3
Waveguides

3.1. Introduction

Hollow waveguides are generally used as transmission lines at frequencies around 1GHz and above. Waveguides have certain advantages in comparison to coaxial lines. These are

(1) Higher power handling capability.

(2) Lower loss per unit length.

(3) A simpler, lower cost structure.

Moreover, the reflections caused by the flanges used in connecting waveguide sections is much less than that associated with coaxial connectors. It is precisely because of the low loss factor, that waveguides

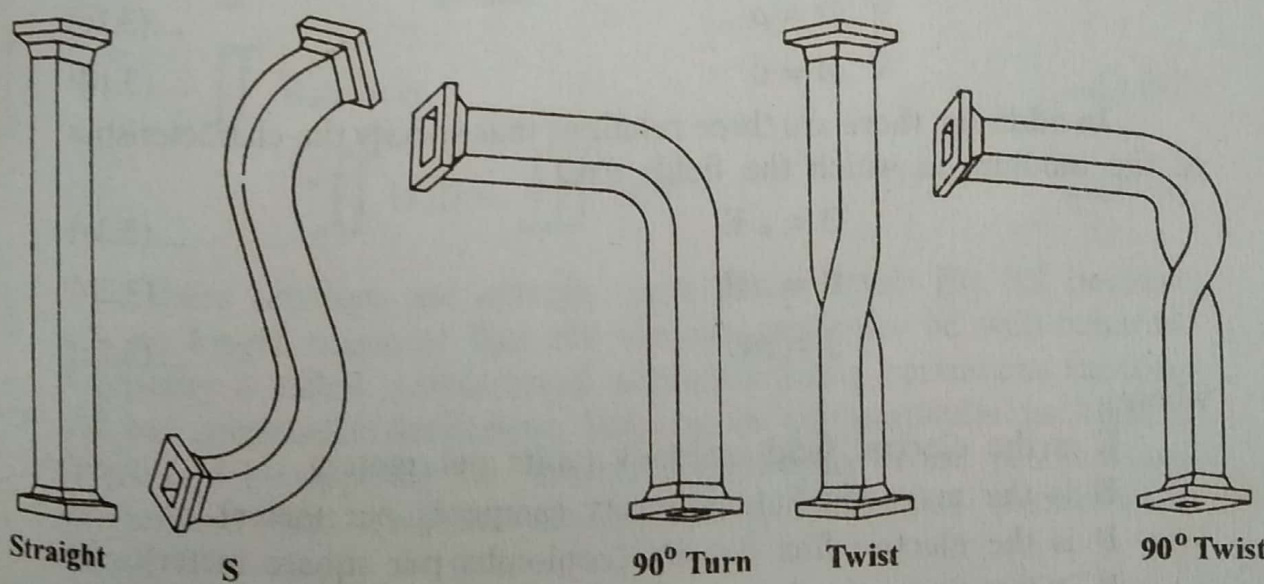


Fig. 3.1 : Typical wave guide shapes

have edge over other kinds of transmission lines at higher frequencies. By definition, a waveguide is simply a pipe of virtually any consistent cross-sectional shape through which an em wave travels by reflection, and not by conduction. It is because of this reason that we always speak of electric and magnetic fields rather than voltage and current as was the case with transmission lines. Since the method of propagation in a waveguide is by means of reflection, hence the interior surfaces should be smooth, often silvered, free of moisture and abrupt changes in shape or direction otherwise they will cause reflections to occur back towards the source. A few typical waveguide shapes are shown in Fig. 3.1.

In this chapter brief discussion of the boundary conditions is given, propagation in parallel plate guide, rectangular and cylindrical guides supporting both TE and TM modes is discussed. Attenuation in these guides is also considered. Finally rectangular and cylindrical cavities are described.

3.2. Maxwell's Equations

Maxwell's equations, sometimes called electromagnetic equations, express the fundamental relations between electromagnetic fields and their sources, *viz.* currents and charges.

The four Maxwell's equations are

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \qquad \text{....(3.1a)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{....(3.1b)}$$

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{....(3.1c)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{....(3.1d)}$$

In addition, there are three relations that specify the characteristics of the medium in which the fields exist.

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \text{....(3.2a)}$$

$$\mathbf{B} = \mu \mathbf{H} \qquad \text{....(3.2b)}$$

$$\mathbf{J} = \sigma \mathbf{E} \qquad \text{....(3.2c)}$$

where,

E is the electric field intensity (volts per meter)
H is the magnetic field intensity (amperes per meter)
D is the electric flux density (coulombs per square meter)
B is the magnetic flux density (webers per square meter)
J is the electric current density (amperes per square meter)
ρ is the electric charge density (coulombs per cubic meter)

ε is the capacitivity (permittivity) of the medium
μ is the inductivity (permeability) of the medium
σ is the conductivity of the medium.

Matter is often classified according to the values of σ, ε and μ. Materials having very large values of σ are called conductors and those having small values of σ are called insulators or dielectrics. For analysis we generally approximate good conductors by perfect conductors having $\sigma = \infty$, and approximate perfect dielectrics by $\sigma = 0$. The permittivity of any material is never less than that of vacuum ε_0 which in MKS units has the value :

$$\varepsilon_0 = 8.854 \times 10^{-12} \cong \frac{1}{36\pi} \times 10^{-9} \text{ farads per meter}$$

The ratio $\varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0}$ is called the dielectric constant or relative capacitivity. For most linear materials permeability is approximately that of free space μ_0. The ratio $\mu_r = \mu/\mu_0$, called the relative permeability is essentially unity for non-magnetic materials.

The value of μ_0 is

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry per meter}$$

Corresponding to each of Eq. 3.1, there are the integral forms of Maxwell's equations

$$\oint \mathrm{H}.dl = \frac{d}{dt}\iint \mathrm{D}.ds + \iint \mathrm{J}.ds \qquad \text{....(3.3a)}$$

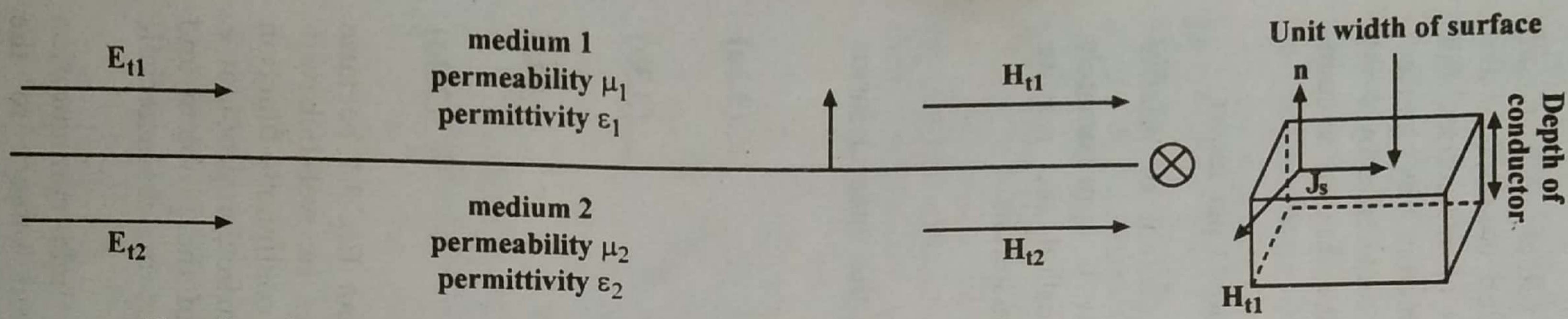
$$\oint \mathrm{E}.dl = -\frac{d}{dt}\iint \mathrm{B}.ds \qquad \text{....(3.3b)}$$

$$\iint \mathrm{B}.ds = 0 \qquad \text{....(3.3c)}$$

$$\iint \mathrm{D.ds} = \iiint q_v dt \qquad \text{....(3.3d)}$$

These relations are actually more general than Eq. 3.1 because it is no longer required that the various quantities be well-behaved. A quantity is called well-behaved whenever it is a continuous function and has continuous derivatives. Solution to any particular problem is determined by applying the boundary conditions of the problem and the solution thus obtained are called eigenvalues or characteristic values.

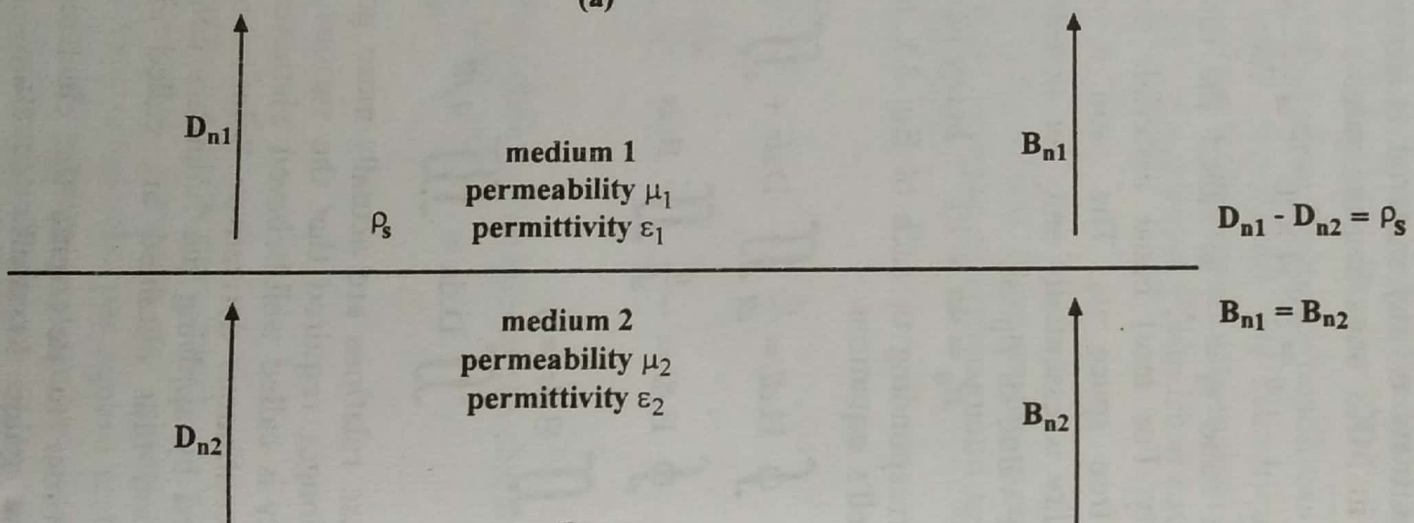
In order to determine the electromagnetic field configuration within the guide Maxwell's equations are solved subject to the appropriate boundary conditions at the walls of the guide.



