CHAPTER

4

Sinusoidal Steady State Analysis of RLC Circuits

4.1 INTRODUCTION

This chapter deals with the *sinusoidal steady state analysis* of networks involving R, L and C in different configurations (*series* and or *parallel*). This chapter also leads to the concept of *complex impedance* and *power factor*.

At first, the analytical concept of series and parallely *a.c.* circuits being introduced, the performance of different combinations of R, L and C circuits will be analysed.

Series and parallel *a.c.* circuits are extensively used in electrical and electronics engineering and hence their analysis is very important. It may be noticed that all the *a.c.* quantities being vector quantities, they have both *magnitudes* and *directions* (or *angles*). Either *polar* or *rectangular*, both forms of representations are extensively used in analysing *a.c.* circuits.

The common symbols used throughout the chapter are as follows :

R = Pure resistance element (ohm),

L = Pure inductance element (H),

C = Pure capacitance element (F or μF),

Z = Impedance (ohms),

X_L = Inductive reactance (ohms) $= 2\pi f L$,

X_C = Capacitive reactance (ohms) $= \dfrac{1}{2\pi f C}$,

f = Supply frequency (Hz),

ω = Angular frequency (rad./sec),

$\phi°$ or $\theta°$ = angle between voltage and current or reactance and resistance.

note In this chapter, the capital letters like I, V etc. represent *r.m.s. quantities*, capital letters with suffix "max" represent maximum values and small alphabets like v, i represent *instantaneous* values.

4.2 SERIES AND PARALLEL A.C. CIRCUITS

In a *series* circuit (Fig. 4.1), the current being same through each of the impedances, the voltage phasors are related to the current by the respective drops across each impedance vectorially added together.

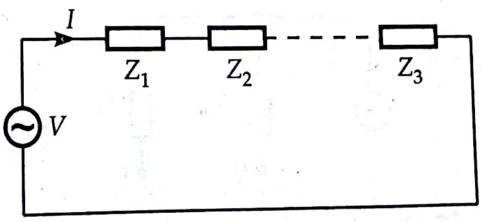

Fig. 4.1 Series a.c. circuit.

i.e., $V = IZ_1 + IZ_2 + IZ_3 + ... + Z_n = IZ_{eq}$ ...(4.1)

where Z_{eq} = equivalent impedance

$$= Z_1 + Z_2 + ... + Z_n \qquad ...(4.1(a))$$

On the other hand, in a *parallel* circuit (Fig. 4.2), the voltage drop across each element being same,

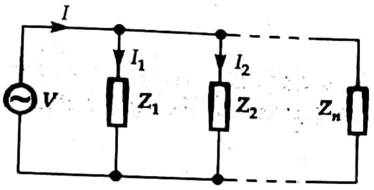

Fig. 4.2 Parallel a.c. circuit.

the currents through each branch are different. The branch currents are to be vectorially added to give the total current.

i.e., $I_1 = \frac{V}{Z_1} = VY_1$

$I_2 = \frac{V}{Z_2} = VY_2$ and so on.

i.e., $I = I_1 + I_2 + ... + I_n$

or, $VY_{eq.} = VY_1 + VY_2 + ... + VY_n + V$

$= V(Y_1 + Y_2 + Y_3 + ... + Y_n)$

i.e., $Y_{eq} = Y_1 + Y_2 + ... + Y_n$ ...(4.1(b))

4.3 CURRENTS AND VOLTAGE DIVISION IN A.C. CIRCUITS

In an *a.c.* parallel circuit (Fig. 4.3),

$$I_a Z_a = I_b Z_b$$

i.e., $\frac{I_a}{I_b} = \frac{Z_b}{Z_a}$

or $\frac{I_a + I_b}{I_b} = \frac{Z_b + Z_a}{Z_a}$

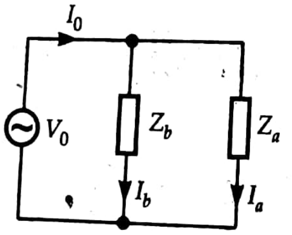

Fig. 4.3

or, $I_b = \frac{Z_a}{Z_a + Z_b}(I_a + I_b) = I_0 \frac{Z_a}{Z_a + Z_b}$ ...(4.2)

Similarly, ...(4.2(a))

$$I_a = I_0 \frac{Z_b}{Z_a + Z_b}$$

Again, in *a.c.* series circuit (Fig. 4.4),

$$I_0 = \frac{V_0}{Z_a + Z_b}$$

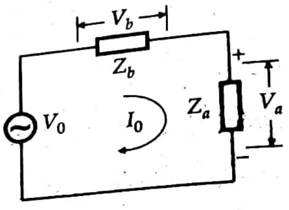

Fig. 4.4

But $V_a = I_0 Z_a = \frac{Z_a}{Z_a + Z_b} V_0$ ...(4.3)

Similarly,

$V_b = I_0 Z_b = \frac{Z_b}{Z_a + Z_a} V_0$ ...(4.3(a))

[Also, for current division,

$$I_a = \frac{Y_a}{Y_a + Y_b} I_0 \;;\; I_b = \frac{Y_b}{Y_a + Y_b} I_0$$

and for voltage division

$$V_a = \frac{Y_b}{Y_a + Y_b} V_0 \text{ and } V_b = \frac{Y_a}{Y_a + Y_b} \cdot V_0 \Bigg]$$

EXAMPLE 4.1 *A 4 ohm resistor is connected to a 10 mH inductor across a 100 V, 50 Hz voltage source. Find*

* *(i) Impedance of the circuit.*
* *(ii) Input current.*
* *(iii) Drop across the resistor and inductance.*
* *(iv) Power factor of the circuit.*
* *(v) Real power consumed in the circuit.*
* *(vi) Total power supplied.*

SOLUTION. In this circuit (Fig. E4.1),

$$X_L = 2\pi f L$$
$$= 2\pi \times 50 \times 10 \times 10^{-3} = 3.14\Omega,$$

and it is given that $R = 4\Omega$

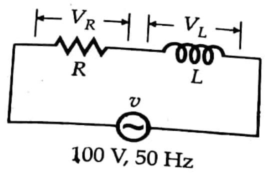

Fig. E4.1

$$\therefore \quad Z = R + j\,X_L = 4 + j\,3.14$$
$$= 5.085\,\Omega\ \angle 38.13^\circ \text{ ohms}$$

i.e., impedance of the circuit is **5.085 Ω ∠38.13°.**

(Ans. of (*i*))

$$\therefore\ I\,(\text{input current}) = \frac{V}{Z} = \frac{100}{5.085}\angle -38.13^\circ\ \text{A}$$

[assuming source voltage as $v = 100\ \angle 0^\circ$ V]

i.e., magnitude input current = **19.66 A**

(Ans. of (*ii*))

Drop across the resistance

$$(= V_R) = IR = 19.66\ \angle -38.13^\circ \times 4$$
$$= 78.64\ \angle -38.13^\circ\ \text{V}$$

while the drop across the reactance

$$(= V_L) = j\,I\,X_L = j\,19.66\ \angle -38.13^\circ \times 3.14$$
$$= (19.66 \times 3.14)\ \angle 90^\circ - 38.13^\circ$$
$$= 61.73\ \angle 51.87\ \text{V}$$

Thus, $\left.\begin{aligned} V_R &= \mathbf{78.64\ \angle -38.13^\circ\ V} \\ V_L &= \mathbf{61.73\ \angle 51.87^\circ\ V} \end{aligned}\right\}$ (Ans. of (*iii*))

Since the power factor of the circuit is the cosine of angle between the voltage and current, here, power factor (p.f.) would be cos (−38.13°) *i.e.,* **0.787 lag** (the p.f. is lagging as the sign of angle between V and I is −ve, *i.e.,* current is lagging the voltage. Also it may be noted that in *R-L* series circuit, the p.f. is lagging) (Ans. of (*iv*))

The real power consumed by the circuit is $VI\cos\phi$

i.e.,
$$P = 100 \times 19.66 \times 0.787$$
$$= \mathbf{1547\ W}\ \text{(Ans. of } (v)\text{).}$$

Check

> Also, power consumed is $I^2\,R$ *i.e.,* here $(19.66)^2 \times 4 = 1546$ W.

Total power supplied is

$$S = VI = 100 \times 19.66$$
$$= \mathbf{1966\ VA}\ \text{(Ans. of } (vi)\text{).}$$

Check

> Also $S = \sqrt{(VI\cos\phi)^2 + (VI\sin\phi)^2}$
> $= [(100 \times 19.66 \times 0.787)^2 + (100 \times 19.66 \times 0.617)^2]^{1/2}$
> $= 1966$ VA.

The phasor diagram of the parameters of the given problem is shown in Fig. E4.2.

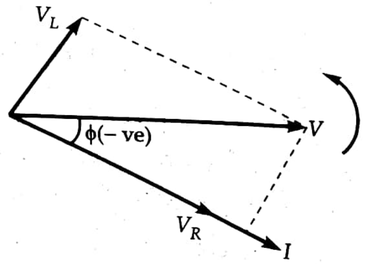

Fig. E4.2

It may be noted here, that the real power (P) is only *consumed* in *resistive part* of the circuit. This is the dissipation of heat in the resistance in form of *energy* (*i.e.,* power multiplied by time). On the other hand, an inductive reactance draws only *reactive power* (Q) which is stored in form of *magnetic energy.*

The real power consumed is either I^2R or $VI\cos\phi$ while the reactive power is $VI\sin\phi$ Obviously S (total power) is given by

$$S = P + j\,Q = \sqrt{P^2 + Q^2}\ \angle \tan^{-1}\frac{Q}{P}$$

where $|S| = |VI|$]

CHAPTER 4

EXAMPLE 4.2 *In a series R-L circuit, R = 20 ohms while L=60 mH. The input current lags the supply voltage by 60°. Obtain the value of applied frequency.*

SOLUTION. $\tan\phi = \dfrac{\omega L}{R}$

[where ϕ is the angle of lag and $\omega = 2\pi f$]

or,
$$\tan 60^\circ = \frac{\omega \times 0.06}{20}$$

[$\because$ R = 20 Ω, L = 60 mH = 0.06 H]

$$\therefore \quad \omega = \frac{20 \times 1.732}{0.06} = 577.33\ \text{rad/sec.}$$

This gives $f = \dfrac{\omega}{2\,\pi} = \dfrac{577.33}{2\,\pi} = \mathbf{91.93\ Hz.}$

Example 4.3 *In a series R-L circuit, the inductance being 20 mH, the impedance is 17.85 Ω. The angle of lag of the input current from the applied voltage being 63.5°, find the values of angular frequency and the resistance of the circuit.*

Solution.

$$\tan\phi = \frac{\omega L}{R}$$

or, $\tan 63.5° = \frac{0.02\,\omega}{R}$ $\quad [\because \phi = 63.5°;\ L = 20 \text{ mH}]$

or, $2R = 0.02\,\omega$ *i.e.,* $R = 0.01\,\omega$ ...(1)

But $Z = \sqrt{R^2 + (\omega L)^2}$

$= \sqrt{(0.01\,\omega)^2 + (0.02\,\omega)^2}$

[∵ in (1) we have obtained $R = 0.01\,\omega$ and $L = 0.02$ H]

$\therefore$ $Z = 0.0224\,\omega$ or $17.85 = 0.0224\,\omega$

$\therefore$ $\omega = 796.875$ rad / sec.

and $R = 0.01\,\omega = 7.98\,\Omega \simeq 8\,\Omega$

Thus we have obtained,

$R = 8\,\Omega$; $\omega = 796.875$ rad / sec.

Example 4.4 *A R-L series circuit draws a current of 1 A when connected across a 10 V, 50 Hz a.c. supply. Assuming the resistance to be 5 ohms, find the inductance of the circuit. What is its power factor ?*

Solution. $Z = \frac{V}{I} = \frac{10}{1} = 10\,\Omega$

$[\because V = 10 \text{ V},\ I = 1 \text{ A}]$

But $Z = \sqrt{R^2 + (\omega L)^2}$

$\therefore$ $(\omega L)^2 = Z^2 - R^2 = 10^2 - 5^2 = 75$

Thus $L = \sqrt{\frac{75}{\omega^2}} = 0.0276$ H *i.e.,* 27.6 mH

The inductance of the circuit is 27.6 mH.

The p.f. of the given circuit is,

$$\cos\phi = \frac{R}{Z} = \frac{5}{10} = 0.5.$$

Then, for the given circuit,

$L = 27.6$ mH; $\cos\phi = 0.5$ (lag).

Example 4.5 *A R-L series circuit has resistance of 20 ohms and inductance of 0.02 Henry. If the net impedance of the given circuit be $40\angle\phi°$ ohm, find ϕ and the frequency of the circuit.*

Solution. $Z\angle\theta = R + jX_L = 20 + j\,0.02\,(2\pi f)$

or, $40\angle\phi = 20 + j\,0.1256\,f$ ...(1)

Again, $\phi = \cos^{-1}\frac{R}{Z} = \cos^{-1}\frac{20}{40} = 60°$

Also, $X_L = Z\sin\phi = 40\sin 60° = 34.64\,\Omega$

But from (1),

$X_L = 0.1256\,f$

$\therefore$ $f = \frac{34.64}{0.1256} = 276$ Hz

Thus we find $\phi = 60°$ (lag)

$f = 276$ Hz.

Example 4.6 *A series circuit has $R = 4\,\Omega$ and $L = 0.01$ H. Find the impedance at 100 Hz and 500 Hz.*

Solution. $Z = R + jX_L$ while $X_L = 2\pi f L$.

for $f = 100$ Hz, $X_{L_{100}} = 2\pi \times 100 \times 0.01 = 6.28\,\Omega$

for $f = 500$ Hz, $X_{L_{500}} = 2\pi \times 500 \times 0.01 = 31.4\,\Omega$

$\therefore$ $Z_{\text{for 100 Hz}} = \sqrt{(4)^2 + (6.28)^2}$

$= 7.45\,\Omega\ \angle\tan^{-1}\frac{6.28}{4}$

$= 7.45\ \angle 57.5°\,\Omega$

$Z_{\text{for 500 Hz}} = \sqrt{(4)^2 + (31.4)^2}$

$= 31.65\ \angle 82.75°\,\Omega.$

[It may be noted that Z, the impedance changes with system frequency]

Example 4.7 *A series RLC circuit has R = 10 Ω, L = 1 H, C = 20 μF. A 100 V, 50 Hz supply is applied across the circuit. Find the input current and voltage across the elements.*

Solution. $\omega = 2\pi f = 2\pi \times 50 = 314$ rad / sec.

$X_L = \omega L = 314 \times 1 = 314\,\Omega$

$X_C = \frac{1}{\omega C}$

$= \frac{1}{314 \times 20 \times 10^{-6}} = \frac{10^6}{6280} = 159.24\,\Omega$

$Z = R + j(X_L - X_C)$

$= 10 + j\,(314 - 159.24)$

$= (10 + j\,155)\,\Omega = 155.33\,\Omega\ \angle 86°$

$\therefore$ $I = \frac{V}{Z} = \frac{100\ \angle 0°}{155.33\ \angle 86°} = 0.644\ \angle -86°$ A

Also, $V_R = IR = 0.644\angle -86° \times 10$

$= \mathbf{6.44\angle -86°\ V.}$

$V_L = jX_L\ I = 314\angle 90° \times 0.644\angle -86°$

$= \mathbf{202.15\angle 4°\ V}$

$V_C = -jX_C\ I$

$= 159.24\angle -90° \times 0.644\angle -86°$

$= \mathbf{102.55\angle -176\ V.}$

EXAMPLE 4.8 *In a series RLC circuit, an AC voltage of 120 ∠0° V is applied at a frequency of 400 rad/sec. The input current leads the voltage by 63.5°. Find the value of R if L=25 mH and C=50 μF. What are the drops across L and C ?*

SOLUTION.

$$X_L = j\omega L = j\ 400 \times 25 \times 10^{-3} = j\ 10\,\Omega.$$

$$X_C = \frac{1}{j\omega C} = -\frac{j}{400 \times 50 \times 10^{-6}} = -j\ 50\,\Omega.$$

However, $Z = R + j(X_L - X_C) = (R - j\ 40)\,\Omega.$

Also, $\tan^{-1}\dfrac{X_L - X_C}{R} = \phi$

(where ϕ is the angle between V and I in the circuit)

or, $\tan(-63.5°) = \dfrac{10 - 50}{R}$

or, $-40 = R \times (-2)\ \therefore\ R = 20\,\Omega$

[Here, ϕ is leading hence the sign of ϕ is –ve]

Also, $I = \dfrac{V}{Z} = \dfrac{120\angle 0°}{20 - j\ 40}$

$= \dfrac{120\angle 0°}{44.7\angle -63.5°} = 2.68\angle 63.5°\ A$

However,

$V_L = j\ IX_L = j \times 2.68\angle 63.5° \times 10$

$= 26.8\angle(90° + 63.5°)\ V$

or, $V_L = \mathbf{26.8\angle 153.5°\ V}$

and $V_C = -j\ I X_c = (2.68\angle 63.5° - 90°)(50)$

$= \mathbf{134\angle -26.5°\ V.}$

EXAMPLE 4.9 *In a series RLC circuit, R=8 Ω, X_C=5 Ω, X_L = 10 Ω. A voltage V is applied across the combination such that the series current is 2 A and it lags the system voltage by 10°. Assuming the system frequency to be 50 Hz, find the drops across each of the units. What is the supply voltage ?*

SOLUTION.

V_R = drop across $R = IR = 2\angle -10° \times 8$

$= 16\angle -10° = (15.76 - j\ 2.78)\ V$

V_L = drop across $L = j\ I\ X_L$

$= (2\angle -10° + 90°) \times 10$

$= 20\angle 80° = (3.47 + j\ 19.7)\ V$

V_C = drop across $C = I(-j\ X_C)$

$= 2\angle -10°(5\angle -90°) = 10\angle -100°$

$= (-0.347 - j\ 1.969)\ V$

$\therefore$ Supply voltage = vector sum of $(V_R + V_L + V_C)$

$= 15.76 - j\ 2.78 + 3.47 + j\ 19.7 - 0.347 - j\ 1.969$

$= 18.883 + j\ 14.951$

$\simeq \mathbf{24.08\angle 38.37°\ volts.}$

EXAMPLE 4.10 *In the circuit shown in Fig. E4.3, r_1=8 Ω, r_2=3.5 Ω, C_1=800 μF, C_2=250 μF. If the supply frequency is 60 Hz, find*

(i) drop across each circuit element

(ii) total resistive and total capacitive drop

(iii) supply voltage

(iv) impedance angle of each branch

(v) power factor of the circuit

Draw the vector diagram.

