Numerical solution of an ordinary differential equation :→

Here we find the solution of the differential eqn $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ — (1)

by using the following methods,

(1) Taylor series method

Here soln of (1) is given by the Taylor series, about $x = x_0$ as,

$$y(x) = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \cdots$$

where $(y^n)_0 = \left(\frac{d^n y}{dx^n}\right)_{(x_0,y_0)}$

Eg Solve $\frac{dy}{dx} = x - y^2$ given $y(0) = 1$ &

find $y(0.1)$ correct to 4 decimal places.

Soln Here $x_0 = 0$, $y_0 = 1$ ∴ Taylor series is given by,

$$y(x) = y_0 + (x-0)(y')_0 + \frac{(x-0)^2}{2!}(y'')_0 + \frac{(x-0)^3}{3!}(y''')_0 + \cdots \quad (2)$$

$$\Rightarrow y(x) = 1 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + \cdots \quad (14)$$

where $y_0 = 1$

$$(y')_0 = (2x - y^2)_0 = 0 - y_0^2 = -1$$

$$(y'')_0 = (1 - 2y \cdot y')_0 = 1 - 2y_0 y_0' = 3$$

$$(y''')_0 = (-2y'^2 - 2y \cdot y'')_0 = -2y_0'^2 - 2y_0 y_0''$$

$$= -2(-1)^2 - 2(1)(3)$$

$$= -8.$$

$$(y^{iv})_0 = (-4y'y'' - 2y'y'' - 2yy''')_0$$

$$= -4y_0'y_0'' - 2y_0'y_0'' - 2y_0 y_0'''$$

$$= 34$$

$$\therefore (2) \Rightarrow y(x) = 1 + x(-1) + \frac{x^2}{2!}(3) + \frac{x^3}{3!}(-8) + \frac{x^4}{4!}(34) + \cdots$$

$$= 1 - x + \frac{3x^2}{2!} - \frac{8x^3}{3!} + \frac{34x^4}{4!} - \cdots$$

$$y(0.2) = 1 - (0.2) + \frac{3}{2!}(0.2)^2 - \frac{8}{3!}(0.2)^3 + \frac{34}{4!}(0.2)^4 = \underline{0.9138}$$ (upto 4th deg., neglecting higher powers)

Eg $y' = 4y$, $y(0) = 1$, find $y(0.2)$.

Taylor series soln is given by,

$$y(x) = y_0 + x y_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{4!} y_0^{iv} + \cdots$$

where $y_0 = 1$

$(y_0)' = 4y_0 = 4$

$(y_0)'' = 4y_0' = 4(4) = 16$

$(y_0)''' = 4y_0'' = 4(16) = 64$ & so on—

$$\therefore y(x) = 1 + 4x + \frac{16x^2}{2!} + \frac{64x^3}{3!} + \cdots$$

$$y(0.2) = 1 + 4(0.2) + 8(0.2)^2 + \frac{64}{6}(0.2)^3$$

$= 2.2053$ (neglecting higher order terms)

(2) Picard's method of successive approximation

Consider the differential eqn $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$. Here each approximate solution is given by,

$$\boxed{y_n = y_0 + \int_{x_0}^{x_n} f(x, y_{n-1})\cdot dx} \quad \text{—(1)} \quad \forall\, n = 1, 2, 3, \ldots$$

Eg. Use Picard's app. method to solve $\frac{dy}{dx} = 2x - y$ upto 3rd approximation, given $y(0) = 0.9$. Also check the result with exact value.

Sol^n Here $f(x, y) = 2x - y$. , $x_0 = 0$, $y_0 = 0.9$

First appx. sol^n (Put n = 1 in (1))

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\cdot dx$$

$$= 0.9 + \int_0^x (2x - y_0)\cdot dx$$

$$= 0.9 + \int_0^x (2x - 0.9)\, dx$$

$$= 0.9 + x^2 - 0.9x.$$

Second appx. sol^n (Put n = 2 in (1))

