SOLVED EXAMPLES

Example 3.15. Three urns contain 6 red, 4 black; 4 red, 6 black, and 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the first urn.

Solution. Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = urn first is chosen, E_2 = urn second is chosen,

E_3 = urn third is chosen, and A = ball drawn is red.

Since there are three urns and one of the three urns is chosen at random, therefore

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}.$$

If E_1 has already occurred, then urn first has been chosen which contains 6 red and 4 black balls. The probability of drawing a red ball from it is 6/10.

So,
$$P(A \mid E_1) = \frac{6}{10}.$$

Similarly $P(A \mid E_2) = \frac{4}{10}$ and $P(A \mid E_3) = \frac{5}{10}$

We are required to find $P(E_1 \mid A)$, *i.e.*, given that the ball drawn is red, what is the probability that it is drawn from the first urn.

By Baye's theorem, we have

$$P(E_1 \mid A) = \frac{P(E_1).P(A \mid E_1)}{P(E_1).P(A \mid E_1) + P(E_2).P(A \mid E_2) + P(E_3).P(A \mid E_3)}$$

$$= \frac{\frac{1}{3} \times \frac{6}{10}}{\frac{1}{3} \times \frac{6}{10} + \frac{1}{3} \times \frac{4}{10} + \frac{1}{3} \times \frac{5}{10}} = \frac{6}{15} = \frac{2}{5}.$$

Example 3.16. Urn A contains 2 white, 1 black and 3 red balls, urn B contains 3 white, 2 black and 4 red balls and urn C contains 4 white, 3 black and 2 red balls. One urn is chosen at random and 2 balls are drawn at random from the urn. If the chosen balls happen to be red and black what is the probability that both balls come from urn B?

Solution. Let E_1, E_2, E_3 and A denote the following events.

E_1 = urn A is chosen, E_2 = urn B is chosen, E_3 = urn C is chosen, and A = two balls drawn at random are red and black. Since one of the urns is chosen at random, therefore

$$P(E_1) = \frac{1}{3} = P(E_2) = P(E_3)$$

If E_1 has already occurred, then urn A has been chosen. The urn A contains 2 white, 1 black and 3 red balls. Therefore the probability of drawing a red and a black ball is $\frac{{}^3C_1 \times {}^1C_1}{{}^6C_2}$.

So,
$$P(A \mid E_1) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{3}{15} = \frac{1}{5}$$

Similarly,
$$P(A \mid E_2) = \frac{{}^4C_1 \times {}^2C_1}{{}^9C_2} = \frac{2}{9}$$

and
$$P(A \mid E_3) = \frac{{}^2C_1 \times {}^3C_1}{{}^9C_2} = \frac{1}{6}$$

We are required to find $P(E_2 \mid A)$. By Baye's theorem, we have

$$(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2) + P(E_3)\,P(A \mid E_3)}$$

$$= \frac{\frac{1}{3} \times \frac{2}{9}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{2}{9} + \frac{1}{3} \times \frac{1}{6}} = \frac{\frac{2}{9}}{\frac{1}{5} + \frac{2}{9} + \frac{1}{6}} = \frac{20}{53}$$

Example 3.17. A factory has three machines, X, Y and Z, producing 1000, 2000 and 3000 bolts per day respectively. The machine X produces 1% defective bolts, Y produces 1.5% and Z produces 2% defective bolts. At the end of the day, a bolt is drawn at random and it is found to be defective. What is the probability that this defective bolt has been produced by the machine X?

Solution. Total number of bolts produced in a day

$$= (1000 + 2000 + 3000) = 6000$$

Let E_1, E_2 and E_3 be the events of drawing a bolt produced by machine X, Y and Z respectively. Then,

$$P(E_1) = \frac{1000}{6000} = \frac{1}{6};\ P(E_2) = \frac{2000}{6000} = \frac{1}{3} \text{ and } P(E_3) = \frac{3000}{6000} = \frac{1}{2}$$

Let A be the event of drawing a defective bolt. Then,

$P(A / E_1)$ = Probability of drawing a defective bolt, given that it is produced by the machine X

$$= \frac{1}{100}$$

$P(A \mid E_2)$ = Probability of drawing a defective bolt, given that it is produced by the machine Y

$$= \frac{1.5}{100} = \frac{15}{1000} = \frac{3}{200}$$

$P(A / E_3)$ = Probability of drawing a defective bolt, given that it is produced by the machine Z

$$= \frac{2}{100} = \frac{1}{50}$$

Required probability = $P(E_1 \mid A)$

= Probability that the bolt drawn is produced by X, given that it is defective

2. The joint distribution of X and Y is

$$f(x, y) = K(x^2 + y^2),\ 0 \le x \le y \le 1$$
$$= 0 \qquad \text{elsewhere}$$

Determine K and find marginal densities of X and Y. [Ans. $K = \frac{3}{2}$]

3. The joint probability density function of the two dimensional random variable (X, Y) is given below:

$$f(x, y) = \begin{cases} \frac{8}{9}xy & 1 \le x \le y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find marginal density function of X and Y.
(b) Find conditional density function of Y given $X = x$ and conditional density function of X given $Y = y$.

[Ans. $f_X(x) = \frac{4}{9}x(4 - x^2),\ 1 \le x \le 2$

$f_Y(y) = \frac{4}{9}y(y^2 - 1),\ 1 \le y \le 2$

$f_{Y/X}(y|x) = \frac{2y}{4 - x^2},\ x \le y \le 2$

$f_{X/Y}(x|y) = \frac{2x}{y^2 - 1},\ 1 \le x \le y$]

4. If X and Y are two random variables having joint density function

$$f(x, y) = \frac{1}{8}(6 - x - y);\quad 0 < x < 2,\ 2 < y < 2$$
$$= 0 \qquad \text{otherwise}$$

Find: (a) $P(X < 1 \cap y < 3)$, (b) $P(X + Y < 3)$, (c) $P(X < 1 \mid 4 < 3)$

[Ans. (a) $\frac{3}{8}$, (b) $\frac{5}{24}$, (c) $\frac{1}{3}$]

5. Joint distribution of X and Y is given by

$$f(x, y) = 4xye^{-(x^2+y^2)},\ x \ge 0,\ y \ge 0$$

Test whether X and Y are independent. Also find the conditional density of X given $Y = y$.

[Ans. Yes, $f(X = x \mid Y = y) = 2xe^{-x^2}$]

3.9. BAYE'S THEOREM

Statement. Let S be the sample space and let $E_1, E_2, \ldots E_n$ be n mutually exclusive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then

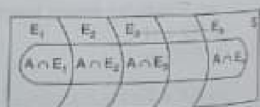

Fig. 3.1

$$P(E_i / A) = \frac{P(E_i)\, P(A / E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A / E_i)},\ i = 1, 2, \ldots, n$$

Proof. Since $E_1, E_2, \ldots, E_n$ are n mutually exclusive and exhaustive events, we have

$$S = E_1 \cup E_2 \cup \ldots \cup E_n, \text{ where } E_i \cap E_j = \phi \text{ for } i \ne j$$
$$A = A \cap S$$
$$\Rightarrow\ A = (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)$$
$$\Rightarrow\ P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n) \quad \text{[By add. theorem]}$$
$$\Rightarrow\ P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$
$$\Rightarrow\ P(A) = \sum_{i=1}^{n} P(E_i)\, P(A / E_i) \quad \ldots(i)\ [\because P(A \cap E_i) = P(E_i)\, P(A/E_i)]$$

Now, using multiplication theorem of probability, we have

$$P(A \cap E_i) = P(A)\, P(E_i / A) \quad \text{for } i = 1, 2, \ldots, n$$
$$\Rightarrow\ P(E_i / A) = \frac{P(A \cap E_i)}{P(A)}$$
$$\Rightarrow\ P(E_i / A) = \frac{P(E_i)\, P(A / E_i)}{P(A)} \quad [\because P(A \cap E_i) = P(E_i)\, P(A / E_i)]$$
$$\Rightarrow\ P(E_i / A) = \frac{P(E_i)\, P(A / E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A / E_i)} \quad \text{[Using (i)]}$$

Hence,

$$P(E_i / A) = \frac{P(E_i)\, P(A / E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A / E_i)},\ i = 1, 2, \ldots, n$$

Note 1. The events $E_1, E_2, \ldots, E_n$ are usually referred to as 'hypothesis' and the probabilities $P(E_1), P(E_2), \ldots P(E_n)$ are known as the 'priori' probabilities as they exist before we obtain any information from the experiment.

Note 2. The probabilities $P(A/E_i)$; $i = 1, 2, \ldots, n$ are called the 'likelyhood probabilities' as they tell us how likely the event A under consideration occurs, given each and every priori probabilities.

Note 3. The probabilities $P(E_i / A)$; $i = 1, 2, \ldots, n$ are called the 'posterior probabilities' as they are determined after the results of the experiment are known.

The significance of Baye's theorem may be understood in the following manner.