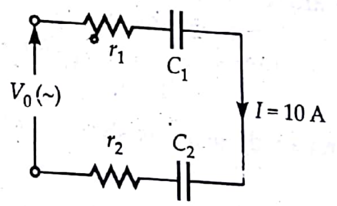

Fig. E4.3

SOLUTION. (*i*) Drop across $r_1 = V_{r_1} = Ir_1 = \mathbf{80\ V}$

Drop across

$C_1 = V_{C_1} = I\,X_{C_1}$

$= I(2\pi f C_1)^{-1} = \dfrac{10}{2\pi \times 60 \times 800 \times 10^{-6}}$

$= \mathbf{33.2\ V}$

Drop across

$r_2 = V_{r_2} = Ir_2 = \mathbf{35\ V}$

Drop across

$C_2 = \dfrac{I}{2\pi f C_2}$

$= \dfrac{10}{2\pi \times 60 \times 250 \times 10^{-6}} = \mathbf{106\ V.}$

(*ii*) Total resistive drop

$V_{r_1} + V_{r_2} = \mathbf{115\ V\ (= V_r)}$

Total capacitive drop

$= 33.2 + 106 = \mathbf{139.2\ V\ (= V_c)}$.

(*iii*) Drop in branch 1 (comprising of r_1 and C_1)

$$V_1 = I(r_1 - j X_{C_1})$$
$$= V_{r_1} - j V_{c_1} = (80 - j\,33.2)\ \text{V}$$

Similarly, drop in branch 2 (comprising of r_2 and C_2)

$$V_2 = I(r_2 - j X_{C_2})$$
$$= V_{r_2} - j V_{c_2} = (35 - j\,106)\ \text{V}$$

∴ Supply voltage

$$V_0 = I[(r_1 + r_2) - j(X_{C_1} + X_{C_2})]$$
$$= \mathbf{115 - j\,139.3\ V.}$$

(*iv*) Impedance angle of

$$Br_1 = \theta_1 = \tan^{-1}\frac{X_{C_1}}{r_1} = \mathbf{22°33'\ (-ve).}$$

Impedance angle of

$$Br_2 = \theta_2 = \tan^{-1}\frac{X_{C_2}}{r_2} = \mathbf{71°45'\ (-ve).}$$

(*v*) PF angle

$$= \tan^{-1}\frac{X_{C_1} + X_{C_2}}{r_1 + r_2} = \mathbf{50°20'}.$$

The phasor diagram is shown in Fig. E4.4.

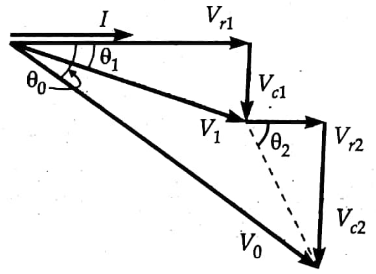

Fig. E4.4

EXAMPLE 4.11 *In the series RLC circuit of Fig. E4.5 $R = 4.2\,\Omega$, $L = 0.03$ H, $C = 450\,\mu$F. If $I = 10$ A, find the drop across each element, supply voltage and power factor angle. Also draw the vector diagram. Assume $f = 50$ Hz.*

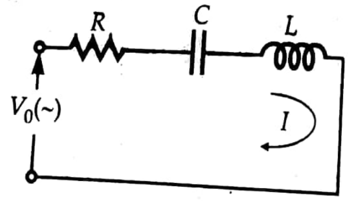

Fig. E4.5

SOLUTION.

Drop across $R\,(V_R) = IR = \mathbf{42\ V}$

Drop across $L\,(V_L) = I\,X_L = I\,(2\,\pi\,f\,L) = \mathbf{94.2\ V}$

Drop across $C\,(V_C) = I\,X_C = \dfrac{I}{2\,\pi\,f\,C} = \mathbf{70.7\ V}$

$$V_0 = I\,[R + j\,(X_L - X_C)]$$
$$= V_R + j\,(V_L - V_C) = \mathbf{42 + j\,23.5\ V}$$

and PF angle $= \tan^{-1}\dfrac{X_L - X_C}{R} = \mathbf{29°22'}$

Figure E4.6 represents the vector diagram.

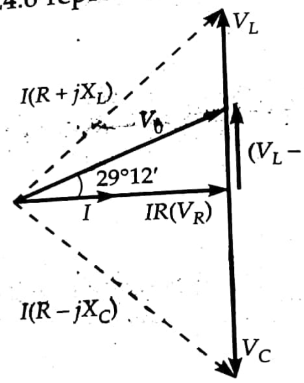

Fig. E4.6

EXAMPLE 4.12 *In the circuit of Fig. E4.7, given the values of each of the elements, the circulating current being 10 A, find the supply voltage and power factor.*

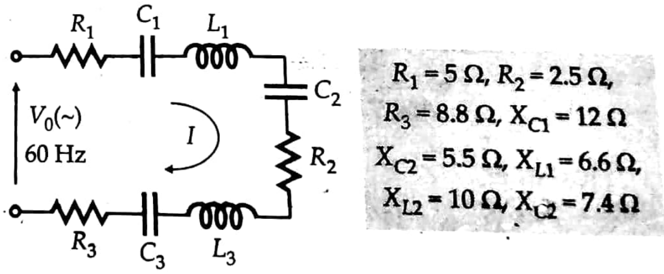

Fig. E4.7

SOLUTION. By inspection,

$$V_0 = I\,[(R_1 + R_2 + R_3) + j\,(X_{L_1} + X_{L_2}) - j\,(X_{C_1} + X_{C_2} + X_{C_3})]$$
$$= 10\,[16.3 + j\,(16.6 - 25)] = (163 - j\,83)$$
$$= 182.9\,\angle\tan^{-1}\left(-\frac{83}{163}\right) = 182.9\,\angle -26.9°\ \text{V}$$

The P.F. is then $\cos(26.9°)$ lead $=$ **0.89 (lead)** and the supply voltage is **182.9∠ – 26.9° V.**

and in r.m.s. values also,

$$I = \frac{V}{R + j\,\omega L}\angle -\tan^{-1}\frac{\omega L}{R} \quad ...(4.13)$$

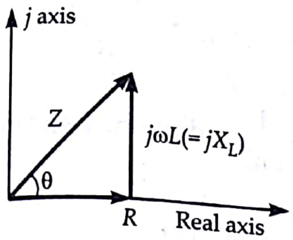

Fig. 4.7 Complex impedance triangle.

Assuming the voltage phasor to the reference $[V = V\angle 0°]$, equations (4.12) and (4.13) clearly show the current *lagging* the voltage by an angle θ given by $\theta = -\tan^{-1}\frac{\omega L}{R}$ (the presence of negative sign indicates the *angle of lag*). The instantaneous current, with voltage as reference phasor, is then given as

$$i = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}\sin\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right) \quad ...(4.14)$$

4.5 SINUSOIDAL RESPONSE OF SERIES *RC* CIRCUIT

In Fig. 4.8, a *RC* series circuit is shown to be excited by an alternating sinusoidal voltage source in steady state and is given by

$$v = V_m \sin\omega t \quad ...(4.15)$$

where V_m, the peak value is given by $(\sqrt{2}\,V)$.

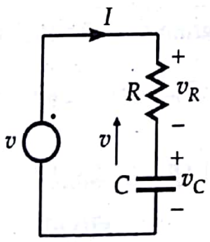

Fig. 4.8 R.C. series circuit excited by voltage sinusoid.

Obviously, in the series circuit of Fig. 4.8,

$$V = I\left(R + \frac{1}{j\,\omega C}\right) \quad ...(4.16)$$

$$= I\left(R - j\frac{1}{\omega C}\right) = IZ \quad ...(4.17)$$

where, $Z = R - \frac{j}{\omega C}$

$$= \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}\angle -\tan^{-1}\frac{1}{\omega RC} \quad ...(4.18)$$

i.e.,

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}\angle -\tan^{-1}\frac{1}{\omega RC}}$$

$$= \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\angle \tan^{-1}\frac{1}{\omega RC} \quad ...(4.19)$$

Expression (4.19) shows that the current in the *RC* series circuit leads the voltage reference by an angle θ given by $\theta = \angle\tan^{-1}\frac{1}{\omega RC}$ [+ ve sign of angle represents angle of lead].

The corresponding instantaneous current is then given by

$$i = I_m \sin(\omega t + \theta)$$

$$= \frac{V_m}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}\sin\left(\omega t + \tan^{-1}\frac{1}{\omega RC}\right) \quad ...(4.20)$$

The phasor diagram of the *RC* series circuit, being energized by sinusoidal voltage source, at steady state, has been shown in Fig. 4.9 while the waveform sketch is given in Fig. 4.10.

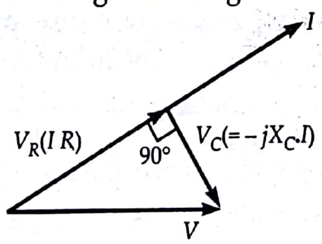

Fig. 4.9 Phasor diagram of *RC* series circuit.

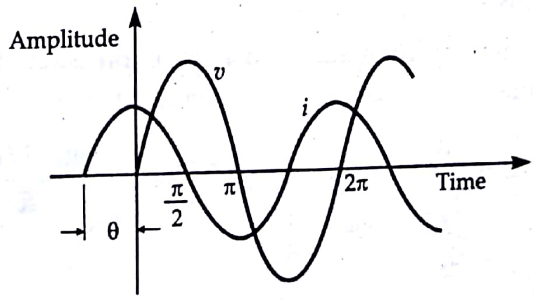

Fig. 4.10 Sketch of current and voltage in series *RC* circuit.

CHAPTER 4

4.6 SINUSOIDAL RESPONSE OF SERIES *RLC* CIRCUIT

With reference to Fig. 4.11, the application of Kirchhoff's law results

$$V = IR + j\,I\,X_L + I\frac{1}{j\omega.C}$$

$$\left[\text{where } v_R = IR\,;\; v_L = j\,I\,X_L\,;\; v_C = I\frac{1}{j\,\omega C}\right]$$

or,
$$V = I\left[R + j\,\omega\,L + \frac{1}{j\,\omega\,C}\right] \qquad \text{...(4.21)}$$

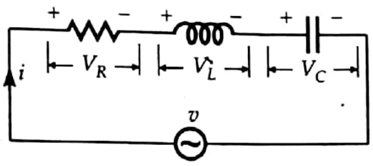

Fig. 4.11 Voltage sinusoid applied at *RLC* series circuit.

Let Z be the net impedance of the circuit. This gives, from equation (4.21),

$$V = IZ = I\left[R + j\,\omega L + \frac{1}{j\,\omega C}\right]$$

or,
$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$= \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\,\angle \tan^{-1}\frac{(\omega\,L) - \left(\frac{1}{\omega C}\right)}{R}. \qquad \text{...(4.22)}$$

Expression (4.22) also gives the expression of the *complex impedance* (Z) which indicates that the circuit will become *inductive* if $\omega L > \frac{1}{\omega C}$ and then the sign of the angle of Z is +ve. On the other hand, for $\omega L < \frac{1}{\omega C}$, the circuit will become *capacitive* and the sign of the angle of the impedance will then be negative.

The instantaneous current expression then becomes (with voltage as reference phasor)

$$i = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\,\angle -\tan^{-1}\frac{\omega L - 1/\omega C}{R} \qquad \text{...(4.23)}$$

Expression (4.23) shows that the current lags the voltage by an angle θ given by

$$\theta = -\tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}$$

provided $\omega L > \frac{1}{\omega C}$ and the current leads the voltage by an angle θ given by $\theta = \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}$ provided $\omega L < \frac{1}{\omega C}$. Figs. 4.12 and 4.13 exhibit the phasor diagrams of *RLC* series circuit.

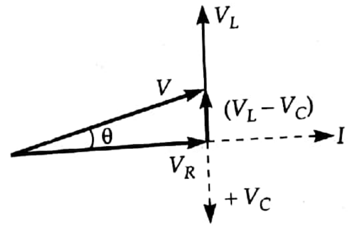

Fig. 4.12 Phasor diagram of series *RLC* circuit when $\omega L > \frac{1}{\omega C}$ *i.e.*, $V_L > V_C$. Here *I* lags *V* (θ is –ve)

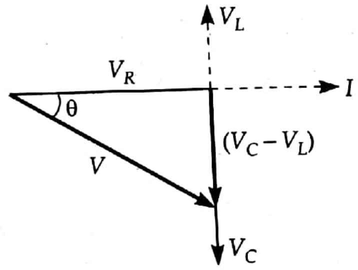

Fig. 4.13 Phasor diagram of series *RLC* circuit when $\omega L < \frac{1}{\omega C}$ *i.e.*, $V_L < V_C$. Here *I* leads *V* (θ is +ve)

EXAMPLE 4.33 *A 50 Hz sinusoidal voltage $v = 311 \sin \omega t$ is applied to a RL series circuit. If the magnitude of resistance is 5 Ω and that of inductance is 0.02 H,*

(*a*) *Calculate the r.m.s. or effective value of steady state current and relative phase angle*

(*b*) *Obtain the expression for the instantaneous current*

(*c*) *Compute the effective magnitude and phase of voltage drops appearing across each circuit element.*

SOLUTION. (*a*) Let V represent the r.m.s. value and be taken as reference

$$V = \frac{V_m}{\sqrt{2}} = \frac{311}{\sqrt{2}} = 220 \text{ V}$$

$\therefore$

$$V = 220 + j\,0 = 220\ \angle 0° \text{ V}$$

i.e.,

But $\quad Z = R + j\,\omega L$

$$= 5 + j\,.2\,\pi.50.0.02 = (5 + j\,6.28)\,\Omega$$

$$= \sqrt{5^2 + 6.28^2}\ \angle \tan^{-1}\frac{6.28}{5.0}$$

$$= 8.03\ \angle 51.47°\,\Omega$$

$\therefore$

$$I = \frac{V}{Z} = \frac{220\ \angle 0°}{8.03\ \angle 51.47°}$$

$$= \mathbf{27.39\ \angle -51.47°\ A.}$$

Thus, the effective value of the circuit current is 27.39 A while the angle of lag of the current from the voltage is 51.47°.

(*b*) The instantaneous current is then given by

$$i = I_m\ \sin(\omega t + \theta)$$

$$= (27.39) \times \sqrt{2}\ \sin(\omega t - 51.47°)$$

$$= \mathbf{38.73\ \sin\,(\omega t - 51.47°)\ A}$$

(*c*) The voltage drop across the resistor is given by

$$V_R = IR = (27.39\ \angle -51.47) \times 5$$

$$= \mathbf{136.95\ \angle -51.47°\ V.}$$

The voltage drop across the inductance is

$$V_L = j\,I\,X_L$$

$$= (j\,27.39\ \angle -51.47°)(2\,\pi.50.0.02)$$

$$= 1\ \angle 90° \cdot 27.39\ \angle -51.47°.6.28$$

$$= 172\ \angle 90° - 51.47°$$

$$= \mathbf{172\ \angle 38.43°\ V.}$$

Check

$V = V_R + V_L$ (vectorially)

$= 136.95\ \angle -51.47° + 172\ \angle 38.53° = 220 + j\,0$ V]

EXAMPLE 4.34 *In a series R-L circuit, $R = 10\,\Omega$, $L = 20$ mH. The circuit current being 10 sin 314 t A, find the total voltage drop (v) across the elements R and L. Also obtain the phase angle between i and v.*

SOLUTION. The phase angle θ between v and i is given by

$$\theta = \tan^{-1}\frac{\omega L}{R}$$

$$= \tan^{-1}\frac{2\,\pi \times 50 \times 20 \times 10^{-3}}{10} = \mathbf{32.13°.}$$

[$\because\ i = 10\sin 314\,t\ \therefore\ \omega = 314 = 2\,\pi\,f$ *i.e.,* $f = 50$ Hz]

$$v = I_m\sqrt{R^2 + (\omega L)^2}\ \sin(\omega t + \theta)$$

$$= 10\sqrt{10^2 + (314 \times 20 \times 10^{-3})^2}\ \sin(314\,t + \theta)$$

$$= \mathbf{118.08\ \sin\,(314\,t + 32.13°)\ V.}$$

[here, i lags v by 32.13°]

EXAMPLE 4.35 *In a series R-L circuit, the current and voltage are given as*

$$i = 1\cos(314\,t - 20°),\ v = 10\cos(314\,t + 10°).$$

Find the values of R and L.

SOLUTION. Let v be the reference vector, i lags v by $(20 + 10) = 30°$.

Also, $\qquad \dfrac{V_m}{I_m} = \dfrac{10}{1} = 10.$

Hence, $\qquad \dfrac{10}{1} = \sqrt{R^2 + (314.\,L)^2}$

or, $\qquad R^2 + (314.\,L)^2 = 100 \qquad$ ...(1)

and $\qquad \tan 30° = 0.577 = \dfrac{314\,L}{R} \qquad$ ...(2)

Solving (1) and (2),

$$L = \mathbf{15.9\ mH};\quad R = \mathbf{8.66\,\Omega}.$$

EXAMPLE 4.36 *In a series RC circuit, the values of $R = 10\,\Omega$ and $C = 25$ nF. A sinusoidal voltage of 50 MHz is applied and the maximum voltage across the capacitance is 2.5 V. Find the maximum voltage across the series combination.*

SOLUTION. $\omega = 2\,\pi\,f$

$$= 2\,.\,\pi\,.\,50 \times 10^6 \text{ rad/sec}$$

$$= 314 \times 10^6 \text{ rad/sec.}$$

The current through the capacitor being given by

$$I = \omega C\,V, \text{ in this problem,}$$

$$I_{max} = \omega C\,V_{max}$$

$$= 314 \times 10^6 \times 25 \times 10^{-9} \times 2.5$$

$$= 19.625 \text{ A}$$

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Again,

$$V_{max} = I_{max}\, Z = I_{max}\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$= 19.625\sqrt{10^2 + \left(\frac{1}{2\,.\,\pi\,.\,50 \times 10^6 \times 25 \times 10^{-9}}\right)^2}$$

$$= \mathbf{196.59\ V.}$$

EXAMPLE 4.37 *In a series LCR circuit, the maximum inductor voltage is twice the capacitor voltage maximum. However, the circuit current lags the applied voltage by 30° and the instantaneous drop across the inductance is given by $v_L = 100 \sin 377\, t$ V. Assuming the resistance being 20 Ω, find the values of the inductance and capacitance.*

SOLUTION. The inductor drop is 100 V (max),

i.e., $$v_{L(max)} = 100\ \text{V} = \omega L\, I_{max}$$

However, the inductor drop is twice the capacitor drop.

$$\therefore\quad \frac{1}{2}.100 = I_{max}\, X_C = \frac{I_{max}}{\omega C}$$

Thus,

$$50\,\omega = \frac{I_{max}}{C} \qquad \text{...(1)}$$

and $$\frac{100}{\omega} = I_{max}\,.\,L \qquad \text{...(2)}$$

Also,

$$\tan\theta = \frac{\omega L - \frac{1}{\omega C}}{R}$$

or, $$\tan 30° = \frac{\omega L - \frac{1}{\omega C}}{20} = 0.577$$

i.e., $$0.577 = \frac{\omega^2 LC - 1}{20\,\omega C} \qquad \text{...(3)}$$

Solving for (1) and (2),

$$\omega^2 LC = 2 \qquad \text{...(4)}$$

Substituting the value of $\omega^2 LC$ in (3), we get,

$$0.577 = \frac{1}{20\,\omega C}$$

or, $$C = \frac{1}{20 \times 0.577 \times \omega}$$

$$= \frac{1}{20 \times 0.577 \times 377} = \mathbf{230\ \mu F.}$$

Thus from (4),

$$L = \frac{2}{\omega^2 C} = \frac{2}{(377)^2 \times 230 \times 10^{-6}}$$

$$= \mathbf{61.18\ mH.}$$

4.7 SINUSOIDAL RESPONSE OF PARALLEL *R-L* CIRCUIT

Figure 4.14 represents a parallel R-L circuit excited by a sinusoid.