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$\Rightarrow y_2 = y_0 + \int_{x_0}^{x} (2x - y_1)\cdot dx$$

$$= y_0 + \int_{x_0}^{x} \left[2x - (0.9 + x^2 - 0.9x)\right] dx$$

$$= 0.9 - 0.9x + \frac{2.9}{2}x^2 - \frac{x^3}{3}$$

<u>Third appx. sol^n</u> (Put $n = 3$ in (1))

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\cdot dx$$

$$= 0.9 + \int_{0}^{x} (2x - y_2)\, dx$$

$$= 0.9 + \int_{0}^{x} 2x - \left(0.9 + 0.9x - \frac{2.9}{2}x^2 + \frac{x^3}{3}\right) dx$$

$$= 0.9 - 0.9x + 1.45x^2 - 1.45\frac{x^3}{3} + \frac{x^4}{12} \quad \text{--- (2)}$$

For exact value,

Solving $\frac{dy}{dx} = 2x - y$.

$$\Rightarrow \frac{dy}{dx} + y = 2x \quad , \quad I.f. = e^{\int 1 \cdot dx} = e^x$$

$$\Rightarrow y(e^x) = \int 2x e^x + c = -2e^x + 2xe^x + c$$

$$\Rightarrow y = -2 + 2x + ce^{-x}$$

$y(0) = 0.9 \Rightarrow 0.9 = -2 + 2(0) + c$

$\Rightarrow c = 2.9$

$\therefore y = -2 + 2x + 2.9e^{-x}$

$= -2 + 2x + 2.9\left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - - -\right]$

$= -2 + 2x + 2.9 - 2.9x + \frac{2.9}{2}x^2 - \frac{2.9}{3!}x^3$

$+ \frac{2.9}{4!}x^4 - - - -$

$= 0.9 - 0.9x + 1.45x^2 - \frac{1.45}{3}x^3 + \frac{1.45}{12}x^4$

$- - - \rightarrow (3)$

Approximate solⁿ (2) is same as exact solⁿ (3) upto the term in x^3.

Eg: Solve by Picard's approximate method upto 3rd approximate.

$\frac{dy}{dx} = xy \quad y(1) = 2$.

Solⁿ: Here $x_0 = 1$, $y_0 = 2$.

$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0) = 2 + \int_1^x 2x\,dx$

$= 1 + x^2$.

$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1).dx = 2 + \int_1^x x(x^2+1)\,dx$

$$= \frac{5}{4} + \frac{x^2}{2} + \frac{x^4}{4}.$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2) \cdot dx$$

$$= y_0 + \int_{x_0}^{x} x \left(\frac{5}{4} + \frac{x^2}{2} + \frac{x^4}{4}\right) dx$$

$$= \frac{29}{24} + \frac{5x^2}{8} + \frac{x^4}{8} + \frac{x^6}{24}$$

③ <u>Euler's method of successive approx.</u>

<u>Solution</u>

Consider the ordinary differential eqn $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ then each app.x. sol^n is given by,

$$y_{n+1} = y_n + h\, f(x_n, y_n) \quad \forall\ n = 0, 1, 2, \ldots$$

where $y_{n+1} = y[x_0 + (n+1)h]$,

$$y_n = y(x_0 + nh)$$

Eg. Solve $\frac{dy}{dx} = x^3 + y$, $y(0) = 1$, $h = .01$ using Euler's method. Also find $y(0.02)$.

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Solⁿ: Here $f(x,y) = x^3+y$

$x_0 = 0 \quad y_0 = 1 \quad , h = 0.01$

$x_1 = x_0 + h = 0.01$

$x_2 = x_0 + 2h = 0.02$

$y_1 = y_0 + h f(x_0, y_0)$

$= 1 + (0.01)[0^3+1] = 1.01$

$y_2 = y_1 + h f(x_1, y_1)$

$= 1.01 + (0.01)(0.01^3 + 1.01)$

$= 1.0201$

$y_2 = y(x_0 + 2h) = y(0.02) = 1.0201.$

Eg: Use Euler's method to find $y(0.1)$ from the following diff. eqn:

$$\frac{dy}{dx} = \frac{y-x}{y+x}, \quad y(0) = 1, \ h = 0.02.$$

Solⁿ $x_0 = 0 \ , \ y_0 = 1$

$x_0 + h = 0 + 0.02 = 0.02$

$x_0 + 2h = 0 + 0.04 = 0.04$

$x_0 + 3h = 0.06, \quad x_0 + 4h = 0.08$

$x_0 + 5h = 0.1$

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 1 + (0.02)\left(\frac{1-0}{1+0}\right) = 1.02$$