An experiment can be performed in n mutually exclusive and exhaustive ways $E_1, E_2, \ldots E_n$. The probability $P(E_i)$ of the occurrence of event E_i; $i = 1, 2, \ldots, n$ is known. The experiment is performed and we are told that the event A has occurred. With this information the probability $P(E_i)$ is changed to $P(E_i / A)$. Baye's theorem enables us to evaluate $P(E_i / A)$ if all the $P(E_i)$ (priori probabilities) and $P(A / E_i)$ (likelyhood probabilities) are known as explained in the following examples.

Example 3.23. Suppose the 5% of men and 0.25% of women have a grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

Solution. Let us define the following events:

E_1 : a male is chosen

E_2 : a female is chosen

A : a grey haired person is chosen

Then,

$$P(E_1) = \frac{1}{2},\ P(E_2) = \frac{1}{2}$$

$P(A/E_1)$ = Probability that a grey haired person is chosen, when it is known that a person is a male

$$= 5\%$$

$$= \frac{5}{100} = 0.05$$

$P(A/E_2)$ = Probability that a grey haired person is chosen, when it is known that a person is a female

$$= 0.25\%$$

$$= \frac{0.25}{100} = 0.0025$$

Hence, by Baye's theorem, we have

$P(E_1/A)$ = Probability that the person is a male when it is known that the person chosen is a grey haired

$$= \frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times 0.05}{\frac{1}{2}\times 0.05 + \frac{1}{2}\times 0.0025} = \frac{0.05}{0.05 + 0.0025}$$

$$= \frac{0.05}{0.0525} = \frac{5\times 100}{525} = \frac{20}{21}.$$

Example 3.24. If a machine is correctly set up, it produces 90% acceptable items. If it is incorrectly set up, it produces only 40% acceptable items. Past experience shows that 80% of the set ups are correctly done. If after a certain set up, the machine produces 2 acceptable items find the probability that the machine is correctly set up.

Solution. Let us define the events as:

E_1 : the machine set up is correct

E_2 : the machine set up is incorrect

A : the machine produces 2 acceptable items

Then,

$P(E_1)$ = Probability that the machine set up is correct

$$= 80\% = \frac{80}{100} = 0.8$$

$P(E_2)$ = Probability that the machine set up is incorrect

$$= 20\% = \frac{20}{100} = 0.2$$

$P(A/E_1)$ = Probability that the machine produces 2 acceptable items given that the machine set up is correct

$$= \frac{90}{100}\times\frac{90}{100} = 0.81$$

$P(A/E_2)$ = Probability that the machine produces 2 acceptable items given that the machine set up is incorrect

$$= \frac{40}{100}\times\frac{40}{100} = 0.16$$

Then by Baye's theorem,

$P(E_1/A)$ = Probability that the machine is correctly set up given that the machine produces 2 acceptable items

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{0.8\times 0.81}{0.8\times 0.81 + 0.2\times 0.16}$$

$$= \frac{0.648}{0.648 + 0.032} = \frac{0.648}{0.680} = \frac{648}{680} = \frac{81}{85}$$

Example 1.25. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?

Solution. Let us define the following events:

E_1 : Getting 5 or 6 in a single throw of a die

E_2 : Getting 1, 2, 3 or 4 in a single throw of a die

A : Getting exactly one head

Then,

$$P(E_1) = \frac{2}{6} = \frac{1}{3},\ P(E_2) = \frac{4}{6} = \frac{2}{3}$$

$P(A/E_1)$ = Probability of getting exactly one head given that a coin is tossed three times

$$= {}^3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{3-1} = {}^3C_1\times\left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$= \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)+P(E_3).P(A/E_3)}$$

$$= \frac{\left(\frac{1}{6}\times\frac{1}{100}\right)}{\left(\frac{1}{6}\times\frac{1}{100}\right)+\left(\frac{1}{3}\times\frac{3}{200}\right)+\left(\frac{1}{2}\times\frac{1}{50}\right)}$$

$$= \left(\frac{1}{600}\times\frac{600}{10}\right) = \frac{1}{10} = 0.1$$

Hence, the required probability is 0.1

Example 3.18. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter driver, car driver and a truck is 0.01, 0.03 and 0.15 respectively. One of the insured person meets with an accident. What is the probability that he is a scooter driver?

Solution. Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = person chosen is a scooter driver,

E_2 = person chosen is a car driver,

E_3 = person chosen is a truck driver, and

A = person meets with an accident.

Since there are 12000 persons, therefore

$$P(E_1) = \frac{2000}{12000} = \frac{1}{6}$$

and $$P(E_2) = \frac{4000}{12000} = \frac{1}{3}$$

and $$P(E_3) = \frac{6000}{12000} = \frac{1}{2}$$

It is given that $P(A/E_1)$ = Probability that a person meets with an accident given that he is a scooter driver = 0.01.

Similarly, $P(A/E_2) = 0.03$ and $P(A/E_3) = 0.15$

We are required to find $P(E_1/A)$, i.e., given that the person meets with an accident, what is the probability that he was a scooter driver.

By Baye's rule, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)+P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{6}\times 0.01}{\frac{1}{6}\times 0.01+\frac{1}{3}\times 0.03+\frac{1}{2}\times 0.15}$$

$$= \frac{1}{1+6+45} = \frac{1}{52}.$$

Example 3.19. A company has two plants to manufacture scooters. Plant I manufactures 70% of the scooters and Plant II manufactures 30%. At Plant I, 80% of the scooters are rated as of standard quality and at Plant II, 90% of the scooters are rated as of standard quality. A scooter is chosen at random and is found to be of standard quality. What is the probability that it has come from Plant II?

Solution. Let E_1, E_2 and A be the following events.

E_1 = Plant I is chosen, E_2 = Plant II is chosen, and A = Scooter is of standard quality

Then, $$P(E_1) = \frac{70}{100},\ P(E_2) = \frac{30}{100},$$

$$P(A/E_1) = \frac{80}{100} \text{ and } P(A/E_2) = \frac{90}{100}$$

We are required to find $P(E_2/A)$. By Baye's theorem, we have

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{30}{100}\times\frac{90}{100}}{\frac{70}{100}\times\frac{80}{100}+\frac{30}{100}\times\frac{90}{100}} = \frac{27}{56+27} = \frac{27}{83}$$

Example 3.20. In a test, an examine either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct, given that he copies it, is 1/8. Find the probability that he knew the answer to the question, given that he correctly answered it.

Solution. Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = the examinee guesses the answer, E_2 = the examinee copies the answer, E_3 = the examinee knows the answer, and A = the examinee answers correctly.

We have $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{6}$. Since E_1, E_2, E_3 are mutually exclusive and exhaustive events, therefore

$$P(E_1) + P(E_2) + P(E_3) = 1$$

$$\Rightarrow \quad P(E_3) = 1 - (P(E_1) + P(E_2))$$

$$= 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}.$$

If E_2 has already occurred, then the examinee guesses. Since there are four choices out of which only one is correct, therefore, the probability that he answers correctly given that he has made a guess is 1/4 i.e., $P(A/E_2) = 1/4$. It is given that $P(A/E_1) = 1/8$, and

$P(A/E_3)$ = Probability that he answers correctly given that he knew the answer

$= 1$.

By Baye's theorem, we have

Required probability $= P(E_3/A)$

$$= \frac{P(E_3)\,P(A/E_3)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1} = \frac{24}{29}.$$

Example 3.21. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probability that he will be late are $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

Solution. Let E_1, E_2, E_3, E_4 be the events that the doctor comes by train, bus, scooter and other means of transport respectively. Then,

$$P(E_1) = \frac{3}{10},\ P(E_2) = \frac{1}{5},\ P(E_3) = \frac{1}{10} \text{ and } P(E_4) = \frac{2}{5}$$

Let A be the event that the doctor visits the patient late. Then,

$P(A/E_1)$ = Probability that the doctor will be late if he comes by train

$= \frac{1}{4}$

$P(A/E_2)$ = Probability that the doctor will be late if he comes by bus

$= \frac{1}{3}$

$P(A/E_3)$ = Probability that the doctor will be late if he comes by scooter

$= \frac{1}{12}$

$P(A/E_4)$ = Probability that the doctor will be late if he comes by other means of transport

$= 0$

We have to find $P(E_1/A)$ (Given)

By Baye's theorem, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3) + P(E_4)\,P(A/E_4)}$$

$$= \frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0} = \frac{3}{40}\times\frac{120}{18} = \frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$.

Example 3.22. By examining the chest X-ray, the probability that a person is diagnosed with TB when he is actually suffering from it, is 0.99. The probability that the doctor incorrectly diagnoses a person to be having TB, on the basis of X-ray reports, is 0.001. In a certain city, 1 in 1000 persons suffers from TB. A person is selected at random and is diagnosed to have TB. What is the chance that he actually has TB?

Solution. Let

E_1 = event that the person selected in suffering from TB.

E_2 = event that the person selected is not suffering from TB.

A = event that the doctor diagnoses TB.

