CHAPTER

1 Successive Differentiation, Expansion of Functions and Approximation

Successive Differentiation, Leibnitz's Theorem, Expansion of Functions by Maclaurin's and Taylor's Series. Errors and Approximation.

INTRODUCTION

If a function $f(x)$ is differentiable we can obtain its derivative and denote it by $f'(x)$. If $f'(x)$ is also differentiable we can differentiate it and get its derivative which is called second order derivative of $f(x)$ and is denoted by $f''(x)$. Continuing this process we get third and higher order derivatives and denote them by $f'''(x)$, $f^{iv}(x)$, ..., $f^{(n)}(x)$, ... This notation is somewhat cumbersome in practice and a better notation is given by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}, \dots$$

or, alternately, the suffix notation $y_1, y_2, y_3, \dots, y_n, \dots$

where $y = f(x)$ and the suffix n is a positive integer giving the order of derivative.

Yet another notation is the operator notation as

$$Dy,\ D^2y,\ D^3y,\ \dots,\ D^ny, \dots$$

where D symbolizes the operation of differentiation.

n^{th} Order Derivatives of some Standard Functions

1. Let $y = e^{ax}$,

   then $y_1 = a e^{ax}$, $y_2 = a^2 e^{ax}, \dots, y_n = a^n e^{ax}$, that is, $\frac{d^n}{dx^n} e^{ax} = a^n e^{ax}$

2. Let $y = (ax+b)^m$

   then, $y_1 = m a (ax+b)^{m-1}$, $y_2 = m(m-1) a^2 (ax+b)^{m-2}, \dots$

   In general, $y_n = m(m-1)(m-2)\dots(m-n+1)\, a^n (ax+b)^{m-n}$.

   In particular, if m is a positive integer greater than n, then we can write,

   $$y_n = \frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$$

   and if m is a positive integer less than n, then $y_n = 0$.

Further, if $m = n$, then $y_n = \dfrac{d^n}{dx^n}(ax+b)^n = n\,!\,a^n$.

and if $m = -1$ $y_n = \dfrac{d^n}{dx^n}(ax+b)^{-1} = (-1)(-2)(-3)\ldots(-n)\,a^n\,(ax+b)^{-1-n}$

or we can write $y_n = \dfrac{d^n}{dx^n}\dfrac{1}{ax+b} = \dfrac{(-1)^n\,n!\,a^n}{(ax+b)^{n+1}}$.

3. Let $y = \log(ax+b)$,

then $y_1 = \dfrac{a}{ax+b} = a\,(ax+b)^{-1}$

and $y_n = \dfrac{d^n}{dx^n}\log(ax+b) = \dfrac{d^{n-1}}{dx^{n-1}}\left(\dfrac{a}{ax+b}\right) = a.\,(-1)^{n-1}\,(n-1)!\,a^{n-1}\,(ax+b)^{-1-(n-1)}$

or $y_n = \dfrac{d^n}{dx^n}\log(ax+b) = (-1)^{n-1}\,\dfrac{(n-1)!\,a^n}{(ax+b)^n}$.

4. Let $y = a^x$

then $y_1 = a^x\log a,\; y_2 = a^x(\log a)^2, \ldots,\; y_n = \dfrac{d^n}{dx^n}a^x = a^x(\log a)^n$.

5. Let $y = \sin(ax+b)$,

then, $y_1 = a\cos(ax+b) = a\sin(ax+b+\pi/2)$

$y_2 = a^2\cos(ax+b+\pi/2) = a^2\sin(ax+b+2.\,\pi/2)$

....

....

$y_n = \dfrac{d^n}{dx^n}\sin(ax+b) = a^n\sin(ax+b+n\,\pi/2)$

Similarly, if $y = \cos(ax+b)$,

then, $y_1 = -a\sin(ax+b) = a\cos(ax+b+\pi/2)$

$y_2 = -a^2\sin(ax+b+\pi/2) = a^2\cos(ax+b+2\,\pi/2)$

....

....

$y_n = \dfrac{d^n}{dx^n}\cos(ax+b) = a^n\cos(ax+b+n\pi/2)$.

The above two formulae, in brief, can be written as

$$\frac{d^n}{dx^n}\begin{matrix}\sin\\ \cos\end{matrix}(ax+b) = a^n\begin{matrix}\sin\\ \cos\end{matrix}(ax+b+n\pi/2).$$

6. Let $y = e^{ax}\sin(bx+c)$,

then, $y_1 = a\,e^{ax}\sin(bx+c) + e^{ax}\,b\cos(bx+c) = e^{ax}\,[a\sin(bx+c) + b\cos(bx+c)]$

If we put $a = r\cos\alpha$, $b = r\sin\alpha$ then

$y_1 = e^{ax}\,[\,r\sin(bx+c)\cos\alpha + r\cos(bx+c)\sin\alpha] = r\,e^{ax}\sin(bx+c+\alpha)$

where $r = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}(b/a)$.

Again differentiating and making the above substitution, gives

$$y_2 = r^2 e^{ax} \sin(bx + c + 2\alpha)$$

and continuing the process, in general, we get

$$y_n = r^n e^{ax} \sin(bx + c + n\alpha).$$

Similarly, if $y = e^{ax} \cos(bx + c)$, and proceeding as above, we get

$$y_n = r^n e^{ax} \cos(bx + c + n\alpha).$$

The above two results can be together written as

$$\frac{d^n}{dx^n} e^{ax} \begin{matrix}\sin\\ \cos\end{matrix} (bx + c) = r^n e^{ax} \begin{matrix}\sin\\ \cos\end{matrix} (bx + c + n\alpha)$$

where $r = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1} b/a$.

[redacted] (a) Find the n^{th} derivative of $\dfrac{x^4}{(x-1)(x-2)}$.

[GGSIPU I Sem End Term 2011, End Term 2012]

(b) If $y = (2 - 3x)^{10}$ find y_9.

[GGSIPU I Sem End Term 2013]

SOLUTION: (a) Let $y = \dfrac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 + \dfrac{15x - 14}{(x-1)(x-2)}$

$$= x^2 + 3x + 7 - \frac{1}{x-1} + \frac{16}{x-2}$$

then n^{th} derivative of $y = y_n = 0 - \dfrac{(1)^n\, n!}{(x-1)^{n+1}} + \dfrac{16(-1)^n\, n!}{(x-2)^{n+1}}$, $(n \geq 3)$

$$= (-1)^n\, n!\left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right] \quad \textbf{Ans.}$$

(b) $$y_9 = \frac{d^9}{dx^9}(2-3x)^{10} = (-3)^9\, 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2\, (2 - 3x)$$

$$= 10!\,(3^9)\,(-1)^9\,(2 - 3x) = 10!\,(3^9)\,(3x - 2) \quad \textbf{Ans.}$$

[redacted] (a) Find the n^{th} derivative of $\sin^2 x \cos^3 x$.

(b) If $y = e^{5x} \sin(2x + 3) \sin(x - 1) \cos(x + 1)$, find y_n.

SOLUTION: (a) Let $y = \sin^2 x \cos^3 x$, we can write

$$y = \left(\frac{1 - \cos 2x}{2}\right)\cdot\left(\frac{\cos 3x + 3\cos x}{4}\right)$$

$$= \frac{1}{8}\left[\cos 3x + 3\cos x - \cos 2x \cos 3x - 3\cos 2x \cos x\right]$$

$$= \frac{1}{8}\left[\cos 3x + 3\cos x - \frac{1}{2}(\cos 5x + \cos x) - \frac{3}{2}(\cos 3x + \cos x)\right]$$

$$= \frac{1}{16}\,[2\cos x - \cos 3x - \cos 5x]$$

$$\therefore \qquad y_n = \frac{1}{16}\left[2\cos\left(x+\frac{n\pi}{2}\right) - 3^n \cos\left(3x+\frac{n\pi}{2}\right) - 5^n \cos\left(5x+\frac{n\pi}{2}\right)\right] \qquad \textbf{Ans.}$$

(b) We have
$$y = \frac{1}{2}\, e^{5x} \sin(2x+3)\,(\sin 2x - \sin 2)$$
$$= \frac{1}{2}\, e^{5x} \{\sin(2x+3)\sin 2x - \sin(2x+3)\cdot \sin 2\}$$
$$= \frac{1}{4}\, e^{5x} \{\cos 3 - \cos(4x+3)\} - \frac{1}{2}\sin 2 \cdot e^{5x}\sin(2x+3)$$
$$= \frac{1}{4}\cos 3 \cdot e^{5x} - \frac{1}{4}\, e^{5x}\cos(4x+3) - \frac{1}{2}\sin 2\cdot e^{5x}\sin(2x+3)$$

Therefore,
$$y_n = \frac{1}{4}\cos 3\cdot 5^n e^{5x} - \frac{e^{5x}}{4} r_1^n \cos(4x+3+n\alpha) - \frac{1}{2}\sin 2\cdot r_2^n\, e^{5x}\sin(2x+3+n\beta)$$

where $r_1 = \sqrt{5^2+4^2} = \sqrt{41}$, $r_2 = \sqrt{5^2+2^2} = \sqrt{29}$, $\alpha = \tan^{-1}\frac{4}{5}$ and $\beta = \tan^{-1}\frac{2}{5}$. **Ans.**

[redacted] **(a)** If $y = \dfrac{x}{x^2+a^2}$, find y_n, the n^{th} derivative of y.

(b) Find the nth derivative of $\tan^{-1}(x/a)$. *[GGSIPU Ist Sem End Term 2008]*

SOLUTION: (a) Here $y = \dfrac{x}{(x+ai)(x-ai)} = \dfrac{1}{2}\left\{\dfrac{1}{x-ai} + \dfrac{1}{x+ai}\right\}$

$$\therefore \qquad y_n = \frac{1}{2}\frac{(-1)^n n!}{(x-ai)^{n+1}} + \frac{1}{2}\frac{(-1)^n n!}{(x+ai)^{n+1}}$$

To simplify the above expression we apply De Moivre's theorem for which, as in previous example, we put $x = r\cos\theta$, $a = r\sin\theta$ so that $\tan\theta = a/x$, and get

$$y_n = \frac{(-1)^n n!}{2}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} + \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$
$$= \frac{(-1)^n n!}{2\,r^{n+1}}\,[(\cos\theta - i\sin\theta)^{-n-1} + (\cos\theta + i\sin\theta)^{-n-1}]$$
$$= \frac{(-1)^n n!}{2\,r^{n+1}}\,[\{\cos(n+1)\theta + i\sin(n+1)\theta\} + \{\cos(n+1)\theta - i\sin(n+1)\theta\}]$$
$$= \frac{(-1)^n n!}{r^{n+1}}\cos(n+1)\theta = \frac{(-1)^n n!}{a^{n+1}}\cos(n+1)\theta\,\sin^{n+1}\theta \quad (\text{using } a = r\sin\theta)$$

where $\cot\theta = x/a$ or $\theta = \cot^{-1}(x/a) = \tan^{-1}(a/x)$. **Ans.**

(b) Let $y = \tan^{-1}(x/a)$ then $y_1 = \dfrac{1}{a\left(1+\dfrac{x^2}{a^2}\right)} = \dfrac{a}{x^2+a^2}$

or $$y_1 = \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai}\left\{\frac{1}{x-ai} - \frac{1}{x+ai}\right\} \quad \text{where } i = \sqrt{-1}.$$

Differentiating the above relation $(n-1)$ times w.r.t. x, gives

$$y_n = \frac{1}{2i}\left\{\frac{d^{n-1}}{dx^{n-1}}\frac{1}{x-ai} - \frac{d^{n-1}}{dx^{n-1}}\frac{1}{x+ai}\right\} = \frac{1}{2i}\left\{\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right\}$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\left\{\frac{1}{(x-ai)^n} - \frac{1}{(x+ai)^n}\right\}.$$

The above expression can be simplified by using De Moivre's theorem, for which we put $x = r\cos\theta,\ a = r\sin\theta$, then

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\sin\theta)^n} - \frac{1}{r^n(\cos\theta + i\sin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i\,r^n}\left[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i\,r^n}\left[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta\right] = \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{(a/\sin\theta)^n}\sin n\theta = \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\sin^n\theta \qquad (\text{using } a = r\sin\theta)$$

where $\tan\theta = \dfrac{a}{x}$ or $\theta = \tan^{-1}(a/x) = \cot^{-1}(x/a)$. **Ans.**

Ans

[redacted] If $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, show that $p + \dfrac{d^2p}{d\theta^2} = \dfrac{a^2b^2}{p^3}$.

SOLUTION: Differentiating $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$

$p^4 + p^3\dfrac{d^2p}{d\theta^2} = a^2b^2$ ...(1)

we get $$2p\frac{dp}{d\theta} = -2a^2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta$$

or $$p\frac{dp}{d\theta} = (b^2 - a^2)\sin\theta\cos\theta \quad \text{(A)} \qquad \ldots(2)$$

Differentiating w.r.t. θ again, we get

$$p\frac{d^2p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)^2 = (b^2 - a^2)(\cos^2\theta - \sin^2\theta)$$

Multiplying throughout by p^2 and using (2), gives then (A).

$$p^3\frac{d^2p}{d\theta^2} + (b^2-a^2)^2\sin^2\theta\cos^2\theta = (b^2-a^2)p^2(\cos^2\theta - \sin^2\theta)$$

$$= p^2\{(b^2\cos^2\theta + a^2\sin^2\theta) - (a^2\cos^2\theta + b^2\sin^2\theta)\}$$

$$= p^2(b^2\cos^2\theta + a^2\sin^2\theta) - p^4;$$

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or $$p^3\frac{d^2p}{d\theta^2}+p^4=(a^2\cos^2\theta+b^2\sin^2\theta)(b^2\cos^2\theta+a^2\sin^2\theta)-(b^2-a^2)^2\sin^2\theta\cos^2\theta$$

$$=a^2b^2(\cos^4\theta+\sin^4\theta)+(a^4+b^4)\sin^2\theta\cos^2\theta-(a^4+b^4-2a^2b^2)\sin^2\theta\cos^2\theta$$

$$=a^2b^2(\cos^4\theta+\sin^4\theta+2\sin^2\theta\cos^2\theta)=a^2b^2(\cos^2\theta+\sin^2\theta)^2=a^2b^2$$

or $$\frac{d^2p}{d\theta^2}+p=\frac{a^2b^2}{p^3}.$$ **Hence Proved.**

[redacted]) If $y=\log(x+\sqrt{x^2}+1)$ prove that $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$.

[GGSIPU I Sem End Term 2009]

(b) If $y=x\log\left(\frac{x-1}{x+1}\right)$, show that

$$y_n=(-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n}-\frac{x+n}{(x+1)^n}\right].$$

SOLUTION: (a) $y=\log(x+\sqrt{1+x^2})$ hence $$\frac{dy}{dx}=\frac{1+\frac{1}{2}\cdot\frac{2x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}$$

or $(\sqrt{1+x^2})\frac{dy}{dx}=1$. Differentiating w.r.t. x, both sides, we get

$$\sqrt{1+x^2}\frac{d^2y}{dx^2}+\frac{2x}{2\sqrt{1+x^2}}\frac{dy}{dx}=0 \quad\text{or}\quad (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0.$$ **Hence proved.**

(b) Here $y=x\log\left(\frac{x-1}{x+1}\right)=x\log(x-1)-x\log(x+1)$

$$\therefore\quad y_1=\frac{x}{x-1}+\log(x-1)-\frac{x}{x+1}-\log(x+1)$$

$$=1+\frac{1}{x-1}+\log(x-1)-1+\frac{1}{x+1}-\log(x+1)=\frac{1}{x-1}+\frac{1}{x+1}+\log(x-1)-\log(x+1)$$

Differentiating both sides $(n-1)$ times with respect to x, we get

$$y_n=\frac{(-1)^{n-1}(n-1)!}{(x-1)^n}+\frac{(-1)^{n-1}(n-1)!}{(x+1)^n}+\frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}}-\frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}}$$

$$=(-1)^{n-2}(n-2)!\left\{\frac{-(n-1)+x-1}{(x-1)^n}\right\}+(-1)^{n-2}(n-2)!\left\{\frac{-(n-1)-(x+1)}{(x+1)^n}\right\}$$

$$=(-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n}-\frac{x+n}{(x+1)^n}\right]$$

[redacted] **(a)** Prove that the value of n^{th} derivative of $\dfrac{x^3}{x^2-1}$ for $x = 0$, is 0 if n is even and $-n!$ if n is odd and greater than 1. *[GGSIPU I Sem Ist Term 2004]*

(b) Find the nth derivative of $\dfrac{1}{1+x+x^2}$. *[GGSIPU I Sem End III Term 2007]*

SOLUTION: (a) Let
$$y = \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1} = x + \frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right)$$

Then
$$y_n = 0 + \frac{1}{2}\frac{d^n}{dx^n}\frac{1}{x-1} + \frac{1}{2}\frac{d^n}{dx^n}\frac{1}{x+1} \quad \text{when } n > 1.$$
$$= \frac{1}{2}\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{1}{2}\frac{(-1)^n n!}{(x+1)^{n+1}}$$

$\therefore$
$$y_n(0) = \frac{(-1)^n n!}{2}\left[\frac{1}{(-1)^{n+1}} + 1\right] = \frac{(-1)^n n!}{2(-1)^{n+1}}[1 + (-1)^{n+1}]$$
$$= \frac{-n!}{2}[1 + (-1)^{n+1}] = \begin{bmatrix} 0 & \text{if } n \text{ is even} \\ -n! & \text{if } n \text{ is odd} \end{bmatrix}$$

Hence Proved.

(b)
$$y = \frac{1}{1+x+x^2} = \frac{1}{(x-w)(x-w^2)} \quad \text{where } w = \frac{-1+i\sqrt{3}}{2},\ w^2 = \frac{-1-i\sqrt{3}}{2}$$
$$= \left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)\frac{1}{w-w^2} = \frac{-i}{\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

$\therefore$
$$y = \frac{d^n}{dx^n}\frac{1}{1+x+x^2} = \frac{-i}{\sqrt{3}}\left[\frac{d^n}{dx^n}\frac{1}{x-w} - \frac{d^n}{dx^n}\frac{1}{x-w^2}\right]$$
$$= \frac{-i}{\sqrt{3}}\left[\frac{(-1)^n n!}{(x-w)^{n+1}} - \frac{(-1)^n n!}{(x-w^2)^{n+1}}\right]$$
$$= i(-1)^{n+1}\frac{n!}{\sqrt{3}}\left[\frac{1}{(x-w)^{n+1}} - \frac{1}{(x-w^2)^{n+1}}\right]$$
$$= \frac{(-1)^{n+1}}{\sqrt{3}}n!\,i\left[\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{n+1}}\right]$$
$$= (-1)^{n+1}\frac{2^{n+1}}{\sqrt{3}}n!\,i\left[\frac{1}{(2x+1-i\sqrt{3})^{n+1}} - \frac{1}{(2x+1+i\sqrt{3})^{n+1}}\right]$$

Putting $2x + 1 = r\cos\theta$, $\sqrt{3} = r\sin\theta$, we get
$$y_n = \frac{(-1)^{n+1}2^{n+1}n!\,i}{\sqrt{3}\cdot r^{n+1}}\left[(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\right]$$

$$= \frac{(-1)^{n+1} 2^{n+1} n! i}{\sqrt{3}\left(\frac{\sqrt{3}}{\sin\theta}\right)^{n+1}} [\cos (n+1)\theta + i \sin (n+1)\theta - \cos (n+1)\theta + i \sin (n+1)\theta]$$

$$= \frac{(-1)^{n+1} 2^{n+1} n! \cdot i \sin^{n+1}\theta}{(\sqrt{3})^{n+2}} . 2i \sin (n+1)\theta$$

$$= (-1)^{n+2} \frac{2^{n+2} n!}{(\sqrt{3})^{n+2}} \sin^{n+1}\theta \sin (n+1)\theta = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin (n+1)\theta$$

where $\tan\theta = \frac{\sqrt{3}}{2x+1}$ and $a = \frac{\sqrt{3}}{2}$. **Ans.**

EXERCISE 1A

1. Find the n^{th} derivative of the following functions

(i) $\dfrac{x^2}{(x+2)(2x+3)}$ (ii) $\dfrac{x^3}{(x-1)(2x+3)}$

2. Obtain the n^{th} derivative of

(i) $\sin^4 x$ (ii) $\dfrac{4x}{(x-1)^2(x+1)}$

3. Determine $\dfrac{d^2y}{dx^2}$ if $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

4. If $y = e^{-kt}\cos(bt + c)$ then prove that

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + n^2y = 0 \quad \text{where } n^2 = k^2 + b^2.$$

5. If $y = \sin ax + \cos ax$, show that

$$y_n = a^n\sqrt{1+(-1)^n \sin 2ax}.$$

[GGSIPU Ist Sem End Term 2008; I Term 2009]

6. If $y = e^{-ax}\cos(bx + c)$, show that

$$y_n = (-1)^n (a^2 + b^2)^{n/2} e^{-ax} \cos(bx + c + n\tan^{-1} b/a)$$

7. Find the nth derivative of $\tan^{-1}\dfrac{1-x}{1+x}$.

8. If $y = \tan^{-1}x$, prove that

$$y_n = (n-1)!\cos\{ny + (n-1)\pi/2\}\cos^n y.$$

9. If $(1 - x^2)\tan y = 2x$, show that

$$y_n = 2(-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta, \text{ where } \cot\theta = x.$$

10. Find the n^{th} derivative of $e^x \sin 4x \cos 6x$.

LEIBNITZ'S THEOREM

This theorem provides the nth derivative of the product of two functions. It states as follows:

If $y = u.v.$ where u and v are functions of x and their nth derivatives are separately known, then $y_n = (uv)_n = u_n v + {}^nC_1\, u_{n-1}\, v_1 + {}^nC_2\, u_{n-2}\, v_2 + \ldots + {}^nC_r\, u_{n-r}\, v_r + \ldots + u.\, v_n.$

Proof: The theorem will be proved by the principle of Mathematical induction.

$$y_1 = \frac{du}{dx}v + u\frac{dv}{dx} = u_1 v + u\, v_1 \text{ which shows that the theorem is true for } n = 1.$$

Now, let us suppose that the statement of the theorem is true for $n = K$, *i.e.* suppose

$$(uv)_K = u_K\, v + {}^KC_1\, u_{K-1}\, v_1 + {}^KC_2\, u_{K-2}\, v_2 + \ldots + u\, v_K \text{ is true.}$$

Then differentiating both sides w.r.t. x, gives

$$(uv)_{K+1} = (u_{K+1}\, v + u_K\, v_1) + {}^KC_1\,(u_K\, v_1 + u_{K-1}\, v_2) + {}^KC_2\,(u_{K-1}\, v_2 + u_{K-2}\, v_3) + \ldots + (u_1\, v_K + u\, v_{K+1})$$

$$= u_{K+1}\, v + (1 + {}^KC_1)\, u_K\, v_1 + ({}^KC_1 + {}^KC_2)\, u_{K-1}\, v_2 + \ldots + ({}^KC_{K-1} + {}^KC_K)\, u_1\, v_K + \ldots + u\, v_{K+1}$$

$$= u_{K+1}\, v + {}^{K+1}C_1\, u_K\, v_1 + {}^{K+1}C_2\, u_{K-1}\, v_2 + \ldots + {}^{K+1}C_K\, u_1\, v_K + u\, v_{K+1}$$

$$\left(\text{on using the result } {}^KC_r + {}^KC_{r+1} = {}^{K+1}C_{r+1} \text{ and } 1 = {}^KC_0\right)$$

This shows that the statement of the theorem is true for $n = K + 1$ under the supposition that it is true for $n = K$. Therefore, by the principle of Mathematical induction the theorem is true for every positive integral value of n.

EXAMPLE 1.7. (a) If $y = x^2 e^x$ show that $y_n = \dfrac{n(n-1)}{2}\, y_2 - n(n-2)\, y_1 + \dfrac{(n-1)(n-2)}{2}\, y.$

[GGSIPU I Sem End Term 2004]

(b) Find the n^{th} derivative of $x^2 e^{3x} \sin 4x$.

SOLUTION: (a) $y = x^2 e^x$ so $y_1 = 2xe^x + x^2 e^x = 2xe^x + y$

Differentiating again, we get $y_2 = 2xe^x + 2e^x + y_1$

or $y_2 = y_1 - y + 2e^x + y_1$ or $2e^x = y_2 - 2y_1 + y.$

Now applying Lebneitz theorem on $x^2 e^x$, we get

$$y_n = x^2 e^x + n\,.2xe^x + \frac{n(n-1)}{2}.2.e^x$$

$$= y + n\,(y_1 - y) + \frac{n(n-1)}{2}(y_2 - 2y_1 + y)$$

$$= \frac{n(n-1)}{2}\, y_2 + y_1\,[n - n(n-1)] + y\left[1 - n + \frac{n(n-1)}{2}\right]$$

$$= \frac{n(n-1)}{2}\, y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)\, y.$$

Hence the result.

(b) Let $y = u\, v$ where $u = e^{3x} \sin 4x$ and $v = x^2$

Hence, $u_n = e^{3x}\, r^n \sin(4x + n\alpha)$ where $r = \sqrt{3^2 + 4^2} = 5$ and $\tan\alpha = 4/3,$

Here $v_1 = 2x,\ v_2 = 2,\ v_3 = v_4 = \ldots = v_n = 0.$

Therefore, by Leibnitz's theorem

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2$$

$$= 5^n e^{3x} \sin(4x + n\alpha) x^2 + {}^nC_1 5^{n-1} e^{3x} \sin(4x + (n-1)\alpha)(2x) + {}^nC_2 5^{n-2} e^{3x} \sin(4x + (n-2)\alpha) \cdot (2)$$

$$= 5^n e^{3x} \left[x^2 \sin(4x + n\alpha) + \frac{2nx}{5} \sin(4x + (n-1)\alpha) + \frac{n(n-1)\cdot(2)}{2.25} \sin(4x + (n-2)\alpha) \right]$$

or $$y_n = 5^n e^{3x} \left[x^2 \sin(4x + n\alpha) + \frac{2nx}{5} \sin(4x + n\alpha - \alpha) + \frac{n(n-1).}{25} \sin(4x + n\alpha - 2\alpha) \right].$$

where $\alpha = \tan^{-1}(4/3)$. **Ans.**

EXAMPLE 1.8. Imp (a) If $y = \dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$, prove that

$(1 - x^2) y_{n+1} - (2n + 1) x y_n - n^2 y_{n-1} = 0$ and hence find $(y_n)_0$

[*GGSIPU I Sem I Term 2010*]

M. Imp (b) If $\log y = \tan^{-1} x$, show that

$(1 + x^2) y_{n+2} + \{2(n+1)x - 1\} y_{n+1} + n(n+1) y_n = 0$

and hence find y_3, y_4 and y_5 at $x = 0$. [*GGSIPU I Sem I Term Sp. 2009*]

SOLUTION: **(a)** The given relation can be written as $y\sqrt{1-x^2} = \sin^{-1} x$ ...(1)

On differentiating w.r.t. x on both sides of (1), gives

$$\sqrt{1-x^2}\, y_1 - \frac{2x}{2\sqrt{1-x^2}} y = \frac{1}{\sqrt{1-x^2}} \quad \text{or} \quad (1-x^2) y_1 - xy = 1 \qquad ...(2)$$

Differentiating n times using Leibnitz's theorem, we get

$$(1-x^2) y_{n+1} + {}^nC_1(-2x) y_n + {}^nC_2(-2) y_{n-1} - x y_n - {}^nC_1 . 1 . y_{n-1} = 0$$

or $$(1-x^2) y_{n+1} - (2n+1) x y_n - n^2 y_{n-1} = 0,$$ **hence the result.**

At $x = 0$ above relation becomes $(y_{n+1})_0 = n^2 (y_{n-1})_0$

or $$(y_n)_0 = (n-1)^2 (y_{n-2})_0$$

$$= (n-1)^2 (n-3)^2 (y_{n-4})_0 = (n-1)^2 (n-3)^2 (n-5)^2 (y_{n-6})_0 = \ldots\ldots$$

$$\Rightarrow \quad (y_n)_0 = \begin{cases} (n-1)^2 (n-3)^2 (n-5)^2 \ldots\ldots 3^2 . 1^2 (y)_0 & \text{when } n \text{ is even} \\ (n-1)^2 (n-3)^2 (n-5)^2 \ldots\ldots 4^2 . 2^2 (y_1)_0 & \text{when } n \text{ is odd} \end{cases}$$

Since $y(0) = 0$ we have $(y_n)_0 = 0$ when n is even,

and since from (2) $(y_1)_0 = 1$, we have

$$(y_n)_0 = (n-1)^2 (n-3)^2 (n-5)^2 \ldots\ldots 2^2 . 1 \text{ when } n \text{ is odd.}$$ **Ans.**

(b) We have $\log y = \tan^{-1} x$ or $y = e^{\tan^{-1} x}$ ...(1)

hence, $y_1 = e^{\tan^{-1} x} . \dfrac{1}{1+x^2}$ or $(1 + x^2) y_1 = y$ ...(2)

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Differentiating (2) $(n+1)$ times using Leibnitz's theorem, gives

$$(1+x^2)y_{n+2}+{}^{n+1}C_1 . 2xy_{n+1}+{}^{n+1}C_2 . 2y_n = y_{n+1}$$

or $$(1+x^2)y_{n+2}+\{2(n+1)x-1\}y_{n+1}+n(n+1)y_n=0 \quad ...(3)$$

At $x=0$ relation (3) becomes $(y_{n+2})_0-(y_{n+1})_0+n(n+1)(y_n)_0=0 \quad ...(4)$

Taking $n=0$ in (4), we get $(y_2)_0-(y_1)_0+0(y)_0=0$ or $(y_2)_0=(y_1)_0$

But from (2), $(y_1)_0=y(0)=1$ hence $(y_2)_0=1$

Taking $n=1$ in (4), we get

$$(y_3)_0-(y_2)_0+2(y_1)_0=0 \Rightarrow (y_3)_0=-1$$

Taking $n=2$ in (4), we get

$$(y_4)_0-(y_3)_0+2.3(y_2)_0=0 \Rightarrow (y_4)_0=-7$$

and lastly, taking $n=3$ in (4), we get

$$(y_5)_0-(y_4)_0+3.4(y_3)_0=0 \Rightarrow (y_5)_0=5.$$ **Ans.**

EXAMPLE 1.9. (a) If $x+y=1$, prove that

$$D^n(x^ny^n)=n!\left[y^n-({}^nC_1)^2y^{n-1}x+({}^nC_2)^2y^{n-2}x^2-....+(-1)^nx^n\right]$$

where $D=\frac{d}{dx}$.

(b) If $V_n=\frac{d^n}{dx^n}(x^n\log x)$, establish $V_n=nV_{n-1}+(n-1)!$

and hence show that $V_n=n!\left(\log x+1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)$.

SOLUTION: (a) Here $x+y=1$ therefore, $\frac{dy}{dx}=Dy=-1$.

Applying Leibnitz's theorem, to $D^n(x^ny^n)$, we get

$$D^n(x^ny^n)=y^nD^n(x^n)+{}^nC_1D(y^n)D^{n-1}(x^n)+{}^nC_2D^2(y^n)D^{n-2}(x^n)+...+D^n(y^n).x^n.$$

$$=y^nn!+{}^nC_1.ny^{n-1}(-1).n!x+{}^nC_2n(n-1)(-1)^2y^{n-2}n!\frac{x^2}{2!}+...+n!(-1)^nx^n$$

$$=n!\left[y^n-{}^nC_1\cdot ny^{n-1}x+{}^nC_2\cdot\frac{n(n-1)}{2!}y^{n-2}x^2-...+(-1)^nx^n\right]$$

or $$D^n(x^ny^n)=n!\left[y^n-({}^nC_1)^2\cdot y^{n-1}x+({}^nC_2)^2y^{n-2}x^2-({}^nC_3)^2y^{n-3}x^3+...+(-1)^nx^n\right].$$

hence the result.

(b) $$V_n=\frac{d^n}{dx^n}(x^n\log x)=\frac{d^{n-1}}{dx^{n-1}}\left\{\frac{d}{dx}(x^n\log x)\right\}=\frac{d^{n-1}}{dx^{n-1}}\left\{x^n\frac{1}{x}+nx^{n-1}\log x\right\}$$

$$=\frac{d^{n-1}}{dx^{n-1}}x^{n-1}+n\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x)=(n-1)!+nV_{n-1} \quad ...(1)$$

Dividing throughout by $n!$, we get

$$\frac{V_n}{n!}=\frac{(n-1)!}{n!}+\frac{n}{n!}V_{n-1}=\frac{1}{n}+\frac{V_{n-1}}{(n-1)!} \quad ...(2)$$

Replacing n by $n-1$ in the above relation, we get

$$\frac{V_{n-1}}{(n-1)!} = \frac{1}{n-1} + \frac{V_{n-2}}{(n-2)!}$$

Therefore, (2) becomes

$$\frac{V_n}{n!} = \frac{1}{n} + \frac{1}{n-1} + \frac{V_{n-2}}{(n-2)!} \quad \text{(again using (2) replacing } n \text{ by } n-2, \text{ we have)}$$

$$= \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \frac{V_{n-3}}{(n-3)!}$$

$$= \dots\dots\dots\dots\dots\dots$$

$$= \dots\dots\dots\dots\dots\dots$$

$$= \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \frac{1}{n-3} + \dots + \frac{1}{2} + \frac{1}{1} + \frac{V_0}{0!}$$

But $\quad V_0 = x^0 \log x = \log x$

Therefore, $\quad V_n = n!\left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right]$ **Hence Proved.**

EXAMPLE 1.10. (a) If $y = e^{x^2/2} \cos x$, show that

$$y_{n+2}(0) - 2n\, y_n(0) + n(n-1)\, y_{n-2}(0) = 0.$$

(b) If $y = \sqrt{\frac{1+x}{1-x}}$ prove that $y = (1-x^2)\, y_1$ and hence show that

$$(1-x^2)\, y_n - \{2(n-1)x + 1\}\, y_{n-1} - (n-1)(n-2)\, y_{n-2} = 0$$

SOLUTION: (a) We have $y = e^{x^2/2} \cos x$ which on differentiation, gives

$$y_1 = e^{x^2/2}\, x \cos x - e^{x^2/2} \sin x = xy - e^{x^2/2} \sin x$$

and $\quad y_2 = xy_1 + y - e^{x^2/2}\, x \sin x - e^{x^2/2} \cos x$

$$= xy_1 + y + x(y_1 - xy) - y \quad \text{or} \quad y_2 = 2xy_1 - x^2 y.$$

Now, differentiating the above relation n times and using Leibnitz's theorem, gives

$$y_{n+2} = 2xy_{n+1} + {}^nC_1 . 2 . y_n - (x^2 y_n + {}^nC_1\, 2xy_{n-1} + {}^nC_2\, 2\, y_{n-2})$$

or $\quad y_{n+2} - 2xy_{n+1} + (x^2 - 2n)\, y_n + 2nxy_{n-1} + n(n-1)\, y_{n-2} = 0$

Putting $x = 0$ throughout, we get

$$y_{n+2}(0) - 2n\, y_n(0) + n(n-1)\, y_{n-2}(0) = 0,$$ hence the result.

(b) The given relation can be written as $\quad (1-x)\, y^2 = 1 + x$ ...(1)

This, on differentiation w.r.t. x, gives $(1-x)\, 2yy_1 - y^2 = 1$

or $\quad (1-x)\, 2yy_1 = 1 + y^2 = 1 + \frac{1+x}{1-x} = \frac{2}{1-x}$

Multiplying throughout by y, gives

$$(1-x)\, 2\, y^2\, y_1 = \frac{2y}{1-x}. \quad \text{Using (1) here, we get} \quad (1-x)^2 \left(\frac{1+x}{1-x}\right) y_1 = y$$

or $\quad (1-x^2)\, y_1 = y$ **Hence Proved.**

Now, differentiating this relation n times, with respect to x and using Lebneitz's theorem, we get

$$(1-x^2)\, y_{n+1} + {}^nC_1(-2x)\, y_n + {}^nC_2(-2)\, y_{n-1} = y_n$$

or $\quad (1-x^2)y_{n+1}-(2nx+1)y_n-n(n-1)y_{n-1}=0$

Replacing n by $n-1$ here, gives

$(1-x^2)y_n-\{2(n-1)x+1\}y_{n-1}-(n-1)(n-2)y_{n-2}=0$ hence the desired result

EXAMPLE 1.11. If $y^{1/m}+y^{-1/m}=2x$ show that $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$

[GGSIPU I Sem End Term 2004 Reappear; End Term 2011]

SOLUTION: The given relation can be written as

$$y^{2/m}-2xy^{1/m}+1=0 \text{ whose roots are } y^{1/m}=\frac{2x\pm\sqrt{4x^2-4}}{2}$$

or $\quad y=(x\pm\sqrt{x^2-1})^m \quad$ or $\quad \log y=m\log\left((x\pm\sqrt{x^2-1}\right)$

Differentiating $w.r.t.$ x, we get

$$\frac{1}{y}y_1=\frac{m}{x\pm\sqrt{x^2-1}}\left[1\pm\frac{x}{\sqrt{x^2-1}}\right]=\pm\frac{m}{\sqrt{x^2-1}}$$

or $\quad (x^2-1)y_1^2=m^2y^2$. This on further differentiation $w.r.t.$ x, gives

$(x^2-1)\,2y_1y_2+2xy_1^2=2m^2yy_1 \quad$ or $\quad (x^2-1)y_2+xy_1=m^2y.$

Differentiating the above relation n times, using Lebneitz theorem, we get

$$(x^2-1)y_{n+2}+n.2x\,y_{n+1}+\frac{n(n-1)}{2}.2.\,y_n+x\,y_{n+1}+n.\;y_n=m^2y_n$$

or $\quad (x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$ Hence the result.

EXAMPLE 1.12. (a) If $y=(\sin^{-1}x)^2$, prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$ and hence find $y_n(0)$. [GGSIPU I Sem End Term 2003; I Term 2011]

(b) If $y=e^{m\sin^{-1}x}$, show that $(1-x^2)xy_{n+2}-(2n+1)y_{n+1}-(m^2+n^2)y_n=0$ and hence find $y_n(0)$.

[GGSIPU I Sem I Term 2004; 2007; End Term Jan. 2011, 2012]

SOLUTION: (a) We are given $y=(\sin^{-1}x)^2$ hence $y_1=\dfrac{2\sin^{-1}x}{\sqrt{1-x^2}}$ ...(1)

or $\quad (1-x^2)y_1^2=4y$ which on differentiation again gives

$(1-x^2)\,2y_1y_2-2xy_1^2=4y_1 \quad$ or $\quad (1-x^2)y_2-xy_1=2$ ... (2)

Now differentiating (2) n times w.r.t. x using Lebneitz's theorem, gives

$$(1-x^2)y_{n+2}+n(-2x)y_{n+1}+\frac{n(n-1)}{2}.(-2)y_n-xy_{n+1}-n\,.\,y_n=0 \quad \text{... (3)}$$

or $\quad \boxed{(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0}$ ... (4)

Next, at $x=0$ relation (3) becomes $y_{n+2}(0)=n^2y_n(0)$

or $\quad y_n(0)=(n-2)^2y_{n-2}(0)=(n-2)^2(n-4)^2y_{n-4}(0)$

$\quad =(n-2)^2(n-4)^2(n-6)^2y_{n-6}(0)=\ldots$

$$= \begin{cases} (n-2)^2 (n-4)^2 (n-6)^2 \ldots\ldots 2^2 y_2(0) & \text{when } n \text{ is even} \\ (n-2)^2 (n-4)^2 (n-6)^2 \ldots\ldots 1^2 y_1(0) & \text{when } n \text{ is odd} \end{cases}$$

But from (1), $y_1(0) = 0$ and from (2), $y_2(0) = 2$, therefore

$$y_n(0) = \begin{cases} 0 & \text{when } n \text{ is odd} \\ (n-2)^2 (n-4)^2 (n-6)^2 \ldots\ldots 2^2 . 2 & \text{when } n \text{ is even.} \end{cases}$$ **Ans.**

(b) Given $y = e^{m \sin^{-1} x}$ or $\log y = m \sin^{-1} x$. Differentiating w.r.t. x, we get

$$\frac{1}{y} y_1 = \frac{m}{\sqrt{1-x^2}} \qquad \text{or} \qquad y_1^2 (1 - x^2) = m^2 y^2 \qquad \ldots (1)$$

On further differentiation, we get

$$(1 - x^2)\, 2y_1 y_2 - 2xy_1^2 = 2m^2 y\, y_1 \qquad \text{or} \qquad (1 - x^2)\, y_2 - xy_1 - m^2 y = 0 \qquad \ldots (2)$$

Differentiating the above relation n times, using Lebneitz's theorem, gives

$$(1 - x^2) y_{n+2} + n(-2x) y_{n+1} + \frac{n(n-1)}{2}(-2) y_n - xy_{n+1} - n.1 y_n - m^2 y_n = 0$$

or $$(1 - x^2) y_{n+2} - (2n+1) xy_{n+1} - (m^2 + n^2)\, y_n = 0 \qquad \ldots (3)$$

At $x = 0$ the above relation becomes

$$y_{n+2}(0) = (m^2 + n^2)\, y_n(0)$$

$$\Rightarrow \quad y_n(0) = [m^2 + (n-2)^2]\, y_{n-2}(0) = [m^2 + (n-2)^2]\, [m^2 + (n-4)^2]\, y_{n-4}(0) = \ldots$$

$$= \begin{cases} [m^2 + (n-2)^2][m^2 + (n-4)^2][m^2 + (n-6)^2] \ldots [m^2 + 1] y_1(0) & \text{when } n \text{ is odd} \\ [m^2 + (n-2)^2][m^2 + (n-4)^2][m^2 + (n-6)^2] \ldots [m^2 + 2^2]\, y_2(0) & \text{when } n \text{ is even} \end{cases}$$

But from (1) $y_1(0) = my(0) = m$ and from (2), $y_2(0) = m^2 y(0) = m^2$.

$$\therefore \quad y_n(0) = \begin{cases} [m^2 + (n-2)][m^2 + (n-4)^2][m^2 + (n-6)^2] \ldots [m^2 + 1] m & \text{when } n \text{ is odd} \\ [m^2 + (n-2)^2][m^2 + (n-4)^2][m^2 + (n-6)^2] \ldots [m^2 + 2^2]\, m^2 & \text{when } n \text{ is even} \end{cases}$$

Ans.

EXAMPLE 1.13. If $y = \cos (m \sin^{-1} x)$ (or $= \sin (m \sin^{-1} x)$), show that

$$(1 - x^2)\, y_{n+2} - (2n + 1) x\, y_{n+1} + (m^2 - n^2)\, y_n = 0$$

and hence find $y_n(0)$.

[GGSIPU I Sem III Term Sp. 2007; End Term 2009; I Term 2013, End Term 2013]

SOLUTION: $y = \cos (m \sin^{-1} x)$, $\therefore$ $y_1 = \sin (m \sin^{-1} x). \dfrac{-m}{\sqrt{1-x^2}}$ ...(i)

or $$(1 - x^2)\, y_1^2 = m^2 (1 - y^2). \text{ Again differentiation gives}$$

$$(1 - x^2)\, 2y_1 y_2 - 2xy_1^2 = -2m^2 y\, y_1 \qquad \text{or} \qquad (1 - x^2)\, y_2 - xy_1 + m^2 y = 0 \qquad \ldots\text{(ii)}$$

Differentiating (2) n times, using Lebneitz theorem, gives

$$(1 - x^2) y_{n+2} + n(-2x) y_{n+1} + \frac{n(n-1)}{2}(-2)\, y_n - xy_{n+1} - n.1.y_n + m^2 y_n = 0$$

or $(1 - x^2) y_{n+2} - (2n + 1)\, xy_{n+1} + (m^2 - n^2) y_n = 0$ **which is the required result.**

Now to find y_n (0) we put $x = 0$ here and get

$$y_{n+2}(0) = (n^2 - m^2)\, y_n(0) \quad \text{or} \quad y_n(0) = [(n-2)^2 - m^2]\, y_{n-2}(0) \qquad \text{...(4)}$$

$$\therefore \qquad y_n(0) = [(m-2)^2 - m^2]\,[(n-4)^2 - m^2]\, y_{n-4}(0)$$

$$= \begin{cases} [n-2)^2 - m^2][(n-4)^2 - m^2] \ldots. (1^2 - m^2) \quad y(0) & \text{if } n \text{ is odd} \\ [(n-2)^2 - m^2][(n-4)^2 - m^2] \ldots. (2^2 - m^2) y_2(0) & \text{if } n \text{ is even} \end{cases}$$

From (1) we have $y_1(0) = 0$ and from (2) $y_2(0) = -m^2 y(0) = -m^2$.

$$\text{Therefore } y_n(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ [(n-2)^2 - n^2][n-4)^2 - m^2] \ldots [2^2 - m^2](-m^2) & \text{if } n \text{ is even} \end{cases} \qquad \textbf{Ans.}$$

EXERCISE 1B

1. Find the n^{th} derivative of

   (*i*) $x^2 \sin x$ (*ii*) $e^x \log x$.

2. If $y = x \log(x+1)$, show that $y_n = \dfrac{(-1)^{n-2}(n-2)!(x+n)}{(x+1)^n}$

3. If $y = a\cos(\log x) + b\sin(\log x)$, show that
   $x^2 y_{n+2} + (2n+1)x\, y_{n+1} + (n^2+1)y_n = 0$ [GGSIPU Ist Sem End Term 2008]

4. If $f(x) = x^2 e^{-x/a}$, show that $f_n(0) = \dfrac{n(n-1)(-1)^n}{a^{n-2}}$.

5. If $y = \tan^{-1}\left(\dfrac{a+x}{a-x}\right)$, prove that $(a^2+x^2)y_{n+2} + 2(n+1)x y_{n+1} + n(n+1)y_n = 0$

6. Given that $y = \left\{\log\left(x + \sqrt{a^2+x^2}\right)\right\}^2$, show that

   (*i*) $(a^2+x^2)y_2 + xy_1 = 2$ (*ii*) $(a^2+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$.

7. Given that $y = \left(x \pm \sqrt{x^2-1}\right)^m$, show that

   $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

8. If $f(x) = \cot x$, prove that

   ${}^nC_1 f_{n-1}(0) - {}^nC_3 f_{n-3}(0) + {}^nC_5 f_{n-5}(0) - \ldots = \cos n\pi/2$

9. If $x = \sin t$, $y = \cos at$, show that

   $(1-x^2)y_1^2 = a^2(1-y^2)$ and hence deduce that

   $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-a^2)y_n = 0$

10. If $\cos^{-1}(y/b) = \log(x/n)^n$, show that $x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0$.

11. If $x = \tan(\log y)$ show that

    $(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$.

12. If $x = \tan y$, prove that $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$.

13. If $y = (x^2-1)^n$, prove that $(x^2-1)y_{n+2} - 2xy_{n+1} - n(n+1)y_n = 0$.

EXPANSION OF FUNCTIONS

MACLAURIN'S THEOREM

If a function $f(x)$ is differentiable any number of times and can be expanded in a convergent series of terms of positive integral powers of x, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

PROOF: Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$ ...(1)

where $a_0, a_1, a_2, a_3, \ldots$ are to be determined.

Differentiating (1) successively, we get

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots$$

$$f''(x) = 2a_2 + 3.2\, a_3 x + 4.3\, a_4 x^2 + \ldots + n(n-1) a_n x^{n-2} + \ldots$$

$$f'''(x) = 3\cdot 2\, a_3 + 4\cdot 3\cdot 2\, a_4 x + 5\cdot 4\cdot 3\, a_5 x^2 + \ldots + n(n-1)(n-2) a_n x^{n-3} + \ldots$$

and, in general, $f^{(n)}(x) = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1\cdot a_n +$ terms with positive powers of x.

At $x = 0$, we have $f(0) = a_0$, $f'(0) = a_1$, $f''(0) = 2.1 a_2$, $f'''(0) = 3.2.1.\ a_3, \ldots$

and, in general, $f^{(n)}(0) = n!\ a_n$.

Substituting these values of $a_0, a_1, a_2, \ldots$ in (1), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

or $$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$ ...(2)

which is the expansion of $f(x)$ in ascending powers of x and is known as Maclaurin's expansion of $f(x)$. The conditions under which the expansion (2) is valid, are

(*i*) *$f(x)$ and its successive derivatives must be finite and continous in the range of x in which $f(x)$ is defined.*

(*ii*) *the series on the right hand side of (2) must be convergent.*

For the condition of convergence of the power series (2), if we write

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

then the remainder R_n should tend to 0 as n tends to ∞. The Lagrangian form of the remainder R_n is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) \quad \text{where} \quad 0 < \theta < 1.$$

Maclaurin's Expansion of Some Standard Functions

1. *Expansion of* $\sin x$

Let $f(x) = \sin x$, then $f^{(n)}(x) = \sin(x + n\pi/2)$ and $f^{(n)}(0) = \sin n\pi/2$

$$\therefore \quad \sin x = \sum_{1}^{\infty} \frac{x^n}{n!} \sin\frac{n\pi}{2}$$

or $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

2. *Expansion of* $\cos x$

Since $\cos x = \frac{d}{dx}\sin x$, the expansion can also be obtained by differentiating the expansion of $\sin x$ term by term, hence $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

3. *Expansion of* e^x

If $f(x) = e^x$, $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$

therefore, $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots$$

4. *Expansion of* $\log(1+x)$

If $f(x) = \log(1+x)$, *then* $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

hence $f^{(n)}(0) = (-1)^{n-1}(n-1)!$.

Thus, $$\log(1+x) = \log 1 + \sum_1^{\infty}(-1)^{n-1}(n-1)!\frac{x^n}{n!} = \sum_1^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

or $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \ (|x| < 1)$$

$$\Rightarrow \quad \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$$

[GGSIPU I Sem End Term 2011]

5. *Expansion of* $\tan x$ [GGSIPU I Sem I Term 2012]

If $f(x) = \tan x$, then $f'(x) = \sec^2 x$,

$f''(x) = 2\sec^2 x \tan x$, $f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$,

hence $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 2$, $f^{(iv)}(0) = 0$, $f^{(v)}(0) = 16$, and so on.

Therefore, $$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

6. *Binomial Expansion of* $(1+x)^n$ for $|x| < 1$.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \ (|x| < 1).$$

TAYLOR'S THEOREM

Let $f(x)$ be a function of x and h be small. If the function $f(x+h)$ is capable of being expanded in a convergent series of terms of positive integral powers of h, then this expansion is given by

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots\ldots$$

PROOF: Assume that $f(x+h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \ldots + A_n h^n + \ldots$ ...(1)

where A's are functions of x

Differentiating successively w.r.t. h, we get

$$f'(x+h) = A_1 + 2A_2 h + 3A_3 h^2 + 4A_4 h^3 + \dots + nA_n h^{n-1} + \dots$$

$$f''(x+h) = 2 \cdot A_2 + 3 \cdot 2 \cdot A_3 h + 4 \cdot 3 \cdot A_4 h^2 + \dots + n(n-1) A_n h^{n-2} + \dots$$

$$f'''(x+h) = 3 \cdot 2 \cdot A_3 + 4 \cdot 3 \cdot 2 \cdot A_4 h + \dots + n(n-1)(n-2) A_n h^{n-3} + \dots$$

and, in general, $f^{(n)}(x+h) = n(n-1)(n-2) \dots 3.2\, A_n$ + terms in ascending powers of h.

Putting $h = 0$, we get $f^{(n)}(x) = n!\, A_n$ so that $A_n = \dfrac{f^{(n)}(x)}{n!}$

Substituting these values of A_0, A_1, A_2, ... in (1), we get

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x) + \dots \quad \dots(2)$$

Its another form can be obtained by replacing x by a and h by $x - a$, so as to get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \dots \quad (3)$$

The conditions under which the above expansion is valid, are

(i) *the function $f(x)$ and its derivatives must be finite and continuous in the range of definition of $f(x)$.*

(ii) *the series on the right hand side of (2) must be convergent for which the remainder term $R_n \to 0$ as $n \to \infty$ where $R^n = \dfrac{h^n}{n!} f^{(n)}(x + \theta h)$ and $0 < \theta < 1$.*

In the form (3) of Taylor's expansion, if we take $a = 0$ then we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

which is nothing but the **Maclaurin's expansion of** $f(x)$. Thus Maclurin's expansion is a particular case of Taylor's expansion. **With a slightly different approach we can show here that Taylor's series can be derived from Maclaurin's series.**

The Maclaurin's series for a function $g(x)$, is

$$g(x) = g(0) + x g'(0) + \frac{x^2}{2!} g''(0) + \dots + \frac{x^n}{n!} g^{(n)}(0) + \dots$$

If we replace here $g(x)$ by $f(x+h)$, then

$$g'(x) = f'(x+h), \quad g''(x) = f''(x+h), \quad \dots\ g^{(n)}(x) = f^{(n)}(x+h), \dots$$

Therefore, $g(0) = f(h)$, $g'(0) = f'(h)$, $g''(0) = f''(h)$, ...

Thus, we get $\quad f(x+h) = f(h) + x f'(h) + \dfrac{x^2}{2!} f''(h) + \dots + \dfrac{x^n}{n!} f^{(n)}(h) + \dots$

Now, in this relation we interchange x and h so as to get

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x) + \dots$$

which is Taylor's expansion of $f(x+h)$.

Thus, it can be concluded that Taylor's and Maclaurin's series are not essentially different.

EXAMPLE 1.14. (a) Expand $\tan^{-1}x$ in powers of $(x-1)$. [GGSIPU I Sem Term 2011]
(b) Expand $\log(1+\sin x)$ by Maclaurin's theorem.
[GGSIPU I Sem End Term 2012] (3 Marks)

SOLUTION: (a) The Taylor's expansion of $f(x)$ in powers of $(x-a)$, is

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\ldots$$

Here $f(x)=\tan^{-1}x$ and $a=1$ therefore, $f(1)=\pi/4$,

and $f'(x)=\frac{1}{1+x^2}$ hence $f'(1)=\frac{1}{2}$.

Next $f''(x)=\frac{-2x}{(1+x^2)^2}$ hence $f''(1)=-\frac{1}{2}$,

and $f'''(x)=\frac{(1+x^2)^2(-2)-(-2x)\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4}=\frac{6x^2-2}{(1+x^2)^3}$, $\therefore\ f'''(1)=\frac{1}{2}$.

Therefore, we have $\tan^{-1}x=\frac{\pi}{4}+(x-1)\left(\frac{1}{2}\right)+\frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right)+\frac{(x-1)^3}{3!}\left(\frac{1}{2}\right)+\ldots\ldots$

or $\tan^{-1}x=\frac{\pi}{4}+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2+\frac{1}{12}(x-1)^3+\ldots\ldots$ **Ans.**

(b) Let $f(x)=\log(1+\sin x)$

$$f(0)=0,\quad f'(x)=\frac{\cos x}{1+\sin x}=\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}=\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}$$

or
$$f'(x)=\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\quad\therefore\ f'(0)=1,$$

$$f''(x)=-\frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\quad\therefore\ f''(0)=-\frac{1}{2}\sec^2\frac{\pi}{4}=-1.$$

$$f'''=-\sec\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot\sec\left(\frac{\pi}{4}-\frac{x}{2}\right)\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot\left(\frac{-1}{2}\right)$$

$$=\frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\tan\left(\frac{\pi}{4}-\frac{x}{2}\right),\quad f'''(0)=1 \text{ and so on.}$$

$\therefore$ As Maclaurin's's expression we have

$$f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$$

or $\log(1+\sin x)=x+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}\cdot(1)+\ldots=x-\frac{x^2}{2}+\frac{x^3}{6}+\ldots$ **Ans.**

EXAMPLE 1.15 (a) Show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots$

[GGSIPU I Sem End Term 2012]

(b) Find the Maclawin's series expansion of $e^x \log(1+x)$

[GGSIPU I Sem End Term 2012]

SOLUTION: (a) Let $f(x) = \log(1+e^x)$. By Maclaurin's theorem.

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!} + f^{iv}(0) + \ldots$$

Now $\quad f(0) = \log 2,\quad f'(x) = \dfrac{e^x}{1+e^x} \quad \therefore\ f'(0) = \dfrac{1}{2}$

$$f''(x) = \frac{e^x}{(1+e^x)^2} \quad \therefore\ f''(0) = \frac{1}{4}$$

$$f'''(x) = \frac{e^x(e^x-1)}{(e^x+1)^3} \quad \therefore\ f'''(0) = 0$$

and $\quad f^{iv}(x) = \dfrac{e^x(1+e^x)(2e^x-1) - 3e^{2x}(e^x-1)}{(e^x+1)^4} \quad \therefore\ f^{iv}(0) = \dfrac{1}{8}, \ldots$

Hence $\quad f(x) = \log(1+e^x) = \log 2 + \dfrac{x}{2} + \dfrac{x^2}{8} - \dfrac{x^4}{192} + \ldots \quad$ **Hence proved.**

(b) $$e^x \log(1+x) = \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \ldots\right]\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\right]$$

$$= x + x^2\left(1 - \frac{1}{2}\right) + x^3\left(\frac{1}{3} + \frac{1}{2} - \frac{1}{2}\right) + x^4\left(-\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{6}\right) + x^5\left(\frac{1}{5} - \frac{1}{4} + \frac{1}{6} - \frac{1}{12} + \frac{1}{24}\right) + \ldots$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + x^4(0) + x^5\left(\frac{3}{40}\right) + \ldots \quad \textbf{Ans.}$$

EXAMPLE 1.16 (a) Expand the polynomial $2x^3 + 7x^2 + x - 1$ in powers of $(x-2)$.

[GGSIPU I Sem I Term 2010]

(b) Expand $\tan x$ in powers of $\left(x - \frac{\pi}{4}\right)$ upto first four terms.

[GGSIPU I Sem. End Term 2009]

SOLUTION: (a) Let $f(x) = 2x^3 + 7x^2 + x - 1$, then

$$f'(x) = 6x^2 + 14x + 1,\quad f''(x) = 12x + 14,\quad f'''(x) = 12.$$

Now, by Taylor's theorem for $f(x)$ about the point $x = 2$, we have

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \ldots$$

Here $\quad f(2) = 45,\quad f'(2) = 53,\quad f''(2) = 38,\quad f'''(2) = 12$ hence

$$f(x) = 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3. \quad \textbf{Ans.}$$

(b) By Taylor's theorem the expansion of $f(x)$ about $x = a$, is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

Here $f(x) = \tan x$ and 'a' $= \frac{\pi}{4}$ and $f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1.$

$f'(x) = \sec^2 x$, $f''(x) = 2\sec^2 x \tan x$ and $f'''_{(x)} = 2\sec^4 x + 4\tan^2 x \sec^2 x$,

$\therefore \quad f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = 2, \quad f''\left(\frac{\pi}{4}\right) = 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 4,$

$f'''\left(\frac{\pi}{4}\right) = 2\sec^4\frac{\pi}{4} + 4\tan^2\frac{\pi}{4}\sec^2\frac{\pi}{4} = 8 + 4 \cdot 1 \cdot 2 = 16$

$\therefore \quad \tan x = 1 + \left(x - \frac{\pi}{4}\right)2 + \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 \cdot 4 + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 \cdot 16$ upto first four terms

$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3.$ **Ans.**

EXAMPLE 1.17. **(a)** Prove that $\log(1 + x + x^2 + x^3 + x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \frac{4}{5}x^5 + \frac{x^6}{6} + \ldots$

(b) Show that $(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \frac{33}{40}x^6 + \ldots$

SOLUTION: **(a)** The result will be derived by using the method of expansion of standard functions.

Since $1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$ (as sum of a geometric series)

we have, $\log(1 + x + x^2 + x^3 + x^4) = \log(1 - x^5) - \log(1 - x)$

$$= \left[-x^5 - \frac{(x^5)^2}{2} - \frac{(x^5)^3}{3} - \frac{(x^5)^4}{4} - \ldots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \ldots\right]$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + x^5\left(\frac{1}{5} - 1\right) + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + x^{10}\left(\frac{1}{10} - \frac{1}{2}\right) + \ldots$$

or $\log(1 + x + x^2 + x^3 + x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \frac{4x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} - \frac{2}{5}x^{10} + \ldots$

Hence Proved.

(b) We can write $(1+x)^x = e^{\log(1+x)^x} = e^{x\log(1+x)} = e^t$ where $t = x\log(1+x)$.

Now, $t = x\log(1+x) = x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \ldots$

$\therefore \quad (1+x)^x = e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots$

$$= 1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \ldots\right) + \frac{1}{2!}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \ldots\right)^2 + \frac{1}{3!}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \ldots\right)^3 + \ldots\ldots$$

$$= 1 + x^2 - \frac{x^3}{2} + x^4\left(\frac{1}{3} + \frac{1}{2}\right) + x^5\left(-\frac{1}{4} - \frac{1}{2}\right) + x^6\left(\frac{1}{5} + \frac{1}{8} + \frac{1}{3} + \frac{1}{6}\right) + \ldots$$

or $$(1 + x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \frac{33}{40}x^6 + \ldots$$ **Hence Proved.**

EXAMPLE 1.18. Expand $\sin(m \sin^{-1} x)$ in ascending powers of x. *[GGSIPU I Sem End Term 2012]*

SOLUTION: Let $y = \sin(m \sin^{-1} x)$ then $y_1 = \cos(m \sin^{-1} x) \cdot \dfrac{m}{\sqrt{1-x^2}}$ ...(1)

or $$(1 - x^2)\, y_1^2 = m^2 \cos^2(m \sin^{-1} x) = m^2 (1 - y^2)$$

$\Rightarrow$ $$(1 - x^2)\, 2y_1 y_2 - 2x\, y_1^2 = -2m^2 y y_1 \quad \text{if} \quad (y_1)_0 \neq 0$$

or $$(1 - x^2)\, y_2 - x y_1 + m^2 y = 0 \qquad \ldots(2)$$

Now differentiating (2) on both sides, n times w.r.t. x, we get

$$(1 - x^2) y_{n+2} + n(-2x)\, y_{n+1} - \frac{2n(n-1)}{2} y_n - [x\, y_{n+1} + n \cdot 1 \cdot y_n] + m^2 y_n = 0$$

or $$(1 - x^2)\, y_{n+2} - (2n + 1)x\, y_{n+1} - (n^2 - m^2)\, y_n = 0$$

At $x = 0$ we have $(y_{n+2})_0 = (n^2 - m^2)\,(y_n)_0$ ...(3)

Putting $x = 0$ in (1), we get $y_1(0) = m$

Putting $x = 0$ in (2), we get $(y_2)_0 = -m^2\, y_1(0)$ $\therefore$ $(y_2)_0 = -m^3$.

From (3) we can write

$$\begin{aligned}(y_n)_0 &= [(n-2)^2 - m^2]\,(y_{n-2})_0 = [(n-2)^2 - m^2]\,[(n-4)^2 - m^2]\,(y_{n-4})_0 \\ &= [(n-2)^2 - m^2]\,[(n-4)^2 - m^2]\,[(n-6)^2 - m^2]\,(y_{n-6})_0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \begin{cases}[(n-2)^2 - m^2][(n-4)^2 - m^2]\ldots(2^2 - m^2)\,(y_2)_0 & \text{when } n \text{ is even} \\ [(n-2)^2 - m^2][(n-4)^2 - m^2]\ldots(1^2 - m^2)\,(y_1)_0 & \text{when } n \text{ is odd}\end{cases} \\ &= \begin{cases}[(n-2)^2 - m^2][(n-4)^2 - m^2]\ldots(4 - m^2)\,(-m^3) & \text{when } n \text{ is even} \\ [(n-2)^2 - m^2][(n-4)^2 - m^2]\ldots(1 - m^2)\, m & \text{when } n \text{ is odd}\end{cases}\end{aligned}$$

Thus, $(y_3)_0 = (1 - m^2)m,\quad (y_5)_0 = (3^2 - m^2)\,(1 - m^2)\, m, \ldots$

and $(y_4)_0 = (2^2 - m^2)\,(-m^3),\quad (y_6)_0 = (4^2 - m^2)\,(2^2 - m^2)\,(-m^3), \ldots$

Therefore,

$$y = 0 + mx + \frac{x^2}{2!}(-m^3) + \frac{x^3}{3!} m(1 - m^2) + \frac{x^4}{4!}(4 - m^2)(-m^3) + \frac{x^5}{5!} m(1 - m^2)(3^2 + m^2) + \ldots \quad \textbf{Ans.}$$

EXAMPLE 1.19. Show that $\tan\left(\dfrac{\pi}{4} + x\right) = 1 + 2x + 2x^2 + \dfrac{8}{3}x^3 + \dfrac{10}{3}x^4 + \ldots$

and hence calculate the value of $\tan 46°$ correct to four decimal places, given that $\pi = 3.14159$.

SOLUTION: Let $f(x) = \tan\left(\dfrac{\pi}{4} + x\right)$, then $f(0) = 1$.

Next, $$f'(x) = \sec^2\left(\frac{\pi}{4} + x\right) = 1 + \tan^2\left(\frac{\pi}{4} + x\right) = 1 + f^2(x) \quad \therefore \quad f'(0) = 1 + 1 = 2$$

Next $\quad f''(x) = 2f(x)f'(x)$, hence $f''(0) = 2f(0)f'(0) = 4$.

and $\quad f'''(x) = 2f(x)f''(x) + 2\{f'(x)\}^2$

hence $\quad f'''(0) = 2f(0)f''(0) + 2\{f'(0)\}^2 = 16$

Now, $\quad f^{iv}(x) = 2f(x)f'''(x) + 2f'(x)f''(x) + 4f'(x)f''(x)$,

hence $\quad f^{iv}(0) = 2f(0)f'''(0) + 6f'(0)f''(0) = 80$

Therefore, $\quad \tan\left(\frac{\pi}{4}+x\right) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots$

$$= 1 + 2x + \frac{4}{2!}x^2 + \frac{16}{3!}x^3 + \frac{80}{4!}x^4 + \ldots = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \ldots$$

Hence the result.

Now taking $x = 1^\circ = \frac{\pi}{180} \doteqdot \frac{3.14159}{180} = 0.01745$

we have $\quad x^2 = 0.000345$ and $x^3 = 0.000005, \ldots$

To achieve the desired accuracy we need consider only first four terms, hence

$$\tan 46^\circ = 1 + 2(0.01745) + 2(0.000345) + \frac{8}{3}(0.000005)$$

$$= 1 + 0.03490 + 0.00069 = 1.0355$$

which is correct to four places of decimal.

EXAMPLE 1.20. Show that $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots$

and hence find the approximate value of π.

SOLUTION: Let $y = \sin^{-1} x$ hence $y_1 = \frac{1}{\sqrt{1-x^2}}$ or $(1-x^2)y_1^2 = 1$.

Differentiating both sides w.r.t. x, gives

$$(1-x^2)\,2y_1y_2 - 2xy_1^2 = 0 \quad \text{or} \quad (1-x^2)y_2 - xy_1 = 0 \text{ as } y_1 \neq 0.$$

Differentiating the above relation n times using Leibnitz's theorem, we get

$$(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n - x\,y_{n+1} - {}^nC_1\,1.\,y_n = 0$$

or $\quad (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0$

At $x = 0$, we have $\quad y_{n+2}(0) = n^2 y_n(0)$ or $y_n(0) = (n-2)^2 y_{n-2}(0)$.

and $\quad y_n(0) = (n-2)^2(n-4)^2 y_{n-4}(0)$.

Continuing the process, we get

$$y_n(0) = (n-2)^2(n-4)^2(n-6)^2 \ldots 2^2 y_2(0), \text{ when } n \text{ is even}$$

$$= (n-2)^2(n-4)^2(n-6)^2 \ldots 1^2 y_1(0). \text{ when } n \text{ is odd.}$$

But $\quad y_1(0) = 1$ and $y_2(0) = 0$,

therefore, $\quad y_n(0) = 0$ when n is even

$$= 1^2 \cdot 3^2 \cdot 5^2 \cdot \ldots (n-2)^2 \text{ when } n \text{ is odd.}$$

Now, by Maclaurin's theorem, we have

$$y = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \ldots$$

or $\quad \sin^{-1} x = 0 + x + \frac{1^2}{3!}x^3 + \frac{1^2 \cdot 3^2}{5!}x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!}x^7 + \ldots\ldots$

$$= x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots\ldots$$

Here $-1 < x < 1$ and to obtain the value of π we put $x = 1/2$, to get

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{8} + \frac{3}{40} \cdot \frac{1}{32} + \frac{5}{112} \cdot \frac{1}{128} + \ldots$$

or $\quad \pi = 3 + \frac{1}{8} + \frac{9}{640} + \frac{30}{14336} = 3.141$ approximately. **Ans.**

EXAMPLE 1.21. (a) Estimate the value of $\sqrt{10}$ correct to four places of decimal using Taylor's theorem. [GGSIPU Ist Sem. End Term 2003]

(b) Apply Taylor's theorem to estimate the value of $f(11/10)$ where $f(x) = x^3 + 3x^2 + 15x - 10$.

[GGSIPU Ist Sem Ist Term 2007; Ist Term Sp. 2009]

SOLUTION: (a) Let $f(x) = \sqrt{x}$, then using Taylor's expansion

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

we get $\quad (x+h)^{1/2} = x^{1/2} + h\frac{d}{dx}x^{1/2} + \frac{h^2}{2!}\frac{d^2}{dx^2}x^{1/2} + \frac{h^3}{3!}\frac{d^3}{dx^3}x^{1/2} + \ldots$

Taking $x = 9$ and $h = 1$ in the above expansion, gives

$$\sqrt{10} = \left[3 + h\left(\frac{1}{2}x^{-1/2}\right) + \frac{h^2}{2!}\left(\frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}\right) + \frac{h^3}{3!}\left(\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}\right) + \ldots\right]_{x=9,\ h=1}$$

$$= 3 + \frac{1}{2.3} - \frac{1}{8.27} + \frac{3}{3.2.8.243} - \ldots\ldots$$

$= 3.16227$ approximately. **Ans.**

(b) By Taylor's expansion, we have

$$f(a+\delta x) = f(a) + \delta x\, f'(a) + \frac{(\delta x)^2}{2!}f''(a) + \frac{(\delta x)^3}{3!}f'''(a) + \ldots \quad \ldots(1)$$

Here $\quad f(x) = x^3 + 3x^2 + 15x - 10$ and $a = 1$ and $\delta x = 0.1$

$f(1) = 1^3 + 3.1^2 + 15.1 - 10 = 9$

$f'(x) = 3x^2 + 6x + 15 \quad \therefore \quad f'(1) = 3 + 6 + 15 = 24$

$f''(x) = 6x + 6, \quad \therefore \quad f''(1) = 12$

$f'''(x) = 6 \quad \therefore \quad f'''(1) = 6$

Using (1) here we have

$$\therefore \quad f\left(\frac{11}{10}\right) = f\left(1 + \frac{1}{10}\right) = 9 + 0.1\,(24) + \frac{(0.1)^2}{2}12 + \frac{(0.1)^3}{3!}(6)$$

$= 9 + 2.4 + 0.06 + 0.001 = 11.461.$ **Ans.**

EXAMPLE 1.22. Apply Taylor's theorem to show that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\alpha\cdot\frac{\sin\alpha}{1} - (h\sin\alpha)^2\cdot\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\cdot\frac{\sin 3\alpha}{3} - \ldots$$

where $\alpha = \cot^{-1}x$. [GGSIPU I Sem I Term 2004]

SOLUTION: Let $f(x) = \tan^{-1}x$,

hence $f'(x) = \dfrac{1}{x^2+1} = \dfrac{1}{(x+i)(x-i)} = \dfrac{1}{2i}\left[\dfrac{1}{(x-i)} - \dfrac{1}{(x+i)}\right]$

Differentiating w.r.t. x, $(n-1)$ times, gives

$$f^{(n)}(x) = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

Since $\alpha = \cot^{-1}x$ or $x = \cot\alpha$, we have

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{\sin^n\alpha}{(\cos\alpha - i\sin\alpha)^n} - \frac{\sin^n\alpha}{(\cos\alpha + i\sin\alpha)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\sin^n\alpha\,[(\cos n\alpha + i\sin n\alpha) - (\cos n\alpha - i\sin n\alpha)]$$

(using de Moivre's Theorem)

$$= (-1)^{n-1}(n-1)!\,\sin^n\alpha\,\sin n\alpha$$

We know, by Taylor's Theoem that

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

$$\therefore \tan^{-1}(x+h) = \tan^{-1}x + \frac{h}{1!}\sin\alpha\sin\alpha - \frac{h^2}{2!}\sin^2\alpha\sin 2\alpha + \frac{h^3}{3!}2!\cdot\sin^3\alpha\sin 3\alpha - \frac{h^4}{4!}3!\sin^4\alpha\sin 4\alpha + \ldots$$

$$= \tan^{-1}x + (h\sin\alpha)\frac{\sin\alpha}{1} - (h\sin\alpha)^2\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\frac{\sin 3\alpha}{3} - (h\sin\alpha)^4\frac{\sin 4\alpha}{4} + \ldots$$

Hence the result.

EXAMPLE 1.23. Expand $\cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right)$ in ascending powers of x. $(x > 0)$

SOLUTION: Let us put $x = \cot\theta$, then

$$\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\frac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta}\right) = \cos^{-1}\left(\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta = 2\cot^{-1}x = 2\left(\frac{\pi}{2} - \tan^{-1}x\right) = \pi - 2\tan^{-1}x.$$

We know that $\tan^{-1}x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$

and general value of $\tan^{-1} x$ is written as

$$\tan^{-1} x = n\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

therefore, $$\cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right) = \pi - 2\left(n\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)$$

$$= -(2n-1)\pi - 2x + \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{2}{7}x^7 - \ldots\ldots \quad \textbf{Ans.}$$

APPROXIMATE ERROR

Let y be a function of x, given by $y = f(x)$. Now, if x suffers a small change δx, it is often required to find how much change takes place in y. Let this change in y be denoted by δy.

As such $y + \delta y = f(x + \delta x)$ or $\delta y = f(x + \delta x) - f(x)$

Using Taylor's expansion, we get

$$\delta y = \left\{f(x) + \delta x\, f'(x) + \frac{\delta x^2}{2!} f''(x) + \ldots\right\} - f(x) = \delta x\, f'(x) + \frac{(\delta x)^2}{2!} f''(x) + \ldots$$

Here, δx is small and if we neglet its square and higher powers, then

$$\boxed{\delta y = f'(x)\, \delta x = \frac{dy}{dx} \delta x} \text{ approximately.}$$

Further, if δx is the error in x, then $\frac{\delta x}{x}$ is called *relative error* and $\frac{\delta x}{x} \times 100$ is called the *percentage error* in x.

EXAMPLE 1.24 (a) Find the change in the total surface area of a right circular cone when

(*i*) the radius r is constant and the altitude h changes by a small amount δh

(*ii*) the altitude is constant and the radius changes by a small amount δr.

[*GGSIPU Ist Sem. Ist Term 2007*]

(b) If $T = 2\pi\sqrt{(l/g)}$ find the error in T corresponding to the error of 2% in l where g is constant.

[*GGSIPU I Sem End Term 2013*]

SOLUTION: (a) The total surfade area S of a right circular cone with radius of the base as r and altitude h, is given by $S = \pi r^2 + \pi r l = \pi r^2 + \pi r\sqrt{r^2 + h^2}$

Now, (*i*) if r is constant and altitude h changes by δh, then

$$\frac{dS}{dh} = 0 + \frac{\pi r}{2}\left(r^2 + h^2\right)^{-1/2} .2h = \frac{\pi r h}{\sqrt{r^2 + h^2}}$$

therefore, the consequential change δS in S will be given by

$$\delta S = \frac{dS}{dh}\delta h = \frac{\pi r h}{\sqrt{r^2 + h^2}}\, \delta h \text{ approximately.} \quad \textbf{Ans.}$$

(*ii*) if h is constant and the radius r changes by δr, then

$$\frac{dS}{dr} = 2\pi r + \pi\sqrt{r^2 + h^2} + \frac{\pi r . 2r}{2\sqrt{r^2 + h^2}} = 2\pi r + \frac{\pi\left(2r^2 + h^2\right)}{\sqrt{r^2 + h^2}}$$

therefore, the resulting change δS, in S will be given by

$$\delta S = \left[2\pi r + \frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}\right]\delta r \quad \text{approximately.} \qquad \textbf{Ans.}$$

(b) We have $T = \dfrac{2\pi}{\sqrt{g}}\sqrt{l}$ $\therefore$ $\dfrac{dT}{dl} = \dfrac{2\pi}{\sqrt{g}}\left(\dfrac{1}{2\sqrt{l}}\right) \Rightarrow \delta T = \dfrac{dT}{dl}\delta l = \dfrac{2\pi}{\sqrt{g}}\left(\dfrac{\delta l}{2\sqrt{l}}\right)$

$$\therefore \quad \frac{\delta T}{T} = \frac{2\pi}{\sqrt{g}}\left(\frac{\sqrt{g}}{2\pi\sqrt{l}}\right)\left(\frac{\delta l}{2\sqrt{l}}\right) = \frac{1}{2}\frac{\delta l}{l} \quad \text{or} \quad \frac{\delta T}{T}\times 100 = \frac{1}{2}\left(\frac{\delta l}{l}\times 100\right).$$

Since $\dfrac{\delta l}{l}\times 100 = 2$ hence percentage error in T is equal to $\dfrac{1}{2}\cdot(2) = 1\%$ **Ans.**

EXAMPLE 1.25. If Δ is the area of a triangle ABC having sides equal to a, b, c and S is the semi-perimeter, prove that the error $\delta\Delta$ in Δ resulting from a small error δc in the measurement of c, is given by

$$\delta\Delta = \frac{\Delta}{4}\left[\frac{1}{S} + \frac{1}{S-a} + \frac{1}{S-b} - \frac{1}{S-c}\right]\delta c.$$

SOLUTION: We know that $\Delta^2 = S(S-a)(S-b)(S-c)$ where $S = (a+b+c)/2$.

or $\quad 2\log\Delta = \log S + \log(S-a) + \log(S-b) + \log(S-c)$.

Differentiating both sides w.r.t. c, gives

$$\frac{2}{\Delta}\frac{d\Delta}{dc} = \frac{1}{S}\frac{dS}{dc} + \frac{1}{S-a}\frac{d}{dc}(S-a) + \frac{1}{S-b}\frac{d}{dc}(S-b) + \frac{1}{S-c}\frac{d}{dc}(S-c)$$

$$= \frac{1}{S}\cdot\frac{1}{2} + \frac{1}{2(S-a)} + \frac{1}{2(S-b)} + \frac{1}{(S-c)}\left(\frac{1}{2}-1\right)$$

or $$\frac{d\Delta}{dc} = \frac{\Delta}{4}\left[\frac{1}{S} + \frac{1}{S-a} + \frac{1}{S-b} - \frac{1}{S-c}\right]$$

Therefore the error $\delta\Delta$ in the area, is given by

$$\delta\Delta = \frac{d\Delta}{dc}\delta c = \frac{\Delta}{4}\left[\frac{1}{S} + \frac{1}{S-a} + \frac{1}{S-b} - \frac{1}{S-c}\right]\delta c.$$ **Hence Proved.**

EXAMPLE 1.26. A heavy string is suspended from two poles of equal height, taking the shape of a catenary with sag a and equation $y = a\cosh(x/a)$. If the absolute value of x is small, show that the shape of the string can be approximated by the parabola $y = a + x^2/(2a)$.

SOLUTION: Here $y = a\cosh(x/a)$, hence $y(0) = a$ and

$$y_1 = a\sinh(x/a)\cdot\frac{1}{a} = \sinh(x/a), \text{ hence } y_1(0) = 0$$

Further $y_2 = \dfrac{1}{a}\cosh(x/a)$, hence $y_2(0) = \dfrac{1}{a}$

Again $y_3 = \dfrac{1}{a^2}\sinh(x/a)$, hence $y_3(0) = 0$

Further $y_4 = \frac{1}{a^3}\cosh(x/a)$, hence $y_4(0) = \frac{1}{a^3}$, and so on.

Therefore, by Maclaurin's expansion of y, we have

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \ldots$$

$$= a + 0 + \frac{x^2}{2a} + 0 + \frac{x^4}{24a^3} + \ldots$$

As $|x|$ is small, neglecting terms beyond x^2, we get

$y = a + \frac{x^2}{2a}$ which is the equation of a parabola. **Hence Proved.**

EXAMPLE 1.27. Expand $\log(x+\sqrt{x^2+1})$ upto first four terms by Maclaurin's theorem. Putting $x = 0.75$ in the expansion, calculate the value of log 2 to four place of decimal and find the percentage error. *[GGSIPU I Sem End Term 2004 Reappear]*

SOLUTION: $f(x) = \log(x+\sqrt{x^2+1})$, $f(0) = 0$.

$$f'(x) = \frac{1 + \frac{1}{2}\frac{2x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}} \quad \text{hence } f'(0) = 1$$

$$f''(x) = \frac{-1}{2}(x^2+1)^{-3/2}\cdot 2x = -x(x^2+1)^{-3/2} \quad \text{hence } f''(0) = 0$$

$$f'''(x) = -1\cdot(x^2+1)^{-3/2} - x\left(\frac{-3}{2}\right)(x^2+1)^{-5/2}\cdot 2x$$

$$= -(x^2+1)^{-3/2} + 3x^2(x^2+1)^{-5/2}, \quad \text{hence } f'''(0) = -1.$$

By Maclaurin's theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3}f'''(0) \text{ (retaining upto four terms)} \quad \ldots(1)$$

Putting here $x = 0.75$ we get $f\left(\frac{3}{4}\right) = \log\left(\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right) = \log 2.$

Thus from (1), we get $\log(2) = \left[0 + x.1 + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1)\right]_{x=0.75}$

$$= 0.75 - \frac{1}{6}(0.75)^3 = \frac{3}{4} - \frac{1}{6}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{87}{128}$$

$= 0.6626$ approximately.

Actual value of $\log 2 = 0.6931$ hence error $= 0.0305$

Therefore the percentage error $= \frac{0.0305}{0.6931} \times 100 = 4.4\%$ **Ans.**

EXAMPLE 1.28. The angles of a triangle are calculated from the sides $a, b, c,$ if small errors $\delta a, \delta b, \delta c$, are made in the sides. Show that $\delta A = \dfrac{a}{2\Delta}\,[\delta a - \delta b \cos C - \delta c \cos B]$ approximately where Δ is the area of the triangle and hence verify that $\delta A + \delta B + \delta C = 0.$

[*GGSIPU I Sem End I Term 2004*]

SOLUTION: We know that $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$. Since $\delta a, \delta b, \delta c$ are the errors made in measuring the sides a, b, c then as a result δA is the error made in measuring angle A.

$$\therefore \quad -\sin A\,\delta A = \frac{-2a\delta a}{2bc} + \delta b\left(\frac{1}{2c} - \frac{c}{2b^2} + \frac{a^2}{2cb^2}\right) + \delta c\left(\frac{1}{2b} - \frac{b}{2c^2} + \frac{a^2}{2bc^2}\right)$$

$$= \frac{-a}{bc}\delta a + \frac{\delta b}{2cb^2}(b^2 - c^2 + a^2) + \frac{\delta c}{2bc^2}(c^2 + a^2 - b^2)$$

$$= \frac{-a}{bc}\delta a + \frac{\delta b}{2cb^2}2ab\cos C + \frac{\delta c}{2bc^2}2ac\cos B$$

$$= \frac{-a}{bc}\delta a + \frac{a}{bc}\delta b\cos C + \frac{a}{bc}\delta c\cos B$$

or
$$\frac{bc}{a}\sin A\,\delta A = \delta a - \delta h\cos C - \delta c\cos B$$

But $\Delta = \dfrac{1}{2}bc\,\sin A$ hence we have

$$\frac{2\Delta}{a}\delta A = \delta a - \delta b\cos C - \delta c\cos B$$

or
$$\delta A = \frac{a}{2\Delta}\,[\delta a - \delta b\cos C - \delta c\cos B]$$

$$\therefore \quad \delta A + \delta B + \delta C = \frac{1}{2\Delta}\,[a\delta a - a\cos c\,\delta b - a\cos B\,\delta c + b\delta b - b\cos A\,\delta c - b\cos C\,\delta a + c\delta c - c\cos A\,\delta b - c\cos B\,\delta a]$$

$$= \frac{1}{2\Delta}\,[(a - b\cos c - c\cos B)\,\delta a + (b - c\cos A - a\cos C)\,\delta b + (c - a\cos B - b\cos A)\,\delta c]$$

$$= 0.$$

Hence the result.

CHAPTER 2

Convergence and Divergence of Infinite Series, Conditional and Absolute Convergence

Convergence and Divergence of Infinite Series, Positive Term Series, Comparison Test D' Alembert's Ratio Test, Integral Test, Cauchy's Root Test, Raabe's Test, Logarithmic Test, Alternating Series, Leibnitz Test, Absolute Convergence and Conditional Convergence.

In various problems of Science and Engineering we come across series with infinite number of terms and the question comes up whether these can be summed up or not. If the sum of the infinite series can be obtained it comes under convergent series otherwise under divergent or oscillatory series. In practice, this problem is encountered very frequently and that is why the concept of convergence and divergence of infinite series gains importance. First, we define sequences which is a prerequisite for the study of infinite series.

SEQUENCES

An ordered set of numbers such as $u_1, u_2, u_3, \ldots, u_n$ is called a *sequence* and is usually designated briefly by $\{u_n\}$. If the sequence possesses unlimited number of terms it is called an *infinite sequence.* For example, $2, 7, 12, 17, \ldots, 32$ is a finite sequence whereas $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \infty$ is an infinite sequence with nth term $u_n = \frac{1}{2n-1}$, $n = 1, 2, 3, \ldots$.

Limit of a Sequence

A number l is called the limit of an infinite sequence $u_1, u_2, u_3, \ldots$ if, for any given small number $\epsilon > 0$, we can find a positive integer N depending upon ϵ such that $|u_n - l| < \epsilon$ for all $n \geq N$. In such a case we write $\lim_{n\to\infty} u_n = l$. If the limit of a sequence exists the sequence is called ***convergent***; otherwise it is called ***divergent***. For example, the sequence $\left\{\frac{3n+1}{n}\right\}$, that is, $4, \frac{7}{2}, \frac{10}{3}, \ldots$ is convergent since $\lim_{n\to\infty} \frac{3n+1}{n} = 3$, whereas the sequence with terms $1, 5, 9, 13, 17, \ldots$ is divergent. For any convergent sequence, the limit is always *unique.*

Theorems on Limits of Sequences

It is easy to show that

(*i*) If $\lim_{n\to\infty} u_n$ exists, it must be unique.

(*ii*) If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$.

(*iii*) If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then $\lim_{n\to\infty} a_n b_n = A \cdot B$.

(*iv*) If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B\ (\neq 0)$ then $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$.

Bounded, Monotonic Sequence

If $u_n \leq M$ for $n = 1, 2, 3, \ldots$, where M is constant (not depending on n), we say that the sequence $\{u_n\}$ is *bounded above* and M is called an upper bound of $\{u_n\}$. If $u_n \geq m$ for all $n \in N$ the sequence is *bounded below* and m is called a *lower bound* of $\{u_n\}$.

If $m \leq u_n \leq M$ for all $n \in N$ the sequence $\{u_n\}$ is called *bounded.* Every convergent sequence is bounded but its converse is not necessarily true.

If $u_{n+1} \geq u_n$ for all $n \in N$, the sequence is called *monotonically increasing* and if $u_{n+1} \leq u_n$ for all $n \in N$, the sequence is called *monotonically decreasing.* Remember that *every bounded and monotonic* (increasing or decreasing) *sequence is convergent.*

Cauchy's Convergence Criterion

It states that *a sequence $\{u_n\}$ of real numbers converges if and only if for every small $\in > 0$ we can find a number N such that $|u_p - u_q| < \in$ for all $p, q > N$.* The criterion has the advantage that one need not know the limit l in order to demonstrate its convergence.

INFINITE SERIES

Let $u_1, u_2, u_3, \ldots$ be a given sequence. Form a new sequence $S_1, S_2, S_3, \ldots$ where

$$S_1 = u_1,\quad S_2 = u_1 + u_2,\quad S_3 = u_1 + u_2 + u_3, \ldots,\quad S_n = u_1 + u_2 + \ldots + u_n;$$

where S_n is called ***nth partial sum*** and is the sum of the first n terms of the sequence $\{u_n\}$.

Here the series $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ denoted by $\sum_1^\infty u_n$ is an *infinite series.* If $\lim_{n\to\infty} S_n$ exists and is equal to S then the series is called ***convergent*** and S is called its sum, otherwise the series is called ***divergent***. However, if $\lim_{n\to\infty} S_n$ is not unique, the series $\sum u_n$ is known as ***oscillatory.***

For example, consider the convergence of the series $1^2 + 2^2 + 3^2 + \ldots + n^2 + \ldots$.

Since $S_n = \sum_{k=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$ we have $\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} = \infty$.

As $\lim_{n\to\infty} S_n$ is infinite, the given series is divergent.

Next, let us consider the convergence of the series $4 - 3 - 1 + 4 - 3 - 1 + 4 - 3 - 1 + \ldots\ldots$

Here $S_n = (4 - 3 - 1) + (4 - 3 - 1) + (4 - 3 - 1) + \ldots$ upto n terms. $= 0, 4,$ or 1 according as the number of terms is of the type, $3m, 3m + 1$ or $3m + 2$. In this case S_n does not tend to a unique limit, hence the given series is *oscillatory.*

EXAMPLE 2.1. *Geometric Series*. Show that the series $\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + ... + ar^{n-1} + ...$ is convergent if $|r| < 1$ and divergent if $|r| \geq 1$.

SOLUTION:

CASE I: $|r| < 1$

Here $S_n = \dfrac{a(1-r^n)}{1-r}$ and $\lim\limits_{n\to\infty} r^n = 0$ as $|r| < 1$.

Therefore, $\lim\limits_{n\to\infty} S_n = \dfrac{a}{1-r}$ which is finite and unique, thus the given series converges to $\dfrac{a}{1-r}$.

CASE II: $r \geq 1$

Here $S_n = \dfrac{a(r^n - 1)}{r-1}$ and $\lim\limits_{n\to\infty} r^n = \infty$.

Therefore, $S_n \to \infty$ as $n \to \infty$ and the series diverges.

Also when $r = 1$, the series becomes $a + a + a + a + ...$

$\therefore \quad S_n = a + a + a + a + ... = na.$

Thus, $S_n \to \infty$ as $n \to \infty$ and the series diverges.

CASE III: $r \leq -1$

Here when $r = -1$ the series becomes $a - a + a - a + ...$

Therefore, $S_n = a - a + a - a + a - a + ... = 0$ when n is even.

$= a$ when n is a odd

Thus, the given series is oscillatory. And when $r < -1$ let $r = -K$ so that $K > 1$

then $r^n = (-1)^n K^n$ and $S_n = \dfrac{a(1-(-1)^n K^n)}{1+K}$

Thus, the n^{th} partial sum $S_n \to -\infty$ as $n \to \infty$, and the series diverges. **Ans.**

FUNDAMENTAL THEOREM ON INFINITE SERIES

Addition or deletion of a finite number of terms does not affect the nature of the infinite series.

To explain this concept, let a series be convergent hence its sum is finite. Now, if we add to it a finite number of terms it will remain finite and also, if we delete from it a finite number of terms it still remains finite hence convergent.

Similarly, if a series is divergent its sum is infinite and, if we add or delete a finite number of terms, the series will remain divergent.

POSITIVE TERM SERIES

An infinite series in which from and after certain term, all the terms are positive, is called positive term series.

For example, the series $-6 - 5 + 4 - 3 + 2 - 1 + 1 + 2 + 3 + 4 + 5 + ...$ is a positive term series since from and after seventh term all the terms are positive.

Necessary condition for the convergence of a positive term series

[GGSIPU I Sem End Term 2004, End Term 2013]

If a positive term series $\sum_{1}^{\infty} u_n$ *is convergent then* $\lim_{n\to\infty} u_n = 0$.

To prove it, consider the n^{th} partial sum $S_n = u_1 + u_2 + ... + u_n$.

Since $\sum_{1}^{\infty} u_n$ is convergent $\lim_{n\to\infty} S_n = S$, say. It follows that $\lim_{n\to\infty} S_{n-1} = S$.

Now $u_n = S_n - S_{n-1}$ hence $\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$.

Remember that it is a necessary condition and not sufficient for the convergence of $\sum u_n$. However, it leads to a **simple test for the divergence of a series:**

According to it if $\lim_{n\to\infty} u_n \neq 0$, **then the positive term series** $\sum u_n$ **must be divergent.**

For example, for the infinite series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...+\frac{1}{n}+...\infty$

$u_n = \frac{1}{n}$ and hence $\lim_{n\to\infty} u_n = 0$ still Σu_n is not convergent.

But, for the series $\frac{3}{2}+\frac{5}{4}+\frac{7}{6}+\frac{9}{8}+...$

$u_n = \frac{2n+1}{2n} = 1+\frac{1}{2n}$ hence $\lim_{n\to\infty} u_n = 1 \neq 0$

Therefore, $\sum u_n$ is not convergent.

TESTS FOR THE CONVERGENCE AND DIVERGENCE OF POSITIVE TERM SERIES

We have stated earlier that the convergence and divergence of a series is established by taking the limit of its n^{th} partial sum. But in reality it is not easy to find the sum of the first n terms of every series and in such situations we resort to various other tests as given below.

COMPARISION TEST

In this test we compare the given series with another series whose behaviour is already known to us. Let the positive term series under test be $\sum u_n$ and let $\sum v_n$ be another positive term series whose nature is already known.

(i) *If* $\sum v_n$ *is convergent and* $u_n \leq v_n$ *for all* n*, then* $\sum u_n$ *is also convergent.*

$\sum v_n$ is convergent implies that $\lim_{n\to\infty} (v_1 + v_2 + ... + v_n) = S$, say.

Also, since $u_1 \leq v_1$, $u_2 \leq v_2$, ..., $u_n \leq v_n$ we have $u_1 + u_2 + ...+ u_n \leq v_1 + v_2 + ...+ v_n$

$\Rightarrow \lim_{n\to\infty} (u_1 + u_2 + ...+ u_n) \leq \lim_{n\to\infty} (v_1 + v_2 + ...v_n) = S$, this implies that $\sum u_n$ is also convergent.

(ii) *If the series* $\sum v_n$ *is divergent and* $u_n \geq v_n$ *for all* n*, then* $\sum u_n$ *is also divergent.*

$\sum v_n$ is divergent implies that $\lim_{n\to\infty} (v_1 + v_2 + ... + v_n) = \infty$

and since $u_1 + u_2 + ... + u_n \geq v_1 + v_2 + ...v_n$ we have

$\lim_{n\to\infty} (u_1 + u_2 + ...+ u_n) \geq \lim_{n\to\infty} (v_1 + v_2 + ... + v_n)$

which establishes that $\sum u_n$ is also divergent.

It should be noted here that on account of the property that the nature of an infinite series is unaffected by addition or deletion of a finite number of terms, in (i) we can easily modify the above condition "$u_n \le v_n$ for all n" as "$u_n \le v_n$ from and after certain term" and in (ii) we can modify the condition "$u_n \ge v_n$ for all n" as "$u_n \ge v_n$ from and after certain term."

(iii) *If two positive term series Σu_n and Σv_n are such that $\lim\limits_{n\to\infty} \frac{u_n}{v_n}$ is non-zero and finite then Σu_n* behaves *as Σv_n does, that is, either both converge or both diverge.*

Let $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$, a non-zero and finite quantity, it follows by the definition of limit that, given a small $+$ve number $\in$, there exists a positive integer m such that

$$\left|\frac{u_n}{v_n} - l\right| < \in \quad \text{for all} \quad n \ge m \quad \text{or} \quad -\in \le \frac{u_n}{v_n} - l \le \in$$

$$\text{or} \qquad l - \in \le \frac{u_n}{v_n} \le l + \in \quad \text{for all} \quad n \ge m \qquad \text{...(1)}$$

Deleting first m terms of both the series, we can write (1) as $l - \in \le \frac{u_n}{v_n} \le l + \in$ for all n.

Now if Σu_n is corvegent and converges to sum S then from (1)

$$\frac{u_1}{v_1} \le l + \in; \quad \frac{u_2}{v_2} < l + \in, \ldots, \frac{u_n}{v_n} - l \le \in$$

$$\therefore \quad \lim_{n\to\infty} (u_1 + u_2 + \ldots + u_n) \le (l + \in) \lim_{n\to\infty} (v_1 + v_2 + \ldots + v_n) = (l + \in)\, S$$

This implies that Σu_n is convergent if Σv_n is convergent.

Similarly, if Σv_n is divergent then $v_1 + v_2 + \ldots + v_n$ tends to infinity as $n \to \infty$ and from (1)

$$\frac{u_1}{v_1} \ge l - \in, \quad \frac{u_2}{v_2} \ge l - \in, \ldots, \quad \frac{u_n}{v_n} \ge l - \in.$$

$$\Rightarrow \quad \lim_{n\to\infty} (u_1 + u_2 + \ldots + u_n) \ge (1 - \in) \lim_{n\to\infty} (v_1 + v_2 + \ldots + v_n)$$

$\Rightarrow \quad u_1 + u_2 + \ldots + u_n \to \infty$ as $n \to \infty$, hence Σu_n diverges as Σv_n is aleady divergent.

Thus the comparision test is fully established.

To apply the comparison test we need to have some series whose behaviour is already known. In this category, next to the geometric series we have the harmonic series, more popularly known as p-series.

THE HARMONIC SERIES (OR "P-SERIES")

The infinite series $\Sigma \frac{1}{n^p}$ is known as harmonic series or simply 'p-series.' The series converges for $p > 1$ and diverges for $p \le 1$

CASE I: $p > 1$.

We must remember that regroupping of terms does not affect the nature of the series. Thus, the given series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$ can be regroupped as

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\frac{1}{10^p}+\frac{1}{11^p}+\frac{1}{12^p}+\frac{1}{13^p}+\frac{1}{14^p}+\frac{1}{15^p}\right)+\cdots$$

Now, we form a new series as

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)+\left(\frac{1}{8^p}+\frac{1}{8^p}+\frac{1}{8^p}+\frac{1}{8^p}+\frac{1}{8^p}+\frac{1}{8^p}+\frac{1}{8^p}+\frac{1}{8^p}\right)+\cdots$$

Comparing the corresponding terms of the two series, gives

$$\frac{1}{1^p}=\frac{1}{1^p},\quad \frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p},\quad \frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p},\ldots$$

Therefore, each term of the given series is less than or equal to the corresponding term of the new series.

The new series is $1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\ldots$ or $1+\frac{1}{2^{p-1}}+\frac{1}{2^{2p-2}}+\frac{1}{2^{3p-3}}+\ldots$

which is a geometric series with common ratio $\frac{1}{2^{p-1}}$ which is less than 1 as $p>1$, therefore, the new series is convergent and in turn the given series is also convergent by Part (*i*) of the Comparison test.

CASE II: $p = 1$. The given series for $p = 1$ is

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$ It can be regroupped as

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots$$

Now, consider a new series

$$1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\ldots$$

Obviously, $\frac{1}{2}=\frac{1}{2}$, $\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$, ...

The new series is $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$ which is divergent and each term of the given series is greater than the corresponding term of the new series. Therefore, the given series by Part (*ii*) of the comparison test is also divergent.

CASE III: $p < 1$.

One can easily note that for $p < 1$

$$\frac{1}{1^p}=1,\quad \frac{1}{2^p}>\frac{1}{2},\quad \frac{1}{3^p}>\frac{1}{3},\quad \frac{1}{4^p}>\frac{1}{4},\ \text{and so on.}$$

Since $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ is divergent as established in case II hence by the Part (*ii*) of the comparison test, the given series is also divergent for $p < 1$.

EXAMPLE 2.2 (a) Test the convergence of the series $\frac{1}{1+x}+\frac{1}{2+x}+\frac{1}{3+x}+...$

(b) Discuss the convergence of the series $\frac{3}{1^p}+\frac{5}{2^p}+\frac{7}{3^p}+\frac{9}{4^p}+...$

SOLUTION: (a) $u_n = \frac{1}{n+x}$. Here, we take $v_n = \frac{1}{n}$.

Hence $\frac{u_n}{v_n} = \frac{n}{n+x} = \frac{1}{1+\frac{x}{n}}$ and $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{1+0} = 1$

which is non-zero and finite, hence Σu_n behaves as Σv_n does; but $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent as it is a 'p-series' with $p = 1$, Therefore, Σu_n is also divergent.

(b) Here $u_n = \frac{2n+1}{n^p}$. Consider the auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^{p-1}}$.

Then $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{2n+1}{n^p}\cdot\frac{n^{p-1}}{1} = \lim\limits_{n\to\infty} \frac{2n+1}{n} = 2$ which is non-zero and finite,

hence Σu_n behaves as Σv_n does. But $\Sigma \frac{1}{n^{p-1}}$ is the harmonic series which is convergent when $p-1>1$, *i.e.* $p>2$ and divergent when $p-1 \le 1$, *i.e.,* $p \le 2$. Therefore, Σu_n is convergent when $p>2$ and divergent when $p \le 2$.

EXAMPLE 2.3. Show that the series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+....$ is convergent.

SOLUTION: Here $u_n = \frac{1}{(n-1)!} = \frac{1}{1.2.3.4......(n-1)} < \frac{1}{2.2.2.......(n-2)\text{ times}} = \frac{1}{2^{n-2}} = v_n$

But $\sum\limits_1^\infty \frac{1}{2^{n-2}}$ is an infinite geometric series with common ratio $\frac{1}{2}$ which is less than 1 hence $\sum\limits_1^\infty \frac{1}{2^{n-2}}$ is convergent and $u_n \le v_n$ for all n, therefore the given series Σu_n is also convergent by the comparison test.

EXAMPLE 2.4. Test the convergence of the series (i) $\sum\limits_2^\infty \frac{1}{\log n}$. (ii) $\sum\limits_{n=1}^\infty \frac{1}{\log(n+7)}$

[GGSIPU I Sem End Term 2004; End Term 2011; I Term 2013]

SOLUTION: (i) Here $u_n = \frac{1}{\log n}$. Now since $\log n < n$ we have $\frac{1}{\log n} > \frac{1}{n}$ and we choose $v_n = \frac{1}{n}$. As such $u_n > v_n$ and $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent hence by comparison test, Σu_n is also divergent. **Ans.**

(ii) $u_n = \frac{1}{\log(n+7)}$, since $\frac{1}{\log(n+7)} > \frac{1}{n+7}$ and $\Sigma \frac{1}{n+7}$ is divergent hence by the comparison test Σu_n is divergent. **Ans.**

EXAMPLE 2.5. (a) Test the convergence of the series $\left(\sqrt{1^2+1}-1\right)+\left(\sqrt{2^2+1}-2\right)+\left(\sqrt{3^2+1}-3\right)+\ldots$

AS K (b) Discuss the convergence of the series $\sum_{1}^{\infty}\left(\sqrt[3]{n^3+1}-n\right)$.

SOLUTION: (a) Here $u_n = \sqrt{n^2+1}-n = \dfrac{\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n} = \dfrac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \dfrac{1}{\sqrt{n^2+1}+n}$

If we choose $v_n = \dfrac{1}{n}$ then, $\lim_{n\to\infty}\dfrac{u_n}{v_n} = \lim_{n\to\infty}\dfrac{n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty}\dfrac{1}{\sqrt{1+\dfrac{1}{n^2}}+\dfrac{1}{n}} = 1$

which is non-zero and finite, By comparison test $\sum u_n$ behaves as $\sum v_n$ does and $\sum\dfrac{1}{n}$ is known to be divergent therefore $\sum u_n$ also diverges.

Ans (a)

$$u_n = \sqrt[3]{n^3+1}-n = (n^3+1)^{1/3}-n = n\left(1+\frac{1}{n^3}\right)^{1/3}-n$$

$$= n\left[1+\frac{1}{3\cdot n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{1}{n^6}+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\frac{1}{n^9}+\ldots\right]-n$$

$$= \frac{1}{3n^2}-\frac{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}{2!}\frac{1}{n^5}+\frac{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)}{3!}\frac{1}{n^8}-\ldots$$

Now we choose $v_n = \dfrac{1}{n^2}$, then

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty} n^2\left[\frac{1}{3.n^2}-\frac{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}{2!}\frac{1}{n^5}+\frac{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)}{3!}\frac{1}{n^8}-\ldots\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{3}-\frac{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}{2!}\frac{1}{n^3}+\ldots\right] = \frac{1}{3}$$

which is non-zero and finite, therefore by comparison test $\sum u_n$ behaves as $\sum v_n$ does. But $\sum v_n = \sum\dfrac{1}{n^2}$ is convergent being p-series with $p = 2\ (>1)$, hence $\sum u_n$ is convergent. **Ans.**

D' ALEMBERT'S RATIO TEST

If Σu_n is a positive term series such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = K$ *then* Σu_n *converges if* $K<1$ *and diverges if* $K>1$.

CASE I: *When* $K < 1$

By definition of limit there must exist a positive number λ ($K < \lambda < 1$) such that

$$\frac{u_2}{u_1} < \lambda,\ \frac{u_3}{u_2} < \lambda,\ \ldots,\ \frac{u_{n+1}}{u_n} < \lambda, \ldots$$

Then $$u_1 + u_2 + u_3 + u_4 + \ldots\infty = u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\frac{u_4}{u_1}+\ldots\right]$$

$$= u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots\right]$$

$$< u_1\left[1+\lambda+\lambda^2+\lambda^3+\ldots\infty\right] = \frac{u_1}{1-\lambda}$$

which is finite, hence Σu_n is convergent.

CASE II : *When* $K > 1$

By definition of limit there must exist a positive number m such that $\frac{u_{n+1}}{u_n} > 1$ for all $n \geq m$.

If we delete the first $m-1$ terms from the series, then we shall have

$$\frac{u_2}{u_1} > 1,\ \frac{u_3}{u_2} > 1,\ \frac{u_4}{u_3} > 1, \ldots$$

$$\therefore \quad u_1 + u_2 + u_3 + \ldots + u_n = u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots n \text{ terms}\right]$$

$$\geq u_1\ [1 + 1 + 1 + 1 \ldots \text{upto } n \text{ terms}] = nu_1$$

Hence, $\lim_{n\to\infty} (u_1 + u_2 + u_3 + \ldots + u_n) = \infty$

Thus the given series Σu_n in this case diverges.

It is to be observed that the test is silent over the case when $K = 1$, still it should not be interpreted as 'test fails when $K = 1$'.

This ratio test is many times stated as follows :

In a positive term series Σu_n *if* $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lambda$, *then the series is convergent when* $\lambda > 1$ *and divergent, when* $\lambda < 1$.

EXAMPLE 2.6. (a) Test the convergence of the series $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots$

(b) Test the convergence of the series $\frac{1}{1.2.3.}+\frac{1}{2.3.4.}+\frac{1}{3.4.5.}+\ldots.$

SOLUTION: (a) Here $u_n=\frac{n^2}{n!}=\frac{n}{(n-1)!}$

$\therefore \quad \frac{u_{n+1}}{u_n}=\frac{(n+1)}{n!}\cdot\frac{(n-1)!}{n}=\frac{n+1}{n^2}=\frac{1}{n}+\frac{1}{n^2}$ and $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=0$ which is less than 1.

Therefore, by *D 'Alembert's* ratio test the given series is convergent. **Ans.**

(b) Here $u_n=\frac{1}{n(n+1)(n+2)}$ and $u_{n+1}=\frac{1}{(n+1)(n+2)(n+3)}$

hence, $\frac{u_{n+1}}{u_n}=\frac{n(n+1)(n+2)}{(n+1)(n+2)(n+3)}=\frac{n}{n+3}=\frac{1}{1+3/n}$ and $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1}{1+3/n}=1$

Therefore, the ratio test in this case does not yield anything. Let us apply comparison test by taking

$v_n=\frac{1}{n^3}$ then, $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{n^3}{n(n+1)(n+2)}=\lim_{n\to\infty}\frac{1}{(1+1/n)(1+2/n)}=1$

which is non-zero and finite hence Σu_n behaves as Σv_n does. But $\Sigma v_n=\sum\frac{1}{n^3}$ is convergent being '*p*-series' with $p=3\ (>1)$ therefore, Σu_n is also convergent. **Ans.**

EXAMPLE 2.7. (a) Discuss the convergence of the series $\frac{x}{1+x}+\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}+\ldots\ (x>0)$

v.and

(b) Test the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{x^n+x^{-n}};\ x>0$

[*GGSIPU I Sem End Term January 2011*]

SOLUTION: (a) Here, $u_n=\frac{x^n}{1+x^n}$ and $\frac{u_{n+1}}{u_n}=\frac{x^{n+1}}{1+x^{n+1}}\cdot\frac{1+x^n}{x^n}$

$\therefore \quad \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=x\lim_{n\to\infty}\frac{1+x^n}{1+x^{n+1}}$ Now, three possibilities arise.

First, if $x<1$, $\lim_{n\to\infty}x^n=0=\lim_{n\to\infty}x^{n+1}$ hence $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=x$

Thus, Σu_n is convergent when $x<1$.

Next, if $x>1$, $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{x^n}{1+x^n}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{x^n}}=1$ and,

since $\lim_{n\to\infty}u_n\neq 0$, Σu_n is divergent.

Further, when $x=1$ the series beocmes $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$ which is obviously divergent.

Thus, Σu_n is convergent for $x < 1$ and divergent for $x \geq 1$.

(b) The given series is Σu_n where $u_n = \dfrac{1}{x^n + x^{-n}}$, $x > 0$

Here $\lim\limits_{n\to\infty} u_n = 0$ when $0 < x < 1$ and also when $x > 1$.

Next, $$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{x^{2n+2}+1} \cdot \frac{x^{2n}+1}{x^n} = \frac{x(x^{2n}+1)}{x^2 \cdot x^{2n}+1}$$

When $0 < x < 1$, we have $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = x \mathop{\text{Lt}}\limits_{n\to\infty} \dfrac{x^{2n}+1}{x^2 \cdot x^{2n}+1} = \dfrac{x(0+1)}{x^2(0)+1} = x < 1.$

Hence Σu_n is convergent when $0 < x < 1$.

When $x > 1$, we have $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = x \mathop{\text{Lt}}\limits_{n\to\infty} \dfrac{1 + \dfrac{1}{x^{2n}}}{x^2 + \dfrac{1}{x^{2n}}} = x\left(\dfrac{1+0}{x^2+0}\right) = \dfrac{1}{x} < 1$

hence Σu_n is convergent for $x > 1$.

However when $x = 1$ we have $\Sigma u_n = \sum \dfrac{1}{2}$ which divergent. **Ans.**

[redacted] Discuss the convergence of the series

(*i*) $\sum\limits_{1}^{\infty} \sin\left(\dfrac{1}{n}\right)$ (*ii*) $\sum\limits_{1}^{\infty} \dfrac{1}{\sqrt{n}} \tan\left(\dfrac{1}{n}\right)$ (*iii*) $\sum\limits_{n=1}^{\infty} \log \dfrac{n}{n+1}$.

[GGSIPU I Sem I Term 2011][End Term 2012]

SOLUTION: (*i*) Here, $u_n = \sin\dfrac{1}{n}$ and we take $v_n = \dfrac{1}{n}$

Then $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{\sin 1/n}{1/n} = \lim\limits_{x\to 0} \dfrac{\sin x}{x}$ (taking $x = 1/n$) $= 1$ which is non-zero and finite.

Therefore, by comparison test Σu_n behaves as Σv_n does.

But $\Sigma v_n = \sum \dfrac{1}{n}$ is divergent, so is Σu_n. **Ans.**

(*ii*) Here $u_n = \dfrac{1}{\sqrt{n}} \tan\dfrac{1}{n}$ and we take $v_n = \dfrac{1}{n^{3/2}}$

then $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{\tan\frac{1}{n} \cdot n^{3/2}}{\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{\tan\frac{1}{n}}{\frac{1}{n}} = 1$ (since $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$).

which is non-zero and finite, hence by comparison test Σu_n behaves as Σv_n does.

But $\Sigma v_n = \sum \dfrac{1}{x^{3/2}}$ is convergent being 'p-series' with $p = 3/2$ (> 1),

therefore Σu_n is also convergent. **Ans.**

(iii) $\sum u_n = \sum_{1}^{\infty} \log \frac{n}{n+1}$ Here $\lim_{n\to\infty} u_n = 0$

Next, n^{th} partial sum $= s_n = \sum_{r=1}^{n} [\log r - \log(r+1)]$

$= (\log 1 - \log 2) + (\log 2 - \log 3) + \dots + (\log n - \log (n+1))$

$= -\log (n+1)$

Sum of the series $= \lim_{n\to\infty} s_n = -\infty$, hence Σu_n is divergent. **Ans.**

EXAMPLE 2.9. (a) Test the convergence of the series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\dots+\frac{n}{n+1}+\dots$

[GGSIPU I Sem End Term 2007]

(b) Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{2n-1}}{(2n+1)(2n+4)}$.

[GGSIPU I Sem End Term 2009]

SOLUTION: (a) The given series is $\sum U_n = \sum_{1}^{\infty} \frac{n}{n+1}$.

Here $U_n = \frac{n}{n+1}$ and $\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{n}{n+1} = 1$ which is not zero. The necessary condition for the convergence of the series $\sum U_n$ is $\underset{n\to\infty}{Lt} U_n = 0$.

Therefore the given series is divergent.

(b) The given series is $\sum U_n$ where $U_n = \frac{\sqrt{2n-1}}{(2n+2)(2n+4)}$.

If is of the order of $\frac{1}{n^{3/2}}$ hence we take $V_n = \frac{1}{n^{3/2}}$

$$\therefore \quad \frac{U_n}{V_n} = \frac{\sqrt{2n-1}\cdot n^{3/2}}{(2n+2)(2n+4)} = \frac{\sqrt{2-\frac{1}{n}}}{\left(2+\frac{2}{n}\right)\left(2+\frac{4}{n}\right)}$$

Hence $\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \frac{1}{2\sqrt{2}}$ which is non-zero and finite.

Since $\sum V_n$ is convergent being a p-series with $p = \frac{3}{2} > 1$, therefore by comparison test $\sum U_n$ is also convergent. **Ans.**

EXAMPLE 2.10. Test the convergence of the series $\sum n^{\log x}$. [GGSIPU I Sem End Term 2007]

SOLUTION: The given series is $\sum U_n$ where $U_n = n^{\log x}$

or $$U_n = \frac{1}{n^{-\log x}} = \frac{1}{n^{\log 1/x}}$$ which gives a p-series with $p = \log\left(\frac{1}{x}\right)$.

Therefore $\sum U_n$ is convergent when $\log\frac{1}{x} > 1$ or $\frac{1}{x} > e$ or $x < \frac{1}{e}$

and divergent when $x \geq \frac{1}{e}$. **Ans.**

EXAMPLE 2.11. (a) Test the convergence of the series $\sum_1^{\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}}$

[GGSIPU I Sem I Term 2004; End Term 2008]

(b) Test the convergence of the series

$$\frac{\sqrt{2}-1}{2^3-1} + \frac{\sqrt{3}-1}{3^3-1} + \frac{\sqrt{4}-1}{4^3-1} + \ldots$$ [GGSIPU I Sem I Term 2009]

SOLUTION: (a) The series is $\sum U_n$ where $U_n = \frac{1}{\sqrt{n}+\sqrt{n-1}}$

We apply the comparison test here and take $V_n = \frac{1}{\sqrt{n}}$

Hence $\underset{n\to\infty}{\text{Lt}} \frac{U_n}{V_n} = \underset{n\to\infty}{\text{Lt}} \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n-1}} = \frac{1}{2}$ which is non-zero and finite. Hence $\sum U_n$ behaves as $\sum V_n$ does, by comparison test. Here $\sum V_n = \sum \frac{1}{\sqrt{n}}$ is divergent as p-series, $p = \frac{1}{2} < 1$

Therefore $\sum U_n$ is divergent. **Ans.**

(b) Here $U_n = \frac{\sqrt{n}-1}{n^3-1}$ then $\underset{n\to\infty}{\text{Lt}} U_n = 0$. U_n is of the order of $\frac{1}{n^{5/2}}$ and take $V_n = \frac{1}{n^{5/2}}$.

Hence $\underset{n\to\infty}{\text{Lt}} \frac{U_n}{V_n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{\sqrt{n}-1}{n^3-1}\right) n^{5/2} = 1$ which is non-zero and finite. Therefore by comparison test $\sum U_n$ behaves as $\sum V_n$ does and since $\sum V_n$ is convergent being p-series with $p = \frac{5}{2} > 1$.

Hence $\sum U_n$ is convergent. **Ans.**

EXAMPLE 2.12. (a) Test the convergence of the series

$$\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\ldots$$

[GGSIPU I Sem End Term 2007]

(b) Test for convergence the series $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\ldots,$

[GGSIPU I sem III Term Sp. 2007]

SOLUTION: (a) For the given series $\sum U_n$ we have

$$U_n=\left[\frac{1.2.3.\ldots n}{3.5.7.\ldots(2n+1)}\right]^2 \text{ and } U_{n+1}=\left[\frac{1.2.3.\ldots n(n+1)}{3.5.7.\ldots(2n+1)(2n+3)}\right]^2$$

hence $$\underset{n\to\infty}{Lt}\frac{U_{n+1}}{U_n}=\underset{n\to\infty}{Lt}\frac{(n+1)^2}{(2n+3)^2}=\underset{n\to\infty}{Lt}\frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{3}{n}\right)^2}=\frac{1}{4}.$$

and $\frac{1}{4}<1$, therefore by ratio test the given series is convergent. **Ans.**

(b) Given series is $\sum_1^\infty U_n$ where $U_n=\frac{n}{1+2^n}$

$$\underset{n\to\infty}{Lt}\frac{U_{n+1}}{U_n}=\underset{n\to\infty}{Lt}\frac{n+1}{1+2^{n+1}}\cdot\frac{1+2^n}{n}=\underset{n\to\infty}{Lt}\frac{n+1}{n}\underset{n\to\infty}{Lt}\cdot\frac{1+2^n}{1+2^{n+1}}=1\underset{n\to\infty}{Lt}\frac{1+\frac{1}{2^n}}{2+\frac{1}{2^n}}$$

$$=\frac{1}{2} \text{ which is less than 1}$$

Therefore by ratio test $\sum U_n$ is convergent. **Ans.**

EXAMPLE 2.13. Test for convergence the series $\frac{2}{1}+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\ldots$

[GGSIPU I Sem III Term Spl. 2007]

SOLUTION: The given series can be written as $\frac{2.5}{1.5}+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\ldots=\sum U_n$

Then $$U_n=\frac{2.5.8.11\ldots(3n+2)}{1.5.9.13\ldots(4n+1)} \text{ and } U_{n+1}=\frac{2.5.8.11\ldots(3n+2)(3n+5)}{1.5.9.13\ldots(4n+1)(4n+5)}$$

$$\therefore \underset{n\to\infty}{Lt}\frac{U_n+1}{U_n}=\underset{n\to\infty}{Lt}\frac{3n+5}{4n+5}=\frac{3}{4}$$

which is less than 1, therefore, by ratio test, the given series is convergent. **Ans.**

EXAMPLE 2.14. Test the convergence of the series $\sum_{1}^{\infty}(\sqrt{n^4+1}-\sqrt{n^4-1})$

[GGSIPU I Sem End Term 2007]

SOLUTION: Given series is $\sum U_n = \sum_{1}^{\infty}(\sqrt{n^4+1}-\sqrt{n^4-1})$.

Here $U_n = \sqrt{n^4+1}-\sqrt{n^4-1} = \dfrac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4+1}} = \dfrac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$

If we take $V_n = \dfrac{1}{n^2}$, then

$$\underset{n\to\infty}{\text{Lt}}\frac{U_n}{V_n} = \underset{n\to\infty}{\text{Lt}}\frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \underset{n\to\infty}{\text{Lt}}\frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}} = 1$$

which is finite and non-zero hence by comparison test $\sum U_n$ behaves as $\sum V_n$ does. Since $\sum\frac{1}{n^2}$ is convergent so is $\sum U_n$. **Ans.**

EXAMPLE 2.15. (a) Discuss the convergence of the series

$$1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!x^3}{4^3}+\frac{4!}{5^4}x^4+\ldots$$

[GGSIPU I Sem End Term 2007]

(b) Test the convergence of $\dfrac{1}{2\sqrt{1}}+\dfrac{x^2}{3\sqrt{2}}+\dfrac{x^4}{4\sqrt{3}}+\dfrac{x^6}{5\sqrt{4}}+\ldots\infty$

[GGSIPU I Sem I Term 2012]

SOLUTION: (a) The given series is $\sum U_n$ where

$$U_n = \frac{n!}{(n+1)^n}x^n \text{ after ignoring the first term.}$$

hence $$\frac{U_{n+1}}{U_n} = \frac{(n+1)\,x\,(n+1)^n}{(n+2)^{n+1}} = \left(\frac{n+1}{n+2}\right)^{n+1}x.$$

Next $$\underset{n\to\infty}{Lt}\frac{U_{n+1}}{U_n} = \underset{n\to\infty}{Lt}\frac{x}{\left(\frac{n+2}{n+1}\right)^{n+1}} = x\underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}} = \frac{x}{e}$$

By Ratio Test $\sum U_n$ is convergent when $x<e$ and divergent when $x>e$. **Ans.**

(b) The series $\sum_{1}^{\infty}U_n$ where $U_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$

Here $U_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{n+1}}$. Applying D'Alembert's ratio test we get

$$\underset{n\to\infty}{\text{Lim}}\frac{U_{n+1}}{U_n} = \underset{n\to\infty}{\text{Lt}}\frac{x^2(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}} = x^2$$

$\therefore$ $\sum U_n$ is convergent when $x^2 < 1$, i.e., $|x| < 1$.

and is divergent when $x^2 > 1$, i.e., $|x| > 1$.

