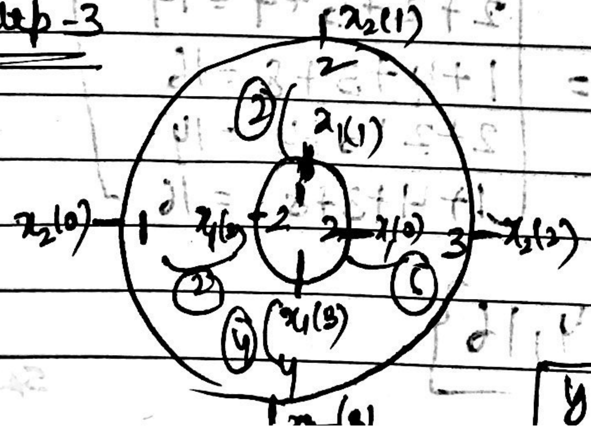


Outer circle → shifted by one step in anticlockwise Direction as compared to step 1.

Inner circle → As it was in Step 1

$\boxed{y(1) = 1 + 4 + 3 + 8 = 16}$

Step-3

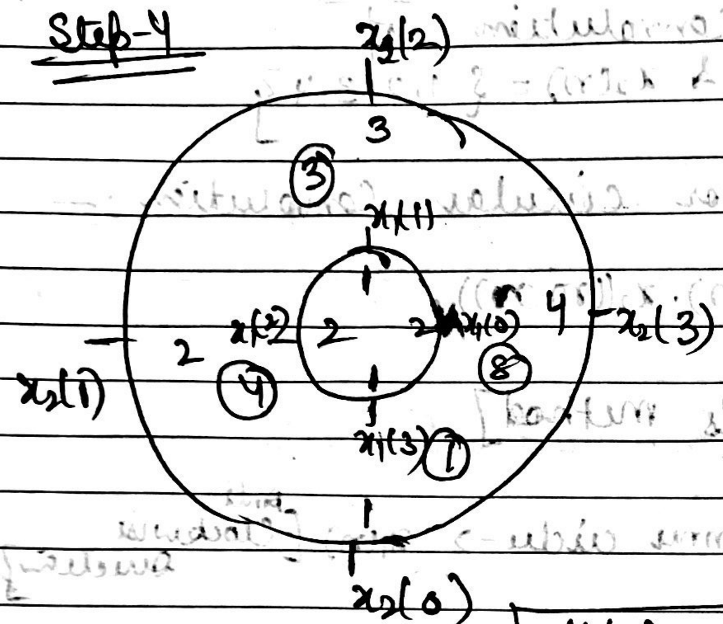


Outer circle → shifted by one step in anticlockwise Direction as compared to step

Inner circle → As it was in step 2.

$y(2) = 2 + 2 + 6 + 4 = 14$

Step-4



outer-circle → shifted by one step as compared to Step-3.

Inner circle → as it was in step3.

$$\boxed{y(3) = 4+3+8+1 = 16}$$

Hence.

$$y(n) = \{14, 16, 14, 16\}$$ **Ans**

method(2)

Matrix method:-

writing $x_2(n)$ in matrix form:-

$$\begin{array}{c} x_2(n) \\ \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \end{array} \begin{array}{c} x_1(n) \\ \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \end{array} = \begin{bmatrix} 2+4+6+2 = 14 \\ 4+1+8+3 = 16 \\ 6+2+2+4 = 14 \\ 8+3+4+1 = 16 \end{bmatrix}$$

So $y(n) = \{14, 16, 14, 16\}$

Or

$$\begin{array}{c} x_1(n) \\ \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} x_2(n) \\ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \end{array} = \begin{bmatrix} 2+2+6+4 = 14 \\ 1+4+3+8 = 16 \\ 2+2+6+4 = 14 \\ 1+4+3+8 = 16 \end{bmatrix}$$

So $y(n) = \{14, 16, 14, 16\}$

Q.25 Compute the circular convolution of two discrete-time sequences

$$s_1(n) = \{1, 2, 1, 2\} \quad \& \quad s_2(n) = \{3, 2, 1, 4\}$$

Sol^n

Circular convolution is defined as:—

$$y(m) = \sum_{n=0}^{N-1} s_1(n) \cdot s_2((m-n))_N$$

There are four steps in computation of circular convolution. These 4 steps are:—

1.) folding one of the two sequences.
2.) Shifting the folded sequence.
3.) multiplying the two sequences for obtaining the product sequence.
4. Summing the values of product sequence

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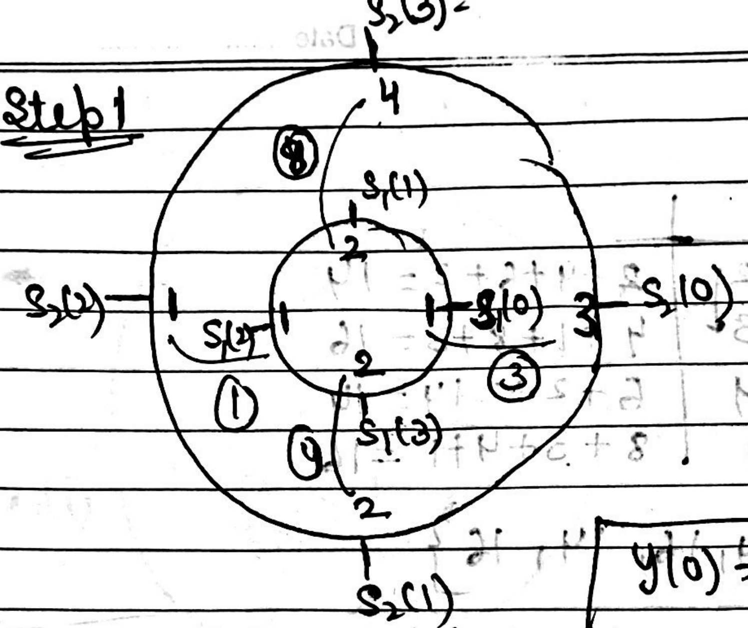
Step 1

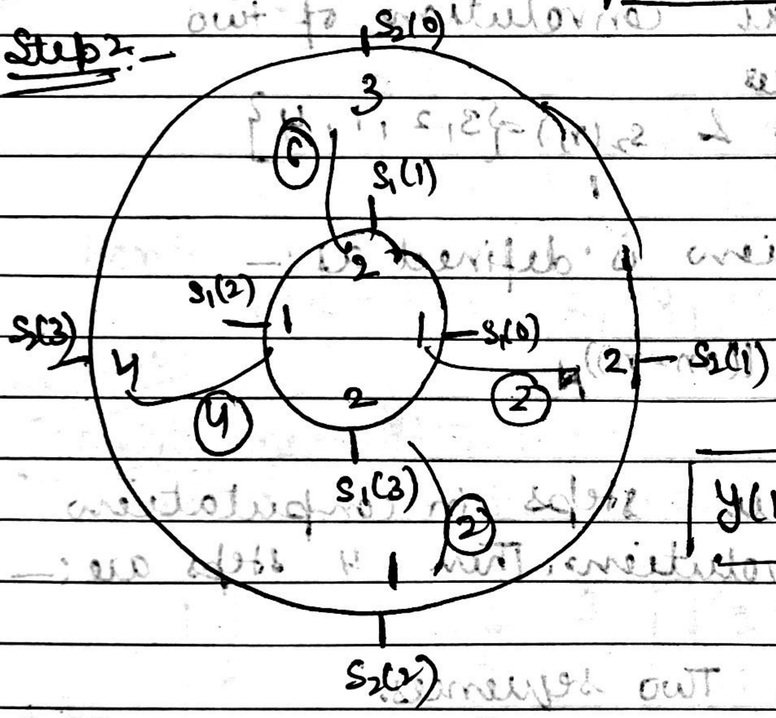


outer circle → ~~anti~~ clockwise Direction

Inner circle → Anticlockwise Direction

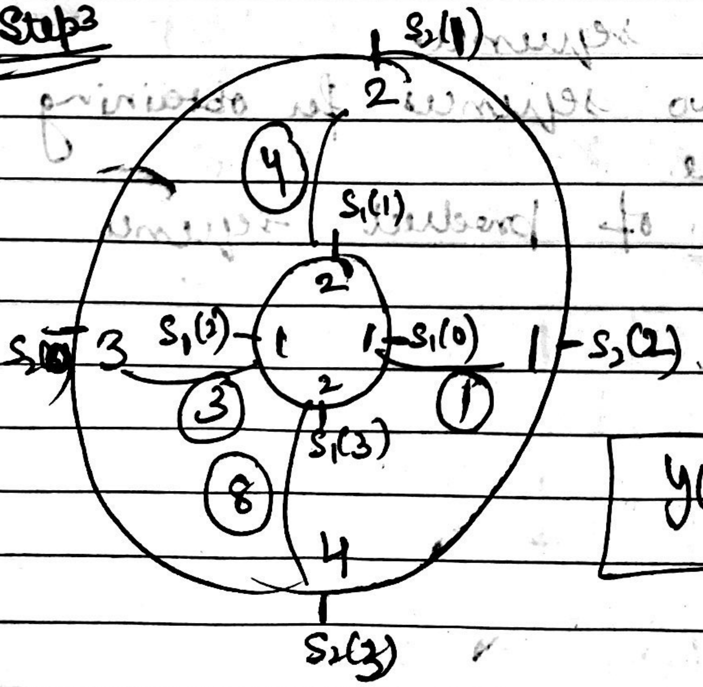
$$\boxed{y(0) = 8+1+4+3 = 16}$$

Step 2



$$\boxed{y(1) = 6+4+2+2 = 14}$$

Step 3



$$\boxed{y(2) = 4+3+8+1 = 16}$$

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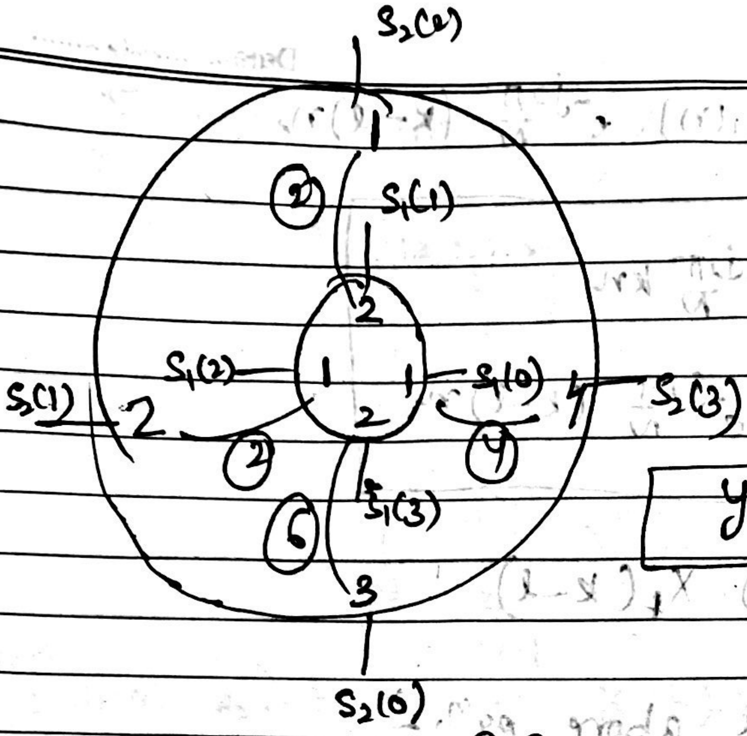

$$\boxed{y(3) = 2 + 2 + 6 + 4 = 14}$$

So $y(m) = \{16, 14, 16, 14\}$

(7) Multiplication or modulation Property of DFT:-

If $DFT\{x_1(n)\} = X_1(k)$

& $DFT\{x_2(n)\} = X_2(K)$

Then,

$$DFT\{x_1(n) \cdot x_2(n)\} = \frac{1}{N}\left[X_1(K) \; (N) \; X_2(K)\right]$$

Proof:-

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n) \, e^{-j\frac{2\pi}{N}kn} \quad \text{—(1)}$$

$$DFT\{x_1(n) \cdot x_2(n)\} = \sum_{n=0}^{N-1} \left[x_1(n) \cdot x_2(n)\right] e^{-j\frac{2\pi}{N}kn} \quad \text{—(2)}$$

Let $x_2(n) = \frac{1}{N} \sum_{l=0}^{N-1} X_2(l) \, e^{+j\frac{2\pi}{N}ln}$ —(3)

Putting value of $x_2(n)$ in eqn (2):-

$$DFT\{x_1(n) \cdot x_2(n)\} = \sum_{n=0}^{N-1} x_1(n) \cdot \frac{1}{N} \sum_{l=0}^{N-1} X_2(l) \, e^{j\frac{2\pi}{N}ln} \cdot e^{-j\frac{2\pi}{N}kn}$$

Rearranging Summation in above eqn

Date ....

