2. ANALYTIC FUNCTIONS

If $w = f(z)$ possesses a derivative at $z = z_0$ and at every point in some neighbourhood of z_0, then $f(z)$ is said to be **analytic at z_0** and z_0 is called a **regular point** of $f(z)$.

- If $f(z)$ is not analytic at z_0 and every neighbourhood of z_0 contains points at which $f(z)$ is analytic then z_0 is called a **singular point** of $f(z)$. (Derivative doesnot exists at that point).

- A function which is analytic at every point of a region R is called **analytic in R** or **holomorphic in R** or **regular in R**

- If a function is analytic at every point in the entire plane, it is said to be an **entire function**.

For example, a polynomial is an entire function since its derivative exists everywhere.

(Differentiability) $\Rightarrow$ (CR-eq^n are satisfied)

CAUCHY- RIEMANN EQUATIONS.

let $f(z) = u + iv$ be a function of z defined in a domain D. Then if $f'(z)$ exists at $z = z_0$ then $f(z)$ satisfy the CR-equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

But the converse is not necessarily true, i.e. the function may satisfy the CR - equations still may not be differentiable.

Thus, the C-R - equations are not necessary conditions for the differentiability (or analyticity) of a complex function.

Ex Show that $f(z) = \sqrt{|xy|}$ is not analytic at the origin, even though CR - equations are satisfied.

Solution let $f(z) = \sqrt{|xy|} = u + iv$

then $u = \sqrt{|xy|}$ and $v = 0$.

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \frac{0-0}{x} = 0.$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \frac{0-0}{y} = 0.$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = 0.$$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x}$.

$\therefore$ CR- equations are satisfied at the origin.

However, $f'(0) = \lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z-0} = \lim\limits_{z\to 0} \dfrac{\sqrt{|xy|}-0}{x+iy}$

$= \lim\limits_{x\to 0} \dfrac{\sqrt{|mx^2|}-0}{x(1+im)}$ along line $y = mx$

$= \dfrac{\sqrt{|m|}}{1+im}$ which is not unique

$\therefore$ $f'(0)$ doesnot exist.

<u>Ex</u>. Function $f(z)$ is defined as

$$f(z) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} + i\,\dfrac{x^3+y^3}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Test the continuity and differentiability of $f(z)$ at the origin.

<u>Solution</u> let $f(z) = u+iv$ where $u = \dfrac{x^3-y^3}{x^2+y^2}$ & $v = \dfrac{x^3+y^3}{x^2+y^2}$

To test the continuity of $f(z)$ at $z=0$, we use the polar coordinates.

Then $u = \dfrac{r^3(\cos^3\theta - \sin^3\theta)}{r^2(\cos^2\theta+\sin^2\theta)} = r(\cos^3\theta - \sin^3\theta)$.

and $v = \dfrac{r^3(\cos^3\theta+\sin^3\theta)}{r^2(\cos^2\theta+\sin^2\theta)} = r(\cos^3\theta + \sin^3\theta)$.

Now $z \to 0 \Rightarrow x \to 0,\ y \to 0$

$\Rightarrow r \to 0,\ \theta \to 0$

Clearly, $\underset{z\to 0}{\text{Lt}}\ f(z) = \underset{\substack{r\to 0\\ \theta\to 0}}{\text{Lt}}\ f(z) = 0 = f(0)$

because u and v tends to zero as $r \to 0$ irrespective of the values of θ.

Thus $f(z)$ is continuous at $(0,0)$.

Next,
$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x-0} = \underset{x\to 0}{\text{Lt}} \frac{\frac{x^3}{x^2}-0}{x}$$
$$= 1.$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y-0} = -1.$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x-0} = \underset{x\to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y-0} = \underset{y\to 0}{\text{Lt}} \frac{y}{y} = 1.$$

Thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

This shows the CR- equations are satisfied at the origin.

Now
$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$
$$= \lim_{x,y\to 0} \frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}.$$

Along the path $y = mx$,

$$= \lim_{x \to 0} \frac{x^3 - m^3x^3 + i(x^3 + m^3x^3)}{(x^2 + m^2x^2)(x + imx)}$$

$$= \lim_{x \to 0} \frac{(1-m^3) + i(1+m^3)}{(1+m^2)(1+im)}$$

which is not unique.