When $|x| = 1$ we have $U_n = \dfrac{1}{(n+1)\sqrt{n}}$

Let us take $V_n = \dfrac{1}{n^{3/2}}$ then $\underset{n\to\infty}{\text{Lt}} \dfrac{U_n}{V_n} = \underset{n\to\infty}{\text{Lt}} \dfrac{n^{3/2}}{(n+1)\sqrt{n}} = 1$

which is non=zero and finite. Therefore, by comparison test $\sum U_n$ behaves as $\sum V_n$ does.

But $\sum V_n = \dfrac{1}{n^{3/2}}$ is convergent as $p = 3/2$ which is greater than 1.

Thus, $\sum U_n$ is convergent when $|x| \le 1$ and is divergent when $|x| > 1$. **Ans.**

EXAMPLE 2.16. Test the convergence of the series

$$1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)} + \ldots, \text{ where } \alpha, \beta > 0.$$

[GGSIPU I Sem End term 2004]

SOLUTION: The series is $\sum_1^\infty U_n$ where $U_n = \dfrac{(\alpha+1)(2\alpha+1)\ldots,(n\alpha+1)}{(\beta+1)(2\beta+1)\ldots,(n\beta+1)}$ on ignoring the first term in the given series. Applying ratio test here

$\underset{n\to\infty}{Lt} \dfrac{U_{n+1}}{U_n} = \underset{n\to\infty}{Lt} \dfrac{(n+1)\alpha+1}{(n+1)\beta+1} = \dfrac{\alpha}{\beta}$. Therefore the given series is convergent when $\dfrac{\alpha}{\beta} < 1$, *i.e.*, $\alpha < \beta$ and divergent when $\dfrac{\alpha}{\beta} > 1$, *i.e.*, $\alpha > \beta$.

When $\alpha = \beta$ the given series becomes $1 + 1 + 1 + 1 + \ldots \infty$ which is divergent.

Thus U_n is convergent for $\alpha < \beta$ and divergent for $\alpha \ge \beta$. **Ans.**

EXERCISE 2A

Test for convergence the series

1. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots$

2. $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{4}}+\ldots$

3. $\frac{1.2}{3.4.5}+\frac{2.3}{4.5.6.}+\frac{3.4.}{5.6.7}+\ldots$

4. $\frac{1}{2}+\frac{3!}{2.4}+\frac{5!}{2.4.6}+\ldots$

5. $\frac{1}{1^2+x^2}+\frac{1}{2^2+x^2}+\frac{1}{3^2+x^2}+\ldots$

6. $\frac{1}{1+x}+\frac{1}{1+2x^2}+\frac{1}{1+3x^3}+\ldots\ (x>1)$

7. $\frac{x}{1^2+1}+\frac{2^3x^2}{2^2+1}+\frac{3^3x^3}{3^2+1}+\frac{4^3x^4}{4^2+1}+\ldots$ (for $x>0$)

8. $\frac{1}{2}+\frac{\sqrt[3]{2}}{3\sqrt{2}}+\frac{\sqrt[3]{3}}{4\sqrt{3}}+\ldots$

9. $1+3x+5x^2+7x^3+\ldots$

10. $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots$

11. $1+\frac{1^2.2^2}{1.3.5}+\frac{1^2.2^2.3^2}{1.3.5.7.9}+\ldots$

12. $\frac{x}{x+1}+\frac{x^2}{x+2}+\frac{x^3}{x+3}+\ldots$

13. $\sum_{n=1}^{\infty}\frac{\log n}{2n^3-1}$.

14. $\sum_{n=1}^{\infty}\frac{1}{n^2\cdot 3^n}$

15. $\frac{1}{3}+\frac{3}{5}+\frac{7}{9}+\frac{15}{17}+\ldots$

16. $\sum^{\infty}\frac{n^2+2^n}{2^n\cdot n^2}$

17. Discuss the convergence of the series $\sum\frac{x^n}{n},\ x>0$

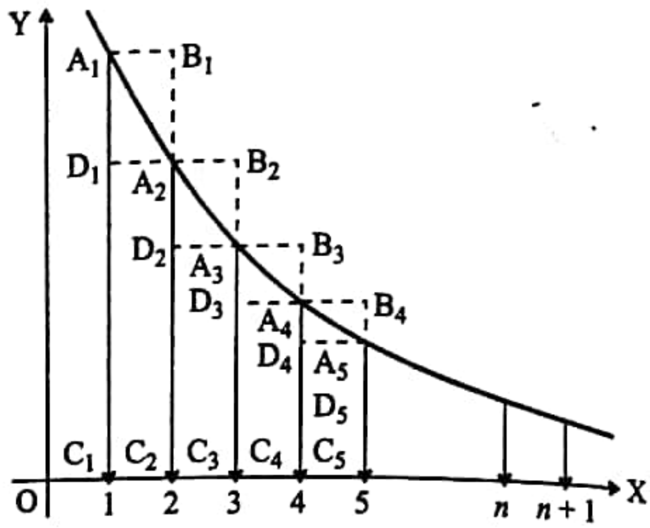
18. Discuss the convergence of the series $\sum\frac{n^2(n+1)^2}{n!}$

19. Test the convergence of the series $\frac{1}{\sqrt{1\cdot 2}}+\frac{1}{\sqrt{2\cdot 3}}+\frac{1}{\sqrt{3\cdot 4}}+\ldots$

INTEGRAL TEST FOR POSITIVE TERM SERIES

[GGSIPU I SEM I TERM 2013]

If $f(x)$ is positive, continuous and monotonic and is such that $f(x) = u_n$ then Σu_n converges or diverges according as the integral $\int_1^{\infty} f(x)dx$ converges or diverges. In other words, Σu_n converges or diverges according as the value of the integral $\int_1^{\infty} f(x)dx$ is finite and unique or infinite.



PROOF:

The area under the curve $y = f(x)$ and between the ordinates at A_1 and A_2 lies between the areas of the rectangles $A_1B_1C_2C_1$ and $D_1A_2C_2C_1$. Similarly, the area under the curve and between the ordinates A_2 and A_3 lies between the areas of the rectangles $A_2B_2C_3C_2$ and $D_2A_3C_3C_2$, and so on, as shown in the adjoining figure.

Thus, $(2-1)f(1) + (3-2)f(2) + (4-3)f(3) + ... + \{n-(n-1)\}f(n)$

$$\geq \int_1^{n+1} f(x)\,dx \geq (3-2)f(2) + (4-3)f(3) + ... + (n+1-n)f(n+1)$$

or $$f(1) + f(2) + ... + f(n) \geq \int_1^{n+1} f(x)dx \geq f(2) + f(3) + ... + f(n+1)$$

or $$u_1 + u_2 + ... + u_n \geq \int_1^{n+1} f(x)dx \geq u_2 + u_3 + ... + u_{n+1}$$

or $$S_n \geq \int_1^{n+1} f(x)dx \geq S_{n+1} - u_1 ... \qquad ...(1)$$

where S_n is the nth partial sum.

Now, taking limit as $n \to \infty$, the second inequality of (1) yields

$$\lim_{n\to\infty} S_{n+1} \leq \int_1^{\infty} f(x)dx + u_1$$

which implies that if $\int_1^{\infty} f(x)dx$ is finite, so is $\lim_{n\to\infty} S_{n+1}$.

Also, the first inequality of (1) yields

$$\lim_{n\to\infty} S_n \geq \int_1^{\infty} f(x)dx$$

which implies that if $\int_1^{\infty} f(x)dx$ is infinite, so is $\lim_{n\to\infty} S_n$. Therefore, we conclude that the given series converges or diverges accroding as the integral $\int_1^{\infty} f(x)dx$ is finite or infinite.

EXAMPLE 2.17. Test the convergence of the series

(a) $\sum_{1}^{\infty} \frac{n}{n^2+1}$. (b) $\sum_{1}^{\infty} \frac{(\log n)^2}{n}$. [GGSIPU I Sem I Term 2011]

SOLUTION: (a) Let as apply the integral test to this series.

Consider $\int_1^M \frac{x\,dx}{x^2+1} = \left[\frac{1}{2}\log(x^2+1)\right]_1^M = \frac{1}{2}\log\frac{M^2+1}{2}$

Since $\int_1^\infty \frac{x\,dx}{x^2+1} = \lim_{M\to+\infty} \frac{1}{2}\log\frac{M^2+1}{2}$ is infinite. Hence by integral test the given series is divergent.

Ans.

(b) . Let us apply here the integral test.

$$u_n = \frac{1}{n}(\log n)^2. \quad \text{Consider the integral } I = \int_1^\infty \frac{1}{x}(\log x)^2\,dx$$

or $$I = \frac{1}{3}\left[(\log x)^3\right]_1^\infty \quad \text{which is infinite hence } \sum u_n \text{ is divergent.}$$ **Ans.**

EXAMPLE 2.18. Discuss the convergence of the series $\sum_{2}^{\infty} \frac{1}{n\log n}$.

SOLUTION: The given series converges or diverges according as $\int_2^\infty \frac{dx}{x\log x}$ is finite or infinite.

$$\int_2^\infty \frac{dx}{x\log x} = \lim_{m\to+\infty}\int_2^m \frac{dx}{x\log x} = \lim_{m\to+\infty} \log\log(x)\Big|_2^m = \lim_{m\to+\infty}(\log\log m - \log\log 2) = \infty$$

Therefore, the given series in divergent. **Ans.**

RAABE'S TEST (OR HIGHER RATIO TEST)

If Σu_n is a positive term series such that $\lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lambda$ then Σu_n converges when $\lambda > 1$ and diverges when $\lambda < 1$.

CASE I: $\lambda > 1$.

Let us choose a number p such that $\lambda > p > 1$. Now, compare the given series with the series $\Sigma v_n = \Sigma \frac{1}{n^p}$ which is already known to be convergent as $p > 1$.

Then Σu_n converges if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p$

or if $\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots$

or $n\left(\frac{u_n}{u_n+1} - 1\right) > p + \frac{p(p-1)}{2!}\frac{1}{n} + \ldots$

or $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) > \lim_{n\to\infty}\left(p+\frac{p(p-1)}{2!}\frac{1}{n}+\ldots\right) = p$

or if $\lambda > p$ which is already true, hence Σu_n is convergent.

CASE II: $\lambda < 1$.

Let as choose a number p such that $\lambda < p < 1$ and compare the given series with the known series $\Sigma v_n = \Sigma\frac{1}{n^p}$ which is divergent for $p < 1$.

Now, Σu_n diverges if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

that is, if $\frac{u_n}{u_{n+1}} < \left(\frac{n+1}{n}\right)^p = \left(1+\frac{1}{n}\right)^p = 1+\frac{p}{n}+\frac{p(p-1)}{2!}\frac{1}{n^2}+\ldots$

or $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) < \lim_{n\to\infty}\left(p+\frac{p(p-1)}{2!}\frac{1}{n}+\ldots\right) = p$

or $\lambda < p$ which is true, hence Σu_n is divergent.

EXAMPLE 2.19. Test for the convergence of the series $\left(\frac{1}{3}\right)^2+\left(\frac{1.4}{3.6}\right)^2+\left(\frac{1.4.7}{3.6.9}\right)^2+\ldots\ldots$

SOLUTION: Here $u_n = \left(\frac{1.4.7\ldots(3n-2)}{3.6.9\ldots 3n}\right)^2$

Hence $\frac{u_{n+1}}{u_n} = \left(\frac{3n+1}{3n+3}\right)^2$ and $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{3+\frac{1}{n}}{3+\frac{3}{n}}\right)^2 = 1$

and the ratio test does not predict anything.

Applying now the Raabe's test, we get

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left\{\left(\frac{3n+3}{3n+1}\right)^2-1\right\} = \lim_{n\to\infty}\frac{n(6n+4)(2)}{(3n+1)^2} = \lim_{n\to\infty}\frac{2\left(6+\frac{4}{n}\right)}{\left(3+\frac{1}{n}\right)^2} = \frac{4}{3}$$

which is greater than 1 hence by Raabe's test the given series is convergent. **Ans.**

EXAMPLE 2.20. Test the convergence of the series

$$\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\ldots\infty,\ (x>0).$$

[GGSIPU I Sem End Term 2004 Reappear; End Term 2011; End Term 2012]

SOLUTION: The given series is $\sum U_n$ where

$$U_n = \frac{3.6.9\ldots(3n)}{7.10.13\ldots(3n+4)}x^n.$$ Let us apply the ratio test here.

$$\underset{n\to\infty}{Lt}\frac{U_{n+1}}{U_n} = \underset{n\to\infty}{Lt}\frac{(3n+3)x}{(3n+7)} = x.$$

Therefore $\sum U_n$ converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$ let us apply the higher ratio test. Thus,

$$\underset{n\to\infty}{Lt}\ n\left(\frac{U_n}{U_{n+1}}-1\right) = \underset{n\to\infty}{Lt}\ n\left(\frac{3n+7}{3n+3}-1\right) = \underset{n\to\infty}{Lt}\ \frac{4n}{3n+3} = \frac{4}{3}$$

which is greater than 1 hence $\sum U_n$ converges for $x = 1$.

Thus the given series is convergent for $x \le 1$ and diverges otherwise. **Ans.**

EXAMPLE 2.21. Test the convergence of the series

$$x^2 + \frac{2^2}{3.4}x^4 + \frac{2^2.4^2}{3.4.5.6}x^6 + \frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8 + ...$$

[GGSIPU I Sem End Term 2003]

SOLUTION: Let us delete the first term because it will not affect the nature of the series, then the series is $\sum_{n=1}^{\infty} U_n$ where $U_n = \frac{2^2.4^2.6^2 ...(2n)^2}{3.4.5.6.7.8...(2n+1)(2n+2)}x^{2n+2}$

Therefore $$\frac{U_{n+1}}{U_n} = \frac{(2n+2)^2 x^2}{(2n+3)(2n+4)} \text{ and } \underset{n\to\infty}{Lt}\ \frac{U_{n+1}}{U_n} = x^2$$

Thus, by ratio test the series $\sum U_n$ will be convergent for $|x| < 1$ and divergent for $|x| > 1$. However, for the case when $|x| = 1$ we apply the higher ratio test.

$$\underset{n\to\infty}{Lt}\ n\left(\frac{U_n}{U_{n+1}}-1\right) = \underset{n\to\infty}{Lt}\ n\left(\frac{(2n+3)(2n+4)}{(2n+2)^2}-1\right) = \underset{n\to\infty}{Lt}\ \frac{n(6n-8)}{(2n+2)^2} = \frac{3}{2}$$

which is greater than 1 hence convergent,. Therefore, the given series is convergent for $|x| \le 1$ and divergent for $|x| > 1$. **Ans.**

LOGARITHMIC TEST

If Σu_n is a positive term series such that $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lambda$ *then Σu_n converges or diverges according as $\lambda > 1$ or $\lambda < 1$.*

CASE I: *When $\lambda > 1$.*

Let p be a positive number such that $\lambda > p > 1$ then we compare the given series with another series $\Sigma v_n = \Sigma \frac{1}{n^p}$ $\therefore$ $\frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p}$

Now, Σu_n is convergent if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

or $$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p$$

or $$\log\frac{u_n}{u_{n+1}} > p\log\left(1+\frac{1}{n}\right) = p\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-...\right)$$

or $\lim_{n\to\infty} n\log\dfrac{u_n}{u_{n+1}} > p$ or $\lambda > p$ which is already true,

therefore the given series $\Sigma\, u_n$ is convergent.

CASE II: *When* $\lambda < 1$.

Here again we can establish that Σu_n is divergent, on similar lines as in case I.

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EXAMPLE 2.22. Test the convergence of the series $1+\dfrac{2x}{2!}+\dfrac{3^2x^2}{3!}+\dfrac{4^3x^3}{4!}+\ldots\ldots$

SOLUTION: Here $u_n = \dfrac{n^{n-1}x^{n-1}}{n!}$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{(n+1)^n x^n}{(n+1)!}\,\frac{n!}{n^{n-1}.x^{n-1}} = \frac{x(n+1)^n}{(n+1).n^{n-1}} = \frac{nx}{n+1}\left(1+\frac{1}{n}\right)^n$$

$$\text{hence} \quad \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x\lim_{n\to\infty}\frac{n}{n+1}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = x.\ 1.\ e. = ex.$$

Thus, by Ratio test $\Sigma\, u_n$ is convergent when $x < 1/e$ and divergent when $x > \dfrac{1}{e}$.

Next, when $x = 1/e$ we have $u_n = \dfrac{n^{n-1}}{n!\,e^{n-1}}$.

Now, let us apply logarithmic test

$$n\log\frac{u_n}{u_{n+1}} = n\log\frac{n^{n-1}(n+1)!\,e^n}{n!\,e^{n-1}(n+1)^n} = n\log\frac{e}{\left(1+\frac{1}{n}\right)^{n-1}}$$

$$\text{Therefore,} \quad \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\left\{\log e-(n-1)\log\left(1+\frac{1}{n}\right)\right\}$$

$$= \lim_{n\to\infty} n\left\{1-(n-1)\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)\right\} = \lim_{n\to\infty}\left\{n-(n-1)\left(1-\frac{1}{2n}+\frac{1}{3n^2}-\ldots\right)\right\}$$

$$= \lim_{n\to\infty}\left\{1+\frac{n-1}{2n}-\frac{n-1}{3n^2}+\ldots\right\} = 1+\frac{1}{2}+0 = \frac{3}{2} \text{ which is greater than 1} \qquad \textbf{Ans.}$$

Hence by logarithmic test Σu_n is convergent.

CAUCHY'S ROOT TEST

In a positive term series $\Sigma\, u_n$ *if* $\lim_{n\to\infty} u_n^{1/n} = \lambda$ *then the series converges for* $\lambda < 1$ *diverges for* $\lambda > 1$.

CASE I: *when* $\lambda < 1$.

By the definition of $\lim_{n\to\infty} u_n^{1/n}$ we can choose a positive number p such that $\lambda < p < 1$ then $u_n^{1/n} < p$ for all n.

Since $p < 1$, $u_n < p^n$ and the geometric series $\Sigma\, p^n$ is convergent hence by comparison test $\Sigma\, u_n$ is also convergent.

CASE II: *when* $\lambda > 1$.

Again by definition of limit we can find a number m such that $u_n^{1/n} > 1$ for all $n \geq m$

or $\quad u_n > 1^n = 1$ for all $n > m$.

Deleting the first $m-1$ terms, for the new series Σu_n

$$u_1 + u_2 + u_3 + \ldots + u_n > n,$$

hence $\lim\limits_{n\to\infty} (u_1 + u_2 + \ldots + u_n) = \infty$ and the series Σu_n is divergent.

EXAMPLE 2.23. (a) Discuss the convergence of the series $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\ldots\ldots$

[*GGSIPU Ist Sem End Term 2008*]

(b) Test the convergence of the series $\sum\limits_{1}^{\infty} \frac{n!\,2^n}{n^n}$. [*GGSIPU Ist Sem End Term 2009*]

SOLUTION : (a) Here $u_n = \left(\frac{n}{2n+1}\right)^n$ The form of u_n suggests the use of Cauchy's root test.

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n}{2n+1}\right)^n\right]^{1/n} = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2}$$

which is less than 1 hence, by root test the series Σu_n is convergent. **Ans.**

(*b*) Given series is Σu_n where $u_n = \frac{n!\,2^n}{n^n}$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{(n+1)!\,2^{n+1}\,n^n}{(n+1)^{n+1}.n!\,2^n} = \frac{2(n+1)}{(n+1)\left(\frac{n+1}{n}\right)^n} = \frac{2}{\left(1+\frac{1}{n}\right)^n}$$

Hence $\mathop{\mathrm{Lt}}\limits_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{2}{e}$ which is less than 1, hence the given series is covergent by ratio test.

Ans.

EXAMPLE 2.24. Test the convergence of the series $\frac{1}{2}x+\left(\frac{2}{3}\right)^4 x^2+\left(\frac{3}{4}\right)^9 x^3+\ldots$

SOLUTION: Here $u_n = \left(\frac{n}{n+1}\right)^{n^2} x^n$. The form of u_n attracts the use of Cauchy's root test.

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{n^2}.x^n\right]^{1/n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n.x$$

$$= \lim_{n\to\infty}\frac{x}{\left(\frac{n+1}{n}\right)^n} = x\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{x}{e}.$$

Therefore, by the root test the series Σu_n converges if $\frac{x}{e} < 1$, *i.e.*, $x < e$

and diverges if $\frac{x}{e} > 1$, *i.e.*, $x > e$.

Further when $x = e$, $u_n = \left(\frac{n}{n+1}\right)^{n^2} e^n$

Obviously $\lim_{n\to\infty} u_n \neq 0$ hence Σu_n is divergent.

Therefore, Σu_n is convergent for $x < e$ and divergent for $x \geq e$. **Ans.**

EXAMPLE 2.25. (a) Test the convergence of the series $\sum_{n\geq 2} \frac{1}{(\log n)^n}$

[*GGSIPU I Sem End Term 2007; I Term 2009*]

(b) Test the convergence of the series $\sum_1^\infty \left(\frac{n}{n+1}\right)^{n^2}$

[*GGSIPU I Sem I Term 2007; I Term Sp. 2009*]

SOLUTION: (a) The given series is $\sum U_n$ where $U_n = \frac{1}{(\log n)^n}$, $n \geq 2$, we apply Cauchy's's root test $\underset{n\to\infty}{Lt} U_n^{1/n} = \underset{n\to\infty}{Lt} \frac{1}{\log n} = 0$ which is less than 1.

Hence the given series is convergent. **Ans.**

(b) The given series is $\sum U_n$ where $U_n = \left(\frac{n}{n+1}\right)^{n^2}$

Applying Cauchy's root test, we have

$$\underset{n\to\infty}{Lt} U_n^{1/n} = \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)^{n^2/n} = \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)^n = \underset{n\to\infty}{Lt} \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \frac{1}{e} \text{ which is less than 1}$$

Therefore $\sum U_n$ is convergent. **Ans.**

EXAMPLE 2.26. Test the convergence of the series $\sum_1^\infty \frac{n^{n^2}}{\left(n+\frac{1}{4}\right)^{n^2}}$ [*GGSIPU I Sem End Term 2004*]

SOLUTION: $\sum U_n = \sum_1^\infty \frac{n^{n^2}}{\left(n+\frac{1}{4}\right)^{n^2}}$ Applying root test, we have

$$U_n^{1/n} = \frac{n^n}{\left(n+\frac{1}{4}\right)^n} = \frac{1}{\left(\frac{n+\frac{1}{4}}{n}\right)^n}$$

$$\therefore \quad \underset{n\to\infty}{Lt}\, U_n^{1/n} = \underset{n\to\infty}{Lt} \frac{1}{\left(\frac{n+\frac{1}{4}}{n}\right)^n} = \underset{n\to\infty}{Lt} \frac{1}{\left(1+\frac{1}{4n}\right)^n}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{\left(1+\frac{1}{4n}\right)^{4n/4}} = \frac{1}{e^{1/4}} \text{ which is less than 1.}$$

Hence the given series is convergent. **Ans.**

EXAMPLE 2.27. Test the convergence of the infinite series

(a) $\sum \frac{[(n+1)x]^n}{n^{n+1}}$ (b) $\sum_1^{\infty} \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \left(\frac{n+1}{n}\right)\right]^{-n}$

[*GGSIPU I Sem I Term 2004*] [*GGSIPU I Sem I Term 2011*]

SOLUTION: (a) The given series is $\sum U_n$ where $U_n = \frac{[(n+1)\,x]^n}{n^{n+1}}$

Here we apply the Cauchy's root test, then

$$U_n^{1/n} = \frac{(n+1)x}{n^{\frac{n+1}{n}}} = \frac{(n+1)x}{n\cdot n^{1/n}}.$$

$$\therefore \quad \underset{n\to\infty}{Lt}\, U_n^{1/n} = \frac{x\, \underset{x\to\infty}{Lt} \frac{n+1}{n}}{\underset{x\to\infty}{Lt}\, n^{1/n}} = \frac{x.1}{1}. \quad \text{(Since } \underset{x\to\infty}{Lt}\, n^{1/n} = 1\text{, by } l\text{'Hospital's rule).}$$

Therefore by root test $\sum U_n$ is convergent when $x < 1$ and divergent when $x > 1$. **Ans.**

(b) Here $u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n} \quad \therefore \quad u_n^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1}$

and $\underset{n\to\infty}{\text{Lt}}\, u_n^{1/n} = \underset{n\to\infty}{\text{Lt}} \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1} = \underset{n\to\infty}{\text{Lt}} \left(\frac{n+1}{n}\right)^{-1} \underset{n\to\infty}{\text{Lt}} \left[\left(\frac{n+1}{n}\right)^n - 1\right]^{-1}$

$$= 1\cdot \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1} = \frac{1}{e-1} < 1 \quad \text{as} \quad 2<e<3.$$

Therefore, by Cauchy's root $\sum u_n$ is convergent. **Ans.**

EXERCISE 2B

1. Discuss the convergence and divergence of the harmonic series $\Sigma \frac{1}{n^p}$ using Integral test.

Test the convergence of the following series.

2. $\sum_{1}^{\infty} n e^{-n^2}$.

3. $\frac{2^2}{3 \cdot 4} + \frac{2^2 \cdot 4^2}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \ldots$

4. $\frac{1}{3}x + \frac{1 \cdot 2}{3 \cdot 5}x^2 + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7}x^3 + \ldots \ (x > 0)$ [GGSIPU I Sem I Term 2009]

5. $2 + \left(\frac{3}{2}\right)^4 + \left(\frac{4}{3}\right)^9 + \left(\frac{5}{4}\right)^{16} + \ldots$

6. $\frac{1}{4} + \frac{4}{27} + \frac{27}{256} + \ldots + \frac{(n-1)^{n-1}}{n^n} + \ldots$

7. $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \ldots \quad x > 0$ [GGSIPU I Sem End Term Jan 2011; End Term 2013]

8. $\frac{4}{1!}x + \frac{4.7}{2!}x^2 + \frac{4.7.10}{3!}x^3 + \ldots$

9. $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ [GGSIPU I Sem. End Term 2008; I Term Sp. 2009; I Term 2010]

10. $\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \ldots \ (x > 0)$ [GGSIPU I Sem I Term 2010]

11. $\frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots$

12. $1 + \frac{a.b}{1.c}x + \frac{a(a+1)b(b+1)}{1.2.c.(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c.(c+1)(c+2)}x^3 + \ldots$

13. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$

14. $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots \quad (x > 0)$

15. $\frac{a}{x+a} + \frac{a^2}{x^2+a^2} + \frac{a^3}{x^3+a^3} + \ldots$

ALTERNATING SERIES

An infinite series, in which from and after certain term all the terms are alternately positive and negative, is called an *alternating series*.

Such a series can be written as $u_1 - u_2 + u_3 - u_4 + \ldots$ where u's are all positive.

LEIBNITZ'S TEST

The alternating series $u_1 - u_2 + u_3 - u_4 + \ldots$ *is convergent if each term is numerically less than its preceding term and* $\lim\limits_{n\to\infty} u_n = 0$, *that is, if*

(*i*) $u_1 > u_2 > u_3 > \ldots$ (*ii*) $\lim\limits_{n\to\infty} u_n = 0.$ ...(1)

If the alternating series is not convergent, it is called *oscillatory*.

To establish the test, first consider S_{2n}, the sum of first $2n$ terms, as

$$S_{2n} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ldots + (u_{2n-1} - u_{2n}) \quad \ldots(2)$$

$$= u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \ldots (u_{2n-2} - u_{2n-1}) - u_{2n} \quad \ldots(3)$$

Now, because of (1) each term in the brackets in (3) is positive hence S_{2n} is positive and increases as n increases. Also, since each term in the brackets of (3) is positive hence S_{2n} is always less than u_1. Thus $\{S_{2n}\}$ is monotonically increasing and bounded, therefore $\lim\limits_{n\to\infty} S_{2n}$ exists and is finite.

Further, $\lim\limits_{n\to\infty} S_{2n+1} = \lim\limits_{n\to\infty} (S_{2n} + u_{2n+1}) = \lim\limits_{n\to\infty} S_{2n} + \lim\limits_{n\to\infty} u_{2n+1} = \lim\limits_{n\to\infty} S_{2n} + 0$

This implies that $\lim\limits_{n\to\infty} S_n$ exists and is finite, hence the given alternating series is convergent.

EXAMPLE 2.28. (a) Test the convergence of the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

[*GGSIPU Ist Sem End Term 2008*]

(b) Discuss the convergence of the series $\sum\limits_{1}^{\infty} \frac{(-1)^{n-1} 2^n}{n^2}$.

SOLUTION: (a) It is an alternating series. Since $\frac{1}{(n+1)^2} < \frac{1}{n^2}$, each term is numerically less than its preceding term. Also, $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{(-1)^{n-1}}{n^2} = 0$, therefore, the given series is convergent. **Ans.**

(b) The given series is $2 - \frac{2^2}{2^2} + \frac{2^3}{3^2} - \frac{2^4}{4^2} + \ldots$

Here $$\left|\frac{u_{n+1}}{u_n}\right| = \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = 2\left(\frac{n}{n+1}\right)^2 = 2\left(1 - \frac{1}{n+1}\right)^2$$

$\therefore$ $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = 2$ hence $|u_{n+1}| \nless |u_n|$ for all n.

Moreover, $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{2^n \log 2}{2n} \quad \text{(using L' Hospital's rule)}$$

$$= \lim_{n\to\infty} \frac{2^n (\log 2)^2}{2} \neq 0$$

Therefore, Σu_n is not convergent and hence is oscillatory. **Ans.**

EXAMPLE 2.29. Test the convergence of the series

(i) $\log\left(\frac{2}{1}\right)-\log\left(\frac{3}{2}\right)+\log\left(\frac{4}{3}\right)-\log\left(\frac{5}{4}\right)+\ldots$ [GGSIPU I Sem End Term 2000]

(ii) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\log(n+1)}{(n+1)^3}$. [GGSIPU I Sem End Term 2007]

SOLUTION: (i) The given series is $\sum_{1}^{\infty}(-1)^{n-1}\log\frac{n+1}{n}=\Sigma U_n$.

Clearly $\underset{n\to\infty}{Lt}\, U_n=0$. Now, since $\frac{n+1}{n}<\frac{n}{n-1}$ as $n^2-1<n^2$ hence $\log\frac{n+1}{n}<\log\frac{n}{n-1}$

that is $\left|\frac{U_{n+1}}{V_n}\right|<1$, and the given series is convergent. **Ans.**

(ii) The given series is $\sum_{1}^{\infty}U_n=\frac{\log 2}{2^3}-\frac{\log 3}{3^3}+\frac{\log 4}{4^3}-\frac{\log 5}{5^3}+\ldots$

where $U_n=(-1)^{n-1}\frac{\log(n+1)}{(n+1)^3}$

Obviously $|U_n+1|<|U_n|$ and $\underset{n\to\infty}{Lt}\, U_n=0$.

Therefore the given series is convergent by Lebneitz test. **Ans.**

EXAMPLE 2.30. (a) Test the convergence of the series $\sum_{1}^{\infty}\frac{(-1)^n x^n}{\log(n+1)}$.

(b) Test the convergence of the series $x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\ldots$

[GGSIPU I Sem End Term 2004, End Term 2011, End Term 2012]

SOLUTION: (a) Here $|u_n|=\frac{x^n}{\log(n+1)}$

$$\therefore\quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\frac{\log(n+1)}{\log(n+2)}|x|=|x|.$$

Therefore the given series converges absolutely for $|x|<1$ and not so for $|x|>1$.

Next, for $x=1$ the series becomes $-\frac{1}{\log 2}+\frac{1}{\log 3}-\frac{1}{\log 4}+\ldots$ [GGSIPU I Sem I Term 2010]

which is convergent by Leibnitz's test since

$$\frac{1}{\log(n+1)}<\frac{1}{\log n}\text{ for all } n \text{ and } \lim_{n\to\infty}\frac{1}{\log n}=0.$$

For $x=-1$ the series becomes $\frac{1}{\log 2}+\frac{1}{\log 3}+\frac{1}{\log 4}+\ldots.$

which has been earlier shown to be divergent by comparison test after comparing it with $\Sigma\frac{1}{n}$. **Ans.**

(b) Case I. When $x < 0$. Let $x = -y$ where $y > 0$.

Then the series is $-y - \frac{y^2}{2^2} - \frac{y^3}{3^2} - \frac{y^4}{4^2} + \ldots$ or $y + \frac{y^2}{2^2} + \frac{y^3}{3^2} + \frac{y^4}{4^2} + \ldots$

which is a positive term series. Applying ratio test here

we get $\underset{n\to\infty}{Lt} \frac{U_{n+1}}{U_n} = \underset{n\to\infty}{Lt} \frac{y^{n+1}}{(n+1)^2} \cdot \frac{n^2}{y^n} = y \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)^2 = y.$

Thus, in this case the series is convergent for $y < 1$, *i.e.,* $x > -1$ and divergent for $y > 1$, *i.e.,* $x < -1$. However, for $y = 1$, *i.e.,* $x = -1$ the given series is $\Sigma \frac{1}{n^2}$ which is convergent as p-series.

Case II: When $x > 0$. ΣU_n is an alternating series.

$$Lt \left|\frac{U_n + 1}{U_n}\right| = \underset{n\to\infty}{Lt} \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} = x \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)^2 = x.$$

Now, when $0 < x < 1$ by Lebneitz test the series is convergent and when $x > 1$ the series is oscillatory.

However when $x = 1$ the series is $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

which is known to be convergent by Lebneitz test.

Thus the given series is convergent for $0 > x \geq 1$ and for $0 < x < 1$ and oscillatory for $x \geq 1$.

Ans.

DONE

EXAMPLE 2.31. Find the interval of convergence of the series

(*i*) $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \ldots$ [*GGSIPU I Sem I Term 2007*]

(*ii*) $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ [*GGSIPU I Sem End Term 2011*]

SOLUTION: (*i*) **Case I:** When $x < 0$. Let $x = -y$ where $y > 0$.

Then the series is $-y - \frac{y^2}{\sqrt{2}} - \frac{y^3}{\sqrt{3}} - \frac{y^4}{\sqrt{4}} - \ldots$ or $y + \frac{y^2}{\sqrt{2}} + \frac{y^3}{\sqrt{3}} + \frac{y^4}{\sqrt{4}} + \ldots$

which is a positive term series. Applying ratio test here, we get

$$\underset{n\to\infty}{Lt} \frac{U_{n+1}}{U_n} = \underset{n\to\infty}{Lt} \frac{y^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{y^n} = y \underset{n\to\infty}{Lt} \frac{\sqrt{n}}{\sqrt{n+1}} = y.$$

Thus is this case the series is convergent for $y < 1$, *i.e.,* $x > -1$ and divergent for $y > 1$, *i.e.,* $x < -1$. However for $y = 1$, *i.e.,* for $x = -1$ the given series is $\sum \frac{1}{\sqrt{n}}$ which is divergent p-series.

Case II: When $x > 0$, ΣU_n is an alternating series.

$$\underset{n\to\infty}{Lt} \left|\frac{U_{n+1}}{U_n}\right| = \underset{n\to\infty}{Lt} \frac{\sqrt{n}}{\sqrt{n+1}} x = x \Rightarrow |u_{n+1}| < |u_n| \quad \text{as} \quad 0 < x < 1$$

Now when $0 < x < 1$ by Lebneitz test the series is convergent and when $x > 1$ series is oscillatory. However when $x = 1$ the series is $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$ which is convergent by Lebneitz test. Thus the given series is convergent for $0 < x \le 1$ and oscillatory for $x > 1$. **Ans.**

(*ii*) The given series is an alternating series Σu_n where $u_n = \frac{(-1)^{n-1} x^{2n-1}}{2n-1}$.

Here, clearly $\lim\limits_{n\to\infty} u_n = 0$ and $\left|\frac{u_{n+1}}{u_n}\right| = \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} = \frac{2n-1}{2n+1} x^2$.

Therefore $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = x^2 < 1$ when $-1 < x < 1$

> 1 when $-\infty < x < -1$ and $1 < x < \infty$

Thus, Σu_n is convergent for $-1 < x < 1$ hence the radius of convergence is 2. **Ans.**

EXAMPLE 2.32. (*a*) Explain Lebneitz's test and examine the character of the series

$$(i)\ \sum_2^{\infty} \frac{(-1)^{n-1} x^n}{n(n-1)},\ 0 < x < 1 \quad (ii)\ \sum \frac{(-1)^n x^n}{n(n+3)}$$

[*GGSIPU I Sem End Term 2007; I Term 2013*]

(*b*) Test the convergence of the series

$$\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots \infty$$

[*GGSIPU I Sem I Term 2012*]

SOLUTION: (*a*) (*i*) Given series is an alternating series. Here $\text{Limit}_{n\to\infty} u_n = 0$ as $0 < x < 1$

$$\left|\frac{U_{n+1}}{U_n}\right| = \frac{x^{n+1}\, n(n-1)}{(n+1)\cdot n \cdot x^n} = \frac{n-1}{n+1} x$$

Since $\underset{n\to\infty}{Lt} \left|\frac{U_{n+1}}{U_n}\right| = x < 1 \Rightarrow |u_{n+1}| < |u_n|$, thus ΣU_n is convergent as $0 < x < 1$ given. **Ans.**

(*ii*) Here $\left|\frac{u_{n+1}}{u_n}\right| = \frac{n\,(n+3)}{(n+1)\,(n+4)} |x| < 1$ when $|x| < 1$, i.e., when $-1 < x < 1$.

$\therefore$ Σu_n is convergent when $|x| < 1$ since $\underset{n\to\infty}{\text{Lt}}\ u_n = 0$.

However, when $|x| \ge 1$ the series is oscillatory. **Ans.**

(*b*) $\sum U_n$ is an alternating series

Here $|U_n| = \frac{1+2+3+\ldots+n}{(n+1)^3} = \frac{n(n+1)}{2(n+1)^3} = \frac{n}{2(n+1)^2}$ and $\underset{n\to\infty}{\text{Lim}}\ U_n = 0$.

Next $\frac{|U_{n+1}|}{|U_n|} = \frac{n+1}{2(n+2)^2} \cdot \frac{2(n+1)^2}{n} = \frac{(n+1)^3}{n(n+2)^2} = \frac{n^3+3n^2+3n+1}{n^3+4n^2+4n}$

Let us check if $3n^2 + 3n + 1 < 4n^2 + 4n$ or $n^2 + n > 1$ which is obviously true.

Hence $|U_{n+1}| < |U_n|$ and $\underset{n\to\infty}{\text{Lt}}\ U_n = 0$

Therefore, by Lebneitz's test ΣU_n is convergent. **Ans.**

ABSOLUTE CONVERGENCE AND CONDITIONAL CONVERGENCE

A series Σu_n is said to be *absolutely convergent* if the series

$$\sum_1^{\infty}|u_n| = |u_1| + |u_2| + |u_3| + \ldots \quad \text{is convergent.}$$

For example, the series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ is absolutely convergent since the series

$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ is convergent.

On the other hand, if a series converges but not absolutely it is called *conditionally convergent.*

For example, the series $\Sigma u_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\ldots$ is convergent by Leibnitz's test,

but $\Sigma|u_n| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$ is divergent. Therefore, the series

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$ is conditionally convergent. *[GGSIPU I Sem End Term 2011]*

An important note at this stage is that *if a series is absolutely convergent it has to be convergent.* It can be established simply by making use of the comparison test part (*i*)

since $u_n \le |u_n|$ for all n and $\Sigma |u_n|$ is convergent hence Σu_n is also convergent.

EXAMPLE 2.33. (a) Test the convergence and absolute convergence, of the series

$$\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\ldots$$ *[GGSIPU I Sem I Term 2009]*

(b) Discuss the absolute convergence of the series $\sum\frac{(-1)^n x^n}{n+1}$

[GGSIPU I Sem End Term 2013]

SOLUTION: (a) $\sum U_n = \sum\frac{(-1)^{n-1}\sin nx}{n}$

Here $\underset{n\to\infty}{\text{Lt}}\ U_n = 0$ and $\frac{\sin(n+1)x}{n+1} < \frac{\sin nx}{n}$

Therefore $\sum U_n$ is convergent by Lebneitz Test. Further, $|U_n| = \left|\frac{\sin nx}{n}\right|$

Now for absolute convergence of $\sum U_n$ we apply the integral test on $\sum |U_n|$

$$\int_0^{\infty}\frac{\sin nx}{n}dx = \int_0^{\infty}\frac{\sin y}{y}dy \quad (\text{on putting } y = nx) = \frac{\pi}{2}, \text{ a well known result.}$$

$\therefore\ \sum U_n$ is convergent as well as absolutely convergent. **Ans.**

(b) Here $u_n = \frac{(-1)^n x^n}{n+1}$ and $\lim_{n\to\infty} u_n = 0$ for all x.

Further, $|u_n| = \frac{x^n}{n+1}$ and $\left|\frac{u_{n+1}}{u_n}\right| = \frac{n+1}{n+2}|x| < 1$ when $|x| < 1$.

$\therefore\ \Sigma u_n$ converges absolutely when $|x| < 1$ and is oscillatory when $|x| \ge 1$. **Ans.**

EXAMPLE 2.34. (*a*) Find, if the series $\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{3}\cdot\frac{1}{2^3}-\ldots$ converges absolutely or conditionally.
[GGSIPU I Sem End Term 2004]

(*b*) Examine the character of the series $\sum_1^\infty \frac{(-1)^{n-1}n}{2n-1}$.
[GGSIPU I Sem End Term 2009]

SOLUTION: (*a*) The given series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\cdot\frac{1}{2^n} = \Sigma U_n$ which is convergent by Lebneitz test.

For absolute convergence of ΣU_n, we have $|U_n| = \frac{1}{n\cdot 2^n}$.

Hence $\left|\frac{U_{n+1}}{U_n}\right| = \frac{n\cdot 2^n}{(n+1)2^{n+1}} = \frac{n}{2(n+1)}$

$\therefore$ $\underset{n\to\infty}{Lt}\left|\frac{U_{n+1}}{U_n}\right| = \frac{1}{2}$ which is less than 1 therefore the given series is absolutely convergent. **Ans.**

(*b*) $U_n = \frac{(-1)^{n-1}n}{2n-1}$, Here ΣU_n is alternating series. Let us apply Lebneitz's test.

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{-(n+1)(2n-1)}{n(2n+1)}\right| = \frac{n+1}{n}\cdot\frac{2n-1}{2n+1} = \frac{2n^2+n-1}{2n^2+n}$$

$= 1-\frac{1}{n(2n+1)}$ which is less than 1 for all n $\Rightarrow$ $|U_{n+1}| < |U_n|$.

Further, $\underset{n\to\infty}{\text{Lt}}|U_n| = \underset{n\to\infty}{\text{Lt}}\frac{n}{2n-1} = \frac{1}{2} \neq 0$.

Hence the given series is oscillatory by Lebneitz's Test. **Ans.**

EXERCISE 2C

Test the following series for convergence and absolute convergence.

1. $1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\ldots$

2. $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\frac{9}{16}-\ldots$

3. $1-\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{3}-\frac{\sqrt{4}}{4}+\ldots$

4. $1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\ldots$

Test for convergence, the series

5. $\sum_{1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$.

6. $\sum_{1}^{\infty}(-1)^n\frac{1+n^2}{1+n^3}$.

7. $\sum_{1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$.

8. $\frac{1}{2}-\frac{1}{2.3}+\frac{1}{2.3.4}-\frac{1}{2.3.4.5}+\ldots$

9. $\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\ldots$

10. State with reason, the value of x for which the series $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ converges.

11. Prove that the series $\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\ldots$ converges absolutely.

12. Test for convergence and absolute convergence of the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\log n)^2}$.

[GGSIPU Ist Sem End Term 2008]

UNIFORM CONVERGENCE OF SERIES OF FUNCTIONS

The infinite series of functions as

$$\sum_1^{\infty} u_n(x) = u_1(x) + u_2(x) + u_3(x) + \dots$$

is said to be uniformly convergent in $[a, b]$ *if the sequence of partial sums* $\{S_n(x)\}$, $n = 1, 2, 3, \dots$ *where* $S_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$, *is convergent in* $[a, b]$.

In such a case we write $\lim_{n\to\infty} S_n(x) = S(x)$ and $S(x)$ is the sum of the infinite series. In other words, $\sum_1^{\infty} u_n$ *is said to be uniformly convergent in* $[a, b]$ *if, for given a small positive number* $\in$, *we can find a number* N *not depending on* x, *such that for every* x *in* (a, b)

$$|S(x) - S_n(x)| < \in \quad \text{for all } n > N.$$

Note here that N depends on $\in$ only and not on x.

WEIRSTRASS M TEST FOR UNIFORM CONVERGENTCE OF SERIES

If there exists a convergent series ΣM_n *of positive constants* $M_1, M_2, M_3, \dots$ *such that* $|u_n(x)| \le M_n$ *for all* x *in the interval* (a, b) *then* $\Sigma u_n(x)$ *converges uniformly in that interval.*

EXAMPLE 2.35. Show that both the series of functions $\sum_1^{\infty} \frac{\cos nx}{n^2}$ and $\sum_1^{\infty} \frac{\sin nx}{n^2}$ are uniformly convergent and also absolutely convergent in $[0, 2\pi]$.

SOLUTION: Since $\left|\frac{\cos nx}{n^2}\right| \le \frac{1}{n^2}$ for all x and $\Sigma \frac{1}{n^2}$ is convergent as 'p-series' with $p = 2\ (> 1)$ therefore, $\Sigma \frac{\cos nx}{n^2}$ converges uniformly in $(0, 2\pi)$ as well as absolutely.

Similarly $\Sigma \frac{\sin nx}{n^2}$ in uniformly convergent.

At this stage we make one very important observation that *if a series is uniformly convergent then it is not necessarily absolutely convergent*. Actually the two properties are independent, i.e., a series can be uniformly convergent without being absolutely convergent and vice-versa.

THEOREMS ON UNIFORMLY CONVERGENT SERIES

We know that the sum of a finite number of continuous functions is continuous. Secondly, the integral of the sum of a finite number of integrable functions over some interval (a, b) is equal to the sum of their integrals separately over (a, b). And thirdly, the derivative of the sum of a finite number of differentiable functions is equal to the sum of their derivatives.

We investigate as to what happens to these properties when the number of functions is infinite.

THEOREM 1. *If* $\{u_n(x)\}$, $n = 1, 2, 3, \dots$ *are continuous in* $[a, b]$ *and if the series* $\Sigma u_n(x)$ *converges uniformly to the sum* $S(x)$ *in* $[a, b]$ *then* $S(x)$ *is also continuous in* $[a, b]$.

In other words a uniformly convergent series of continuous functions is also continuous.

THEOREM 2. *If* $\{u_n\}$, $n = 1, 2, 3, \dots$ *are continuous in* $[a, b]$ *and* $\Sigma u_n(x)$ *converges uniformly to the sum* $S(x)$ *in* $[a, b]$, *then*

$$\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$$

that is
$$\int \left(\sum_{n=1}^{\infty} u_n(x) \right) dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$$

In short, a uniformly convergent series of continuous functions can be integrated term by term.

THEOREM 3. *If* $\{u_n\}$, $n = 1, 2, 3, \ldots$ *are continuous and have continuous derivatives in* $[a, b]$ *and if* $\Sigma u_n(x)$ *converges uniformly to* $S(x)$ *while* $\Sigma u_n'(x)$ *is uniformly convergent in* $[a, b]$, *then in* $[a, b]$ *we have*

$$S'(x) = \sum_{n=2}^{\infty} u_n'(x) \quad \text{that is,} \quad \frac{d}{dx}\left(\sum_{1}^{\infty} u_n(x) \right) = \sum_{1}^{\infty} \frac{d}{dx} u_n(x).$$

Thus, a uniformly convergent series of functions can be differentiated term by term provided the series of the derivatives of the functions also converges uniformly.

EXAMPLE 2.36. Test for uniform convergence, the series

$$(a)\ \sum_{1}^{\infty} \frac{x^n}{n^{3/2}},\ (|x| \le 1) \qquad (b)\ \sum_{1}^{\infty} \frac{\cos nx}{n}.$$

SOLUTION: (*a*) Applying D' Alembert's ratio test we find that the series $\Sigma \frac{x^n}{n^{3/2}}$ converges in $[-1, 1]$. For all x in this interval $\left| \frac{x^n}{n^{3/2}} \right| = \frac{|x^n|}{n^{3/2}} \le \frac{1}{n^{3/2}}$

Therefore, if we choose $M_n = \frac{1}{n^{3/2}}$ we see that $\Sigma M_n = \Sigma \frac{1}{n^{3/2}}$ is convergent being '*p*-series' with $p = 3/2$ (> 1). Thus, by Weierstrass M-test the given series is uniformly convergent in $[-1, 1]$. **Ans.**

(*b*) Here $u_n = \frac{\cos nx}{n}$ and $\left| \frac{\cos nx}{n} \right| \le \frac{1}{n}$.

But $\Sigma M_n = \Sigma \frac{1}{n}$ is not convergent, therefore M-test cannot be applied in this case and as such nothing can be concluded about the uniform convergence of the given series. **Ans.**

EXERCISE 2D

Test for uniform convergence the following series

1. $\frac{\cos x}{1^3}+\frac{\cos 2x}{2^3}+\frac{\cos 3x}{3^3}+\frac{\cos 4x}{4^3}+\ldots$

2. $\frac{\sin x}{2}+\frac{\sin 2x}{2^2}+\frac{\sin 3x}{2^3}+\frac{\sin 4x}{2^4}+\ldots$

3. $\frac{1}{1^3+1^4x^2}+\frac{1}{2^3+2^4x^2}+\frac{1}{3^3+3^4x^2}+\ldots$

4. $\Sigma\frac{1}{n^2+x^2}$.

5. Prove that the series $\frac{1}{1+x^2}-\frac{1}{2+x^2}+\frac{1}{3+x^2}-\frac{1}{4+x^2}+\ldots$ is uniformly convergent for all x but not absolutely.

6. Show that the series $\frac{x}{1(1+x^2)}+\frac{x}{2(1+2x^2)}+\frac{x}{3(1+3x^2)}+\ldots$ converges uniformly for all x.

❑❑❑

CHAPTER

3 Asymptotes and Curve Tracing

Asymptotes of Cartesian Curves, Curve Tracing for Cartesian, Parametric and Polar Curves

ASYMPTOTE

We shall consider here only those curves whose branches extend upto infinity, like parabola and hyperbola. At a point on the curve, tangent is drawn and then the point of contact is moved farther and farther away from the origin. If this tangent in the process tends to some definite straight line, it is called *asymptote* of the curve. In other words, **the *limiting position of the tangent line as the point of contact recedes to infinity is known as asymptote.*** In some typical cases these asymptotes are curves also. However, we shall consider here only those curves which have straight lines as asymptotes. Some asymptotes are parallel to x-axis or to y-axis and those asymptotes which are not parallel to any of the coordinate axes are called *oblique asymptotes*.

DETERMINING THE ASYMPTOTES

Let the equation of the curve be algebraic in x and y of degree n, its general form being,

$$(a_0x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + ... + a_n y^n) + (b_1 x^{n-1} + b_2 x^{n-2} y + ... + b_n y^{n-1}) +$$
$$(c_2 x^{n-2} + c_3 x^{n-3} y + ... + c_n y^{n-2}) + ... + (l_{n-1} x + l_n y) + k_n = 0. \quad ...(1)$$

Since the expressions in brackets are homogeneous, the above equation can be written as

$$x^n \phi_n (y/x) + x^{n-1} \phi_{n-1} (y/x) + x^{n-2} \phi_{n-2} (y/x) + ... + x \phi_1 (y/x) + k_n = 0$$

where $\phi_n (y/x)$ is a function of (y/x) of degree n, etc.

or $$\phi_n(y/x) + \frac{1}{x}\phi_{n-1}(y/x) + \frac{1}{x^2}\phi_{n-2}(y/x) + ... + \frac{1}{x^{n-1}}\phi_1(y/x) + \frac{k_n}{x^n} = 0 \quad ...(2)$$

Now, let $y = mx + c$ or $y/x = m + \dfrac{c}{x}$ ...(3)

be an an asymptote to (1), then $\lim_{x\to\infty} (y/x) = m$.

Taking limit in (2) as $x \to \infty$, we get $\phi_n (m) = 0$ ...(4)

The real roots of the equation (4) in m, give the slopes of the asymptotes.

Using (3), the equation (2) becomes

$$\phi_n\left(m+\frac{c}{x}\right)+\frac{1}{x}\phi_{n-1}\left(m+\frac{c}{x}\right)+\frac{1}{x^2}\phi_{n-2}\left(m+\frac{c}{x}\right)+\ldots=0$$

Since x is very large, $\frac{c}{x}$ will be very small and we can expand $\phi_n\left(m+\frac{c}{x}\right), \phi_{n-1}\left(m+\frac{c}{x}\right),\ldots$ by Taylor's theorem, to get

$$\left[\phi_n(m)+\frac{c}{x}\phi_n'(m)+\frac{c^2}{2!\,x^2}\phi_n''(m)+\ldots\right]+\frac{1}{x}\left[\phi_{n-1}(m)+\frac{c}{x}\phi_{n-1}'(m)+\frac{c^2}{2!\,x^2}\phi_{n-1}''(m)+\ldots\right]$$
$$+\frac{1}{x^2}\left[\phi_{n-2}(m)+\frac{c}{x}\phi_{n-2}'(m)+\frac{c^2}{2!\,x^2}\phi_{n-2}''(m)+\ldots\right]=0 \quad \ldots(5)$$

But, by (4), $\phi_n(m)=0$, hence the above equation (5) on multiplying throughout by x, becomes

$$\left[c\,\phi_n'(m)+\phi_{n-1}(m)\right]+\frac{1}{x}\left[\frac{c^2}{2!}\phi_n''(m)+c\,\phi_{n-1}'(m)+\phi_{n-2}(m)\right]$$
$$+\frac{1}{x^2}\left[\frac{c^3}{3!}\phi_n'''(m)+\frac{c^2}{2!}\phi_{n-1}''(m)+c\,\phi_{n-2}'(m)+\phi_{n-3}(m)\right]+\ldots=0$$

As $x\to\infty$, we get $c\,\phi_n'(m)+\phi_{n-1}(m)=0$ ...(6)

or $$c=\frac{-\phi_{n-1}(m)}{\phi_n'(m)}.$$

Now, if the roots of the equation $\phi_n(m)=0$ are real and distinct as $m=m_1, m_2, \ldots, m_n$, the corresponding values of c as $c_1, c_2, \ldots, c_n$ are easily obtained from (6). However, in case two roots of (4) are equal, say $m_1=m_2$, then it is a case of two parallel asymptotes and two values of c cannot be obtained from (6) since $\phi_{n-1}(m_1)=0$ and $\phi_n'(m_1)=0$.

In such a case we take into account $\frac{c^2}{2!}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)=0$ which gives two values of c, say c_1 and c_2 for one value of m. Similarly, in the case when three roots of equation (4) are equal, we get three parallel asymptotes and the corresponding three values of c can be obtained from the equation

$$\frac{c^3}{3!}\phi_n'''(m)+\frac{c^2}{2!}\phi_{n-1}''(m)+\frac{c}{1!}\phi_{n-2}'(m)+\phi_{n-3}(m)=0$$

To sum up, the working procedure for obtaining the equations of the asymptotes, will be as follows: In the given algebraic equation we first replace y by $(mx+c)$, and equate to zero the coefficient of the highest degree term in x to get the values of m, the slopes of the asymptotes. Then equate to zero the coefficient of the next to the highest degree term in x to get 'c' of the asymptote.

As a further short cut, the polynomials $\phi_n(m), \phi_{n-1}(m), \ldots$ are obtained by replacing x by 1 and y by m in the nth degree term, then in the $(n-1)^{th}$ degree term, and so on, respectively in the equation of the curve.

Asymptotes Parallel to Coordinate Axes

The asymptotes parallel to x-axis can be obtained by the method just explained, but the asymptotes parallel to y-axis cannot be determined by this method simply because the slope m of such asymptotes is infinite. For the asymptotes, parallel to y-axis we make slight change in the approach as follows.

The general algebraic equation of n^{th} degree can be written as

$$y^n \phi_0(x) + y^{n-1}\phi_1(x) + y^{n-2}\phi_2(x) + ... = 0 \qquad ...(1)$$

where $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$,... are polynomials in x of degree 0, 1, 2, ... respectively. Dividing throughout by y^n, (1) becomes $\phi_0(x)+\frac{1}{y}\phi_1(x)+\frac{1}{y^2}\phi_2(x)+.... = 0.$ ...(2)

Now we take limit as y tends to infinity and x tends to some finite value k, then we get

$\phi_0(k) = 0$ where k is the root of the equation $\phi_0(x) = 0$.

If the roots of the equation $\phi_0(x) = 0$ are k_1, k_2,... then the asymptotes parallel to y-axis are

$x = k_1$, $x = k_2$,...

Thus, the ***asymptotes parallel to y-axis are obtained by equating to zero the coefficient of the highest degree term in y*** and if the cofficient of the highest degree term in y is constant, then obviously, there is no asymptote parallel to y-axis.

Following on the similar lines we can conclude **that the *asymptotes parallel to x-axis can be easily obtained by equating to zero the cofficient of the highest degree term in x*.** However, if the coefficient of the highest degree term in x is constant there is no asymptote parallel to x-axis.

EXAMPLE 3.1. (a) Find the asymptotes of the curve $x^2y^2 - a^2.(x^2 + y^2) = 0$.

[*GGSIPU I Sem I Term 2010*]

(b) Find the asymptotes parallel to the axes of the curve $x^2y^3 + x^3y^2 = x^3 + y^3$

[*GGSIPU I Sem End Term 2012*]

SOLUTION: **(a)** The equation of the curve is of fourth degree therefore, there cannot be more than four asymptotes of the curve.

First, let us find the asymptotes parallel to x-axis for which we rewrite the given equation as

$$x^2(y^2 - a^2) - a^2y^2 = 0$$

Equating to zero the cofficient of the highest degree term in x, we get

$$y^2 - a^2 = 0 \quad \text{or} \quad y = \pm a$$

Thus there are two asymptotes of the curve parallel to x-axis as $y = \pm a$. Similarly, to find the asymptotes parallel to y-axis we rewrite the given equation as

$$y^2(x^2 - a^2) - a^2x^2 = 0.$$

Equating to zero the coefficient of the highest degree term in y, we get

$$x^2 - a^2 = 0 \quad \text{or} \quad x = \pm a.$$

Therefore, there are two asymptotes of the curve parallel to y-axis as $x = \pm a$. **Ans.**

(b) The equation is $x^3(y^2 - 1) + y^3(x^2 - 1) = 0$

Equating to zero the coefficient of the highest degree term in x, we get asymptote parallel to x-axis. Thus $y^2 - 1 = 0$ or $y = \pm 1$ are asymptotes parallel to x-axis.

Equating to zero the coefficient of the highest degree term in y, we get the asymptote parallel to y-axis as $x^2 - 1 = 0$ or $x = \pm 1$. **Ans.**

EXAMPLE 3.2. Find the asymptotes of the curve (i) $y^3 - x^2y + 2y^2 + 4y + x = 0$.

(ii) $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$ [*GGSIPU I Sem I Term 2012*]

SOLUTION : (i) The given equation of the curve is of degree 3. Replacing y by m and x by 1 in the third degree terms in the equation of the curve, we get

$$\phi_3(m) = m^3 - m = m(m^2 - 1).$$

Therefore, $\phi_3(m) = 0$ gives the slopes of the asymptotes as $m = 0, 1, -1$ which are all distinct.

Next, replacing y by m and x by 1 in the second degree terms of the equation of the curve, we get $\phi_2(m) = 2m^2$.

The 'c' of the asymptotes are given by $c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = -\dfrac{2m^2}{3m^2-1}$

Hence, for $m = 0$ we have $c = 0$, and for $m = \pm 1$ we have $c = -1$ and thus there are three asymptotes of the curve whose equations are $y = 0$, $y = x - 1$ and $y = -x - 1$. **Ans.**

It is to be observed here that the asymptote $y = 0$ parallel to x-axis could be obtained directly also, by equating to zero the coefficient of the highest degree term in x. The highest degree term in x here is x^2 whose coefficient y when equated to 0 gives $y = 0$, the x-axis itself. Further, since the cofficient of the highest degree term in y is constant there is no asymptote parallel to y-axis.

(ii) $\phi_3(m) = 1 + 2m - m^2 - 2m^3$, $\phi_2(m) = m - m^2, \phi_1(m) = 0$

Now $\phi_3(m) = 0$ gives $2m^3 + m^2 - 2m - 1 = 0$ or $(2m+1)(m^2-1) = 0$ hence $m = 1, -1, -1/2$.

Next $C = \dfrac{-\phi_2(m)}{\phi_3'(m)} = \dfrac{-(m-m^2)}{-6m^2-2m+2} = \dfrac{-m^2+m}{6m^2+2m-2}$

For $m = \dfrac{-1}{2}, C = \dfrac{1}{2}$ and for $m = 1$, $C = 0$ and for $m = -1$, $C = \dfrac{-1-1}{6-2-2} = -1$.

Thus the asymptotes are

$y = x$, $y = -x - 1$ and $y = -\dfrac{1}{2}x + \dfrac{1}{2}$ **Ans.**

EXAMPLE 3.3. Find the asymptotes of the curve (i) $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$ [*GGSIPU I Sem I Term 2007*]

(ii) $x^3 + x^2y + xy^2 + y^3 + 2x^2 + 3xy - 4y^2 + 7x + 2y = 0$. [*GGSIPU I Sem I Term 2013*]

SOLUTION: (i) The given equation is of degree three hence not more than three asymptotes can be there for this curve. Here

$\phi_3(m) = m^3 - 6m^2 + 11m - 6 = (m-1)(m^2-5m+6) = (m-1)(m-2)(m-3)$,

$\phi_2(m) = 0$ and $\phi_1(m) = 1 + m$.

The slopes of the asymptotes are the roots of the equation $\phi_3(m) = 0$ which are $m = 1, 2, 3$

The 'c' of the asymptotes are given by

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = 0 \quad \text{as} \quad \phi_2(m) = 0.$$

Therefore, there are three asymptotes of the curve, with equations $y = x$, $y = 2x$, $y = 3x$. **Ans.**

(ii) Here $\phi_3(m) = 1 + m + m^2 + m^3 = (m+1)(m^2+1)$,

$\phi_2(m) = 2 + 3m - 4m^2$, $\phi_1(m) = 7 + 2m$.

For slope of asymptotes $\phi_3(m) = 0$ which gives $m = -1$.

Corresponding 'c' is given by $c = \dfrac{-\phi_2(m)}{\phi_3'(m)} = \dfrac{4m^2-3m-2}{3m^2+2m+1} = \dfrac{5}{2}$ for $m = -1$.

$\therefore$ Asymptote is $y = -x + \dfrac{5}{2}$. **Ans.**

EXAMPLE 3.4. (a) Find the asymptotes, if any, of the curve $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=1$

[GGSIPU I Sem III End Term Sep. 2007; End Term 2011]

(b) $y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0.$

[GGSIPU I Sem End Term 2009]

SOLUTION: (a) Equation of the curve is $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=1$ or $x^2y^2 - a^2y^2 - b^2x^2 = 0$ ...(1)

For asymptotes parallel to X-axis we equate to zero the coefficient of x^2, to get

$$y^2 - b^2 = 0 \quad \text{or} \quad y = \pm b.$$

For asymptotes parallel to Y-axis we equate to zero the coefficient of y^2, to get

$$x^2 - a^2 = 0 \quad \text{or} \quad x = \pm a.$$

Thus, we have four asymptotes of the curve as $x = \pm b$ and $y = \pm a$. **Ans.**

(b) In the equation of the given curve

$\phi_3(m) = m^3 - 2m^2 - m + 2,\quad \phi_2(m) = -7m + 3m^2 + 2,\quad \phi_1(m) = 2 + 2m.$

For slopes of the asymptotes we have $\phi_3(m) = 0$ or $m^3 - 2m^2 - m + 2 = 0$

or $(m + 1)(m^2 - 3m + 2) = 0 \quad \therefore \quad m = -1, 1, 2.$

For 'c' of the asymptotes, we have $c = \dfrac{-\phi_3(m)}{\phi_3(m)} = \dfrac{-(3m^2-7m+2)}{3m^2-4m-1}$

When $m = -1$, $\quad c = \dfrac{-(3+7+2)}{3+4-1} = -2$

When $m = 1$, $\quad c = \dfrac{-(3-7+2)}{3-4-1} = -1$

and when, $m = 2$, $\quad c = -\dfrac{(12-14+2)}{12-8-1} = 0$

$\therefore$ Asymptotes are $y = -x - 2$, $y = x - 1$ and $y = 2x$. **Ans.**

EXAMPLE 3.5. Determine the asymptotes, if any, of the curve

(i) $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0.$ [GGSIPU I Sem End Term 2007; End Term 2003]

(ii) $x^2y + xy^2 - xy + y^2 + 3x = 0.$ [GGSIPU I Sem I Term 2013]

SOLUTION: (i) The coefficient of the highest degree term, that is, x^3 is 4 hence no asymptote is there parallel to x-axis. Similarly, the coefficient of the highest degree term in y is y^3 whose coefficient is -1, hence no asymptote parallel to y-axis.

Next, here $\phi_3(m) = 4\cdot 1^3 - 3\cdot 1\cdot m^2 - m^3 = -m^3 - 3m^2 + 4$

and $\phi_2(m) = 2\cdot 1^2 - 1\cdot m - m^2 = -m^2 - m + 2$ and $\phi_1(m) = 0.$

The slopes of the asymptotes are given by the equation $\phi_3(m) = 0$

or $m^3 + 3m^2 - 4 = 0$ or $(m-1)(m+2)^2 = 0$ hence $m = 1, -2, -2$

The 'c' of the asymptote corresponding to $m = 1$, is given by

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \left(\frac{m^2+m-2}{-3m^2-6m}\right)_{m=1} = 0$$

Therefore one asymptote has equation $y = x$. Since the other two roots of the equation $\phi_3(m) = 0$, are equal, this is the case of a pair of parallel asymptotes. The values of 'c' for these parallel asymptotes are given by $\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$

or $\quad \frac{c^2}{2}(-6m-6) + c(-2m-1) + 0 = 0 \quad$ or $\quad 3c^2(m+1) + c(2m+1) = 0$

Putting $m = -2$, in this equation, we get $-3c^2 - 3c = 0$ which gives $c = 0, -1$

Therefore, the equations of the parallel asymptotes are $y = -2x + 0$ and $y = -2x - 1$

Thus, the given curve has three asymptotes with equations

$$y = x, \quad y = -2x \quad \text{and} \quad y = -2x - 1. \qquad \textbf{Ans.}$$

(ii) Here $\phi_3(m) = m + m^2$, $\phi_2(m) = m + m^2$, $\phi_1(m) = 3$.

For slope of the asymptotes of the curve we have $\phi_3(m) = 0$

or $m^2 + m = 0$ which gives $m = 0, -1$.

For 'c' of the asymptote we have $c = \frac{-\phi_2(m)}{\phi_3'(m)} = -\frac{(m+m^2)}{2m+1} = 0$ for $m = 0$

and $c = -\frac{(-1+1)}{-2+1} = 0$ for $m = -1$.

Hence, there are two asymptotes as $y = 0$ and $y = -x$. **Ans.**

EXAMPLE 3.6. Determine the asymptotes, if any, of the curve

(i) $y(x-y)^2 = x + y$. [*GGSIPU I Sem I Term 2011*]

(ii) $(x+y)^2(x+2y+2) = x + 9y - 2$. [*GGSIPU I Sem I Term 2009; End Term 2004*]

SOLUTION: (i) The given curve is of degree three, hence

$$\phi_3(m) = m(1-m)^2, \quad \phi_2(m) = 0 \quad \text{and} \quad \phi_1(m) = -1 - m.$$

Now $\phi_3(m) = 0$ gives $m = 0, 1, 1$ as the slopes of the asymptotes.

Since $\quad C = \frac{-\phi_2(m)}{\phi_3'(m)} = 0$, the asymptote corresponding to $m = 0$, is $y = 0$.

For 'C', corresponding to equal roots $m = 1, 1$, we have

$$\frac{C^2}{2}\phi_3''(m) + \frac{C}{1}\phi_2'(m) + \phi_1(m) = 0. \qquad \text{...(1)}$$

Here $\quad \phi_3'(m) = 3m^2 - 4m + 1$ and $\phi_3''(m) = 6m - 4$.

Therefore (1) becomes $\frac{C^2}{2}(6m-4) + C(0) - m - 1 = 0$

At $m = 1$ we have $C^2 = 2$ hence $C = \pm\sqrt{2}$.

Thus, the asymptotes are $y = 0$, $y = x + \sqrt{2}$ and $y = x - \sqrt{2}$. **Ans.**

(ii) Equation of the curve can be written as $(x+y)^2(x+2y)+2(x+y)^2-x-9y+2=0$

Here $\phi_3(m)=(1+m)^2(1+2m)$, $\phi_2(m)=2(1+m)^2$ and $\phi_1(m)=-1-9m$.

The slopes of the asymptotes are obtained from $\phi_3(m)=0$

or $(1+m)^2(1+2m)=0$ hence $m=-1,-1,-1/2$.

For $m=-1/2$, we have $c\,\phi'_3(m)+\phi_2(m)=0$

or $$c=\left[\frac{-\phi_2(m)}{\phi'_3(m)}\right]_{m=\frac{-1}{2}}=\left[\frac{-2(1+m)^2}{2(1+m)^2+2(1+m)(1+2m)}\right]_{m=\frac{-1}{2}}=-1$$

For $m=-1$, the repeated root, we have $\frac{c^2}{2}\phi''_3(m)+c\phi'_2(m)+\phi_1(m)=0$

or $$\frac{c^2}{2}\phi''_3(-1)+c\phi'_2(-1)+\phi_1(-1)=0 \quad\text{or}\quad \frac{c^2}{2}\,2(-3+2)+c\,4(1+m)-1+9=0$$

or $$-c^2+0+8=0,\quad c=\pm 2\sqrt{2}$$

Hence the asymptote corresponding to $m=\frac{-1}{2}$ is $y=\frac{-1}{2}x-1$ and the two asymptotes corresponding to $m=-1$, are $y=-x\pm 2\sqrt{2}$. **Ans.**

EXAMPLE 3.7. Find the asymptotes of the curve $x^3+3x^2y-4y^3-x+y+3=0$.

[GGSIPU I Sem I Term 2004, I Term 2010]

SOLUTION: Equation of the curve is $x^3+3x^2y-4y^3-x+y+3=0$.

Here $\phi_3(m)=1+3m-4m^3$, $\phi_2(m)=0$, $\phi_1(m)=-1+m$, $\phi_0(m)=3$.

To find the slopes 'm' of asymptotes we write $\phi_3(m)=0$

or $4m^3-3m-1=0$ or $(m-1)(4m^2+4m+1)=0$

which gives $m=1,-1/2,-1/2$.

For 'c' of the asymptote corresponding to $m=1$, we have

$c\phi'_3(m)+\phi_2(m)=0$ or $c(3-12m^2)-0=0$ hence $c=0$

For $m=-1/2$, we have two parallel asymptotes whose 'c' is given by

$$\frac{c^2}{2!}\phi''_3(m)+\frac{c}{1!}\phi'_2(m)+\phi_1(m)=0$$

or $$\frac{c^2}{2}(-24m)+c(0)+(-1+m)=0 \quad\text{or}\quad -c^2(12m)-1+m=0.$$

For $m=\frac{-1}{2}$ we have $6c^2-\frac{3}{2}=0$ or $c=\pm\frac{1}{2}$.

Thus we have three asymptotes as $y=x$, $y=\frac{-1}{2}x+\frac{1}{2}$ and $y=\frac{-1}{2}x-\frac{1}{2}$. **Ans.**

EXAMPLE 3.8 Find the asymptotes of the curve given by the equation

$$y^4-2xy^3+2x^3y-x^4-3x^3+3x^2y+3xy^2-3y^3-2x^2+2y^2-1=0$$

[GGSIPU I Sem End Term Jan. 2011]

SOLUTION: The equation of the curve is of degree 4 hence it cannot have more than four asymptotes. From the equation of the curve, we have

$$\phi_4(m) = m^4-2\cdot 1\, m^3+2\cdot 1^3 m-1^4 = m^4-2m^3+2m-1$$

$$= (m-1)(m^3-m^2-m+1) = (m-1)^2(m^2-1) = (m-1)^3(m+1),$$

$$\phi_3(m) = -3\cdot 1^3+3\cdot 1^2\cdot m+3\cdot 1\cdot m^2-3m^3 = -3m^3+3m^2+3m-3,$$

$$\phi_2(m) = -2\cdot 1^2+2m^2 = 2(m^2-1), \quad \text{and} \quad \phi_1(m)=0.$$

For the slopes of the asymptote, we have $\phi_4(m)=0$ or $(m-1)^3(m+1)=0$

whose roots are $m=1,\ 1,\ 1$ and -1. Thus, of the four asymptotes, three are parallel.

The 'c' of the asymptote with slope -1, is given by

$$c = -\frac{\phi_3(m)}{\phi_4'(m)} = \frac{-3(-m^3+m^2+m-1)}{3(m-1)^2(m+1)+(m-1)^3\cdot 1} = 0 \quad \text{at} \quad m=-1.$$

Therefore, the asymptote corresponding to $m=-1$ is $y=-x$.

To find the values of 'c' for the three parallel asymptotes corresponding to $m=1$, we have

$$\frac{c^3}{3!}\phi_4'''(m)+\frac{c^2}{2!}\phi_3''(m)+c\,\phi_2'(m)+\phi_1(m)=0$$

or $$\frac{c^3}{6}(24m-12)+\frac{c^2}{2}(-18m+6)+c\cdot 4m+0=0$$

or $$c[(4m-2)c^2+(-9m+3)c+4m]=0.$$

Putting $m=1$ we get $c=0$ and $2c^2-6c+4=0$ or $(c-1)(c-2)=0$

Thus, the three values of c for $m=1$, are $c=0,\ 1,\ 2.$

Therefore, the four asymptotes are

$$y=-x, \quad y=x, \quad y=x+1 \quad \text{and} \quad y=x+2.$$

Ans.

EXERCISE 3A

Find the asymptotes of the following curves

1. $x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0.$
2. $x^3-4y^3+3x^2y+y-x+3=0.$
3. $y^3+2xy^2+x^2y-y+1=0.$
4. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1.$
5. $(x^3+a^3)\,y=bx^3.$
6. $y\,(x-y)^3=y\,(x-y)+2.$
7. $x^3-5x^2y+8xy^2-4y^3+x^2-3xy+2y^2-1=0.$
8. $r\theta=a.$
9. Find all the asymptotes of the curve $y^2\,(x-2a)=x^3-a^3.$
10. Find all the asymptotes of the curve $x^2y^2-x^2y-xy^2+x+y+1=0.$

[*GGSIPU I Sem End Term 2008, I Sem End Term 2009*]

11. Find the asymptotes of the curve $y^3-xy^2-x^2y+x^3+x^2-y^2=0.$

CURVE TRACING

For computing the areas, lengths, volume of solids of revolution and surface of solids of revolution, it is very important and useful to know the shape of the curve represented by the given equation. Curve tracing tit bits help us in obtaining the rough shape of the curve without plotting a large number of points on it. In addition to it, the properties of some of the standard curves which frequently occur in engineering applications, will be discussed here.

TRACING OF CARTESIAN CURVES

We give below some of the main rules which definitely help in determining the shape of the curves from their equations.

1. Symmetry

(*a*) If powers of y in the equation of the curve, are all even, then the curve is symmetrical about x-axis, *i.e.*, whatever is the shape of the curve on one side of x-axis, same should be on the other side of it, because for one value of x we get two values of y which differ in sign only. For example, the curve $y^2 = 4ax$ is symmetrical about x-axis.

(*b*) If all the powers of x in the equation are even, the curve is symmetrical about y-axis, *i.e.*, whatever is the shape of the curve on one side of y-axis, same should be on the other side of it, because for one value of y we get two values of x which differ in sign only. For example, the curve $x^2 = 4ay$ is symmetrical about y-axis.

(*c*) If on interchanging x and y, the equation of the curve remains unchanged, the curve is symmetrical about the line $y = x$. For example, the curves $xy = c^2$ and $x^3 + y^3 = 3\,axy$ both are symmetrical about the line $y = x$.

(*d*) If x is changed to $-x$ and y to $-y$ and the equation of the curve remains unchanged, then the curve is symmetrical in the opposite quadrants, *i.e.*, whatever is the shape of the curve in the first quadrant same is in the third quadrant and similarly, for the second and fourth quadrants.

For example, the curves $xy = c^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are symmetrical in the opposite quadrants.

2. The Origin

(*a*) If there is no constant term in the equation, the curve passes through the origin.

(*b*) If the curve is passing through the origin, we can find the equation of the tangent to the curve at the origin, by equating to zero the lowest degree term in the equation. For example, the curve $y^2 = 4ax$ has y-axis as tangent at the origin and the curve $y = x^3$ has x-axis as tangent to the curve at the origin.

(*c*) If there are two or more tangents to the curve at the origin, it is called a *multiple point*. Further, the origin is called a *node*, a *cusp* or an *isolated point according* as the tangents there are real and distinct, real and coincident or imaginary. For example, in the curve $y^2(a^2 + x^2) + x^2(a^2 - x^2) = 0$, the origin is an isolated point and for the curve $y^2(a - x) = x^3$ the origin is a cusp.

3. Intersection with the Co-ordinate Axes

(*a*) Find the points where the curve intersects the x-axis and the y-axis separately.

(*b*) Find the tangent to the curve at its point of intersection with the coordinate axes by first shifting the origin to this point and then equating to zero the lowest degree term.

4. Region (s) of Absence of Curve

If possible, express y in terms of x or x in terms of y explicitly, and examine how y varies as x varies and how x varies as y varies. On solving for y, say, in terms of x, suppose we find that between $x = a$ and $x = b$, y is imaginary, then the curve does not exist in the region between the lines $x = a$ and $x = b$. For example, in the curve $y^2(1-x) = x^3$, y is imaginary when $x < 0$ and when $x > 1$ hence no curve lies on the right of the line $x = 1$ and on the left of y-axis.

5. Maxima, Minima and Points of Inflexion.

Find $\frac{dy}{dx}$ and equate it to zero to get the critical points at which y may be maximum or minimum.

Also find $\frac{d^2y}{dx^2}$ and equate it to zero to get the possible points of inflexion.

6. Asymptotes

Find the asymptotes parallel to x-axis (or y-axis) by equating to zero the coefficient of the highest degree term in x (or y). For example, in the curve $y^2(1-x) = x^3$ the asymptote parallel to y-axis is the line $x = 1$.

To obtain oblique asymptotes put $y = mx + c$ in the equation of the curve and then equate to zero the coefficients of highest and the next highest degree terms in x, this will give equations to find m and c of the equations of asymptotes.

7. Conversion to Polar Form

If needed, transform the given equation into polar coordinates r and θ by putting $x = r\cos\theta$, $y = r\sin\theta$. The polar form of some curves is more convinient to handle than their cartesian form.

EXAMPLE 3.9. Trace the curve $a^2x^2 = y^3(2a - y)$.

SOLUTION: (*a*) The equation of the curve is $a^2x^2 = y^3(2a-y)$.

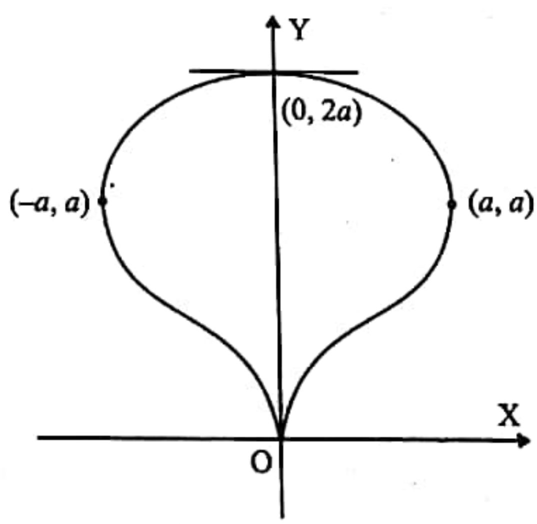
(*i*) In the given equation of the curve all the powers of x are even, hence the curve is symmetrical about y-axis.

(*ii*) Since there is no constant term in the equation, the curve passes through the origin. Now, equating to zero the lowest degree term, gives the tangent to the curve at the origin as $x^2 = 0$, *i.e.*, y-axis. Here origin is a cusp.

(*iii*) The curve meets the y-axis at $(0, 0)$ and at $(0, 2a)$. Shifting the origin to the point $(0, 2a)$ the equation of the curve becomes $a^2x^2 = -(y + 2a)^3 y$. Equating to zero the lowest degree term gives $y = 0$. Therefore, tangent at $(0, 2a)$ is parallel to x-axis.

(*iv*) Since $x^2 < 0$ for $y > 2a$ and for $y < 0$, there is no portion of the curve below the x-axis and above the line $y = 2a$.

(*v*) Putting $y = a$ we get $x = \pm a$. Giving suitable values to y like $y = \frac{a}{2}$ and $\frac{3a}{2}$ we can find corresponding values of x and such points give the shape of the curve, as shown in the adjoining figure.



EXAMPLE 3.10. Trace the curve $9ay^2 = x(x-3a)^2$ [GGSIPU I Sem I Term 2013]

SOLUTION: The equation of the curve is $4ay^2 = x(x-2a)^2$.

(*i*) Since all powers of y in the given equation are even, the curve is symmetrical about the x-axis.

(*ii*) There is no constant term in the equation hence the curve passes through the origin. Equating to zero the lowest degree term we get $x = 0$ as the tangent to the curve at the origin, *i.e.*, y-axis.

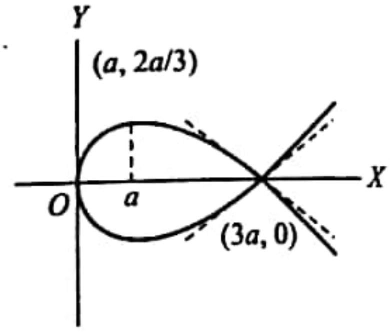
(*iii*) The curve meets the y-axis at the origin only and the x-axis at $A(3a, 0)$. Shifting the origin to the point $(3a, 0)$ the equation of the curve becomes $9ay^2 = (x+3a)x^2$ and now equating to zero the lowest degree term we get $y = \pm \dfrac{x}{\sqrt{3}}$ as the tangent to the curve at the point A.

(*iv*) From the given equation of the curve, we note that y is imaginary when x is negative hence no portion of the curve lies on the left of y-axis.

(*v*) From the equation of the curve we have

$$y = \pm \frac{1}{3}\frac{(x-3a)}{3}\sqrt{\frac{x}{a}}$$

$$\therefore \quad \frac{dy}{dx} = \pm\left[\frac{x-3a}{3}\,\frac{1}{2\sqrt{ax}} + \frac{1}{3}\sqrt{\frac{x}{a}}\right] = \frac{x-a}{2\sqrt{x}}$$



Now $\dfrac{dy}{dx} = 0$ at $x = a$ for which y is maximum.

The maximum value of y is given by $y = \dfrac{2a}{3}$.

(*vi*) Beyond $x = 3a$, y increases monotonically as x increases.

(*vii*) Give suitable values to x like $x = a, 2a, 4a$ and find the corresponding values of y.

EXAMPLE 3.11. Trace the curve $y^2(x+a) = x^2(3a-x)$

SOLUTION: The equation of the curve is $y^2(x+a) = x^2(3a-x)$.

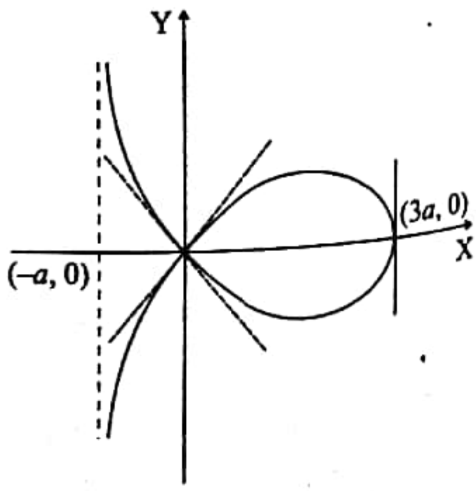
(*i*) The curve is symmetrical about the axis of x.

(*ii*) The curve passes through the origin. Equating to 0 the lowest degree term, we get $ay^2 = 3ax^2$ or $y = \pm x\sqrt{3}$ as two tangents to the curve at the origin.

(*iii*) The curve meets the y-axis only at the origin while it meets x-axis at $(3a, 0)$ also. Shifting the origin to this point the equation of the curve becomes

$$y^2(x+4a) = -(x+3a)^2 x,$$

and now equating to 0 the lowest degree term we get $x = 0$, the y-axis, as the tangent there. Thus, at $(3a, 0)$ tangent is parallel to y-axis.



(*iv*) y is imaginary for $x < -a$ and for $x > 3a$, therefore there is no portion of the curve on the left of the line $x = -a$ and on the right of the line $x = 3a$.

(*v*) There is no asymptote parallel to x-axis, and for asymptote parallel to y-axis we equate to zero the coefficient of the highest degree term in y so as to get $x + a = 0$ as the asymptote parallel to y-axis.

(*vi*) Differentiating the given equation, we get

$$2y\frac{dy}{dx}(x+a) + y^2\cdot 1 = 6ax - 3x^2 \quad \text{or} \quad 2y(x+a)^2\frac{dy}{dx} + y^2(x+a) = (6ax-3x^2)(x+a)$$

or $$2y(x+a)^2\frac{dy}{dx} = 3x(2a-x)(a+x) - x^2(3a-x) = 2x(3a^2-x^2).$$

For maxima $\dfrac{dy}{dx} = 0$ and we get $x = \pm a\sqrt{3}$ and $x = 0$.

Clearly, y is maximum at $x = a\sqrt{3}$.

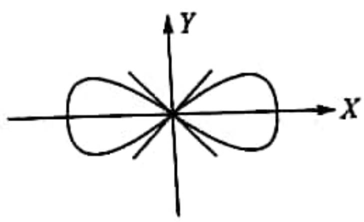
(*vii*) Give suitable values to x like $x = -a/2,\ a,\ 2a$ and find the corresponding values of y, plot such points and get the rough shape of the curve as shown in the above figure.

EXAMPLE 3.12. (a) Trace the curve $a^2y^2 = a^2x^2 - x^4$. [*GGSIPU I Sem End Term 2011*]

(b) Trace the curve $x^2(x^2 + y^2) = a^2(x^2 - y^2)$ or $y^2(a^2 + x^2) = x^2(a^2 - x^2)$.

[*GGSIPU I Sem I Term 2012; End Term 2013*]

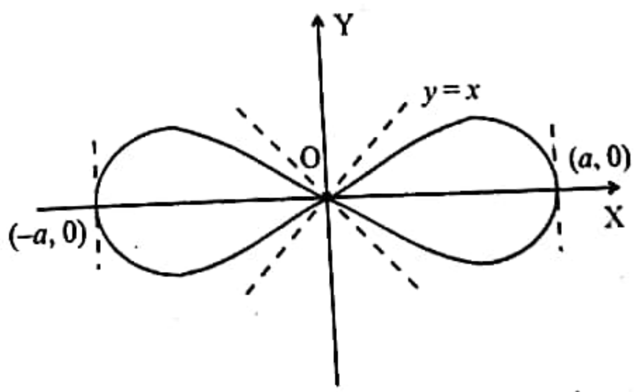
SOLUTION: (a) $a^2y^2 = x^2(a^2 - x^2)$ is symmetrical about X-axis and Y-axis and passes through the origin. Tangents at $(0, 0)$ are $y \pm x$ and curve meets X-axis at $x = \pm a$. The tangent at $(a, 0)$ is Y-axis.



The shape of the curve is given in the adjoining figure:

(b) (*i*) The curve is symmetrical about both the axes.

(*ii*) It passes through the origin and equating to zero the lowest degree term we get $x^2 - y^2 = 0$, or $y = \pm x$ as tangents to the curve at the origin.



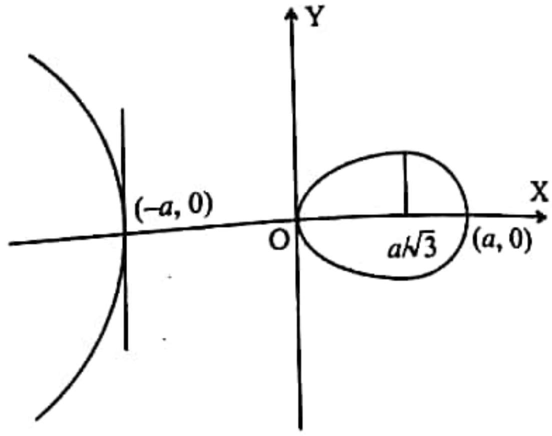
(*iii*) The curve does not meet the y-axis except at the origin but meets x-axis at the $(a, 0)$ and at $(-a, 0)$. Shifting the origin to these points we find that the tangents at $(\pm a, 0)$ are parallel to y-axis.

(*iv*) We can rewrite the equation of the curve as $y^2(a^2+x^2) = x^2(a^2-x^2)$. Now y^2 is negative for $x > a$ and for $x < -a$, therefore, there is no portion of the curve on the right of the line $x = a$ and on the left of the line $x = -a$.

(*v*) Give suitable values to x like $x = a/3,\ a/2,\ 2a/3$ and get the corresponding values of y and plot the points to get the shape of the curve as given in the above figure having two loops.

EXMAPLE 3.13. Trace the curve $ay^2 = x(a^2 - x^2)$.

SOLUTION: The equation of the curve is $ay^2 = x(a^2 - x^2)$.



(i) The curve is symmetrical about x-axis.

(ii) It passes through the origin and equating to 0 the lowest degree term we get $x = 0$, the y-axis, as the tangent to the curve at the origin.

(iii) The curve does not meet the y-axis except at the origin. It meets the x-axis at $(-a, 0)$ and at $(a, 0)$.

Shifting the origin to the point $(a, 0)$, the equation of the curve becomes $ay^2 = (x + a)(-2ax - x^2)$. Now, equating to 0 the lowest degree term, we get $x = 0$, therefore the tangent at $(a, 0)$ is parallel to y-axis.

Similarly, shifting the origin to the point $(-a, 0)$, the equation of the curve becomes $ay^2 = (x - a)(2ax - x^2)$ and equating to 0 the lowest degree term here, we get $x = 0$, hence tangent to the curve at $(-a, 0)$ is also parallel to y-axis.

(iv) Since y is imaginary for $x > a$ and also for $-a < x < 0$, hence there is no portion of the curve on the right of the line $x = a$ and between the lines $x = -a$ and $x = 0$.

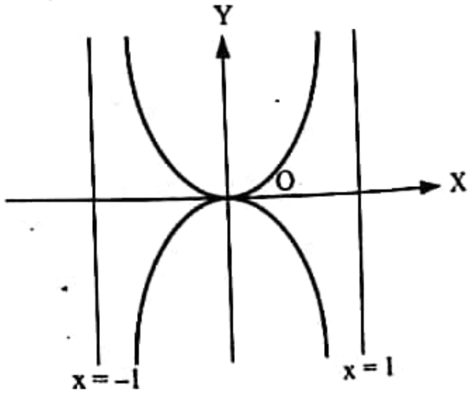
(v) Differentiating the given equation, gives

$$2ay\frac{dy}{dx} = a^2 - 3x^2 \quad \text{and,} \quad \frac{dy}{dx} = 0 \text{ gives } x = \frac{a}{\sqrt{3}} \text{ which is the maxima.}$$

(vi) Now, give suitable values to x like $x = \frac{a}{2}, -2a, -3a$ and find the corresponding values of y and plot these points and the shape of the curves is as shown in the above figure.

EXAMPLE 3.14. Trace the curve $y^2(1 - x^2) = x^2(1 + x^2)$ [GGSIPU I Sem End Term 2012]

SOLUTION: The curve is symmetrical about X-axis and Y-axis. Curve passes through the origin and tangents to the curve at the origin are $y = \pm x$.



Two asymptotes parallel to y-axis are $x = \pm 1$.

No curve is there when $x > 1$ and also when $x < -1$.

Shape of the curve is shown in the adjoining figure.

EXAMPLE 3.15. Trace the curve $y^2(a^2 - x^2) = a^3 x$.

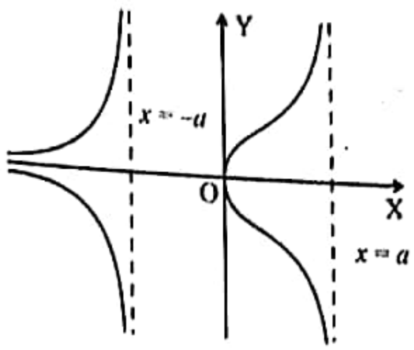
SOLUTION: The equation of the curve is $y^2(a^2 - x^2) = a^3 x$.

(i) The curve is symmetrical about the axis of x.

(ii) It passes through the origin and the tangent to the curve at the origin is $x = 0$, the y-axis.

(iii) The curve does not meet the coordinate axes except at the origin.

(iv) For the asymptotes parallel to x-axis we equate to zero the coefficient of the highest degree term in x and get $y^2 = 0$, hence x-axis itself is the asymptote. For asymptotes parallel to y-axis, we equate to zero the coefficient of the highest degree term in y and get $a^2 - x^2 = 0$ therefore both the lines $x = a$ and $x = -a$ are asymptotes to the curve parallel to y-axis.



(v) Value of y is imaginary for $x > a$ and for $-a < x < 0$, therefore there is no portion of the curve on the right of the line $x = a$ and between the lines $x = -a$ and $x = 0$.

(vi) Give the suitable values to x like $x = -3a, -2a, -3a/2, a/2$ and get the corresponding values of y and plot the points to get the rough shape of the curve as shown in the above figure.

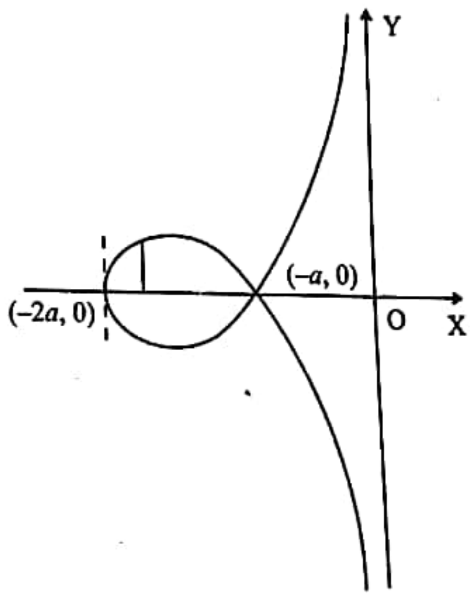
EXAMPLE 3.16. Trace the curve $xy^2 + (x + a)^2 (x + 2a) = 0$ [GGSIPU I Sem II Term 2009]

SOLUTION:

(i) The curve is symmetrical about x-axis.

(ii) The curve does not pass through the origin and meets the x-axis at $(-2a, 0)$ and at $(-a, 0)$.

Shifting the origin to $(-a, 0)$ the equation of the curve becomes $(x - a) y^2 + x^2 (x + a) = 0$ and equating to zero the lowest degree term we get $y = \pm x$ as tangents at the new origin. Similarly, shifting the origin to the point $(-2a, 0)$, the given equation of curve becomes $(x - 2a) y^2 + (x - a)^2 x = 0$ and then equating to zero the lowest degree term we get $x = 0$ the new y-axis as tangent there.



(iii) For asymptotes parallel to y-axis, we equate to zero the coefficient of the highest degree term in y and get y-axis as the asymptote.

(iv) From the given equation of the curve we observe that

$y = \pm (x + a)\sqrt{-\dfrac{(x + 2a)}{x}}$ and hence y is imaginary when $x < -2a$ and also when $x > 0$. Thus, there is no portion of the curve on the left of $x = -2a$ and on the right of y-axis.

(v) Differentiating the given equation and equating $\dfrac{dy}{dx}$ to zero, we get maxima at $x = \dfrac{-a\left(1 + \sqrt{5}\right)}{2}$.

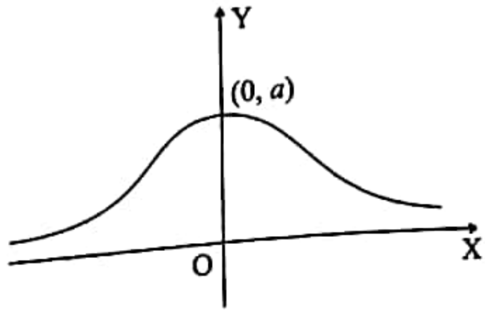
(vi) Give suitable values to x, like $x = -3a/2, -a/2$ and find the corresponding points on the curve and get the shape of the curve as shown in the above figure.

EXAMPLE 3.17. Trace the curve $y(x^2 + a^2) = a^3$.

SOLUTION:

(*i*) The curve is symmetrical about y-axis.

(*ii*) The curve does not meet x-axis but meets y-axis at $(0, a)$. Shifting the origin to $(0, a)$ the equation becomes $(y + a)(x^2 + a^2) = a^3$ or $y(x^2 + a^2) + ax^2 = 0$ and now equating to zero the lowest degree term we get $y = 0$, the x-axis as the tangent there.



(*iii*) Solving the given equation for x we get, $x = \pm a\sqrt{\frac{a-y}{y}}$ which shows that x is imaginary when $y<0$ and also when $y>a$. Thus, there is no curve below x-axis and above the line $y=a$.

(*iv*) Equating to 0 the coefficient of the highest degree term in x we get $y=0$, the x-axis, as asymptote of the curve.

(*v*) Give suitable values to x, like $x=a/2$, a, $2a$ and obtain the corresponding points on the curve to get the rough shape of the curve as shown in the above figure.

EXAMPLE 3.18. Trace the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.

SOLUTION:

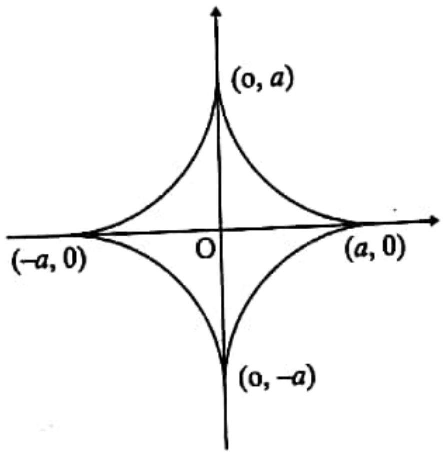
(*i*) If we write the equation of the curve as $(x^{1/3})^2 + (y^{1/3})^2 = a^{2/3}$ it is clear that the curve is symmetrical about x-axis and y-axis both.

(*ii*) The curve does not pass through the origin but meets the x-axis at $(\pm a, 0)$ and the y-axis at $(0, \pm a)$.

(*iii*) Differentiating the given equation, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

At $(\pm a, 0)$ the slope of the tangent is 0 hence x-axis is the tangent at these points. At $(0, \pm a)$ the slope of the tangent is infinity, hence y-axis is tangent at these points.



(*iv*) We can write the given equation of the curve in parametric form as $x = a\cos^3 t$, $y = a\sin^3 t$ which shows that $-a \le x \le a$ and $-a \le y \le a$. Therefore, no point of the curve lies outside the square within the lines $x = \pm a$, $y \pm a$.

(*v*) Give suitable values to x like $x = a/3$, $a/2$, $2a/3$ and find points on the curve and get the shape of the curve as shown in the adjoining figure.

CURVES IN PARAMETRIC FORM

Following few points will help in tracing the curves whose equations are given in parametric form or can be expressed in parametric form: $x = f_1(t)$, $y = f_2(t)$.

1. (*a*) If $x = f_1(t)$ is an even function of t and $y = f_2(t)$ is an odd function of t, the curve is symmetrical about x-axis.

   (*b*) If $x = f_1(t)$ is an odd function of t and $y = f_2(t)$ is an even function of t, the curve is symmetrical about y-axis, because if (x, y) is a point on the curve then $(-x, y)$ also lies on the curve.

2. If, for some value of t, both x and y become zero then the curve passes through the origin.
3. (a) Find the value of t for which $f_2(t) = 0$ and for this value of t, find $x = f_1(t)$ and get the point where the curve meets the x-axis.
   (b) Find the value of t for which $f_1(t) = 0$ and for this value of t, find $y = f_2(t)$ and get the point where the curve meets the y-axis
4. If possible find the greatest and the least values of x and y which give the region or regions in which the curve lies.
5. Find $\frac{dy}{dx}\left(=\frac{dy}{dt}\Big/\frac{dx}{dt}\right)$ and find the points on the curve where the tangent to the curve is parallel to either coordinate axis.
6. Find the range of the parameter t for which x or y cannot be obtained and this will indicate the region in which the curve does not lie.
7. Find the possible asymptotes of the curve, parallel to the coordinate axes.

EXAMPLE 3.19. Trace the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

SOLUTION:

(i) Here θ is the parameter and x is an odd function of θ, and y an even function of θ hence the curve is symmetrical about y-axis.

(ii) For $\theta = 0$ we have $x = 0$ and $y = 0$ hence the curve passes through the origin.

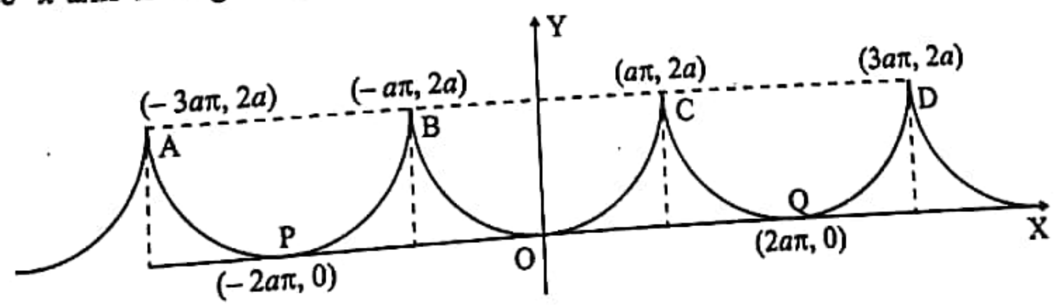
(iii) The curve meets y-axis at the origin only while it meets the x-axis where $1 - \cos\theta = 0$ *i.e.*, at $\theta = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$ Thus the curve meets the x-axis at $(0, 0)$, $(\pm 2a\pi, 0)$, $(\pm 4a\pi, 0)$, $(\pm 6a\pi, 0)$, etc.

(iv) Observe here that $0 \le y \le 2a$ therefore the curve lies only within the strip $y = 0$ and $y = 2a$.

(v) From the equation of the curve, we have $\frac{dx}{d\theta} = a(1 + \cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$

hence $\frac{dy}{dx} = \tan\theta/2$. Now $\frac{dy}{dx} = 0$ at $\theta = 0, \pm 2\pi, \pm 4\pi, \ldots$

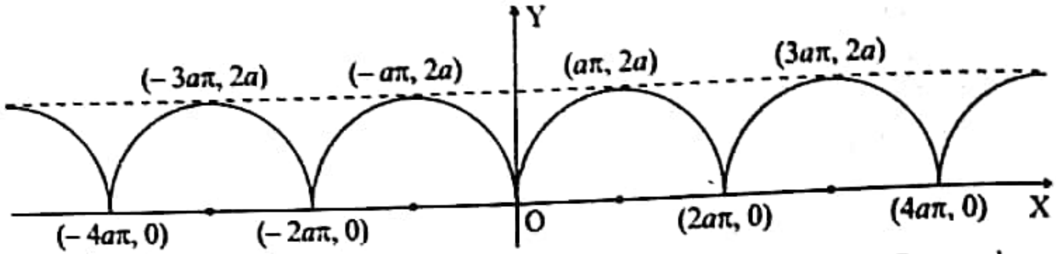
hence x-axis is tangent to the curve at its points $(0, 0)$, $(\pm 2a\pi, 0)$ $(\pm 4a\pi, 0)$, ...



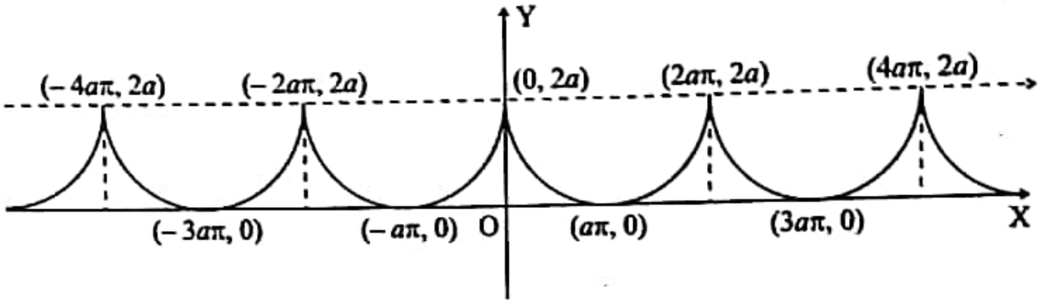
At $\theta = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ $\frac{dy}{dx}$ is infinite, hence at the point $(\pm a\pi, 2a)$, $(\pm 3a\pi, 2a)$, $(\pm 5a\pi, 2a)$, ... tangent is parallel to y-axis.

Curve APB is one cycloid, BOC is another cycloid, CQD is yet another cycloid, and so on as shown in the above figure. **There are other cycloids also as given below:**

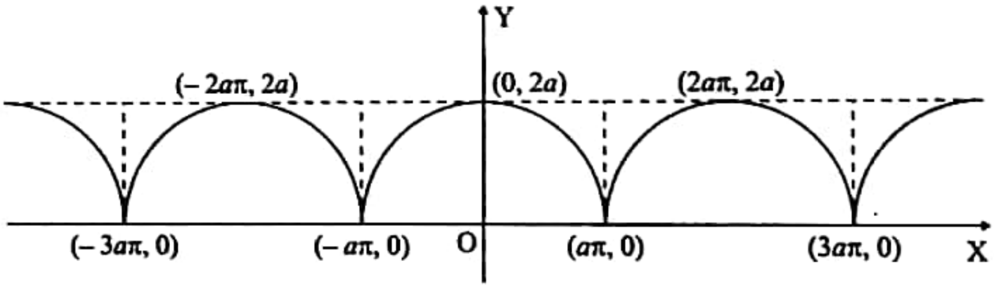
Similarly, the shape of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is as shown in figure given below and these are called inverted cycloids.



Also, the shape of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$ is shown in the figure given below.

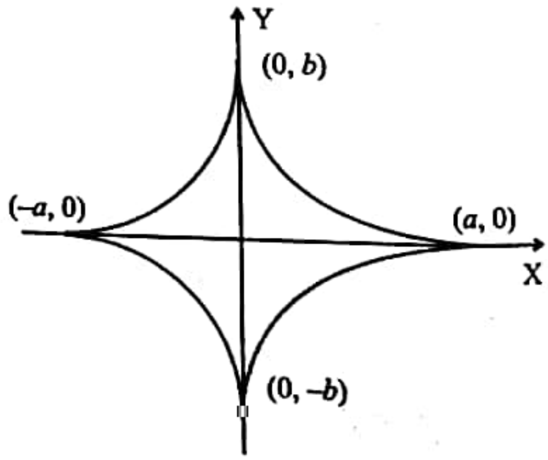


And the shape of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ is as shown in the figure given below.



EXAMPLE 3.20. Trace the hypocycloid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1.$

SOLUTION: The given equation of the curve can be written in parametric form as $x = a\cos^3\theta$, $y = b\sin^3\theta$ where θ is the parameter.



(*i*) x is an even function of θ and y is an odd function of θ hence the curve is symmetrical about x-axis.

Also, if θ is changed to $(\pi - \theta)$, y remains the same but x becomes $-x$, therefore the curve is symmetical about y-axis.

The symmetry of the curve about both the axes can be observed also by rewriting the given equation as

$$\left[(x/a)^{1/3}\right]^2 + \left[(y/b)^{1/3}\right]^2 = 1.$$

(*ii*) Since $-a \le x \le a$ and $-b \le y \le b$, the entire curve lies within the rectangle $x = \pm a$, $y = \pm b$.

(*iii*) The curve does not pass through the origin and meets the x-axis at $(\pm a, 0)$ and y-axis at $(0, \pm b)$.

(*iv*) $$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = \frac{-b}{a}\tan\theta.$$

which is 0 at $(\pm a, 0)$ and is infinite at $(0, \pm b)$. Hence x-axis is the tangent to the curve at $(\pm a, 0)$ and y-axis is the tangent to the curve at $(0, \pm b)$.

(v) Next, give suitable values to the parameter θ as $\theta = \pi/6,\ \pi/4,\ \pi/3$ and get the corresponding points on the curve and get the rough shape of the curve as given above.

EXAMPLE 3.21. Trace the tractrix $x = a\,(\cos t + \log|\tan(t/2)|),\quad y = a \sin t.$

SOLUTION: We can rewrite the given equation as $x = a\left(\cos t + \frac{1}{2}\log\tan^2\frac{t}{2}\right),\ y = a\sin t.$

(i) The curve is symmetrical about x-axis, as x is an even function of t and y is an odd function of the parameter t.

(ii) When we put $t = \pm\pi/2$, we get $x = 0$ and $y = \pm a$, hence the curve meets the y-axis at the points $(0, \pm a)$.

(iii) Since $-a \le y \le a$, therefore the curve lies wholly within the strip $y = \pm a$.

(iv) As t approaches to 0, x tends to infinity and y tends to 0, therefore the x-axis is an asymptote to the curve.

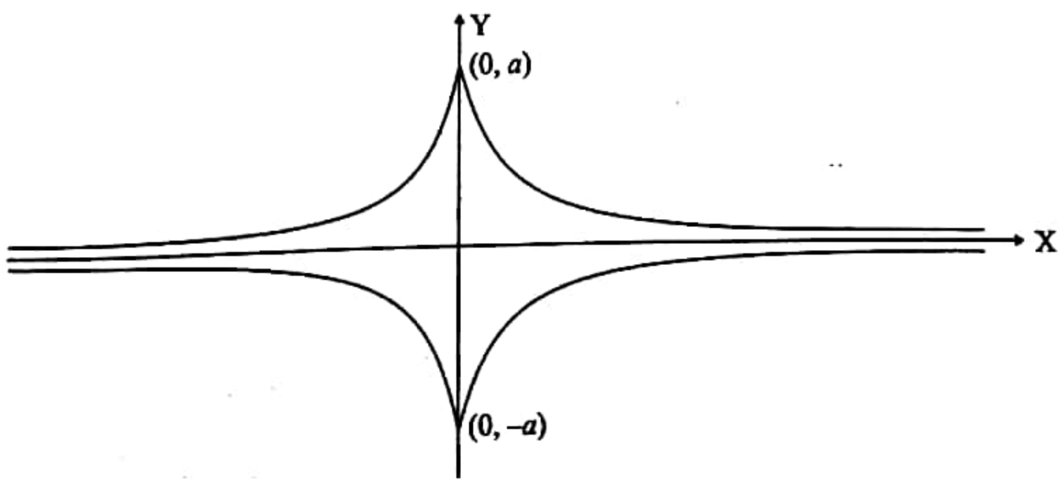
(v) Next, $\dfrac{dx}{dt} = a\left(-\sin t + \dfrac{\sec^2 t/2}{2\tan t/2}\right) = a\left(-\sin t + \dfrac{1}{\sin t}\right) = \dfrac{a\cos^2 t}{\sin t}$ and $\dfrac{dy}{dt} = a\cos t$

hence $$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{a\cos t}{a\cos^2 t}\sin t = \tan t.$$

Also, at $t = \pm\pi/2$. that is, at the points $(0 \pm a)$, $\dfrac{dy}{dx}$ is infinite hence y-axis is the tangent there. And, $\dfrac{dy}{dx} = 0$ at $t = 0$,

that is, at $y = 0,\ x = \infty$ which further establishes that x-axis is an asymptote to the curve.

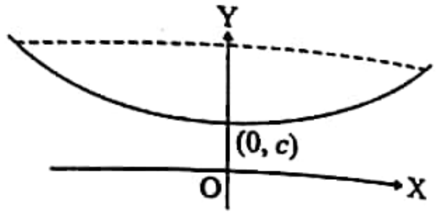
(vi) Give suitable values to t and get the corresponding points on the curve and the rough shape of the curve comes out as shown in the adjoining figure.



EXAMPLE 3.22. Trace the catenary $y = c\cosh(x/c)$

SOLUTION: The equation of the curve can also be written as

$$y = \frac{c}{2}\left(e^{x/c} + e^{-x/c}\right).$$



(*i*) Changing x to $-x$ the equation remains unchanged, hence the curve is symmetrical about y-axis.

(*ii*) The curve does not meet the axis of x but meets the y-axis at $(0, c)$.

(*iii*) Differentiating the given equation, we get

$$\frac{dy}{dx} = \frac{c}{2}\left(\frac{e^{x/c}}{c} - \frac{e^{-x/c}}{c}\right) = \sinh x/c.$$

As $\frac{dy}{dx} = 0$ at $x = 0$, the tangent to the curve at $(0, c)$ is parallel to x-axis.

(*iv*) Since $\cosh(x/c) \geq 1$ there is no part of the curve below the line $y = c$.

(*v*) Give suitable values to x and find the corresponding values of y and plot the points on the curve and get the shape of the curve as shown in the figure.

As physical illustration, consider a chain hanging from two fixed points at the same horizontal level. The shape of the chain is nothing but the above catenary.

EXERCISE 3B

Trace the following curves:

1. $y = x^3$ (cubical parabola)
2. $y^2 = (x - a)^3$ (Semi cubical parabola)
3. (a) $3ay^2 = x^2 (a - x)$
   (b) $3ay^2 = x (x - a)^2$. [GGSIPU Ist Sem End Term 2009]
4. $ay^2 = x (a^2 + x^2)$
5. $a^2y^2 = x^2 (2a - x) (x - a)$
6. $y^2x = a^2 (a - x)$
7. $y(x^2 + 4a^2) = 8a^3$ [GGSIPU I Sem I Term 2010]
8. $x^2y^2 = a^2 (y^2 - x^2)$ 2K15
9. $a^2y^2 = x^3 (2a - x)$
10. $y = \dfrac{x}{1+x^2}$
11. $x = a \cos \theta, y = b \sin \theta$
12. $x = a \sec t, y = b \tan t$
13. $x = a \cos^3 \theta, y = a \sin^3 \theta$.
14. $y^2x = a^2 (a - y)$
15. Trace the curve $y^2 (2a - x) = x^3$. 2K16
16. Trace the curve $x^2y^2 = x^2 + a^2$. [GGSIPU I Sem End Term 2008]

TRACING OF POLAR CURVES

Curves in polar coordinates r and θ are traced basically in the same manner as the cartesian curves. The hints given below would definitely help in sketching the polar curves.

1. Symmetry

(*a*) If θ is changed to $-\theta$ and the equation remains unchanged, the curve is symmetrical about the initial line (x-axis in cartesian).

(*b*) If θ is changed to $\pi - \theta$ and the equation remains unchanged, the curve is symmetrical about the line $\theta = \pi/2$ (y-axis in cartesian).

(*c*) If θ is changed to $\pi + \theta$ and the equation remains unchanged, the curve is symmetrical about the pole (same as symmetry in opposite quadrants). In other words, we can say that if r is changed to $-r$ and the equation remains unchanged, the curve is symmetrical about the pole.

(*d*) If θ is changed to $(\pi/2 - \theta)$ and the equation remains unaltered, the curve is symmetrical about the line $\theta = \pi/4$ (same as symmetry about the line $y = x$).

2. Pole

We find whether the pole lies on the curve or not. If $r = 0$ for $\theta = \alpha$, then the curve passes through the pole and the tangent to the curve at the pole is the line $\theta = \alpha$.

3. Direction of the tangent

Differentiate the equation of the curve with respect to θ and using $\tan\phi = r\dfrac{d\theta}{dr}$ we obtain ϕ which is the angle between the tangent to the curve at a point and the radius vector of this point. This angle ϕ gives the direction of the tangent to the curve at the point.

4. Regions

Solve the given equation of the curve for r in terms of θ and if we can find the range of θ, say $\alpha < \theta < \beta$ in which r becomes imaginary then we conclude that no portion of the curve lies in the triangular strip $\theta = \alpha$ and $\theta = \beta$.

Also, if possible, observe the greatest and the least values of r. If the least value of r is a then no part of the curve lies within the circle with radius a and centre at the pole. Similarly, if the greatest value of r is b then no part of the curve lies outside the circle with centre at the pole and radius b.

5. Asymptotes

If r becomes infinite for $\theta = \theta_1$, say, then there may exist asymptote of the curve.

6. Intersection with axes

Determine where the curve meets the initial line, $\theta = 0$ and the lines $\theta = \pi/2$, $\theta = \pi$, $\theta = 3\pi/2$ etc., and obtain the tangents, at these points. Then give some other suitable values to θ and find the corresponding points on the curve and the tangents there.

The above mentioned points would enable us to determine rough shape of the curve.

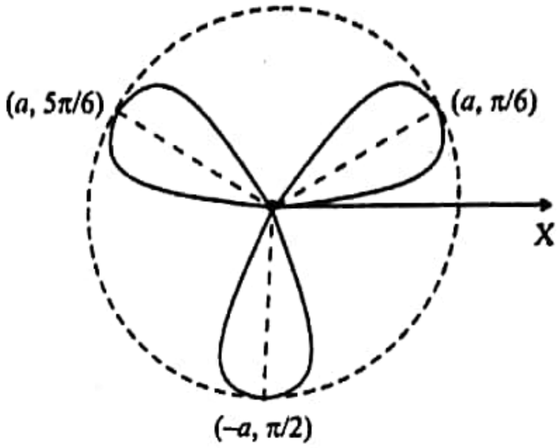
EXAMPLE 3.23. Trace the curve $r = a \sin 3\theta$ **(three leaved rose).**

SOLUTION:

(i) On changing θ to $(\pi - \theta)$ the equation of the curve remains unchanged hence the curve is symmetrical about the line $\theta = \pi/2$.

(ii) The greatest value of r is 'a', as the maximum value of $\sin 3\theta$ is 1, therefore the curve wholly lies within a circle of radius a and centre at the pole.

(iii) The curve meets the pole for $\theta = 0, \ \pi/3, \ 2\pi/3, \ \pi$. The variation of r with the variation of θ, is shown in the following table:



θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
r	0	a	0	$-a$	0	a	0

Thus, the tangents to the curve at the pole are having the equations. $\theta = 0$, $\theta = \pi/3$, $\theta = 2\pi/3$ and $\theta = \pi$.

(iv) We shall get three loops as shown in the adjoining figure. It is more popularly known as a rose with three leaves.

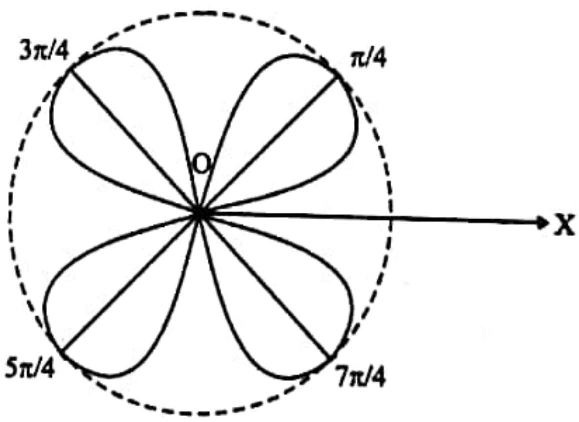
EXAMPLE 3.24. Trace the curve $r = a \sin 2\theta$ **(four leaved rose).**

SOLUTION:

(i) On changing θ to $\pi + \theta$ the equation of the curve remains unaltered therefore, it is symmetrical about the pole.

(ii) On changing θ to $(\pi/2 - \theta)$ the equation remains unchanged hence the curve is symmetrical about the line $\theta = \pi/4$.

(iii) The curve meets the pole at $\theta = 0, \pi/2, \pi, 3\pi/2$. Therefore, the tangents to the curve at the pole are the lines $\theta = 0$, $\theta = \pi/2$, $\theta = \pi$ and $\theta = 3\pi/2$.



(iv) The maximum value of r is 'a' as the maximum value of $\sin 2\theta$ is 1, hence the curve wholly lies within a circle with centre at the pole and radius a. The variation of r with the variation of θ is shown in the following table.

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	0	a	0	$-a$	0	a	0	$-a$	0

Thus, we get four loops as shown in the above figure. This is more popularly known as four leaved rose.

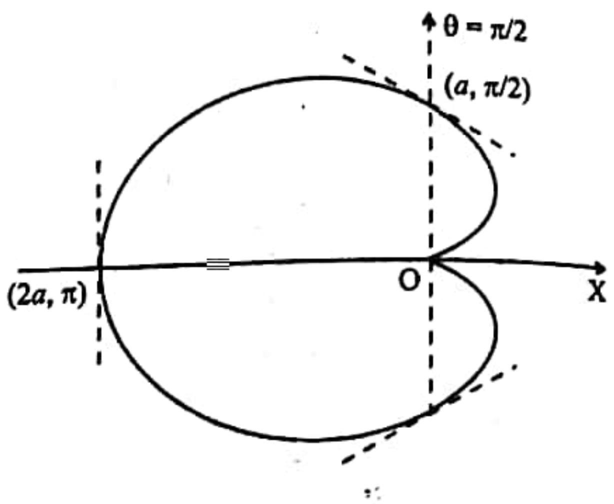
EXAMPLE 3.25. Trace the cardioide $r = a(1 - \cos\theta)$.