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_2(l) \cdot \sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi}{N}(k-l)n}$$

we know that

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{so} \quad X(k-l) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}(k-l)n}}$$

$$DFT\{x_1(n) x_2(n)\} = \frac{1}{N} \sum_{l=0}^{N-1} X_2(l) \cdot X_1(k-l)$$

Rearranging the above eq^n.:-

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_1(k-l) \cdot X_2(l)$$ ← formula of convolution

$$DFT\{x_1(n) \cdot x_2(n)\} = \frac{1}{N}\left[X_1(k) \, (N) \, X_2(k)\right]$$

(8) <u>Parseval's Theorem</u>:-

If $DFT\{x_1(n)\} = X_1(k)$

& $DFT\{x_2(n)\} = X_2(k)$

then

$$\sum_{n=0}^{N-1} x_1(n) \cdot x_2^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2^*(k)$$

<u>Proof</u>:- As $x_2(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_2(k)\, e^{j\frac{2\pi}{N}kn}$

$$x_2^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_2^*(k)\, e^{-j\frac{2\pi}{N}kn}$$

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$$\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = \sum_{n=0}^{N-1} x_1(n) \cdot \frac{1}{N} \sum_{k=0}^{N-1} X_2^*(k)\, e^{-j\frac{2\pi}{N}kn}$$

Rearranging the above eqn :→

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_2^*(k) \sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_2^*(k) \cdot X_1(k)$$

$$\sum_{n=0}^{N-1} x_1(n) \cdot x_2^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k)\, X_2^*(k)$$ ← This is General form of Parseval's theorem

If $x_1(n) = x_2(n)$

then

$$\sum_{n=0}^{N-1} x_1(n) \cdot x_1^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_1^*(k)$$

$$X_1(k) \cdot X_1^*(k) = |S(k)|^2$$

so

$$\underbrace{\sum_{n=0}^{N-1} |x_1(n)|^2}_{\text{Represents Energy of } x(n)} = \frac{1}{N} \sum_{k=0}^{N-1} |X_1(k)|^2$$ ← Special form of Parseval's theorem

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Circular Correlation Property:-

If $DFT\{x(n)\} = X(k)$

& $DFT\{y(n)\} = Y(k)$

then

$$DFT\{\underbrace{r_{xy}(l)}_{\text{Cross-Correlation}}\} = R_{xy}(k) = X(k) \cdot Y^*(k)$$

Proof:- $$r_{xy}(l) = \underbrace{\sum_{n=0}^{N-1} x(n) \cdot y^*((n-l))_N}$$ Circular convolution Property.

$$r_{xy}(l) = x(l) \, N \, y^*((-l))_N \quad — (1)$$

As $DFT\{x^*((-n))_N\} = X^*(k)$

So $DFT\{y^*((-l))_N\} = Y^*(k)$

Taking DFT on both sides in eqn (1) :-

$$DFT\{r_{xy}(l)\} = DFT\{x(l) \, N \, y^*((-l))_N\}$$

$$\boxed{R_{xy}(k) = X(k) \cdot Y^*(k)}$$ ← Cross-Correlation

if $x(n) = y(n)$

$R_{xx}(k) = X(k) \cdot X^*(k)$

$$\boxed{R_{xx}(k) = |X(k)|^2}$$ ← Auto Correlation

Date ....

(10) Symmetry Property:-

Real as well as Imaginary value.

$$x(n) \xrightarrow{DFT} X(k) \leftarrow \text{Complex value.}$$

$$x(n) = x_R(n) + j\, x_I(n) \quad —(1)$$

$$X(k) = X_R(k) + j\, X_I(k) \quad —(2)$$

As

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \le k \le N-1$$

$$X_R(k) + j\, X_I(k) = \sum_{n=0}^{N-1} \left[x_R(n) + j\, x_I(n)\right] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left[x_R(n) + j\, x_I(n)\right]\left[\cos\frac{2\pi}{N}kn - j \sin\frac{2\pi}{N}kn\right]$$

$$= \sum_{n=0}^{N-1} x_R(n) \cos\frac{2\pi}{N}kn - j\, x_R(n) \sin\frac{2\pi}{N}kn$$

$$+ j\, x_I(n) \cos\frac{2\pi}{N}kn - j^2 \sin\frac{2\pi}{N}kn\Big]$$

$$X_R(k) + j\, X_I(k) = \sum_{n=0}^{N-1} \left\{x_R(n) \cos\frac{2\pi}{N}kn + x_I(n) \cdot \sin\frac{2\pi}{N}kn\right\}$$

$$-j\left\{x_R(n) \sin\frac{2\pi}{N}kn - x_I(n) \cos\frac{2\pi}{N}kn\right\}$$

on comparing real & imaginary terms:-

$$\boxed{X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n) \cos\frac{2\pi}{N}kn + x_I(n) \sin\frac{2\pi}{N}kn\right]}$$

$$\boxed{X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n) \sin\frac{2\pi}{N}kn - x_I(n) \cos\frac{2\pi}{N}kn\right]}$$

Similarly, we can define

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{+j\frac{2\pi}{N}kn} \quad \text{---(3)}$$

Putting the values of eqⁿ (1) & (2) in eqⁿ (3) :—

$$x_R(n) + j\,x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k) + j\,X_I(k)\right] \cdot e^{+j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_R(k)\cos\frac{2\pi}{N}kn + j\,X_R(k)\sin\frac{2\pi}{N}kn$$

$$+ j\,X_I(k)\cos\frac{2\pi}{N}kn - X_I(k)\sin\frac{2\pi}{N}kn$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left\{\left[X_R(k)\cos\frac{2\pi}{N}kn - X_I(k)\sin\frac{2\pi}{N}kn\right] + j\left[X_R(k)\sin\frac{2\pi}{N}kn + X_I(k)\cos\frac{2\pi}{N}kn\right]\right\}$$

So

$$\boxed{x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\frac{2\pi}{N}kn - X_I(k)\sin\frac{2\pi}{N}kn\right]}$$

$$\boxed{x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\frac{2\pi}{N}kn + X_R(k)\cos\frac{2\pi}{N}kn\right]}$$

$$X(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi}{N}kn + x_I(n)\sin\frac{2\pi}{N}kn\right]$$

$$-j\sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi}{N}kn - x_I(n)\cos\frac{2\pi}{N}kn\right]$$

if $x(n)$ is

i) Real & Even Sequence

Date ....

$X_I(n) = 0$

$x(n) = x_R(n)$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) \cos \frac{2\pi}{N} kn}$$

ii) If $x(n)$ is Real & odd Sequence:-

$x_I(n) = 0$

$x(n) = x_R(n)$

$$\boxed{X(k) = -j \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi}{N} kn}$$

iii) If $x(n)$ is purely Imaginary sequence:-

$x_R(n) = 0$

& $x(n) = +j\, x_I(n)$

$$X_R(k) + j X_I(k) = \sum_{n=0}^{N-1} x_I(n) \sin \frac{2\pi}{N} kn + j \sum_{n=0}^{N-1} x_I(n) \cos \frac{2\pi}{N} kn$$

$$\boxed{\begin{array}{l} X_R(k) = \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi}{N} kn \\ X_I(k) = \sum_{n=0}^{N-1} x(n) \cos \frac{2\pi}{N} kn \end{array}}$$

Date ....

Q. find the linear convolution of

$x_1(n) = \{2, 1, 2, 1\}$ & $x_2(n) = \{1, 2, 3, 4\}$

Sol^n

No. of samples will be $= [4+4-1] = 7$.

$x_1(n) \Rightarrow$				2	1	2	1
$x_2(n) \Rightarrow$				1	2	3	4
				8	4	8	4
			6	3	6	3	x
		4	2	4	2	x	x
	2	1	2	1	x	x	x
	2	5	10	16	12	11	4

So $y(n) = \{2, 5, 10, 16, 12, 11, 4\}$

Circular Convolution from:-
linear Convolution

$y(n) = \{2, 5, 10, 16, 12, 11, 4\}$

So $y(n) = \{14, 16, 14, 16\}$ Ans

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★ Linear filtering using DFT & IDFT :-

Linear filter operation is implemented with the help of Linear Convolution

$x(n)$ of Length L $\longrightarrow$ $h(n)$ of Length M (LTI) $\longrightarrow$ $y(n)$ Length $\Rightarrow N = L + M - 1$

[Fig. (a) Linear Convolution to obtain $y(n)$]

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k) \quad \text{--- (1)}$$

Taking Fourier Transform Both Sides :-

$$Y(\omega) = F\left\{ \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k) \right\} \quad \text{--- (2)}$$

Using convolution property of Fourier Transform :-

$$F\left\{ x_1(n) * x_2(n) \right\} = X_1(\omega) \cdot X_2(\omega)$$

So, $Y(\omega) = H(\omega) \cdot X(\omega)$ --- (3)

we know that

$$Y(k) = Y(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$$

$$X(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$$

$$H(k) = H(\omega)\Big|_{\omega = \frac{2\pi k}{N}} \quad \text{where } k = 0, 1 \text{ --- } N-1.$$

So, $Y(k) = H(k) \cdot X(k)$; $k = 0, 1 \cdots N-1$ --- (4)

$$y(n) = IDFT\left\{ Y(k) \right\} = IDFT\left\{ X(k) \cdot H(k) \right\} \quad \text{--- (5)}$$

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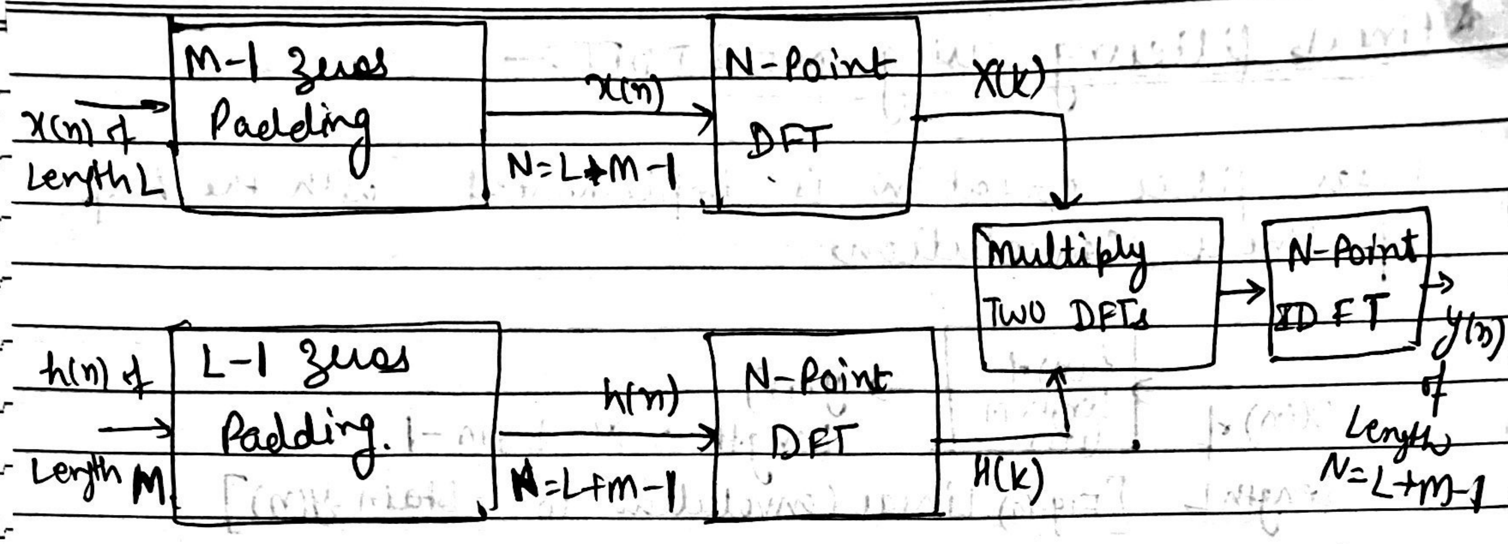


Fig(b) y(n) obtained through DFT & IDFT.

