Laplace Transform

Introduction →

→ It is used for the solution of linear ordinary integro-differential equation.

→ In comparison with classical method of solving Linear integro diff. eqn Lapalace transform has following two attractive features:

1> homogenous eqn and P. I. of the sol'n are obtained in one operation

2> The L.T. converts the integro diff. eqn into an algebric eqn in s (Laplace operator). It is then possible to manipulate the algebric eqn by simple algebric rules to obtain the expression in suitable form. The final sol'n is obtained by taking I.L.T.

Definition of L.T. →

for f(t) which is zero for $t<0$ and that satisfy the condition

$$\int_0^{\infty} |f(t) e^{-\sigma t}| dt < \infty$$

for some real and positive σ

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$s \rightarrow$ Laplace operator which is a complex variable

$$s = \sigma + j\omega$$

Inverse L.T. →

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s)\, e^{st}\, dt$$

Properties of L.T. →

1> Multiplication by a const.

$$\mathcal{L}[K f(t)] = K F(s)$$

2> Sum & Difference

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

3. Differentiation w.r.t 't' →

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - \lim_{t\to 0} f(t) = sF(s) -$$

$f(t)$ & $f(0^+)$ be the value of $f(t)$ as t

Proof:- $F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$

Let $f(t) = U$

$$\left[\frac{df(t)}{dt}\right]dt = dU$$

$$e^{-st}dt = dv$$

then $v = \frac{-1}{s}e^{-st}$

On integration

$$F(s) = \int_0^\infty U\,dv = UV\Big|_0^\infty - \int_0^\infty V\,dU$$

$$= f(t)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right)\left[\frac{df(t)}{dt}\right]dt$$

$$= \frac{1}{s}f(0^+) + \frac{1}{s}\int_0^\infty e^{-st}\left[\frac{df(t)}{dt}\right]dt$$

$$= \frac{1}{s}f(0^+) + \frac{1}{s}\mathcal{L}\left[\frac{df(t)}{dt}\right]$$

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$$

L.T of second derivative of $f(t)$ is

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = \mathcal{L}\left[\frac{d}{dt}\left\{\frac{df(t)}{dt}\right\}\right]$$

$$= s\mathcal{L}\left[\frac{df(t)}{dt}\right] - \frac{df(t)}{dt}\Big|_{t=0}$$

$$= s\left[sF(s) - f(0^+)\right] - f'(0^+)$$

$$= s^2F(s) - sf(0^+) - f'(0^+)$$

Integration by 't' →

If $\mathcal{L}[f(t)] = F(s)$

then L.T. of first integral of $f(t)$ is

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Proof :- $\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \int_0^\infty \left[\int_0^t f(t)\,dt\right] e^{-st}\,dt$

Let $u = \int_0^t f(t)\,dt$ then $du = f(t)\,dt$

$dv = e^{-st}\,dt \Rightarrow v = -\frac{1}{s}e^{-st}$

On Integrating $\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = -\frac{1}{s}e^{-st}\int_0^t f(t)\,dt\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)\,dt$$

$$= 0 - 0 + \frac{1}{s}\mathcal{L}f(t) = \frac{F(s)}{s}.$$

In general

$$\mathcal{L}\left[\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_n} f(t)\,dt_1\,dt_2\cdots dt_n\right] = \frac{F(s)}{s^n}$$

5. Differentiation w.r.t. 's'

$$\mathcal{L}[t.f(t)] = -\frac{dF(s)}{s}$$

6. Integration by 's'

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

7 Shifting theorem

(a) Time shifting →

$$\mathcal{L}[f(t-a).u(t-a)] = e^{-as}F(s)$$

(b) frequency shifting

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

8) Initial Value Theorem →

$$f(0^+) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} [s . F(s)]$$

9) final Value Theorem →

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} [s . F(s)]$$

10) Time Scaling

$$\mathcal{L}[f(at)] = \frac{1}{a} F(s/a)$$

11) L.T. of Sinusoidal fn. →

$$f(t) = \sin\omega t$$

$$F(s) = \int_0^\infty \sin\omega t \, e^{-st} \, dt$$

$$= \int_0^\infty \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t}) e^{-st} \, dt$$

$$= \frac{1}{2j} \int_0^\infty [e^{(j\omega - s)t} - e^{-(j\omega + s)t}] \, dt$$

$$= \frac{1}{2j} \left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right]$$

$$= \frac{1}{2j} \left[\frac{2j\omega}{s^2 + \omega^2} \right] = \frac{\omega}{s^2 + \omega^2}.$$