Similarly

$$y_2 = y(0.04) = y_1 + h\, f(x_1, y_1) = 1.0392$$

$$y_3 = y(0.06) = y_2 + h\, f(x_2, y_2) = 1.0577$$

$$y_4 = y(0.08) = y_3 + h\, f(x_3, y_3) = 1.0756$$

$$y_5 = y(0.1) = y_4 + h\, f(x_4, y_4) = 1.0928.$$

(4) **Modified Euler's method :-**

Consider the differential eqn, $\frac{dy}{dx} = f(x, y)$;

$y(x_0) = y_0$,.

By Euler's method, first approximate soln is given by, $y_1 = y_0 + h\, f(x_0, y_0)$ — (1)

we modify this value to a better value, using Euler's modified method, as

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1)\right] \quad \text{— (2)}$$

(1) is known as "predictor" & (2) is known as "corrector".

Again, when corrector is applied, we find better value of $y_1^{(1)}$ as $y_1^{(2)}$, given by,

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_1^{(1)})\right]$$

After obtaining y_1 upto desired accuracy, we begin finding y_2 as

$$y_2 = y_1 + h\, f(x_1, y_1) \rightarrow \text{predictor}$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_0+h, y_1) + f(x_0+2h, y_2)\right]$$

↳ corrector

where $x_1 = x_0 + h$.

and so on for $y_2^{(2)}, y_2^{(3)}, \ldots$ till desired accuracy.

Similarly we find $y_3, y_4, \ldots$, & their modified values, whenever required.

Eg:-> Use modified Euler's method to solve the eqn

$$\frac{dy}{dx} = \log(x+y) \; ; \; y(1) = 2$$

for (i) $x = 1.2$

(ii) $x = 1.4$, correct to 3 decimal places.

Solⁿ (i). To find $y(1.2)$.

given $x_0 = 1$, $y_0 = 2$, $f(x,y) = \log(x+y)$

Let $h = 0.2$ then

$x_0 = 1 \Rightarrow y_0 = y(x_0) = y(1)$

$x_1 = x_0 + h = 1.2 \Rightarrow y_1 = y(x_0+h) = y(1.2)$

$\therefore$ we have to modify y_1 upto desired accuracy.

$$y_1 = y(1.2) = y_0 + h f(x_0, y_0)$$
$$= 2 + 0.2 \log(x_0 + y_0)$$
$$= 2 + 0.2 \log 3 = 2.2197$$

By corrector formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0+h, y_1)]$$
$$= 2 + \frac{0.2}{2}[\log(1+2) + \log(1.2 + 2.2197)]$$
$$= 2 + 0.1[\log 3 + \log(3.4197)]$$
$$= 2.2328.$$

Again applying corrector,

$$y_1^{(2)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_0+h, y_1^{(1)})\}$$

$$y_1^{(2)} = 2 + \frac{0.2}{2}\left[\log(3) + \log(1.2 + 2.2328)\right]$$ (19)

$$= 2.2332.$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_1^{(2)})\right]$$

$$= 2 + \frac{0.2}{2}\left[\log(1+2) + \log(1.2 + 2.2332)\right]$$

$$= 2.2332.$$

Since $y_1^{(2)}$, $y_1^{(3)}$ are same upto 4 places of decimals, value will repeat further.

So modified value of $y(1.2)$ is 2.2332.