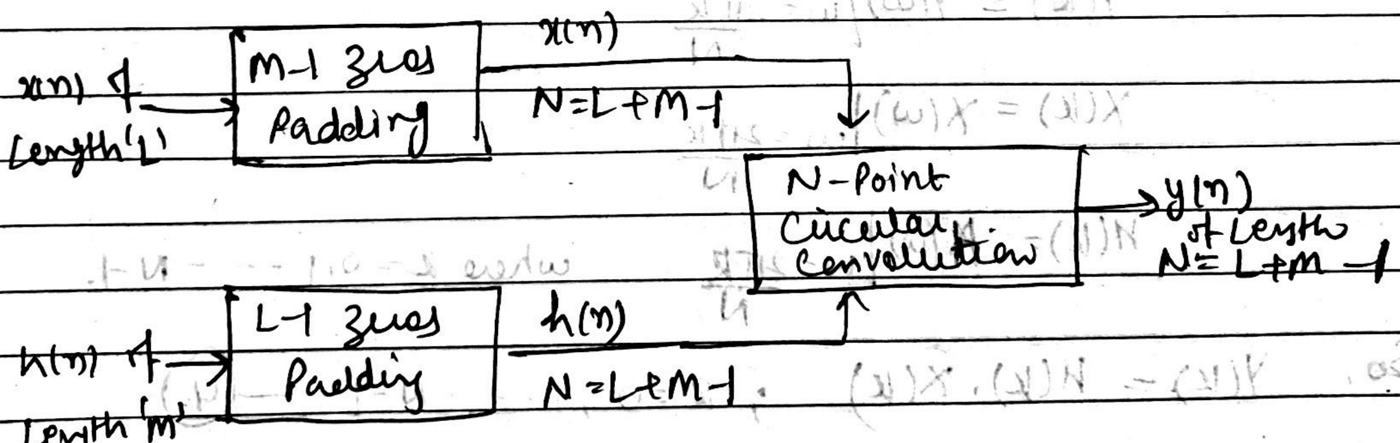
Linear filtering using Circular Convolution.

$$Y(K) = X(K) \cdot H(K) \quad \text{--- (4)}$$

from circular convolution we know that

$$x_1(n) \; (N) \; x_2(n) = X_1(k) \cdot X_2(k)$$

$$y(n) = x(n) \; (N) \; h(n)$$



fig(c) {Linear filtering using circular Convolution}

Q Perform the following on the sequences $x(n) = \{1, 2, 3, 1\}$ and $h(n) = \{1, 1, 1\}$

(i) Linear Convolution
(ii) Circular Convolution
(iii) Linear Convolution using Circular Convolution

Sol^n (i) <u>Linear Convolution</u>

Length of $x(n) \Rightarrow L = 4$

" " $h(n) \Rightarrow M = 3$

$\because$ After linear convolution, No. of samples should be

$\therefore \quad N = L + M - 1 = 4 + 3 - 1 = 6$

$$
\begin{array}{cccccc}
 & & 1 & 2 & 3 & 1 \\
 & & & 1 & 1 & 1 \\
\hline
 & & 1 & 2 & 3 & 1 \\
 & 1 & 2 & 3 & 1 & \times \\
1 & 2 & 3 & 1 & \times & \times \\
\hline
1 & 3 & 6 & 6 & 4 & 1
\end{array}
$$

$y(n) = \{1, 3, 6, 6, 4, 1\}$

6 Samples

(ii) Linear Convolution using Circular Convolution :—

$h(n) = \{1, 1, 1, 0\}$ ← Appended zero to make length of sequences equal.

$$
\begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix}
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}
=
\begin{bmatrix} 5 \\ 4 \\ 6 \\ 6 \end{bmatrix}
$$

So $y(n) = \{5, 4, 6, 6\}$

4 samples.

As $y(n) = \{1, 3, 6, 6, 4, 1\}$ ← using Linear convolution

$= \{5, 4, 6, 6\}$ Ans.

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(iii) Linear Convolution using Circular Convolution-

$N = L + M - 1$ ← Length of Both Sequences should be

Append $M-1$ zeros to $x(n)$

$L-1$ " " $h(n)$

$M - 1 = 3 - 1 = 2$

$L - 1 = 4 - 1 = 3$

2 zeros.

$x(n) = \{1, 2, 3, 1, 0, 0\}$

$h(n) = \{1, 1, 1, 0, 0, 0\}$ 3 zeros.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 2 \\ 2 & 1 & 0 & 0 & 1 & 3 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 4 \\ 4 \\ 1 \end{bmatrix}$$

$y(n) = \{1, 3, 6, 4, 4, 1\}$

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* DFT for Linear filtering of long Duration sequence :-

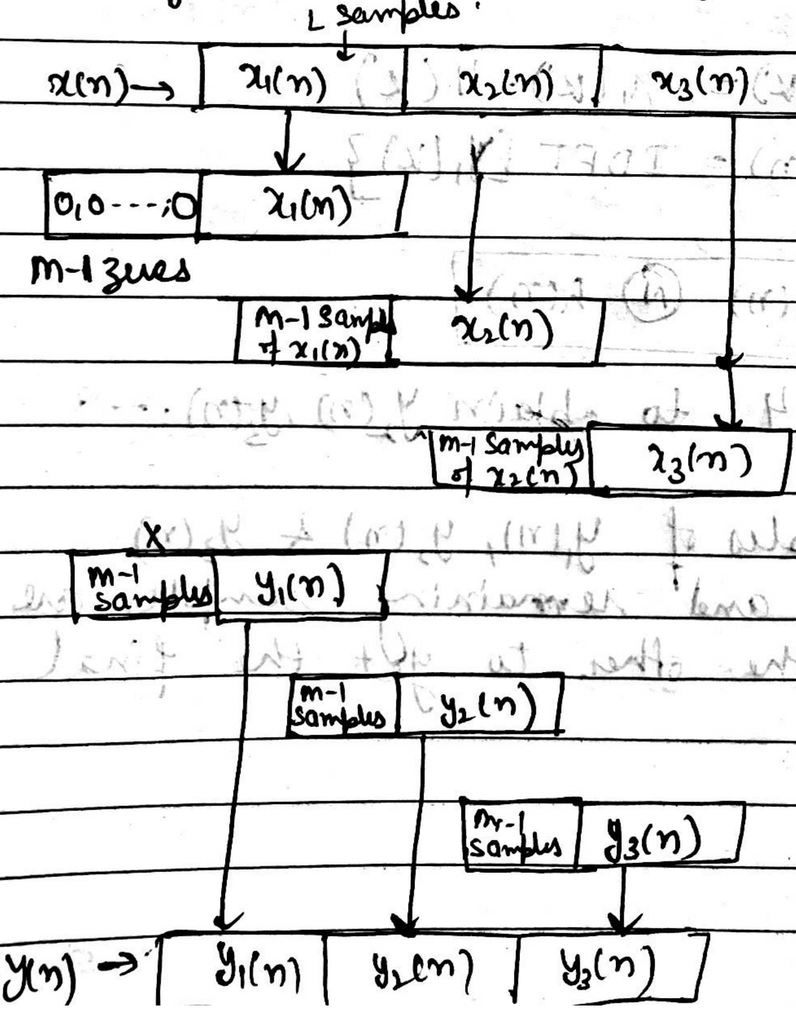
In last topic we have seen that linear convolution can be obtained using DFT. But if the sequence is of long duration then it becomes time consuming process.

So there are two method for linear filtering of long duration sequences :-

① Overlap Save method
② Overlap Add Method.

① Overlap Save Method :-

Let Unit Impulse response $h(n)$ of length 'M'
$x(n) \rightarrow$ is long duration sequence, which is segmented into the blocks of length 'L'.



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Step 1:- Select value of $N = 2^m$

Step 2:- The length of $h(n)$ is made 'N' by padding L-1 zeros [As $N = M+L-1$]

$$h(n) = \{h(0), h(1), \ldots h(M-1), 0, 0 \ldots 0\}$$

Step 3:- The sequence $x(n)$ is divided into sub-sequences of length 'N' as:-

$$x_1(n) = \{\underbrace{0, 0 \ldots 0}_{M-1 \text{ zeros}}, x(0), x(1) \ldots, x(L-1)\}$$

$$x_2(n) = \{\underbrace{x(L-M+1) + \ldots x(L-1)}_{M-1 \text{ Samples of } x_1(n)}, x(L), x(L+1) \ldots \ldots x(2L-1)\}$$

$$x_3(n) = \{\underbrace{x(L-M+1) \ldots x(2L-1)}_{M-1 \text{ Samples of } x_2(n)}, x(2L), x(2L+1) \ldots \ldots x(3L-1)\}$$

Step 4:- Calculate $Y_1(K) = X_1(K) \cdot H(K)$

$$y_1(n) = \text{IDFT}\{Y_1(K)\}$$

or

$$\boxed{y_1(n) = x_1(n) \; N \; h(n)}$$

Step 5:- Repeat Step 4 to obtain $y_2(n), y_3(n) \ldots$

Step 6:- First M-1 samples of $y_1(n)$, $y_2(n)$ & $y_3(n)$ are discarded and remaining samples are fitted one after the other to get the final sequence.

Date ....

Q. find $y(n)$ for $h(n) = \{1, 1, 1\}$ and $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$ using overlap - save method.