12) $f(t) = \cos\omega t$

$$F(s) = \frac{s}{s^2 + \omega^2}$$

13) $e^{-at} \sin\omega t = f(t)$

$$F(s) = \int_0^\infty e^{-at} \sin\omega t \, e^{-st} \, dt$$

$$= \int_0^\infty \left[\frac{1}{2j} e^{(j\omega - s - a)t} - e^{-(j\omega + s + a)t} \right] dt$$

$$= \frac{1}{2j} \left[\frac{1}{j\omega - s - a} + \frac{1}{j\omega + s + a} \right]_0^\infty$$

$$= \frac{1}{2j} \left[\frac{1}{s + a - j\omega} - \frac{1}{s + a + j\omega} \right]$$

$$= \frac{\omega}{(s+a)^2 + \omega^2}$$

tain the L.T. of the given w/f.

$f(t) = K\,U(t+t_1)$

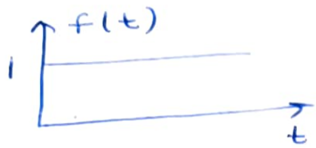
$F(s) = \dfrac{K e^{t_1 s}}{s}$

$\rightarrow F(s) = \int_0^\infty K\,U(t+t_1)e^{-st}\,dt$

$= \int_{-t_1}^{\infty} K e^{-st}\,dt = K\left[\dfrac{e^{-st}}{-s}\right]_{-t_1}^{\infty}$

$= \dfrac{K e^{t_1 s}}{s}$

2. Unit step fn. $U(t) \rightarrow$



$f(t) = U(t)$

$F(s) = \int_0^\infty U(t)e^{-st}\,dt$

by definition of $U(t)$

$F(s) = \int_0^\infty e^{-st}\,dt = \dfrac{1}{s}$

3. Delayed step fn. $K U(t-a)$

$f(t) = K U(t-a)$

$F(s) = \int_0^\infty K\,U(t-a)\,e^{-st}\,dt$

$= \int_a^\infty K e^{-st}\,dt = \dfrac{K e^{-as}}{s}$

4. Ramp fn. $K r(t) = K t\,U(t) = f(t)$

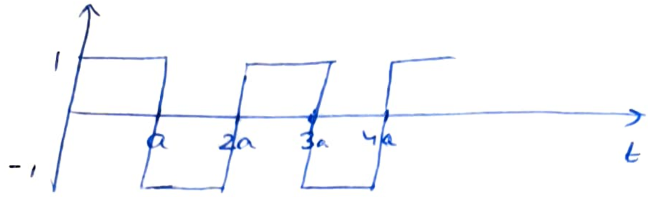
$F(s) = \int_0^\infty K t\,U(t) e^{-st}\,dt$

$= \int_0^\infty K t\, e^{-st}\,dt = K\left[t\dfrac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty 1.\dfrac{e^{-st}}{-s}\,dt\right]$

$= K[0-0] + \dfrac{K}{s^2}e^{-st}\Big|_0^\infty = K/s^2$

5.

6.



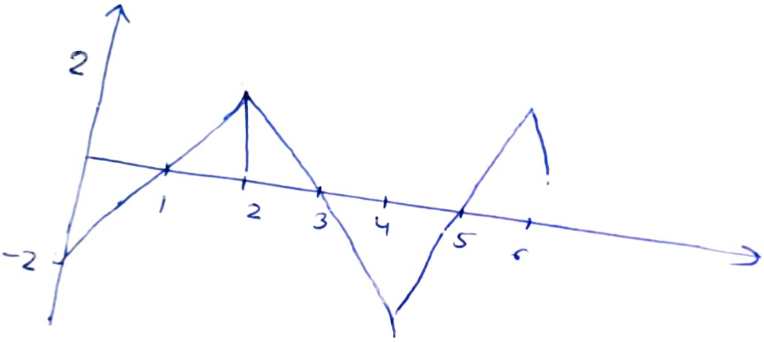
$$f(t) = 1[U(t) - U(t-a)] + (-1)[U(t-a) - U(t- \ldots] + 1[U(t-2a) - U(t \ldots$$

$$= U(t) - 2U(t-a) + 2U(t-2a)$$

$$F(s) = \frac{1}{s} - \frac{2e^{-as}}{s} + \frac{2e^{-2as}}{s} + \cdots$$

$$= \frac{1}{s}\left[1 - 2e^{-as} + 2e^{-2as} + \cdots\right]$$

7.