SOLUTION:

(*i*) When θ is changed to $-\theta$ the equation of the curve remains unaltered, hence there is symmetry about the initial line and no other point of symmetry is satisfied here.

(*ii*) When $\theta = 0$, $r = 0$ hence the curve passes through the pole and the tangent to the curve at the pole is the initial line.

(*iii*) The minimum value of $\cos\theta$ is -1 hence the maximum value of r is $2a$ and as such the whole curve lies within the circle with centre at the pole and radius $2a$.



(*iv*) From $r = a(1 - \cos\theta)$ we have $\dfrac{dr}{d\theta} = a\sin\theta$.

Hence $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a(1-\cos\theta)}{a\sin\theta} = \tan\theta/2$, so $\phi = \theta/2$.

Now, at $\theta = \pi/2$, $\phi = \pi/4$, hence the tangent at the point $(a, \pi/2)$ makes an angle $\pi/4$ with line $\theta = \pi/2$. At $\theta = \pi$, $\phi = \pi/2$ hence at the point $(2a, \pi)$ the tangent is perpendicular to the line $\theta = \pi$.

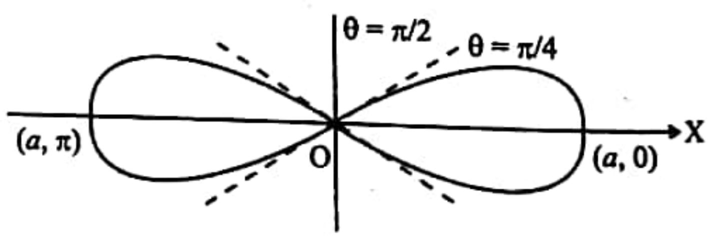
(*v*) Give suitable values to θ, like $\theta = \pi/4, 3\pi/4$ and find the corresponding points on the curve. The rough shape of the curve is as given in the above figure.

EXAMPLE 3.26. Trace the Leminiscate of Bernoulli $r^2 = a^2\cos 2\theta$.

SOLUTION:

(*i*) On changing θ to $-\theta$ the equation remains unchanged hence the curve is symmetrical about the initial line.

(*ii*) When θ is changed to $\pi - \theta$ the equation remains the same hence there is symmetry about the line $\theta = \pi/2$.



(*iii*) When θ is changed to $\pi + \theta$ the equation remains unchanged hence the curve is also symmetrical about the pole.

(*iv*) $r = 0$ when $\theta = \pm\pi/4$, therefore curve passes through the pole and the tangents to the curve at the pole are the lines $\theta = \pm\pi/4$.

(*v*) The maximum value of r is 'a' when $\theta = 0$ and when $\theta = \pi$, hence the curve lies wholly within the circle with centre at the pole and radius a.

(*vi*) When θ increases from $\pi/4$, r^2 becomes negative hence r is imaginary, and it remains so till θ becomes $3\pi/4$. Thus, there is no portion of the curve in the region between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

(*vii*) Differentiating the equation of the curve, we have $2r\dfrac{dr}{d\theta} = -2a^2\sin 2\theta$.

hence $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{r^2}{-a^2\sin 2\theta} = \dfrac{a^2\cos 2\theta}{-a^2\sin 2\theta} = -\cot 2\theta = \tan\left(\dfrac{\pi}{2}+2\theta\right)$

Thus, $\phi = \dfrac{\pi}{2}+2\theta.$

Therefore, $\phi = \pi/2$ when $\theta = 0$, and the tangent to the curve at the point $(a, 0)$ is perpendicular to the initial line. Similarly, when $\theta = \pi$, we have $\phi = 5\pi/2$ hence the tangent to the curve at the point (a, π) is perpendicular to the line $\theta = \pi$.

(*viii*) Give suitable values to θ like $\theta = \pm\pi/6$ and get the corresponding points on the curve. The final shape of the curve is as shown in the above figure.

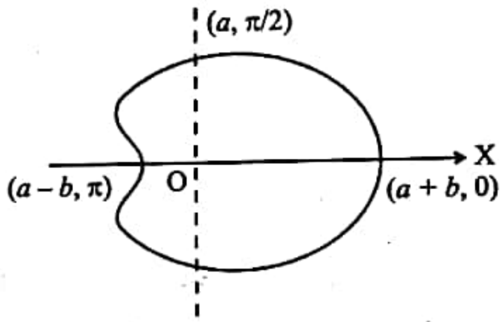
EXAMPLE 3.27. Trace the curve $r = a + b\cos\theta$ **(lamicon),** a **and** b **being positive.**

SOLUTION: When θ is changed to $-\theta$ the equation remains unaltered hence the curve is symmetrical about the initial line. Here we shall get three types of curve in the following three cases :

CASE I. $a > b$. Here r is never negative.

At $\theta = 0$, $r = a + b$ and at $\theta = \pi$, $r = a - b$.

Also, $\dfrac{dr}{d\theta} = -b\sin\theta$, hence $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a+b\cos\theta}{-b\sin\theta}$.

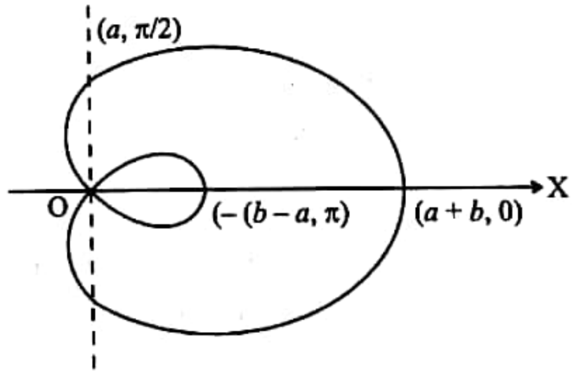


When $\theta = 0$, we have $\phi = \pi/2$.

and when $\theta = \pi$, again $\phi = \pi/2$, therefore, tangents to the curve at $(a + b, 0)$ and $(a - b, \pi)$ are perpendicular to the initial line. Then give suitable values to θ like $\theta = \pi/3, \pi/2, 2\pi/3$ and get corresponding points on the curve. The shape of the curve is given in the above figure.

CASE II. $a < b$.

Here r is negative for some values of θ and positive for other values of θ. Actually, we get one small loop inside a bigger loop as shown in the adjoining figure.



CASE III. $a = b$.

The equation of the curve here becomes $r = a(1 + \cos\theta)$ which is a cardioide.

(*i*) The curve passes through the pole when $\theta = \pi$ hence the tangent to the curve at the pole is along the negative direction of the initial line.

(*ii*) Maximum value of r is $2a$ at $\theta = 0$, therefore the curve wholly lies within a circle of radius $2a$ and centre at the pole.

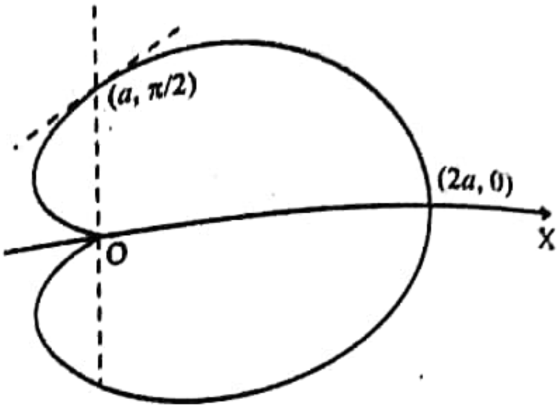
(*iii*) Differentiating the given equation, we get $\dfrac{dr}{d\theta} = -a\sin\theta$, hence

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\cot\theta/2 = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

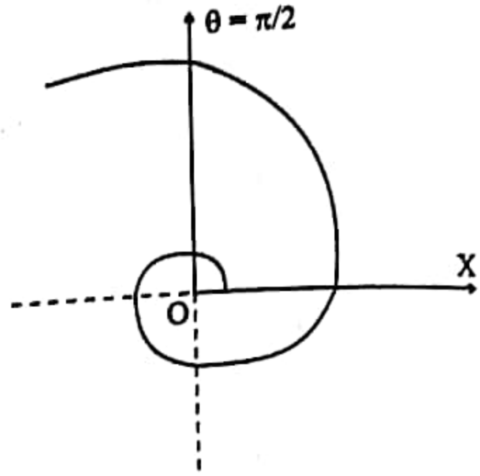
Therefore, $\phi = \pi/2 + \theta/2$.

Thus, at the point $(2a, 0)$. $\phi = \pi/2$ hence the tangent there is perependicular to the initial line and at the point $(a, \pi/2)$, $\phi = 3\pi/4$ and the tangent there would make angle $3\pi/4$, with the line, $\theta = \pi/2$.

(*iv*) Finally, give suitable values to θ like $\theta = \pi/6$, $\pi/3$, $2\pi/3$ and get the corresponding points on the curve. The shape of the curve is given in the adjoining figure.



EXAMPLE 3.28. **Trace the equiangular spiral $r = ae^{\theta \cot \alpha}$.**



SOLUTION: There is no symmetry in this curve.

At $\theta = 0$, we have $r = a$.

The shape of this curve is mainly guided by an important characteristic property it possesses.

Here $$\frac{dr}{d\theta} = a e^{\theta \cot \alpha} \cot \alpha = r \cot \alpha.$$

Hence $$\tan \phi = r \frac{d\theta}{dr} = \frac{r}{r \cot \alpha} = \tan \alpha.$$

Thus, $\phi = \alpha$, meaning thereby that the tangent to the curve at any point of it always makes a constant angle α with the radius vector of that point. Give suitable value to θ, like $\theta = \pi/2, \pi, 3\pi/2, 2\pi, 5\pi/2$ and get corresponding points on the curve. The bend of the curve at every point is guided by the above mentioned property. The shape of the curve is as shown in the adjoining figure.

EXAMPLE 3.29. **Trace the spiral $r\theta = a$.**

SOLUTION: The equation of the curve is $r = a/\theta$.

(*i*) The curve is symmetrical about the line $\theta = \pi/2$ because when θ is changed to $-\theta$ and r to $-r$ equation of the curve remains unchanged, that is, for every point (r, θ) on the curve we shave $(-r, -\theta)$ also on the curve.

(*ii*) As $\theta \to 0$, $r \to \infty$ hence an asymptote may exist here. We know that the equation of asymptote for the polar curve $\frac{1}{r} = f(\theta)$, is $r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$ and $r \to \infty$ as $\theta \to \alpha$.

In the present case, $\alpha = 0$ and $f(\theta) = \theta/a$.

$\therefore$ $f'(\theta) = 1/a$, therefore, the asymptote here is

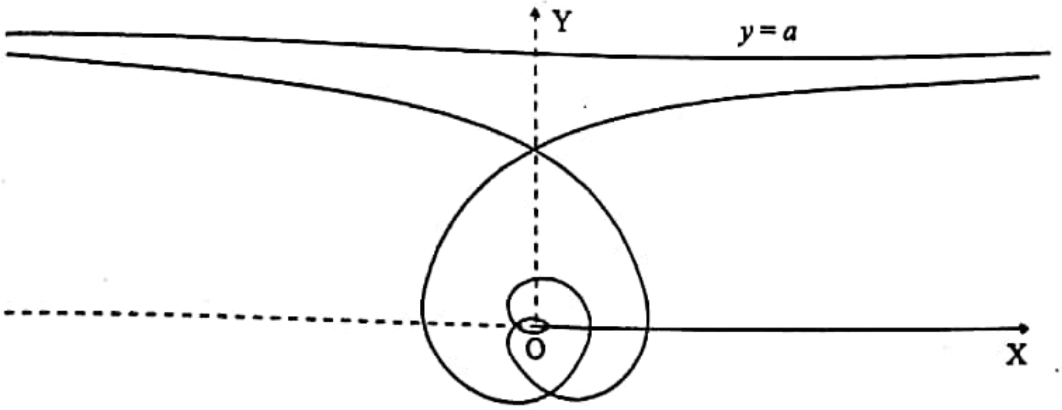
$$r \sin(\theta - 0) = \frac{1}{1/a} \quad \text{or} \quad r \sin \theta = a$$

or $y = a$ (in cartesian coordinates).

Further, since $\sin \theta < \theta$, r for the curve is always less than the r for the asymptote. Thus, the curve lies completely below the asymptote.

(iii) The way r changes with the change in θ, is shown in the table below.

θ	0	$\pi/2$	π	$3\pi/2$	2π	∞
r	∞	$2a/\pi$	a/π	$2a/3\pi$	$a/2\pi$	0



The shape of the curve is as shown in the above figure. It is in two parts, one when θ and r both are positive and the other when θ and r both are negative. Both the parts are symmetrical about the line $\theta = \pi/2$.

EXAMPLE 3.30. **Trace the Follum of Descartes $x^3 + y^3 = 3axy$.**

SOLUTION:

(i) On interchanging x and y the equation remains unchanged, hence the curve is symmetrical about the line $y = x$.

(ii) The curve passes through the origin and equating to zero the lowest degree term, we find that both x-axis and y-axis are tangents to the curve at the origin.

(iii) The curve does not meet the coordinate axes except at the origin.

(iv) The curve meets the line of symmetry $y = x$ at the point $(3a/2, 3a/2)$. To find the tangent to the curve at the point $(3a/2, 3a/2)$. shift the origin to this point and equation becomes

$$(x+3a/2)^3+(y+3a/2)^3=3a\,(x+3a/2)\,(y+3a/2)$$

or $$x^3+y^3-3axy+\frac{9a}{2}\left(x^2+y^2\right)+\frac{27a^2}{4}(x+y)-\frac{9a^2}{2}(x+y)=0$$

or $$x^3+y^3-3axy+\frac{9a}{2}\left(x^2+y^2\right)+\frac{9a^2}{4}(x+y)=0$$

Equating to 0 the lowest degree term we get the tangent to the curve at the new origin as $x + y = 0$. Therefore, the tangent to the given curve at $(3a/2, 3a/2)$ is $x + y = 3a$.

(v) No asymptote parallel to x-axis or y-axis exists. However an oblique asymptote does exist. By the method discussed in the early part of this chapter we get the equation of this asymptote as $x + y + a = 0$.

(vi) Observe here that x and y cannot be simultaneously negative hence no portion of the curve lies in the third quadrant.

(*vii*) Now, for convenience, we convert the given equation to polar coordinates so as to get

$$r^3\cos^3\theta + r^3\sin^3\theta = 3a\,r^2\cos\theta\sin\theta$$

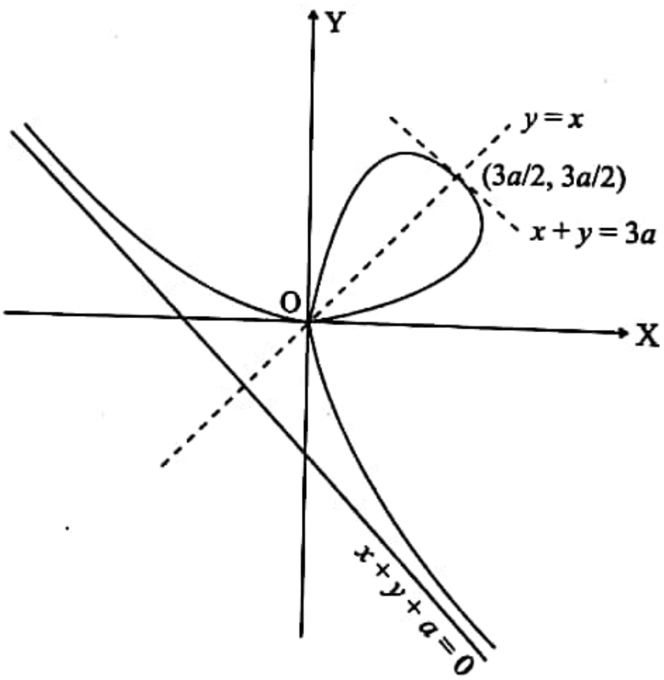
or $$r = \frac{3a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$$ [GGSIPU I Sem End Term Jan 2011]

Next, we give suitable values to θ and find the corresponding values of r and put them in tabular form as

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	0	$\frac{3a\sqrt{2}}{2}$	$\frac{6a\sqrt{3}}{1+3\sqrt{3}}$	0	$\frac{-6a\sqrt{3}}{3\sqrt{3}-1}$	0

We observe here that as θ increases from 0 to $\pi/4$, r increases, and when θ goes from $\pi/4$ to $\pi/2$, r decreases. When θ goes from $\pi/2$ to π, r is negative.

The shape of the curve is as shown in the figure given below.



EXERCISE 3C

Trace the following curves:

1. (i) $r = 2a \cos \theta$ (ii) $r = 2a \cos (\theta - \alpha)$.
2. $r = a\theta$ (spiral of Archamedes).
3. $r = a \cos 2\theta$.
4. $r = a \cos 3\theta$.
5. $r^2 = a^2 \sin 2\theta$.
6. $r = a \sin 5\theta$.
7. $r^2 \cos 2\theta = a^2$.
8. $x^6 + y^6 = a^2 x^2 y^2$.
9. $x^5 + y^5 = 5ax^2y^2$.

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CHAPTER

4

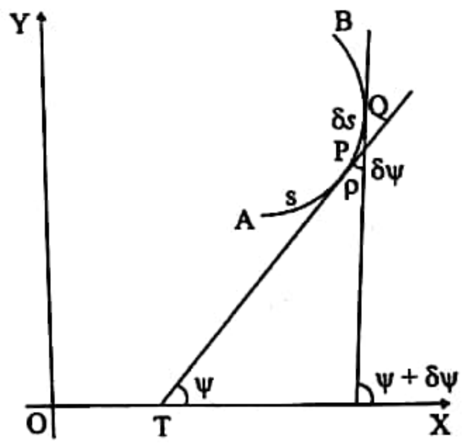
Curvature and Radius of Curvature

Curvature and Radius of Curvature, Concavity and Convexity of curves and Points of Inflexion.

CURVATURE

Of any two curves one may be bending more sharply than the other. Also the bending may be different at different points of curve. For example, in case of parabola the bending is different at its different points, being maximum at its vertex. Actually, the curvature is a numerical measure of bending of the curve.

Let P be any point of a curve and s be the length of the arc from a fixed point A of the curve to the point P. Let the tangent at P be inclined at an angle ψ with the axis of x and arc AP = s. (See the figure). The direction of the curve at a point is determined by the slope of the tangent at that point. Let Q be another point on the curve close to P so that arc AQ = $s + \delta s$, and the tangent at Q make an angle $\psi + \delta\psi$ with the axis of x. The measure of the rapidity with which the curve is turning (or bending) at the point is the rate of change of ψ with respect to s. This rate of change is called ***the curvature at the particular point***. Thus, the curvature at P



$$= \lim_{Q\to P}\frac{\delta\psi}{\delta s} = \lim_{\delta s\to 0}\frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}.$$

For example, consider a circle of radius a and centre on the y-axis at $(0, a)$, then we have $s = a\psi$ and therefore the curvature $= \dfrac{d\psi}{ds} = \dfrac{1}{a}$.

This shows that the curvature of a circle is same at every point of its circumference and is reciprocal of the radius of the circle. Idea is extended to any curve and this is the reason that the reciprocal of the curvature of a curve at a point is defined as its **radius of curvature** at that point and is denoted by ρ. Therefore, we have

$$\boxed{\frac{1}{\rho} = \frac{d\psi}{ds} \quad \text{or} \quad \rho = \frac{ds}{d\psi}}$$

Obviously, the radius of curvature of circle at any point is same as its radius.

RADIUS OF CURVATURE (in Cartesian Coordinates):

We know that $\tan\psi = \dfrac{dy}{dx}$. ...(1)

Differentiating on both sides with respect to s, gives

$$\sec^2\psi\,\frac{d\psi}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}\cdot\frac{dx}{ds}$$

or $$\frac{ds}{d\psi} = \frac{\sec^2\psi}{\frac{d^2y}{dx^2}} \cdot \frac{ds}{dx} = \frac{(1+\tan^2\psi)}{\frac{d^2y}{dx^2}} \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \text{since} \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Now, using the relation (1), we have

$$\rho = \frac{ds}{d\psi} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad \text{where} \quad \frac{d^2y}{dx^2} \neq 0.$$

In case, the tangent to the curve at the point under consideration is parallel to y-axis then $\frac{dy}{dx}$ will be infinite, hence $\frac{dx}{dy} = 0$ then we consider $\frac{dx}{dy} = \cot\psi$.

Differentiating both sides w.r.t. s, gives

$$\frac{d}{ds}\left(\frac{dx}{dy}\right) = \frac{d}{ds}\cot\psi$$

or $$\frac{d}{dy}\left(\frac{dx}{dy}\right)\frac{dy}{ds} = -\operatorname{cosec}^2\psi\,\frac{d\psi}{ds} \quad \text{or} \quad \frac{d^2x}{dy^2}\frac{dy}{ds} = -(1+\cot^2\psi)\frac{d\psi}{ds}$$

or $$\rho = \left|\frac{ds}{d\psi}\right| = \left|\frac{-(1+\cot^2\psi)}{\frac{d^2x}{dy^2}}\frac{ds}{dy}\right| = \left|\frac{-\left[1+\left(\frac{dx}{dy}\right)^2\right]\sqrt{1+\left(\frac{dx}{dy}\right)^2}}{\frac{d^2x}{dy^2}}\right|$$

or $$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} \quad \text{where} \quad \frac{d^2x}{dy^2} \neq 0.$$

RADIUS OF CURVATURE FOR PARAMETRIC CURVES

Let the equation of the curve be $x = f(t)$ and $y = g(t)$ where t is the parameter,

then $$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{y'}{x'}$$ where the dashe represents the derivative w.r.t. t.

Further, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx} = \frac{x'y''-y'x''}{x'^2 \cdot \frac{dx}{dt}}$$

$$\therefore \quad \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\frac{y'^2}{x'^2}\right)^{\frac{3}{2}}}{\frac{x'y''-y'x''}{x'^3}} = \frac{\left(x'^2+y'^2\right)^{\frac{3}{2}}}{x'y''-y'x''}.$$

RADIUS OF CURVATURE AT THE ORIGIN BY NEWTON'S METHOD

This is a very fundamental method but has limited use because it is applicable *only when the curve passes through the origin and has x-axis or y-axis as the tangent there.*

Let us first consider the case where the curve passes through the origin and x-axis is the tangent to the curve at the origin. Thus, $y_1 = 0$ at the origin and hence $\rho = \dfrac{\left(1+y_1^2\right)^{3/2}}{y_2} = \dfrac{1}{y_2}$.

Now let us consider $\lim\limits_{x\to 0} \dfrac{x^2}{2y}$ which is of the form $\dfrac{0}{0}$.

Applying L' Hospital's rule, we get

$$\lim_{x\to 0} \frac{x^2}{2y} = \lim_{x\to 0} \frac{2x}{2y_1} \quad \text{which is again of } \frac{0}{0} \text{ form}$$

Again applying L' Hospital's rule, we get $\lim\limits_{x\to 0} \dfrac{x^2}{2y} = \lim\limits_{x\to 0} \dfrac{2}{2y_2} = \rho$ {where $y_2 \neq 0$ at $(0, 0)$ }

Therefore, we have $\rho = \lim\limits_{x\to 0} \dfrac{x^2}{2y}$, when x-axis is tangent to curve at the origin.

Similarly, in the case when curve passes through the origin and y-axis is the tangent to the curve at the origin, we have $\rho = \lim\limits_{y\to 0} \dfrac{y^2}{2x}$.

EXAMPLE 4.1. Find the radius of curvature at the origin for the following curves.

(*i*) $x^2 = 4ay$ (*ii*) $y^2(a^2 - x^2) = a^3x$.

SOLUTION: (*i*) The parabola $x^2 = 4ay$ passes through the origin and tangent at the origin, is $y = 0$, the axis of x. Therefore, by Newton's method, $\rho = \lim\limits_{x\to 0} \dfrac{x^2}{2y} = \lim\limits_{x\to 0} \dfrac{4ay}{2y} = 2a$ **Ans.**

(*ii*) The given curve $y^2(a^2 - x^2) = a^3x$ passes through the origin and by equating to 0 the lowest degree term we get the tangent at the origin as $x = 0$, the axis of y. Therefore, by Newton's method,

$$\rho = \lim_{y\to 0} \frac{y^2}{2x} = \lim_{\substack{y\to 0\\ x\to 0}} \frac{a^3 x}{2\left(a^2 - x^2\right)\cdot x} = \lim_{\substack{y\to 0\\ x\to 0}} \frac{a^3}{2a^2 - 2x^2} = \frac{a}{2}. \qquad \textbf{Ans.}$$

EXAMPLE 4.2. If ρ is the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S is its focus then show that ρ^2 varies as SP^3. Also, show that the radius of curvature at the vertex is equal to the length of the semi-latus rectum.

[*GGSIPU I Sem End Term 2012*]

SOLUTION: Differentiating $y^2 = 4ax$ we get $2y\,y_1 = 4a$ or $y_1 = \dfrac{2a}{y}$.

and $$y_2 = \frac{-2a}{y^2}\cdot\frac{dy}{dx} = \frac{-2a}{y^2}\cdot\frac{2a}{y} = \frac{-4a^2}{y^3}$$

$$\therefore \qquad \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{\left(1+4\dfrac{a^2}{y^2}\right)^{3/2}}{\dfrac{-4a^2}{y^3}} = \frac{\left(y^2+4a^2\right)^{3/2}}{-4a^2}$$

$$= \frac{\left(4ax+4a^2\right)^{3/2}}{4a^2} = \frac{(4a)^{3/2}}{4a^2}(x+a)^{3/2} = \frac{2}{\sqrt{a}}(x+a)^{3/2} \quad \text{(in absolute value).}$$

Since the focus S is at $(a, 0)$ and $P(x, y)$ is on the parabola, hence

$$SP = \sqrt{(x-a)^2+(y-0)^2} = \sqrt{(x-a)^2+4ax} = x+a$$

$$\therefore \qquad \rho = \frac{2}{\sqrt{a}} SP^{3/2} \quad \text{or} \quad \rho^2 = \frac{4}{a} SP^3$$

This establishes that ρ^2 varies as SP^3.

Next, to obtain ρ at the vertex $(0, 0)$, we find that $\dfrac{dy}{dx}$ is infinite at the vertex since tangent there, is parallel to y-axis. Therefore, using the formula

$$\rho = \frac{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}{\dfrac{d^2x}{dy^2}}$$

and, from the equation of the curve, $\dfrac{dx}{dy} = \dfrac{y}{2a} = 0$ at $(0, 0)$ and $\dfrac{d^2x}{dy^2} = \dfrac{1}{2a}$

we get, $\rho(0, 0) = \dfrac{(1+0)^{3/2}}{\left(\dfrac{1}{2a}\right)} = 2a =$ length of the semi latus rectum. **Hence Proved.**

EXAMPLE 4.3. Find the radius of curvature at any point 'θ' of the cycloid

$$x = a(\theta+\sin\theta), \quad y = a(1-\cos\theta).$$

[GGSIPU Ist Sem End Term 2007; End Term 2009; End Term 2013]

SOLUTION: $x' = \dfrac{dx}{d\theta} = a(1+\cos\theta), \quad y' = \dfrac{dy}{d\theta} = a\sin\theta$

$$x'' = \frac{d^2x}{d\theta^2} = -a\sin\theta, \quad y'' = \frac{d^2y}{d\theta^2} = a\cos\theta$$

For parametric equation of the curve, we have

$$\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''} = \frac{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}^{3/2}}{a^2(1+\cos\theta)\cos\theta+a^2\sin^2\theta}$$

$$= \frac{a(1+\cos^2\theta+2\cos\theta+\sin^2\theta)^{3/2}}{\cos\theta+\cos^2\theta+\sin^2\theta} = \frac{a(2+2\cos\theta)^{3/2}}{1+\cos\theta}$$

$$= 2\sqrt{2}\,a\sqrt{1+\cos\theta} = 2\sqrt{2}\,a\sqrt{2\cos^2\frac{\theta}{2}} = 4a\cos\frac{\theta}{2}$$

Therefore at the point 'θ' we have $\rho = 4a\cos\dfrac{\theta}{2}$. **Ans.**

EXAMPLE 4.4. (a) Show that the radius of curvature at any point (x, y) of the rectangular hyperbola $xy = c^2$, is given by $\rho = \frac{(x^2+y^2)^{3/2}}{2c^2}$.

(b) If ρ_1 and ρ_2 are the radii of curvature at the extremities of a focal chord of a parabola, show that $\rho_1^{-2/3} + \rho_2^{-2/3}$ is a constant.

[GGSIPU I Sem I Term 2011]

SOLUTION: (a) The equation of the curve can be written as $y = c^2/x$

$$\therefore \quad y_1 = \frac{-c^2}{x^2} \quad \text{and} \quad y_2 = \frac{2c^2}{x^3}.$$

$$\therefore \quad \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{c^4}{x^4}\right)^{3/2}}{\frac{2c^2}{x^3}} = \frac{(x^4+c^4)^{3/2}\, x^3}{x^6 \cdot 2c^2}$$

$$= \frac{(x^4+c^4)^{3/2}}{2c^2\, x^3} = \frac{(x^4+x^2\, y^2)^{3/2}}{2c^2\, x^3} = \frac{(x^2+y^2)^{3/2}}{2c^2} \quad (\text{using } c^2 = xy)$$

Hence Proved.

(b) Let the parabola be $y^2 = 4ax$, any point on it is $(at^2, 2at)$.

We know that if $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are the extremities of a focal chord of the parabola, then $t_1 t_2 = -1$.

We have $x = at^2$, $y = 2at$ hence $x' = 2at$, $y' = 2a$ and $x'' = 2a$, $y'' = 0$ therefore at the point $(at^2, 2at)$

$$\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''} = \frac{(4a^2t^2+4a^2)^{3/2}}{0-4a^2} = 2a\,(1+t^2)^{3/2} \quad (\text{not taking} - \text{sign}).$$

or $$\rho^{-2/3} = (2a)^{-2/3}(1+t^2)^{-1}$$

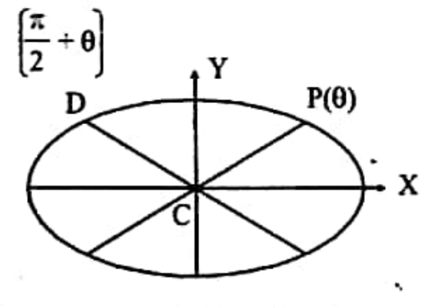
Therefore $$\rho_1^{-2/3}+\rho_2^{-2/3} = (2a)^{-2/3}\left(\frac{1}{1+t_1^2}+\frac{1}{1+t_2^2}\right) = (2a)^{-2/3}\,\frac{(2+t_1^2+t_2^2)}{(1+t_1^2+t_2^2+t_1^2t_2^2)}$$

$$= (2a)^{-2/3} \quad \text{since } t_1\, t_2 = -1 = \text{Constant.}$$

Ans.

EXAMPLE 4.5. If ρ and ρ' are the radii of curveture at the extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, prove that $(\rho^{2/3} + \rho'^{2/3})\,(ab)^{2/3} = a^2 + b^2$.

[GGSIPU I Sem End Term 2003]



SOLUTION: Let CP and CD be two conjugate semi diameters of the given ellipse. Parametric equation of the ellipse is $x = a\cos\theta$, $y = b\sin\theta$.

Let the coordinates of the point P be $(a\cos\theta, b\sin\theta)$ then those of the point D are obtained by replacing 'θ' by $\frac{\pi}{2}+\theta$, so as to get $D = \{a\cos(\theta+\pi/2), b\sin(\theta+\pi/2)\}$.

To find the radius of curvature at P, the point 'θ' of the ellipse, we have $x' = -a\sin\theta$, $y' = b\cos\theta$ and $x'' = -a\cos\theta$, $y'' = -b\sin\theta$.

$$\therefore \quad \rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''} = \frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{(-a\sin\theta)(-b\sin\theta)-(b\cos\theta)(-a\cos\theta)}$$

$$\text{or} \quad \rho = \frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab}. \qquad \text{...(1)}$$

For the radius of curvature ρ' at D, we have to replace θ by $(\theta + \pi/2)$ in (1) and get

$$\rho' = \frac{(a^2\cos^2\theta+b^2\sin^2\theta)^{3/2}}{ab}$$

$$\therefore \quad \rho^{2/3}+\rho'^{2/3} = \frac{(a^2\sin^2\theta+b^2\cos^2\theta)}{(ab)^{2/3}} + \frac{(a^2\cos^2\theta+b^2\cos^2\theta)}{(ab)^{2/3}} = \frac{a^2+b^2}{(ab)^{2/3}}$$

$$\text{or} \quad (ab)^{2/3}(\rho^{2/3}+\rho'^{2/3}) = a^2+b^2.$$

Hence Proved.

EXAMPLE 4.6. Find the radius of curvature at the origin, for the curve

$$2x^3-3x^2y+4y^3+y^2-3x=0.$$

SOLUTION: The given curve is passing through the origin and on equating to zero the lowest degree term, we get the equation of the tangent at the origin as $x = 0$, the axis of y.

Therefore, by Newton's method $\rho = \lim_{y\to 0}\frac{y^2}{2x}$.

Dividing the given equation throughout by x, we get

$$2x^2-3xy+4y\cdot\frac{y^2}{x}+\frac{y^2}{x}-3=0.$$

At the origin, we have $0+0+0+2\rho-3=0$ or $\rho = \frac{3}{2}$. **Ans.**

EXAMPLE 4.7. Find the point on the curve $y = e^x$ at which the curvature is maximum and show that the tangent to the curve at that point forms with the coordinate axes, a triangle whose sides are in the ratio $1 : \sqrt{2} : \sqrt{3}$.

SOLUTION: We have $y = e^x$, hence $y_1 = e^x = y_2 = y$.

$$\therefore \quad \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} \quad \text{or} \quad \rho^2 = \frac{\left(1+y_1^2\right)^3}{y_2^2}.$$

Curvature is maximum when radius of curvature is minimum, *i.e.*, when $\frac{\left(1+y_1^2\right)^3}{y_2^2}$ is minimum.

If we write $y^2 = t$ then $y_1^2 = y_2^2 = t$, hence

$$\rho^2 = \frac{(1+t)^3}{t} = t^2+3t+3+\frac{1}{t}$$

Now, $\frac{d\rho^2}{dt} = 0$ gives $2t+3-\frac{1}{t^2}=0$ or $2t^3+3t^2-1=0$

or $(t+1)(2t^2+t-1)=0$ or $(t+1)^2(2t-1)=0.$

$$\Rightarrow \quad t=-1, \frac{1}{2} \quad \text{or} \quad y^2=-1, \frac{1}{2}$$

Obviously, $y^2 \neq -1$ as $y^2 \geq 0$, hence $y^2 = \frac{1}{2}$ or $y = \frac{1}{\sqrt{2}}$ since e^x is never negative.

For ρ^2 to be minimum $\frac{d\rho^2}{dt} = 0$ and $\frac{d^2\rho}{dt^2} > 0$

$$\frac{d^2\rho^2}{dt^2} = 2 + \frac{2}{t^3} = 2 + \frac{2}{y^6} > 0 \quad \text{hence } \rho^2 \text{ is minimum at } y = \frac{1}{\sqrt{2}}.$$ **Ans.**

For $y = \frac{1}{\sqrt{2}}$ we have $x = \log y = -\frac{1}{2}\log 2.$

Thus, the required point is $\left(-\frac{1}{2}\log 2, \frac{1}{\sqrt{2}}\right)$ where the curveture is maximum.

The tangent at the point $\left(-\frac{1}{2}\log 2, \frac{1}{\sqrt{2}}\right)$ has the equation

$$y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x + \frac{1}{2}\log 2\right) \quad \text{or} \quad \frac{y}{1} - \frac{x}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2}\log 2\right) = k, \text{ say.}$$

Then intercepts on the coordinate axes are $k\sqrt{2}$ and k therefore the tangent is of length

$$= \sqrt{\left(k\sqrt{2}\right)^2 + k^2} = k\sqrt{3}.$$

Hence the sides of the triangle are in the ratio $1 : \sqrt{2} : \sqrt{3}$. **Hence Proved.**

EXAMPLE 4.8. Show that the radius of curvature at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is equal to three times the length of the perpendicular from the origin to the tangent at that point.

[*GGSIPU I Sem End Term 2012; I Term 2004; I Sem I Term 2010*]

SOLUTION: The given curve, in the parametric form, is $x = a\cos^3\theta$, $y = a\sin^3\theta$ hence $x' = -3a\cos^2\theta\sin\theta$, $y' = 3a\sin^2\theta\cos\theta$

and $x'' = 6a\cos\theta\sin^2\theta - 3a\cos^3\theta$, $y'' = 6a\sin\theta\cos^2\theta - 3a\sin^3\theta$.

$$\therefore \quad \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{27a^3\cos^3\theta\sin^3\theta}{-9a^2\sin^2\theta\cos^2\theta} = 3a\sin\theta\cos\theta \text{ (on simplification).}$$

The equation of the tangent to the curve at $(a\cos^3\theta, a\sin^3\theta)$, is

$$y - a\sin^3\theta = \frac{-\sin\theta}{\cos\theta}(x - a\cos^3\theta) \quad \text{or} \quad x\sin\theta + y\cos\theta = a\sin\theta\cos\theta$$

$\therefore$ The length of the perpendicular from the origin upon this perpendicular is $p = a\sin\theta\cos\theta$

Therefore we have $\rho = 3p$. **hence the result.**

EXAMPLE 4.9. Find the curvature of $x = 4\cos t$, $y = 3\sin t$. At what point on this ellipse does the curvature have the greatest and the least values? What are magnitudes?

[GGSIPU I Sem I Term 2007; End Term Jan 2011]

SOLUTION: The curve is $x = 4\cos t$, $y = 3\sin t$, hence $x' = -4\sin t$, $y' = 3\cos t$,

$x'' = -4\cos t$, $y'' = -3\sin t$.

Therefore the radius of curvature ρ at any point 't' on the curve, is given by

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(16\sin^2 t + 9\cos^2 t)^{3/2}}{-4\sin t(-3\sin t) - 3\cos t(-4\cos t)} = \frac{1}{12}(9\cos^2 t + 16\sin^2 t)^{3/2}$$

or $$\rho^{2/3} = \frac{1}{12^{2/3}}(9\cos^2 t + 16\sin^2 t)$$

Since the curvature is reciprocal of the radius of curvature, the curvature will be greatest and least when the radius of curvature is least and greatest respectively, in other words, when $(12\rho)^{2/3}$ is least and greatest respectively.

For maximum and minimum of $(12\rho)^{2/3}$ we have $\frac{d}{dt}(16\sin^2 t + 9\cos^2 t) = 0$

or $14\sin t\ \cos t = 0$ which gives $t = 0$ and $t = \pi/2$.

Hence $(12\rho)^{2/3} = 9$ at $t = 0$, *i.e.* at $(4, 0)$ and is equal to 16 at $t = \pi/2$, *i.e.* at $(0, 3)$

Thus the curvature is least at $(0, 3)$ and greatest at $(4, 0)$, and least value of curvature is 3/16 and greatest is 4/9. **Ans.**

EXAMPLE 4.10. Find the radius of curvature of the curve $y^2 = \frac{4a^2(2a - x)}{x}$ at a point where the curve meets the axis of x.

[GGSIPU I Sem III Term Sp. 2007]

SOLUTION: The given curve is $y^2 = \frac{4a^2(2a-x)}{x}$ which meets the axis of x at the point $(2a, 0)$. Since the tangent to the curve at $(2a, 0)$ is parallel to y-axis hence $y_1 = \infty$ at this point. Therefore we consider the other formula $\rho = \frac{(1 + x_1^2)^{3/2}}{x_2}$ for finding the radius if curvature at $(2a, 0)$

The equation of the curve can be written as $$x = \frac{8a^3}{y^2 + 4a^2}$$

Hence $$x_1 = \frac{-16a^3 y}{(y^2 + 4a^2)^2}$$ which is 0 at $(2a, 0)$.

Also $$x_2 = \frac{-16a^3(y^2 + 4a^2)^2 + 16a^3 y \cdot 2(y^2 + 4a^2)\cdot 2y}{(y^2 + 4a^2)^4}$$

$$= \frac{-16a^3(y^2 + 4a^2) + 64a^3 y^2}{(y^2 + 4a^2)^3} = -\frac{1}{a} \quad \text{at } (2a, 0).$$

Therefore $$\rho = \frac{(1 + x_1^2)^{3/2}}{x_2} = a \quad \text{at } (2a, 0).$$ **Ans.**

EXERCISE 4A

1. Find the radius of curvature at the point (x, y) of the curves

   (*i*) $y = c \cosh (x/c)$ [GGSIPU I Sem I Term 2012]

   (*ii*) $y = a \log \sec (x/a)$. [GGSIPU I Sem I Term 2013]

2. Obtain the radius of curvature at the point (x, y) of the curves

   (*i*) $x^{2/3} + y^{2/3} = a^{2/3}$. (*ii*) $\sqrt{x} + \sqrt{y} = \sqrt{a}$

3. Find the radius of curvature at any point '*t*' on the curve

$$x = a\,(\cos t + \log \tan t/2), \quad y = a \sin t$$

4. The tangents at two points P and Q on the cycloid

$$x = a\,(\theta - \sin \theta), \quad y = a\,(1 - \cos \theta)$$

are at right angles. If ρ_1 and ρ_2 are the radii of curvature at these points then show that

$$\rho_1^2 + \rho_2^2 = 16a^2.$$

5. Apply Newton's method to find the radius of curveture at the origin for the curve

$$x^3 + y^3 - 2x^2 + 6y = 0.$$

6. Find the radius of curvature at the origin for the curve $x^3 + y^3 = 3axy$.

7. Show that the radius of curvature at the end of the major axis of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is equal to the semi-latus rectum of the ellipse.

8. Apply Newton's method to obtain the radius of curvature of the curve $y^2 = (2a - x)/x$ at the point $(2a, 0)$.

9. Find the radius of curvature of the curve $y^2 = x^2\,(a + x)/(a - x)$ at the origin.

10. For the curve $s = ce^{x/c}$ show that $c\rho = s\,(s^2 - c^2)^{1/2}$

11. For the curve $y = \dfrac{ax}{a+x}$, prove that

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$$

[GGSIPU I Sem End Term 2008; End Term 2011]

where ρ is the radius of curvature of the curve at its point (x, y).

12. Show that the ratio of radii of curvature at the points on the curves $xy = c^2$, $x^3 = 3c^2y$ which have same abscissa, varies as the square root of the ratio of the ordinates.

13. Find the radius of curvature at the origin for the curve

$$x^3y - xy^3 + 2x^2y + xy - y^2 + 2x = 0$$

14. For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, prove that $\rho = \dfrac{a^2b^2}{p^3}$, p being the perpendicular distance from the centre on the tangent at (x, y).

15. If ρ_1 and ρ_2 are the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$ then show that $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$. [GGSIPU I Sem End Term 2011]

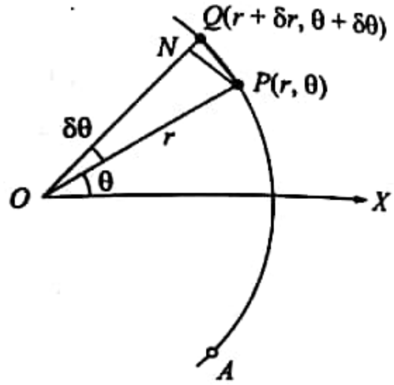
RADIUS OF CURVATURE (POLAR CURVES)

Let $P(r, \theta)$ be a point on the curve with equation $r = f(\theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be a neighbouring point on the curve.

Let s denote the length of the arc AP where A is some fixed point on the curve and then the arc AQ of length $s + \delta s$ so that arc PQ $= \delta s$. If N is the foot of the perpendicular from P on OQ, then $PN = r \sin \delta\theta$ and $NQ = r + \delta r - r \cos \delta\theta$.

Since $\delta\theta$ is very small we can write $PN = r\delta\theta$ and $NQ = \delta r$.

Therefore $$\delta s = \frac{\text{arc } PQ}{\text{chord } PQ} \cdot \text{chord } PQ = \frac{\text{arc } PQ}{\text{chord } PQ} \sqrt{(r\delta\theta)^2 + (\delta r)^2}$$

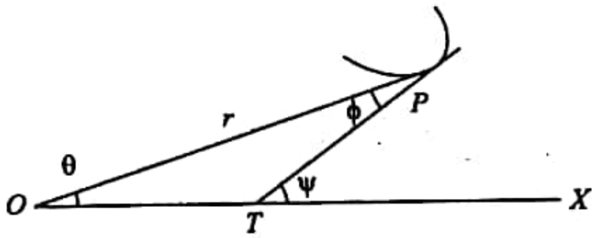


$$\Rightarrow \quad \frac{\delta s}{\delta\theta} = \frac{\text{arc } PQ}{\text{chord } PQ} \sqrt{r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2}$$

Taking limit as $\delta\theta \to 0$ we have

$$\frac{ds}{d\theta} = 1 \cdot \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r_1^2} \quad \text{where } r_1 = \frac{dr}{d\theta}.$$

Further, let PT be the tangent to the curve at P, making an angle ψ with the initial line OX and ϕ is the angle between OP, the radius vector of P and the tangent there, then, as in the figure $\psi = \theta + \phi$.



Hence $$\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

We know that $\tan \phi = \dfrac{r d\theta}{dr} = \dfrac{r}{r_1}$

Differentiating the above relation w.r.t. θ, we get

$$\sec^2 \phi \frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - r r_2}{r_1^2} \quad \text{where } r_2 = \frac{d^2 r}{d\theta^2}$$

$$\therefore \quad \frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2(1 + \tan^2\phi)} = \frac{r_1^2 - r r_2}{r_1^2\left(1 + \dfrac{r^2}{r_1^2}\right)} = \frac{r_1^2 - r r_2}{r_1^2 + r^2}$$

$$\Rightarrow \quad \frac{d\psi}{d\theta} = 1 + \frac{r_1^2 - r r_2}{r_1^2 + r^2} = \frac{2r_1^2 + r^2 - r r_2}{r_1^2 + r^2}$$

Therefore $$\rho = \frac{ds}{d\psi} = \frac{ds/d\theta}{d\psi/d\theta} = \frac{\sqrt{r^2 + r_1^2}\,(r^2 + r_1^2)}{2r_1^2 + r^2 - r r_2} = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - r r_2}.$$

CENTRE OF CURVATURE AND CIRCLE OF CURVATURE

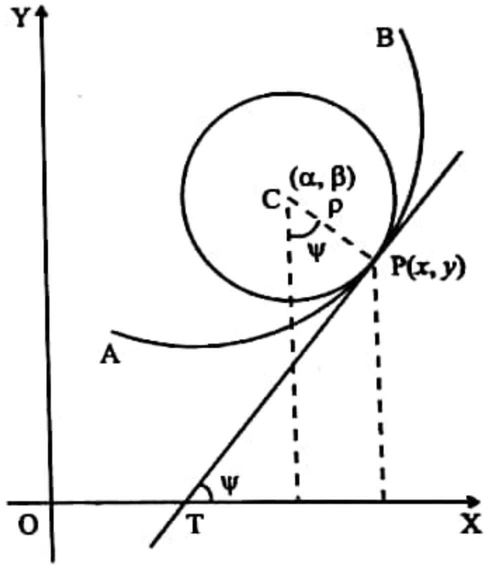
Let APB be a portion of the curve and P (x, y) be any point on it and PT the tangent to the curve at P. If we draw a circle passing through P on the same side of the tangent as the curve, with centre at a point C on the normal to the curve at P so that the radius CP = ρ, the radius of curvature at P, then such a circle is known as *circle of curvature* at P (See figure).

The point C is called the *centre of curvature*. Let C = (α, β) then from the figure, we have

$$\alpha = x - \rho \sin\psi \quad \text{and} \quad \beta = y + \rho\cos\psi.$$

Since $\tan\psi = \dfrac{dy}{dx} = y_1$, we have $\sin\psi = \dfrac{y_1}{\sqrt{1+y_1^2}}$ and $\cos\psi = \dfrac{1}{\sqrt{1+y_1^2}}$,

Already we have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$



therefore $\alpha = x - \dfrac{(1+y_1^2)^{3/2}}{y_2}\dfrac{y_1}{\sqrt{1+y_1^2}} = x - \dfrac{y_1(1+y_1^2)}{y_2}$

and $\beta = y + \dfrac{(1+y_1^2)^{3/2}}{y_2}\dfrac{1}{\sqrt{1+y_1^2}} = y + \dfrac{(1+y_1^2)}{y_2}$

Thus, the circle of curveture has the equation $(x-\alpha)^2 + (y-\beta)^2 = \rho^2$.

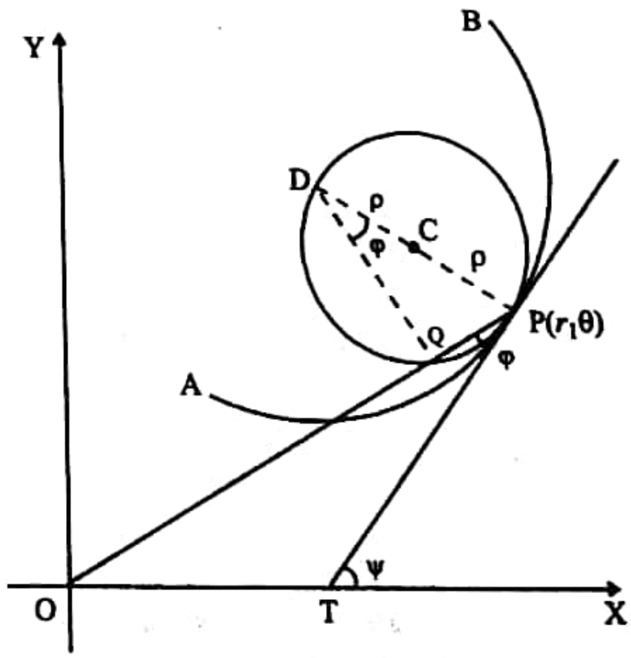
For parametric curves, we have

$$\alpha = x - \frac{y'(x'^2+y'^2)}{x'y''-y'x''}, \quad \beta = y + \frac{x'(x'^2+y'^2)}{x'y''-y'x''}$$

Further, the locus of the centre of curvature of the curve is called its **evolute** and the curve is referred to as **involute** of the evolute.

CHORD OF CURVATURE (THROUGH THE ORIGIN)

At the point P of the curve APB we draw the circle of curvature PQD with centre C and radius ρ. PT is the tangent to the curve at P making angle ψ with the axis of x. Joining the origin (pole) O to P we get the chord PQ called chord of curvature on the circle of curvature. Since $\angle$PDQ = $\angle$OPT = ϕ, as is clear from the adjoining figure, we have



Chord of curvature = PQ = 2ρ sin ϕ.

Next, *for the chord of curvature parallel to x-axis, we have* $\phi = \psi$ *hence* ***the chord of curvature parallel to x-axis*** $= 2\rho \sin\psi$ *and for the chord of curvature parallel to y-axis we have* $\phi = \pi/2 - \psi$ *hence* ***the chord of curvature parallel to y-axis*** $= 2\rho\cos\psi$.

EXAMPLE 4.11. Find the radius of curvature of the curve $r = a(1 - \cos\theta)$ at the point (r, θ).

SOLUTION: We have $r = a(1 - \cos\theta)$, hence $r_1 = \dfrac{dr}{d\theta} = a\sin\theta$ and $r_2 = \dfrac{d^2r}{d\theta^2} = a\cos\theta$.

$$\therefore\ \rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2} = \frac{\left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{2a^2\sin^2\theta + a^2(1-\cos\theta)^2 - a^2(1-\cos\theta)\cos\theta}$$

$$= \frac{a\left[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right]^{3/2}}{2\sin^2\theta + 1 + 2\cos^2\theta - 3\cos\theta} = \frac{a\left[2(1-\cos\theta)\right]^{3/2}}{3(1-\cos\theta)} = \frac{2a\sqrt{2}}{3}\sqrt{1-\cos\theta} = \frac{4a}{3}\sin\theta/2$$

Also, we can write $\rho = \dfrac{2a}{3}\sqrt{2(1-\cos\theta)} = \dfrac{2}{3}\sqrt{2ar}$ which implies that ρ varies as $\sqrt{r}$. **Ans.**

EXAMPLE 4.12. Show that the radius of curvature at the point (r, θ) of the curve

$$r^n = a^n \cos n\theta, \quad \text{is} \quad \frac{a^n r^{1-n}}{1+n}.$$

SOLUTION: We have $r^n = a^n\cos n\theta$ hence $n r^{n-1} r_1 = -n a^n \sin n\theta$.

or $\quad r_1 = \dfrac{-a^n \sin n\theta}{r^n}\, r = -\dfrac{-r a^n \sin n\theta}{a^n \cos n\theta} = -r\tan n\theta.$

and $\quad r_2 = -r_1\tan n\theta - r\sec^2 n\theta \cdot n = r\tan^2 n\theta - n r\sec^2 n\theta.$

$$\therefore\quad \rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2} = \frac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{2r^2\tan^2 n\theta + r^2 - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{r\sec^3 n\theta}{\tan^2 n\theta + 1 + n\sec^2 n\theta} = \frac{r\sec n\theta}{1+n} = \frac{r^n \sec n\theta}{(1+n)r^{n-1}} = \frac{a^n r^{1-n}}{n+1}.$$

Hence the result.

EXAMPLE 4.13. Show that at the point of intersection of the curves $r = a\theta$ and $r\theta = a$, the curvatures are in the ratio $3 : 1$ $(0 < \theta < 2\pi)$.

SOLUTION: The points of intersection of the curves $r = a\theta$ and $r\theta = a$, are given by

$a\theta^2 = a \quad$ or $\quad \theta = \pm 1.$

Now for the curve $r = a\theta$ we have $r_1 = a$ and $r_2 = 0$

$$\therefore\ at\ \theta = \pm 1,\quad \rho = \left[\frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2}\right]_{\theta=\pm 1} = \left[\frac{(a^2\theta^2 + a^2)^{3/2}}{2a^2 + a^2\theta^2 - 0}\right]_{\theta=\pm 1} = \frac{a(2\sqrt{2})}{3} = \rho_1, \text{ say.}$$

Next for the curve $r\theta = a$, we have $r_1 = \dfrac{-a}{\theta^2}$ and $r_2 = \dfrac{2a}{\theta^3}$, hence at $\theta = \pm 1$ $\quad \rho = \left[\dfrac{\left(\dfrac{a^2}{\theta^2}+\dfrac{a^2}{\theta^4}\right)^{3/2}}{2\dfrac{a^2}{\theta^4}+\dfrac{a^2}{\theta^2}-\dfrac{2a^2}{\theta^4}}\right]_{\theta=\pm 1} = \left[\dfrac{a\left(1+\theta^2\right)^{3/2}}{\theta^4}\right]_{\theta=\pm 1} = 2a\sqrt{2} = \rho_2$, say.

$\therefore \quad \dfrac{\rho_2}{\rho_1} = \dfrac{2a\sqrt{2}}{2a\sqrt{2}/3} = \dfrac{3}{1}$ Thus, $\rho_2 : \rho_1 = 3 : 1$ **Hence Proved.**

EXAMPLE 4.14. Find the centre of curvature of the parabola $x = at^2,\ y = 2at$ at the point 't' and hence find its evolute.

SOLUTION: As $x = at^2,\ y = 2at$ we have $y_1 = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt} = \dfrac{2a}{2at} = \dfrac{1}{t}$

and $\quad y_2 = \dfrac{d}{dx}\left(\dfrac{1}{t}\right) = -\dfrac{1}{t^2}\cdot\dfrac{dt}{dx} = -\dfrac{1}{t^2\cdot 2at} = -\dfrac{1}{2at^3}$

The coordinates of the centre of curvature $(\bar{x}, \bar{y})$ are given by

$$\bar{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2} \quad \text{and} \quad \bar{y} = y + \frac{1+y_1^2}{y_2}$$

or $\quad \bar{x} = at^2 - \dfrac{1}{t}\left(1+\dfrac{1}{t^2}\right)\left(-2at^3\right) = at^2 + 2a(1+t^2) = a(2+3t^2)$ ...(1)

and $\quad \bar{y} = 2at + \left(1+\dfrac{1}{t^2}\right)\left(-2at^3\right) = 2at - 2at(1+t^2) = -2at^3.$ ...(2)

Thus, the centre of curvature $= (\bar{x}, \bar{y}) = \{a(2+3t^2),\ -2at^3\}$.

For evolute, we find the locus of the centre of curvature, for which, eliminating t in (1) and (2), gives

$$(\bar{x}-2a)^3 = (3a)^3\left(\frac{\bar{y}}{-2a}\right)^2 \quad \text{or} \quad 4(\bar{x}-2a)^3 = 27a\,\bar{y}^2$$

$\therefore$ Evolute is $4(x-2a)^3 = 27ay^2$. **Ans.**

EXAMPLE 4.15. Find the centre of curvature of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ and also find the evolute of the curve.

SOLUTION: The parametric form of the equation of the given curve is $x = a\cos^3\theta,\ y = a\sin^3\theta$.

$\therefore \quad y_1 = \dfrac{dy}{dx} = \dfrac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta.$

and $\quad y_2 = \dfrac{d^2y}{dx^2} = -\sec^2\theta\,\dfrac{d\theta}{dx} = \dfrac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \dfrac{1}{3a\cos^4\theta\sin\theta}.$

Hence, the centre of curvature $(\bar{x}, \bar{y})$ at the point 'θ', is given by

$$\bar{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2} = a\cos^3\theta + \tan\theta\,(1+\tan^2\theta)\,3a\cos^4\theta\sin\theta = a(\cos^3\theta + 3\sin^2\theta\cos\theta)$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = a\sin^3\theta + (1+\tan^2\theta)\,3a\cos^4\theta\sin\theta = a(\sin^3\theta + 3\cos^2\theta\sin\theta).$$

Further, $\bar{x} + \bar{y} = a(\cos\theta + \sin\theta)^3$ and $\bar{x} - \bar{y} = a(\cos\theta - \sin\theta)^3$

or $(\bar{x}+\bar{y})^{2/3} = a^{2/3}(\cos\theta+\sin\theta)^2$ and $(\bar{x}-\bar{y})^{2/3} = a^{2/3}(\cos\theta - \sin\theta)^2$

$\therefore$ $(\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = 2a^{2/3}$

Therefore the evolute of the given curve, as locus of $(\bar{x}, \bar{y})$, is

$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$. **Ans.**

EXAMPLE 4.16 Prove that the length of the chord of curvature through the pole of the equiangular spiral $r = ae^{\theta\cot\alpha}$ is $2r$.

SOLUTION: The equation of the curve is $r = ae^{\theta\cot\alpha}$

hence $r_1 = a\cot\alpha\, e^{\theta\cot\alpha} = r\cot\alpha$ and $r_2 = r_1\cot\alpha = r\cot^2\alpha$.

Next, $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{r}{r_1} = \tan\alpha \Rightarrow \phi = \alpha$.

$$\therefore \quad \rho = \frac{\left(r^2+r_1^2\right)^{3/2}}{2r_1^2+r^2-rr_2} = \frac{\left(r^2+r^2\cot^2\alpha\right)^{3/2}}{2r^2\cot^2\alpha + r^2 - r^2\cot^2\alpha} = \frac{r\left(1+\cot^2\alpha\right)^{3/2}}{1+\cot^2\alpha} = r\operatorname{cosec}\alpha.$$

Now, we know that the length of the chord of curvature through the pole is equal to

$2\rho\sin\phi = 2r\operatorname{cosec}\alpha.\sin\alpha = 2r.$ **Hence Proved.**

EXAMPLE 4.17. Find the chord of curvature parallel to y-axis for the curve $y = a\log(\sec(x/a))$.

SOLUTION: We have $y = a\log(\sec(x/a))$

hence $\dfrac{dy}{dx} = y_1 = \dfrac{a}{\sec(x/a)}\cdot\sec(x/a)\tan(x/a)\cdot\dfrac{1}{a} = \tan(x/a)$ and $\dfrac{d^2y}{dx^2} = y_2 = \dfrac{1}{a}\sec^2(x/a)$

$$\therefore \quad \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{\left[1+\tan^2(x/a)\right]^{3/2}}{\frac{1}{a}\sec^2\frac{x}{a}} = a\sec(x/a)$$

Next, $\tan\psi = \dfrac{dy}{dx} = \tan(x/a)$, hence $\psi = x/a$.

Therefore, the chord of curvature parallel to y-axis

$= 2\rho\cos\psi = 2a\sec(x/a)\cos(x/a) = 2a.$ **Ans.**

EXERCISE 4B

1. Find the radius of curvature at the point (r,θ) of the following curves

   (*i*) $r = a\theta$ (*ii*) $r = a\sin\theta$.

2. Find the radius of curvature at the point (r, θ) of the parabola

$$\frac{2l}{r} = 1 + \cos\theta.$$

3. Show that the radius of curvature at the point (r, θ) of the curve

$$r^2 \cos 2\theta = a^2 \quad \text{is} \quad \frac{r^3}{a^2}.$$

4. Find the radius of curvature at the point (r, θ) of the curve

$$\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}(a/r).$$

5. Show that the radius of curvature at any point 't' of the curve

   $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ is given by $\rho = at$.

6. Find the radius of curvature at the origin for the curve $r = a\sin n\theta$.

7. If ρ_1 and ρ_2 are the radii of curvature at the extremities of a chord of the cardioide $r = a(1 + \cos\theta)$ passing through the pole, prove that $\rho_1^2 + \rho_2^2 = \frac{16}{9}a^2$.

8. Find the centre of curvature at the point (x, y) of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and also obtain its evolute.

9. Find the radius of curvature and the centre of curvature for the curve $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

10. Show that the centre of curvature at any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{is} \quad \left(\frac{a^2 - b^2}{a^4}x^3, \frac{a^2 - b^2}{-b^4}y^3\right).$$

11. Find the chord of curvature through the pole of the cardioide $r = a(1 + \cos\theta)$.

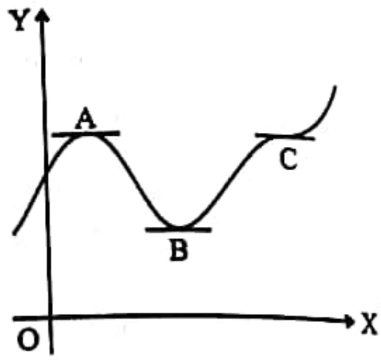
12. For the curve $r^2 = a^2\cos 2\theta$ show that

$$\rho = \frac{a^2}{3r}$$

CONCAVITY, CONVEXITY AND POINT OF INFLEXION

We have discussed earlier that in an interval in which $\frac{dy}{dx} > 0$, y is an increasing function of x, *i.e.*, the curve is rising to the right, while if $\frac{dy}{dx} < 0$, y is a decreasing function of x, *i.e.*, the curve is falling to the right. It has also been seen that at the points where the curve changes its course (from rising to falling or from falling to rising) $\frac{dy}{dx} = 0$ and hence tangent there is parallel to x-axis. However, there may exist points on the curve at which $\frac{dy}{dx} = 0$, but these points need not necessarily be the points of maxima or minima.

In the adjoining figure we notice that at the point A the curve wholly lies below the tangent there and we say that the curve is ***concave downwards*** (or ***convex upwards***) at A, whereas at the point B the curve wholly lies above the tangent there and we say that the curve is ***concave upwards*** (or ***convex downwards***). However, at the point C the curve crosses the tangent (or the tangent crosses the curve). Here on the immediate left of C the curve is concave downwards and on the immediate right of C the curve is concave upwards. Such a point is called the ***point of inflexion***. Thus, *at the point of inflexion the curve changes its course either from concave downwards to concave upwards or from concave upwards to concave downwards.*



We observe here that at the point A curve changes its course from increasing to decreasing, therefore, $\frac{dy}{dx}$ changes its sign from positive to negative which clearly means that $\frac{dy}{dx}$ itself is a decreasing function of x and hence $\frac{d^2y}{dy^2} < 0$ and the curve there is concave downwards. Similarly, at the point B the curve changes its course from decreasing to increasing therefore $\frac{dy}{dx}$ changes its sign from negative to positive which clearly means that $\frac{dy}{dx}$ itself is an increasing function of x and hence $\frac{d^2y}{dx^2} > 0$ and the curve there is concave upwards. However, at a point where the direction of bending of the curve changes from concave upwards to concave downwards or vice versa, is called the point of inflexion and $\frac{d^2y}{dx^2} = 0$ there. In the above figure the point C on the curve is a point of inflexion as the curve changes its course from concave downwards to concave upwards.

Thus, we can state that "A point $x = a$ is a point of inflexion on the curve $y = f(x)$, if (*i*) $\frac{d^2y}{dx^2} = 0$ at $x = a$ and (*ii*) $\frac{d^3y}{dx^3} \neq 0$ at $x = a$."

Generalising the above assertion we say that at the point of inflexion the even order derivative should be zero and the next odd order derivative there should not be zero.

EXAMPLE 4.18. Show that the origin is a point of inflexion on the curve $a^{n-1}y = x^n$ if n is odd and greater than 2.

SOLUTION: The curve $a^{n-1}y = x^n$ passes through the origin. On differentiating *w.r.t.* x, we get

$$a^{n-1}\frac{dy}{dx} = n\,x^{n-1} \quad \text{and} \quad a^{n-1}\frac{d^2y}{dx^2} = n\,(n-1)\,x^{n-2}.$$

For the point of inflexion the even order derivative should be zero at that point and the next odd derivative there should not be zero. Clearly, if n is an odd positive integer greater than or equal to 3 then the even order derivative will be zero at $x = 0$ and the first next odd order derivative will not be zero there, therefore the origin is the point of inflexion of the given curve. **Hence Proved.**

EXAMPLE 4.19. Find the point of inflexion of the curve $y^2 = (x-1)^2(x-2)$.

SOLUTION: We have $y = (x-1)\sqrt{x-2}$ $\quad\therefore\quad \dfrac{dy}{dx} = \dfrac{(x-1)}{2\sqrt{x-2}} + 1\cdot\sqrt{x-2} = \dfrac{3x-5}{2\sqrt{x-2}}$

and $$\frac{d^2y}{dx^2} = \frac{\sqrt{x-2}\cdot 3 - (3x-5)\dfrac{1}{2\sqrt{x-2}}}{2(x-2)} = \frac{3x-7}{4(x-2)^{3/2}}$$ which is zero at $x = 7/3$.

The next odd order derivative $\dfrac{d^3y}{dx^3}$ is given by

$$\frac{d^3y}{dx^3} = \frac{(x-2)^{3/2}\cdot 3 - (3x-7)\frac{3}{2}(x-2)^{1/2}}{4\,(x-2)^3} = \frac{3[2\,(x-2)-(3x-7)]}{8(x-2)^{5/2}}$$

$$= \frac{-3x+9}{8\,(x-2)^{5/2}}$$ which is not zero at $x = \dfrac{7}{3}$.

Therefore, the point $\left(\dfrac{7}{3}, \dfrac{4}{3\sqrt{3}}\right)$ on the curve, is a point of inflexion. **Ans.**

EXAMPLE 4.20. Find the range of values of x in which the curve $x^2 - xy + 4 = 0$ is concave upwards or concave downwards.

SOLUTION: The given equation can be written as $y = x + \dfrac{4}{x}$.

$$\therefore \quad \frac{dy}{dx} = 1 - \frac{4}{x^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{8}{x^3}.$$

Therefore, in the interval $(-\infty, 0)$, $\dfrac{d^2y}{dx^2}$ is negative, hence the curve is concave downwards and in the interval $(0, \infty)$, $\dfrac{d^2y}{dx^2}$ is positive, hence the curve is concave upwards. **Ans.**

EXAMPLE 4.21. Obtain the range of values of x for which the curve $y = 3x^5 - 40x^3 + 3x - 20$ is concave upwards or concave downwards. Also find the points of inflexion, if any.

SOLUTION: We have $y = 3x^5 - 40x^3 + 3x - 20$,

hence $\dfrac{dy}{dx} = 15x^4 - 120x^2 + 3$ and $\dfrac{d^2y}{dx^2} = 60x^3 - 240x = 60x\ (x-2)(x+2)$

and $\dfrac{d^3y}{dx^3} = 180x^2 - 240 = 60\ (3x^2 - 4)$. Now, $\dfrac{d^2y}{dx^2} = 0$ at $x = 0, -2, 2$

Since $\dfrac{d^3y}{dx^3}$ does not vanish at any of the points $x = 0, -2, 2$ hence $x = 0, \pm 2$ are three points of inflexion of the given curve. Now, let us consider the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, $(2, \infty)$.

In $(-\infty, -2)$ $\dfrac{d^2y}{dx^2} < 0$ hence the curve is concave downwards;

in $(-2, 0)$ $\dfrac{d^2y}{dx^2} > 0$ hence the curve is concave upwards;

in $(0, 2)$ $\dfrac{d^2y}{dx^2} < 0$ hence the curve is concave downwards, and

in $(2, \infty)$ $\dfrac{d^2y}{dx^2} > 0$ hence the curve is concave upwards. **Ans.**

EXAMPLE 4.22. Find the points of inflexion of the curve $y = (x-2)^6 (x-3)^5$.

SOLUTION: Equation of the curve is $y = (x-2)^6 (x-3)^5$

hence $\dfrac{dy}{dx} = 6(x-2)^5 (x-3)^5 + 5(x-2)^6 (x-3)^4 = (x-2)^5 (x-3)^4 (11x-28)$

and
$$\frac{d^2y}{dx^2} = 11\,(x-2)^5 (x-3)^4 + (11x-28)\,[5\,(x-2)^4 (x-3)^4 + 4(x-2)^5 (x-3)^3]$$
$$= (x-2)^4 (x-3)^3\,[11\,(x-2)(x-3) + (11x-28)(5x-15+4x-8)]$$
$$= (x-2)^4 (x-3)^3\,[110x^2 - 560x + 710]$$

Now $\dfrac{d^2y}{dx^2} = 0$ when $x = 2, x = 3$ and when $11x^2 - 56x + 71 = 0$

which gives $x = \left(28 \pm \sqrt{3}\right)/11$.

But $\dfrac{d^3y}{dx^3} \neq 0$ at $x = \left(28 \pm \sqrt{3}\right)/11$.

therefore, $x = \left(28 \pm \sqrt{3}\right)/11$ are two points of inflexion also, $\dfrac{d^3y}{dx^3} = 0$ at $x = 2$ and at $x = 3$

At $x = 3$, $\dfrac{d^4y}{dx^4} = 0$ and $\dfrac{d^5y}{dx^5} \neq 0$

hence $x = 3$ is yet another point of inflexion.

Next, at $x = 2$, $\dfrac{d^5y}{dx^5} = 0$ and $\dfrac{d^6y}{dx^6} \neq 0$ hence $x = 2$ is not a point of inflexion. **Ans.**

EXERCISE 4C

1. Find the points of inflexion of the curve $y = \dfrac{2x}{1+x^2}$.

2. Find the points of inflexion of the curve $y = \dfrac{x}{x^2+2x+2}$.

3. Obtain the angle subtended by the line joining two points of inflexion of the curve

   $y^2 (x - a) = x^2 (x + a)$ at the origin.

4. Show that the curve $y = e^{-x}$ is concave upwards at every point of it.

5. Find the points of inflexion of the curve $x = a (2\theta - \sin \theta)$, $y = a (2 - \cos \theta)$.

6. Obtain the points of inflexion, if any, of the curve $x^2 y = a^2 (x - y)$.

7. Show that every point in which the sine curve $y = k \sin (x/a)$

   meets the x-axis is a point of inflexion.

8. Find the range of values of x in which the curve $y = e^{-x^2}$ is concave upwards or concave downwards.

❑❑❑

CHAPTER

5 Properties of Definite Integrals and Reduction Formulae

Definite Integrals, Properties, Reduction Formulae for Integrals of the Form $\int_0^{\pi/2} \sin^n \theta \, d\theta$, $\int_0^{\pi/2} \cos^n \theta \, d\theta$, $\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta$

DEFINITE INTEGRALS AND THEIR PROPERTIES

Generally, we define *indefinite integrals* in terms of anti-derivative (or primitive) of a function. If F (x) is the primitive of $f(x)$, then $\frac{d}{dx}$ F $(x) = f(x)$ so that $\int f(x)\,dx = \mathrm{F}(x)$.

If $f(x)$ is defined in the interval $[a, b]$, then the *definite integral* of $f(x)$ is written as

$$\int_a^b f(x)\,dx = \mathrm{F}(x)\Big|_a^b = \mathrm{F}(b) - \mathrm{F}(a).$$

Actually, the definite integral, defined above, denotes the area bounded by the curve $y = f(x)$, the x-axis and two ordinates at $x = a$ and $x = b$. Given below are some important properties of the definite integrals. Since these are simple consequences of the definition, proofs of some of the obvious properties will be omitted here.

I. The process of definite integration is a linear operation. Thus, if $f_1(x)$ and $f_2(x)$ are continuous and bounded functions over the range $[a, b]$ and k_1 and k_2 are two constants, then

$$\int_a^b \left[k_1 f_1(x) + k_2 f_2(x)\right] dx = k_1 \int_a^b f_1(x)\,dx + k_2 \int_a^b f_2(x)\,dx$$

This is called *linearity property.*

II. In a definite integral, the variable of integration is dummy,

hence $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(y)\,dy$

III. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

IV. If $f(x)$ in integrable over $[a, b]$ and c is any point such that $a < c < b$, then $f(x)$ is integrable over $[a, c]$ and $[c, b]$ and hence

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

V. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

It is known as *invariance property* and can be proved very easily.

Putting $x = a - t$, so that $dx = -dt$, we have

$$\int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt = \int_0^a f(a-t)\,dt \quad \text{by property III.}$$

$$= \int_0^a f(a-x)\,dx \quad \text{by property II.}$$

VI. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even, *i.e.*, $f(-x) = f(x)$.

$= 0$ if $f(x)$ is odd, *i.e.*, $f(-x) = -f(x)$.

To establish this property, we write

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \quad \text{by property IV.}$$

In the first integral on the R.H.S., we write $x = -t$, so $dx = -dt$,

hence $\int_{-a}^0 f(x)\,dx = -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt$ by property III

$$= \int_0^a f(-x)\,dx \quad \text{by property II}$$

$$\therefore \quad \int_{-a}^a f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$$

$$\begin{cases} = 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even, } i.e., \text{ if } f(-x) = f(x) \\ = 0 & \text{if } f(x) \text{ is odd, } i.e., f(-x) = -f(x). \end{cases}$$

VII. $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

To derive this property, we write

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \quad \text{by property IV.}$$

In the second integral on the R.H.S, we put $x = 2a - t$, so $dx = -dt$,

then $\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt$ by property III

$$= \int_0^a f(2a-x)\,dx \quad \text{by property II}$$

$$\therefore \quad \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = \int_0^a [f(x)+f(2a-x)]\,dx$$

$$\begin{cases} = 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ = 0 & \text{if } f(2a-x) = -f(x). \end{cases}$$

VIII. Mean Value Theorem of Integrals

Let m and M be the least and the greatest values of $f(x)$ in $[a, b]$ and let the interval (a, b) be divided into n parts δx_k, $k = 1, 2, \ldots, n$, then

$$m\,\delta x_k \le f(x_k)\,\delta x_k \le M\,\delta x_k$$

which means that

$$\sum_1^n m\,\delta x_k \le \sum_{k=1}^n f(x_k)\,\delta x_k \le \sum_1^n M\,\delta x_k$$

or $$m(b-a) \le \sum_{k=1}^n f(x_k)\,\delta x_k \le M(b-a)$$

Since $f(x)$ is integrable over $[a, b]$, the above relation becomes

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \quad \text{when } n \to \infty.$$

or $$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

If we assume that $f(x)$ is continuous in $[a, b]$ then $f(x)$ assumes every value between m and M. Therefore, there exists a number ξ such that $a < \xi < b$ for which

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

This is called first mean value theorem of Integral Calculus.

IX. First Fundamental Theorem of Integral Calculus:

Let $f(x)$ be a continuous function over the closed interval $[a, b]$ and bounded then the function

$F(x) = \int_a^x f(x)\,dx$ is continuous on $[a, b]$, differentiable in (a, b) and $F'(x) = f(x)$.

Second Fundamental Theorem of Integral Calculus:

Let $F(x)$ be an anti derivative of a continuous function $f(x)$ on $[a, b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

EXAMPLE 5.1. Evaluate $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\,dx$.

SOLUTION: Let $I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$ ...(1)

or $$I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}\,dx}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}$$ (using property V)

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}}$$ ...(2)

Adding (1) and (2), gives $2I = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$ hence $I = \frac{\pi}{4}$. **Ans.**

EXAMPLE 5.2. Evaluate $\int_0^{\pi/2} \log \sin\theta\,d\theta$.

SOLUTION: Let $I = \int_0^{\pi/2} \log \sin\theta\,d\theta$ then by property V, ...(1)

$$I = \int_0^{\pi/2} \log\ \sin\left(\frac{\pi}{2}-\theta\right)d\theta = \int_0^{\pi/2} \log \cos\theta\ d\theta$$ ...(2)

Adding (1) and (2), gives

$$2I = \int_0^{\pi/2} (\log \sin\theta + \log \cos\theta)\,d\theta = \int_0^{\pi/2} \log(\sin\theta\ \cos\theta)\,d\theta = \int_0^{\pi/2} \log\left(\frac{1}{2}\sin 2\theta\right)d\theta$$

$$= \int_0^{\pi/2} (\log \sin 2\theta - \log 2)\,d\theta = \int_0^{\pi/2} \log \sin 2\theta\,d\theta - \frac{\pi}{2}\log 2 = I_1 - \frac{\pi}{2}\log 2$$

where $I_1 = \int_0^{\pi/2} \log \sin\ 2\theta\ d\theta$. (Putting $2\theta = \phi$ so $d\theta = \frac{1}{2}\,d\phi$)

$$= \frac{1}{2}\int_0^{\pi} \log \sin\phi\,d\phi = \frac{1}{2}\int_0^{\pi} \log \sin\theta\,d\theta$$ (by Property II)

$$= \frac{1}{2}\cdot 2\int_0^{\pi/2} \log \sin\theta\,d\theta$$ (by Property VII)

$\therefore$ $2I = I - \frac{\pi}{2}\log 2$ or $I = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$. **Ans.**

EXAMPLE 5.3. Evaluate $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx.$

SOLUTION: Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)\, dx}{\sec(\pi - x) + \tan(\pi - x)}$ (by Property V)

$$= \int_0^{\pi} \frac{-(\pi - x)\tan x\, dx}{-\sec x - \tan x} = \pi \int_0^{\pi} \frac{\tan x\, dx}{\sec x + \tan x} - I$$

$\therefore$
$$2I = \pi \int_0^{\pi} \frac{\tan x\, dx}{\sec x + \tan x} = \pi \int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} dx = \pi \int_0^{\pi} \frac{\sec x \tan x - \tan^2 x}{1} dx$$

$$= \pi \int_0^{\pi} \left(\sec x \tan x - \sec^2 x + 1\right) dx = \pi\left[\sec x - \tan x + x\right]_0^{\pi} = \pi(\pi - 2)$$

or
$$I = \frac{\pi}{2}(\pi - 2).$$

Ans.

EXAMPLE 5.4. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x\, dx}{\sin x + \cos x}.$

SOLUTION : Let $I = \int_0^{\pi/2} \frac{\sin^2 x\, dx}{\sin x + \cos x}$ ...(1)

Using the property V, we have

$$I = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x)\, dx}{\sin(\pi/2 - x) + \cos(\pi/2 - x)} = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx \qquad ...(2)$$

Adding (1) and (2), gives

$$2I = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin x \cos \pi/4 + \cos x \sin \pi/4}$$

or
$$I = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin(x + \pi/4)} = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx = \frac{1}{2\sqrt{2}}\left[\log \tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{2\sqrt{2}}\left[\log \tan \frac{3\pi}{8} - \log \tan \frac{\pi}{8}\right] = \frac{1}{2\sqrt{2}} \log \frac{\tan(3\pi/8)}{\tan(\pi/8)}$$

$$= \frac{1}{2\sqrt{2}} \log \frac{2\sin(3\pi/8)\cos(\pi/8)}{2\cos(3\pi/8)\sin(\pi/8)} = \frac{1}{2\sqrt{2}} \log \frac{\sin\frac{\pi}{2} + \sin\frac{\pi}{4}}{\sin\frac{\pi}{2} - \sin\frac{\pi}{4}} = \frac{1}{2\sqrt{2}} \log\left(\frac{1 + 1/\sqrt{2}}{1 - 1/\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}} \log \frac{\sqrt{2} + 1}{\sqrt{2} - 1} = \frac{1}{2\sqrt{2}} \log \frac{\left(\sqrt{2} + 1\right)^2}{2 - 1} = \frac{2}{2\sqrt{2}} \log\left(\sqrt{2} + 1\right)$$

or
$$I = \frac{1}{\sqrt{2}} \log\left(1 + \sqrt{2}\right).$$

Ans.

Example 5.5. Evaluate $\int_0^1 \frac{\sin^{-1} x}{x} dx$.

Solution: Let $I = \int_0^1 \frac{\sin^{-1} x}{x} dx$. Putting $x = \sin\theta$ so $dx = \cos\theta\, d\theta$, then

$$I = \int_0^{\pi/2} \frac{\theta.\cos\theta\, d\theta}{\sin\theta} = \int_0^{\pi/2} \theta\cot\theta\; d\theta. \text{ (Integrating it by parts)}$$

$$= [\theta\ \log\ \sin\theta]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\log\sin\theta\ \ d\theta = 0 - \lim_{\theta\to 0}(\theta\log\sin\theta) - \int_0^{\pi/2}\log\sin\theta\ \ d\theta$$

From Example 5.2 we have $\int_0^{\pi/2} \log\sin\theta\, d\theta = -\frac{\pi}{2}\log 2$ and

$\lim_{\theta\to 0}(\theta\log\sin\theta) = \lim_{\theta\to 0}\frac{\log\sin\theta}{1/\theta}\left(\text{form } \frac{\infty}{\infty}\right)$. Now applying L' Hospitals' rule

$$= \lim_{\theta\to 0}\frac{\cot\theta}{-\frac{1}{\theta^2}} = -\lim_{\theta\to 0}\frac{\theta^2}{\tan\theta} = -\lim_{\theta\to 0}\frac{\theta}{\tan\theta}\cdot\lim_{\theta\to 0}\theta = 0$$

Therefore $I = \frac{\pi}{2}\log 2.$ **Ans**

Example 5.6. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$.

Solution: Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$. Putting $x = \tan\theta$, so $dx = \sec^2\theta\, d\theta$, we have

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta\, d\theta = \int_0^{\pi/4}\log(1+\tan\theta)\, d\theta$$

$$= \int_0^{\pi/4}\log(1+\tan(\pi/4-\theta))\, d\theta \quad \text{(by property V)}$$

$$= \int_0^{\pi/4}\log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta = \int_0^{\pi/4}\log\frac{2}{1+\tan\theta}\ \ d\theta$$

$$= \int_0^{\pi/4}\log 2\ \ d\theta - \int_0^{\pi/4}\log(1+\tan\theta)\ \ d\theta = \frac{\pi}{4}\log 2 - I$$

or $2I = \frac{\pi}{4}\log 2$ or $I = \frac{\pi}{8}\log 2.$ **Ans.**

EXAMPLE 5.7. Evaluate $\int_0^1 \cot^{-1}(1-x+x^2)\,dx$.

SOLUTION: The given integral can be written as

$$I = \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = \int_0^1 \tan^{-1}\left(\frac{1}{1+x(x-1)}\right)dx$$

$$= \int_0^1 \tan^{-1}\left\{\frac{x-(x-1)}{1+x(x-1)}\right\}dx = \int_0^1\left[\tan^{-1}x - \tan^{-1}(x-1)\right]dx.$$

$$= \int_0^1 \tan^{-1}x\ dx - \int_0^1 \tan^{-1}(x-1)\ dx$$

But $\int_0^1 \tan^{-1}(x-1)\ dx = \int_0^1 \tan^{-1}(1-x-1)\ dx = -\int_0^1 \tan^{-1}x\ dx$ (by property V)

$\therefore$ $I = 2\int_0^1 \tan^{-1}x\ dx$. Integrating it by parts taking 1 as second function, we get

$$I = 2\left[\tan^{-1}x \cdot x\right]_0^1 - 2\int_0^1 \frac{x}{1+x^2}\ dx = 2\cdot 1\cdot\frac{\pi}{4} - \left[\log(1+x^2)\right]_0^1$$

or $I = \frac{\pi}{2} - \log 2$ **Ans.**

EXAMPLE 5.8. Show that $\int_0^{\pi/2} \cos^3 2x \cdot \sin^4 4x\ dx = 0$

SOLUTION: Let $I = \int_0^{\pi/2} \cos^3 2x \cdot \sin^4 4x\ dx$ Putting $2x = t$ we get

$$I = \frac{1}{2}\int_0^{\pi} \cos^3 t \cdot \sin^4 2t\ dt = \frac{1}{2}\int_0^{\pi} 2^4 \cos^3 t \sin^4 t \cos^4 t\ dt$$

$$= 8\int_0^{\pi} \sin^4 t \cos^7 t\ \ dt = 0 \quad \text{(by property VII)}$$ **Hence Proved.**

EXAMPLE 5.9. Show that $\int_0^{\pi} x\,f(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$ [GGSIPU I Sem End Term 2007]

SOLUTION: Let $I = \int_0^{\pi} x\,f(\sin x)dx.$ Applying the property

$\int_0^a g(x)dx = \int_0^a g(a-x)dx$ here, we get

$$I = \int_0^{\pi}(\pi-x)\,f[\sin(\pi-x)]dx = \int_0^{\pi}(\pi-x)\,f(\sin x)dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi}(x+\pi-x)\,f(\sin x)dx = \pi\int_0^{\pi} f(\sin x)dx$$

or $I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$ **Hence the result.**

EXERCISE 5A

Evaluate

1. $\displaystyle\int_0^2 |1-x|\, dx.$

2. $\displaystyle\int_0^\pi \sqrt{\frac{1+\cos 2x}{2}}\, dx.$

3. $\displaystyle\int_0^\pi \sin^5 x\, dx.$

4. $\displaystyle\int_0^\pi \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}.$

5. $\displaystyle\int_0^\pi \frac{x\, dx}{a^2-\cos^2 x}\ (a>1).$

6. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{dx}{5+7\cos x+\sin x}.$

7. Show that $\displaystyle\int_0^\pi \frac{dx}{13+3\cos x+4\sin x} = \frac{1}{6}\left[\frac{\pi}{2}-\tan^{-1}\frac{1}{3}\right].$

8. Evaluate $\displaystyle\int_0^\pi \frac{2+3\cos x\ dx}{\sin x+2\cos x+3}.$

9. Show that $\displaystyle\int_0^\pi \frac{d\theta}{a^2-2a\cos\theta+1} = \frac{\pi}{1-a^2}$ or $\dfrac{\pi}{a^2-1}$ according as $a<1$ or $a>1$.

10. Evaluate $\displaystyle\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}.$

11. Evaluate $\displaystyle\int_0^\pi \sin^6 x\cos^7 x\ dx.$

12. Evaluate $\displaystyle\int_0^\pi \log(1+\cos x)\ dx.$

REDUCTION FORMULAE

Suppose we wish to evaluate $\quad I_n = \int x^n e^{ax}\, dx \quad$ where $n \in I^+$ ...(1)

Integrating it by parts taking x^n as first function and e^{ax} as second, we get

$$I_n = x^n \frac{e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a}\, dx$$

or $$I_n = \frac{x^n}{a} e^{ax} - \frac{n}{a} I_{n-1}. \qquad ...(2)$$

Formulae like the one above are called *reduction formulae.* These can be used repeatedly to reduce the integral. Let us illustrate the use of relation (2), taking $n = 5$, and $a = 2$.

Then $$I_5 = \int x^5 e^{2x}\, dx = x^5 \frac{e^{2x}}{2} - \frac{5}{2} I_4$$

Now applying the above formula for $n = 4$ and $a = 2$, we get

$$I_4 = \frac{x^4}{2} e^{2x} - \frac{4}{2} I_3.$$

Therefore $$I_5 = \frac{x^5}{2} e^{2x} - \frac{5}{2}\left[\frac{x^4}{2} e^{2x} - \frac{4}{2} I_3\right] = \frac{x^5}{2} e^{2x} - \frac{5}{4} x^4 e^{2x} + 5 I_3$$

$$= \frac{x^5}{2} e^{2x} - \frac{5}{4} x^4 e^{2x} + 5\left[x^3 \frac{e^{2x}}{2} - \frac{3}{2} I_2\right]$$

$$= \frac{x^5}{2} e^{2x} - \frac{5x^4}{4} e^{2x} + \frac{5}{2} x^3 e^{2x} - \frac{15}{2} I_2$$

$$= e^{2x}\left[\frac{x^5}{2} - \frac{5}{4} x^4 + \frac{5}{2} x^3\right] - \frac{15}{2}\left[\frac{x^2}{2} e^{2x} - \frac{2}{2} I_1\right]$$

$$= e^{2x}\left(\frac{x^5}{2} - \frac{5}{4} x^4 + \frac{5}{2} x^3 - \frac{15}{4} x^2\right) + \frac{15}{2} I_1$$

$$= e^{2x}\left(\frac{1}{2} x^5 - \frac{5}{4} x^4 + \frac{5}{2} x^3 - \frac{15}{4} x^2\right) + \frac{15}{2}\left(x \frac{e^{2x}}{2} - \frac{1}{2} I_0\right)$$

$$= e^{2x}\left(\frac{1}{2} x^5 - \frac{5}{4} x^4 + \frac{5}{2} x^3 - \frac{15}{4} x^2 + \frac{15}{4} x\right) - \frac{15}{4} \int x^0 e^{2x}\, dx$$

or $$I_5 = e^{2x}\left(\frac{1}{2} x^5 - \frac{5}{4} x^4 + \frac{5}{2} x^3 - \frac{15}{4} x^2 + \frac{15}{4} x - \frac{15}{8}\right) + C$$

As a special case, consider the reduction formula (2) when $a = -1$, then we have

$$I_n = \int x^n e^{-x}\, dx = -x^n e^{-x} + n \int x^{n-1} e^{-x}\, dx = -x^n e^{-x} + n I_{n-1} \qquad ...(3)$$

Now, if we consider the definite integral $I_n = \int_0^\infty e^{-x} x^n\, dx$ ...(4)

then the formula (3), gives

$$I_n = \left[-x^n e^{-x}\right]_0^\infty + n I_{n-1} = 0 + n I_{n-1} \qquad ...(5)$$

The function defined by the integral in (4) is called **Gamma Function** and is denoted by $\overline{|n+1}$.

Therefore, we have $\overline{|n+1} = n\overline{|n}$ as the reduction formula

Thus, $\overline{|1} = 0! = 1,\ \overline{|2} = 1\overline{|1} = 1,\ \overline{|3} = 2!,\ \overline{|4} = 3!, \ldots$

It will be discussed in detail in a latter chapter.

Reduction Formula for $\int \sin^n x\ dx\ (n \in N)$

Let $I_n = \int \sin^n x\ dx\ (n \geq 2) = \int \sin^{n-1} x \sin x\, dx$. Integrating it by parts, we get

$$I_n = \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)dx$$

$$= -\sin^{n-1} x\ \cos x + (n-1)\int \sin^{n-2} x(1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

or $$(1+n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

or $$I_n = -\frac{1}{n}\sin^{n-1} x\ \cos x + \frac{n-1}{n} I_{n-2} \qquad \ldots(5)$$

which is the reduction formula for $\int \sin^n x\ dx$.

Now, if we consider the definite integral $I_n = \int_0^{\pi/2} \sin^n x\, dx$ Using (5), gives us

$$I_n = \int_0^{\pi/2} \sin^n x\, dx = -\frac{1}{n}[\sin^{n-1} x\ \cos x]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

or $$I_n = 0 + \frac{n-1}{n} I_{n-2}.$$

or $$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx \qquad \ldots(6)$$

Reduction Formula for $\int \cos^n x\ dx\ (n \in N)$

Let $I_n = \int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx$. Integrating by parts, we get

$$I_n = \cos^{n-1} x\ \sin x - \int (n-1)\cos^{n-2} x(-\sin x)\sin x\, dx$$

$$= \cos^{n-1} x\ \sin x + (n-1)\int \cos^{n-2} x(1 - \cos^2 x)\ dx$$

$$= \cos^{n-1} x\ \sin x + (n-1)\int \cos^{n-2} x\ dx - (n-1)\int \cos^n x\ dx$$

$$= \cos^{n-1} x\ \sin x + (n-1) I_{n-2} - (n-1) I_n$$

or $$(1+n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

or $$I_n = \frac{1}{n}\cos^{n-1}x \;\sin x + \frac{n-1}{n} I_{n-2}. \qquad \text{...(7)}$$

Now, if we consider the definte integral

$$I_n = \int_0^{\pi/2} \cos^n x \; dx \text{ then, relation (7), gives}$$

$$I_n = \frac{1}{n}\Big[\cos^{n-1} x \; \sin x\Big]_0^{\pi/2} + \frac{n-1}{n} \; I_{n-2} = 0 + \frac{n-1}{n} \; I_{n-2}$$

Thus, $$\int_0^{\pi/2} \cos^n x \; dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x \; dx \qquad \text{...(8)}$$

[GGSIPU I Sem II Term 2001]

The above two reduction formulae (6) and (8) can be together written as

$$I_n = \int_0^{\pi/2} \begin{matrix}\sin\\ \cos\end{matrix}^n x \, dx = \frac{n-1}{n}\int_0^{\pi/2} \begin{matrix}\sin\\ \cos\end{matrix}^{n-2} x \; dx = \frac{n-1}{n} I_{n-2} \qquad \text{...(9)}$$

Applying the reduction formula (9) repeatedly, we get

$$I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2} I_{n-4} = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4} I_{n-6} = \dots$$

Two cases arise here for n being even or odd.

$$I_n = \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{3}{4}\cdot\dfrac{1}{2} \; I_0 & \text{if } n \text{ is even} \\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{4}{5}\cdot\dfrac{2}{3} \; I_1 & \text{if } n \text{ is odd} \end{cases}$$

Now $$I_0 = \int_0^{\pi/2} \begin{matrix}\sin\\ \cos\end{matrix}^0 x \, dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2} \quad \text{and} \quad I_1 = \int_0^{\pi/2} \begin{matrix}\sin\\ \cos\end{matrix} x \, dx = 1$$

Therefore, $$I_n = \int_0^{\pi/2} \begin{matrix}\sin\\ \cos\end{matrix}^n x \, dx = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\dots 3\cdot 1}{n(n-2)(n-4)\dots 4\cdot 2}\dfrac{\pi}{2} & \text{if } n \text{ is even} \\[2ex] \dfrac{(n-1)(n-3)(n-5)\dots 4\cdot 2}{n(n-2)(n-4)\dots 5\cdot 3}\cdot 1 & \text{if } n \text{ is odd} \end{cases}$$

Reduction Formula for $\int \sin^m x \cos^n x \; dx$ $(m, n \in N)$

Let $I_{m,n} = \int \sin^m x \cos^n x \; dx$. Then, we have

$$I_{m,n} = \int \sin^m x \cos^{n-1} x \cos x \, dx = \int \cos^{n-1} x \left(\sin^m x \; \cos x\right) dx$$

Integrating it by parts taking ($\sin^m x \cos x$) as second function, gives

$$I_{m,n} = \cos^{n-1} x \; \frac{\sin^{m+1} x}{m+1} - \int (n-1)\cos^{n-2} x\,(-\sin x)\frac{\sin^{m+1} x}{m+1} \; dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \left(1 - \cos^2 x\right) dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \ dx - \frac{n-1}{m+1} \int \sin^m x \cos^n x \ dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \mathrm{I}_{m,\, n-2} - \frac{n-1}{m+1} \mathrm{I}_{m,\, n}$$

or $$\left(1 + \frac{n-1}{m+1}\right) \mathrm{I}_{m,\, n} = \frac{\cos^{n-1} x \ \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \mathrm{I}_{m,n-2}$$

or $$\mathrm{I}_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} \mathrm{I}_{m,\, n-2}. \quad ...(10)$$

In this reduction formula the power of cosine gets reduced by two. Similarly, if the power of sine is to be reduced by two, the corresponding reduction formula will be :

$$\mathrm{I}_{m,n} = \int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,\, n}. \quad ...(11)$$

In case of definite integral with limits 0 to $\pi/2$, we have

$$\mathrm{I}_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx = \left[\frac{\cos^{n-1} x \ \sin^{m+1} x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m x \cos^{n-2} dx$$

or $$\mathrm{I}_{m,n} = 0 + \frac{n-1}{m+n} \mathrm{I}_{m,\, n-2}. \quad ...(12)$$

Similarly, we can have $$\mathrm{I}_{m,\, n} = \frac{m-1}{m+n} \mathrm{I}_{m-2,\, n} \quad ...(13)$$

[*GGSIPU I Sem II Term 2004, End Term 2007*]

Replacing n by $n-2$ in (12), we get

$$\mathrm{I}_{m,n-2} = \frac{n-3}{m+n-2} \mathrm{I}_{m,\, n-4},$$ hence (12) becomes

$$\mathrm{I}_{m,n} = \frac{(n-1)(n-3)}{(m+n)(m+n-2)} \mathrm{I}_{m,\, n-4}$$

$$= \frac{(n-1)(n-3)(n-5)}{(m+n)(m+n-2)(m+n-4)} \mathrm{I}_{m,\, n-6},$$ and so on.

Continuing the process, finally there arise two cases

CASE I: When n is an even integer

$$\mathrm{I}_{m,n} = \frac{(n-1)(n-3)(n-5)\ldots 3.1}{(m+n)(m+n-2)(m+n-6)\ldots(m+2)} \mathrm{I}_{m,0}$$

But $$\mathrm{I}_{m,0} = \int_0^{\pi/2} \sin^m x \, dx = \mathrm{I}_m$$

$$\therefore \quad I_{m,n} = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\ldots 1}{(m+n)(m+n-2)\ldots(m+2)} \cdot \dfrac{(m-1)(m-3)\ldots 1}{m(m-2)\ldots 2} \cdot \dfrac{\pi}{2} & \text{if } m \text{ is even} \\ \dfrac{(n-1)(n-3)(n-5)\ldots 1}{(m+n)(m+n-2)\ldots(m+2)} \cdot \dfrac{(m-1)(m-3)\ldots 2}{m(m-2)\ldots 3} \cdot 1 & \text{if } m \text{ is odd} \end{cases}$$

CASE II: When n is an odd integer

$$I_{m,n} = \frac{(n-1)(n-3)(n-5)\ldots 2}{(m+n)(m+n-2)\ldots(m+3)} I_{m,1}.$$

But $$I_{m,1} = \int_0^{\pi/2} \sin^m x \cos x \, dx = \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}.$$

$$\therefore \quad I_{m,n} = \frac{(n-1)(n-3)\ldots 4.2}{(m+n)(m+n-2)\ldots(m+3)} \frac{1}{(m+1)}.$$

EXAMPLE 5.10. (a) Evaluate $\int \cos^6 x \, dx$ *[GGSIPU I Sem End Term 2009]*

(b) Evaluate $\int_0^{\pi/4} \sin^7 x \, dx$ *[GGSIPU I Sem I Term 2013]*

SOLUTION: (a) We know that $\int \cos^n x \, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx$ ...(1)

Taking $n = 6$, we have $\int \cos^6 x \, dx = \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\int \cos^4 x \, dx$

Now, using (1) here for $n = 4$, we get

$$\int \cos^6 x \, dx = \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\left[\frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x \, dx\right]$$

Again, using (1) here for $n = 2$, we have

$$\int \cos^6 x \, dx = \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{8}\left[\frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1 \, dx\right]$$

$$= \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{16}\cos x \sin x + \frac{5x}{16}. \quad \textbf{Ans.}$$

(b) We know that $\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$

$$\therefore \quad \int_0^{\pi/4} \sin^7 x \, dx = -\frac{1}{7}\left[\sin^6 x \cos x\right]_0^{\pi/4} + \frac{6}{7}\int_0^{\pi/4} \sin^5 x \, dx \quad \text{...(1)}$$

and $$\int_0^{\pi/4} \sin^5 x \, dx = -\frac{1}{5}\left[\sin^4 x \cos x\right]_0^{\pi/4} + \frac{4}{5}\int_0^{\pi/4} \sin^3 x \, dx \quad \text{...(2)}$$

and $$\int_0^{\pi/4} \sin^3 x \, dx = -\frac{1}{3}\left[\sin^2 x \cos x\right]_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} \sin x \, dx \quad \text{...(3)}$$

Using (2) and (3) in (1), we get

$$\int_0^{\pi/4} \sin^7 x\, dx = -\frac{1}{7}\sin^6\frac{\pi}{4}\cos\frac{\pi}{4} + \frac{6}{7}\left[-\frac{1}{5}\sin^4\frac{\pi}{4}\cos\frac{\pi}{4} + \frac{4}{5}\left\{-\frac{1}{3}\sin^2\frac{\pi}{4}\cos\frac{\pi}{4} + \frac{2}{3}(-\cos x)_0^{\pi/4}\right\}\right]$$

$$= -\frac{1}{7}\sin^6\frac{\pi}{4}\cos\frac{\pi}{4} - \frac{6}{35}\sin^4\frac{\pi}{4}\cos\frac{\pi}{4} + \frac{24}{35}\left[-\frac{1}{3}\sin^2\frac{\pi}{4}\cos\frac{\pi}{4} - \frac{2}{3}\cos\frac{\pi}{4} + \frac{2}{3}\right]$$

$$= -\frac{1}{7}\left(\frac{1}{\sqrt{2}}\right)^7 - \frac{6}{35}\left(\frac{1}{\sqrt{2}}\right)^5 - \frac{8}{35}\left(\frac{1}{\sqrt{2}}\right)^3 - \frac{16}{35}\frac{1}{\sqrt{2}} + \frac{16}{35} \quad \textbf{Ans.}$$

EXAMPLE 5.11. Evaluate $\int_0^{\pi/2} \sin^5 x \cos 2x\, dx$ [GGSIPU I Sem End Term 2013]

SOLUTION:

$$I = \int_0^{\pi/2} \sin^5 x \cos 2x\, dx = \int_0^{\pi/2} \sin^5 x\,(\cos^2 x - \sin^2 x)\, dx$$

$$= \int_2^{\pi/2} \sin^5 x \cos^2 x\, dx - \int_0^{\pi/2} \sin^7 x\, dx$$

Using here the reduction formulae

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2} x \cos^n x\, dx \quad \text{and} \quad \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx,$$

we get

$$I = \frac{4}{7}\int_0^{\pi/2} \sin^3 x \cos^2 x\, dx - \frac{6}{7}\int_0^{\pi/2} \sin^5 x\, dx$$

$$= \frac{4}{7}\cdot\frac{2}{5}\int_0^{\pi/2} \sin x \cos^2 x\, dx - \frac{6}{7}\cdot\frac{4}{5}\int_0^{\pi/2} \sin^3 x\, dx$$

$$= \frac{8}{35}\left[-\frac{1}{3}\cos^3 x\right]_0^{\pi/2} - \frac{24}{35}\cdot\frac{2}{3}\int_0^{\pi/2} \sin x\, dx = \frac{8}{105} + \frac{16}{35} = \frac{56}{105} \quad \textbf{Ans.}$$

EXAMPLE 5.12. Evaluate $\int_{-\pi}^{\pi} \sin^4 x \cos^2 x\, dx$.

SOLUTION: Since the integrand $\sin^4 x \cos^2 x$ is an even function of x, hence by the properties of definite integrals, we have $\int_{-\pi}^{\pi} \sin^4 x \cos^2 x\, dx = 2\int_0^{\pi} \sin^4 x \cos^2 x\, dx = 4\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$ on using property VII.

Now, applying the reduction formula, the given integral is equal to

$$= 4\,\frac{(2-1)}{4+2}\int_0^{\pi/2} \sin^4 x\, dx = \frac{4}{6}\cdot\frac{(4-1)(4-3)}{4(4-2)}\cdot\frac{\pi}{2} = \frac{\pi}{8}. \quad \textbf{Ans.}$$

EXAMPLE 5.13 Evaluate $\int_0^{\pi} \sin^m x \cos^n x\, dx$ (GGSIPU Ist Sem. IInd Term 2004)

SOLUTION: Let $I = \int_0^{\pi} \sin^m x \cos^n x\, dx = \int_0^{\pi/2} \sin^m x \cos^n x\, dx + \int_{\pi/2}^{\pi} \sin^m x \cos^n x\, dx$ (Putting $x = \pi - t$)

$$= \int_0^{\pi/2} \sin^m x \cos^n x\, dx - \int_{\pi/2}^{0} \sin^m (\pi - t) \cos^n (\pi - t) dt$$

$$= \int_0^{\pi/2} \sin^m x \cos^n x\, dx + (-1)^n \int_0^{\pi/2} \sin^m t \cos^n t\, dt$$

$$= \begin{cases} 2\int_0^{\pi/2} \sin^m x \cos^n x\, dx & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Thus, $I = 0$ when n is odd.

When n is even, $I = \int_0^{\pi/2} \sin^m \cos^n x\, dx = \dfrac{2(n-1)(n-3)\dots 3 \cdot 1}{(m+n)(m+n-2)\dots(m-2)} \int_0^{\pi/2} \sin^m x\, dx$

$$= \begin{cases} \dfrac{2(n-1)(n-3)\dots 5 \cdot 3 \cdot 1}{(m+n)(m+n-2)\dots(m-2)} \cdot \dfrac{(m-1)(m-3)\dots(5 \cdot 3 \cdot 1)}{m(m-2)(m-4)\dots 4 \cdot 2} \cdot \dfrac{\pi}{2} & \text{if } m \text{ is even} \\ \dfrac{2(n-1)(n-3)\dots 5 \cdot 3 \cdot 1}{(m+n)(m+n-2)\dots(m-2)} \cdot \dfrac{(m-1)(m-3)\dots 5 \cdot 3 \cdot 1}{m(m-2)\dots 4 \cdot 2} \cdot 1 & \text{if } m \text{ is odd} \end{cases}$$

EXAMPLE 5.14 Evaluate $\int_0^{\pi} x \sin^7 x \cos^4 x\, dx$ [GGSIPU I Sem II Term 2011]

SOLUTION: Let $I = \int_0^{\pi} x \sin^7 x \cos^4 x\, dx$

Using property V of definite integrals, we have

$$I = \int_0^{\pi} (\pi - x) \sin^7 (\pi - x) \cos^4 (\pi - x)\, dx = \int_0^{\pi} (\pi - x) \sin^7 x \cos^4 x\, dx$$

$$= \pi \int_0^{\pi} \sin^7 x \cos^4 x\, dx - I$$

or $\quad 2I = \pi \int_0^{\pi} \sin^7 x \cos^4 x\, dx = 2\pi \int_0^{\pi/2} \sin^7 x \cos^4 x\, dx$ (by property VII of definite integrals)

or $\quad I = \pi \int_0^{\pi/2} \sin^7 x \cos^4 x\, dx$. Now applying reduction formula, we have

$$I = \pi \cdot \frac{(4-1)(4-3)}{(7+4)(7+2)} \int_0^{\pi/2} \sin^7 x\, dx = \pi \cdot \frac{3 \cdot 1}{11 \cdot 9} \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)} \cdot 1 = \frac{16\pi}{1155}.$$ **Ans.**

EXAMPLE 5.15. Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^5}$.

SOLUTION: Let $I = \int_0^\infty \frac{dx}{(1+x^2)^5}$. Putting $x = \tan\theta$, so $dx = \sec^2\theta\, d\theta$, we get

$$I = \int_0^{\pi/2} \frac{\sec^2\theta\ d\theta}{(1+\tan^2\theta)^5} = \int_0^{\pi/2} \frac{\sec^2\theta\ d\theta}{\sec^{10}\theta}$$

$$= \int_0^{\pi/2} \cos^8\theta\ d\theta = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}.$$ **Ans.**

EXAMPLE 5.16. Evaluate $\int_0^4 x^3\sqrt{4x-x^2}\ dx$.

SOLUTION: Let $I = \int_0^4 x^3\sqrt{4x-x^2}\ dx = \int_0^4 x^{\frac{7}{2}}(4-x)^{\frac{1}{2}}\ dx$

Putting $x = 4\sin^2\theta$, so $dx = 8\sin\theta\cos\theta\ d\theta$, we have

$$I = \int_0^{\pi/2} 4^{7/2}\sin^7\theta\,(4-4\sin^2\theta)^{1/2}\cdot 8\sin\theta\cos\theta\, d\theta$$

$$= 2^7\cdot 16\int_0^{\pi/2} \sin^8\theta\cos^2\theta\ d\theta = 128.16\cdot\frac{1}{10}\int_0^{\pi/2}\sin^8\theta\ d\theta$$

$$= 128\cdot\frac{16}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 28\,\pi.$$ **Ans.**

REDUCTION FORMULA FOR $\int \tan^n x\ dx$

Let $I_n = \int \tan^n x\ dx$, then we can write

$$I_n = \int \tan^{n-2} x\ \tan^2 x\, dx = \int \tan^{n-2} x\left(\sec^2 x - 1\right) dx$$

$$= \int \tan^{n-2} x\ \sec^2 x\ dx - \int \tan^{n-2} x\ dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\ dx$$

or $\quad I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$ which is the required reduction formula.

REDUCTION FORMULA FOR $\int \sec^n x\ dx$ [GGSIPU I Sem II Term 2010]

Let $I_n = \int \sec^n x\ dx = \int (\sec^{n-2} x)\sec^2 x\ dx$

Integrating it by parts, gives

$$= (\sec^{n-2} x)\tan x - \int (n-2)\,(\sec^{n-3})x\ \sec x\ \tan x\ \tan x\ dx$$

$$= (\sec^{n-2} x)\tan x - (n-2)\int (\sec^{n-2} x)\tan^2 x\ dx$$

$$= (\sec^{n-2} x) \tan x - (n-2) \int (\sec^{n-2} x)(\sec^2 x - 1)\, dx$$
$$= (\sec^{n-2} x) \tan x - (n-2) \int \sec^n x \; dx + (n-2) \int (\sec^{n-2} x) \; dx$$
$$= (\sec^{n-2} x) \tan x - (n-2)\, I_n + (n-2)\, I_{n-2}$$

or $\quad I_n (1 + n - 2) = \sec^{n-2} x \tan x + (n-2)\, I_{n-2}$

or $\quad I_n = \dfrac{1}{n-1} \sec^{n-2} x \tan x + \dfrac{n-2}{n-1} I_{n-2}$

which is the required reduction formula.

EXAMPLE 5.17. Evaluate $\int \tan^4 x\, dx$.

[*GGSIPU I Sem End Term 2004 Reappear*]

SOLUTION: Using the reduction formula $I_n = \int \tan^n x\, dx = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$ ...(1)

for $n = 4$, we have $I_4 = \dfrac{1}{3} \tan^3 x - I_2$ ...(2)

and on taking $n = 2$ in (1), we get $I_2 = \tan x - I_0 = \tan x - \int \tan^0 x\, dx = \tan x - x$

Therefore from (2) we get $I_4 = \dfrac{1}{3} \tan^3 x - (\tan x - x) + C$ **Ans.**

where C is an constant of integration.

EXAMPLE 5.18. Evaluate $\displaystyle\int_0^{2a} x^3 (2ax - x^2)^{3/2}\, dx$.

SOLUTION: Let $\displaystyle I = \int_0^{2a} x^3 (2ax - x^2)^{3/2}\, dx = \int_0^{2a} x^{9/2} (2a - x)^{3/2}\, dx$.

Putting $x = 2a \sin^2 \theta$, so $dx = 4a \sin\theta \cos\theta\; d\theta$, we get

$$I = \int_0^{\pi/2} (2a)^{9/2} \sin^9 \theta\, (2a - 2a \sin^2 \theta)^{3/2}\, 4a \sin\theta \cos\theta \; d\theta$$

$$= (2a)^6\, 4a \int_0^{\pi/2} \sin^{10}\theta \cos^4 \theta \; d\theta = 64 \times 4a^7 \cdot \frac{3}{14} \cdot \frac{1}{12} \int_0^{\pi/2} \sin^{10} \theta \; d\theta$$

$$= \frac{32}{7} a^7\, \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{9\pi a^7}{16}.$$ **Ans.**

EXAMPLE 5.19. Evaluate $\displaystyle\int_0^{\infty} \frac{x^6 - x^3}{\left(1 + x^3\right)^5} x^2\; dx$.

SOLUTION: Let $\displaystyle I = \int_0^{\infty} \frac{x^6 - x^3}{\left(1 + x^3\right)^5} x^2\, dx$. Putting $x^3 = \tan^2 \theta$, so $3x^2\, dx = 2 \tan\theta \sec^2 \theta\, d\theta$

then $$I = \int_0^{\pi/2} \frac{\left(\tan^4 \theta - \tan^2 \theta\right)}{\left(1 + \tan^2 \theta\right)^5} \frac{2}{3} \tan\theta \sec^2 \theta \; d\theta = \frac{2}{3} \int_0^{\pi/2} \frac{\tan^5 \theta \; d\theta}{\sec^8 \theta} - \frac{2}{3} \int_0^{\pi/2} \frac{\tan^3 \theta \; d\theta}{\sec^8 \theta}$$

$$= \frac{2}{3}\int_0^{\pi/2} \sin^5\theta\cos^3\theta \; d\theta - \frac{2}{3}\int_0^{\pi/2} \sin^3\theta\cos^5\theta \; d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \sin^5\theta\cos^3\theta \; d\theta - \frac{2}{3}\int_0^{\pi/2} \sin^3\left(\frac{\pi}{2}-\theta\right)\cos^5(\pi/2-\theta) \; d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \sin^5\theta\cos^3\theta \; d\theta - \frac{2}{3}\int_0^{\pi/2} \cos^3\theta\sin^5\theta \; d\theta = 0.$$ **Ans.**

EXAMPLE 5.20. If $I_n = \int x^n \left(a^2-x^2\right)^{\frac{1}{2}} dx$, establish the recurrence relation
$(n+2)\,I_n = -x^{n-1}(a^2-x^2)^{3/2} + a^2(n-1)\,I_{n-2}$.

SOLUTION: We can write $I_n = \int x^{n-1}\left(x\sqrt{a^2-x^2}\right)dx.$ ...(1)

Consider $J = \int x\sqrt{a^2-x^2}\,dx$. (On putting $a^2-x^2=t^2$, so $-2x\,dx = 2t\,dt$),

$$= -\int t.t\,dt = \frac{-t^3}{3} = \frac{-1}{3}(a^2-x^2)^{3/2}.$$

Now integrating (1) by parts taking x^{n-1} as first function, we get

$$I_n = -\frac{1}{3}x^{n-1}\left(a^2-x^2\right)^{3/2} - \int (n-1)\,x^{n-2}\left(\frac{-1}{3}\right)\left(a^2-x^2\right)^{3/2} dx$$

$$= -\frac{x^{n-1}}{3}\left(a^2-x^2\right)^{3/2} + \frac{n-1}{3}\int x^{n-2}\left(a^2-x^2\right)^{3/2} dx$$

$$= -\frac{x^{n-1}}{3}\left(a^2-x^2\right)^{3/2} + \frac{n-1}{3}\int x^{n-2}\left(a^2-x^2\right)\left(a^2-x^2\right)^{1/2} dx$$

$$= -\frac{x^{n-1}}{3}\left(a^2-x^2\right)^{3/2} + \frac{(n-1)a^2}{3}\int x^{n-2}\left(a^2-x^2\right)^{1/2} dx - \frac{n-1}{3}\int x^n\left(a^2-x^2\right)^{1/2} dx$$

$$= -\frac{x^{n-1}}{3}\left(a^2-x^2\right)^{3/2} + \frac{(n-1)a^2}{3}I_{n-2} - \frac{n-1}{3}I_n$$

or $$\left(1+\frac{n-1}{3}\right)I_n = -\frac{1}{3}x^{n-1}\left(a^2-x^2\right)^{3/2} + (n-1)\frac{a^2}{3}I_{n-2}$$

or $$(n+2)\,I_n = -x^{n-1}(a^2-x^2)^{3/2} + a^2(n-1)\,I_{n-2}$$ **Hence Proved.**

EXAMPLE 5.21. If $I_n = \int x^n (a-x)^{1/2} dx$, prove that
$(2n+3)\,I_n = 2an\,I_{n-1} - 2x^n(a-x)^{3/2}$

and hence evaluate $\int_0^a x^2(a-x)^{1/2}\,dx$.

SOLUTION: Integrating I_n by parts taking x^n as first function, we get

$$I_n = -x^n\frac{2}{3}(a-x)^{3/2} - \int n x^{n-1}(a-x)^{3/2}\left(\frac{-2}{3}\right)dx$$

$$= -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2n}{3}\int x^{n-1}(a-x)(a-x)^{1/2}\,dx$$

$$= -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2an}{3}\int x^{n-1}(a-x)^{1/2}\,dx - \frac{2n}{3}\int x^n(a-x)^{1/2}\,dx$$

$$= -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2an}{3}\mathrm{I}_{n-1} - \frac{2n}{3}\mathrm{I}_n$$

or $$\left(1+\frac{2n}{3}\right)\mathrm{I}_n = -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2an}{3}\mathrm{I}_{n-1}$$

or $$(2n+3)\,\mathrm{I}_n = -2x^n(a-x)^{3/2} + 2an\,\mathrm{I}_{n-1} \qquad ...(1)$$

Hence Proved.

Now, we consider the definite integral $\mathrm{I}_2 = \int_0^a x^2(a-x)^{1/2}\,dx$

Putting $n = 2$ in (1), we get

$$7\,\mathrm{I}_2 = -\left[2x^2(a-x)^{3/2}\right]_0^a + 4a\,\mathrm{I}_1 = 0 + 4a\,\mathrm{I}_1$$

Again taking $n = 1$ in (1), we get

$$5\,\mathrm{I}_1 = -\left[2x(a-x)^{3/2}\right]_0^a + 2a\,\mathrm{I}_0 = 0 + 2a\,\mathrm{I}_0$$

$\therefore$ $$7\,\mathrm{I}_2 = \frac{4a}{5}2a\,\mathrm{I}_0 = \frac{8a^2}{5}\mathrm{I}_0$$

and $$\mathrm{I}_0 = \int_0^a x^0(a-x)^{1/2}\,dx = -\frac{2}{3}\left[(a-x)^{3/2}\right]_0^a = \frac{2}{3}a^{3/2}$$

Therefore, $$\mathrm{I}_2 = \frac{8a^2}{35}\cdot\frac{2}{3}a^{3/2} = \frac{16}{105}a^{7/2}.$$ **Ans.**

[illegible] (a) If $U_n = \int_0^{\pi/2}\theta\sin^n\theta\,d\theta$, prove that $\mathrm{U}_n = \frac{n-1}{n}U_{n-2} + \frac{1}{n^2}$ hence evaluate U_3 and U_5. [*GGSIPU I Sem II Term 2002; II Term 2007*]

(b) If $\mathrm{I}_n = \int_0^{\pi/2}\theta\cos^n\theta\,d\theta$, prove that

$$\mathrm{I}_n = -\frac{1}{n^2} + \frac{n-1}{n}\mathrm{I}_{n-2},$$ and hence evaluate I_4.

[*GGSIPU I Sem End Term 2008*]

Solution: (a) Given $U_n = \int_0^{\pi/2}\theta\sin^n\theta\,d\theta = \int_0^{\pi/2}\theta\sin^{n-1}\theta\sin\theta\,d\theta$

Integrating by parts taking $\sin\theta$ as second function, we get

$$U_n = \left[\theta\sin^{n-1}\theta\cdot(-\cos\theta)\right]_0^{\pi/2} - \int_0^{\pi/2}[\sin^{n-1}\theta + \theta.(n-1)\sin^{n-2}\theta\cdot\cos\theta](-\cos\theta)\,d\theta$$

$$= 0 + \int_0^{\pi/2} \sin^{n-1}\theta \cos\theta \, d\theta + (n-1)\int_0^{\pi/2} \theta \sin^{n-2}\theta \cos^2\theta d\theta$$

$$= \left[\frac{1}{n}\sin^n\theta\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \theta \sin^{n-2}\theta\,(1-\sin^2\theta)\, d\theta$$

$$= \frac{1}{n} + (n-1)\int_0^{\pi/2} \theta \sin^{n-2}\theta\, d\theta - (n-1)\int_0^{\pi/2} \theta \sin^n\theta\, d\theta$$

$$= \frac{1}{n} + (n-1)U_{n-2} - (n-1)U_n$$

or $\quad (1+n-1)\,U_n = \dfrac{1}{n} + (n-1)\,U_{n-2}$

or $$U_n = \frac{1}{n^2} + \frac{n-1}{n}U_{n-2} \qquad \text{... (1)}$$

Putting $n = 3$, $U_3 = \dfrac{1}{3^2} + \dfrac{2}{3}U_1 = \dfrac{1}{9} + \dfrac{2}{3}\displaystyle\int_0^{\pi/2} \theta \sin\theta d\theta$

$$= \frac{1}{9} + \frac{2}{3}[\theta(-\cos\theta)]_0^{\pi/2} - \frac{2}{3}\int_0^{\pi/2} 1\,(-\cos\theta) d\theta$$

$$= \frac{1}{9} + 0 + \frac{2}{3}[\sin\theta]_0^{\pi/2} = \frac{1}{9} + \frac{2}{3} = \frac{7}{9}$$

In (1) we put next $n = 5$ to get $U_5 = \dfrac{1}{5^2} + \dfrac{4}{5}U_3 = \dfrac{1}{25} + \dfrac{4}{5}\cdot\dfrac{7}{9} = \dfrac{149}{225}$

Thus, $\quad U_3 = \dfrac{7}{9}$ and $U_5 = \dfrac{149}{225}$. **Ans.**

(b) We can write $I_n = \displaystyle\int_0^{\pi/2} \theta \cos^{n-1}\theta \cdot \cos\theta \; d\theta$.