Solⁿ $h(n) = \{1, 1, 1\}$, $\therefore M = 3$

Step-1 $N = 2^M = 2^3 = 8$

So $\boxed{N=8}$

As we know $N = M + L - 1$ So $L = N - M + 1$

$L = 8 - 3 + 1$

$\boxed{L = 6}$, $\boxed{L-1 = 5}$

Step 2 $h(n) = \{1, 1, 1, \underbrace{0, 0, 0, 0, 0}_{L-1 \text{ zeros}}\}$

Steps 3:- ~~x(n) = {3, -1, 0, 1, 3, 2}~~

$x_1(n) = \{\underbrace{0, 0}_{M-1 \text{ zeros}}, \underbrace{3, -1, 0, 1, 3, 2}_{L \text{ samples}}\}$

$x_2(n) = \{\underbrace{3, 2}_{M-1 \text{ samples of } x_1(n)}, 0, 1, 2, 1, 0, 0\}$

Step 4:- $y_1(n) = x_1(n) \; N \; x_2(n)$

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Date ....

$$\begin{array}{c} x_1(n) \\ \begin{bmatrix} 0 & 2 & 3 & 1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 & -1 & 3 \\ 3 & 0 & 0 & 2 & 3 & 1 & 0 & -1 \\ -1 & 3 & 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 2 & 3 & 1 \\ 1 & 0 & -1 & 3 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & -1 & 3 & 0 & 0 & 2 \\ 2 & 3 & 1 & 0 & -1 & 3 & 0 & 0 \end{bmatrix}_{8\times 8} \end{array} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 2 \\ 2 \\ 0 \\ 4 \\ 6 \end{bmatrix}$$

So $y_1(n) = \{ \underbrace{5, 2}_{\times}, 3, 2, 2, 0, 4, 6 \}$

Step 5:- $y_2(n) = x_2(n) \, (N) \, h(n)$

$$\begin{bmatrix} 3 & 0 & 0 & 1 & 2 & 1 & 0 & 2 \\ 2 & 3 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 3 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 & 3 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 3 \\ 3 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

So $y_2(n) = \{ \underbrace{3, 5}_{\times}, 5, 3, 3, 4, 3, 1 \}$

Step 6:- discard $M-1$ samples of $y_1(n)$ & $y_2(n)$

$\boxed{M-1 = 2}$

$y(n) = \{ 3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1 \}$ Ans

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Overlap- Add Method:-

$h(n) \rightarrow$ length 'm'

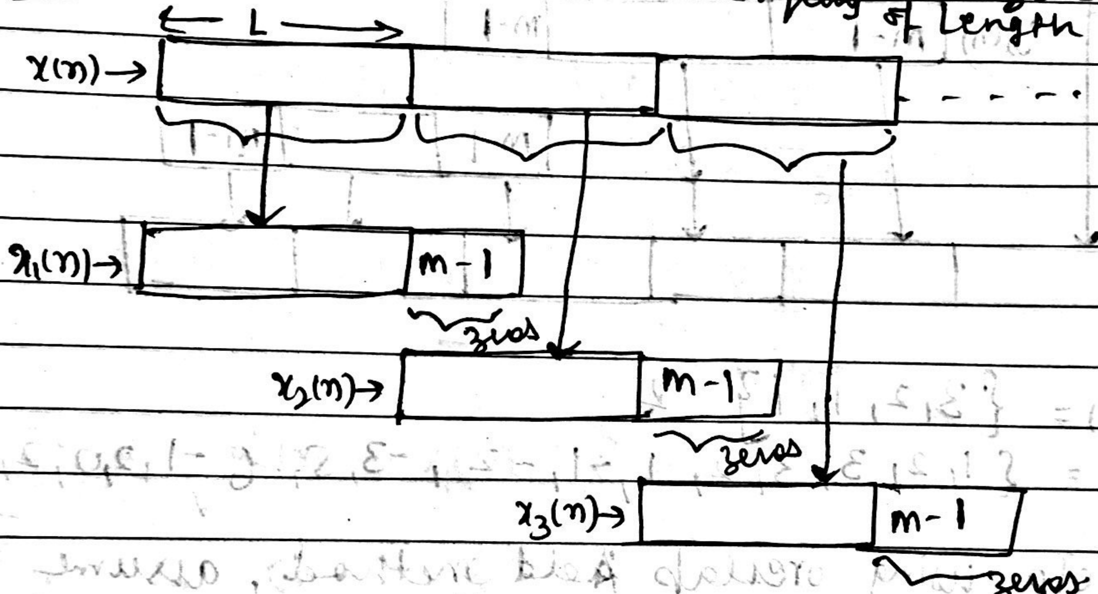
$x(n) \rightarrow$ segmented in blocks

Step-1 : Select $N = 2^m$

Step 2 : Length of $h(n)$ is made 'N' by padding L-1 zeros. $[N = L+m-1]$

$$h(n) = \{h(0), h(1) \cdots \underbrace{, 0, 0, \cdots 0}_{m-1 \text{ zeros}}\}$$

Step 3 :- $x(n)$ is divided into subsequence of Length 'N' as:-



$$x_1(n) = \{x(0), x(1), \cdots x(L-1), \underbrace{0, 0 \cdots 0}_{m-1 \text{ zeros}}\}$$

$$x_2(n) = \{x(L), x(L+1) \cdots x(2L-1), \underbrace{0, 0 \cdots 0}_{m-1 \text{ zeros}}\}$$

$$x_3(n) = \{x(2L), x(2L+1) \cdots x(3L-1), \underbrace{0, 0 \cdots 0}_{m-1 \text{ zeros}}\}$$

Step 4: Calculate: $Y_1(K) = X_1(K) \cdot H(K)$

$$y_1(n) = \text{IDFT}\{Y_1(K)\}$$

OR

$$\boxed{y_1(n) = x_1(n) \; N \; h(n)}$$

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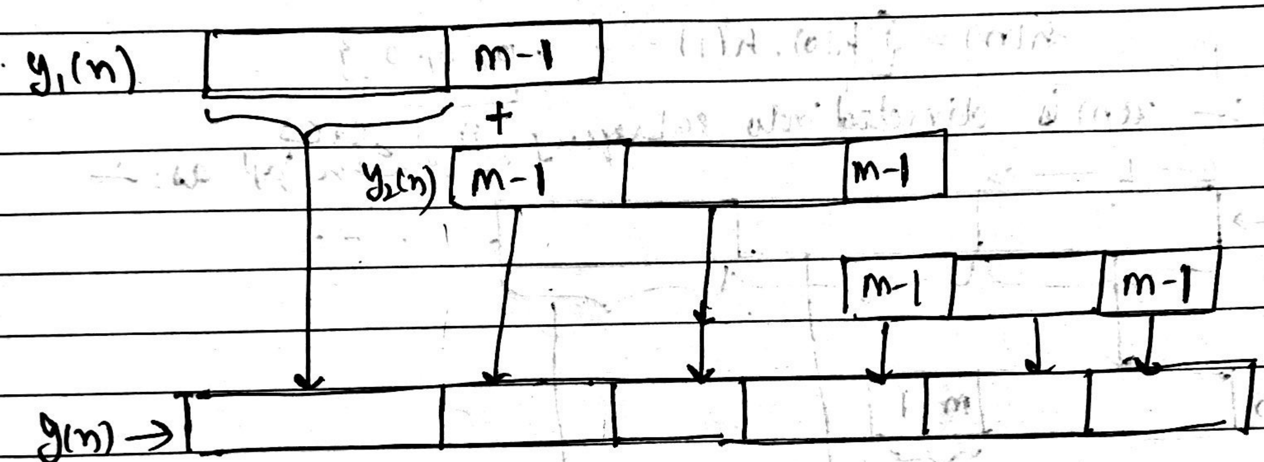
Step 5 :- Repeat Step 4 to obtain $y_2(n), y_3(n)$ —

Step 6 :- Add all M-1 Samples of each o/p sequence to first M-1 samples of succeeding o/p sequence. Such sequences are fitted one after another to get final sequence.



Q. for $h(n) = \{3, 2, 1, 1\}$ &

$x(n) = \{1, 2, 3, 3, 2, 1, -1, -2, -3, 5, 6, -1, 2, 0, 2, 1\}$

find the o/p using overlap Add method, assume block length as 7.

Sol^n

$h(n) = \{3, 2, 1, 1\}$, $\therefore M = 4$

Step 1 :- $N = 7$ ← This is given in Question. [$N = 2^M$ ← if N is not given]

$N = M + L - 1$

$L = 7 - 4 + 1 = 4$

$\boxed{L = 4}$

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Step 2:- $h(n) = \{3, 2, 1, 1, \underbrace{0, 0, 0}_{L-1 \text{ zeros}}\}$

Step 3:- Since $L = 4$, we will take 4 samples at a time.

$x_1(n) = \{1, 2, 3, 3, \underbrace{0, 0, 0}_{m-1 \text{ zeros}}\}$

$x_2(n) = \{2, 1, -1, -2, \underbrace{0, 0, 0}_{m-1 \text{ zeros}}\}$

$x_3(n) = \{-3, 5, 6, -1, \underbrace{0, 0, 0}_{m-1 \text{ zeros}}\}$

$x_4(n) = \{2, 0, 2, 1, \underbrace{0, 0, 0}_{m-1 \text{ zeros}}\}$

Step 4 $y_1(n) = x_1(n) \; (N) \; h(n)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 & 3 & 2 \\ 2 & 1 & 0 & 0 & 0 & 3 & 3 \\ 3 & 2 & 1 & 0 & 0 & 0 & 3 \\ 3 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 18 \\ 14 \\ 18 \\ 11 \\ 6 \\ 3 \end{bmatrix}$$

So, $y_1(n) = \{3, 8, 14, 18, 11, 6, 3\}$

Date ....

Step 5:- $y_2(n) = x_2(n) \, (N) \, h(n)$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & -1 & 1 \\ 1 & 2 & 0 & 0 & 0 & -2 & -1 \\ -1 & 1 & 2 & 0 & 0 & 0 & -2 \\ -2 & -1 & 1 & 2 & 0 & 0 & 0 \\ 0 & -2 & -1 & 1 & 2 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 1 \\ -5 \\ -4 \\ -3 \\ -2 \end{bmatrix}$$

$y_2(n) = \{6, 7, 1, -5, -4, -3, -2\}$

$y_3(n) = x_3(n) \, (N) \, h(n)$

$$\begin{bmatrix} -3 & 0 & 0 & 0 & -1 & 6 & 5 \\ 5 & -3 & 0 & 0 & 0 & -1 & 6 \\ 6 & 5 & -3 & 0 & 0 & 0 & -1 \\ -1 & 6 & 5 & -3 & 0 & 0 & 0 \\ 0 & -1 & 6 & 5 & -3 & 0 & 0 \\ 0 & 0 & -1 & 6 & 5 & -3 & 0 \\ 0 & 0 & 0 & -1 & 6 & 5 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 9 \\ 25 \\ 11 \\ 9 \\ 5 \\ -1 \end{bmatrix}$$

$y_3(n) = \{-9, 9, 25, 11, 9, 5, -1\}$

Similarly.

$y_4(n) = x_4(n) \, (N) \, h(n)$

$y_4(n) = \{2, 0, 2, 1, 0, 0, 0\} \, (N) \, \{3, 2, 1, 1, 0, 0, 0\}$

So $y_4(n) = \{6, 4, 8, 9, 4, 3, 1\}$

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$\boxed{M-1=3}$ Date ....

Step 6:-

$y_1(n) = 3, 8, 14, 18, 11, 6, 3$

$y_2(n) = 6, 7, 1, -5, -4, -3, -2$

$y_3(n) = -9, 9, 25, 11, 9, 5, -1$

$y_4(n) = 6, 4, 8, 9, 4, 3, 1$

$y(n) = 3, 8, 14, 18, 17, 13, 4, -5, -13, 6, 23, 11, 15, 9, 7, 9, 4, 3, 1$

Ans

Q. Compute linear convolution of sequence

$x(n) = \{7, 6, 4, 5, 2, 4, 5, 2, 3\}$

$h(n) = \{1, 2, 3\}$ using overlap save method.