$$f(t) = 2(t-1)U(t) - 4(t-2)U(t-2) + 4(t-4)U(t-4) + \cdots$$

$$= 2tU(t) - 2U(t) - 4(t-2)U(t-2) - \cdots$$

$$F(s) = \frac{2}{s^2} - \frac{2}{s} - \frac{4}{s^2}e^{-2s} + \cdots$$

Theorem for periodic functions

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-Ts}} F_1(s)$$

where

$$f(t) = f_1(t) + f_2(t) + f_3(t)$$

$$= f_1(t) + f_1(t-T)U(t-T) + f_1(t-2T)U(t-2T) + \cdots$$

Determine $f(0^+)$ if

$$F(s) = \frac{2(s+1)}{s^2+2s+5}$$

$$f(0^+) = \lim_{s\to\infty} [s.F(s)]$$

$$= \lim_{s\to\infty}\left[\frac{2s(s+1)}{s^2+2s+5}\right] = \lim_{s\to\infty}\left[\frac{2+2/s}{1+2/s+5/s^2}\right]$$

$$= 2.$$

eg. $i(t) = 5U(t) - 3e^{-2t}$, find $I(s)$ and hence to determine $i(0^+)$ & $i(\infty)$

Sol $$I(s) = 5/s - \frac{3}{s+2} = \frac{5s+10-3s}{s(s+2)} = \frac{2s+10}{s(s+2)}$$

$$i(0^+) = \lim_{s\to\infty}[s.I(s)]$$

$$= \lim_{s\to\infty}\left[\frac{\cancel{s}(2s+10)}{\cancel{s}(s+2)}\right] = \lim_{s\to\infty}\left[\frac{2+10/s}{1+2/s}\right]$$

$$= 2$$

$$i(\infty) = \lim_{s\to 0}[s\,I(s)] = \cancel{\lim_{s\to 0}}\ \cancel{10}\ 5$$

eg. Without find I.L.T. of F(s) determine $f(0^+)$ and $f(\infty)$ for following fn.

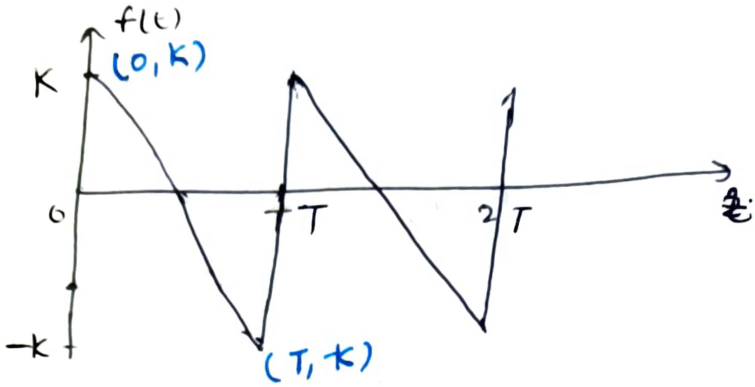
(i) $\dfrac{4e^{-2s}(s+50)}{s}$

$$f(0^+) = \lim_{s\to\infty} \frac{s.4e^{-2s}(s+50)}{s}$$

$$= \lim_{s\to\infty} 4e^{-2s}(s+50) = 0$$

$$f(\infty) = \lim_{s\to 0} \frac{s.4e^{-2s}(s+50)}{s} = 200$$

Find L.T. of the w/f →



for first cycle

$$f_1(t) = \left\{ \frac{-k-K}{T} \cdot t + K \right\} G_{0,T}(t)$$

$$= \left\{ -\frac{2k}{T} t + k \right\} G_{0,T}(t)$$

$$= -\frac{2k}{T} \left\{ t - T/2 \right\} \left[u(t) - u(t-T) \right]$$

$$= -\frac{2K}{T} \left(t - T/2 \right) u(t) + \frac{2k}{T} \left(t - T/2 \right) u(t-T)$$

