Method of Variation of Parameters

Consider a second order linear differential eq^n with constant co-efficients

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

or

$$y'' + Py' + Qy = R \quad — (1)$$

where P, Q are constants and R is function of x.

Suppose that the C.F. of (1) i.e, the general sol^n of

$$y'' + Py' + Qy = 0 \quad — (2)$$

is
$$y = c_1 y_1 + c_2 y_2 \quad — (3)$$

where c_1 & c_2 are arbitrary constants.

The general solution of (1) is

$$y = C.F. + P.I.$$

Now, we shall discuss a general method of finding the particular integral (P.I.) of (1). This method is known as variation of parameters.

In this method we replace the constants c_1 and c_2 given in (3) by two functions of x (called parameters) say u_1 and u_2 respectively.

Let $y = u_1 y_1 + u_2 y_2 \quad — (4)$

Since y_1 & y_2 are two solutions of (2), therefore

$$y_1'' + Py_1' + Qy_1 = 0 \ \& \ y_2'' + Py_2' + Qy_2 = 0 \quad — (5)$$

Differentiating (4),

$$y' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2' \quad —— (6)$$

Suppose $u_1'y_1 + u_2'y_2 = 0$ — (7)

From (6) & (7), $y' = u_1 y_1' + u_2 y_2'$ — (8)

Differentiating again, $y'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$ — (9)

Substituting (9), (8) and (4) in (1), we obtain

$$(u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2') + P(u_1 y_1' + u_2 y_2') + Q(u_1 y_1 + u_2 y_2) = R$$

or

$$u_1(y_1'' + Py_1' + Qy_1) + u_2(y_2'' + Py_2' + Qy_2) + (u_1' y_1' + u_2' y_2') = R \quad \text{— (10)}$$

Using (5) in (10), we obtain

$$u_1' y_1' + u_2' y_2' = R \quad \text{— (11)}$$

To solve eq^n (7) & (11) for u_1' & u_2', we proceed as follows:

The wronskian of the solutions y_1 & y_2 is defined as

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Since y_1 & y_2 are linearly independent solutions, $W \neq 0$.

Solving equations (7) & (11) for u_1' & u_2' by Cramer's rule, we obtain

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ R & y_2' \end{vmatrix}}{W} = \frac{-y_2 R}{W}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & R \end{vmatrix}}{W} = \frac{y_1 R}{W}$$

$$\therefore \quad u_1' = \frac{-y_2 R}{W}, \quad u_2' = \frac{y_1 R}{W} \quad \text{— (12)}$$

Integrating the above relations in (12) wrt x we obtain u_1 & u_2 and so

P.I. $= u_1 y_1 + u_2 y_2$. Hence the complete sol^n of (1) is $y = C.F. + P.I.$

Rule:- The solution of $y''+Py'+Qy=R$ by the method of variation of parameters is given by the following steps:

Step1 Find the C.F.: $y=c_1y_1+c_2y_2$ of the given equation.

Step2 Find the wronskian of the solutions y_1 & y_2:

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Step3 $P.I. = u_1y_1+u_2y_2$, where u_1 & u_2 are given by $u_1' = \frac{-y_2R}{W}$, $u_2' = \frac{y_1R}{W}$

or $u_1 = -\int \frac{y_2R}{W}dx$, $u_2 = \int \frac{y_1R}{W}dx$

Step4 The complete solution of $y''+Py'+Qy=R$ is given by

$$y = C.F. + P.I.$$

Question 1: Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2}+4y = 4\sec^2 2x.$$

Solution: $(D^2+4)y = 4\sec^2 2x$

Its A.E. is $m^2+4=0$

$m = \pm 2i$

$\therefore$ C.F. is

$$y = c_1\cos 2x + c_2\sin 2x$$

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$ & $X = 4\sec^2 2x$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0$$

$$P.I. = -y_1\int \frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx$$

$$= -\cos 2x\int\frac{\sin 2x \cdot 4\sec^2 2x}{2}dx + \sin 2x\int\frac{\cos 2x\cdot 4\sec^2 2x}{2}dx$$

$$= -1 + \sin 2x \log(\sec 2x + \tan 2x)$$

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Hence the complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x).$$