If medium 1 and medium 2 are both dielectrics, $E_{t1} = E_{t2}$ and $H_{t1} = H_{t2}$

If medium 2 is perfect conductor $E_{t1} = 0 = E_{t2}$ and $H_{t1} = J_s$

(a)



(b)

Fig. 3.2. Boundary conditions at the interface of two media : (a) for the tangential E and H fields, and (b) for the normal D and B fields.

3.3. Electromagnetic Boundary conditions

To solve electromagnetic problems involving continuous region of different constitutive parameters, it is necessary to know the boundary conditions that the field vectors E, D, B and H must satisfy at the interfaces. Boundary conditions are derived by applying the integral form of Maxwell's equations 3.3 to a small region at an interface of two media. The application of integral form of a curl equation to a flat closed path at the boundary with top and bottom sides in the two touching media yields the boundary conditions for the tangential components while the application of the integral form of divergence equations to a shallow pillbox at an interface with top and bottom faces in the two media gives the boundary conditions for the normal components. Fig. 3.2 gives the boundary surface between two media.

The boundary condition for the tangential components between two dielectric media are obtained from Eqs. 3.3a and 3.3b as

$$a_{n2} \times (H_1 - H_2) = J_s \quad (A/m) \qquad \text{....(3.4a)}$$

$$E_{1t} = E_{2t} \quad (V/m) \qquad \text{....(3.4b)}$$

Similarly the boundary conditions for the normal components of D and B are obtained from Eqs. 3.3c and 3.3d as

$$a_{n2} \cdot (D_1 - D_2) = \rho_s \quad (C/m^2) \qquad \text{....(3.4c)}$$

$$B_{1n} - B_{2n} = 0 \quad (T) \qquad \text{....(3.4d)}$$

where a_{n1}, a_{n2} are the outward normal unit vectors in medium 1 and 2.

Consider two important special cases of (1) a boundary between two lossless media, and (2) a boundary between a good dielectric and a good conductor.

A lossless linear medium can be specified by a permittivity ε and a permeability μ with $\sigma = 0$. There are usually no free charges and no surface currents at the interface between two lossless media. That is $\rho_s = 0$ and $J_s = 0$. Hence the boundary conditions are listed as

$$E_{1t} = E_{2t} \Rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\varepsilon_1}{\varepsilon_2} \qquad \text{....(3.5a)}$$

$$H_{1t} = H_{2t} \Rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2} \qquad \text{....(3.5b)}$$

$$D_{1n} = D_{2n} \Rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \qquad \text{....(3.5c)}$$

$$B_{1n} = B_{2n} \Rightarrow \mu_1 H_{1n} = \mu_{2n} H_{2n} \qquad \text{....(3.5d)}$$

When interface between a dielectric and a perfect conductor is encountered the above mentioned boundary conditions are slightly modified. Practical cases will only have good conductors, for ease of solution of field problems good conductors are often considered as perfect conductors. For the boundary conditions if the second medium is conductor $E_2=0$, $H_2=0$, $D_2=0$ and $B_2=0$,

$$E_{1t} = 0, \qquad E_{2t} = 0 \qquad \text{....(3.6a)}$$

$$a_{n2} \times H_1 = J_s, \quad H_{2t} = 0 \qquad \text{....(3.6b)}$$

$$a_{n2} \cdot D_1 = \rho_s, \quad D_{2n} = 0 \qquad \text{....(3.6c)}$$

$$B_{1n} = 0, \qquad B_{2n} = 0 \qquad \text{....(3.6d)}$$

In chapter two, we learned that transmission lines consist of two or more conductors and they support transverse electromagnetic (TEM) waves. This shows that they do not have longitudinal field components. TEM waves have a uniquely defined voltage, current and characteristic impedance. Waveguides, often consist of a single conductor and support transverse electric (TE) or Transverse Magnetic (TM) waves, they support one/or both the longitudinal field components. Also these guides do not have a unique definition for characteristic impedance.

We assume time harmonic fields with an $e^{j\omega t}$ dependence, and wave propagation along the z-axis. For understanding wave propagation in TE or TM mode we first discuss the case of wave propagation in parallel conducting planes.

3.3.1 Wave equation in source free region

The Maxwell's equations in source-free region become

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} \qquad \text{....(3.1a)}$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \qquad \text{....(3.1b)}$$

$$\nabla . \mathbf{D} = 0 \qquad \text{....(3.1c)}$$

$$\nabla . \mathbf{B} = 0 \qquad \text{....(3.1d)}$$

Take the curl of Eq. 3.1b

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times (j\omega\mu\mathbf{H})$$

LHS from the vector identity is given as

$$\nabla \times \nabla \times \nabla . \mathbf{E} = \nabla \nabla \mathbf{E} - \nabla^2\mathbf{E}$$

Using 3.1a, we have

$$\nabla^2 E = -(j\omega\mu)(j\omega\varepsilon)\mathbf{E}$$

putting

$$\gamma = \sqrt{(j\omega\varepsilon)\,(j\omega\mu)}$$

$$\nabla^2\mathbf{E} - \gamma^2\mathbf{E} = 0 \qquad \text{....(3.7a)}$$

Similarly from Eq. 3.1a, we get

$$\nabla^2\mathbf{H} - \gamma^2\mathbf{H} = 0 \qquad \text{....(3.7b)}$$

Eq. 3.7 are the wave equations in **E** and **H**.

The two curl equations can be written in the rectangular coordinates as

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\,j\omega\mu H_x \qquad \text{....(3.8a)}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\,j\omega\mu H_y \qquad \text{....(3.8)}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\,j\omega\mu H_z \qquad \text{....(3.8c)}$$

and

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \qquad \text{....(3.9a)}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \qquad \text{....(3.9b)}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \qquad \text{....(3.9c)}$$

As mentioned earlier it is assumed that the propagation is in the z-direction and the variation of all field components in this direction will be expressed as $e^{-\overline{\gamma}z}$, where in general, propagation constant $\overline{\gamma} = \overline{\alpha} + j\overline{\beta}$ is a complex quantity. When the time variation factor is combined with the z-variation factor, it gives

$$e^{j\omega t}\, e^{-\overline{\gamma}z} = e^{(j\omega t - \overline{\gamma}z)} = e^{-\overline{\alpha}z}\, e^{j(\omega t - \overline{\beta}z)} \qquad \text{....(3.10)}$$

which represents a wave propagating in z-direction. If γ is an imaginary quantity, that is if $\overline{\alpha} = 0$, expression 3.10 represent a wave without attenuation. On the other hands if $\overline{\gamma}$ is real so that $\overline{\beta} = 0$, there is no wave motion but only an exponential decrease in amplitude.

The six components of Eq. 3.8 and 3.9 can be solved for the four transverse field components in terms of H_z and E_z as

$$H_x = -\frac{\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial x} + j\frac{\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y} \quad \text{....(3.10a)}$$

$$H_y = -\frac{\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial y} - \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} \quad \text{....(3.10b)}$$

$$E_x = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial x} - j\frac{\omega\mu}{h^2}\frac{\partial H_z}{\partial y} \quad \text{....(3.10c)}$$

$$E_y = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial y} + j\frac{\omega\mu}{h^2}\frac{\partial H_z}{\partial x} \quad \text{....(3.10d)}$$

where

$$h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon \quad \text{....(3.11a)}$$

putting $\quad k_0^2 = \omega^2\mu\varepsilon \quad , \quad$ then $h^2 = \bar{\gamma}^2 + k_0^2$ ....(3.11b)

From 3.10 it is clear that there must be *a z* component of either E or H; otherwise all the components would be zero and there would be no field at all in the region considered. Wave propagation in parallel conducting planes is discussed below.

3.4. Parallel Conducting Planes

3.4.1. Transverse electric modes (TE, $E_z = 0$)

When we consider the case of parallel conducting plates shown in Fig 3.3 the space between the planes is of infinite extent in the *y* direction. Hence it can be assumed that the field is constant in the *y* direction. In other words, derivation with respect to *y* in Eqs. 3.8 and 3.9 can be equated

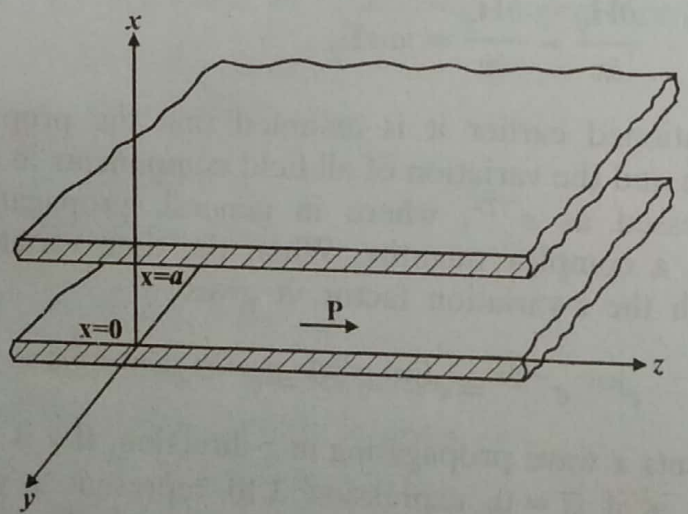


Fig. 3.3 Parallel plate guide

These expressions are same as for wave propagation in free space. Unlike the TE and TM waves, the velocity of the TEM wave is independent of frequency and has the value, $c \approx 3 \times 10^8$ m/s. Also the cutoff frequency of the TEM wave from Eq. 3.30 is zero, *i.e.*, all frequencies down to zero frequency can propagate along the line.

3.5. Rectangular Guides

Rectangular waveguides were one of the earliest types of transmission lines used to carry microwave signals. The hollow rectangular waveguides can propagate in TE or TM modes, but not in TEM mode, since only one conductor is present. As mentioned earlier the rectangular waveguides have cut-off frequencies below which propagation is not possible. The waveguide shown in Fig. 3.4a is filled with a source free lossless dielectric material ($\sigma_d = 0$) and its walls are perfectly conducting ($\sigma_c \approx \infty$).