Then,

$$P(E_1) = \frac{1}{1000} \text{ and } P(E_2) = \left(1-\frac{1}{1000}\right) = \frac{999}{1000}$$

$P(A/E_1)$ = probability that TB is diagnosed, when the person actually has TB

$= \frac{99}{100}$

$P(A/E_2)$ = probability that TB is diagnosed, when the person has no TB

$= \frac{1}{1000}$

Using Bayes' theorem, we have

$P(E_1/A)$ = probability of a person actually having TB, if it is known that he is diagnosed to have TB

$$= \frac{P(E_1).\,P(A/E_1)}{P(E_1).\,P(A/E_1) + P(E_2).\,P(A/E_2)}$$

$$= \frac{\frac{1}{1000}\times\frac{99}{100}}{\frac{1}{1000}\times\frac{99}{100}+\frac{1}{1000}\times\frac{99}{100}} = \frac{110}{221}.$$

Hence, the required probability is $\frac{110}{221}$.

$$P(B_1) = \frac{3}{7}$$

Then

Let B_2 be the event that the second marble drawn in black. Then

$P(B_2|B_1)$ = Conditional probability of the event B_2 given that B_1 has occurred

$$= \frac{2}{6}$$

Hence by multiplication rule, we get

$$P(B_1 \text{ and } B_2) = P(B_1 \cap B_2) = P(B_1)\,P(B_2|B_1)$$

$$= \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$$

Example 1.28. A card is drawn from a well shuffled deck of 52 cards and then second card is drawn, find the probability that the first card is a spade and then second card is a club if the first card is not replaced.

Solution. We have

$$P(\text{first card spade}) = P(S) = \frac{13}{52} = \frac{1}{4}$$

After the event of drawing a spade the deck has 51 cards 13 of which are clubs (C)

Therefore,

$$P(C|S) = \frac{13}{51}$$

Hence,

$$P(S \text{ and } C) = P(S)\,P(C|S)$$

$$= \frac{1}{4} \cdot \frac{13}{51} = \frac{13}{204}$$

Example 1.29. If $P(A) = \frac{1}{3}, P(B) = \frac{1}{4}, P(A \cup B) = \frac{1}{2}$ determine : (i) $P(B|A)$ (ii) $P(A|B')$.

Solution. Given that: $P(A) = \frac{1}{3}, P(B) = \frac{1}{4}, P(A \cup B) = \frac{1}{2}$ gives $P(B') = \frac{3}{4}$,

From the addition theorem on the probability,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore \quad P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}$$

(i) $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$$

(ii) $$P(A \cap B') = P(A) - P(A \cap B) \qquad \text{Divide by } P(B')$$

(i) $$\frac{P(A \cap B')}{P(B')} = \frac{P(A)}{P(B')} - \frac{P(A \cap B)}{P(B')}$$

$$\Rightarrow \quad P(A/B') = \frac{\frac{1}{3}}{1-\frac{1}{4}} - \frac{\frac{1}{12}}{1-\frac{1}{4}} = \frac{\frac{1}{3}}{\frac{3}{4}} - \frac{\frac{1}{12}}{\frac{3}{4}}$$

$$= \frac{4}{3}\left[\frac{1}{3} - \frac{1}{12}\right] = \frac{4}{3}\left(\frac{4-1}{12}\right) = \frac{1}{3}$$

$$\therefore \quad P(A/B') = \frac{1}{3}$$

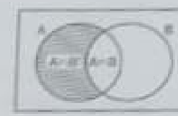

Fig. 1.1.

Example 1.30. A dice is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once ?

Solution. Consider the events

A = number 4 appears atleast once

B = the sum of the number appearing is 6

Then A = {(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6) (6, 4) (5, 4) (3, 4) (2, 4) (1, 4)}

and B = {(1, 5) (2, 4) (3, 3) (4, 2) (5, 1)}

$$P(A \cap B) = \{(2, 4)\ (4, 2)\}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2}{5}$$

Example 1.31. A market survey was conducted in four cities to find out the preference for brand A soap. The responses are shown below:

	Delhi	*Kolkata*	*Chennai*	*Mumbai*
Yes	45	55	60	50
No	35	45	35	45
No opinion	5	5	5	5

(a) What is the probability that a consumer preferred brand A, given that he was from Chennai?

(b) Given that a consumer preferred brand A, what is the probability that he was from Mumbai?

$P(A/E_2)$ = Probability of getting exactly one head given that a coin is tossed once (whether a head or tail is obtained)

$$= \frac{1}{2}$$

Hence, by Baye's theorem,

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{2}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{3}\left(\frac{3}{8}+1\right)}$$

$$= \frac{8}{3+8} = \frac{8}{11}.$$

Example 3.27. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Solution. Let E_1, E_2 and A be the events defined as follows:

E_1 = six occurs, E_2 = six does not occur, and A = the man reports that it is a six

We have,

$$P(E_1) = \frac{1}{6},\ P(E_2) = \frac{5}{6}$$

Now, $P(A/E_1)$ = Probability that the man reports that there is a six on the die given that six has occurred on the die

= Probability that the man speaks truth $= \frac{3}{4}$

and $P(A/E_2)$ = Probability that the man reports that there is a six on the die given that six has not occurred on the die

= Probability that the man does not speak truth $= 1-\frac{3}{4}=\frac{1}{4}$

We have to find $P(E_1/A)$ i.e., the probability that there is six on the die given that the man has reported that there is six. By Baye's theorem, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}} = \frac{3}{8}$$

Example 3.28. A letter is known to have come either from TATANAGAR or CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter has come from: (i) Calcutta, (ii) Tatanagar?

Solution. Let E_1 be the event that the letter came from Calcutta and E_2 be the event that the letter came from Tatanagar. Let A denote the event that two consecutive letters visible on the envelope are TA.

Since the letters have come either from Calcutta or Tatanagar, therefore,

$$P(E_1) = \frac{1}{2} = P(E_2)$$

If E_1 has occurred, then it means that the letter came from Calcutta. In the word CALCUTTA there are 8 letters in which TA occurs in the end. Considering TA as one letter there are seven letters out of which one can be in 7 ways. Therefore,

$$P(A/E_1) = \frac{1}{7}$$

If E_2 has occurred, then the letter came from Tatanagar. In the word TATANAGAR there are 9 letters in which TA occurs twice. Considering one of the two TA's as one letter there are 8 letters. Therefore,

$$P(A/E_2) = \frac{2}{8}$$

By Baye's Theorem, we have

(i)
$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{1}{7}}{\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{2}{8}} = \frac{4}{11}$$

(ii)
$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{2}{8}}{\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{2}{8}} = \frac{7}{11}$$

Example 3.29. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart.

Solution. Let E_1, E_2, E_3, E_4 and A be the events as defined below:

E_1 = the missing card is a heart card.

E_2 = the missing card is a spade card.

E_3 = the missing card is a club card.

E_4 = the missing card is a diamond card, and

A = Drawing two heart cards from the remaining cards.

Solution. The information from responses during market survey is as follows:

	Delhi	Kolkata	Chennai	Mumbai	Total
Yes	45	55	60	50	210
No	35	45	35	45	160
No opinion	5	5	5	5	20
Total	85	105	100	100	390

Let X denote the event that a consumer selected at random preferred brand A. Then :

(a) The probability of a consumer preferred brand A, given that he was from Chennai :

$$P(X|C) = \frac{P(X \cap C)}{P(C)} = \frac{\frac{60}{390}}{\frac{100}{390}} = \frac{3}{5}$$

(b) The probability that the consumer belongs to Mumbai, given that he preferred brand A:

$$P(M|X) = \frac{P(M \cap X)}{P(X)} = \frac{\frac{50}{390}}{\frac{210}{390}} = \frac{5}{21}$$

Example 1.32. Data on the readership of a certain magazine show that the proportion of male readers under 35 is 0.40 and over 35 is 0.20. If the proportion of readers under is 0.70, find the proportion of subscribers that are 'females over 35 years'. Also calculate the probability that a randomly selected male subscriber is under 35 years of age.

Solution. Let us define the following events:

A : Reader of the magazine is a male

B : Reader of the magazine is over 35 years of age

Then in usual notations, we are given :

(i) The proporation of subscribers that are females over 35 years is :

$$P(A \cap B) = 0.20,\ P(A \cap \bar{B}) = 0.40$$

and
$$P(\bar{B}) = 0.70 \Rightarrow P(B) = 0.30$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$
$$= 0.30 - 0.20 = 0.10$$

(ii) The probability that a randomly selected male subscriber is under 35 years is :

$$P(\bar{B}|A) = \frac{P(A \cap \bar{B})}{P(A)} = \frac{0.40}{0.60} = \frac{2}{3}$$

$$[\because P(A) = P(A \cap B) + (A \cap \bar{B}) = 0.20 + 0.40 = 0.60]$$

Example 1.33. If the probability that a communication system will have high fidelity is 0.81 and the probability that it will have high fidelity and selectivity is 0.18, what is the probability that a system with high fidelity will also have selectivity?