(ii) To find $y(1.4)$

$x_0 = 1$ $\Rightarrow y_0 = y(1)$

$x_0 + h = 1 + 0.2 = 1.2$ $\Rightarrow y_1 = y(1.2)$

$x_0 + 2h = 1 + 0.4 = 1.4$ $\Rightarrow y_2 = y(1.4)$

$\therefore$ we have to modify y_2 upto desired result.

$$y_2 = y(1.4) = y_1 + h\, f(x_1, y_1)$$

$$= 2.2332 + 0.2 \log(1.2 + 2.2332)$$

$$= 2.47917 \rightarrow \text{Predicted value.}$$

Corrector formula gives,

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_1+h, y_2)\right]$$

$$= y_1 + \frac{h}{2}\left[f(x_0+h, y_1) + f(x_0+2h, y_2)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[\log(1.2 + 2.2332) + \log(1.4 + 2.47917)\right]$$

$$= \underbrace{2.4924}$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_0+h, y_1) + f(x_0+2h, y_2^{(1)})\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[\log(1.2 + 2.2332) + \log(1.4 + 2.4924)\right]$$

$$= \underbrace{2.4924}.$$

Again as by part (ii), accurate value of $y_2 = y(1.4)$ is 2.4924

Eg Use modified Euler's method to obtain $y(0.2)$, $y(0.4)$ correct to three decimal places, given that $\frac{dy}{dx} = y - x^2$ with $y(0) = 1$.

Sol^n $x_0 = 0,\ y_0 = 1 \quad f(x,y) = y - x^2$

(i) ~~x_0~~ $y(0.2) = ?$

Let $h = 0.2$

then $x_1 = x_0 + h = 0.2$

$\Rightarrow y_1 = y(x_0+h) = y(0.2)$

$\therefore$ Predicted value of y_1,

$$y_1 = y(0.2) = y_0 + h\, f(x_0, y_0)$$
$$= 1 + 0.2\, f(0,1)$$
$$= 1 + 0.2\,(1^2 - 0) = 1.2$$

Improved value,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_0+h, y_1)\right]$$
$$= 1 + \frac{0.2}{2}\left[(1-0) + (1.2 - 0.2^2)\right]$$
$$= 1.216.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_0+h, y_1^{(1)})\right]$$
$$= 1 + \frac{0.2}{2}\left[(1-0) + (1.216 - (0.2)^2)\right]$$
$$= \underline{1.217}6.$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_0+h, y_1^{(2)})\right]$$
$$= \underline{1.217}$$

$\therefore$ $y_1^{(2)}$ & $y_1^{(3)}$ are same ~~corr~~ upto 3 places of decimals

$\therefore$ $\boxed{y(0.2) = 1.217}$ Ans.

Similarly we can find $\boxed{y_2 = y(0.4) = 1.461}$

⑤ Runge-Kutta method of 4th order.

Let the eqn be, $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

Calculate, $K_1 = h f(x_0, y_0)$

$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$K_4 = h f(x_0 + h, y_0 + K_3)$

then Compute, $K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

So we obtain $\boxed{y_1 = y_0 + K = y(x_0 + h)}$

K is known as weighted mean of K_1, K_2, K_3 & K_4.

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Eg. Using Runge Kutta method of 4th order determine $y(0.1)$ & $y(0.2)$ correct to 4 decimal places given that $\frac{dy}{dx} = y - x$ where $y(0) = 2$

$h = 0.1$.

Sol^n given, $x_0 = 0$, $y_0 = 2$, $h = 0.1$

$f(x,y) = y - x$

$y(0.1) = y(x_0 + h) = y_1$

$\therefore\ K_1 = h f(x_0, y_0) = 0.1(2-0) = 0.2$

$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= 0.1\left(y_0 + \frac{K_1}{2} - \left(x_0 + \frac{h}{2}\right)\right)$

$= 0.1\left[\left(2 + \frac{0.2}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right]$

$= 0.205$

$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= h\left\{y_0 + \frac{K_2}{2} - \left(x_0 + \frac{h}{2}\right)\right]$

$\Rightarrow K_3 = 0.1\,[2.1025 - 0.05] = 0.20525$

$K_4 = h\,f(x_0+h,\ y_0+K_3) = h\,[y_0+K_3 - (x_0+h)]$

$= 0.1\,[2.205 - 0.1]$

$= 0.2105$

$\therefore K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}[0.2 + 0.41 + 0.~~4105~~ + .2105]$