Fig. 3.4b shows a parallel plate transmission line similar to that in Fig. 3.3. The width of the metal strip is w and their spacing is b. Power is delivered to the load via longitudinal current flow along the two plates. If now a pair of shorted stubs of lengths l_s are connected to the parallel strip as shown in the Fig. 3.4b, and the stubs are quarter wavelength long, they will have no effect on the transmission of power since they present an infinite impedance in shunt with the line. The current flow in the stub is reactive as no real power can be delivered to the shorts. Power can only be delivered by longitudinal currents, which is why they are called power currents. Note that the direction of power and reactive currents are perpendicular to each other. We can connect an infinite set of such quarter wavelength shorted stubs along the length of the parallel plate transmission line. The resultant configuration will be exactly similar to the rectangular waveguides shown in Fig. 3.4a, where $a = w + 2l_s$.

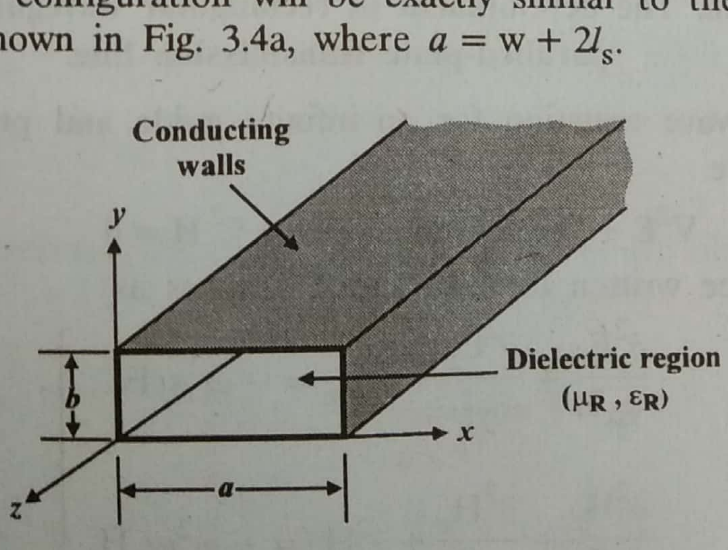


Fig. 3.4a. Rectangular waveguide.

Since $l_s = \lambda/4$ and the width w must be finite so that power currents can flow, the condition for power transmission is

$$a > \lambda/2$$

In other words, electromagnetic wave propagation in rectangular waveguides can only occur at frequencies high enough to satisfy this inequality. As the frequency decreases, λ increases and hence $\lambda/4$ stubs take up a greater portion of the guide width a, leaving less room for w the parallel-plate line. Decreasing the frequency to the point where $a = 2l_s = \lambda/2$ results in $w = 0$, which prevents the flow of power currents. The frequency at which this occurs is called the cutoff frequency (f_c) of the guide.

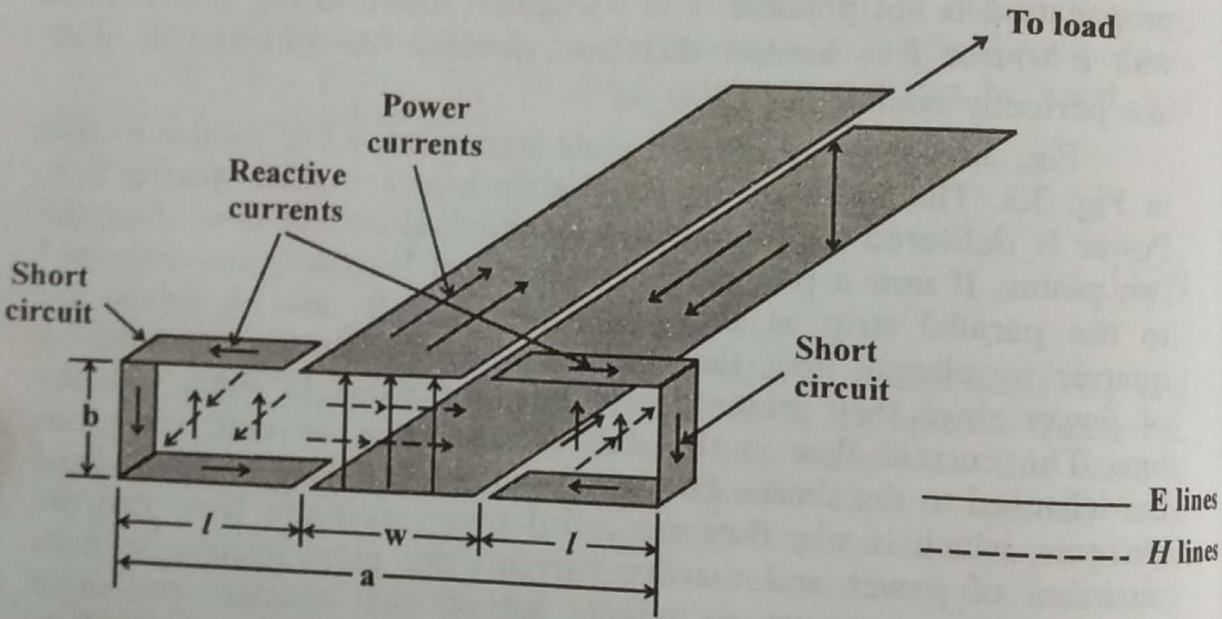


Fig. 3.4b. The development of rectangular waveguide from a paralled-plate transmission line.

The wave equation for an infinite guide and propagating in z direction are

$$\nabla^2 \mathbf{E} + \bar{\gamma}^2 \mathbf{E} = 0 \text{ and } \nabla^2 \mathbf{H} + \bar{\gamma}^2 \mathbf{H} = 0 \qquad \text{....(3.31)}$$

which can be written for TM and TE waves as

$$\left.\begin{aligned} \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \bar{\gamma}^2 E_z &= -\omega^2 \mu\varepsilon E_z \\ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \bar{\gamma}^2 H_z &= -\omega^2 \mu\varepsilon H_z \end{aligned}\right\} \qquad \text{....(3.32)}$$

3.5.1. Transverse magnetic (TM) modes in rectangular guides

The wave equations in 3.32 are partial differential equations and can be solved by the method of separation of variables. For the case of TM modes $H_z = 0$ using the wave equation in E_z, we can write

$$E_z(x, y, z) = E_z^0(x, y)\, e^{-\bar{\gamma} z} \qquad \text{....(3.33)}$$

Let $$E_z^0 = XY \qquad \text{....(3.34)}$$

where X is a function of x alone and Y is a function of y alone. Substituting Eq. 3.34 in Eq. 3.32, one obtains

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + \bar{\gamma}^2 XY = -\omega^2 \mu\varepsilon XY$$

Using $h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon$ two equations one only in X and other only in Y are obtained.

$$\frac{1}{X}\frac{d^2X}{dx^2} + h^2 = A^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -A^2$$

where A^2 is a constant solution in X and Y are

$$X = C_1 \cos Bx + C_2 \sin Bx$$

where $B^2 = h^2 - A^2$

$$Y = C_3 \cos A_y + C_4 \sin A_y$$

This gives

$$E_z^0 = XY$$

The solution for this equation on applying the following boundary conditions,

$E_z = 0$, when $x = 0, x = a, y = 0, y = b$ gives

$$E_z^0 = C \sin\left(\frac{m\pi}{a}\right) x \sin\left(\frac{n\pi}{b}\right) y \qquad \text{....(3.35)}$$

where C is a constant. The other field components are obtained from Eq. 3.10 as

$$E_x^0 = -\frac{j\bar{\beta}C}{h^2}\left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi}{a}\right) x \ \sin\left(\frac{n\pi}{b}\right) y \qquad \text{....(3.36a)}$$

$$E_y^0 = -\frac{j\bar{\beta}C}{h^2}\left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi}{a}\right) x \ \cos\left(\frac{n\pi}{b}\right) y \qquad \text{....(3.36b)}$$

$$H_x^0 = \frac{j\omega\varepsilon C}{h^2}\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}\right)x\cos\left(\frac{n\pi}{b}\right)y \quad \text{....(3.36c)}$$

$$H_y^0 = -\frac{j\omega\varepsilon C}{h^2}\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}\right)x\sin\left(\frac{n\pi}{b}\right)y \quad \text{....(3.36d)}$$

The propagation constant γ in this case is given by

$$\bar{\gamma} = \bar{\alpha} + j\bar{\beta} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon} \quad \text{....(3.37)}$$

When the attenuation constant $\bar{\alpha}$ is equal to zero, the phase constant $\bar{\beta}$ will have the value

$$\bar{\beta} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad \text{....(3.38)}$$

The cut-off frequency ω_c is obtained when $\bar{\beta} = 0$

$$\omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad \text{....(3.39)}$$

This is the frequency below which wave propagation will not occur.

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad \text{....(3.40)}$$

The corresponding cut-off wavelength is

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad \text{....(3.41)}$$

from which it can be seen that

$$f_c\lambda_c = v_0$$

The phase velocity is given by

$$\bar{v}_p = \frac{\omega}{\bar{\beta}} = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \quad \text{....(3.42)}$$

This expression indicates that the velocity of propagation of the wave in the guide is greater than the phase velocity in the free space. As the value of frequency is increased above cut-off, the phase velocity decreases from an infinitely large value and approaches c, the velocity of free space. Since the wavelength in the guide is given by

$\bar{\lambda} = \bar{v}_p/f$, it will be longer than the corresponding free space wavelength. The lowest possible value for m and n for TM mode is 1. Substituting for $m = n = 1$ in Eq. 3.36 we get fields for the lowest particular mode called the TM_{11}. Higher-order modes require higher frequencies in order to be propagated along the guide of given dimensions.

The wave configuration for different TM modes in a waveguide are shown in Fig. 3.5. The subscripts indicate the number of half sine wave variations of the field components in the x and y directions respectively.

