6.1 SERIES OF COMPLEX TERMS. 6

Let $(u_1+iv_1)+(u_2+iv_2)+\dots+(u_n+iv_n)+\dots$ — (1) be an infinite series of complex terms where u's and v's are real numbers. The series (1) is stb convergent if Σu_n and Σv_n both are convergent.

If Σu_n converges to the sum A and Σv_n converges to the sum B then the series (1) converges to the sum $A+iB$.

- We write $\lim_{n\to\infty} \Sigma(u_n+iv_n) = A+iB$

- Also if the series (1) is convergent, then $\lim_{n\to\infty} (u_n+iv_n)=0$.

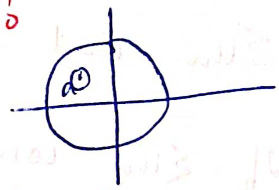
- The series (1) is said to be absolutely convergent if the series $|u_1+iv_1|+|u_2+iv_2|+\dots+|u_n+iv_n|$ is convergent.

- If a power series $a_0+a_1z+a_2z^2+\dots+a_nz^n+\dots$ converges for $z=z_1$, then it converges absolutely for every value of z satisfying $|z|<|z_1|$.

TAYLOR'S SERIES

If a function $f(z)$ is analytic inside a circle C with centre at $z=a$ then it can be expanded in the series

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$



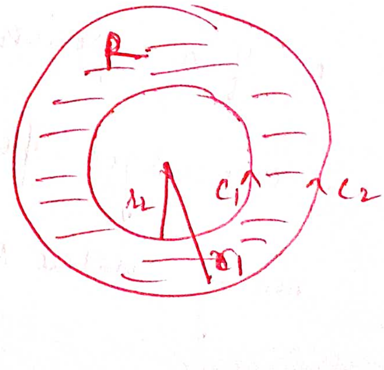
which is convergent at every point inside C.

- In expanding a function $f(z)$ by Taylor's series at a point 'a' we require that $f(z)$ be analytic at $z=a$.

In next series (Laurent series) $f(z)$ need not to be analytic at point a.

LAURENT SERIES

If $f(z)$ is analytic in an annular region R bounded by concentric circles C_1 and C_2 of radii r_1 and r_2 $(r_1 > r_2)$ and centre at 'a', then $\forall z$ in R



we have

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^n + \cdots + \frac{a_{-1}}{(z-a)}$$

$$+ \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + \cdots$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int \frac{f(t)\,dt}{(t-a)^{n+1}}$

- In other words, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} \quad b_n = a_{-n}$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int \frac{f(t)\,dt}{(t-a)^{n+1}}$ $\quad n = 0, 1, 2 \ldots$

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(is called Regular part)

$b_n = \dfrac{1}{2\pi i} \displaystyle\int \frac{f(t)\,dt}{(t-a)^{-n+1}}$ $\quad n = 1, 2 \ldots$

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(Principal part)

Ex Find the Laurent's expansion of

(a) $f(z) = \dfrac{1}{z(z-1)^2}$ at the point $z=1$

(b) Expand $\dfrac{1}{z(z^2-3z+2)}$ for the regions $1 \le |z| \le 2$.

Solution (a) To obtain the expansion of $f(z)$ about the point $z=1$, we put $z_1 = z-1$

$\Rightarrow z = 1+z_1$

$$\therefore \quad f(z) = \frac{1}{(1+z_1)(z_1)^2} = \frac{1}{z_1^2}(1+z_1)^{-1}$$

$$= \frac{1}{z_1^2}\left[1 - z_1 + z_1^2 - z_1^3 - - - -\right]$$

$$= \frac{1}{z_1^2} - \frac{1}{z_1} + 1 - z_1 + z_1^2 - - -$$

$$= \frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 - (z-1) + (z-1)^2 - -$$

$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^{n-2}$$

which is valid for $0 < |z-1| < 1$.