Integrating it by parts taking $\cos\theta$ as second function, we get

$$I_n = \left[\theta \cos^{n-1}\theta \sin\theta\right]_0^{\pi/2} - \int_0^{\pi/2} \left\{1\cdot\cos^{n-1}\theta - (n-1)\theta\cos^{n-2}\theta\sin\theta\right\}\sin\theta \; d\theta$$

$$= 0 - \int_0^{\pi/2} \cos^{n-1}\theta \sin\theta \; d\theta + (n-1)\int_0^{\pi/2} \theta\cos^{n-2}\theta\sin^2\theta \; d\theta$$

$$= \left[\frac{\cos^n\theta}{n}\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \theta \cos^{n-2}\theta\left(1-\cos^2\theta\right) \; d\theta$$

$$= -\frac{1}{n} + (n-1)\,I_{n-2} - (n-1)\,I_n$$

or $\quad (1+n-1)\,I_n = -\frac{1}{n}+(n-1)\,I_{n-2}$ or $I_n = -\frac{1}{n^2}+\frac{n-1}{n}I_{n-2}$ ...(1)

Now, first taking $n=4$ in (1) and then $n=2$, we get

$$I_4 = -\frac{1}{16}+\frac{3}{4}I_2 = -\frac{1}{16}+\frac{3}{4}\left[-\frac{1}{4}+\frac{1}{2}I_0\right]$$

$$\therefore \quad I_4 = -\frac{1}{4}+\frac{3}{8}I_0 = -\frac{1}{4}+\frac{3}{8}\int_0^{\frac{\pi}{2}}\theta\, d\theta = -\frac{1}{4}+\frac{3\pi^2}{64}$$ **Ans.**

EXAMPLE 5.23. If $I_{m,n} = \int_0^1 \left(1-x^m\right)^n dx \;\; (n \in N)$ show that $(1+mn)\,I_{m,n} = m\,n\,I_{m,n-1}$

SOLUTION: Given $I_{m,n} = \int_0^1 \left(1-x^m\right)^n dx.$

Integrating it by parts taking 1 as second function, we get

$$I_{m,n} = \left[\left(1-x^m\right)^n x\right]_0^1 - \int_0^1 n\left(1-x^m\right)^{n-1}(-m)\,x^{m-1}\,x\;dx$$

$$= 0 - mn\int_0^1 \left(1-x^m\right)^{n-1}\left(-x^m\right)dx$$

$$= -mn\int_0^1 \left(1-x^m\right)^{n-1}\left(1-x^m-1\right)dx$$

$$= -mn\int_0^1 \left(1-x^m\right)^n dx + mn\int_0^1 \left(1-x^m\right)^{n-1} dx = -mn\,I_{m,n} + mn\,I_{m,n-1}$$

or $\quad (1+mn)\,I_{m,n} = mn\,I_{m,n-1},$ **Hence the result.**

EXAMPLE 5.24. By considering the integral $\int_0^1 x^{2n+1}\left(1-x^2\right)^{-1/2} dx$ show that

$$\frac{1}{2n+2}+\frac{1}{2}\frac{1}{(2n+4)}+\frac{1}{2}\cdot\frac{3}{4}\frac{1}{2n+6}+\ldots = \frac{2\cdot4\cdot6\ldots2n}{3\cdot5\cdot7\ldots(2n+1)}.$$

SOLUTION: Let $I = \int_0^1 x^{2n+1}\left(1-x^2\right)^{-1/2} dx$

Putting $x = \sin\theta$, $dx = \cos\theta\, d\theta$, we get

$$I = \int_0^{\pi/2} \sin^{2n+1}\theta\left(\cos^2\theta\right)^{-1/2}\cos\theta\, d\theta = \int_0^{\pi/2}\sin^{2n+1}\theta\, d\theta$$

Applying reduction formula here, we get

$$I = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{2\cdot4\cdot6\ldots\ldots\ldots2n}{3\cdot5\cdot7\ldots\ldots\ldots(2n+1)}.$$

Next, using Binomial expansion $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots,$ we have

$$(1-x^2)^{-1/2} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(-x^2)^2}{2!} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{(-x^2)^3}{3!} + \ldots$$

$$= 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2^2\cdot 2!}x^4 + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^6 + \ldots$$

Then $$I = \int_0^1 x^{2n+1}\left[1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2^2\cdot 2!}x^4 + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^6 + \ldots\right]dx$$

$$= \int_0^1 \left[x^{2n+1} + \frac{1}{2}x^{2n+3} + \frac{1\cdot 3}{2^2\cdot 2!}x^{2n+5} + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^{2n+7} + \ldots\right]dx$$

$$= \left[\frac{x^{2n+2}}{2n+2} + \frac{1}{2}\cdot\frac{x^{2n+4}}{2n+4} + \frac{1\cdot 3}{2^2\cdot 2!}\frac{x^{2n+6}}{2n+6} + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}\frac{x^{2n+8}}{2n+8} + \ldots\right]_0^1$$

$$= \frac{1}{(2n+2)} + \frac{1}{2(2n+4)} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{2n+6} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{2n+8} + \ldots$$

Hence the result.

EXAMPLE 5.25. Prove the following $\int_0^\infty x^{n-1}e^{-ax}\cos bx\,dx = \frac{\Gamma n}{(a^2+b^2)^{n/2}}\cos\left(n\tan^{-1}\frac{b}{a}\right)$

[*GGSIPU III Sem End Term 2003, 2007*]

SOLUTION: Consider the integral $I = \int_0^\infty x^{n-1}e^{-ax}\cdot e^{-ibx}dx = \int_0^\infty x^{n-1}e^{-(a+ib)x}dx$

Putting $(a+ib)x = t$ we get $I = \int_0^\infty \frac{t^{n-1}}{(a+ib)^{n-1}}e^{-t}\frac{dt}{a+ib} = \int_0^\infty t^{n-1}e^{-t}\frac{dt}{(a+ib)^n}.$

Next, let $a = r\cos\alpha$, $b = r\sin\alpha$ then $(a+ib)^n = r^n(\cos n\alpha + i\sin n\alpha)$

where $r = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}(b/a)$.

$$\therefore \quad \frac{1}{(a+ib)^n} = \frac{1}{r^n}(\cos n\alpha - i\sin n\alpha) = \frac{1}{\sqrt{(a^2+b^2)^n}}[\cos(n\tan^{-1}b/a) - i\sin(n\tan^{-1}b/a)]$$

Hence $I = \frac{\Gamma n}{r^n}(\cos n\alpha - i\sin n\alpha)$ and the given integral is the real part of I, so its value is $\frac{\Gamma n}{(a^2+b^2)^{n/2}}\cos(n\tan^{-1}b/a)$.

Hence Proved.

EXERCISE 5B

1. Evaluate $\int_0^{\pi/6} \cos^4 3x \sin^3 6x \; dx$ [GGSIPU I Sem End Term 2011]

2. Show that $\int_0^{\infty} \frac{x^3}{(1+x^2)^{\frac{9}{2}}} dx = \frac{2}{35}$

3. Show that $\int_0^{a} \frac{x^n \, dx}{\sqrt{ax-x^2}} = \frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n} \pi a^n$

4. Evaluate $\int_{-\pi/2}^{\pi/2} \cos^3\theta \,(1+\sin\theta)^2 \, d\theta$

5. Evaluate $\int_0^{\pi/2} \frac{x \, dx}{\sin x + \cos x}$

6. Evaluate $\int_0^{\pi} \frac{\sin^4\theta\sqrt{1-\cos\theta}}{(1+\cos\theta)^2} d\theta$

7. Show that $\int_0^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \cos\theta \; d\theta = \frac{3\pi}{16\sqrt{2}}$.

8. Show that $\int_0^{\infty} \frac{dx}{(1+x^2)^n} = \frac{(2n-3)(2n-5)\ldots3\cdot1}{(2n-2)(2n-4)\ldots4\cdot2}\frac{\pi}{2} = \frac{(2n-2)!}{2^{2n-2}((n-1)!)^2}\frac{\pi}{2}$

[GGSIPU I Sem End Term 2006]

9. Evaluate $\int_0^{\infty} (1+x^2)^{-n-\frac{1}{2}} dx$

10. Show that $\int_0^{1} x^{4n+1}\sqrt{\frac{1-x^2}{1+x^2}}\, dx = \frac{1}{2}\left[\frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\cdots\frac{1}{2}\frac{\pi}{2} - \frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{2}{3}\right]$

11. Show that $\int_0^{\pi} \theta\sin^3\theta\cos\theta \;\; d\theta = \frac{-3\pi}{32}$.

12. Show that $\int_0^{1} x^5 \sin^{-1} x \, dx = \frac{11\pi}{192}$.

13. Evaluate $\int_0^{2\pi} \sin^2\theta\,(1+\cos\theta)^4 \, d\theta$

14. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos^2\theta \ d\theta}{\cos^2\theta + 4\sin^2\theta}$

15. If $\displaystyle I_n = \int_0^{\pi/2} \cos^n x \sin nx \ dx$, prove that

$\displaystyle I_n = \frac{1}{2n} + \frac{1}{2} I_{n-1}$ and hence, show that $\displaystyle I_n = \frac{1}{2^{n+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \ldots \frac{2^n}{n}\right]$

16. If $\displaystyle I_n = \int_0^{\infty} e^{-x} x^n \ dx$, show that $I_n = n\, I_{n-1} = n!$ where n is a natural number.

17. If $\displaystyle I_n = \int \frac{x^n \ dx}{(x^2+a^2)^{3/2}}$ $(n \in N)$, show that $\displaystyle (2-n)\, I_n = -\frac{x^{n-1}}{\sqrt{x^2+a^2}} = +(n-1)\, a^2\, I_{n-2}$

and hence evaluate $\displaystyle\int_0^1 \frac{x^5 \ dx}{(x^2+4)^{3/2}}$.

18. If $\displaystyle I_n = \int_0^{\pi/2} x^n \cos ax \ dx$, show that $\displaystyle I_n = \frac{1}{a}\left(\frac{\pi}{2}\right)^n \left[\sin\frac{a\pi}{2} + \frac{2n}{a\pi}\cos\frac{a\pi}{2}\right] - \frac{n(n-1)}{a^2} I_{n-2}$

19. If $\displaystyle I_n = \int_0^{\pi/4} \sin^{2n} \ dx$, show that $\displaystyle I_n = \left(1 - \frac{1}{2n}\right) I_{n-1} - \frac{1}{n \cdot 2^{n+1}}$.

20. If $\displaystyle I_n = \int_0^{\pi/2} x^n \sin(2a+1)x \ dx$, show that $\displaystyle (2a+1)^2 I_n + n(n-1)\, I_{n-2} = (-1)^a n \left(\frac{\pi}{2}\right)^{n-1}$.

where a and n are positive integers.

21. (a) If $\displaystyle I_n = \int_{\pi/4}^{\pi/2} \cot^n x \ dx \ (n > 2)$, show that $\displaystyle I_n = \frac{1}{n-1} - I_{n-2}$ and hence evaluate $\displaystyle\int_{\pi/4}^{\pi/2} \cot^6 x \ dx$.

(b) If $\displaystyle I_n = \int_0^{\pi/4} \tan^n\theta \, d\theta$, show that $n\,(I_{n-1} + I_{n+1}) = 1$ $(n \in N)$, and hence evaluate

$\displaystyle\int_0^a x^5 (2a^2 - x^2)^{-3} \ dx$.

22. By considering the integral $\displaystyle\int_0^1 (1-x^2)^n \ dx$, show that

$$1 - \frac{n}{1!\,3} + \frac{n(n-1)}{2!\,5} + \frac{n(n-1)(n-2)}{3!\,7} + \ldots = \frac{2 \cdot 4 \cdot 6 \ldots 2n}{3 \cdot 5 \cdot 7 \ldots (2n+1)}.$$

□□□

CHAPTER 6

Applications of Integration—Area, Length, Volume, Surface Area, and Beta, Gamma Functions

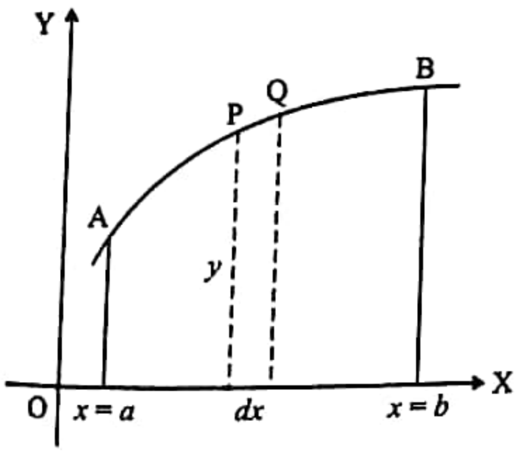
Applications of Integration in Finding Area and Length under the Curve, Volume and Surface Area of Solid of Revolution, Beta and Gamma Functions.

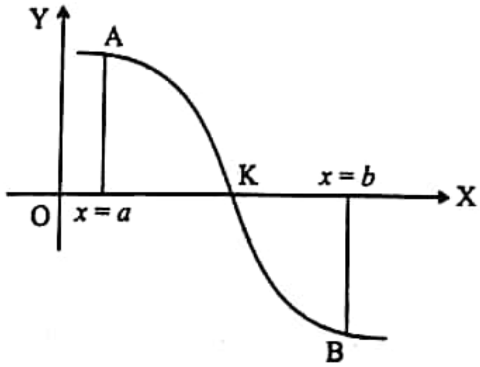
In this chapter, we shall study the applications of Integral Calculus to obtain the important physical quantities like area under the curve, length of an arc of a curve, numerical integration, volume and surface area of solids generated by revolving an area or an arc of a curve about an axis, etc.

AREA UNDER A CURVE (QUADRATURE) IN CARTESIAN CO-ORDINATES

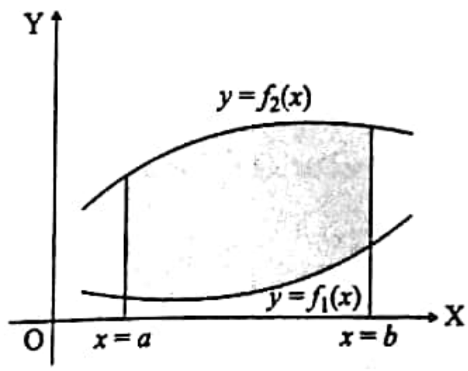
Area A bounded by the curve $y = f(x)$, the X-axis and two ordinates $x = a$ and $x = b$, is given by (see adjoining figure)

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx \qquad ...(1)$$





Note that when the portion of the curve under consideration is above the X-axis, y is positive and, hence the area will be positive; on the other hand, when the portion of the curve under consideration, is below the X-axis, y will be negative and, hence the area will be negative and in this case we have to take its absolute value. However, when the curve crosses the axis of X at a point K, say, as shown in the above figure, the area under the portion AK of the curve, is positive while the area under the portion KB of the curve, is negative. As such the area A as calculated from the



integral in (1) will be the difference between the two areas. To compute the actual area we find the absolute values of the two areas and add them up.

Similarly, the area A under the curve $x = f(y)$, the y-axis and the two abscissas $y = c$ and $y = d$, is given by $A = \int_c^d x\,dy = \int_c^d f(y)\,dy$

Further, the area A bounded by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = a$ and $x = b$, as depicted by the shaded portion in the above figure, is given by

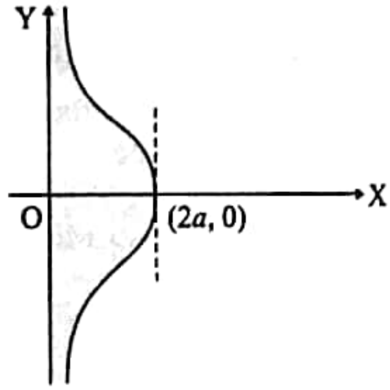
$$A = \int_a^b (y_2 - y_1)\,dx = \int_a^b [f_2(x) - f_1(x)]\,dx$$

where y_1 and y_2 are the ordinates of the lower and the upper curves respectively.

EXAMPLE 6.1. Find the area between the curve $xy^2 = 4a^2(2a - x)$ and its asymptote.

SOLUTION: The curve is symmetrical about the axis of x. It lies between $x = 0$ and $x = 2a$ since y^2 is negative for $x < 0$ and for $x > 2a$. The curve has the asymptote $x = 0$ (y-axis) as shown in the adjoining figure. Thus, the required area A is given by

$$A = 2\int_0^{2a} y\,dx = 2\int_0^{2a} 2a\sqrt{\frac{2a-x}{x}}\,dx.$$



Putting $x = 2a\sin^2\theta$, so $dx = 4a\sin\theta\cos\theta\,d\theta$, we have

$$A = 4a\int_0^{\frac{\pi}{2}} \sqrt{\frac{2a - 2a\sin^2\theta}{2a\sin^2\theta}} \cdot 4a\sin\theta\cos\theta\,d\theta$$

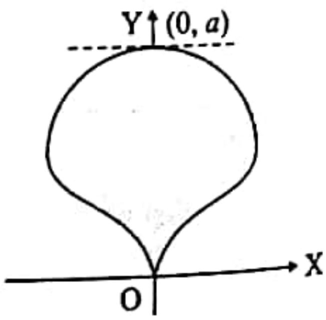
$$= 16a^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 16a^2\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) = 4\pi a^2$$

$\therefore$ Required area $= 4\pi a^2$ sq. units. **Ans.**

EXAMPLE 6.2. Calculate the area of the curve (i) $a^2x^2 = y^3(a - y)$. (ii) $a^2y^2 = x^3(2a - x)$

[GGSIPU I Sem IInd Term 2004; GGSIPU I Sem End Term 2013]

SOLUTION: (i) The curve is symmetrical about the axis of y. Since x^2 is negative for $y < 0$ and for $y > a$, the curve lies within the strip $y = 0$ and $y = a$. Y-axis is the tangent to the curve at the origin and $y = a$ is the tangent to the curve at the point $(0, a)$. The baloonic shape of the curve is shown in the adjoining figure.



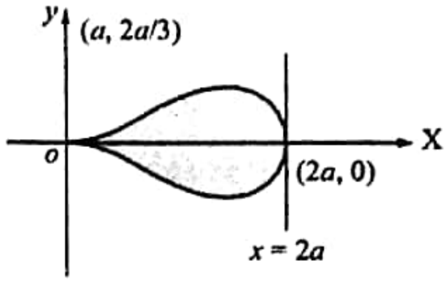
The required area $A = 2\int_0^a x\,dy = 2\int_0^a \frac{y^{3/2}}{a}\sqrt{a-y}\,dy$

Substituting $y = a\sin^2 t$, so $dy = 2a\sin t\cos t\,dt$, we get

$$A = 2\int_0^{\pi/2} \frac{a^{3/2}}{a}\sin^3 t \cdot \sqrt{a\cos^2 t} \cdot 2a\sin t\cos t\,dt = 4a^2\int_0^{\pi/2} \sin^4 t \cdot \cos^2 dt$$

$$= 4a^2 \frac{\Gamma\frac{5}{2}\Gamma\frac{3}{2}}{2\Gamma\frac{4+2+2}{2}} = \frac{2a^2 \cdot \frac{3}{2} \cdot \left(\Gamma\frac{3}{2}\right)^2}{\Gamma 4} = \frac{3a^2}{3!}\left(\frac{1}{2}\Gamma\frac{1}{2}\right)^2 = \frac{\pi a^2}{8} \text{ sq. units.} \quad \textbf{Ans.}$$

(ii) The curve $a^2y^2 = x^3(2a - x)$ is symmetrical about x-axis, passes through the origin at which x-axis is tangent, meets the x-axis at $(2a, 0)$. Y-axis is tangent to the curve at $(2a, 0)$. Curve lies in between $X = 0$ and $X = 2a$. Required area is the shaded portion in the figure.



$$\text{Area} = 2\int_0^{2a} y\,dx = 2\int_0^{2a} \frac{1}{a}\sqrt{x^3(2a-x)}\,dx = \frac{2}{a}\int_0^{2a} x^{3/2}(2a-x)^{1/2}\,dx$$

(Putting $x = 2a\sin^2\theta$, so $dx = 4a\sin\theta\cos\theta\,d\theta$)

$$= \frac{2}{a}\int_0^{\pi/2} (2a)^{3/2}\sin^3\theta\,(2a)^{1/2}\cos\theta \,.\, 4a\sin\theta\cos\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta = 32a^2\,\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \pi a^2 \quad \textbf{Ans.}$$

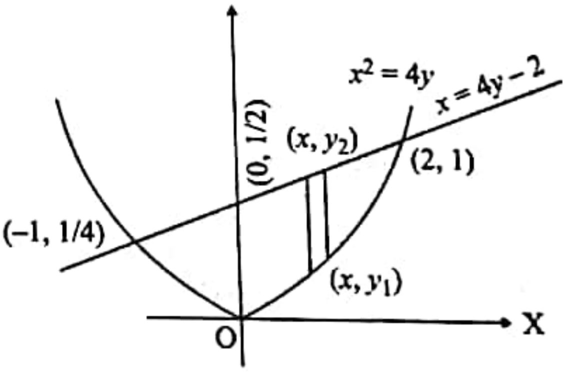
EXAMPLE 6.3. Find the area enclosed between the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

[*GGSIPU I Sem II Term 2010*]

SOLUTION: The straight line $x = 4y - 2$ meets the parabola $x^2 = 4y$ at the points given by $x^2 = x + 2$

or $(x + 1)(x - 2) = 0$ hence $x = -1$ and $x = 2$.

The required area A, as shown by the shaded area in the adjoining figure, is given by



$$A = \int_{-1}^{2}(y_2 - y_1)\,dx$$

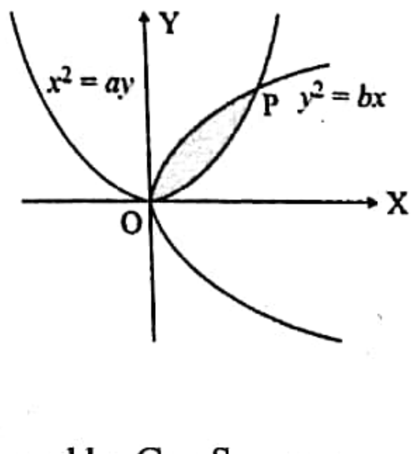
where y_2 and y_1 are the ordinates from the equations of the straight line and the parabola respectively.

$$\therefore \quad A = \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx = \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{4}\left[\frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}\right] = \frac{9}{8} \text{ sq. units.} \quad \textbf{Ans.}$$

EXAMPLE 6.4. (a) Find the area common to the parabolas $x^2 = ay$ and $y^2 = bx$.

(b) Find the area of the region bounded by the parabola $y = 4x^2$, the axis of Y and two abscissas at $y = 1$ and $y = 4$.

[*GGSIPU I Sem II Term 2013*]

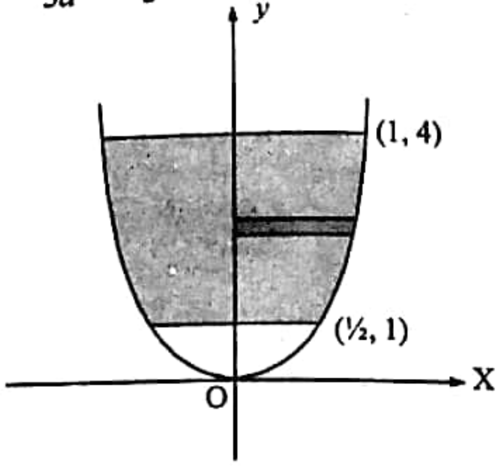


SOLUTION: (a) The curves are standard parabolas. Their common area is depicted by the shaded portion in the adjoining figure. Their points of intersection are given by $\frac{x^4}{a^2} = bx$ or $x(x^3 - a^2b) = 0$

$\Rightarrow$ $x = 0$ and $x = a^{2/3}b^{1/3}$.

$\therefore$ The required area $A = \int_0^{a^{2/3}b^{1/3}} (y_2 - y_1)\, dx = \int_0^{a^{2/3}b^{1/3}} \left(\sqrt{bx} - \frac{x^2}{a}\right) dx$

$$= \left[\sqrt{b}\cdot\frac{2}{3}x^{3/2} - \frac{x^3}{3a}\right]_0^{a^{2/3}b^{1/3}} = \frac{2}{3}\sqrt{b}\, a\sqrt{b} - \frac{a^2 b}{3a} = \frac{ab}{3} \text{ sq. units.} \qquad \textbf{Ans.}$$

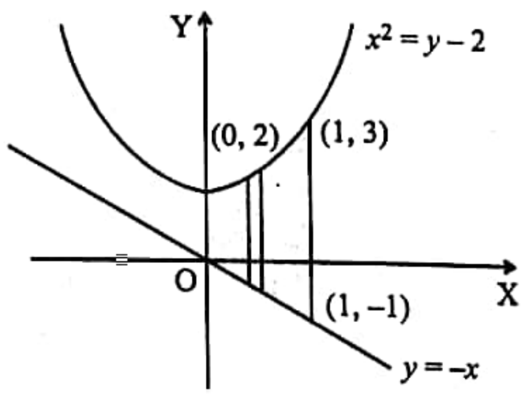


(b) Required area $A = 2\int_1^4 x\, dy$

$$= 2\int_1^4 \frac{1}{2}\sqrt{y}\, dy$$

$$= \frac{2}{3}\left[y^{3/2}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3} \text{ sq. units.}$$

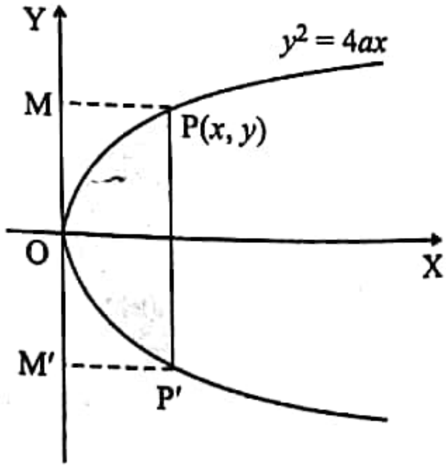
EXAMPLE 6.5. Compute the area bounded by the parabola $y = x^2 + 2$ and the straight lines $x = 0$, $x = 1$ and $x + y = 0$.



SOLUTION: The parabola $x^2 = y - 2$ has the vertex at $(0,2)$ and axis along y-axis. The line $x = 1$ meets the parabola at $(1,3)$ and the line $x + y = 0$ at $(1, -1)$. The required area A, as depicted by the shaded portion in the adjoining figure, is given by $A = \int_0^1 (y_2 - y_1)\, dx = \int_0^1 \left[x^2 + 2 - (-x)\right] dx$

$$= \left[\frac{x^3}{3} + 2x + \frac{x^2}{2}\right]_0^1 = \frac{17}{6} \text{ sq. units. } \textbf{Ans.}$$

EXAMPLE 6.6. Show that the area cut off a parabola by any double ordinate is two-third of the area of the rectangle formed by the double ordinate and the tangent at the vertex.



SOLUTION: Choose the co-ordinate system such that the origin is at the vertex and the axis of the parabola along X-axis. Let the equation of the parabola be $y^2 = 4ax$ and P (x, y) be any point on the parabola so that the double ordinate is PP'. The shaded area as shown in the adjoining figure, is equal to

$$= 2\int_0^x y\, dx = 2\int_0^x \sqrt{4ax}\, dx = 4\sqrt{a}\cdot\frac{2}{3}\left[x^{3/2}\right]_0^x$$

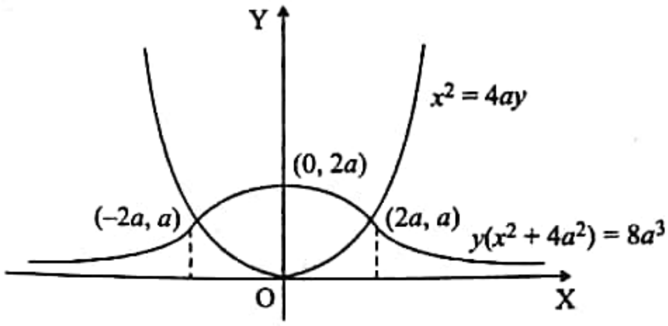
$$= \frac{2}{3}x\cdot 2\sqrt{4ax} = \frac{2}{3}\cdot(x\cdot 2y)$$

$$= \frac{2}{3} \text{ area of the rectangle PMM'P'}$$

Hence Proved.

EXAMPLE 6.7. Find the area enclosed by the curves $x^2 = 4ay$ and $y = \dfrac{8a^3}{x^2+4a^2}$.

SOLUTION: The curve $x^2 = 4ay$ is the parabola with vertex at the origin and axis along y-axis. The curve $y = \dfrac{8a^3}{x^2+4a^2}$ is symmetrical about y-axis, meets y-axis at $(0, 2a)$ and has $y = 0$, *i.e.*, x-axis as asymptote. The two curves meet at $(-2a, a)$ and $(2a, a)$. The required area is the shaded portion as shown in the adjoining figure and is equal to



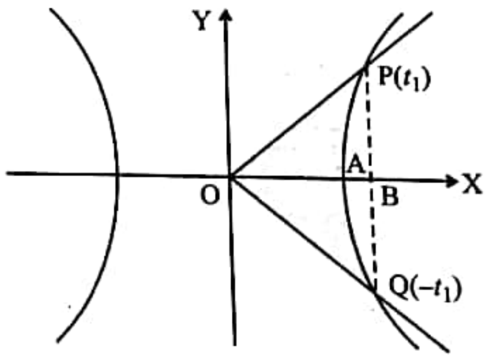
$$\int_{-2a}^{2a}(y_2 - y_1)\,dx = \int_{-2a}^{2a}\left(\frac{8a^3}{x^2+4a^2} - \frac{x^2}{4a}\right)dx = 2\int_0^{2a}\left(\frac{8a^3}{x^2+4a^2} - \frac{x^2}{4a}\right)dx$$

$$= \frac{16a^3}{2a}\left[\tan^{-1}\frac{x}{2a}\right]_0^{2a} - \frac{1}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = 8a^2\frac{\pi}{4} - \frac{8a^3}{6a} = 2a^2\left(\pi - \frac{2}{3}\right). \quad \textbf{Ans.}$$

EXAMPLE 6.8. For any real t, $x = \dfrac{e^t + e^{-t}}{2}$, $y = \dfrac{e^t - e^{-t}}{2}$ is a point of the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by this hyperbola and the lines joining its centre to the points corresponding to 't_1' and '$-t_1$' is t_1.

SOLUTION: Let $P(t_1)$ and $Q(-t_1)$ be two points on the hyperbola. The area bounded by the hyperbola and two lines OP and OQ is depicted by the shaded portion as shown in the adjoining figure.

Required area = area ΔOPQ – area PAQBP



$$\text{Area of } \Delta OPQ = PB \cdot OB = \frac{e^{t_1}+e^{-t_1}}{2}\cdot\frac{e^{t_1}-e^{-t_1}}{2}$$

$$= \frac{1}{4}\left(e^{2t_1} - e^{-2t_1}\right).$$

$$\text{Area (PAQBP)} = 2\int_A^B y\,dx = 2\int_0^{t_1} y\frac{dx}{dt}\,dt = 2\int_0^{t_1}\frac{e^t - e^{-t}}{2}\cdot\frac{e^t - e^{-t}}{2}\,dt$$

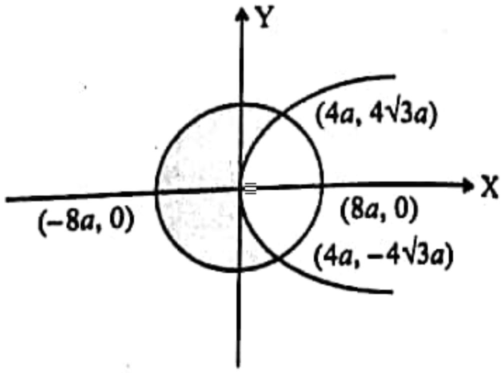
$$= \frac{1}{2}\int_0^{t_1}\left(e^{2t} + e^{-2t} - 2\right)dt = \frac{1}{2}\left[\frac{e^{2t}}{2} + \frac{e^{-2t}}{-2} - 2t\right]_0^{t_1} = \frac{1}{4}\left[e^{2t_1} - e^{-2t_1} - 4t_1\right]$$

Therefore, the required area $= \dfrac{1}{4}\left(e^{2t_1} - e^{-2t_1}\right) - \dfrac{1}{4}\left(e^{2t_1} - e^{-2t_1} - 4t_1\right) = t_1.$ **Ans.**

EXAMPLE 6.9. (a) Find the larger of the two areas in which the circle $x^2 + y^2 = 64a^2$ is divided by the parabola $y^2 = 12\,ax$.

(b) Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$. [GGSIPU Ist Sem End Term 2009]

SOLUTION: (a) The required area is the area of the shaded portion as shown in the adjoining figure. Let us first find the points of intersection of the circle $x^2 + y^2 = 64a^2$ with the parabola $y^2 = 12ax$, thereby



$x^2 + 12ax - 64a^2 = 0$ which gives $x = -16\,a$ and $x = 4a$.

Rejecting $x = -16\,a$, we get the points as $(4a, \pm 4\sqrt{3}\,a)$.

The unshaded area

$$= 2\int_0^{4a\sqrt{3}} (x_2 - x_1)\,dy = 2\int_0^{4a\sqrt{3}} \left(\sqrt{64a^2 - y^2} - \frac{y^2}{12\,a}\right) dy$$

$$= 2\left[\frac{y}{2}\sqrt{64a^2 - y^2}\right]_0^{4a\sqrt{3}} + \frac{2.64a^2}{2}\left(\sin^{-1}\frac{y}{8a}\right)_0^{4a\sqrt{3}} - 2\left[\frac{y^3}{36\,a}\right]_0^{4a\sqrt{3}}$$

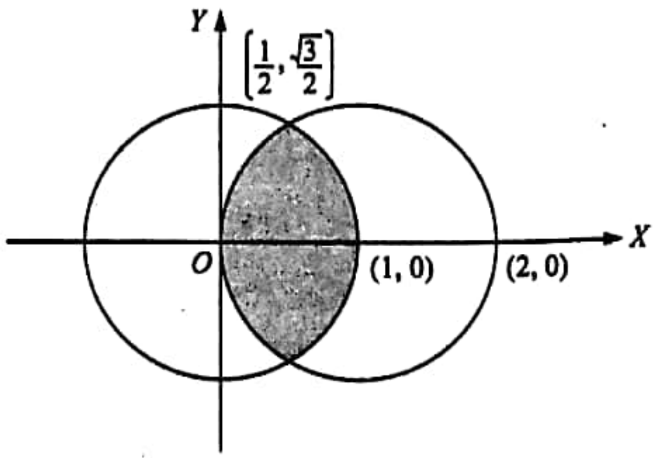
$$= 4a\sqrt{3}\sqrt{64\,a^2 - 48\,a^2} - 0 + 64a^2 \sin^{-1}\frac{\sqrt{3}}{2} - \frac{2\cdot 64\cdot 3\sqrt{3}\,a^3}{36\,a} = a^2\left(16\sqrt{3}\right) + 64\,a^2\,\frac{\pi}{3} - \frac{32\sqrt{3}a^2}{3}$$

$\therefore$ Required area = Area of the circle – unshaded area

$$= 64\pi a^2 - 16a^2\sqrt{3} - \frac{64\pi a^2}{3} + \frac{32\sqrt{3}}{3}a^2 = \frac{16a^2}{3}\left[8\pi - \sqrt{3}\right].$$ **Ans.**

(b) The required area is the shaded portion in the figure and is equal to

$$2\int_{1/2}^{1} \sqrt{1 - x^2}\,dx + 2\int_0^{1/2} \sqrt{1 - (x-1)^2}\,dx$$



$$= 4\int_{1/2}^{1} \sqrt{1 - x^2}\,dx \quad \text{(by symmetry)}$$

$$= 4\left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x\right]_{1/2}^{1}$$

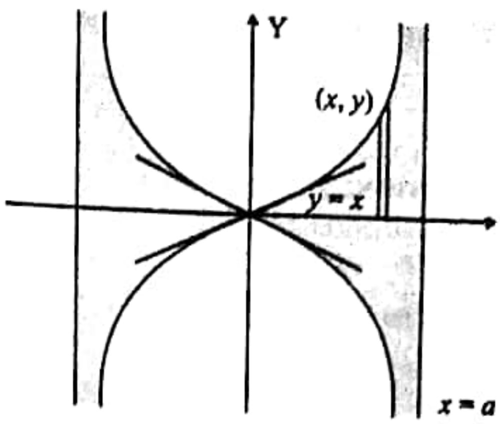
$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$ **Ans.**

EXAMPLE 6.10. Find the area between the curve $x^2 y^2 = a^2 (y^2 - x^2)$ and its asymptotes.

[GGSIPU I Sem End Term 2004 Reappear]

SOLUTION: The curve $x^2 y^2 = a^2 (y^2 - x^2)$ is symmetrical about both the co-ordinate axes. It passes through the origin and tangents to the curve at the origin are $y = \pm x$. The curve has two asymptotes parallel to y-axis as $x = \pm a$.

The required area A is the area of the shaded region as shown in the adjoining figure and is four times the area in the first quadrant bounded by the curve, the x-axis and the asymptote $x = a$.



$$\therefore \quad A = 4\int_0^a y\,dx = 4\int_0^a \frac{ax\,dx}{\sqrt{a^2 - x^2}}. \quad \text{(now putting } x = a\sin\theta)$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{a^2 \sin\theta \cdot a\cos\theta\,d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = 4a^2 \int_0^{\pi/2} \sin\theta\,d\theta$$

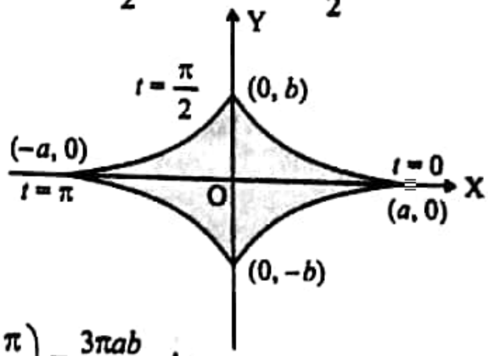
$$= 4a^2. \qquad \textbf{Ans.}$$

EXAMPLE 6.11. Find the area of the hypocycloid $x = a\cos^3 t,\ \ y = b\sin^3 t.$

SOLUTION: The shape of the figure is as shown in the adjoining figure. The parametric equation of the curve is $x = a\cos^3 t,\ y = b\sin^3 t$.

It meets the X-axis at $t = 0$ and $t = \pi$ and the Y-axis at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

Curve is symmetrical about both the axes.



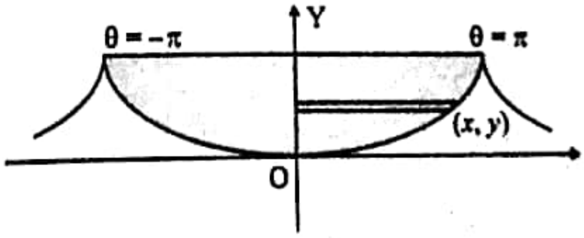
$$\text{The required area} = 4\int_0^a y\,dx = 4\int_{\pi/2}^{0} b\sin^3 t \cdot \frac{dx}{dt}\,dt$$

$$= 4b\int_{\pi/2}^{0} \sin^3 t \cdot 3a\cos^2 t\,(-\sin t)\,dt$$

$$= 12\,ab\int_0^{\pi/2} \sin^4 t\,\cos^2 t\,dt = 12ab\left(\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{3\pi ab}{8} \quad \textbf{Ans.}$$

EXAMPLE 6.12. For the cycloid $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$ obtain the area between its base and the portion of the curve from cusp to cusp.

SOLUTION: The shape of the curve $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$ is as shown in the adjoining figure. It is symmetrical about y-*axis*. The required area A is the area of the shaded portion from the cusp at $\theta = -\pi$ to the cusp at $\theta = \pi$ and is given by



$$A = 2\int_0^{2a} x\,dy = 2\int_0^{\pi} a(\theta + \sin\theta)\,a\sin\theta\,d\theta$$

$$= 2a^2\int_0^{\pi} \theta\sin\theta\,d\theta + 2a^2\int_0^{\pi}\sin^2\theta\,d\theta$$

$$= 2a^2\left[-\theta\cos\theta\right]_0^{\pi} - 2a^2\int_0^{\pi} 1(-\cos\theta)\,d\theta + a^2\int_0^{\pi}(1 - \cos 2\theta)\,d\theta$$

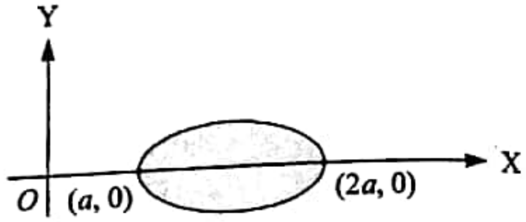
$$= 2a^2(0 + \pi) + 2a^2\left[\sin\theta\right]_0^{\pi} + a^2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = 3\pi a^2. \qquad \textbf{Ans.}$$

EXAMPLE 6.13. Show that the area of the loop of the curve

$$a^2y^2 = x^2\,(2a - x)\,(x - a) \quad \text{is} \quad 3\pi\, a^2/8.$$

[GGSIPU I Sem End Term 2007]

SOLUTION: The equation $a^2y^2 = x^2(x - a)\,(2a - x)$ represents a curve which consists of a loop as shown in the adjacent figure.



$$\text{The area A of the loop} = 2\int_a^{2a} y\,dx$$

$$= \frac{2}{a}\int_a^{2a} x\sqrt{(2a-x)\,(x-a)}\,dx$$

Putting $x = a\cos^2\theta + 2a\sin^2\theta$ so $dx = 2a\sin\theta\cos\theta\,d\theta$ and $x - a = a\sin^2\theta$ and $2a - x = a\cos^2\theta$, we get

$$A = \frac{2}{a}\int_0^{\pi/2}(a\cos^2\theta + 2a\sin^2\theta)\,2a^2\sin^2\theta\cos^2\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2}(\cos^4\theta\sin^2\theta + 2\sin^4\theta\cos^2\theta)\,d\theta$$

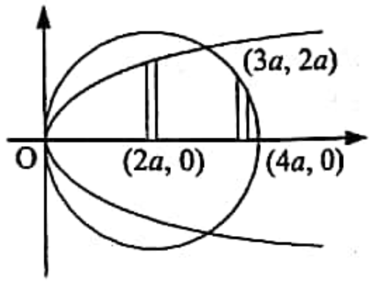
$$= 4a^2\left[\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 2\cdot\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= 12a^2\left(\frac{\pi}{32}\right) = \frac{3\pi a^2}{8} \quad \textbf{Ans.}$$

EXAMPLE 6.14. Find the area common the two curves $y^2 = ax$, and $x^2 + y^2 = 4ax$.

[GGSIPU I Sem End Term 2007; II Term 2011]

SOLUTION: Required area is the shaded portion in the adjoining figure. Because of symmetry about X-axis, the required area



$$= 2\int_0^{3a} y\,dx + 2\int_{3a}^{4a}\sqrt{4ax - x^2}\,dx$$

$$= 2\int_0^{3a}\sqrt{ax}\,dx + 2\int_{3a}^{4a}\sqrt{(2a)^2 - (x-2a)^2}\,dx$$

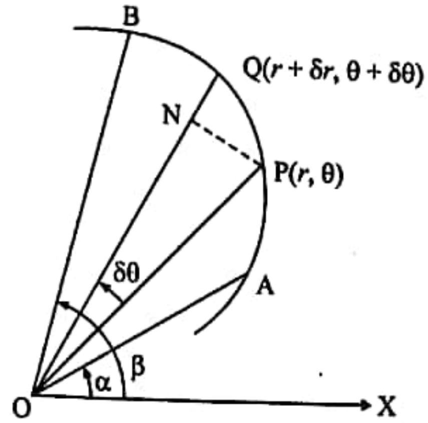
$$= 2\sqrt{a}\,\frac{2}{3}[x^{3/2}]_0^{3a} + 2\left[\frac{x-2a}{2}\sqrt{4ax - x^2} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= \frac{4}{3}\sqrt{a}\;3a\sqrt{3a} + 2\left[a\cdot 0 - \frac{a}{2}\cdot a\sqrt{3} + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\right]$$

$$= 4a^2\sqrt{3} + 2\left[-\frac{a^2\sqrt{3}}{2} + 2a^2\frac{\pi}{3}\right] = 3a^2\sqrt{3} + \frac{4\pi a^2}{3} \qquad \textbf{Ans.}$$

AREA UNDER THE CURVE IN POLAR CO-ORDINATES

Let us consider the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$, *i.e.*, the area OABO (see figure). Divide the area OABO into sectors of the type OPQO where P (r, θ) and Q $(r + \delta r, \theta + \delta\theta)$ are two points on the curve such that Q is quite close to P. If δA is the area of the elementary triangular strip OPQ then



$$\delta A = \frac{1}{2} OQ \cdot PN$$

where the arc PQ is approximated by chord PQ.

$$\therefore \quad \delta A = \frac{1}{2}(r + \delta r)\, r \sin \delta\theta$$

then
$$\frac{dA}{d\theta} = \lim_{\delta\theta \to 0} \frac{\delta A}{\delta\theta} = \lim_{\delta\theta \to 0} \frac{1}{2}\left(r^2 + r\,\delta r\right)\frac{\sin \delta\theta}{\delta\theta} = \frac{1}{2} r^2$$

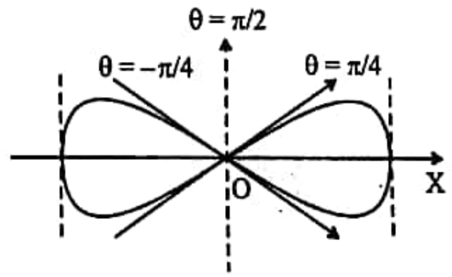
$$\therefore \quad A = \int_\alpha^\beta \frac{dA}{d\theta}\, d\theta = \int_\alpha^\beta \frac{1}{2} r^2\, d\theta.$$

Thus, **the area bounded by the curve $r = f(\theta)$ and the two radii vectors $\theta = \alpha$ and $\theta = \beta$ is** $\frac{1}{2}\int_\alpha^\beta r^2\, d\theta$

EXAMPLE 6.15. Compute the area bounded by a loop of the leminiscate $r^2 = a^2 \cos 2\theta$.

[*GGSIPU IInd Term 2007*]

SOLUTION: The curve $r^2 = a^2 \cos 2\theta$ has been traced in the Chapter on curve tracing. The area A of one loop of the curve as depicted by the shaded region in the adjoining figure, is given by



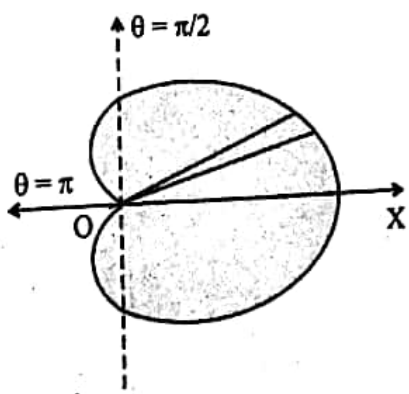
$$A = 2\int_0^{\pi/4} \frac{1}{2} r^2\, d\theta = \int_0^{\pi/4} a^2 \cos 2\theta\, d\theta$$

$$= \frac{a^2}{2}\left[\sin 2\theta\right]_0^{\frac{\pi}{4}} = \frac{a^2}{2}(1 - 0) = \frac{a^2}{2}. \qquad \textbf{Ans.}$$

EXAMPLE 6.16. Find the area of the cardioide $r = a\,(1 + \cos\theta)$.

[*GGSIPU I Sem. II Term 2009; II Term 2011*]

SOLUTION: The cardioide $r = a\,(1 + \cos\theta)$ has already been traced in the Chapter on curve tracing. The required area A, shown as shaded region in the adjoining figure, is given by



$$A = 2\int_0^\pi \frac{1}{2} r^2\, d\theta = \int_0^\pi a^2 (1 + \cos\theta)^2\, d\theta$$

$$= 4a^2 \int_0^\pi \cos^4 (\theta/2)\, d\theta.$$

Putting $\frac{\theta}{2} = \phi$, $\quad d\theta = 2\, d\phi$, $\quad$ we get

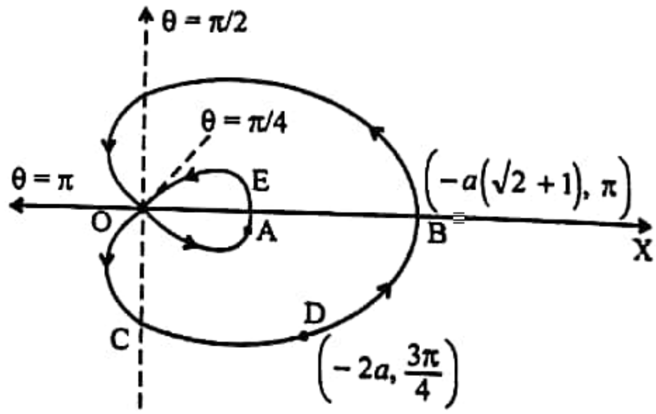
$$A = 4a^2 \int_0^{\pi/2} \cos^4 \phi \cdot 2d\phi = 8a^2 \frac{\Gamma(1/2)\,\Gamma(5/2)}{2\,\Gamma(3)} = \frac{4a^2}{2!}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \frac{3}{2}\pi a^2. \qquad \textbf{Ans.}$$

EXAMPLE 6.17. Determine the area included between the two loops of the curve $r = a\left(\sqrt{2}\cos\theta - 1\right)$.

SOLUTION: Let us first trace the curve $r = a\left(\sqrt{2}\cos\theta - 1\right)$. It is symmetrical about the initial line. Following table gives the values of r for some suitable values of θ from $\theta = 0$ to $\theta = \pi$. For the values of θ from π to 2π the curve, can be drawn by symmetry.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	$a\left(\sqrt{2}-1\right)$	0	$-a$	$-2a$	$-a\left(\sqrt{2}+1\right)$

The curve starts at A where $\theta = 0$ and $r = a\left(\sqrt{2}-1\right)$. As θ increases, r decreases and becomes 0 at $\theta = \frac{\pi}{4}$. When θ increases from $\frac{\pi}{4}$, r becomes negative, so it is in opposite direction. For $\theta = \frac{\pi}{2}$, $r = -a$ hence the point C. For $\theta = \frac{3\pi}{4}$, $r = 2a$ hence point D and at $\theta = \pi$, $r = -a\left(\sqrt{2}+1\right)$, hence point B. Obviously OB > OA. As shown in the adjoining figure we get the curve AOCDB as θ goes from 0 to π and by symmetry we can get the complete curve.



Half of the inner loop OEA is obtained when θ goes from 0 to $\pi/4$ and half of the outer loop is obtained when θ goes from $\pi/4$ to π.

Therefore, area A_1 of the inner loop is given by

$$A_1 = 2\int_0^{\pi/4} \frac{1}{2} r^2\, d\theta = \int_0^{\pi/4} a^2\left(\sqrt{2}\cos\theta - 1\right)^2 d\theta$$

$$= a^2 \int_0^{\pi/4} \left(2\cos^2\theta - 2\sqrt{2}\cos\theta + 1\right) d\theta$$

$$= a^2 \int_0^{\pi/4} \left(1 + \cos 2\theta - 2\sqrt{2}\cos\theta + 1\right) d\theta = a^2\left[2\theta + \frac{1}{2}\sin 2\theta - 2\sqrt{2}\sin\theta\right]_0^{\pi/4}$$

$$= a^2\left[\frac{\pi}{2} + \frac{1}{2} - 2\sqrt{2}\,\frac{1}{\sqrt{2}}\right] = \frac{a^2}{2}(\pi - 3).$$

And the area of the outer loop $= A_2 = 2\int_{\pi/4}^{\pi} \frac{1}{2} r^2\, d\theta$

$$= \int_{\pi/4}^{\pi} a^2(\sqrt{2}\cos\theta - 1)^2 d\theta = a^2\left[2\theta + \frac{1}{2}\sin 2\theta - 2\sqrt{2}\sin\theta\right]_{\pi/4}^{\pi}$$

$$= a^2\left[2\pi + 0 - 0 - 2.\frac{\pi}{4} - \frac{1}{2} + 2\sqrt{2}.\frac{1}{\sqrt{2}}\right] = a^2\left[\frac{3\pi}{2} + \frac{3}{2}\right] = \frac{3a^2}{2}(\pi + 1)$$

Therefore, the area between the two loops $= A_2 - A_1$

$$= \frac{3a^2}{2}(\pi + 1) - \frac{a^2}{2}(\pi - 3) = a^2(\pi + 3) \text{ sq. units.}$$

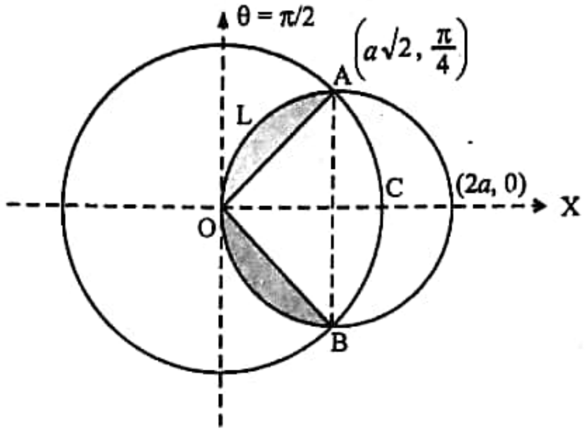
Ans.

EXAMPLE 6.18. Obtain the area common to two circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$.

[*GGSIPU I Sem End Term 2011*]

SOLUTION: The equation $r = a\sqrt{2}$. represents the circle with centre at the pole and radius $a\sqrt{2}$ while $r = 2a\cos\theta$ represents the circle with centre at $(a, 0)$, passing through the pole and having radius a. Both the circles are symmetrical about the initial line OX.

The points of intersection of two circles are $A\left(a\sqrt{2}, \frac{\pi}{4}\right)$ and $B\left(a\sqrt{2}, \frac{-\pi}{4}\right)$. As shown in adjoining figure, the required area



$$= 2[\text{area OCAO} + \text{area OALO}]$$

$$= 2\left[\int_0^{\pi/4} \frac{1}{2} r^2\, d\theta\right]_{r=a\sqrt{2}} + 2\left[\int_{\pi/4}^{\pi/2} \frac{1}{2} r^2\, d\theta\right]_{r=2a\cos\theta}$$

$$= \int_0^{\pi/4} \left(a\sqrt{2}\right)^2 d\theta + \int_{\pi/4}^{\pi/2} (2a\cos\theta)^2\, d\theta$$

$$= 2a^2\frac{\pi}{4} + 2a^2 \int_{\pi/4}^{\pi/2} (1+\cos 2\theta)\, d\theta$$

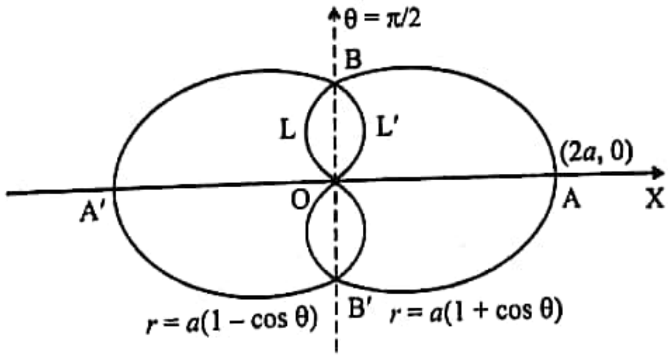
$$= 2a^2\frac{\pi}{4} + 2a^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2} = \frac{\pi a^2}{2} + 2a^2\left[\frac{\pi}{2} + 0 - \frac{\pi}{4} - \frac{1}{2}\right] = (\pi - 1)\,a^2 = \text{required area.} \quad \textbf{Ans.}$$

EXAMPLE 6.19. Find the area common to the two cardioides $r = a(1+\cos\theta)$, and $r = a(1-\cos\theta)$

[*GGSIPU I Sem II Term 2003, I Term 2012*]

SOLUTION : The shapes of these curve have already been discussed in the chapter on curve tracing. Their points of intersection are B and B'.

Both curves are symmetrical about the initial line as shown in the adjoining figure.



∴ Required area = 2 × area under OL′BLO

$$= 2\,[\text{area OL}'\text{BO} + \text{area OBLO}]$$

$$= 2\int_0^{\pi/2} \frac{1}{2} r_1^2\, d\theta + 2\int_{\pi/2}^{\pi} \frac{1}{2} r_2^2\, d\theta \quad \text{where } r_1 = a(1-\cos\theta) \text{ and } r_2 = a(1+\cos\theta).$$

$$= \int_0^{\pi/2} a^2 (1-\cos\theta)^2\, d\theta + \int_{\pi/2}^{\pi} a^2 (1+\cos\theta)^2\, d\theta$$

$$= a^2 \int_0^{\pi/2} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta + a^2 \int_{\pi/2}^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= a^2 \int_0^{\pi} \left(1 + \cos^2\theta\right) d\theta - 2a^2 \int_0^{\pi/2} \cos\theta\, d\theta + 2a^2 \int_{\pi/2}^{\pi} \cos\theta\, d\theta$$

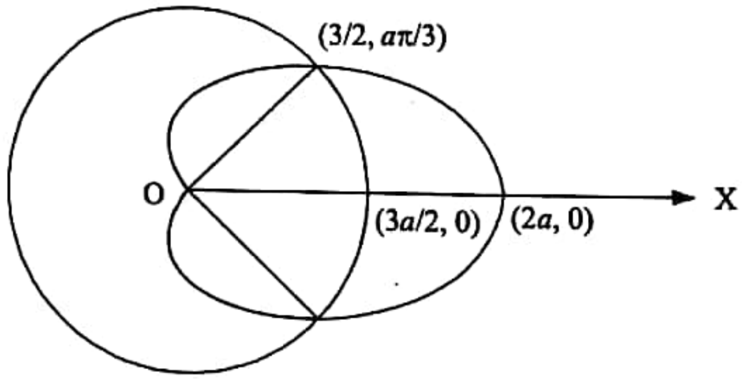
$$= a^2\int_0^{\pi}\left(1+\frac{1+\cos 2\theta}{2}\right)d\theta - 2a^2\left[\sin\theta\right]_0^{\pi/2} + 2a^2\left[\sin\theta\right]_{\pi/2}^{\pi}$$

$$= a^2\left[\frac{3\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\pi} - 2a^2(1-0) + 2a^2(0-1) = \frac{3}{2}\pi a^2 - 4a^2 = \frac{a^2}{2}(3\pi-8).$$ **Ans.**

EXAMPLE 6.20. Find the area common to the cardiode $r = a(1+\cos\theta)$ and the circle $r = 3a/2$ and also the area of the remainder of the cardiode.

[*GGSIPU I Sem End Term 2003*]

SOLUTION: Shaded area in the figure is the required area common to the cardiode $r = a(1+\cos\theta)$ and the circle $r = 3a/2$. The curves intersect each other at $\theta = \pm\pi/3$.



$$\text{Area} = 2\int_0^{\pi/3}\frac{1}{2}r_1^2\,d\theta + 2\int_{\pi/3}^{\pi}\frac{1}{2}r_2^2\,d\theta \quad \text{where } r_1 = \frac{3a}{2} \text{ and } r_2 = a(1+\cos\theta).$$

$$= \int_0^{\pi/3}\left(\frac{3a}{2}\right)^2 d\theta + \int_{\pi/3}^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= \frac{9a^2}{4}\cdot\frac{\pi}{3} + a^2\int_{\pi/3}^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{3\pi a^2}{4} + a^2\left[\frac{3\theta}{2}+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_{\pi/3}^{\pi}$$

$$= \frac{3\pi a^2}{4} + a^2\left[\frac{3}{2}\pi + 2\sin\pi + \frac{1}{4}\sin 2\pi - \frac{3}{2}\cdot\frac{\pi}{3} - 2\sin\frac{\pi}{3} - \frac{1}{4}\sin\frac{2\pi}{3}\right]$$

$$= \frac{3}{4}\pi a^2 + a^2\left[\pi - 2\cdot\frac{\sqrt{3}}{2} - \frac{1}{4}\frac{\sqrt{3}}{2}\right] = a^2\left[\frac{7\pi}{4} - \frac{9\sqrt{3}}{8}\right]$$

The area of the cardiode $= 2\int_0^{\pi}\frac{1}{2}r_2^2\,d\theta = \int_0^{\pi}a^2(1+\cos\theta)^2\,d\theta = a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$

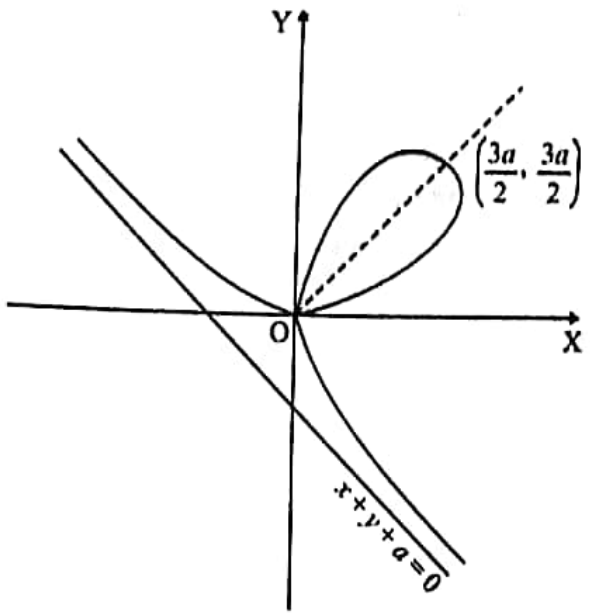
$$= a^2\int_0^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta = a^2\left[\frac{3\theta}{2}+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi} = \frac{3}{2}\pi a^2$$

Therefore the required remainder of the area of cardiode

$$= \frac{3}{2}\pi a^2 - a^2\left(\frac{7\pi}{4} - \frac{9\sqrt{3}}{8}\right) = a^2\left(\frac{9\sqrt{3}}{8} - \frac{\pi}{4}\right)$$ **Ans.**

EXAMPLE 6.21. Find the area of the loop of the curve $x^3 + y^3 = 3axy$.

SOLUTION: The shape of the curve is given in the adjoining figure and has been discussed earlier in the chapter on curve tracing. The computation of the area of the loop will be easier if we transform the cartesian form into polar form.



Putting $x = r\cos\theta,\ \ y = r\sin\theta,$ the equation of the curve becomes $r^3(\cos^3\theta + \sin^3\theta) = 3a\,r^2\sin\theta\cos\theta$

or $\quad r = \dfrac{3a\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$

The loop is completed when θ goes from $\theta = 0$ to $\dfrac{\pi}{2}$, therefore the area A of the loop,

$$A = \int_0^{\pi/2} \frac{1}{2} r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{9a^2\sin^2\theta\cos^2\theta\, d\theta}{(\cos^3\theta + \sin^3\theta)^2} = \frac{9a^2}{2}\int_0^{\pi/2} \frac{\tan^2\theta\sec^2\theta\, d\theta}{(1+\tan^3\theta)^2}$$

Now substituting $1 + \tan^3\theta = t$, so $3\tan^2\theta\sec^2\theta\, d\theta = dt$, we get

$$A = \frac{3a^2}{2}\int_1^{\infty}\frac{dt}{t^2} = \frac{3a^2}{2}\left[\frac{-1}{t}\right]_1^{\infty} = \frac{3a^2}{2}. \qquad \textbf{Ans.}$$

EXAMPLE 6.22. Find the area of the loop of the curve $(x^2 + y^2)(3ay - x^2 - y^2) = 4ay^3$.

SOLUTION: In the given equation $(x^2 + y^2)(3ay - x^2 - y^2) = 4ay^3$, the presence of the term $x^2 + y^2$ suggests the conversion to polar form. The curve passes through the pole, hence putting $x = r\cos\theta,\ \ y = r\sin\theta,$ it becomes $r^2(3ar\sin\theta - r^2) = 4ar^3\sin^3\theta$.

or $\quad r = a(3\sin\theta - 4\sin^3\theta) = a\sin 3\theta$

Here $r = 0$ when $\theta = 0$ and also when $\theta = \pi/3$. As such for one loop θ varies from 0 to $\pi/3$.

$$\therefore \text{ Area of one loop} = \int_0^{\pi/3}\frac{1}{2}r^2\, d\theta = \int_0^{\pi/3}\frac{1}{2}a^2\sin^2 3\theta\, d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi/3}(1-\cos 6\theta)\, d\theta = \frac{a^2}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{a^2}{4}\left[\frac{\pi}{3} - 0\right] = \frac{\pi a^2}{12}. \qquad \textbf{Ans.}$$

EXERCISE 6A

1. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
2. Show that the curve $a^2 y^2 = x^2 (a^2 - x^2)$ consists of two loops and find the area of each loop.
3. Find the area bounded by the parabolas $y^2 = 9x$ and $x^2 = 9y$.
4. Find the area bounded by the parabolas $y^2 = 5x + 6$ and $x^2 = y$.
5. Find the area between the curve $y^2 = \frac{x^3}{a - x}$ and its asymptote.
6. Show that the area of the loop of the curve $a y^2 = x (x - a)^2$ is $\frac{8a^2}{15}$.
7. Find the area between the curves $y = 2x^2$ and $y = 1 - \frac{x^2}{2}$.
8. Show that the area enclosed by the curve $|x| + |y| = 2a$, $(a > 0)$ is $8a^2$.
9. Determine the area common to the two ellipses $a^2x^2 + b^2y^2 = 1$ and $b^2x^2 + a^2y^2 = 1$.
10. Obtain the area bounded by the rectangular hyperbola $xy = c^2$, X-axis and the ordinates $x = c$, $x = 2c$.
11. Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.
12. Show that the area cut off the parabola $y^2 = 2x$ by the straight line $y = 4x - 1$ is $9/32$ sq. units.
13. Find the area under one arc of the curve $y = \sin x$.
14. Find the area bounded by $y = x^3$ and the lines $x + 2 = 0$, $x = 1$ and $y = 0$.
15. Find the area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.
16. Find the area included between the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and its base.
(GGSIPU I Sem End Term 2008)
17. Trace the tractrix $x = a(\cos t + \log|\tan t/2|)$, $y = a \sin t$ and find the area enclosed.
18. Find the area of one loop of the curve $r = a \sin n\theta$, $n \in N$. What is the total area of all the loops ?
19. Determine the area of one loop of the curve $r = a \cos 2\theta$.
20. Show that the area bounded by the spiral $r = ae^{m\theta}$ and two radii vectors is proportional to the difference of the squares of these radii.
21. Find the area included between two loops of the curve $r = a\left(2\cos\theta + \sqrt{3}\right)$.
22. Trace the curve $r = a(\sec\theta + \cos\theta)$ and find the area between the curve and its asymptote.
23. Trace the curve $r^2 = a^2 \sin 2\theta$ and find the area of one loop of the curve.
24. Find the area lying between the cardioide $r = a(1 - \cos\theta)$ and its double tangent.
25. Find the area of the loop of the curve $x^4 + y^4 = 2a^2 xy$.
26. Find the area outside the circle $r = 2a\cos\theta$ and inside the cardioide $r = a(1 + \cos\theta)$.
27. Determine the area of the loop of the curve $x^5 + y^5 = 5ax^2 y^2$.

LENGTH UNDER A CURVE (RECTIFICATION)

Curves in Cartesian Coordinates

We know that $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ and $\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$ for the carterian curves

and $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ for the curves having equations in parametric form.

Therefore, the length of the arc of the curve $y = f(x)$ between $x = a$ and $x = b$, is given by

$$S = \int_a^b \frac{ds}{dx}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

And the length of the arc of the curve $x = f(y)$ between $y = c$ and $y = d$, is given by

$$S = \int_c^d \frac{ds}{dy}\,dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Also, if the equation of the curve is given or can be expressed in parametric form $x = f_1(t)$, $y = f_2(t)$, then the length S of the arc between the points $t = t_1$ and $t = t_2$, is given by

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Curves in Polar Coordinates

White discussing curvature for polar curves, we had the formulae

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \quad \text{and} \quad \frac{ds}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$$

Therefore, the length S of the arc of the curve $r = f(\theta)$ between the points $\theta = \alpha$ and $\theta = \beta$, is given by

$$S = \int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

and if the equation of the curve is of the form $\theta = f(r)$, the length S of the arc of the curve between the points $r = r_1$ and $r = r_2$, is given by

$$S = \int_{r_1}^{r_2} \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr$$

Intrinsic Equation of the Curve

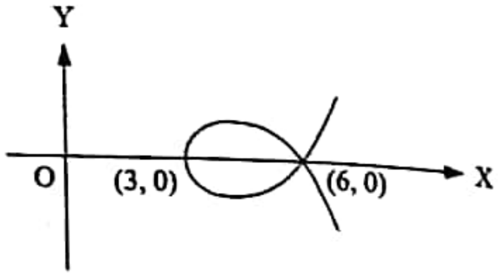
If 'S' is the length of the arc of a curve measured from some fixed point A on the curve to a variable point P and ψ is the angle between the tangents at A and tangent at P then the relation between S and ψ is called the intrinsic equation of the curve and S and ψ are called intrinsic coordinates.

EXAMPLE 6.23. Determine the length of the loop of the curve $9y^2 = (x-3)(x-6)^2$

[GGSIPU End Term 2003]

SOLUTION: The shape of the curve $9y^2 = (x-3)(x-6)^2$ is shown in the adjacent figure. The required length of the loop is given by

$$L = 2\int_3^6 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$



From the equation of the curve we have

$$18y\frac{dy}{dx} = 2(x-3)(x-6) + (x-6)^2$$

$$= (x-6)(3x-12)$$

$$\therefore \qquad \frac{dy}{dx} = \frac{(x-6)(x-4)}{6y} \text{ and } \left(\frac{dy}{dx}\right)^2 = \frac{(x-6)^2(x-4)^2}{4(x-3)(x-6)^2} = \frac{(x-4)^2}{4(x-3)}$$

Thus

$$L = 2\int_3^6 \sqrt{1+\frac{(x-4)^2}{4(x-3)}}\, dx = \int_3^6 \frac{\sqrt{x^2-8x+16+4x-12}}{\sqrt{x-3}}\, dx$$

$$= \int_3^6 \frac{x-2}{\sqrt{x-3}}\, dx = \int_3^6 \frac{x-3+1}{\sqrt{x-3}}\, dx = \int_3^6 \left(\sqrt{x-3}+\frac{1}{\sqrt{x-3}}\right) dx$$

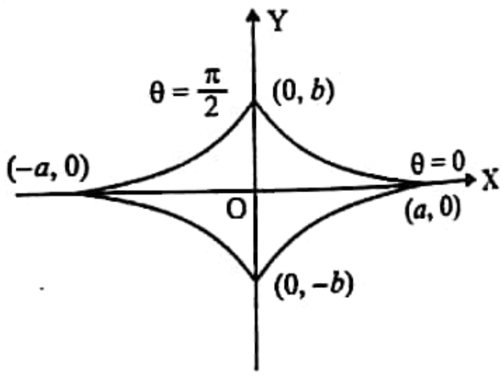
$$= \left[\frac{2}{3}(x-3)^{3/2} + 2\sqrt{x-3}\right]_3^6 = \frac{2}{3}.3\sqrt{3} + 2\sqrt{3} = 4\sqrt{3}$$ **Ans.**

EXAMPLE 6.24. Find the total length (perimeter) of the hypocycloid whose parametric equation is
$x = a\cos^3\theta, \quad y = b\sin^3\theta.$

SOLUTION: The curve $x = a\cos^3\theta$, $y = b\sin^3\theta$ is symmetrical about x-axis as well as about y-axis. The shape of the curve is shown in the adjoining figure and has already been discussed in the chapter on curve tracing.

Because of symmetry of the curve, its total length S which can be termed as perimeter, is given by

$$S = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$



$$= 4\int_0^{\pi/2} \sqrt{\left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3b\sin^2\theta\cos\theta\right)^2}\, d\theta$$

$$= 12\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\cdot\sin\theta\cos\theta\, d\theta$$

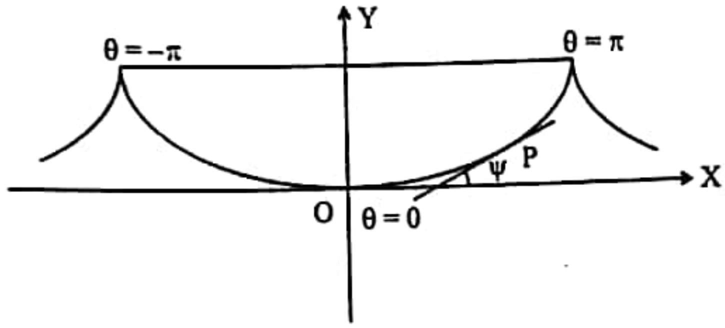
Putting here $a^2\cos^2\theta + b^2\sin^2\theta = t^2$, so $2\left(b^2-a^2\right)\sin\theta\cos\theta\, d\theta = 2t\, dt$, we get

$$S = 12\int_a^b t\cdot\frac{t\, dt}{\left(b^2-a^2\right)} = \frac{12}{b^2-a^2}\left[\frac{t^3}{3}\right]_a^b = 4\frac{\left(b^3-a^3\right)}{b^2-a^2} = \frac{4\left(a^2+ab+b^2\right)}{a+b}.$$ **Ans.**

EXAMPLE 6.25. For the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ show that the length S of the cycloid measured from the vertex is given by $S = 4a\sin\frac{\theta}{2}$. Also find its intrinsic equation.

SOLUTION : We have $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

$$\therefore \frac{dx}{d\theta} = a(1+\cos\theta), \frac{dy}{d\theta} = a\sin\theta$$



Vertex of the cycloid is at the origin where tangent is along X-axis, hence

$$\tan\psi = \frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2}$$

$$\therefore \psi = \frac{\theta}{2}.$$

Required length S along the curve from the vertex O, is given by

$$S = \int_0^\theta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^\theta \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= a\int_0^\theta \sqrt{2+2\cos\theta}\, d\theta = a\int_0^\theta \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta = 2a\int_0^\theta \cos\frac{\theta}{2}\, d\theta = 4a\sin\frac{\theta}{2}.$$

Since $\psi = \frac{\theta}{2}$ hence the intrinsic equation of the curve is $S = 4a\sin\psi$. **Ans.**

EXAMPLE 6.26. Find the length of the arc of the tractrix $x = a\left(\cos t + \log\left|\tan\frac{t}{2}\right|\right)$, $y = a\sin t$ from the fixed point $(0, a)$ on the curve. Also find the intrinsic equation of the curve.

SOLUTION : From the given parametric equation, we have

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{2}\frac{\sec^2\frac{t}{2}}{\tan\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a\cos^2 t}{\sin t}, \quad \frac{dy}{dt} = a\cos t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{a\cos t}{a\cos^2 t}\sin t = \tan t.$$

At the fixed point $(0, a)$, $t = \frac{\pi}{2}$ and $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}} = \infty$, therefore y-axis is the tangent to the curve at the fixed point $(0, a)$.

$$\therefore \quad \tan\psi = \frac{1}{dy/dx} = \cot t = \tan\left(\frac{\pi}{2} - t\right) \quad \therefore \psi = \frac{\pi}{2} - t.$$

$$\therefore \quad S = \int_{\pi/2}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{\pi/2}^{t} \sqrt{\frac{a^2\cos^4 t}{\sin^2 t} + a^2\cos^2 t}\, dt = \int_{\pi/2}^{t} a\cos t\sqrt{\cot^2 t + 1}\, dt$$

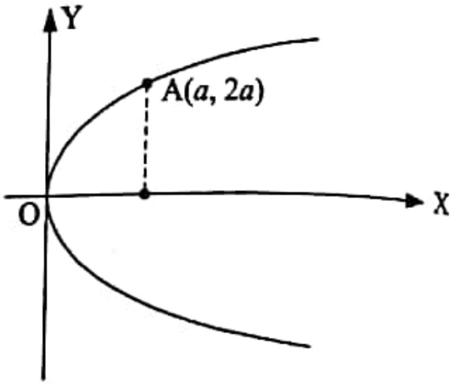
$$= a\int_{\pi/2}^{t} \cot t\, dt = a\left[\log\sin t\right]_{\pi/2}^{t} = a\log\sin t$$

But $\psi = \frac{\pi}{2} - t$ or $t = \frac{\pi}{2} - \psi$, therefore, the intrinsic equation is

$$S = a\log\sin\left(\frac{\pi}{2} - \psi\right) = a\log\cos\psi. \quad \textbf{Ans.}$$

EXAMPLE 6.27. Find the length of the arc of the parabola $y^2 = 4ax$ from the vertex to one extremity of the latus rectum. [GGSIPU I Sem II term 2003]

SOLUTION: Parabola is $y^2 = 4ax$ and A is one extremity of the latus rectum.



$$\text{Length of the arc, } OA = \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^a \sqrt{1+\left(\frac{4a}{2y}\right)^2}\,dx = \int_0^a \sqrt{1+\frac{4a^2}{4ax}}\,dx$$

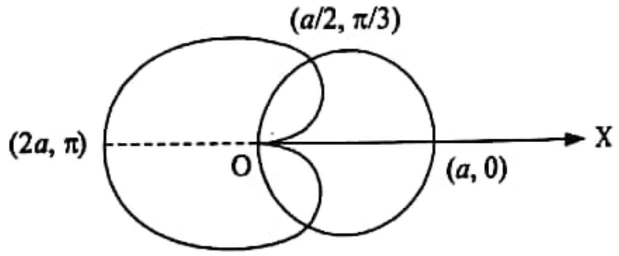
$$= \int_0^a \sqrt{\frac{a+x}{x}}\,dx \quad \text{(putting } x = at^2\text{)}$$

$$= 2a\int_0^1 \sqrt{t^2+1}\,dt = 2a\left[\frac{t}{2}\sqrt{t^2+1}+\frac{1}{2}\log\left(t+\sqrt{t^2+1}\right)\right]_0^1$$

$$= a\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right] \qquad \textbf{Ans.}$$

EXAMPLE 6.28. Determine the length of the cardiode $r = a(1 - \cos\theta)$ lying outside the circle $r = a\cos\theta$ [GGSIPU I Sem II Term 2007; II Term 2010]

SOLUTION: Points of intersection of the circle $r = a\cos\theta$ and the cardiode $r = a(1 - \cos\theta)$ are $(a/2, \pm\pi/3)$. The length of the cardiode outside the circle equals



$$= 2\int_{\pi/3}^{\pi} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 2\int_{\pi/3}^{\pi} \sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}\,d\theta = 2a\int_{\pi/3}^{\pi}\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta$$

$$= 2a\sqrt{2}\int_{\pi/3}^{\pi}\sqrt{1-\cos\theta}\,d\theta = 2a\sqrt{2}\int_{\pi/3}^{\pi}\sqrt{2\sin^2\frac{\theta}{2}}\,d\theta = 4a\int_{\pi/3}^{\pi}\sin\frac{\theta}{2}\,d\theta$$

$$= 4a\left[-2\cos\frac{\theta}{2}\right]_{\pi/3}^{\pi} = 4a\sqrt{3} \qquad \textbf{Ans.}$$

EXAMPLE 6.29. Show that the length of an arc of the equiangular spiral $r = a\,e^{m\theta}$ varies as the difference of the radii vectors of the extremities of the arc.

SOLUTION: Let $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ be the extremities of an arc of the given curve $r = ae^{m\theta}$, then the length S of this arc, is given by

$$S = \int_{r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr$$

From the equation of the curve we have $\frac{dr}{d\theta} = m\,a\,e^{m\theta} = m\,r$ or $r\frac{d\theta}{dr} = \frac{1}{m}$

$$\therefore \quad S = \int_{r_1}^{r_2} \sqrt{1+\frac{1}{m^2}}\,dr = \sqrt{1+\frac{1}{m^2}}\,(r_2-r_1).$$

Therefore, S, the length of the arc PQ, varies as (r_2-r_1) **Hence Proved.**

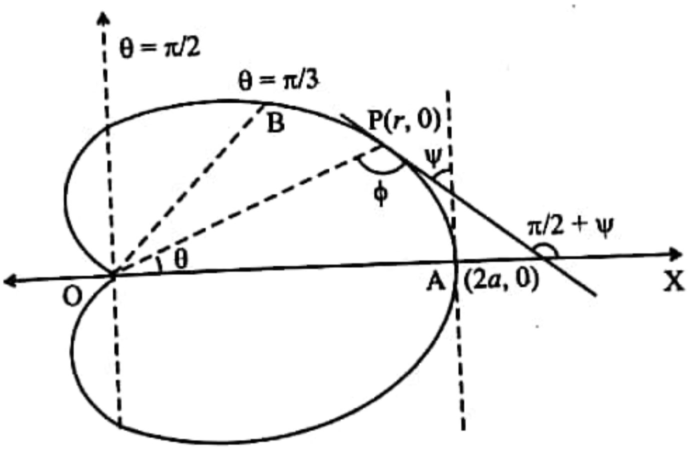
EXAMPLE 6.30. Find the total length of the cardioide $r = a(1+\cos\theta)$ and show that its upper half is bisected by the line $\theta = \frac{\pi}{3}$. Also find the intrinsic equation of the curve taking $\theta = 0$ as the fixed point. *[GGSIPU I Sem II Term 2004]*

SOLUTION: From the equation $r = a(1+\cos\theta)$ we have $\frac{dr}{d\theta} = -a\sin\theta$.

Therefore the length S of the cardioide from $\theta = 0$ to any point $P(r,\theta)$, is given by

$$S = \int_0^{\theta} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{\theta}\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,d\theta = a\int_0^{\theta}\sqrt{2(1+\cos\theta)}\,d\theta$$

$$= a\int_0^{\theta}\sqrt{4\cos^2\frac{\theta}{2}}\,d\theta = 4a\left[\sin\frac{\theta}{2}\right]_0^{\theta} = 4a\sin\frac{\theta}{2} \quad \text{...(1)}$$



The total length of the cardioid = 2 . length of the upper half of the cardioid

$$= 2\left[4a\sin\frac{\theta}{2}\right]_0^{\pi} = 8a.$$

$\therefore$ Length of the upper half cardioid = $4a$

Now, the length of the cardioid from $\theta = 0$ to $\theta = \frac{\pi}{3}$, $= [4a\sin\theta/2]_{\theta=0}^{\pi/3} = 2a$.

Therefore the line OB having equation $\theta = \pi/3$ bisects the upperhalf of the curve.

Next, for the intrinsic equation, from the above figure it is clear that $\frac{\pi}{2} + \psi = \theta + \phi$ ...(2)

and $\quad \tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \Rightarrow \phi = \frac{\pi}{2}+\frac{\theta}{2}.$

$\therefore$ (2) becomes $\frac{\pi}{2}+\psi = \theta+\frac{\pi}{2}+\frac{\theta}{2}$ or $\psi = \frac{3\theta}{2}$

Then, in turn (1) becomes $\quad S = 4a\sin\frac{\theta}{2} = 4a\sin\left(\frac{\psi}{3}\right).$

Ans.

Thus, $\quad S = 4a\sin\frac{\psi}{3}$ is the required intrinsic equation

EXAMPLE 6.31. Find the length of the arc of the hyperbolic spiral $r\theta = a$ from the point $r = a$ to $r = 2a$.

SOLUTION: From the equation $r\theta = a$ we have $r\dfrac{d\theta}{dr} + \theta = 0$

Hence the required length S of the curve, is given by

$$S = \int_a^{2a} \sqrt{1 + \left(\frac{r\,d\theta}{dr}\right)^2}\, dr = \int_a^{2a} \sqrt{1+\theta^2}\, dr = \int_a^{2a} \sqrt{1 + \frac{a^2}{r^2}}\, dr = \int_a^{2a} \frac{\sqrt{r^2+a^2}}{r^2}\, r\, dr$$

Putting $r^2 + a^2 = t^2$, so $2r\,dr = 2t\,dt$ we get

$$S = \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t \cdot t\, dt}{t^2 - a^2} = \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{t^2 - a^2 + a^2}{t^2 - a^2}\, dt = \int_{a\sqrt{2}}^{a\sqrt{5}} dt + a^2 \int_{a\sqrt{2}}^{a\sqrt{5}} \frac{dt}{t^2 - a^2}$$

$$= a\sqrt{5} - a\sqrt{2} + \frac{a^2}{2a}\left[\log \frac{t-a}{t+a}\right]_{a\sqrt{2}}^{a\sqrt{5}} = a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\left[\log \frac{a\sqrt{5}-a}{a\sqrt{5}+a} - \log \frac{a\sqrt{2}-a}{a\sqrt{2}+a}\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\log\left(\frac{\sqrt{5}-1}{\sqrt{5}+1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = a\left(\sqrt{5} - \sqrt{2}\right) + \frac{a}{2}\log\left[\frac{5-1}{\left(\sqrt{5}+1\right)^2} \cdot \frac{\left(\sqrt{2}+1\right)^2}{2-1}\right]$$

$$= a\left(\sqrt{5} - \sqrt{2}\right) + a\log\frac{2\left(\sqrt{2}+1\right)}{\left(\sqrt{5}+1\right)}.$$

Ans.

EXERCISE 6B

1. A cable from an antenna tower has the equation $3y = 4x^{3/2}$ from $x = 0$ to $x = 20$ metres. Find the total length of the cable.
2. Find the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$.
3. Find the length of the arc of the curve $4ax = y^2 - 2a^2 \log\left(\frac{y}{a}\right) - a^2$ from the point $(0, a)$ to the point (x, y).
4. If S is the length of the arc of the catenary $y = c \cosh\left(\frac{x}{c}\right)$ from the vertex to any point (x, y) on it, then prove that $y^2 = c^2 + s^2$. [GGSIPU I Sem. End Term 2011]
5. The parabolic chain of a suspension bridge has the form $x^2 = \frac{b^2 y}{h}$ where $2b$ is the span and h is the sag. Show that the length of the chain when h/b is small, is $2b + (4h^2/3b)$ approximately.
6. Find the perimeter of the loop of the curve $9ay^2 = (x - 2a)(x - 5a)^2$. [GGSIPU I Sem. II Term 2009]
7. Find the length of the arc of the curve $x^2 = a^2\left(1 - e^{\frac{y}{a}}\right)$ measured from the origin to any point (x, y) on it. [GGSIPU I Sem End Term 2012]
8. (a) Find the length of the curve $y^2 = (2x - 1)^3$ cut off by the line $x = 4$.

   (b) Show that the perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ may be expressed as

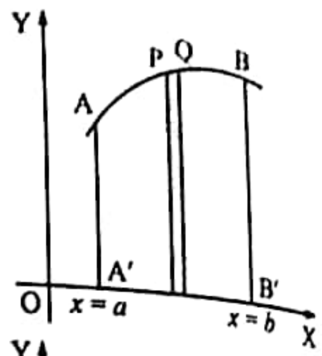
$$2a\pi\left[1 - \frac{1}{2^2}e^2 - \frac{1^2 \cdot 3}{2^2 \cdot 4^2}e^4 - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2}e^6 - \dots\right]$$

9. Obtain the length of the arc of the curve $x = e^\theta \sin\theta$, $y = e^\theta \cos\theta$, from $\theta = 0$ to $\theta = \pi/2$.
10. Find the length of one cycloid of the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.
11. Determine the perimeter of both the loops of the curve $8a^2y^2 = x^2(a^2 - x^2)$.
12. Show that the intrinsic equation of the curve $x = a(3\cos\theta - \cos 3\theta)$, $y = a(3\sin\theta - \sin 3\theta)$, length being measured from the point $\theta = \pi/2$ on it, is $S = 6a\cos^2\psi$.
13. Show that the length of the curve $y = \log\sec x$ from $x = 0$ to $x = \frac{\pi}{3}$, is $\log\left(2 + \sqrt{3}\right)$. [GGSIPU I Sem End Term Jan 2011]
14. Find the length of the arc of the parabola $y^2 = 4a(a - x)$ cut off by the y-axis.
15. Find the length of the arc of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$.
16. (a) Find the full length of the cardioid $r = a(1 - \cos\theta)$ and prove that the upper half of the curve is bisected by the line $\theta = 2\pi/3$. Also find the intrinsic equation of the curve.

    (b) Find the length of the arc of the parabola $\frac{2a}{r} = 1 + \cos\theta$ from its vertex, and also obtain the intrinsic equation of the curve.
17. Show that the length of one loop of the leminiscate $r^2 = a^2 \cos 2\theta$ is $\frac{a\sqrt{\pi}}{2}\frac{\left|\overline{1/4}\right.}{\left|\overline{3/4}\right.}$.

VOLUMES OF SOLIDS OF REVOLUTION

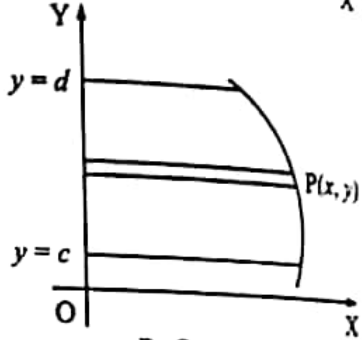
Let $y = f(x)$ be a curve and the area bounded by the curve from A to B and the X-axis be revolved about X-axis. An elementary strip of width dx at P (x, y) of the curve, when revolved about the X-axis, generates elementary solid of volume $\pi y^2\,dx$. (see the adjoining figure.)



Summing up the volumes of revolution of all such strips from $x = a$ to $x = b$, we get the volume of solid of revolution of the area A A′ B′ B about X-axis, equal to

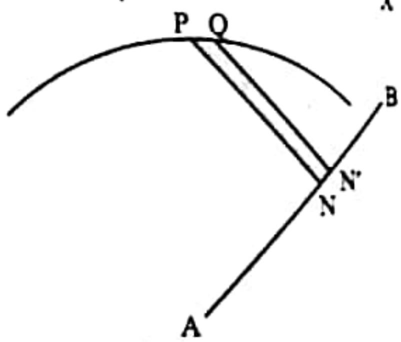
$$\int_a^b \pi y^2\,dx.$$

Similarly, if the area bounded by the curve, the y-axis and two extreme abscissas, $y = c$ and $y = d$, is revolved about y-axis, (see the the adjacent figure), then the volume of the solid of revolution, is equal to



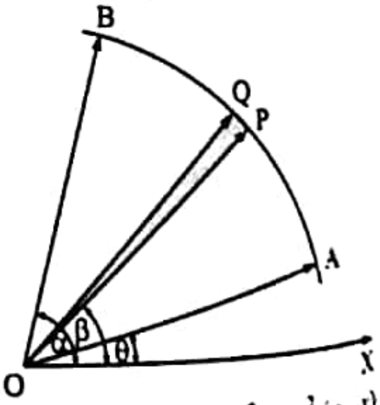
$$\int_c^d \pi x^2\,dy.$$

If the curve revoles about any other line say, AB. take any point on the line as a fixed point and let the perpendiculars from P and Q, two adjacent points on the curve, meet AB in N and N′. Then as NN′ = d (AN), the volume of the elementary disc of revolution about AB is $\pi\,(\mathrm{PN})^2\,d(\mathrm{AN})$ (see the adjoining figure), and hence the total amount of solid of revolution is equal to $\pi\int \mathrm{PN}^2\,d(\mathrm{AN})$ with proper limits of integration.



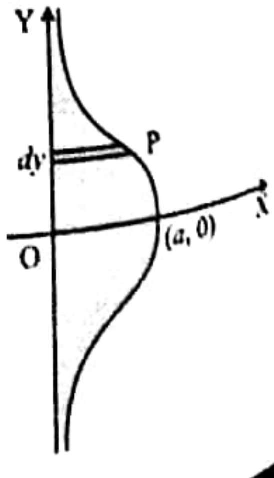
In case of curve in polar co-ordinates, a sectorial element OAB bounded by the radii vectors $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$ revolves about the initial line, then the volume of the solid of revolution, is

$$\int_\alpha^\beta \frac{2}{3}\pi r^2 \cdot r\sin\theta\,d\theta = \frac{2\pi}{3}\int_\alpha^\beta r^3\sin\theta\,d\theta$$



Example 6.32 Find the volume of the solid formed by the revolution of the curve $xy^2 = a^2(a-x)$ through four right angles about the axis of Y.

Solution: The curve $xy^2 = a^2(a-x)$ is symmetrical about the x-axis, meets x-axis at $(a, 0)$ and has y-axis as asymptote. The shape of the curve is shown in the adjoining figure. The shaded area revolves about the axis of y.



Required volume $V = \int_{-\infty}^{\infty} \pi x^2\,dy = 2\pi\int_0^{\infty} x^2\,dy$

(because of symmetry about X-axis).

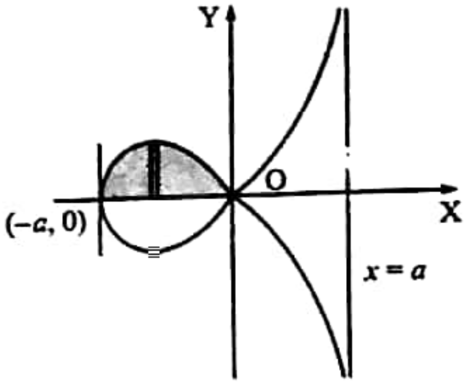
From the equation of the curve we have $x = \dfrac{a^3}{y^2 + a^2}$

$$\therefore \quad V = 2\pi \int_0^{\infty} \frac{a^6}{\left(y^2 + a^2\right)^2}\, dy. \quad \text{Putting } y = a \tan\theta, \text{ hence } dy = a\sec^2\theta\, d\theta,$$

$$V = 2\pi \int_0^{\pi/2} \frac{a^6\, a\sec^2\theta\, d\theta}{\left(a^2 + a^2\tan^2\theta\right)^2} = \frac{2\pi a^7}{a^4}\int_0^{\pi/2} \cos^2\theta\, d\theta = \pi a^3 \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= \pi a^3 \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi^2 a^3}{2}. \qquad \textbf{Ans.}$$

EXAMPLE 6.33. Obtain the volume of the solid of revolution of the loop of the curve $y^2 = \dfrac{x^2(a+x)}{a-x}$ about X-axis.



SOLUTION: The curve is symmetrical about X-axis, passes through the origin and the tangents at the origin are $y = \pm x$. It meets the X-axis at $(-a, 0)$ and has $x = a$ as an asymptote parallel to y-axis. For $x < -a$ as well as for $x > a$, y^2 is negative hence no part of the curve lies in the regions $x < -a$ and in $x > a$. As shown in the adjoining figure, there is one loop lying in the region $-a \le x \le a$.

The volume of the solid of revolution generated by the revolution of the loop about X-axis is given by

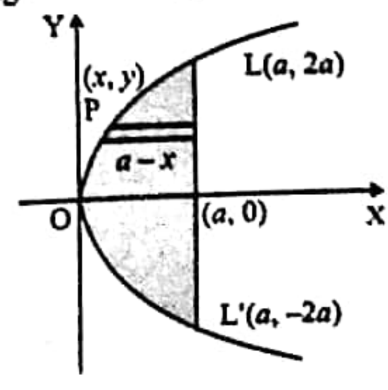
$$V = \int_{-a}^{0} \pi y^2\, dx = \pi \int_{-a}^{0} \frac{x^2(a+x)\, dx}{(a-x)} \quad \text{Putting } (a-x) = t \text{ so that } dx = -dt, \text{ we get}$$

$$V = -\pi \int_{2a}^{a} \frac{(a-t)^2(2a-t)\, dt}{t} = \pi \int_a^{2a} \frac{1}{t}\left[2a^3 - t\left(a^2 + 4a^2\right) + t^2(2a + 2a) - t^3\right] dt$$

$$= \pi \int_a^{2a} \left[\frac{2a^3}{t} - 5a^2 + 4at - t^2\right] dt = \pi\left[2a^3 \log t - 5a^2 t + 2at^2 - \frac{t^3}{3}\right]_a^{2a}$$

$$= \pi a^3\left[2\log 2 - 10 + 5 + 8 - 2 - \frac{8}{3} + \frac{1}{3}\right] = 2\pi a^3\left(\log 2 - \frac{2}{3}\right) \qquad \textbf{Ans.}$$

EXAMPLE 6.34. Calculate the volume of the solid generated by revolving the area bounded by the parabola $y^2 = 4ax$ and its latus rectum about the latus rectum of the parabola.



SOLUTION: The tangent at the vertex $(0, 0)$ of the parabola $y^2 = 4ax$ is Y-axis. The latus rectum is LL′ where $L = (a, 2a)$ and $L'(a, -2a)$. The shaded portion as shown in the adjoining figure is revolved about Y-axis.

$$\text{Required volume} = \int_{-2a}^{2a} \pi (a-x)^2\, dy = 2\pi \int_0^{2a} (a^2 - 2ax + x^2)\, dy$$

(because of symmetry about X-axis)

$$= 2\pi \int_0^{2a} \left(a^2 - \frac{y^2}{2} + \frac{y^4}{16a^2} \right) dy$$

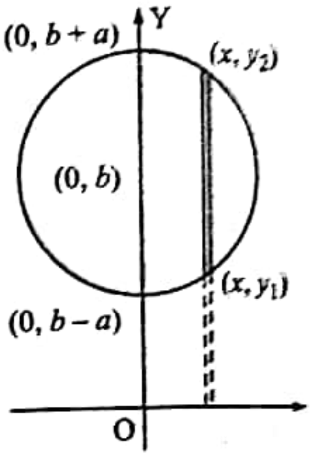
$$= 2\pi \left[a^2 y - \frac{y^3}{6} + \frac{y^5}{80a^2} \right]_0^{2a} = \frac{32\pi a^3}{15}. \quad \textbf{Ans.}$$

EXAMPLE 6.35. Find the volume of the solid in the form of a Torus formed by the revolution of the circle $x^2 + (y - b)^2 = a^2$ $(b > a)$ about the axis of X.

SOLUTION: From the given equation of the circle, we have $y = b \pm \sqrt{a^2 - x^2}$

Let us denote $y_2 = b + \sqrt{a^2 - x^2}$ and $y_1 = b - \sqrt{a^2 - x^2}$.

The shaded area, as shown in the adjoining figure, is revolved about the axis of x then the required volume V of the torus, because of symmetry about Y-axis, is given by



$$V = 2\left[\int_0^a \pi y_2^2 \, dx - \int_0^a \pi y_1^2 \, dx \right] = 2\pi \int_0^a \left(y_2^2 - y_1^2 \right) dx$$

$$= 2\pi \int_0^a (y_2 + y_1)(y_2 - y_1) \, dx = 2\pi \int_0^a 2b \cdot 2\sqrt{a^2 - x^2} \, dx$$

$$= 8\pi b \int_0^a \sqrt{a^2 - x^2} \, dx = 8\pi b \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

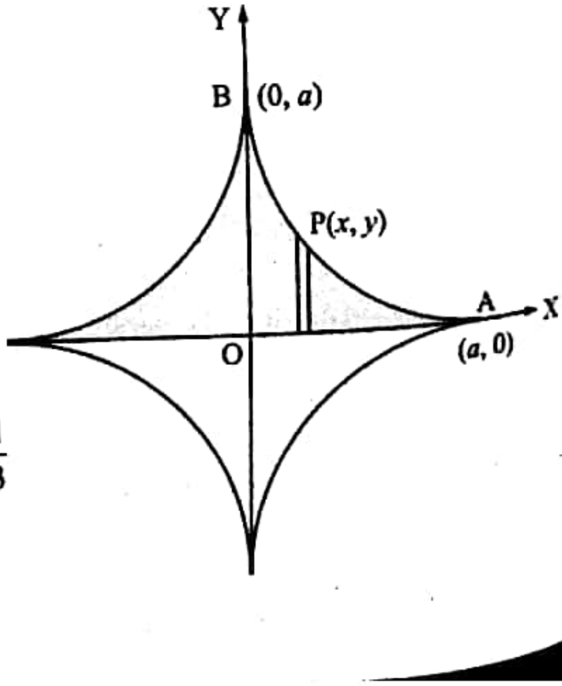
or $\quad V = 2\pi^2 a^2 b. \quad$ **Ans.**

EXAMPLE 6.36. The area enclosed by the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is revolved about X-axis. Find the volume of the spindle shaped solid of revolution.

SOLUTION: The given equation can be written in the parametric form as $x = a\cos^3\theta$, $y = a\sin^3\theta$.

The curve meets the x-axis at $(\pm a, 0)$ and y-axis at $(0, \pm a)$. The required volume will be generated by revolving the upper-half-area about x-axis as shown in the adjoining figure.

At the point A $(a, 0)$ $\theta = 0$ and at B $(0, a)$ $\theta = \frac{\pi}{2}$.



Required volume $= \int_{-a}^{a} \pi y^2 \, dx = 2\pi \int_0^a y^2 \, dx$

(because of symmetry about y-axis)

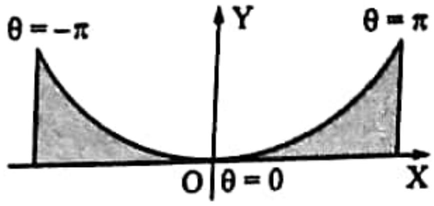
$$= 2\pi \int_{\pi/2}^{0} a^2 \sin^6\theta \cdot \left(-3a\cos^2\theta \sin\theta \right) d\theta$$

$$= 6\pi a^3 \int_0^{\frac{\pi}{2}} \sin^7\theta \cos^2\theta \, d\theta = 6\pi a^3 \, \frac{6}{9} \cdot \frac{4}{7} \cdot \frac{2}{5} \cdot \frac{1}{3}$$

$$= \frac{32\pi a^3}{105}. \quad \textbf{Ans.}$$

Find the volume of solid of revolution in the form of a reel by revolving the area enclosed by one cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about X-axis.

SOLUTION: The shaded region in the figure represents the area being revolved about the X-axis. The required volume V is given by



$$V = \int_{-a\pi}^{a\pi} \pi y^2\, dx = 2\int_0^{a\pi} \pi y^2\, dx$$

$$= 2\pi \int_0^{\pi} a^2 (1-\cos\theta)^2 a(1+\cos\theta)\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^2 \cdot 2\cos^2\frac{\theta}{2}\, d\theta$$

$$V = 16\pi a^3 \int_0^{\frac{\pi}{2}} 2\sin^4 t \cos^2 t\, dt \qquad (\text{On puting } \theta = 2t)$$

$$= 32\pi a^3 \cdot \frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \pi^2 a^3. \qquad \textbf{Ans.}$$

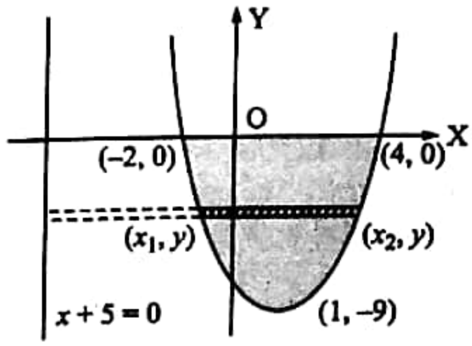
Find the volume of the solid generated by revolving the area included between the curve $\dfrac{y+8}{x} = x - 2$ and the X-axis, about the line $x + 5 = 0$.

SOLUTION: The given curve is the parabola $x^2 - 2x = y + 8$ or $(x - 1)^2 = y + 9$ ... (1)

with vertex at $(1, -9)$ and axis of the parabola being parallel to Y-axis. It meets X-axis at $(-2, 0)$ and $(4, 0)$. (see figure). From (1), we have $x = 1 \pm \sqrt{y+9}$

Let us write $x_1 = 1 - \sqrt{y+9}$, $x_2 = 1 + \sqrt{y+9}$.

Required volume when the shaded portion is revolved about the line $x = -5$, is given by



$$V = \int_{-9}^{0} \pi (x_2 + 5)^2\, dy - \int_{-9}^{0} \pi (5 + x_1)^2\, dy$$

$$= \pi \int_{-9}^{0} \left[(x_2+5)^2 - (5+x_1)^2\right] dy$$

$$= \pi \int_{-9}^{0} (10 + x_1 + x_2)(x_2 - x_1)\, dy = \pi \int_{-9}^{0} (10+2)\, 2\sqrt{y+9}\, dy$$

$$= 24\pi \frac{2}{3}\left[(y+9)^{3/2}\right]_{-9}^{0} = 16\pi\left[9\sqrt{9} - 0\right] = 432\pi. \qquad \textbf{Ans.}$$

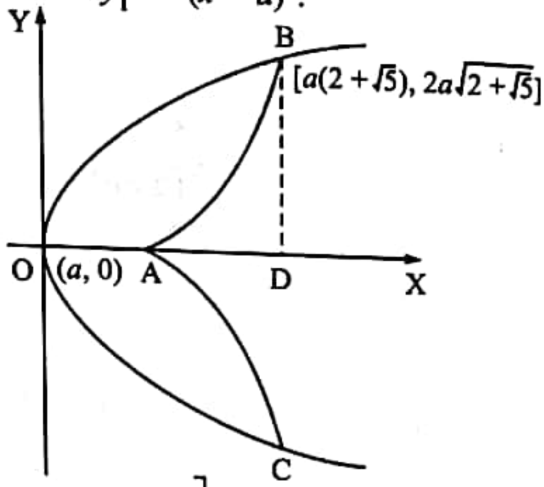
Example 6.39. Find the volume of the solid formed by revolving about X-axis, the area enclosed between the curve $2ay^2 = (x-a)^3$, the X-axis and the parabola $y^2 = 4ax$.

Solution: The curve $2ay^2 = (x-a)^3$ is a semi-cubical parabola with vertex at $(a, 0)$ and is symmetrical about X-axis. The curve (1) meets the parabola $y^2 = 4ax$ when $2a \cdot 4ax = (x-a)^3$ or $x^3 - 5a^2x - 3ax^2 - a^3 = 0$ which gives $x = -a$ and $x = a\left(2 \pm \sqrt{5}\right)$. Rejecting $x = -a$ and $x = a\left(2-\sqrt{5}\right)$, since x is not negative, we have $x = a\left(2+\sqrt{5}\right)$. (see the figure)

Required volume = Volume obtained by revolving the area OBDO about X-axis

− Volume obtained by revolving the area ABDA about X-axis

$$= \int_0^{a(2+\sqrt{5})} \pi\left(y_2^2\right)dx - \int_a^{a(2+\sqrt{5})} \pi\left(y_1^2\right)dx \quad \text{where } y_2^2 = 4ax \text{ and } 2ay_1^2 = (x-a)^3.$$



$$= \pi \int_0^{a(2+\sqrt{5})} 4ax\,dx - \pi \int_a^{a(2+\sqrt{5})} \frac{1}{2a}(x-a)^3\,dx$$

$$= 2\pi a\left[x^2\right]_0^{a(2+\sqrt{5})} - \frac{\pi}{2a}\left[\frac{(x-a)^4}{4}\right]_a^{a(2+\sqrt{5})}$$

$$= 2\pi a^3\left(4+5+4\sqrt{5}\right) - \frac{\pi a^3}{8}\left(1+\sqrt{5}\right)^4$$

$$= 2\pi a^3\left(9+4\sqrt{5}\right) - \frac{\pi a^3}{8}\left(6+2\sqrt{5}\right)^2 = \pi a^3\left[(18+8\sqrt{5}) - \frac{1}{2}(9+5+6\sqrt{5})\right]$$

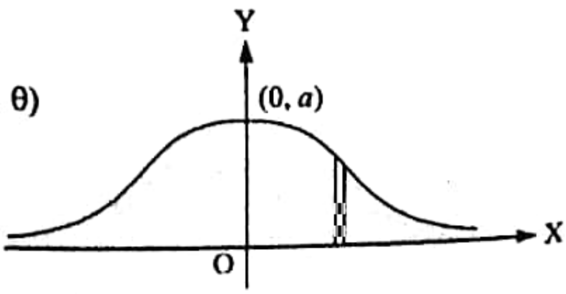
$$= \pi a^3\left(11+5\sqrt{5}\right). \qquad \textbf{Ans.}$$

Example 6.40. Find the volume of the solid generated by revolving the area under the curve $y = \dfrac{a^3}{x^2+a^2}$ about its asymptote. [*GGSIPU I Sem Term 2007; II Term 2009; II Term 2011*]

Solution: The curve has x-axis as asymptote and the shape of the curve is as shown in the figure here. The shaded area is revolved about x-axis and volume of the solid formed

$$= \int_{-\infty}^{\infty} \pi y^2\,dx = \pi \int_{-\infty}^{\infty} \frac{a^6\,dx}{(x^2+a^2)^2} = 2\pi a^6 \int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$$

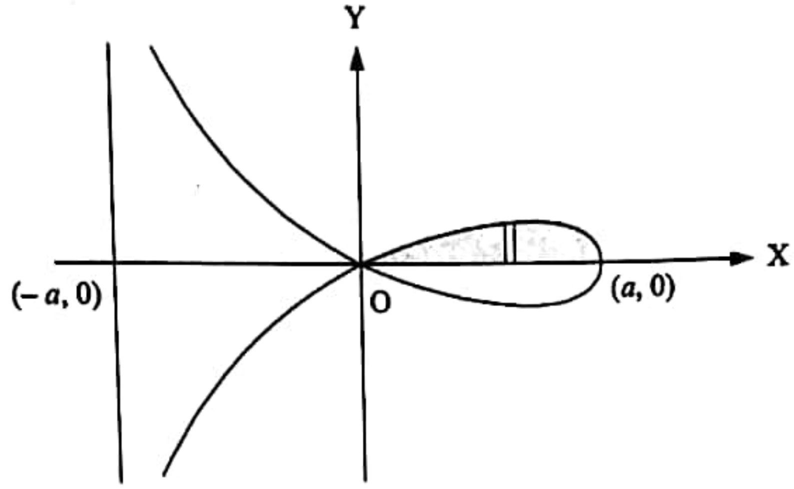
$$= 2\pi a^6 \int_0^{\pi/2} \frac{a\sec^2\theta\,d\theta}{(a^2+a^2\tan^2\theta)^2} \quad (\text{on putting } x = a\tan\theta)$$



$$= 2\pi a^3 \int_0^{\pi/2} \cos^2\theta\,d\theta = \pi a^3 \int_0^{\pi/2} (1+\cos 2\theta)\,d\theta$$

$$= \pi a^3\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi^2 a^3}{2} \qquad \textbf{Ans.}$$

EXAMPLE 6.41. Find the volume of the solid formed by the revolution about the x-axis of the loop of the curve $y^2(a+x) = x^2(a-x)$ [*GGSIPU I Sem II Term 2002*]



SOLUTION: The curve $y^2(a+x) = x^2(a-x)$ has an asymptote $x = -a$. (see the figure).

Required volume V of the solid formed is given by

$$V = \int_0^a \pi y^2 dx = \int_0^a \pi x^2 \frac{(a-x)dx}{a+x} \quad \text{(Now putting } x + a = t\text{)}$$

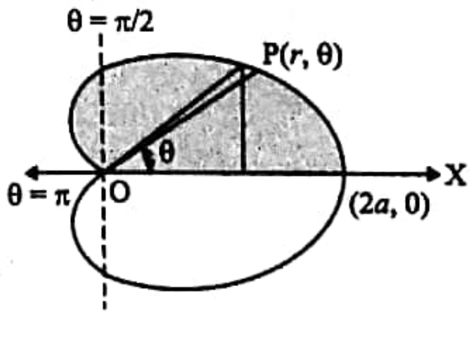
$$= \pi \int_a^{2a} \frac{(t-a)^2(2a-t)dt}{t} = \pi \int_a^{2a} \frac{(-t^3 + 4at^2 - 5a^2t + 2a^3)}{t} dt$$

$$= \pi \left[-\frac{t^3}{3} + 2at^2 - 5a^2t + 2a^3 \log t \right]_a^{2a}$$

$$= 2\pi a^3 \left(\log 2 - \frac{2}{3} \right)$$ **Ans.**

EXAMPLE 6.42. Find the volume of the solid generated when the area bounded by the cardioide $r = a(1 + \cos\theta)$ is revolved about the initial line.

SOLUTION: The shape of the cardioide $r = a(1+\cos\theta)$ is as shown in the adjoining figure. To obtain the required volume, the upper half of the cardioide is revolved about the initial line.



The required volume V is given by

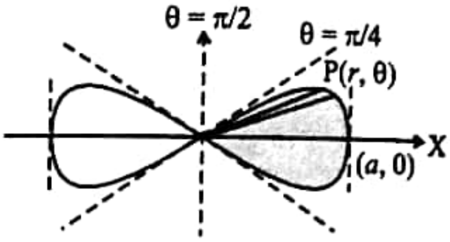
$$V = \int_0^\pi \frac{2}{3}\pi r^2 \cdot r \sin\theta \, d\theta = \frac{2\pi}{3}\int_0^\pi a^3 (1+\cos\theta)^3 \sin\theta \, d\theta$$

Putting $1 + \cos\theta = t$, $-\sin\theta \, d\theta = dt$, we get

$$V = \frac{2\pi a^3}{3}\int_2^0 -t^3 \, dt = \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2 = \frac{8}{3}\pi a^3.$$ **Ans.**

EXAMPLE 6.43. The area bounded by one loop of the leminiscate of Bernoulli $r^2 = a^2 \cos 2\theta$ is revolved about the line $\theta = \pi/2$. Find the volume of the solid of revolution.
[*GGSIPU I Sem II Term 2004*]

SOLUTION: The shape of the given curve $r^2 = a^2 \cos 2\theta$ is as shown in the figure. The shaded area is revolved about the line $\theta = \pi/2$. The loop lies between the lines $\theta = -\pi/4$ and $\theta = \pi/4$.



The required volume V is given by

$$V = \int_{-\pi/4}^{\pi/4} \frac{2}{3}\pi r^2 \cdot r \cos\theta \, d\theta = \frac{4\pi}{3}\int_0^{\pi/4} a^3 \cos^{3/2}(2\theta)\cos\theta \, d\theta$$

(because of symmetry about initial line)

$$= \frac{4\pi a^3}{3}\int_0^{\pi/4} \left(1 - 2\sin^2\theta\right)^{3/2} \cos\theta \, d\theta$$

Now, Putting $\sqrt{2}\sin\theta = \sin\phi$, hence $\sqrt{2}\cos\theta \, d\theta = \cos\phi \, d\phi$, we have

$$V = \frac{4\pi a^3}{3}\int_0^{\pi/2} \left(1 - \sin^2\phi\right)^{3/2} \cdot \frac{1}{\sqrt{2}} \cos\phi \, d\phi = \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2} \cos^4\phi \, d\phi$$

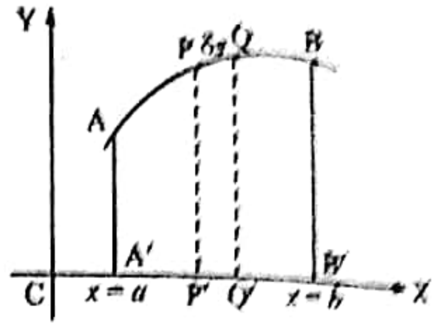
$$= \frac{4\pi a^3}{3\sqrt{2}} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^3}{4\sqrt{2}}.$$ **Ans.**

EXERCISE 6C

1. Find the volume of the solid generated by revolving the area included between the parabola $y^2 = 4ax$ and its latus rectum about the axis of X.
2. Find the volume of the solid generated by the revolution of the area of the ellipse $\frac{x^2}{16}+\frac{y^2}{9} = 1$ about the X-axis.
3. Find the volume of the solid formed when the area bounded by the curve $y^2 (a - x) = a^2 x$ and its asymptote, is revolved about the asymptote.
4. Find the volume of the solid generated by revolving the area included between the curves $ay^2 = x^3$ and $ax^2 = y^3$ about the X-axis.
5. (*a*) Find the volume of the spherical cap of height h cut off a sphere of radius a.
   (*b*) A quadrant of a circle of radius a is revolved about its chord. Find the volume of the spindle, thus generated.
6. Find the volume of the solid formed by the revolution of the area enclosed between the curve $27ay^2 = 4(x - 2a)^3$, the axis of X and the parabola $y^2 = 4ax$ about X-axis.
7. Find the volume generated by revolving the area cut off the parabola $9y = 4(9 - x^2)$ by the line $4x + 3y = 12$ about X-axis
8. Find the volume generated by revolving the area bounded by the circle $x^2 + y^2 = 25$ cut off by the lines $3x - 4y = 0$ and $y = 0$ about the axis of X.
9. The area included between the X-axis and the upper half of the circle $x^2 + y^2 - 4x = 0$ is rotated about Y-axis. Find the voume thus generated.
10. The area bounded by one arc of the sine wave $y = \sin x$ and the X-axis is revolved about the axis of X. Find the volume of the solid thus generated.
11. Obtain the volume generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about its base.
12. The area bounded by a quadrant of a circle of radius a and the tangents at the extremities revolves about one of the tangents. Find the volume of the solid generated.
13. The area enclosed by the curves $y = \sin x$, $y = \cos x$ and the axis of X between $x = 0$ and $x = \frac{\pi}{2}$, is revolved about X-axis. Find the volume of the solid so formed.
14. If the area bounded by the parabolas $y^2 = 4x$ and $y^2 = 5 - x$ is revolved about each of the co-ordinate axes, calculate the respective volumes.
15. The area of the loop of the curve $r = a\cos\theta$ lying between $\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$, revolves about the initial line. Find the volume of the solid of revolution.
16. The arc of the cardioid $r = a(1 + \cos\theta)$ included between $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$, is rotated about the line $\theta = \frac{\pi}{2}$. Show that the volume so generated is $2\left(2+\frac{5\pi}{8}\right)\pi a^3$.
17. Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the tangent at the end of (i) major axis (ii) minor axis.

SURFACE OF SOLIDS OF REVOLUTION

When the arc AB given by the equation $y = f(x)$, is rotated about the axis of X through four right angles, the curved surface area of the slice may be found by considering the revolution of the chord PQ instead of the arc PQ about X-axis. (see figure.)



The elementary surface area δS is approximately equal to the circumference of the circle with centre P' and radius y, multiplied by arc PQ, *i.e.*, $\delta S = 2\pi y\, \delta s$ where δs is the arc PQ.

Therefore, the total curved surface area S of the solid of revolution about X-axis , is given by

$$S = \int 2\pi y\, ds = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{where} \quad A = (a, c), B = (b, d).$$

Similarly, if the area is rotated about Y-axis, the surface of solid of revolution, is given by

$$S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dy.$$

Further, if the equation of the curve is in parametric form, the curved surface area S of the solid generated by the revolution about X-axis, is given by

$$S = \int_{t_1}^{t_2} 2\pi y \frac{ds}{dt}\, dt = \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Similarly, if the arc is revolved about Y-axis, then

$$S = \int_{t_1}^{t_2} 2\pi x \frac{ds}{dt}\, dt = \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Next, if the equation of the curve is given in polar coordinates, the curved surface area S of the solid of revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$ about the initial line, is given by

$$S = \int 2\pi y\, ds = \int_\alpha^\beta 2\pi y \frac{ds}{d\theta}\, d\theta = \int_\alpha^\beta 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

And if the area bounded by the curve $r = f(\theta)$, the radii vectors $\theta = \alpha$ and $\theta = \beta$ is revolved about the line $\theta = \pi/2$, the surface of revolution S is given by

$$S = \int 2\pi x\, ds = \int_\alpha^\beta 2\pi x \frac{ds}{d\theta}\, d\theta = \int_\alpha^\beta 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

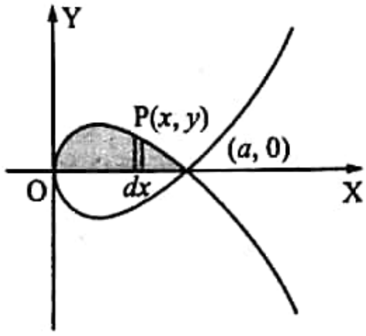
[illegible] Find the area of the surface generated by revolving the loop of the curve $3ay^2 = x(x-a)^2$ about the axis of X.

SOLUTION: The loop of the given curve $3ay^2 = x(x-a)^2$, as shown in the adjoining figure, extends from $x = 0$ to $x = a$. Therefore, revolving the upper half of the loop about X-axis, we get the required surface area S, given by

$$S = \int 2\pi y\, ds = 2\pi \int_0^a y \frac{ds}{dx}\, dx = 2\pi \int_0^a y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

From the given equation, we have $6\,ay\dfrac{dy}{dx} = (x-a)^2 + 2x(x-a)$

or $$\frac{dy}{dx} = \frac{(x-a)(3x-a)}{6\,ay}$$



$$\therefore \quad S = 2\pi\int_0^a y\sqrt{1+\frac{(x-a)^2(3x-a)^2}{36\,a^2y^2}}\,dx$$

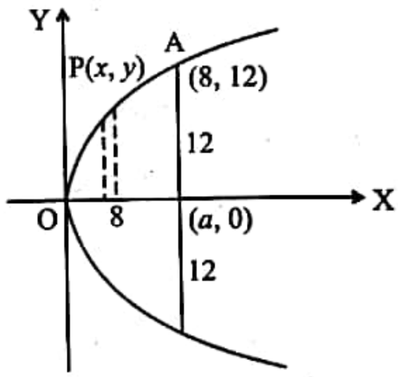
$$= 2\pi\int_0^a \sqrt{y^2+\frac{(x-a)^2(3x-a)^2}{36\,a^2}}\,dx$$

$$= 2\pi\int_0^a \sqrt{\frac{x(x-a)^2}{3a}+\frac{(x-a)^2(3x-a)^2}{36a^2}}\,dx = 2\pi\int_0^a \frac{(x-a)}{6a}\sqrt{12ax+(3x-a)^2}\,dx$$

$$= \frac{2\pi}{6a}\int_0^a (x-a)(3x+a)\,dx = \frac{\pi}{3a}\int_0^a \left(3x^2-2ax-a^2\right)dx$$

$$= \frac{\pi}{3a}\left[x^3-ax^2-a^2x\right]_0^a = \frac{\pi a^2}{3} \text{ as absolute value.}$$ **Ans.**

EXAMPLE 6.45. A parabolic reflector of an automobile headlight is 24 cm in diameter and 8cm in depth. Find the cost of plating the front portion of the reflector if the cost of plating is Rs. 4 per sq. cm.