Solution. Let A be the event that represent a communication system will have high fidelity

$\therefore$
$$P(A) = 0.81$$

Let $(A \cap B)$ be the event that represents high fidelity and selectivity

$\therefore$
$$P(A \cap B) = 0.18$$

$\therefore$ The probability that a system will have high fidelity will also high selectivity (by using conditional probability) is

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.18}{0.81} = \frac{2}{9}$$

Example 1.34. A couple has two children. Find the probability that both children are boys, if it is known that at least one of the children is a boy.

Solution. Let B_i and G_i stands for i^{th} child be a boy and girl respectively. Then sample space can be expressed as

$$S = \{B_1 B_2, B_1 G_2, G_1 B_2, G_1 G_2\}$$

Consider the following events

$A =$ both the children are boys

$B =$ at least one of the children is a boy

Then $A = \{B_1 B_2\}$

$B = \{B_1 G_2, G_1 B_2, B_1 B_2\}$

So $A \cap B = \{B_1 B_2\}$

Required
$$P(B/A) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example 1.35. The probability that a student selected at random from a class will pass in a Mathematics is $\frac{4}{5}$ and the probability that he/she passes in Mathematics and Computer Science is $\frac{1}{2}$. What is the probability that he/she will pass in computer science, if it is known that he has passed in mathematics?

Solution. Probability (Pass in Mathematics)

$$= \frac{4}{5} = P(M)$$

Probability (Passes in Mathematics and Computer Science)

$$= \frac{1}{2} = P(M \cap C)$$

$$P(C) = ?$$

$$P(C|M) = \frac{P(M \cap C)}{P(M)} = \frac{\frac{1}{2}}{\frac{4}{5}}$$

$$P(C|M) = \frac{1}{2} \times \frac{5}{4} = \frac{5}{8}$$

Example 1.36. **A and B are two independent witnesses (i.e., there is no collision between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statement. Show that the probability that this statement is true is $\frac{xy}{1-x-y+2xy}$.**

Solution. Let A_1 be the event that A and B agree in a statement and A_2 be the event that their statement is correct.

Then $P(A_1) = xy + (1-x)(1-y) = 1 - x - y + 2xy$

$$P(A_1 \cap A_2) = xy$$

Now

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)}$$

$\therefore$

$$= \frac{xy}{1-x-y+2xy}$$

Example 1.37. **A bag contains 19 tickets numbered from 1 to 19. A ticket is drawn and then another ticket is drawn without replacement. Find the probability that both ticket will show even numbers.**

Solution. Let A be the event of drawing an even number ticket in first draw and B be the event of drawing an even numbered ticket in the second draw then

Required probability $= P(A \cap B) = P(A)\, P(B|A)$

Since there are 19 tickets numbered 1 to 19 in the bag out of which 9 are even numbered *i.e.*, 2, 4, 6, 8, 10, 12, 16, 18. Therefore,

$$P(A) = \frac{9}{19}$$

Since the ticket drawn in the first draw is not replaced therefore second drawn in from the remaining 18 tickets out of the which 8 are even numbered

$$P(B|A) = \frac{8}{18} = \frac{4}{9}$$

Hence, required probability $= P(A \cap B)$

$$P(A) \cdot P(B|A) = \frac{9}{19} \times \frac{4}{9} = \frac{4}{19}$$

Example 1.38. **Two integers are selected at random from 1 to 11. If the sum is even, find the probability that both the numbered are odd.**

Solution. Consider the following events

A = Both the numbers choosen are odd

B = The sum of the numbers chosen is even

Since the sum two integers is even if either both are even or both are odd therefore,

$$P(A) = \frac{{}^6C_2}{{}^{11}C_2}$$

$$P(B) = \frac{{}^6C_2 + {}^5C_2}{{}^{11}C_2}$$

$$P(A \cap B) = \frac{{}^6C_2}{{}^{11}C_2}$$

Now required probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{{}^6C_2}{{}^{11}C_2}}{\frac{{}^6C_2 + {}^5C_2}{{}^{11}C_2}} = \frac{{}^6C_2}{{}^6C_2 + {}^5C_2} = \frac{15}{15+10} = \frac{15}{25} = \frac{3}{5}$$

Example 1.39. **In a certain college 25% of the students failed in Probability and 15% of the students failed in Statistics. A student is selected at random and 10% of the students failed both in Probability and Statistics.**

(a) If he failed in Statistics, what is the probability that he failed in Probability?

(b) If he failed in Probability, what is the probability that he failed in Statistics?

(c) What is the probability that he failed in Probability or Statistics?

Solution. A : Student failed in Probability

B : Student failed in Statistics

$$P(A) = \frac{25}{100}$$

$$P(B) = \frac{15}{100}$$

$$P(A \cap B) = \frac{10}{100}$$

(a) Require probability $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{10}{100}}{\frac{15}{100}} = \frac{2}{3}$

(b) Required probability $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{10}{100}}{\frac{25}{100}} = \frac{2}{5}$

1.21. MODE OF BINOMIAL DISTRIBUTION

Mode is the value of r at which $p(r)$ has maximum value. Let r be the mode of binomial distribution, then

$$p(r) \geq p(r+1) \quad \text{and} \quad p(r) \geq p(r-1)$$

Consider

$$p(r) \geq p(r+1) \quad \Rightarrow \quad \frac{p(r)}{p(r+1)} \geq 1$$

$$\Rightarrow \quad \frac{{}^nC_r p^r q^{n-r}}{{}^nC_{r+1} p^{r+1} q^{n-r-1}} \geq 1 \quad \Rightarrow \quad \frac{(r+1)}{(n-r)} \frac{q}{p} \geq 1$$

$$\Rightarrow \quad (r+1)q \geq (n-r)p \quad \Rightarrow \quad (p+q)r \geq np - q$$

$$\Rightarrow \quad r \geq np - 1 + p \quad \Rightarrow \quad [(n+1)p - 1] \leq r \qquad \ldots(i)$$

Again consider

$$p(r) \geq p(r-1) \quad \Rightarrow \quad \frac{p(r)}{p(r-1)} \geq 1$$

$$\Rightarrow \quad \frac{{}^nC_r p^r q^{n-r}}{{}^nC_{r-1} p^{r-1} q^{n-r+1}} \geq 1 \quad \Rightarrow \quad \frac{n-r+1}{r} \frac{p}{q} \geq 1$$

$$\Rightarrow \quad (n-r+1)p \geq qr \quad \Rightarrow \quad r \leq (n+1)p \qquad \ldots(ii)$$

From equations (i) and (ii)

$$[(n+1)p - 1] \leq r \leq (n+1)p$$

Case (i) : If $(n+1)p$ is not an integer, then mode is the integral part of $(n+1)p$. In this case the distribution is called 'unimodal'.

Case (ii) : If $(n+1)p$ is an integer then both $(n+1)p$ and $[(n+1)p - 1]$ will represent modes. In this case the distribution is called 'bimodal'.

Constants of Binomial Distribution

$$\text{Mean} = np$$

$$\text{Standard deviation} = \sqrt{npq}$$

$$\text{First moment or } \mu_1 = 0$$

$$\text{Second moment or } \mu_2 = npq$$

$$\text{Third moment or } \mu_3 = npq\,(q-p)$$

$$\text{Fourth moment or } \mu_4 = 3n^2p^2 + npq\,(1-6pq)$$

$$\beta_1 = \frac{(q-p)^2}{npq}$$

$$\beta_2 = 3 + \frac{1-6pq}{npq}$$

1.22. CONDITIONS FOR APPLICATION OF BINOMIAL DISTRIBUTION

1. The variable should be discrete *i.e.*, defectives should could be 1, 2, 3, 4 or 5 etc., and never 1.5, 2.1 or 3.41 etc.
2. A dichotomy exists. In other words, the happening of events must be of two alternative. It must be either a success or failure.
3. The number of trials n should be finite and small.
4. The trials or events must be independent. The happening of one event must not affective happening of other events. In other words, statistical independence must exist.
5. The trial or events must be repeated under identical conditions.

1.23. RECURSION FORMULA OR RECURRENCE RELATION FOR BINOMIAL DISTRIBUTION

We known that for the Binomial distribution

$$P(X=r) = {}^nC_r p^r q^{n-r}$$

and

$$P(X=r+1) = {}^nC_{r+1} p^{r+1} q^{n-r-1}$$

$$\Rightarrow \quad \frac{P(X=r+1)}{P(X=r)} = \frac{{}^nC_{r+1} p^{r+1} q^{n-r-1}}{{}^nC_r p^r q^{n-r}}$$

$$= \frac{n!}{(r+1)!(n-r-1)!} \times \frac{r!(n-r)!}{n!} \times \frac{p^{r+1}q^{n-r-1}}{p^r q^{n-r}} = \frac{n-r}{r+1} \frac{p}{q}$$

$$\Rightarrow \quad P(X=r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q} P(X=r); \; r = 1, 2, 3 \ldots$$

which is the required recurrence formula. Applying this formula successively, we can find $P(X=1)$, $P(X=2)$, $P(X=3)$ .... if $P(X=0)$ is known.

