The Van Kampen theorem.

Hatcher's Problems.

1. Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

   Solution. Consider the two tori separately.

$$\begin{tikzcd} * \arrow[r, "a"] & * \\ * \arrow[u, "b"] \arrow[r, "a"] & * \arrow[u, "b"] \end{tikzcd}$$

$$\begin{tikzcd} * \arrow[r, "c"] & * \\ * \arrow[u, "d"] \arrow[r, "c"] & * \arrow[u, "d"] \end{tikzcd}$$

   So, the fundamental group of each torus is

$$\pi_1(T_1) = \langle a, b \mid aba^{-1}b^{-1} \rangle$$
$$\pi_1(T_1) = \langle c, d \mid cdc^{-1}d^{-1} \rangle$$

   where the generator a is identified with $S^1 \times \{x_0\}$ in one of the tori and the generator c is identified with $S^1 \times \{x_0\}$ in the other. By Van Kampen's theorem, for $X = T_1 \sqcup T_2/\sim$ we take all the generators and relations from above and add the relation that $a = c$. Thus,

$$\pi_1(X) = \langle a, b, d \mid aba^{-1}b^{-1}, ada^{-1}d^{-1} \rangle$$

2. The mapping torus T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$.

   (a) In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$.
   Solution. Notice that the 1-skeleton of $(S^1 \vee S^1)/\sim$ is simply $X \vee S^1 = S^1 \vee S^1 \vee S^1$ [include picture]. Let a, b be generators of π_1 coming from the original $S^1 \vee S^1$ and let c be the generator coming from this new S^1. Since f is basepoint preserving a, b get mapped to two loops $f(a)$ and $f(b)$. Thus to obtain the two-skeleton of T_f we attach two cells in the following manner.

$$\begin{tikzcd} * \arrow[r, "f(a)"] & * \\ * \arrow[u, "c"] \arrow[r, "a"] & * \arrow[u, "c"] \end{tikzcd}$$

$$\begin{tikzcd} * \arrow[r, "f(b)"] & * \\ * \arrow[u, "c"] \arrow[r, "b"] & * \arrow[u, "c"] \end{tikzcd}$$

   So, by Van Kampen's theorem:

$$\pi_1(T_f) = \langle a, b, c \mid acf_*(a)^{-1}c^{-1}, bcf_*(b)^{-1}c^{-1} \rangle$$

(b) Do the same when $X = S^1 \times S^1$. [One way to do this is to regard T_f as built from $X \vee S^1$ by attaching cells.]

Solution. One can approach this similarly by constructing T_f from $X \vee S^1$. Let's track what happens to the 0-cell e_0 of $S^1 \times S^1$ when we pass to the mapping torus. This 0-cell e^0 becomes $e^0 \times I/(e^0, 0) \sim (f(e^0), 1)$, i.e. it becomes a loop, just like in the previous example. Pictorially, we have this one cell with the ends identified.

$$\begin{tikzcd} * \\ * \arrow[u, "e^0\times I"] \end{tikzcd}$$

Since we only started with a single 0-cell, every 1-cell a of X is a loop, i.e.

$$\begin{tikzcd} * & \\ * \arrow[u, "e^0\times I"] \arrow[r, "a"] & * \end{tikzcd}$$

where the endpoints of a are identified. When we consider what happens to this 1-cell when we pass on to the mapping torus T_f, we have the following picture, which is the same as before.

$$\begin{tikzcd} * \arrow[r, "f(a)\times\{1\}"] & * \\ * \arrow[u, "e^0\times I"] \arrow[r, "a\times\{0\}"] & * \arrow[u, "e^0\times I"'] \end{tikzcd}$$

So every 1-cell of X corresponds to a 2-cell in T_f attached along the word $(a \times \{0\})(e^0 \times I)(f(a) \times \{1\})^{-1}(e^0 \times I)^{-1}$. We can keep on building T_f by tracking the effect on higher dimensional cells on passing to the mapping torus, but as previously proved attaching higher dimensional cells to the 2-skeleton has no effect on the fundamental group of a space, so in our case we can stop here. The 1-skeleton of $S^1 \times S^1$ is the same as the 1-skeleton of $S^1 \vee S^1$, therefore the fundamental group is the same as before along with the additional relation $aba^{-1}b^{-1}$ coming from the torus:

$$\pi_1(T_f) = \langle a, b, c \mid acf_*(a)^{-1}c^{-1}, bcf_*(b)^{-1}c^{-1}, aba^{-1}b^{-1} \rangle$$

3. The Klein bottle is usually pictured as a subspace of $\mathbb{R}^3$ like the subspace $X \subset \mathbb{R}^3$ shown in the first figure at the right. If one wanted a model that could actually function as a bottle, one would delete the open disk bounded by the circle of self-intersection of X, producing a subspace $Y \subset X$.