$$= -\frac{2K}{T} \left(t - T/2 \right) u(t) + \frac{2k}{T} \left(t - T + T/2 \right) u(t-T)$$

$$= -\frac{2k}{T} \left[t\,u(t) - T/2\, u(t) - (t-T)\,u(t-T) - T/2\, u(t-T) \right]$$

$$= -\frac{2k}{T} \left[t\,u(t) - (t-T)\,u(t-T) - T/2 \left[u(t) + u(t-T) \right] \right]$$

$$\mathcal{L}\{ f_1(t) = \frac{2K}{TS} \left\{ \frac{1}{s^2} - \frac{e^{-Ts}}{s^2} - T/2 \left[\frac{1}{s} + \frac{e^{-Ts}}{s} \right] \right.$$

$$m = \frac{-k-k}{T-0} = \frac{-2k}{T}$$

$$f_1(t) = \frac{-2k}{T} \left(t - T/2 \right) \left[u(t) - u(t-T) \right]$$

Circuit Analysis by Laplace Transform

Solution of Linear differential equation →

1. Partial fraction expansion when all roots of determinator are simple →

eg. $x'' + 3x' + 2x = 0$, $x(0^+) = 2$, $x'(0^+) = -3$

by taking L.T

$$s^2 X(s) - s x(0^+) - x'(0^+) + 3s X(s) - 3x(0^+) + 2X(s) = 0$$

$$X(s)(s^2 + 3s + 2) = s x(0^+) + x'(0^+) + 3x(0^+)$$

$$(s^2 + 3s + 2) X(s) = 2s - 3 + 6$$
$$= 2s + 3$$

$$X(s) = \frac{2s+3}{s^2+3s+2} = \frac{2s+3}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

$$K_1 = (s+1)X(s)\Big|_{s=-1}$$
$$= \frac{2s+3}{s+2}\Big|_{s=-1} = \frac{-2+3}{-1} = 1$$

$$K_2 = (s+2)X(s)\Big|_{s=-2}$$
$$= \frac{2s+3}{s+1}\Big|_{s=-2} = \frac{-4+3}{-1} = 1$$

$$X(s) = \frac{2s+3}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{1}{s+2}$$

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{1}{s+2}\right]$$

$$x(t) = e^{-t} + e^{-2t}$$

2. Partial fraction expansion when some roots of ~~quad. eqn~~ denominator are of multiple order

eg. $$I(s) = \frac{1}{s(s+1)^2(s+2)}$$

$$I(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{(s+1)^2} + \frac{k_4}{s+2}$$

$$k_1 = s\,I(s)\Big|_{s=0} = \frac{1}{(s+1)^2(s+2)}\Big|_{s=0} = \frac{1}{2}$$

$$k_2 = \cancel{(s+1)^2 I(s) = \frac{1}{s(s+1)(s+2)}\Big|_{s=-1}}$$

$$= \frac{d}{ds}\left[(s+1)^2 I(s)\right]\Big|_{s=-1}$$

$$= \frac{d}{ds}\left[\frac{1}{(s+2)s}\right]\Big|_{s=-1} = \frac{\cancel{s^2+2s} - 2s+2}{s^2(s+2)^2}\Big|_{s=-1}$$

$$= 0$$

$$k_3 = (s+1)^2 I(s)\Big|_{s=-1} = \frac{1}{s(s+2)}\Big|_{s=-1}$$

$$= -1$$

$$k_4 = (s+2)\,I(s)\Big|_{s=-2} = \frac{1}{s(s+1)^2} = \frac{1}{2}$$

$$I(s) = \frac{1}{2s} - \frac{1}{(s+1)^2} + \frac{1}{2(s+2)}$$

$$i(t) = \frac{1}{2} - te^{-t} + \frac{1}{2}e^{-2t}$$

3 Partial fraction expansion when two roots of denominator are of complex conjugate.

eg $I(s) = \dfrac{s^2+5s+9}{s^3+5s^2+12s+8}$; find $i(t)$

$$= \frac{s^2+5s+9}{(s+1)(s^2+4s+8)} = \frac{s^2+5s+9}{(s+1)\left[(s+2)^2-(j2)^2\right]}$$

$$= \frac{s^2+5s+9}{(s+1)(s+2+j2)(s+2-j2)}$$