At steady state,

$$I_R = \frac{V}{R} \qquad \text{...(4.24)}$$

and $$I_L = \frac{V}{j\,X_L} = \frac{V}{j\omega L} \qquad \text{...(4.25)}$$

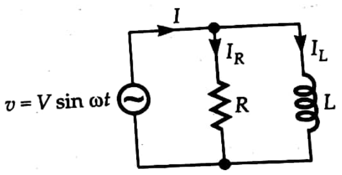


Fig. 4.14 A parallel R-L circuit being energized by sinusoid

However, applying Kirchhoff's Current Law,

$$I = I_R + I_L$$

$$= V\left(\frac{1}{R} + \frac{1}{j\omega L}\right) = VY \qquad \text{...(4.26)}$$

[∵ Y = admittance = G + jB;
where, G = conductance = 1/R mho
B = inductive susceptance = $1/X_L$ mho].

Here, $$Y = \frac{1}{R} + \frac{1}{j\omega L} = \frac{1}{R} - j\frac{1}{\omega L} \qquad \text{...(4.27)}$$

[it may be noted that, actual sign of B is –ve].

Also, the total current supplied by the source in steady state *lags* the voltage by impedance angle given by

$$\theta = \tan^{-1}\frac{-1/\omega L}{1/R} = \tan^{-1}\left(-\frac{R}{\omega L}\right) \qquad \text{...(4.28)}$$

Again, as $i = i_R + i_L$, we can further write,

$$i = \frac{v}{R} + \frac{1}{L}\int v\,dt = \frac{V_m}{R}\cos\omega t + \frac{V_m}{\omega L}\sin\omega t$$

[assuming $v = V_m \cos\omega t$]

$$\therefore \quad i = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L}\right)^2}\; V_m \cos(\omega t - \theta),$$

...(4.29)

when θ is given by equation (4.28)

If $R >> \omega L$, $\theta \to 90°$ and $i = \frac{V_m}{\omega L}\cos(\omega t - 90°)$.

With this relatively high resistance, the *current drawn by the resistance branch is low* and then $i \simeq i_L$.

If, on the other hand,

$R << \omega L$, $\theta \to 0°$ and $i = \frac{V_m}{R}\cos\omega t$.

Here, the *current drawn by the inductive branch is negligible* and then $i \simeq i_R$.

4.8 SINUSOIDAL RESPONSE OF PARALLEL *RC* CIRCUIT

Figure 4.15 represents a parallel *RC* circuit at steady state and excited by sinusoidal voltage source $v = V_m \sin\omega t$.

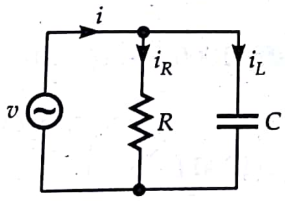


Fig. 4.15 Sinusoid applied to parallel *RC* circuit.

Here, $I_R = \frac{V}{R}$; $I_C = \frac{V}{1/j\omega C}$ $\left[\because |X_C| = \frac{1}{\omega C}\right]$

However, vectorially,

$$I = I_R + I_C$$

which gives, $I = V\left[\frac{1}{R} + \frac{1}{1/j\omega C}\right]$

or,

$$I = V\left[\frac{1}{R} + j\,\omega C\right] = V[G + jB]$$

...(4.30)

where, $G = 1/R$ mho and $B = +\omega C$ mho

and

$$\theta = \tan^{-1}\frac{\omega C}{1/R} = \tan^{-1}(\omega RC)$$

...(4.31)

Thus, it is evident that the *current leads the voltage* by an angle given by expression (4.31).

Also, $i = i_R + i_C$

$$= \frac{v}{R} + C\frac{dv}{dt}$$

$$= \frac{V_m}{R}\sin\omega t + \omega C V_m \cos\omega t$$

Then, $i = \sqrt{\left(\frac{1}{R}\right)^2 + (\omega C)^2}\; V_m \sin(\omega t + \theta),$

...(4.32)

θ being given by expression (4.31).

If $R >> \frac{1}{\omega C}$, $\theta \to 90°$

and $i \simeq i_c = \omega C V_m \sin(\omega t + 90°)$

and if $R << \frac{1}{\omega C}$, $\theta \to 0°$

and $i \simeq i_R = \frac{V_m}{R}\sin\omega t.$

4.9 SINUSOIDAL RESPONSE OF PARALLEL *RLC* CIRCUIT

Figure 4.16 represents a steady state parallel *RLC* circuit being energized by a voltage sinusoid

$$v = V_m \sin\omega t.$$

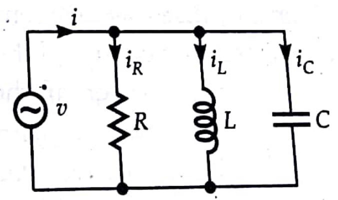


Fig. 4.16 Parallel *RLC* circuit.

Here, $i = i_R = i_L + i_C = \frac{v}{R} + \frac{1}{L}\int v\,dt + C\frac{dv}{dt}$

$$= \frac{V_m}{R}\sin\omega t - \frac{V_m}{\omega L}\cos\omega t + \omega C V_m \cos\omega t$$

...(4.33)

Let $i = A\sin(\omega t + \theta)$

$$= A\sin\omega t\cos\theta + A\cos\omega t\sin\theta$$

...(4.34)

CHAPTER 4

Equating the coefficients of $\sin \omega t$ and $\cos \omega t$ in (4.33) and (4.34), we get

$$\frac{V_m}{R} = A \cos \theta$$

and $$\left(\omega C - \frac{1}{\omega L}\right) V_m = A \sin \theta$$

Then $$\tan \theta = \frac{\omega C - \dfrac{1}{\omega L}}{1/R} \quad ...(4.35)$$

$$\cos \theta = \frac{1/R}{\sqrt{\left(\dfrac{1}{R}\right)^2 + \left(\omega C - \dfrac{1}{\omega L}\right)^2}} \quad ...(4.35(a))$$

$$A = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2} \cdot V_m \quad ...(4.36)$$

and $$i = \sqrt{\left(\frac{1}{R}\right)^2 - \left(\omega C - \frac{1}{\omega L}\right)^2} \times V_m \sin\left[\omega t + \tan^{-1}\left(\omega C - \frac{1}{\omega L}\right) R\right] \quad ...(4.37)$$

Obviously, the sign of the phase angle θ depends on the relative values of ωC and $\dfrac{1}{\omega L}$.

In this context it may be noted that the inductive branch current I_L is at 90° lagging the supply voltage while the capacitive branch current I_C is at 90° leading the supply voltage. With proper selection of L and C, these two currents I_L and I_C may mutually cancel each other and the net current can be *resistive* only. However, if the capacitive current is predominant, *i*, the net current would be capacitive and if the inductive current I_L is predominant, the net current *i* would be inductive.

EXAMPLE 4.38 *The current in the resistive branch of a parallel RC circuit is given by $i_r = 10 \cos (1000\, t - 10°)$ amp. What is the current in the capacitance ? Assume $R = 10\,\Omega$, $C = 10\,\mu F$.*

SOLUTION. In the parallel *RC* circuit,

$$v_c = v_r \quad ...(A)$$

However,

$$v_r = Ri_R = 10 \times 10 \cos (1000\, t - 10°)\text{ V}$$
$$= 100 \cos (1000\, t - 10°) \quad ...(B)$$

The capacitive current i_c will obviously lead the voltage v_r (and thus v_c) by 90°.

$$\therefore\ i_c = \omega CV \cos (1000\, t - 10° + 90°) \quad ...(C)$$
$$= 1000 \times 10 \times 10^{-6} \times 100 \cos (1000\, t + 80°)\text{ A}$$
$$= \mathbf{1 \cos (1000\, t + 80°)\ A}$$

[Here *V* in eqn. (*C*) has been shown to be equivalent to the r.m.s. value of the expression of v_r in eqn. (*B*)].

EXAMPLE 4.39 *Fig. E4.23 represents a parallel RL circuit being energized by a sinusoidal a.c. voltage of*

$$v = 100 \sin (1000\, t + 36°)\ V$$

Obtain the instantaneous value of currents through R and L. Hence obtain the total current in term of R.M.S. *values.*

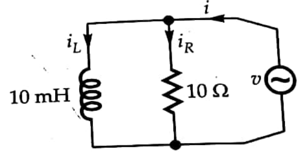


Fig. E4.23

SOLUTION.

$$i_R = \frac{V_m}{R} \sin (1000\, t + 36°)$$
$$= 10 \sin (1000\, t + 36°)\text{ A}$$
$$= 10 \cos (1000\, t - 54°) \quad ...(A)$$

$$i_L = \frac{V_m}{\omega L} \sin (1000\, t + 36° - 90°)$$

[$\because$ i_L would lag v by 90°]

$$= 10 \sin (1000\, t - 54°)$$

$$\left[\because \frac{V_m}{\omega L} = \frac{100}{1000 . 10 \times 10^{-3}} = 10\right]$$

$$= 10 \cos (1000\, t - 144°) \quad ...(B)$$

From (*A*) and (*B*) we get,

$I_R = 10\angle -54°$ [Please note that there are the r.m.s. values]

and $I_L = 10\angle -144°$

$\therefore$ net r.m.s. current

$$I = I_R + I_L = 10\angle -54° + 10\angle -144°$$
$$= 10 (\cos 54° - j \sin 54°) + 10 [\cos (-144°) + j \sin (-144°)]$$
$$= 5.88 - j\,8.1 - 8.1 - j\,5.88 = (-2.22 - j\,13.97)\text{ A}$$

i.e., $\mathbf{I = 14.14\angle -99°\ A}$

EXAMPLE 4.40 *The applied voltage in a parallel RLC circuit is given by*

$$v=\left[50\sin\left(5000\,t+\frac{\pi}{4}\right)\right]V$$

If the values of R, L and C be given as 20 Ω, 1.6×10^{-3} H and 20 μF, find the total current supplied by the source.

SOLUTION. In the parallel RLC circuit, the net current from the source will be vector sum of the branch currents I_R, I_L and I_C.

Now, $$I_R=\frac{V}{R}=\frac{(50/\sqrt{2})\angle 45^\circ}{20}$$

$$=1.768\angle 45^\circ \text{ A}$$

$$\left[\text{from } v=50\sin\left(5000\,t+\frac{\pi}{4}\right),\; V=\frac{50}{\sqrt{2}}\angle\frac{\pi}{4}\text{ volts}\right]$$

$$I_L=\frac{V}{X_L}$$

$$=\frac{(50/\sqrt{2})\angle 45^\circ}{5000\times1.6\times10^{-3}\angle 90^\circ}$$

$$=4.421\angle -45^\circ \text{ A}$$

$$[\because X_L=j\omega L=j\times5000\times1.6\times10^{-3}\text{ ohms}]$$

$$I_C=\frac{V}{X}=j\,V\omega C$$

$$=\left(\frac{50}{\sqrt{2}}\right)\angle 45^\circ\times5000\times20\times10^{-6}\angle 90^\circ$$

$$=3.54\angle 135^\circ \text{ A.}$$

$$\therefore\; I=I_R+I_L+I_C$$

$$=(1.768\angle 45^\circ+4.421\angle -45^\circ+3.54\angle 135^\circ)$$

$$=1.768(\cos 45^\circ+j\sin 45^\circ)+4.421(\cos 45^\circ-j\sin 45^\circ)+3.54(\cos 135^\circ+j\sin 135^\circ)$$

$$=1.768\times0.707+j\,1.768\times0.707+4.421\times0.707-j\,4.421\times0.707+3.54\times(-0.707)+j\,3.54\times0.707$$

$$=(1.873+j\,0.627)\text{ A}$$

$$\therefore\; \mathbf{I=1.975\angle 18.5^\circ\text{ A (lead).}}$$

[I is the net current from source]

4.10 ADDITIONAL EXAMPLES*

EXAMPLE 4.41 *In a series RLC circuit $R=3\,\Omega$, $X_L=j\,6\,\Omega$, $X_C=-j\,2\,\Omega$. What is the voltage applied across the combination if the series current is $10\angle -143^\circ$ A?*

SOLUTION.

$$Z=R+j(X_L-X_C)$$
$$=3+j(6-2)=3+j\,4$$
$$=5\angle 53^\circ\,\Omega.$$

Let the supply voltage be $V\angle\phi^\circ$.

Here

$$V\angle\phi^\circ=(Z)(I)=5\angle 53^\circ\times10\angle -143^\circ.$$
$$=\mathbf{50\angle -90^\circ\text{ V}}$$

i.e., the voltage leads the current by

$$\mathbf{(143^\circ-90^\circ)=53^\circ.}$$

EXAMPLE 4.42 *Find the magnitude and direction of current through Z if $Z=(2+j\,1-j\,0.5)\,\Omega$ [Fig. E4.24].*

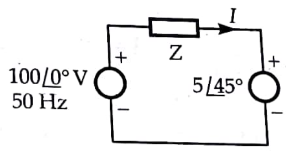


Fig. E4.24

SOLUTION.

$$Z=2+j(1-0.5)=(2+j\,0.5)\,\Omega$$
$$=2.06\angle 14^\circ\,\Omega.$$

$\therefore$ I, the current through Z is given by

$$I=\frac{10\angle 0^\circ-5\angle 45^\circ}{2.06\angle 14^\circ}\text{ A}$$

$$=\frac{7.37\angle -28.7^\circ}{2.06\angle 14^\circ}=3.58\angle -42.7^\circ\text{ A}$$

Here, it is seen that the current lags the voltage by **42.7°** while the magnitude of current is **3.58 A.**

EXAMPLE 4.43 *In Fig. E4.25, find Z.*

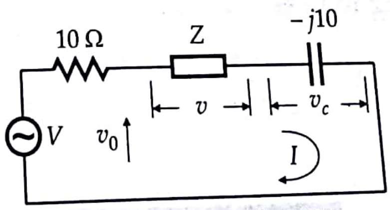


Fig. E4.25

*Also ref. to "An Introduction to Network, Filters and Transmission Lines" by the same author.

SOLUTION.

$$v_0 = 5\angle 60° = (2.5 + j\,4.33)\text{ V}$$

$$v_C = 2\angle -30° = (1.732 - j\,1)\text{ V}$$

$$v = v_0 - v_C \text{ (vectorial subtraction)}$$

also $$I = \frac{v_C}{X_C} = \frac{v_C}{-j\,10} = \frac{2\angle -30°}{10\angle -90°} = 0.2\angle 60°\text{ A}$$

$$Z = \frac{v}{I} = \frac{5\angle 60° - 2\angle -30°}{0.2\angle 60°} = \frac{0.768 + j\,5.33}{0.2\angle 60°}$$

$$= \mathbf{26.92\angle 21.8°\,\Omega}.$$

EXAMPLE 4.44 *A voltage of 230 V ∠45°, 50 Hz is applied across an RLC series circuit where the input current is 10 ∠−30° A. If* $R = 5\,\Omega$, $X_L = j\,8\,\Omega$, *find the value of capacitive reactance.*

SOLUTION.

$$Z_{\text{total}} = \frac{V}{I} = \frac{230\angle 45°}{10\angle -30°}$$

$$= 23\angle 75°\,\Omega = (5.95 + j\,22.21)\,\Omega$$

However,

$$Z = R + j\,(X_L - X_C) = 5 + j\,(8 - X_C)$$

or, $$23^2 = 5^2 + (8 - X_C)^2 \qquad [\because\ Z = 23\,\Omega \text{ and } Z^2 = \sqrt{R^2 + (X_L - X_C)^2}\,]$$

or, $$23^2 - 5^2 = 64 - 16\,X_C + X_C^2$$

or, $$X_C^2 - 16\,X_C + 64 - 23^2 + 5^2 = 0$$

or, $$X_C^2 - 16\,X_C - 440 = 0$$

$$\therefore\ X_C = \frac{16 \pm \sqrt{16^2 + 4 \times 440}}{2}$$

$$= 8 \pm 22.45 = \text{either } 30.45 \text{ or } -14.45$$

Taking the +ve root only

$$X_C = \frac{1}{2\,\pi\,fC}$$

or, $$C = \frac{1}{2\,\pi\,f\,X_C} = \frac{1}{2\,\pi \times 50 \times 30.45}$$

$$= 1.05 \times 10^{-4}\text{ F} = 105\,\mu\text{F}.$$

Thus $X_C = \mathbf{30.45\,\Omega}$ [or $C = 105\,\mu$F]

EXAMPLE 4.45 *An unknown impedance of Z Ω is connected in series with (5 + j 8) Ω coil (Fig. E4.26). If I = 2.5 ∠ − 15° A, find the value of Z.*

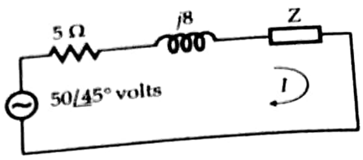


Fig. E4.26

SOLUTION. Let Z_0 be the net impedance of the circuit.

Obviously

$$Z_0 = \frac{V}{I} = \frac{50\angle 45°}{2.5\angle -15°}$$

or, $$Z_0 = 20\angle 60° = (10 + 17.3)\,\Omega$$

However, $$Z_0 = (5 + j\,8)\,\Omega + Z$$

or, $$10 + j\,17.3 = 5 + j\,8 + Z.$$

$$\therefore\quad Z = (5 + j\,9.3)\,\Omega$$

$$= \mathbf{10.56\angle 61.74°\,\Omega}.$$

EXAMPLE 4.46 *A heater takes 10 A at 50 V. Calculate the impedance of a choke of 5 Ω resistance to be placed in series with it in order that it may work at 200 V, 50 Hz supply. Find the power factor of the circuit.*

SOLUTION. In Fig. E4.27, it is given that $I = 10$ A, V_H (drop across heater) $= 50$ V, R_C (resistance of choke) $= 5\,\Omega$, $V_S = 200$ V.