Step 1:- $N = 2^m = 2^3 = 8$

Step 2:-

$N = L + M - 1$

$8 = L + 3 - 1$

$\boxed{L = 6}$ $\boxed{L-1 = 5 \text{ zeros}}$

So $h(n) = \{1, 2, 3, \underbrace{0, 0, 0, 0, 0}_{(L-1)\text{ zeros}}\}$

Step 3:-

$x_1(n) = \{\underbrace{0, 0}_{\text{Padding }(M-1)\text{ zeros}}, 7, 6, 4, 5, 2, 4\}$

$x_2(n) = \{\underbrace{2, 4}_{M-1 \text{ samples of } x_1(n)}, 5, 2, 3, 0, 0, 0\}$

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Step-4 : $y_1(n) = x_1(n) \; (N) \; h(n)$

$$\begin{bmatrix} 0 & 4 & 2 & 5 & 4 & 6 & 7 & 0 \\ 0 & 0 & 4 & 2 & 5 & 4 & 6 & 7 \\ 7 & 0 & 0 & 4 & 2 & 5 & 4 & 6 \\ 6 & 7 & 0 & 0 & 4 & 2 & 5 & 4 \\ 4 & 6 & 7 & 0 & 0 & 4 & 2 & 5 \\ 5 & 4 & 6 & 7 & 0 & 0 & 4 & 2 \\ 2 & 5 & 4 & 6 & 7 & 0 & 0 & 4 \\ 4 & 2 & 5 & 4 & 6 & 7 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 12 \\ 7 \\ 20 \\ 37 \\ 31 \\ 24 \\ 23 \end{bmatrix}$$

$y_2(n) = x_2(n) \; (N) \; h(n)$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 3 & 2 & 5 & 4 \\ 4 & 2 & 0 & 0 & 0 & 3 & 2 & 5 \\ 5 & 4 & 2 & 0 & 0 & 0 & 3 & 2 \\ 2 & 5 & 4 & 2 & 0 & 0 & 0 & 3 \\ 3 & 2 & 5 & 4 & 2 & 0 & 0 & 0 \\ 0 & 3 & 2 & 5 & 4 & 2 & 0 & 0 \\ 0 & 0 & 3 & 2 & 5 & 4 & 2 & 0 \\ 0 & 0 & 0 & 3 & 2 & 5 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 19 \\ 24 \\ 22 \\ 12 \\ 9 \\ 0 \end{bmatrix}$$

FFT [Fast Fourier Transform]

Date ....

Although DFT plays a vital role in several applications of DSP. These applications may include linear filtering, correlation analysis & spectrum analysis.

In the previous chapter we have studied how to obtain DFT of a sequence by using direct computation. Basically, the direct computation of DFT requires large number of computations. Hence more processing time is required.

For the computation of N-point DFT, N^2 complex multiplications and (N^2-N) complex additions are required. If the value of N is large then the no. of computations will go into lakhs. This proves inefficiency of direct-DFT computation.

So, to solve the computation issues an efficient algorithm is used to compute DFT, which is named as FFT [fast fourier Transform]

* **Radix-2 FFT Algorithms:-**

These algorithms are the most widely used FFT algorithms

Types:-

· While computing FFT, divide number of input samples by 2, till we reach minimum two samples. Based on this division, there are two algorithms as under:-

1.) Radix-2 Decimation in Time (DIT) Algorithms.

2.) Radix-2 Decimation in frequency (DIF) Algorithm.

(DIT FFT)

Date ....

* **Radix-2 DIT (Decimation In Time) Algorithm**

The word 'decimate' means to break into parts. Therefore, DIT indicates dividing (splitting) the sequence in time domain.

Let sequence $x(n)$ of length N.

$x(n) = \{x(0), x(1), x(2), x(3) \ldots, x(N-2), x(N-1)\}$

lets decimate $x(n)$ into two sequences of length $\frac{N}{2}$.

even indexed sequence : $\{x(0), x(2), x(4), \ldots x(N-2)\}$

odd indexed sequence : $\{x(1), x(3), x(5) \ldots x(N-1)\}$

we know that N-Point DFT can be given as:-

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \le K \le N-1 \quad \text{--- (1)}$$

Decimating eq^n (1) into even & odd indexed sequences.

$$X(K) = \underbrace{\sum_{n=0}^{N-2} x(n) W_N^{kn}}_{\text{even indexed}} + \underbrace{\sum_{n=1}^{N-1} x(n) W_N^{kn}}_{\text{odd indexed}} \quad \text{--- (2)}$$

Put $n = 2l$ in first term of eq^n (2) and $n = 2l+1$ in second term.

So,

$$X(K) = \sum_{l=0}^{\frac{N}{2}-1} x(2l) W_N^{2kl} + \sum_{l=0}^{\frac{N}{2}-1} x(2l+1) W_N^{k(2l+1)} \quad \text{--- (3)}$$

1st term →
$2l = 0$
$l = 0$ ← lower limit
$2l = N-2$
$l = \frac{N}{2} - 1$ ← upper limit

2nd term
$n = 2l+1 = 1$
$2l = 0$
$l = 0$ ← lower limit
$2l+1 = N-1$ upper limit
$2l = N-2$, $l = \frac{N}{2} - 1$

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$$X(K) = \sum_{l=0}^{\frac{N}{2}-1} x(2l)\, W_N^{2Kl} + \sum_{l=0}^{\frac{N}{2}-1} x(2l+1)\, W_N^{K(2l+1)} \quad \text{--- (3)}$$

Lets replace $x(2l)$ with $g(l)$ and $x(2l+1)$ with $h(l)$

$$X(K) = \sum_{l=0}^{\frac{N}{2}-1} g(l)\, W_N^{2Kl} + \sum_{l=0}^{\frac{N}{2}-1} h(l)\, W_N^{K(2l+1)}$$

$$X(K) = \sum_{l=0}^{\frac{N}{2}-1} g(l)\, W_N^{2Kl} + \sum_{l=0}^{\frac{N}{2}-1} h(l) \cdot W_N^{2Kl} \cdot W_N^{K}$$

As $W_N = e^{-j\frac{2\pi}{N}} \Rightarrow W_N^2 = e^{-j\frac{2\pi}{N}\cdot 2} = e^{-j\frac{2\pi}{N/2}} = W_{N/2}$

$\boxed{\text{So } W_N^2 \text{ can be written as } W_{N/2}}$

Rearranging the above eqn.

So $X(K) =$

$$X(K) = \underbrace{\sum_{l=0}^{N/2-1} g(l)\, W_{\frac{N}{2}}^{Kl}}_{\frac{N}{2}\text{ DFT of Even Seq.}} + W_N^{K} \underbrace{\sum_{l=0}^{\frac{N}{2}-1} h(l)\, W_{N/2}^{Kl}}_{\frac{N}{2}\text{ DFT of odd Sequence}} \quad \text{--- (4)}$$

$$X(K) = G(K) + W_N^K H(K) \;;\; 0 \le K \le \frac{N}{2}-1 \quad \text{--- (5)}$$

$G(K)$ & $H(K)$ are Periodic Signals with Period $\frac{N}{2}$

So

$$X(K) = G\left(K-\frac{N}{2}\right) + W_N^K H\left(K-\frac{N}{2}\right) \;;\; \frac{N}{2} \le K \le \frac{N}{2}-1 \quad \text{--- (6)}$$

for example:

$N=8$, $\therefore K \to 0$ to 7

$K=0$ to 3 in eqn (5) & $K=4$ to 7 in eqn (6)

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Taking eqⁿ (5) :—

for $k=0$, $X(0) = G(0) + W_8^0 H(0)$

" $k=1$, $X(1) = G(1) + W_8^1 H(1)$

" $k=2$, $X(2) = G(2) + W_8^2 H(2)$

" $k=3$, $X(3) = G(3) + W_8^3 H(3)$

Taking eqⁿ (6) :—

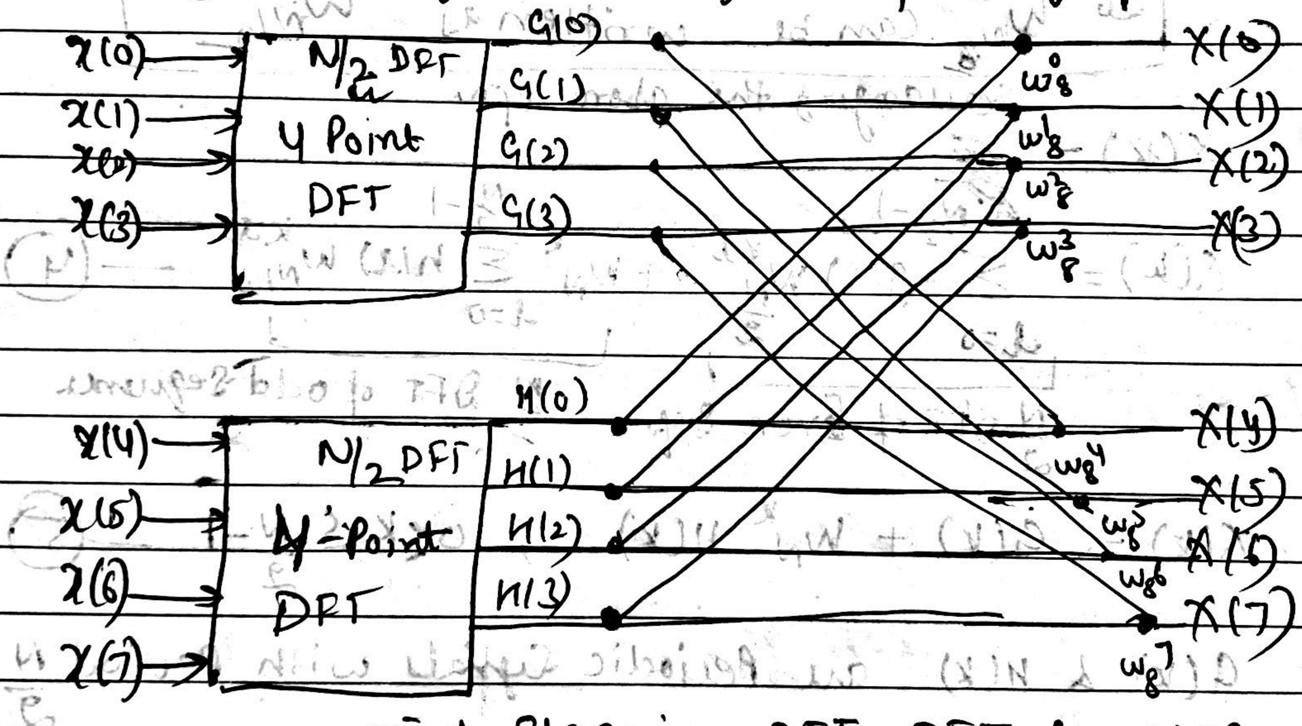
for $k=4$, $X(4) = G(0) + W_8^4 H(0)$

" $k=5$, $X(5) = G(1) + W_8^5 H(1)$

" $k=6$, $X(6) = G(2) + W_8^6 H(2)$

" $k=7$, $X(7) = G(3) + W_8^7 H(3)$

Representing these eqⁿs in signal flow graph.