SOLUTION: Let the reflector be obtained by revolving the upper half of the parabola $y^2 = 4ax$ about X-axis. As given in the problem points $(8, \pm 12)$ should lie on the parabola, $\therefore\ a = 9/2$.

Thus, the parabola under revolution about X-axis has the equation

$$y^2 = 18\,x, \text{ hence } \frac{dy}{dx} = \frac{9}{y}. \text{ (See the figure.)}$$

Required surface of revolution S is given by

$$S = 2\pi\int y\,ds = 2\pi\int_0^8 y\frac{ds}{dx}\,dx = 2\pi\int_0^8 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^8 y\sqrt{1+\frac{81}{y^2}}\,dx$$

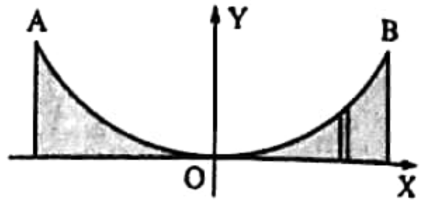
$$= 2\pi\int_0^8 \sqrt{y^2+9^2}\,dx = 2\pi\int_0^8 \sqrt{18x+81}\,dx$$

$$= \frac{6\pi}{2}\cdot\frac{2}{3}\left[(2x+9)^{3/2}\right]_0^8 = 2\pi(125-27) = 196\,\pi.$$ **Ans.**

Hence, the cost of plating $= 4 \times 196\,\pi =$ Rs. 2464/- approximately.

[redacted] Find the surface area of the solid generated by revolving one arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, about the tangent at the vertex.
[GGSIPU I Sem II Term 2007]

SOLUTION: One cycloid is completed when θ varies from $\theta = -\pi$ to $\theta = \pi$ in the equation $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ as given in the figure.



Required surface area S is given by

$$S = \int 2\pi y\, ds = \int_{-\pi}^{\pi} 2\pi y \frac{ds}{d\theta}\, d\theta \quad \text{(by symmetry about Y-axis)}$$

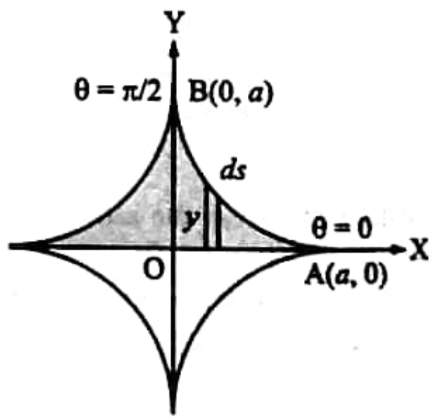
$$= 4\pi \int_0^{\pi} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = 4\pi \int_0^{\pi} a(1-\cos\theta)\sqrt{a^2(1+\cos\theta)^2 + a^2 \sin^2\theta}\, d\theta$$

$$= 4\pi a^2 \int_0^{\pi} (1-\cos\theta)\sqrt{2+2\cos\theta}\, d\theta = 4\pi a^2 \int_0^{\pi} 2\sin^2\frac{\theta}{2}\sqrt{4\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta = 16\pi a^2\left[\frac{2}{3}\sin^3\frac{\theta}{2}\right]_0^{\pi} = \frac{32}{3}\pi a^2. \qquad \textbf{Ans.}$$

[redacted] Find the surface of the solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the axis of X or axis of Y.
[GGSIPU I Sem II Term 2003; 2004]

SOLUTION: The parametric equation of the given astroid is $x = a\cos^3\theta$, $y = a\sin^3\theta$. (see the figure.) Revolving the upper half of the curve about the X-axis, the surface area S of the solid generated, is given by



$$S = \int 2\pi y\, ds = 2\pi \int_0^{\pi} y \frac{ds}{d\theta}\, d\theta$$

$$= 4\pi \int_0^{\pi/2} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

(because of symmetry about Y-axis)

$$= 4\pi \int_0^{\pi/2} a\sin^3\theta \sqrt{\left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2}\, d\theta$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^3\theta\sqrt{\sin^2\theta\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta = 12\pi a^2 \int_0^{\pi/2} \sin^4\theta\cos\theta\, d\theta$$

$$= 12\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} d\theta = \frac{12\pi a^2}{5}. \qquad \textbf{Ans.}$$

The cardioide $r = a(1 - \cos\theta)$ revolves about the initial line. Find the surface area of the solid so generated.

SOLUTION: The upper half of the cardioid $r = a(1 - \cos\theta)$ revoles about the initial line. (see adjoining figure.) Then the required surface area S of the solid generated, is given by

$$S = \int 2\pi y\, ds = 2\pi \int_0^{\pi} y \frac{ds}{d\theta} d\theta = 2\pi \int_0^{\pi} r \sin(\pi - \theta) \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\pi \int_0^{\pi} r \sin\theta \sqrt{r^2 + a^2 \sin^2\theta}\, d\theta$$

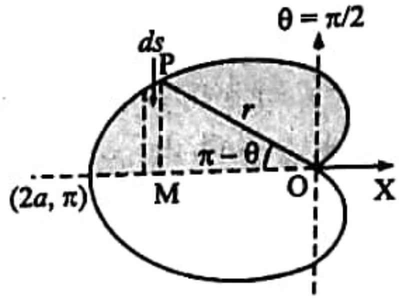
$$= 2\pi a \int_0^{\pi} (1 - \cos\theta)\sin\theta \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= 2\pi a^2 \int_0^{\pi} (1 - \cos\theta)\sin\theta \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= 2\pi a^2 \int_0^{\pi} 2\sin^2\frac{\theta}{2} \cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \cdot \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta = 16\pi a^2 \int_0^{\pi} \sin^4\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \left[\frac{2\sin^5\frac{\theta}{2}}{5}\right]_0^{\pi} = \frac{32\pi a^2}{5}.$$

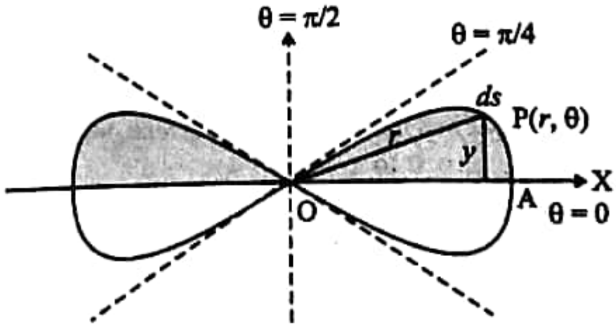
Ans.



Determine the surface area of the solid generated when the leminiscate of Bernoulli $r^2 = a^2 \cos 2\theta$ revolves about the initial line. [*GGSIPU I Sem End Term Jan 2011*]

SOLUTION: The shape of the curve is as shown in the adjoining figure. The upper half of both the loops are being revolved about the initial line.

Required surface area S is twice the surface area obtained on revolving the upper half of one loop for which θ varies from $\theta = 0$ to $\theta = \pi/4$.



$$\therefore \quad S = 2\int 2\pi y\, ds = 2\int_0^{\pi/4} 2\pi y \frac{ds}{d\theta} d\theta$$

$$= 4\pi \int_0^{\pi/4} r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

From the equation of the curve $r^2 = a^2\cos 2\theta$ we have $2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta$

Therefore, $$S = 4\pi \int_0^{\pi/4} r\sin\theta \sqrt{r^2 + \frac{a^4\sin^2 2\theta}{r^2}}\, d\theta = 4\pi \int_0^{\pi/4} \sin\theta\sqrt{r^4 + a^4\sin^2 2\theta}\, d\theta$$

$$= 4\pi \int_0^{\pi/4} \sin\theta \sqrt{a^4\cos^2 2\theta + a^4 \sin^2 2\theta}\, d\theta = 4\pi a^2 \int_0^{\pi/4} \sin\theta\, d\theta$$

$$= 4\pi a^2 [-\cos\theta]_0^{\frac{\pi}{4}} = 2\sqrt{2}\,\pi a^2\left(\sqrt{2} - 1\right).$$

Ans.

EXERCISE 6D

1. Determine the surface area of the cone generated by revolving the line $y = 2x$ from $x = 0$ to $x = 2$ about X-axis.
2. The portion of the parabola $y^2 = 4ax$ cut off the latus rectum revolves about the tangent at the vertex. Find the surface area of the reel thus generated.
3. Find the curved surface area generated by revolving the cubical parabola $x = y^3$ from $y = 0$ to $y = 2$ about the axis of Y.
4. Find the area of the surface generated by the revolution of the arc of the catenary $y = c \cosh x/c$ from the vertex to any point (x, y) of the curve, about the X-axis.
5. Show that the area of the curved surface generated when one loop of the curve $x^2 (a^2 - x^2) = 8a^2y^2$ is revolved about X-axis, is $\pi a^2/4$.
6. Find the surface area of the solid generated by revolving the loop of the curve

   $x = t^2, \quad y = t - \dfrac{t^3}{3}$ about the axis of X.
7. The area under the ellipse of eccentricity e is revolved about its major axis. Find the ratio of the surface area of the spheroid generated to the area of the ellipse.
8. (*a*) A quadrant of a circle of radius a revolves about its chord so that a solid is formed. Find the surface area of the spindle generated.

   (*b*) A circular arc revolves about its chord. Show that the area of the surface generated is equal to $4\pi a^2 (\sin \alpha - \alpha \cos \alpha)$ where 2α is the angle subtended by the arc at the centre.
9. Find the surface area of the solid generated by revolving the area bounded by the circle $x^2 + y^2 = a^2$ about the line $y = a$.
10. The leminiscate of Bernoulli is revolved about the tangent at the pole. Find the surface area of the solid generated.
11. Find the area of the curved surface of the cup formed by the revolution about its axis of the smaller part of the parabola $y^2 = 4ax$ cut-off by the line $x = 3a$.
12. Find the area of the surface formed by revolving a catenary $y = a \cosh (x/a)$ about X-axis from $x = 0$ to $x = a$.
13. Show that the volume and the surface generated by the revolution of an arc of the cycloid

$$x = a (\theta + \sin \theta), y = a (1 - \cos\theta)$$

   about the Y-axis are respectively $\pi a^3\left(\dfrac{3}{2}\pi^2 - \dfrac{8}{3}\right), 4\pi a^2\left(2\pi - \dfrac{8}{3}\right)$.

BETA AND GAMMA FUNCTIONS

GAMMA FUNCTION of n, denoted by Γn, is defined as

$$\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx, \quad n > 0} \qquad \text{...(1)}$$

Clearly, $\Gamma 1 = \int_0^\infty e^{-x}.1\, dx = -\left[e^{-x}\right]_0^\infty = 1.$

If we put $x = at$ in (1) we get $\Gamma n = \int_0^\infty e^{-at}(at)^{n-1} a\, dt = a^n \int_0^\infty e^{-at}\, t^{n-1} dt$. [GGSIPU III Sem End Term 2007]

And, by definition $\Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx$. Integrating it by parts, we get

$$\Gamma(n+1) = -\left[x^n e^{-x}\right]_0^\infty + \int_0^\infty n\, x^{n-1} e^{-x}\, dx = -\lim_{x\to\infty} \frac{x^n}{e^x} + n\Gamma n$$

Applying L' Hospital's rule n times to $\lim_{x\to\infty} \frac{x^n}{e^x}$ its value comes out to be zero, therefore we have the reduction formula $\boxed{\Gamma(n+1) = n\,\Gamma n.}$

If n is a positive integer, then

$$\Gamma(n+1) = n\Gamma n = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) = ... = n(n-1)(n-2)...3.2.1\,\Gamma 1$$

$$= n(n-1)(n-2)...3.2.1 \qquad \text{as } \Gamma 1 = 1$$

Thus, $\boxed{\Gamma(n+1) = n! \quad (n \in \mathrm{N}).}$

There is another representation of Γn as follows.

Putting $x = t^2$ in (1), gives $\Gamma n = \int_0^\infty e^{-t^2}.t^{2n-2}.2t\, dt = 2\int_0^\infty e^{-t^2}.t^{2n-1} dt$

or $$\boxed{\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx} \qquad \text{...(2)}$$

[GGSIPU III Sem End Term 2008]

In yet another representation of Γn we put $x = at^2$ in (1) and get

$$\Gamma n = \int_0^\infty e^{-at^2}(at^2)^{n-1} 2at\, dt$$

[GGSIPU III Sem II Term 2003, End Term 2007]

$$= 2a^n \int_0^\infty e^{-at^2} t^{2n-1} dt$$

Gamma of a negative number can also be found. For that we use the formula

$\Gamma(n+1) = n\Gamma(n)$ as $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$.

For example, let us evaluate $\Gamma\left(\frac{-3}{2}\right)$ and $\Gamma\left(\frac{-5}{2}\right)$. [GGSIPU II Sem End Term 2013]

$\Gamma\left(\frac{-5}{2}\right) = \dfrac{\Gamma(-3/2)}{-5/2} = \dfrac{-2}{5}\Gamma\left(\frac{-3}{2}\right)$. Similarly $\Gamma\left(\frac{-3}{2}\right) = \dfrac{\Gamma(-1/2)}{-3/2} = \dfrac{-2}{3}\Gamma\left(\frac{-1}{2}\right)$

and also $\Gamma\left(\frac{-1}{2}\right) = \dfrac{\Gamma(1/2)}{-1/2} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$

$$\therefore \qquad \Gamma\left(\frac{-3}{2}\right) = \frac{-2}{3}\left(-2\Gamma\left(\frac{1}{2}\right)\right) = \frac{4}{3}\Gamma\left(\frac{1}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

$$\text{and} \qquad \Gamma\left(\frac{-5}{2}\right) = \frac{-2}{5}\left(\frac{4}{3}\Gamma\left(\frac{1}{2}\right)\right) = \frac{-8}{15}\Gamma\left(\frac{1}{2}\right) = \frac{-8}{15}\sqrt{\pi}.$$

Next, the **BETA FUNCTION,** denoted by $\beta(m, n)$, is defined as

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \qquad ...(3)$$

Applying the property of definite integrals. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ in (3), we get

$$\beta(m, n) = \int_0^1 (1-x)^{m-1} x^{n-1}\, dx = \beta(n, m)$$

Thus, we have $\beta(m, n) = \beta(n, m)$. ...(4)

There is yet another representation of $\beta(m, n)$. Putting $x = \dfrac{t}{1+t}$ in (3), gives

$$\beta(m, n) = \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1}\left(1-\frac{t}{1+t}\right)^{n-1}\frac{dt}{(1+t)^2} = \int_0^\infty \frac{t^{m-1}\,dt}{(1+t)^{m+n}}.$$

We can also write $\beta(m, n) = \displaystyle\int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}}\,dt$ (on using (4)).[GGSIPU III Sem End Term 2011]

Further, in the relation (3) if we substitute $x = \sin^2\theta$, so $dx = 2\sin\theta\cos\theta\, d\theta$, we get

$$\beta(m, n) = \int_0^{\pi/2} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$\text{or} \quad \beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta \qquad ...(5)$$

RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

[GGSIPU III Sem End Term 2009; End Term 2004; End Term 2006; End Term 2004; End Term 2012; I Term 2013; End Term 2013]

PROOF: We know that $\Gamma(m) = 2\int_0^{\infty} e^{-x^2} x^{2m-1}\, dx$ and $\Gamma(n) = 2\int_0^{\infty} e^{-y^2} y^{2n-1}\, dy$

$$\therefore \quad \Gamma(m)\,\Gamma(n) = 4\int_0^{\infty} e^{-x^2} x^{2m-1}\, dx \int_0^{\infty} e^{-y^2} y^{2n-1} dy = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx dy$$

Now, converting into polar coordinates, we get

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r^{2m+2n-2} \cos^{2m-1}\theta \sin^{2n-1}\theta \;\; r\, dr d\theta$$

$$= 4\int_0^{\infty} e^{-r^2} r^{2m+2n-1} dr \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

$$= \Gamma(m+n)\;\beta(m, n) \quad \text{on using (2) and (5).}$$

Hence, $$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad \ldots(6)$$

In relation (5), writing $2m - 1 = p$ and $2n - 1 = q$ and using (6), we get

$$2\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

or $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} \qquad \ldots(7)$$

This relation is a big facility and is used very frequently in the evaluation of the definite integrals.

In (7) if we take $p = q = 0$, we get

$$\int_0^{\pi/2} \sin^0\theta \cos^0\theta\, d\theta = \frac{\Gamma(1/2)\,\Gamma(1/2)}{2\,\Gamma(1)} \quad \text{or} \quad \frac{\pi}{2} = \frac{\left(\Gamma(1/2)\right)^2}{2} \quad \text{since } \Gamma(1) = 1.$$

Therefore, we have yet another very important and useful result

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

[GGSIPU III Sem I Term 2007; End Term 2006; End Term 2011, 2012]

DUPLICATION FORMULA: $\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$ or $\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}}$

[GGSIPU III Sem I Term 2005; I Term 2007; End Term 2010]

In relation (6) putting $m=n$, we get

$$\beta(n,n)=\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)}=\frac{\Gamma(n)\Gamma(n)}{(2n-1)(2n-2)(2n-3)\ldots3.2.1}$$

$$\therefore\quad \beta(n,n)=\frac{\Gamma(n)\Gamma(n)}{2^{2n-1}\left(n-\frac{1}{2}\right)(n-1)\left(n-\frac{3}{2}\right)(n-2)\ldots2\cdot\frac{3}{2}\cdot1\cdot\frac{1}{2}}$$

$$=\frac{\Gamma(n)\Gamma(n)}{2^{2n-1}\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}(n-1)(n-2)\ldots2.1}$$

or
$$\frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)}=\frac{\Gamma(n)\,\Gamma(n)\,\Gamma(1/2)}{2^{2n-1}\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)\,\Gamma(n)}=\frac{\Gamma(n)\sqrt{\pi}}{2^{2n-1}\,\Gamma\left(n+\frac{1}{2}\right)}\quad\left(\text{as } \Gamma(1/2)=\sqrt{\pi}\right)$$

$$\Rightarrow\quad \Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$$ **which called is the duplication formula.**

Replacing n by $n+\frac{1}{2}$ on both sides, we can get $\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{n}}$

EXAMPLE 6.50. Evaluate (i) $\int_0^\infty x^{1/4}e^{-\sqrt{x}}\,dx$ (ii) $\int_0^\infty x^2e^{-a^2x^2}\,dx$.

SOLUTION: (i) Let $I=\int_0^\infty x^{1/4}e^{-\sqrt{x}}\,dx$. Substituting $x=t^2$, so $dx=2t\,dt$, gives

$$I=\int_0^\infty t^{\frac{1}{2}}e^{-t}\,2t\,dt=2\int_0^\infty e^{-t}\,t^{\frac{5}{2}-1}\,dt=2\,\Gamma\left(\frac{5}{2}\right)=2\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)=\frac{3\sqrt{\pi}}{2}.$$ **Ans.**

(ii) Let $I=\int_0^\infty x^2e^{-a^2x^2}\,dx$. Putting $x=\frac{\sqrt{t}}{a}$, so $dx=\frac{1}{2a\sqrt{t}}\,dt$, we have

$$I=\int_0^\infty\frac{t}{a^2}e^{-t}\frac{1}{2a\sqrt{t}}\,dt=\frac{1}{2a^3}\int_0^\infty e^{-t}\,t^{\frac{3}{2}-1}\,dt=\frac{1}{2a^3}\Gamma\left(\frac{3}{2}\right)=\frac{1}{2a^3}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{4a^3}.$$ **Ans.**

EXAMPLE 6.51. Evaluate (i) $\int_{-\infty}^\infty e^{-x^2}\,dx$ (ii) $\int_0^\infty\frac{x^a}{a^x}\,dx$. [GGSIPU III Sem I Term 2005]

SOLUTION: (i) Let $I=\int_{-\infty}^\infty e^{-x^2}\,dx=2\int_0^\infty e^{-x^2}\,dx$ (by property of definite integrals)

Putting $x=\sqrt{t}$, so $dx=\frac{1}{2\sqrt{t}}\,dt$, we get

$$I=2\int_0^\infty e^{-t}\frac{1}{2\sqrt{t}}\,dt=\int_0^\infty e^{-t}\,t^{\frac{1}{2}-1}\,dt=\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$ **Ans.**

It also implies that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

(ii) Let $I = \int_0^\infty \frac{x^a}{a^x}\,dx$. Substituting $a^x = e^t$ or $x\log a = t$ so that $dx = \frac{dt}{\log a}$, we get

$$I = \int_0^\infty e^{-t}\left(\frac{t}{\log a}\right)^a \frac{dt}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}\, t^{a+1-1}\,dt = \frac{\Gamma(a+1)}{(\log a)^{a+1}}.$$ **Ans.**

EXAMPLE 6.52. Evaluate $\int_0^\infty x^{n-1}\cos ax\,dx$ and $\int_0^\infty x^{n-1}\sin ax\,dx$.

SOLUTION: Let us consider the integral

$$I = \int_0^\infty x^{n-1}e^{-aix}\,dx = \int_0^\infty x^{n-1}(\cos ax - i\sin ax)\,dx \quad (i = \sqrt{-1})$$

Putting $aix = t$, $dx = \frac{dt}{ai}$, we get

$$I = \int_0^\infty e^{-t}\left(\frac{t}{ai}\right)^{n-1}\frac{dt}{ai} = \frac{1}{a^n i^n}\int_0^\infty e^{-t}\,t^{n-1}\,dt = \frac{\Gamma(n)}{a^n}(-i)^n$$

$$= \frac{\Gamma(n)}{a^n}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^n = \frac{\Gamma(n)}{a^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

Thus, $$\int_0^\infty x^{n-1}(\cos ax - i\sin ax)\,dx = \frac{\Gamma(n)}{a^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

Equating real and imaginary parts on both sides, we get

$$\int_0^\infty x^{n-1}\cos ax\,dx = \frac{\Gamma(n)}{a^n}\cos\frac{n\pi}{2} \quad \text{and} \quad \int_0^\infty x^{n-1}\sin ax\,dx = \frac{\Gamma(n)}{a^n}\sin\frac{n\pi}{2}.$$ **Ans.**

EXAMPLE 6.53. Evaluate

(i) $\int_0^2 x^{(m-1)}(2-x)^{n-1}\,dx$ *[GGSIPU III Sem I Term 2010]*

(ii) $\int_0^1 x^{(m-1)}\left(\log\frac{1}{x}\right)^{n-1}dx$, $m>0, n>0$. *[GGSIPU III Sem I Term 2010]*

SOLUTION: *(i)* Let $I = \int_0^2 x^{(m-1)}(2-x)^{n-1}\,dx$. Putting $x = 2\sin^2\theta$ in the given integral, we get

$$I = \int_0^{\pi/2} 2^{(m-1)}\sin^{2m-2}\theta\cdot 2^{n-1}\cos^{2n-2}\theta\cdot 2\sin\theta\cos\theta\,d\theta = 2^{m+n-1}\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

$$= 2^{m+n-1}\beta(m,n) = 2^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$ **Ans.**

(*ii*) Let $I=\int_0^1 x^{m-1}\left(\log\frac{1}{x}\right)^{n-1}dx$. Putting $\log\frac{1}{x}=t$ or $x=e^{-t}$, so $dx=-e^{-t}dt$, we have

$$I=\int_\infty^0 e^{-(m-1)t}\,t^{n-1}\left(-e^{-t}\right)dt=\int_0^\infty e^{-mt}\,t^{n-1}\,dt$$

Again putting $mt=y$, so $dt=\frac{1}{m}dy$, we get

$$I=\int_0^\infty e^{-y}\left(\frac{y}{m}\right)^{n-1}\frac{1}{m}dy=\frac{1}{m^n}\int_0^\infty e^{-y}\,y^{n-1}\,dy=\frac{\Gamma n}{m^n}.$$ **Ans.**

Example 6.54. (*a*) Show that $n\,\beta(m+1,n)=m\,\beta(m,n+1)$.

[*GGSIPU III Sem I Term 2005; End Term 2013*]

(*b*) $\frac{\beta(m+1,n)}{m}=\frac{\beta(m,n+1)}{n}=\frac{\beta(m,n)}{m+n+1}$ [*GGSIPU III Sem End Term 2010*]

Solution: (*a*) Using the property $\beta(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma m+n}$, we have

$$n\,\beta(m+1,n)=\frac{n\,\Gamma m+1\,\Gamma n}{\Gamma m+n+1}=\frac{n\cdot m\,\Gamma m\,\Gamma n}{\Gamma m+n+1}=\frac{m\,\Gamma m\,\Gamma n+1}{\Gamma m+n+1}$$

$=m\,\beta(m,n+1)$ by the above mentioned property. **Hence Proved.**

(*b*) $$\frac{\beta(m+1,n)}{m}=\frac{\Gamma m+1\,\Gamma n}{m\,\Gamma m+n+1}=\frac{\Gamma m\,\Gamma n}{(m+n+1)\,\Gamma m+n}=\frac{\beta(m,n)}{m+n+1}$$

Similarly $$\frac{\beta(m,n+1)}{n}=\frac{\Gamma m\,\Gamma n+1}{n\,\Gamma m+n+1}=\frac{\Gamma m\,\Gamma n}{(m+n+1)\,\Gamma m+n}=\frac{\beta(m,n)}{m+n+1}.$$ **Hence Proved.**

Example 6.55. (a) Evaluate $\int_0^\pi \sin^2\theta\,(1+\cos\theta)^4\,d\theta$.

(b) Show that $\int_0^{\pi/2}\frac{1}{\sqrt{\sin x}}dx\cdot\int_0^{\pi/2}\sqrt{\sin x}\,dx=\pi$. [*GGSIPU III Sem I Term 2011*]

Solution : (a) Let $I=\int_0^\pi \sin^2\theta\,(1+\cos\theta)^4\,d\theta$

$$=\int_0^\pi\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2\left(2\cos^2\frac{\theta}{2}\right)^4d\theta=64\int_0^\pi\sin^2\frac{\theta}{2}\cos^{10}\frac{\theta}{2}\,d\theta.$$

Putting $\frac{\theta}{2}=t$ so $d\theta=2dt$, we get

$$I=64\int_0^{\pi/2}\sin^2t\cos^{10}t.\,2dt=64.2\,\frac{\Gamma\frac{(2+1)}{2}\,\Gamma\frac{(10+1)}{2}}{2\,\Gamma\frac{(2+10+2)}{2}}=64\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{11}{2}}{\Gamma 7}$$

$$= \frac{64 \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \left(\Gamma\left(\frac{3}{2}\right) \right)^2}{6!} = \frac{4 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \left(\frac{1}{2} \Gamma\left(\frac{1}{2}\right) \right)^2}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = \frac{21\pi}{16}$$

Thus, $\displaystyle\int_0^{\pi} \sin^2\theta\,(1+\cos\theta)^4\,d\theta = \frac{21\pi}{16}.$ **Ans.**

(b) $\quad \text{LHS} = \displaystyle\int_0^{\pi/2} \sin^{-1/2} x\,dx \cdot \int_0^{\pi/2} \sin^{1/2} x\,dx$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+2}{2}\right)} \cdot \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{4\,\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)} = \frac{\pi\,\Gamma\left(\frac{1}{4}\right)}{4\,\Gamma\left(\frac{5}{4}\right)} = \frac{\pi\,\Gamma\left(\frac{1}{4}\right)}{4 \cdot \frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \pi.$$ **Hence Proved.**

[illegible] Express the integral $\displaystyle\int_0^1 x^m (1-x^n)^p\,dx$ in terms of Gamma function

[GGSIPU III Sem I Term 2012; I Sem End Term 2013]

and evaluate (*i*) $\displaystyle\int_0^1 x^2 (1-x^2)^4\,dx.$ (ii) $\displaystyle\int_0^1 x^{3/2}\left(1-\sqrt{x}\right)^{1/2} dx$

[GGSIPU III Sem End Term 2009; I Term 2011]

Solution: Let $\displaystyle I = \int_0^1 x^m (1-x^n)^p\,dx$, then putting $x^n = t$ so $nx^{n-1}\,dx = dt$, we get

$$I = \int_0^1 t^{\frac{m}{n}} (1-t)^p \frac{t^{1/n}\,dt}{nt} = \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1} (1-t)^{p+1-1}\,dt$$

$$= \frac{1}{n}\beta\left(\frac{m+1}{n},\, p+1\right) = \frac{1}{n}\,\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}. \qquad \text{...(1)}$$

(*i*) Now, taking $m = 2, n = 2$ and $p = 4$ in the above relation, we get

$$\int_0^1 x^2\left(1-x^2\right)^4 dx = \frac{1}{2}\,\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma(4+1)}{\Gamma\left(\frac{2+1}{2}+4+1\right)} = \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(5)}{\Gamma\left(\frac{13}{2}\right)} = \frac{\Gamma\left(\frac{3}{2}\right)\cdot 4!}{2 \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)} = \frac{128}{3465}$$ **Ans.**

(*ii*) Next taking $m = \frac{3}{2}$, $n = \frac{1}{2}$ and $p = \frac{1}{2}$ in (1), we get

$$\int_0^1 x^{3/2}\left(1-\sqrt{x}\right)^{1/2} dx = \frac{1}{1/2}\beta\left(5, \frac{3}{2}\right) = \frac{2\Gamma(5)\Gamma(3/2)}{\Gamma\left(5+\frac{3}{2}\right)}$$

$$= \frac{2\cdot(4\cdot 3\cdot 2\cdot 1)\Gamma(3/2)}{(11/2)(9/2)(7/2)(5/2)\,\frac{3}{2}\Gamma\left(\frac{3}{2}\right)} = \frac{512}{3465} \qquad \textbf{Ans.}$$

EXAMPLE 6.57. Show that $\int_a^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1}\,\beta(m+1, n+1)$.

[*GGSIPU I Sem II Term 2003*]

SOLUTION: Let $I = \int_a^b (x-a)^m (b-x)^n \, dx$ Putting $x-a=y$, so $dx=dy$, we get

$I = \int_0^{b-a} y^m (b-a-y)^n \, dy$ and now putting $y=(b-a)\,t$, so $dy=(b-a)\,dt$, we get

$$I = \int_0^1 (b-a)^m t^m (b-a)^n (1-t)^n (b-a)\,dt$$

$$= (b-a)^{m+n+1}\int_0^1 t^m (1-t)^n \, dt = (b-a)^{m+n+1}\,\beta(m+1, n+1). \qquad \textbf{Ans.}$$

EXAMPLE 6.58. (*a*) Show that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}$.

[*GGSIPU III Sem I Term 2005*]

(*b*) Show that $\int_0^{\pi/2} \sqrt{\tan\theta} + \sqrt{\sec\theta}\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[\Gamma\left(\frac{3}{4}\right) + \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right]$

[*GGSIPU III Sem I Term 2010*]

SOLUTION: (*a*) Using the property of definite integrals, we have

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\tan\left(\frac{\pi}{2}-\theta\right)}\, d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$$

Next, $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{-\frac{1}{2}}\theta \, d\theta = \dfrac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \dfrac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{4}\right)}{2}$, as $\Gamma(1) = 1$.

Now, let us recall the duplication formula $2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2n)$.

Taking $n = \frac{1}{4}$ here, we get

$$2^{\frac{-1}{2}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right) \quad \text{or} \quad \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2} \quad \text{as} \quad \Gamma\left(\frac{1}{2}\right) - \sqrt{\pi}.$$

$$\therefore \quad \int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \frac{1}{2}\pi\sqrt{2} = \frac{\pi}{\sqrt{2}} = \int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta.$$

Hence Proved.

(*b*) $$\int_0^{\pi/2}\sqrt{\tan\theta}+\sqrt{\sec\theta}\,d\theta = \int_0^{\pi/2}(\sin^{1/2}\theta\cos^{-1/2}\theta+\cos^{-1/2}\theta)\,d\theta$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[\Gamma\left(\frac{3}{4}\right)+\frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right]$$

since $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Hence Proved.

Example 6.59. Show that $\displaystyle\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \beta(m, n)$.

[*GGSIPU III Sem End Term 2008; I Sem II Term 2003*]

Solution: Let $\displaystyle I = \int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1\frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1\frac{x^{n-1}}{(1+x)^{m+n}}\,dx = I_1 + I_2$, say.

Putting $x = \tan^2\theta$, $dx = 2\tan\theta\sec^2\theta\,d\theta$, in the above integrals, I_1 and I_2, we get

$$I_1 = \int_0^1\frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^{\pi/4}\frac{\tan^{2m-2}\theta\cdot 2\tan\theta\sec^2\theta\,d\theta}{(1+\tan^2\theta)^{m+n}}$$

$$= 2\int_0^{\pi/4}\frac{\tan^{2m-1}\theta\sec^2\theta\,d\theta}{\sec^{2m+2n}\theta} = 2\int_0^{\pi/4}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

Similarly, $\displaystyle I_2 = \int_0^1\frac{x^{n-1}\,dx}{(1+x)^{m+n}} = 2\int_0^{\pi/4}\sin^{2n-1}\theta\,\cos^{2m-1}\theta\,d\theta$

In this integral, substituting $\theta = \frac{\pi}{2} - \phi$, so $d\theta = -\,d\phi$, we get

$$I_2 = -2\int_{\pi/2}^{\pi/4}\sin^{2n-1}\left(\frac{\pi}{2}-\phi\right)\cos^{2m-1}\left(\frac{\pi}{2}-\phi\right)d\phi = 2\int_{\pi/4}^{\pi/2}\cos^{2n-1}\phi\,\sin^{2m-1}\phi\,d\phi$$

$$= 2\int_{\pi/4}^{\pi/2}\cos^{2n-1}\theta\,\sin^{2m-1}\theta\,d\theta \quad \text{by property of definite integrals.}$$

$$\therefore \quad I = 2\int_0^{\pi/4}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta + 2\int_{\pi/4}^{\pi/2}\cos^{2n-1}\theta\,\sin^{2m-1}\,d\theta$$

$$= 2\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m, n).$$

Hence Proved.

EXAMPLE 6.60. (*i*) Using Gamma function, evaluate

$$\int_0^2 (4-x^2)^{3/2}\,dx$$ [GGSIPU I Sem II Term 2013]

(*ii*) Evaluate $\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}\,dx$ [GGSIPU III Sem End Term 2006]

SOLUTION: (*i*) Let $I=\int_0^2 (4-x^2)^{3/2}\,dx$. Putting $x = 2\sin\theta$ here, we get

$$I=\int_0^{\pi/2} 4^{3/2}\cos^3\theta\cdot 2\cos\theta\,d\theta = 16\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$=16\cdot\frac{\Gamma(5/2)\,\Gamma(1/2)}{2\,\Gamma(3)} = 4\cdot\frac{3}{2}\cdot\frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 3\pi \quad \textbf{Ans.}$$

(*ii*) Let $I=\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}\,dx$. Putting $1 + x = y$, gives

$$I=\int_0^2 y^{p-1}(2-y)^{q-1}dy.$$ Again putting $y = 2t$, we get

$$I=\int_0^1 2^{p-1}t^{p-1}\,2^{q-1}(1-t)^{q-1}2dt = 2^{p+q-1}\int_0^1 t^{p-1}(1-t)^{q-1}dt$$

$$=2^{p+q-1}\beta(p,q) = 2^{p+q-1}\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}.$$ **Ans.**

EXAMPLE 6.61. (*i*) Prove that $\int_0^{\frac{\pi}{2}}\sin^3 x\cos^{5/2}x\,dx = \frac{8}{77}$. [GGSIPU III Sem End Term 2000]

(*ii*) Show that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ $(0 < n < 1)$. [GGSIPU III Sem End Term 2006]

SOLUTION: (*i*) Using the result $\int_0^{\pi/2}\sin^p x\cos^q x\,dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$ we have

$$\int_0^{\pi/2}\sin^3 x\cos^{\frac{5}{2}}x\,dx = \frac{\Gamma(2)\,\Gamma\left(\frac{7}{4}\right)}{2\,\Gamma\left(\frac{15}{4}\right)} = \frac{1\cdot\Gamma\left(\frac{7}{4}\right)}{2\times\frac{11}{4}\cdot\frac{7}{4}\Gamma\left(\frac{7}{4}\right)} = \frac{8}{77}.$$ **Hence Proved.**

(*ii*) We know that $\beta(m,n) = \int_0^{\infty}\frac{x^{n-1}dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$. Writing $m = 1 - n$, gives

$$\int_0^{\infty}\frac{x^{n-1}dx}{1+x} = \frac{\Gamma(n)\,\Gamma(1-n)}{1}$$

In complex integration chapter we shall establish that

$$\int_{-\infty}^{\infty} \frac{e^{ax}\,dx}{e^x+1} = \frac{\pi}{\sin a\pi},\ 0 < a < 1$$

Now substituting $e^x = t$ here, we get $\displaystyle\int_0^{\infty} \frac{t^{a-1}dt}{1+t} = \frac{\pi}{\sin a\pi}$

$$\Rightarrow \qquad \int_0^{\infty} \frac{x^{n-1}dx}{x+1} = \frac{\pi}{\sin n\pi}, \quad 0 < n < 1$$

Therefore, $\quad \Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}, \quad 0 < n < 1.$ **Hence Proved.**

EXAMPLE 6.62 Show that $\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = \left(\frac{1}{4}-x^2\right)\pi \sec \pi x$ provided $-1 < 2x < 1$.

[*GGSIPU I Sem End Term 2003*]

SOLUTION: LHS $= \Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = \left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}-x\right)\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}+x\right) = \left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right)$

Now we shall apply the well known result $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$. Replacing n by $\left(\frac{1}{2}+x\right)$ here, we get

$$\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(1-\left(\frac{1}{2}+x\right)\right) = \frac{\pi}{\sin \pi\left(\frac{1}{2}+x\right)} = \frac{\pi}{\sin\left(\frac{\pi}{2}+\pi x\right)}$$

or $$\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \frac{\pi}{\cos \pi x} = \pi \sec \pi x.$$

Therefore, $\quad$ L.H.S. $= \left(\frac{1}{4}-x^2\right)\pi \sec \pi x =$ R.H.S. **Hence Proved.**

EXAMPLE 6.63 Prove that $\beta\left(m, \frac{1}{2}\right) = 2^{2m-1}\,\beta(m, m)$. [*GGSIPU III Sem I Term 2005*]

SOLUTION: By definition, $\quad \beta(m, 1/2) = \dfrac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}.$

Using the duplication formula $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}$ here, we get

$$\beta\left(m, \frac{1}{2}\right) = \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)\cdot 2^{2m-1}}{\sqrt{\pi}\,\Gamma(2m)}\Gamma(m) = \frac{\Gamma(m)\,\Gamma(m)\cdot 2^{2m-1}}{\Gamma(2m)} \quad \left(\text{since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right).$$

$$= 2^{2m-1}\,\beta(m, m) = \text{RHS.}$$ **Hence Proved.**

[redacted] (a) Show that $\int_0^{\pi/2} \tan^n x\,dx = \frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right),\ 0 < n < 1.$

(b) Evaluate $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2} dx$ using Beta-Gamma function.

[GGSIPU II Sem End Term 2015]

SOLUTION: (a) $\int_0^{\pi/2} \tan^n x\,dx = \int_0^{\pi/2} \sin^n x \cos^{-n} x\,dx$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma\left(\frac{n-n+2}{2}\right)} = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\left(\frac{n+1}{2}\right)\right)$$

Using the result $\Gamma m\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}$ where $0 < m < 1$, we can write

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\left(\frac{n+1}{2}\right)\right) = \frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi} \quad \text{since} \quad \frac{1}{2} < \frac{n+1}{2} < 1.$$

$$\therefore \quad \int_0^{\pi/2} \tan^n x\,dx = \frac{\pi}{2\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right)} = \frac{\pi}{2\cos\left(\frac{n\pi}{2}\right)} = \frac{\pi}{2}\sec\frac{n\pi}{2}$$

Hence Proved

(b) Let $\quad I = \int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2} dx$ Put here $x^3 = \sin^2\theta \quad \therefore \quad x = \sin^{2/3}\theta$, then

$$I = \int_0^{\pi/2} \frac{\sin\theta \cdot \frac{2}{3}\sin^{-1/3}\theta\cos\theta\,d\theta}{\cos\theta} = \frac{2}{3}\int_0^{\pi/2} \sin^{2/3}\theta\,d\theta$$

$$= \frac{\Gamma\left(\frac{2/3+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{\left(\frac{2}{3}+2\right)}{2}\right)} = \frac{\Gamma(5/6)\,\Gamma(1/2)}{2\Gamma(4/3)}.$$

Ans.

EXERCISE 6E

1. Evaluate (i) $\int_0^\infty x e^{-ax} \cos bx \, dx$. (ii) $\int_0^\infty x e^{-ax} \sin bx \, dx$.

2. Evaluate $\int_0^\infty \frac{x^4 \, dx}{4^x}$.

3. Show that $\int_0^1 (x \log x)^3 \, dx = \frac{-3}{128}$.

4. Evaluate $\int_0^1 x^m (\log x)^n \, dx$.

5. Show that $\beta(m, n+1) + \beta(m+1, n) = \beta(m, n)$. [GGSIPU III Sem End Term 2009]

6. Evaluate $\int_0^\infty \frac{x^4 (1+x^5) \, dx}{(1+x)^{15}}$.

7. Show that $\int_0^1 \frac{x^{m-1} (1-x)^{n-1} \, dx}{(a+x)^{m+n}} = \frac{\beta(m,n)}{a^n (a+1)^m}$.

8. Evaluate $\int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(1+x)^{m+n}} \, dx$.

9. Show that $\int_1^\infty \frac{dx}{x^{m+1} (x-1)^n} = \beta(m+n, 1-n), \; -m < n < 1$.

10. If $\Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$, show that $\int_0^\infty \frac{x^{n-1} \, dx}{(1+x)} = \frac{\pi}{\sin n\pi}$.

11. Show that $\beta(m+1, n) = \frac{m}{m+n} \beta(m,n)$.

12. Prove that $\beta(n, n) = 2 \int_0^{1/2} t^{n-1} (1-t)^{n-1} \, dt$.

13. If $I_n = \frac{\sqrt{\pi} \, \Gamma(n+1)}{2 \, \Gamma(n/2+1)}$, show that $I_{n+2} = (n+1) I_n$.

14. Evaluate $\int_0^1 (1-x^{1/n})^{m-1} \, dx$

15. Prove the following

$$\int_0^\infty x^{n-1} e^{-ax} \cos bx \, dx = \frac{\Gamma n}{(a^2+b^2)^{n/2}} \cos\left(n \tan^{-1} \frac{b}{a}\right)$$

[GGSIPU III Sem End Term 2003]

Matrices, Determinants, Hermitian and Skew-Hermitian Matrices, Inverse and Rank of Matrix

Type of Matrix, Inverse of Matrix of Gauss-Jordan Method, Orthogonality, Rank of a Matrix, Linear Dependence and Independence of Vectors, Hermitian and Skew-Hermitian Matrix.

MATRICES

Matrices have variety of applications, for example, in the solution of linear algebraic system of simultaneous equations, in the solution of ordinary and partial differential equations, etc.

DEFINITIONS

A matrix is a rectangular arrangement of $(m \times n)$ elements real or complex in m rows and n columns. Such a matrix A is usually denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}] \quad \text{where} \quad i = 1, 2, .., m; \quad j = 1, 2, ..., n.$$

and is said to be of *order* $m \times n$. Matrices as a whole are denoted by capital letters while corresponding small letters with suffices represent its **elements**. The elements $a_{11}, a_{12}, a_{13} \cdots a_{mn}$ may be real or complex. Here we have a **double suffix notation** in which the first suffix indicates the row and the second suffix indicates column in which the element is located. Thus, a_{ij} is an element of i^{th} row and j^{th} column in matrix A.

The matrices are usually denoted by bold face upper case letters A, B, C, ... etc. If all the elements of a matrix are real it is called a **real matrix**, whereas if one or more elements of the matrix are complex, it is called a **complex matrix**.

Unlike determinants, a matrix does not have any value. If $m = n$ the matrix is said to be a **square matrix** of order n.

ROW AND COLUMN MATRICES

A matrix may also have only one row or only one column. Thus, $[a_1 \; a_2 \; a_3 \dots a_n]$ is known as **row matrix or row vector of order n**, whereas $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is known as **column matrix or column vector** of order n.

DIAGONAL MATRIX

In a square matrix A the elements $a_{11}, a_{22}, \dots, a_{nn}$ form the **leading diagonal** (or **main diagonal** or **principal diagonal**). The sum of the diagonal elements, that is, $\sum_{i=1}^{n} a_{ii}$ is called the **trace** of the square matrix A.

If all the elements of a matrix are zero it is called **zero matrix or null matrix**. Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be equal when they are of the same order and their corresponding elements are equal, that is, $a_{ij} = b_{ij}$. Further, a square matrix in which all the elements except those in the leading diagonal are zero is called a **diagonal matrix**. In other words, if $a_{ij} = 0$ for all $i \neq j$ then A is a diagonal matrix. For example,

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ is a diagonal matrix.}$$

SCALAR MATRIX AND UNIT MATRIX

In particular, a diagonal matrix in which all the leading diagonal elements are equal, is called a **scalar matrix**. If, in a diagonal matrix all the elements in the leading diagonal are 1, *i.e.*, if

$$\begin{aligned} a_{ij} &= 0 \quad \text{for} \quad i \neq j \\ &= 1 \quad \text{for} \quad i = j \end{aligned}$$

it is called a **unit matrix or Identity matrix of order n** and is generally denoted by I_n.

For example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is a unit matrix of order 3.}$$

TRIANGULAR MATRICES

A square matrix is called an **upper triangular matrix** if all its elements below the leading diagonal are zero, *i.e.*, if $a_{ij} = 0$ for $i > j$. Similarly, a square matrix is called a **lower triangular matrix** if all its elements above the leading diagonal are zero, *i.e.* if $a_{ij} = 0$ for $i < j$.

Thus, the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ is an upper triangular matrix

and $\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -3 & 4 & 0 \end{bmatrix}$ is a lower triangular matrix.

MATRIX OPERATIONS

1. Addition and Subtraction.

Two matrices of the same order can be added (or subtracted) by adding (or subtracting) their corresponding elements.

Thus, $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 5 & 1 \\ 3 & 2 & -1 \\ 4 & 1 & 2 \\ -3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 & -1 \\ 7 & 2 & 1 \\ 4 & 3 & 5 \\ -2 & 1 & 6 \end{bmatrix}$ and $\begin{bmatrix} 7 & 5 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 7 & -2 \end{bmatrix}$.

2. Multiplication of a Matrix by a Scaler (or Scalar Multiplication).

When a matrix is multiplied by a scalar, say k, then its every element gets multiplied by k.

Thus, $3\begin{bmatrix} 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & 6 \\ 0 & 5 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -3 & 3 \\ 12 & -3 & 9 & 18 \\ 0 & 15 & 6 & 12 \end{bmatrix}$.

Further, if A, B, C are any three matrices of same order then the following properties hold good.

(*i*) A + (B + C) = (A + B) + C (Associativity of addition)

(*ii*) A + B = B + A (Commutativity of addition)

(*iii*) k (A + B) = k A + kB, k being scalar (distributivity of addition with respect to a scalar)

3. Matrix Multiplication.

Two matrices A and B can be multiplied as A . B only when the number of columns in A is equal to the number of rows in B. This is known as **compatibility condition** *for the product AB* to exist. Thus, the product AB of matrices A($l \times m$) and B($p \times n$) can exist only when $p = m$. Such matrices are also called **conformable**, and the product matrix C(= AB) would be of order $l \times n$.

We write $C = (c_{ij})$ where $c_{ij} = a_{i1}\,b_{1j} + a_{i2}\,b_{2j} + \ldots + a_{im}\,b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj}$

For example,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \\ r_1 & r_2 \end{bmatrix} = \begin{bmatrix} a_1p_1 + b_1q_1 + c_1r_1 & a_1p_2 + b_1q_2 + c_1r_2 \\ a_2p_1 + b_2q_1 + c_2r_1 & a_2p_2 + b_2q_2 + c_2r_2 \\ a_3p_1 + b_3q_1 + c_3r_1 & a_3p_2 + b_3q_2 + c_3r_2 \\ a_4p_1 + b_4q_1 + c_4r_1 & a_4p_2 + b_4q_2 + c_4r_2 \end{bmatrix}$$

In the product AB, the matrix A is said to be *post - multiplied* by the matrix B and the matrix B is said to be *premultiplied* by A. In general, AB ≠ BA; in other words, *the matrices* AB *and* BA *are not necessarily equal even if they are conformable.*

If AB = 0 then it does not always imply that either A = 0 or B = 0. For example, let

$A = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

and $BA = \begin{bmatrix} 0 & 0 \\ ax+by & 0 \end{bmatrix} \neq AB$. Thus, matrix multiplication is **not necessarily commutative.**

Further if $AB = AC$, it does not always imply that $B = C$, that is, **cancellation law does not** apply here.

Define $A^k = A \times A \times ... \times A$ (k times). Then, a matrix A, such that $A^k = 0$ for some positive integer k, is said to be **nilpotent**. The smallest value of k for which $A^k = 0$, is called the **index of nilpotency** of the matrix A.

Also, if $A^2 = A$, then A is called an **idempotent matrix**.

SOME MORE PROPERTIES ON MATRIX MULTIPLICATION:

1. If A, B, C are matrices of orders $m \times n, n \times p, p \times q$ respectively, then $(AB)C = A(BC)$ is a matrix of order $m \times q$. (associative property)
2. If A is a matrix of order $m \times n$ and B, C are matrices of order $n \times p$, then $A(B + C) = AB + AC$ (distributive property).
3. If A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$, then
$$\alpha(AB) = A(\alpha B) = (\alpha A) B \text{ for any scalar } \alpha.$$

4. Transpose of a Matrix

A matrix obtained by interchanging the rows and columns of a matrix A is defined as the *transpose* of A and is denoted by by A' or by A^T. Thus, if A is of order $m \times n$, its transpose A' will be of order $n \times m$. For example,

$$\text{if} \quad A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 5 & 3 \end{bmatrix} \quad \text{then} \quad A' = \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 1 & 3 \end{bmatrix}.$$

SOME PROPERTIES ON TRANSPOSE OF MATRICES:

1. The transpose of a row matrix is a column matrix and the transpose of a column matrix is a row matrix.
2. $(A^T)^T = A$ or we could write $(A')' = A$.
3. $(A + B)^T = A^T + B^T$ provided A and B are compatible for addition.
4. $(AB)^T = B^T A^T$ if A and B are compatible for multiplication.

SYMMETRIC AND SKEW-SYMMETRIC MATRICES

A real matrix $A = (a_{ij})$ is said to be **symmetric** if $a_{ij} = a_{ji}$ for all i and j, that is, if $A = A^T$ (or A').

And A is said to be **skew-symmetric** if $a_{ij} = -a_{ji}$ for all i, j, that is, if $A = -A^T$ (or $-A'$).

For example, the matrix $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is symmetric while the matrix $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -5 \\ 3 & 5 & 0 \end{bmatrix}$ is skew-symmetric.

SOME IMPORTANT DEDUCTIONS:

1. In a skew-symmetric matrix $A = (a_{ij})$, all its diagonal elements are zero, that is, $a_{ii} = 0$ for all i.
2. A matrix which is both symmetric and skew-symmetric must be a null matrix.
3. For any real matrix A, the matrix $A + A^T$ is always symmetric whereas the matrix $A - A^T$ is always skew-symmetric.

   Therefore, a real matrix A can be written as the sum of a symmetric matrix and a skew-symmetric matrix, that is, $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

EXAMPLE 7.1. (a) For any two square matrices A and B conformable for multiplication, are the following assertions true ? If not, correct them.

(*i*) $(A+B)^2 = A^2+2AB+B^2$

(*ii*) $(A+B)(A-B) = A^2-B^2$.

(b) Show by means of an example that the matrix $AB = 0$ does not necessarily mean that either $A = 0$ or $B = 0$ where 0 stands for the null matrix.

[*GGSIPU I Sem End Term 2011*]

SOLUTION: **(a)** (*i*) **Not true**, because $BA \neq AB$, in general.

The correct form is $(A+B)^2 = (A+B)(A+B) = A^2+AB+BA+B^2$.

(*ii*) **Not true,** because $BA \neq AB$. The form should be

$$(A+B)(A-B) = A(A-B)+B(A-B) = A^2-AB+BA-B^2.$$

(b) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $A \neq 0$ and $B \neq 0$. **Ans.**

EXAMPLE 7.2. Express the matrix $\begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & 7 \\ -4 & 5 & 3 \end{bmatrix}$ as the sum of two matrices, one symmetric and the other skew-symmetric.

SOLUTION: Let $A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & 7 \\ -4 & 5 & 3 \end{bmatrix}$ then $A' = \begin{bmatrix} 3 & 2 & -4 \\ 1 & 1 & 5 \\ -2 & 7 & 3 \end{bmatrix}$

Now, $$\frac{1}{2}(A+A') = \begin{bmatrix} 3 & 3/2 & -3 \\ 3/2 & 1 & 6 \\ -3 & 6 & 3 \end{bmatrix} = C, \text{ say.}$$

and $$\frac{1}{2}(A-A') = \begin{bmatrix} 0 & -1/2 & 1 \\ 1/2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} = D, \text{ say.}$$

Then $$C+D = \begin{bmatrix} 3 & 3/2 & -3 \\ 3/2 & 1 & 6 \\ -3 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & 1 \\ 1/2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & 7 \\ -4 & 5 & 3 \end{bmatrix} = A$$

$\therefore$ $A = C+D$ where C is symmetric and D is skew-symmetric. **Ans.**

DETERMINANTS

With every square matrix of order n, we associate a determinant of order n, denoted by **det (A)**, given by

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

The determinant has a value which is real if the matrix is real and is complex if the matrix is complex.

Now let us find ways to find the value of the determinant. A determinant of order 2 has two rows and two columns. Its value is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

MINORS AND COFACTORS

Let a_{ij} be the general element of a determinant $|A|$. If we suppress the i^{th} row and j^{th} column from the determinant, we get a new determinant of order $(n-1)$, which is called **minor** of the element a_{ij}. Let us denote this minor by M_{ij}.

The **cofactor** of the element a_{ij}, denoted by A_{ij}, is given by

$$A_{ij} = (-1)^{i+j}\, M_{i,j}$$

We can expand a determinant of order n through the elements of any row or column of the determinant and is the sum of the products of the elements of the i^{th} row (or j^{th} column) and their corresponding cofactors. Thus, when we expand $|A|$ by the elements of the i^{th} row, we have

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\, M_{ij} = \sum_{j=1}^{n} a_{ij}\, A_{ij}\,.$$

However, if we expand det (A) through the elements of j^{th} column, we get

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot M_{ij} = \sum_{i=1}^{n} a_{ij}\, A_{ij}$$

This is known as **Laplace expansion** of the determinant.

We prefer to expand a determinant through that row or column which has a number of zeros.

PROPERTIES OF DETERMINANTS

1. If the corresponding rows and columns are interchanged the value of the determinant remains unchanged, that is $|A| = |A^T|$.
2. If any two rows (or columns) are interchanged, the value of the determinant is multiplied by (-1).

   As a consequence of this property we can say that if the corresponding elements of two rows (or columns) are same, in other words two rows (or columns) are identical, the value of the determinant is zero.

3. If all the elements of any row (or column) are zero, then the value of the determinants is zero.
4. If each element of any row (or column) is multiplied by a scalar α then the value of the determinant becomes α times its previous value. Note here that when a matrix is multiplied by a scalar α then every element of the matrix gets multiplied by α. Therefore

$$|\alpha A| = \alpha^n |A| \text{ where A is a matrix of order } n.$$

5. If α times the elements of some row (or column) is added to the corresponding elements of some other row (or column) then the value of the determinant remains unchanged.
6. The value of a determinant of a diagonal matrix, or a lower triangular matrix or an upper triangular matrix is the product of its diagonal elements.

Further, it is standard practice to denote i^{th} row as R_i and j^{th} column as C_j. Also, when α times elements of i^{th} row are added to the elements of j^{th} row, it is denoted by $R_j \to R_j + \alpha R_i$. This is called elementary row operation. Similarly, the operation $C_j \to C_j + \alpha C_i$ is called elementary column operation.

7. The sum of the products of any row (or column) with their corresponding cofactors gives the value of the determinant, whereas, the sum of the products of the elements of any row (or column) with the corresponding cofactors of any other row (or column) is zero. Thus,

$$a_{i1} A_{j1} + a_{i2} A_{j2} + \dots + a_{in} A_{jn} = \begin{cases} |A| & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases}$$

8. If A and B are two square matrices of the same order then $|AB| = |A|\,|B|$.

EXAMPLE 7.3. Find the value of the determinant

For finding the determinant by Row operation we've to make two zeroes in 1st coloumn.

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 2 & 4 \\ -1 & 3 & 2 \end{vmatrix}$$

(i) using row operations (ii) using column operations

SOLUTION: (i) Applying the operations $R_1 \to R_1 + 2R_3$ and $R_2 \to R_2 + 3R_3$, we get

$$|A| = \begin{vmatrix} 0 & 7 & 3 \\ 0 & 11 & 10 \\ -1 & 3 & 2 \end{vmatrix} \quad \text{(expanding it by } C_1\text{)}$$

and by coloumn operation, we've to make two zeroes in 1st Row.

$$= 0 + 0 - 1\,(70 - 33) = -37.$$

(ii) Applying the operations $C_1 \to C_1 - 2C_2$ and $C_3 \to C_3 + C_2$, we get

$$|A| = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 2 & 6 \\ -7 & 3 & 5 \end{vmatrix} \quad \text{(expanding it by } R_1\text{)}$$

$$= 0 - 1\,(-5 + 42) + 0 = -37. \quad \textbf{Ans.}$$

EXAMPLE 7.4. Show that

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)\,(b-c)\,(c-a).$$

SOLUTION: Applying the operations $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ on $|A|$, we get

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} = (b-a)\,(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$$= (b-a)\,(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} = (b-a)\,(c-a)\,(c+a-b-a)$$

$$= (a-b)\,(b-c)\,(c-a). \quad \textbf{Ans.}$$

EXAMPLE 7.5. If $A = \begin{bmatrix} -6 & 3 & 7 \\ 12 & -5 & -9 \\ 2 & 4 & -6 \end{bmatrix}$ evaluate $|A|$.

SOLUTION: Expanding $|A|$ by column 1, we get

$$|A| = \sum_{i=1}^{3} (-1)^{i+1} a_{i1} M_{i1} = -6 \begin{vmatrix} -5 & -9 \\ 4 & -6 \end{vmatrix} - 12 \begin{vmatrix} 3 & 7 \\ 4 & -6 \end{vmatrix} + 2 \begin{vmatrix} 3 & 7 \\ -5 & -9 \end{vmatrix}$$

$$= -6\,(30+36) - 12\,(-18-28) + 2\,(-27+35) = 172$$

Expanding $|A|$ by column 2, we get

$$|A| = \sum_{i=1}^{3} (-1)^{i+2} a_{i2} M_{i2} = -3 \begin{vmatrix} 12 & -9 \\ 2 & -6 \end{vmatrix} + 5 \begin{vmatrix} -6 & 7 \\ 2 & -6 \end{vmatrix} - 4 \begin{vmatrix} -6 & 7 \\ 12 & -9 \end{vmatrix}$$

$$= -3\,(-72+18) + 5(36-14) - 4\,(54-84) = 172. \quad \textbf{Ans.}$$

EXAMPLE 7.6. Given the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -5 & 2 \\ 1 & 7 & 1 \end{bmatrix}$$

find the value of $|AB|$ and verify that $|AB| = |A|\,|B|$.

SOLUTION: Here, $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 1 & -5 & 2 \\ 1 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 13 & 8 \\ 23 & 25 & 20 \\ 22 & 117 & 0 \end{bmatrix}$

Expanding $|AB|$ by R_3, we get $|AB| = 22(260-200) + 117\,(184-160) = 4128.$ **Ans.**

Next, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = 1(45+48) + 2(42-36) + 3\,(-32-35) = -96$

and $|B| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -5 & 2 \\ 1 & 7 & 1 \end{vmatrix} = 3(-19) + 2(1) + 1(12) = -43$

$\therefore$ $|A|\,|B| = (-96)\,(-43) = 4128$

Hence verified

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EXAMPLE 7.7. Without evaluating the determinant, show that

$$D = \begin{vmatrix} \cos(x-\alpha) & \cos(x-\beta) & \cos(x-\gamma) \\ \cos(y-\alpha) & \cos(y-\beta) & \cos(y-\gamma) \\ \cos(z-\alpha) & \cos(z-\beta) & \cos(z-\gamma) \end{vmatrix} = 0$$

SOLUTION: Expanding all the terms, we get

$$D = \begin{vmatrix} \cos x\cos\alpha+\sin x\sin\alpha & \cos x\cos\beta+\sin x\sin\beta & \cos x\cos\gamma+\sin x\sin\gamma \\ \cos y\cos\alpha+\sin y\sin\alpha & \cos y\cos\beta+\sin y\sin\beta & \cos y\cos\gamma+\sin y\sin\gamma \\ \cos z\cos\alpha+\sin z\sin\alpha & \cos z\cos\beta+\sin z\sin\beta & \cos z\cos\gamma+\sin z\sin\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & \sin x & 0 \\ \cos y & \sin y & 0 \\ \cos z & \sin z & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \sin\alpha & \sin\beta & \sin\gamma \\ 0 & 0 & 0 \end{vmatrix}$$

$= 0 \times 0 = 0.$ **Hence Proved.**

SINGULAR MATRIX

Let A b a square matrix of order n, then A is called a **singular matrix** if $|A| = 0$ and **non-singular matrix** if $|A| \neq 0$.

ADJOINT OF A MATRIX

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order $n \times n$, then the corresponding determinant is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

Let $A_{11}, A_{12}, A_{13}, \ldots$ be the co-factors of $a_{11}, a_{12}, a_{13}, \ldots,$ in $|A|$, then the adjoint of A, denoted by **adj (A)** is the transpose of the cofactor matrix and is given by

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & \cdots & A_{nn} \end{bmatrix}$$

For example, if $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

then $A_{11} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3,$ $A_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = -1,$ $A_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5,$

$$A_{21} = -\begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = 1, \qquad A_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, \qquad A_{23} = -\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 1,$$

$$A_{31} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7, \qquad A_{32} = -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 5, \qquad A_{33} = \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -13.$$

$$\therefore \quad \text{adj}(A) = \begin{vmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{vmatrix}^T = \begin{vmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{vmatrix}.$$

INVERSE OF SQUARE A MATRIX

Let A be a non-singular matrix of order $n \times n$ and B be another matrix of the same order such that $BA = I = AB$ where I is the unit matrix of order n, then B is said to be the inverse of A and is denoted by A^{-1} such that $AA^{-1} = I = A^{-1}A$.

Here A^{-1} is unique and $A^{-1} = \dfrac{adj\,(A)}{|A|}$.

To establish this relation,

$$A.\,\text{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & \cdots & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} = |A|\, I_n$$

Since $|A| \neq 0$, dividing both sides by $|A|$, we get

$$A\frac{adj(A)}{|A|} = I \quad \Rightarrow \quad A^{-1} = \frac{adj(A)}{|A|}.$$

Another important result $(AB)^{-1} = B^{-1}A^{-1}$ can be shown to be true as follows :

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Similarly $\quad (B^{-1}A^{-1})(AB) = (B)^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

Therefore, we have $\quad (AB)^{-1} = B^{-1}A^{-1}$.

ORTHOGONAL MATRIX

An **orthogonal matrix** is one whose transpose is its inverse. Thus, a square matrix A for which $AA' = A'A = I$, is called orthogonal.

For example, let
$$A = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

then $A^{-1} = \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} = A'$ hence A is orthogonal.

Also $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ can be easily verified to be an orthogonal matrix.

ADDITIONAL PROPERTIES

(*i*) $(A')^{-1} = (A^{-1})'$

We know that $(AA^{-1}) = I$. Taking transpose on both sides, we get $(AA^{-1})' = I' = I$

$\Rightarrow \quad (A^{-1})' A' = I$. Hence, $(A')^{-1} = (A^{-1})'$.

(*ii*) $(A^{-1})^{-1} = A$

(*iii*) If A is a matrix of order $n \times n$ then $|adj\,(A)| = |A|^{n-1}$.

Earlier we have already established that

$$A\; adj\,(A) = |A|\, I = \begin{bmatrix} |A| & 0 & 0 & \cdots & \cdots & 0 \\ 0 & |A| & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & |A| \end{bmatrix}$$

Using the property $|AB| = |A|\,|B|$ here, we get

$$|A.\, adj\,(A)| = |A|\,|adj\,(A)| = \begin{vmatrix} |A| & 0 & 0 & \cdots & \cdots & 0 \\ 0 & |A| & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & |A| \end{vmatrix} = |A|^n$$

$$\Rightarrow \quad |adj\,(A)| = |A|^{n-1}.$$

(*iv*) If $D = \text{diag}\,(d_{11}, d_{22}, ..., d_{nn}),\ d_{ii} \neq 0$, then

$D^{-1} = \text{diag}\,(1/d_{11}, 1/d_{22}, ..., 1/d_{nn})$.

EXAMPLE 7.8. Find the adjoint and the inverse of the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}.$$

SOLUTION: The co-factors of the elements of $A\ (= a_{ij})$ are

$$A_{11} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3, \qquad A_{12} = -\begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = 2, \qquad A_{13} = \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} = -2,$$

$$A_{21} = -\begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix} = 10, \qquad A_{22} = \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = -8, \qquad A_{23} = -\begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} = 4,$$

$$A_{31} = \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = 1, \qquad A_{32} = -\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = -2, \qquad A_{33} = \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = 2$$

$$\therefore \quad adj\,A = \begin{bmatrix} -3 & 2 & -2 \\ 10 & -8 & 4 \\ 1 & -2 & 2 \end{bmatrix}' = \begin{bmatrix} -3 & 10 & 1 \\ 2 & -8 & -2 \\ -2 & 4 & 2 \end{bmatrix}. \qquad \textbf{Ans.}$$

Also $\quad |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 2(-3) + 4(2) + 3(-2) = -4$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\,adj\,A = -\frac{1}{4}\begin{bmatrix} -3 & 10 & 1 \\ 2 & -8 & -2 \\ -2 & 4 & 2 \end{bmatrix}. \qquad \textbf{Ans.}$$

EXAMPLE 7.9. Show that the matrix $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is orthogonal.

SOLUTION: Let $\quad A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ then $\quad A' = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

Now $\quad AA' = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Therefore A is an orthogonal matrix.

Hence Proved.

Solving the $n \times n$ Linear System of Simultaneous Equations

Consider the following n equations with n unknowns $x_1, x_2 \dots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots &= \dots \\ \dots\dots\dots\dots\dots\dots\dots &= \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

We can write the above equations in matrix equation form as $AX = B$ ...(1)

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Here A is called the coefficient matrix, B on the right hand side the column vector and X the solution vector.

If $B \neq 0$, that is, atleast one of the elements $b_1, b_2, ..., b_4$ is not zero, then the system of equations is called **non-homogeneous**. But if $B = 0$ the system of equations is called **homogeneous**. The system of equations is called **consistent** if it has atleast one solution and **Inconsistent** if it has no solution.

SOLVING NON-HOMOGENEOUS SYSTEM OF EQUATIONS BY MATRIX METHOD.

Premultiplying $AX = B$ by A^{-1}, we get

$A^{-1}(AX) = A^{-1}B$. Applying associative property of multiplication, we get

$(A^{-1}A)X = A^{-1}B$ or $X = A^{-1}B$.

or
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Equating the values of $x_1, x_2, ..., x_n$ to the corresponding elements in the product on the right hand side of the above equation, we get the desired solution.

SOLVING NON-HOMOGENEOUS SYSTEM OF EQUATIONS BY DETERMINANT METHOD (CRAMER'S RULE)

Let A be a non-singular matrix. As per Cramer's rule the solution of $AX = B$, is given by

$$x_i = \frac{|A_i|}{|A|}, \quad i = 1, 2, ..., n.$$

where $|A_i|$ is the determinant of the matrix A_i which is obtained by replacing the i^{th} column of A by the right hand side column vector B.

Following cases arise here:

Case I. When $|A| \neq 0$ the system of equations is consistent and the solution obtained is unique.

Case II. When $|A| = 0$, and one or more of $|A_i|$, $i = 1, 2, ..., n$ are not zero, then the system of equations has **no solution** and the system is **inconsistent.**

Case III. When $|A| = 0$ and all $|A_i| = 0$, $i = 1, 2, ..., n$, then the system of equations is consistent and has infinite number of solutions (or many solutions).

HOMOGENEOUS SYSTEM OF EQUATIONS

Consider the homogeneous system of equations $AX = 0$ then $X = 0$ is always a solution and is known as **Trivial** solution. Let us take up the following two possibilities.

(i) If A is non-singular, i.e., $|A| \neq 0$ then
$X = A^{-1}0 = 0$ so trivial solution exists.

(ii) If A singular, i.e., $|A| = 0$ then non-trivial solution exists and the system has infinite number of solutions.

We also conclude that a homogeneous system of equations is always consistent.

EXAMPLE 7.10. Solve the following system of equations

$$x + 2y + 3z = 0$$
$$2x + 3y - 2z = 0$$
$$4x + 7y + 4z = 0.$$

$x = y = z = 0$

SOLUTION: The system of equations can be written as $AX = 0$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -2 \\ 4 & 7 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Here $|A| = 1(12 + 14) - 2(8 + 8) + 3\,(14 - 12) = 0$ hence A is singular matrix. Therefore, the system bears infinite member of solutions.

Taking $z = t$ the first two equations give

$$x + 2y + 3t = 0 \quad \text{and} \quad 2x + 3y - 2t = 0$$

$\Rightarrow$ $x = 13t$ and $y = -8t$ where t is arbitrary.

Also, this solution satisfies the third equation as well. **Ans.**

EXAMPLE 7.11. If possible, solve the following system of equations

$$4x + 9y + 3z = 6$$
$$2x + 3y + z = 2$$
$$2x + 6y + 2z = 7$$

SOLUTION: The system of equations is $AX = B$ where

$$A = \begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Here $$|A| = 4(6 - 6) - 9(4 - 2) + 3(12 - 6) = 0$$

and $$|A_1| = \begin{vmatrix} 6 & 9 & 3 \\ 2 & 3 & 1 \\ 7 & 6 & 2 \end{vmatrix} = 6(6 - 6) - 9(4 - 7) + 3(12 - 21) = 0$$

$$|A_2| = \begin{vmatrix} 4 & 6 & 3 \\ 2 & 2 & 1 \\ 2 & 7 & 2 \end{vmatrix} = 4(4 - 7) - 6(4 - 2) + 3(14 - 4) = 6$$

and $$|A_3| = \begin{vmatrix} 4 & 9 & 6 \\ 2 & 3 & 2 \\ 2 & 6 & 7 \end{vmatrix} = 4(21-12) - 9(14-4) + 6(12-6) = -18$$

Since $|A| = 0$ and $|A_2| \neq 0$ the system of equations is inconsistent. **Ans.**

EXAMPLE 7.12. Investigate the following system of equations, for solution

$$x - y + z = 4$$
$$2x + y - 3z = 0$$
$$x + y + z = 2$$

using (i) matrix method (ii) Cramer's rule.

SOLUTION: The given system of equations is $AX = B$ where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Here $$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1+3) + 1(2+3) + 1(2-1) = 10$$

Hence the matrix A is non-singular and the system of equations has a unique solution.

(i) Solution by matrix method.

$$A^{-1} = \frac{1}{10} \text{ [Transpose of the cofactor matrix of } A]$$

$$= \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$\therefore$ $$X = A^{-1}B = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Thus $x = 2, y = -1, z = 1$ is the required solution.

(ii) Solution by Cramer's rule.

We have $$|A_1| = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 4(4) + 0 + 2(2) = 20$$

$$|A_2| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 1(6) - 2(2) + 1(-12) = -10$$

$$|A_3| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(4) - 2(-2) + 1(2) = 10$$

$\therefore$ $$x = \frac{|A_1|}{|A|} = \frac{20}{10} = 2, \quad y = \frac{|A_2|}{|A|} = \frac{-10}{10} = -1, \quad z = \frac{|A_3|}{|A|} = \frac{10}{10} = 1. \quad \textbf{Ans.}$$

EXAMPLE 7.13 Solve the following system of equations, if possible,

$$x - y + 3z = 3$$
$$2x + 3y + z = 2$$
$$3x + 2y + 4z = 5$$

SOLUTION: The system of equations is $AX = B$ where

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Here
$$|A| = 1(12 - 2) + 1(8 - 3) + 3(4 - 9) = 10 + 5 - 15 = 0.$$

$$|A_1| = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix} = 3(12 - 2) + 1(8 - 5) + 3(4 - 15) = 0$$

$$|A_2| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 1(8 - 5) - 3(8 - 3) + 3(10 - 6) = 0$$

$$|A_3| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 1(15 - 4) + 1(10 - 6) + 3(4 - 9) = 0$$

This means that the system of equations has infinite number of solutions. From the first two equations, taking $z = t$, we have

$$x - y = 3 - 3t \quad \text{and} \quad 2x + 3y = 2 - t$$

These, on solving for x and y, give

$$x = \frac{(11 - 10t)}{5} \quad \text{and} \quad y = \frac{(5t - 4)}{5}, \quad z = t$$

where t is arbitrary. Taking various values of t we can have various values of x and y. **Ans.**

EXERCISE 7A

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ show that $A^2 - 5A = 2I$ where I is the unit matrix of order 2, and hence find A^4.

2. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ verify that $(AB)' = B'A'$.

3. Let $A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 3 & -1 \\ 4 & 5 & 1 \end{bmatrix}$ show that $(A^2)' = (A')^2$.

4. Express the matrix $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

5. Let $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ show that $A^{-1} = \dfrac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}$ and hence find the inverse of the matrix $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Is B orthogonal ?

6. Determine the inverse of the matrix $\begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$.

7. Given that $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & -1 \end{bmatrix}$, verify

   (*i*) $(A^2)' = (A')^2$ (*ii*) $(AB)^{-1} = B^{-1}A^{-1}$.

8. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ show that $adj\ A = 3A'$ and find $adj\ (adj\ A)$.

9. Find $\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1}$.

10. Solve by matrix inversion method the following set of equations

$$x + 3y + 3z = 1$$
$$x + 4y + 3z = 0$$
$$x + 3y + 4z = 2.$$

SOLUTION OF GENERAL SYSTEM OF EQUATIONS

Uptil now we have taken into account the solution of only linear system of n equations with n unknowns through matrix method and Cramer's rule. In the matrix method we required to evaluate n^2 determinants each of order $(n-1)$ to generate the cofactor matrix and one determinant of order n whereas in Cramer's rule we require evaluation of $(n+1)$ determinants each of order n. Evaluating a high order determinant is quite time consuming and that is the reason that these methods are not used for large values of $n \geq 4$.

In this section we shall discuss how to solve a general system of m equations in n unknowns, given by $AX = B$

where
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Here the coefficient matrix A is of order $m \times n$, the column vector is of order $m \times 1$ and the solution vector X is of order $n \times 1$.

The matrix [A/B] or [A : B] given by

$$[A/B] = \left[\begin{array}{cccc|c} a_{11} & a_{22} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & b_2 \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** having m rows and $(n+1)$ columns. Actually, the augmented matrix describes the system of equations completely. The solution vector X is an n-tuple $(x_1, x_2, ..., x_n)$ which satisfies all equations. There are three possibilities here.

(i) the system has a unique solution.

(ii) the system has no solution.

(iii) the system has infinitely many solutions.

The system of equations is said to be **consistent** if it has atleast one solution and is said to be **inconsistent** if it has no solution. For all these we need to introduce the concepts of elementary row **operations** and **rank** of a matrix.

ELEMENTARY ROW AND COLUMN OPERATIONS

Following three operations on a matrix are called row and column operations:

(i) **Interchange of any two rows (or columns).** If we interchanging i^{th} row and j^{th} row, it is denoted by $R_i \leftrightarrow R_j$ or by R_{ij}. Similarly, interchanging i^{th} column and j^{th} column is denoted by $C_i \leftrightarrow C_j$ or by C_{ij}.

(ii) **Multiplication of any row by a scalar.** If we multiply i^{th} row by a scalar α it is denoted by αR_i. Similarly, for column.

(iii) **Adding a scalar multiple of any row to another row.** If we add α times j^{th} row to i^{th} row, it is denoted by $R_i \to R_i + \alpha R_j$. Similarly, for columns.

ELEMENTARY MATRICES

When an elementary transformation or operation is applied to a unit matrix the resulting matrix is called elementary matrix.

For instance, consider

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Applying the operation } R_2 \leftrightarrow R_3, \text{ on it, we get}$$

$$I_3 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{which is an elementry matrix.}$$

Similarly applying $R_1 \to R_1 + kR_3$ on I_3 we get

$$I_3 \sim \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{which is also an elementary matrix.}$$

Now, we illustrate the case where more than one row operation is carried on a given matrix.

As yet another example let $A = \begin{bmatrix} 6 & -1 & 1 & 4 \\ 9 & 3 & 7 & -7 \\ 0 & 2 & 1 & 5 \end{bmatrix}$

On A we perform the following operations in succession.

O_1 : add (–2) times row 2 to row 3.

O_2 : add now 2 (row 1) to row 2.

O_3 : interchange row 1 and row 3.

O_4 : multiply row 2 by (–4) and obtain the resulting matrix B.

We produce a 3×3 matrix C such that $B = CA$.

Starting with I_3 perform the given sequence of operations.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[O_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \xrightarrow[O_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow[O_3]{} \begin{bmatrix} 0 & -3 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[O_4]{} \begin{bmatrix} 0 & -3 & 1 \\ -8 & -4 & 0 \\ 1 & 0 & 0 \end{bmatrix} = C$$

$$\therefore \quad CA = \begin{bmatrix} 0 & -3 & 1 \\ -8 & -4 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & -1 & 1 & 4 \\ 9 & 3 & 7 & -7 \\ 0 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -27 & -7 & -20 & 26 \\ -84 & -4 & -36 & -4 \\ 6 & -1 & 1 & 4 \end{bmatrix} = B.$$

Two matrices are called **row equivalent** if one can be obtained from the other by a sequence of elementary row operations.

These follow the following properties:

1. Every matrix is row equivalent to itself (reflexive property)
2. If A is row equivalent to B then B is row equivalent to A (symmetric property)
3. If A is row equivalent to B and B is row equivalent to C then A is row equivalent to C (Transitive property).

The elementary matrices are interesting because **each elementary row operation on a matrix A can be performed by pre-multiplying A by an elementary matrix.** Thus, let A be a $m \times n$ matrix and B be another matrix formed by applying an elementary row operation on A, let E be the elementary matrix formed by applying this elementary row operation on I_n. Then $B = EA$.

For example, let $A = \begin{bmatrix} 1 & -5 \\ 9 & 4 \\ -3 & 2 \end{bmatrix}$. Suppose, we form B from A by interchanging second and third row, so that

$$B = \begin{bmatrix} 1 & -5 \\ -3 & 2 \\ 9 & 4 \end{bmatrix}.$$

But we can also form an elementary matrix by applying this operation on I_3 to form

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{then} \quad EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 9 & 4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -3 & 2 \\ 9 & 4 \end{bmatrix} = B.$$

As another example,

let $$A = \begin{bmatrix} -6 & 14 & 2 \\ 4 & 4 & -9 \\ -3 & 2 & 13 \end{bmatrix}$$

Get B from A by adding 6 times first row to second row, so

$$B = \begin{bmatrix} -6 & 14 & 2 \\ -32 & 88 & 3 \\ -3 & 2 & 13 \end{bmatrix}$$

Next, applying 6 times first row to second row to I_3, we get

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then}$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & 14 & 2 \\ 4 & 4 & -9 \\ -3 & 2 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 14 & 2 \\ -32 & 88 & 3 \\ -3 & 2 & 13 \end{bmatrix} = B$$

ECHELON FORM AND ROW-REDUCED ECHELON FORM OF A MATRIX

A matrix A is said to be in **Echelon form** if

(i) The first non-zero element in each row, called its leading entry, is 1. However, if it is not 1 we scale the row to make its leading entry 1.

(ii) In any two successive rows i^{th} and $(i+1)^{\text{th}}$ that do not consist entirely of zeros, the leading element in the $(i+1)^{\text{th}}$ row lies to the right of the leading element in i^{th} row.

(iii) Any row or rows that consist entirely of zeros, lie at the bottom of the matrix.

The matrix A is said to be in **row-reduced echelon form** if, in addition to conditions (i) to (iii), it is also true that

(iv) In a column that contains the leading entry of a row, all the other elements are zero.

In essence, it means that a matrix A is in **echelon form** if the first non-zero entry in any row is 1 and the entry appears to the right of the first non-zero entry in the row above and all rows of zeros lie at the bottom of the matrix. Furthermore, matrix A is in **row-reduced echelon form** if, in addition to these conditions, the first non zero entry in any row is the only non-zero entry in the column containing that entry.

For example, the matrices

$$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ are in } \textbf{echelon form}$$

1

and the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 5 & 4 \\ 0 & 1 & 0 & 2 & 7 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ are all in } \textbf{row-reduced echelon form.}$$

The process of reducing a matrix to echelon form and to row-reduced echelon form, will be more clear after undertaking following example.

EXAMPLE 7.14 Reduce the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 2 & 4 & 8 & 2 & 4 \\ 1 & 2 & 4 & 2 & 2 \\ 1 & 3 & 6 & 1 & 5 \end{bmatrix}$$

to its echelon form and its row-reduced echelon form.

SOLUTION: Interchanging rows 1 and 2, that is, applying $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 2 & 4 & 8 & 2 & 4 \\ 0 & 1 & 2 & 0 & 3 \\ 1 & 2 & 4 & 2 & 2 \\ 1 & 3 & 6 & 1 & 5 \end{bmatrix} \quad \left(\text{now applying } R_1 \to \frac{1}{2} R_1\right)$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 1 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 1 & 2 & 4 & 2 & 2 \\ 1 & 3 & 6 & 1 & 5 \end{bmatrix} \quad \text{(applying } R_3 \to R_3 - R_1 \text{ and } R_4 \to R_4 - R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 1 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 3 \end{bmatrix} \quad \text{(applying } R_4 \to R_4 - R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 1 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and this matrix is in echelon form. Having obtained the echelon form of the given matrix, we use it to get the row-reduced echelon form. Thus, we have

$$A \sim \begin{bmatrix} 1 & 2 & 4 & 1 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(apply } R_1 \to R_1 - 2R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(apply } R_1 \to R_1 - R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{Ans.}$$

and this matrix is now in its row-reduced echelon form.

INVERSE OF A MATRIX BY GAUSS-JORDAN METHOD

The method can be stated as follows:

"The elementary row operations which reduce a given matrix $A\ (n \times n)$ to a unit matrix $I\ (n \times n)$, when applied to the unit matrix I_n, give the inverse of A."

Let the successive row operations which reduce the given square matrix $A\ (n \times n)$ to I_n result from the premultiplication of A by the elementary matrices $R_1, R_2, ..., R_k$, so that

$$R_k R_{k-1} R_{k-2} \dots R_2 R_1 A = I$$

Post-multiplying this by A^{-1}, we obtain $(R_k R_{k-1} \dots R_2 R_1 A) A^{-1} = IA^{-1}$

or $\quad (R_k R_{k-1} \dots R_2 R_1)(AA^{-1}) = A^{-1}$ or $R_k R_{k-1} \dots R_2 R_1 I = A^{-1}$

or $\quad A^{-1} = R_k R_{k-1} \dots R_2 R_1 I.$

Thus, to find A^{-1} we write the augmented matrix $[A, I]$ where I is the unit matrix of the same order as that of A. Then perform the same row operations on both A and I, till such time when A reduces to I, the other matrix represents A^{-1}. However, this method fails when $\det(A) = 0$.

This can be symbolically written as $(A/I) \xrightarrow[\text{row operation}]{\text{elementary}} (I/B)$.

EXAMPLE 7.15. (*a*) Applying Gauss-Jordan technique, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$ [*GGSIPU I Sem End Term 2013*]

(*b*) By elementary transformations find the inverse of $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 2 \end{bmatrix}$

[*GGSIPU I Sem End Term January 2011*]

SOLUTION: **(a)** The augmented matrix $[A, I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$

Applying $R_2 - 2R_1$ and $R_3 - 3R_1$, we get

$$[A, I] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

Next, applying $R_2 \leftrightarrow R_3$ and $R_2 \to -R_2$, we get

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

Now, applying $R_1 \to R_1 - 2R_2$ and $R_3 \to -R_3$, we get

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

Finally, applying $R_1 \to R_1 + 3R_3$ and $R_2 \to R_2 - 3R_3$, we get

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

Hence $$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$ **Ans.**

(*b*) Let $A=\begin{bmatrix}1&3&5\\3&2&4\\5&4&2\end{bmatrix}$ and $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, then

$$AI=\begin{bmatrix}1&3&5\\3&2&4\\5&4&2\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$ (Applying $R_2 \to R_2-3R_1$ and $R_3\to R_3-5R_1$ on both A and I, we get

$$=\begin{bmatrix}1&3&5\\0&-7&-11\\0&-11&-23\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\-5&0&1\end{bmatrix}$$ (Applying $R_3\to R_3-2R_2$)

$$=\begin{bmatrix}1&3&5\\0&-7&-11\\0&3&-1\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\1&-2&1\end{bmatrix}$$ (Next applying $R_1\to R_1-R_3$ and $R_2\to R_2-2R_3$)

$$=\begin{bmatrix}1&0&6\\0&-1&-13\\0&3&-1\end{bmatrix}\begin{bmatrix}0&2&-1\\-1&-3&2\\1&-2&1\end{bmatrix}$$ (Now applying $R_3\to R_3+3R_2$)

$$=\begin{bmatrix}1&0&6\\0&-1&-13\\0&0&-40\end{bmatrix}\begin{bmatrix}0&2&-1\\-1&-3&2\\-2&-11&7\end{bmatrix}$$ (Next applying $R_2\to -R_2$ and $R_3\to -R_3$)

$$=\begin{bmatrix}1&0&6\\0&1&13\\0&0&40\end{bmatrix}\begin{bmatrix}0&2&-1\\1&3&-2\\2&11&-7\end{bmatrix}$$ (Now applying $R_3\to \frac{1}{40}R_3$)

$$=\begin{bmatrix}1&0&6\\0&1&13\\0&0&1\end{bmatrix}\begin{bmatrix}0&2&-1\\1&3&-2\\\frac{2}{40}&\frac{11}{40}&\frac{-7}{40}\end{bmatrix}=\frac{1}{40}\begin{bmatrix}1&0&6\\0&1&13\\0&0&1\end{bmatrix}\begin{bmatrix}0&80&-40\\40&120&-80\\2&11&-7\end{bmatrix}$$

(Next applying $R_1\to R_1-6R_3$ and $R_2\to R_2-13R_3$)

$$=\frac{1}{40}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-12&14&2\\14&-23&11\\2&11&-7\end{bmatrix}\quad \therefore\ A^{-1}=\frac{1}{40}\begin{bmatrix}-12&14&2\\14&-23&11\\2&11&-7\end{bmatrix}$$ **Ans.**

EXAMPLE 7.16. Using the method of elementary row transformations to compute the inverse of

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

[GGSIPU I Sem End Term 2007]

SOLUTION: Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We apply row transformations on A as well as I simultaneously till A becomes I and fate of I will be A^{-1}.

Applying $R_2 \to R_2 + 2R_1$ and $R_3 \to + R_3 + R_1$, we get

$$A \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 3 & 6 \end{bmatrix} \quad \text{and} \quad I \sim \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$ Now applying $R_3 \to R_3 + 3R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 0 & -21 \end{bmatrix} \quad \text{and} \quad I \sim \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}$$ Now applying $R_2 \to -R_2$, $R_3 \to -\frac{1}{21}R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad I \sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ \frac{5}{21} & \frac{-3}{21} & \frac{-1}{21} \end{bmatrix}$$

Next applying $R_2 \to R_2 - 9R_3$ and $R_1 \to R_1 - 5R_3$,

$$A \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I \sim \begin{bmatrix} \frac{-4}{21} & \frac{15}{21} & \frac{5}{21} \\ \frac{-3}{21} & \frac{6}{21} & \frac{9}{21} \\ \frac{5}{21} & \frac{-3}{21} & \frac{-1}{21} \end{bmatrix} = \frac{1}{21}\begin{bmatrix} -4 & 15 & 5 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

Finally applying $R_1 \to R_1 - 2R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I \sim \frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

Thus, $$A^{-1} = \frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$ **Ans.**

EXAMPLE 7.17. Find the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by E – row operations.

[*GGSIPU I Sem Term I 2006; End Term 2011; End Term 2012*]

SOLUTION: Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

These we write as $\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ (Applying $R_1 \leftrightarrow R_2$)

$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ (Next applying $R_3 \to R_3 - 3R_1$)

$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right]$ (Next $R_3 \to R_3 + 5R_2$)

$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right]$ (Next $R_2 \to R_2 - R_3$)

$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -4 & 3 & 1 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right]$ (Next $R_1 \to R_1 - 2R_2$)

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 8 & -5 & 2 \\ 0 & 1 & 0 & -4 & 3 & 1 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right]$ (Next $R_3 \to \frac{1}{2} R_3$)

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 8 & -5 & 1/2 \\ 0 & 1 & 0 & -4 & 3 & 1 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$ (Next $R_1 \to R_1 - 3R_3$)

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -4 & 3 & 1 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$

Then $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & 1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & 2 \\ 5 & -3 & 1 \end{bmatrix}$ **Ans.**

EXAMPLE 7.18. Find the inverse of the matrix (*i*) $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 0 & 4 \\ 2 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix}$

by using elementary row operations.

[*GGSIPU I Sem II Term 2007; End Term 2009*]

SOLUTION: (*i*) $A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = AI$

Applying $R_3 \to R_3 - 3R_2$ on A and I both $= \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

Next applying $R_2 \to R_2 - 2R_1$ on A and I, we get $\begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

Next applying $R_1 \to R_1 - 2R_2$ and then $R_2 \to R_2 - 3R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix} \quad \therefore \quad A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$ **Ans.**

(*ii*) Let $A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix} = AI = \begin{bmatrix} 1 & 0 & 4 \\ 2 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$, gives

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -7 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Further applying $R_2 \to 2R_2$ and $R_3 \to R_3 + R_2$, we get

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Next, applying $R_1 \to R_1 + 4R_3$ and $R_2 \to R_2 - 7R_3$, gives

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 8 \\ -2 & -6 & -14 \\ 0 & 1 & 2 \end{bmatrix}$ Finally applying $R_2 \to -\frac{1}{2} R_2$ and $R_3 \to -R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 8 \\ 1 & 3 & 7 \\ 0 & -1 & -2 \end{bmatrix} \quad \therefore \quad A^{-1} = \begin{bmatrix} 1 & 4 & 8 \\ 1 & 3 & 7 \\ 0 & -1 & -2 \end{bmatrix}$$ **Ans.**

RANK OF A MATRIX

A matrix is said to be of *rank* r if

(*i*) it has at least one non-zero minor of order r, and

(*ii*) every minor of order greater than r vanishes.

In other words, the rank of a matrix A is equal to the order of the highest ordered non-zero minor in A and is denoted by $r(A)$ or by $\rho(A)$.

Thus, *the rank of a non-singular square matrix of order* n is n.

For example, let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $\rho(A) = 2$.

and if $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then $\rho(A) = 1$

Further, if $A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 2 \\ 3 & 6 & 1 \end{bmatrix}$, we have $|A| = 0$ and $\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} \neq 0$ hence $\rho(A) = 2$.

Two matrices A and B are said to be *equivalent* if they are of the same order and of same rank and we write $A \sim B$. It is important to note here that when a matrix is subjected to one or more elementary operations, its rank does not change.

EXAMPLE 7.19. Find the rank of the matrix

(*a*) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ (*c*) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

[*GGSIPU Ist Sem End Term 2009; II Term 2010, End Term 2012*]

SOLUTION: (*a*) Let $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$. (Applying $R_2 - 2R_1$ and $R_3 - R_1$)

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}. \text{ (Applying } R_3 + R_2)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Clearly rank } (A) = 2$$

Ans.

(*b*) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$. (Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$)

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -4 \\ 0 & 2 & -1 \end{bmatrix}. \quad \text{(Next, applying } R_3 \to R_3 - R_2\text{)}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}. \quad \text{Hence rank of } A \text{ is 3.} \quad \textbf{Ans.}$$

(c) $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix} \quad \text{(Applying here } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Its rank} = 2 \quad \text{since} \quad \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} \neq 0 \qquad \therefore \ \rho(A) = 2 \qquad \textbf{Ans.}$$

EXAMPLE 7.20. Find the rank of the matrix

(i) $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$. (ii) $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 5 \\ 5 & 0 & 14 \end{bmatrix}$ [*GGSIPU I Sem II Term 2011*]

SOLUTION: (i) $$A \sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 2 & 4 & 6 & -1 \\ 3 & 10 & 9 & 7 \\ 4 & 16 & 12 & 15 \end{bmatrix} \quad \text{(on operating } C_{12} \text{ on A)}$$

$$\sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \end{bmatrix} \text{(on operating } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1, R_4 \to R_4 - 4R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \end{bmatrix} \text{(on operating } C_2 \to C_2 - 6C_1, C_3 \to C_3 - 3C_1, C_4 \to C_4 - 8C_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{(on operating } C_2 \to -\frac{1}{8}C_2 \text{ and then } C_4 \to C_4 - 17C_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (on operating } R_3 \to R_3 - R_2 \text{ and } R_4 \to R_4 - R_2)$$

which clearly establishes that rank $(A) = 2$. **Ans.**

(ii) $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 5 \\ 5 & 0 & 14 \end{bmatrix}$. Applying $R_2 - 2R_1$ and $R_3 - 5R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 5 & -1 \\ 0 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \; (R_3 - R_2)$$

$\therefore$ Rank$(A) = 2$. **Ans.**

EXAMPLE 7.21. Find the values of a and b such that the rank of matrix $A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & a & b \end{bmatrix}$ is 2.

[GGSIPU I Sem II Term 2007]

SOLUTION: $A \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & -5 & a+3 & b-6 \end{bmatrix}$ (on applying $R_3 \to R_3 - 3R_2$ and $R_2 \to R_2 - 2R_1$)

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & a-4 & b-6 \end{bmatrix} \text{ (on applying } R_3 \to R_3 + R_2)$$

Since rank of A is given to be 2 we have $a = 4, b = 6$. **Ans.**

EXAMPLE 7.22. If A is a non-zero column vector $(n \times 1)$ then the rank of the matrix $A A^T$ is

(i) 0 (ii) 1 (iii) $n-1$ (iv) n

[GGSIPU I Sem End Term 2004]

SOLUTION: A is of order $n \times 1$ then A^T is is of order $1 \times n$

If $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ then $A^T = (a_1, a_2 \dots a_n]$ hence $AA^T = \begin{bmatrix} a_1^2 & a_1a_2 & \dots & a_1a_n \\ a_1a_2 & a_2^2 & \dots & a_2a_n \\ \dots & \dots & \dots & \dots \\ a_na_1 & a_na_2 & \dots & a_n^2 \end{bmatrix}$

$\therefore$ $A A^T$ is of order $n \times n$. Since A is non zero, i.e., at least one of $a_1, a_2, \dots, a_n$ is not zero. Therefore rank of AA^T is 1.

Hence correct choice is (ii). **Ans.**

EXAMPLE 7.23. The rank of matrix $\begin{bmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ -1 & 0 & \mu \end{bmatrix}$ is 2, for μ equal to

(i) any row number (ii) 3 (iii) 1 (iv) 2 [*GGSIPU I Sem End Term 2004*]

SOLUTION: Let $A = \begin{bmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ -1 & 0 & \mu \end{bmatrix}$ Operating $R_1 \leftrightarrow R_3$, we have

$$A \sim \begin{bmatrix} -1 & 0 & \mu \\ 0 & \mu & -1 \\ \mu & -1 & 0 \end{bmatrix} \quad \text{Then applying } R_3 \to R_3 + \mu R_1, \text{ gives}$$

$$A \sim \begin{bmatrix} -1 & 0 & \mu \\ 0 & \mu & -1 \\ 0 & -1 & \mu^2 \end{bmatrix} \quad (\text{Now applying } R_2 \to R_2 + \mu R_3)$$

$$A \sim \begin{bmatrix} -1 & 0 & \mu \\ 0 & 0 & \mu^3 - 1 \\ 0 & -1 & \mu^2 \end{bmatrix}$$

Since rank of A is 2 we must have $\mu^3 = 1$ $\therefore$ $\mu = 1$. **Ans (iii)**

EXAMPLE 7.24. Find the rank of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$. [*GGSIPU I Sem End Term 2007*]

SOLUTION:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \quad (\text{on applying } R_2 - R_1, R_3 - 2R_1, R_4 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \quad (\text{on applying } R_3 + R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{on applying } R_4 - \tfrac{3}{8} R_3) \quad \therefore \ \rho(A) = 3 \qquad \textbf{Ans.}$$

NORMAL FORM OF A MATRIX

Every non-zero matrix A of rank r can be reduced by a sequence of elementary transformations, to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ which is called the **normal form** of A.

In other words, we can perform any number of row or column transformations such that the top left element becomes 1 and the other elements on the leading diagonal should be zero or one. Count the numbers 1 on the leading diagonal and that is the rank of the matrix.

Because each elementary transformation can be affected by pre-multiplication or post-multiplication with a suitable elementary matrix and each elementary matrix is non-singular, therefore, we have an important result as follows:

Corresponding to every matrix A of rank r there exist non-singular matrices P and Q such that PAQ = $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Note that if A is a matrix of order $m \times n$ then P and Q are square matrices of order $m \times m$ and $n \times n$ respectively.

EXAMPLE 7.25. Reduce the following matrix into normal form and find its rank

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

[*GGSIPU I Sem End Term January 2011*]

SOLUTION: Applying $R_2 \to R_2 - 4R_1$ in A, we get

$$A \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -8 & -10 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$ Applying $R_2 \to R_2 - 4R_4$ and $R_3 \to R_3 - 3R_4$, gives

$$A \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -8 & -18 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 8 & 20 \end{bmatrix}.$$ Next, applying $R_4 \to R_4 - 8R_3$, gives

$$A \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -8 & -18 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 36 \end{bmatrix}.$$ Now, applying $R_4 \to \frac{1}{36} R_4$ and $R_3 \to R_3 + 2R_4$, we get

$$A \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -8 & -18 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ Next, applying $R_2 \to R_2 + 8R_3$ and $R_2 \to R_2 + 18R_4$, gives

$$A \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ Now applying $R_1 \to R_1 + R_2,\ R_1 \to R_1 - 2R_3,\ R_1 \to R_1 - 3R_4,$ gives

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4.$$ Hence, the rank of $A = 4$. **Ans.**

EXAMPLE 7.26. Find two non-singular matrices P and Q such that PAQ is in normal form

where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ [GGSIPU I Sem II End Term 2007]

SOLUTION: We write $IAI = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(On applying } R_2 \to R_2 - 2R_1\text{)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(On applying } R_3 \to R_3 + R_2\text{)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(On applying } C_3 \to C_3 - C_2\text{)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(On applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1\text{)}$$

Therefore $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

and the normal form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **Ans.**

EXAMPLE 7.27. Find the rank of the matrix $A = \begin{bmatrix} -1 & 2 & -1 & -2 \\ -2 & 5 & 3 & 0 \\ 1 & 0 & 1 & 10 \end{bmatrix}$ by reducing it to normal form.

[GGSIPU I Sem II Term 2012]

SOLUTION: $A = \begin{bmatrix} -1 & 2 & -1 & -2 \\ -2 & 5 & 3 & 0 \\ 1 & 0 & 1 & 10 \end{bmatrix}$. Applying $R_1 \leftrightarrow R_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 10 \\ -2 & 5 & 3 & 0 \\ -1 & 2 & -1 & -2 \end{bmatrix}$$

Next, performing $R_2 + 2R_1$ and $R_3 + R_1$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 10 \\ 0 & 5 & 5 & 20 \\ 0 & 2 & 0 & 8 \end{bmatrix}$$

Next, applying $\frac{1}{5}R_1$ and then $R_3 - 2R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

Now applying $R_3 \to \frac{-1}{2}R_3$ and $R_2 - R_3$ and $R_1 - R_4$, we get

$$A = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & \cdot 1 & 0 \end{bmatrix}$$

Next, operating $C_4 \to C_4 - 10C_1$ and $C_4 \to C_4 - 4C_2$, gives

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which is in normal form hence the rank of A is 3. **Ans.**

LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

We are familiar with the vectors like $2i - 3j + 5k$ having components $2, -3, 5$ along the coordinate axes. In general, any quantity having n components is *called* a vector of order n. As such the co-efficients of unknowns in a linear equation form a vector. Elements of a row matrix or a column matrix also, form vectors.

The set of vectors $X_1, X_2, \ldots X_n$ is said to be **linearly dependent** if there exist scalars $c_1, c_2, c_3, \ldots, c_n$, not all zero, such that

$$c_1X_1 + c_2X_2 + \ldots + c_nX_n = O \qquad \ldots(1)$$

and, if no such scalars other than zero exist, the vectors $X_1, X_2, \ldots, X_n$ are **called linearly independent.**

In the above definition, out of $c_1, c_2, ..., c_n$ if at least one of them is non-zero, say $c_1 \neq 0$ then the relation (1) can be written as

$$X_1 = -\frac{c_2}{c_1}X_2 - \frac{c_3}{c_1}X_3 ... -\frac{c_n}{c_1}X_n$$

and we say that vector X_1 is a linear combination of the vectors $X_2, X_3, ..., X_n$. In other words, if atleast one vector can be expressed as a linear combination of the other vectors then the vectors $X_1, X_2, .. X_n$ are said to be **linearly dependent**; and if not a single vector can be expressed as a linear combination of the others then the vectors are *called* **linearly independent**.

EXAMPLE 7.28. (a) Investigate if the following set of vectors are linearly independent or dependent:

$X_1 = [1, 2, 3]'$, $X_2 = [3, -2, 1]'$, $X_3 = [1, -6, -5]'$.

(b) Show that the system of vector $[1, 3, 2]', [1, -7, -8]'$, and $[2, 1, -1]'$ is linearly dependent. [*GGSIPU I Sem II Term 2013*]

SOLUTION: (a) Let us assume that the given vectors are linearly dependant then there must exist scalars c_1, c_2, c_3, not all zero, such that $c_1X_1 + c_2X_2 + c_3X_3 = 0$ ... (1)

or $$c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}3\\-2\\1\end{bmatrix} + c_3\begin{bmatrix}1\\-6\\-5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow$
$$\begin{aligned} c_1 + 3c_2 + c_3 &= 0 \\ 2c_1 - 2c_2 - 6c_3 &= 0 \\ 3c_1 + c_2 - 5c_3 &= 0 \end{aligned}$$

which are homogeneous having trivial solution $c_1 = c_2 = c_3 = 0$. However, for non-trivial solution let us find the rank of the coefficient matrix.

$$\begin{bmatrix}1 & 3 & 1\\2 & -2 & -6\\3 & 1 & -5\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 1\\0 & -8 & -8\\0 & -8 & -8\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 1\\0 & -8 & -8\\0 & 0 & 0\end{bmatrix}$$

The rank of this matrix is 2 which is less than the number of unknowns therefore, non-trivial solution exists. Thus, we have two independent equations as $c_1 + 3c_2 + c_3 = 0$ and $c_2 + c_3 = 0$.

If we take $c_3 = \lambda$ then $c_2 = -\lambda$ and hence $c_1 = 2\lambda$.

Putting these values of c_1, c_2, c_3 in (1), we have

$$2\lambda X_1 - \lambda X_2 + \lambda X_3 = 0 \quad \text{or} \quad 2X_1 - X_2 + X_3 = 0.$$

Therefore, the vectors are dependent.

(b) $X_1 = \begin{bmatrix}1\\3\\2\end{bmatrix}$, $X_2 = \begin{bmatrix}1\\-7\\-8\end{bmatrix}$ and $X_3 = \begin{bmatrix}2\\1\\-1\end{bmatrix}$

As done in part (a), X_1, X_2, X_3 will be linearly dependent if

$$\begin{vmatrix}1 & 1 & 2\\3 & -7 & 1\\2 & -8 & -1\end{vmatrix} = 0$$

or $1(+7 + 8) - 1(-3 - 2) + 2) - 2(-24 + 14) = 0.$ **Hence Proved.**

COMPLEX MATRIX

COMPLEX CONJUGATE MATRIX: Let $A = (a_{ij})$ be a complex matrix and $\bar{a}_{ij}$ denote the complex conjugate of a_{ij}, then the matrix $\bar{A} = (\bar{a}_{ij})$ is called the complex conjugate of A.

HERMITIAN AND SKEW-HERMITIAN MATRICES

Hermitian matrix is a generalisation of symmetric matrix whereas skew-Hermitian matrix is that of skew-symmetrix matrix, when the elements of the matrix are complex quantities.

Consider the matrix Z of complex numbers z, given by

$$Z = (z_{mn}) \quad \text{where} \quad z_{mn} = a_{mn} + i\,b_{mn}, \quad i = \sqrt{-1}$$

The matrix $\bar{Z}$, conjugate of the matrix Z, is given by

$$\bar{Z} = (\bar{z}_{mn}) \quad \text{where} \quad \bar{z}_{mn} = a_{mn} - i\,b_{mn}.$$

Now, let us define a matrix Z^* as transpose of $\bar{Z}$, that is, $Z^* = (\bar{Z})$.

The matrix Z is called **Hermitian** if $Z^* = Z$ then $\bar{z}_{nm} = z_{mn}$ for all m, n

$\Rightarrow \quad a_{nm} - i\,b_{nm} = a_{mn} + i\,b_{mn}$ hence $a_{nm} = a_{mn}$ and $b_{mn} = -b_{nm}$ for all m, n.

When $m = n$ we have $b_{nn} = -b_{nn}$, i.e., $b_{nn} = 0$ for all n.

Thus, in a Hermitian matrix the leading diagonal elements are all real.

Next, the matrix Z is called **skew-Hermitian** if $Z^* = -Z$ which means that

$\bar{z}_{nm} = -z_{mn}$, i.e., $a_{nm} = -a_{nm}$ and $b_{nm} = b_{mn}$ for all m, n.

When $m = n$ we have $a_{nn} = -a_{nn}$ hence $a_{nn} = 0$ for all n.

Thus, in a skew-Hermitian matrix all the leading diagonal elements are purely imaginary.

For example, the matrix

$$\begin{bmatrix} 2 & 1-i & -2-3i \\ 1+i & -3 & 3+2i \\ -2+3i & 3-2i & 5 \end{bmatrix}$$ is a third order Hermitian matrix.

[*GGSIPU I Sem II Term 2011*]

And the matrix $$\begin{bmatrix} 2i & 3+i & 1-i \\ -3+i & 3i & 2+3i \\ -1-i & -2+3i & 4i \end{bmatrix}$$ is a third order skew-Hermitian matrix.

[*GGSIPU I Sem II Term 2011*]

UNITARY MATRIX

A square matrix A is called *unitary* if $AA^* = I = A^*A$, where $A^* = (\bar{A})'$, that is, if A^* is the transpose of the conjugate of A. In otherwords, a complex square matrix A is **unitary** if $A^{-1} = (\bar{A})'$ or $(\bar{A})^{-1} = A'$ where A' denotes the transpose of A. Obviously, if A is real then its unitary matrix is same as orthogonal matrix.

Note that, if A and B are Hemitian matrices, then $aA + bB$ is also Hermitian for any real scalars a and b, since $(\overline{aA+bB})' = (a\bar{A} + b\bar{B})' = a\bar{A}' + b\bar{B}' = aA + bB$

Also one can note that the inverse of a unitary (orthogonal matrix is unitary (orthogonal), since

$$A^{-1} = (\bar{A})'. \text{ Let } B = A^{-1} \text{ then,}$$

Imp

$$B^{-1} = A = (\bar{A}')^{-1} = \left[(\bar{A})^{-1}\right]' = \left[\overline{(A^{-1})}\right]' = \bar{B}'$$

EXAMPLE 7.29. If A and B are any two complex matrices show that

(*i*) $(A^*)^* = A$ (*ii*) $(A+B)^* = A^* + B^*$ (*iii*) $(AB)^* = B^*A^*$

SOLUTION: (*i*) $A^* = (\bar{A})'$ hence $(A^*)^* = (\bar{\bar{A}}')' = A$

(*ii*) $(A+B)^* = (\overline{A+B})' = (\bar{A}+\bar{B})' = \bar{A}'+\bar{B}' = A^*+B^*$

(*iii*) $(AB)^* = (\overline{AB})' = (\bar{A}\bar{B})' = (\bar{B}'\bar{A}') = \bar{B}'\bar{A}' = B^*A^*$

Hence Proved.

Imp

EXAMPLE 7.30. Show that every complex square matrix A can be expressd as a sum R + *i*S uniquely where R and S are both Hermitian.

SOLUTION: We can write $A = \left(\dfrac{A+A^*}{2}\right) + i\left(\dfrac{A-A^*}{2i}\right) = R + iS$

where $R = \dfrac{1}{2}(A+A^*)$, and $S = \dfrac{1}{2i}(A-A^*)$.

Now we are to show that R and S are Hermitian.

$$R^* = \frac{1}{2}(A+A^*)^* = \frac{1}{2}[A^* + (A^*)^*] = \frac{1}{2}[A^* + A] = R \text{ hence R is Hermitian}$$

Next $$S^* = \left[\frac{1}{2i}(A-A^*)\right]^* = \frac{1}{2(-i)}(A-A^*)^* = -\frac{1}{2i}[A^*-(A^*)^*] = -\frac{1}{2i}[A^*-A] = S$$

Hence S is also Hermitian

Hence Proved.

EXAMPLE 7.31. Express the matrix $A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$

as the sum of Hermitian and skew Hermitian matrix.

SOLUTION:

$$A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix} \text{ hence } \bar{A} = \begin{bmatrix} 1-i & 2 & 5+5i \\ -2i & 2-i & 4-2i \\ -1-i & -4 & 7 \end{bmatrix}$$

$$\therefore \quad A^* = (\bar{A})' = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}$$

Now $$\frac{A+A^*}{2} = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix} = P, \text{ say, which is Hermitian.}$$

and $$\frac{A-A^*}{2} = \begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix} = Q, \text{ say, which is skew Hermitian.}$$

Therefore, $$A = \frac{A+A^*}{2} + \frac{A-A^*}{2} = P+Q. \quad \textbf{Ans.}$$

EXAMPLE 7.32. (*a*) Show that the matrix $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ is a Hermitian matrix.

[GGSIPU I Sem End Term 2008; I Term 2012]

(*b*) Prove that eigen values of Hermitian matrix are always real.

[GGSIPU I Sem End Term January 2011]

(*c*) Prove that $U = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

[GGSIPU I Sem End Term 2013]

SOLUTION: (*a*) $$\bar{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$$

and $$\bar{A}' = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix} = A$$

Hence A is a Hermitian matrix.

(*b*) We know that if A is a Hermitian matrix then $A^* = (\bar{A})' = A$.

If λ stands for the eigen values of A then $|A-\lambda I| = 0 \Rightarrow |\bar{A}-\bar{\lambda}I| = 0 \Rightarrow |(\bar{A}-\bar{\lambda}I)'| = 0$

or $|\bar{A}'-\bar{\lambda}I'| = 0$ or $|\bar{A}'-\bar{\lambda}I| = 0$ or $|A^*-\bar{\lambda}I| = 0$

or $|A-\bar{\lambda}I| = 0$ as $A^* = A$. Also we have $|A-\lambda I| = 0$.

Therefore, $\bar{\lambda} = \lambda$, means λ must be real. **Hence proved.**

(*c*) $U = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ $\therefore$ $U^* = (\bar{U})' = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}' = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

Then $$UU^* = \frac{1}{4}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2+2 & 2-2 \\ 2-2 & 2+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore U is a unitary matrix. **Hence Proved**

EXERCISE 7B

1. Test whether the vectors $(1, 1, 1, 3)$, $(1, 2, 3, 4)$ and $(2, 3, 4, 7)$ are linearly dependent or not. If dependent, find the relation between them.

2. Show that the square matrix

$$\begin{bmatrix} \cos\phi\cos\theta & \sin\phi & \cos\phi\sin\theta \\ -\sin\phi\cos\theta & \cos\phi & -\sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \text{ is orthogonal.}$$

3. Find the rank of the matrix (i) $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$, (ii) $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ *[GGSIPU I Sem II Term 2004]*

4. Find the rank of the matrix

(i) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ *[GGSIPU I Sem End Term 2008]* (ii) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$.

5. (a) Using row operations compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$ *[GGSIPU I Sem End Term 2004]*

(b) Compute the inverse of the following matrix by using elementary transformations

$$\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

6. Find the value of θ for which the system of equations

$$(2\sin\theta)x + y - 2z = 0$$
$$3x + (2\cos 2\theta)y + 3z = 0$$
$$5x + 3y - z = 0$$

has a non-trivial solution. *(GGSIPU I Sem End Term 2008)*

7. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$, find non- singular matrixes P and Q such that PAQ is in normal form.

8. Reduce the following matrix into its normal form and hence find its rank :

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

9. Find two non-singular matrix P and Q such the PAQ is in the normal form where

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

10. Find the rank of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & -4 \\ -3 & 1 & -1 \end{bmatrix}$ by reducing it in its normal form.

[GGSIPU I Sem II Term 2013]

11. Are the vectors $X_1 = (1, 3, 4, 2)$, $X_2 = (3, -5, 2, 2)$, $X_3 = (2, -1, 3, 2)$ linearly dependent? If so,express one of these as a linearly combination of others.

12. Examine the following system of vectors for linear dependence. If dependent, find the relation between them $X_1 = (1, 1, -1, 1)$, $X_2 = (1, -1, 2, -1)$, $X_3 = (3, 1, 0, 1)$.

❑❑❑

CHAPTER 8

Linear Transformations, Solution of General System of Equations, Eigen Values, Eigen Vectors, Diagonalizing a Matrix, etc.

Linear Transformations, Application of Matrices in finding Solution of a General System of Homogeneous and Non-homogeneses Equations, Eigen Values and Eigen Vectors and their Properties, Cayley-Hamilton Theorem, Diagonalizing a Matrix, Quadratic form and Reduction of Quadratic form to Canonical Form.

LINEAR TRANSFORMATIONS

Consider three co-ordinate systems in a plane denoted by x_1x_2 – system, y_1y_2 – system and w_1w_2– system. Let us assume that these are related by the transformations

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 \\ y_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned}\right\} \quad ...(1)$$

and
$$\left.\begin{aligned} x_1 &= b_{11}w_1 + b_{12}w_2 \\ x_2 &= b_{21}w_1 + b_{22}w_2 \end{aligned}\right\} \quad ...(2)$$

which are particular linear transformations. On putting (2) in (1) we see that the y_1y_2 – system can be obtained directly from the w_1w_2 – system by a single linear transformation

$$\left.\begin{aligned} y_1 &= c_{11}w_1 + c_{12}w_2 \\ y_2 &= c_{21}w_1 + c_{22}w_2 \end{aligned}\right\} \quad ...(3)$$

This yields that

$$\left.\begin{aligned} y_1 &= a_{11}(b_{11}w_1 + b_{12}w_2) + a_{12}(b_{21}w_1 + b_{22}w_2) \\ y_2 &= a_{21}(b_{11}w_1 + b_{12}w_2) + a_{22}(b_{21}w_1 + b_{22}w_2) \end{aligned}\right\} \quad ...(4)$$

Comparing (4) with (3), gives

$$\left.\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21}, \qquad & c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21}, \qquad & c_{22} &= a_{21}b_{12} + a_{22}b_{22}) \end{aligned}\right\} \quad ...(5)$$

Our above calculation provides motivation for matrix multiplication. The transformation (1) can be written in compact form as $y = Ax$ ...(6)

where $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

and (2) can be written as $x = Bw$ ...(7)

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

Thus, substituting the linear transformation (7) in (6) we get the co-efficient matrix C of the composite transformation as $y = Ax = A(Bw) = ABw = Cw$ where $C = AB$.

The idea can be easily extended to m variables $y_1, y_2, ..., y_m$ and n variables $x_1, x_2, ..., x_n$ and p variables $w_1, w_2, ..., w_p$. The matrix A is of order $m \times n$, B of order $n \times p$ therefore C is of order $m \times p$.

For example, let (x, y) be the co-ordinates of a point P referred to a set of rectangular axes OX, OY. Then its co-ordinates (x', y') referred to another set of rectangular axes OX′, OY′ obtained by rotating the former axes OX, OY through an angle θ, are given by

$$x' = x\cos\theta + y\sin\theta, \quad y' = -x\sin\theta + y\cos\theta$$

Similarly, the relations of the form

$$x' = l_1x + m_1y + n_1z$$
$$y' = l_2x + m_2y + n_2z$$
$$z' = l_3x + m_3y + n_3z$$

define a linear transformation from (x, y, z) to (x', y', z') in 3D.

In general, the relation $Y = AX$

$$\text{where} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

gives an example of a linear transformation from n variables $x_1, x_2, ..., x_n$ to n variables $y_1, y_2, ..., y_n$ *i.e.*, the transformation of the vector X to the vector Y. This transformation is called **linear** because the linear relations $A(x_1 + x_2) = Ax_1 + Ax_2$ and $A(bx) = bA(x)$ hold good.

If the determinant of the coefficient matrix is zero, *i.e.*, $|A| = 0$, the transformation is called **singular** otherwise it is called **non-singular**. For a non-singular transformation $Y = AX$ we can invert it to get the inverse transformation $X = A^{-1}Y$. A non-singular transformation is sometimes referred to as *regular transformation.*

The linear transformation $Y = AX$ is called orthogonal if it transforms $x_1^2 + x_2^2 + ... + x_n^2$ into $y_1^2 + y_2^2 + ... + y_n^2$.

If $\quad X = [x_1, x_2, ..., x_n]$ then $X' = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ hence $XX' = x_1^2 + x_2^2 + ... + x_n^2$

Similarly, let $Y = [y_1, y_2, ..., y_n]$ then $Y' = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ hence $YY' = y_1^2 + y_2^2 + ... + y_n^2$.

Thus, if $Y = AX$ is an orthogonal transformation, then

$$X'X = Y'Y = (AX)'(AX) = X'A'AX$$

which is possible only when $A'A = I$ so that $A' = A^{-1}$ for orthogonal transformation.

SOLUTION OF GENERAL SYSTEM OF EQUATIONS

CONSISTENCY OF SIMULTANEOUS EQUATIONS

Consider first the following general non-homogeneous system of m linear equations in n unknowns :

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

The above system can be written in matrix form as AX = B ...(1)

where
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

If $B = 0$, *i.e.*, if $b_1 = b_2 = \dots = b_m = 0$ then (1) is called the **system of homogeneous equations.**

When (1) has a solution it is said to be ***consistent,*** otherwise the system is *called* **inconsistent.** A consistent system of equations may have only one solution called ***unique solution,*** or **may have infinitely many solutions.**

Consider the matrix = $\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} & b_m \end{bmatrix}$ where B is appended to A.

It is called ***augmented matrix*** written as [A : B] or as [A/B].

We find here the ranks of the co-efficinent matrix A and of the augmented matrix [A : B]. The golden rule for the consistency of the given system of equations, is as follows :

(*i*) rank (A) $\neq$ rank (A : B) then the system is **inconsistent** hence no solution.

(*ii*) rank (A) = rank (A : B) then the system is **consistent** and following two cases arise.

(*a*) if rank (A) = rank (A : B) = n, the number of unknowns, we have a **unique solution.**

(*b*) if rank (A) = rank (A : B) < n, there are **infinitely many solutions.**

In the **case of linear homogeneous system of simultaneous equations** AX = 0 we first find the rank of the co-efficient matrix A as r, say. If $r = n$ the above homogeneous system has trivial solution $x_1 = x_2 = \dots = x_n = 0$. But if $r < n$, the above system gives non-trivial solutions. Actually, in this case there would exist r solutions where r unknowns are expressed as a linear combination of the remaining $(n - r)$ unknowns.