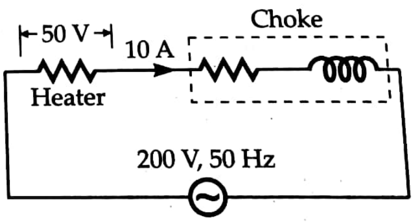


Fig. E4.27

$$\therefore\ R_H \text{ (Resistance of heater)} = \frac{50}{10} = 5\,\Omega$$

Hence, net resistance of the circuit is

$$R = R_H + R_C = 5 + 5 = 10\,\Omega$$

Obviously, V_{R_H} (drop across R_H)

$$= R_H \times I = 5 \times 10 = 50\text{ V}$$

and $$V_{R_C} = R_C \times I = 50\text{ V}$$

Also, V_{R_L} (drop across X_L, the inductive reactance of the choke)

$$= IX_L = 10\, X_L \text{ V}$$

However, the supply voltage being the vector sum of the drops V_R and V_L,

$$V = (V_{R_C} + V_{R_H}) + j\, V_L$$

$$(200)^2 = (50+50)^2 + V_L^2$$

$$\therefore \quad V_L^2 = 200^2 - 100^2$$

$$V_L = 173.21 \text{ V}$$

or,

$$X_L = \frac{173.21}{I} = \frac{173.21}{10} = 17.32\ \Omega$$

This gives the impedance of the choke as

$$Z = \sqrt{R_C^2 + X_L^2} = \sqrt{5^2 + (17.32)^2} = 18\ \Omega$$

Thus the impedance of the coil $= \mathbf{18\ \Omega}$.

$\therefore$ The impedance of the whole circuit

$$= \sqrt{(R_C + R_H)^2 + X_L^2)}$$

$$= \sqrt{10^2 + (17.32)^2} = 20\ \Omega.$$

$$\text{Power factor} = \frac{R}{Z} = \frac{10}{20} = 0.5$$

Thus for the given problem,

Z (coil) = 18 Ω, P. F. = 0.5.

Example 4.47 *A coil takes a current of 1 ∠60° A (lag) from 100 V, 60 Hz supply. Calculate its inductance, resistance and impedance.*

Solution. Given, $I = 1\angle -60^\circ$ A

$$\therefore \quad \phi = 60^\circ \text{ (lag)}$$

$$V = 100 \text{ V}, \ f = 50 \text{ Hz}$$

$$\therefore \quad Z = \frac{V}{I} = \frac{100}{1\angle -60^\circ} = 100\ \angle 60^\circ\ \Omega$$

However, $\cos\phi\,(\text{p.f.}) = \dfrac{R}{Z}$

$$\therefore \quad R = Z\cos\phi = 100\cos 60^\circ = 50\ \Omega$$

Again, $Z = \sqrt{R^2 + X_L^2}$

$$\therefore \quad X_L^2 = Z^2 - R^2 = (100)^2 - (50)^2$$

$$\therefore \quad X_L = 86.6\ \Omega$$

and

$$L = \frac{X_L}{2\pi f} = \frac{86.6}{2 \times 3.14 \times 60} = 0.23 \text{ H}$$

Thus, for the given problem,

$$\mathbf{R = 50\ \Omega;\ L = 230\ mH;\ Z = 100\ \Omega.}$$

Example 4.48 *In Fig. E4.28, "r" is a pure resistance and "CH" the choke coil, connected in series. Power dissipated in r being 250 W and that in choke being 50 W, find the value of the reactance in the choke and the value of the supply voltage.*

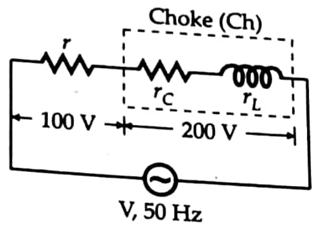


Fig. E4.28

Solution. Power dissipated $= \dfrac{V^2}{R}$

$\therefore$ For the non inductive resistor,

$$r = \frac{V^2}{W} = \frac{100^2}{250} = 40\ \Omega$$

and current through r,

$$I = \frac{\text{drop across } r}{r}$$

i.e.,

$$I = \frac{100}{40} = 2.5 \text{ A}$$

Obviously, the current flowing through the choke would also be 2.5 A.

As the power loss in choke can only take place in the resistive part of the choke, hence,

$$I^2 r_C = 50$$

or,

$$r_C = \frac{50}{(2.5)^2} = 8\ \Omega$$

But the drop across the coil is 200 V, while through current is 2.5 A.

$$\therefore \quad Z_{Ch} = \frac{V}{I} = \frac{200}{2.5} = 80\ \Omega$$

and

$$X_L = \sqrt{(Z_{Ch})^2 - r_C^2}$$

$$= \sqrt{(80)^2 - (8)^2} = 79.6\ \Omega$$

8

Analysis of Transient Response in Passive Circuits
(*Differential Equation Approach*)

8.1 INTRODUCTION

Analysis of behaviour of electric circuit reveals that as soon as a circuit is switched from one condition to another either by change of source or by alteration of circuit elements, branch currents and voltage drops change from their initial values to new values. These changes take a short spell of time to settle to permanent values (steady state) till further switching or circuit alteration is attempted. This brief spell of time is called *transient time* and the value of the variables (current and voltage drop) during this period is called *transient value.* This chapter deals with transient analysis of *passive* electric circuits using initial conditions and differential equation approach.

8.2 PREAMBLE OF DIFFERENTIAL EQUATION APPROACH

The *first order circuit,* during its transient state of operation, is governed by a first order linear differential equation and in reality the first order circuit contains resistance(s) and one inductor or capacitor *i.e.,* the resistance and one energy storing component. On the other hand, a *second order circuit* contains two independent energy storage elements with or without addition to resistance(s).

The first order circuit equation is given by

$$\frac{dy}{dt} + ay = bx$$

where x is the input, y the output and a and b are the constants. Superposition and linearity, both are applicable in first order passive circuits. The *natural response* of the first order circuit is obtained when the governing equation of the circuit, at its transient state, is given by

$$\frac{dy}{dt} + ay = 0$$

[*i.e.*, by a *homogeneous differential equation*]

The solution for such a equation is given by

$$y = Ke^{-\alpha t},$$

where K is a constant and if evaluated using initial conditions, the solution y becomes a *particular solution.* At $t = 0$, if $y = y_0$, then the natural response of the circuit is given by the following equation :

$$y_n = y_0\, e^{-\alpha t} \quad \text{for } t > 0.$$

[for $t = 0$, y at t_0 $(= y_0)$ becomes K in the genera solution $y = Ke^{-\alpha t}$].

When x (input) is specified, it is called *forcir function* and the solution y is called the *forc* *response* (or the *particular integral* y_p). The complete solution is given by

$$y = y_p + y_n$$
$$= y_p + Ke^{-\alpha t} \quad \text{for } t > 0.$$

K is determined with given initial conditions.

Elaborating further we can say that if the first order circuit is governed by a *non-homogeneous* differential equation

$$\frac{dy}{dt} + ay = bx = Q\,(\text{say}),$$

where Q is either a function of independent variable or a constant, the solution is then given by

$$y = e^{-\alpha t}\int Qe^{\alpha t}\,dt + Ke^{-\alpha t} = y_p + y_c$$

Here $y_p\ (= e^{-\alpha t}\int Qe^{\alpha t}\,dt)$ is known as *particular integral* and $y_c\ (= Ke^{-\alpha t})$ is known as *complementary function.* It may be noted here that y_p does not contain the arbitrary constant K and the complementary function does not depend on the forcing function Q.

If Q is constant, we can write

$$y = e^{-\alpha t}\,.\,Q.\frac{e^{\alpha t}}{a} + Ke^{-\alpha t}$$
$$= \frac{Q}{a} + Ke^{-\alpha t}$$

A second order differential equation may be expressed as

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0$$

[A, B, C being constants]

The general solution of this second order differential equation is given by

$$y = K_1e^{\alpha_1 t} + K_2e^{\alpha_2 t},$$

where K_1 and K_2 are the constants and α_1 and α_2 are the roots of the *characteristic equation.*

$$Ap^2 + Bp + C = 0$$

α_1 and α_2 are given by

$$\alpha_1, \alpha_2 = -\frac{B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

If $\alpha_1 = \alpha_2$, the roots are said to be repetitive type and the general solution is given by

$$y = K_1e^{-\frac{B}{2A}.t} + K_2e^{-\frac{B}{2A}.t}$$

Let us now explore applications of the discussions we have in the preceeding paragraphs. A series RC or RL circuit energized by a voltage source or a parallel RC or RL circuit energized by a current source are the examples of linear first order circuits. When the voltage in such a series circuit or current in such a parallel circuit is applied by switching at $t = 0$, the governing differential equation is given by

$$\frac{dy}{dt} + ay = bx\ (= Q)$$

where the forcing function is either voltage or current. The forced response is obtained at $t > 0$ as $y = y_p + y_c$ (as discussed earlier). Next, we suppose that this series or parallel circuit attains steady state and the voltage or current source is suddenly shorted at $t = 0$. The differential equation of the circuit at $t = 0+$ is obtained as $\frac{dy}{dt} + ay = 0$ as there is no forcing function (*i.e.*, voltage or current source).

The natural response of circuit is obtained as

$$y = Ke^{-\alpha t} \quad \text{for } t > 0.$$

We now explore another possibility ; let a first order circuit be at steady state with a constant source and at $t = 0$, switching takes place and the circuit now jumps to another constant source supply. Obviously, the solution will have the form

$$y = Y_s + Ke^{-\alpha t}, \quad t > 0$$

where Y_s is the new steady state value.

Let us assume that at $t = 0^+$,

$$y = Y_0.$$

$$\therefore \quad y\,(\text{at } t = 0^+) = Y_s + Ke^{-\alpha.0} = Y_s + K$$

or

$$Y_0 = Y_s + K$$

[$\because$ we assumed $y = Y_0$ at $t = 0^+$]

$$\therefore \quad K = (Y_0 - Y_s)$$

Then finally,

$$y = Y_s + (Y_0 - Y_s)\,e^{-\alpha t}, \quad t > 0.$$

If the switching takes place at $t \neq 0$ but at $t = t_0$, we can write

$$y = Y_s + (Y_0 - Y_s)\,e^{-\alpha(t - t_0)}$$

In reality, a first order circuit may face this situation when switching takes place from one d.c. source to another d.c. source.

$t = 0^+$ indicates the instant just after t = 0th instant of time. Similarly, $t = 0^-$ indicates the instant just before the t = 0 th instant of time.

The second order circuit being containing R, L and C elements, for convenience we assume the constants A, B and C of the second order differential equation as

$$A = 1, \quad B = 2\alpha \text{ and } C = \omega_0^2$$

[α is the *damping factor* and ω_0 the *frequency of oscillation*]

$\therefore$ The homogeneous second order differential equation becomes

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \omega_0^2\, y = 0$$

and the characteristic equation becomes

$$p^2 + 2\alpha p + \omega_0^2 = 0$$

or

$$(p+\alpha)^2 + \omega_0^2 - \alpha^2 = 0$$

The roots of the above equation are p_1 and p_2 and are called *natural frequencies*. Depending on the relative values of α and ω_0 there are *three* different cases for the roots :

(*i*) $\alpha > \omega_0$ **[overdamped condition]**

$$p_1 = -\alpha + (\alpha^2 - \omega_0^2)^{1/2} = -s_1$$

$$p_2 = -\alpha - (\alpha^2 - \omega_0^2)^{1/2} = -s_2$$

s_1 and s_2 being real and positive.

The natural response is then given by

$$y_n = K_1 e^{-s_1 t} + K_2 e^{-s_2 t}, \quad t > 0.$$

(*ii*) $\alpha < \omega_0$ **[underdamped condition]**

Let $\omega_0^2 - \alpha^2 = \omega_d^2$

The roots are complex conjugate with negative real part.

$$p_1 = -\alpha + j\omega_d \; ; \; p_2 = -\alpha - j\omega_d$$

The natural response is given by

$$y_n = e^{-\alpha t}(K_1 \cos \omega_d t + K_2 \sin \omega_d t)$$

(*iii*) $\alpha = \omega_0$ **[critically damped condition]**

$$(p+\alpha)^2 = 0 \; ; \; p = -\alpha = -\omega_0$$

and two roots p_1 and p_2 coincide.

The natural response is given by

$$y_n = (K_1 + K_2 t)\, e^{-\alpha t}, \quad t > 0.$$

In all the above cases, K_1 and K_2 are determined from initial conditions, usually the values of y and (dy/dt) at $t = 0^+$.

8.3 TRANSIENT RESPONSE OF SERIES *R-L* CIRCUIT HAVING D.C. EXCITATION (FIRST ORDER CIRCUIT)

Let a d.c. voltage V be applied suddenly (*i.e.*, at $t = 0$) by closing a switch K in a series R-L circuit as shown in Fig. 8.1

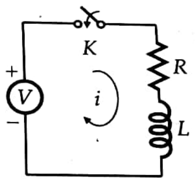


Fig. 8.1 Series RL circuit.

Applying Kirchhoff's voltage law (KVL) yields

$$Ri + L\frac{di}{dt} = V \qquad \text{...(8.1)}$$

or,

$$\frac{di}{dt} + i\frac{R}{L} = \frac{V}{L}$$

or,

$$\left(p + \frac{R}{L}\right) i = \frac{V}{L} \quad \left[\text{where } p = \frac{d}{dt}\right] \qquad \text{...(8.2)}$$

Equation (8.2) is a *non-homogeneous differential equation* and the forced response is obtained from its solution ; the solution is given by

$$i = i_C + i_p \; ;$$

where i_C is the *complementary function* that always goes to zero value in a relatively short time (transient solution) and is given by

$$i_C = c\,e^{-(R/L)t} \; ;$$

c being a constant ; i_p is the particular solution of i that provides the steady state response.

$$i_p = e^{(-R/L)t} \int e^{(R/L)t} \left(\frac{V}{L}\right) dt = \frac{V}{R}.$$

Hence the net solution i is given by

$$i = ce^{-(R/L).t} + \frac{V}{R} \qquad \text{...(8.3)}$$

An inductance, due to its "electrical inertia" does not allow sudden change of current through it following the rules of electromagnetic induction and hence at current though it just before switching is same to the current just after the switching. This is represented as

$$i(0^-) \equiv i(0^+)$$

However, before switching, there was no current through the inductor and hence at time $t=0^+$ (*i.e.*, just after the switching) the current through the inductor will also be zero.

i.e., $\quad i(0^+)=0$

With the initial condition, equation (8.3) at $t=0^+$ becomes

$$0 = c\,e^{-(R/L).0} + \frac{V}{R}$$

i.e., $\quad c = -\frac{V}{R}$

This gives,

$$i = -\frac{V}{R}e^{-(R/L).t} + \frac{V}{R}$$

or, $$i = \frac{V}{R}(1-e^{-(R/L).t})\ \text{A} \qquad ...(8.4)$$

Expression (8.4) clearly shows the exponential rise of current *i* charging the inductor. The profile of *i* vs *t* has been shown in Fig. 8.2.

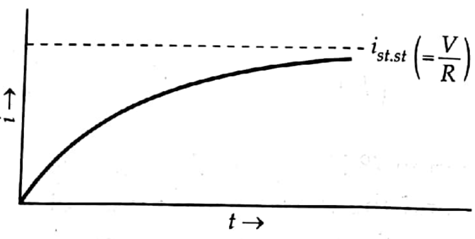


Fig. 8.2 Charging current profile in *R-L* circuit.

It may be noted that once the transient dies out within the short spell of time (which can be observed by substituting large values of *t*) the *steady state current* (final current) remains in the circuit given by (*V*/*R*).

It may also be observed that if we put $t=\frac{L}{R}$, expression (8.4) becomes

$$i=\frac{V}{R}(1-e^{-1})=\frac{V}{R}(1-0.368)=0.632\ I_S \qquad ...(8.5)$$

[I_S being the steady state current $= V/R$]

Expression (8.5) means that at time $t=\frac{L}{R}$, the current through the *R-L* circuit rises to 63.2% of the final value. Conventionally, this time is known as "time constant" (TC) and is the ratio of *L* and *R* in the *L-R* circuit. (*R*/*L*) is the inverse of TC and is called "damping ratio".

The voltage drops across the resistance and inductance during the transient period is given by

$$v_R = iR = V(1-e^{-(R/L).t})$$

and $$v_L = L\frac{di}{dt} = L\frac{d}{dt}\left[\frac{V}{R}(1-e^{-(R/L).t})\right]$$

$$= V\,e^{-(R/L).t}.$$

It may now be observed that the voltage transient across the resistor is an *exponential rise* while that across the inductance is *exponentially decaying.*

Obviously

$$v_R + v_L = V(1-e^{-(R/L).t}) + V\,e^{-(R/L).t} = V.$$

The instantaneous powers in the resistor and inductor are given as

$$p_R = v_R\, i = V(1-e^{-(R/L).t}).\frac{V}{R}(1-e^{-(R/L).t})$$

$$= \frac{V^2}{R}(1-2\,e^{-(R/L).t} + e^{-2(R/L).t}) \qquad ...(8.6)$$

and $$p_L = v_L\, i = V\,e^{-(R/L).t}.\frac{V}{R}(1-e^{-(R/L).t})$$

$$= \frac{V^2}{R}(e^{-(R/L).t} - e^{-2(R/L).t}) \qquad ...(8.6(a))$$

The total power (*p*) being

$$p_R + p_L = \frac{V^2}{R}(1-2e^{-(R/L).t} + e^{-2(R/L).t}) + \frac{V^2}{R}(e^{-(R/L).t} - e^{-2(R/L).t})$$

$$= (V^2/R)(1-e^{-(R/L).t})\ \text{VA}$$

Let us now analyse another transient condition (natural response) of the *R-L* circuit assuming that following the closing of the switch, the circuit reaches at steady state (at $t=\infty$) and suddenly the voltage is withdrawn by opening the switch *K* and throwing it to *K'*. Application of KVL yields (Fig. 8.3)

$$Ri + L\frac{di}{dt} = 0$$

[Voltage source is withdrawn by opening the switch *K*]

or, $$(p + R/L)\,i = 0 \qquad ...(8.7)$$

Equation (8.7) is a *homogeneous differential* equation having only complementary function in the solution of *i*, particular function is zero.

$$\therefore\ i = i_C + 0 = i_C = c'\,e^{-(R/L).t} \qquad ...\mathbf{(8.8)}$$

However, at $t=0^+$, the inductor will keep the steady state current (V/R) even the switch is thrown to position K' (Fig. 8.3) short circuiting the charged R-L circuit and withdrawing the voltage source.

$$\therefore \quad i(0^+) = i_{st.st}(0^-) = \frac{V}{R}.$$

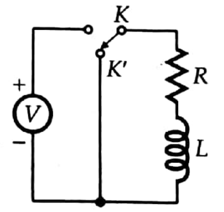


Fig. 8.3 R-L discharging circuit.

This gives from equation (8.8)

$$\frac{V}{R} = c' e^{-(R/L).0} = c'$$

Hence the final solution becomes

$$i = \frac{V}{R} e^{-(R/L).t} = I_0 e^{-(R/L).t} \text{ A} \quad ...(8.9)$$

[I_0 being (V/R)]

The decaying current i is thus a exponentially decaying function, the profile being shown in Fig. 8.4.