$= 0.205$

$\therefore \underline{y(0.1) = y_0 + K = 2.205.} = y_1$

<u>To find</u> $y(0.2) = y(x_0+2h) = y_2$

Here $K_1 = h\,f(x_1, y_1) = 0.1\,f(0.1, 2.205)$

$= 0.2105.$

$K_2 = h\,f(x_1+\frac{h}{2},\ y_1+\frac{K_1}{2}) = 0.1\left[y_1+\frac{K_1}{2} - (x_1+\frac{h}{2})\right]$

$= 0.21525.$

$K_3 = h\,f\left(x_1+\frac{h}{2},\ y_1+\frac{K_2}{2}\right)$

$= 0.21625$

$K_4 = h\,f(x_1+h,\ y_1+K_3) = 0.22125$

$\Rightarrow K = 0.2158$ Hence $y_2 = y(0.2) = y_1 + K = \underline{\underline{2.4208}}$

⑥ Adams Method

$$y_{i+1}^{p} = y_i + \frac{h}{24}\left(55 f_i - 59 f_{i-1} + 37 f_{i-2} - 9 f_{i-3}\right)$$

← Predictor formula

$$y_{i+1}^{c} = y_i + \frac{h}{24}\left(f_{i-2} - 5 f_{i-1} + 19 f_i + 9 f_{i+1}\right)$$

where $x_i = x_0 + ih$

$y_i = y(x_i)$ — obtain by RK. method

$f_i = f(x_i, y_i)$

Eg. Given $\frac{dy}{dx} = \frac{2y}{x}$, $y(1) = 2$. estimate $y(2)$, using the Adams method, $h = 0.25$.

Sol^n By Adams predictor formula,

$x_1 = 1 + .25 = 1.25$ $\quad y(x_1) = y_1 = 3.13$

$x_2 = 1 + 2(.25) = 1.5$ $\quad y(x_2) = y_2 = 4.50$

$x_3 = 1 + 3(.25) = 1.75$ $\quad y(x_3) = y_3 = 6.13$

$x_4 = 1 + 4(.25) = 2$ $\quad y(x_4) = y_4$.

we have to determine y_4.

$$y_4^{p} = y_3 + \frac{0.25}{24}\left(55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right)$$

$f_1 = 5.01$, $f_2 = 6$, $f_3 = 7.01$, $f_0 = 4$

$\therefore \quad y_4^p = 8.0146$

$f_4 = \dfrac{2 \times 8.0146}{2} = 8.0146$

$y_4^c = y_3 + \dfrac{h}{24}(f_1 - 5f_2 + 19f_3 + 9f_4^p)$

$= 6.13 + \dfrac{0.25}{24}(5.01 - 5 \times 6 + 19 \times 7.01 + 9 \times 8.0146)$

$= 8.0086.$ Ans.

(7) Milne - Simpson method.

$y_{i+1}^p = y_{i-3} + \dfrac{4h}{3}(2f_{i-2} - f_{i-1} + 2f_i)$

$y_{i+1}^c = y_{i-1} + \dfrac{h}{3}(f_{i-1} + 4f_i + f_{i+1})$

Eg Solve above question by Milne - Simpson method.

Sol^n In order to estimate $y(2)$, we need to find y_4.

$\therefore \quad y_4^p = y_0 + \dfrac{4h}{3}(2f_1 - f_2 + 2f_3)$

$= 2 + \dfrac{4(0.25)}{3}[2(5.01) - 6 + 2(7.01)]$

$= 8.01$

$f_4 = \dfrac{2 \times 8.01}{2} = 8.01$

Numerical Solution of Partial differential equations (P.d.e.)

General form of Pde:-

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \quad \text{--- (1)}$$

If $B^2 - 4AC < 0$ then (1) is s.t.b. elliptic.

If $B^2 - 4AC > 0$ " " " " Hyperbolic.