SOLVED EXAMPLES

Example 1.54. Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.

Solution. When one coin is thrown,

The probability of getting a head $= \frac{1}{2}$

$$\therefore \quad p = \frac{1}{2}$$

The probability of not getting a head $= 1 - \frac{1}{2} = \frac{1}{2}$

$$\therefore \quad q = \frac{1}{2}$$

Then P (at least 7 heads) $= P$ (7 heads) $+ P$ (8 heads) $+ P$ (9 heads) $+ P$ (10 heads)

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$$

$$= \frac{1}{2^{10}}\left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right]$$

$$= \frac{120+45+10+1}{1024} = \frac{176}{1024} = \frac{11}{64}$$

Example 1.55. In a lot of 200 articles 10 are defective, find the probability of : (*i*) no defective article, (*ii*) one defective article, (*iii*) at least one defective article, in a random sample of 20 articles.

Solution. The probability of defective article is $\frac{10}{200} = \frac{1}{20}$

$$\therefore \quad p = \frac{1}{20}$$

The probability of non-defective article $= 1 - \frac{1}{20} = \frac{19}{20} \Rightarrow q = \frac{19}{20}$

(*i*) The probability of no defective article out of 20

$$= {}^{20}C_0 (p)^0 q^{20} = \left(\frac{19}{20}\right)^{20} \qquad \left[\because {}^{20}C_0 = 1\right]$$

(*ii*) The probability of exactly one defective article

$$= {}^{20}C_1 (p)^1 (q)^{19} = 20 \times \frac{1}{20} \times \left(\frac{19}{20}\right)^{19} = \left(\frac{19}{20}\right)^{19}$$

(*iii*) The probability of at least one will be defective

$$= 1 - [\text{probability that none will be defective}]$$

$$= 1 - {}^{20}C_{20}\left(\frac{19}{20}\right)^{20} = 1 - \left(\frac{19}{20}\right)^{20}$$

Example 1.56. If on an average, one ship out of 10 is wrecked, find the probability that out of 5 ships expected to arrive the port, at least four will arrive safely.

Solution. p be the probability of a ship arriving safely $= 1 - \frac{1}{10} = \frac{9}{10}$

$$q = 1 - \frac{9}{10} = \frac{1}{10}$$

Binomial distribution is $\left(\frac{1}{10} + \frac{9}{10}\right)^5$

Probability that at least four ships out of five arrive safely

$$= P(4) + P(5) = {}^5C_4\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^4 + {}^5C_5\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^5$$

$$= \left(\frac{9}{10}\right)^4 \frac{14}{10} = \left(\frac{9}{10}\right)^4 \frac{7}{5} = 0.91854$$

Example 1.57. The probability that a man aged 60 will live to be 70 is 0.65. What is the probability that out of 10 men aged 60 now, at least 7 would live to be 70?

Solution. Probability of survival upto the age of 70

$$= p = 0.65$$

Probability of non-survival upto the age of 70

$$= q = 1 - p = 1 - 0.65 = 0.35$$

Probability that out of 10 such men at least 7 would survive as desired

= Probability that exactly 7 would survive +
Probability that exactly 8 would survive +
Probability that exactly 9 would survive +
Probability that exactly 10 would survive

$$= P(7) + P(8) + P(9) + P(10)$$

$$= {}^{10}C_7 p^7 q^3 + {}^{10}C_8 p^8 q^2 + {}^{10}C_9 p^9 q + {}^{10}C_{10} p^{10}$$

$$= 120\, p^7 q^3 + 45\, p^8 q^2 + 10\, p^9 q + p^{10}$$

$$= p^7 (120\, q^3 + 45\, pq^2 + 10\, p^2 q + p^3)$$

$$= (0.65)^7 \left[120 \times (0.35)^3 + 45\,(0.65)\,(0.35)^2 + 10\,(0.65)^2\,(0.35) + (0.65)^3\right]$$

$= 0.514$, the required result.

Example 1.58. Six dice are thrown together at a time, the process is repeated 729 times. How many times do you expect at least three dice to have 4 to 6?

Solution. The chance of getting 4 or 6 with one dice is

$$\frac{2}{6} \text{ i.e., } p = \frac{1}{3} \quad \text{and} \quad q = 1 - \frac{1}{3} = \frac{2}{3}$$

In one throw of six dice together, we have probability of getting at least 3 dice to have 4 or 6.

$$= P(3) + P(4) + P(5) + P(6)$$

$$= {}^6C_3 p^3 q^3 + {}^6C_4 p^4 q^2 + {}^6C_5 p^5 q + {}^6C_6 p^6$$

$$= 20\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + 15\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + 6\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \left(\frac{1}{6}\right)^6$$

$$= \frac{1}{(3)^6}[160 + 60 + 12 + 1] = \frac{233}{(3)^6}$$

Now the process is repeated 729 times

$\therefore$ Required number of times at least 3 dice have 4 or 6

$$= 729 \times \frac{233}{(3)^6} = 233, \text{ the required result.}$$

Note. In the above case the binomial distribution is $N(q + p)^n$ where $N = 729$, $n = 6$

Example 1.59. If the sum of the mean and the variance of binomial distribution of 5 trials is 4.8, find the distribution.

Solution. Let the required binomial distribution be $^nC_r p^r q^{n-r}$ where n = number of trials = 5

Mean of the distribution = np

and the variance of the distribution = npq

By the given condition

$$np + npq = 4.8$$

$$5p + 5pq = 4.8$$

$$\Rightarrow \quad 5p(1+q) = 4.8 \qquad [\because p = 1-q]$$

$$50(1-q^2) = 48 \quad \Rightarrow \quad 50 - 50q^2 = 48$$

$$\Rightarrow \quad 50q^2 = 2 \Rightarrow q = \frac{1}{5}$$

$$\Rightarrow \quad p = 1 - q = 1 - \frac{1}{5} = \frac{4}{5}$$

$\therefore$ Hence, the required binomial distribution is $^5C_r\left(\frac{4}{5}\right)^r\left(\frac{1}{5}\right)^{5-r}$

Example 1.60. The probability that a bomb dropped from a place will strike the target is $\frac{1}{5}$. If six bombs are dropped, find the probability that : (*i*) exactly two will strike the target, (*ii*) at least two will strike the target.

Solution. The probabilities of 0, 1, 2 ..., successes are given by the respective terms in the expansion of

$$(q+p)^6 = \left(\frac{4}{5}+\frac{1}{5}\right)^6, \text{ since } p = \frac{1}{5}, q = \frac{4}{5} \text{ and } n = 6.$$

$\therefore P(2)$ = The probability that exactly two bombs will strike the target

$$= {}^6C_2 p^2 q^4 = \frac{6.5}{1.2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^4 = 0.246$$

The probability that at least 2 bombs will strike the target

$$= 1 - [P(0) + P(1)]$$

$$= 1 - q^6 - {}^6C_1 q^5 p = 1 - (0.8)^6 - 6(0.2)\,(0.8)^5$$

$$= 1 - 0.2621 - 0.3932 = 0.345$$

Example 1.61. Assuming that half the population are consumers of rice so that the chance of an individual being a rice consumer is $\frac{1}{2}$ and assuming that 100 investigations each take 10 individuals to see whether they are rice consumers. How many investigations would you expect to report that three people or less consumers?

Solution. Here $p = \frac{1}{2}, q = \frac{1}{2}, n = 10, N = 100$

$\therefore$ The probability that r persons out of 10 persons are consumers of rice is given by

$$P(r) = {}^{10}C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{10-r}$$

$\therefore$ The expected number of investigators (*i.e.*, expected frequencies) who would report that three or less people were consumers of rice.

$$= 100\,[P(0) + P(1) + P(2) + P(3)]$$

$$= 100\left[{}^{10}C_0\left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10}\right]$$

$$= \frac{100}{2^{10}}[1 + 10 + 45 + 120] = \frac{17600}{1024} = 17 \text{ approx.}$$

Example 1.62. A die is thrown 5 times. Getting an even number greater than 2 is considered a success. Calculate $P(X = r)$ for $r = 1, 2, 3, 4, 5$ from recurrence formula.