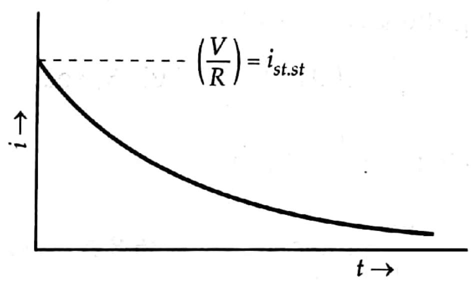


Fig. 8.4 i-t profile for discharging condition of R-L circuit.

Assuming $t = \frac{L}{R}$, from equation (8.9),

$$i = I_0 e^{-1} = 0.368 I_0 \simeq 0.37 I_0$$

$\frac{L}{R}$ being called "time constant" it is observed that the decaying current will reach to 37% of the initial steady state current at one time constant.

The corresponding voltages across the resistance and inductance are then given as

$$v_R = iR = V e^{-(R/L).t}$$

and
$$v_L = L\frac{di}{dt} = -V e^{-(R/L).t}$$

The instantaneous powers are given by

$$p_R = v_R\, i = \frac{V^2}{R} e^{-2(R/L).t} \text{ W}$$

and
$$p_L = v_L\, i = -\frac{V^2}{R} e^{-2(R/L).t} \text{ VA.}$$

Example 8.1 *In a series R-L circuit, the application of a direct voltage results a steady state current of 0.632 I in 1 second. I being the final steady state value of the current. However, after the current has reached its final value, a sudden short circuit is applied against the source. What would be the value of the current after one second ?*

Solution. The charging current in the R-L circuit is given by

$$i = I(1 - e^{-t/T});$$

T being the time constant.

Here, in the first case,

$$0.632\, I = I(1 - e^{-t/T})$$

or, $\quad 0.632 = (1 - e^{-t/T}) \qquad [\because t = 1 \text{ sec., given}]$

or, $\quad e^{-1/t} = 1 - 0.632 = 0.368.$

As soon as short circuit is applied in steady state, current will start decaying from I. The decay current i' is given by

$$i' = I e^{-t/T}$$

After a decay for 1 sec,

$$i = I e^{-t/T} = I \times 0.368 = 0.368\, I.$$

$\therefore$ The value of the decay current after 1 sec. of application of short circuit is **0.368 I.**

Example 8.2 *Find the current in a series R-L circuit having $R = 2\,\Omega$ and $L = 10$ H while a d.c. voltage of 100 V is applied. What is the value of this current after 5 sec. of switching on ?*

Solution. Time constant $= \frac{L}{R} = \frac{10}{2} = 5$ sec.

Charging constant is given by

$$i = I(1 - e^{-t/T});$$

I being the steady state current.

Here, $\quad i = \frac{100}{2}(1 - e^{-t/5}) = 50(1 - e^{-t/5})$

$$\left[\because I = \frac{100 \text{ V}}{2\,\Omega}\right.$$

$\therefore \quad i_{\text{steady state}} = 50\text{ A}; \quad i_{\text{transient}} = -50\,e^{-t/5}\text{ A}$

After $\quad t = 5$ sec, $\quad i_{\text{transient}} = -50\,e^{-t/5}$

$$= -50 \times \frac{1}{e} = -18.518\text{ A}$$

Thus i (after 5 sec.)

$$= i_{\text{st. state}} + i_{\text{transient}}$$

$$= 50 - 18.518 = \mathbf{31.482\ A.}$$

EXAMPLE 8.3 *A series R-L circuit has R = 25 ohm and L = 5 Henry. A d.c. voltage of 100 V is applied at t = 0. Find (a) the equations for charging current, voltage across R and L and (b) the current in the circuit 0.5 second later and (c) the time at which the drops across R and L are same.*

SOLUTION. (*a*) $\frac{L}{R}$ = time constant $(T) = \frac{5}{25} = \frac{1}{5}$ sec.

The charging current is given by

$$i = I(1 - e^{-t/T});$$

T being the final steady state current.

Here, $\quad i = \frac{100}{25}(1 - e^{-t/(1/5)})$ or $\mathbf{4\,(1 - e^{-5t})\ A}$

Voltage drop across R is

$$v_R = iR = 4 \times 25\,(1 - e^{-5t})$$

$$= \mathbf{100\,(1 - e^{-5t})\ V}$$

Voltage drop across L is,

$$v_L = L\frac{di}{dt}$$

or $\quad v_L = 5 \times \frac{d}{dt}[4\,(1 - e^{-5t})] = \mathbf{100\,e^{-5t}\ V}$

(*b*) At $t = 0.5$ sec.,

$$i = 4\,(1 - e^{-5(0.5)}) = 4\,(1 - e^{-2.5}) = \mathbf{3.67\ A.}$$

(*c*) To satisfy the condition of $v_R \equiv v_L$, $v_R = v_L = 50$ V, since applied voltage is 100 V.

$\therefore \quad 50 = L\frac{di}{dt} = 100\,e^{-5t}$

or, $\quad 0.5 = e^{-5t}$ or $t = \mathbf{0.139\ sec.}$

EXAMPLE 8.4 *A d.c. voltage of 100 V is applied to a coil having R = 100 Ω and L = 10 H. What is the value of the current 0.1 sec later the switching on ? What is the time taken by the current to reach half of its final value ?*

SOLUTION. Final current $I = \frac{V}{R} = \frac{100}{10} = 10$ A.

T (time constant) $= \frac{L}{R} = \frac{10}{10} = 1$ sec.

The charging current is given by

$$i = I(1 - e^{-t/T}) \text{ or } i = 10\,(1 - e^{-t})$$

The value of current 0.1 sec. later is

$$i = 10\,(1 - e^{-0.1}) = \mathbf{0.95\ A.}$$

Again, when the current will be half of final value,

$$5 = 10\,(1 - e^{-t})$$

or, $\quad 0.5 = (1 - e^{-t}) \quad \therefore\ t = 0.69$ sec.

Hence, after **0.69 sec.** of switching, the current will be just half the final value.

EXAMPLE 8.5 *A coil having resistance of 10 Ω and inductance of 1 H is switched on to a direct voltage of 100 V. Calculate the rate of change of the current (a) at the instant of closing the switch and (b) when t = L/R (c) Also find the steady state value of the current.*

SOLUTION. The charging current,

$$i = \frac{V}{R}(1 - e^{-t/T})$$

where T is the time constant $= \frac{1}{10} = 0.1$ sec.

$\therefore \quad i = \frac{100}{10}(1 - e^{-t/0.1})$ or $i = 10\,(1 - e^{-10t})$

$\therefore \quad \frac{di}{dt} = 100\,e^{-10t}$

Thus the rate of change of current at the instant of closing the switch $(t = 0)$ is

$$\frac{di}{dt} = 100\,e^{-10\times 0} = \mathbf{100\ A/sec.}$$

(*b*) When $t = \frac{L}{R} = 0.1$ sec.,

$$\frac{di}{dt} = 100\,e^{-10\times 0.1} = \mathbf{36.8\ A/sec.}$$

(*c*) The final steady state current

$$i_{\text{st.}} = \frac{100\text{ V}}{10\,\Omega} = \mathbf{10\ A.}$$

EXAMPLE 8.6 *Determine the voltage at the terminals of a coil having R = 10 Ω and L = 15 H at the instant when the current is 10 A and increasing @ 5 A/sec. Also find the stored energy in the inductor. Compute the same exercise for the case when the current decreases @ 5 A/sec.*

SOLUTION. In the L-R circuit,

$$E = iR + L\frac{di}{dt}$$

[i = current, E = voltage at coil terminals]

$$= 10i + 15\frac{di}{dt} = 10 \times 10 + 15 \times 5 = \mathbf{175\ V.}$$

$$\text{Energy stored} = \frac{1}{2} LI^2 = \frac{1}{2} \times 15 \times 10^2 = \mathbf{750\ J}$$

When the current is decaying,

$$E = iR - L\frac{di}{dt} = 10 \times 10 - 15 \times 5 = \mathbf{25\ V.}$$

$$\text{Energy stored} = \frac{1}{2} LI^2 = \frac{1}{2} 15 \times 10^2 = \mathbf{750\ J.}$$

EXAMPLE 8.7 *In Fig. E8.1 the switch K is kept first at position 1 and steady state condition is reached. At $t = 0$, the switch is moved to position 2. Find the current in both the cases.*

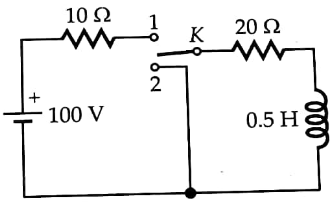


Fig. E8.1

SOLUTION. When the switch is at 1, the steady state current becomes

$$i = \frac{100}{10 + 20} = \mathbf{3.33\ A}$$

As soon as the switch is moved to position 2 at $t = 0$, the LR circuit starts decaying and the decay current is given by

$$i = I\,e^{-t/T}$$

where I is the initial steady state current and T is time constant (L/R)

or,
$$i = 3.33\,e^{-t/0.025} = \mathbf{3.33\,e^{-40\,t}\ A.}$$

EXAMPLE 8.8 *In Fig. E8.2 steady state condition is reached with 100 V d.c. source. At $t = 0$, switch K is suddenly opened. Find the expression of current through the inductor after $t = 1/2$ sec.*

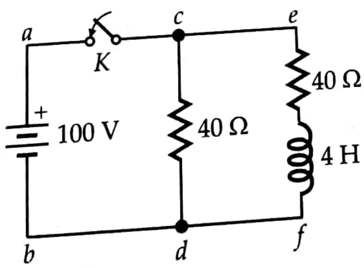


Fig. E8.2

SOLUTION. At steady state condition of close position of switch K,

$$i_{\text{st. st.}} = \frac{100}{40 \,||\, 40} = 5\ \text{A}$$

Thus $i_{\text{st. st.}}$, the current from 100 V d.c. source is 5 A when K is closed. As the circuit paths cd and ef, both have same resistance, hence, the current in each branch is 2.5 A.

As soon as K is opened, the source is removed and the LR circuit starts discharging. The decay current is given by

$$i = 2.5\,e^{-t/T} \quad \text{where } T = \frac{L}{R}$$

or,
$$i = 2.5\,e^{-t/0.05} = 2.5\,e^{-20\,t}\ \text{A.}$$

at $t = 1/2$ sec.,

$$= 2.5\,e^{-20 \times 1/2} = \mathbf{1.14 \times 10^{-4}\ A.}$$

EXAMPLE 8.9 *In Fig. E8.3, the switch is closed at position 1 at $t = 0$. At $t = 0.5$ m sec. the switch is moved to position 2. Find the expression for the current in both the conditions and sketch the transient.*

*

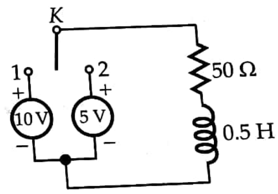


Fig. E8.3

SOLUTION. At position 1 of the switch,

$$50i + 0.5\frac{di}{dt} = 10$$

or,
$$500\,i + 5\frac{di}{dt} = 100$$

$$(p + 100)\,i = 20 \qquad \text{...(a)}$$

The complete solution of equation (a) consists of the complementary function (i_C) and the particular solution (i_p)

$$\therefore \quad i = i_C + i_p$$

But
$$i_C = c\,e^{-100\,t}$$

$$i_p = e^{-(R/L).\,t} \int e^{(R/L).\,t} \left(\frac{V}{L}\right) dt = \frac{V}{R}.$$

$$\therefore \quad i = ce^{-100\,t} + \frac{V}{R} = ce^{-100\,t} + 0.2$$

However, at $t = 0$, $i = 0$ (Due to presence of inductor)

Using this initial condition,

$$i = c + 0.2 \text{ or } 0 = c + 0.2$$

$$\therefore \quad c = -0.2$$

$$i = \frac{V}{R}(1 - e^{-t/T})$$

or,
$$i = \frac{200}{20}(1 - e^{-t/(L/R)}) = 10(1 - e^{-100t})$$

at $t = 0.02$ sec.,

$$i = 10(1 - e^{-100 \times 0.02}) = 8.646 \text{ A}$$

The voltage across the inductor after lapse of 0.02 sec from switching is

$$L\frac{di}{dt} = 200 - iR = 200 - 8.646 \times 20 = \mathbf{27\ V.}$$

The rate of change of current at $t = 0.01$ sec. is then

$$\frac{di}{dt} = \frac{27\text{ V}}{L} = \frac{27}{0.2} = \mathbf{135\ A/sec.}$$

EXAMPLE 8.12 *A d.c. source of V volts supplies a steady current of I A to a series L-R circuit having resistance of R ohms and inductance of L Henry. A part of total resistance is suddenly shorted. Derive an expression for the current flowing from the d.c. source subsequent to this operation.*

SOLUTION. Let R_x be the part of the resistance that is shorted at $t = 0$. If i be current after t sec. in the circuit, we can write by KVL

$$(R - R_X)i + L\frac{di}{dt} = V$$

or,
$$i = ce^{-\left(\frac{R - R_X}{L}\right)t} + \frac{V}{R - R_X}$$

$$\left[i \text{ being equal to } i_C + i_p,\ i_C = ce^{-\left(\frac{R-R_X}{L}\right)t},\quad i_p = \frac{V}{R - R_X} \right]$$

However, at $t = 0$, $i = \dfrac{V}{R}$

$$\therefore\quad c = \frac{V}{R} - \frac{V}{R - R_X}$$

$$= V\left[\frac{R - R_X - R}{R(R - R_X)}\right] = -\frac{V R_X}{R(R - R_X)}$$

Hence
$$i = \frac{V}{R - R_X} - \frac{V R_X}{R(R - R_X)} e^{-\left(\frac{R - R_X}{L}\right).t}$$

$$= \frac{V}{R - R_X}\left[1 - \frac{R_X}{R} e^{-\left(\frac{R - R_X}{L}\right)t}\right]$$

Thus the expression for the current at t sec. later then the switching of s.c to a part of R has been derived.

8.4 TRANSIENT RESPONSE IN SERIES *RC* CIRCUIT HAVING D.C. EXCITATION (FIRST ORDER CIRCUIT)

Let a d.c. voltage V be applied (at $t = 0$) by closing a switch S in a series R-C circuit (Fig. 8.5). The current at $t > 0$ being i, application of KVL leads to

$$Ri + \frac{1}{C}\int i\,dt = V \qquad ...(8.10)$$

Differentiation of equation (8.10) results,

$$R\frac{di}{dt} + \frac{i}{C} = 0 \quad \text{or} \quad \left(p + \frac{1}{RC}\right)i = 0 \qquad ...(8.11)$$

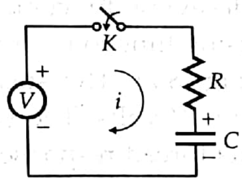


Fig. 8.5 Series *RC* circuit.

Equation (8.11) is a *homogeneous differential equation* whose solution will contain only complementary function, the particular function being zero. The solution will give natural response.

$$\therefore \quad i = i_C = Ke^{-t/RC} \qquad ...(8.12)$$

With application of voltage and assuming no initial charge across the capacitor, the capacitor will not produce any voltage across it but acts as a short circuit causing the circuit current to be (V/R).

i.e., at $t = 0^+$, $\quad i(0^+) = \dfrac{V}{R}$

Hence, from equation (8.12),

at $t = 0^+$ $\quad \dfrac{V}{R} = K$

Finally, we then obtain,

$$i = \frac{V}{R}e^{-t/RC} \text{ A} \qquad ...(8.13)$$

It may be observed that the charging current is a decaying function, the plot being shown in Fig. 8.6. As the capacitor is getting charged, the charging current dies out.

The corresponding voltage drops across the resistor and capacitor can be obtained as follows :

$$v_R = iR = Ve^{-t/RC} \qquad ...(8.14)$$

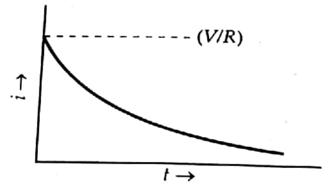


Fig. 8.6 Profile of current in RC charging circuit.

and $$v_C = \frac{1}{C}\int i\,dt = \frac{1}{C}\int_0^t \frac{V}{R}e^{-t/RC}\,dt$$

$$= V(1 - e^{-t/RC}). \quad ..(8.15)$$

Observing equations (8.14) and (8.15) it reveals that v_R is a decaying function while v_C is an exponentially rising function (profiles of v_R and v_C are shown in Fig. 8.7). The steady state voltage across capacitor is V volts. The voltage across the capacitor exhibits forced response.

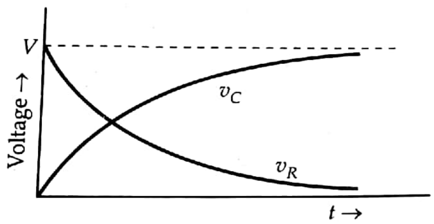


Fig. 8.7 Profiles of v_R and v_C in RC charging circuit.

Time constant is obtained by putting $t = RC$ which gives $v_C = V(1 - 0.368) = 0.632$ V *i.e.*, the time by which the capacitor attains 63.2% of steady state voltage.

The instantaneous powers are given by

$$p_R = i\,v_R = \frac{V^2}{R}e^{-2\frac{t}{RC}}\ \text{W}$$

and $$p_C = i\,v_C = \frac{V^2}{R}(e^{-t/RC} - e^{-2t/RC})\ \text{W}$$

Let us now study the discharging case when the switch S is thrown to a contact S' such that the R-C circuit is shorted and the voltage source is withdrawn (Fig. 8.8.)

Application of KVL yields

$$Ri + \frac{1}{C}\int i\,dt = 0 \quad ...(8.16)$$

Differentiating (8.16), we get

$$R\frac{di}{dt} + \frac{i}{C} = 0 \quad \text{or} \quad \left(p + \frac{1}{RC}\right)i = 0 \quad ...(8.17)$$

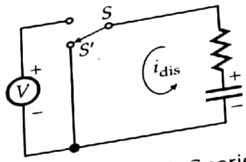


Fig. 8.8 Discharging in RC series circuit.

Equation (8.17) being a homogeneous differential equation, its solution reveals

$$i = i_C \text{ (complementary function)}$$

$$= K' e^{-t/RC} \quad ...(8.18)$$

However at $t = 0^+$, the voltage across the capacitor will start discharging current through the resistor in *opposite* to the original current direction (shown by i_{dis} in Fig. 8.8). Hence the *direction of i during discharge is negative* and its *magnitude is given by (V/R).*

Thus $$i(0^+) = -\frac{V}{R}$$

Hence, from equation (8.18), we get

$$-\frac{V}{R} = K' \text{ (at } t = 0^+)$$

The complete solution is then

$$i = -\frac{V}{R}e^{-t/RC}\ \text{A.}$$

The decay transient is plotted in Fig. 8.9 (natural response of the circuit).