First Stage in DIT FFT for $N=8$

As $G(k)$ & $H(k)$ are $\frac{N}{2}$ point sequences, so these can be represented as combination of $\frac{N}{4}$ points.

$$G(k) = \sum_{l=0}^{N/2-1} g(l) W_{N/2}^{kl} \quad \text{—(7)}$$

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$$G(k) = \sum_{l=0}^{\frac{N}{2}-1} g(l)\, W_{N/2}^{kl} + \sum_{l=0}^{N/2-1} g(l)\, W_{N/2}^{kl}$$

$$g(l) = \{g(0), g(1), g(2), \ldots, g\left(\tfrac{N}{2}-2\right), g\left(\tfrac{N}{2}-1\right)$$

Put $l = 2l$ in I^{st} term, $l = 2l+1$ in second term of eq^n (7):—

$$G(k) = \sum_{l=0}^{N/4-1} \underbrace{g(2l)}_{a(l)} W_{N/2}^{2kl} + \sum_{l=0}^{N/4-1} \underbrace{g(2l+1)}_{b(l)} W_{N/2}^{k(2l+1)}$$

$$G(k) = \underbrace{\sum_{l=0}^{\frac{N}{4}-1} a(l) \cdot W_{N/4}^{4kl}}_{A(k)} + W_{N/2}^{k} \underbrace{\sum_{l=0}^{\frac{N}{4}-1} b(l)\, W_{N/4}^{4kl}}_{B(k)}$$

$$\boxed{\text{As } W_{\frac{N}{2}}^{2} = W_{\frac{N}{4}}}$$

$$G(k) = A(k) + W_{N/2}^{k} B(k) \;;\quad 0 \le k \le \frac{N}{4}-1 \quad \text{—— (8)}$$

similary for $H(k)$:—

$$H(k) = C(k) + W_{N/2}^{k} D(k) \;;\quad 0 \le k \le \frac{N}{4}-1 \quad \text{—— (9)}$$

Here $A(k)$, $B(k)$, $C(k)$ & $D(k)$ are periodic with period $\frac{N}{4}$

Hence.

$$G(k) = A\left(k-\tfrac{N}{4}\right) + W_{N/2}^{k} B\left(k-\tfrac{N}{4}\right) \;;\quad \frac{N}{4} \le k \le \frac{N}{2}-1 \quad \text{—— (10)}$$

$$H(k) = C\left(k-\tfrac{N}{4}\right) + W_{N/2}^{k} D\left(k-\tfrac{N}{4}\right) \;;\quad \frac{N}{4} \le k \le \frac{N}{2}-1 \quad \text{—— (11)}$$

Put $k=0$ & 1 for eq^n (8) & (9):—

$k=2$ & 3 for eq^n (10) & (11):—

$k=0$ $G(0) = A(0) + W_4^0 B(0)$	$H(0) = C(0) + W_4^0 D(0)$
$k=1$ $G(1) = A(1) + W_4^1 B(1)$	$H(1) = C(1) + W_4^1 D(1)$
$k=2$ $G(2) = A(0) + W_4^2 B(0)$	$H(2) = C(0) + W_4^2 D(0)$
$k=3$ $G(3) = A(1) + W_4^3 B(1)$	$H(3) = C(1) + W_4^3 D(1)$

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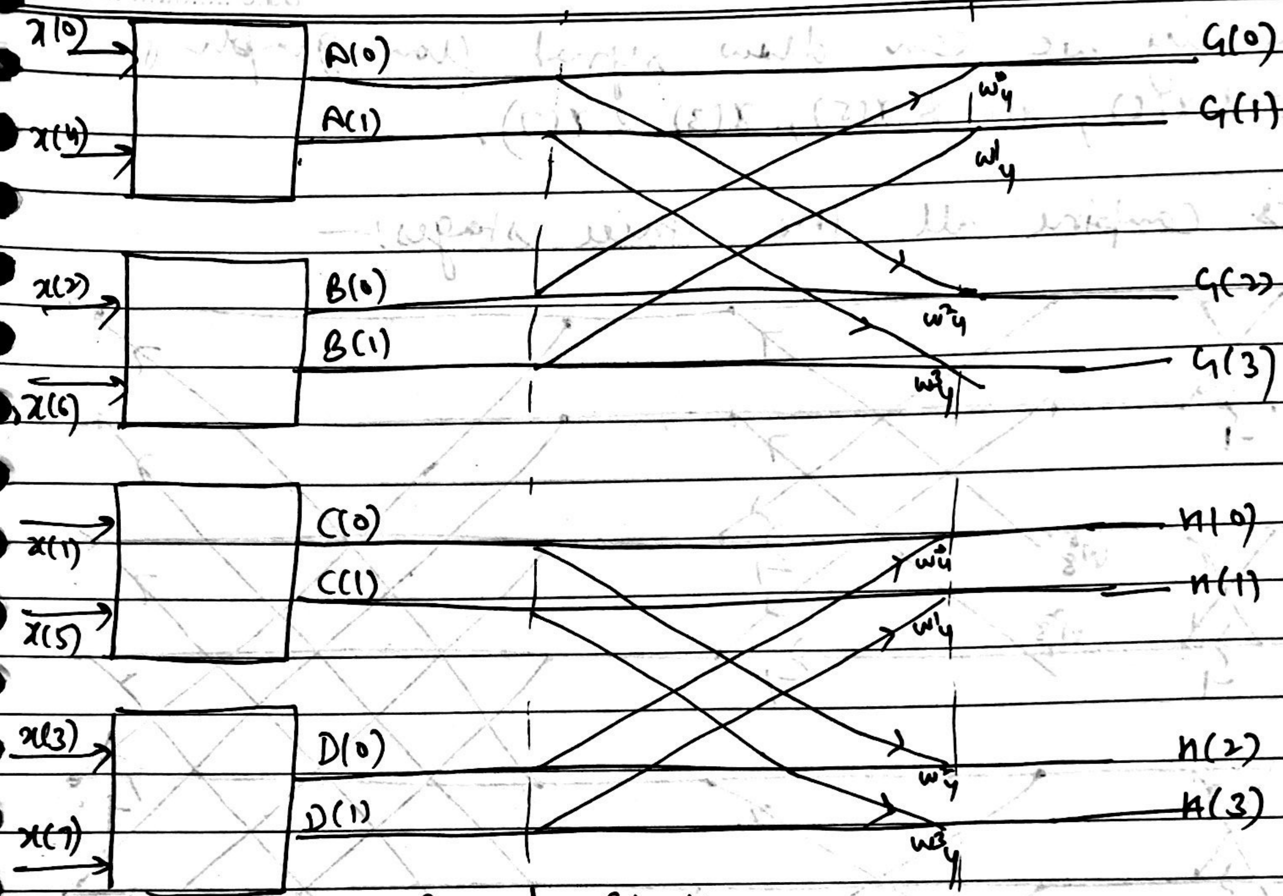


Second Stage

Each $\frac{N}{4}$ DFT as two $\frac{N}{8}$ Point DFTs.

Considering 2-Point DFT of $x(0)$ & $x(4)$

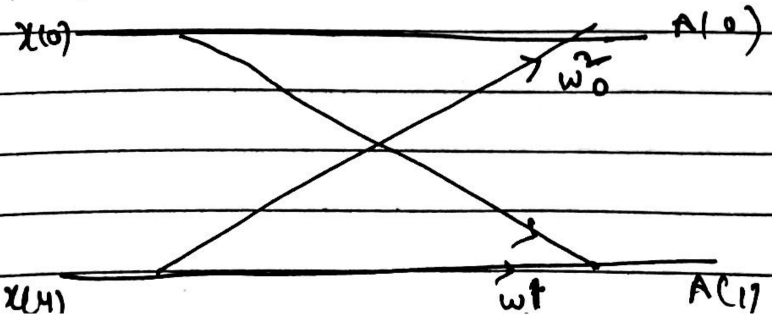
$$A(K) = \sum_{n=0}^{\frac{N}{4}-1} x(n)\, W_{\frac{N}{4}}^{kn} \quad ; \quad 0 \le K \le \frac{N}{4} - 1$$

if $N=8$

$$A(k) = \sum_{n=0}^{1} x(n)\, W_2^{kn} \quad ; \quad 0 \le k \le 1$$

for; $k=0 \Rightarrow A(0) = x(0) + W_2^0\, x(4)$

for; $K=1 \Rightarrow A(1) = x(1) + W_2^1\, x(4)$



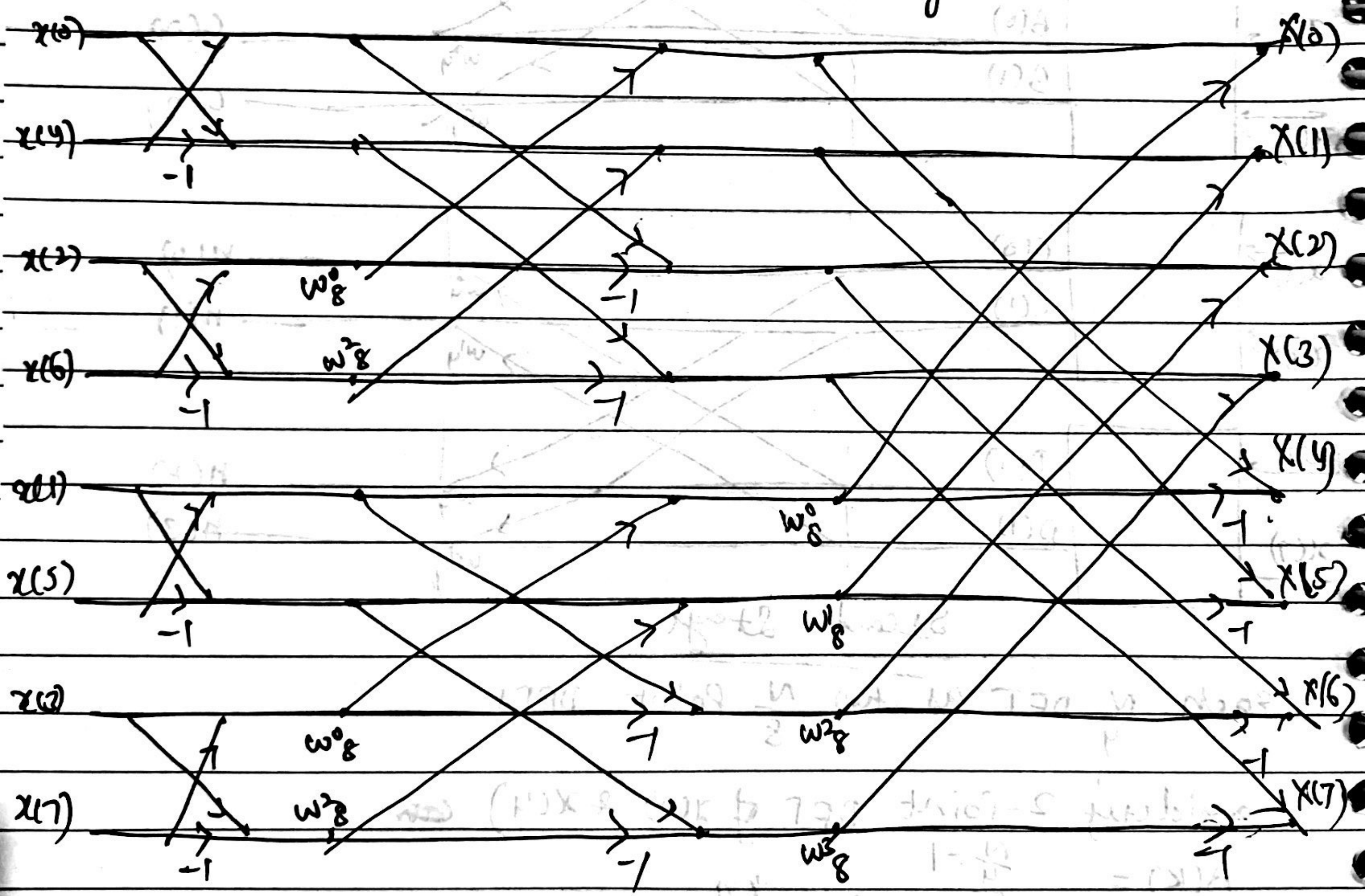
← Signal flow Graph

3rd Stage.

Date ....

Similarly we can draw signal flow graph for $x(2)$ & $x(6)$, $x(1)$ & $x(5)$, $x(3)$ & $x(7)$.

Lets combine all the three stages:—



(Complete signal flow Graph for Radix-2 DIT-FFT Algorithm)

FFT :- (Fast fourier transform) :- Date ....

→ The fast fourier transform is a highly efficient procedure for computing the DFT of a finite series.

→ It requires less no. of computations than that of direct evaluation of DFT.

→ FFT is based on decomposition or breaking the transform into smaller transforms and combining them to get the total transform.

→ FFT reduces the computation time required to compute DFT & improves the performance by a factor 100 or more over direct evaluation of the DFT.

★ **Comparison of DFT & FFT :-**

Direct evaluation of DFT :-

for N-point DFT, N^2 multiplication & $N(N-1)$ additions are required.