(b) $f(z) = \dfrac{1}{z(z^2-3z+2)} = \dfrac{1}{z(z-1)(z-2)}$

$$= \frac{1}{2z} - \frac{1}{(z-1)} + \frac{1}{2(z-2)}$$

(Using partial fractions)

$\le |z| \le 2$

$\frac{1}{|z|} \le 1$ | $\frac{|z|}{2} \le 1$

$$= \frac{1}{2z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{4}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{1}{4}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right)$$

$$= \frac{-1}{2z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} \cdots - \frac{1}{4}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}\cdots\right).$$

Ex Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent series

valid for

(a) $1 < |z| < 3$ $\qquad \frac{1}{|z|} < 1$ and $\frac{|z|}{3} < 1$

(b) $|z| > 3$ $\qquad \frac{3}{|z|} < 1$.

(c) $0 < |z+1| < 2$ $\qquad \frac{|z+1|}{2} < 1$.

(d) $|z| < 1$

Solution (a) $f(z) = \dfrac{1}{(z+1)(z+3)} = \dfrac{1}{2}\left(\dfrac{1}{z+1} - \dfrac{1}{z+3}\right)$

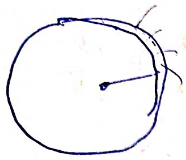
$$= \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{6\left(1+\frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}\cdots\right) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{3^2}\cdots\right)$$

(b). for $|z| > 3$. $3 < |z| < \infty$

$$f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right] = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$



$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$|z| > 3$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} - - -\right) - \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - - -\right)$$

① $\frac{3}{|z|} < 1$

Also. $\frac{1}{|z|} < \frac{1}{3} < 1$

$$= \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - - -\right) - \frac{1}{2}\left(\frac{1}{z} - \frac{3}{z^2} + \frac{9}{z^3} - -\right)$$

$(1+x)^n = 1 + nx$

$x < 1$

$$= \frac{1}{2}\left(\frac{2}{z^2} - \frac{8}{z^3} - - - -\right)$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - - - -$$

(c) for $|z+1| < 2$

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+1+2}\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4\left(1 + \frac{z+1}{2}\right)}$$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\left(1 + \left(\frac{z+1}{2}\right)\right)^{-1}$$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\left[1-\left(\frac{z+1}{2}\right)+\left(\frac{z+1}{2}\right)^2-\left(\frac{z+1}{2}\right)^3 - -\right].$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \left(\frac{z+1}{8}\right) - \frac{(z+1)^2}{16} + \frac{(z+1)^3}{32} - - -$$

$\left(\frac{|z|}{3} < \frac{1}{3} < 1\right)$

(d) For $|z|<1$

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right) = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left(1-z+z^2 - z^3 - - -\right) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27} - - -\right)$$

$$= \frac{1}{3} - \frac{4z}{9} + \frac{13z^2}{27} - - - -$$

Ex. Find the Taylor's series and Laurent series of $f(z) = \frac{-2z+3}{z^2-3z+2}$ with centre at origin.

$|z|=2$

Solution $f(z) = \frac{-2z+3}{z^2-3z+2} = \frac{-2z+3}{(z-1)(z-2)} = \frac{-1}{z-1} - \frac{1}{z-2}$

For Taylor's expansion of $f(z)$, we take $|z|<1$

$\frac{|z|}{2} < \frac{1}{2} < 1$

then $f(z) = \frac{-1}{-1}(1-z)^{-1} + \frac{1}{2\left(1-\frac{z}{2}\right)}$

$$= (1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \left(1+z+z^2+z^3+\cdots\right) + \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}\cdots\right)$$

$$= \frac{3}{2} + \frac{5z}{4} + \frac{9z^2}{8} + \frac{17z^3}{16} + \cdots$$

For laurent's expansion, we consider the annular region $1 \le |z| \le 2$, then

$$f(z) = \frac{-1}{z-1} - \frac{1}{z-2} = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{-1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\cdots\right] + \frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}\cdots\right]$$

Ex. (a) Expand the function $\frac{1-\cos z}{z^3}$ in Laurent series about the point $z=0$.

Solution. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \cdots$

$$\therefore \quad 1-\cos z = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} \cdots$$

$$\therefore \quad \frac{1-\cos z}{z^3} = \frac{1}{2z} - \frac{z}{4!} + \frac{z^3}{6!} - \cdots$$

Ex Expand $\frac{e^{2z}}{(z-1)^3}$ about $z=1$ in Laurent's series.