EXAMPLE 8.1. Discuss the consistency of the following system of equations

$$x+y+2z+w=5$$
$$2x+3y-z-2w=2$$
$$4x+5y+3z=7$$

SOLUTION: The matrix equation is $AX = B$ where

$$A=\begin{bmatrix}1&1&2&1\\2&3&-1&-2\\4&5&3&0\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\\w\end{bmatrix},\quad B=\begin{bmatrix}5\\2\\7\end{bmatrix}.$$

The augmented matrix $[A:B]=\left[\begin{array}{cccc:c}1&1&2&1&5\\2&3&-1&-2&2\\4&5&3&0&7\end{array}\right].$

$$[A:B]\sim\left[\begin{array}{cccc:c}1&1&2&1&5\\0&1&-5&-4&-8\\0&-1&5&4&3\end{array}\right].\quad \text{(On operating } R_3\to R_3-2R_2 \text{ and } R_2\to R_2-2R_1)$$

$$[A:B]\sim\left[\begin{array}{cccc:c}1&1&2&1&5\\0&1&-5&-4&-8\\0&0&0&0&-5\end{array}\right].\quad \text{(On applying } R_3\to R_3+R_2)$$

$\therefore$ Rank $[A:B]=3$ since $\begin{vmatrix}2&1&5\\-5&-4&-8\\0&0&-5\end{vmatrix}\neq 0$

and Rank $(A)=$ rank of $\begin{bmatrix}1&1&2&1\\0&1&-5&-4\\0&0&0&0\end{bmatrix}=2.$

Since $\rho\,[A:B]\neq\rho\,(A)$ the given system of equations is inconsistent. **Ans.**

EXAMPLE 8.2. If possible, solve the following system of equations

$$2x+3y+4z=11$$
$$x+5y+7z=15$$
$$3x+11y+13z=25.$$

SOLUTION: The given system can be written in matrix equation form as $AX=B$

where $A=\begin{bmatrix}2&3&4\\1&5&7\\3&11&13\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad B=\begin{bmatrix}11\\15\\25\end{bmatrix}.$

$$[A:B]=\left[\begin{array}{ccc:c}2&3&4&11\\1&5&7&15\\3&11&13&25\end{array}\right]$$

$$\sim\left[\begin{array}{ccc:c}1&5&7&15\\0&-7&-10&-19\\0&-4&-8&-20\end{array}\right]\quad \text{(on applying } R_1\leftrightarrow R_2, R_2\to R_2-2R_1 \text{ and } R_3\to R_3-3R_1)$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & \vdots & 15 \\ 0 & 1 & 2 & \vdots & 5 \\ 0 & -7 & -10 & \vdots & -19 \end{bmatrix} \text{ (on applying } R_3 \to -\frac{1}{4} R_3 \text{ and } R_2 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & \vdots & 15 \\ 0 & 1 & 2 & \vdots & 5 \\ 0 & 0 & 4 & \vdots & 16 \end{bmatrix} \text{ (on operating } R_3 \to R_3 + 7R_2)$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & \vdots & 15 \\ 0 & 1 & 0 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 4 \end{bmatrix} \text{ (on applying } R_3 \to \frac{1}{4} R_3 \text{ and } R_2 \to R_2 - 2R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 4 \end{bmatrix} \text{ (on applying } R_1 \to R_1 - 5R_2 - 7R_3)$$

$\therefore$ $\rho[A : B] = 3$ and $\rho(A) = 3$. Thus $\rho(A : B) = \rho(A) = 3$ = number of unknowns, hence unique solution as $x = 2, y = -3, z = 4$. **Ans.**

EXAMPLE 8.3. Discuss the consistency of the following system of equations for various values of λ:

$$2x_1 - 3x_2 + 6x_3 - 5x_4 = 3$$
$$x_2 - 4x_3 + x_4 = 1$$
$$4x_1 - 5x_2 + 8x_3 - 9x_4 = \lambda.$$

And, if consistent, solve it.

SOLUTION: The matrix equation is $AX = B$. Hence, the augmented matrix is

$$[A:B] = \begin{bmatrix} 2 & -3 & 6 & -5 & \vdots & 3 \\ 0 & 1 & -4 & 1 & \vdots & 1 \\ 4 & -5 & 8 & -9 & \vdots & \lambda \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 6 & -5 & \vdots & 3 \\ 0 & 1 & -4 & 1 & \vdots & 1 \\ 0 & 1 & -4 & 1 & \vdots & \lambda - 6 \end{bmatrix} \text{ (on operating } R_3 \to R_3 - 2R_1)$$

$$\sim \begin{bmatrix} 2 & -3 & 6 & -5 & \vdots & 3 \\ 0 & 1 & -4 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & \vdots & \lambda - 7 \end{bmatrix} \text{ (on operating } R_3 \to R_3 - R_2)$$

When $\lambda \neq 7$, rank $(A : B) = 3$ and rank $(A) = 2$ hence the system is inconsistent.

And when $\lambda = 7$, rank $(A : B) = 2$ and rank $(A) = 2$ hence consistent. Also, since rank $(A : B)$ = rank $(A) < 4$, the number of unknowns, therefore we shall have infinitely many solutions.

To obtain many solutions when $\lambda = 7$, we take two independent equations as :

$$x_2 - 4x_3 + x_4 = 1 \qquad ...(1)$$
$$2x_1 - 3x_2 + 6x_3 - 5x_4 = 3 \qquad ...(2)$$

Here the number of arbitrary solutions = number of unknowns – rank of A

Therefore we take $x_3 = t_1$ and $x_4 = t_2$ then from (1), we get $x_2 = 4t_1 - t_2 + 1$

and from (2) $2x_1 = 3(4t_1 - t_2 + 1) - 6t_1 + 5t_2 + 3$ or $x_1 = 3t_1 + t_2 + 3$

Giving arbitrary values to t_1 and t_2 we get infinitely many values of x_1 and x_2. **Ans.**

EXAMPLE 8.4. (*a*) Test for consistency of the following set of equations
$2x-3y+5z=1,\quad 3x+y-z=2,\quad$ and $\quad x+4y-6z=1$
and, if consistent, solve the system.

(*b*) Find the values of a, b for which the equations
$$x + ay + z = 3,\quad x + 2y + 2z = b,\quad x + 5y + 3z = 9$$
are consistent. When will the equations have a unique solution?

[*GGSIPU Ist Sem End Term 2009*]

SOLUTION: (*a*) Given system of equations can be written as $AX = B$ where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -1 \\ 1 & 4 & -6 \end{bmatrix},\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

The Augmented matrix $(A:B) = \left[\begin{array}{ccc:c} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{array}\right]$ (applying $R_1 \leftrightarrow R_3$ and then R_2-3R_1 and R_3-2R_1)

$$\sim \left[\begin{array}{ccc:c} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & -11 & 17 & -1 \end{array}\right] \text{ (operating } R_3 \to R_3-R_2)$$

$$\sim \left[\begin{array}{ccc:c} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad ...(1)$$

Obviously here rank $(A : B) = 2$, rank $(A) = 2$ hence the system is consistent. Also since the common value of the ranks is less than the number of unknowns, we can have infinitely many solutions.

From (1) it follows that we have two independent equations,

$$x + 4y - 6z = 1 \qquad ...(2)$$

and $$-11y + 17z = -1 \qquad ...(3)$$

Here number of arbitrary solutions = number of unknowns – rank (A)

Therefore, we set $z = t$, then from (3), we get $y = \dfrac{(1+17t)}{11}$.

and from (2), we get $x = 1 - 4y + 6z = 1 - \dfrac{4}{11}(1 + 17t) + 6t = \dfrac{(7-2t)}{11}$.

Thus, the infinitely many solutions are given by

$$x = \frac{7-2t}{11},\quad y = \frac{1+17t}{11},\quad z = t$$ where t is an arbitrary quantity. **Ans.**

(*b*) The coefficient matrix $A = \begin{bmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{bmatrix}$ and augmented matrix $D = \left[\begin{array}{ccc:c} 1 & a & 1 & 3 \\ 1 & 2 & 2 & b \\ 1 & 5 & 3 & 9 \end{array}\right]$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$D \sim \left[\begin{array}{ccc|c} 1 & a & 1 & 3 \\ 0 & 2-a & 1 & b-3 \\ 0 & 5-a & 2 & 6 \end{array}\right]$$

If $a = 5$ then
$$D \sim \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 0 & -3 & 1 & b-3 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

which shows that Rank (A) = Rank $(D) = 3$. Hence consistent.

Further, since here rank (A) = rank $(D) = 3$ = number of unknowns

The equations have a unique solution when $a = 5$ and b may have any value. **Ans.**

EXAMPLE 8.5. Test for consistency of the following system of equations and, if yes, solve them:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ 3x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 - 2x_2 + 3x_3 &= 2 \\ x_1 - x_2 + x_3 &= -1. \end{aligned}$$

SOLUTION: The given system in the matrix equation form is $AX = B$;

where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$

$$(A:B) = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right]$$

(on operating $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 - R_1$)

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & -3 & 2 & -4 \end{array}\right]$$ (on performing $R_2 \to R_2 - R_3$ and $R_3 \to R_3 - 2R_4$)

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{array}\right]$$ (on performing $R_2 \to -R_2$ and $R_4 \to R_4 + 3R_2$)

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ (on performing $R_4 \to R_4 - 2R_3$)

Therefore, rank (A : B) = rank (A) = 3 = number of unknowns, hence unique solution. To obtain this unique solution, we have

$$(A:B) \sim \begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 4 \\ 0 & 0 & 1 & \vdots & 4 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \text{ (on operating } R_1 \to R_1 + R_3 \text{ and } R_1 \to R_1 - 2R_2\text{)} .$$

Therefore, the unique solution is $x = -1, \quad y = 4, \quad z = 4$. **Ans.**

EXAMPLE 8.6. (*a*) Determine the value(s) of λ for which the following homogeneous set of equations possess a non trivial solution and find the solutions for real λ :

$$3x+y-\lambda z=0, \quad 4x-2y-3z=0, \quad 2\lambda x+4y+\lambda z=0.$$

(*b*) Find the value of λ for which the system of equations

$$x + y + 4z = 1$$
$$x + 2y - 2z = 1$$
$$\lambda x + y + z = 1$$

will have unique solution. [*GGSIPU I Sem II Term 2010*]

SOLUTION: (*a*) Given equations are homogeneous and these will have non-trivial solution when the determinant of the coefficient matrix is equal to 0, *i.e*; when

$$\begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$$

or $\qquad 3(-2\lambda + 12) - (4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0 \quad \Rightarrow \quad \lambda = -9, 1.$

For $\lambda = -9$, the set of equations are

$$3x + y + 9z = 0, \qquad 4x - 2y - 3z = 0, \quad -18x + 4y - 9z = 0.$$

Taking any two of the above equations, say first two, the solution is

$$\frac{x}{15} = \frac{y}{45} = \frac{z}{-10} \quad \text{or} \quad \frac{x}{3} = \frac{y}{9} = \frac{z}{-2}.$$

$\Rightarrow \qquad x = \dfrac{-3k}{2}, \; y = \dfrac{-9k}{2}, \; z = k$ where k is an arbitrary quantity.

Similarly, for $\lambda = 1$ the set of equations are

$$3x+y-z = 0, \quad 4x-2y-3z = 0, \quad 2x+4y+z = 0$$

Taking any two of the above equations, say first two, the solution is $\dfrac{x}{-5} = \dfrac{y}{5} = \dfrac{z}{-10}$

$\Rightarrow \qquad x = \dfrac{k'}{2}, \quad y = -\dfrac{k'}{2}, \quad z = k'$ where k' is another arbitrary quantity.

(b) $$(A:B) = \left[\begin{array}{ccc|c} 1 & 1 & 4 & 1 \\ 1 & 2 & -2 & 1 \\ \lambda & 1 & 1 & 1 \end{array}\right].$$ Applying $R \to R_2 - R_1$ and $R_3 \to R_3 - \lambda R_1$ we get

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 4 & 1 \\ 0 & 1 & -6 & 0 \\ 0 & 1-\lambda & 1-4\lambda & 1-\lambda \end{array}\right].$$ Applying $R_3 \to R_3 - (1-\lambda)R_2$, gives

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 4 & 1 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 7-10\lambda & 1-\lambda \end{array}\right].$$

Now $\rho(A) = 3$ if $\lambda \neq \frac{7}{10}$. Also $\rho(A:B) = 3$ if $\lambda \neq \frac{7}{10}$.

$\therefore$ Unique solution when $\lambda \neq \frac{7}{10}$. **Ans.**

EXAMPLE 8.7. (a) Find for what values of a and b, the equations

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$

have (i) no solution (ii) a unique solution (iii) more than one solution.

[*GGSIPU I Sem II Term 2006*]

(b) Find the values of a and b for which the equations

$x + ay + z = 3,\ x + 2y + 2z = b,\ x + 5y + 3z = 9$

are consistent. Will these have unique solution? [*GGSIPU I Sem End Term 2013*]

SOLUTION: (a) The given system of equations is $AX = B$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & a \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 6 \\ 9 \\ b \end{bmatrix}$$

The augmented matrix $$D = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 1 & 3 & 5 & : & 9 \\ 2 & 5 & a & : & b \end{bmatrix}$$ (On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$)

$$D \sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & a-6 & : & b-12 \end{bmatrix}$$ (On applying $R_3 \to R_3 - R_2$)

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & a-8 & : & b-15 \end{bmatrix}$$

If $a = 8, b \neq 15$ we have $\rho(A) = 2,\ \rho(D) = 3$ so $\rho(A) \neq \rho(D)$ hence system has no solution.

Similarly if $a \neq 8$ we have $\rho(A) = 3, \quad \rho(D) = 3$

$\therefore \quad \rho(A) = \rho(D) =$ number of unknowns hence the system will have unique solution.

Further, if $a = 8$ and $b = 15$ we have $\rho(A) = 2, \quad \rho(D) = 2$

$\therefore \quad \rho(A) = \rho(D) < 3$ hence more than one solution. **Ans.**

(b) Given system of equations is AX = B, where

$$A = \begin{bmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ b \\ 9 \end{bmatrix}.$$

$$\therefore \quad [A : B] = \left[\begin{array}{ccc|c} 1 & a & 1 & 3 \\ 1 & 2 & 2 & b \\ 1 & 5 & 3 & 9 \end{array}\right] \text{ (applying } R_3 \to R_3 - R_2 \text{ and } R_2 \to R_2 - R_1)$$

$$\sim \left[\begin{array}{ccc|c} 1 & a & 1 & 3 \\ 0 & 2-a & 1 & b-3 \\ 0 & 3 & 1 & 9-b \end{array}\right]$$

When $a = 2$, $[A : B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & b-3 \\ 0 & 3 & 1 & 9-b \end{array}\right]$ we have $\rho(A) = \rho(A : B) = 3$ hence unique solution.

When $a = -1$, $[A : B] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & b-3 \\ 0 & 3 & 1 & 9-b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & b-3 \\ 0 & 0 & 0 & 12-2b \end{array}\right]$

Therefore when $a = -1$ and $b = 6$

$$[A : B] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

then $\rho(A) = \rho(A : B) = 2$ hence many solutions.

and when $a = -1$ and $b \neq 6$

$$\rho(A) = 2 \text{ and } \rho(A : B) = 3$$

Thus, system of equation is inconsistent when $a = -1, b \neq 6$, system of equations is consistent. **Ans.**

EXAMPLE 8.8. (a) For what values of η the equations

$$x + y + z = 1$$
$$x + 2y + 4z = \eta$$
$$x + 4y + 10z = \eta^2.$$

have a solution and solve them completely in each case.

[GGSIPU I Sem II End Term 2007]

(b) Determine the values of λ for which the following equations are consistent. Also solve the system for these values of λ.

$$x + 2y + z = 3$$
$$x + y + z = \lambda$$
$$3x + y + 3z = \lambda^2$$

[GGSIPU I Sem II Term 2012]

SOLUTION: (a) The given equation can be written as matrix equation $AX = B$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix}.$

Augmented matrix $(A : B) = D = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & \eta \\ 1 & 4 & 10 & : & \eta^2 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$D \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \eta - 1 \\ 0 & 2 & 6 & : & \eta^2 - \eta \end{bmatrix} \text{(Now, applying } R_3 \to R_3 - 2R_2)$$

$$D \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \eta - 1 \\ 0 & 0 & 0 & : & \eta^2 - 3\eta + 2 \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \eta - 1 \\ 0 & 0 & 0 & (\eta - 1)(\eta - 2) \end{array}\right]$$

For $\eta = 1$ or $2, \rho(A) = 2$ and $\rho(D) = 2$ hence consistent. Since $\rho(A) = \rho(D) = 2 < 3$, the number of unknowns, hence for $\eta = 1$ and for $\eta = 2$ we have many solutions.

For $\eta = 1$, if we take $z = t$ then $x = 2t + 1$ and $y = -3t$

For $\eta = 2$, if we take $z = t$, then $x = 2t$ and $y = 1 - 3t$ where t is arbitrary. **Ans.**

(b) The matrix equation is $AX = B$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ \lambda \\ \lambda^2 \end{bmatrix}.$$

The augmented matrix $A : B = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & \lambda \\ 3 & 1 & 3 & \lambda^2 \end{array}\right]$

(on applying $R_1 \leftrightarrow R_2, R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$]

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & \lambda \\ 0 & 1 & 0 & 3-\lambda \\ 0 & -2 & 0 & \lambda^2-3\lambda \end{array}\right] \text{ (on applying } R_3 \to R_3 + 2R_2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & \lambda \\ 0 & 1 & 0 & 3-\lambda \\ 0 & 0 & 0 & \lambda^2-5\lambda+6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2\lambda-3 \\ 0 & 1 & 0 & 3-\lambda \\ 0 & 0 & 0 & (\lambda-3)(\lambda-2) \end{array}\right] \text{ (on applying } R_1 - R_2)$$

When $\lambda = 3$ $A : B = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Hence $\rho(A) = 2$ and $\rho(A : B) = 2$. The system is consistent having many solutions.
Here $y = 0$ and $x + z = 3$. The solutions are $(2, 0, 1)$ and in general $(3 - k, 0, k), k \in I$

When $\lambda = 2$ $A : B = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\therefore$ system is consistent having many solution.

Here $y = 1$ and $x + z = 1$ solution is $(0, 1, 1)$ or in general $(1 - k, 1, k), k \in I$. **Ans.**

EXAMPLE 8.9. Consider the following system of equations in which p is a real parameter.

$$x_1 - 2x_2 + 3x_3 = -1$$
$$2x_1 - 3x_2 + 4x_3 = -1$$
$$px_1 + x_2 + (p^2 - p - 3)\, x_3 = 2p + 2$$

By using row operations on the associated matrix obtain a row equivalent triangular form.

(i) Find the values of p (if any) for which the system is inconsistent.

(ii) Find the values of p (if any) for which the system has an infinite number of solutions.

(iii) Solve the system in the case when $p = 2$, explaining carefully how you obtain the solution from the triangular form. *[GGSIPU I Sem End Term 2004]*

SOLUTION: The given equations can be written as $AX = B$

where $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 4 \\ p & 1 & p^2-p-3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} -1 \\ -1 \\ 2p+2 \end{bmatrix}.$

The augmented matrix $D = [A : B] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 2 & -3 & 4 & -1 \\ p & 1 & p^2-p-3 & 2p+2 \end{array}\right]$

To convert it into triangular form, we apply $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - pR_1$ and get:

$$D \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 1+2p & p^2-4p-3 & 3p+2 \end{array}\right]$$

Next, applying $R_3 \to R_3 - (1 + 2p)\,R_2$, we get

$$D \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & p^2-1 & p+1 \end{array}\right] = D', \text{ say}$$

(i) Now if $p = 1$ then $\rho(A) = 2$ and $\rho(D) = 3$. Since $\rho(A) \neq \rho(D)$ then system is inconsistent.

(ii) And if $p = -1$ then $\rho(A) = 2$ and $\rho(D) = 2$. Since $\rho(A) = \rho(D) < 3$, the number of unknowns, we get many solutions.

(iii) If $p = 2$, we have $D' = \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right]$. On applying $R_3 \to \frac{1}{3} R_3$ and $R_2 \to R_2 + 2R_3$ we get

$$D' \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right].$$ Lastly applying $R_1 \to R_1 - 3R_3$ and $R_1 \to R_1 + 2R_2$ we get

$$D' \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right].$$ Hence the solution, for $p = 2$ is $x_1 = 2,\ x_2 = 3,\ x_3 = 1$ **Ans.**

EXERCISE 8A

1. Consider the linear transformations

$$x_1 = y_1 - 2y_2 + y_3$$
$$x_2 = 2y_1 + y_2 - 3y_3,$$

and
$$y_1 = z_1 + 2z_2$$
$$y_2 = 2z_1 - z_2$$
$$y_3 = 2z_1 + 3z_2.$$

Using matrix method find the linear transformation from (z_1, z_2) to (x_1, x_2).

2. Given the linear transformation $Y = AX$ where

$$Y = [y_1, y_2, y_3]', \quad X = [x_1, x_2, x_3]' \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix},$$

find the coordinates (x_1, x_2, x_3) corresponding to $(2, 0, 5)$ in Y.

3. Show that the transformation

$$y_1 = x_1 - x_2 + x_3$$
$$y_2 = 3x_1 - x_2 + 2x_3$$
$$y_3 = 2x_1 - 2x_2 + 3x_3$$

is non-singular. Find the inverse transformation.

4. Using the rank concept investigate the consistency of the system of equations

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21.$$

and, if consistent, solve it. [GGSIPU I Sem End Term 2012]

5. Show that the system of equations

$$x + y + 3z = 6$$
$$2x + 3y + z = 8$$
$$x + 5y + 7z = 20$$
$$x + z = 10$$

has no solution.

6. (a) Investigate whether the set of equations

$$2x - y - z = 2$$
$$x + 2y + z = 2$$
$$4x - 7y - 5z = 2$$

is consistent or not, and, if consistent, solve it.

(b) Find the value of λ so that the following equations have a non-trivial solution

$$2x + 3y + 4z = 0$$
$$x + 2y - 5z = 0$$
$$3x + 5y - \lambda z = 0$$

[GGSIPU I Sem II Term 2011]

7. Investigate if the following system of equations is consistent or not.

$$x_1 + x_2 + x_3 = 3$$
$$2x_1 - x_2 + 3x_3 = 1$$
$$4x_1 + x_2 + 5x_3 = 3$$
$$3x_1 - 2x_2 + x_3 = 4.$$

8. Determine the value(s) of λ for which the following equations

$$\begin{aligned} 2x - 3y + z &= \lambda x \\ 2x - 3y + 2z &= \lambda y \\ -x + 2y + 0z &= \lambda z \end{aligned}$$

can possess a non-trivial solution.

9. Determine the values of λ and μ so that the equations

$$\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + \lambda z &= \mu \end{aligned}$$

have (*i*) no solution, (*ii*) a unique solution and (*iii*) infinitely many solutions.

[*GGSIPU I Sem End Term 2011*]

10. Solve by matrix method, the system of equations

$$\begin{aligned} x + 3y - z - \omega &= 0 \\ 3x + y + 2z - 2\omega &= 4 \\ 4x - 3y - 4z - 3\omega &= -13. \end{aligned}$$

11. Test for consistency the following system of equations

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 3z &= 4 \\ x + 4y + 9z &= 4 \\ 2x + 3y + 9z &= 6. \end{aligned}$$

12. Solve the following system of equations using elementary transformations

$$\begin{aligned} 2x + y - z + 3\omega &= 11 \\ x - 2y + z + \omega &= 8 \\ 4x + 7y + 2z - \omega &= 0 \\ 3x + 5y + 4z + 4\omega &= 17. \end{aligned}$$

13. Find the values of a and b such that the system of equations

$$\begin{aligned} 3x - 2y + z &= 6 \\ 5x - 8y + 9z &= 3 \\ 2x + y + az &= -b \end{aligned}$$

may have unique solution.

14. Find the value of λ for which the equations

$$\begin{aligned} (\lambda - 1)x + (3\lambda + 1)y + 2\lambda z &= 0 \\ (\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z &= 0 \\ 2x + (3\lambda + 1)y + 3(\lambda - 1)z &= 0 \end{aligned}$$

are consistent and find the ratios of $x : y : z$ when λ has the smallest of these values. What happens when λ has the greatest of these values.

15. For what values of the parameters λ and μ do the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$, have (i) no solution (ii) unique solutions (iii) more then one solution?

[*GGSIPU I Sem End Term 2011; II Term 2013*]

CHARACTERISTIC EQUATION, EIGEN VALUES AND EIGEN VECTORS

Consider the linear transformation $Y = AX$ which transforms the column vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ into the column vector $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ where A, the transformation matrix, is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix}.$$

It may be interesting to consider another transformation $Y = \lambda X$ (where λ is scalar) under which a vector X is transformed into a vector Y having same direction as that of X but having different magnitude. This implies that $\lambda X = AX$

or $(A - \lambda I)X = 0$ where I is an identity matrix of order n. ...(1)

which represents the following system of homogeneous equations in $x_1, x_2, ..., x_n$

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots \quad \cdots \cdots \quad \cdots \quad \cdots \quad \cdots \cdots \cdots \\ \cdots \quad \cdots \cdots \quad \cdots \quad \cdots \quad \cdots \cdots \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n = 0 \end{array}\right\} \quad ...(2)$$

This system of equations has trivial solution $X = 0$ for any λ, but we are interested only in the non-trivial solutions of (2) which is possible only when $|A - \lambda I| = 0$

or

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad ...(3)$$

This is called the **characteristic equation** of the square matrix A.

Expanding the determinant in (3) we get a polynomial of degree n in λ which is of the form

$$P_n(\lambda) = |A - \lambda I| = (-1)^n [\lambda^n - C_1\lambda^{n-1} + C_2\lambda^{n-2} - ... + (-1)^n C_n] = 0$$

or $$\lambda^n - C_1 \lambda^{n-1} + C_2 \lambda^{n-2} - ... + (-1)^n C_n = 0$$

where $C_1, C_2, ..., C_n$ can be expressed in terms of the elements of the matrix A.

The roots $\lambda_1, \lambda_2, ..., \lambda_n$ of the equation $P_n(\lambda) = 0$ are called the **Eigenvalues or Characteristic values or characteristic roots or latent roots** of the matrix A. The roots can be single or repeated real or complex.

By using the relation between the roots and the coefficients, we can write

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = C_1 = a_{11} + a_{22} + \ldots + a_{nn}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{n-1}\lambda_n = C_2$$

....

....

$$\lambda_1\lambda_2\lambda_3 \ldots \lambda_n = C_n$$

If we set $\lambda = 0$ in equation (3), we get $|A| = (-1)^{2n} C_n = C_n = \lambda_1\lambda_2 \ldots \lambda_n$ ... (3A)

Thus, **sum of the eigen values = TRACE (A) and the product of the eigenvalues = $|A|$**

The set of eigen values is called the **spectrum** of A and the largest eigenvalue in magnitude, is called the **spectral radius of** A and is denoted by $\rho(A)$. If $|A| = 0$ the matrix is singular then from equation (3A) we find that one of the eigenvalues is zero. Conversely if one of the eigen values is zero then $|A| = 0$. Also, one can note that if A is a diagonal matrix or an upper triangular matrix or a lower triangular matrix then the diagonal elements of the matrix are the eigenvalues of A.

Corresponding to n eigen values we can get n values of the column vector X by solving the equations (1), that is, $AX = \lambda X$.

These column vectors are called **eigen vectors or characteristic vectors** of the matrix A.

Let us take up the frequently occurring case when the matrix A is of the order 3×3.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the characteristic equation $|A - \lambda I| = 0$ becomes

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0 \quad \text{which, on expansion, gives}$$

$$\lambda^3 - (a_{11} + a_{22} + a_{33})\,\lambda^2 + \left\{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\right\}\lambda - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

An important *property* to note here, is **"If A is a symmetric matrix the eigen vectors of the equation $(A - \lambda I)X = 0$ are pairwise orthogonal"** Thus, if $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$ are two eigen vectors of a symmetric matrix A then $x_1x_1' + x_2x_2' + x_3x_3' + \ldots + x_nx_n' = 0$.

The sum of the eigen values of a matrix is called *Trace*. The product of the eigen values of a matrix is equal to determinant of the matrix. [*GGSIPU I Sem End Term January 2011*]

SOME USEFUL DEDUCTIONS

If λ is a characteristic value of a square matrix A, then

(i) $\lambda + k$ is a characteristic value of the matrix $A + kI$

(ii) $k\lambda$ is a characteristic value of kA

(iii) $\frac{1}{\lambda}$ is a characteristic value of A^{-1}. In otherwords $\frac{|A|}{\lambda}$ is the eigen value of the matrix adj. (A). [GGSIPU I Ist Sem End Term 2009; I Sem End Term 2011]

(iv) λ^2 is a characteristic value of A^2. [GGSIPU I Sem II Term 2012]

In general, λ^k is an eigen value of A^k.

Let us deduce the above results.

Since λ is a eigen value of A we have $(A - \lambda I)X = 0$ ... (I)

(i) We can write (1) as $(A + kI - kI - \lambda I)X = 0$ or $[A + kI - (\lambda + k)]X = 0$ which establishes that $A + kI$ has eigen value $\lambda + k$.

(ii) Multiplying (*i*) by k we get $k(A - \lambda I)X = 0$ or $[kA - (\lambda k)I]X = 0$ which establishes that the matrix kA has eigen value as λk.

(iii) Premultiplying (1) by A^{-1}, we get

$$\{A^{-1}(A - \lambda I)\}X = 0 \text{ or } (I - \lambda A^{-1})X = 0$$

multiplying throughout by λ^{-1}, we get

$$(\lambda^{-1} I - A^{-1})X = 0 \quad \text{or} \quad (A^{-1} - \lambda^{-1} I)X = 0$$

which establishes that $\frac{1}{\lambda}$ is eigen value of A^{-1}.

(iv) Premultiplying (1) by A, we get $\{A(A - \lambda I)\}X = 0$ or $(A^2 - \lambda A)X = 0$

or $(A^2 - \lambda(A - \lambda I) - \lambda^2 I)X = 0$ or $(A^2 - \lambda^2 I)X - \lambda(A - \lambda I)X = 0$

But $(A - \lambda I)X = 0$ hence $(A^2 - \lambda^2 I)X = 0$

which show that λ^2 is the eigen value of the matrix A^2.

The deduction (iv) can be extended to the fact that λ^k will be the eigen value of the square matrix A^k if λ is the given value of the square matrix A.

Note that eigen values and eigen vectors of $\overline{A}$, the conjugate of the matrix A, are the conjugate of the eigen values and eigen vectors of A since $AX = \lambda X$ gives $\overline{A}\overline{X} = \overline{\lambda}\overline{X}$

Also we establish following important properties:

The eigen values of

(i) a Hermitian matrix are real,

(ii) a skew-Hermitian matrix are zero or purely imaginary,

(iii) an unitary matrix are of magnitude 1.

Proof: Let λ be an eigen value and X be the corresponding eigen vector of the matrix A then $AX = \lambda X$. Premultiplying both sides by $\overline{X}'$, we get $\overline{X}'AX = \lambda\overline{X}'X$ or $\lambda = \frac{\overline{X}'AX}{\overline{X}'X}$. ...(1)

Here one can note that $\overline{X}'AX$ and $\overline{X}'X$ are scalars and $\overline{X}'X$ is always real and positive. Therefore the nature of λ is governed by the scalar $\overline{X}'AX$.

(i) Let A be Hermitian then $\overline{A} = A'$

$$\therefore \quad \overline{(\overline{X}'AX)} = X'\overline{A}\overline{X} = X'A'\overline{X} = (X'A' \cdot \overline{X})' = \overline{X}'AX.$$

Since $\overline{X}'A'X$ is a scalar, hence $\overline{X}'AX$ is real.

From equation (1) above, we conclude that λ is real.

(ii) Let A be a skew-Hermitian matrix, then $A' = -\overline{A}$.

Now $\overline{(\overline{X}'AX)} = X'\overline{A}\overline{X} = -(X'A'\overline{X})' = -\overline{X}'AX$ as $X'A'\overline{X}$ is a scalar, hence $\overline{X}'AX$ is zero or purely imaginary. From equation (1) above, we can infer that λ is zero or purely imaginary.

(iii) Let A be a unitary matrix then $\overline{A}' = (\overline{A})'$. Also since $AX = \lambda X$ we get $\overline{A}\overline{X} = \overline{\lambda}\overline{X}$.

Therefore $(\overline{A}\,\overline{X})' = (\overline{\lambda}\overline{X})'$ or $\overline{X}'\overline{A}' = \overline{\lambda}\,\overline{X}'$.

$$\Rightarrow \quad (\overline{X}'\overline{A}')AX = (\overline{\lambda}\,\overline{X}')\,(\lambda X) = |\lambda|^2\,\overline{X}X \quad \text{or} \quad \overline{X}'X \qquad = |\lambda^2|\,\overline{X}'X$$

Since $X \neq 0$ we have $\overline{X}'X \neq 0$ $\therefore$ $|\lambda^2| = 1$ or $\lambda = +1$. **Hence Proved**

Obvious conclusion from the above results is that the eigen values of

(i) a symmetric marix are real

(ii) a skew-symmetric matrix are zero or purely imaginary.

(iii) an orthogonal matrix are of magnitude 1 and are either real or complex conjugate pairs.

[illegible] **(a)** Find the sum of the characteristic roots of the matrix A^2, given that

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$$

[GGSIPU I Sem II Term 2007]

(b) If matrix A is defined as $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ then the eigen values of A^2 are

(i) $-1, -9, -4$ (ii) $1, 9, 4$ (iii) $-1, -3, 2$ (iv) $1, 3, -2$

[GGSIPU I Sem End Term 2004]

Solution: (a) The characteristic equation for the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$ is $|A - \lambda I| = 0$

$$\text{or} \quad \begin{bmatrix} 3-\lambda & 0 & 0 \\ 8 & 4-\lambda & 0 \\ 6 & 2 & 5-\lambda \end{bmatrix} = 0$$

whose roots as characteristic roots of A are $\lambda = 3, 4$ and 5.

We know that the characteristic roots of the matrix A^2 are λ^2 as 9, 16, and 25, whose sum $= 9 + 16 + 25 = 50$. **Ans.**

(b) The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{or} \quad \begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & -3-\lambda & 0 \\ 1 & 4 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)(-3-\lambda)(2-\lambda) = 0$$

Hence $\lambda = -1, \lambda = -3$ and $\lambda = 2$ are eigen values of A. We know that the eigen values of A^2 are the squares of the eigen values of A, that is 1, 9, 4.

Hence (ii) **Ans.**

EXAMPLE 8.11. If the eigen values of matrix A are 2, 3, 6 then write the eigen values of A^T, A^{-1}, A^m $(m \in N)$, adj A and of kA. [GGSIPU I Sem II Term 2012]

SOLUTION: Eigen values of A are 2, 3 and 6 and since eigen values of A^T are same as these of A, hence eigen values of A^T are 2, 3 and 6.

Next, the eigen values of A^{-1} are 1/2, 1/3 and 1/6.

and eigen values of A^m are $2^m, 3^m$ and 6^m.

Further, the eigen values of adj (A) are $\frac{1}{2}(2 \cdot 3 \cdot 6)$, $\frac{1}{3}(2 \cdot 3 \cdot 6)$ and $\frac{1}{6}(2 \cdot 3 \cdot 6)$, that is, are 18, 12 and 6.

Also, the eigen values of kA are $2k, 3k$ and $6k$ where k is constant. **Ans.**

EXAMPLE 8.12. Find the eigen values and eigen vectors of the matrix

(i) $A = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}$. (ii) $\begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$. [GGSIPU I Sem II Term 2011]

SOLUTION: (i) The characteristic equation of the given matrix A, is $|A - \lambda I| = 0$

or $\begin{vmatrix} 4-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda - 3 = 0$ which gives $\lambda = -1$ and 3.

Therefore, the eigen values of A are $\lambda = -1$, $\lambda = 3$.

Now, when $\lambda = -1$, the equation $(A - \lambda I)X = 0$ is equivalent to the system of linear equations

$$5x_1 - 5x_2 = 0 \quad \text{and} \quad x_1 - x_2 = 0.$$

The obvious solution of this system is the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Of course, this vector multiplied by any scalar is also a solution. Thus, for any number $c_1 \neq 0$, $X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = -1$.

Next, when $\lambda = 3$ the equation $(A - \lambda I)X = 0$ is equivalent to one equation only $x_1 - 5x_2 = 0$. which has $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ as obvious solution. In general for any non-zero number c_2, $X = c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is the other eigen vector of A corresponding to $\lambda = 3$.

(ii) $A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$. Characteristic equation is $|A - \lambda I| = 0$

or $\begin{vmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = 0$ or $\lambda^2 - 7\lambda + 12 = 0 \Rightarrow \lambda = 3, 4$ are eigen values of A.

For eigen vectors we have $(A - \lambda I)X = 0$.

When $\lambda = 3$ we have $-2x + 3y = 0$ $\therefore$ eigen vector is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

and when $\lambda = 4$ we have $-3x + 3y = 0$ or $y = x$ $\therefore$ eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. **Ans.**

EXAMPLE 8.13. Find the eigen values and eigen vectors of the matrix

$$A=\begin{bmatrix}2 & 1 & 1\\ 2 & 3 & 2\\ 3 & 3 & 4\end{bmatrix}.$$

SOLUTION: The characteristic equation $|A-\lambda I|=0$ is $\begin{vmatrix}2-\lambda & 1 & 1\\ 2 & 3-\lambda & 2\\ 3 & 3 & 4-\lambda\end{vmatrix}=0$

which on expansion gives $\lambda^3-9\lambda^2+15\lambda-7=0$ or $(\lambda-1)(\lambda-1)(\lambda-7)=0$.

$\therefore$ The eigen values of A are $\lambda_1=1,\ \lambda_2=1,\ \lambda_3=7,$ here $\lambda_1=\lambda_2$.

Now the eigen vectors are given by $(A-\lambda I)X=0$. ... (1)

Let us first take-up the non-repeated eigen value $\lambda_3=7$.

Equation (1) gives $-5x_1+x_2+x_3=0 \quad 2x_1-4x_2+2x_3=0 \quad 3x_1+3x_2-3x_3=0$.

From the last two equations, we get $\dfrac{x_1}{1}=\dfrac{x_2}{2}=\dfrac{x_3}{3}$

Therefore, the eigen vector of A for $\lambda=7$, is $X_1=k_1\begin{bmatrix}1\\2\\3\end{bmatrix}(k_1\neq 0)$

Next, for $\lambda_1=\lambda_2=1$ the equation (1), gives only one independent equation as

$$x_1+x_2+x_3=0$$

Since the matrix A is not symmetric we can get two linearly independent vectors X_2 and X_3 by taking $x_1=0$ and $x_2=0$ respectively. When $x_1=0$ we have $x_2+x_3=0$

On setting $x_2=k_2$ we have $x_3=-k_2$ and hence one eigen vector of A, for $\lambda=1$ is

$$X_2=k_2\begin{bmatrix}0\\1\\-1\end{bmatrix}(k_2\neq 0).$$

When we take $x_2=0$ we have $x_1+x_3=0$ and on setting $x_3=-k_3$ we have $x_1=k_3$ hence the other eigen vector of A for $\lambda=1$, is $X_3=k_3\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

Thus, the eigen vectors of the given matrix A are

$$X_1=\begin{bmatrix}1\\2\\3\end{bmatrix}, X_2=\begin{bmatrix}0\\1\\-1\end{bmatrix}, X_3=\begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

It can be easily shown that X_1, X_2, X_3 are linearly independent.

EXAMPLE 8.14. Find the eigen values and eigen vectors of the matrix

$$\text{(i)} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$ [GGSIPU I Sem End Term 2012, End Term 2013]

SOLUTION: (i) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$ then the characteristic equation is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 2 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$$

or $(2-\lambda)[(3-\lambda)(2-\lambda) - 2] - (2 - \lambda + 2) + (1 + 3 - \lambda) = 0$

or $(2-\lambda)(4-\lambda)(1-\lambda) = 0$ hence $\lambda = 1, 2, 4$ are the eigen values of the given matrix.

The eigen vectors are given by $(A - \lambda I)X = 0$

or $$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 2 \\ -1 & 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

For $\lambda = 1$ we have $x + y + z = 0,\ x + 2y + 2z = 0, -x + y + z = 0$

which give $x = 0, y = 1, z = -1$, hence $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is the eigen vector.

For $\lambda = 2$, we have $0x + y + z = 0,\ x + y + 2z = 0,\ -x + y = 0.$

These give $y + z = 0$ and $y = x$ hence eigen vector is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

For $\lambda = 4$, we have $-2x + y + z = 0,\ x - y + 2z = 0, -x + y - 2z = 0$

or $-2x + y + z = 0$ and $x - y + 2z = 0 \Rightarrow \dfrac{x}{3} = \dfrac{y}{5} = \dfrac{z}{1}$ $\therefore$ eigen vector is $\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$ **Ans.**

(ii) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. The characteristic equation is $|A - \lambda I| = 0$

or $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$ or $(1-\lambda)[(2-\lambda)^2 - 1] = 0$

$\therefore$ Eigen values of A are $\lambda = 1, 1, 3.$

Eigen vectors for $\lambda = 3$ are given by $(A - \lambda I)X = 0$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

or $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ that is, $-x_1 + x_2 + x_3 = 0$, $x_1 - x_2 + x_3 = 0$, $-2x_3 = 0$

Hence the eigen vector for $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Next, the eigen vectors of A for $\lambda = 1$ are given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{or} \quad x_1 + x_2 + x_3 = 0$$

Hence, the eigen vectors for $\lambda = 1$ are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. **Ans.**

EXAMPLE 8.15. Determine the characteristic roots and the corresponding characteristic vectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

[*GGSIPU I Sem End Term 2007*]

SOLUTION: The characteristic equation is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad (2-\lambda)^3 = 0$$

Hence the characteristic roots are $\lambda = 2, 2, 2$.

For characteristic vectors we write $(A - \lambda I)X = 0$

$$\Rightarrow \quad \begin{aligned} (2-\lambda)x + y &= 0 \\ (2-\lambda)y + z &= 0 \\ (2-\lambda)z &= 0 \end{aligned}$$

On taking $\lambda = 2$ we have $y = 0, z = 0$

Thus the eigen vector can be taken as $x = t, y = 0, z = 0$ where t is arbitrary.

That is $\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$. Thus three characteristic vectors can be obtained by taking three different values of t as (1, 0, 0), (2, 0, 0), (3, 0, 0) as illustration. **Ans.**

EXAMPLE 3.16. Obtain the eigen values and eigen vectors for the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$ and verify that the eigen vectors are orthogonal.

[GGSIPU I Sem End Term 2004; End Term 2011]

SOLUTION: The characteristic equation for the matrix A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad (2-\lambda)^3 - (2-\lambda) = 0$$

or $$(2-\lambda)(2-\lambda+1)(2-\lambda-1) = 0 \quad \text{or} \quad (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

Hence the eigen values of A are $\lambda = 1, \ 2, \ 3.$ **Ans.**

Now for eigen vectors X of A, we have $(A - \lambda I)X = 0$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

or $$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

or $$(2-\lambda)x + z = 0, \quad (2-\lambda)y = 0, \quad x + (2-\lambda)z = 0$$

Now, when $\lambda = 1$ we have $x + z = 0$, $y = 0$, that is if $x = 1$, $z = -1$, $y = 0$

and the eigen vector for $\lambda = 1$, is $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Next, when $\lambda = 2$ we have $z = 0, x = 0, y = y$

and the eigen vector for $\lambda = 2$ is $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Next, when $\lambda = 3$ we have $-x + z = 0$, $y = 0$ and $x - z = 0$.

Thus if we take $x = 1$ then $z = 1, y = 0$, and the eigen vector for $\lambda = 3$ is $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Now, for orthogonality of eigen vectors

$$X_1 \cdot X_2 = 1.0 + 0.1 + (-1)\,0 = 0$$
$$X_1 \cdot X_3 = 1.1 + 0.0 + (-1)\,1 = 0$$

and $$X_2 \cdot X_3 = 0.1 + 1.0 + 0.1 = 0$$

Therefore X_1. X_2. X_3 are orthogonal to each other.

Hence Proved.

EXAMPLE 8.17. Find the eigen values and eigen vectors of the matrix

$$A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ [GGSIPU Ist Sem. IInd Term 2007]

and test for the orthogonality of the vectors.

SOLUTION: The characteristic equation of A is $|A-\lambda I|=0$

or $$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}=0 \quad \text{or} \quad \lambda^3-18\lambda^2+45\lambda=0 \quad \text{or} \quad \lambda(\lambda-3)(\lambda-15)=0$$

Thus, the eigen values of A are $\lambda_1 = 0,\ \lambda_2 = 3,\ \lambda_3 = 15$.

The eigen vector X_1 of A corresponding to $\lambda_1 = 0$ is the solution of the system of equations $(A-\lambda_1 I)X = 0$, where $X=[x_1 \quad x_2 \quad x_3]$.

that is,
$$8x_1-6x_2+2x_3 = 0$$
$$-6x_1+7x_2-4x_3 = 0$$
$$2x_1-4x_2+3x_3 = 0$$

Eliminating x_1 from the last two equations, gives $-5x_2+5x_3=0$ or $x_2=x_3$

Setting $x_3=k_1$, we get $x_2=k_1$ and $x_1=\dfrac{k_1}{2}$.

$\therefore$ The eigen vector of A corresponding to $\lambda=0$, is $X_1=k_1\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}=k_1\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $(k_1\neq 0)$

Similarly, for $\lambda = 3$ we get $X_2 = k_2\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ $(k_2 \neq 0)$

and for $\lambda=15$, we get $X_3=k_3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ $(k_3\neq 0)$.

Further, since A is symmetric matrix, the eigen vectors X_1, X_2, X_3 should be mutually orthogonal.

$$X_1.X_2 = (1)(2)+(2)(1)+(2)(-2)=0$$
$$X_2.X_3 = (2)(2)+(1)(-2)+(-2)(1)=0$$
$$X_3.X_1 = (2)(1)+(-2)(2)+(1)(2)=0.$$

Hence verified.

CALEY HAMILTON THEOREM

Every square matrix satisfies its own characteristic equation. [GGSIPU I Sem II Term 2011]

PROOF: For a square matrix A of order n the characteristic equation is $|A - \lambda I| = 0$

or $(-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n] = 0$

Replacing λ by A, gives $A^n + a_1A^{n-1} + a_2 A^{n-2} + ... + a_n I = 0.$ ...(1)

We know that $[A - \lambda I]\ adj\ [A - \lambda I] = |A - \lambda I|\ I$ ...(2)

Since adj $(A - \lambda I)$ has elements as cofactors of the elements of $|A - \lambda I|$ and the elements of $[A - \lambda I]$ are at the most of first degree in λ. the elements of adj $(A - \lambda I)$ are polynomials in λ of degree $n - 1$ or less.

Therefore, adj $(A - \lambda I)$ can be written as a matrix polynomial in λ, as

$$adj(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + ... + B_{n-1}.$$ ...(3)

Using (1) and (3), equation (2) becomes

$$(-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n]\ I = (A - \lambda I)\ [B_0\lambda^{n-1} + B_1\lambda^{n-2} + ... + B^{n-1}]$$

Comparing the co-efficients of terms of same powers of λ on both sides, we get

$$(-1)^n I = -B_0 I$$
$$(-1)^n a_1 I = AB_0 - B_1 I$$
$$(-1)^n a_2 I = AB_1 - B_2 I$$
$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$
$$(-1)^n a_n I = AB_{n-1}$$

Premultiplying the above equations by $A^n, A^{n-1}, A^{n-2}, ..., I$ respectively, and then adding them, we get

$$(-1)^n[A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_nI] = -B_0A^n + A^nB_0 - B_1A^{n-1} + A^{n-1}B_1 - B_2A^{n-2} + ... + AB_{n-1} = 0$$

$\Rightarrow$ $A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_nI = 0,$ hence the theorem.

The theorem can be used to find A^{-1} as given below :

Premultiplying the above equation by A^{-1}, gives

$$A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + ... + a_{n-1}I + a_nA^{-1} = 0$$

$$\therefore \quad A^{-1} = \frac{-1}{a_n}[A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + ... + a_{n-1}I].$$

EXAMPLE 8.18. (*a*) Use Caley-Hamilton theorem to express

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A where $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

[GGSIPU I Sem II Term 2011]

(*b*) Using Caley Hamilton theorem, find A^8, if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

[GGSIPU Ist Sem End Term 2009]

SOLUTION: (*a*) Characteristic equation of A is $|A - \lambda I| = 0$

or $\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$ or $\lambda^2 - 4\lambda - 5 = 0$ ... (1)

By Caley-Hamilton theorem, A satisfies (1) hence $A^2 - 4A - 5I = 0$. ... (2)

Therefore the given expression $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$$= A^3(A^2 - 4A - 5I) - 2A^3 + 11A^2 - A - 10I = A^3(O) - 2A^3 + 11A^2 - A - 10I$$
$$= -2A(A^2 - 4A - 5I) + 3A^2 - 11A - 10I = -2A(O) + 3A^2 - 11A - 10I$$
$$= 3(A^2 - 4A - 5I) + A + 5I = O + A + 5I$$
$$= \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}.$$

(*b*) $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. Hence the characteristic equation is $\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

or $-(1 - \lambda^2) - 4 = 0$ or $\lambda^2 - 5 = 0$.

By Caley-Hamilton theorem we have $A^2 - 5I = 0$ or $A^2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$\therefore$ $A^4 = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$ and $A^8 = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$. **Ans.**

EXAMPLE 8.19. (*a*) Verify Caley-Hamilton theorem for the matrix

$$\text{(i) } A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

[GGSIPU I Sem End Term 2011, II Term 2012]

SOLUTION: (i) Characteristic equation of the matrix A is $|A - \lambda I| = 0$.

or $\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$ or $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

By Caley-Hamilton theorem, we get $A^3 - 6A^2 + 9A - 4I = 0$.

Here $A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$ and hence $A^3 = A^2 A = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$

$\therefore$ $A^3 - 6A^2 + 9A - 4I = 0$. **Hence Proved.**

(ii) Characteristic equation is $|A - \lambda I| = 0$

or $\begin{vmatrix} 3-\lambda & 1 & -1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$ or $\lambda^3 - 11\lambda^2 + 40\lambda - 48 = 0$

By Caley Hamilton theorem $A^3 - 11A^2 + 40A - 48I = 0$.

$$A^2 = \begin{bmatrix} 7 & 9 & -7 \\ -9 & 25 & -7 \\ 7 & -7 & 9 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 5 & 59 & -37 \\ -59 & 123 & -37 \\ 37 & -37 & 27 \end{bmatrix}$$

$\therefore \quad A^3 - 11A^2 + 40A - 48I = 0$

Hence proved

EXAMPLE 8.20. Verify Caley Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

and use it to find A^{-1} and A^4.

SOLUTION: The characteristic equation of A is $|A - \lambda I| = 0$

or $\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$ or $\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$.

This means, we are to verify $A^3 - 5A^2 + 9A - I = 0$ ...(1)

Now $\quad A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$

and $\quad A^3 = A^2A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$.

$$\therefore \quad A^3 - 5A^2 + 9A - I = \begin{bmatrix} -13+5+9-1 & 42-60+18-0 & -2+20-18-0 \\ -11+20-9-0 & 9-35+27-1 & 10-10+0-0 \\ 10-10+0-0 & -22+40-18-0 & -3-5+9-1 \end{bmatrix} = 0$$

Hence the theorem is verified.

Now, multiplying (1) throughout by, A^{-1}, we get $A^2 - 5A + 9I - A^{-1} = 0$

$$\therefore \quad A^{-1} = A^2 - 5A + 9I = \begin{bmatrix} -1-5+9 & 12-10+0 & -4+10+0 \\ -4+5+0 & 7-15+9 & 2+0+0 \\ 2+0+0 & -8+10+0 & 1-5+9 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Next, to find A^4 we multiply (1) by A and get $A^4 - 5A^3 + 9A^2 - A = 0$

$$\therefore \quad A^4 = 5A^3 - 9A^2 + A = \begin{bmatrix} -65+9+1 & 210-108+2 & -10+36-2 \\ -55+36-1 & 45-63+3 & 50-18+0 \\ 50-18+0 & -110+72-2 & -15-9+1 \end{bmatrix} = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{bmatrix} \text{ Ans.}$$

EXAMPLE 8.22. Verify Caley-Hamiltion theorem for the matrix following matrices and find their inverse.

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

[GGSIPU I Sem End Term 2007, End Term 2013]

SOLUTION: $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. The characteristic equation is $|A - \lambda I| = 0$

or $\begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$ or $(1-\lambda)(\lambda^2 - 2\lambda - 3) - 2\lambda + 6 = 0$

or $\lambda^3 - 3\lambda^2 + \lambda - 3 = 0$

By Caley-Hamilton theorem, we have $A^3 - 3A^2 + A - 3I = 0$ ...(i)

Now $A^2 = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$ and $A^3 = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$

$\therefore$ $A^3 - 3A^2 + A = 3I = 0$ hence Caley-Hamilton theorem is verified.

Next, multiplying (i) throughout by A^{-1}, we get

$A^2 - 3A + I - 3A^{-1} = 0$

or $$A^{-1} = \frac{1}{3}(A^2 - 3A + I) = \frac{1}{3}\left[\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} + \begin{bmatrix} -3 & -6 & 0 \\ 3 & -3 & -6 \\ -3 & -6 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix} \quad \textbf{Ans.}$$

EXAMPLE 8.23. Find the characteristic equation of the matrix

$A = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$. Hence find A^{-1}. [GGSIPU I Sem II Term 2004]

SOLUTION: The characteristic eqaution of A is $|A - \lambda I| = 0$

or $\begin{vmatrix} 1-\lambda & -1 & 1 \\ 4 & 1-\lambda & 0 \\ 8 & 1 & 1-\lambda \end{vmatrix} = 0$ or $(1-\lambda)^3 + 4 - 4\lambda + 4 - 8 + 8\lambda = 0$

or $\lambda^3 - 3\lambda^2 - \lambda - 1 = 0$ is the required characteristic equation. ...(1)

Now, by Caley-Hamiton theorem, A satisfies (1) hence $A^3 - 3A^2 - A - I = 0$

Multiplying throughout by A^{-1}, we get $A^2 - 3A - I = A^{-1}$.

Now $$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 8 & -3 & 4 \\ 20 & -6 & 9 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 5 & -1 & 2 \\ 8 & -3 & 4 \\ 20 & -6 & 9 \end{bmatrix} + \begin{bmatrix} -3 & 3 & -3 \\ -12 & -3 & 0 \\ -24 & -3 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix} \qquad \textbf{Ans.}$$

DIAGONALISING A SQUARE MATRIX

A square matrix A of order $n \times n$, is called diagonalizable if there exists a matrix P of order $n \times n$ such that $P^{-1}AP$ is a diagonal matrix. When such a matrix P exists, we say that P diagonalizes A.

THEOREM: For any square matrix A of order n having linearly independent eigen vectors, there exists a matrix P such that $P^{-1}AP$ is a diagonal matrix.

PROOF: For convenience, we choose A as a 3×3 square matrix. Let its eigen values be $\lambda_1, \lambda_2, \lambda_3$ and corresponding eigen vectors be

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}, \text{ so that } AX_1 = \lambda_1 X_1,\ AX_2 = \lambda_2 X_2,\ AX_3 = \lambda_3 X_3.$$

Now consider the square matrix

$$[X_1\ X_2\ X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = P \text{ say, then we have}$$

$$A.P = A[X_1 X_2 X_3] = [AX_1, AX_2, AX_3] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$= PD \quad \text{where D is the diagonal matrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\Rightarrow P^{-1}(AP) = P^{-1}(PD) = (P^{-1}P)D = ID = D.$$

The matrix P which diagonalizes the given matrix A, is called the **MODAL MATRIX** and the resulting diagonal matrix D is called the **SPECTRAL MATRIX** of A. The modal matrix P is found by grouping the eigen vectors of A into square matrix and diagonal matrix has the eigen values of A as its elements.

SIMILAR MATRICES

Let A and B be square matrices of same order, the matrix A is said to be similar to B if there exists an invertible matrix P such that $A = P^{-1}BP$ or $PA = BP$.

Premultiplying by P^{-1} on both sides of the above relation, we get

$$(PA)\ P^{-1} = (BP)\ P^{-1} \quad \text{or} \quad PAP^{-1} = B$$

Therefore, A is similar to B if and only if B is similar to A. Such a matrix P is called the **similarity** matrix.

A deduction: Similar matrices have the same characteristic equation and hence same eigen values. However, the converse of it is not true. Thus, two matrices which have same characteristic equation need not always be similar.

Also, if A is similar to B and B is similar to C then A is similar to C.

For example, consider two matrices $A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$.

A and B will be similar if there exists an invertible matrix P such that

$$A = P^{-1}\ BP \quad \text{or} \quad PA = BP$$

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We are to find a, b, c, d such that $PA = BP$ and then check whether P is non-singular.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

or

$$\begin{bmatrix} 5a-2b & 5a \\ 5c-2d & 5c \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -3a+4c & -3b+4d \end{bmatrix}$$

$\Rightarrow$

$$\begin{aligned} 5a - 2b &= a + 2c \quad &\text{or} \quad 4a - 2b - 2c &= 0 \\ 5a &= b + 2d \quad &\text{or} \quad 5a - b - 2d &= 0 \\ 5c - 2d &= -3a + 4c \quad &\text{or} \quad 3a + c - 2d &= 0 \\ 5c &= -3b + 4d \quad &\text{or} \quad 3b + 5c - 4d &= 0 \end{aligned}$$

whose solution is $a = 1,\ b = 1,\ c = 1,\ d = 2$.

Therefore, $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ which is non-singular, hence A and B are similar matrices.

EXAMPLE 8.24 If possible, diagonalize $A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}_{2\times 2}$.

2

SOLUTION: Characteristic equation of A is $|A - \lambda I| = 0$

or $\begin{vmatrix} -1-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = -1,\ 3$ which are eigen values of A.

Then, the eigen vectors of A for $\lambda = -1$ and 3 are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

$\therefore$ The modal matrix $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Since the eigen vectors are linearly independent, this matrix is non-singular.

Next, $$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

which has eigen values down the main diagonal, corresponding to the order in which the eigen vectors were written as columns of P.

However, if we use the other order in writing the eigen vectors as columns and write

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ then we get } Q^{-1}AQ = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Note: It is not necessary that A have n distinct eigen values in order to have n linearly independent eigen vectors. See the next example.

Example 8.25. Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Find the modal matrix M such that $M^{-1}AM$ is a diagonal matrix. *[GGSIPU I Sem IInd Term 2004; End Term Jan 2011]*

Solution: The characteristic equation of matrix A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

or $(\lambda - 2)^2 (\lambda - 8) = 0$ Hence, the eigen values are 2, 2 and 8.

For eigen vectors, we have $(A - \lambda I)X = 0$

or
$$(6-\lambda)x - 2y + 2z = 0,$$
$$-2x + (3-\lambda)y - z = 0,$$
$$2x - y + (3-\lambda)z = 0.$$

Therefore, for $\lambda = 8$, we have
$$-2x - 2y + 2z = 0,$$
$$-2x - 5y - z = 0,$$
$$2x - y - 5z = 0.$$

which give $x = 2, \quad y = -1, \quad z = 1,$ hence one eigen vector is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Next, for $\lambda = 2$, we have
$$4x - 2y + 2z = 0$$
$$-2x + y - z = 0$$
$$2x - y + z = 0$$

but actually we have here only one equation as $2x - y + z = 0$

Here, let us take $x = t$ then we have $y - z = 2t$

Now for $z = 0$, $y = 2t$, and for $y = 0$, $z = -2t$

Thus the eigen vectors for $\lambda = 2$ are $\begin{bmatrix} t \\ 2t \\ 0 \end{bmatrix}$ and $\begin{bmatrix} t \\ 0 \\ -2t \end{bmatrix}$, that is, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

Therefore the modal matrix $M = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Now we calculate M^{-1}, and then get

$$M^{-1}AM = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \text{ the diagonal matrix.} \qquad \textbf{Ans.}$$

Example 8.26. Diagonalize the matrix $\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

[GGSIPU I Sem End Term 2013]

Solution: Let $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Characteristic equation is $|A - \lambda I| = 0$.

or $\begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$ or $(3 - \lambda)(\lambda^2 - 3\lambda) + 4 - 2\lambda + 2 = 0$

or $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$ which gives the eigen values of A as $\lambda = 1, 2, 3$.

For $\lambda = 1$ the eigen vector is the solution of $(A - \lambda I)X = 0$

or $2x_1 + x_2 - x_3 = 0$, $-2x_1 + 2x_3 = 0$, $x_2 + x_3 = 0$.

$\therefore$ Eigen vector for $\lambda = 1$ is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$, the eigen vector is the solution of $(A - 2I)X = 0$.

or $x_1 + x_2 - x_3 = 0$, $-2x_1 - x_2 + 2x_3 = 0$, $x_2 = 0$.

$\therefore$ The eigen vector for $\lambda = 2$ is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = 3$, the eigen vector is the solution of $(A - 3I)X = 0$

or $x_2 - x_3 = 0$, $x_1 + x_2 - x_3 = 0$, $x_2 = x_3$

$\therefore$ Eigen vector for $\lambda = 3$ is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Hence the modal matrix $M = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

and $$M^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Therefore, the diagonal matrix $D = M^{-1}AM$.

$$= \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 4 & 2 & -2 \\ -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ **Ans.**

EXAMPLE 8.27. The eigen vectors of a matrix A corresponding to the eigen values 1, 1, 3 are $[1, 0, -1]^T$, $[0, 1, -1]^T$ and $[1, 1, 0]^T$ respectively. Find the matrix A.

[GGSIPU I Sem End Term 2008; End Term 2011]

SOLUTION: The modal matrix of A is $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Here $|P| = 2$ and $\text{adj}(P) = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$ $\quad \therefore \quad P^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

We know that $P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow P(P^{-1}AP) = PD$ or $AP = PD \Rightarrow A = PDP^{-1}$

$$\therefore \quad A = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ **Ans.**

POWER OF A SQUARE MATRIX

We can use the method of diagonalization of a matrix to obtain powers of the matrix.

Let A be a square matrix then we can obtain a non-singular matrix P so that $D = P^{-1}AP$

Hence $D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP$

$= P^{-1}(AIA)P = P^{-1}A^2P$

Similarly $D^3 = P^{-1}A^3P$ and in general $D^n = P^{-1}A^nP$.

Now to find A^n we premultiply D^n by P and postmultiply by P^{-1}.

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1} = PP^{-1}A^nPP^{-1} = IA^nI = A^n$.

Therefore, $A^n = PD^nP^{-1}$ where $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$.

QUADRATIC FORMS

A quadratic form in n variables $x_1, x_2, ..., x_n$ is a homogeneous expression of the form

$Q = \sum_{i=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ where A is the coefficient matrix and the total power in each term is 2.

Expanding the above expression, we have

$$\begin{aligned} Q = {} & a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + (a_{13}+a_{31})x_1x_3 + \dots + (a_{1n}+a_{n1})x_1x_n \\ & + a_{22}x_2^2 + (a_{23}+a_{32})x_2x_3 + (a_{24}+a_{42})x_2x_4 + \dots + (a_{2n}+a_{n2})x_2x_n \\ & + \dots + a_{nn}x_n^2 \\ = {} & X^TAX \end{aligned}$$

using the definition of matrix multiplication.

Now, if we set $c_{ij} = (a_{ij} + a_{ji})/2$ then the matrix $C = (c_{ij})$ is symmetric since $c_{ij} = c_{ji}$.
Further, $c_{ij} + c_{ji} = a_{ij} + a_{ji}$. Therefore, we can write
$Q = X^TCX$, where C is a symmetric matrix and $c_{ij} = (a_{ij} + a_{ij})/2$.
In particular, for $n = 2$, we have

$$c_{11} = a_{11}, \quad c_{12} = c_{21} = (a_{12}+a_{21})/2 \quad \text{and} \quad c_{22} = a_{22}.$$

For example, consider the quadratic form $Q = 2x_1^2 + 3x_1x_2 + x_2^2$.

Here, $a_{11} = 2$, $a_{12} + a_{21} = 3$ and $a_{22} = 1$, let us obtain the symmetric matrix C for the above quadratic form.

$\therefore \quad c_{11} = a_{11} = 2, \quad c_{12} = c_{21} = \frac{1}{2}(a_{12}+a_{21}) = \frac{3}{2}$ and $c_{22} = a_{22} = 1$

$\therefore \quad c = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}$.

CANONICAL FORM (OR STANDARD FORM)

A quadratic form in which mixed product terms are missing and only squared terms are there, is called **canonical form** or sometimes called **standard form.**

REDUCTION OF QUADRATIC FORM TO CANONICAL FORM: Let A be a $n \times n$ real symmetric matrix having eigen values $\lambda_1, \lambda_2, ..., \lambda_n$ and P be an orthogonal matrix that diagonalizes A, so that $P^T AP = D$ where D is a diagonal matrix with the eigen values of A as the leading diagonal elements. Then the change of variable X = PY, involving the column vectors $X = [x_1, x_2, ..., x_n]^T$ and $Y = [y_1, y_2, ..., y_n]^T$ transforms the real quadratic form $Q = X^Y AX$ into the standard form $\lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2$.

Proof: Since P is an orthogonal matrix, $P^T AP = D$.

Putting $X = PY$ in $Q = X^T AX$, we get

$$X^T AX = (PY)^T A(PY) = Y^T P^T A PY = Y^T DY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2.$$

EXAMPLE 8.28. Reduce the quadratic form $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2$ to the canonical form.

[*GGSIPU I Sem End Term 2013*]

SOLUTION: Here the coefficient matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

The eigen values of A are $\lambda = 2, 3, 6$. Here $Q = X^T AX$.

So the standard form of Q is $P = 2y_1^2 + 3y_2^2 + 6y_3^2$.

For $\lambda = 2, 3, 6$ the eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

$\therefore$ Normalized modal matrix $M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ so that $M^{-1} AM = D$.

Therefore the canonical form $= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 2y_1^2 + 3y_2^2 + 6y_3^2$ **Ans.**

EXERCISE 8B

1. Determine the eigen values and eigen vectors of the matrix $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$.

2. Show that any square matrix A and its transpose A' have same characteristic equation. [GGSIP I Sem II Term 201.]

3. If λ is an eigen value of an orthogonal matrix then show that $1/\lambda$ is also its eigen value.

4. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ 16 & 4 & 11 \end{bmatrix}$

5. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

6. Verify Caley-Hamilton theorem for the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ and hence find A^{-1}.

7. Verify Caley-Hamilton theorem for the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and use it to simplify the expression

$$A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I.$$

8. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non-singular matrices P and Q such that $PAQ = I$ and hence find A^{-1}.

[GGSIP I Sem II Term 2009]

9. Find the inverse of the matrix $A = \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ using Caley Hamilton theorem.

10. Verify Caley-Hamilton theorem and find the inverse of the matrix $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

11. Verify Caley-Hamilton theorem for the matrix $A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. Also, obtain A^{-1} and A^3.

[GGSIPU I Sem End Term 2008]

12. Find the eigen values and eigen vectors of the matrix A given by $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. Also, diagonalise A.

13. Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ can be reduced to the diagonal form with diagonal elements as eigen values of A.

[GGSIP I Sem II Term 2012]

14. (a) Diagonalise the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

[GGSIP I Sem II Term 2013]

(b) Diagonalize the matrix $A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$

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CHAPTER

9

Ordinary Differential Equations of Order One, Linear, Reducible to Linear, Exact and Reducible to Exact Form

Ordinary Differential Equations of First Order: Linear Differential Equation, Leibnitz and Bernoulis Equation. Exact and Reducible to Exact Form. Applications to Physical Problem.

INTRODUCTION AND DEFINITIONS

An equation involving differential coefficients and independent and dependent variables is called a *differential equation.* If a differential equation involves ordinary differential coefficients only, it is called *ordinary differential equation* and if it involves partial derivatives it is called *partial differential equation.* For example,

$$\frac{dy}{dx}+\sin x=0 \quad ...(1) \qquad y=x\frac{dy}{dx}+\frac{K}{\frac{dy}{dx}} \quad ...(2)$$

$$\frac{d^2y}{dx^2}+\frac{7dy}{dx}+16y=0 \quad ...(3) \qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2x-7y \quad ...(4)$$

$$\left.\begin{aligned}\frac{dx}{dt}+5x-2y=2t\\ \frac{dy}{dt}+2x+5y=e^{-t}\end{aligned}\right\} \quad ...(5)$$

are all differential equations out of which (4) is a partial differential equation and all others are ordinary differential equations.

Order and degree of a differential equation

The order of a differential equation is the order of the highest order derivative occurring in the differential equation, Thus (3) is a second order differential equation while (1), (2) and (5) are of first order. *The degree* of a differential equation is the degree of the highest order derivative when the differential coefficients are rational and free from fractions. Thus, (1) and (3) are of first degree while (2) is of second degree.

FORMATION OF A DIFFERENTIAL EQUATION

If a relation between certain variables involves arbitrary constant(s) then a differential equation can be formed on eliminating the arbitrary constants.

For instance, consider a relation

$$x = Ae^{3t}+Be^{-5t} \quad \text{where A and B are arbitrary constants.} \quad ...(1)$$

To eliminate the arbitrary constants we need two more equations which can be easily obtained by differentiating the above relation twice, that is ,

$$\frac{dx}{dt} = 3Ae^{3t} - 5Be^{-5t} \quad ...(2)$$

and $$\frac{d^2x}{dt^2} = 9Ae^{3t} + 25Be^{-5t} \quad ...(3)$$

Eliminating A and B amongst (1), (2) and (3), we get $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 15x = 0$ ...(4)

The relation (1) is called the solution of the differential equation (4). Similarly, the elimination of three arbitrary constants will give rise to third order differential equation and so on.

EXAMPLE 9.1. Form the differential equation from the equation $y = e^x$ (A cos x + B sin x).

SOLUTION: Here $y = e^x (A\cos x + B\sin x)$ ...(1)

where A and B are parameters to be eliminated. Differentiating (1) w. r. t. x, we get

$$\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$$

or $$\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + y \quad ...(2)$$

Again differentiating (2) on both sides $w.r.t. x$, gives

$$\frac{d^2y}{dx^2} = e^x(-A\cos x - B\sin x) + e^x(-A\sin x + B\cos x) + \frac{dy}{dx}.$$

or $$\frac{d^2y}{dx^2} = -y + e^x(-A\sin x + B\cos x) + \frac{dy}{dx} \quad ...(3)$$

Subtracting (2) from (3), gives

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = -y + \frac{dy}{dx} - y \quad \text{or} \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which is the required differential equation. **Ans.**

EXAMPLE 9.2. Form the differential equation of all circles of radius 'r'.

SOLUTION: Equation of any circle with radius 'r' is $(x-h)^2 + (y-k)^2 = r^2$ ...(1)

where h and k, the coordinates of the centre of the circle, are parameters. Differentiating (1) w.r.t. x on both sides, we get

$$2(x-h) + 2(y-k)\frac{dy}{dx} = 0 \quad \text{or} \quad (x-h) + (y-k)\frac{dy}{dx} = 0 \quad ...(2)$$

Differentiating (2) $w.r.t.$ x, gives

$$1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad \text{or} \quad y - k = \frac{-(1+y'^2)}{y''} \quad ...(3)$$

Using (4) in (2), we get $$x - h = \frac{y'(1+y'^2)}{y''} \quad ...(4)$$

Substituting for $(x - h)$ and $(y - k)$ from (4) and (3) in (1), gives

$$\frac{y'^2(1+y'^2)^2}{y''^2} + \frac{(1+y'^2)^2}{y''^2} = r^2$$

or $\quad (1 + y'^2)^3 = r^2 y''^2 \quad$ or $\quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$ **Ans.**

is the required differential equation of second order and second degree.

Solution of a Differential Equation

Any relation between the dependent and the independent variables not involving the derivatives, which satisfies a differential equation is called *a solution* of the differential equation.

The solution which involves as many arbitrary constants as the order of the differential equation is called the *general solution*. Thus, $x = Ae^{3t} + Be^{-5t}$ is the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 15x = 0.$$

The solution in which arbitrary constants are given specific numerical values, is called a *particular solution*. Thus, in the above relation if we take $A = 3$ and $B = 4$, we have

$$x = 3e^{3t} + 4e^{-5t}$$

which satisfies the above differential equation and so is a particular solution of it.

There is yet another type of solution called a *singular solution*. The singular solution is a relation between the variables, which has no arbitrary constants and still satisfies the differential equation. A singular solution can not be obtained from the general solution by giving particular numerical values to the arbitrary constants. However, in engineering problems the singular solution will not be of much interest.

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

Simplest type of differential equation is the equation of first order and first degree. Such an equation can be written as $\frac{dy}{dx} = f(x, y)$ or $M(x, y)dx + N(x, y)dy = 0$.

All the differential equations of first order and first degree cannot be solved in the closed form. Only those which belong to one of the following categories can be solved by standard methods :

1. Variable separable form
2. Homogeneous equations
3. Reducible to homogeneous form
4. Linear equations
5. Reducible to linear form
6. Exact differential equations
7. Reducible to exact form
8. Method of substitution to reduce the equation to one of the above forms.

VARIABLE SEPARABLE FORM

It is of the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ where $g(y) \neq 0$ ...(1)

and can be written as $g(y)\,dy = f(x)dx$.

Integrating both sides, we get $\int g(y)\,dy = \int f(x)\,dx + c$.

For example, consider the equation $\frac{dy}{dx} = (y^2)e^{-x}$ which is of variable separable form.

We can write $\frac{1}{y^2}dy = e^{-x}dx$. This on integration, gives $-\frac{1}{y} = -e^{-x} + k$ which implicitly defines the general solution. Here, we could explicitly solve for y and get the general solution as

$$y = \frac{1}{e^{-x} - k} \quad \text{where } k \text{ is an arbitrary constant.}$$

Let us recall, here that we required that $y \neq 0$ in order to separate the variables by dividing by y^2. Actually, $y = 0$ is a solution of $y' = y^2e^{-x}$ although it cannot be from the general solution by any choice of k. Because of this, $y(x) = 0$ is called a **singular solution** of this given equation.

Note here that whenever we use the separation of variables we must be alert to the solutions potentially lost through conditions imposed by the algebra used to make the separation of variables.

EXAMPLE 9.3. (*a*) Solve $x^2\frac{dy}{dx} = 1 + y$.

(*b*) $\frac{dy}{dx} = \frac{x-y+3}{2(x-y)+5}$

[*GGSIPU Ist Sem End Term 2009*]

SOLUTION: (*a*) Given equation is separable and can be written as $\frac{1}{1+y}dy = \frac{dx}{x^2}$, where $x \neq 0$ and $y \neq -1$. Even if we put $x = 0$ and $y = -1$ in the given equation we get $0 = 0$.

Integrating the separated equation, we obtain

$$\ln|1 + y| = -\frac{1}{x} + k$$

This is implicit general solution. Explicitly we can write

$$|1 + y| = e^k\, e^{-1/x} = A\, e^{-1/x}, \text{ where } A = e^k.$$

Since k could be any number, A is always positive.

Thus, $1 + y = \pm A\, e^{-1/x}$ where $A \neq 0$. **Ans.**

(*b*) In the given equation we put $x - y = t$ so that

$$1 - \frac{dy}{dx} = \frac{dt}{dx} \qquad \therefore \quad \frac{dt}{dx} = 1 - \frac{t+3}{2t+5} = \frac{t+2}{2t+5}$$

$$\Rightarrow \quad x + c = \int \frac{2t+5}{t+2}\,dt = \int\left(2 + \frac{1}{t+2}\right)dt = 2t + \log(t+2)$$

or $\quad x + c = 2(x - y) + \log(x - y + 2) \quad$ or $\quad 2y - x + c = \log(x - y + 2)$ **Ans.**

EXAMPLE 9.4. (a) Solve $xy(1+y)\,dy - (1-x^2)(1-y)\,dx = 0$.

(b) Solve $(x+y+1)\dfrac{dy}{dx} = 1$.

SOLUTION: (a) The given equation can be written as $\dfrac{y(1+y)dy}{1-y} = \dfrac{1-x^2}{x}dx$.

Integrating both sides, gives $\displaystyle\int \frac{y(1+y)dy}{1-y} = \int\left(\frac{1}{x} - x\right)dx$

Substituting $y = t+1$ in the integral on the left, we get

$$\int \frac{(t+1)(t+2)\,dt}{-t} = \log x - \frac{x^2}{2} + c \quad \text{or} \quad -\int\left(t+3+\frac{2}{t}\right)dt = \log x - \frac{x^2}{2} + c$$

or $$-\frac{t^2}{2} - 3t - 2\log t = \log x - \frac{x^2}{2} + c$$

or $$\log x + 2\log(y-1) = \frac{x^2}{2} - \frac{1}{2}(y-1)^2 - 3(y-1) - c$$

where c is an arbitrary constant of integration. **Ans.**

(b) A simple substitution will reduce the given differential equation into variable separable form. Let us put $x+y+1 = t$ hence $1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$

and the given equation becomes $\dfrac{dt}{dx} = \dfrac{1}{t} + 1$ or $\dfrac{tdt}{t+1} = dx$

Integrating both sides, gives

$$x + c = \int \frac{t\,dt}{t+1} = \int\left(1 - \frac{1}{t+1}\right)dt = t - \log(t+1)$$

or $$x + c = (x+y+1) - \log(x+y+2)$$

or $$\log(x+y+2) = y + 1 - c = y + \log c' \quad \text{or} \quad \log\frac{x+y+2}{c'} = y$$

or $x+y+2 = c'e^y$ where c' is an arbitrary constant of integration. **Ans.**

EXAMPLE 9.5. (a) Solve the differential equation $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

[GGSIPU I Sem End Term 2007]

(b) Solve $\sqrt{1+x^2+y^2+x^2y^2} + xy\dfrac{dy}{dx} = 0$ [GGSIPU I Sem End Term 2003]

SOLUTION: (a) The equation $(e^y+1)\cos x\,dx + e^y\sin x\,dy = 0$ is of variable separable form,

$$\therefore \quad \frac{\cos x\,dx}{\sin x} + \frac{e^y\,dy}{1+e^y} = 0$$

which on integration, gives $\log\sin x + \log(1+e^y) = \log C$

or $\sin x(1+e^y) = C$ where C is arbitrary constant of integration. **Ans.**

(b) The equation is $\sqrt{(1+x^2)(1+y^2)} + xy\dfrac{dy}{dx} = 0$ which is of variable separable form.

or $$\frac{\sqrt{1+x^2}}{x}dx + \frac{ydy}{\sqrt{1+y^2}} = 0$$

Its solution is $\int \frac{\sqrt{1+x^2}}{x}dx + \int \frac{y\,dy}{\sqrt{1+y^2}} = C$ or $\int \frac{\sqrt{1+x^2}\,.\,x}{x^2}dx + \sqrt{1+y^2} = C$

In the above integral, put $1 + x^2 = t^2$ so $x\,dx = t\,dt$

$$\therefore \quad \int \frac{\sqrt{1+x^2}\,.\,x}{x^2}dx = \int \frac{t\,.\,t\,dt}{t^2-1} = \int\left(1+\frac{1}{t^2-1}\right)dt$$

$$= t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| = \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|.$$

Therefore, the solution is

$\sqrt{1+x^2} + \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) + \sqrt{1+y^2} = C$ where C is a constant of integration. **Ans.**

EXAMPLE 9.6. Solve $\frac{dy}{dx} = \frac{x(x^2+y^2-1)}{y(x^2+y^2+1)}$.

SOLUTION: The given differential equation can be written as

$$(x^2+y^2-1)x\,dx = (x^2+y^2+1)y\,dy \qquad ...(1)$$

Let us put $x^2 = u$ and $y^2 = v$ hence $2x\,dx = du$ and $2y\,dy = dv$, then (1) becomes

$$(u+v-1)\,du = (u+v+1)dv \quad \text{or} \quad \frac{dv}{du} = \frac{u+v-1}{u+v+1} \qquad ...(2)$$

Again, put $u + v = t$ hence $1 + \frac{dv}{du} = \frac{dt}{du}$, then (2) becomes $\frac{dt}{du} = \frac{t-1}{t+1} + 1 = \frac{2t}{t+1}$

$$\Rightarrow \quad \int 2du = \int \frac{t+1}{t}dt = \int\left(1+\frac{1}{t}\right)dt$$

or $\quad 2u = t + \log t + c = u + v + \log(u+v) + c \quad$ or $\quad 2x^2 = x^2 + y^2 + \log(x^2+y^2) + \log c'$

$\Rightarrow \quad \log[c'(x^2+y^2)] = x^2 - y^2 \quad$ or $\quad x^2 + y^2 = c''\, e^{(x^2-y^2)}$

where c'' is an arbitrary constant of integration. **Ans.**

EXAMPLE 9.7. (a) Solve $(1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$. [GGSIPU I Sem End Term 2004]

(b) Solve the differential equation $(xy^2 + x)\,dx + (yx^2 + y)dy = 0$ [GGSIPU I Sem End Term 2013]

SOLUTION: (a) Let us put $\frac{x}{y} = t$ or $x = ty$ so $\frac{dx}{dy} = t + y\frac{dt}{dy}$ and the given equation becomes

$$(1+e^t)\left(t + y\frac{dt}{dy}\right) + e^t(1-t) = 0$$

or $\quad y\frac{dt}{dy} = \frac{te^t - e^t}{1+e^t} - t = \frac{-(e^t+t)}{e^t+1} \quad$ or $\quad -\frac{dy}{y} = \frac{e^t+1}{e^t+t}dt$

Integrating both sides, we get

$$-\int \frac{dy}{y} = \int \frac{e^t+1}{e^t+t}dt \quad \text{or} \quad \log(e^t+t) = -\log y + \log c$$

or $\quad e^t + t = \dfrac{c}{y}$ or $e^{x/y} + \dfrac{x}{y} = \dfrac{c}{y}$ or $ye^{x/y} + x = c$

is the required general solution where c is an arbitrary constant. **Ans.**

(b) Given equation is $x(y^2 + 1)\,dx + y(x^2 + 1)\,dy = 0$.

This can be written as $\dfrac{x\,dx}{x^2+1} + \dfrac{y\,dy}{y^2+1} = 0$

which on integration gives $\ln(x^2 + 1) + \ln(y^2 + 1) = \log k$ where k is an arbitrary constant.

or $\quad (x^2 + 1)(y^2 + 1) = k$. **Ans.**

HOMOGENEOUS EQUATION

A first order differential equation if it has the form $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$ is called homogeneous.

A homogeneous equation is always transformed into a variable separable form by the substitution $y = vx$.

Then $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v\cdot 1 = f(v)$

This can be written as $\dfrac{1}{f(v)-v}\dfrac{dv}{dx} = \dfrac{1}{x}$ or $\dfrac{1}{f(v)-v}dv = \dfrac{1}{x}dx$

and the variables (now v and x) have been separated. Upon integrating this equation, we get the general solution of the transformed equation. Then replacing v by y/x gives the general solution of the original differential equation.

For example, consider $xy' = \dfrac{y^2}{x} + y$, which can be written as $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right)^2 + \dfrac{y}{x}$ if $x \neq 0$.

Let $y = vx$, then $x\dfrac{dv}{dx} + v = v^2 + v$ or $\dfrac{1}{v^2}dv = \dfrac{dx}{x}$.

This on integration, gives $-\dfrac{1}{v} = \ln|x| + C$ where C is an arbitrary constant.

Hence $v(x) = -\dfrac{1}{\ln|x| + C}$ is the general solution of the transformed equation.

and the general solution of the original equation is $y = \dfrac{-x}{\ln|x| + C}$.

EXAMPLE 9.8. (a) Solve $x^2y\,dx - (x^3 + y^3)\,dy = 0$.

[GGSIPU I Sem II Term 2010; End Term Jan 2011]

(b) Solve $\quad x^3\dfrac{dy}{dx} = y^3 + y^2\sqrt{y^2 - x^2}$.

SOLUTION: (a) The given differentail equation can be written as

$$\frac{dy}{dx} = \frac{x^2y}{x^3 + y^3} \quad \text{which is of homogeneous form.}$$

Substituting $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, the given equation becomes

$$v + x\frac{dv}{dx} = \frac{vx^3}{x^3 + v^3x^3} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{-v^4}{1+v^3} \quad \therefore \quad -\int \frac{1+v^3}{v^4}\,dv = \int \frac{dx}{x}$$

or $$-\int \frac{dv}{v^4} - \int \frac{1}{v}\,dv = \int \frac{1}{x}\,dx + \log c \quad \text{or} \quad \log(cxv) = \frac{1}{3v^3}$$

or $cy = e^{x^3/3y^3}$ where c is an arbitrary constant. **Ans.**

(b) Here we have $\frac{dy}{dx} = \frac{y^3 + y^2\sqrt{y^2 - x^2}}{x^3}$

It is homegeneous hence putting $y = vx$, we get

$$v + x\frac{dv}{dx} = v^3 + v^2\sqrt{\left(\frac{y}{x}\right)^2 - 1} \quad \text{or} \quad \frac{xdv}{dx} = v^3 - v + v^2\sqrt{v^2 - 1}$$

or $$\frac{dv}{v(v^2-1) + v^2\sqrt{(v^2-1)}} = \frac{dx}{x}$$

Integrating both sides, we get

$$\log x + \log c = \int \frac{dv}{v\sqrt{v^2-1}\left(\sqrt{v^2-1} + v\right)}$$ where c is an arbitrary constant.

Rationalising the integrand here, we get

$$\log(cx) = \int \frac{\sqrt{v^2-1} - v\ dv}{v\sqrt{(v^2-1)}(v^2 - 1 - v^2)} = -\int \frac{\sqrt{v^2-1} - v}{v\sqrt{v^2-1}}\,dv$$

$$= \int \left(\frac{1}{\sqrt{v^2-1}} - \frac{1}{v}\right) dv = \log\left(v + \sqrt{v^2-1}\right) - \log v$$

or $$cx = \frac{v + \sqrt{v^2-1}}{v} = 1 + \sqrt{1 - \frac{x^2}{y^2}} \quad \text{or} \quad cxy = y + \sqrt{y^2 - x^2}$$ **Ans.**

EXAMPLE 9.9. Solve $y^2dx + (x^2 - xy + y^2)\,dy = 0$

[GGSIPU I Sem End Term 2003]

SOLUTION: The given equation is $\frac{dy}{dx} = \frac{-y^2}{x^2 - xy + y^2}$

This is of homogeneous form, hence putting $y = vx$, gives

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2 - x^2v + v^2x^2} = \frac{-v^2}{1 - v + v^2}$$

or $$\frac{xdv}{dx} = \frac{-v^2}{1-v+v^2} - v = \frac{-v^3 - v}{1 - v + v^2}$$

or $$\frac{dx}{x} = \frac{1 + v^2 - v}{-v(v^2+1)}\,dv = \left(-\frac{1}{v} + \frac{1}{v^2+1}\right)dv$$

On integration we get $\log x = -\log v + \tan^{-1} v + \log C$

Replacing v by (y/x), gives $\log x = -\log y + \log x + \tan^{-1}(y/x) + \log C$

or $\log(y/C) = \tan^{-1}(y/x)$ where C is constant of integration. **Ans.**

REDUCIBLE TO HOMOGENEOUS FORM

The differential equations under this category are of the form

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \quad \text{...(1)}$$

Substituting $x = X + h$, $y = Y + k$ hence $\dfrac{dy}{dx} = \dfrac{dY}{dX}$, then (1) becomes

$$\boxed{\frac{dY}{dX} = \frac{a_1 X + b_1 Y + a_1 h + b_1 k + c_1}{a_2 X + b_2 Y + a_2 h + b_2 k + c_2}} \quad \text{...(2)}$$

Now choose h and k such that $a_1 h + b_1 k + c_1 = 0 = a_2 h + b_2 k + c_2$

which on solving for h and k, give

$$h = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \quad k = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}.$$ Now two cases arise.

CASE I: $a_1 b_2 - a_2 b_1 \neq 0$, *i.e.,* $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

In this case (2) becomes $\dfrac{dY}{dX} = \dfrac{a_1 X + b_1 Y}{a_2 X + b_2 Y}$

which is homogeneous and can be solved by putting $Y = vX$, etc.

CASE II: $a_1 b_2 - a_2 b_1 = 0$ In this case we have $\dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} = \lambda$ (say)

then we can write $a_2 = \lambda a_1$ and $b_2 = \lambda b_1$ and hence (1) becomes

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{\lambda(a_1 x + b_1 y) + c_2} \quad \text{...(3)}$$

Here we substitute $a_1 x + b_1 y = t$ which gives $a_1 + b_1 \dfrac{dy}{dx} = \dfrac{dt}{dx}$

and therefore (3) becomes $\dfrac{dt}{dx} = a_1 + b_1 \left(\dfrac{t + c_1}{\lambda t + c_2} \right)$

in which the variables are separable hence can be easily solved.

EXAMPLE 9.10. Solve $\dfrac{dy}{dx} = \dfrac{2x + 3y - 4}{4x + y - 3}$, given that $y = 1$ when $x = 1$.

SOLUTION: Substituting $x = X + h$ and $y = Y + k$, the given equation becomes

$$\frac{dY}{dX} = \frac{2X + 3Y + 2h + 3k - 4}{4X + Y + 4h + k - 3} \quad \text{...(1)}$$

Let us chose h and k such that $2h + 3k - 4 = 0$, $4h + k - 3 = 0$

which give $h = \dfrac{1}{2}$, $k = 1$. Then (1) becomes $\dfrac{dY}{dX} = \dfrac{2X + 3Y}{4X + Y}$ which is homogeneous.

Putting $Y = VX$, we get

$$V + X\frac{dV}{dX} = \frac{2+3V}{4+V} \quad \text{or} \quad X\frac{dV}{dX} = \frac{2+3V}{4+V} - V = \frac{2-V-V^2}{4+V}$$

or $$\frac{(V+4)\,dV}{(V-1)(V+2)} = -\frac{dX}{X}.$$

On integrating both sides, we get

$$\int -\frac{dX}{X} = \int\left[\frac{(-2/3)}{V+2} + \frac{(5/3)}{V-1}\right] dV$$

or $$-\log X + \log C = -\frac{2}{3}\log(V+2) + \frac{5}{3}\log(V-1)$$

Replacing V by $\frac{Y}{X}$ gives $-3\log X + 3\log C = -2\log\left(\frac{Y+2X}{X}\right) + 5\log\left(\frac{Y-X}{X}\right)$

or $$5\log(Y-X) - 2\log(Y+2X) = 3\log C.$$

Putting back $X = x - h = x - \frac{1}{2}$ and $Y = y - k = y - 1$, we get

$$5\log\left(y - x - \frac{1}{2}\right) - 2\log(y+2x-2) = 3\log C$$

or $$\frac{\left(y-x-\frac{1}{2}\right)^5}{(y+2x-2)^2} = C^3 \qquad ...(2)$$

To find the constant of integration C we have $y = 1$ at $x = 1$, therefore $C^3 = \frac{-1}{2^5}$

and, in turn, (2) becomes $\frac{\left(y-x-\frac{1}{2}\right)^5}{(y+2x-2)^2} = \frac{-1}{2^5}$

or $$(y+2x-2)^2 = -2^5\left(y-x-\frac{1}{2}\right)^5 = (2x-2y+1)^5$$

Thus, the solution is $(y + 2x - 2)^2 = (2x - 2y + 1)^5$. **Ans.**

EXAMPLE 9.11. Solve $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$ [GGSIPU I Sem End Term 2004]

SOLUTION: The given equation is of the reducible to homogeneous form

$$\frac{dy}{dx} = \frac{-(2x+y+1)}{4x+2y-1} \qquad ...(1)$$

Putting $2x + y = t$ so that $2 + \frac{dy}{dx} = \frac{dt}{dx}$, (1) becomes

$$\frac{dt}{dx} - 2 = \frac{-(t+1)}{(2t-1)} \quad \text{or} \quad \frac{dt}{dx} = 2 - \frac{t+1}{2t-1} = \frac{3t-3}{2t-1}$$

or $$dx = \frac{2t-1}{3t-3}\,dt = \left[\frac{2}{3} + \frac{1}{3(t-3)}\right]dt$$

On integration, we get $x = \frac{2t}{3} + \frac{1}{3}\log(t-3) + C$

or $$x = \frac{2}{3}(2x+y) + \frac{1}{3}\log(2x+y-3) + C$$

or $x + 2y + \log(2x+y-3) = -3C$ where C is arbitrary constant of integration. **Ans.**

LINEAR DIFFERENTIAL EQUATION

A differential equation is said to be *linear* if the dependent variable and its derivative occur only in first degree and are not multiplied together. Its general form is

$$\frac{dy}{dx} + Py = Q \quad \text{where P and Q are functions of } x \text{ only.} \qquad ...(1)$$

It was introduced by Leibneitz and is therefore better known as **Leibneitz's linear differential equation.**

Multiplying (1) on both sides by a factor $e^{\int P dx}$, we get

$$\frac{dy}{dx}e^{\int P dx} + y\left(e^{\int P dx}.P\right) = Qe^{\int P dx} \quad \text{or} \quad \frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$$

Integrating both sides, gives $ye^{\int P dx} = \int Qe^{\int P dx}\,dx + c$ ...(2)

which is the required solution. The factor $e^{\int P\,dx}$ is called the **Integrating Factor** (I.F.).

Therefore the solution (2) can be written as $y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$

Similarly, the equation $\frac{dx}{dy} + Px = Q$ where P and Q are functions of y only, ...(3)

is linear in x. Here the integrating factor is $e^{\int P dy}$ and the general solution will be written as

$$x\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dy + c \quad \text{where I.F.} = e^{\int P dy}.$$

EXAMPLE 9.12. Solve $x(1-x^2)\frac{dy}{dx} + (2x^2-1)y = x^3$.

SOLUTION: Dividing throughout by $x(1-x^2)$, we get

$$\frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}y = \frac{x^2}{1-x^2} \quad \text{where } x \neq 0, \pm 1. \qquad ...(1)$$

which is linear in y. Here $P = \frac{2x^2-1}{x(1-x^2)}$ and $Q = \frac{x^2}{1-x^2}$.

$$\therefore \int P dx = \int \frac{(2x^2-1)\,dx}{x(1-x)(1+x)} = \int\left[-\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\right]dx$$

$$= -\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x) = -\log\left(x\sqrt{1-x^2}\right)$$

$$\therefore \text{I.F.} = e^{\int P dx} = \left[e^{-\log x(\sqrt{1-x^2})}\right] = \frac{1}{x\sqrt{1-x^2}}$$

Therefore, the solution of (1) is

$$y.\frac{1}{x\sqrt{1-x^2}}=\int\frac{x^2}{1-x^2}.\frac{1}{x\sqrt{1-x^2}}dx=\int\frac{x\,dx}{(1-x^2)^{3/2}}=\frac{1}{\sqrt{1-x^2}}+C$$

or $\quad y=x+Cx\sqrt{1-x^2}$ where C is the arbitrary constant of integration. Ans.

EXAMPLE 9.13. Solve $(2xy\cos x^2-2xy+1)\,dx+(\sin x^2-x^2)\,dy=0$

[GGSIPU I Sem End Term 20..]

SOLUTION: The equation is $\dfrac{dy}{dx}+\dfrac{2x\cos x^2-2x}{\sin x^2-x^2}y=\dfrac{-1}{\sin x^2-x^2}$ which is linear in y.

Here $\quad$ I.F. $=e^{\int\frac{2x\cos x^2-2x}{\sin x^2-x^2}dx}=e^{\log(\sin x^2-x^2)}=\sin x^2-x^2$

$\therefore$ The solution is $y(\sin x^2-x^2)=\int(\sin x^2-x^2)\dfrac{(-1)}{\sin x^2-x^2}dx=\int-1\,dx=-x+C$

or $y(\sin x^2-x^2)+x=C$ where C is constant of integration. **Ans.**

EXAMPLE 9.14. Solve $dr+(2r\cot\theta+\sin 2\theta)\,d\theta=0$

SOLUTION: The equation is $\dfrac{dr}{d\theta}+2r\cot\theta=-\sin 2\theta$ which is linear in r. (1)

Here $\quad P=2\cot\theta$ and $Q=-\sin 2\theta$.

and $\quad$ I.F. $=e^{\int 2\cot\theta d\theta}=e^{2\log\sin\theta}=\sin^2\theta.$

Therefore, the solution of (1) is $r\,\text{I.F.}=\int Q.(\text{I.F.})\,d\theta=\int-\sin 2\theta.\sin^2\theta d\theta$

or $\quad r\sin^2\theta=-2\int\sin^3\theta\cos\theta d\theta=-\dfrac{2}{4}\sin^4\theta+c$

or $\quad r=-\dfrac{1}{2}\sin^2\theta+c\,\text{cosec}^2\theta$, where c is an arbitrary constant. **Ans.**

EXAMPLE 9.15. Solve $ye^y=(y^3+2xe^y)\dfrac{dy}{dx}$.

SOLUTION: The given equation can be written as

$\dfrac{dx}{dy}-\dfrac{2}{y}x=y^2e^{-y}$ which is linear in x. Here $P=-\dfrac{2}{y}$ and $Q=y^2e^{-y}$.

$\therefore\quad$ I.F. $=e^{\int P dy}=e^{\int-\frac{2}{y}dy}=e^{-2\log y}=\dfrac{1}{y^2}$.

Therefore the general solution is $x\,\text{I.F.}=\int Q(\text{I.F.})dy$

or $\quad \dfrac{x}{y^2}=\int\dfrac{1}{y^2}y^2e^{-y}dy=-e^{-y}+c$

or $\quad x+y^2e^{-y}=cy^2$ where c is an arbitrary constant. **Ans.**

EQUATIONS REDUCIBLE TO LINEAR FORM (BERNOULLI'S EQUATION)

The differential equation $\frac{dy}{dx} + Py = Qy^n$ ...(1)

where P and Q are functions of x and n is a real number, is known as Bernoulli's' equation and can be reduced to Leibneitz's linear form. Dividing throughout by y^n, we get

$$y^{-n}\frac{dy}{dx} + P \cdot y^{1-n} = Q \quad ...(2)$$

Putting here $y^{1-n} = t$ so that $(1-n)\, y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$.

Substituting these in (2), we get

$$\frac{1}{1-n}\frac{dt}{dx} + Pt = Q \quad \text{or} \quad \frac{dt}{dx} + (1-n)Pt = (1-n)Q.$$

which is linear in t and can be solved easily.

Another type of equation, which can be reduced to the linear form, is

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \quad ...(3)$$

where P and Q are functions of x alone or constants.

The substitution $t = f(y)$, so $\frac{dt}{dx} = f'(y)\frac{dy}{dx}$, transforms the equation (3) into linear in t, as

$$\frac{dt}{dx} + Pt = Q.$$

EXAMPLE 9.16. Solve (i) $\frac{dy}{dx} - x^3y^3 + xy = 0$ (ii) $\frac{dy}{dx} + \frac{y}{x} = y^2$. [*GGSIPU I Sem II Term 2013*]

SOLUTION: (i) The given equation can be written as $\frac{dy}{dx} + xy = x^3y^3$ ...(1)

which is of Bernoulli's form. Dividing throughout by y^3, gives

$$y^{-3}\frac{dy}{dx} + xy^{-2} = x^3 \quad ...(2)$$

Now substituting $y^{-2} = t$, so $-2y^{-3}\frac{dy}{dx} = \frac{dt}{dx}$, therefore, (2) becomes

$$\frac{dt}{dx} - 2tx = -2x^3 \quad \text{which is linear in } t.$$

Here $\text{I.F.} = e^{\int -2x\,dx} = e^{-x^2}$

hence the solution is $t\,(\text{I.F.}) = \int -2x^3 (\text{I.F.})\,dx$

$$\text{or} \quad te^{-x^2} = \int -2x^3 e^{-x^2}\,dx. \quad (\text{Putting here } -x^2 = u)$$

$$= -\int ue^u\,du \quad (\text{now integrating by parts})$$

$$= -ue^u + \int e^u\,du = (1-u)\,e^u + c$$

$$\text{or} \quad y^{-2}e^{-x^2} = (1+x^2)\,e^{-x^2} + c \quad \text{or} \quad \frac{1}{y^2} = 1 + x^2 + ce^{x^2}$$

which is the required solution. **Ans.**

(ii) Dividing the given equation by y^2 on both sides, we get

$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=1$ Putting $-\frac{1}{y}=t$, the equation becomes $\frac{dt}{dy}-\frac{t}{x}=1$

It is linear in t, hence I.F. $= e^{\int\frac{-1}{x}dx} = e^{-\log x} = \frac{1}{x}$

$\therefore$ The solution is $t\left(\frac{1}{x}\right) = \int 1\cdot\frac{1}{x}dx = \log x + c$ where c is an arbitrary constant.

or $$-\frac{1}{xy} = \log x + \log k \text{ where } c = \log k.$$

or $$xy \log (kx) + 1 = 0. \quad \textbf{Ans.}$$

[redacted] (a) $y(2xy + e^x)\, dx - e^x dy = 0$ [GGSIPU I Sem End Term 2012]

(b) Solve $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)\, e^x \sec y.$ [GGSIPU I Sem End Term 2003]

SOLUTION: (a) Equation is $e^x\frac{dy}{dx}-ye^x=2xy^2$ or $\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}=2x\,e^{-x}$

Putting $-\frac{1}{y}=t$ or $y=-\frac{1}{t}$ the above equation becomes

$$\frac{dt}{dx}+t=2x\,e^{-x} \text{ which is linear in } t.$$

$\therefore$ I.F. $=e^{\int dx}=e^x$ Hence the solution is $t\,e^x=\int e^x\cdot 2xe^{-x}dx=x^2+c$

or $$-\frac{1}{y}e^x=x^2+c \quad\text{or}\quad y(x^2+c)+e^x=0 \quad \textbf{Ans.}$$

(b) Equation can be written as $\cos y\,\frac{dy}{dx}-\frac{\sin y}{1+x}=(1+x)e^x$

Putting $\sin y=t$ we get $\frac{dt}{dx}-\frac{t}{1+x}=(1+x)\,e^x$ which is linear in t.

Here $$\text{I.F.} = e^{\int\frac{-1\,dx}{1+x}} = e^{-\log(1+x)} = \frac{1}{1+x}$$

$\therefore$ Solution is $\frac{1}{1+x}t = \int\frac{1}{1+x}(1+x)\,e^x\,dx = \int e^x\,dx = e^x + C$

or $\sin y = (1 + x)\,(e^x + C)$ is the required solution where C is constant of integration. **Ans.**

[illegible] (*a*) Solve $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2.$ [*GGSIPU I Sem End Term Jan 2011*]

(*b*) $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$ [*GGSIPU Ist Sem End Term 2009*]

SOLUTION: (*a*) Dividing the given equation throughout by $y(\log y)^2$, we get

$$\frac{1}{y(\log y)^2}\frac{dy}{dx}+\frac{1}{x\log y}=\frac{1}{x^2}$$

Let us substitute $\frac{1}{\log y}=t$, so $-\frac{1}{y(\log y)^2}\frac{dy}{dx}=\frac{dt}{dx}$,

hence we have $\frac{dt}{dx}-\frac{t}{x}=\frac{-1}{x^2}$ which is linear in t.

Here $\text{I.F.}=e^{\int\frac{-1}{x}dx}=e^{-\log x}=\frac{1}{x}$

$\therefore$ Solution is $t\cdot\frac{1}{x}=\int\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)dx$ or $\frac{1}{x\log y}=\frac{1}{2x^2}+c$

or $\frac{1}{\log y}=\frac{1}{2x}+cx$ where c is an arbitrary constant of integration. **Ans.**

(*b*) In the given equation, putting $y=vx$, we get

$$v+x\frac{dv}{dx}+v=v^2 \quad \text{or} \quad \frac{dv}{v^2-2v}=\frac{dx}{x}$$

which on integration, gives

$$\int\frac{dv}{v(v-2)}=\log x+\log c \quad \text{or} \quad \log cx=\frac{1}{2}\int\left(\frac{1}{v-2}-\frac{1}{v}\right)dv=\frac{1}{2}\log\frac{v-2}{v}$$

or $c^2x^2=\frac{v-2}{v}=1-\frac{2x}{y}$ or $y-2x=c^2x^2y$ **Ans.**

9.19 Solve $r\sin\theta\,d\theta+(r^3-2r^2\cos\theta+\cos\theta)dr=0.$

SOLUTION: The given equation can be written as $\sin\theta\frac{d\theta}{dr}-\left(2r-\frac{1}{r}\right)\cos\theta=-r^2$ ...(1)

Now putting $-\cos\theta=t$ so that $\sin\theta\frac{d\theta}{dr}=\frac{dt}{dr}$, (1) becomes

$$\frac{dt}{dr}+\left(2r-\frac{1}{r}\right)t=-r^2 \text{ which is linear in } t. \quad ...(2)$$

Here $\text{I.F.}=e^{\int\left(2r-\frac{1}{r}\right)dr}=e^{r^2-\log r}=e^{r^2}e^{\log r^{-1}}=\frac{1}{r}e^{r^2}$

Therefore the solution is $t\frac{1}{r}e^{r^2}=\int-r^2\cdot\frac{1}{r}e^{r^2}dr$ or $\frac{\cos\theta}{r}e^{r^2}=\frac{1}{2}e^{r^2}+c$

or $\cos\theta=r\left(\frac{1}{2}+ce^{-r^2}\right)$ where c is an arbitrary constant. **Ans.**

EXERCISE 9A

1. Form the differential equation whose general solution is $ax^2 + by^2 = 1$.
2. Form the differential equation for the relation $y = ax + \frac{b}{x}$ where a and b are arbitrary constants.
3. Eliminate the arbitrary constants a and α by way of obtaining the differential equation for $y = a \cos (nx + \alpha)$.
4. Obtain the differential equation for the general solution $(y - k)^2 = 4a\,(x - h)$ where h and k are arbitrary constants.
5. Solve $\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$.
6. Solve $\frac{y}{x}\frac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2 y^2}$.
7. Solve $\cos x\, dy - (\sin x + x \sec x) \cot y\, dx = 0$.
8. Solve $\frac{dy}{dx} = \frac{x \cos x}{2e^y \sinh y}$.
9. Solve $\left(\frac{x + y - a}{x + y + b}\right)\frac{dy}{dx} = \left(\frac{x + y + a}{x + y - b}\right)$.
10. Solve $\frac{dy}{dx} = (4x + y + 1)^2$
11. Solve $x^3 \frac{dy}{dx} = y^3 + y^2 \sqrt{y^2 - x^2}$.
12. Solve $x dy - y dx = \sqrt{x^2 + y^2}\, dx$.
13. Solve $x \sin\left(\frac{y}{x}\right)\frac{dy}{dx} = y \sin\left(\frac{y}{x}\right) + x$.
14. Find the equation to the curve for which the tangent at any point on it cuts the x-axis at a point which is equidistant from the origin and the point on the curve.
15. Solve $(3y - 7x + 7)\, dx + (7y - 3x + 3) dy = 0$.
16. Solve $(6x + 9y + 6)\, dy - (2x + 3y - 1) dx = 0$.
17. (a) Solve $(1 + x^2)\frac{dy}{dx} + 2xy = \cos x$.

    (b) Solve the linear differential equation $y' + y = e^{-x} \tan x$.
18. (a) Solve $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$. *[GGSIPU I Sem End Term 2007]*

    (b) Solve $\frac{dy}{dx} = y \tan x - 2 \sin x$.

    [GGSIPU I Sem End Term 2004 Reappear]

19. Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1.$

20. Solve $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$

21. Solve $ye^y = (y^3 + 2x\,e^y)\,\frac{dy}{dx}.$

22. Solve $\frac{dy}{dx} - y\tan x = y^4 \sec x.$

23. Solve $\frac{dy}{dx} = e^{x-y}\,(e^x - e^y).$

24. Solve $xy\frac{dy}{dx} + y^2 = 4x$ if $y = 0$ at $x = 0.$

25. Solve $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}.$

26. Solve $\frac{dy}{dx} - x^2 y = y^2 e^{-x^3/3}$

27. Solve $y\,(2xy + e^x)\,dx - e^x\,dy = 0$ [GGSIPU I Sem End Term 2008; End Term 2011]

28. Solve $(x^2 y - 2xy^2)\,dx - (x^3 - 3x^2 y)\,dy = 0.$

$y = vx$

$\frac{x}{y} = t$

EXACT DIFFERENTIAL EQUATION

Any first order differential equation $y' = f(x, y)$ can be written in the form $M(x, y)\,dx + N(x, y)dy = 0$.

An interesting thing happens if there exists a function ϕ such that

$$\frac{\partial \phi}{\partial x} = M(x, y) \text{ and } \frac{\partial \phi}{\partial y} = N(x, y). \quad ...(1)$$

Then the given differential equation becomes

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0 \text{ which is nothing but } \frac{d}{dx}\phi(x, y) = 0.$$

This means $\phi(x, y(x)) = C$ where C is constant.

The above equation implicitly defines a function $y(x)$ which is the general solution of the differential equation. Thus, finding a function ϕ which satisfies equation (1) is equivalent to solving the differential equation. Such a function ϕ is called the **potential function** for the differential equation $Mdx + Ndy = 0$ over a region R, if for each (x, y) in R, we have

$$\frac{\partial \phi}{\partial x} = M(x, y) \text{ and } \frac{\partial \phi}{\partial y} = N(x, y).$$

TEST FOR EXACTNESS:

[GGSIPU I Sem II Term 2013]

Let $M(x, y)$, $N(x, y)$, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ be continuous within a rectangle R in the xy-plane, then $M(x, y)dx + N(x, y)dy = 0$ is exact on R, if and only if, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Proof: If $M\,dx + N\,dy = 0$ is exact, then there is a potential function ϕ such that $\frac{\partial \phi}{\partial x} = M(x, y)$ and $\frac{\partial \phi}{\partial y} = N(x, y)$.

Then
$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y\,\partial x} = \frac{\partial^2 \phi}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial N}{\partial x}.$$

Conversely, suppose $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous. Choose any (x_0, y_0) in R and define, for (x, y) in R.

$$\phi(x, y) = \int_{x_0}^{x} M(\xi, y_0)\,d\xi + \int_{y_0}^{y} N(x, \eta)\,d\eta \quad ...(2)$$

Hence, from fundamental theorem of calculus, we have

$\frac{\partial \phi}{\partial y} = 0 + N(x, y)$ since the first integral in equation (2) is independent of y.

Next,
$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\int_{x_0}^{x} M(\xi, y_0)d\xi + \frac{\partial}{\partial x}\int_{y_0}^{y} N(x, \eta)\,d\eta$$

$$= M(x, y_0) + \int_{y_0}^{y} \frac{\partial N}{\partial x}(x, \eta)\,d\eta = M(x, y_0) + \int_{y_0}^{y} \frac{\partial M}{\partial y}(x, \eta)\,d\eta$$

$$= M(x, y_0) + M(x, y) - M(x, y_0) = M(x, y)$$

and hence the theorem is proved.

If the condition is satisfied, the general solution of (1) is obtained as given below :

"First integrate M *w.r.t.* x treating y as constant, then integrate *w.r.t.* y only those terms in N which do not involve x. The sum of the two expressions thus obtained equated to an arbitrary constant, is the required solution." That is,

$$\int(\text{all terms in M treating } y \text{ as constant})\,dx + \int(\text{only those terms in N not involving } x)\,dy = c.$$

The general solution may also be written as

$$\int(\text{all terms in N})\,dy + \int(\text{only those terms in M not involving } y)\,dx = c.$$

EXAMPLE 9.20. Solve $(1+e^{x/y})\,dx + e^{x/y}\left(1-\dfrac{x}{y}\right)dy = 0.$ [*GGSIPU I Sem End Term*]

SOLUTION: Comparing the given equation with $Mdx + Ndy = 0$, we have

$$M = 1+e^{x/y} \quad \text{and} \quad N = e^{x/y}\left(1-\frac{x}{y}\right).$$

Here $\dfrac{\partial M}{\partial y} = e^{x/y}\left(-\dfrac{x}{y^2}\right)$ and $\dfrac{\partial N}{\partial x} = e^{x/y}\left(\dfrac{1}{y}\right)\left(1-\dfrac{x}{y}\right)+e^{x/y}\left(-\dfrac{1}{y}\right) = e^{x/y}\left(-\dfrac{x}{y^2}\right)$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ the equation is exact. Therefore the general solution is

$\int(1+e^{x/y})\,dx + \int 0\,dy = c$ as there is no term in N not involving x.

or $x + ye^{x/y} = c$ where c is an arbitrary constant. **Ans.**

[illegible] Solve $(y\sin xy + xy^2\cos xy)\,dx + (x\sin xy + x^2y\cos xy)\,dy = 0.$

SOLUTION: Comparing the given equation with $Mdx + Ndy = 0$, we have

$$M = y\sin xy + xy^2\cos xy \quad \text{and} \quad N = x\sin xy + x^2y\cos xy.$$

$\therefore$ $\dfrac{\partial M}{\partial y} = xy\cos xy + \sin xy + 2xy\cos xy - x^2y^2\sin xy$

and $\dfrac{\partial N}{\partial x} = xy\cos xy + \sin xy + 2xy\cos xy - x^2y^2\sin xy$

Here $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ hence the equation is exact.

The general solution is $\int(y\sin xy + xy^2\cos xy)\,dx + \int 0\,dy = c$

or $\dfrac{-y\cos xy}{y} + y^2\int x\cos xy\,dx = c$ or $-\cos xy + y^2\left[\dfrac{x\sin xy}{y} - \int 1.\dfrac{\sin xy}{y}\,dx\right] = c$

or $-\cos xy + xy\sin xy + \dfrac{y^2\cos xy}{y^2} = c$ where c is an arbitrary constant.

or $xy\sin xy = c$ which is the required solution. **Ans.**

EXAMPLE 9.22. Solve the differential equation [GGSIPU I Sem End Term 2007]

$$y \sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0$$

SOLUTION: The given equation is of the form $Mdx + Ndy = 0$

where $M = y \sin 2x$ and $N = -y^2 - \cos^2 x$.

$$\frac{\partial M}{\partial y} = \sin 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2\cos x \sin x = \sin 2x$$

Hence the given equation is exact and its solution is

$$\int M dx + \int N \text{ (only those terms not involving } x)\, dy = 0$$

or $\int y \sin 2x\, dx + \int -y^2\, dy = C$ or $3y \cos 2x + 2y^3 = C$

where C is an arbitrary constant of integration. **Ans.**

REDUCIBLE TO EXACT DIFFERENTIAL EQUATION

Sometimes a differential equation which is not exact, can be converted to exact differential equation by multiplying it by some suitable factor called *Integrating Factor* (I. F.)

In what follows, we list some rules for finding the integrating factors for the equation

$Mdx + Ndy = 0$ which is not exact.

RULE 1. ***I. F. for a homegeneous equation*: If $Mdx + Ndy = 0$ is a homogeneous equation in x and y then $\dfrac{1}{Mx + Ny}$ is the integrating factor.**

EXAMPLE 9.23. (a) Solve $(y^3 - 3xy^2)\, dx + (2x^2y - xy^2)\, dy = 0$.

(b) $y \sin 2x\, dx - (1 + y^2 + \cos^2 x)\, dy = 0$. [GGSIPU I Sem End Term 2011]

SOLUTION: (a) The given equation $Mdx + Ndy = 0$ is homogeneous.

where $M = y^3 - 3xy^2$ and $N = 2x^2y - xy^2$. Hence $\dfrac{\partial M}{\partial y} = 3y^2 - 6xy$ and $\dfrac{\partial N}{\partial x} = 4xy - y^2$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ the equation is not exact.

$$\therefore \quad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{xy^3 - 3x^2y^2 + 2x^2y^2 - xy^3} = \frac{1}{-x^2y^2}$$

Multiplying the given equation by $\dfrac{-1}{x^2y^2}$, gives

$$\left(\frac{3}{x} - \frac{y}{x^2}\right)dx + \left(\frac{1}{x} - \frac{2}{y}\right)dy = 0 \text{ which is exact.}$$

$\therefore$ The solution is $\int\left(\dfrac{3}{x} - \dfrac{y}{x^2}\right)dx + \int \dfrac{-2}{y}\, dy = C$ or $3\log x + \dfrac{y}{x} - 2\log y = \log C_1$

or $\dfrac{x^3}{y^2} = C_1 e^{-y/x}$ which is the required solution.

Ans.

(b) Equation is $M\,dx + N\,dy = 0$ where $M = y \sin 2x$, $N = -(1 + y^2 + \cos^2 x)\,dy$

$$\frac{\partial M}{\partial y} = \sin 2x, \quad \frac{\partial N}{\partial x} = +2\cos x \sin x \text{ hence exact.}$$

The solution is $-\int (1 + y^2 + \cos^2 x)\,dy + \int 0\,dx = C$ or $y + \dfrac{y^3}{3} + y\cos^2 x = C$. **Ans.**

RULE 2. ***I.F. for an equation of the type $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$:***

If the equation $Mdx + Ndy = 0$ is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$ then $\dfrac{1}{Mx - Ny}$ is the integrating factor, provided $Mx \neq Ny$.

EXAMPLE 9.24. Solve $(x^2y^3 + 2y)dx + (2x - 2x^3y^2)\,dy = 0$.

SOLUTION: The given equation which is not exact, can be written as

$$(x^2y^2 + 2)\,y\,dx + (2 - 2x^2y^2)\,x\,dy = 0 \text{ which is of the form } f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$$

Here $\text{I.F.} = \dfrac{1}{Mx - Ny} = \dfrac{1}{xy\,(x^2y^2 + 2) - xy(2 - 2x^2y^2)} = \dfrac{1}{3x^3y^3}$

$\therefore$ Multiplying the given equation by $\dfrac{1}{x^3y^3}$, we get

$$\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)dx + \left(\frac{2}{x^2y^3} - \frac{2}{y}\right)dy = 0$$

which can be verified to be exact. Therefore the solution is

$$\int\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)dx + \int \frac{-2}{y}\,dy = C \quad \text{or} \quad \log x - \frac{1}{x^2y^2} - 2\log y = C$$

Replacing C by $\log c$ here, we get

$$x = cy^2\, e^{\frac{1}{x^2y^2}} \text{ where } c \text{ is an arbitrary constant.}$$ **Ans.**

EXAMPLE 9.25. Solve $(xy^2 \sin xy + y\cos xy)\,dx + (x^2y \sin xy - x\cos xy)\,dy = 0$.

SOLUTION: The given equation is not exact and can be written as

$$(xy\sin xy + \cos xy)\,ydx + (xy\sin xy - \cos xy)\,xdy = 0$$

which is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$.

Here $\text{I.F.} = \dfrac{1}{Mx - Ny} = \dfrac{1}{xy\,[xy\sin xy + \cos xy - xy\sin xy + \cos xy]} = \dfrac{1}{2xy\cos xy}$

Multiplying the given equation throughout by $\dfrac{1}{xy\cos xy}$, gives

or $$\left(y\tan xy + \frac{1}{x}\right)dx + \left(x\tan xy - \frac{1}{y}\right)dy = 0$$

which can be verified to be exact. Therefore, the solution is

$$\int\left(y\tan xy+\frac{1}{x}\right)dx\int-\frac{1}{y}dy=C,\quad \text{or}\quad y\log\sec(xy)\cdot\frac{1}{y}+\log x-\log y=C$$

$$\text{or}\qquad \log\sec xy+\log\left(\frac{x}{y}\right)=C,\qquad \text{or}\qquad x\sec xy=c_1 y$$

is the required solution where c_1 is an arbitrary constant. **Ans.**

RULE 3. *In the equation* $Mdx + Ndy = 0$ *if* $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ *is a function of x only* $= f(x)$, *say, then* $I.F. = e^{\int f(x)dx}$;

EXAMPLE 9.26. Solve $(x^2+y^2+x)\,dx+xy\,dy=0$.

SOLUTION: Here $M=x^2+y^2+x$, $N=xy$

$$\therefore\quad \frac{\partial M}{\partial y}=2y,\quad \frac{\partial N}{\partial x}=y\quad \text{hence the equation is not exact.}$$

$$\text{But}\quad \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2y-y}{xy}=\frac{1}{x},\quad \text{function of } x \text{ alone.}$$

Therefore, $\text{I.F.}=e^{\int\frac{1}{x}dx}=e^{\log x}=x.$

Multiplying the given equation throughout by x, we get

$$(x^3+xy^2+x^2)\,dx+x^2y\,dy=0\quad \text{which is exact.}$$

Hence the general solution is

$$\int(x^3+xy^2+x^2)dx+\int 0\,dy=c\qquad \text{or}\qquad \frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{x^3}{3}=c$$

or $\quad 3x^4+4x^3+6x^2y^2=c$ which is the required solution. **Ans.**

EXAMPLE 9.27. Solve $\left(xy^2-e^{\frac{1}{x^3}}\right)dx-x^2y\,dy=0.$

SOLUTION: Comparing the given equation with $Mdx+Ndy=0$, we have

$$M=xy^2-e^{\frac{1}{x^3}},\quad N=-x^2y\quad \therefore\quad \frac{\partial M}{\partial y}=2xy\quad \text{and}\quad \frac{\partial N}{\partial x}=-2xy$$

$$\text{But}\quad \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{4xy}{-x^2y}=\frac{-4}{x},\quad \text{a function of } x \text{ alone.}$$

$$\therefore\quad \text{I.F.}=e^{\int-\frac{4}{x}dx}=e^{-4\log x}=\frac{1}{x^4}.$$

Multiplying the given equation throughout by $\frac{1}{x^4}$, gives

$$\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)dx-\frac{y}{x^2}dy=0\quad \text{which is exact.}$$

Hence the general solution is

$$\int\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)dx+\int 0\,dy = C \quad \text{or} \quad -\frac{y^2}{2x^2}+\frac{1}{3}e^{\frac{1}{x^3}} = C$$

or $\quad 3y^2-2x^2 e^{\frac{1}{x^3}} = -Cx^2 \quad$ which is the required solution. **Ans.**

RULE 4. ***In the equation*** $M\,dx + N\,dy = 0$ ***if*** $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ ***is a function of*** y ***only,*** $= f(y)$ ***say, then*** *I.F.* $= e^{\int f(y)\,dy}$.