(a) Show that $\pi_1(X) \approx \mathbb{Z} * \mathbb{Z}$.

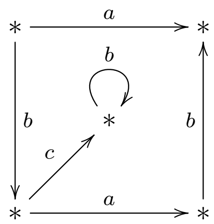
Using pictures, it is not terrible difficult to see that there exists a deformation retraction of X onto $S^1 \vee S^1 \vee S^2$. From this it follows that $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$.

(b) Show that $\pi_1(Y)$ has the presentation $\langle a, b, c \mid aba^{-1}b^{-1}cb^{\epsilon}c^{-1} \rangle$ for $\epsilon = \pm 1$. Changing the sign of ϵ gives an isomorphic group, as it happens.)

Pictorially, what we want to do is remove an open disk from the usual polygonal representation of the Klein bottle

$$\begin{tikzcd} * \arrow[r, "a"] \arrow[d, "b"] & * \\ * \arrow[r, "a"] & * \arrow[u, "b"'] \end{tikzcd}$$

and then give it a cell structure. The following is an example of how one could do this.



By Van Kampen's theorem, the fundamental group of this space Y is

$$\pi_1(Y) = \langle a, b, c \mid aba^{-1}b^{-1}cbc^{-1} \rangle$$

where the choice of epsilon comes from picking the orientation of the loop in the middle.

4. The space Y in the preceding exercise can be obtained from a disk with two holes by identifying its three boundary circles. there are only two essentially different ways of identifying the three boundary circles. Show that the other way yields a space Z with $\pi_1(Z)$ not isomorphic to $\pi_1(Y)$. [Abelianize the fundamental groups to show they are not isomorphic.]

   Once again, this problem can be approached by giving a cell structure to this space and using Van Kampen's theorem to figure out what the fundamental group is. Firstly, the abelianization of the $\pi_1(Y)$ from before is simply

$$aba^{-1}b^{-1}cbc^{-1}$$
$$aa^{-1}bb^{-1}bcc^{-1}$$
$$b$$

   Therefore

$$\pi_1(Y)^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

   The following is a cell structure for a disk with two holes with the boundary circles identified. [I can't draw this with XYPIC anymore. It's a simple picture though. Take a base point. Draw a big loop around it (this is the boundary of the disk itself. Then poke two holes in this disk, draw 1-cells from the basepoint to these holes, then identify the loops going around the wholes with the boundary of the disk. There is a choice of orientation here.]

   If you choose the orientation of the loops that corresponds to the fundamental group which is

$$\pi_1(X) = \langle a, b, c \mid acac^{-1}ba^{-1}b^{-1} \rangle$$

   then the abelianization of this group is

$$\pi_1(X)^{ab} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$$

   If you choose the other distinct orientation you will get the space Y from the previous example.

UT Austin Problems

5. Let X be the 2-complex obtained from a Klein bottle K by attaching a 2-cell D along an essential orientation-preserving curve.

   (a) Compute the fundamental group of X.

   In this case we can split the space X up as the union of the Klein Bottle and the disk and use Van Kampen's theorem.

$$\pi_1(K) = \langle ab \mid aba^{-1}b \rangle$$

$$\pi_1(D^2) = \langle | \rangle$$

$$\pi_1(K \cap D^2 = S^1) = \langle \gamma \, | \rangle$$

   Now, note that $i_*(\gamma) = a$, so by Van Kampen we have

$$\pi_1(X) = \langle a, b \mid aba^{-1}b, a \rangle$$

$$\pi_1(X) = \mathbb{Z}_2$$

6. Let K be the 2-complex be a punctured double torus with boundary curve J with a disk added along a meridian curve. Let L be a Klein bottle with a meridian curve A. Let X be the complex obtained from $K \cup L$ identifying J with A via a piecewise linear homeomorphism.

   (a) What are the fundamental groups of K and L?

   The fundamental group of K is the fundamental group of the 1-skeleton of the double torus, with a two cell attached to one of the meridian curves that kills the homotopy class generated by one of the one cells, $\pi_1(K) = \langle a, b, c, d \mid d \rangle$.

   The fundamental group of L is that of the Klein bottle, $\pi_1(L) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta \rangle$, where a is the generator that corresponds to the loop going oce around the meridian curve A.

   (b) Using the decomposition of X as $K \cup L$, compute the fundamental group of the complex X using van Kampen's Theorem.

   Since $K \cap L = S^1$, let γ be the generator of $\pi_1(K \cap L)$. Note that $i_*(\gamma) = \alpha$ and $i_*(\gamma) = aba^{-1}b^{-1}cdc^{-1}d^{-1}$. Therefore, the fundamental group of X is

$$\pi_1(X) = \langle \alpha, \beta, a, b, c, d \mid \alpha\beta\alpha^{-1}\beta, \alpha^{-1}aba^{-1}b^{-1}cdc^{-1}d^{-1}, d \rangle$$