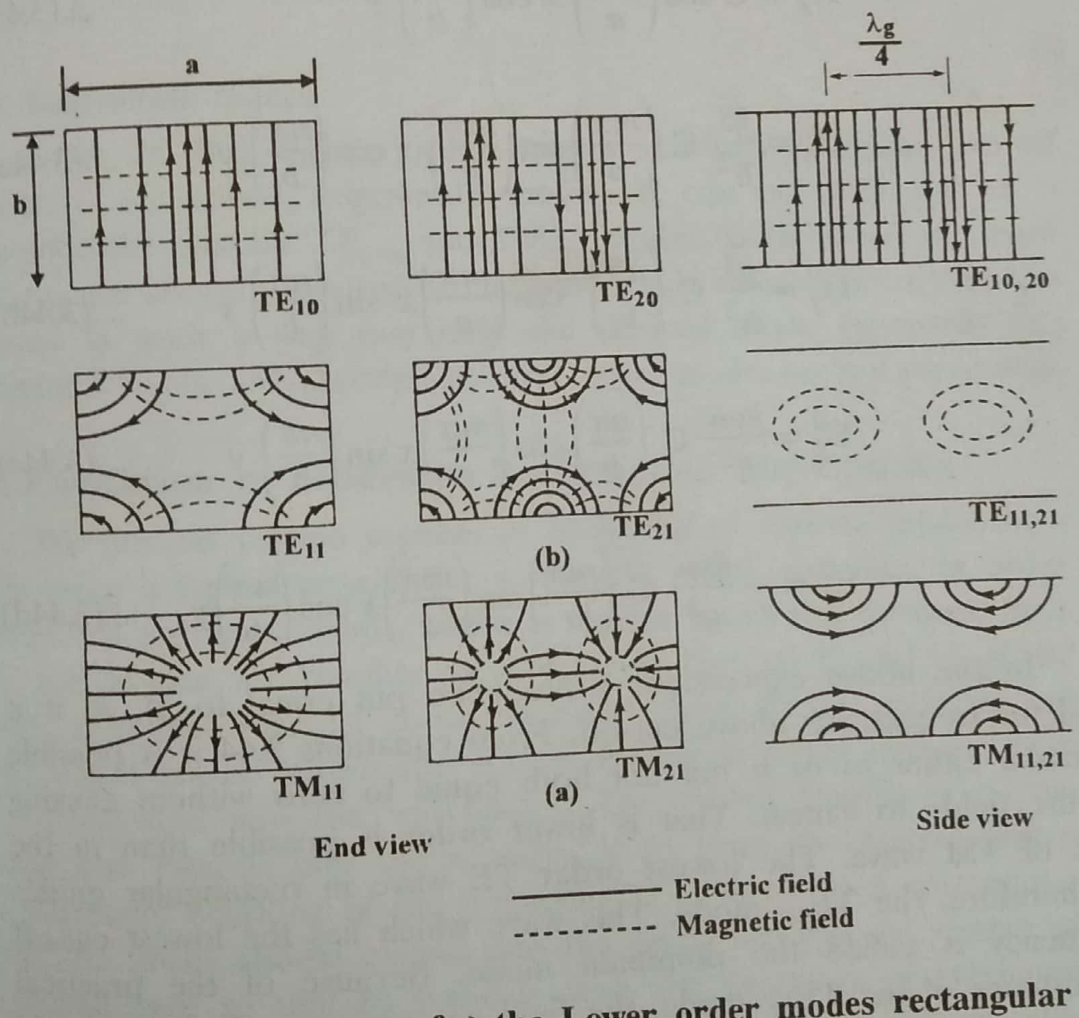


Fig. 3.5. Field patterns for the Lower order modes rectangular waveguide (TM and TE)

3.5.2. Transverse electric (TE) modes

The solutions for transverse electric waves are derived in similar manner to that for transverse magnetic waves. The H_z component will be given as

$$H_z(x,y,z) = H_z^{\,0}(x,y)\, e^{-\bar{\gamma} z}$$

Let $\quad H_z^{\,0} = XY$

Boundary conditions cannot be applied to H_z directly. Hence it is differentiated with respect to x and y to obtain $E_x^{\,0}$, $E_y^{\,0}$, $H_x^{\,0}$ and $H_y^{\,0}$. The boundary conditions are then applied to these field components as

$$E_x^0 = 0 \text{ at } y = 0, b$$

$$E_y^0 = 0 \text{ at } x = 0, a$$

The field expressions result in the following relations :

$$H_z^0 = C \cos\left(\frac{m\pi}{a}\right) x \cos\left(\frac{n\pi}{b}\right) y \qquad \text{....(3.43)}$$

$$H_x^0 = \frac{j\bar{\beta}}{h^2}\, C \left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi}{a}\right) x \cos\left(\frac{n\pi}{b}\right) y \qquad \text{....(3.44a)}$$

$$H_y^0 = \frac{j\bar{\beta}}{h^2}\, C \left(\frac{n\pi}{b}\right) \cos\left(\frac{m\pi}{a}\right) x \sin\left(\frac{n\pi}{b}\right) y \qquad \text{....(3.44b)}$$

$$E_x^0 = \frac{j\omega\mu}{h^2}\, C \left(\frac{n\pi}{b}\right) \cos\left(\frac{m\pi}{a}\right) x \sin\left(\frac{n\pi}{b}\right) y \qquad \text{....(3.44c)}$$

$$E_y^0 = -\frac{j\omega\mu}{h^2}\, C \left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi}{a}\right) x \cos\left(\frac{n\pi}{b}\right) y \qquad \text{....(3.44d)}$$

In the above expressions $\bar{\gamma}$ has been put equal to $j\bar{\beta}$, as it is valid for frequencies above cut-off. From equations 3.44 it is possible to make either m or n but not both equal to zero without causing all the fields to vanish. That is lower order is possible than in the case of TM wave. The lowest order TE wave in rectangular guides is, therefore, the TE_{10} mode. This wave which has the lowest cut-off frequency is called the dominant mode. Because of the practical importance of the TE_{10} mode, the field components, when m=1 and n=0 case are given as follows

$$H_z = C \cos\frac{\pi x}{a} \qquad \text{....(3.45a)}$$

$$H_x^{\,0} = \frac{j\bar{\beta} a C}{\pi} \sin\frac{\pi x}{a} \qquad \text{....(3.45b)}$$

$$E_y^0 = -\frac{j\omega\mu a C}{\pi} \sin\frac{\pi x}{a} \quad \text{....(3.45c)}$$

$$E_x^0 = H_y = 0$$

and
$$\bar{\beta} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{\pi}{a}\right)^2} \quad \text{....(3.47)}$$

$$f_c = \frac{c}{2a}, \qquad \lambda_c = 2a \quad \text{and } h = \frac{\pi}{a}$$

For the TE_{10} mode the cutoff frequency is independent of the dimension b. Field configurations for some of the TE modes are shown in Fig. 3.5.

3.5.3. Degenerate modes

Some of the higher order modes, having the same cut-off frequency, are called degenerate modes. It can be seen that in a waveguide the possible TE_{mn} and TM_{mn} modes (both m and n cannot be zero) are always degenerate. The wave guide dimensions are always selected in such a way that only the desired mode (generally the dominant TE_{10} or TM_{11}) propagate and higher modes are not supported.

3.6. Excitation of Modes in Rectangular Waveguides

We present certain aspects of excitation of various modes in a guide using a typical coaxial line-waveguide probe coupling. In order to launch a particular mode, probe is chosen which will produce lines of E and H that are roughly parallel to the lines of E and H of the particular mode. Fig. 3.6 shows that the short circuit position and probe depth can be adjusted to achieve maximum power transfer from the coaxial line into the waveguides. The centre conductor of the coaxial line extends into the waveguides to form an electric probe. For maximum coupling of the dominant TE_{10} mode in a rectangular guide, the probe should extend into the guide through the centre of the broad face so that to coincide with the positions of maximum electric field for the TE_{10} mode. The evanescent modes that are also excited are localized fields that store reactive energy. These give the junction its reactive properties.

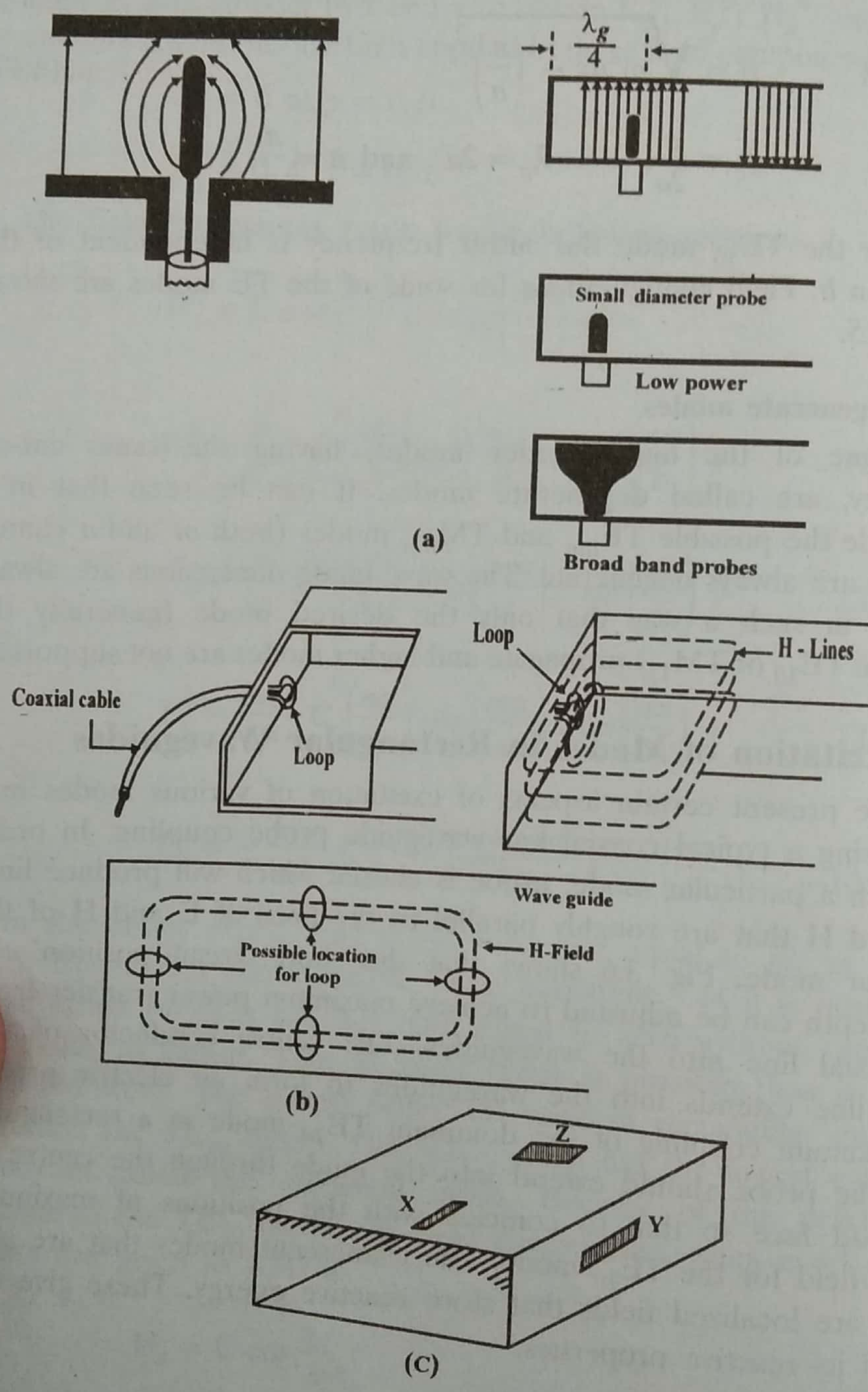


Fig. 3.6. Methods of coupling to and from a rectangular waveguide.