[redacted] Solve $\quad (x+2y^3)\,\dfrac{dy}{dx} = y+2x^3y^2.$

SOLUTION: The given equation can be written as $\quad (y+2x^3y^2)\,dx+(-x-2y^3)\,dy=0$

Here $\quad M = y+2x^3y^2 \quad$ and $\quad N = -x-2y^3 \quad \therefore \quad \dfrac{\partial M}{\partial y} = 1+4x^3y \quad$ and $\quad \dfrac{\partial N}{\partial x} = -1$

Now $\quad \dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \dfrac{-2-4x^3y}{y+2x^3y^2} = \dfrac{-2}{y}, \quad$ a function of y alone.

$\therefore$ Integrating factor $= e^{\int \frac{-2}{y}dy} = e^{-2\log y} = \dfrac{1}{y^2}$

Multiplying the given equation throughout by $\dfrac{1}{y^2}$, we get

$$\left(\frac{1}{y}+2x^3\right)dx+\left(-\frac{x}{y^2}-2y\right)dy = 0 \quad \text{which can be easily verified to be exact.}$$

The general solution is

$$\int\left(\frac{1}{y}+2x^3\right)dx+\int -2y\,dy = C \quad \text{or} \quad \frac{x}{y}+\frac{2x^4}{4}-y^2 = C$$

where c is an arbitrary constant. **Ans.**

or $\quad 2x+x^4y-2y^3 = cy \quad$ is the required solution

[GGSIPU I Sem II Term 2012]

[redacted] $x\,e^x\,(dx-dy)+e^x\,dx+y\,e^y dy = 0.$

SOLUTION: Equation is $M\,dx+N\,dy=0$ where $M=(x+1)\,e^x$ and $N=-x\,e^x+y\,e^y$.

Here $\quad \dfrac{\partial M}{\partial y} = 0 \quad$ and $\quad \dfrac{\partial N}{\partial x} = 0-x\,e^x-e^x.$

Now $\quad \dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \dfrac{-(x+1)e^x}{(x+1)e^x} = -1 \quad$ which can be taken as a function of y.

$\therefore$ I.F. $= e^{\int -1\,dy} = e^{-y}$. Multiplying both sides by I.F. we get

$$(x+1)\,e^x\,e^{-y}\,dx+(y-x\,e^{x-y})\,dy = 0 \quad \text{which is exact.}$$

Its solution is $e^{-y}\int (x+1)e^{x}dx + \int y\,dy = 0$

or $$e^{-y}\left[(x+1)e^{x} - e^{x}\right] + \frac{y^2}{2} = C$$ where C is an arbitrary constant.

or $$xe^{x-y} + \frac{y^2}{2} = C$$ **Ans.**

RULE 5. *I.F. for the equations of the type* $x^a y^b\,(pydx + qxdy) + x^c y^d\,(rydx + sxdy) = 0$ *is* $x^h y^k$. *The constants h and k are so chosen that the new equation (after multiplying by* $x^h y^k$*) becomes exact.*

EXAMPLE 9.30. Solve $(3xy + 8y^5)dx + (2x^2 + 24\,xy^4)dy = 0.$

SOLUTION: The given equation can be written as $x(3ydx + 2xdy) + 8y^4\,(ydx + 3xdy) = 0$ ...(1)

It is of the form mentioned in the rule 5 above, therefore the integrating factor is $x^h y^k$.

Multiplying the given equation throughout by $x^h y^k$, we get

$$(3x^{h+1}y^{k+1} + 8x^h y^{k+5})dx + (2x^{h+2}y^k + 24x^{h+1}y^{k+4})\,dy = 0$$

It is of the form $Mdx + Ndy = 0$ ...(2)

$$\therefore \quad \frac{\partial M}{\partial y} = 3(k+1)x^{h+1}y^k + 8(k+5)x^h y^{k+4},$$

and $$\frac{\partial N}{\partial x} = 2(h+2)x^{h+1}y^k + 24\,(h+1)x^h y^{k+4}$$

For the equation to be exact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore, equating the coefficients of like terms, gives

$$3\,(k+1) = 2\,(h+2) \quad \text{and} \quad 8\,(k+5) = 24\,(h+1)$$

which on solving for h and k, yield $h = 1$, $k = 1$.

With $h = 1$, $k = 1$, (2) becomes $(3x^2y^2 + 8xy^6)dx + (2x^3y + 24x^2y^5)dy = 0$ which is exact. Its solution is

$$\int (3x^2y^2 + 8xy^6)dx + \int 0dy = C$$ where C is the constant of integration.

or $$x^3y^2 + 4x^2y^6 = C$$ is the required solution. **Ans.**

EXAMPLE 9.31. Solve $x\,(3y\,dx + 2x\,dy) + 8y^4\,(y\,dx + 3x\,dy) = 0$

[GGSIPU I Sem End Term 2004 (Reappear)]

SOLUTION: The given differential equation is of the form

$$x^a y^b\,(py\,dx + qx\,dy) + x^c y^d\,(ry\,dx + sx\,dy) = 0$$

We multiply the given equation throughout of $x^h y^k$ so that the resulting equation becomes exact.

Thus we have $x^{h+1}y^k\,(3ydx + 2xdy) + 8x^h y^{k+4}\,(ydx + 3xdy) = 0$

or $$(3x^{h+1}y^{k+1} + 8x^h y^{k+5})\,dx + (2x^{h+2}y^k + 24x^{h+1}y^{k+4})\,dy = 0 \quad \text{...(1)}$$

For (1) to be exact $\frac{\partial}{\partial y}(3x^{h+1}y^{k+1} + 8x^h y^{k+5}) = \frac{\partial}{\partial x}(2x^{h+2}y^k + 24x^{h+1}y^{k+4})$

or $3(k+1)\,x^{h+1}y^k + 8(k+5)\,x^h y^{k+4} = 2(h+2)\,x^{h+1}y^k + 24(h+1)\,x^h y^{k+4}$

$$3(k+1) = 2(h+2) \quad \text{and} \quad 8(k+5) = 24(h+1).$$

$\Rightarrow$
$$3k+3 = 2h+4 \quad \text{and} \quad k+5 = 3(h+1)$$

or

These equations on solving for h and k, give $h = 1, k = 1$

Therefore, (1) becomes $(3x^2y^2 + 8xy^6)\, dx + (2x^3y + 24x^2y^5)\, dy = 0$ which is exact.

Its solution is $\int (3x^2y^2 + 8xy^6)\, dx + \int 0\, dy = C$ where C is an arbitrary constant.

or
$$x^3y^2 + 4x^2y^6 = C. \qquad \textbf{Ans.}$$

RULE 6. ***Transformation to Polar co-ordinates*:** Sometimes conversion from cartesian to polar co-ordinates helps solve the differential equation easily.

We have $x = r\cos\theta$, $y = r\sin\theta$ so $x^2 + y^2 = r^2$ and $\frac{y}{x} = \tan\theta$

$\therefore$ $dx = \cos\theta\, dr - r\sin\theta\, d\theta$ and $dy = \sin\theta\, dr + r\cos\theta\, d\theta$

Therefore $x\,dy - y\,dx = r\cos\theta(\sin\theta\, dr + r\cos\theta\, d\theta) - (r\sin\theta)(\cos\theta\, dr - r\sin\theta\, d\theta) = r^2 d\theta$

and $x\,dx + y\,dy = r\,dr$.

EXAMPLE 9.32. (*a*) Solve $\dfrac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\dfrac{a^2 - x^2 - y^2}{x^2 + y^2}}$

[*GGSIPU I Sem II Term 2012*]

(*b*) $x\,dy - y\,dx + 2x^3\,dx = 0$.

SOLUTION: (*a*) Conversion to polar coordinates, converts the given equation to

$$\frac{r\,dr}{r^2 d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}} \quad \text{or} \quad \frac{dr}{d\theta} = \sqrt{a^2 - r^2}$$

or $\dfrac{dr}{\sqrt{a^2 - r^2}} = d\theta$. Integrating both sides, gives $\sin^{-1}\dfrac{r}{a} = \theta + C$

or $\sin^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{a}\right) = \tan^{-1}\left(\dfrac{y}{x}\right) + C$ is the required solution. **Ans.**

(*b*) Equation is $\dfrac{x\,dy - y\,dx}{x^2} + 2x\,dx = 0$

or
$$d\left(\frac{y}{x}\right) + d(x^2) = 0$$

whose solution is $\dfrac{y}{x} + x^2 = C$ where C is an arbitrary constant. **Ans.**

Example 9.33. (a) Solve the logistic equation $\frac{dy}{dt} = ky(a-y)$ where $k > 0$ and $0 \le y \le a$.

(b) Solvve $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$.

[GGSIPU I Sem End Term 2012]

Solution: (a) Equation is separable and can be written as

$$\frac{dy}{y(a-y)} = kdt \quad \text{or} \quad \left(\frac{1}{y} + \frac{1}{a-y}\right)dy = akdt$$

which on integration, gives $\ln\left|\frac{y}{a-y}\right| = akt + C$ where C is an arbitrary constant.

Since $0 \le y \le a$, the above result simplifies to

$$y = \frac{ba}{b + \exp(-akt)}$$

where b is an arbitrary constant and the arbitrary constant b is related to c by $b = e^c$. Since C is arbitrary, the constant b is also arbitrary.

(b) Equation is $(x^2 + y^2)\,dx + 2x\,dx + 2y\,dy = 0$

or $$dx + \frac{2x\,dx + 2y\,dy}{x^2 + y^2} = 0 \quad \text{or} \quad dx + \frac{d(x^2 + y^2)}{x^2 + y^2} = 0$$

whose solution is $x + \log(x^2 + y^2) = C$ **Ans.**

EXERCISE 9B

Solve the following differential equations:

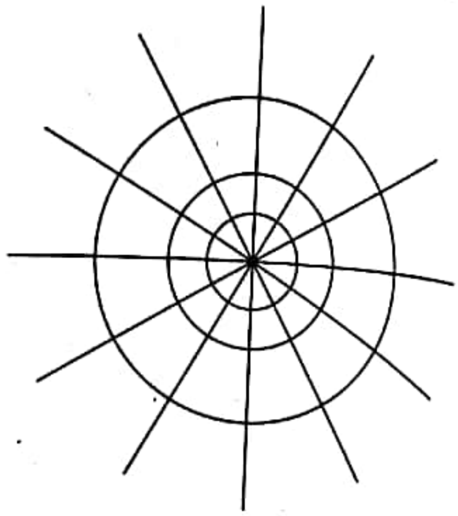
1. $(2x + y - 1)dy - (x - 2y + 5)dx = 0$
2. $(2xy + 2y^2 e^{2x})dx + (x^2 + 2y\, e^{2x})\, dy = 0$
3. $(y\, e^{xy} - \tan x)\, dx + (x\, e^{xy} - \sec y)\, dy = 0.$
4. $[y \sin (xy) + xy^2 \cos (xy)]\, dx + [x \sin (xy) + x^2 y \cos (xy)]dy = 0.$
5. $(x^4 e^x - 2a\, xy^2)dx + 2a\, x^2 y\, dy = 0.$
6. $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$
7. $y(x^2 y + e^x)dx - e^x\, dy = 0.$
8. $x^2 y\, dx - (x^3 + y^3)\, dy = 0.$
9. $(x^2y^2 + 5xy + 2)\, y\, dx + (x^2y^2 + 4xy + 2)\, x\, dy = 0.$
10. $(3xy^2 - y^3)dx + (xy^2 - 2x^2 y)dy = 0$
11. $(x + 2y^3)\dfrac{dy}{dx} = y + 2x^3 y^2$
12. $\left(xy \log \dfrac{x}{y}\right)dx + \left(y^2 - x^2 \log \dfrac{x}{y}\right)dy = 0.$
13. $(2x^2y^2 + y)\, dx - (x^3 y - 3x)\, dy = 0.$
14. $\left(2x \sinh \dfrac{y}{x} + 3y \cosh \dfrac{y}{x}\right)dx - 3x \cosh \dfrac{y}{x}\, dy = 0.$
15. $(y^2 + 2yx^2)dx + (2x^3 - xy)dy = 0.$
16. $x^2 (x\, dx + y\, dy) + y(x\, dy - y\, dx) = 0.$
17. $x\, dy - y\, dx = (x^2 + y^2)\, (x\, dx + y\, dy).$
18. $(2x^2 + 3y^2 - 7)x\, dx - (3x^2 + 2y^2 - 8)y\, dy = 0.$
19. $(x+y)^2 \left(x\dfrac{dy}{dx} + y\right) = xy\left(1 + \dfrac{dy}{dx}\right).$
20. $\left(x^4 \dfrac{dy}{dx} + x^3 y\right) = -\sec (xy).$
21. $\left[\cos x \log (2y-8) + \dfrac{1}{x}\right]dx + \dfrac{\sin x}{y-4}dy = 0, \quad y(1) = \dfrac{9}{2}$
22. $(2x \log x - xy)\, dy + 2y\, dx = 0$
23. $y(xy + 2x^2y^2)\, dx + x(xy - x^2y^2)\, dy = 0$
24. $(y^3 + x^3)\, dx - xy^2\, dy = 0$
25. $(x^2 + y^2 - a^2)\, x\, dx + (x^2 - y^2 - b^2)\, y\, dy = 0.$
26. Solve the equation $\left[y\left(1 + \dfrac{1}{x}\right) + \cos y\right]dx + (x + \log x - x \sin y)\, dy = 0.$
27. Solve $(xy^3 + y)\, dx + 2\, (x^2y^2 + x + y^4)\, dy = 0.$
28. Solve $(x^2y^2 + xy + 1)\, y\, dx + (x^2y^2 - xy + 1)\, x\, dy = 0.$
29. Solve $y\, (2xy + 1)\, dx + x\, (1 + 2xy - x^3y^3)\, dy = 0.$

APPLICATIONS OF DIFFERENTIAL EQUATIONS OF FIRST ORDER

Let us consider here only those physical and geometrical problems which lead to a differential equation of first order and first degree.

APPLICATIONS TO GEOMETRICAL PROBLEMS

ORTHOGONAL TRAJECTORIES: Two curves intersecting at a point P are said to be **orthogonal** if their tangents at P are perpendicular to each other. Two families of curves (or trajectories) are orthogonal if each curve of the first family is orthogonal to each curve of the second family, wherever an intersection takes place. Orthogonal families are formed in many situations. On the globe the parallels and meridians (also called latitudes and longitudes) are orthogonal families. Also the equipotential lines and the electrical lines of force are mutually orthogonal.



In the early days when calculus was being developed Newton the great, was occupied with the problem of finding the family of orthogonal trajectories of a given family of curves. In other words, suppose we are given a family S of curves in the plane and we want to construct another family T of curves so that every curve in S is orthogonal to every curve in T provided they intersect each other. As a simple example, suppose S consists of all circles about the origin. Then T consists of all straight lines through the origin as shown in the adjacent figure. Clearly each straight line is orthogonal to each circle whenever they intersect.

In general, suppose we are given a family S of curves given by the equation $F(x, y, k) = 0$, giving different curve for each choice of the constant k. Think of these curves as integral curves of a differential equation $\frac{dy}{dx} = f(x, y)$ which we determine from the equation $F(x, y, k) = 0$ by differentiation. At a point (x_0, y_0) the slope of the curve C in S through this point must have slope $-1/f(x_0, y_0)$.
Then the family T of orthogonal trajectories of S, consists of the integral curves of the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}.$$

Then we solve this differential equation for the curves in T.

EXAMPLE 9.34. Obtain the equation of the curve, the length of whose any tangent between its point of contact and its x-intercept is equal to the abscissa of the point of contact.

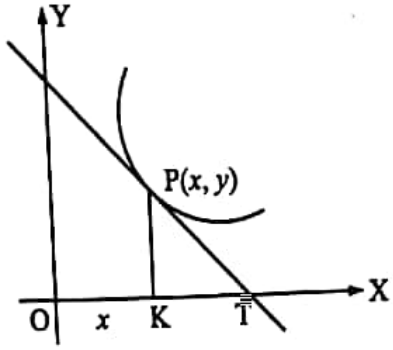
SOLUTION: The curve is such that the abscissa of the point $P(x, y) = x = PT$ = length of tangent between P, the point of contact, and the x-axis.

Equation of the tangent at P is $Y - y = \frac{dy}{dx}(X - x)$

It meets X-axis at T hence putting $Y = 0$, we get

$$X = OT = x - \frac{y}{dy/dx}, \quad KT = OT - x = -\frac{y}{dy/dx}.$$ (See the figure.)

$$PT=\sqrt{PK^2+KT^2} = \sqrt{y^2+\frac{y^2}{(dy/dx)^2}} = \frac{y\sqrt{1+(dy/dx)^2}}{dy/dx}$$



$\therefore$ Since $PT = x$ we have $y\sqrt{1+(dy/dx)^2} = x\dfrac{dy}{dx}$

or $$y^2+y^2\left(\frac{dy}{dx}\right)^2 = x^2\left(\frac{dy}{dx}\right)^2 \quad \text{or} \quad \frac{dx}{dy} = \sqrt{\frac{x^2}{y^2}-1}$$

Putting $v=\dfrac{x}{y}$ we get $v+y\dfrac{dv}{dy} = \sqrt{v^2-1}$ or $\dfrac{dv}{\sqrt{v^2-1}-v} = \dfrac{dy}{y}$

$$\therefore \quad \int\frac{dy}{y} = \int\frac{dv}{\sqrt{v^2-1}-v} = \int\frac{\sqrt{v^2-1}+v}{v^2-1-v^2}dv = -\int\left(\sqrt{v^2-1}+v\right)dv$$

$$= -\frac{v}{2}\sqrt{v^2-1}+\frac{1}{2}\log\left(v+\sqrt{v^2-1}\right)-\frac{v^2}{2}+\frac{C}{2}$$

or $$2\log y = -\frac{x}{y}\sqrt{\frac{x^2}{y^2}-1}-\frac{x^2}{y^2}+\log\left(\frac{x}{y}+\sqrt{\frac{x^2}{y^2}-1}\right)+C$$

or $$\frac{x}{y^2}\sqrt{x^2-y^2}+\frac{x^2}{y^2} = \log\frac{x+\sqrt{x^2-y^2}}{y^3}+C$$ which is the required solution. **Ans.**

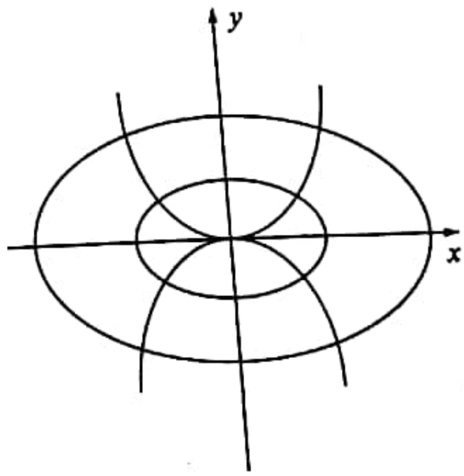
EXAMPLE 9.35. Given the family S of the curves given by $F(x, y, k) = y - kx^2 = 0$ which is a family of parabolas, determine the family T of orthogonal trajectories.

SOLUTION: First we obtain the differential equation of S by differentiating $y - kx^2 = 0$ and get $\dfrac{dy}{dx} = 2kx$. To eliminate k we use the equation $y - kx^2 = 0$ to have $k = \dfrac{y}{x^2}$.

Then we have $\dfrac{dy}{dx}-2\left(\dfrac{y}{x^2}\right)x = 0$ or $\dfrac{dy}{dx} = \dfrac{2y}{x} = f(x, y)$ is the differential equation of the family S.

Curves in S are integral curves of this differential equation which is of the form $\dfrac{dy}{dx} = f(x, y)$ with

$f(x, y) = \dfrac{2y}{x}$. The family T of orthogonal trajectories therefore



has the differential equation $\dfrac{dy}{dx} = \dfrac{-1}{f(x, y)} = \dfrac{-x}{2y}$.

This equation is of variable separable form, since $2y\,dy = -x\,dx$

which on integration gives $y^2 = -\dfrac{1}{2}x^2+C$

This is family of ellipses $\dfrac{1}{2}x^2+y^2 = C$

Some of the parabolas and ellipses from S and T are shown in the adjoining figure. Each parabola in S is orthogonal to each ellipse in T wherever these curves intersect.

APPLICATION TO PHYSICAL PROBLEMS

EXAMPLE 9.36. An object of mass m is falling under gravity and facing air resistance proportional to the square of the velocity. If after falling through a distance x it possesses a velocity v at that instant prove that $\frac{2kx}{m} = \log\frac{a^2}{a^2 - v^2}$ where k is the constant of proportionality and $\frac{mg}{k} = a^2$.

SOLUTION: Let the weight of the object be mg and the air resistance be kv^2 where k is the constant of proportionality then by Newton's second law of motion

$$m\frac{d^2x}{dt^2} = mg - kv^2 \text{ where } v = \frac{dx}{dt}$$

Here the acceleration $\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ $\therefore$ $mv\frac{dv}{dx} = mg - kv^2$ ...(1)

Putting $v^2 = z$, hence $2v\frac{dv}{dx} = \frac{dz}{dx}$, we get

$$\frac{m}{2}\frac{dz}{dx} = mg - kz \quad \text{or} \quad \frac{m}{2k}\frac{dz}{dx} = \frac{mg}{k} - z = a^2 - z \quad \text{where} \quad a^2 = \frac{mg}{k}$$

or $$\frac{dz}{a^2 - z} = \frac{2k}{m}dx$$

Integrating both the sides, we get $-\log(a^2 - z) = \frac{2kx}{m} + C$

or $-\log(a^2 - v^2) = \frac{2kx}{m} + C$ where C is an arbitrary constant. ...(2)

Initially $v = 0$ at $x = 0$ $\therefore$ $C = -\log a^2$ hence (2) becomes $\frac{2kx}{m} = \log a^2 - \log(a^2 - v^2)$

or $\frac{2kx}{m} = \log\frac{a^2}{a^2 - v^2}$ is the desired result. **Ans.**

EXAMPLE 9.37. If $\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$ where R is the radius of the earth and $r\ (\geq R)$ is the distance of an object at any time t, projected vertically upwards from the centre of the earth with velocity $v_0 = \sqrt{2gR}$, show that the object will never return to the earth.

SOLUTION: We have $$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}v \quad \text{where } v = \frac{dr}{dt}$$
$$= \frac{dv}{dr}\cdot\frac{dr}{dt} = v\frac{dv}{dr}$$

$\therefore$ The given equation becomes $v\frac{dv}{dr} = -\frac{gR^2}{r^2}$ or $2v\,dv = \frac{-2gR^2\,dr}{r^2}$

Integrating both sides, we get $v^2 = \frac{2gR^2}{r} + C$

We are given that $v = v_0 = \sqrt{2gR}$ when $r = R$, therefore $C = 0$

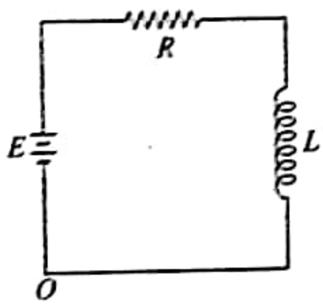
$\therefore$ $v^2 = \frac{2gR^2}{r}$ or $v = R\sqrt{\frac{2g}{r}}$

Thus, the object will not come to rest as $v = 0$ only when $r = \infty$, meaning thereby that the object will never come back to earth.

Hence Proved.

APPLICATIONS TO ELECTRICAL CIRCUITS

EXAMPLE 9.38. Consider an **LR-circuit** as shown in the figure. Starting at a point O, move clockwise around the circuit, first crossing the battery, where there is an increase in potential of E volts. Next there is a decrease in potential of iR volts across the resistance. Finally there is a decrease of $L\frac{di}{dt}$ across the inductance and we come back to A. By Kirchoff's voltage law $E - iR - L\frac{di}{dt} = 0$. Find the current i in the circuit at any time t.



SOLUTION: Given equation can be written as

$$\frac{di}{dt}+i\frac{R}{L} = \frac{E}{L}$$ which is linear in i. I.F. $= e^{\int \frac{R}{L}dt} = e^{Rt/L}$.

Hence the solution is $ie^{Rt/L} = \frac{E}{L}e^{Rt/L} + k$ or $i = \frac{E}{L} + ke^{-Rt/L}$ where k is some constant.

To determine the constant k we need to be given the current at sometime. Even without this it can be observed from the equation that as $t \to \infty$ the current i approaches the limit E/L. This is actually the steady-state value of the current in the circuit.

APPLICATIONS TO HEAT CONDUCTION (OR HEAT FLOW) PROBLEMS

Following are the basic rules to be followed while dealing with the heat flow problems :

(*i*) Heat flows from higher temperature to lower temperature

(*ii*) The amount of heat in a body is proportional to its mass and temperature

(*iii*) The amount of heat Q flowing per second through a slab of surface area A and thickness δx whose faces are at temperatures t and $t + \delta t$, is given by

$$Q = -kA\frac{dt}{dx}$$ where k is the co-efficient of thermal conductivity of the material of the body.

EXAMPLE 9.39. A pipe 20 cm in diameter contains steam at 160°C and is protected with a covering 5 cm thick for which k, the coefficient of thermal conductivity = 10025. If the temperature of the outer surface of the covering is 50°C, find the temperature at half way through the covering under steady conditions.

SOLUTION: We have $Q = -kA\frac{dt}{dx} = -k(2\pi x)\frac{dt}{dx}$,

since A is the surface area of the pipe of length 1cm. or $dt = \frac{-Q}{2k\pi}\frac{dx}{x}$ ...(1)

We are given that $t = 100°$ C at $x = 10$ cm and $t = 50°$ C at $x = 15$ cm.

Integrating (1) throughout, gives $\int_{160}^{50} dt = \frac{-Q}{2\pi k}\int_{10}^{15}\frac{dx}{x}$

or $\quad -110° = \dfrac{-Q}{2\pi k}\log\dfrac{15}{10}$ $\quad$ or $\quad \dfrac{Q}{2\pi k} = \dfrac{110°}{\log 1.5}$. ...(2)

Let $t = T$ when $x = 12.5$ (that is, half way through the covering), we have

$$\int_{160°}^{T} dt = \frac{-Q}{2\pi k}\int_{10}^{12.5}\frac{dx}{x} \quad \Rightarrow \quad T - 160° = \frac{-Q}{2\pi k}[\log x]_{10}^{12.5}$$

Using (2), we get $T = 160° - 110° \dfrac{\log 1.25}{\log 1.5} = 99.5°$ C. **Ans.**

APPLICATIONS TO PROBLEMS ON CHEMICAL REACTIONS

EXAMPLE 9.40. (Radioactive Decay and Carbon Dating)

In the phenomenon of radio activity, mass of radioactive substance gets converted to energy by radiation and hence the mass decays. Actually, the rate of change of mass of a radioactive substance is proportional to the mass itself. Thus, if $m(t)$ is the mass at time t, then

$$\frac{dm}{dt} = km \qquad ...(1)$$

where k is the constant of proportionality which depends on the substance.

or $\quad \dfrac{dm}{m} = k\,dt$ which gives $\log m = kt + C$ as $m > 0$

or $\quad m(t) = e^{kt + C} = A\,e^{kt}$ where A is some positive quantity.

Suppose at zero time mass is M then $m(0) = A = M$, hence $m(t) = M\,e^{kt}$.

Next, if at some time T we find that mass is M_T, then $m(T) = M_T = M\,e^{kT}$.

$$\Rightarrow \quad \ln\left(\frac{M_T}{M}\right) = kT \quad \text{or} \quad k = \frac{1}{T}\ln\left(\frac{M_T}{M}\right)$$

Thus k is known and we have $m(t) = M\,e^{\ln(M_T/M)t/T}$.

A more convenient formula for the mass can be obtained if we choose the time of the second measurement at the time $T = H$ at which exactly half of the mass has radiated, that is, at this time half of the mass remains. Thus, $M_T = M/2$ or $\dfrac{M_T}{M} = \dfrac{1}{2}$ hence we have $m(t) = M\,e^{\ln(1/2)\,t/H}$.

or $\quad m(t) = M\,e^{-(\ln 2)\,t/H}$. ...(2)

This number H is called the half-life of the element.

Infact, between any times t_1 and $t_1 + H$ exactly half of the mass of the element present at t_1 will radiate away. For this we write

$$m(t_1 + H) = M\,e^{-(\ln 2)(t_1+H)/H} = M\,e^{-\ln(2)\,t_1/H}\,e^{-(\ln 2)\,H/H}$$

$$= M\,e^{-\ln(2)}\,m(t_1) = \frac{1}{2}m(t_1)$$

Equation (2) is the basis for an important technique used to estimate the ages of certain ancient artifacts. The concept is also used to the study of carbon dating.

EXERCISE 9C

1. A particle is projected vertically upwards in the gravitational field under a resistance equal to k times the square of its velocity per unit mass. If V_0 is the initial velocity find the maximum height attained.

2. The differential equation for the motion of a particle moving in a central orbit is
$$\frac{d^2x}{dt^2} = -\mu\left(x + \frac{a^4}{x^3}\right).$$
If it starts from test at a distance 'a', find the time it will take to arrive at the origin.

3. A car starts from rest and its acceleration at a time t is equal to $k\left(1 - \frac{t}{T}\right)$ where k is a constant and T is the time taken to attain the highest speed which is maintained constant after that time. Find the highest speed and the distance travelled till this speed is attained.

4. Show that the curve for which the portion of the tangent intercepted between the co-ordinate axes is bisected at the point of contact, is a rectangular hyperbola.

5. Find the equation of the curve whose subtangent is twice the abscissa of the point of contact and passes through the point (1, 2).

6. Find the orthogonal trajectories of a system of confocal and co-axial parabolas.

7. In an electric circuit containing an inductance L, a resistance R and a battery E, the current i builds up at a rate given by $L\frac{di}{dt} + Ri = E$. Find the current i in terms of time t. Also, calculate the time taken before the current takes half its peak value.

8. A voltage Ee^{-at} is applied initially to an electric circuit containing inductance L and resistance R. Obtain the current developed at any time t.

9. The temperature θ of a body satisfies the equation $\frac{d\theta}{dt} = -k(\theta - \theta_0)$ where θ_0 is the temperature of the surrounding medium. A thermometer which reads 75° F indoors is taken outdoors. After 5 minutes it reads 65°F and after another 5 minutes, it reads 60°F.

What is the outside temperature ?

10. The rate of which certain mass of radium decomposes is at any instant proportional to the mass then present and the mass decreases to half its original values in 1500 years. If the initial mass be 100 mg, in how many years will it be 80 mg ?

11. The equation of e.m.f. in terms of current for an electric circuit having resistance R and a condenser of capacity C in series, is $E = Ri + \int \frac{i\,dt}{c}$. Find the current i at any time t when $E = E_0 \sin \omega t$.

□□□

CHAPTER

10 Linear Differential Equations of Higher Order, Method of Variation of Parameters and Applications

Linear Differential Equations of Higher Order with Constant Coefficients, Homogeneous and Non-homogeneous Differential Equations Reducible to Linear Form with Constant Coefficients, Method of Variation of Parameters. Applications to Physical Problem.

INTRODUCTION

In this section we shall discuss the techniques for solving the ordinary differential equations of order higher than one. For instance, the general second order linear differential equation is of the form

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)\,y=S(x). \qquad ...(1)$$

An important example of this type, is the equation of motion of a mass on a spring:

$m\dfrac{d^2u}{dt^2}+c\dfrac{du}{dt}+ku=F(t)$ where m, c, k are constants and $F(t)$ is some given function.

Similarly, the equations $(1-x^2)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+n(n+1)\,y=0$

and $\quad x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+(x^2+n^2)y=0.$

which are respectively the Legendre's equation and the Bessel's equation, arise in many engineering problems.

In equation (1) if the functions P, Q, R and S are continuous in some interval (a, b) and $P(x)$ is nowhere zero in (a, b) then it can be written in the form

$\dfrac{d^2y}{dx^2}+p(x)\dfrac{dy}{dx}+q(x)y=r(x)$ and it will have a unique solution.

The general linear equation of n^{th} order is of the form

$$\frac{d^ny}{dx^n}+P_1(x)\frac{d^{n-1}y}{dx^{n-1}}+P_2(x)\frac{d^{n-2}y}{dx^{n-2}}+...+P_n(x)y=X$$

where $P_1, P_2, ... P_n$ and X are functions of x or are constants.

LINEAR DIFFERENTIAL EQUATION WITH CONSTANT CO-EFFICIENTS

The general form of such an equation is

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = X \quad \dots(1)$$

where X is a function of the independent variable x and a_i, $(i = 1, \dots, n)$ are constants.

Using the differential operator D for $\frac{d}{dx}$ (so that $\boxed{\frac{dy}{dx} = Dy}$, $\boxed{\frac{d^2y}{dx^2} = D^2y}$, ..., $\frac{d^ny}{dx^n} = D^ny$)

equation (1) can be written as $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n)\ y = X \quad \dots(2)$

or $\quad f(D)y = X$

where $f(D) \equiv D^n + a_1 D^{n-1} + \dots + a_n$, is a polynomial differential operator.

The operator D behaves much the same as an algebraic quantity to a limited extent. Some properties concerning D, are as follows:

$$D^n (ay + bz) = aD^n y + bD^n z,$$

$$D^m (D^n y) = D^n (D^m y) = D^{m+n} y,$$

$$(D^m + D^n)\, y = D^m y + D^n y,$$

$$(D - m_1)(D - m_2)\, y = (D - m_2)(D - m_1) y,$$

$$[f_1(D) + f_2(D)]y = f_1(D)y + f_2(D)y,$$

$$f_1(D) f_2(D)\, y = f_1(D)\, [f_2(D)y].$$

The indices can be negative as well, then $D^{-1}y = \frac{1}{D}y = z$ so that $Dz = y$.

Now, an obvious, but very important and useful result follows :

" If y_1 and y_2 are two solutions of the differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = 0 \quad \dots(3)$$

then $c_1 y_1 + c_2 y_2$ is also its solution where c_1 and c_2 are arbitrary constants."

In general, the general solution of a differential equation of order n will have n arbitrary constants, therefore it follows that if $y_1, y_2, \dots, y_n$ are n independent solutions of (3) then $u = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is the complete general solution of (3).

Therefore, if $y = u$ is a solution of the differential equation (3), then

$$\frac{d^n u}{dx^n} + a_1 \frac{d^{n-1}u}{dx^{n-1}} + a_2 \frac{d^{n-2}u}{dx^{n-2}} + \dots + a_n u = 0 \quad \dots(4)$$

And, if $y = v$ is a particular solution of

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = X \quad \dots(5)$$

then we have $\quad \frac{d^n v}{dx^n} + a_1 \frac{d^{n-1}v}{dx^{n-1}} + a_2 \frac{d^{n-2}v}{dx^{n-2}} + \dots + a_n v = X \quad \dots(6)$

Adding (4) and (6) we get

$$\frac{d^n (u+v)}{dx^n} + a_1 \frac{d^{n-1}(u+v)}{dx^{n-1}} + a_2 \frac{d^{n-2}(u+v)}{dx^{n-2}} + \dots + a_n (u+v) = X$$

which shows that $y = u + v$ is the complete solution of (5). The part u is called the ***Complementary Function*** (C.F.) and part v is called the ***Particular Integral*** (P.I.)

Thus, the complete solution of (5) is $y = \text{C.F.} + \text{P.I.}$

THE COMPLEMENTARY FUNCTION (C.F.)

The differential equation under consideration is

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n y = 0 \quad \ldots(1)$$

which, in the symbolic operator form, is $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)\ y = 0$

or $\quad f(D)\, y = 0 \quad \ldots(2)$

Since $f(D)$ is a polynomial in D of degree n and D behaves the same as an algebraic quantity we can, in general, factorize $f(D)$ into n linear factors and equation (2) can be written as

$$(D - m_1)(D - m_2) \ldots (D - m_n)\, y = 0$$

where $m_1, m_2, \ldots, m_n$ are the roots of the *Auxiliary Equation* (A.E.) $f(m) = 0$

or $\quad m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0 \quad \ldots(3)$

CASE I. Roots of the auxiliary equation are all real and distinct.

Let us first find the solution of $(D - m)\, y = 0$, *i.e.*, $\frac{dy}{dx} - my = 0$

or $\quad \frac{dy}{y} = m\, dx$ which on integration, gives

$\log y = mx + \log c \quad$ or $\quad y = ce^{mx}$ where c is an arbitrary constant.

Therefore, the solutions corresponding to $(D - m_1)y = 0,\ (D - m_2)\, y = 0,\ \ldots,\ (D - m_n)\, y = 0$ are $y = c_1 e^{m_1 x},\ \ y = c_2 e^{m_2 x},\ \ldots,\ \ y = c_n e^{m_n x}$

Thus, the complementary function which is also the complete solution of (1) in this case, is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \quad \ldots(4)$$

where $c_1, c_2, \ldots, c_n$ are n arbitrary constants.

CASE II. The auxiliary equation having multiple roots.

Suppose $m_2 = m_1$, i.e., m_1 is a double root of the A.E. then the solution (4) becomes

$$y = (c_1 + c_2)\ e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

or $\quad y = c\, e^{m_1 x} + c_3\, e^{m_3 x} + \cdots + c_n\, e^{m_n x}$

where $c\ (= c_1 + c_2)$ is one arbitrary constant. Thus, it has only $(n - 1)$ arbitrary constants and therefore is not a complete solution of (1) since a complete solution should have exactly the same number of arbitrary constants as the order of the differential equation. To solve this hurdle consider the part

$$(D - m_1)(D - m_1)\, y = 0 \quad \ldots(5)$$

Writing $(D - m_1)\, y = t$ equation (5) becomes $(D - m_1)\, t = 0 \quad \ldots(6)$

whose solution is $\quad t = c_1\, e^{m_1 x} \quad$ or $\quad \frac{dy}{dx} - m_1 y = c_1\, e^{m_1 x}.$

Therefore, (6) becomes $\quad (D - m_1)\, y = c_1\, e^{m_1 x}$

It is linear in y hence the I.F. $= e^{\int -m_1 dx} = e^{-m_1 x}$

and the solution of (6) is $\quad ye^{-m_1 x} = \int c_1\, e^{-m_1 x} e^{m_1 x}\, dx = c_1 x + c_2 \quad$ or $\quad y = (c_1 x + c_2)\ e^{m_1 x}.$

Thus, when $m_2 = m_1$ the solution of (1) is $\quad y = (c_1 x + c_2)\ e^{m_1 x} + c_3\, e^{m_3 x} + \cdots + c_n\, e^{m_n x}.$

Similarly, if m_1 is a three fold root, *i.e.*, $m_1 = m_2 = m_3$ then the corresponding part in the solution is $\quad (c_1 x^2 + c_2 x + c_3)\ e^{m_1 x}.$

CASE III. A.E. having Complex Roots

Since the complex roots occur in pairs, let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, the corresponding part of the solution of the equation $f(D)y = 0$ takes the form

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} = e^{\alpha x}[c_1 e^{i\beta x} + c_2 e^{-i\beta x}]$$

$$= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)]$$

$$= e^{\alpha x}[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x]$$

If we write $c_1' = c_1 + c_2$ *and* $c_2' = i(c_1 - c_2)$ as two new arbitrary constants, the corresponding part of the solution can be written as $y = e^{\alpha x}[c_1'\cos\beta x + c_2'\sin\beta x]$.

In case, there are two pairs of equal complex roots, *i.e.*, if

$m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta,\ m_3 = \alpha + i\beta,\ m_4 = \alpha - i\beta$ then the solution corresponding to the factors $[D - (\alpha + i\beta)]^2 [D - (\alpha - i\beta)]^2$, is

$$y = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x].$$

EXAMPLE 10.1. Solve $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} - 3y = 0$

SOLUTION: The given equation can be written as $(D^3 - D^2 - 5D - 3)y = 0$ where $D \equiv \dfrac{d}{dx}$.

The auxiliary equation is $m^3 - m^2 - 5m - 3 = 0$

or $(m + 1)(m^2 - 2m - 3) = 0$ or $(m + 1)^2(m - 3) = 0$

Therefore, the roots of the auxiliary equation are $m = -1, -1, 3$.

Then the complete solution of the given equation, is $y = (c_1 x + c_2)e^{-x} + c_3 e^{3x}$.

Ans.

EXAMPLE 10.2. Solve $\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} - 14\dfrac{dy}{dx} - 12y = 0.$

SOLUTION: The equation can be written as

$$(D^4 + D^3 - 6D^2 - 14D - 12)y = 0 \quad \text{where} \quad D \equiv \frac{d}{dx}.$$

The auxiliary equation is $m^4 + m^3 - 6m^2 - 14m - 12 = 0$

or $(m - 3)(m^3 + 4m^2 + 6m + 4) = 0$

or $(m - 3)(m + 2)(m^2 + 2m + 2) = 0$ Thus, the roots of the auxiliary equation (A.E.) are $m = 3, -2, -1 + i, -1 - i$.

The solution corresponding to the roots $3, -2$ is $c_1 e^{3x} + c_2 e^{-2x}$ and the solution corresponding to the roots $-1 \pm i$ is $e^{-x}(c_3\cos x + c_4\sin x)$

Therefore the complete solution of the given equation, is

$$y = c_1 e^{3x} + c_2 e^{-2x} + e^{-x}(c_3\cos x + c_4\sin x)$$

Ans.

EXAMPLE 10.3. Solve (*a*) $(D^4 + 2a^2D^2 + a^4)y = 0$ where $D \equiv \frac{d}{dx}$.

[GGSIPU I Sem II Term 2012]

(*b*) $(D - 2)(D^2 + D + 1)^2 y = 0$

[GGSIPU I Sem End Term 2012]

SOLUTION: (*a*) Equation is $\frac{d^4y}{dx^4} + 2a^2\frac{d^2y}{dx^2} + a^4y = 0$

Its A.E. is $m^4 + 2a^2m^2 + a^4 = 0$ or $(m^2 + a^2)^2 = 0$ $\therefore$ $m = \pm ai, \pm ai$.

$\therefore$ C.F. $= (C_1x + C_2)\cos ax + (C_3x + C_4)\sin ax$.

Then $y = (C_1x + C_2)\cos ax + (C_3x + C_4)\sin ax$. **Ans.**

(*b*) $(D - 2)(D^2 + D + 1)^2y = 0$ where $D \equiv \frac{d}{dt}$

The roots of its A.E. are $m = 2, \ \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}, \ \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$.

Hence the solution of the given equation, is

$$y = e^{-t/2}\left[(C_1t + C_2)\cos\frac{\sqrt{3}t}{2} + (C_3t + C_4)\sin\frac{\sqrt{3}t}{2}\right] + C_5e^{2t} \quad \textbf{Ans.}$$

EXAMPLE 10.4. Solve $\frac{d^4y}{dx^4} + y = 0$.

SOLUTION: Here the auxiliary equation is $m^4 + 1 = 0$ which can be written as

$(m^2 + 1)^2 - 2m^2 = 0$ or $(m^2 - \sqrt{2}m + 1)(m^2 + \sqrt{2}m + 1) = 0$.

Its roots are $\frac{1 \pm i}{\sqrt{2}}$ and $\frac{-1 \pm i}{\sqrt{2}}$.

Therefore the general solution of the given equation, is

$$y = e^{-x/\sqrt{2}}\left(c_1\cos\frac{x}{\sqrt{2}} + c_2\sin\frac{x}{\sqrt{2}}\right) + e^{x/\sqrt{2}}\left(c_3\cos\frac{x}{\sqrt{2}} + c_4\sin\frac{x}{\sqrt{2}}\right) \quad \textbf{Ans.}$$

[GGSIPU I Sem End Term 2007]

EXAMPLE 10.5. Solve $y'' + 4y' + 4y = 0$, $y(0) = 1, y'(0) = 1$.

SOLUTION: The equation can be written as $(D^2 + 4D + 4)y = 0$ where $D = \frac{d}{dx}$.

The auxiliary equation is $m^2 + 4m + 4 = 0$ whose roots are $m = -2, -2$.

Therefore the complementary function $= (c_1x + c_2)e^{-2x}$

and the general solution is $y = (c_1x + c_2)e^{-2x}$ where c_1, c_2 are constants.

The initial conditions are $y(0) = 1$ and $y'(0) = 1$

Therefore we have $c_2 = 1$ and $y' = -2(c_1x + c_2)e^{-2x} + c_1e^{-2x}$

Putting $x = 0$ here, gives $1 = -2c_2 + c_1$ hence $c_1 = -3$

Thus the required solution is $y = (1 + 3x)e^{-2x}$ **Ans.**

THE PARTICULAR INTEGRAL. (P.I.)

Before discussing ways to find Particular Integral (P.I.) we have to introduce *inverse operator* $\frac{1}{f(D)}$.

$\frac{1}{f(D)}X(x)$ is that function of x which when operated upon by $f(D)$ gives $X(x)$ only.

Thus, $f(D)\left(\frac{1}{f(D)}X\right) = X.$

Therefore $\frac{1}{f(D)}X$ satisfies the equation $f(D)\,y = X$ and hence it is the Particular Integral (P.I.) of the equation $f(D)\,y = X$ and is symbolically written as $Yp = \frac{1}{f(D)}X.$

Let us first consider the inverse operator $\frac{1}{D}$.

By definition $D\left\{\frac{1}{D}X\right\} = X, \quad \Rightarrow \quad \frac{1}{D}X = \int X\,dx.$

Here no constant of integration is to be added as all the arbitrary constants are taken care of, in the complementary function. Next, consider $\frac{1}{D-a}X$.

Let $\frac{1}{D-a}X = v$ hence $(D-a)\,v = X$

or $\frac{dv}{dx} - av = X$ which is linear in v.

I.F. $= e^{\int -a\,dx} = e^{-ax}$, therefore the solution is

$e^{-ax}v = \int Xe^{-ax}\,dx$ or $v = e^{ax}\int Xe^{-ax}\,dx.$

Thus, we have $\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}\,dx.$

Next, if $f(D) \equiv (D-m_1)(D-m_2)\ldots(D-m_n)$

then $\text{P.I.} = \frac{1}{f(D)}X = \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_n)}X$

Resolving the operator $\frac{1}{f(D)}$ into partial fractions, we get

$$\text{P.I.} = \left(\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \ldots + \frac{A_n}{D-m_n}\right)X$$

$$= A_1\frac{1}{D-m_1}X + A_2\frac{1}{D-m_2}X + A_n\frac{1}{D-m_n}X$$

$$= A_1e^{m_1x}\int Xe^{-m_1x} + A_2e^{m_2x}\int Xe^{-m_2x} + \ldots + A_ne^{m_nx}\int Xe^{-m_nx}\,dx.$$

SHORT CUT METHODS FOR FINDING P.I.

Although the general method given above will always work, it, many times, may lead to laborious and difficult integration. To avoid this, short cut methods for getting the P.I. without actually integrating, have been developed depending upon the particular form of the function X, as follows :

I. When X = e^{ax}

Since $D\, e^{ax} = a\, e^{ax}$, $D^2 e^{ax} = a^2 e^{ax}$, ..., $D^n e^{ax} = a^n e^{ax}$,

we immediately have $f(D)\, e^{ax} = f(a)\, e^{ax}$.

Operating both sides of the above equation by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)}\{f(D)\, e^{ax}\} = \frac{1}{f(D)}\{f(a)\, e^{ax}\} \quad \text{or} \quad e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$$

$$\therefore \quad \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, \quad \textbf{provided} \quad f(a) \neq 0.$$

If $f(a) = 0$ it implies that $(D - a)$ is a factor of $f(D)$.

Let $f(D) = (D - a)\,\phi(D)$ where $\phi(a) \neq 0$. Then

$$\frac{1}{f(D)}e^{ax} = \frac{1}{D-a}\frac{1}{\phi(D)}e^{ax} = \frac{1}{D-a}\frac{1}{\phi(a)}e^{ax}$$

$$= \frac{1}{\phi(a)}\frac{1}{D-a}e^{ax} = \frac{1}{\phi(a)}e^{ax}\int e^{-ax}e^{ax}dx$$

$$= \frac{1}{\phi(a)}e^{ax}\int 1\, dx = \frac{x\, e^{ax}}{\phi(a)}, \quad \text{provided } \phi(a) \neq 0.$$

Similarly, if $(D - a)^2$ is a factor of $f(D)$,

let $f(D) = (D - a)^2\,\phi(D)$ say, where $\phi(a) \neq 0$, then

$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)^2\,\phi(D)}e^{ax} = \frac{1}{(D-a)^2\,\phi(a)}e^{ax} = \frac{1}{\phi(a)}\frac{1}{(D-a)}\left(\frac{1}{D-a}e^{ax}\right)$$

$$= \frac{1}{\phi(a)}\cdot\frac{1}{(D-a)}xe^{ax} = \frac{1}{\phi(a)}e^{ax}\int e^{-ax}xe^{ax}dx$$

$$= \frac{1}{\phi(a)}e^{ax}\int x\, dx = \frac{x^2}{2!}\frac{e^{ax}}{\phi(a)}.$$

and, in general, it can be shown that if $(D - a)^r$ occurs as a factor of $f(D)$, so that $f(D) = (D - a)^r\,\phi(D)$ say, then

$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)^r}\frac{1}{\phi(D)}e^{ax} = \frac{x^r}{r!}\frac{e^{ax}}{\phi(a)} \quad \text{where } \phi(a) \neq 0.$$

Another way of writing the above result is

$$\frac{1}{f(D)}e^{ax} = \frac{x^r}{\left[\frac{d^r}{D^r}f(D)\right]_{D=a}}e^{ax} \quad \text{if } f(a) = f'(a) = f''(a) = \ldots = f^{(r-1)}(a) = 0.$$

EXAMPLE 10.6. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^{-x}$ [GGSIPU I Sem End Term 2003]

SOLUTION: Given equation can be written as $(D^2 - 2D + 5)y = e^{-x}$ where $D = \frac{d}{dx}$

For C.F. the Auxiliary equation is $m^2 - 2m + 5 = 0$ whose roots are $1 \pm 2i$.

$\therefore$ C.F. $= e^x[c_1 \cos 2x + c_2 \sin 2x]$

And P.I. $= \dfrac{1}{D^2 - 2D + 5} e^{-x} = \dfrac{1}{(-1)^2 - 2(-1) + 5} e^{-x} = \dfrac{1}{8} e^{-x}$

Therefore, the complete solution of the given equation, is

$$y = \text{C.F.} + \text{P.I.} = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{8}e^{-x}.$$ **Ans.**

EXAMPLE 10.7. Find the complete solution of the equation

$$\frac{d^4y}{dx^4} + 5\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 8y = e^{-2x} + 2e^{-x} + 3e^x - 3.$$

SOLUTION: The given equation can be written as

$$(D^4 + 5D^3 + 6D^2 - 4D - 8)y = e^{-2x} + 2e^{-x} + 3e^x - 3 \quad \text{where } D \equiv \frac{d}{dx}.$$

The auxiliary equation $m^4 + 5m^3 + 6m^2 - 4m - 8 = 0$ has the roots $m = -2, -2, -2, 1$

$\therefore$ C.F. is $(c_1 + c_2x + c_3x^2)\, e^{-2x} + c_4e^x$

and P.I. $= \dfrac{1}{(D+2)^3 (D-1)} (e^{-2x} + 2e^{-x} + 3e^x - 3)$

$$= \frac{1}{(D+2)^3(D-1)}e^{-2x} + \frac{2}{(D+2)^3(D-1)}e^{-x} + \frac{3}{(D+2)^3(D-1)}e^x - \frac{3}{(D+2)^3(D-1)}e^{0x}$$

$$= \frac{1}{(D+2)^3}\left(\frac{1}{-2-1}e^{-2x}\right) + \frac{2}{(-1+2)^3(-1-1)}e^{-x} + \frac{3}{(1+2)^3}\left(\frac{1}{D-1}e^x\right) - \frac{3}{(0+2)^3(0-1)}e^{0x}$$

$$= -\frac{1}{3}\cdot\frac{x^3}{3!}e^{-2x} - e^{-x} + \frac{1}{9}\cdot\frac{x}{1!}e^x + \frac{3}{8}.1 = -\frac{x^3}{18}e^{-2x} - e^{-x} + \frac{x}{9}e^x + \frac{3}{8}.$$

Therefore, the complete solution is

$$y = \text{C.F.} + \text{P.I} = (c_1 + c_2x + c_3x^2)e^{-2x} + c_4e^x - \frac{x^3}{18}e^{-2x} - e^{-x} + \frac{x}{9}e^x + \frac{3}{8}.$$

or $$y = \left(c_1 + c_2x + c_3x^2 - \frac{x^3}{18}\right)e^{-2x} - e^{-x} + \frac{3}{8} + \left(c_4 + \frac{x}{9}\right)e^x$$

which is the required solution.

Ans.

II When X = sin (ax + b) or cos (ax + b)

We have $D^2 \sin(ax+b) = -a^2 \sin(ax+b)$ and $D^2 \cos(ax+b) = -a^2 \cos(ax+b)$

$D^4 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$ and $D^4 \cos(ax+b) = (-a^2)^2 \cos(ax+b)$

$D^6 \sin(ax+b) = (-a^2)^3 \sin(ax+b)$ and $D^6 \cos(ax+b) = (-a^2)^3 \cos(ax+b)$, etc.

$\Rightarrow$ $f(D^2) \sin(ax+b) = f(-a^2) \sin(ax+b)$ and $f(D^2) \cos(ax+b) = f(-a^2) \cos(ax+b)$

or $f(D^2)\,{}^{\sin}_{\cos}(ax+b) = f(-a^2)\,{}^{\sin}_{\cos}(ax+b)$.

Operating both sides by $\dfrac{1}{f(D^2)}$, we get

$$\frac{1}{f(D^2)}\left\{f(D^2)\,\underset{\cos}{\sin}(ax+b)\right\} = \frac{1}{f(D^2)}\left\{f(-a^2)\,\underset{\cos}{\sin}(ax+b)\right\}$$

or $$\underset{\cos}{\sin}(ax+b) = f(-a^2)\cdot\frac{1}{f(D^2)}\underset{\cos}{\sin}(ax+b)$$

$$\Rightarrow \quad \frac{1}{f(D^2)}\underset{\cos}{\sin}(ax+b) = \frac{1}{f(-a^2)}\underset{\cos}{\sin}(ax+b), \quad \text{provided } f(-a^2) \neq 0$$

CASE OF FAILURE. when $f(-a^2) = 0$.

In this case, to evaluate $\frac{1}{f(D^2)}\underset{\cos}{\sin}(ax+b)$, let us consider

$$\frac{1}{D^2+a^2}\,[\cos(ax+b) + i\sin(ax+b)] = \frac{1}{D^2+a^2}\,e^{i(ax+b)} \quad \text{where } i = \sqrt{-1}$$

$$= e^{ib}\frac{1}{D^2+a^2}e^{aix} = e^{ib}\frac{1}{(D-ai)(D+ai)}e^{aix} = e^{ib}\frac{1}{D-ai}\left(\frac{1}{(D+ai)}.e^{aix}\right)$$

$$= e^{ib}\frac{1}{(D-ai)}\left(\frac{1}{2ai}e^{aix}\right) = \frac{e^{ib}}{2ai}\cdot\frac{1}{D-ai}.e^{aix} = \frac{e^{ib}}{2ai}\cdot\frac{x}{1}.e^{aix} = \frac{x}{2ai}e^{i(ax+b)}$$

$$= \frac{x}{2ai}[\cos(ax+b) + i\sin(ax+b)] = \frac{x}{2a}[-i\cos(ax+b) + \sin(ax+b)]$$

Equating real and imaginary parts on both sides, we get

$$\frac{1}{D^2+a^2}\sin(ax+b) = -\frac{x}{2a}\cos(ax+b)$$

$$\frac{1}{(D^2+a^2)}\cos(ax+b) = \frac{x}{2a}\sin(ax+b).$$

Actually when $f(-a^2) = 0$ then (D^2+a^2) must be a factor of $f(D^2)$, let $f(D^2) = (D^2+a^2)\,\phi(D^2)$.

$$\therefore \quad \frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{(D^2+a^2)\,\phi(D^2)}\sin(ax+b), \quad \text{provided } \phi(-a^2) \neq 0$$

$$= \frac{1}{(D^2+a^2)}\frac{1}{\phi(-a^2)}\sin(ax+b)$$

$$= \frac{1}{\phi(-a^2)}\frac{1}{D^2+a^2}\sin(ax+b) = \frac{-x}{\phi(-a^2).2a}\cos(ax+b).$$

Similarly $$\frac{1}{f(D^2)}\cos(ax+b) = \frac{x}{\phi(-a^2).2a}\sin(ax+b)$$

Now we deal with the problem of finding $\frac{1}{f(D)}\underset{\cos}{\sin}(ax+b)$ where $f(D)$ will have even as well as odd powers of D.

It will be made clear through the following problems :

EXAMPLE 10.8. Solve $(D^3 + D^2 - D - 1)\,y = \sin(2x + 3)$ where $D \equiv \dfrac{d}{dx}$

SOLUTION: The auxiliary equation is $m^3 + m^2 - m - 1 = 0$

or $(m + 1)(m^2 - 1) = 0$ whose roots are $-1, -1, 1$.

$\therefore$ C.F. $= c_1 e^x + (c_2 x + c_3) e^{-x}$

Next, P.I. $= \dfrac{1}{D^3 + D^2 - D - 1} \sin(2x + 3)$ (Now replacing D^2 by -2^2)

$$= \frac{1}{-2^2 D + (-2^2) - D - 1} \sin(2x+3) = -\frac{1}{5}\left(\frac{1}{D+1}\right)\sin(2x+3)$$

$$= -\frac{1}{5}\left(\frac{D-1}{D^2-1}\right)\sin(2x+3) = -\frac{1}{5}\left(\frac{D-1}{-2^2-1}\right)\sin(2x+3) = \frac{1}{25}[D\sin(2x+3) - 1.\sin(2x+3)]$$

$$= \frac{1}{25}[2\cos(2x+3) - \sin(2x+3)].$$

Therefore, the complete solution of the given equation, is

$$y = c_1 e^x + (c_2 x + c_3) e^{-x} + \frac{2}{25}\cos(2x+3) - \frac{1}{25}\sin(2x+3).$$ **Ans.**

EXAMPLE 10.9. Solve $\dfrac{d^2y}{dx^2} + y = \cos x \cos 2x$

SOLUTION: The auxiliary equation $m^2 + 1 = 0$ has the roots $0 \pm i$.

$\therefore$ C.F $= e^{0x}[c_1 \sin x + c_2 \cos x] = c_1 \sin x + c_2 \cos x$

and P.I. $= \dfrac{1}{D^2+1}\cos x \cos 2x = \dfrac{1}{2}\dfrac{1}{D^2+1}[\cos x + \cos 3x]$

$$= \frac{1}{2}\frac{1}{D^2+1}\cos x + \frac{1}{2}\frac{1}{D^2+1}\cos 3x = \frac{1}{2}\cdot\frac{x}{2D}\cos x + \frac{1}{2}\frac{1}{-3^2+1}\cos 3x$$

$$= \frac{x}{4}\int \cos x\,dx - \frac{1}{16}\cos 3x = \frac{x}{4}\sin x - \frac{1}{16}\cos 3x$$

Therefore, the solution of the given differential equation, is

$$y = c_1 \sin x + c_2 \cos x + \frac{x}{4}\sin x - \frac{1}{16}\cos 3x.$$

III. When X = sinh (ax + b) or cosh (ax + b)

Since $D^2 \underset{\cosh}{\sinh}(ax+b) = a^2 \underset{\cosh}{\sinh}(ax+b)$, $D^4 \underset{\cosh}{\sinh}(ax+b) = a^4 \underset{\cosh}{\sinh}(ax+b)$, etc, therefore we have $f(D^2)\underset{\cosh}{\sinh}(ax+b) = f(a^2)\underset{\cosh}{\sinh}(ax+b)$

$$\Rightarrow \quad \frac{1}{f(D^2)}\underset{\cosh}{\sinh}(ax+b) = \frac{1}{f(a^2)}\underset{\cosh}{\sinh}(ax+b), \text{ provided } f(a^2) \neq 0.$$

EXAMPLE 10.10. Solve $\dfrac{d^3y}{dx^3} + 5\dfrac{dy}{dx} = \sinh 2x.$

SOLUTION: The equation is $D(D^2 + 5)y = \sinh 2x.$

The A.E. $m(m^2 + 5) = 0$ has roots $m = 0, \pm i\sqrt{5}$

$\therefore$ C.F. $= c_1 + c_2 \cos\sqrt{5}\,x + c_3 \sin\sqrt{5}\,x.$

Next, $\quad$ P.I. $= \dfrac{1}{D^3+5D}\sinh 2x = \dfrac{1}{D^2.D+5D}\sinh 2x$

$$= \frac{1}{2^2D+5D}\sinh 2x = \frac{1}{9D}\sinh 2x = \frac{1}{9}\int \sinh 2x\,dx = \frac{1}{18}\cosh 2x.$$

Therefore, the given equation has complete solution as

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2\cos\sqrt{5}x + c_3\sin\sqrt{5}x + \frac{1}{18}\cosh 2x. \qquad \textbf{Ans.}$$

IV. When $X = x^m$

If X is a polynomial in x, let us consider $\dfrac{1}{f(D)}x^m = [f(D)]^{-1}x^m$.

We expand $\{f(D)\}^{-1}$ in ascending powers of D upto the m^{th} power then evaluate the above expression through differentiation of x^m. Obviously the derivatives of order higher than m are not to be considered. The procedure will be illustrated in the following problems.

EXAMPLE 10.11. Solve $(D^3 - 3D^2 + 2D)\,y = x^2$ where $D \equiv \dfrac{d}{dx}$.

SOLUTION: The auxiliary equation is $m^3 - 3m^2 + 2m = 0$, or $m(m^2 - 3m + 2) = 0$

Its roots are $m = 0, 1, 2$, therefore, C.F. $= C_1e^{0x} + C_2e^x + C_3e^{2x}$.

Next, $\quad$ P.I. $= \dfrac{1}{D^3-3D^2+2D}x^2 = \dfrac{1}{2D}\left[1 - \dfrac{3}{2}D + \dfrac{1}{2}D^2\right]^{-1}x^2$

$$= \frac{1}{2D}\left[1 + \frac{3}{2}D - \frac{1}{2}D^2 + \left(\frac{3}{2}D - \frac{1}{2}D^2\right)^2 + \left(\frac{3}{2}D - \frac{1}{2}D^2\right)^3 + \ldots\right]x^2$$

$$= \frac{1}{2D}\left[1 + \frac{3}{2}D + D^2\left(-\frac{1}{2} + \frac{9}{4}\right) + D^3\left(-\frac{3}{2} + \frac{27}{8}\right) + \ldots\right]x^2$$

$$= \left[\frac{1}{2D} + \frac{3}{4} + \frac{7}{8}D + \frac{15}{16}D^2\right]x^2 \qquad \text{(retaining terms up to } D^2 \text{ here)}$$

$$= \frac{1}{2}\frac{x^3}{3} + \frac{3}{4}x^2 + \frac{7}{8}(2x) + \frac{15}{16}.(2) = \frac{x^3}{6} + \frac{3}{4}x^2 + \frac{7}{4}x + \frac{15}{8}.$$

$\therefore$ The complete solution is $\quad y = \text{C. F.} + \text{P.I.} = C_1 + C_2e^x + C_3e^{2x} + \dfrac{x^3}{6} + \dfrac{3}{4}x^2 + \dfrac{7}{4}x$

where $\dfrac{15}{8}$ has been absorbed in the arbitrary constant C_1. **Ans.**

EXAMPLE 10.12. Solve $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + x^2 + x$. [GGSIPU I Sem End Term 2004]

SOLUTION: The given equation is $(D^3 + 2D^2 + D)\,y = e^{2x} + x^2 + x$ where $D = \dfrac{d}{dx}$.

For C.F., the A.E. is $m^3 + 2m^2 + m = 0$ whose roots are $m = 0, -1, -1$.

Hence $\quad$ C.F. $= c_1 + (c_2 + c_3x)e^{-x}$

and $\quad$ P.I. $= \dfrac{1}{D^3+2D^2+D}(e^{2x} + x^2 + x)$

$$= \frac{1}{D^3+2D^2+D}e^{2x} + \frac{1}{D^3+2D^2+D}(x^2+x)$$

$$= \frac{1}{2^3 + 2.2^2 + 2} e^{2x} + \frac{1}{D}(1+D)^{-2}(x^2 + x)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D}\left[1 - 2D + \frac{(-2)(-3)}{2} D^2\right](x^2 + x)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D}[x^2 + x - 2(2x+1) + 6] \quad \text{since} \quad \frac{1}{D} \equiv \int$$

$$= \frac{1}{18} e^{2x} + \left[\frac{x^3}{3} - \frac{3x^2}{2} + 4x\right]$$

Therefore the complete solution is

$$y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{1}{18} e^{2x} + \frac{x^3}{3} - \frac{3}{2}x^2 + 4x. \qquad \textbf{Ans.}$$

EXAMPLE 10.13. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$ [GGSIPU I Sem End Term 2003]

SOLUTION: Given equation can be written as $(D^2 + D - 2)\, y = x + \sin x$, $D \equiv \frac{d}{dx}$

For *C.F.*, the auxiliary equation is $m^2 + m - 2 = 0$ whose roots are $m = 1, -2$

$$\therefore C.F. = c_1 e^x + c_2 e^{-2x}.$$

and $$P.I. = \frac{1}{D^2 + D - 2}(x + \sin x) = \frac{1}{D^2 + D - 2} \sin x + \frac{1}{-2 + D + D^2} x$$

$$= \frac{1}{-1 + D - 2} \sin x + \frac{1}{(-2)}\left[1 - \frac{D}{2} - \frac{D^2}{2}\right]^{-1} x = \frac{D+3}{D^2 - 9} \sin x - \frac{1}{2}\left[1 + \frac{D}{2} + \frac{D^2}{2} + \ldots\right] x$$

$$= \frac{D+3}{-1-9} \sin x - \frac{1}{2}\left[x + \frac{1}{2}\right] = \frac{-1}{10}(\cos x + 3\sin x) - \frac{1}{4}(2x + 1)$$

Hence the complete solution is $y = C.F + P.I.$

or $$y = c_1 e^x + c_2 e^{-2x} - \frac{1}{10}(\cos x + 3\sin x) - \frac{1}{4}(2x + 1). \qquad \textbf{Ans.}$$

V When $X = e^{ax}.V$ where V is any function of x.

By successive differentiation, we have

$$D(e^{ax} V) = e^{ax} DV + a e^{ax} V = e^{ax}(D + a) V$$

$$D^2(e^{ax} V) = a e^{ax}(D + a) V + e^{ax} D(D + a) V$$

$$= e^{ax}(D^2 + 2aD + a^2)V = e^{ax}(D + a)^2 V, \text{ etc.}$$

It follows that $f(D)(e^{ax} V) = e^{ax} f(D + a) V$

and hence $$\frac{1}{f(D)}(e^{ax} V) = e^{ax} \frac{1}{f(D+a)} V.$$

[redacted] $(D^2 - 1)y = \cosh x \cos x$ where $D \equiv \dfrac{d}{dx}$. [GGSIPU I Sem End Term 2012]

SOLUTION: The A.E. is $m^2 - 1 = 0$ hence $m = \pm 1$

$\therefore$ C.F. $= C_1 e^x + C_2 e^{-x}$

and
$$\text{P.I.} = \frac{1}{D^2-1}\cosh x \cos x = \frac{1}{D^2-1}\left(\frac{e^x+e^{-x}}{2}\right)\cos x$$

$$= \frac{1}{2}\frac{1}{D^2-1}(e^x \cos x) + \frac{1}{2}\frac{1}{D^2-1}(e^{-x}\cos x)$$

$$= \frac{e^x}{2}\frac{1}{(D+1)^2-1}\cos x + \frac{e^{-x}}{2}\frac{1}{(D-1)^2-1}\cos x$$

$$= \frac{e^x}{2}\left(\frac{1}{D^2+2D}\right)\cos x + \frac{e^{-x}}{2}\left(\frac{1}{D^2-2D}\right)\cos x$$

$$= \frac{e^x}{2}\left(\frac{1}{-1+2D}\right)\cos x + \frac{e^{-x}}{2}\left(\frac{1}{-1-2D}\right)\cos x$$

$$= \frac{e^x}{2}\left(\frac{2D+1}{4D^2-1}\right)\cos x - \frac{e^{-x}}{2}\left(\frac{2D-1}{4D^2-1}\right)\cos x = \frac{e^x}{2}\left(\frac{2D+1}{-4-1}\right)\cos x - \frac{e^{-x}}{2}\left(\frac{2D-1}{-4-1}\right)\cos x$$

$$= \frac{e^x}{-10}(-2\sin x + \cos x) + \frac{e^{-x}}{10}(-2\sin x - \cos x) = \frac{2\sin x}{10}(e^x - e^{-x}) - \frac{\cos x}{10}(e^x + e^{-x})$$

$$= \frac{2}{5}\sin x \sinh x - \frac{1}{5}\cos x \cosh x.$$

$\therefore$ Complete solution is $y = C_1 e^x + C_2 e^{-x} + \dfrac{2}{5}\sin x \sinh x - \dfrac{1}{5}\cos x \cosh x$ **Ans.**

[redacted] $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 12y = (x-1)\,e^{2x}$. [GGSIPU I Sem End Term 2011]

SOLUTION: Equation is $(D^2 + 4D - 12)\,y = (x-1)\,e^{2x}$ where $D \equiv \dfrac{d}{dx}$.

A.E. is $m^2 + 4m - 12 = 0$ which gives $m = -6, 2$. $\therefore$ C.F. $= C_1 e^{-6x} + C_2 e^{2x}$

and
$$\text{P.I.} = \frac{1}{D^2+4D-12}(x-1)\,e^{2x} = e^{2x}\frac{1}{(D+2)^2+4(D+2)-12}(x-1) = \frac{e^{2x}}{D^2+8D}(x-1)$$

$$= \frac{e^{2x}}{8D}\left(1+\frac{D}{8}\right)^{-1}(x-1) = \frac{e^{2x}}{8D}\left[1-\frac{D}{8}+\ldots\right](x-1) = \frac{e^{2x}}{8D}\left[(x-1)-\frac{1}{8}\right] = \frac{e^{2x}}{8}\left[\frac{x^2}{2}-\frac{9}{8}x\right]$$

$\therefore$ Complete solution is $y = C_1 e^{-6x} + C_2 e^{2x} + \dfrac{e^{2x}}{8}\left(\dfrac{x^2}{2} - \dfrac{9}{8}x\right)$. **Ans.**

EXAMPLE 10.16. Solve $(D^3 - 3D - 2)\,y = x^2 e^{-x}$ where $D \equiv \dfrac{d}{dx}$.

[GGSIPU I Sem End Term 2007]

SOLUTION: Given equation is $(D^3 - 3D - 2)y = x^2 e^{-x}$

For complementary function the auxiliary equation is $m^3 - 3m - 2 = 0$

or $(m+1)(m^2 - m - 2) = 0$. Its roots are -1 2, -1.

$\therefore$ C.F. $= (c_1 x + c_2)\, e^{-x} + c_3 e^{2x}$.

and
$$P.I. = \frac{1}{D^3 - 3D - 2} e^{-x} \cdot x^2 = e^{-x}\left[\frac{1}{(D-1)^3 - 3(D-1) - 2}\right]x^2$$

$$= e^{-x}\frac{1}{D^3 - 3D^2}x^2 = e^{-x}\left(-\frac{1}{3D^2}\right)\left(1 - \frac{D}{3}\right)^{-1} x^2$$

$$= -\frac{1}{3}e^{-x}\frac{1}{D^2}\left[1 + \frac{D}{3} + \frac{D^2}{9} + \ldots\right]x^2 = -\frac{1}{3}e^{-x}\frac{1}{D^2}\left[x^2 + \frac{2}{3}x + \frac{2}{9}\right]$$

$$= \frac{-1}{3}e^{-x}\frac{1}{D}\left[\frac{x^3}{3} + \frac{x^2}{3} + \frac{2x}{9}\right] = \frac{-1}{3}e^{-x}\left[\frac{x^4}{12} + \frac{x^3}{9} + \frac{x^2}{9}\right]$$

$\therefore$ Required complete solution is $y = C.F. + P.I.$

or
$$y = (c_1 x + c_2)\, e^{-x} + c_3 e^{2x} - \frac{1}{3}e^{-x}\left[\frac{x^4}{12} + \frac{x^3}{9} + \frac{x^2}{9}\right] \qquad \textbf{Ans.}$$

VI. When X = x V where V is any function of x

Repeated differentiation gives

$$D(xV) = xDV + 1.V$$
$$D^2(xV) = xD^2V + 2DV$$
$$D^3(xV) = xD^3V + 3D^2V$$

....

....

$$D^n(xV) = xD^nV + nD^{n-1}V = xD^nV + \left(\frac{d}{dD}D^n\right)V$$

$$\Rightarrow \quad f(D)(xV) = xf(D)V + f'(D)V \quad \text{where } f'(D) = \frac{d}{dD}f(D). \qquad \ldots(1)$$

Now let $f(D)\,V = V_1$ hence $V = \dfrac{1}{f(D)}V_1$ ...(2)

Putting (2) in (1), we get

$$f(D)\left(x\frac{1}{f(D)}V_1\right) = xV_1 + \frac{f'(D)}{f(D)}V_1$$

Operating both sides by $\dfrac{1}{f(D)}$, we get

$$x\frac{1}{f(D)}V_1 = \frac{1}{f(D)}(xV_1) + \frac{f'(D)}{[f(D)]^2}V_1 \quad \text{or} \quad \frac{1}{f(D)}(xV_1) = x\frac{1}{f(D)}V_1 - \frac{f'(D)}{[f(D)]^2}V_1$$

Since V_1 is any function of x, we can also write the above relation as

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V \quad \text{or} \quad \frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \left(\frac{d}{dD}\frac{1}{f(D)}\right)V$$

If $X = x^2V$ or, in general, $x^n V$, we can use the above formula repeatedly.

EXAMPLE 10.17. Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x\,e^x \sin x.$ [*GGSIPU Sem I Sem End Term 2008*]

SOLUTION : The auxiliary equation $m^2 - 2m + 1 = 0$ has roots as $m = 1, 1$

$\therefore$ $\quad$ C.F. $= (C_1x + C_2)e^x$

and $\quad$ P.I. $= \dfrac{1}{D^2 - 2D + 1}x\,e^x \sin x = e^x \dfrac{1}{(D+1)^2 - 2(D+1) + 1}x \sin x$

$$= e^x \frac{1}{D^2}(x \sin x) = e^x \frac{1}{D}\int x \sin x\,dx = e^x \frac{1}{D}[-x\cos x + \sin x]$$

$$= e^x \int(-x\cos x + \sin x)\,dx = e^x \int -x\cos x\,dx + e^x\int \sin x\,dx$$

$$= e^x[-2\cos x + x \sin x]$$

$\therefore$ Complete solution is

$$y = \text{C.F.} + \text{P.I.} = (C_1x + C_2)e^x + e^x(-2\cos x - x\sin x)$$ **Ans.**

EXAMPLE 10.18. Find the P.I. for the equation $(D^4 + 2D^2 + 1)y = x^2 \cos x$.

SOLUTION : Here P.I. $= \dfrac{1}{D^4 + 2D^2 + 1}.x^2\cos x = \dfrac{1}{(D^2+1)^2}x^2\cos x$

It is the real part of $\dfrac{1}{(D^2+1)^2}x^2 e^{ix}$. Now consider

$$\frac{1}{(D^2+1)^2}x^2 e^{ix} = e^{ix}\frac{1}{[(D+i)^2+1]^2}x^2$$

$$= e^{ix}\frac{1}{[D^2 + 2iD]^2}x^2 = e^{ix}\frac{1}{(2iD)^2\left(1 + \frac{D}{2i}\right)^2}x^2$$

$$= -\frac{e^{ix}}{4}\frac{1}{D^2}\left(1 + \frac{D}{2i}\right)^{-2}x^2 = -\frac{e^{ix}}{4}\frac{1}{D^2}\left[1 + (-2)\frac{D}{2i} + \frac{(-2)(-3)}{2!}\left(\frac{D^2}{-4}\right) + \dots\right]x^2$$

(retaining terms upto D^2 only)

$$= -\frac{e^{ix}}{4}\frac{1}{D^2}\left[1 - \frac{D}{i} - \frac{3D^2}{4}\right]x^2$$

$$= -\frac{e^{ix}}{4}\frac{1}{D^2}\left[x^2 - \frac{2x}{i} - \frac{3}{2}\right] = -\frac{e^{ix}}{4}\frac{1}{D}\left[\frac{x^3}{3} - \frac{x^2}{i} - \frac{3x}{2}\right]$$

$$= -\frac{e^{ix}}{4}\left[\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3x^2}{4}\right] = -\frac{1}{4}(\cos x + i\sin x)\left[\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3x^2}{4}\right]$$

Its real part is $-\frac{1}{4}\cos x\left(\frac{x^4}{12}-\frac{3}{4}x^2\right)+\frac{1}{12}x^3\sin x$

$\therefore$ $$\text{P.I.} = -\frac{x^4}{48}\cos x+\frac{3}{16}x^2\cos x+\frac{x^3}{12}\sin x$$ **Ans.**

VI. General Method

If X does not fall under any of the forms discussed in I – V, we make use of the result $\frac{1}{D-a}X = e^{ax}\int e^{-ax}X\,dx$ which always works.

EXAMPLE 10.19. (a) Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx} = (1+e^x)^{-1}$. [GGSIPU Ist Sem II Term 2012]

(b) Solve $(D^3+1)y = (e^x+1)^2$ [GGSIPU Ist Sem End Term 2009]

SOLUTION: (a) The equation can be written as $D(D+1)y=(1+e^x)^{-1}$ where $D \equiv \frac{d}{dx}$.

Here C.F. $= C_1 e^{0x}+C_2 e^{-x}$ where C_1, C_2 are arbitrary constants.

and $$\text{P.I.} = \frac{1}{D(D+1)}(1+e^x)^{-1} = \left(\frac{1}{D}-\frac{1}{D+1}\right)(1+e^x)^{-1} = \frac{1}{D}(1+e^x)^{-1}-\frac{1}{D+1}(1+e^x)^{-1}$$

$$= \int\frac{dx}{1+e^x}-e^{-x}\int\frac{e^x\,dx}{1+e^x}$$ (now putting $1+e^x=t \therefore e^x dx=dt$)

$$= \int\frac{dt}{t(t-1)}-e^{-x}\int\frac{dt}{t} = \int\left(\frac{1}{t-1}-\frac{1}{t}\right)dt - e^{-x}\log t$$

$$= \log(t-1)-\log t-e^{-x}\log t = \log e^x-\log(1+e^x)-e^{-x}\log(1+e^x)$$

$$= x-(1+e^{-x})\log(1+e^x)$$

$\therefore$ Complete solution is $y=C_1+C_2e^{-x}+x-(1+e^{-x})\log(1+e^x)$. **Ans.**

(b) The A.E. is $m^3+1=0$ $\therefore$ $m=-1,\ (1\pm i\sqrt{3})/2$

$\therefore$ $$\text{C.F.} = c_1e^{-x}+e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)$$

and $$\text{P.I.} = \frac{1}{D^3+1}(e^x+1)^2 = \frac{e^{2x}}{D^3+1}+\frac{2}{D^3+1}e^x+\frac{1}{D^3+1}e^{0x}$$

$$= \frac{e^{2x}}{2^3+1}+\frac{2e^x}{1^3+1}+\frac{1}{0+1}e^{0x} = \frac{1}{9}e^{2x}+e^x+1$$

$\therefore$ Solution $y = \text{C.F.}+\text{P.I.} = c_1e^{-x}+e^{x/2}+e^{x/2}\left(c_2\cos\frac{\sqrt{3}x}{2}+c_3\sin\frac{\sqrt{3}x}{2}\right)+1+e^x+\frac{1}{9}e^{2x}$ **Ans.**

Evaluate $\dfrac{1}{D^2+a^2}\tan ax$ where $D \equiv \dfrac{d}{dx}$.

SOLUTION: $\dfrac{1}{D^2+a^2}\tan ax = \dfrac{1}{(D+ai)(D-ai)}\tan ax = \dfrac{1}{2ai}\left[\dfrac{1}{D-ai}-\dfrac{1}{D+ai}\right]\tan ax$

(on resolving into partial fractions)

$$= \frac{1}{2ai}\frac{1}{D-ai}\tan ax - \frac{1}{2ai}\frac{1}{D+ai}\tan ax$$

Now, consider $\dfrac{1}{D-ai}\tan ax = e^{aix}\int e^{-aix}\tan ax\, dx$

$$= e^{aix}\int(\cos ax - i\sin ax)\frac{\sin ax}{\cos ax}dx = e^{aix}\int\left[\sin ax - i\frac{(1-\cos^2 ax)}{\cos ax}\right]dx$$

$$= e^{aix}\int(\sin ax + i\cos ax - i\sec ax)\,dx$$

$$= e^{aix}\left[-\frac{1}{a}\cos ax + \frac{i}{a}\sin ax - \frac{i}{a}\log(\sec ax + \tan ax)\right]$$

Replacing i by $-i$ in the above result, we get

$$\frac{1}{D+ai}\tan ax = e^{-aix}\left[-\frac{1}{a}\cos ax - \frac{i}{a}\sin ax + \frac{i}{a}\log(\sec ax + \tan ax)\right]$$

$$\therefore \quad \frac{1}{D^2+a^2}\tan ax = \frac{1}{2a^2 i}[-\cos ax\,(e^{aix}-e^{-aix}) + i\sin ax(e^{aix}+e^{-aix})$$

$$-i\log(\sec ax + \tan ax)\,(e^{aix}+e^{-aix})]$$

$$= \frac{1}{2ia^2}[-2i\cos ax\sin ax + 2i\sin ax\cos ax - 2i\cos ax\log(\sec ax + \tan ax)]$$

$$= -\frac{1}{a^2}\cos ax\log(\sec ax + \tan ax)$$ **Ans.**

10.21. Solve $(D^2+5D+6)\,y = e^{-2x}\sec^2 x\,(1+2\tan x)$.

SOLUTION: The roots of auxiliary equation are $-2, -3$ hence C.F. $= C_1 e^{-2x} + C_2 e^{-3x}$.

and P.I. $= \dfrac{1}{(D+2)(D+3)}\, e^{-2x}\sec^2 x\,(1+2\tan x)$

$$= \frac{1}{(D+3)}\left[\frac{1}{D+2}\{e^{-2x}\sec^2 x\,(1+2\tan x)\}\right]$$

$$= \frac{1}{(D+3)}\left[e^{-2x}\int e^{2x}.e^{-2x}\sec^2 x\,(1+2\tan x)\,dx\right]$$

$$= \frac{1}{(D+3)}\left[e^{-2x}\int(1+2\tan x)\sec^2 x\,dx\right] = \frac{1}{(D+3)}[e^{-2x}(\tan x + \tan^2 x)]$$

$$= e^{-3x}\int e^{3x}e^{-2x}(\tan x + \tan^2 x)\,dx = e^{-3x}\left[\int[e^x\tan x + e^x(\sec^2 x - 1)]\right]dx$$

$$= e^{-3x}\left[\int e^x(\tan x + \sec^2 x)\,dx - \int e^x dx\right] = e^{-3x}[e^x\tan x - e^x] = e^{-2x}(\tan x - 1)$$

$\therefore$ The complete solution is $y =$ C.F.+ P.I. $= C_1 e^{-2x} + C_2 e^{-3x} + e^{-2x}(\tan x - 1)$ where C_1, C_2 are arbitrary constants. **Ans.**

or $\quad y = e^{-2x}(\tan x + C_1) + C_2 e^{-3x}$

FINDING P.I. BY THE METHOD OF VARIATION OF PARAMETERS

In this method P.I. is obtained from the C. F. itself by considering the arbitrary constants as functions of the independent variable. The method is applicable to the equations of the form

$$y'' + py' + qy = X \quad \text{...(1)}$$

where p, q and X are functions of x.

Let y_1 and y_2 be the solutions of the equation $y'' + py' + qy = 0$ ...(2)

then C.F. of (1) is $C_1 y_1 + C_2 y_2$.

Now for P.I. we replace C_1 and C_2 (regarded as parameters) by unknown functions $u(x)$ and $v(x)$ so that P.I. is given by $y = uy_1 + vy_2$ ...(3)

then $y' = uy'_1 + vy'_2 + u' y_1 + v' y_2$

Let u and v be chosen in such a way that $u' y_1 + v' y_2 = 0$ ...(4)

then we have $y' = uy_1' + vy_2'$ ...(5)

and hence $y'' = u'y_1' + v'y_2' + uy_1'' + vy_2''$ ...(6)

Substituting for y, y' and y'' from (3) (5) and (6) respectively in (1), we get

$$uy_1'' + u'y_1' + vy_2'' + v'y_2' + p(uy_1' + vy_2') + q(uy_1 + vy_2) = X$$

or $$u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + u'y_1' + v'y_2' = X.$$

But y_1 and y_2 satisfy (2), hence we have $u'y_1' + v'y_2' = X$ ...(7)

Solving (4) and (7) for u' and v', we get

$$u' = \frac{-y_2X}{y_1y_2' - y_1'y_2} = \frac{-y_2X}{W}$$

$$v' = \frac{y_1X}{y_1y_2' - y_1'y_2} = \frac{y_1X}{W}$$

where $W = y_1y_2' - y_1'y_2$ which is called **WRONSKIAN** of y_1 and y_2.

$$\Rightarrow \quad u = -\int \frac{y_2X}{W}\,dx \quad \text{and} \quad v = \int \frac{y_1X}{W}\,dx.$$

Substituting for u and v in (3), we get the required P.I.

Point to be noted here is that if the wronskian $w = 0$, y_1 and y_2 are linearly dependent and if $w \neq 0$, y_1 and y_2 are linearly independent.

EXAMPLE 10.22. Solve, using the method of variation of parameters, the equation

$$\frac{d^2y}{dx^2} + 4y = \tan 2x.$$

[*GGSIPU I Sem End Term January 2011, End Term 2012, II Term 2013*]

SOLUTION: The given equation is $(D^2 + 4)y = \tan 2x$ where $D \equiv \frac{d}{dx}$

$\therefore$ C.F. $= C_1 \cos 2x + C_2 \sin 2x$. Let us write $y_1 = \cos 2x$ and $y_2 = \sin 2x$.

Here Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$

By the method of variation of parameters, we know that

$$y_1' = \frac{-y_2 X}{W} \quad \text{and} \quad y_2' = \frac{y_1 X}{W} \quad \text{where } X = \tan 2x \text{ here.}$$

$$\therefore \quad y_1' = -\frac{1}{2}\sin 2x \tan 2x = -\frac{1}{2}\frac{\sin^2 2x}{\cos 2x} = -\frac{1}{2}\frac{(1-\cos^2 2x)}{\cos 2x} = \frac{1}{2}(\cos 2x - \sec 2x)$$

and

$$y_2' = \frac{1}{2}\cos 2x \tan 2x = \frac{1}{2}\sin 2x.$$

$$\Rightarrow \quad y_1 = \frac{1}{4}[\sin 2x - \log(\sec 2x + \tan 2x)] \quad \text{and} \quad y_2 = -\frac{1}{4}\cos 2x.$$

Thus, $$\text{P.I.} = y_1 \cos 2x + y_2 \sin 2x = \frac{1}{4}[\sin 2x \cos 2x - \cos 2x \log(\sec 2x + \tan 2x) - \sin 2x \cos 2x]$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x).$$

$\therefore$ Complete solution is $y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$. **Ans.**

EXAMPLE 10.23. Apply the method of variation of parameters to solve

(i) $y'' + a^2 y = \cos ax$ *[GGSIPU I Sem End Term 2007]*

(ii) $y'' + a^2 y = \sec ax$ *[GGSIPU I Sem End Term 2008]*

SOLUTION: For complementary function we have $(D^2 + a^2)\, y = 0$ where $D \equiv \frac{d}{dx}$.

C.F. $= C_1 \cos ax + C_2 \sin ax$.

Thus, here two solutions are $y_1 = \cos ax$ and $y_2 = \sin ax$.

By the method of variation of parameters we replace C_1 and C_2 by $u(x)$ and $v(x)$ and we get P.I. $= uy_1 + vy_2$

Here, the Wronskian $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a.$$

In **problem** (i) X = cos ax

$$\therefore \quad u' = \frac{-y_2 X}{W} = -\frac{1}{a}\sin ax \cdot \cos ax = -\frac{1}{2a}\sin 2ax \quad \text{and} \quad v' = \frac{y_1 X}{W} = \frac{1}{a}\cos^2 ax = \frac{1}{2a}(1 + \cos 2ax)$$

Therefore, $$u(x) = \frac{-1}{2a}\int \sin 2ax\, dx = \frac{1}{4a^2}\cos 2ax.$$

and $$v(x) = \frac{1}{2a}\int (1 + \cos 2ax)\, dx = \frac{1}{2a}\left[x + \frac{1}{2a}\sin 2ax\right]$$

Thus $$\text{P.I.} = uy_1 + vy_2 = \frac{1}{4a^2}\cos 2ax \cdot \cos ax + \frac{1}{2a}\left(x + \frac{1}{2a}\sin 2ax\right)\sin ax$$

$$= \frac{x \sin ax}{2a} + \frac{1}{4a^2}(\cos 2ax \cos ax + \sin 2ax \sin ax) = \frac{x}{2a}\sin ax + \frac{1}{4a^2}\cos ax$$

and the solution is $y = C_1 \cos ax + C_2 \sin ax + \frac{x}{2a}\sin ax + \frac{1}{4a^2}\cos ax$ **Ans.**

For **problem (ii)** $X = \sec ax$, hence

$$u' = \frac{-y_2 X}{W} = \frac{-1}{a}\sin ax \sec ax = \frac{-1}{a}\tan ax$$

and $$v' = \frac{y_1 X}{W} = \frac{1}{a}\cos ax \sec ax = \frac{1}{a}$$

Therefore, $$u = \frac{-1}{a}\int \tan ax\, dx = \frac{1}{a^2}\log\cos ax \quad \text{and} \quad v = \frac{x}{a}.$$

Thus, $$\text{P.I.} = u\,y_1 + v\,y_2 = \frac{1}{a^2}(\log\cos ax)\cos ax + \frac{x}{a}\sin ax$$

$\therefore$ The complete solution is $y = C_1\cos ax + C_2\sin ax + \dfrac{\cos ax}{a^2}\log\cos ax + \dfrac{x}{a}\sin ax$ **Ans.**

EXAMPLE 10.24 By the method of variation of parameters, solve

(i) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \log x$ *[GGSIPU I Sem End Term 2009]*

(ii) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \dfrac{e^x}{1+e^x}$. *[GGSIPU I Sem End Term 2007]*

SOLUTION: (i) The A.E. is $m^2 - 2m + 1 = 0 \quad \therefore \quad m = 1, 1$

$\therefore$ C.F. $= (C_1 + C_2 x)e^x$, hence here $y_1 = e^x$ and $y_2 = xe^x$.

Wronskian $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x}\begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix} = e^{2x}.$$

By the method of variation of parameters

$$\text{P.I.} = u(x)\,y_1 + v(x)\,y_2$$

$\therefore$ $$u' = \frac{-y_2 X}{W} \quad \text{and} \quad v' = \frac{y_1 X}{W} \quad \text{where } X = e^x \ln(x)$$

or $$u' = -\frac{xe^x \cdot e^x \ln(x)}{e^{2x}} = -x\ln(x) \quad \text{and} \quad v' = \frac{e^x \cdot e^x \ln(x)}{e^{2x}} = \ln(x).$$

$\Rightarrow$ $$u(x) = -\int x\ln(x)\,dx = -\frac{x^2}{2}\ln(x) + \int \frac{1}{x}\frac{x^2}{2}dx = \frac{-x^2}{2}\ln(x) + \frac{x^2}{4} = \frac{x^2}{4}(1 - 2\ln(x))$$

and $$v(x) = \int \ln(x)\,dx = x\ln(x) - \int \frac{1}{x}\cdot x\,dx = x(\ln(x) - 1).$$

$\therefore$ $$\text{P.I.} = \frac{x^2}{4}[1 - 2\ln(x)]\,e^x + x[\ln(x) - 1]\,xe^x = \frac{x^2}{4}e^x[2\ln(x) - 3]$$

Solution is $y = (c_1 + xc_2)\,e^x + \dfrac{x^2}{4}e^x(2\ln(x) - 3)$. **Ans.**

(ii) The A.E. is $m^2 - 3m + 2 = 0$ whose roots are 1, 2.

$\therefore$ C.F. $= C_1 e^x + C_2 e^{2x}$. Here $y_1 = e^x$ and $y_2 = e^{2x}$.

Wronskian $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}.$$

By the method of variation of parameters P.I. $= uy_1 + vy_2$.

Now $$u' = \frac{-y_2 X}{W} \quad \text{and} \quad v' = \frac{y_1 X}{W} \quad \text{where } X = \frac{e^x}{1+e^x}.$$

or $$u' = \frac{-e^{2x}}{e^{3x}} \cdot \frac{e^x}{1+e^x} = \frac{-1}{1+e^x} \quad \text{and} \quad v' = \frac{e^x}{e^{3x}} \frac{e^x}{1+e^x} = \frac{1}{e^x(1+e^x)}$$

$\therefore$ $$u = -\int \frac{dx}{1+e^x} = -\int \frac{e^{-x}}{1+e^{-x}} dx = \ln(1+e^{-x})$$

and $$v = \int \frac{dx}{e^x(1+e^x)} = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x} \right) dx = -e^{-x} + \ln(1+e^{-x}).$$

$\therefore$ $$\text{P.I.} = uy_1 + vy_2 = e^x \ln(1+e^{-x}) + e^{2x}[-e^{-x} + \ln(1+e^{-x})].$$

Hence the complete solution is
$$y = C_1 e^x + C_2 e^{2x} + e^x \ln(1+e^{-x}) - e^x + e^{2x} \ln(1+e^{-x}). \quad \textbf{Ans.}$$

EXAMPLE 10.25. Apply the method of variation of parameters, to solve [GGSIPU I Sem End Term 2007]
$$x^2 y'' + xy' - y = x^2 e^x. \quad \text{... (1)}$$

SOLUTION: The given equation is $x^2 y'' + xy' - y = x^2 e^x$
which is of Cauchy's homogeneous form. Putting $x = e^t$ it becomes
$$D(D-1)y + Dy - y = e^{2t}.e^{e^t} \quad \text{where } D \equiv \frac{d}{dt}. \quad \text{... (2)}$$

or $$(D^2 - 1)y = e^{2t}.\ e^{e^t}$$

For C.F. of (2) the A.E. is $m^2 - 1 = 0$ Hence $m = \pm 1$

$\therefore$ C.F. $= c_1 e^t + c_2 e^{-t}$ ...(3)

To find P.I. of (2) we take $y = u e^t + v e^{-t}$ where u and v are functions of t. ...(4)

$\therefore$ $y' = u' e^t + ue^t + v'e^{-t} - ve^{-t}$

Now, choose u and v such that $u'e^t + v'e^{-t} = 0$

Then $y' = ue^t - ve^{-t}$, hence $y'' = u'e^t + ue^t - v'e^{-t} + ve^{-t}$

Putting these values of y and y' and y'' in (2), we get
$$u'e^t + ue^t - v'e^{-t} + ve^{-t} - ue^t - ve^{-t} = e^{2t}.\ e^{e^t}$$

or $$u' e^t - v' e^{-t} = e^{2t} e^{e^t}$$

Solving (4) and (5) for u' and v', gives

$$u' = \frac{1}{2} e^t e^{e^t} \quad \text{and} \quad v' = \frac{-1}{2} e^{3t} e^{e^t}$$

Hence $$u = \frac{1}{2} \int e^t e^{e^t}\, dt = \frac{1}{2} e^{e^t} = \frac{1}{2} e^x.$$

and $$v = \frac{-1}{2} \int e^{3t} e^{e^t}\, dt = \frac{-1}{2} \int x^2 e^x\, dx$$

$$= \frac{-1}{2}\left[x^2 e^x - \int 2x e^x\, dx\right] = \frac{-1}{2} x^2 e^x + \int x e^x\, dx = \frac{-x^2}{2} e^x + x e^x - e^x$$

$\therefore$ $$P.I. = xu + \frac{1}{x} v = \frac{x}{2} e^x - \frac{x}{2} e^x + e^x - \frac{1}{x} e^x = e^x - \frac{1}{x} e^x$$

Therefore the complete solution is $y = C.F + P.I. = c_1 x + \dfrac{c_2}{x} + e^x - \dfrac{1}{x} e^x$ **Ans.**

EXERCISE 10A

1. Solve $(D^3 + 7D^2 + 16D + 10)\, y = 0$ where $D \equiv \frac{d}{dx}$.

2. Solve $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$.

3. (a) Solve $(D^4 + D^2 + 1)\, y = 0$ where $D \equiv \frac{d}{dx}$.

   (b) Solve $(D^3 + 2D^2 - 5D - 6)y = e^{4x}$ where $D \equiv \frac{d}{dx}$.

4. Solve $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 18y = 2^x$.

5. Solve $\frac{d^3y}{dx^3} - y = (1 + e^x)^2$.

6. Solve $\frac{d^4y}{dx^4} - y = \cosh x \sinh x$.

7. (a) Solve $(D - 1)^2 (D^2 + 1)^2\, y = \sin^2 \frac{x}{2} + e^x$ where $D \equiv \frac{d}{dx}$.

   (b) Solve $(D^2 + D + 1)\, y = \sin 2x$ [GGSIPU I Sem End Term 2007]

8. Solve $\operatorname{cosec} x \frac{d^4y}{dx^4} + y \operatorname{cosec} x = \sin 2x$.

9. (a) Solve $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 3x^2 - 5x + 2$.

   (b) Find the particular integral of the equation $(D^2 + 2D - 5)y = 2x^2 - 3x + 1$ where $D \equiv \frac{d}{dx}$.

10. (a) Solve $\frac{d^3y}{dx^3} - \frac{7dy}{dx} - 6y = e^{2x}(1 + x) + \cosh x \cos x$.

    (b) Solve $(D^2 + 3D + 2)\, y = x \cos 2x$ where $D \equiv \frac{d}{dx}$.

11. Solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-t} \log t$.

12. Find the particular integral of the equation $\frac{d^4y}{dx^4} - y = \cos x \cosh x$.

13. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x \sin x$.

14. Find the particular Integral of the equation $\frac{d^2y}{dx^2} + 4y = x \sin^2 x$.

15. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin(e^x)$.

16. Find the P.I. of the equation $\frac{d^2y}{dx^2} - y = (1 - e^{-x})^{-2}$.

17. Solve $(D^2 + 3D + 2)\, y = e^{e^x}$.

18. Apply the method of variation of parameters, to solve

$$\frac{d^2y}{dx^2} - y = e^{-x} \sin (e^{-x}) + \cos (e^{-x}).$$

19. By the method of variation of parameters, solve $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$.

[GGSIPU I Sem II Term 2010]

20. Solve $(D^2 - 1) y = \text{sech } x$ by the method of variation of parameters.

21. Solve $y'' - 2y' + 2y = e^x \tan x$ by the method of variation of parameters.

22. Solve $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$.

23. Solve $(D-2)^2 y = 8 (e^{2x} + \sin 2x + x^2$ where $D \equiv \frac{d}{dx}$

24. Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x.$ [GGSIPU I Sem End Term 2013]

25. Solve $(D-1)^2 (D+1)^2 y = \sin^2\left(\frac{x}{2}\right) + e^x + x$ where $D \equiv \frac{d}{dx}$

26. Solve $(D^3 + 2D^2 + D) y = x^2 e^{2x} + \sin^2 x$ where $D \equiv \frac{d}{dx}$

27. Solve $\frac{d^2y}{dx^2} - 4y = x \sinh x$.

28. Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x} / x^2.$$

[GGSIPU I Sem End Term 2013]

29. Solve $\frac{d^2y}{dx^2} + y = x \sin x$ by the method of variation of parameters.

30. Find the general solution of $y'' + 4y = \sec x, -\frac{\pi}{4} < x < \frac{\pi}{4}$.

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

Here we shall study two types of homogeneous linear differential equations which can be reduced to the linear differential equation with constant coefficients by suitable substitution.

I. CAUCHY'S HOMOGENEOUS LINEAR EQUATION

It is of the form $x^n \dfrac{d^n y}{dx^n} + a_1 x^{n-1} \dfrac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \dfrac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = X(x)$

where $a_1, a_2, \ldots, a_n$ are constants and X is a function of x.

This can be reduced to linear differential equation with constant co-efficients by the substitution

$$x = e^t \quad \text{or} \quad \log x = t \quad \text{then} \quad \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

or $$x\frac{dy}{dx} = \frac{dy}{dt} = Dy \quad \text{where } D \equiv \frac{d}{dt}.$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{dx} = \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2} \cdot \frac{1}{x}$$

or $$x^2\frac{d^2y}{dx^2} = \frac{-dy}{dt} + \frac{d^2y}{dt^2} = -Dy + D^2y = D(D-1)y.$$

Similarly, we can have $x^3\dfrac{d^3y}{dx^3} = D(D-1)(D-2)\,y$, and so on.

This way the substitution reduces the given homogeneous equation to linear equation with constant co-efficients.

II. LEGENDRE'S HOMOGENEOUS LINEAR EQUATION

It is of the form $(ax+b)^n \dfrac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \dfrac{d^{n-1} y}{dx^{n-1}} + a_2 (ax+b)^{n-2} \dfrac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = X$ ...(1)

where $a_1, a_2, \ldots, a_n$ are constants and X is a function of x. If we substitute $ax + b = t$ the equation (1) will reduce to the Canchy's form discussed above and then a further substitution $t = e^z$ reduces it to the linear differential equation with constant co-efficients. This suggests that in place of two, a single substitution $ax + b = e^z$ will reduce (1) to the linear differential equation with constant co-efficients, as will be clear in the following examples :

EXAMPLE 10.26. Solve $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$. [GGSIPU I Sem End Term January 2011]

SOLUTION: The given equation is $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 12\log x$ which is of homogeneous form of Cauchy.

Putting $x = e^t$ in (i) we get $D(D-1)y + Dy = 12t$ where $D \equiv \dfrac{d}{dt}$.

or $D^2y = 12t \Rightarrow Dy = 6t^2 + C_1$ and $y = 2t^3 + C_1 t + C_2$.

$\therefore$ Solution is $y = 2(\log x)^3 + C_1 \log x + C_2$.

EXAMPLE 10.27. Solve $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = \dfrac{x^3}{1+x^2}$.

SOLUTION: Putting $x = e^t$, *i.e.*, $t = \log x$ the given equation becomes

$$[D(D-1)+D-1]y = \frac{e^{3t}}{1+e^{2t}} \quad \text{where } D \equiv \frac{d}{dt}. \quad \text{or} \quad (D+1)(D-1)y = \frac{e^{3t}}{1+e^{2t}}$$

The A.E. has the roots $-1, 1$, therefore $\text{C.F.} = C_1 e^{-t} + C_2 e^t = \dfrac{C_1}{x} + C_2 x$

and $$\text{P.I.} = \frac{1}{D+1}\left[\frac{1}{D-1}\frac{e^{3t}}{1+e^{2t}}\right] = \frac{1}{D+1}\left[e^t \int e^{-t}.\frac{e^{3t}}{1+e^{2t}}dt\right] = \frac{1}{D+1}\left[e^t \int \frac{e^{2t}}{1+e^{2t}}dt\right]$$

$$= \frac{1}{D+1}\left[\frac{e^t}{2}\log(1+e^{2t})\right] = e^{-t}\int e^t.\frac{e^t}{2}\log(1+e^{2t})dt$$

$$= \frac{1}{2}e^{-t}\int e^{2t}\log(1+e^{2t})\,dt \qquad \text{(on putting } 1+e^{2t}=u\text{)}$$

$$= \frac{1}{2}e^{-t}\int\frac{1}{2}(\log u)\,du \qquad \text{(integrating by parts)}$$

$$= \frac{1}{4}e^{-t}(u\log u - u) = \frac{1}{4}e^{-t}[(1+e^{2t})\log(1+e^{2t}) - (1+e^{2t})]$$

$$= \frac{1}{4}e^{-t}[(1+x^2)\log(1+x^2) - 1]$$

$$= \frac{1}{4}\frac{1}{x}(1+x^2)[\log(1+x^2)-1] = \frac{1}{4}\left(x+\frac{1}{x}\right)\log(1+x^2) - \frac{1}{4}\left(x+\frac{1}{x}\right)$$

Therefore, the solution is $y = \dfrac{C_1}{x} + C_2 x + \dfrac{1}{4}\left(x+\dfrac{1}{x}\right)\log(1+x^2)$

after adjusting the arbitrary constants. **Ans.**

EXAMPLE 10.28. Solve $(2x+3)^2\dfrac{d^2y}{dx^2} - (2x+3)\dfrac{dy}{dx} - 12y = 6x$.

SOLUTION: Putting $2x + 3 = e^t$ the given equation becomes

$$[2^2 D(D-1) - 2D - 12]y = 3(e^t - 3) \quad \text{where } D \equiv \frac{d}{dt}.$$

or $$(2D^2 - 3D - 6)y = \frac{3}{2}(e^t - 3).$$

A.E. is $2m^2 - 3m - 6 = 0$ whose roots are k_1 and k_2 where $k_1 = \dfrac{3+\sqrt{57}}{4}$, $k_2 = \dfrac{3-\sqrt{57}}{4}$

$\therefore$ $\text{C.F.} = C_1 e^{k_1 t} + C_2 e^{k_2 t} = C_1 (e^t)^{k_1} + C_2 (e^t)^{k_2} = C_1(2x+3)^{k_1} + C_2(2x+3)^{k_2}$

and $$\text{P.I.} = \frac{1}{2D^2-3D-6}\frac{3}{2}(e^t-3) = \frac{3}{2}\frac{1}{2D^2-3D-6}e^t - \frac{9}{2}\frac{1}{2D^2-3D-6}e^{0t}$$

$$= \frac{3}{2}.\frac{1}{2-3-6}e^t - \frac{9}{2}\left(\frac{1}{0-0-6}\right)1 = -\frac{3}{14}(2x+3) + \frac{3}{4}$$

$\therefore$ The complete solution is

$$y = C_1(2x+3)^{k_1} + C_2(2x+3)^{k_2} - \frac{3}{14}(2x+3) + \frac{3}{4} \quad \text{where } k_1 = \frac{3+\sqrt{57}}{4},\ k_2 = \frac{3-\sqrt{57}}{4}. \quad \textbf{Ans.}$$

10.29 Solve $x^2 \dfrac{d^2 y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$.

SOLUTION: Putting $x = e^z$ in the given equation, we get

$[D(D-1) + 4D + 2]\,y = e^{e^z}$ where $D \equiv \dfrac{d}{dz}$.

The A.E. is $m^2 + 3m + 2 = 0$ whose roots are $m = -1, -2$

Therefore C.F. $= C_1 e^{-z} + C_2 e^{-2z}$.

Next,
$$\text{P.I.} = \frac{1}{D^2+3D+2} e^{e^z} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z}$$
$$= \frac{1}{D+1} e^{e^z} - \frac{1}{D+2} e^{e^z} = e^{-z}\int e^z \cdot e^{e^z}\,dz - e^{-2z}\int e^{2z} \cdot e^{e^z}\,dz$$
$$= e^{-z} e^{e^z} - e^{-2z}(e^z - 1)e^{e^z} = e^{-2z} e^{e^z}.$$

Hence the complete sulution is $y = \text{C.F.} + \text{P.I.} = C_1 e^{-z} + C_2 e^{-2z} + e^{-2z} e^{e^z}$

or
$$y = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{1}{x^2} e^x. \qquad \textbf{Ans.}$$

EXERCISE 10B

1. Solve $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x.$

2. (a) Solve $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = -x^4 \sin x.$

   (b) Find the general solution of $x^2y'' - 4xy' + 4y = x^4 + x^2.$

3. Solve $x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}.$

4. Solve $\dfrac{d^2u}{dr^2} + \dfrac{1}{r}\dfrac{du}{dr} - \dfrac{u}{r^2} + kr = 0$ given that $u = 0$ for $r = 0,\ r = a.$

5. Solve $x^3 \dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = x + \log x.$

6. Solve $(x+a)^2 \dfrac{d^2y}{dx^2} - 4(x+a)\dfrac{dy}{dx} + 6y = x.$

7. Solve $(x+1)^2 \dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} + y = 2 \sin [\log (x+1)].$

8. Solve $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1.$ [GGSIPU I Sem II Term 2010]

9. Solve the equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin (\log x^2)$$

[GGSIPU I Sem End Term 2009]

10. Solve the equation

$$x^2 \frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 \log x$$

11. Show that the solution of the equation

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin [2 \log (1+x)]$$

is $y = c_1 \cos \log (1+x) + c_2 \sin \log (1+x) - \dfrac{1}{3} \sin [2 \log (1+x)]$

where c_1 and c_2 are arbitrary constants.

12. Solve the differential equation $(x^2D^2 - xD + 2)\, y = x \log x.$

[GGSIPU I Sem End Term 2008; End Term 2011]

SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

The simultaneous linear differential equations are of two types.

TYPE I. Form : $f_1(D)\,x + f_2(D)\,y = \phi_1(t)$

$$\psi_1(D)\,x + \psi_2(D)\,y = \phi_2(t)$$

Here we have two or more dependent variables and a single independent variable. By the process of elimination we obtain a differential equation involving one dependent variable x or y with the independent variable t. It will be solved by the methods studied just before and, as illustrated below.

EXAMPLE 10.30. Solve the simultaneous equation

$$\frac{dx}{dt} + x = y + e^t,$$

$$\frac{dy}{dt} + y = x + e^t.$$

[*GGSIPU I Sem End Term 2007, End Term 2012*]

SOLUTION: The given equations can be written as

$$(D+1)\,x - y = e^t, \qquad ...(1)$$

$$(D+1)\,y - x = e^t \quad \text{where } D \equiv \frac{d}{dt} \qquad ...(2)$$

Operating (1) on both sides by $(D+1)$ and adding in (2), we get

$$(D+1)^2 x - x = (D+1)e^t + e^t$$

or $$(D^2 + 2D)x = e^t + e^t + e^t = 3e^t \qquad ...(3)$$

For the $C.F$, for x, the $A.E.$ is $m^2 + 2m = 0 \Rightarrow m = 0, -2$

$\therefore$ $$\text{C.F. (for } x) = C_1 + C_2\, e^{-2t}$$

and $$P.I. \text{ (for } x) = \frac{1}{D^2 + 2D} 3e^t = \left(\frac{3}{1^2 + 2.1}\right)e^t = e^t$$

$\therefore$ $$x = c_1 + c_2\, e^{-2t} + e^t$$

Substituting for x in (1), we get

$$y = (D+1)x - e^t = C_1 - C_2\, e^{-2t} + e^t$$

Thus, the general solution is $x = C_1 + C_2\, e^{-2t} + e^t, \quad y = C_1 - C_2\, e^{-2t} + e^t.$ **Ans.**

EXAMPLE 10.31. Solve $$\frac{2dx}{dt} - \frac{dy}{dt} + 2x + y = 11t$$

$$\frac{2dx}{dt} + 3\frac{dy}{dt} + 5x - 3y = 2$$

[*GGSIPU I Sem End Term 2003*]

SOLUTION: Given simultaneous equations are

$$(2D+2)x - (D-1)y = 11t \qquad ...(1)$$

$$(2D+5)x + 3(D-1)y = 2 \quad \text{where } D \equiv \frac{d}{dt} \qquad ...(2)$$

Eliminating y in (1) and (2), we get

$$(6D + 6 + 2D + 5)x = 33t + 2 \quad \text{or} \quad (8D + 11)x = 33t + 2$$

or $$\frac{dx}{dt}+\frac{11x}{8} = \frac{33}{8}t+\frac{1}{4} \quad \text{which is linear in } x. \qquad \text{... (3)}$$

Its I.F. $= e^{11t/8}$. Hence the solution of (3) is

$$x\,e^{11t/8} = \int\left(\frac{33}{8}t+\frac{1}{4}\right)e^{11t/8}\,dt = \left(\frac{33}{8}t+\frac{1}{4}\right)e^{11t/8}\frac{8}{11}-\int\frac{33}{8}e^{11t/8}\frac{8}{11}\,dt$$

or $$x\,e^{11t/8} = \left(3t+\frac{2}{11}\right)e^{11t/8}-3\cdot\frac{8}{11}e^{11t/8}+C_1$$

or $$x = 3t+\frac{2}{11}-\frac{24}{11}+C_1\,e^{-11t/8} = 3t-2+C_1\,e^{-11t/8}. \qquad \text{... (4)}$$

To find y, we substitute for x from (4) in (1), to get

$$(D-1)y = (2D+2)x-11t$$

or $$\frac{dy}{dt}-y = 2\left[3-\frac{11}{8}C_1\,e^{-11t/8}\right]+2\left[3t-2+C_1\,e^{-11t/8}\right]-11t$$

or $$\frac{dy}{dt}-y = 2-5t-\frac{3}{4}C_1\,e^{-11t/8} \quad \text{which is linear in } y.$$

$I.F = e^{-t}$ hence its solution is

$$ye^{-t} = \int e^{-t}\left[2-5t-\frac{3}{4}C_1\,e^{-11t/8}\right]dt = \int(2-5t)e^{-t}\,dt-\frac{3}{4}C_1\int e^{-19t/8}\,dt$$

$$= -(2-5t)e^{-t}+\int(-5)e^{-t}\,dt+\frac{3}{4}\cdot\frac{8}{19}C_1e^{-19t/8}+C_2$$

$$= (5t-2)e^{-t}+5e^{-t}+\frac{6}{19}C_1e^{-19t/8}+C_2$$

or $$y = 5t+3+\frac{6}{19}C_1e^{-11t/8}+C_2e^{t}$$

$\therefore$ The solution is $x = 3t-2+C_1e^{-11t/8}$, and $y = 5t+3+\frac{6}{19}C_2e^{-11t/8}+C_2e^{t}$ **Ans.**

Type II. Symmetrical Simultaneous Equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

where P, Q, R are functions of x, y, z.

The solution of this type of simultaneous equations consists of two independent relations u = constant, v = constant where u and v are functions of x, y, z. To find such solutions of this type of simultaneous equations, following two methods are used :

(*a*) **Method of Grouping** as illustrated below.

Example 10.32. Solve $\dfrac{dx}{y^2z/x} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$.

Solution: The given equations can be written as

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x} \qquad \text{...(1)}$$

Taking the first two members and cancelling z, we get $\dfrac{dx}{y^2} = \dfrac{dy}{x^2}$ or $x^2\,dx - y^2\,dy = 0$

which on integration gives $\dfrac{x^3}{3} - \dfrac{y^3}{3} = C_1$ or $x^3 - y^3 = C_1'$ ...(2)

Next, taking first and third members and cancelling y^2, we get $\dfrac{dx}{z} = \dfrac{dz}{x}$ or $x\,dx - z\,dz = 0$

which on integration, gives $\dfrac{x^2}{2} - \dfrac{z^2}{2} = C_2$ or $x^2 - z^2 = C_2'$ ...(3)

Thus, equations (2) and (3) taken together represent the required solution. **Ans.**

Example 10.33. Solve $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z - a\sqrt{x^2 + y^2 + z^2}}$.

Solution: Taking first two members in the given equation, we have $\dfrac{dx}{x} = \dfrac{dy}{y}$

which, on integration, gives $\log x - \log y = \log C_1$ or $x = C_1 y$ ...(1)

Now, take last two members of the given equation and use $x = C_1 y$ to get

$$\frac{dy}{y} = \frac{dz}{z - a\sqrt{(C_1^2 + 1)\,y^2 + z^2}} \quad \text{or} \quad \frac{dz}{dy} = \frac{z - a\sqrt{(C_1^2+1)\,y^2 + z^2}}{y}$$

This is of homogeneous form, hence putting $z = vy$, we get

$$v + y\frac{dv}{dy} = \frac{vy - a\sqrt{(C_1^2+1)\,y^2 + v^2y^2}}{y} = v - a\sqrt{(C_1^2+1) + v^2}$$

or $$\frac{dv}{\sqrt{(C_1^2+1) + v^2}} + \frac{a\,dy}{y} = 0$$ which, on integration, gives

$$\log\left(v + \sqrt{(C_1^2+1) + v^2}\right) + a\log y = \log C_2$$

or $$y^a\left(v + \sqrt{(C_1^2+1) + v^2}\right) = C_2 \quad \text{or} \quad y^{a-1}\left(vy + \sqrt{(C_1^2+1)\,y^2 + v^2y^2}\right) = C_2$$

Since $vy = z$ we have $z + \sqrt{(C_1^2+1)\,y^2 + z^2} = C_2 y^{1-a}$.

Also, putting back $C_1 = \dfrac{x}{y}$ here, we get

$$z + \sqrt{x^2 + y^2 + z^2} = C_2\, y^{1-a} \qquad \text{...(2)}$$

The relations (1) and (2) together represent the solution of the given equation. **Ans.**

(b) Method of multipliers

Let l, m, n be the multipliers , not necessarily constants, then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

Now, if possible, choose l, m, n in such a way that $lP + mQ + nR = 0$, then it implies that

$$l\,dx + m\,dy + n\,dz = 0$$

which on integration, may provide one solution as

$$f_1(x, y, z) = C_1 \qquad ...(1)$$

Next, if it is possible to find another set of multipliers l', m', n' , not necessarily constant, such that $l'P + m'Q + n'R = 0$

then we have $l'\,dx + m'\,dy + n'\,dz = 0$

and which, in turn, on integration, may result in another solution as $f_2(x, y, z) = C_2$ ...(2)

Then, the solutions (1) and (2) together represent the complete solution of the given equation.

EXAMPLE 10.34. Solve $\dfrac{dx}{y - xz} = \dfrac{dy}{yz + x} = \dfrac{dz}{x^2 + y^2}$.

SOLUTION: Considering first two members, we have $\dfrac{dx}{y - xz} = \dfrac{dy}{yz + x} = \dfrac{y\,dx + x\,dy}{x^2 + y^2}$

$$\Rightarrow \quad y\,dx + x\,dy = dz \quad \text{or} \quad d(xy) = dz \quad \Rightarrow \quad xy - z = C_1 \qquad ...(1)$$

Next, taking multipliers as $x, -y, z$, we get

$$\frac{dx}{y - xz} = \frac{dy}{yz + x} = \frac{dz}{x^2 + y^2} = \frac{x\,dx - y\,dy + z\,dz}{xy - x^2z - y^2z - xy + x^2z + y^2z}$$

$$\Rightarrow \quad x\,dx - y\,dy + z\,dz = 0 \quad \Rightarrow \quad x^2 - y^2 + z^2 = C_2 \qquad ...(2)$$

Thus, (1) and (2) together represent the solution of the given simultaneous equations. **Ans.**

EXAMPLE 10.35. Solve $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$.

SOLUTION: The last two members give $\dfrac{dy}{y} = \dfrac{dz}{z}$, which, on integration, gives

$$\log y = \log z + \log C_1 \quad \text{or} \quad y = C_1 z. \qquad ...(1)$$

Next, each member of the given equation is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$$

$$\Rightarrow \quad \frac{dy}{2xy} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} \quad \text{or} \quad \frac{dy}{y} = \frac{d(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2}$$

which, on integration, gives $\log y = \log(x^2 + y^2 + z^2) - \log C_2$ or $x^2 + y^2 + z^2 = C_2 y$ ...(2)

Therefore (1) and (2) together represent the required solution. (1) and (2) can also be written as

$$\frac{y}{z} = C_1 \quad \text{and} \quad \frac{x^2 + y^2 + z^2}{y} = C_2.$$

The solution is sometimes written in any one of the following forms:

$$\frac{y}{z} = f_1\left(\frac{x^2 + y^2 + z^2}{y}\right) \quad \text{or} \quad \frac{x^2 + y^2 + z^2}{y} = f_2\left(\frac{y}{z}\right) \quad \text{or} \quad f\left(\frac{x^2 + y^2 + z^2}{y}, \frac{y}{z}\right) = 0. \qquad \textbf{Ans.}$$

EXAMPLE 10.36. Solve $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$.

SOLUTION: Each member of the given equation, is equal to

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy} = \frac{dz - dx}{z^2 - xy - x^2 + yz}$$

or $$\frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)} = \frac{dz - dx}{(z - x)(x + y + z)}$$

or $$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x} \qquad \text{...(1)}$$

The first two members of (1), on integration, give

$$\log (x - y) = \log (y - z) + \log C_1 \quad \text{or} \quad \frac{x - y}{y - z} = C_1 \qquad \text{...(1)}$$

where c_1 is an arbitrary constant.

Each member of the given equation, is also equal to $\dfrac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$

and taking x, y, z as multipliers, each member of the given equation, is also equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\Rightarrow \quad \frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$\Rightarrow \quad \frac{x\,dx + y\,dy + z\,dz}{x + y + z} = dx + dy + dz$$

or $$x\,dx + y\,dy + z\,dz = (x + y + z)(dx + dy + dz)$$

which on integration, gives $x^2 + y^2 + z^2 = (x + y + z)^2 - 2C_2$ ...(2)

or $$x y + yz + zx = C_2$$

Relations (1) and (2) together constitute the solution of the given equation. The solution can also be written as

$$f_1\left(xy + yz + zx, \frac{x - y}{y - z}\right) = 0 \quad \text{or} \quad xy + yz + zx = f_2\left(\frac{x - y}{y - z}\right) \quad \text{or} \quad \frac{x - y}{y - z} = f_3\,(xy + yz + zx).$$ **Ans.**

EXERCISE 10C

1. Solve $\frac{dx}{dt} + 5x + y = e^t$, $\frac{dy}{dt} + 3y - x = e^{2t}$. [GGSIPU I Sem End Term 2011]

2. Two coupled electrical circuits have currents i_1 and i_2 given by

$$L\frac{di_1}{dt} + Ri_1 + R(i_1 - i_2) = E$$

$$L\frac{di_2}{dt} + Ri_2 - R(i_1 - i_2) = 0$$

where the inductance L, resistance R and E.M.F. E are constants. Find i_1, i_2 as functions of time t, if initially $i_1 = i_2 = 0$ at $t = 0$.