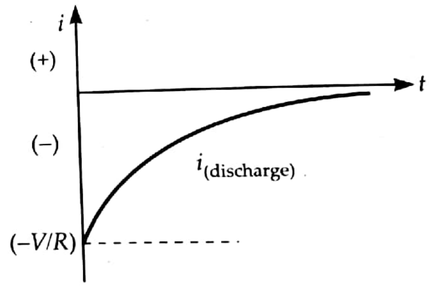


Fig. 8.9 Current decay transient in RC discharging circuit.

The corresponding transient voltages are given by

v_R (voltage drop across R)

$$= iR = -V e^{-t/RC}$$

and v_C (voltage drop across C)

$$= \frac{1}{C}\int i\,dt = Ve^{-t/RC}$$

Obviously, $v_R + v_C = 0$

Figure 8.10 represents the profiles of v_R and v_C with t.

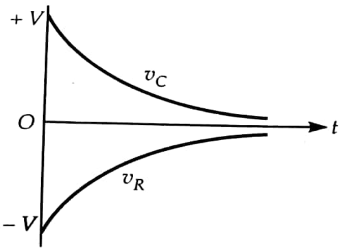


Fig. 8.10 v_R and v_C in RC discharging circuit.

In the discharging circuit, the time constant is given by the product of R and C such that

$$v_C = V e^{-1} = 0.369 \text{ V} \simeq 0.37 \text{ V}$$

i.e., the time by which the capacitor discharges to 37% of its initial voltage.

The instantaneous powers are given by

$$p_R = v_R\, i = \frac{V^2}{R} e^{-2t/RC} \text{ W.}$$

and $$p_C = v_C\, i = -\frac{V^2}{R} e^{-2t/RC} \text{ W.}$$

The charge stored in the capacitor during charging is given by

$$q = C v_C = CV(1 - e^{-t/RC}) \text{ or } q = Q(1 - e^{-t/RC})$$

while that during discharging is given by

$$q = C v_C = C V e^{-t/RC} \text{ columbs.}$$

or, $$q = Q e^{-t/RC} \text{ columbs.}$$

EXAMPLE 8.13 *Calculate the time taken by a capacitor of 1 μF and in series with a 1 MΩ resistance to be charged upto 80% of the final value.*

SOLUTION. The time constant T is given by

$$T = RC = 1 \times 10^6 \times 1 \times 10^{-6} = 1 \text{ sec.}$$

The charging of capacitor is expressed by the following equation

$$q = Q_0 (1 - e^{-t/RC)}$$

Here, $q = 0.8\, Q_0$; $R = 1$ sec.

$\therefore$ $0.8 = 1 - e^{-t}$

$0.2 = e^{-t}$

$\therefore$ $t = \mathbf{1.61\ sec.}$

EXAMPLE 8.14 *A d.c. constant voltage source feeds a resistance of 2000 kΩ in series with a 5 μF capacitor. Find the time taken for the capacitor when the charge retained will be decayed to 50% of the initial value, the voltage source being short circuited.*

SOLUTION. Time constant

$$T = RC = 2 \times 10^6 \times 5 \times 10^{-6} = 10 \text{ sec.}$$

The decaying condition is represented by the following expression

$$q = Q_0\, e^{-t/T}$$

However, $q = 0.5\, Q_0$

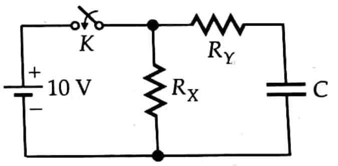
$\therefore$ $0.5\, Q_0 = Q_0\, e^{-t/T}$

or, $0.5 = e^{-t/T} = e^{-t/10}$

or, $-t/10 = \log_e (0.5)$

$\therefore$ $t = \mathbf{6.94\ sec.}$

EXAMPLE 8.15 *In Fig. E8.6, the switch K is closed. Find the time when the current from the battery reaches to 500 mA.*



$[R_X = 50\ \Omega\,;\ R_Y = 70\ \Omega,\ C = 100\ \mu F]$

Fig. E8.6

SOLUTION. Let current through R_X be I_X and through C by I_Y after the switch K is closed.

$$I_X = \frac{10}{50} = 0.2 \text{ A} = 200 \text{ mA.}$$

However,

$I = I_X + I_Y$ [I being the current from the supply]

or, $500 = 200 + I_Y$

[$\because$ Supply current is 500 mA]

$\therefore$ $I_Y = 300$ mA

But $I_Y = \dfrac{V}{R_Y} e^{-t/T}$

$[T = RC = 70 \times 100 \times 10^{-6} = 0.007$ sec

or, $0.3 = \dfrac{10}{70} e^{-\frac{t}{0.007}}$

or, $-\dfrac{t}{0.007} = \log_e (2.1)$

$\therefore$ $t = \mathbf{5.2\ m\,sec.}$

This is the time required when the d.c. sou[rce] current flow will be 500 mA.

Example 8.16 *A 10 μF capacitor is initially charged to 100 volts d.c. It is then discharged through a resistance of R ohms for 20 seconds when the p.d. across the capacitor is 50 V. Calculate the value of R.*

Solution. In the discharging condition of the capacitor,

$$q = Q_0\, e^{-t/RC} \quad \text{or} \quad v = V_0\, e^{-t/RC}$$

As per the question, capacitor p.d. gets discharged to 50 V from the initial p.d. of 100 V.

$$\therefore \qquad v = 0.5\, V_0$$

Hence we obtain,

$$0.5 = e^{-t/R \times 10 \times 10^{-6}}$$

$$\text{or,} \quad \log_e 0.5 = -\frac{t}{R \times 10^{-5}} = -\frac{20}{R \times 10^{-5}}$$

$$\text{or,} \quad -0.7 = -\frac{20}{R \times 10^{-5}}$$

$$\therefore \qquad R = 28.57 \times 10^5 = \mathbf{2.86\ M\Omega}.$$

Example 8.17 *A resistance R and 5 μF capacitor are connected in series across a 100 V d.c. supply. Calculate the value of R such that the voltage across the capacitor becomes 50 V in 5 sec after the circuit is switched on.*

Solution. In case of charging

$$q = Q_0\,(1 - e^{-t/T})$$

$$[T = RC = (5 \times 10^{-6}\ R)\ \text{sec.}]$$

$$\text{or,} \quad v = V_0\,(1 - e^{-t/T})$$

As per the question, the p.d. across the capacitor is 50 V within 5 sec.

$$\therefore \qquad v = 0.5\, V_0 \qquad [V_0 = \text{final p.d. in steady state} = 100\ \text{V}]$$

$$\text{and} \qquad t = 5\ \text{sec.}$$

$$\therefore \qquad 0.5 = 1 - e^{-5/5 \times 10^{-6} R}$$

$$\text{or,} \quad -0.5\, v = -e^{-10^6/R}$$

$$\therefore \qquad R = \mathbf{1.45\ Mohm}.$$

Example 8.18 *A 5 μF capacitor is initially charged with 500 μC. At t = 0, the switch K is closed (Fig. E8.7). Determine the voltage drop across the resistor at t < T and at t = ∞.*

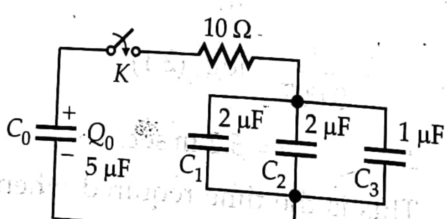


Fig. E8.7

Solution. The equivalent capacitance of bank of parallel capacitors is 5 μF. As soon as K is closed, this equivalent 5 μF capacitor is in series with C_0 and the net capacitance becomes 2.5 μF.

$$\therefore\ T\ (\text{time const.})$$

$$= RC_{net} = 10 \times 2.5 \times 10^{-6} = 25\ \mu\ \text{sec.}$$

The initial voltage V_0 across capacitor C_0 is given by

$$V_0 = \frac{Q_0}{C_0} = \frac{500\ \mu C}{5 \times 10^{-6}} = 100\ \text{V}$$

With closing of K, the capacitor C_0 will start discharging, however at $t = 0^+$, there will be no voltage across C_1, C_2 or C_3.

Thus the entire voltage drop will be across R only (v_R) at $t = 0^+$ time.

i.e.,

$$v_R = V_0\ \text{(decaying)}$$

$$= V_0\, e^{-t/RC} = 100\, e^{-t/25 \times 10^{-6}}$$

$$= \mathbf{100\, e^{-40 \times 10^4 t}\ V}$$

At $t = \infty$, v_R **becomes zero.**

It is also evident that in steady state ($t = \infty$), the charge of C_0 will be distributed through C_1, C_2 and C_3 and no current will flow through the circuit. Hence $i = 0$, $v_R = iR = 0$.

Example 8.19 *In Fig. E8.8, a capacitor of capacitance C is charged to a voltage V_0 (d.c.) and is allowed to discharge through a resistance R while charging another capacitor of capacitance αC. Determine the final voltage at terminal "a-b" under steady state condition.*

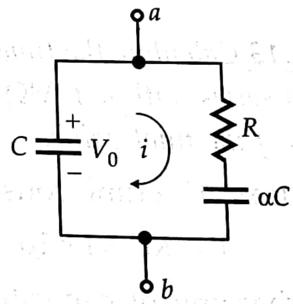


Fig. E8.8

Solution. Let V_f be the final voltage appearing across *a-b* after discharging of C charging αC through R.

Equating the charge of the two capacitors

$$V_f\,(\alpha C) = V_0 C - V_f C$$

$$\text{or,} \quad V_f\,(1 + \alpha) = V_0$$

$$\therefore \qquad \mathbf{V_f = V_0 / (1 + \alpha)}.$$

[It may be noted that the final voltage across *a-b* is independent of *R*]

Again, the final charge Q being (CV) the magnitude of this charge becomes, at steady state, $(1 \times 10^{-6} \times 5)$ *i.e.*, $5\ \mu$ C.

Also $\quad q = K' e^{-t/RC} + 5$

at $t = 0^+$, $\quad q \equiv q(0^+) \equiv q(0^-) = -10\ \mu$ C

$\therefore \quad -10 = K' \times 1 + 5$

$\therefore \quad K' = -15\ \mu$ C

i.e., $\quad q = (-15e^{-t/T} + 5)\ \mu$ C

and
$$i = \frac{dq}{dt} = 15 \times \frac{1}{RC} e^{-t/T}$$
$$= 15 \times 10^{-4}\ e^{-10^4 t}$$
$$= 1.5\ e^{-10^4 t}\ \text{mA}$$

Profile of Q is drawn in **Fig. E8.15.**

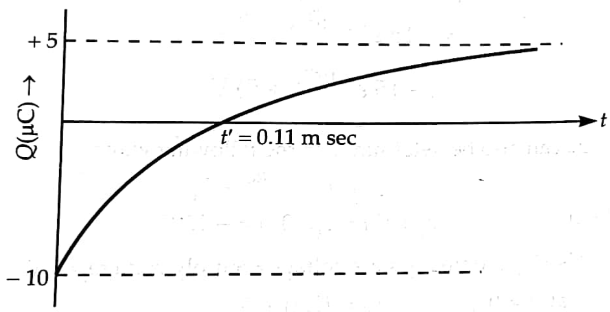


Fig. E8.15

8.5 TRANSIENT RESPONSE IN SERIES *RL* CIRCUIT WITH SINUSOIDAL EXCITATION (FIRST ORDER CIRCUIT)

Let $v = V_m \sin(\omega t + \phi)$ where ϕ varies from $0 - 2\pi$ depending on the switching instant. Application of KVL at $t = 0^+$ in Fig. 8.11 after the switch is closed, gives

$$Ri + L\frac{di}{dt} = V_m \sin(\omega t + \phi)$$

or,
$$\frac{R}{L}i + \frac{di}{dt} = \frac{V_m}{L}\sin(\omega t + \phi)$$

or,
$$\left(p + \frac{R}{L}\right)i = \frac{V_m}{L}\sin(\omega t + \phi) \qquad \ldots(8.19)$$

where "p" represents first derivative of i.

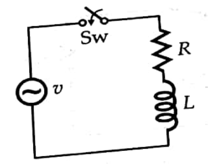


Fig. 8.11 Series RL circuit.

The complementary function of equation 8.19 is
$$i_C = C\,e^{-(R/L)t} \qquad \ldots(8.20)$$

Next we are to obtain the particular solution of current (i_p) such that the net current solution i is given by
$$i = i_C + i_p \text{ (forced response)}$$

Let
$$i_p = A\cos(\omega t + \phi) + B\sin(\omega t + \phi) \qquad \ldots(8.21)$$

where A and B are unknown constants.

$\therefore$ The first derivative of i_p becomes
$$pi_p = -A\omega\sin(\omega t + \phi) + B\omega\cos(\omega t + \phi)$$

Substituting these expressions for i_p and pi_p in equation 8.19,
$$[-A\omega\sin(\omega t + \phi) + B\omega\cos(\omega t + \phi)] + \frac{R}{L}[A\cos(\omega t + \phi) + B\sin(\omega t + \phi)] = \frac{V_m}{L}\sin(\omega t + \phi) \qquad \ldots(8.22)$$

or,
$$\left(-A\omega + \frac{BR}{L}\right)\sin(\omega t + \phi) + \left(B\omega + \frac{AR}{L}\right)\cos(\omega t + \phi) = \frac{V_m}{L}\sin(\omega t + \phi)$$

Equating coefficients of like terms,
$$\left[-A\omega + \frac{BR}{L}\right] = \frac{V_m}{L}; \qquad A = \frac{-\omega L V_m}{R^2 + \omega^2 L^2}$$
$$\left[B\omega + \frac{AR}{L}\right] = 0; \qquad B = \frac{RV_m}{R^2 + \omega^2 L^2}$$

Substituting the values of A and B in the expression for i_p (equation 8.21)
$$i_p = \frac{-\omega L V_m}{R^2 + \omega^2 L^2}\cos(\omega t + \phi) + \frac{RV_m}{R^2 + \omega^2 L^2}\sin(\omega t + \phi)$$
$$= \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}[-\omega L\cos(\omega t + \phi) + R\sin(\omega t + \phi)]$$

Let $R = \cos\theta$ and $\omega L = \sin\theta$; θ being $\tan^{-1}\frac{\omega L}{R}$.

Thus,

$$i = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}[\sin(\omega t+\phi)\cos\theta - \cos(\omega t+\phi)\sin\theta]$$

or, $$i_p = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t+\phi-\theta)]$$

$$= \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\left[\sin\left(\omega t+\phi-\tan^{-1}\frac{\omega L}{R}\right)\right]$$

This gives the net current solution as

$$i = i_C + i_p$$

or, $$i = Ce^{-\left(\frac{R}{L}\right)\cdot t} + \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t+\phi-\tan^{-1}\frac{\omega L}{R}\right) \quad ...(8.23)$$

However, the inductance will oppose any sudden change of current though it.

As there was no current in the circuit before the switch was closed, hence $i_0 = 0$.

Then at $t = 0$,

$$i_0 = 0 = C + \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin\left(\phi-\tan^{-1}\frac{\omega L}{R}\right).$$

or, $$C = -\frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin\left(\phi-\tan^{-1}\frac{\omega L}{R}\right)$$

Substituting of value of C, in equation (8.23),

$$i = e^{-(R/L)t}\left[-\frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin\left(\phi-\tan^{-1}\frac{\omega L}{R}\right)\right] + \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t+\phi-\tan^{-1}\frac{\omega L}{R}\right) \quad ...(8.24)$$

The final part of equation (8.24) contains the exponential factor $e^{-(R/L)t}$ which becomes zero at relatively short time as with increasing t, $e^{-(R/L)t}$ decreases. The expression within the bracket depends on the value of ϕ *i.e.*, the instant when the switch is closed.

The second part of equation (8.24) is the steady state current which lags the applied voltage by an angle given by $\tan^{-1}\frac{\omega L}{R}$.

EXAMPLE 8.25 *A 50 Hz 400 V (peak value) sinusoidal voltage is applied at $t = 0$ to a series R-L circuit having resistance 5 Ω and inductance 0.2 H. Obtain an expression of current at any instant "t". Calculate the value of the transient current 0.01 sec after switching on.*

SOLUTION. Z (impedance of L-R circuit)

$$= 5 + j \times 2\pi \times 50 \times 0.2 = 63\angle 85.44°\,\Omega$$

Applying KVL in the R-L circuit,

$$Ri + L\frac{di}{dt} = 400\sin 314\,t$$

or, $$i\frac{R}{L} + \frac{di}{dt} = \frac{400}{L}\sin 314\,t$$

or, $$\left(p+\frac{R}{L}\right)i = \frac{400}{0.2}\sin 314\,t$$

$$= 2000\sin 314\,t$$

The complementary function i_C is given by

$$i_C = Ce^{-(R/L)t} = Ce^{-25t}$$

and the particular solution is

$$i_p = \frac{400}{Z}\sin\left(\omega t+\phi-\tan^{-1}\frac{\omega L}{R}\right)$$

$$= 6.35\sin(314\,t - 85.44°)$$

$$= 6.35\sin(314\,t - 1.49)$$

$$[\because 85.44° \equiv 1.49 \text{ rad}]$$

$\therefore$ The complete solution is,

$$i = i_C + i_p$$

i.e., $$i = Ce^{-25t} + \mathbf{6.35\sin(314\,t - 1.49)}$$

at $t = 0$, $i = 0 \quad \therefore \quad 0 = C + 6.35\sin(-1.49)$

$\therefore$ $C = 6.33$

The current becomes

$$i = 6.33e^{-25t} + 6.35\sin(314\,t - 1.49)$$

at $t = 0.01$ sec.

$$i_C = 6.33\,e^{-25\times 0.01} = \mathbf{4.93\ A.}$$

[Since i_C is the transient part that is only wanted in the problem, hence we need not calculate i_p which is a steady state part.]

EXAMPLE 8.26 *An a.c. voltage of $v = V\sin 100\pi t$ is applied to a series R-L circuit. If the L-R circuit has $R = 10\,\Omega$ and L=0.1 H, calculate the ratio of maximum value to which the current rise to the steady state maximum value when the voltage is applied at an instant $t = 0.002$ sec.*

CHAPTER 8

Solution. The complementary function for the current in this *R-L* circuit where a.c. voltage is applied at t time is given by

$$i_C = Ce^{-(R/L)t}$$

and particular solution is given by

$$i_p = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$$

where ϕ is the phase angle voltage at time $t=0$.