All TE_{m0} modes with m even has a zero field at the centre and field polarities reverse alternately in the transverse direction. These modes can be excited by two probes fed with opposite phase currents. Fig. 3.6b illustrates loop or inductive coupling. The loop is placed at a point where magnetic field (H) is maximum. There are a number of locations possible for H to be maximum. Aperture coupling is shown in Fig. 3.6c, slot X is at a position of maximum E field while slot Y is in a region of maximum H field density.

We have seen that only TE ($E_z = 0$) and TM ($H_z = 0$) modes can propagate in hollow rectangular and cylindrical waveguides. The familiar TEM mode for which there in no axial component of either E or H does not propagate. Suppose a TEM wave is assumed to exist within a hollow guide of any shape. Then lines of H must lie entirely in the transverse plane. In a non-magnetic material,

$$\nabla \cdot H = 0$$

which requires that lines of H should be closed loops. Hence, if a TEM wave exists in a hollow guide then the lines will be closed loops in the plane perpendicular to the axis. But by Maxwell's first equation the magnetomotive force around each of these closed loops must equal to the axial current through the loop. In the case of a coaxial transmission line, this axial current through the H loops is the conduction current in the inner conductor. But as the hollow waveguides do not have an inner conductor, this axial current must be a displacement current. But an axial displacement current requires an axial component of E field, which is not present in a TEM wave. Therefore the TEM wave does not exist in a single conductor waveguide.

3.7. Power Flow in Rectangular Waveguides

The power flown through the guide for the TE_{10} mode is calculated as

$$P_{10} = \frac{1}{2}\text{Re}\int_{x=0}^{a}\int_{y=0}^{b} E \times H \,.\, z.dy\, dx \qquad \text{....(3.48)}$$

$$= \frac{1}{2}\text{Re}\int_{x=0}^{a}\int_{y=0}^{b} E_y H_x^* \, dy\, dx$$

$$= \frac{\omega\mu a^2}{2\pi^2}\text{Re}\,(\bar{\beta})\left|C_{10}\right|^2 \int_{x=0}^{a}\int_{y=0}^{b} \sin^2\frac{\pi x}{a}\, dx \qquad \text{....(3.49)}$$

$$= \frac{\omega\mu a^3 \left|C_{10}\right|^2 b}{4\pi^2}\text{Re}\,(\bar{\beta})$$

3.8. Attenuation in Rectangular Waveguide

Attenuation in the waveguides occurs due to (1) dielectric loss (2) conductor loss.

The power lost per unit length due to finite wall conductivity is given as

$$P_l = \frac{R_s}{2} \int_C |J_s|^2 \, dl \quad \text{....(3.50)}$$

where R_s is the wall surface resistance, and the integration contour C encloses the perimeter of the guide walls. The surface current on all four walls contribute to losses. We compute the power lost in the walls at $x = 0$ and $y = 0$, and double their sum to get the contribution from other two walls. The surface current on the $x = 0$ (left) walls is

$$J_s = \hat{n} \times H\Big|_{x=0} = -\hat{y} H\Big|_{x=0} = -\hat{y} C e^{-\bar{\beta} z} \quad \text{....(3.51)}$$

The surface current on the bottom wall at $y = 0$ is

$$J_s = \hat{n} \times H\Big|_{y=0} = \hat{y} \times \left(\hat{x} H_x\Big|_{y=0} + \hat{z} H_z\Big|_{y=0} \right)$$

$$= \hat{z} \frac{j\bar{\beta} a}{\pi} C \sin \frac{\pi x}{a} e^{-j\beta z} + \hat{x} C \sin \frac{\pi x}{a} e^{-j\bar{\beta} z} \quad \text{....(3.52)}$$

On substituting Eq. 3.52 in Eq 3.50, we obtain power loss as

$$P_l = R_s \int_{y=0}^{b} |J_{sy}|^2 \, dy + R_s \int_{x=0}^{a} \left[|J_{sx}|^2 + |J_{sz}|^2 \right] dx$$

$$= R_s \, |C|^2 \left(b + \frac{a}{2} + \frac{\bar{\beta}^2 a^3}{2\pi^2} \right) \quad \text{......(3.53)}$$

The attenuation due to conductor loss is obtained by perturbation method. According to this method the field magnitude decreases as $e^{-\alpha z}$ and the power decreases as $P_0 e^{-2\bar{\alpha} z}$, where P_0 is the power at the sending end $z = 0$.

The rate of decrease of power for any mode (TE_{mn} or TM_{mn}) is given by

$$-\frac{dP_{mn}}{dz} = 2\alpha_c P_{mn} = P_l \quad \text{....(3.53)}$$

where P_l is the power loss per unit length of the guide and P_{mn} is the power transmitted through the guide. Hence the attenuation constant is given by

$$\alpha_c = \frac{P_l}{2P_{mn}} \quad \text{....(3.54)}$$

We have already obtained P_1 and P_{10}, substituting these in Eq. 3.54 the attenuation for TE_{10} mode is given as

Attenuation versus frequency curves are drawn in Fig. 3.7.

$$\alpha_c = \frac{P_1}{2P_{10}} = \frac{2\pi^2 R_s\,[b + a/2 + (\beta^2 a^3/2\pi^2]}{\omega\mu a^3 b\beta} \qquad ....(3.55)$$

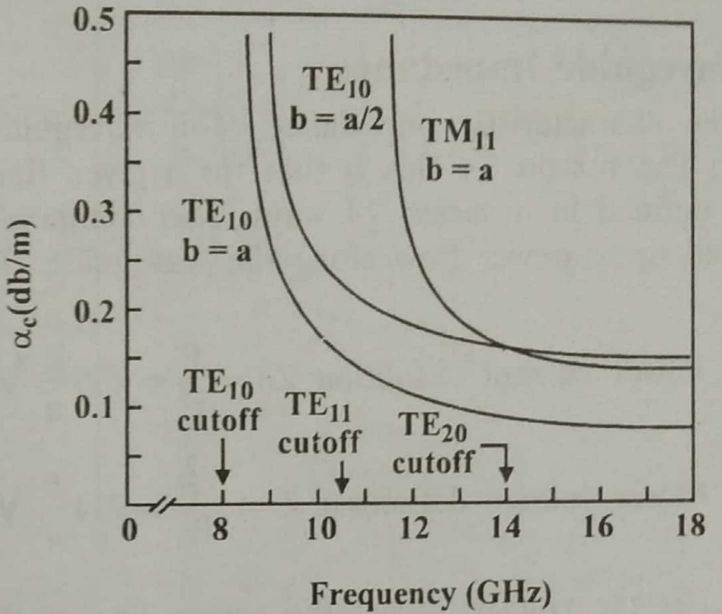


Fig. 3.7 Attenuation vs. frequency curves for various modes in a typical rectangular brass guide.

Attenuation due to dielectric loss is computed from the propagation constant, using complex dielectric constant

$$\bar{\gamma} = \bar{\alpha}_d + j\bar{\beta} = \sqrt{h^2 - \omega^2\mu\varepsilon}$$

$$= \sqrt{h^2 - \omega^2\mu\varepsilon\,(1 - j\tan\delta)} \qquad ....(3.56)$$

For most dielectric materials $\tan\delta << 1$, Eq. 3.56 can be written as

$$\bar{\gamma} = \sqrt{h^2 - \omega^2\mu\varepsilon + j\omega^2\mu\varepsilon\tan\delta}$$

$$= \sqrt{h^2 - \omega^2\mu\varepsilon}\left(1 + \frac{j\omega^2\mu\varepsilon\tan\delta}{\sqrt{h^2 - \omega^2\mu\varepsilon}}\right)^{1/2}$$

$$= \sqrt{h^2 - \omega^2\mu\varepsilon} + \frac{j\omega^2\mu\varepsilon\tan\delta}{2\sqrt{h^2 - \omega^2\mu\varepsilon}}$$

$$= \frac{\omega^2 \mu\varepsilon \tan\delta}{2/\bar{\beta}} + j\bar{\beta} \quad ....(3.57)$$

where $j\bar{\beta} = \sqrt{h^2 - \omega^2 \mu\varepsilon}$

From Eq. 3.57 it is observed that when the loss is small the phase constant $\bar{\beta}$ is unchanged, while the attenuation constant due to dielectric loss is given by

$$\bar{\alpha}_d = \frac{\omega^2 \mu\varepsilon \tan\delta}{2\bar{\beta}} \text{ Np/m} \quad \text{for both TE and TM modes.}$$

3.9. Waveguide Impedance

The characteristic impedance of a waveguide is not uniquely defined. The reason for this is that for a given field pattern, voltage can be defined in a variety of ways. The commonly used definitions are based upon power flow along the waveguide. These are

1. Power current definition $Z_0 = \frac{P}{I_z^2} = 465 \frac{b}{a} \sqrt{\frac{\mu}{\varepsilon}} \frac{\lambda_g}{\lambda}$ ohm

2. Power voltage definition $Z = \frac{V_{cl}^2}{P} = 754 \frac{b}{a} \sqrt{\frac{\mu}{\varepsilon}} \frac{\lambda_g}{\lambda}$ ohm

Here V_{cl} is the voltage across the center line of the waveguide and I_z is the longitudinal current.