3. Solve $\frac{dx}{dt} = x^2 + xy$, $\frac{dy}{dt} = y^2 + xy$ given that $x = 1$, $y = 2$ at $t = 0$.

4. Velocity components of a particle are given by

$$\frac{dx}{dt} + 2x - y = 1 \quad \text{and} \quad \frac{dy}{dt} + 2y - x = 0.$$

If the particle was at the origin initially, find its path.

5. The equations of motion of a particle, are given by

$$\frac{dx}{dt} + \omega y = 0, \quad \frac{dy}{dt} - \omega x = 0$$

Find the path of the particle and show that it is a circle.

6. A mechanical system satisfies the simultaneous equations

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4, \quad 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0.$$

Obtain x and y in terms of t, given that $x, y, \frac{dx}{dt}, \frac{dy}{dt}$ all vanish at $t = 0$.

7. Solve $\frac{l\,dx}{(m-n)\,yz} = \frac{m\,dy}{(n-l)\,zx} = \frac{n\,dz}{(l-m)\,xy}$.

8. Solve $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$.

9. Solve the simultaneous linear equations $\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}$.

10. Solve $\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{4xy^2 - 2z}$.

11. Solve $\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)}$.

12. Solve $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$.

13. Solve $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{(x+y)(1+2xy+3x^2y^2)}$

14. Solve the simultaneous equations

$$\frac{dx}{dt} + 4x + 3y = t, \quad \frac{dy}{dt} + 2x + 5y = e^t$$

15. Solve the system of simultaneous equations

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=\sin t,\quad \frac{dx}{dt}+x-3y=0$$

16. Solve the simultaneous equations

$$\frac{dx}{dt}+5x-2y=t,\quad \frac{dy}{dt}+2x+y=0$$

given that $x = y = 0$ at $t = 0$.

17. (a) Solve the following simultaneous equations

$$\frac{dx}{dt}+2x+3y=0,\quad 3x+\frac{dy}{dt}+2y=2e^{2t}.$$

(b) Solve $2\frac{dx}{dt}+3\frac{dy}{dt}+x+y=\sin t,$

$\frac{dx}{dt}+\frac{dy}{dt}+5x+7y=2,$ given that $x=y=0$ at $t=0.$

[GGSIPU I Sem End Term

18. Solve $\frac{dx}{dt}=7x-y$ and $\frac{dy}{dt}=2x+5y.$

APPLICATIONS TO MECHANICAL AND ELECTRICAL OSCILLATIONS CIRCUITS

EXAMPLE 10.37. An uncharged condenser of capacitance C is charged by applying an e.m.f. equal to $E\sin\left(\frac{t}{\sqrt{LC}}\right)$ through the leads of inductance L of negligible resistance, the charge Q on the plate of the condenser satisfies the equation $L\frac{di}{dt}+\frac{Q}{C}=E\sin\frac{t}{\sqrt{LC}}$, where current $i=\frac{dQ}{dt}$. Find the charge at any time t.

SOLUTION: The given equation can be written as

$$\frac{d^2Q}{dt^2}+n^2Q=\frac{E}{L}\sin nt \quad \text{where} \quad n=\frac{1}{\sqrt{LC}}.$$

or $(D^2+n^2)Q=\frac{E}{L}\sin nt$ where $D\equiv\frac{d}{dt}$.

Its C.F. $=C_1\cos nt+C_2\sin nt$

and P.I. $=\frac{E}{L}\cdot\frac{1}{D^2+n^2}\sin nt=\frac{E}{L}\frac{t}{\frac{d}{dD}(D^2+n^2)}\sin nt=\frac{E}{L}t\cdot\frac{1}{2D}\sin nt=-\frac{E}{2nL}t\cos nt.$

Therefore, the general solution is

$$Q=C_1\cos nt+C_2\sin nt-\frac{E}{2nL}t\cos nt$$

Since $Q=0$ at $t=0$ we have $0=C_1$

Hence $Q=C_2\sin nt-\frac{E}{2nL}t\cos nt.$

Then $i=\frac{dQ}{dt}=n\,C_2\cos nt-\frac{E}{2nL}(\cos nt-nt\sin nt)$

Since $i=0$ at $t=0$ we have $0=n.C_2-\frac{E}{2nL}$ or $C_2=\frac{E}{2Ln^2}$

$\therefore$ $Q=\frac{E}{2n^2L}\sin nt-\frac{Et}{2nL}\cos nt$

or $Q=\frac{E}{2n^2L}[\sin nt-nt\cos nt]$ where $n^2=\frac{1}{LC}$. **Ans.**

EXAMPLE 10.38. Give a physical interpretation of the equation

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+\omega^2x=a\cos pt$$

Integrate the equation and state what each term in the solution represents. When t is very large, show that the solution takes the form

$$x=\frac{a\cos(pt-\theta)}{\sqrt{(\omega^2-p^2)^2+4k^2p^2}} \quad \text{where} \quad \tan\theta=\frac{2kp}{\omega^2-p^2}.$$

SOLUTION: The given equation represents the case of forced oscillations with damping in the mechanical system as well as in the electric circuit system. The given equation can be written as

$$(D^2+2kD+\omega^2)x=a\cos pt \quad \text{where} \quad D\equiv\frac{d}{dt}.$$

The roots of A.E, are $-k \pm \sqrt{k^2 - \omega^2} = -k \pm i\,n$ where $n^2 = \omega^2 - k^2$ as $\omega > k$.

$\therefore$ C.F. $= e^{-kt}(A \cos nt + B \sin nt)$

and P.I. $= a \dfrac{1}{D^2 + 2kD + \omega^2} \cos pt = a \dfrac{1}{2kD + \omega^2 - p^2} \cos pt$

$$= a \frac{\{\omega^2 - p^2 - 2kD\}}{(\omega^2 - p^2)^2 - 4k^2D^2} \cos pt = \frac{a\,\{\omega^2 - p^2 - 2kD\}}{(\omega^2 - p^2)^2 + 4k^2p^2} \cos pt$$

$$= \frac{a\,\{(\omega^2 - p^2)\cos pt + 2kp \sin pt\}}{(\omega^2 - p^2)^2 + 4k^2p^2}.$$

Now, if we write $\omega^2 - p^2 = r\cos\theta$ and $2kp = r\sin\theta$

so that $r = \sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}$ and $\tan\theta = \dfrac{2kp}{\omega^2 - p^2}$

then P.I. $= \dfrac{a\,[r\cos\theta\cos pt + r\sin\theta\sin pt]}{r^2} = \dfrac{a}{r}\cos(pt - \theta)$

Therefore the complete solution is

$$x = e^{-kt}(A\cos nt + B\sin nt) + \frac{a}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}\cos(pt - \theta) \quad \text{where} \quad n = \sqrt{\omega^2 - k^2}.$$

The first term in the above solution represents free oscillations whereas the second term, the forced oscillation. When t is very large the free oscillations die down because of the presence of the term e^{-kt} whereas the forced oscillations alone persist because in the expression for forced oscillations there is no diminishing term, only a periodic term is there.

Therefore for large t the solution is

$$x = \frac{a\cos(pt - \theta)}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}} \quad \text{where} \quad \tan\theta = \frac{2kp}{(\omega^2 - p^2)}.$$ **Ans.**

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CHAPTER 11

Bessel's Differential Equation and Legendre's Differential Equation

Bessel's Differential Equation, Bessel's Functions, Legendre's Differential Equation and Legendre's Polynomials.

In this section we shall introduce mainly two functions-Bessel's and Legendre's. They gain importance by their frequent appearence in a variety of practical problems. They are needed in the evaluation of some typical integrals and in the solution of certain differential equations in vibrations, electric fields, heat conduction, fluid flow, etc.

BESSEL'S EQUATION – ITS SOLUTION

[*GGSIPU III Sem I Term 2007*]

We shall use Frobenius method for solving one of the most important differential equations in Applied Mathematics, the Bessel's differential equation

$$x^2y'' + xy' + (x^2 - n^2)\, y = 0 \quad \text{...(1)}$$

Or, in the standard form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

for n a non-negative real number.

Assuming its solution, in series, as $y(x) = \sum_{r=0}^{\infty} a_r x^{m+r}$

and substituting it with undetermined coefficients and its derivatives into (1), yields

$$\Sigma(m+r)(m+r-1)\, a_r x^{m+r} + \Sigma(m+r)\, a_r x^{m+r} + \Sigma a_r x^{m+r+2} - n^2 \,\Sigma a_r x^{m+r} = 0$$

$$\text{or} \quad \Sigma\{(m+r)^2 - n^2\}\, a_r x^{m+r} + \Sigma a_r x^{m+r+2} = 0 \quad \text{...(2)}$$

This is an identity in x. Equating to zero the coefficient of the lowest degree term x^m in (2), we get the indicial equation as $a_0\,(m^2 - n^2) = 0$ which gives $m = -n, n$ since $a_0 \neq 0$.

And equating to zero the coefficient of x^{m+1} in (2), gives

$$\{(m+1)^2 - n^2\}\, a_1 = 0 \quad \text{which gives} \quad a_1 = 0.$$

Next, equating to zero the coefficient of x^{m+r}, we get the recurrence relation as

$$(m+r+n)\,(m+r-n)\, a_r + a_{r-2} = 0 \text{ ...}$$

It follows that $a_1 = a_3 = a_5 = a_7 = \ldots = 0$

and $a_2 = \dfrac{-a_0}{(m+2+n)(m+2-n)}$,

$$a_4 = \frac{-a_2}{(m+4+n)(m+4-n)} = \frac{a_0}{(m+2+n)(m+4+n)(m+2-n)(m+4-n)}, \text{ etc.}$$

Therefore, the solution is

$$y = a_0 x^m \left[1 - \frac{x^2}{(m+2+n)(m+2-n)} + \frac{x^4}{(m+2+n)(m+4+n)(m+2-n)(m+4-n)} - \ldots\ldots\right].$$

When $m = n$ we have one solution as

$$y_1 = a_0 x^n \left[1 - \frac{x^2}{2\cdot 2(n+1)} + \frac{x^4}{2\cdot 4\cdot 2^2(n+1)(n+2)} - \frac{x^6}{2\cdot 4\cdot 6\cdot 2^3(n+1)(n+2)(n+3)} + \ldots\right] \quad \ldots(3)$$

and when $m = -n$ the other solution is

$$y_2 = a_0 x^{-n} \left[1 - \frac{x^2}{2\cdot 2(-n+1)} + \frac{x^4}{2\cdot 4\cdot 2^2(-n+1)(-n+2)} - \frac{x^6}{2\cdot 4\cdot 6\cdot 2^3(-n+1)(-n+2)(-n+3)} + \ldots\right] \quad \ldots(4)$$

Now arise the following three cases in respect of values of n.

CASE I: *n is not an integer and not zero.*

In this case the general solution is $y = Ay_1 + By_2$ where A and B are arbitrary constants.

Let us choose $a_0 = \dfrac{1}{2^n \lceil n+1}$ then the solution y_1 as given in (3), is called the ***Bessel's function of first kind of order*** n and is denoted by $J_n(x)$. Thus, we have

$$J_n(x) = \left(\frac{x}{2}\right)^n \left[\frac{1}{\lceil n+1} - \frac{x^2}{2\cdot 2^1 \lceil n+2} + \frac{x^4}{2\cdot 4\cdot 2^2 \lceil n+3} - \frac{x^6}{2\cdot 4\cdot 6\cdot 2^3 \lceil n+4} + \ldots\right]$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\lceil n+1} - \frac{(x/2)^2}{1!\lceil n+2} + \frac{(x/2)^4}{2!\lceil n+3} - \frac{(x/2)^6}{3!\lceil n+4} + \ldots\right]$$

or

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\lceil n+r+1}\left(\frac{x}{2}\right)^{n+2r}. \quad \ldots(5)$$

Also, from (4), we get

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\lceil -n+r+1}\left(\frac{x}{2}\right)^{-n+2r} \quad \ldots(6)$$

which is called the ***Bessel's function of first kind of order*** $-n$.

Thus, the complete solution of Bessel's equation, when n is neither zero nor an integer, is $y = AJ_n(x) + BJ_{-n}(x)$ where A and B are arbitrary constants.

For many purposes it is convenient to take the linear combination

$$Y_n = \frac{\cos n\pi\, J_n(x) - J_{-n}(x)}{\sin n\pi} \qquad (n \text{ not integer})$$

instead of $J_{-n}(x)$ as the second independent solution of the Bessel's equation. $Y_n(x)$ is known as **Bessel function of the second kind of order** n and we can write the complete solution of Bessel's equation in the alternative form as $y = AJ_n(x) + B\,Y_n(x)$ (n not an integer)

Case II: $n = 0$ [GGSIPU III Sem I Term 2010]

Consider the Bessel's equation of order zero, as obtained from (1).

$$xy'' + y' + xy = 0 \quad \text{...(7)}$$

Let us assume its solution in series, as $\quad y = \sum_{0}^{\infty} a_r x^{m+r}$ ...(8)

Substituting it in (7), gives

$$\Sigma(m+r)(m+r-1)\, a_r x^{m+r-1} + \Sigma(m+r)\, a_r x^{m+r-1} + \Sigma a_r x^{m+r+1} = 0$$

or
$$\sum_{0}^{\infty}(m+r)^2\, a_r x^{m+r-1} + \sum_{0}^{\infty} a_r x^{m+r+1} = 0$$

Equating the coefficient of the lowest degree term x^{m-1} to zero, we get the indicial equation as

$$m^2 a_0 = 0 \qquad \text{hence} \qquad m = 0, 0 \qquad \text{as } a_0 \neq 0.$$

Equating to zero the coefficient of x^m, we get

$$(m+1)^2 a_1 = 0 \qquad \text{hence} \qquad a_1 = 0 \text{ since } m \neq -1.$$

The recurrence relation of coefficients is $\quad (m+r+2)^2 a_{r+2} + a_r = 0$

It follows that $\quad a_1 = a_3 = a_5 = a_7 = \ldots = 0$

and
$$a_{r+2} = -\frac{a_r}{(m+r+2)^2}, \quad r = 0, 1, 2, \ldots$$

$$\therefore \quad a_2 = \frac{-a_0}{(m+2)^2}, \quad a_4 = \frac{-a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2(m+4)^2},$$

$$a_6 = \frac{-a_0}{(m+2)^2(m+4)^2(m+6)^2}, \ldots.$$

Therefore, (8) becomes

$$y = x^m \Sigma \frac{(-1)^r x^{2r}}{(m+2)^2(m+4)^2 \ldots (m+2r)^2} \quad \text{...(9)}$$

One solution is obtained directly by taking $m = 0$ in (9), as

$$y_1 = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r} = J_0(x). \quad \text{...(10)}$$

For the second independent solution we differentiate (9) partially w.r.t. m to get

$$\frac{\partial y}{\partial m} = x^m \log x \sum_{0}^{\infty} \frac{(-1)^r x^{2r}}{(m+2)^3(m+4)^2 \ldots (m+2r)^2}$$
$$+ x^m \sum_{0}^{\infty}(-1)^r x^{2r}\left[\frac{-2}{(m+2)^2(m+4)^2 \ldots (m+2r)^2} + \frac{-2}{(m+2)^2(m+4)^3(m+6)^2 \ldots (m+2r)^2} + \ldots + \frac{-2}{(m+2)^2(m+4)^2 \ldots (m+2r-2)^2(m+2r)^3}\right]$$

or
$$\frac{\partial y}{\partial m} = x^m \log x \sum_{0}^{\infty} \frac{(-1)^r x^{2r}}{(m+2)^2(m+4)^2 \ldots (m+2r)^2}$$
$$+ x^m \sum_{0}^{\infty} \frac{(-1)^r x^{2r}(-2)}{(m+2)^2(m+4)^2 \ldots (m+2r)^2}\left[\frac{1}{m+2} + \frac{1}{m+4} + \ldots + \frac{1}{m+2r}\right]$$

The second independent solution of (7) is $\left(\frac{\partial y}{\partial m}\right)_{m=0}$, written as

$$y_2 = Y_0(x) = J_0(x)\log x + \sum_{0}^{\infty} \frac{(-1)^{r+1}}{(r!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{r}\right)\left(\frac{x}{2}\right)^{2r}$$

Thus, the general solution of (7) is $y = A\,J_0(x) + Y_0(x)$

where A and B are arbitrary constants.

CASE III: *n is a non-zero integer.*

From equations (3) and (4) we have seen that when n is a positive integer y_2 fails to give a solution and when n is a negative integer y_1 fails to given a solution. Thus, when n is an integer we get only one solution and we still require another independent solution to form a general solution of Bessel's equation in this case. The second independent solution can be obtained in one of the several ways. One way is to adopt the standard practice to define the second solution by the function

$$Y_\nu = \frac{J_\nu(x)\cos \nu x - J_{-\nu}(x)}{\sin \nu\pi}, \quad \nu \neq \text{integer}$$

and $$Y_n = \lim_{\nu \to n} Y_\nu(x) = \lim_{\nu \to n} \frac{J_\nu(x)\cos \nu x - J_{-\nu}(x)}{\sin \nu\pi} \quad \left(\frac{0}{0}\text{ form}\right) \quad \text{where } n \text{ is integer.}$$

THEOREM: When n is integer the Bessel's function $J_n(x)$ and $J_{-n}(x)$ are **linearly dependent** and $J_{-n}(x) = (-1)^n J_n(x)$.

[*GGSIPU III Sem I Term 2007; I Term 2011; I Term 2012, I Term 2013*]

PROOF: We know that $$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Since the Gamma functions in the coefficients of the first n terms become infinite, these coefficients become zero and the summation starts with $r = n$ only, hence

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Writing here $r = n + k$ in the above summation, we get

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{(n+k)!\,\Gamma(k+1)}\left(\frac{x}{2}\right)^{-n+2n+2k} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(n+k+1)\,k!}\left(\frac{x}{2}\right)^{n+2k}$$

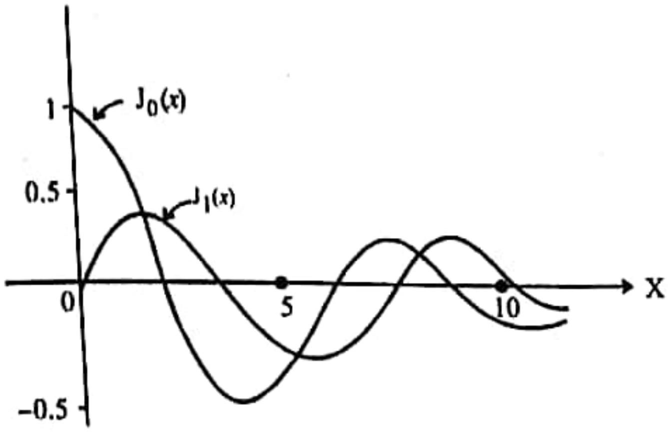
$$= (-1)^n J_n(x) \qquad (\text{since } \Gamma(n+k+1) = (n+k)! \text{ and } \Gamma(k+1) = k!)$$

which completes the proof.

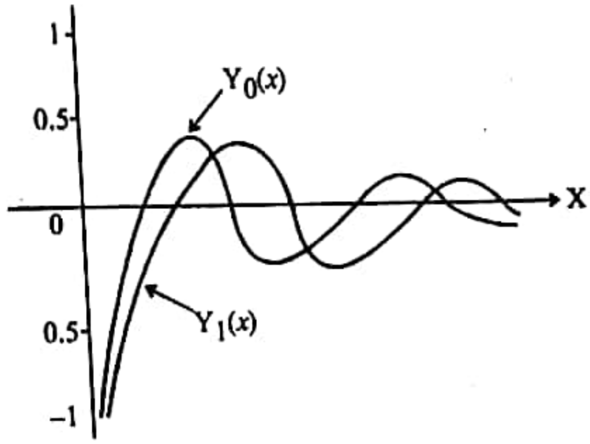
We have $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

and $$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \ldots$$

The graphs of these two functions are given in the adjoining figure. Both the functions are oscillatory with varying period and decreasing amplitude.



The graphs of the functions $Y_0(x)$ and $Y_1(x)$ are shown in the adjoining figure.



BESSEL FUNCTIONS OF ORDER HALF

[*GGSIPU III Sem I Term 2005; I Term 2007; I Term 2010; I Term 2011; I Term 2012*]

From the definition of $J_n(x)$, takin $n = 1/2$, we have

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\left(\frac{3}{2}+r\right)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2r} = \frac{\sqrt{x}}{\sqrt{2}\,\Gamma\left(\frac{3}{2}\right)} \left[1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 4\cdot 3\cdot 5} - \ldots\right]$$

$$= \frac{1}{\sqrt{x}\,\sqrt{2}\,\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right] = \sqrt{\frac{2}{\pi x}}\sin x \qquad \left(\text{since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

Similarly, on taking $n = -1/2$ in $J_n(x)$, we get

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\left(\frac{1}{2}+r\right)} \left(\frac{x}{2}\right)^{-1/2+2r} = \left(\frac{x}{2}\right)^{-1/2} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \left[1 - \frac{x^2}{2\cdot 1} + \frac{x^4}{2\cdot 4\cdot 1\cdot 3} - \ldots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right] = \sqrt{\frac{2}{\pi x}}\cos x.$$ [*GGSIPU III Sem I Term 2013*]

Thus, $J_{1/2} = \sqrt{\dfrac{2}{\pi x}}\sin x$ and $J_{-1/2} = \sqrt{\dfrac{2}{\pi x}}\cos x$.

RECURRENCE RELATIONS OF BESSEL FUNCTIONS

The Bessel functions of successive orders are connected by certain relations as follows.

I. $\dfrac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$ or $\displaystyle\int_0^x x^n J_{n-1}(x)\,dx = x^n J_n(x)$.

[GGSIPU III Sem II Term 2011, I Sem End Term 2013]

From the definition of $J_n(x)$, we have

$$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2n+2r}}{r!\,\overline{|n+r+1}\cdot 2^{n+2r}}$$

$$\therefore \quad \frac{d}{dx}\{x^n J_n(x)\} = \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)\, x^{2n+2r-1}}{r!\,\overline{|n+r+1}\cdot 2^{n+2r}} = x^n \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)\, x^{n+2r-1}}{r!\,\overline{|n+r+1}\cdot 2^{n+2r-1}}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\overline{|n+r}} \left(\frac{x}{2}\right)^{n+2r-1} = x^n J_{n-1}(x), \quad \text{as} \quad \overline{|n+r} = \overline{|n-1+r-1}$$

II. $\dfrac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$.

[GGSIPU III Sem II Term 2007, End Term 2013]

We have $$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{-n}}{r!\,\overline{|n+r+1}} \left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{r!\,\overline{|n+r+1}\; 2^{n+2r}}$$

$$\therefore \quad \frac{d}{dx}\{x^{-n} J_n(x)\} = \sum_{r=0}^{\infty} \frac{2r(-1)^r x^{2r-1}}{r!\,\overline{|n+r+1}\; 2^{n+2r}} = -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^{r-1} r}{r!\,\overline{|n+r+1}} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -x^{-n} \sum_{1}^{\infty} \frac{(-1)^{r-1}}{(r-1)!\,\overline{|n+r+1}} \left(\frac{x}{2}\right)^{n+2r-1} \quad \text{now put } r = k+1$$

$$= -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\overline{|n+2+k}} \left(\frac{x}{2}\right)^{n+2k+1} = -x^{-n} J_{n+1}(x).$$

III. $J_n'(x) = J_{n-1}(x) - \dfrac{n}{x} J_n(x)$

[GGSIPU III Sem II Term 2013]

$\dfrac{d}{dx}\{x^n J_n(x)\} = x^n J_n'(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$ (Using recurrence relation I)

Dividing throughout by x^n, gives: $J_n'(x) = J_{n-1}(x) - \dfrac{n}{x} J_n(x)$.

IV. $J_n'(x) = -J_{n+1}(x) + \dfrac{n}{x} J_n(x)$ [GGSIPU III Sem End Term 2008; End Term 2010; End Term 2012]

$\dfrac{d}{dx}\{x^{-n} J_n(x)\} = x^{-n} J_n'(x) - nx^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$ (Using recurrence relation II)

Multiplying throughout by x^n, gives

$$J_n' = -J_{n+1} + \frac{n}{x} J_n$$

V. $J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

[GGSIPU III Sem End Term 2005]

Adding the recurrence relations III and IV, gives

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

hence $$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)].$$

VI. $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$ [GGSIPU III Sem End Term 2004]

Subtracting recurrence relation IV from III, gives

$$0 = J_{n-1}(x) + J_{n+1}(x) - \frac{2n}{x} J_n(x)$$

or $$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x).$$

This can also be written in any of the following two forms:

or $$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

or $$J_{n-1}(x) = \frac{2n}{x} J_n(x) - J_{n+1}(x).$$

EXAMPLE 11.1. Express $J_{3/2}$, $J_{-3/2}$, $J_{5/2}$ and $J_{-5/2}$ in terms of sine and cosine functions.

SOLUTION: We know that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ ...(1)

Putting $n = 1/2$ in the recurrence relation $\frac{2n}{x} J_n = J_{n-1} + J_{n+1}$

we get $$\frac{1}{x} J_{1/2} = J_{-1/2} + J_{3/2} \qquad ...(2)$$

$$\therefore \quad J_{3/2} = \frac{1}{x} J_{1/2} - J_{-1/2} = \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x}{x} - \cos x\right]$$

Next, putting $n = -1/2$ in (1), we get $-\frac{1}{x} J_{-1/2} = J_{-3/2} + J_{1/2}$ ...(3)

$$\therefore \quad J_{-3/2} = -\frac{1}{x} J_{-1/2} - J_{1/2} = -\sqrt{\frac{2}{\pi x}} \left[\frac{\cos x}{x} + \sin x\right].$$

Further, putting $n = 3/2$ in (1), we get $\frac{3}{x} J_{3/2} = J_{1/2} + J_{5/2}$

$$\therefore \quad J_{5/2} = \frac{3}{x} J_{3/2} - J_{1/2} = \sqrt{\frac{2}{\pi x}} \left[\frac{3}{x}\left(\frac{\sin x}{x} - \cos x\right) - \sin x\right] \quad \text{(using (2))}$$

[GGSIPU III Sem II Term 2009]

or $$J_{5/2} = \sqrt{\frac{2}{\pi x}} \left[\frac{3 - x^2}{x^2} \sin x - \frac{3}{x} \cos x\right],$$

and, on putting $n = -3/2$ in (1), we get $-\frac{3}{x} J_{-3/2} = J_{-5/2} + J_{-1/2}$

$$\therefore \quad J_{-5/2} = -\frac{3}{x} J_{-3/2} - J_{-1/2} = \sqrt{\frac{2}{\pi x}} \left[\frac{3}{x}\left(\frac{\cos x}{x} + \sin x\right) - \cos x\right] \quad \text{(using (3))}$$

Ans.

or $$J_{-5/2} = \sqrt{\frac{2}{\pi x}} \left[\frac{3 - x^2}{x^2} \cos x + \frac{3}{x} \sin x\right].$$

EXAMPLE 11.2. Show that $\frac{d}{dx}[x\, J_n\, J_{n+1}] = x[J_n^2 - J_{n+1}^2]$ or $\int x(J_n^2 - J_{n+1}^1)\, dx = x\, J_n\, J_{n+1}$.

[GGSIPU III Sem II Term 2013]

SOLUTION. $\frac{d}{dx}[x\, J_n\, J_{n+1}] = 1 \cdot J_n\, J_{n+1} + x\,(J_n\, J'_{n+1} + J'_n\, J_{n+1})$ ...(1)

From recurrence relation III we can write: $J'_{n+1}(x) = J_n(x) - \frac{n+1}{x} J_{n+1}$ ...(2)

and recurrence relation IV is $J'_n = -J_{n+1} + \frac{n}{x} J_n$ ...(3)

Using (2) and (3) in (1), we get

$$\frac{d}{dx}[x\, J_n\, J_{n+1}] = J_n\, J_{n+1} + xJ_n\left(J_n - \frac{n+1}{x} J_{n+1}\right) + xJ_{n+1}\left(-J_{n+1} + \frac{n}{x} J_n\right)$$
$$= J_n\, J_{n+1} + xJ_n^2 - (n+1)\, J_n\, J_{n+1} - xJ_{n+1}^2 + n\, J_n\, J_{n+1}$$
$$= x\,(J_n^2 - J_{n+1}^2)$$ **hence the result.**

Aliter

$$\frac{d}{dx}[x\, J_n\, J_{n+1}] = \frac{d}{dx}[(x^{-n}\, J_n)\,(x^{n+1}\, J_{n+1})]$$
$$= x^{-n}\, J_n \frac{d}{dx}(x^{n+1}\, J_{n+1}) + x^{n+1}\, J_{n+1} \frac{d}{dx}(x^{-n}\, J_n)$$
$$= x^{-n}\, J_n\, \{x^{n+1}\, J_n\} + x^{n+1}\, J_{n+1}\, \{-x^{-n}\, J_{n+1}\}$$ (by recurrence relations I and II)
$$= xJ_n^2 - xJ_{n+1}^2 = x\,(J_n^2 - J_{n+1}^2).$$ **hence the result.**

EXAMPLE 11.3. Evaluate $\int x^{-1} J_4(x)\, dx$.

SOLUTION: Using the recurrence relation II, we can write

$$\frac{d}{dx}[x^{-3}\, J_3(x)] = -x^{-3}\, J_4(x) \Rightarrow \int x^{-3} J_4(x)\, dx = -x^{-3}\, J_3(x)$$

$\therefore$ $\int x^{-1} J_4(x)\, dx = \int x^2\, \{x^{-3}\, J_4(x)\}\, dx$, (now integrating it by parts)

$$= x^2\, \{-x^{-3}\, J_3(x)\} - \int 2x\, \{-x^{-3}\, J_3(x)\}\, dx$$

or $\int x^{-1} J_4(x)\, dx = -x^{-1}\, J_3(x) + 2\int x^{-2}\, J_3(x)\, dx$ ...(1)

From recurrence relation II, we have $\int x^{-2} J_3(x)\, dx = -x^{-2}\, J_2(x)$ ...(2)

Therefore, using (2) in (1), gives: $\int x^{-1} J_4(x)\, dx = -x^{-1}\, J_3(x) - 2x^{-2}\, J_2(x)$. **Ans.**

EXAMPLE 11.4. Show that $J_3(x) + 3J_0'(x) + 4J_0'''(x) = 0$. [GGSIPU III Sem II Term 2010]

SOLUTION: From the recurrence relation II, we can write: $J_0'(x) = -J_1(x)$

Differentiating the above relation w.r.t. x, we get

$$J_0''(x) = -J_1'(x) = -\frac{1}{2}[J_0(x) - J_2(x)]$$ (using recurrence relation V)

Differentiating again w.r.t. x, gives

$$J_0'''(x) = -\frac{1}{2}[J_0'(x) - J_2'(x)] = -\frac{1}{2}\left[J_0'(x) - \frac{1}{2}\{J_1(x) - J_3(x)\}\right] \quad \text{(using recurrence relation V)}$$

$$= -\frac{1}{2}\left[J_0'(x) + \frac{1}{2}J_0'(x) + \frac{1}{2}J_3(x)\right] = -\frac{1}{4}[2J_0'(x) + J_0'(x) + J_3(x)]$$

Therefore $4J_0'''(x) = -3J_0'(x) - J_3(x)$. **hence the result.**

16.5. Express $\int x^{-2} J_2(x)\,dx$ in terms of Bessel functions.

SOLUTION: Given integral $= \int x^{-2} J_2(x)\,dx = \int x^{-4}\{x^2 J_2(x)\}\,dx$ (now integrating by parts)

$$= x^2 J_2(x)\frac{x^{-3}}{(-3)} - \int \frac{d}{dx}(x^2 J_2)\cdot\frac{x^{-3}}{(-3)}\,dx$$

$$= -\frac{1}{3x}J_2(x) + \frac{1}{3}\int x^{-3}\{x^2 J_1(x)\}\,dx \quad \text{(by recurrence relation I)}$$

$$= -\frac{1}{3x}J_2(x) + \frac{1}{3}\int x^{-1} J_1 dx.$$

Next, $\int x^{-1}J_1(x)\,dx = \int x^{-2}(xJ_1)\,dx$ (now integrating by parts)

$$= -x J_1(x)\,x^{-1} - \int \frac{x^{-1}}{-1}\frac{d}{dx}(xJ_1)\,dx = -J_1(x) + \int x^{-1} xJ_0(x)\,dx$$

$$= -J_1(x) + \int J_0\,dx$$

$$\therefore \quad \int x^{-2} J_2(x)\,dx = -\frac{1}{3x}J_2(x) - \frac{1}{3}J_1(x) + \frac{1}{3}\int J_0(x)\,dx \qquad \textbf{Ans.}$$

The integral on the right cannot be evaluated. It can only be obtained from the standard table.

16. Show that $\left(\frac{1}{x}\frac{d}{dx}\right)^m [x^{-n}J_n(x)] = \frac{(-1)^m}{x^{n+m}}J_{n+m}(x)$. [GGSIPU III Sem End Term 2007]

SOLUTION: From the recurrence relation $\frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1}(x)$ we have

$$\text{or} \quad \frac{1}{x}\frac{d}{dx}(x^{-n}J_n(x)) = \frac{-x^{-n}}{x}J_{n+1}(x)$$

$$\text{or} \quad \left(\frac{1}{x}\frac{d}{dx}\right)(x^{-n}J_n(x)) = (-1)^1\left(\frac{1}{x^{n+1}}J_{n+1}(x)\right) \qquad \text{(Note this operation)}$$

$$\Rightarrow \quad \left(\frac{1}{x}\frac{d}{dx}\right)^2(x^{-n}J_n(x)) = -\left(\frac{1}{x}\frac{d}{dx}\right)\left[\frac{1}{x^{n+1}}J_{n+1}(x)\right] = (-1)^2\frac{1}{x^{n+2}}J_{n+2}(x)$$

$$\text{Similarly} \quad \left(\frac{1}{x}\frac{d}{dx}\right)^3(x^{-n}J_n(x)) = -\left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{1}{x^{n+2}}J_{n+2}(x)\right) = (-1)^3\frac{1}{x^{n+3}}J_{n+3}(x)$$

Continuing the process, we get

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m(x^{-n}J_n(x)) = (-1)^m\frac{1}{x^{n+m}}J_{n+m}(x)$$

Hence the result.

EXAMPLE 11.7. (*i*) Show that $\int J_5(x)dx = -J_4(x) - \frac{4}{x}J_3(x) - \frac{8}{x^2}J_2(x)$.

[GGSIPU III Sem II Term 2006]

(*ii*) Show that $\int_0^{\pi/2} \sqrt{\pi x}\, J_{\frac{1}{2}}(2x)\, dx = 1$ [GGSIPU III Sem II Term 2006]

SOLUTION: (*i*) In the recurrence relation $\frac{d}{dx}(x^{-n} J_n) = -x^{-n} J_{n+1}$ putting $n = 4$, we have

$$\frac{d}{dx}(x^{-4} J_4) = -x^{-4}J_5 \quad \text{or} \quad J_5 = -x^4 \frac{d}{dx}(x^{-4} J_4)$$

$$\Rightarrow \quad \int J_5\, dx = -\int x^4 \frac{d}{dx}(x^{-4}J_4)dx \quad \text{(on integrating by parts)}$$

$$= -x^4\cdot(x^{-4}J_4) + \int 4x^3\cdot x^{-4}J_4\, dx = -J_4 + 4\int \frac{1}{x} J_4\, dx$$

$$= -J_4 + 4\int x^2\,(x^{-3}\, J_4)dx \quad \text{(now using recurrence relation II)}$$

$$= -J_4 - \int 4x^2 \frac{d}{dx}(x^{-3}J_3)dx \quad \text{(on integration by parts)}$$

$$= -J_4 - 4\left[x^2 . x^{-3}\, J_3 - \int 2x . x^{-3}\, J_3\, dx\right]$$

$$= -J_4 - \frac{4}{x}J_3 + 8\int x^{-2}\, J_3\, dx = -J_4 - \frac{4}{x}J_3 - 8\int \frac{d}{dx}(x^{-2}\, J_2)dx$$

$$= -J_4 - \frac{4}{x}J_3 - \frac{8}{x^2}J_2.$$ **Hence Proved.**

(*ii*) Let $I = \int_0^{\pi/2} \sqrt{\pi x}\, J_{\frac{1}{2}}(2x)dx$. Putting $2x = y$, we get

$$I = \int_0^{\pi} \sqrt{\pi}\sqrt{\frac{y}{2}}\, J_{\frac{1}{2}}(y)\cdot\frac{dy}{2} = \frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^{\pi}\sqrt{y}\, J_{\frac{1}{2}}(y)dy \quad \text{But} \quad J_{\frac{1}{2}}(y) = \sqrt{\frac{2}{\pi y}}\sin y$$

$$\therefore \quad I = \frac{1}{2}\sqrt{\frac{\pi}{2}}\int_0^{\pi}\sqrt{y}\,\frac{\sqrt{2}}{\sqrt{\pi y}}\sin y\, dy = \frac{1}{2}\int_0^{\pi}\sin y\, dy = 1$$ **Hence Proved.**

EXERCISE 11A

1. Show that $J_n''(x) = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$.

2. Show that $\frac{d}{dx}\{J_n^2(x)\} = \frac{x}{2n}\{J_{n-1}^2(x) - J_{n+1}^2(x)\}$.

3. Prove that $\int J_3(x)\,dx = -J_2(x) - \frac{2}{x}J_1(x)$.

4. Express $\int x J_0^2(x)\,dx$ in terms of $J_0(x)$ and $J_1(x)$.

[GGSIPU III Sem End Term 2011, III Sem II Term 201

5. Find the value of the integral $\int_0^1 x^{5/2} J_{3/2}(ax)\,dx$

6. Express $J_4(x)$ in terms of J_0 and J_1.

[GGSIPU I Sem End Term 201

or show that $J_2(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1 + \left(1 - \frac{24}{x^2}\right)J_0(x)$.

7. Show that $J_{-1/2}(x) = J_{1/2}(x)\cot x$.

8. Prove that $\int_0^a x J_0(\lambda x)\,dx = \frac{a}{\lambda}J_1(a\lambda)$.

9. Show that $\int_a^b J_0(x) J_1(x)\,dx = \frac{1}{2}[J_0^2(a) - J_0^2(b)]$.

10. Establish $\frac{d}{dx}\{J_n(x)\} = \frac{n}{x}J_n(x) - J_{n+1}(x)$ and hence show that $x^n J_n(x)$ is a solution o

$$x\frac{d^2y}{dx^2} + (1-2n)\frac{dy}{dx} + xy = 0.$$

GENERATING FUNCTION FOR $J_n(X)$

[GGSIPU III Sem II Term 2006; III Sem II Term 2010; II Term 2012]

Consider the function $f(x, t) = e^{x/2\,(t - 1/t)}$. It is such that when expanded in powers of t, the coefficient of t^n will be nothing but the Bessel function $J_n(x)$.

$$f(x,t) = e^{xt/2}\, e^{-x/2t}$$

$$= \left[1+\frac{xt}{2}+\frac{1}{2!}\left(\frac{xt}{2}\right)^2+\ldots+\frac{1}{n!}\left(\frac{xt}{2}\right)^n+\frac{1}{(n+1)!}\left(\frac{xt}{2}\right)^{n+1}+\frac{1}{(n+2)!}\left(\frac{xt}{2}\right)^{n+2}+\ldots\right]$$

$$\times\left[1-\frac{x}{2t}+\frac{1}{2!}\left(\frac{x}{2t}\right)^2-\frac{1}{3!}\left(\frac{x}{2t}\right)^3+\ldots+\frac{(-1)^n}{n!}\left(\frac{x}{2t}\right)^n+\ldots\right]$$

Multiplying the two series on R.H.S. we get the coefficient of t^n, as

$$\frac{1}{n!}\left(\frac{x}{2}\right)^n-\frac{1}{1!\,(n+1)!}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{2!\,(n+2)!}\left(\frac{x}{2}\right)^{n+4}-\ldots.$$

This can be written as

$$\frac{1}{0!\,\lceil n+1}\left(\frac{x}{2}\right)^n-\frac{1}{1!\,\lceil n+2}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{2!\,\lceil n+3}\left(\frac{x}{2}\right)^{n+4}-\ldots$$

which is nothing but $J_n(x)$.

Similarly, the coefficient of t^{-n} is the same as the above, except for the multiplying factor $(-1)^n$, in other words, the coefficient of t^{-n} is $(-1)^n J_n(x)$. Therefore, we have

$$e^{x/2\,(t-1/t)} = J_0(x)+[t\,J_1(x)+t^2 J_2(x)+t^3 J_3(x)+\ldots]+\left[-\frac{1}{t}J_1(x)+\frac{1}{t^2}J_2(x)-\frac{1}{t^3}J_3(x)+\ldots\right]$$

$$= J_0(x)+\left(t-\frac{1}{t}\right)J_1(x)+\left(t^2+\frac{1}{t^2}\right)J_2(x)+\left(t^3-\frac{1}{t^3}\right)J_3(x)+\ldots$$

$$= J_0(x)+\sum_{n=1}^{\infty}\left\{t^n+\frac{(-1)^n}{t^n}\right\}J_n(x) = J_0(x)+\sum_{n=1}^{\infty}[t^nJ_n(x)+t^{-n}J_{-n}(x)].$$

or
$$e^{x/2\,(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x) \qquad ...(1)$$

Thus, the Bessel functions of various orders can be derived as coefficients of the corresponding powers of t in the expansion of $e^{x/2\,(t-1/t)}$. For this reason, it is known as *generating function of Bessel functions.*

Now, replacing x by $-x$ on both sides of (2), we get

$$\sum_{-\infty}^{\infty} t^n J_n(-x) = e^{-\frac{x}{2}\left(t-\frac{1}{t}\right)}$$

Next, replacing t by $-t$ in the above relation, we get

$$\sum (-1)^n t^n J_n(-x) = \sum e^{x/2\left(t-\frac{1}{t}\right)} \quad \text{which is equal to} \quad \sum_{-\infty}^{\infty} t^n J_n(x)$$

It implies that $J_n(-x) = (-1)^n J_n(x)$. [GGSIPU III Sem II Term 2011]

Now, let $t = \cos\theta + i\sin\theta$ and so $\frac{1}{t} = \cos\theta - i\sin\theta$ and $\left(t - \frac{1}{t}\right) = 2i\sin\theta$.

$$e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta) \equiv \sum_{-\infty}^{\infty} t^n J_n(x)$$

Also $$t^n + \frac{(-1)^n}{t^n} = (\cos n\theta + i\sin n\theta) + (-1)^n(\cos n\theta - i\sin n\theta) = \begin{cases} 2\cos n\theta, & \text{when } n \text{ is even;} \\ 2i\sin n\theta, & \text{when } n \text{ is odd.} \end{cases}$$

Next, equating real and imaginary parts on both sides of (1), we get

$$\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \ldots] \quad \ldots(3)$$

$$\sin(x\sin\theta) = 2[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \ldots] \quad \ldots(4)$$

These series (3) and (4) are known as **JACOBI SERIES.**

If we replace θ by $(\pi/2 - \theta)$ in (3) and (4), we get another form of Jacobi series, as

$$\cos(x\cos\theta) = J_0 + 2[-J_2\cos 2\theta + J_4\cos 4\theta - J_6\cos 6\theta + \ldots]$$

and $$\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \ldots]$$

Now, if we multiply both sides of (3) by $\cos n\theta$ and those of (4) by $\sin n\theta$ and integrate from $\theta = 0$ to $\theta = \pi$ and use orthogonality property of sine and cosine functions, we get

$$\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta = \begin{cases} \pi J_n(x), & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$$

and $$\int_0^\pi \sin(x\sin\theta)\sin n\theta\, d\theta = \begin{cases} \pi J_n(x), & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

On adding these two relations, we get

$$\int_0^\pi [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\, d\theta = \pi J_n(x) \quad \text{where } n \text{ is any integer.}$$

Thus, for every integral value of n, we have

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\, d\theta \quad \ldots(5)$$

This is called ***integral representation of*** $J_n(x)$ given by Bessel himself.

[*GGSIPU III Sem II Term 2006, III Sem II Term 2013*]

The recurrence relations I to VI can also be derived from the generating function as follows:

For instance, differentiating $e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$ partially w.r.t. t, we get ...(6)

$$e^{x/2(t-1/t)}\,\frac{x}{2}\left(1 + \frac{1}{t^2}\right) = \sum_{n=-\infty}^{\infty} n t^{n-1} J_n(x).$$

or $$\frac{x}{2}\left(1 + \frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} t^n J_n(x) = \sum_{-\infty}^{\infty} n t^{n-1} J_n(x) \quad \ldots(7)$$

Equating the coefficients of t^{n-1} on both sides of (7), gives

$$nJ_n(x) = \frac{x}{2}[J_{n-1} + J_{n+1}] \quad \ldots(8)$$

Which is the recurrence relation VI.

Next, differentiating (6) partially w.r.t. x, we get

$$e^{x/2(t-1/t)}\left(\frac{t}{2}-\frac{1}{2t}\right) = \sum_{n=-\infty}^{\infty} t^n J_n'(x),$$

or $$\left(\frac{t}{2}-\frac{1}{2t}\right)\sum_{n=-\infty}^{\infty} t^n J_n(x) = \sum_{n=-\infty}^{\infty} t^n J_n'(x). \quad ...(9)$$

Equating the coefficients of t^n on both sides of (9), we get

$$J_n'(x) = \frac{1}{2} J_{n-1} - \frac{1}{2} J_{n+1} \quad \text{or} \quad 2 J_n' = J_{n-1} - J_{n+1} \quad ...(10)$$

which is the recurrence relation V.

Eliminating J_{n+1} from (8) and (10), we get $xJ_n' = -nJ_n + xJ_{n-1}$
and eliminating J_n from (8) and (10), we get

$$xJ_n' = nJ_{n-1} - xJ_{n+1}$$

which are recurrence relations III and IV.

ORTHOGONALITY OF BESSEL FUNCTIONS

If α and β are the roots of the equation $J_n(x) = 0$, that is, if $J_n(\alpha) = J_n(\beta) = 0$, then

$$\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{1}{2} J_{n+1}^2(\alpha) & \text{if } \beta = \alpha. \end{cases}$$

[GGSIPU III Sem II Term 2009; III Sem II Term 2007; End Term 2010; II Sem End Term 2011]

It can be easily verified that $J_n(kx)$ is a solution of the equation

$$x^2y'' + xy' + (k^2x^2 - n^2)\, y = 0$$

therefore $u = J_n(\alpha x)$ will be the solution of

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)\, u = 0 \quad ...(1)$$

and $v = J_n(\beta x)$ will be the solution of

$$x^2v'' + xv' + (\beta^2x^2 - n^2)\, v = 0 \quad ...(2)$$

Multiplying (1) by $\frac{v}{x}$ and (2) by $-\frac{u}{x}$ and adding, we get

$$x\,(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)\, xuv = 0$$

or $$\frac{d}{dx}\,[x\,(u'v - uv')] + (\alpha^2 - \beta^2)\, xuv = 0$$

On integrating from $x = 0$ to $x = 1$, we get

$$(\alpha^2 - \beta^2)\int_0^1 x\, uv\, dx = -[x\,(u'v - uv')]_0^1 = -[u'v - uv']_{x=1}$$

or $$(\alpha^2 - \beta^2)\int_0^1 xuv\, dx = [uv' - u'v]_{x=1} \quad ...(3)$$

Here $u = J_n(\alpha x)$ so $u' = \alpha J_n'(\alpha x)$ and $v = J_n(\beta x)$ so $v' = \beta\, J_n'(\beta x)$ where $J_n'(\alpha x)$ is derivative of $J_n(\alpha x)$ w.r.t. αx and $J_n'(\beta x)$ is the derivative of $J_n(\beta x)$ w.r.t. βx.

Putting these values in (3), we get

$$(\alpha^2-\beta^2)\int_0^1 xJ_n(\alpha x)\,J_n(\beta x)\,dx = \beta J_n(\alpha)\,J_n'(\beta)-\alpha J_n'(\alpha)\,J_n(\beta) \qquad ...(4)$$

Since α and β are the roots of $J_n(x)=0$ we have $J_n(\alpha)=0=J_n(\beta)$, hence

$$(\alpha^2-\beta^2)\int_0^1 xJ_n(\alpha x)\,J_n(\beta x)\,dx = 0$$

$$\Rightarrow \quad \int_0^1 xJ_n(\alpha x)\,J_n(\beta x)\,dx = 0 \quad \text{provided} \quad \beta\neq\alpha. \qquad ...(5)$$

This property is analogous to the following orthogonal property of sine and cosine functions

$$\left.\begin{aligned}\int_0^\pi \cos mx\cos nx\,dx &= \begin{cases}0 & \text{for } m\neq n\\ \pi & \text{for } m=n\end{cases}\\ \int_0^\pi \sin mx\sin nx\,dx &= \begin{cases}0 & \text{for } m\neq n\\ \pi & \text{for } m=n\end{cases}\end{aligned}\right\} \qquad ...(6)$$

the difference in (5) from (6) is the extra factor x which is known as the weight function and we say that the *functions $J_n(\alpha x)$ and $J_n(\beta x)$ are orthogonal on the interval (0, 1) with respect to the* weight function x.

Next, if $\beta=\alpha$, from (4), we get

$$\left[\int_0^1 xJ_n(\alpha x)\,J_n(\beta x)\,dx\right]_{\beta=\alpha} = \lim_{\beta\to\alpha}\frac{\beta J_n(\alpha)\,J_n'(\beta)-\alpha J_n'(\alpha)\,J_n(\beta)}{\alpha^2-\beta^2} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{\beta\to\alpha}\frac{\beta J_n(\alpha)\,J_n''(\beta)+J_n(\alpha)\,J_n'(\beta)-\alpha J_n'(\alpha)\,J_n'(\beta)}{-2\beta}$$

$$= \frac{\alpha J_n(\alpha)\,J_n''(\alpha)+J_n(\alpha)\,J_n'(\alpha)-\alpha J_n'(\alpha)\,J_n'(\alpha)}{-2\alpha}$$

$$= \frac{1}{2}J_n'^2(\alpha), \quad \text{since } J_n(\alpha)=0.$$

$$\Rightarrow \quad \int_0^1 x\,J_n^2(\alpha x)\,dx = \frac{1}{2}J_n'^2(\alpha). \qquad ...(7)$$

Now, on putting $x=\alpha$ in the recurrence relation $x\,J_n'(x)=n\,J_n(x)-xJ_{n+1}(x)$ we get

$\alpha\,J_n'(\alpha)=n\,J_n(\alpha)-\alpha\,J_{n+1}(\alpha)$

But $J_n(\alpha)=0$ hence we have $J_n'(\alpha)=-J_{n+1}(\alpha)$.

Therefore, result (7) can also be written as

$$\int_0^1 x\,J_n^2(\alpha x)\,dx = \frac{1}{2}J_{n+1}^2(\alpha) \qquad ...(8)$$

The above result can be generalized by replacing α by a α in (7) and (8), to get

$$\int_0^a x J_n^2(a\alpha x)\,dx = \frac{a^2}{2}[J_n'(a\alpha)]^2$$ [GGSIPU III Sem End Term 2003]

$$\int_0^a x J_n^2(a\alpha x)\,dx = \frac{a^2}{2} J_{n+1}^2(a\alpha) \quad ...(9)$$

[GGSIPU III Sem End Term 2003]

where $a\alpha$ is the root of $J_n(x) = 0$.

FOURIER – BESSEL EXPANSION

From the orthogonality property we can expand a function $f(x)$ as Bessel-Fourier series in the range 0 to a. Let $\alpha_1, \alpha_2, ...\alpha_n$ be the roots of $J_n(x) = 0$ and $f(x)$ be continuous having finite number of oscillations in the interval $(0, a)$ then we can write,

$$f(x) = C_1J_n(\alpha_1 x) + C_2J_n(\alpha_2 x) + C_3J_n(\alpha_3 x) + ... + C_nJ_n(\alpha_n x). \quad ...(10)$$

To determine the coefficients $C_k, k = 1, 2, ..., n,$ we multiply both sides of (10) by $xJ_n(\alpha_k x)$ and integrate both sides in $(0, a)$ and use orthogonality property, to get

$$\int_0^a f(x)\, xJ_n(\alpha_k x)\,dx = C_k \int_0^a xJ_n^2(\alpha_k x)\,dx = \frac{a^2}{2} C_k J_{n+1}^2(a\alpha_k) \quad \text{(using (9))}$$

which determines C_k. Thus, we have $f(x) = \sum_{k=1}^{\infty} C_k J_n(\alpha_k x)$

where $$C_k = \frac{2}{a^2 J_{n+1}^2(a\alpha_k)} \int_0^a x f(x) J_n(\alpha_k x)\,dx.$$

EXAMPLE 11.8. Prove that $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)\,d\theta$.

SOLUTION: We have had the Jacobi series

$$\cos(x \sin\theta) = J_0 + 2[J_2 \cos 2\theta + J_4 \cos 4\theta + ...]$$

Substituting here $\theta = \pi/2 - \phi$, we get

$$\cos(x\cos\phi) = J_0 + 2[-J_2\cos 2\phi + J_4 \cos 4\phi - J_6 \cos 6\phi + ...]$$

Now, integrating both sides w.r.t ϕ in $(0, \pi)$, gives

$$\int_0^{\pi} \cos(x\cos\phi)\,d\phi = \int_0^{\pi} J_0\,d\phi - 2J_2\int_0^{\pi}\cos 2\phi\,d\phi + 2J_4\int_0^{\pi}\cos 4\phi\,d\phi - ...$$
$$= \pi J_0 + 0 + 0 + ...$$

Therefore, $$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\phi)\,d\phi = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\theta)\,d\theta$$ **Hence Proved.**

EXAMPLE 11.9. Show that (*i*) $\cos x = J_0 - 2J_2 + 2J_4 - 2J_6 + ...$

and (*ii*) $\sin x = 2J_1 - 2J_3 + 2J_5 - ....$

SOLUTION: We have the Jacobi series as

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{r=1}^{\infty} J_{2r}\cos 2r\theta \quad \text{and} \quad \sin(x\sin\theta) = 2\sum_{r=0}^{\infty} J_{2r+1}\sin(2r+1)\theta$$

Putting $\theta = \pi/2$ in the above results, we get

$$\cos x = J_0 + 2\,(J_2 \cos \pi + J_4 \cos 2\pi + J_6 \cos 3\pi + \ldots) = J_0 - 2J_2 + 2J_4 - \ldots$$

and $$\sin x = 2\left[J_1 \sin\frac{\pi}{2} - J_3 \sin\frac{3\pi}{2} + J_5 \sin\frac{5\pi}{2} - J_7 \sin\frac{7\pi}{2} + \ldots\right] = 2\,[J_1 - J_3 + J_5 - \ldots]$$ **Hence Proved.**

11.10. (a) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{2\,J_0(\lambda_n x)}{\lambda_n J_1(\lambda_n)} = 1.$

(b) Expand $f(x) = x^2$ in $(0, 2)$ in terms of the Bessel functions $J_2(\lambda_n x)$ where λ_n are given by $J_2(2\lambda_n) = 0$.

SOLUTION: (a) Here we take $f(x) = 1$, $n = 0$, and interval as $(0, 1)$. Let the Fourier-Bessel series for $f(x) = 1$ be

$$1 = \sum_{n=1}^{\infty} a_n J_0(\lambda_n x) \qquad \ldots(1)$$

Multiplying throughout by $xJ_0(\lambda_n x)$ and integrating both sides w.r.t. x in $(0, 1)$ and using orthogonality, we get

$$\int_0^1 xJ_0(\lambda_n x)\,dx = a_n \int_0^1 xJ_0^2(\lambda_n x)\,dx = \frac{1}{2} a_n J_1^2(\lambda_n).$$

$$\therefore \quad a_n = \frac{2}{J_1^2(\lambda_n)} \int_0^1 xJ_0(\lambda_n x)\,dx = \frac{2}{J_1^2(\lambda_n)}\left[\frac{xJ_1(\lambda_n x)}{\lambda_n}\right]_0^1. \quad \text{(on using the recurrence relation I)}$$

$$= \frac{2}{J_1^2(\lambda_n)} \frac{J_1(\lambda_n)}{\lambda_n} = \frac{2}{\lambda_n J_1(\lambda_n)}$$

Substituting for a_n in (1), we get

$$1 = \sum_{n=1}^{\infty} a_n J_0(\lambda_n x) = \sum_{n=1}^{\infty} \frac{2\,J_0(\lambda_n x)}{\lambda_n J_1(\lambda_n)}.$$ **Hence Proved.**

(b) Let the Fourier-Bessel expansion of $f(x) = x^2$, be

$$x^2 = \sum_{n=1}^{\infty} a_n J_2(\lambda_n x). \qquad \ldots(1)$$

Multiplying both sides by $xJ_2(\lambda_n x)$ and integrating both sides w.r.t. x in $(0, 2)$ and using orthogonality property on the R.H.S., we get

$$\int_0^2 x^3 J_2(\lambda_n x)\,dx = a_n \int_0^2 x\,J_2^2(\lambda_n x)\,dx.$$

Using recurrence relation I on the L.H.S. and orthogonality property on the R.H.S., we get

$$\left[\frac{x^3 J_3(\lambda_n x)}{\lambda_n}\right]_0^2 = a_n \frac{(2)^2}{2} J_3^2(2\lambda_n) \quad \text{or} \quad \frac{8J_3(2\lambda_n)}{\lambda_n} = 2a_n J_3^2(2\lambda_n)$$

or $$a_n = \frac{4}{\lambda_n J_3(2\lambda_n)} \quad \therefore \quad x^2 = \sum_{n=1}^{\infty} \frac{4\,J_2(\lambda_n x)}{\lambda_n J_3(2\lambda_n)}.$$ **Ans.**

TRANSFORMATION OF BESSEL'S EQUATION

Many second order differential equations can be reduced to Bessel's form by suitable substitutions and then solved in terms of Bessel functions. In order to recognise such differential equations we give below quite a general transformation of Bessels equation.

We transform the Bessel's equation $t^2 \frac{d^2u}{dt^2} + t\frac{du}{dt} + (t^2 - n^2)\, u = 0$ ...(1)

in three phases. The solution of (1) is $u = J_n(t)$. In **first phase** we put $t = \lambda z$, then (1) becomes

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} + (\lambda^2 z^2 - n^2)\, u = 0 \qquad ...(2)$$

whose solution is $u = J_n(\lambda z)$. ...(3)

Equation (2) can be written as

$$z\frac{d}{dz}\left(z\frac{du}{dz}\right) + (\lambda^2 z^2 - n^2)\, u = 0 \qquad ...(4)$$

In the **second phase** we put $z = x^{\beta}$ or $\log z = \beta \log x$, so that $\frac{dz}{z} = \beta\frac{dx}{x}$, then (4) becomes

$$\frac{x}{\beta}\frac{d}{dx}\left(\frac{x}{\beta}\frac{du}{dx}\right) + (\lambda^2 x^{2\beta} - n^2)\, u = 0$$

or $$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \beta^2(\lambda^2 x^{2\beta} - n^2)u = 0 \qquad ...(5)$$

The solution of (5) is $u = J_n(\lambda x^{\beta})$.

In the third and **final phase** we put $u = x^{-\alpha} y$

so that $\frac{du}{dx} = x^{-\alpha}\frac{dy}{dx} - \alpha x^{-\alpha-1} y$ or $x\frac{du}{dx} = x^{1-\alpha}\frac{dy}{dx} - \alpha x^{-\alpha} y$

$$\Rightarrow \quad x\frac{d^2u}{dx^2} + 1\frac{du}{dx} = x^{1-\alpha}\frac{d^2y}{dx^2} + (1-\alpha)\, x^{-\alpha}\frac{dy}{dx} - \alpha x^{-\alpha}\frac{dy}{dx} + \alpha^2 x^{-\alpha-1} y$$

or $$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} = x^{2-\alpha}\frac{d^2y}{dx^2} + (1-2\alpha)\, x^{1-\alpha}\frac{dy}{dx} + \alpha^2 x^{-\alpha} y$$

Hence (5) becomes $$x^{2-\alpha}\frac{d^2y}{dx^2} + (1-2\alpha)\, x^{1-\alpha}\frac{dy}{dx} + \alpha^2 x^{-\alpha} y + \beta^2(\lambda^2 x^{2\beta} - n^2)\, x^{-\alpha} y = 0$$

or $$x^2\frac{d^2y}{dx^2} + (1-2\alpha)\, x\frac{dy}{dx} + (\alpha^2 + \beta^2\lambda^2 x^{2\beta} - n^2\beta^2)\, y = 0$$

or $$x^2\frac{d^2y}{dx^2} + (1-2\alpha)\, x\frac{dy}{dx} + \{\lambda^2\beta^2 x^{2\beta} + (\alpha^2 - n^2\beta^2)\}\, y = 0 \qquad ...(6)$$

Thus, the solution of (6) is of the form $y = x^{\alpha} u = x^{\alpha} J_n(\lambda x^{\beta})$ ...(7)

Equation (6) is sufficiently general and can cover most of the cases met in practice. Actually we compare any given differential equation with (6) and find the values of α, λ and β. Then the solution is obtained as in (7).

EXAMPLE 11.11. (a) Solve the differential equation $\frac{d^2y}{dx^2} + xy = 0$.

(b) Solve the differential equation $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + 4\left(x^2 - \frac{1}{x^2}\right)y = 0$.

SOLUTION: (a) Let us rewrite the given equation as $x^2\frac{d^2y}{dx^2} + x^3 y = 0$

Now compare it with (6) above, and find $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $\lambda = \frac{2}{3}$ and $n = \frac{\alpha^2}{\beta^2} = \frac{1}{9}$.

Therefore the solution of the given equation in terms of Bessel functions is

$$y = x^{1/2} J_{1/9}\left(\frac{2}{3}x^{3/2}\right).$$

Ans.

(b) The given equation can be written as $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 4(x^4 - 1)y = 0$

which is of the form given in (6) above. Comparing the terms, gives

$$(1 - 2\alpha) = -2, \quad \beta = 2, \quad \lambda = 1, \quad \alpha^2 - n^2\beta^2 = -4.$$

$$\therefore \quad \alpha = \frac{3}{2} \text{ and } 4n^2 = \frac{9}{4} + 4 \text{ hence } n = \pm\frac{5}{4}.$$

Therefore the general solution of the given equation, is

$$y = x^{3/2}\,[AJ_{5/4}(x^2) + BJ_{-5/4}(x^2)].$$

where A and B are arbitrary constants. **Ans.**

MODIFIED BESSEL'S FUNCTION

The second order differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + n^2)\,y = 0 \qquad \text{...(1)}$$

is called *modified Bessel's* equation of order n.

Actually, this equation (1) can be easily obtained by replacing x by ix in the Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\,y = 0.$$

$\therefore$ The solution of (1) is

$$J_n(ix) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{ix}{2}\right)^{n+2r} = i^n \sum_{r=0}^{\infty} \frac{(-1)^r (i^2)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} = i^n \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

The series $\sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$ is a positive term series and is denoted by $I_n(x)$. It is a real function and is known as ***modified Bessel function of first kind of order n.*** Therefore, we have

$$I_n(x) = e^{-in\pi/2} J_n(ix) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}.$$

The other independent solution of the differential equation (1) is called the ***modified Bessel functions of the second kind of order n denoted by $K_n(x)$.*** It is given by

$$K_n(x) = \frac{\pi/2}{\sin n\pi}\,[I_{-n}(x) - I_n(x)], \; n \text{ being non-integer.}$$

Hence the complete or general solution of (1) is written in the form. $y = AI_n(x) + BK_n(x)$, where A and B are arbitrary constants. Unlike Bessel functions the function $I_n(x)$ and $K_n(x)$ are non-oscillatory. $I_n(x)$ behaves in a fashion similar to exponential functions.

Here $I_0(x)$ is approximately 1 when x is small, but for large values of x, $I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$.

BER AND BEI FUNCTIONS

We now take up another differential equation which occurs in problems dealing with eddy currents in electrical engineering. This is obtained on replacing x by $i^{3/2}x$ in the Bessel's equation with

$n = 0$, as $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - ixy = 0.$ ...(1)

Multiplying (1) throughout by x and comparing it with the transformed Bessel's equation

$$x^2\frac{d^2y}{dx^2} + (1-2\alpha)\,x\frac{dy}{dx} + \{\lambda^2\beta^2x^{2\beta} + (\alpha^2 - n^2\beta^2)\}\,y = 0 \quad \text{...(2)}$$

we find $\alpha = 0,\ n = 0,\ \beta = 1$ and $\lambda = i^{3/2}$. Therefore, as discussed earlier, the solution of (1) is

$$y = x^{\alpha} J_n(\lambda x^{\beta}) = J_0(i^{3/2}x)$$

$$= \left[\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\lfloor n+r+1}\left(\frac{i^{3/2}x}{2}\right)^{n+2r}\right]_{n=0} = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\cdot\left(\frac{-ix^2}{4}\right)^r = \sum_{r=0}^{\infty} \frac{1}{(r!)^2}\,\frac{i^r x^{2r}}{4^r}$$

$$= 1 + \frac{ix^2}{1!^2\,4^1} - \frac{x^4}{(2!)^2\,4^2} - \frac{ix^6}{(3!)^2\,4^3} + \frac{x^8}{(4!)^2\,4^4} + \frac{ix^{10}}{(5!)^2\,4^5} - \frac{x^{12}}{(6!)^2\,4^6} - \ldots$$

or

$$y = \left[1 - \frac{x^4}{2^2\cdot4^2} + \frac{x^8}{2^2\cdot4^2\cdot6^2\cdot8^2} - \frac{x^{12}}{2^2\cdot4^2\cdot6^2\cdot8^2\cdot10^2\cdot12^2} + \ldots\right]$$

$$+ i\left[\frac{x^2}{2^2} - \frac{x^6}{2^2\cdot4^2\cdot6^2} + \frac{x^{10}}{2^2\cdot4^2\cdot6^2\cdot8^2\cdot10^2} - \ldots\right] \quad \text{...(3)}$$

which is complex for real x. The series in the first bracket of (3) is the real part of the solution and is called *Bessel-real* denoted by 'Ber' and the series in the second bracket of (3) is called *Bessel-imaginary* denoted by 'Bei'. Thus, we have

$$\text{Ber}(x) = 1 - \frac{x^4}{2^2\cdot4^2} + \frac{x^8}{2^2\cdot4^2\cdot6^2\cdot8^2} - \frac{x^{12}}{2^2\cdot4^2\cdot6^2\cdot8^2\cdot10^2\cdot12^2} + \ldots$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)^r\,x^{4r}}{2^2\cdot4^2\cdot6^2\cdot\ldots(4r)^2}$$

and

$$\text{Bei}(x) = \frac{x^2}{2^2} - \frac{x^6}{2^2\cdot4^2\cdot6^2} + \frac{x^{10}}{2^2\cdot4^2\cdot6^2\cdot8^2\cdot10^2} - \ldots$$

$$= -\sum_{r=1}^{\infty} \frac{(-1)^r\,x^{4r-2}}{2^2\cdot4^2\cdot6^2\cdot\ldots(4r-2)^2}$$

Therefore, we have $\mathbf{J_0(i^{3/2}x) = Ber\,(x) + i\,Bei\,(x)}$.

EXAMPLE 11.12. Show that

(i) $\dfrac{d}{dx}\{x\,\text{Ber}'(x)\} = -x\,\text{Bei}(x)$ *[GGSIPU III Sem End Term 2008; II Term 2011]*

(ii) $\dfrac{d}{dx}\{x\,\text{Bei}'(x)\} = x\,\text{Ber}(x)$.

[GGSIPU III Sem End Term 2010; II Term 2012; End Term 2012]

SOLUTION: We know that $\text{Ber}(x) = 1 + \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^r\,x^{4r}}{2^2\cdot4^2\cdot6^2\cdot\ldots(4r)^2}$

and $$\text{Bei}(x) = -\sum_{r=1}^{\infty} \frac{(-1)^r x^{4r-2}}{2^2 \cdot 4^2 \cdot 6^2 \cdot \ldots (4r-2)^2}.$$

$$\therefore \quad x\,\text{Ber}'(x) = \sum_{r=1}^{\infty} \frac{(-1)^r \cdot 4r\, x^{4r}}{2^2 \cdot 4^2 \cdot 6^2 \cdot \ldots (4r)^2}$$

and $$\frac{d}{dx}[x\,\text{Ber}'(x)] = \sum_{r=1}^{\infty} \frac{(-1)^r (4r)^2 x^{4r-1}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4r)^2} = \sum_{r=1}^{\infty} \frac{(-1)^r x^{4r-1}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4r-2)^2} = -x\,\text{Bei}(x)$$

which proves the result (*i*). **Hence Proved.**

Next, $$x\,\text{Bei}'(x) = -\sum_{r=1}^{\infty} \frac{(-1)^r (4r-2)\, x^{4r-2}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4r-2)^2}$$

hence $$\frac{d}{dx}[x\,\text{Bei}'(x)] = -\sum_{r=1}^{\infty} \frac{(-1)^r (4r-2)^2 x^{4r-3}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4r-2)^2} = \sum_{r=1}^{\infty} \frac{(-1)^{r+1} x^{4r-3}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4r-4)^2}$$

$$= x - \frac{x^5}{2^2 \cdot 4^2} + \frac{x^9}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \ldots = x\,\text{ber}(x)$$

which proves the result (*ii*). **Hence Proved.**

EXAMPLE 17.13. Prove that $\int_0^a x[\text{Ber}^2(x) + \text{Bei}^2(x)]\,dx = a[\text{Ber}(a)\,\text{Bei}'(a) - \text{Bei}(a)\,\text{Ber}'(a)]$

[GGSIPU III Sem II Term 2007; End Term 2006; End Term 2010]

SOLUTION: In the last example, we have shown that

$$\frac{d}{dx}[x\,\text{Ber}'(x)] = -x\,\text{Bei}(x) \qquad \ldots(1)$$

and $$\frac{d}{dx}[x\,\text{Bei}'(x)] = x\,\text{Ber}(x) \qquad \ldots(2)$$

Multiplying (2) by $\text{Ber}(x)$ and (1) by $\text{Bei}(x)$ and subtracting, we get

$$x[\text{Ber}^2(x) + \text{Bei}^2(x)] = \text{Ber}(x)\frac{d}{dx}[x\,\text{Bei}'(x)] - \text{Bei}(x)\frac{d}{dx}[x\,\text{Ber}'(x)]$$

$$\therefore \quad \int_0^a x[\text{Ber}^2(x) + \text{Bei}^2(x)]\,dx = \int_0^a [\text{Ber}(x)\frac{d}{dx}[x\text{Bei}'(x)]dx - \int_0^a [\text{Bei}(x)\frac{d}{dx}[x\text{Ber}'(x)]dx$$

$$= [\text{Ber}(x)\, x\,\text{Bei}'(x)]_0^a - \int_0^a \text{Ber}'(x)\, x\,\text{Bei}'(x)\,dx$$

$$- [\text{Bei}(x)\, x\,\text{Ber}'(x)]_0^a + \int_0^a \text{Bei}'(x)\, x\,\text{Ber}'(x)\,dx$$

$$= \text{Ber}(a)\, a\,\text{Bei}'(a) - \text{Bei}(a)\, a\,\text{Ber}'(a)$$

$$= a[\text{Ber}(a)\,\text{Bei}'(a) - \text{Bei}(a)\,\text{Ber}'(a)].$$ **Hence Proved.**

EXERCISE 11B

1. Show that $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)\, d\phi$.
2. Find the value of $J_0 + 2J_2 + 2J_4 + 2J_6 + \ldots$
3. Show that $J_0(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta)\, d\theta = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\cos\theta)\, d\theta$.
4. Use the generating function to show that $J_{-n}(x) = (-1)^n J_n(x)$.
5. Solve the equation $x^2y'' + xy' + (k^2x^2 - n^2)\, y = 0$ in terms of Bessel functions.
6. Find the solution of the equation $xy'' - y' + 4x^2y = 0$ in the Bessel form.
7. Solve the equation $x\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + xy = 0$.
8. Solve the differential equation $4\frac{d^2y}{dx^2} + 9xy = 0$.
9. Expand $f(x) = x^3$ in the interval $(0, 3)$ in terms of Bessels functions $J_1(\lambda_n x)$ where λ_n are the roots of $J_1(3\lambda) = 0$.
10. Expand $f(x) = x^2$ in $(0, a)$ in Fourier Bessel series in $J_2(\lambda_n x)$ where $(a\lambda_n)$ are positive zeros of $J_2(x) = 0$.
11. Using the integral form of $J_0(x)$ and $J_0'(x) = -J_1(x)$, show that

$$J_1(x) = \frac{2}{\pi}\int_0^{\pi/2} \sin(x\sin\theta)\sin\theta\, d\theta.$$

12. Show that $\int_0^x J_0(x)\, dx = \frac{2}{\pi}\int_0^{\pi/2} \frac{\sin(x\sin\theta)}{\sin\theta}\, d\theta$.
13. Using series representation for $J_0(x)$ verify the following

(i) $\int_0^{\pi/2} J_0(x\cos\theta)\cos\theta\, d\theta = \frac{\sin x}{x}$.

(ii) $\int_0^{\pi/2} J_1(x\cos\theta)\, d\theta = \frac{1-\cos x}{x}$.

LEGENDRE'S EQUATION

We now consider the first and foremost equation of Physics, the Legendre's differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)\,y=0 \qquad ...(1)$$

which is encountered most in problems exhibiting spherical symmetry (*e.g.* in electrostatics). The parameter n in (1) is a given non-negative real number. The solutions of (1) are called *Legendre functions*.

Since $x=0$ is an ordinary point of equation (1), we can assume its solution in power series, as

$$y=\sum_{r=0}^{\infty}a_rx^r \qquad ...(2)$$

Substituting (2) and its derivatives in (1), we obtain

$$(1-x^2)\sum_{r=2}^{\infty}r(r-1)\,a_rx^{r-2}-2x\sum_{r=1}^{\infty}ra_rx^{r-1}+n(n+1)\sum_{0}^{\infty}a_rx^r=0$$

or

$$\sum_{r=2}^{\infty}r(r-1)\,a_rx^{r-2}-\sum_{r=2}^{\infty}r(r-1)\,a_rx^r-2\sum_{r=1}^{\infty}ra_rx^r+n(n+1)\sum_{0}^{\infty}a_rx^r=0$$

This must be an identity in x, if (2) is a solution of (1), hence the coefficients of terms of each powers of x should be zero. Equating to zero the coefficients of x^0 and x^1, we get

$$2\cdot 1a_2+n\,(n+1)\,a_0=0$$

and

$$3\cdot 2a_3+\{-2+n\,(n+1)\}\,a_1=0$$

Then equating to zero the coefficient of x^r, we get the coefficient recurrence, as

$$(r+2)\,(r+1)\,a_{r+2}+\{n\,(n+1)-r\,(r-1)-2r\}\,a_r=0$$

or

$$a_{r+2}=-\frac{(n-r)\,(n+r+1)}{(r+2)\,(r+1)}a_r \qquad ...(3)$$

This is also known as recursion formula.

$$\therefore \quad a_2=\frac{-n\,(n+1)}{2!}a_0,\quad a_4=\frac{-(n-2)\,(n+3)}{4\cdot 3}a_2=\frac{(n-2)\,(n)\,(n+1)\,(n+3)}{4!}a_0,$$

$$a_3=\frac{-(n-1)(n+2)}{3!}a_1,\quad a_5=\frac{-(n-3)(n+4)}{5\cdot 4}a_3=\frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1,\text{ etc.}$$

By inserting these values of the coefficients in (2), we obtain

$$y\,(x)=a_0y_1(x)+a_1y_2(x) \qquad ...(4)$$

where

$$y_1(x)=\left(1-\frac{n\,(n+1)}{2!}x^2+\frac{(n-2)\,n\,(n+1)\,(n+3)}{4!}x^4-...\right) \qquad ...(5)$$

and

$$y_2(x)=\left(x-\frac{(n-1)\,(n+2)}{3!}x^3+\frac{(n-3)\,(n-1)\,(n+2)\,(n+4)}{5!}x^5-...\right) \qquad ...(6)$$

These series in (5) and (6) converge for $|x|<1$. Since (5) contains only even powers of x and (6) contains only odd powers of x, $\frac{y_1}{y_2}$ is not constant, so y_1 and y_2 are not proportional and are therefore linearly independent solutions. Hence (4) is a general solution of (1) in $(-1, 1)$.

In most of the applications the parameter n in the Legendre's equation will be a non-negative integer. It can be noticed here that when n is an integer, one or the other of the two series (5) and (6) terminates. **If n is a positive even integer (say $2m$) the series in (5) terminates at the term x^{2m} and hence y_1 becomes a polynomial. Similarly, if n is an odd integer, the series in (6) terminates and y_2 becomes a polynomial. These polynomials, with the coefficients a_0 or a_1 so adjusted that the value of the polynomial becomes 1 at $x = 1$ are called *Legendre polynomials* and are denoted by $P_n(x)$.** The other non-terminating solution (with a_0 or a_1 suitably adjusted) is called *Legendre function of the second kind* and is denoted by $Q_n(x)$.

Now, for convenience, let us express the polynomial $P_n(x)$ **in descending powers of x starting with the term x^n.** For this purpose we rewrite the recurrence formula (3) as

$$a_r = \frac{-(r+2)(r+1)}{(n-r)(n+r+1)} a_{r+2}, \qquad (r \le n-2) \qquad \text{...(7)}$$

This way all the non-vanishing coefficients can be expressed in terms of the coefficient of the highest power of x in the polynomial.

$$\text{Let us chose } a_n \begin{cases} = 1, & \text{when } n = 0 \\ = \dfrac{(2n)!}{2^n (n!)^2} = \dfrac{1\cdot 3\cdot 5\cdot \ldots (2n-1)}{n!}, & \text{when } n = 1, 2, 3, \ldots \end{cases} \qquad \text{...(8)}$$

The reason for this choice of a_n is that, then all the polynomials will have value 1 at $x = 1$. This we shall establish while deriving the Rodrigue's formula in the next article. From (7) and (8), we obtain

$$a_{n-2} = \frac{-n(n-1)}{2\cdot(2n-1)} a_n = \frac{-n(n-1)(2n)!}{2(2n-1)\, 2^n (n!)^2}$$

$$= -\frac{n(n-1)\, 2n(2n-1)(2n-2)!}{2(2n-1)\, 2^n \cdot n(n-1)!\,(n)\cdot(n-1)(n-2)!} = -\frac{(2n-2)!}{2^n (n-1)!\,(n-2)!}$$

Similarly, $$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(2n-4)!}{2!\, 2^n (n-2)!\,(n-4)!}$$

$$a_{n-6} = \frac{-(2n-6)!}{3!\, 2^n (n-3)!\,(n-6)!}, \text{ etc.}$$

and, in general, when $n - 2r \ge 0$, i.e., $n > 2r$

$$a_{n-2r} = \frac{(-1)^r (2n-2r)!}{r!\, 2^n (n-r)!\,(n-2r)!}. \qquad \text{...(9)}$$

The resulting solution of Legendre's differential equation is called the Legendre polynomial of degree n and is denoted by $P_n(x)$.

Thus, $$\mathbf{P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!\, x^{n-2r}}{2^n r!\,(n-r)!\,(n-2r)!}}$$

$$= \frac{(2n)!\, x^n}{2^n (n!)^2} - \frac{(2n-2)!\, x^{n-2}}{2^n 1!\,(n-1)!\,(n-2)!} + \frac{(2n-4)!\, x^{n-4}}{2^n 2!\,(n-2)!\,(n-4)!} - + \ldots \qquad \text{...(10)}$$

where $N = \frac{n}{2}$ or $\frac{n-1}{2}$ according as n is even or odd

Therefore,
$$\left.\begin{array}{ll} P_0(x) = 1, & P_1(x) = x \\ P_2(x) = \frac{1}{2}(3x^2 - 1), & P_3(x) = \frac{1}{2}(5x^3 - 3x) \\ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), & P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \end{array}\right\} \quad ...(11)$$

and so on.

RODRIGUE'S FORMULA

According to this formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n]$. ...(1)

[*GGSIPU III Sem II Term End Term 2009; II Term 2007; II Term 2006; End Term 2005; End Term 2010; II Term 2011; II Term 2012; III Sem II Term 2013*]

To derive it, let $v = (x^2 - 1)^n \quad \therefore \quad \frac{dv}{dx} = n\,(x^2 - 1)^{n-1}\,(2x)$

or $$(1 - x^2)\,v_1 + 2nxv = 0 \quad \text{where} \quad v_1 = \frac{dv}{dx} \quad ...(2)$$

Differentiating (2), $(n + 1)$ times, using Lebneitz's theorem, we get

$$(1 - x^2)\,v_{n+2} + (n + 1)(-2x)\,v_{n+1} + \frac{(n+1)n}{2}(-2)v_n + 2nxv_{n+1} + 2n(n + 1)\,v_n = 0$$

or $$(1 - x^2)\,v_{n+2} - 2xv_{n+1} + n\,(n + 1)\,v_n = 0 \quad \text{where} \quad v_n = \frac{d^n v}{dx^n}.$$

This is Legendre's equation in v_n and its solution is

$$v_n = \frac{d^n}{dx^n}(x^2 - 1)^n = c\,P_n(x)$$

where c is some constant which is determined from the property $P_n(1) = 1$ for all n.

Thus, $$cP_n(x) = \frac{d^n}{dx^n}[(x-1)^n (x+1)^n] \quad \text{(now using Lebreitz's theorem)}$$
$$= (x-1)^n \frac{d^n}{dx^n}(x+1)^n + {}^nC_1\, n\,(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x+1)^n$$
$$+ {}^nC_2\, n\,(n-1)\,(x-1)^{n-2} \frac{d^{n-2}}{dx^{n-2}}(x+1)^n + ... + (x+1)^n \frac{d^n}{dx^n}(x-1)^n$$

Putting $x = 1$ on both sides, gives

$$cP_n(1) = \left[(x+1)^n \frac{d^n}{dx^n}(x-1)^n\right]_{x=1} = 2^n\, n!$$

while rest of the terms vanish because of the factor $(x - 1)$.

$\therefore \quad c = 2^n\, n! \quad$ since $P_n(1) = 1$.

Thus, we have $P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2 - 1)^n$ as required.

From this formula also, we can easily find expressions for Legendre's polynomials of different orders.

For example, $P_0(x) = 1$, $P_1(x) = \frac{1}{2!}\frac{d}{dx}(x^2-1) = x$,

$$P_2(x) = \frac{1}{2^2\,2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}[2(x^2-1)2x] = \frac{1}{2}(3x^2-1),$$

$$P_3(x) = \frac{1}{2^3\,3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^2}{dx^2}[3(x^2-1)^2 2x] = \frac{1}{2}(5x^3-3x), \text{ etc.}$$

THE GENERATING FUNCTION OF LEGENDRE POLYNOMIALS

Expanding the function $f(x, t) = (1-2xt+t^2)^{-1/2}$ in powers of t, we shall show that the coefficient of t^n is Legendre polynomial $P_n(x)$ for $n = 0, 1, 2, \ldots$

$$f(x,t) = (1-2xt+t^2)^{-1/2} = [1-t(2x-t)]^{-1/2}$$

$$= 1 + \frac{t}{2}(2x-t) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2}t^2(2x-t)^2 - \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{3!}t^3(2x-t)^3 + \ldots$$

$$\ldots + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{(2r-1)}{2}\right)\frac{(-1)^r}{r!}t^r(2x-t)^r + \ldots$$

$$= 1 + \frac{2!\,t(2x-t)}{(1!)^2\cdot 2^2} + \frac{4!}{(2!)^2\cdot 2^4}t^2(2x-t)^2 + \frac{6!}{(3!)^2\cdot 2^6}t^3(2x-t)^3 + \ldots$$

$$+ \frac{(2n-2r)!}{[(n-r)!]^2\,2^{2n-2r}}t^{n-r}(2x-t)^{n-r} + \ldots + \frac{(2n-2)!}{[(n-1)!]^2\,2^{2n-2}}t^{n-1}(2x-t)^{n-1}$$

$$+ \frac{(2n)!}{(n!)^2\,2^{2n}}t^n(2x-t)^n + \ldots \qquad \ldots(1)$$

The term of t^n from the term containing $t^{n-r}(2x-t)^{n-r}$

$$= \frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}(t^{n-r})\,{}^{n-r}C_r\,(-1)^r(2x)^{n-2r}\,t^r$$

$$= \frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}\cdot\frac{(n-r)!}{r!(n-2r)!}(-1)^r\,t^n\,(2x)^{n-2r}.$$

$$= \frac{(-1)^r(2n-2r)!}{2^n\,r!(n-r)!(n-2r)!}\cdot x^{n-2r}\,t^n$$

Collecting all terms in t which occur in the term containing $t^r(2x-t)^r$ and in the preceding terms, we see that the terms in t^n equals

$$= \sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!\,x^{n-2r}\,t^n}{2^n\,r!(n-r)!(n-2r)!} = P_n(x)\,t^n$$

where $N = \frac{n}{2}$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Hence (1) may be written as $[1-t(2x-t)]^{-1/2} = \sum_{n=0}^{\infty}P_n(x)\,t^n$.

This shows that $P_n(x)$ is the coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$. That is the reason, it is known as the generating function of $P_n(x)$.

RECURRENCE RELATIONS FOR $P_n(x)$

These relations are obtained from the generating function of $P_n(x)$.

We have $[1 - 2xt + t^2)]^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$. ...(1)

I. $(n+1) P_{n+1} = (2n+1) xP_n - nP_{n-1}$. [GGSIPU IIIrd Sem. End Term 2004; Emd Term 2010]

Differentiating (1) partially w.r.t. t, gives

$$\left(-\frac{1}{2}\right)(-2x+2t)\ (1 - 2xt + t^2)^{-3/2} = \sum_{n=0}^{\infty} nt^{n-1} P_n(x)$$

Now multiplying throuthout by $(1 - 2x + t^2)$, gives

$$(x-t)\ (1 - 2xt + t^2)^{-1/2} = (1-2xt+t^2) \sum_{n=0}^{\infty} nt^{n-1} P_n(x)$$

or $$(x-t) \sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2) \sum_{0}^{\infty} nt^{n-1} P_n(x).$$

Now equating the coefficients of t^n on both sides, we get

$$xP_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2nx P_n(x) + (n-1) P_{n-1}(x)$$

or $$(n+1) P_{n+1}(x) = (2n+1) xP_n(x) - n P_{n-1}(x).$$

II. $P_n = P'_{n+1} - 2xP'_n + P'_{n-1}$.

Differentiating (1) partially w.r.t. x, gives

$$-\frac{1}{2}(1 - 2xt + t^2)^{-3/2}(-2t) = \sum_{0}^{\infty} t^n P_n'(x)$$

or $$t(1 - 2xt + t^2)^{-1/2} = (1-2xt+t^2) \sum_{0}^{\infty} t^n P_n'(x)$$

or $$t \sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2) \sum_{0}^{\infty} t^n P_n'(x)$$

Now, equating on both sides the coefficient of t^{n+1}, we get

$$P_n(x) = P'_{n+1}(x) - 2x P'_n(x) + P'_{n-1}(x)$$

[GGSIPU III Sem End Term 2013]

III. $nP_n = xP_n' - P'_{n-1}$.

Here we differentiate the recurrence relation I partially w.r.t. x, to get

$$(n+1) P'_{n+1} = (2n+1)\, x P'_n + (2n+1) P_n - nP'_{n-1}.$$

Substituting here for P'_{n+1} from recurrence relation II, we get

$$(n+1) P_n(x) = -2(n+1) xP'_n(x) + (n+1) P'_{n-1}(x) + (2n+1) xP'_n(x) + (2n+1) P_n(x) - nP'_{n-1}(x)$$

or $$0 = nP_n(x) - xP'_n(x) + P'_{n-1}(x)$$ or $$nP_n = xP'_n - P'_{n-1}.$$

IV. $(n+1) P_n = P'_{n+1} - xP'_n$

Adding the relations II and III, gives

$$(n+1) P_n = P'_{n+1} - xP'_n.$$

V. $(2n+1)\,P_n = P'_{n+1} - P'_{n-1}$. [GGSIPU IIIrd Sem. End Term 2003]

Adding the relations III and IV, we get

$$(2n+1)\,P_n = P'_{n+1} - P'_{n-1}.$$

VI. $(1-x^2)\,P'_n = n\,[P_{n-1} - xP_n]$.

Replacing n by $(n-1)$ in recurrence relation IV, we get $nP_{n-1} = P'_n - xP'_{n-1}$ ...(i)

and multiplying III by x, gives $nxP_n = x^2P'_n - xP'_{n-1}$. ...(ii)

Subtracting (ii) from (i), gives $(1-x^2)\,P'_n = n\,[P_{n-1} - xP_n]$.

VII. $(1-x^2)\,P'_n = (n+1)\,[xP_n - P_{n+1}]$.

Replacing n by $n+1$ in recurrence relation III, we get

$$(n+1)\,P_{n+1} = xP'_{n+1} - P'_n \quad ...(iii)$$

and then multiplying IV by x, we get $(n+1)\,xP_n = xP'_{n+1} - x^2P'_n$. ...(iv)

Subtracting (iii) from (iv), gives $(1-x^2)\,P'_n = (n+1)\,[x\,P_n - P_{n+1}]$.

EXAMPLE 11.14. (a) Show that $P'_n(1) = \dfrac{n(n+1)}{2}$. [GGSIPU III Sem End Term 2013]

(b) Prove that $(1-x^2)\,P'_n(x) = \dfrac{n(n+1)}{2n+1}\,[P_{n-1}(x) - P_{n+1}(x)]$.

SOLUTION: (a) The generating function of $P_n(x)$ is

$$(1-2xt+t^2)^{-1/2} = \sum_0^{\infty} t^n\,P_n(x) \quad ...(1)$$

Differentiating (1) partially w.r.t. x, we get

$$\sum_0^{\infty} t^n\,P'_n(x) = \frac{-1}{2}(1-2xt+t^2)^{-3/2}\,(-2t) = t(1-2xt+t^2)^{-3/2}$$

Putting $x = 1$ here, we get

$$\sum_0^{\infty} t^n P'_n(1) = t\,(1-2t+t^2)^{-3/2} = t(1-t)^{-3}$$

$$= t\left[1+(-3)(-t)+\frac{(-3)(-4)}{2!}(-t)^2+\frac{(-3)(-4)(-5)(-t)^3}{3!}+\ldots\right.$$
$$\left.+\frac{(-3)(-4)(-5)\ldots(-n-2)(-t)^{n-1}}{(n-1)!}+\ldots\right]$$

Comparing the coefficient of t^n on both sides, we get

$$P'_n(1) = \frac{n(n+1)}{2}$$

Hence Proved

(b) Multiplying the recurrence relation I by n and VI by $(2n+1)$ and adding them, we get

$$n(n+1)\,P_{n+1} + (2n+1)(1-x^2)\,P'_n = -n^2P_{n-1} + n(2n+1)\,P_{n-1}$$

or $$(2n+1)(1-x^2)\,P'_n = n(n+1)\,P_{n-1} - n(n+1)\,P_{n+1}$$

$$(1-x^2)\,P'_n = \frac{n(n+1)}{2n+1}\,[P_{n-1} - P_{n+1}].$$

Hence Proved.

EXAMPLE 11.15. Show that

(*i*) $P_n(1) = 1$ and $P_n(-1) = (-1)^n$

[GGSIPU III Sem II Term 2009; IInd Term 2007; End Term 2012]

(*ii*) $P_{2m}(0) = (-1)^m \dfrac{1 \cdot 3 \cdot 5 \ldots (2m-1)}{2 \cdot 4 \cdot 6 \ldots 2m}$ and $P_{2m+1}(0) = 0.$

SOLUTION: The generating function for $P_n(x)$ is

$$\sum_0^{\infty} P_n(x)\, t^n = (1 - 2xt + t^2)^{-1/2} \qquad \ldots(1)$$

(*i*) Putting $x = 1$ in (1), we get

$$\sum_0^{\infty} P_n(1)\, t^n = (1 - 2t + t^2)^{-1/2} = (1-t)^{-1} = \sum_0^{\infty} t^n \;\Rightarrow\; P_n(1) = 1.$$

Next, putting $x = -1$ in (1), we get

$$\sum_0^{\infty} P_n(-1)\, t^n = (1 + 2t + t^2)^{-1/2} = (1+t)^{-1} = \sum_0^{\infty} (-1)^n t^n \;\Rightarrow\; P_n(-1) = (-1)^n.$$ **Hence Proved.**

(*ii*) In (1) let us put $x = 0$ to get

$$\sum_0^{\infty} P_n(0)\, t^n = (1 + t^2)^{-1/2} = 1 - \frac{1}{2}t^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}t^4 + \ldots + \frac{(-1)^r\, 1 \cdot 3 \cdot 5 \cdot \ldots (2r+1)}{2 \cdot 4 \cdot 6 \cdot \ldots 2r} t^{2r} + \ldots$$

Since on the R.H.S there are only even power terms in t, on comparing the coefficients of t^n on both sides, we have

$$P_{2m+1}(0) = 0 \qquad \text{when } n \text{ is odd } (= 2m+1)$$

$$P_{2m}(0) = \frac{(-1)^m\, 1 \cdot 3 \cdot 5 \ldots (2m-1)}{2 \cdot 4 \cdot 6 \ldots (2m)} \qquad \text{when } n \text{ is even } (= 2m.)$$ **Hence Proved.**

EXAMPLE 11.16. (a) Prove that $P_n(-x) = (-1)^n P_n(x)$. [GGSIPU III Sem II Term 2010; II Term 2012]

(b) Express $f(x) = x^3 - 5x^2 + x + 2$ in terms of Legendre polynomials.

[GGSIPU III Sem II Term 2011]

SOLUTION: (a) Replacing x by $-x$ in the generating function for $P_n(x)$

$$\sum P_n(x) t^n = (1 - 2xt + t^2)^{-1/2}, \quad \text{we get}$$

$$\sum_0^{\infty} P_n(-x)\, t^n = (1 + 2xt + t^2)^{-1/2}.$$

Again replacing t by $-t$ in (1), we have

$$\sum_0^{\infty} P_n(x)\,(-t)^n = (1 + 2xt + t^2)^{-1/2}$$

$$\Rightarrow \quad \sum_0^{\infty} P_n(x)\,(-t)^n = \sum_0^{\infty} P_n(-x)\, t^n$$ **Hence Proved.**

Therefore, we have $P_n(-x) = (-1)^n P_n(x)$.

(b) We know that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$\Rightarrow \quad x = P_1$, $x^2 = \frac{1}{3}(2P_2 + P_0)$ and $x^3 = \frac{1}{5}(2P_3 + 3P_1)$.

Therefore $\quad f(x) = \left(\frac{2P_3 + 3P_1}{5}\right) - 5\left(\frac{2P_2 + P_0}{3}\right) + P_1 + 2P_0 = \frac{2}{5}P_3 - \frac{10}{3}P_2 + \frac{8}{5}P_1 + \frac{1}{3}P_0.$ **Ans.**

EXAMPLE 11.17. Express the following polynomials in terms of Legendre polynomials

(i) $1 - 2x + x^2 + 5x^3$ [GGSIPU III Sem End Term 2008]

(ii) $4x^3 + 6x^2 + 7x + 2$ [GGSIPU III Sem II Term 2010]

(iii) $3x^3 - 2x^2 + 1$ [GGSIPU III Sem End Term 2011]

(iv) $x^3 + 2x^2 - x - 3$ [GGSIPU I Sem End Term 2013]

SOLUTION: We know that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$ and $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

$\therefore \quad x^2 = \frac{1}{3}[1 + 2P_2(x)] = \frac{1}{3}[P_0(x) + 2P_2(x)]$ and $x^3 = \frac{1}{5}(2P_3 + 3P_1)$

(i) The given expression $5x^3 + x^2 - 2x + 1$ can be written as

$$3P_1(x) + 2P_3(x) + \frac{1}{3}[P_0(x) + 2P_2(x)] - 2P_1(x) + P_0(x)$$

$$= 2P_3(x) + \frac{2}{3}P_2(x) + P_1(x) + \frac{4}{3}P_0(x).$$ **Ans.**

(ii) $\quad f(x) = 4x^3 + 6x^2 + 7x + 2 = \frac{8}{5}P_3 + 4P_2 + \frac{47}{5}P_1 + 4P_0.$ **Ans.**

(iii) $\quad 3x^3 + 2x^2 + 1 = \frac{3}{5}(2P_3 + 3P_1) + \frac{2}{3}(2P_2 + P_0) + P_0$

$$= \frac{6}{5}P_3 + \frac{4}{3}P_2 + \frac{9}{5}P_1 + \frac{5}{3}P_0.$$ **Ans.**

(iv) $x^3 - 5x^2 + x + 2 = \frac{1}{5}(2P_3 + 3P_1) - \frac{5}{3}(2P_2 + P_0) + P_1 + 2P_0 = \frac{2}{5}P_3 - \frac{10}{3}P_2 - \frac{8}{5}P_1 + \frac{1}{3}P_0.$

EXAMPLE 11.18. Prove the Christofell's expansion

$$P'_n(x) = (2n - 1)P_{n-1} + (2n - 5)P_{n-3} + (2n - 9)P_{n-5} + (2n - 13)P_{n-7} + \ldots$$

where the last term is $3P_1$ or P_0 according as n is even or odd.

SOLUTION: Replacing n by $(n - 1)$ in the recurrence relation V, we get

$$(2n - 1)P_{n-1}(x) = P'_n(x) - P'_{n-2}(x) \quad \ldots(i)$$

Replacing n by $n - 2, n - 4, n - 6, \ldots$ in the above result consecutively, we have

$$(2n - 5)P_{n-3}(x) = P'_{n-2}(x) - P'_{n-4}(x) \quad \ldots(ii)$$

$$(2n - 9)P_{n-5}(x) = P'_{n-4}(x) - P'_{n-6}(x) \quad \ldots(iii)$$

$$(2n - 13)P_{n-7}(x) = P'_{n-6}(x) - P'_{n-8}(x) \quad \ldots(iv)$$

....

The last relation in the sequence will be

$$3P_1 = P'_2 - P'_0 \quad \text{if } n \text{ is even.}$$

or

$$5P_2 = P'_3 - P'_1 \quad \text{if } n \text{ is odd.}$$

Adding all the above relations (*i*), (*ii*), (*iii*), ..., wet get

$$(2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + (2n-9)\,P_{n-5} + \ldots + 3P_1 = P'_n(x) - P'_0 \quad \text{when } n \text{ is even}$$

and

$$(2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + (2n-9)\,P_{n-5} + \ldots + 5P_2 = P'_n(x) - P'_1 \quad \text{when } n \text{ is odd.}$$

But $P'_0 = 0$ as $P_0(x) = 1$ and $P'_1(x) = 1 = P_0$

$$\therefore \quad P'_n(x) = (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + (2n-9)\,P_{n-5} + \ldots$$

the last term on the RHS being $3P_1$ when n is even and P_0 when n is odd.

11.19. Show that $\dfrac{1+t}{t(1-2xt+t^2)^{1/2}} - \dfrac{1}{t} = \sum\limits_{n=0}^{\infty} (P_n + P_{n+1})\, t^n$.

SOLUTION: L.H.S. $= \left(1+\dfrac{1}{t}\right)(1-2xt+t^2)^{-1/2} - \dfrac{1}{t} = \left(1+\dfrac{1}{t}\right)\sum\limits_0^{\infty} t^n P_n(x) - \dfrac{1}{t}$

$$= \sum_0^{\infty} t^n P_n(x) + \sum_0^{\infty} t^{n-1} P_n(x) - \frac{1}{t}$$

$$= \sum_0^{\infty} t^n P_n(x) + \frac{1}{t} P_0(t) + \sum_0^{\infty} t^n P_{n+1}(x) - \frac{1}{t} P_0(t)$$

$$= \sum_0^{\infty} t^n \left[P_n(x) + P_{n+1}(x)\right] = \text{RHS.}$$

Hence Proved.

11.20. Using Rodrigue's formula, prove the recurrence relation

$$nP_n = xP'_n - P'_{n-1}.$$

SOLUTION: We know that

$$P_n(x) = \frac{1}{2^n n!} D^n (x^2-1)^n \quad \text{where } D \equiv \frac{d}{dx}. \qquad \text{...(1)}$$

$$\therefore \quad xP'_n(x) = \frac{x}{2^n n!} D^{n+1} (x^2-1)^n$$

Now by Lebneitz's theorem

$$D^{n+1}[x(x^2-1)^n] = xD^{n+1}(x^2-1)^n + (n+1)\,D^n(x^2-1)^n$$

or

$$xD^{n+1}(x^2-1)^n = D^{n+1}[x(x^2-1)^n] - (n+1)\,D^n(x^2-1)^n$$

$\therefore$ (1) becomes

$$xP'_n(x) = \frac{1}{2^n n!}\left[D^{n+1}\{x(x^2-1)^n\} - (n+1)\,D^n(x^2-1)^n\right]$$

$$= \frac{1}{2^n n!} D^n \left[D\{x(x^2-1)^n\} - (n+1)(x^2-1)^n\right]$$

$$= \frac{1}{2^n n!} D^n \left[xn(x^2-1)^{n-1}(2x) + (x^2-1)^n - (n+1)(x^2-1)^n\right]$$

$$= \frac{1}{2^n n!} D^n \left[2nx^2(x^2-1)^{n-1} - n(x^2-1)^n\right]$$

$$= \frac{1}{2^n n!} D^n [2n (x^2 - 1 + 1)(x^2 - 1)^{n-1} - n (x^2 - 1)^n]$$

$$= \frac{1}{2^n n!} D^n [2n (x^2 - 1)^n + 2n (x^2 - 1)^{n-1} - n (x^2 - 1)^n]$$

$$= \frac{1}{2^n n!} D^n [n (x^2 - 1)^n + 2n (x^2 - 1)^{n-1}]$$

$$= nP_n + \frac{1}{2^n n!} 2n D^n (x^2 - 1)^{n-1} \qquad \text{(using Rodrigues formula)}$$

$$= nP_n + \frac{1}{2^{n-1}(n-1)!} D \; D^{n-1} (x^2 - 1)^{n-1}$$

or $\quad x P_n'(x) = nP_n + P'_{n-1}$ (on using Rodrigue's formula) **Hence Proved.**

EXAMPLE 11.21. (*i*) Using Rodrigues formula, prove the recurrence relation

$$P_n'(x) = xP'_{n-1}(x) + nP_{n-1}(x).$$

[*GGSIPU III Sem End Term 2007*]

(*ii*) Prove that $\quad P_n'(x) - P'_{n-2}(x) = (2n-1) \,.\, P_{n-1}(x)$

[*GGSIPU III Sem End Term 2003*]

SOLUTION: (*i*) By Rodrigues formula we have $\quad P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n}(x^2 - 1)^n$

$$\therefore \quad P_n'(x) = \frac{1}{2^n \cdot n!} \frac{d^{n+1}}{dx^{n+1}}(x^2 - 1)^n = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n}[n(x^2 - 1)^{n-1} \cdot 2x]$$

$$= \frac{1}{2^n n!}\left[2nx \frac{d^n}{dx^n}(x^2 - 1)^{n-1} + {}^nc_1 \frac{d}{dx}(2nx) \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2 - 1)^{n-1}\right]$$

$$= \frac{2nx}{2^n \cdot n!} \frac{d}{dx} \frac{d^{n-1}}{dx^{n-1}}(x^2 - 1)^{n-1} + \frac{n \cdot 2n}{2^n \cdot n!} \frac{d^{n-1}}{dx^{n-1}}(x^2 - 1)^{n-1}$$

$$= x \cdot \frac{d}{dx}\left[\frac{1}{2^{n-1} \cdot (n-1)!} \frac{d^{n-1}}{dx^{n-1}}(x^2 - 1)^{n-1}\right] + n \cdot \frac{1}{2^{n-1}(n-1)!} \frac{d^{n-1}}{dx^{n-1}}(x^2 - 1)^{n-1}$$

or $\quad P_n'(x) = nP'_{n-1}(x) + nP_{n-1}(x).$ **Hence the result.**

(*ii*) We have the recurrence relation $\quad (n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$

or $\quad nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$

$\Rightarrow \quad nP'_n = (2n-1)xP'_{n-1} + (2n-1)P_{n-1} - (n-1)P'_{n-2}$ ...(1)

Also we have another recurrence relation as

$$P_n = P'_{n+1} - 2xP'_n + P'_{n-1} \quad \text{or} \quad P_{n-1} = P'_n - 2x P'_{n-1} + P'_{n-2} \qquad ...(2)$$

Eliminating xP'_{n-1} in (1) and (2), we get

$$2nP'_n + (2n-1)P_{n-1} = 2(2n-1)P_{n-1} - 2(n-1)P'_{n-2} + (2n-1)P_n' - 2x(2n-1)P'_{n-1} + (2n-1)P'_{n-2}$$

which on simplification, gives $\quad P'_n - P'_{n-2} = (2n-1)P_{n-1}$ **Hence the result.**

EXAMPLE 11.22 (i) Prove that $\int_{-1}^{1} P_n(x)\,dx = 0,\ n \neq 0.$ [GGSIPU III Sem II Term 2007]

(ii) Show that $\int_{0}^{1} P_n(x)\,dx = \frac{1}{n+1} P_{n-1}(0)$

SOLUTION: (i) The Legendre's equation can be written as

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0$$

Integrating both sides in (–1, 1), we get

$$\therefore \quad \int_{-1}^{1} n(n+1)P_n(x)\,dx = -\left[(1-x^2)P_n'(x)\right]_{-1}^{1}$$

$$\Rightarrow \quad \int_{-1}^{1} P_n(x)\,dx = \frac{-1}{n(n+1)}\left[(1-x^2)P_n'(x)\right]_{-1}^{1}$$

$$= 0 \quad \text{for} \quad n \neq 0.$$

Hence the results.

(ii) Integrating on both sides of the above Legendre's equation, in (0, 1), we get

$$n(n+1)\int_{0}^{1} P_n(x)\,dx = -\left[(1-x^2)P_n'(x)\right]_{0}^{1} = P_n'(0) \quad \text{or} \quad \int_{0}^{1} P_n(x)\,dx = \frac{1}{n(n+1)}P'_n(0)$$

Now using the recurrence relation $(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$ at $x = 0$,
we have $P'_n(0) = nP_{n-1}(0)$.

$$\therefore \quad \int_{0}^{1} P_n(x)\,dx = \frac{1}{n(n+1)}n\,P_{n-1}(0) = \frac{1}{n+1}P_{n-1}(0).$$

Hence the result.

ORTHOGONALITY OF LEGENDRE POLYNOMIALS

$$\int_{-1}^{1} \mathbf{P}_m(x)\,\mathbf{P}_n(x)\,dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}$$

[GGSIPU III Sem End Term 2009; End Term 2008; End Term 2004; II Term 2010; II Term 2012; End Term 2012; End Term 2013]

To establish it, consider the following two differential equations

$$(1-x^2)u'' - 2xu' + m(m+1)u = 0 \quad \text{...(1)}$$

and $$(1-x^2)v'' - 2xv' + n(n+1)v = 0 \quad \text{...(2)}$$

Clearly $P_m(x)$ and $P_n(x)$ are the respective solutions of (1) and (2).

Multiplying (1) by v and (2) by u and subtracting, we get

$$(1-x^2)(u''v - uv'') - 2x(u'v - uv') + [m(m+1) - n(n+1)]\,uv = 0$$

or $$\frac{d}{dx}[(1-x^2)(u'v - uv')] + (m-n)(m+n+1)\,uv = 0$$

Now integrating both the sides from -1 to 1, gives

$$(m-n)(m+n+1)\int_{-1}^{1} uv\,dx = -[(1-x^2)(u'v - uv')]_{-1}^{1} = 0$$

$$\Rightarrow \quad \int_{-1}^{1} uv\,dx = 0 \text{ provided } m \neq n \quad i.e. \quad \int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0, \quad m \neq n$$

which establishes the first orthogonality property of Legendre polynomials.

Further, when $m = n$, by Rodrigue's formula $2^n\, n!\, P_n(x) = D^n(x^2-1)^n$ we have

$$(2^n n!)^2 \int_{-1}^{1} P_n^2(x)\,dx = \int_{-1}^{1} [D^n(x^2-1)^n][D^n(x^2-1)^n]\,dx \quad \text{where } D \equiv \frac{d}{dx}. \text{ (Now integrating by parts)}$$

$$= [D^n(x^2-1)^n D^{n-1}(x^2-1)^n]_{-1}^{1} - \int_{-1}^{1} D^{n+1}(x^2-1)^n D^{n-1}(x^2-1)^n\,dx$$

The integrated part on the R.H.S. vanishes at both the limits since $D^{n-1}(x^2-1)^n$ contains a factor (x^2-1). Repeated integration by parts in $(-1, 1)$, gives

$$(2^n n!)^2 \int_{-1}^{1} P_n^2(x)\,dx = (-1)^n \int_{-1}^{1} [D^{2n}(x^2-1)^n]\cdot(x^2-1)^n\,dx = (-1)^n \int_{-1}^{1} (2n)!\cdot(x^2-1)^n\,dx$$

$$= \int_{-1}^{1} (2n)!\,(1-x^2)^n\,dx = 2\,(2n)!\int_{0}^{1} (1-x^2)^n\,dx$$

$$= 2\,(2n)! \int_{0}^{\pi/2} \cos^{2n+1}\theta\,d\theta \qquad \text{(on putting } x = \sin\theta)$$

$$= 2\,(2n)!\,\frac{\Gamma\left(\frac{2n+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{2n+3}{2}\right)} = \frac{(2n)!\,(n!)\sqrt{\pi}}{\Gamma\left(n+\frac{3}{2}\right)}$$

$$= \frac{(2n)!\,(n!)\sqrt{\pi}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

or
$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{(2n)!\,2^{n+1}\cdot n!}{(2n+1)(2n-1)(2n-3)\ldots5\cdot3\cdot1\,(2^n n!)^2} = \frac{2\cdot 2n(2n-2)(2n-4)\ldots4\cdot2}{(2n+1)(2^n n!)}$$

or
$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$$

Alternative approach to prove $\int_{-1}^{1} P_n^2(x)\,dx = \dfrac{2}{2n+1}$.

[GGSIPU I Sem End Term 2013; III Sem II Term 2013]

In this approach we use $(1-2xt+t^2)^{-1/2} = \sum_{r=0}^{\infty} t^r P_r(x)$

Squaring both sides, we get $(1-2xt+t^2)^{-1} = \sum_{m=0}^{\infty} t^m P_m(x) \sum_{n=0}^{\infty} t^n P_n(x)$

Integrating both sides w.r.t. x from -1 to 1, gives

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \int_{-1}^{1}[P_0(x)+tP_1(x)+t^2P_2(x)+...][P_0(x)+tP_1(x)+t^2P_2(x)+...]\,dx$$

$$= \int_{-1}^{1}[P_0^2(x)+t^2P_1^2(x)+t^4P_2^2(x)+t^6P_3^2(x)+...+t^{2n}P_n^2(x)+...]\,dx$$

[on using the orthogonality property]

Thus, $$\sum_{n=0}^{\infty}\left[t^{2n}\int_{-1}^{1}P_n^2(x)\,dx\right] = \int_{-1}^{1}\frac{dx}{1+t^2-2xt} = -\frac{1}{2t}[\log(1+t^2-2xt)]_{-1}^{1}$$

$$= -\frac{1}{2t}[\log(1-t)^2-\log(1+t)^2] = \frac{1}{t}[\log(1+t)-\log(1-t)]$$

$$= \frac{1}{t}\left[\left(t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+...\right)+\left(t+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{4}+...\right)\right]$$

$$= 2\left[1+\frac{t^2}{3}+\frac{t^4}{5}+...\right]$$

Comparing the coefficient of t^{2n} on both sides gives: $\int_{-1}^{1}P_n^2(x)\,dx = \frac{2}{2n+1}$.

FOURIER–LEGENDRE EXPANSION OF AN ARBITRARY FUNCTION $F(X)$

Let $f(x)$ be a function defined in $(-1, 1)$ and assume that it can be expressed in terms of Legendre polynomials of different orders as $f(x) = \sum_{n=0}^{\infty}a_nP_n(x)$ ...(1)

To determine the unknown coefficients a_n we multiply both sides of (1) by $P_n(x)$ and integrate in $(-1, 1)$, to get

$$\int_{-1}^{1}f(x)P_n(x)\,dx = \int_{-1}^{1}[a_0P_0(x)P_n(x)+a_1P_1(x)P_n(x)+a_2P_2(x)P_n(x)+...]\,dx$$

$$= \int_{-1}^{1}a_nP_n^2(x)\,dx \quad \text{(by orthogonality property of Legendre polynomials)}$$

$$= a_n\cdot\frac{2}{2n+1}$$

$$\therefore \quad a_n = \frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)\,dx \qquad ...(2)$$

Substituting for a_n in (1) we get the Fourier–Legendre expansion of $f(x)$.

EXAMPLE 11.23. Obtain the Fourier–Legendre expansion of $f(x)$ defined as

$$f(x) = 0, \quad -1<x<0$$
$$= 1, \quad 0<x<1.$$

SOLUTION: Let $f(x) = \sum_{n=0}^{\infty}a_nP_n(x)$

From equation (2) above, we have

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\,P_n(x)\,dx = \frac{2n+1}{2}\left[\int_{-1}^{0} 0P_n(x)\,dx + \int_{0}^{1} 1P_n(x)\,dx\right] = \frac{2n+1}{2}\int_{0}^{1} P_n(x)\,dx$$

Therefore $\quad a_0 = \frac{1}{2}\int_0^1 P_0(x)\,dx = \frac{1}{2}\int_0^1 1\,dx = \frac{1}{2}$

$$a_1 = \frac{3}{2}\int_0^1 P_1(x)\,dx = \frac{3}{2}\int_0^1 x\,dx = \frac{3}{4},$$

$$a_2 = \frac{5}{2}\int_0^1 P_2(x)\,dx = \frac{5}{2}\int_0^1 \frac{3x^2-1}{2}\,dx = \frac{5}{4}[x^3-x]_0^1 = 0$$

and $\quad a_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx = \frac{7}{2}\int_0^1 \frac{5x^3-3x}{2}\,dx = \frac{7}{4}\left[\frac{5x^4}{4} - \frac{3x^2}{2}\right]_0^1 = \frac{-7}{16}$

$\therefore \quad f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \ldots$ **Ans.**

EXAMPLE 11.24. Show that $\int_{-1}^{1}(1-x^2)\,P'_m(x)\,P'_n(x)\,dx = 0, \quad$ if $m \neq n$

$$= \frac{2n(n+1)}{2n+1}, \quad \text{if } m = n.$$

[GGSIPU III Sem II Term 2009; End Term 2011]

SOLUTION: $\int_{-1}^{1}(1-x^2)\,P'_m\,P'_n\,dx = \int_{-1}^{1}[(1-x^2)\,P'_m]\,P'_n\,dx$ (integrate by parts)

$$= [(1-x^2)\,P'_m\,P_n]_{-1}^{1} - \int_{-1}^{1} P_n \frac{d}{dx}[(1-x)^2\,P'_m]\,dx$$

$$= 0 - \int_{-1}^{1} P_n\,[(1-x^2)\,P''_m - 2xP'_m]\,dx.$$

But $P_m(x)$ is the solution of the equation,

$$(1-x^2)\,P''_m - 2xP'_m + m\,(m+1)\,P_m = 0$$

$$\therefore \quad \int_{-1}^{1}(1-x^2)\,P'_mP'_n\,dx = (-1)^2\,m\,(m+1)\int_{-1}^{1} P_nP_m\,dx$$

Now, by orthogonality property, we have

$$\int_{-1}^{1}(1-x^2)\,P'_m\,P'_n\,dx = 0, \qquad \text{if } m \neq n$$

$$= \frac{2n(n+1)}{2n+1}, \quad \text{if } m = n.$$

Hence Proved.

EXAMPLE 11.25. (i) Evaluate $\int_{-1}^{1} P_5^2(x)\, dx$

[GGSIPU III Sem End Term 2006]

(ii) Prove that $\int_{-1}^{1} x^2 P_{n+1}(x)\, P_{n-1}(x)\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

[GGSIPU III Sem II Term 2007; II Term 2006; Emd Term 2010; II Term 2011]

SOLUTION: (i) We know by orthogonal property of Legendre polynomials, that

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$$

$$\therefore \qquad \int_{-1}^{1} P_5^2(x)\, dx = \frac{2}{2(5)+1} = \frac{2}{11} \qquad \textbf{Ans.}$$

(ii) Let us use the recurrence relation $xP_{n-1} = \frac{nP_n + (n-1)P_{n-2}}{(2n-1)}$

and (on replacing n by $n + 2$) $xP_{n+1} = \frac{(n+2)P_{n+2} + (n+1)P_n}{(2n+3)}$.

Then $\int_{-1}^{1} xP_{n-1} \times x\, P_{n+1} dx = \int_{-1}^{1} \frac{[nP_n + (n-1)\, P_{n-2}][(n+2)P_{n+2} + (n+1)P_n]}{(2n-1)(2n+3)} dx$

$$= \frac{1}{(2n-1)(2n+3)} \int_{-1}^{1} [n(n+2)P_n P_{n+2} + n(n+1)P_n^2 + (n-1)(n+2)P_{n-2}\, P_{n+2} + (n^2-1)P_{n-2}P_n]\, dx$$

$$= \frac{n(n+1)}{(2n-1)(2n+3)} \int_{-1}^{1} P_n^2 dx + 0 \quad \text{(by orthogonality property)}$$

$$= \frac{n(n+1)2}{(2n-1)(2n+3)(2n+1)}.$$

Thus $\int_{-1}^{1} x^2 P_{n-1}\, P_{n+1} dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$. **Hence Proved.**

EXAMPLE 11.26. Let $P_n(x)$ be the Legendre polynomial of degree n. Show that, for any function $f(x)$ for which the nth derivative is continuous,

$$\int_{-1}^{1} f(x)\, P_n(x)\, dx = \frac{(-1)^n}{2^n \cdot n!} \int_{-1}^{1} (x^2-1)^n f^{(n)}(x)\, dx.$$

[GGSIPU III Sem II Term 2010]

SOLUTION: By Rodrigues formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$, we have

$$\int_{-1}^{1} f(x)\,P_n(x)\,dx = \int_{-1}^{1} f(x)\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\,dx \quad \text{(now integrating by parts)}$$

$$= \left[f(x)\frac{1}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1} f'(x)\frac{1}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

$$= 0-\frac{1}{2^n n!}\int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx \quad \left(\text{since } \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \text{ has a factor } (x^2-1)\right)$$

Integrating by parts repeatedly, we get

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x).(x^2-1)^n\,dx. \qquad \textbf{Hence Proved.}$$

Example 11.27. Using Generating function show that

$$P_n\left(-\frac{1}{2}\right) = P_0\left(-\frac{1}{2}\right)P_{2n}\left(\frac{1}{2}\right)+P_1\left(-\frac{1}{2}\right)P_{2n-1}\left(\frac{1}{2}\right)+\ldots+P_{2n}\left(-\frac{1}{2}\right)P_0\left(\frac{1}{2}\right)$$

Solution: In the generating function $(1-2hx+h^2)^{-1/2} = \sum_0^\infty h^n P_n(x)$ ...(1)

putting $x = -\frac{1}{2}$ and $x = \frac{1}{2}$, we get

$$(1+h+h^2)^{-1/2} = \sum_0^\infty h^n P_n\left(-\frac{1}{2}\right) \qquad ...(2)$$

and $$(1-h+h^2)^{-1/2} = \sum_0^\infty h^n P_n\left(\frac{1}{2}\right) \qquad ...(3)$$

Replacing h by h^2 in (2), we get

$$(1+h^2+h^4)^{-1/2} = \sum_0^\infty h^{2n} P_n\left(-\frac{1}{2}\right)$$

But $(1+h^2+h^4) = (1+h+h^2)(1-h+h^2)$

$$\therefore \quad \sum_0^\infty h^{2n}P_n\left(-\frac{1}{2}\right) = (1+h+h^2)^{-1/2}(1-h+h^2)^{-1/2} = \sum_0^\infty h^n P_n\left(-\frac{1}{2}\right)\sum_0^\infty h^n P_n\left(\frac{1}{2}\right)$$

$$= \left[P_0\left(-\frac{1}{2}\right)+hP_1\left(-\frac{1}{2}\right)+h^2P_2\left(-\frac{1}{2}\right)+\ldots+h^n P_n\left(-\frac{1}{2}\right)+\ldots\right]$$

$$\times\left[P_0\left(\frac{1}{2}\right)+h\,p_1\left(\frac{1}{2}\right)+h^2 P_2\left(\frac{1}{2}\right)+\ldots+h^{2n-1}P_{2n-1}\left(\frac{1}{2}\right)+h^{2n}P_{2n}\left(\frac{1}{2}\right)+\ldots\right]$$

Equating the coefficients of h^{2n} on both sides, we get

$$P_n\left(-\frac{1}{2}\right)=P_0\left(-\frac{1}{2}\right)P_{2n}\left(\frac{1}{2}\right)+P_1\left(-\frac{1}{2}\right)P_{2n-1}\left(\frac{1}{2}\right)+P_2\left(-\frac{1}{2}\right)P_{2n-3}\left(\frac{1}{2}\right)+\ldots+P_{2n}\left(-\frac{1}{2}\right)P_0\left(\frac{1}{2}\right).$$

Hence Proved.