Gauss Elimination Method

In this method the unknowns are eliminated one by one so that the system is reduced to an upper triangular system from which the unknowns are obtained by back substitution. The method is quite general and is well adapted for computer operations. The method will be better explained by considering the following 3x3 system of equations.

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

The above system of equations reduced into the following system of equations by using elementary row operations:

$$a_1 x + b_1 y + c_1 z = d_1$$
$$b_2' y + c_2' z = d_2'$$
$$c_3'' z = d_3''$$

We get, $x = \dfrac{d_1 - b_1 y - c_1 z}{a_1}$, $y = \dfrac{d_2' - c_2' z}{b_2'}$, $z = \dfrac{d_3''}{c_3''}$

Note: The method will fail if any one of the a_1, b_2' or c_3'' (known as pivots) becomes zero.

Problem:- Solve the system of equations by Gauss elimination method

$$8x_2 + 2x_3 = -7 \quad \text{———} \quad (1)$$
$$3x_1 + 5x_2 + x_3 = 8 \quad \text{———} \quad (2)$$
$$6x_1 + 2x_2 + 8x_3 = 26 \quad \text{———} \quad (3)$$

Solution:- We apply pivoting since eq^n(1) has no x_1 term and in eq^n(3) has the largest co-eff of x_1, hence we interchange eq^n(1) &

& (3) and then we get

$$\left.\begin{aligned} 6x_1 + 2x_2 + 8x_3 &= 26 \\ 3x_1 + 5x_2 + 2x_3 &= 8 \\ 8x_2 + 2x_3 &= -7 \end{aligned}\right\} \quad \text{———} \quad (4)$$

Step 1 :- Let us write the augmented matrix as

$$\left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right] \quad \text{———} \quad (5)$$

To eliminate x_1 from the other equations we apply $R_2 \to R_2 - \frac{1}{2}R_1$ and get

$$\left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array}\right] \quad \text{———} \quad (6)$$

Step 2 :- Elimination of x_2

The largest co-eff of x_2 is 8 in the third eqn and we interchange R_2 and R_3 to get

$$\left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right] \quad \text{———} \quad (7)$$

To eliminate x_2 we apply $R_3 \to R_3 - \frac{1}{2}R_2$ & get

$$\left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & \frac{-3}{2} \end{array}\right]$$

Step 3 :- We use back substitution to get x_3, x_2, x_1 and get

$x_3 = \frac{1}{2}$, $x_2 = \frac{1}{8}(-7 - 2x_3) = -1$

$x_1 = \frac{1}{6}(26 - 2x_2 - 8x_3) = 4$

Exercise

Solve the system of equations by using Gauss elimination method :

1) $$\begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$$

Soln
$x = 5, y = 6,$
$z = -10, w = 8$

2) $x + y + z = 9$
$2x + 5y + 7z = 52$
$2x + y - z = 0$

Soln
$x = 1, y = 3, z = 5$

3) $2x - y + 3z = 8$
$-x + 2y + z = 4$
$3x + y - 4z = 0$

Soln
$x = y = z = 2$

4) $x + y + 4z = 6$
$3x + 2y - 2z = 9$
$5x + y + 2z = 13$

Soln
$x = 2, y = 2, z = \frac{1}{2}$

5) $2x - y + 3z - 9 = 0$
$x + y + z - 6 = 0$
$x - y + z - 2 = 0$

Soln
$x = 1, x = 2, x = 3$