Thus $f'(0)$ does not exists..

Ex let f be an analytic function in the domain D. If $|f(z)| = \lambda$ where λ is constant, then show that f is constant in D.

Proof let $f(z) = u + iv$ be an analytic function

Then $|f(z)| = \lambda \Rightarrow u^2 + v^2 = \lambda^2 = \text{constant}$.

Differentiating partially wrt x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \quad \text{—— (1)}$$

Since the function $f(z)$ is analytic, $\therefore$ CR-equations are satisfied, that is, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ Solving (1) we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow (u^2+v^2)u_x = 0 \quad \text{and} \quad (u^2+v^2)u_y = 0.$$

$\Rightarrow \quad u_x = \dfrac{0}{u^2+v^2}$ and $u_y = \dfrac{0}{u^2+v^2}$

$\Rightarrow \quad u_x = 0$ and $u_y = 0$

$\Rightarrow \quad u = u(x,y) = c_1$ where c_1 is a constant

Ily $v(x,y) = c_2$

Therefore $f(z) = c_1 + ic_2$, which is constant.

Ex Show that the function $\ln z$ is analytic for all z except when $\operatorname{Re} z \leq 0$.

Solution let $z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Taking natural logarithm on both sides,

$\ln_e z = \ln(r) + i\theta \cancel{\log} \ln e$

$\ln z = \ln(r) + i\theta \quad = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

Taking $u = \frac{1}{2}\ln(x^2+y^2)$ and $v = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \quad \& \quad \frac{\partial v}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2}.$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \quad \& \quad \frac{\partial v}{\partial x} = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2}.$$

Hence CR equations are satisfied

$\therefore \ln z$ is analytic for all z. except for $\operatorname{Re} z \leq 0$.

Ex Show that $f(z) = \frac{z}{z+1}$ is analytic at $z = \infty$

Solution The function $f(z)$ is analytic at $z = \infty$ if the function $f(\frac{1}{z})$ is analytic at $z = 0$.

Since $f(z) = \frac{z}{z+1}$, $f\left(\frac{1}{z}\right) = \frac{\frac{1}{z}}{\frac{1}{z}+1} = \frac{1}{z+1}$

Now, $f\left(\frac{1}{z}\right) = \frac{1}{z+1} = \frac{1}{x+1+iy} * \frac{(x+1)-iy}{(x+1)-iy}$

$$= \frac{(x+1)-iy}{(x+1)^2+y^2}$$

let $u = \frac{x+1}{(x+1)^2+y^2}$, $v = \frac{-y}{(x+1)^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{((x+1)^2+y^2)(1) - (x+1)(2(x+1))}{((x+1)^2+y^2)^2} = \frac{y^2-(x+1)^2}{((x+1)^2+y^2)^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \frac{-1}{1^2} = -1$$

$$\frac{\partial v}{\partial y} = \frac{((x+1)^2+y^2)(-1) + y(2y)}{((x+1)^2+y^2)^2} = \frac{-(x+1)^2+y^2}{((x+1)^2+y^2)^2}$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = -1$$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

lly $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$ (check)

Hence $f(\frac{1}{z})$ is analytic at $z=0$

$\Rightarrow f(z)$ is analytic at $z=\infty$.

Ex Show that $f(z) = \bar{z}$ is not differentiable at $z=0$ and it is nowhere analytic.

Solution let $f(z) = \bar{z} = x - iy$

let $u = x$ and $v = -y$

$\frac{\partial u}{\partial x} = 1$ $\quad \frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial y} = -1$ $\quad \frac{\partial v}{\partial x} = 0$.

Thus $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$.

Hence $f(z)$ is not analytic.

For derivative of $f(z)$ at $z=0$,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{x \to 0} \frac{x - iy}{x + iy}$$

$$= \lim_{x \to 0} \frac{x - imx}{x + imx} \quad \text{(Along } y = mx)$$

$$= \lim_{x \to 0} \frac{x(1-im)}{x(1+im)} = \frac{1-im}{1+im}$$

which being not unique,

Thus, $f(z)$ is not differentiable at the origin.