If $B^2 - 4AC = 0$ " " . Parabolic

Eg a) $2u_{xx} + 4u_{xy} + 3u_{yy} = 2$ -(1)

Here $B^2 - 4AC = 16 - 24 < 0$ $\therefore$ (1) is elliptic.

b) $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 0$

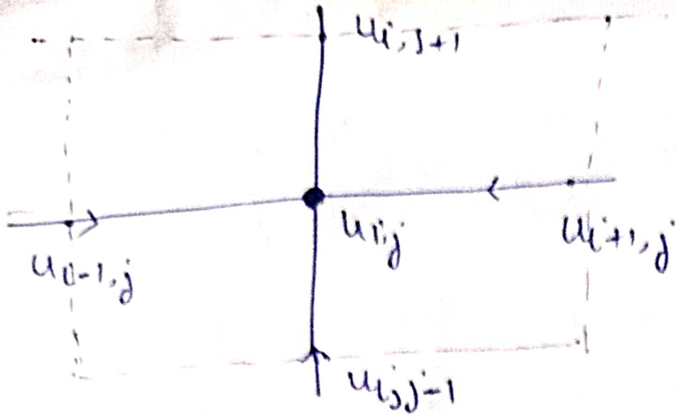
Here $B^2 - 4AC = 16 - 16 = 0$

$\therefore$ Parabolic.

Consider a Laplace eqn $u_{xx} + u_{yy} = 0$, we have;

(1) Standard five pt. formula (SFPF)

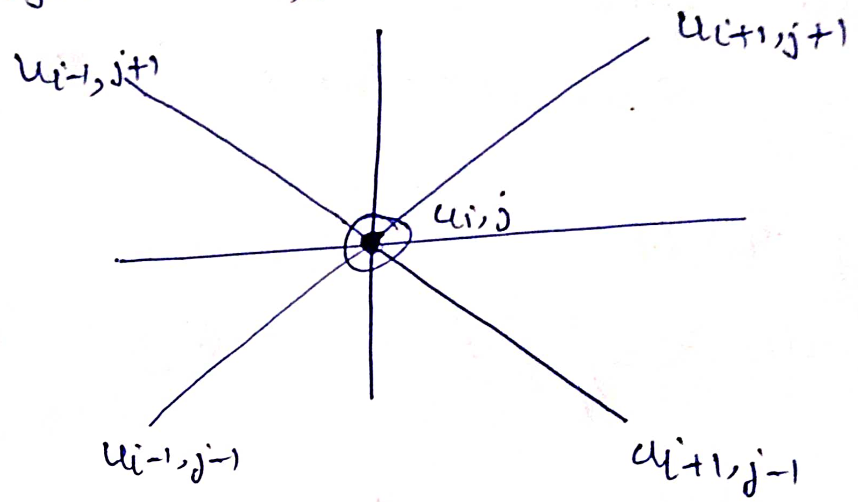
$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$



$u_{i,j}$'s are known as mesh points

② <u>Diagonal five pt. formula</u> (DFPF)

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right]$$



First we find the solution by SFPF or DFPF then we find the app. value by Iterative methods as follows:

1) <u>Jacobi iterative method</u> :-

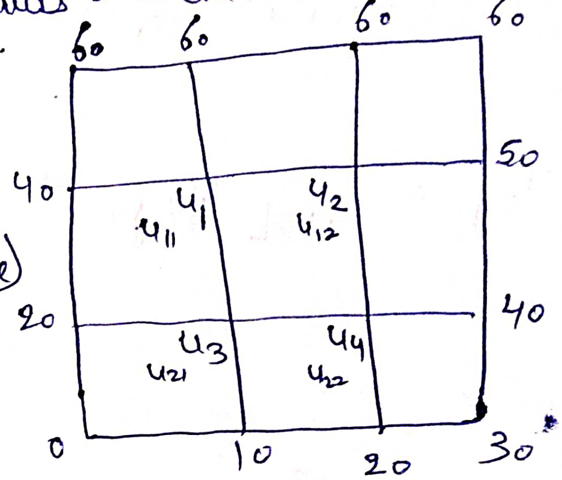
If $u_{i,j}^{(n)}$ denotes the n^{th} iterative value of $u_{i,j}$ for interior mesh points, then iterative formula is,

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)}\right]$$

2) Gauss Seidal method

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right]$$

Eg Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown by Gauss - Seidal method upto 2nd iteration.