Solution Let $z_1 = z-1$

$z = 1+z_1$.

$$\therefore \quad \frac{e^{2z}}{(z-1)^3} = \frac{e^{2(1+z_1)}}{z_1^3} = e^2 \cdot \left(\frac{e^{2z_1}}{z_1^3}\right)$$

$$= \frac{e^2}{z_1^3}\left[1 + 2z_1 + \frac{4z_1^2}{2!} + \frac{8z_1^3}{3!} + \cdots\right]$$

$$= e^2\left[\frac{1}{z_1^3} + \frac{2}{z_1^2} + \frac{4}{2z_1} + \frac{8}{6} + \cdots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \cdots\right].$$

Ex Find the expansion of $f(z) = \frac{1}{z(1-z^2)}$ where $1<|z-1|<2$

Solution $f(z) = \frac{1}{z(1-z^2)}$ is to be expanded in powers of $(z-1)$ where $1<|z-1|<2$.

$$\therefore f(z) = \frac{-1}{z(z-1)(z+1)} = \frac{-1}{(z-1)}\left[\frac{1}{z} - \frac{1}{z+1}\right]$$

$$= \frac{-1}{z(z-1)} + \frac{1}{(z-1)(z+1)}$$

$$= \frac{-1}{(z-1)}\left[\frac{1}{z-1+1}\right] + \frac{1}{(z-1)}\left[\frac{1}{z-1+2}\right]$$

$$\cancel{= \frac{-1}{(z-1)}\left[1+(z-1)\right]^{-1} + \frac{1}{2(z-1)\left[1+\left(\frac{z-1}{2}\right)\right]}}$$

$$\cancel{= \frac{-1}{(z-1)}\left[1-(z-1)\right)}.$$

$$= \frac{-1}{(z-1)^2\left[1+\frac{1}{(z-1)}\right]} + \frac{1}{2(z-1)\left[1+\left(\frac{z-1}{2}\right)\right]}$$

$$= \frac{-1}{(z-1)^2}\left[1+\left(\frac{1}{z-1}\right)\right]^{-1} + \frac{1}{2(z-1)}\left[1+\left(\frac{z-1}{2}\right)\right]^{-1}$$

$$= \frac{-1}{(z-1)^2}\left[1-\left(\frac{1}{z-1}\right)+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}\cdots\right]$$

$$+\frac{1}{2(z-1)}\left[1-\left(\frac{z-1}{2}\right)+\left(\frac{z-1}{2}\right)^2+\cdots\right]$$

$$= \left[\frac{-1}{(z-1)^2}+\frac{1}{(z-1)^3}-\frac{1}{(z-1)^4}\cdots\right]+\left[\frac{1}{2(z-1)}-\frac{1}{4}+\frac{(z-1)}{8}\cdots\right]$$

Ex Find Laurent's series for

$$f(z) = \frac{7z-2}{z^3-z^2-2z} \quad \text{in the region.}$$

(1) $0 < |z+1| < 1$

(2) $1 < |z+1| < 3$.

Solution $f(z) = \dfrac{7z-2}{z(z-2)(z+1)} = \dfrac{1}{z} + \dfrac{2}{z-2} - \dfrac{3}{z+1}$

$$= \frac{1}{z+1-1} + \frac{2}{z+1-3} - \frac{3}{z+1}$$

(1) For $0 < |z+1| < 1$

$$f(z) = -\left[1-(z+1)\right]^{-1} - \frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1} - \frac{3}{(z+1)}$$

$$= -\left[1+(z+1)+(z+1)^2+\cdots\right] - \frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\cdots\right] - \frac{3}{(z+1)}.$$

$$= -\frac{5}{3} - \frac{11}{9}(z+1) - \frac{29}{27}(z+1)^2 + \cdots + \left(\frac{-3}{z+1}\right).$$

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(2) For $1 < |z+1| < 3$.

$$f(z) = \frac{1}{(z+1)\left(1-\frac{1}{(z+1)}\right)} + \frac{2}{-3\left(1-\frac{z+1}{3}\right)} - \frac{3}{(z+1)}$$

$$= \frac{1}{(z+1)}\left[1-\frac{1}{(z+1)}\right]^{-1} - \frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1} - \frac{3}{(z+1)}$$

$$= \frac{-3}{(z+1)} + \frac{1}{(z+1)}\left[1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\cdots\right] - \frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\cdots\right]$$