Ex Find the values of c_1 and c_2 such that

$$f(z) = (x^2 + c_1 y^2 - 2xy) + i(c_2 x^2 - y^2 + 2xy)$$ is analytic.

Hence find $f'(z)$.

Solution let $u = x^2 + c_1 y^2 - 2xy$ and $v = c_2 x^2 - y^2 + 2xy$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x - 2y = -2y + 2x$$ (which is true)

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2c_1 y - 2x = -2c_2 x - 2y$$

on comparing,

$$c_2 = 1, \quad c_1 = -1.$$

$$\therefore f(z) = (x^2 + y^2 - 2xy) + i(x^2 - y^2 + 2xy).$$

Thus, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (2x - 2y) + i(2x + 2y)$

$$= 2(x-y) + 2i(x+y)$$

$$= 2(x + iy) + 2i(x + iy)$$

$$= 2z + 2iz = 2z(1+i).$$

Ex Determine 'p' such that the function

$$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{px}{y} \text{ is analytic.}$$

Solution let $u = \frac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1}\left(\frac{px}{y}\right)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{x}{x^2+y^2} = \frac{\frac{-px}{y^2}}{1+\frac{p^2x^2}{y^2}} = \frac{-px}{y^2+p^2x^2}.$$

and
$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \Rightarrow \frac{y}{x^2+y^2} = \frac{\frac{p}{y}}{1+\frac{p^2x^2}{y^2}} = \frac{py}{y^2+p^2x^2}$$

On comparing, we get

$p^2=1$ and $-p=1$

$$\Rightarrow \boxed{p=-1}$$

Ex Determine whether the CR equations are satisfied for the function $f(z) = 2x + ixy^2$

Solution let $f(z) = u+iv$

hence $u = 2x$ and $v = xy^2$

$$\frac{\partial u}{\partial x} = 2, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = y^2, \quad \frac{\partial v}{\partial y} = 2xy$$

Thus $2 = 2xy \Rightarrow xy = 1$ and $y = 0$.

Hence CR-eqns are not satisfied.

Ex Show that polar form of Cauchy - Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

Solution Let (r,θ) be the coordinates of a point whose cartesian coordinates are (x, y)

Then $z = x+iy = re^{i\theta}$

$\therefore \quad u+iv = f(z) = f(re^{i\theta})$

Differentiating partially wrt r and θ, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\, e^{i\theta}$$

and $$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Comparing real and imaginary parts, we have

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}.$$

Ex Show that the function $f(z) = \begin{cases} e^{-z^4} & z \neq 0 \\ 0 & z = 0 \end{cases}$

is not analytic at the origin though the CR-equations are satisfied.

Solution $\quad e^{-z^4} = e^{-r^4(\cos 4\theta + i\sin 4\theta)}$ (26)

$$= e^{-r^4\cos 4\theta} \cdot e^{-ir^4\sin 4\theta}$$

$$= e^{-r^4\cos 4\theta}\left(\cos(r^4\sin 4\theta) - i\sin(r^4\sin 4\theta)\right)$$

let $\quad u = e^{-r^4\cos 4\theta}\cos(r^4\sin 4\theta)$

$$v = -e^{-r^4\cos 4\theta}\sin(r^4\sin 4\theta)$$

Now, $\dfrac{\partial u}{\partial r} = -e^{-r^4\cos 4\theta}\, 4r^3\cos 4\theta\cos(r^4\sin 4\theta)$

$$- e^{-r^4\cos 4\theta}\sin(r^4\sin 4\theta)(4r^3\sin 4\theta)$$

$$= -r^3 4 e^{-r^4\cos 4\theta}\left[\cos 4\theta\cos(r^4\sin 4\theta) + \sin 4\theta\sin(r^4\sin 4\theta)\right]$$