Sol^n let $u_4 = 0$

we find the initial sol^n by SFPF or DFPF. (as applicable)

By DFPF,

$$u_1 = \frac{1}{4}[60 + 0 + 60 + 20] = 35$$

By SFPF,

$$u_2 = \frac{1}{4}[\overset{u_1}{35} + 50 + 60 + u_4] = 36.25$$

$$u_3 = \frac{1}{4}[u_1 + 10 + 20 + u_4] = 16.25.$$

$$u_4 = \frac{1}{4}[u_2 + u_3 + 40 + 20] = 28.125$$

(As $u_4 = 0$ was initially assumed to compute u_1, u_2, u_3)

$\therefore$ Initial sol^n is,

$$\left.\begin{aligned} u_1 &= u_1^{(0)} = 35 \\ u_2 &= u_2^{(0)} = 36.25 \\ u_3 &= u_3^{(0)} = 16.25 \\ u_4 &= u_4^{(0)} = 28.125 \end{aligned}\right\} \quad -(1)$$

Now we find iterative soln by Gauss Seidel eqn as,

$(n+1)^{th}$ iteration,

$$\left.\begin{aligned}
u_1^{(n+1)} &= \frac{1}{4}\left[40 + 60 + u_2^{(n)} + u_3^{(n)}\right] \\
u_2^{(n+1)} &= \frac{1}{4}\left[60 + u_4^{(n)} + 50 + u_1^{(n+1)}\right] \\
u_3^{(n+1)} &= \frac{1}{4}\left[u_1^{(n+1)} + 10 + 20 + u_4^{(n)}\right] \\
u_4^{(n+1)} &= \frac{1}{4}\left[u_2^{(n+1)} + 20 + 40 + u_3^{(n+1)}\right]
\end{aligned}\right\} \quad \text{--- (2)}$$

1st iteration (Put $n=0$) (By (1) & (2), we have)

$$u_1^{(1)} = \frac{1}{4}\left[40 + 60 + u_2^{(0)} + u_3^{(0)}\right] = 38.125$$

$$u_2^{(1)} = \frac{1}{4}\left[60 + u_4^{(0)} + 50 + u_1^{(0)}\right] = 44.0625$$

$$u_3^{(1)} = \frac{1}{4}\left[u_1^{(1)} + 10 + 20 + u_4^{(0)}\right] = 24.062$$

$$u_4^{(1)} = \frac{1}{4}\left[u_2^{(1)} + 20 + 40 + u_3^{(1)}\right] = 32.0313$$

2nd iteration (Put $n=1$ in (2))

$$u_1^{(2)} = \frac{1}{4}\left[40 + 60 + u_2^{(1)} + u_3^{(1)}\right] = 42.0313$$

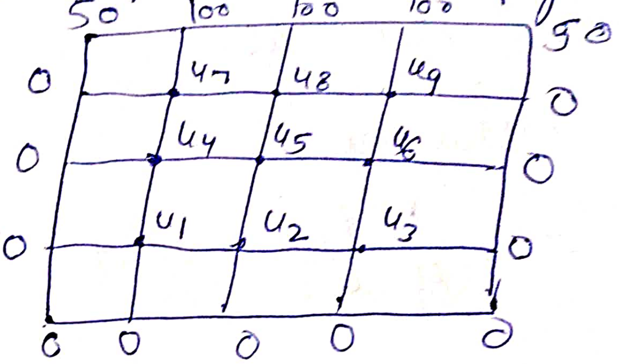
$$u_2^{(2)} = \frac{1}{4}\left[60 + u_4^{(1)} + 50 + u_1^{(2)}\right] = 46.0156$$

$$u_3^{(2)} = \frac{1}{4}\left[u_1^{(2)} + 10 + 20 + u_4^{(1)}\right] = 26.0156$$

$$u_4^{(2)} = \frac{1}{4}\left[u_2^{(2)} + 20 + 40 + u_3^{(2)}\right] = 33.0078$$

Eg Solve $u_{xx}+u_{yy}=0$ for the figure given below:

upto 3rd iteration, using Gauss seidal method.



(Ans $u_1^{(3)} = 7.13$, $u_2^{(3)} = 9.83$,

$u_3^{(3)} = 7.20$, $u_4^{(3)} = 18.81$)