Solution. Let p be the probability of getting an even number greater than 2 on a die.

$$\Rightarrow \quad p = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \quad q = 1 - p \Rightarrow q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\therefore \quad \frac{p}{q} = \frac{1}{2}. \quad \text{Also } n = 5$$

$P(X) = 0$ = Probability of no success in 5 trials

$$= {}^5C_0 (q)^5 = \left(\frac{2}{3}\right)^5 = 0.1317$$

Recurrence formula for binomial distribution is

$$P(X = r+1) = \frac{n-r}{r+1}\cdot\frac{p}{q}\,P(X = r)$$

$$= \frac{5-r}{r+1}\left(\frac{1}{2}\right)P(X = r) \qquad \ldots(i)$$

Putting $r = 0$, in (*i*), $P(X = 1) = 5\left(\frac{1}{2}\right)P(X = 0) = 5\left(\frac{1}{2}\right)(0.1317) = 0.3292$

Putting $r = 1$, in (*i*), $P(X = 2) = 2\left(\frac{1}{2}\right)P(X = 1) = P(X = 1) = 0.3292$

Putting $r = 2$, in (*i*), $P(X = 3) = (1)\left(\frac{1}{2}\right)P(X = 2) = \frac{1}{2}\cdot(0.3292) = 0.1646$

(iii) The probability of hitting the target exactly once

$$P(X=1)\ {}^{4}C_1\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 = \frac{192}{729}$$

(b) If he fires so that the probability of his hitting the target atleast once is greater than 3/4 then

$$P(X \geq 1) = 3/4$$

$\Rightarrow$ $1 - P(X<1) > 3/4$

$\Rightarrow$ $1 - P(X=0) > 3/4$

$\Rightarrow$ $1 - (2/3)^n > 3/4$

$\Rightarrow$ $1/4 > (2/3)^n$

$\Rightarrow$ $\frac{1}{2^2} > \left(\frac{2}{3}\right)^n$

$\Rightarrow$ $3n > 2^{n+2}$

$\Rightarrow$ This inequality is satisfied for $n = 4$

This inequality is satisfied for $n=4$ that probability of hitting the target atleast once is greater than 3/4.

$\therefore$ He must fire 4 times to that probability of hitting the target atleast once is greater than 3/4.

Example 1.67. A student takes a true-false examination consisting of 8 questions. He guesses each answer. The guesses are made at random. Find the smallest value of n so that the probability of guessing atleast n correct answers is less than 1/2.

Solution. For given data, we have to find

$$P(X \geq n) < 1/2$$

Now probability of guessing a correct answer is 1/2 and guessing a wrong answer is 1/2

$\therefore$ $p = 1/2, q = 1/2, n = 8$

Using Binomial distribution

$$P(X \geq n) < 1/2$$

$\Rightarrow$ $1 - P(X<n) < 1/2$

$\Rightarrow$ $1 - [P(X=0) + P(X=1) \ldots P(X=n-1)] < 1/2$

$\Rightarrow$ $P(X=0) + P(X=1) \ldots P(X=n-1) > 1/2$

$\Rightarrow$ ${}^{8}C_0\left(\frac{1}{2}\right)^8 + {}^{8}C_1\left(\frac{1}{2}\right)^8 + \ldots {}^{8}C_{n-1}\left(\frac{1}{2}\right)^8 > \frac{1}{2}$

$\Rightarrow$ $\left(\frac{1}{2}\right)^8\left[{}^{8}C_0 + {}^{8}C_1 + \ldots + {}^{8}C_{n-1}\right] > \frac{1}{2}$

This inequality is satisfied if $n - 1 = 4$

$\Rightarrow$ $n = 5$

Example 1.68. Fit a binomial distribution to following data, when tossing 5 coins.

x	0	1	2	3	4	5
f	2	14	20	34	22	8

Solution.

$$\Sigma f = 100 = N$$

$$n = 5$$

$$\bar{x} = \text{mean}$$

$$= \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{0\times 2 + 1\times 14 \ldots + 5\times 8}{100}$$

$$= 2.84$$

$$\text{Mean} = \bar{x} = np = 2.84$$

$\Rightarrow$ $p = 0.57$ and $q = 0.43$ $[\because n = 5]$

Using binomial distribution

$$P(X=x) = {}^{n}C_x p^x q^{n-x}\ ;\ x = 0, 1, 2 \ldots n$$

and expected frequency obtained from

$$f(x) = N.\ P(x)$$

x	f	$p(x) = {}^{n}C_x p^x q^{n-x}$	$f(x) = N.P(x)$
0	2	${}^{5}C_0\,(0.57)^0\,(0.43)^5 = (0.43)^5$	$100 \times (0.43)^5 = 1$
1	14	${}^{5}C_1\,(0.57)^1\,(0.43)^4 = 0.098$	$100 \times (0.098) = 10$
2	20	${}^{5}C_2\,(0.57)^2\,(0.43)^3 = 0.260$	$100 \times (0.260) = 26$
3	34	${}^{5}C_3\,(0.57)^3\,(0.43)^2 = 0.342$	$100 \times (0.342) = 34$
4	22	${}^{5}C_4\,(0.57)^4\,(0.43) = 0.224$	$100 \times (0.224) = 22$
5	8	$C_5\,(0.57)^5\,(0.43)^0 = 0.059$	$100 \times (0.059) = 6$

Example 1.69. In a Binomial distribution consisting of 5 independent trials, probability of 1 and 2 success are 0.4096 and 0.2048 respectively. Find the parameter 'p' of distribution.

Solution. Given $n = 5$

For Binomial distribution

$$P(X=x) = {}^{n}C_x p^x q^{n-x};\ x = 0, 1, 2, 3 \ldots n$$

$$P(X=1) = {}^{n}C_1\, p^1\, q^{n-1} = 0.4096 \quad \ldots(i)$$

$$P(X=2) = {}^{n}C_2\, p^2\, q^{n-2} = 0.2048 \quad \ldots(ii)$$

Dividing eqn. (ii) with eqn. (i)

$$\frac{{}^{5}C_2 p^2 q^3}{{}^{5}C_1 p^1 q^4} = \frac{10p}{5q} = \frac{0.4096}{0.2048}$$

$\Rightarrow$ $\frac{2p}{1-p} = 1/2$

$\Rightarrow$ $4p = 1 - p$

$\Rightarrow$ $p = 1/5$

EXERCISE 1.4

1. During war, 1 ship out of 9 was sunk on an average in making a certain voyage. What was the probability that exactly 3 out of the convey of 6 ships would arrive safely? $\left[\text{Ans. } \frac{10240}{9^6}\right]$
2. The incidence of occupational disease in an industry is such that the workers have a 20% change of suffering from it. What is the probability that out of six workers chosen at random, four or more will suffer from the disease? $\left[\text{Ans. } \frac{53}{3125}\right]$
3. The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that : (*i*) exactly 2 will be defective, (*ii*) none will defective, (*iii*) at least two will be defective.
4. A dice is thrown. If "getting an odd number" is a "success", what is the probability of :
   (*i*) 5 successes
   (*ii*) at least 5 successes
   (*iii*) almost 5 successes $\left[\text{Ans. } (i)\ \frac{3}{32}, (ii)\ \frac{7}{64}, (iii)\ \frac{63}{64}\right]$
5. If on an average one ship in every ten is wrecked, find the probability that out 5 ships expected to arrive, 4 at least will arrive safely. $\left[\text{Ans. } \frac{7}{5}\left(\frac{9}{10}\right)^4\right]$
6. Five cards are drawn successively with replacement from a well-shuffled pack of 52 cards. What is the probability that:
   (*i*) all the five cards are spades
   (*ii*) only 3 cards are spades
   (*iii*) none is a spade [Hints. Number of spades : 13] $\left[\text{Ans. } (i)\left(\frac{1}{4}\right)^5, (ii)\, 90\left(\frac{1}{4}\right)^5, (iii)\left(\frac{3}{4}\right)^5\right]$
7. State reason to justify whether the following statement is true or false. "The mean of a Binomial distribution is 6 and standard deviation is 7". [Ans. false]
8. *A* and *B* take turns in throwing dice, the first to throw 10 being the winner. If *A* throws firstly, show that they have change of winning as 12 : 11.
9. In a bombing action there is 50% chance that any bomb will strike target. Two direct hits are needed to destroy and target completely. How many bombs are required to be dropped to give a 99% chance of better of completely destroying the target. [Ans. 11]
10. Out of 800 families with 5 children each, how many would you expect to have: (*i*) 3 boys, (*ii*) 5 girls, (*iii*) either 2 or 3 boys? Assume equal probabilities for boys and girls. [Ans. (*i*) 250, (*ii*) 25, (*iii*) 500]
11. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts. [Ans. 458]

12. The probability that a bulb produced by a factory will fuse after 150 days of use is $\frac{1}{20}$. Find the probability that out of 5 such bulbs :
    (*i*) None
    (*ii*) not more than one
    (*iii*) more than one
    (*iv*) at least one, will fuse after 150 days of use.
    $$\left[\text{Ans. } (i)\left(\frac{19}{20}\right)^5, (ii)\,\frac{24}{20}\left(\frac{19}{20}\right)^4, (iii)\,1-\frac{23}{20}\left(\frac{19}{20}\right)^4, (iv)\,1-\left(\frac{19}{20}\right)^5\right]$$
13. Four coins are tossed 160 times. The number of times *r* heads occur ($r = 0, 1, 2, 3, 4$) is given below:

r	0	1	2	3	4
No. of times	8	34	69	43	6

Fit a binomial distribution to this data on the hypothesis that coins are unbiased.
$$\left[\begin{array}{llllll}\text{Ans. } r = & 0 & 1 & 2 & 3 & 4\\ f(r) = & 10 & 40 & 60 & 40 & 10\end{array}\right]$$