3. Modified power voltage definition

$$Z_0 = 377 \frac{b}{a} \sqrt{\frac{\mu}{\varepsilon}} \frac{\lambda_g}{\lambda} \text{ ohms}$$

where $\lambda_g/\lambda = \frac{1}{\sqrt{1 - (f_c/f)^2}}$

Note that all these definitions differ only by a constant. Most commonly used definition is the modified power-voltage one.

3.10. Circular Waveguide

A hollow metal tube of circular cross-section also supports TE and TM waveguide modes. Fig. 3.8 shows a circular waveguide of inner radius *a*. We employ cylindrical coordinates when dealing with cylindrical waveguide with circular cross-section. The wave equation for this geometry is written as

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + \omega^2 \mu\varepsilon\psi = 0$$

For TM modes ψ stands for E_z and for TE modes as H_z.

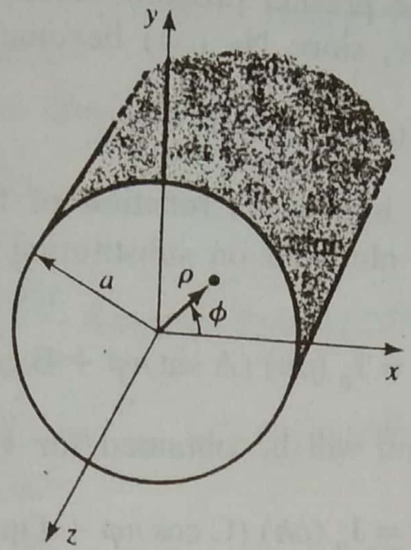


Fig. 3.8. Circular waveguide section

3.10.1. TM wave equations

In this case $H_z = 0$, the wave equation becomes

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \phi} + \omega^2 \mu \varepsilon \, E_z = 0 \quad ....(3.60)$$

A solution can be derived using the method of separation of variables as used for rectangular waveguides.

Hence $\quad E_z = P(\rho)\, Q(\phi)\, e^{-\bar{\gamma} z} = E_z^{\,0} e^{-\bar{\gamma} z}$ ....(3.61)

We obtain

$$Q\frac{d^2P}{d\rho^2} + \frac{1}{\rho P}\frac{dP}{d\rho} + \frac{1}{Q\rho^2}\frac{d^2Q}{d\phi^2} + h^2 = 0 \quad ....(3.62)$$

Equation 3.62 can be broken up into two differential equations

$$\frac{d^2Q}{d\phi^2} = -n^2 Q \quad ....(3.63)$$

$$\frac{d^2P}{d(\rho h)} + \frac{1}{(\rho h)}\frac{dP}{d(\rho h)} + \left[1 - \frac{n^2}{(\rho h)^2}\right] P = 0 \quad ....(3.64)$$

Eq. 3.63 has a solution

$$Q = (A \sin n\phi + B \cos n\phi) \quad ....(3.65)$$

Equation 3.64 is a standard form of Bessel's equation in terms of (ρh). The second order equation as usual will have two solutions $J_n(\rho h)$ and $N_h\,(\rho h)$, called Bessel functions of first and second kind respectively. For the present problem solution that is finite at $(\rho h) = 0$ is only acceptable, since N_h (ρh) becomes infinite at $\rho = 0$.

Hence, $\quad P(\rho h) = J_n(\rho h)$ ....(3.66)

where $J_n(\rho h)$ is Bessel's function of the first kind of order n. The value of E_z is obtained on substituting Eq. 3.65 and 3.66 in Eq. 3.61 as

$$E_z = J_n(\rho h)(A \sin n\phi + B \cos n\phi)\, e^{-\bar{\gamma} z} \quad ....(3.67)$$

Similar solution will be obtained for H_z (TE) mode.

$$H_z = J_n(\rho h)(C \cos n\phi + Dn \sin n\phi)\, e^{-\bar{\gamma} z} \quad ....(3.68)$$

3.10.2. TM modes in circular waveguides

$$E_z(\rho, \phi) = (A \sin n\phi + B \cos n\phi)\, J_n(h\rho)$$

For the TM mode is that the boundary conditions can now be applied directly to E_z, since

$$E_z(\rho, \phi) = 0 \text{ at } \rho = a \quad ....(3.69a)$$

Thus, we must have

$$J_n(ha) = 0 \quad ....(3.69b)$$

or

$$h_{nm} = \frac{P_{nm}}{a} \quad ....(3.70)$$

where P_{nm} is the mth root of $J_n(x)$, that is $J_n(P_{nm}) = 0$. Table 3.1 gives first few values of P_{nm}.

Table 3.1.

Values of P_{nm} for TM modes of a circular guide

n	P_{n1}	P_{n2}	P_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

The propagation constant of TM_{nm} mode is

$$\bar{\beta}_{nm} = \sqrt{k_0^2 - h_{nm}^2} = \sqrt{k_0^2 - (Pnm/a)^2} \quad ....(3.71a)$$

The cut-off frequency is

$$\bar{f}_{cnm} = \frac{h_{nm}}{2\pi\sqrt{\omega\varepsilon}} = \frac{P_{nm}}{2\pi a\sqrt{\omega\varepsilon}} \quad ....(3.71b)$$

The first TM mode to propagate is TM_{01} with $P_{01} = 2.405$. This is greater than $P'_{11} = 1.81$. Hence TE_{11} is the dominant mode of circular waveguide.

The transverse field components for TM_{01} are given as

$$E_\rho = \frac{-j\bar{\beta}}{h_{nm}} A \sin n\phi\, J_1'(h\rho)\, e^{-j\bar{\beta}z} \quad \text{....(3.72a)}$$

$$E_\phi = \frac{-j\bar{\beta}n}{h_{nm}^2 \rho} A \cos n\phi\, J_1(h\rho)\, e^{-j\bar{\beta}z} \quad \text{....(3.72b)}$$

$$H_\rho = \frac{j\omega\varepsilon n}{h_{nm}^2 \rho} A \cos n\phi\, J_1(h\rho)\, e^{-j\bar{\beta}z} \quad \text{....(3.72c)}$$

$$H_\phi = \frac{-j\omega\varepsilon}{h_{nm}} A \sin n\phi\, J_1'(h\rho)\, e^{-j\bar{\beta}z} \quad \text{....(3.72d)}$$

3.10.3. TE modes in circular waveguides

The boundary conditions to be met for TE waves are that $E_\phi = 0$ at $\rho = a$. But E_ϕ is proportional to $\frac{\partial H_z}{\partial \rho}$, and therefore to $J'_n(h\rho)$, where the prime denotes the derivative with respect to $(h\rho)$. Hence, for TE modes the boundary conditions required are

$$\frac{\partial J_n(ha)}{\partial a} = 0 = J_n'(ha) = 0 \quad \text{....(3.73)}$$

3.10.4. TE modes

If the roots of $J_n'(x)$ are defined as P'_{nm}, so that $J_n'(P'_{nm}) = 0$, where $P'_{nm} = 0$ is the m^{th} root of J'_n, then h_{nm} must have the value

$$h_{nm} = \frac{p'_{nm}}{a} \quad \text{....(3.74)}$$

First few values of P'_{nm} are given in table 3.2.

Table 3.2.

Values of P'_{nm} for TE Modes of a circular waveguide

n	P'_{n1}	P'_{n2}	P'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

The TE_{nm} modes are thus defined by the cutoff wave number $h_{nm} = P'_{nm}/a$ where n refers to number of full wave variation in the circumferential (Φ) direction and the second number (m) refers to the Bessel function variation in the radial direction. The propagation constant of the TE_{mn} mode is

$$\bar{\beta}_{nm} = \sqrt{\omega^2\mu\varepsilon - h^2} = \sqrt{k_0^2 - h_{nm}^2}$$

$$= \sqrt{k_0^2 - \left(\frac{P'_{nm}}{a}\right)^2}$$

The cutoff frequency is

$$f_{cnm} = \frac{h_{nm}}{2\pi\sqrt{\mu\varepsilon}} = \frac{P'_{nm}}{2\pi a\sqrt{\mu\varepsilon}} \quad ....(3.75)$$

The first mode (dominant mode) to propagate is the mode with the smallest P'_{nm} which from the table is TE_{11} mode. There is no TE_{10} mode in the circular waveguide but there exists a TE_{01} mode.

The transverse field components for TE_{11} mode are

$$H_z = D \sin\phi \, J_1(h\rho)\, e^{-j\bar{\beta}z} \quad ....(3.76a)$$

$$E_\rho = \frac{-j\omega\mu}{h_{nm}^2\rho} D \cos\phi \, J_1(h\rho)\, e^{-j\bar{\beta}z} \quad ....(3.76b)$$

$$E_\phi = \frac{j\omega\mu}{h_{nm}} D \sin\phi \, J'_1(h\rho)\, e^{-j\bar{\beta}z} \quad ....(3.76c)$$

$$H_\rho = \frac{-j\beta}{h_{nm}} D \sin\phi \, J'_1(h\rho)\, e^{-j\bar{\beta}z} \quad ....(3.76d)$$

$$H_\phi = \frac{-j\beta}{h_{nm}^2\rho} D \cos\phi \, J_1(h\rho)\, e^{-j\bar{\beta}z} \quad ....(3.76e)$$