Question 2 :- Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$$

Solution :- $(D^2 - 6D + 9)y = \dfrac{e^{3x}}{x^2}$

A.E is $m^2 - 6m + 9 = 0$

$m = 3, 3$

C.F. is $y = (c_1 + c_2 x)e^{3x}$

Here $y_1 = e^{3x}$, $y_2 = xe^{3x}$

$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (3x+1)e^{3x} \end{vmatrix} = e^{6x}$$

$$P.I. = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -e^{3x} \int \frac{xe^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}} dx + xe^{3x} \int \frac{e^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}} dx$$

$$= -e^{3x} \int \frac{1}{x} dx + xe^{3x} \int \frac{1}{x^2} dx$$

$$= -e^{3x} \log x + xe^{3x}\left(\frac{-1}{x}\right) = -(1 + \log x)e^{3x}.$$

Hence complete solution is

$$y = (c_1 + c_2 x)e^{3x} - (1 + \log x)e^{3x}$$

$$\Rightarrow y = (C_1 + c_2 x - \log x)\, e^{3x}$$

where $C_1 = c_1 - 1$

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Question 3:- Solve $\frac{d^2y}{dx^2} - y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

Solution:- $(D^2-1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

A.E. is $m^2 - 1 = 0$

$m = \pm 1$

C.F. is $y = c_1 e^{x} + c_2 e^{-x}$

Here $y_1 = e^{x}$, $y_2 = e^{-x}$ & $X = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2$$

$$P.I. = -y_1 \int \frac{y_2 X}{W}\,dx + y_2 \int \frac{y_1 X}{W}\,dx$$

$$= -e^{x} \int \frac{e^{-x}[e^{-x}\sin(e^{-x}) + \cos(e^{-x})]}{-2}\,dx + e^{-x} \int \frac{e^{x}[e^{-x}\sin(e^{-x}) + \cos(e^{-x})]}{-2}\,dx$$

$$= \frac{1}{2}e^{x} I_1 - \frac{1}{2}e^{-x} I_2$$

$$I_1 = \int e^{-x}[e^{-x}\sin(e^{-x}) + \cos(e^{-x})]\,dx$$

$= -\int (t\sin t + \cos t)\,dt$, where $t = e^{-x}$

$$= -\left[t(-\cos t) - \int 1\cdot(-\cos t)\,dt + \sin t\right]$$

$$= -(-t\cos t + 2\sin t) = e^{-x}\cos(e^{-x}) - 2\sin(e^{-x})$$

Also, $I_2 = \int e^{x}[e^{-x}\sin(e^{-x}) + \cos(e^{-x})]\,dx$

$= e^{x}\cos(e^{-x})$ $\quad \left[\int e^{x}(f(x)+f'(x))\,dx = e^{x}f(x)\right]$

$$P.I. = \frac{1}{2}e^{x}[e^{-x}\cos(e^{-x}) - 2\sin(e^{-x})] - \frac{1}{2}e^{-x}[e^{x}\cos(e^{-x})] = -e^{x}\sin(e^{-x})$$

Hence complete solution is

$y = c_1 e^{x} + c_2 e^{-x} - e^{x}\sin(e^{-x})$

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Solve by the method of variation of paramet

Q1) $\frac{d^2y}{dx^2}+y=\operatorname{cosec}x$

Q2) $\frac{d^2y}{dx^2}+y=\sec x$

Q3) $\frac{d^2y}{dx^2}+a^2y=\sec ax$

Q4) $\frac{d^2y}{dx^2}+y=\tan x$

Q5) $\frac{d^2y}{dx^2}+4y=\tan 2x$

Q6) $\frac{d^2y}{dx^2}+y=x\sin x$

Q7) $y''-2y'+2y=e^x\tan x$

Q8) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}=e^x\sin x$

Q9) $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=\frac{e^{-3x}}{x^3}$

Q10) $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+16y=\frac{12e^{4x}}{x^4}$

Q11) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^{2x}\sec^2x$

Q12) $y''-2y'+y=e^x\log x$

Q13) $\frac{d^2y}{dx^2}-y=\frac{2}{1+e^x}$

Q14) $\frac{d^2y}{dx^2}+y=\frac{1}{1+\sin x}$