14. If successive trials are independent and the probability of success on any trial is *p*, shown that the first success occurs on the *n*th trial is
    $$p(1-p)^{n-1},\ n = 1, 2, 3, \ldots$$
15. The mean of a Binomial distribution is 3 and variance is 4. Give your comments.
16. If the chance that one of the ten telephone lines is busy at an instant is 0.2.
    (*i*) What is the chance that 5 of the lines are busy?
    (*ii*) What is the probability that all the lines are busy? [Ans. (*i*) 0.02579, (*ii*) 21.024×10^{-7}]
17. The following data are the number of seeds germinating out of 10 on damp filter for 80 sets of seeds. Fit a binomial distribution of this data :

x	0	1	2	3	4	5	6	7	8	9	10	Total
f	6	20	28	12	8	6	0	0	0	0	0	80

[Ans. $80(0.7825 + 0.2175)^{10}$]

18. Mark the correct answer.
    (*a*) The probability that a man hit a target is given as $\frac{1}{5}$. Then his probability of atleast one hit in 10 shots is
    (*i*) $1-\left(\frac{4}{5}\right)^{10}$ (*ii*) $\left(\frac{1}{5}\right)^{10}$ (*iii*) $1-\left(\frac{1}{5}\right)^{10}$ (*iv*) None
    (*b*) 8 coins are tossed simultaneously. The probability of getting at least 6 heads is
    (*i*) $\frac{57}{64}$ (*ii*) $\frac{229}{256}$ (*iii*) $\frac{7}{64}$ (*iv*) $\frac{37}{256}$
    [Ans. (*a*) (*i*), (*b*) (*iv*)]

Then

$$P(E_1) = \frac{13}{52} = \frac{1}{4},$$

$$P(E_2) = \frac{13}{52} = \frac{1}{4},$$

$$P(E_3) = \frac{13}{52} = \frac{1}{4},$$

$$P(E_4) = \frac{13}{52} = \frac{1}{4},$$

$P(A/E_1)$ = Probability of drawing two heart cards given that one heart card is missing

$$= \frac{^{12}C_2}{^{51}C_2}$$

$P(A/E_2)$ = Probability of drawing two heart cards given that one spade card is missing

$$= \frac{^{13}C_2}{^{51}C_2}$$

Similarly, $P(A/E_3) = \frac{^{13}C_2}{^{51}C_2}$ and $P(A/E_4) = \frac{^{13}C_2}{^{51}C_2}$

By Baye's Theorem, we have

Required probability $= P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3) + P(E_4)\,P(A/E_4)}$$

$$= \frac{\frac{1}{4}\cdot\frac{^{12}C_2}{^{51}C_2}}{\frac{1}{4}\cdot\frac{^{12}C_2}{^{51}C_2} + \frac{1}{4}\cdot\frac{^{13}C_2}{^{51}C_2} + \frac{1}{4}\cdot\frac{^{13}C_2}{^{51}C_2} + \frac{1}{4}\cdot\frac{^{13}C_2}{^{51}C_2}}$$

$$= \frac{^{12}C_2}{^{12}C_2 + {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2} = \frac{66}{66 + 78 + 78 + 78} = \frac{11}{50}.$$

EXERCISE 3.3

1. Bag A contains 2 white and 3 red balls, and bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from bag B. [Ans. $\frac{5}{[illegible]}$]

2. Urn A contains 1 white, 2 black and 3 red balls; urn B contains 2 white, 1 black and 1 red ball; and urn C contains 4 white, 5 black and 3 red balls. One urn is chosen at random and two balls are drawn. These happen to be one white and one red. What is the probability that they come from urn A? [Ans. $\frac{33}{118}$]

3. In a bulb factory, three machines, A, B, C, manufacture 60%, 25% and 15% of the total production respectively. Of their respective outputs, 1%, 2% and 1% are defective. A bulb is drawn at random from the total product and it is found to be defective. Find the probability that it was manufactured by machine C. [Ans. $\frac{2}{25}$]

4. In a certain college, 4% of boys and 1% of girls are taller than 1.75 metres. Furthermore, 60% of the students are girls. If a student is selected at random and is taller than 1.75 metres, what is the probability that the selected student is a girl? [Ans. $\frac{3}{11}$]

5. Two groups are competing for the positions on the board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.8 and when the second group wins, the corresponding probability is 0.3. Find the probability that the new product introduced was by the second group. [Ans. $\frac{3}{5}$]

6. The contents of three urns are as follows:

   Urn 1: 7 white, 3 black balls,
   Urn 2: 4 white, 6 black balls, and
   Urn 3: 2 white, 8 black balls

   One of these urns is chosen at random with probabilities 0.20, 0.60 and 0.20 respectively. From the chosen urn two balls are drawn at random without replacement. If both these balls are white, what is the probability that these came from urn 3? [Ans. $\frac{1}{40}$]

7. Suppose 5 men out of 100 and 25 women out of 1000 are good orator. An orator is chosen at random. Find the probability that a male person is selected. Assume that there are equal number of men and women. [Ans. $\frac{2}{3}$]

8. In a class, 5% of the boys and 10% of the girls have an IQ of more than 150. In this class, 60% of the students are boys. If a student is selected at random and found to have an IQ of more than 150, find the probability that the student is a boy. [Ans. $\frac{3}{7}$]

9. A box contains 2 gold and 3 silver coins. Another box contains 3 gold and 3 silver coins. A box is chosen at random, and a coin is drawn from it. If the selected coin is a gold coin, find the probability that it was drawn from the second box. [Ans. $\frac{5}{9}$]

❑❑❑

Putting $x = 3$, in (i), $P(X=4) = \frac{2}{4}\cdot\frac{1}{2}P(X=3) = \frac{1}{4}(0.1646) = 0.0412$

Putting $x = 4$, in (i), $P(X=5) = \frac{1}{5}\cdot\frac{1}{2}P(X=4) = \frac{1}{10}(0.0412) = 0.0041.$

Example 1.63. Out of 800 families with 4 children each, how many families would be expected to have: (*i*) 2 boys and 2 girls, (*ii*) at least one boy, (*iii*) no girl, (*iv*) at most two girls? Assume equal probabilities for boys and girls.

Solution. Since probability for boys and girls are equal

$$p = \text{Probability of having a boy} = \frac{1}{2}$$

$$q = \text{Probability of having a girl} = \frac{1}{2}$$

$$n = 4,\ N = 800$$

The binomial distribution is $800\left(\frac{1}{2}+\frac{1}{2}\right)^4$.

(*i*) The expected number of families having 2 boys and 2 girls

$$= 800\cdot {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 800\times 6\times\frac{1}{16} = 300$$

(*ii*) The expected number of families having at least one boy

$$= 800\left[{}^4C_1\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4\right]$$

$$= 800\times\frac{1}{16}[4+6+4+1] = 750$$

(*iii*) The expected number of families having no girl having 4 boys

$$= 800\times{}^4C_4\left(\frac{1}{2}\right)^4 = 50$$

(*iv*) The expected number of families having at most two i.e., having at least 2 boys

$$= 800\left[{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4\right]$$

$$= 800\times\frac{1}{16}[6+4+1] = 550.$$

Example 1.64. A student obtained the following answer to a certain problem given to him Mean = 2.4; variance = 3.2 for a binomial distribution. Comment on the result.

Solution. The mean of binomial distribution is np and variance npq. We are given mean $= np = 2.4$

$$\text{Variance} = npq$$

$$2.4q = 3.2$$

$$q = \frac{3.2}{2.4} = 1.333$$

Since the value of q is greater than 1, the given results are inconsistent.

Example 1.65. Ten coins are tossed 1024 times and the following frequencies are observed. Compare these frequencies with the expected frequencies :

Number of heads	0	1	2	3	4	5	6	7	8	9	10
Frequencies	2	10	38	106	188	257	226	128	59	7	3

Solution. Here $n = 10$, $N = 1024$

$$p = \text{The change of getting a head in one toss} = \frac{1}{2}$$

$$\therefore \quad q = 1 - p = \frac{1}{2}$$

The expected frequencies are the respective terms of the binomial $1024\left(\frac{1}{2}+\frac{1}{2}\right)^{10}$

The frequency of r heads $(0 \geq r \geq 10)$ is

$$= 1024\cdot{}^{10}C_r\left(\frac{1}{2}\right)^{10-r}\left(\frac{1}{10}\right) = 1024\times{}^{10}C_r\left(\frac{1}{2}\right)^{10} = {}^{10}C_r$$

Hence, we have the following comparison.

Number of heads	0	1	2	3	4	5	6	7	8	9	10
Observed frequency	2	10	38	106	188	257	226	128	59	7	3
Expected frequency	1	10	45	120	210	252	210	120	45	10	1

$({}^{10}C_0, {}^{10}C_1, {}^{10}C_2$ and so on.)

Example 1.66. Probability of man hitting a target is $\frac{1}{3}$.

(*a*) If he fires of 6 times, what is the probability of hitting: (*i*) at most 5 times, (*ii*) at least 5 times, (*iii*) exactly once

(*b*) If he fires so that the probability of his hitting target atleast once is greater than $\frac{3}{4}$, find n.