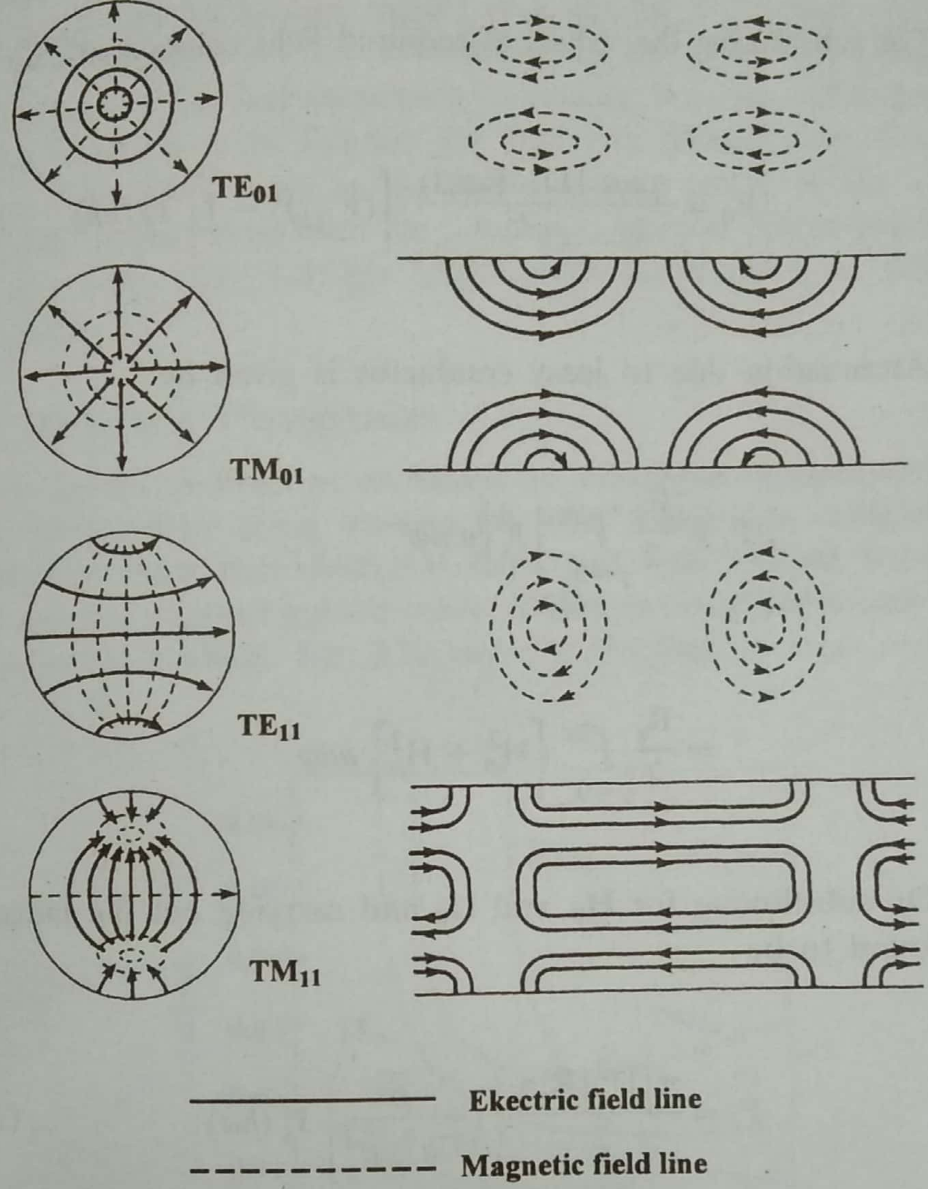
$$E_z = 0 \quad ....(3.76f)$$

Here we have assumed that the coordinate system has been rotated about z axis to give H_z a $\sin\phi$ variation hence, $\cos\phi$ variation is equated to zero.

The wave impedance is

$$Z_{TE} = \frac{H_\rho}{H_\phi} = \frac{-E_\phi}{H_\rho} = \frac{\eta k_0}{\bar{\beta}} = \frac{\eta}{\sqrt{1 - (f_c/f)^2}} \quad ....(3.77)$$

Field paterns for some low order modes in a circular waveguide are shown in Fig. 3.9a.



3.9a. Circular waveguide field patterns for lower order modes

3.11. Attenuation in Circular Guides

For TE modes the power flow down the guide can be computed as follows

$$P_0 = \frac{1}{2} \mathrm{Re} \int_{\rho=0}^{a} \int_{\theta=0}^{2\pi} E \times H^* . \hat{z} \rho \, d\phi \, d\rho$$

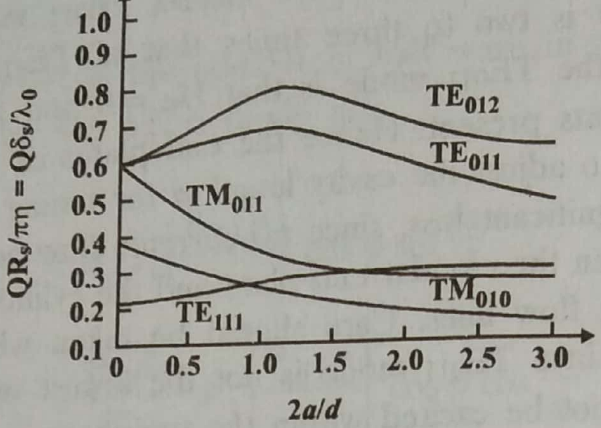


Fig. 3.15 Normalized Q for various cylinderical cavity modes.

Fig. 3.15 gives a plot of $Q\,\delta_s/\lambda_0$ versus $\left(\frac{2a}{d}\right)$ for several modes. It is evident from the figure that considerably higher value of Q is obtained for the TE_{011} mode compared to that for the TE_{111} mode. It is also clear that typical values of Q range from 10,000 to 40,000 or more.

Example 3.1. What is the width (*a*) of a rectangular waveguide whose characteristic impedance Z_0 is 408 ohms and which is required to propagate a 9.6 GHz signal in the TE_{10} mode ? **[4.09 cm.]**

Solution :

$$Z_0 = \frac{k_0\eta}{\beta}$$

$$\beta = \frac{377}{408}k_0 \;;\; k_0 = \frac{\omega}{c} = 2.01\ \text{cm}^{-1}$$

Also $$\beta^2 = k_0^2 - \left(\frac{\pi}{a}\right)^2$$

$$\left(\frac{\pi}{a}\right)^2 = k_0^2\left[1 - \left(\frac{377}{408}\right)^2\right] = 0.1462\,k_0^2$$

$$a^2 = \frac{\pi^2}{0.1462\,k_0^2} = 16.728$$

$$a = 4.09\ \text{cm}$$

Example 3.2. A rectangular waveguide of cross-section 5 cm × 2 cm is used to propagate TM_{11} mode at 9GHz. Determine the cut-off wavelength and wave impedance.

Solution :

For TM_{11} mode, $m = 1$, $n = 1$ at $f = 9\times 10^9$, $\lambda = 3.33$ cm.

$$\lambda_c = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}} = \frac{2ab}{\sqrt{a^2 + b^2}}$$

$$= \frac{2 \times 5 \times 2}{\sqrt{5^2 + 2^2}} = 3.714 \text{ cm}$$

$$Z_0 = 120\pi \sqrt{1 - (\lambda/\lambda_c)^2}$$

$$= 120\pi \sqrt{1 - \left(\frac{3.333}{3.714}\right)^2} = 325\Omega$$

Example 3.3. A waveguide operating in TE_{10} mode has dimensions $a = 2.26$ cm and b = 1 cm. The measured guide wave length is 4 cm. Find (a) the cut-off frequency of the propagating mode (b) the frequency of operation, (c) maximum frequency of propagation in this guide.

Solution :

(a) For TE_{10} mode, cut-off frequency f_c is

$$f_c = \frac{1}{2a\sqrt{\omega\varepsilon}} = \frac{c}{2a} = \frac{3 \times 10^{10}}{2 \times 2.26} = 6.64\text{GHz}$$

(b) $$\beta = \frac{2\pi}{\lambda_g} = \frac{2\pi}{4}$$

$$\beta_{10} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{\pi}{a}\right)^2}$$

$$\beta_{10}^2 + \left(\frac{\pi}{a}\right)^2 = \left(\frac{2\pi f}{c}\right)^2$$

$$f = \frac{c^2}{4\pi^2}\left[\beta^2 + \left(\frac{\pi}{a}\right)^2\right]$$

$$= 10 \text{ GHz.}$$

(c) First higher order mode is TE_{20} mode

Here, $m = 2$, $n = 0$

$$f_{20} = \frac{c}{2\pi}\sqrt{\left(\frac{2\pi}{a}\right)^2} = \frac{c}{a} = 13.27\text{GHz}$$

Example 3.4. A rectangular waveguide with dimensions a =2.5 cm, b =1 cm is to operate below 15.2 GHz. How many TE and TM modes can be propagated in the waveguide if the guide has the medium parameters given by $\sigma = 0$, $\varepsilon = 4\varepsilon_0$, $\mu_r = \mu_0$. (a) Calculate the cut off frequencies of the modes (b) Also calculate the phase constant, phase velocity and the wave impedance for TE_{10} mode operating at 15 GHz.

Solution:

(a) the cut off frequency is given by

$$f_{c_{mn}} = \frac{c'}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

Given $a = 2.5$ cm, $b = 1$ cm, $\varepsilon_r = 4$

$$c' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{2}$$

Hence,

$$f_{cmn} = \frac{c}{4a}\sqrt{m^2 + \frac{a^2}{b^2}n^2}$$

$$= \frac{3 \times 10^8}{4(2.5 \times 10^{-2})}\sqrt{m^2 + 6.25\, n^2}$$

$$= 3\sqrt{m^2 + 6.25\, n^2} \text{ GHz}$$

To find all $f'_{cmn} < 15$ GHz, we fix $m = 0$ and vary n. We obtain

for TE_{01} mode (m =0, n =1) $f_{c_{01}} = 7.5$ GHz

TE_{02} mode (m =0, n =2) $f_{c_{02}} = 15$ GHz

$f_{cmn} < 15.2$ GHz the maximum n =2.

Now fix n and increase m till $f_{c_{mn}} < 15.2$ GHz.

For TE_{10} mode (m =1, n =0) $f_{c_{10}} = 3$ GHz

TE_{20} mode, $f_{c_{20}} = 6$ GHz

TE_{30} mode, $f_{c_{30}} = 9$ GHz

TE_{40} mode, $f_{c_{40}} = 12$ GHz

TE_{50} mode, $f_{c_{50}} = 15$ GHz

TE_{60} mode, $f_{c_{60}} = 18$ GHz

That is for $f_{c_{mn}} < 15.2$ GHz, the maximum m =5. We have found the maximum m and n, other possible combinations in between these values of m and n are.

For TE$_{11}$, TM$_{11}$ (degenarate modes), $f_{c_{11}} = 8.078$ GHz

TE$_{21}$, TM$_{21}$ $f_{c_{21}} = 9 \cdot 6\ GH_z$

TE$_{31}$, TM$_{31}$ $f_{c_{31}} = 11.72$ GHz

TE$_{41}$, TM$_{41}$ $f_{c_{41}} = 14.14\ GH_z$

TE$_{12}$, TM$_{12}$ $f_{c_{12}} = 15.3$ GHz

Hence the modes for which the cut-off frequency is less than 15.2 GHz are eleven TE modes and four TM modes.