In this problem $\phi = 0$.

$$\therefore \quad i = i_C + i_p$$

$$= Ce^{-(R/L)t} + \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$$

or,
$$i = Ce^{-100t} + \frac{V}{32.95}\sin(500\pi t - 1.26)$$

$$\left[\because R/L = 100;\ |Z| = \sqrt{R^2+\omega^2 L^2} = 32.95 \text{ and } \phi = 0;\ \tan^{-1}\frac{\omega L}{R} = 1.26\right]$$

at $t = 0.002$ sec.

$$i = Ce^{-100\times 0.002} + \frac{V}{32.95}\sin(500\times\pi\times 0002 - 1.26)$$

$$= 0.82\,C + \frac{V}{32.95}\sin(\pi - 1.26)$$

But at $t = 0.002$ sec., $i = 0$

$$\therefore \quad 0 = 0.82\,C + \frac{V}{32.95}\sin(\pi - 1.26)$$

$$\therefore \quad 0.82\,C = -\frac{V}{32.95}\sin(\pi - 1.26)$$

$$\therefore \quad C = -\frac{V}{32.95\times 0.82}\sin(\pi - 1.26)$$

$$= -0.037\times 0.952\text{ V} = (-0.0352\text{ V}).$$

Thus

$$\mathbf{i = V\left[-0.0352\,e^{-100t} + 0.03\sin(500\pi t - 1.26)\right] A}$$

It may be observed that i will be maximum if the 2nd term in the bracketed portion is maximum with –ve sign. This is possible if

$$\sin(500\pi t - 1.26) = -1$$

i.e.,
$$500\pi t - 1.26 = \frac{3\pi}{2}$$

$$\therefore \quad t = \frac{1.5\pi + 1.26}{500\pi} = 0.0038 \text{ sec.}$$

With $t = 0.0038$ sec.

$$i_c = -0.0352\,e^{-100t}\text{ V}$$

$$= -0.0352\,e^{-100\times 0.0038}\text{ V}$$

$$= (-0.024\text{ V})\text{ A.}$$

and
$$i = V.0.03\sin(500\pi\times 0.0038 - 1.26)$$

$$= (-0.02999\text{ V})$$

$\therefore$ Max. value of current is

$$i = [(-0.024) + (-0.0299)]\text{ V} = (-0.054\text{ V})$$

Thus, the ratio of maximum value to which the current rises to the steady state maximum value is given by

$$\text{Ratio} = \frac{(0.054\text{ V})}{(0.0299\text{ V})} = \mathbf{1.8.}$$

Example 8.27 *Obtain the current at $t > 0$, if a.c. voltage v is applied when the switch K is moved to 2 from 1 at $t = 0$ (Fig. E8.16). Assume a steady state current of 1 A in the LR circuit when the switch was at position 1.*

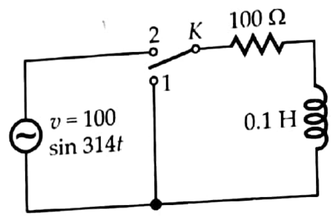


Fig. E8.16

Solution. At position 1 of the switch K, the steady state current in the circuit is 1 A, *i.e.*, $i(0^-) = 1$ A. As soon as the switch is moved to position 2, at $t = 0$, the a.c. voltage v appears across the *LR* series circuit. Due to presence of inductance,

$$i(0^-) = i(0^+) = 1\text{ A.}$$

$$Z = R + jX_L = 100 + j\,2\pi\times 50\times 0.1$$

$$= 104.8\angle 17.47°\,\Omega$$

Applying KVL in the *RL* circuit,

$$R.i + L\frac{di}{dt} = v$$

or,
$$100\,i + 0.1\frac{di}{dt} = 100\sin 314\,t$$

or,
$$10^3\,i + \frac{di}{dt} = 10^3\sin 314\,t$$

i.e.,
$$(p + 10^3)\,i = 10^3\sin 314\,t$$

$$\therefore \quad i_C = Ce^{-R/L\,.\,t}$$

$$= Ce^{-\frac{100}{0.1}t} = Ce^{-1000\,t}$$

$$i_p = \frac{V_m}{Z}\sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$$

$$= \frac{100\sqrt{2}}{104.8}\sin(314\,t - 17.47°)$$

$$[\because \phi = 0°]$$

$$= 1.345\sin(314\,t - 0.304)$$

Then the complete current is given by

$$i = i_C + i_p$$

$$= Ce^{-1000\,t} + 1.345\sin(314\,t - 0.304)$$

However, at $t = 0^+$,

$$i(0^+) = 1\text{ A}$$

$$\therefore \quad 1 = C + 1.345\sin(-0.304) = C + (-0.3)$$

or, $$C = 1.3$$

Hence $\boldsymbol{i = 1.3\,e^{-1000\,t} + 1.345\sin(314\,t - 0.304)}$.

8.6 TRANSIENT RESPONSE IN *RLC* CIRCUIT WITH D.C. EXCITATION (SECOND ORDER CIRCUIT)

Application of KVL in the series RLC circuit $t = 0^+$ after the switch is closed, leads to the following differential equation

$$iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V_0 \text{ (Fig. 8.12)}$$

By differentiation,

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$

or, $$\left(p^2 + \frac{R}{L}p + \frac{1}{LC}\right)i = 0 \qquad \text{...(8.25)}$$

Equation (8.25) is a second order, linear, homogenous differential equation. The characteristic equation then becomes

$$p^2 + \frac{R}{L}p + \frac{1}{LC} = 0$$

where the coefficients are constant. The roots of the characteristic equation then become

$$p_1, p_2 = \frac{-R/L \pm \sqrt{(R/L)^2 - 4/LC}}{2}.$$

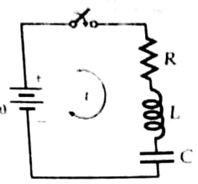


Fig. 8.12 Series RLC circuit.

Let $$\alpha = (-R/2L)$$

and $$\beta = \sqrt{(R/2L)^2 - \frac{1}{LC}}$$

Hence, $$p_1 = \alpha + \beta$$
$$p_2 = \alpha - \beta$$

Also, the solution of differential equation (8.25) becomes

$$i = C_1\,e^{p_1 t} + c_2\,C^{p_2 t} \qquad \text{...(8.26)}$$

C_1 and C_2 being the constants.

Case 1. When $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$

This time β is positive real quantity. Hence, the roots p_1 and p_2 are real but unequal.

$$\therefore \quad p_1 = \alpha + \beta;\ p_2 = \alpha - \beta$$

and $$i = C_1\,e^{(\alpha+\beta)t} + C_2\,e^{(\alpha-\beta)t}$$
$$= e^{\alpha t}(C_1\,e^{\beta t} + C_2\,e^{-\beta t}) \qquad \text{...(8.27)}$$

The current response is *over-damped* (Fig. 8.13).

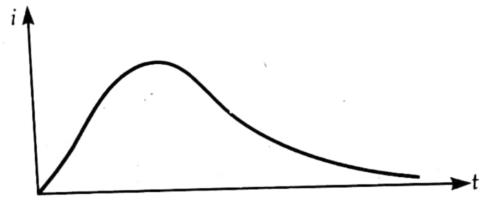


Fig. 8.13 Current over-damping.

Case 2. When $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$

This time β is imaginary and then the roots p_1 and p_2 are complex conjugates.

$$p_1 = \alpha + j\beta; \qquad p_2 = \alpha - j\beta$$

and $$i = C_1\,e^{(\alpha+j\beta)t} + C_2\,e^{(\alpha-j\beta)t}$$
$$= e^{\alpha t}(C_1\,e^{j\beta t} + C_2\,e^{-j\beta t})$$
$$= e^{\alpha t}[C_1\cos(\beta t) + C_2\sin(\beta t)] \qquad \text{...(8.28}$$

Thus the current solution is *underdamped* (or oscillatory) [Fig. 8.14].

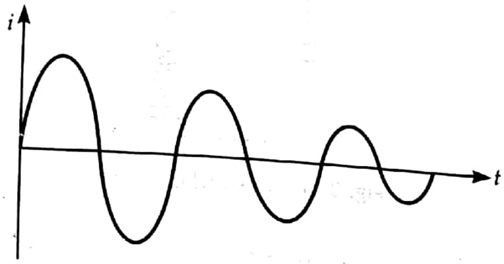


Fig. 8.14 Current oscillation.

Case 3. When $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$

This time β is zero.

Hence, roots p_1 and p_2 are real and equal.

$$p_1 = p_2 = \alpha \quad \text{and} \quad i = C_1 e^{\alpha t} + C_2 te^{\alpha t}$$

$$= e^{\alpha t}(C_1 + C_2 t) \qquad ...(8.29)$$

The current response is then a *critically* damped one (Fig. 8.15).

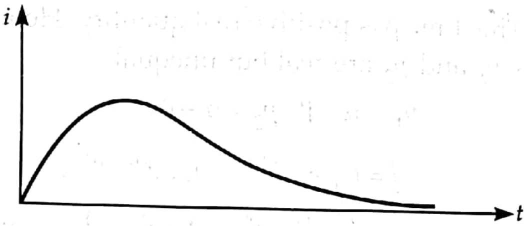


Fig. 8.15 Critical damping of current.

Example 8.28 *In a series R-L-C circuit, $R = 5\,\Omega$, $L = 1$ H and $C = 1$ F. A d.c. voltage of 20 V is applied at $t = 0$. Obtain $i(t)$.*

Solution. Application of KVL yields, in the series *RLC* circuits

$$iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V \qquad ...(1)$$

On differentiation,

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$

or,
$$\left(Lp^2 + Rp + \frac{1}{C}\right)i = 0 \qquad ...(2)$$

Thus the characteristic equation is

$$Lp^2 + Rp + \frac{1}{C} = 0,$$

or,
$$p^2 + 5p + 1 = 0 \qquad ...(3)$$

$$\therefore \quad p_1, p_2 = \frac{-5 \pm \sqrt{25-4}}{2} = \frac{-5 \pm 4.58}{2}$$

$$= -0.21, -4.79.$$

$\therefore$ The solution becomes

$$i(t) = K_1 e^{-0.21t} + K_2 e^{-4.79t} \qquad ...(4)$$

As current in inductance cannot change instantaneously and also the voltage across the capacitor does not change instantaneously,

hence, $i(0^+) = 0$ and $\frac{1}{C}\int i\,dt = 0$

Thus from (1),

$$L\frac{di}{dt} = V \text{ (at } t = 0^+).$$

or,
$$\frac{di}{dt}(0^+) = \frac{V}{L} = \frac{20}{1} = 20 \text{ A/sec.}$$

Again, substituting $i = 0$ at $t = 0^+$ in (4),

$$0 = K_1 e^{-0.21\times 0} + K_2 e^{-4.79\times 0}$$

$$= K_1 + K_2 \qquad ...(5)$$

Again, differentiating equation (4),

$$\frac{di}{dt} = -0.21\,K_1\, e^{-0.21t} - 4.79\, e^{-4.79t}$$

or, at $t = 0^+$,

$$20 = -0.21\,K_1\, e^{-0.21\times 0} - 4.79\,K_2\, e^{-4.7\times 0}$$

$$= -0.21\,K_1 - 4.79\,K_2$$

or,
$$20 = 0.21\,K_2 - 4.79\,K_2 = -4.58\,K_2$$

$$[\because K_1 + K_2 = 0]$$

$\therefore \quad K_2 = -4.37$; Hence $K_1 = 4.37$

$\therefore \quad \boldsymbol{i(t) = (4.37\,e^{-0.21t} - 4.37\,e^{-4.79t})}$ **A**

8.7 TRANSIENT RESPONSE IN SERIES *RC* CIRCUIT WITH SINUSOIDAL EXCITATION (FIRST ORDER CIRCUIT)

$$v = V_m \sin(\omega t + \phi)$$

Application of KVL in the series *R-C* circuit of Fig. 8.16 yields

$$R.i + \frac{1}{C}\int i\,dt = V_m \sin(\omega t + \phi) \qquad ...(8.30)$$

Differentiation of equation (8.29) gives,

$$\left(p + \frac{1}{RC}\right)i = \frac{\omega V_m}{R}\cos(\omega t + \phi)$$

...(8.31)

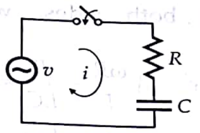


Fig. 8.16 AC voltage application in RC series.

The complementary function being i_C, the current solutions (forced response) of the differential equation (8.31) is given by

$$i = i_C + i_p$$

where $\quad i_C = C' e^{-t/RC}$, C' being a constant

and $\quad i_p = \dfrac{V_m}{\sqrt{(R^2 + (1/\omega C)^2)}} \sin\left(\omega t + \phi + \tan^{-1}\dfrac{1}{\omega CR}\right)$

[The particular function i_p has been obtained in a similar way as has been done for the case with $R-L$ circuit under sinusoidal response].

$$\therefore \quad i = C' e^{-t/2C} + \frac{V_m}{\sqrt{R^2 + \left(\dfrac{1}{\omega C}\right)^2}} \sin\left(\omega t + \phi + \tan^{-1}\frac{1}{\omega CR}\right) \qquad ...(8.32)$$

In order to determine the constant C', the circuit condition at $t = 0$ is considered.

At $t = 0$, the capacitor acts as short circuit.

$$\therefore i_0 \text{ (initial current)} = \frac{V_m}{R} \sin\phi$$

Hence setting $t = 0$, the complete solution becomes

$$i_0\left(= \frac{V_m}{R}\sin\phi\right) = C' + \frac{V_m}{\sqrt{R^2 + (1/\omega C)^2}} \sin\left(\phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

or, $$C' = \frac{V_m}{R}\sin\phi - \frac{V_m}{\sqrt{R^2 + \left(\dfrac{1}{\omega C}\right)^2}} \sin\left(\phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

Substitution of C' in equation (8.32),

$$i = e^{-t/RC}\left[\frac{V_m}{R}\sin\phi - \frac{V_m}{\sqrt{R^2 + \left(\dfrac{1}{\omega C}\right)^2}} \sin\left(\phi + \tan^{-1}\frac{1}{\omega CR}\right)\right]$$

$$+ \frac{V_m}{\sqrt{R^2 + (1/\omega C)^2}} \sin\left(\omega t + \phi + \tan^{-1}\frac{1}{\omega CR}\right) \qquad ...(8.33)$$

The first term being the transient with *decay factor* $e^{-t/RC}$, the second term of equation (8.33) is the *steady state* current which *leads* the applied voltage by $\angle \tan^{-1}(1/\omega CR)$.

EXAMPLE 8.29 *A series R-C circuit has $R = 20\,\Omega$ and $C = 100\,\mu F$. A voltage $v = 200\sin 314t$ is applied at $t = 2.14$ m sec. Obtain an expression for i. Also, find the value of current after time 1 m sec from the switching instant.*

SOLUTION. It may be noted that the voltage is applied **not** at $t = 0$ but at ϕ where

$$t \equiv 2.14 \text{ m sec.} = 314 \times 0.00214 = 0.672 \text{ rad.}$$

$\therefore$ As soon as the switch is closed, the application of KVL leads to

$$iR + \frac{1}{C}\int i\,dt = 200\sin(314t + 0.672)$$

This leads to the solution

$$i = i_C + i_p$$

where i_C = complementary function

$$= Ke^{-t/RC} = Ke^{-500t}$$

$$[RC = 100 \times 10^{-6} \times 20 = 2 \times 10^{-3} \text{ sec.}]$$

and $\quad i_p$ = particular solution

$$= \frac{V_m}{Z}\sin\left(314t + 0.672 + \tan^{-1}\left(\frac{1}{R\omega C}\right)\right)$$

$$= \frac{200}{37.6}\sin(314t + 0.672 + 1.01)$$

$$\left[\text{Here } Z = R - j\frac{1}{\omega C} = 20 - j\frac{10^6}{314 \times 100} = 20 - j\,31.8 = 37.6 \; -1.01\,\Omega\right]$$

Hence the complete solution i is given by

$$i = i_C + i_p$$

$$= Ke^{-500t} + 5.32\sin(314t + 0.672 + 1.01) \text{ A}$$

However, at $t = 0^+$, i will be given by the ratio of voltage at $t = 2.14$ m sec to the resistance only as $t = 0^+$ indicates the moment just after the switching, but the switching is performed at $t = 2.14$ m sec and not $t = 0$. Hence $t = 0^+$ here means $t = (2.14 \text{ m sec.})^+$.

$$\therefore \quad i(0^+) = \frac{200\sin 314 \times 0.00214}{20} = 6.22 \text{ A}$$

Hence we get

$$6.22 = K(1) + 5.32\sin(1.682) = K + 5.29$$

$$\therefore \quad K = 0.93$$

$$\therefore \quad i = 0.93\, e^{-500\,t} + 5.32 \sin(314\,t + 0.672 + 1.01)$$

$$= \mathbf{0.93\, e^{-500\,t} + 5.32 \sin(314\,t + 1.682)\ A}$$

After 1 m sec. the current becomes

$$i = 0.93\, e^{-500 \times 10^{-3}} + 5.32 \sin(314 \times 10^{-3} + 1.682)$$

$$= 0.564 + 4.85 = \mathbf{5.414\ A.}$$

8.8 TRANSIENT RESPONSE IN SERIES *RLC* CIRCUIT WITH SINUSOIDAL EXCITATION (SECOND ORDER CIRCUIT)

$$v = V_m \sin(\omega t + \phi)$$

As soon as the switch is closed, the mesh equation becomes

$$R\,.\,i + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V_m \sin(\omega t + \phi) \qquad ...(8.34)$$

Differentiation leads

$$\left(p^2 + \frac{R}{L}p + \frac{1}{LC}\right) i = \frac{\omega V_m}{L}\cos(\omega t + \phi) \qquad ...(8.35)$$

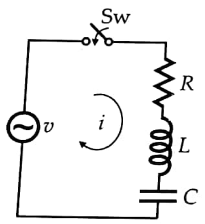


Fig. 8.17 *AC* application in *RLC* series circuit.