FFT :-

for N-point DFT, $\frac{N}{2}\log_2(N)$ multiplications and $N\log_2(N)$ additions are required.

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DFT VS FFT:-

No. of Points (N)	Direct Evaluation		Radix-2 FFT	
	N(N-1) DFT Additions	N^2 multiplications	$N \log_2(N)$ Additions	$\frac{N}{2} \log_2(N)$ multiplications
4	12	16	8	4
8	56	64	24	12
16	240	256	64	32
32	992	1024	160	80
64	4032	4096	384	192

Applications of FFT:-

1.) Digital Spectral Analysis
2.) Filter Simulation
3.) Auto Correlation, Pattern Recognition.

* Types of FFT Algorithms:-

① DIT (Decimation in-Time):-

The seq. for which we need DFT is successively divided into small sequences & DFTs of these subsequences are combined in a certain pattern to obtain the required DFT of the entire sequence.

② DIF (Decimation in frequency):-

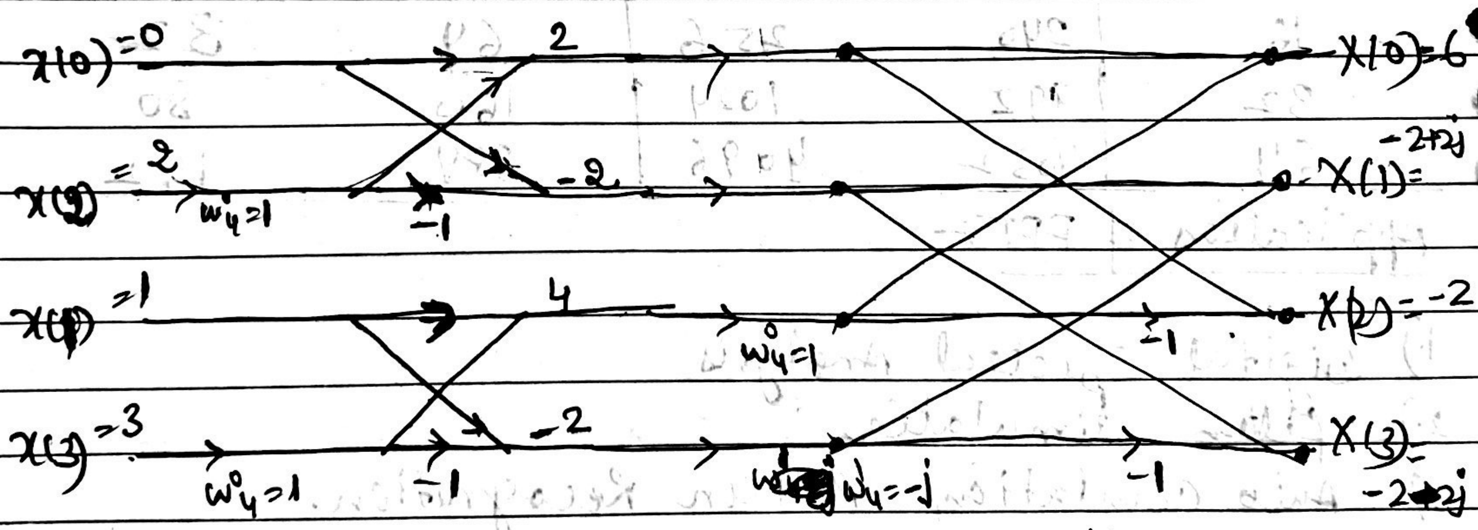
In this approach, the freq. samples of the DFT are decomposed into smaller & smaller subsequences in a similar manner.

$W_4^0 = 1$, $W_4^1 = -j$

Date ....

Q. Given $x(n) = \{0, 1, 2, 3\}$, find $X(k)$ using DIT-FFT algorithm

Sol^n $N = 4$, as there are 4 samples in $x(n)$.



flow Graph for DIT-FFT ; N=4

So, $X(k) = \{6, -2+2j, -2, -2-2j\}$

Q.2 $x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$
find $X(k)$ using DIT-FFT

Sol^n : $N = 8$

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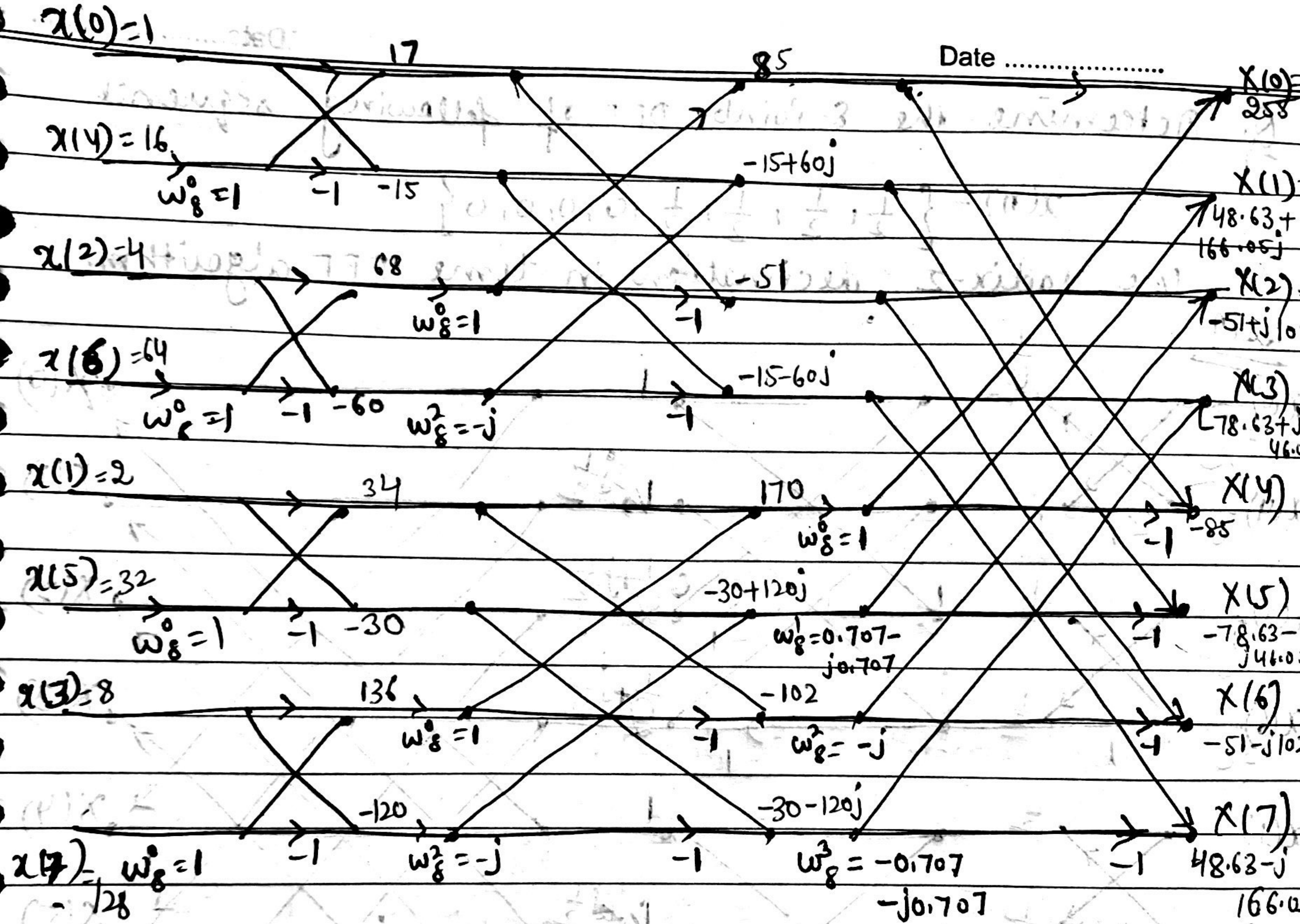


So, $X(K) = \{255,\ 48.63 + j166.05,\ -51 + j102,\ -78.63 + j46.05,$

$-85,\ -78.63 - j46.05,\ -51 - j102,\ 48.63 - j166.05\}$

Ans

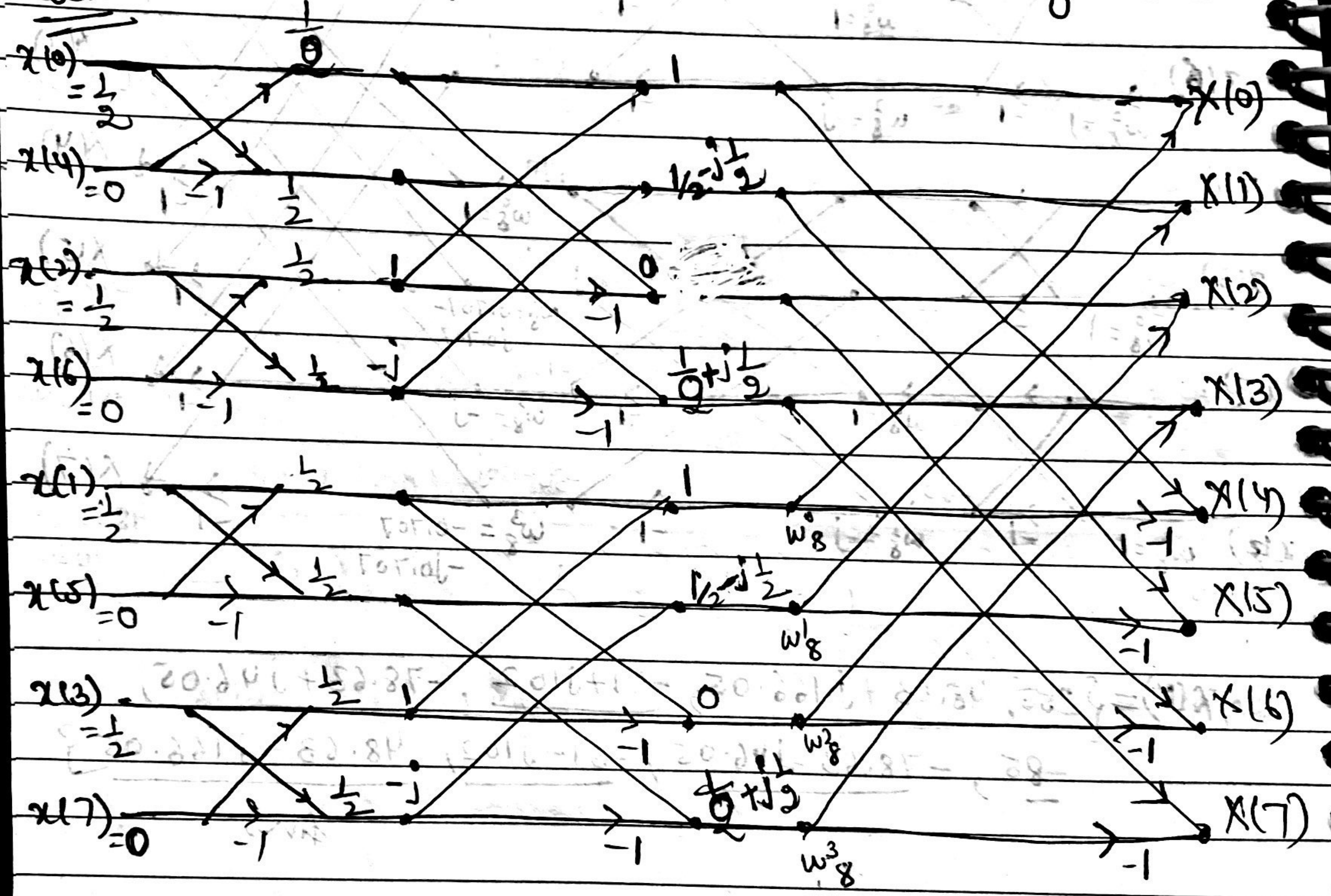
Date ....