(b) For TE10 mode

$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$= \frac{2\pi \times 5 \times 10^9 \times 2}{3 \times 10^8}\sqrt{1 - \left(\frac{3}{15}\right)^2}$$

$$= 200\pi\sqrt{0.96} = 615.6 \text{ rad/m}$$

$$u_p = \frac{\omega}{\beta} = \frac{2\pi \times 15 \times 10^9}{615.6} = 1.531 \times 10^8\ m/s.$$

$$\eta_{TE} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$= 60\pi\sqrt{0.96} = 192.4\ \Omega$$

Example 3.5. A rectangular waveguide has a = 3.0 cm, b = 1.5 cm, $\mu = 1$ and $\varepsilon = 2.25$. Calculate (a) the cutoff wavelength and frequency for TE$_{10}$, TE$_{20}$ and TM$_{11}$ modes, (b) λ_g and Z$_o$ and Z$_o$ at 4.0 GHz.

Solution :

(a) $$\lambda_c = \frac{1}{\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}}$$

for TE$_{10}$, $m = 1,\ n = 0\ \ \lambda_c = 2a$

TE$_{20}$, $m = 2,\ n = 0\ \ \lambda_c = a$

TE$_{11}$, $m = 1,\ n = 1\ \ \lambda_c = \dfrac{2ab}{\sqrt{a^2 + b^2}}$

Thus the cut-off wavelengths :

for

$TE_{10} = 6$ cm

$TE_{20} = 3$ cm

$TM_{11} = 2.68$ cm

The cut-off frequencies are functions of the dielectric properties of the insulating material as well as the guide dimensions.

$$f_c = \frac{c}{\lambda_c \sqrt{\varepsilon\mu}}$$

substitute for $\varepsilon = 2.25$, $\mu = 1$ and different value of λ_c Then

$$f_{c_{10}} = 3.33 \text{ GHz}$$
$$f_{c_{20}} = 6.66 \text{ GHz}$$
$$f_{c_{11}} = 7.46 \text{ GHz}$$

(b) At 4.0 GHz , $\lambda_o = 7.5$ cm

$$\lambda_g = \frac{\lambda_o}{\sqrt{\mu\varepsilon - \left(\lambda_o/\lambda_c\right)^2}}$$

In this case $\varepsilon = 2.25$, $\mu = 1$ hence

$$\lambda_g = \frac{7.5}{\sqrt{2.25 - \left(\lambda_o/\lambda_c\right)^2}}$$

$$= 9.05 \text{ cm.}$$

λ_g for TE_{10} mode is 9.05 cm.

The characteristic impedance of the guide is calculated from the relation

$$Z_o = \frac{377 b/a \sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1 - \left(\frac{f_c}{f_o}\right)^2}} = 377\frac{b}{a}\sqrt{\frac{\mu}{\varepsilon}}\,\frac{\lambda_g}{\lambda}$$

$$= 377 \times \frac{1.5}{3.0}\sqrt{\frac{1}{2.25}} \times \frac{9.05}{7.5/\sqrt{2.25}} = 227 \text{ ohms}$$

Example 3.6. A rectangular waveguide has a cross-section of 1.5 cm $\times$ 0.8 cm, $\sigma = 0$, $\mu = \mu_0$, and $\varepsilon = 4\,\varepsilon_0$. The magnetic field component is given as

$$H_x = 2 \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{3\pi y}{b}\right) \sin(\pi \times 10^{11} t - \beta z) \ \text{A/m}$$

Determine :

(a) The mode of operation.

(b) The cut-off frequency.

(c) The phase constant β.

(d) The propagation constant γ.

(e) The intriasic wave impedance η .

Solution :

(a) From the expression for H_x, it is clear that $m = 1$, $n = 3$ that is the guide is operating in either TM_{13} or TE_{13} mode. We choose TE_{13} mode

(b)
$$f_{c13} = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{3}{b}\right)^2}$$

$$\frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{2}$$

Hence
$$f_{c_{13}} = \frac{c}{4} \sqrt{\left(\frac{1}{1.5\times10^{-2}}\right)^2 + \left(\frac{3}{0.8\times10^{-2}}\right)^2}$$

$$= \frac{3\times10^8}{4} (\sqrt{0.444 + 14.03}) \times 10^2 = 28.57 \text{ GHz}$$

(c) Phase constant
$$\beta = \omega\sqrt{\mu\varepsilon} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

Here, $\omega = \pi \times 10^{11}$ or $f = 50$ GHz

$$\text{or } \beta = \frac{\pi \times 10^{11}(2)}{3\times10^8} \sqrt{1 - \left(\frac{28.57}{50}\right)^2} = 1718.81 \ \text{rad/m}$$

(d) Propogation constant, $\gamma = j\beta = j1718.81/\text{m}$

$$\eta TE_{13} = \frac{\eta}{\sqrt{1 - (f_c/f)^2}} = \frac{377}{\sqrt{\varepsilon_r}} \left[1 - \left(\frac{28.57}{50}\right)^2\right]^{½}$$

$= 229.69 \ \Omega$

Example 3.7. An air-filled rectangular wave with cross-section of 4 cm × 2 cm transports energy in the dominant mode (TE_{10}) at a rate of 2 mW. If the frequency of operations 10GHz, determine the peak value of the electric field in the waveguide.

Solution :

The field expressions corresponding to TE_{10} mode are

$$E_x = 0, \quad E_y = -\frac{j\omega\mu a c}{\pi}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$f_c = \frac{c}{2a} = \frac{3\times10^8}{2\left(4\times10^{-2}\right)} = 3.75 \text{ GHz}$$

$$\eta = \frac{\eta_0}{\sqrt{1-\left(\frac{f_c}{f_0}\right)^2}} = 406.7\ \Omega$$

The average power transmitted is

$$P_{ave} = \int_{y=0}^{b}\int_{x=0}^{a}\frac{\left|E_y\right|^2}{2\eta}\,dx\,dy$$

$$= \frac{E_s^2}{2\eta}\int_0^b dy\int_0^a \sin^2\left(\frac{\pi x}{a}\right)dx$$

where $E_o = \frac{\omega\mu a}{\pi}c$

$$\therefore \quad P_{ave} = \frac{E_o^2 ab}{4\eta}$$

Therefore,

$$E_o^2 = \frac{4\eta P_{ave}}{ab} = 4\left(406.7\right)\times2\times\frac{10^{-3}}{8\times10^{-4}}$$

$$= 4067$$

or $\quad E_0 = 63.77$ V/m

Example 3.8. A copper waveguide has dimensions a = 2.286 cm, b= 1.016 cm and is operating at a frequency f=10GHz in TE_{10} mode. For a guide of 1 m length, find the attenuation in dB.

Solution :

At 10GHz, the propagation constant of the TE_{10} mode is

$$\beta = \sqrt{k_o^2 - \left(\frac{\pi}{a}\right)^2}$$

$$k_o^2 = \omega^2\mu\varepsilon = \left(\frac{2\pi f}{c}\right)^2 = 209.44\ m^{-1}$$

$$a = .02286\ m$$

$$\beta = 158.05\ \text{m}^{-1}$$

For copper walls ($\sigma = 5.8\times10^7$ s/m)

$$\text{Hence} \quad R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = 0.026\ \Omega$$

So the attenuation constant from Eq. 3.55, is

$$\alpha_C = \frac{R_s}{a^3 b\, \beta\, k_o\, \eta}\left(2b\pi^2 + a^3k_o^2\right) = 0.0125\ N_p/m$$

$$\alpha_C(\text{dB}) = -20 \log e^{-\alpha_C} = 0.11\ \text{dB/m.}$$

Example 3.9. An air-filled resonant cavity with dimensions a = 5 cm, b = 4 cm, and c = 10 cm is made of copper $\left(\sigma_c = 5.8\times10^7\text{mhos/m}\right)$. It is filled with a lossless material $(\mu_r = 1)$, $(\varepsilon_r = 3)$. Find the resonant frequency f_r and the quality factor for TE_{101} mode.

Solution :

$$\text{Resonant frequency } f_r = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{l}{c}\right)^2}$$

$$\frac{1}{2\sqrt{\mu\varepsilon}} = \frac{c}{2\sqrt{3}} = \frac{1.5\times10^8}{1.732}$$

$$f_{r_{101}} = \frac{1.5\times10^8}{1.732}\left[\left(\frac{1}{5\times10^{-2}}\right)^2 + \left(\frac{1}{10\times10^{-2}}\right)^2\right]^{1/2}$$

$$= 1.936\text{GHz}$$

(b) The quality factor for TE_{101} mode is given as

$$Q_{TE101} = \frac{\left(a^2 + c^2\right) ab\, c}{\delta\left[2b\left(a^3 + c^3\right) + ac\left(a^2 + c^2\right)\right]}$$

$$= \frac{(25 + 100)\ 200 \times 10^{-2}}{\delta\left[\delta\left(125 + 1000\right) + 50\left(25 + 100\right)\right]}$$

$$= \frac{1}{61\delta} = \frac{\sqrt{\pi f_{101} \mu_o \sigma_c}}{61}$$

$$= 10.93 \times 10^3$$

Select Bibliography

1. Ramo, S., Whinnery, J.R. and Van Duzer, T.; **Fields and waves in communication Electronics,** John Wiley and Sons, 1965.
2. Jordan, E.C. and Balmain, K.G. **Electromagnetic waves and Radiating systems,** Prentice Hali Inc., 1968.
3. Collen, R.E., Foundations for **Microwave Engineering. McGraw-hill Inc. New York, 1992.**
4. Sisodia, M.L., Raghuvanshi, G.S. Microwave Circuits and passive Devices New Age Int. (P) Ltd., 2001

Review Questions and Problems

1. What is meant by a cut off frequency of a waveguide?
2. Explain why single conductor hollow or dielectric filled waveguide cannot support TEM waves.
3. Can a waveguide have more than one cuttoff frequency? On what factors does the cutoff frequency of a waveguide depend.
4. Is the guide wavelength of a propagating wave in a waveguide longer or shorter than the wavelength in the corresponding unbounded dielectric medium?
5. What is meant by the dominant mode of a waveguide? What is the dominant mode of a parallel plate waveguide?
6. Does the attenuation constant due to dielectric losses increase or decrease with frequency for TM and TE modes in a parallel plate waveguide?
7. Discuss the general attenuation behaviour caused by wall losses as a function of frequency for the TE_{10} mode in a rectangular waveguide.