$$= -4r^3 e^{-r^4\cos 4\theta}\cos(4\theta - r^4\sin 4\theta).$$

and $\dfrac{\partial u}{\partial \theta} = e^{-r^4\cos 4\theta}\, 4r^4\sin 4\theta\cos(r^4\sin 4\theta) - e^{-r^4\cos 4\theta}\sin(r^4\sin 4\theta)\, 4r^4\cos 4\theta.$

$$= 4r^4 e^{-r^4\cos 4\theta}\sin(4\theta - r^4\sin 4\theta)$$

$$\frac{\partial v}{\partial r} = -4r^3 e^{-r^4\cos 4\theta}\sin(4\theta - r^4\sin 4\theta)$$

$$\frac{\partial v}{\partial \theta} = -4r^4 e^{-r^4\cos 4\theta}\cos(4\theta - r^4\sin 4\theta)$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Thus CR- equations are satisfied.

However

$$f'(z) = \lim_{z\to 0} \frac{f(z) - f(0)}{z-0} = \lim_{z\to 0} \frac{e^{-z^4}}{z}$$

$$= \lim_{z\to 0} \frac{1}{z\left(1+z^4+\frac{z^8}{2!}---\right)} = \infty$$

Hence $f(z)$ is not analytic at the origin.

Ex If $u(r,\theta) = \left(r-\frac{1}{r}\right)\sin\theta$, $r \neq 0$ find an analytic function $f(z) = u+iv$.

Solution $u = \left(r-\frac{1}{r}\right)\sin\theta$.

Hence $\frac{\partial u}{\partial r} = \left(1+\frac{1}{r^2}\right)\sin\theta$, $\frac{\partial u}{\partial \theta} = \left(r-\frac{1}{r}\right)\cos\theta$

Since $f(z)$ is analytic, $\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

$\Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = \left(1+\frac{1}{r^2}\right)\sin\theta$ and $-r\frac{\partial v}{\partial r} = \left(r-\frac{1}{r}\right)\cos\theta$

$\Rightarrow \frac{\partial v}{\partial \theta} = \left(r+\frac{1}{r}\right)\sin\theta$ and $\frac{\partial v}{\partial r} = \left(-1+\frac{1}{r^2}\right)\cos\theta$

$\therefore v = -\left(r+\frac{1}{r}\right)\cos\theta + c_1$

Thus, $f(z) = u+iv = \left(r-\frac{1}{r}\right)\sin\theta - i\left(r+\frac{1}{r}\right)\cos\theta + c.$

Ex Find an analytic function $f(z) = u + iv$ in terms of z such that

$$v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$$

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Solution $v = r^2\cos 2\theta - r\cos\theta + 2$

$$\frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta \quad \text{and} \quad \frac{\partial v}{\partial \theta} = -2r^2\sin 2\theta + r\sin\theta$$

Since $f(z)$ is analytic, CR - equations are satisfied

Hence $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = -2r\sin 2\theta + \sin\theta$

and $\dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} = -2r^2\cos 2\theta + r\cos\theta$

$$\Rightarrow \quad u = -\int (2r^2\cos 2\theta + r\cos\theta)\, d\theta$$

$$u = \frac{-2r^2\sin 2\theta}{2} + r\sin\theta + C$$

$$= -r^2\sin 2\theta + r\sin\theta + C$$

Thus, $f(z) = u + iv$

$$= -r^2(\sin 2\theta - i\cos 2\theta) + r(\sin\theta - i\cos\theta) + c + 2i \quad (29)$$

$$= +ir^2(\cos 2\theta + i\sin 2\theta) + ri(\cos\theta + i\sin\theta) + c + 2i$$

$$= iz^2 - iz + c + 2i$$

$$= i(z^2 - z) + c + 2i$$

Ex. If $f(z) = u + iv$ is an analytic function of the complex variable z and $u - v = e^x(\cos y - \sin y)$ find $f(z)$ in terms of z.