Solution. (*a*) Given $p = 1/3$, $q = 1 - 1/3 = 2/3$ $n = 6$

(*i*) The probability of hitting the target almost 5 times.

$$P(X \le 5) = 1 - P(X > 5) = 1 - P(X = 6)$$

$$= 1 - \left[{}^6C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^0\right] = 1 - \frac{1}{729} = \frac{728}{729}$$

(*ii*) The probability of hitting the target atleast 5 times

$$P(X \ge 5) = P(X=5) + P(X=6)$$

$$= {}^6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + {}^6C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^0 = \frac{13}{729}$$

(c) Required probability is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{25}{100} + \frac{15}{100} - \frac{10}{100} = \frac{30}{100} = \frac{3}{10}$$

Example 1.40. **A bag contains 6 white and 9 black balls. Four balls are drawn at a time. Find the probability for the first draw to give four white and second draw to give four black balls in each of the following cases:**

(a) The balls are replaced before the second draw.

(b) The balls are not replaced before the second draw.

Solution. A : Getting 4 white ball in first draw

B : Getting 4 black ball in second draw

(a)

$$P(A) = \frac{{}^6C_4}{{}^{15}C_4} = \frac{6\times5\times4\times3}{15\times14\times13\times12} = \frac{360}{32760}$$

$$P(B) = \frac{{}^9C_4}{{}^{15}C_4} = \frac{9\times8\times7\times6}{15\times14\times13\times12} = \frac{3024}{32760}$$

Required probability $= P(A)\,P(B)$

$$= \frac{360}{32760} \times \frac{3024}{32760} = \frac{6}{5915}$$

(b)

$$P(A) = \frac{{}^6C_4}{{}^{15}C_4} = \frac{6\times5\times4\times3}{15\times14\times13\times12} = \frac{360}{32760}$$

$$P(B \mid A) = \frac{{}^9C_4}{{}^{11}C_4} = \frac{9\times8\times7\times6}{11\times10\times9\times8} = \frac{3024}{7920}$$

Required probability $= P(A)\,P(B/A)$

$$= \frac{360}{3270} \times \frac{3024}{7920} = \frac{3}{715}$$

EXERCISE 1.2

1. If $P(A) = \frac{6}{11}, P(B) = \frac{5}{11}, P(A \cup B) = \frac{7}{11}, P(A \cup B) = \frac{7}{11}$

(i) $P(A \cup B)$ (ii) $P\left(\frac{A}{B}\right)$ (iii) $P\left(\frac{B}{A}\right)$ $\left[\textbf{Ans.}\ (i)\ \frac{4}{11}, (ii)\ \frac{4}{5}, (iii)\ \frac{2}{3}\right]$

2. If $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{3}$, find $P\left(\frac{\bar{A}}{B}\right)$ and $P\left(\frac{\bar{B}}{A}\right)$ $\left[\textbf{Ans.}\ \frac{3}{4}, \frac{3}{5}\right]$

3. A pair of dice is thrown. Let E be the event that sum is greater than or equal to 10 and F be the event the 5 appears on the first dice. Find $P(E/F)$. $\left[\textbf{Ans.}\ \frac{1}{3}\right]$

4. A pair of dice is thrown. If the two numbers appearing on them are different find the probability that

(i) the sum of numbers is 6.

(ii) the sum of number of 4 or less. $\left[\textbf{Ans.}\ (i)\ \frac{30}{36}, (ii)\ \frac{2}{15}\right]$

5. A bag contains 10 white and 15 black balls. Two balls are drawn is succession without replacement. What is the probability that first is white and second is black. $\left[\textbf{Ans.}\ \frac{1}{4}\right]$

6. Find the probability of drawing a diamond card in each of the two consecutive draws from a well shuffled pack of cards. If the card drawn in not replaced after the first draw. $\left[\textbf{Ans.}\ \frac{1}{17}\right]$

7. Two dice are thrown that it is known that first dice shows a six. Find the probability that the sum of numbers showing on the dice is 7.

8. A coin in tossed then a dice is thrown. Find the probability of a 6 given that heads came up. $\left[\textbf{Ans.}\ \frac{1}{6}\right]$

9. The probability that a certain person will buy a shirt is 0.2 the probability that he will buy a trouser is 0.3 and the probability that he will buy a shirt given that he buys a trouser is 0.4. Find the probability that he will buy both shirt and trouser. Find also the probability that he will buy a trouser given that the buys a shirt. [**Ans.** 0.06]

10. A bag contain 10 white and 15 black balls. Two balls are drawn in succession without replacement. What is the probability that first is white and second is black. $\left[\textbf{Ans.}\ \frac{1}{4}\right]$

11. A dice is rolled twice and the sum of the numbers appearing on them is observed to be 6. What is the conditional probability that the number 4 has appeared at least once. $\left[\textbf{Ans.}\ \frac{2}{5}\right]$

12. A dice is rolled twice and the sum of the numbers appearing on them is observed to be 7. What is the conditional probability that the number 2 has appeared at least once. $\left[\textbf{Ans.}\ \frac{1}{3}\right]$

13. Find the probability of drawing a diamond card in each of the two consecutive draws from a well shuffled pack of cards. If the card drawn in not replaced after the first draw. $\left[\textbf{Ans.}\ \frac{1}{17}\right]$

26. A speaks truth in 75% cases and B in 80% cases. In what percentage of cases are they likely to contradict each other in stating the same fact? [Ans. 35%]

27. A purse contains 2 silver and 4 copper coins. A second purse contains 4 silver and 3 copper. If a coin is pulled out at random from one of the two purses, what is the probability that is it a silver coin?

28. A student takes his examination in four subjects P, Q, R, S. He estimates his chances of passing in P as $\frac{4}{5}$, in Q as $\frac{3}{4}$, in R as $\frac{5}{6}$ and in S as $\frac{2}{3}$. To qualify, he must pass in P atleast two other subjects. What is the probability that he qualifies? $\left[\text{Ans. } \frac{61}{90}\right]$

29. Define a random experiment, sample space, event and mutually exclusive events. Give example of each.

1.4. CONDITIONAL PROBABILITY

Let A and B be two events associated with the same sample space of a random experiment. Then the probability of occurrence of A under the condition that B has already occurred, at $P(B) \neq 0$, is called conditional probability, denoted by $P(A/B)$.

We define,
$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)}, \quad \text{where } P(B) \neq 0 \text{ and}$$
$$P(B/A) = \frac{P(B \cap A)}{P(A)}$$
$$= \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)}, \text{ where, } P(A) \neq 0$$

1.4.1. Properties of Conditional Probability

Let A and B be events of a sample space S of an experiment, then we have

Property 1. $P(S/B) = P(B/B) = 1$

We know that
$$P(S/B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Also
$$P(B/B) = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Thus, $P(S/B) = P(B/B) = 1$

Property 2. If A and B are any two events of a sample space S and F is any event of S such that $P(F) \neq 0$, then
$$P((A \cup B)/F) = P(A/F) + P(B/F) - P((A \cap B)/F)$$

In particular, if A and B are disjoint events, then
$$P(A \cup B/F) = \frac{P[(A \cup B) \cap F]}{P(F)}$$

$$= \frac{P[(A \cap F) \cup (B \cap F)]}{P(F)}$$

(By distributive law of union of set over intersection)

$$= \frac{P(A \cap F) + P(B \cap F) - P(A \cap B \cap F)}{P(F)}$$
$$= P(A/F) + P(B/F) - P(A \cap B/F)$$

When A and B are disjoint events, then
$$P((A \cap B)/F) = 0$$
$$\Rightarrow \quad P((A \cup B)/F) = P(A/F) + P(B/F)$$

Property 3. $P(E'/F) = 1 - P(E/F)$

From property 1, we know that
$$P(S/F) = 1$$
$$\Rightarrow \quad P(E \cup E'/F) = 1$$
$$\Rightarrow \quad P(E/F) + P(E'/F) = 1 \qquad \text{Since } E \text{ and } E' \text{ are disjoint events}$$

Thus $P(E'/F) = 1 - P(E/F)$

SOLVED EXAMPLES

Example 1.26. A pair of dice is rolled, find $P(A/B)$ if

A: 2 appears on atleast one dice.

B: sum of numbers appearing on dice is 6.

Solution. We have
$$A = \{(2,1)(2,2)(2,3)(2,4)(2,5)(2,6)(1,2)(3,2)(4,2)(5,2)(6,2)\}$$
$$B = \{(1,5)(2,4)(3,3)(4,2)(5,1)\}$$
$$A \cap B = \{(2,4)(4,2)\}$$
$$P(A \cap B) = \frac{2}{36}$$
$$P(B) = \frac{5}{36}$$

Therefore,
$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$

Example 1.27. Two marbles are drawn successively from a box containing 3 black and 4 white marbles. Find the probability that both marbles are black if the first marble is not required before the second drawing.

Solution. Let B_1 is the event of drawing the first black marble.