Ex Expand $\dfrac{z}{(z^2+4)(z^2-1)}$ in $1<|z|<2$

Solution $f(z) = \dfrac{z}{(z^2+4)(z^2-1)}$

$$= \frac{z}{5}\left[\frac{1}{z^2-1} - \frac{1}{z^2+4}\right]$$

$$= \frac{z}{5}\left[\frac{1}{z^2\left[1-\frac{1}{z^2}\right]} - \frac{1}{4\left[1+\frac{z^2}{2^2}\right]}\right]$$

$$= \frac{1}{5z}\left[1-\frac{1}{z^2}\right]^{-1} - \frac{z}{20}\left[1+\frac{z^2}{4}\right]^{-1}$$

$$= \frac{1}{5z}\left[1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}\cdots\right] - \frac{z}{20}\left[1-\frac{z^2}{4}+\frac{z^4}{16}\cdots\right]$$

EX 18C

Ex 1. Expand $\dfrac{1}{z}$ by Taylor's series about the point $z=1$.

Solution. let $z_1 = z-1$

$z = z_1 + 1$

$\therefore\ f(z) = \dfrac{1}{z} = \dfrac{1}{z_1+1} = (1+z_1)^{-1} = 1 - z_1 + z_1^2 - z_1^3 \cdots$

$= 1-(z-1)+(z-1)^2+\cdots$

Ex 2 Find Taylor's expansion of the func. $f(z) = \dfrac{2z^3+1}{z^2+z}$ about $z = i$.

Solution $f(z) = \dfrac{2z^3+1}{z^2+z} = 2z - 2 + \dfrac{2z+1}{z(z+1)}$

$$= 2z - 2 + \frac{1}{z} + \frac{1}{z+1}$$

Now $f'(z) = 2 - \dfrac{1}{z^2} - \dfrac{1}{(z+1)^2}$

$$f''(z) = \frac{2}{z^3} + \frac{2}{(z+1)^3}$$

$$f'''(z) = \frac{-6}{z^4} - \frac{6}{(z+1)^4}.$$

$\therefore$ By Taylor's series expansion.

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots$$

$$f(z) = f(i) + (z-i)f'(i) + \frac{(z-i)^2}{2}f''(i) + \cdots$$

$$= (2i-2) + \frac{1}{i} + \frac{1}{i+1} + (z-i)\left(2 - \frac{1}{i^2} - \frac{1}{(i+1)^2}\right) + \frac{(z-i)^2}{2}\left(\frac{2}{i^3} + \frac{2}{(i+1)^3}\right) \cdots$$

$$= \left(\frac{i-3}{2}\right) + (z-i)\left(3 + \frac{i}{2}\right) + \frac{(z-i)^2}{2}\left(\frac{3i-1}{2}\right) + \cdots$$

Ex. Expand $f(z) = \frac{1}{(z+1)^2}$ about the point $z = i$.

Solution. $f(z) = \frac{1}{(z+1)^2}$, $f(i) = \frac{1}{(i+1)^2} = \frac{1}{2i}$

$f'(z) = \frac{-2}{(z+1)^3}$, $f'(i) = \frac{-2}{(i+1)^3} = \frac{-2}{2i(i+1)}$

$= \frac{i}{i+1} \times \frac{(i-1)}{(i-1)}$

$= \frac{-1-i}{-2}$

$= \frac{1+i}{2}$

$f''(z) = \frac{-6}{(z+1)^4}$, $f''(i) = \frac{-6}{(i+1)^4} = \frac{-6}{(2i)^2} = \frac{-6}{-4} = \frac{3}{2}$

$\therefore f(z) = f(i) + (z-i)f'(i) + \frac{(z-i)^2}{2} f''(i) + \cdots$

$= \frac{1}{2i} + (z-i)\left(\frac{1+i}{2}\right) + \frac{(z-i)^2}{2}\left(\frac{3}{2}\right) + \cdots$

Ex Determine the Laurent's expansion for $f(z) = \dfrac{1}{(1-z)(2-z)}$ valid for $1<|z|<2$.

Solution $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{(z-1)} - \dfrac{1}{(z-2)}$

$$= \frac{1}{-1\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= -\left[1-\frac{1}{z}\right]^{-1} + \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$$

$$= -\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}---\right] + \frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}---\right]$$