Q. Determine the 8-Point DFT of following sequence

$$x(n) = \left\{ \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0 \right\}$$

Use radix-2 decimation in time FFT algorithm.

Sol^n



As

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\frac{\pi}{4}} = 0.707 - j0.707$$

$$W_8^2 = e^{-j\frac{\pi}{2}} = -j$$

$$W_8^3 = -0.707 - j0.707$$

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So

$$\boxed{X(0) = 1 + W_8^0 = 1+1 = 2}$$

$$X(1) = \frac{1}{2} - \frac{j}{2} + \left(\frac{1}{2} - \frac{j}{2}\right) W_8^1$$

$$= \frac{1}{2} - \frac{j}{2} + \left(\frac{1}{2} - \frac{j}{2}\right)(0.707 - j0.707)$$

So

$$\boxed{X(1) = 0.5 - j1.207}$$

$$\boxed{X(2) = 0 + 0 = 0}$$

$$X(3) = \frac{1}{2} + j\frac{1}{2} + W_8^3\left(\frac{1}{2} + j\frac{1}{2}\right)$$

$$= \left(\frac{1}{2} + \frac{j}{2}\right) + (0.707 - j0.707)\left(\frac{1}{2} + \frac{j}{2}\right)$$

$$X(3) = \left(\frac{1}{2} + \frac{j}{2}\right)(0 - j0.707)$$

$$\boxed{X(3) = 0.5 - j\,0.207}$$

$$\boxed{X(4) = 1 - 1 = 0}$$

$$X(5) = \left(\frac{1}{2} - j\frac{1}{2}\right) - W_8^1\left(\frac{1}{2} - \frac{j}{2}\right)$$

$$= \left(\frac{1}{2} - \frac{j}{2}\right) - (0.707 - j0.707)\left(\frac{1}{2} - \frac{j}{2}\right)$$

$$X(5) = \frac{1}{2} - \frac{j}{2} - (-0.707j)$$

So

$$\boxed{X(5) = 0.5 + j0.207}$$

Date ....

$$\boxed{X(6) = 0 + 0 = 0}$$

$$X(7) = \left(\frac{1}{2} + \frac{j}{2}\right) - w^3\left(\frac{1}{2} + \frac{j}{2}\right)$$

$$= \left(\frac{1}{2} + \frac{j}{2}\right) - (-0.707 - j\,0.707)\left(\frac{1}{2} + \frac{j}{2}\right)$$

$$X(7) = \left(\frac{1}{2} + \frac{j}{2}\right) + 0.707\,j$$

$$\boxed{X(7) = 0.5 + j\,1.21}$$

So

$$X(K) = \{2, 0.5 - j1.207, 0, 0.5 - j0.207, 0, 0.5 + j0.207, 0, 0.5 + j1.21\}$$

Ans

Q. Determine 8-Point DFT of following sequence

$$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$$

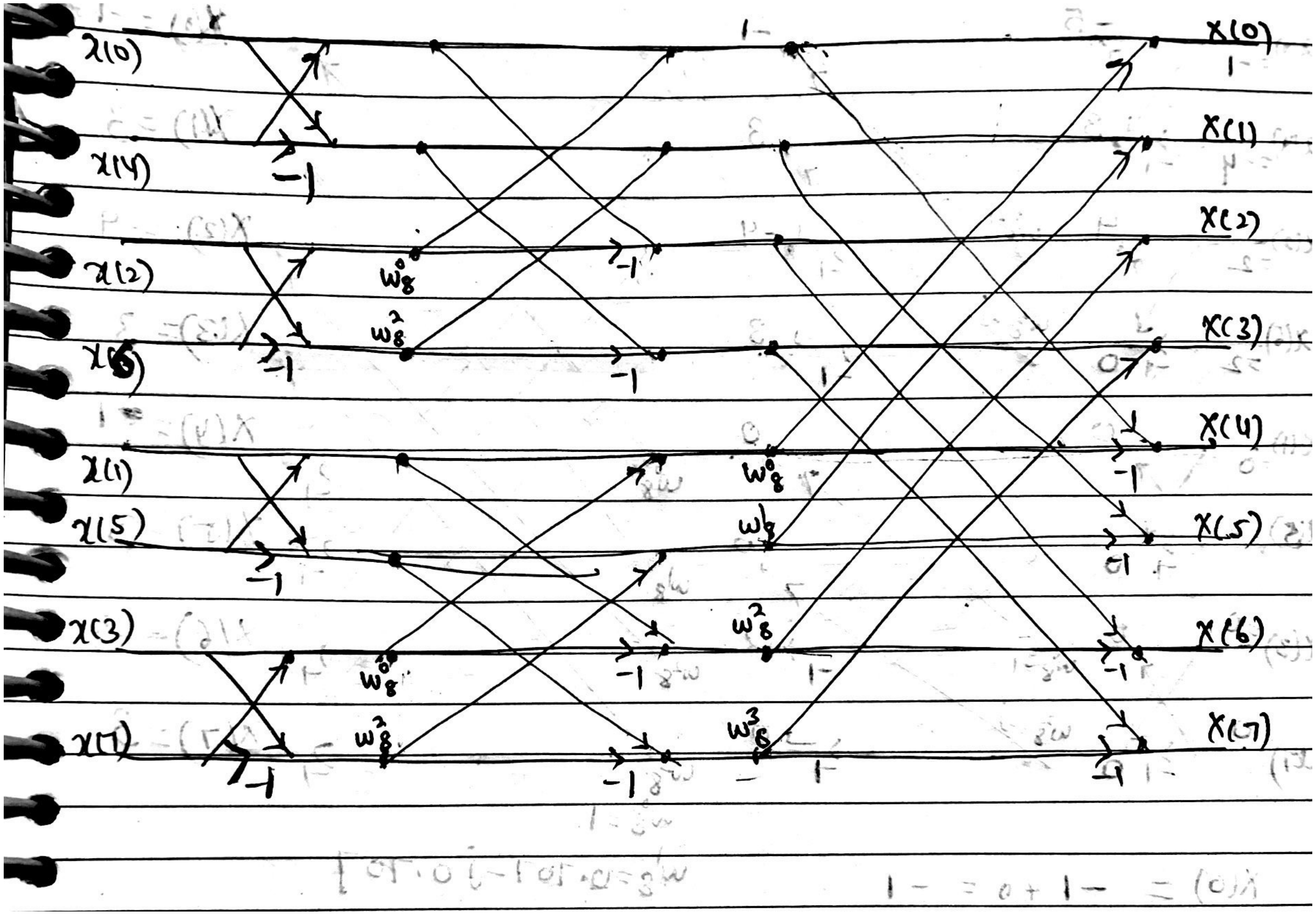
Soln

$$w_8^0 = 1$$

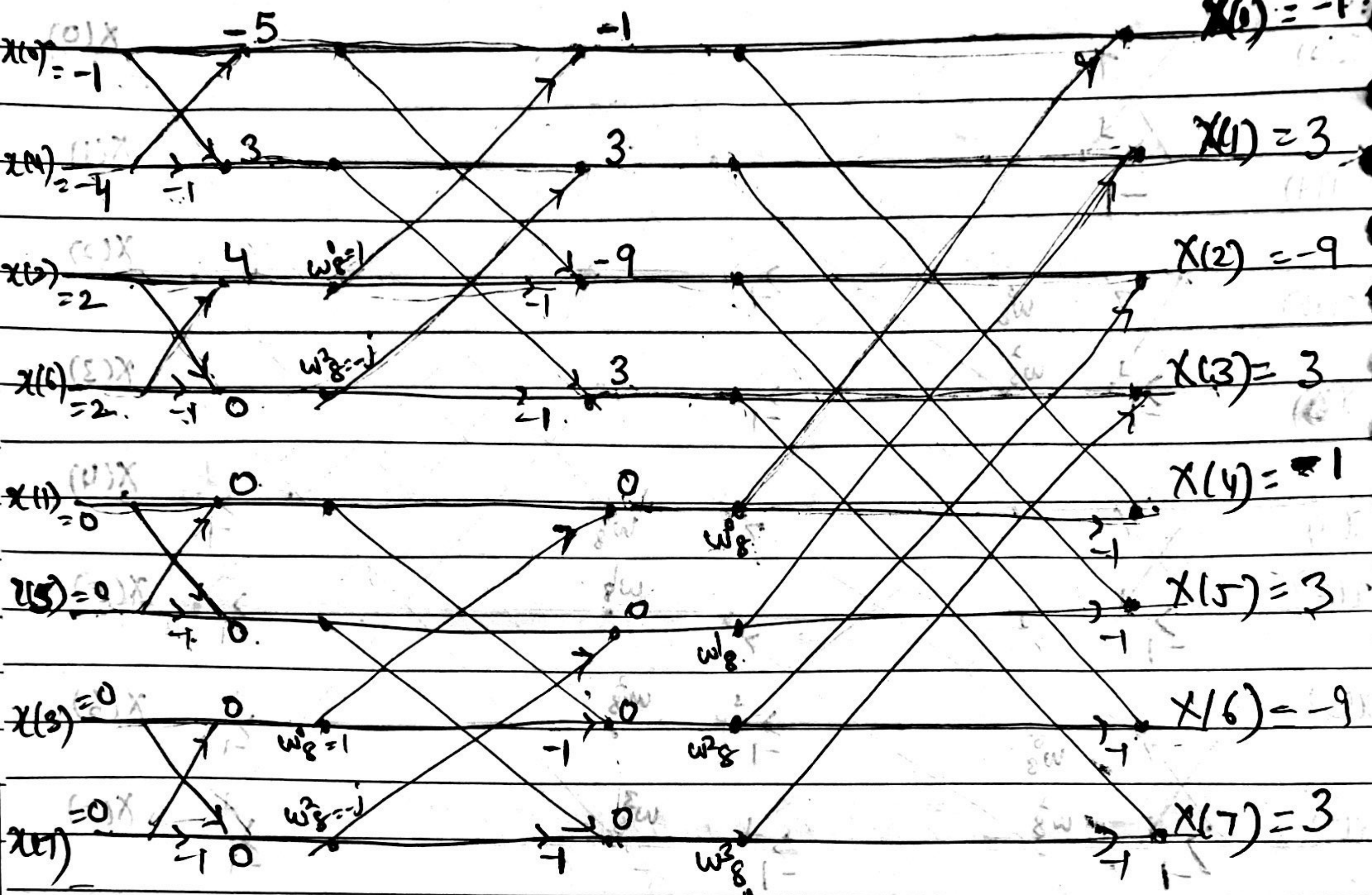
$$w_8^1 = 0.707 - j\,0.707$$

$$w_8^2 = e^{-j\frac{\pi}{2}} = -j$$

$$w_8^3 = -0.707 - j\,0.707$$


x(0)
x(4)
x(2)
x(6)
x(1)
x(5)
x(3)
x(7)
W_8^0
W_8^2
W_8^0
W_8^2
W_8^0
W_8^1
W_8^2
W_8^3
-1
X(0)
X(1)
X(2)
X(3)
X(4)
X(5)
X(6)
X(7)

Date ....



$W_8^0 = 1$

$W_8^1 = 0.707 - j0.707$

$W_8^2 = -j$

$W_8^3 = -0.707 - j0.707$

$X(0) = -1 + 0 = -1$

$X(1) = 3 + 0 = 3$

$X(2) = -9 + 0 = -9$

$X(3) = 3 + 0 = 3$

$X(4) = -1 + 0 = -1$

$X(5) = 3 + 0 = 3$

$X(6) = -9 + 0 = -9$

$X(7) = 3 + 0 = 3$

so

$X(k) = \{-1, 3, -9, 3, -1, 3, -9, 3\}$ Ans.