The particular solution can be obtained as follows :

Let

$$i_p = A\cos(\omega t + \phi) + B\sin(\omega t + \phi)$$

$$i'_p = -A\,\omega\sin(\omega t + \phi) + B\,\omega\cos(\omega t + \phi)$$

$$i''_p = -A\omega^2\cos(\omega t + \phi) - B\,\omega^2\sin(\omega t + \phi)$$

Substitution of values of i'_p and i''_p in (8.35)

$$[-A\,\omega^2\cos(\omega t + \phi) - B\,\omega^2\sin(\omega t + \phi)]$$

$$+ \frac{R}{L}[-A\,\omega\sin(\omega t + \phi) + B\,\omega\cos(\omega t + \phi)]$$

$$+ \frac{1}{LC}[A\cos(\omega t + \phi) + B\sin(\omega t + \phi)]$$

$$= \frac{\omega V_m}{L}\cos(\omega t + \phi)$$

Comparing both sides, we get for sine coefficients,

$$-B\omega^2 - A\frac{\omega R}{L} + \frac{B}{LC} = 0$$

or
$$A\left(\frac{\omega R}{L}\right) + B\left(\omega^2 - \frac{1}{LC}\right) = 0 \qquad ...(8.36)$$

and for cosine coefficients,

$$-A\omega^2 + B\frac{\omega R}{L} + A\frac{1}{LC} = \frac{\omega V_m}{L}.$$

or,
$$A\left(-\omega^2 + \frac{1}{LC}\right) + B\left(\frac{\omega R}{L}\right) = \frac{\omega V_m}{L} \qquad ...(8.37)$$

From equation (8.36),

$$A = -B\frac{\left(\omega^2 - \frac{1}{LC}\right)}{\frac{\omega R}{L}} = B\frac{\left(\frac{1}{LC} - \omega^2\right)}{\frac{\omega R}{L}}$$

Substitution of A in equation (8.37) yields.

$$B\left[\left(-\omega^2 + \frac{1}{LC}\right)^2 + \frac{\omega^2 R^2}{L^2}\right] = \frac{\omega^2 R V_m}{L^2}$$

$$\therefore \quad B = \frac{V_m \frac{\omega^2 R}{L^2}}{\left[\left(-\omega^2 + \frac{1}{LC}\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}$$

$$= \frac{V_m \frac{\omega^2 R}{L^2}}{\left[\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}$$

Thus

$$A = +B\frac{\left(-\omega^2 + \frac{1}{LC}\right)}{\frac{\omega R}{L}} = \frac{V_m\,\omega\left(\frac{1}{LC} - \omega^2\right)}{L\left[\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}$$

Hence with the values of A and B

$$i_p = + \frac{V_m\,\omega\left(-\omega^2 + \frac{1}{LC}\right)}{L\left[\left(\omega^2 + \frac{1}{LC}\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}\cos(\omega t + \phi)$$

$$+ \frac{V_m \frac{\omega^2 R}{L^2}}{\left[\left(-\omega^2 + \frac{1}{LC}\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}\sin(\omega t + \phi)$$

Let $$M\sin\theta = \frac{V_m\,\omega\left(-\omega^2+\frac{1}{LC}\right)}{L\left[\left(-\omega^2+\frac{1}{LC}\right)^2+\frac{\omega^2R^2}{L^2}\right]}$$

and $$M\cos\theta = \frac{V_m\frac{\omega^2R}{L^2}}{\left[\left(-\omega^2+\frac{1}{LC}\right)^2+\frac{\omega^2R^2}{L^2}\right]}$$

Also $$\frac{M\sin\theta}{M\cos\theta} = \tan\theta$$

$$= \frac{\frac{V_m\,\omega}{L}\left(-\omega^2+\frac{1}{LC}\right)}{V_m\frac{\omega^2R}{L^2}} = \frac{L}{\omega R}\left(-\omega^2+\frac{1}{LC}\right)$$

$$= \frac{1}{R}\left(-\omega L+\frac{1}{\omega C}\right)$$

$$\therefore\ \theta = \tan^{-1}\left[\frac{1}{R}\left(-\omega L+\frac{1}{\omega C}\right)\right]$$

Thus,

$$\begin{aligned} i_p &= M\sin\theta\cos(\omega t+\phi) + M\cos\theta\sin(\omega t+\phi)\\ &= [\sin(\omega t+\phi)\cos\theta + \cos(\omega t+\phi)\sin\theta]\,M\\ &= M\sin(\omega t+\phi+\theta)\end{aligned}$$

$$\therefore\ i_p = M\left[\sin\left\{\omega t+\phi+\tan^{-1}\frac{1}{R}\left(\frac{1}{\omega C}-\omega L\right)\right\}\right]$$

However,

$$M = \sqrt{M^2\cos^2\theta + M^2\sin^2\theta}$$

$$= \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}$$

Thus

$$i_p = \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}\sin\left(\omega t+\phi+\tan^{-1}\frac{\left(\frac{1}{\omega C}-\omega L\right)}{R}\right)$$

the complementary function i_C being equal to the d.c. response of RLC circuit.

Then, for overdamped case, when $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$,

$$i = e^{\alpha t}\left(C_1e^{\beta t}+C_2e^{-\beta t}\right) + \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}\sin\left(\omega t+\phi+\tan^{-1}\frac{\frac{1}{\omega C}-\omega L}{R}\right),$$

for underdamped case, when $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$,

$$i = e^{\alpha t}(C_1\cos\beta t + C_2\sin\beta t) + \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}\sin\left(\omega t+\phi+\tan^{-1}\frac{\left(\frac{1}{\omega C}-\omega L\right)}{R}\right).$$

and for the critically damped case, when

$$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC},$$

$$i = e^{\alpha t}(C_1+C_2t) + \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}\sin\left(\omega t+\phi+\tan^{-1}\frac{\left(\frac{1}{\omega C}-\omega L\right)}{R}\right).$$

EXAMPLE 8.30 *In Fig. E8.17 with switch open, steady state is reached with $v = 100\sin 314\,t$ volts. The switch is closed at $t=0$. The circuit is allowed to come to steady state again. Determine the steady state current and complete solution of transient current.*

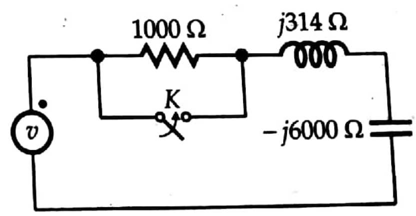


Fig. E8.17

SOLUTION. Before the switch is operated, *i.e.* $t<0$,

$$Z = 1000 + j\,(314-6000)$$
$$= 5773.27\ \angle(-1.39\ \text{rad})\,\Omega$$

Differentiating equation (1)

$$\frac{v}{L}+G\frac{dv}{dt}+C\frac{d^2v}{dt^2}=0.$$

or, $$\left(Cp^2+Gp+\frac{1}{L}\right)v=0 \qquad ...(2)$$

∴ The characteristic equation is then

$$Cp^2+Gp+\frac{1}{L}=0 \ \text{ or } \ p^2+10\,p+2=0.$$

i.e., $$p_1, p_2=\frac{-1\pm\sqrt{100-8}}{2}=\frac{-1\pm 9.6}{2}$$

or, $$p_1=4.3 \ \text{ and } \ p_2=-5.3$$

The general solution is then

$$v(t)=K_1\,e^{+4.3\,t}+K_2\,e^{-5.3\,t} \qquad ...(3)$$

At $t=0^+, v(0^+)=10$ V

(due to presence of initial charge across C)

and $$\frac{1}{L}\int v\,dt=i_L=0.$$

∴ From equation (1), at $t=0^+$,

$$C\frac{dv}{dt}+10\times 10=0 \ \text{ or } \ \frac{dv}{dt}=-\frac{100}{C}=-100 \text{ V/sec.}$$

Substituting $v(0^+)=10$ in equation (3), at $t=0^+$

$$10=K_1+K_2 \qquad ...(4)$$

Again, differentiating equation (3),

$$\frac{dv(t)}{dt}=4.3\,K_1\,e^{4.3\,t}-5.3\,K_2\,e^{-5.3\,t}$$

But, at $t=0^+$, $$\frac{dv(t)}{dt}=-100.$$

∴ $$-100=4.3\,K_1-5.3\,K_2$$

i.e., $$4.3\,K_1-5.3\,K_2=-100$$

or, $$4.3\,K_1-5.3\,(10-K_1)=-100 \qquad \text{[from (4)]}$$

or, $$4.3\,K_1-53+5.3\,K_1=-100$$

or, $$9.6\,K_1=-47$$

i.e., $$K_1=-4.896$$

This gives $K_2=14.896$.

Finally, therefore,

$$\boldsymbol{v(t)=-4.896\,e^{+4.3\,t}+14.896\,e^{-5.3\,t}\ \text{V}.}$$

8.9 ADDITIONAL EXAMPLES

EXAMPLE 8.32 *In Fig E8.19, determine the current expressions i_1 and i_2 at $t>0$ if the switch is closed at $t=0$.*

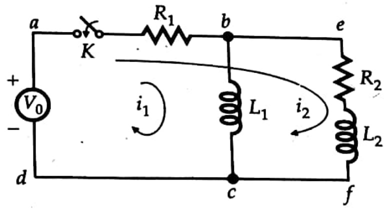


Fig. E8.19

SOLUTION. At $t=0$, the switch being closed, in loop *abcd*,

$$V_0=i_1\,R_1+L_1\frac{di_1}{dt}+i_2\,R_1$$

or, $$i_1\frac{R_1}{L_1}+\frac{di_1}{dt}+i_2\frac{R_1}{L_1}=\frac{V_0}{L_1}$$

or, $$(p+R_1/L_1)\,i_1+(R_1/L_1)\,i_2=V_0/L_1 \qquad ...(1)$$

Similarly, in loop *aefd*,

$$V_0=(i_1+i_2)\,R_1+i_2\,R_2+L_2\frac{di_2}{dt}$$

$$=i_1\,R_1+(R_1+R_2)\,i_2+L_2\frac{di_2}{dt}$$

or, $$i_1\frac{R_1}{L_2}+i_2\frac{R_1+R_2}{L_2}+\frac{di_2}{dt}=\frac{V_0}{L_2}$$

or, $$\left(p+\frac{R_1+R_2}{L_2}\right)i_2+i_1\frac{R_1}{L_2}=\frac{V_0}{L_2}$$

or, $$i_1\frac{R_1}{L_2}+\left(p+\frac{R_1+R_2}{L_2}\right)i_2=\frac{V_0}{L_2} \qquad ...(2)$$

Rearranging equations (1) and (2) in matrix form,

$$\begin{bmatrix} p+\dfrac{R_1}{L_1} & \dfrac{R_1}{L_1} \\ \dfrac{R_1}{L_2} & p+\dfrac{R_1+R_2}{L_2} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}=\begin{bmatrix} V_0/L_1 \\ V_0/L_2 \end{bmatrix}$$

Using Cramer's rule,

$$i_1=\frac{\begin{vmatrix} V_0/L_1 & R_1/L_1 \\ V_0/L_2 & p+\dfrac{R_1+R_2}{L_2} \end{vmatrix}}{\Delta}.$$

However,

$$\begin{vmatrix} V_0/L_1 & R_1/L_1 \\ V_0/L_2 & p+\dfrac{R_1+R_2}{L_2} \end{vmatrix} = p\frac{V_0}{L_1} + V_0\frac{R_1+R_2}{L_1 L_2} - \frac{V_0 R_1}{L_1 L_2},$$

Since $\dfrac{V_0}{L_1}$ is constant, $p\dfrac{V_0}{L_1}\left(i.e., \dfrac{d}{dt}\dfrac{V_0}{L_1}\right)$ is zero.

Thus, $\Delta i_1 = V_0 \dfrac{R_2}{L_1+L_2}$

where $$\Delta = \begin{bmatrix} p+\dfrac{R_1}{L_1} & \dfrac{R_1}{L_1} \\ \dfrac{R_1}{L_2} & p+\dfrac{R_1+R_2}{L_2} \end{bmatrix}$$

$$= \left(p^2 + p\frac{R_1+R_2}{L_2} + p\frac{R_1}{L_1} + \frac{R_1(R_1+R_2)}{L_1 L_2} - \frac{R_1^2}{L_1 L_2}\right)$$

$$= p^2 + \left(\frac{R_1 L_1 + R_2 L_1 + R_1 L_2}{L_1 L_2}\right)p + \frac{R_1 R_2}{L_1 L_2}$$

$$= p^2 + Mp + N.$$

where $$M = \frac{R_1 L_1 + R_2 L_1 + R_1 L_2}{L_1 L_2}$$

$$N = \frac{R_1 R_2}{L_1 L_2}.$$

Hence we finally obtain,

$$(p^2 + Mp + N)\, i_1 = V_0 \frac{R_2}{L_1+L_2}.$$

The characteristic equation is of the form

$$p^2 + M.p + N = 0.$$

Here, $$p_1, p_2 = \frac{-M \pm \sqrt{M^2-4N}}{2},$$

i.e., $$p_1 = \left(-\frac{M}{2} + \frac{\sqrt{M^2-4N}}{2}\right)$$

and $$p_2 = -\frac{M}{2} - \frac{\sqrt{M^2-4N}}{2}$$

Observation reveals the quantity $M^2 - 4N > 0$.

Hence p_1 and p_2 are *real* and *unequal* and the complementary function is given by

$$i_{c_1} = e^{-M/2.t}\left(C_1 e^{\frac{\sqrt{M^2-4N}}{2}t} + C_2 e^{-\frac{\sqrt{M^2-4N}}{2}t}\right)$$

the particular solution being

$$i_{p_1} = \frac{V_0}{R_1}$$

$$\therefore \quad i_1 = i_{c_1} + i_{p_1}$$

$$= e^{-(M/2)t}\left(C_1 e^{\frac{\sqrt{M^2-4N}}{2}.t} + C_2 e^{-\frac{\sqrt{M^2-4N}}{2}.t}\right) + \frac{V_0}{R_1}$$

C_1 and C_2 can be obtained for initial condition.

In a similar way

$$i_2 = \frac{\begin{vmatrix} p+R_1/L_1 & V_0/L_1 \\ R_1/L_2 & V_0/L_2 \end{vmatrix}}{\Delta};$$

Δ being the determinant

or, $$\Delta i_2 = p\frac{V_0}{L_2} + \frac{R_1 V_0}{L_1 L_2} - V_0\frac{R_1}{L_1 L_2} = 0$$

$$\left[\because p\frac{V_0}{L_2} = 0\right]$$

or, $$\left[p^2 + \left(\frac{R_1 L_1 + R_2 L_1 + R_1 L_2}{L_1 L_2}\right)p + \frac{R_1 R_2}{L_1 L_2}\right] i_2 = 0.$$

or, $$(p^2 + Mp + N)\, i_2 = 0.$$

The complementary function i_{c_2} being same as i_{c_1} in this part i_{p_2} is zero since the differential equation with i_2 is of homogeneous nature.

$$\therefore \quad i_2 = i_{c_2} + 0 = e^{(-M/2)t}\left(C_1' e^{\frac{\sqrt{M^2-4N}}{2}t} + C_2' e^{-\frac{\sqrt{M^2-4N}}{2}t}\right)$$

It may be observed that at steady state, L_1 acts as short circuit, the transient being vanished within this time, the current i_1 is given by (V_0/R_1) only. Steady state i_2 will be obviously zero as the entire source current will pass as i_1 getting a short circuited path though *b-c* [Ref. Fig. E8.19(*a*)]

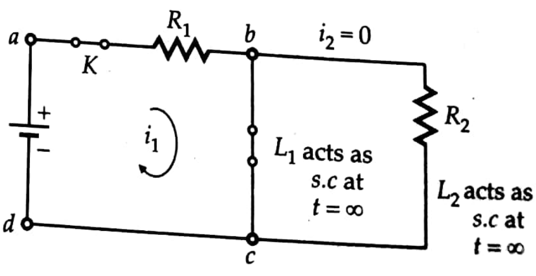


Fig. E8.19(*a*)

EXAMPLE 8.33 *A d.c. voltage of 100 V is suddenly applied in the network shown in Fig. E8.20. Find the transient currents in both the loops and obtain the transient voltage across the capacitor.*

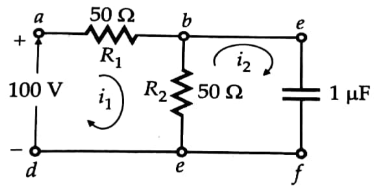


Fig. E8.20

SOLUTION. Applying KVL in loop *abcd*,

$$50\,i_1 + (i_1 - i_2)\,50 = 100$$

or, $$100\,i_1 - 50\,i_2 = 100$$

or, $$2i_1 - i_2 = 2$$

or, $$2\frac{di_1}{dt} - \frac{di_2}{dt} = 0$$

i.e., $$2pi_1 - pi_2 = 0$$

or, $$pi_2 = 2pi_1 \qquad ...(a)$$

Next, applying KVL in loop *befc*,

$$(i_2 - i_1)\,50 + \frac{1}{C}\int i_2\,dt = 0 \qquad ...(b)$$

or, $$-50\,i_1 + 50\,i_2 + \frac{1}{1\times 10^{-6}}\int i_2\,dt = 0$$

or, $$-50\frac{di_1}{dt} + 50\frac{di_2}{dt} + \frac{i_2}{1\times 10^{-6}} = 0$$

or, $$-50\,pi_1 + 50pi_2 + i_2 \times 10^6 = 0 \qquad ...(c)$$

However, $pi_1 = \frac{1}{2}\,pi_2$ (from (*a*))

Thus equation (*c*) becomes

$$-25\,pi_2 + 50\,pi_2 + i_2 \times 10^6 = 0$$

or, $$(25\,p + 10^6)\,i_2 = 0$$

i.e., $$(p + 4\times 10^4)\,i_2 = 0 \qquad ...(d)$$

Since equation (*d*) is a homogeneous differential equation hence the particular solution is zero and the complementary function i_{C_2} is given by

$$i_2 = i_{C_2} = Ke^{-4\times 10^4\,t} \qquad ...(e)$$

At $t = 0$, equation (*b*) becomes

$$(i_2 - i_1)\,50 = 0 \quad \text{or} \quad i_1 = i_2$$

Using $i_1 = i_2$ in the original loop equation of loop *abcd*,

$$50\,i_1 = 100 \quad \text{or} \quad i_1 = 2 \text{ A}$$

This gives

$$i_2 = 2 \text{ A} \qquad [\because i_1 = i_2]$$

Thus from equation (*e*), at $t = 0$,

$$K = 2$$

Thus $$i_2 = \mathbf{2}e^{-4\times 10^4\,t}\ \textbf{A}$$

Again in loop *abcd*, we have already obtained

$$2i_1 - i_2 = 2 \quad \text{or} \quad 2i_1 - (2e^{-4\times 10^{4}t}) = 2$$

$$i_1 - e^{-4\times 10^4\,t} = 1$$

or, $$i_1 = \mathbf{1 +} e^{-4\times 10^4\,t}\ \textbf{A.}$$

The transient voltage across the capacitor is given by

$$v_c = \frac{1}{C}\int i_2\,dt = \frac{1}{1\times 10^{-6}}\int 2\,e^{-4\times 10^4\,t}\,dt$$

$$= \mathbf{50\,(1 - }e^{-4\times 10^4\,t}\mathbf{)\ V.}$$

Also $v_c = V\,(1 - e^{-t/RC})$

where V is the steady state voltage across the capacitor.

V = Supply voltage – drop across R_1

(refer to Fig. E8.20(*a*)).

$= 100 - i_1 \times R_1 = 100 - 1 \times 50 = 50$ V

$[\because$ Steady state $i_1 = 1$ A$]$

$\therefore\ v_c = 50\,(1 - e^{-4\times 10^4\,t})$

$$\left[\because \frac{1}{RC} = \frac{1}{(R_1\,||\,R_2)\,C} = \frac{1}{25\times 10^{-6}} = 4\times 10^4\right]$$

EXAMPLE 8.34 *In a RC series circuit* $R = 1\,\Omega$ *and* $C = 05$ F. *An exponential voltage* $v = 10\,e^{-t}$ *is suddenly applied at* $t = 0$. *Obtain* $i\,(t)$. *Assume no initial charge in the capacitor.*

SOLUTION. Application of KVL in the series RC circuit,

$$iR + \frac{1}{C}\int i\,dt = v = 10\,e^{-t} \qquad ...(1)$$

Differentiating (1),

$$R\frac{di}{dt} + \frac{1}{C}.\,i = -10\,e^{-t}$$

or, $$\frac{di}{dt} + \frac{i}{RC} = -10\,e^{-t}$$

CHAPTER 8