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Solution $\quad u - v = e^x\cos y - e^x\sin y$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x\cos y - e^x\sin y \quad \text{——} \ (1)$$

and $\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial y} = -e^x\sin y - e^x\cos y$

$$\Rightarrow -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = -e^x\sin y - e^x\cos y$$

($f(z)$ is analytic)
$\left(\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \right)$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = e^x\sin y + e^x\cos y. \quad \text{——} \ (2)$$

Adding (1) and (2)

$$\frac{\partial u}{\partial x} = e^x\cos y \quad \Rightarrow u = e^x\cos y + c_1.$$

$$\frac{\partial v}{\partial x} = e^x\sin y \quad \Rightarrow v = e^x\sin y + c_2.$$

$f(z) = u + iv$

$= e^x(\cos y + i\sin y) + (c_1 + ic_2)$

$= e^x e^{iy} + (c_1 + ic_2) = e^z + c_1 + ic_2.$

Ex If $f(z) = u + iv$ is an analytic function $z = (x + iy)$

and $u + v = (x+y)(2 - 4xy + x^2 + y^2)$

then find u and v and function $f(z)$

Solution $u + v = (x+y)(2 - 4xy + x^2 + y^2)$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = (x+y)(-4y+2x) + (2-4xy+x^2+y^2)(1)$$

$$= -4xy - 4y^2 + 2x^2 + 2xy + 2 - 4xy + x^2 + y^2$$

$$= 3x^2 - 3y^2 - 6xy + 2 \quad \text{———} \quad (1)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = (x+y)(-4x+2y) + (2-4xy+x^2+y^2)(1)$$

$$= -4x^2 - 4xy + 2xy + 2y^2 + 2 - 4xy + x^2 + y^2$$

$$= -3x^2 + 3y^2 - 6xy + 2$$

Since $f(z)$ is analytic

$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}.$

$$\Rightarrow \quad -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = -3x^2+3y^2-6xy+2 \quad \text{—— (2)}$$

Adding (1) and (2) we get

$$2\frac{\partial u}{\partial x} = -12xy+4$$

$$\Rightarrow \frac{\partial u}{\partial x} = -6xy+2 \qquad \Rightarrow \quad u = -6y\frac{x^2}{2} + 2x + c$$

$$\therefore \quad u = -3x^2y + 2x + c_1$$

Again

$$\frac{\partial v}{\partial x} = 3x^2 - 3y^2$$

$$v = \frac{3x^3}{3} - 3y^2x + c_2$$

Thus,

$$f(z) = u + iv =$$

$$= (-3x^2y + 2x + c_1) + i(x^3 - 3y^2x + c_2)$$

$$= ix^3 - 3x^2y - i3xy^2 + 2x + (c_1 + ic_2)$$

$$\Rightarrow \quad -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = -3x^2 + 3y^2 - 6xy + 2 \quad \text{—— (2)}$$

Adding (1) and (2) we get

$$2\frac{\partial u}{\partial x} = -12xy + 4$$

$$\Rightarrow \frac{\partial u}{\partial x} = -6xy + 2 \qquad \Rightarrow \quad u = -6y\frac{x^2}{2} + 2x + c$$

$$\therefore \quad u = -3x^2y + 2x + c_1$$

Again $\qquad \frac{\partial v}{\partial x} = 3x^2 - 3y^2$

$$v = \frac{3x^3}{3} - 3y^2x + c_2$$

Thus, $\quad f(z) = u + iv =$

$$= (-3x^2y + 2x + c_1) + i(x^3 - 3y^2x + c_2)$$

$$= ix^3 - 3x^2y - i3xy^2 + 2x + (c_1 + ic_2)$$

EXTRA QUESTIONS.

Ex 1. Determine which of the following functions are analytic :-

(i) $2xy + i(x^2 - y^2)$

(ii) $\dfrac{x - iy}{x^2 + y^2}$

(iii) $\cosh z$

Ex Determine the analytic function whose real part is

(1) $x^3 - 3xy^2 + 3x^2 - 3y^2$

(2) $\cos x \cosh y$.

(Harmonic questions)

Ex Find the analytic function $z = u + iv$, if

(1) $u - v = (x - y)(x^2 + 4xy + y^2)$

(2) $u + v = \dfrac{2\sin 2x}{e^{2y} - e^{-2y} - 2\cos 2x}$

x