## Some facts

If $f : X \to Y$ is a homotopy equivalence then we have a bijection $Vect_n(X) \cong Vect_n(Y)$.

Let $G_n(\mathbb{C}^\infty)$ be the set of $n$ dimensional planes in $\mathbb{C}$. This is a manifold. Let $V_n(\mathbb{C}^\infty)$ be its tautological bundle given by taking the union of all $n$-planes.

The homotopy classes of maps $f : X \to G_n(\mathbb{C}^\infty)$ are in bijection with $Vect_n(X)$ given by $[f] \mapsto f^*(V_n(\mathbb{C}^\infty))$. This is equivalently expressed as $Vect_n(X) \cong [X : G_n(\mathbb{C}^\infty)]$.

We know that $Vect(X) \cong \sqcup_n Vect_n(X)$ forms a semi ring with the Whitney and Tensor product, because elements $[E], [F] \in Vect(X)$ are subject to operations

$$[E] + [F] = [E \oplus F]$$
$$[E] \cdot [F] = [E \otimes F]$$

The additive identity is the rank 0 bundle, and the multiplicative identity is trivial bundle $X \times \mathbb{C}$. This is a semi-ring because we have no additive inverses.

## K-theory

To make this into a ring $K(X)$ we consider elements of $K(X)$ to be $[E]-[F]$, where $[E], [F] \in Vect(X)$ subject to the relation

$$[E] - [F] \simeq [J] - [K] \text{ if there exists } [Q] \in Vect(X) \text{ such that } E \oplus K \oplus Q = J \oplus F \oplus Q.$$

**Theorem.** Every element $[E] - [F]$ in $K(X)$ can be written as $[J] - [\varepsilon^n]$ where $\epsilon^n$ is the trivial bundle.

*Proof.* Let $K$ be such that $F \oplus K \cong \varepsilon^n$ and set $J \cong E \oplus K$. Then

$$\begin{aligned}[J] - [\varepsilon^n] &= [E \oplus K] - [F \oplus K] \\ &= [E] - [F].\end{aligned}$$

Addition in $K(X)$ is defined as follows:

$$([E] - [F]) + ([J] - [K])) = (E \oplus J) - (F \oplus K)$$

while multiplication is

$$([E] - [F]) \cdot ([J] - [K])) = ((E \otimes J) \oplus (F \otimes K)) - ((E \otimes K) \oplus (F \otimes J)).$$

### Reduced K-theory

Let $X$ be a topological space. We begin by defining the **rank function**

$$\rho : \mathrm{Vect}(X) \to \mathbb{N}_0,$$

which sends a vector bundle $E$ over $X$ to its rank:

$$\rho(E) := \mathrm{rank}(E).$$

This is a semiring homomorphism:

$$\begin{aligned} \rho(E \oplus F) &= \rho(E) + \rho(F), \\ \rho(E \otimes F) &= \rho(E) \cdot \rho(F), \\ \rho(\varepsilon^1) &= 1, \end{aligned}$$

We can extend $\rho$ to a ring homomorphism

$$\tilde{\rho} : K(X) \to \mathbb{Z}$$

defined on the Grothendieck group $K(X)$ by:

$$\tilde{\rho}([E] - [F]) := \rho(E) - \rho(F).$$

**Lemma.** $\tilde{\rho}$ is well-defined.

*Proof.* Suppose $[E] - [F] = [J] - [K]$ in $K(X)$. Then, by definition of the Grothendieck group, there exists a vector bundle $Q$ such that

$$E \oplus K \oplus Q \cong J \oplus F \oplus Q.$$

Taking ranks, we obtain:

$$\rho(E) + \rho(K) + \rho(Q) = \rho(J) + \rho(F) + \rho(Q),$$

$$\rho(E) - \rho(F) = \rho(J) - \rho(F).$$

Thus, $\tilde{\rho}$ is well-defined.

We define the reduced $K$-theory of $X$ as the kernel of the rank map:

$$\widetilde{K}(X) := \ker(\tilde{\rho}) \subseteq K(X).$$

That is,

$$[E] - [F] \in \widetilde{K}(X) \quad \Longleftrightarrow \quad \mathrm{rank}(E) = \mathrm{rank}(F).$$

**Definition.** Two vector bundles $E$ and $F$ over $X$ are said to be *stably isomorphic* if there exist integers $m, n \geq 0$ such that:

$$E \oplus \varepsilon^m \cong F \oplus \varepsilon^n.$$

The set of stable isomorphism classes of vector bundles over $X$ is denoted $Vect^s(X)$.

**Theorem.** There is an isomorphism $\varphi : Vect^s(X) \to \widetilde{K}(X)$.

*Proof.* Let $\{E\}$ be an equivalence class of stable vector bundles over $X$, then the isomorphism sends

$$\{E\} \mapsto [E] - [\epsilon^{\mathrm{rank}(E)}].$$

We want to show that $\varphi$ is surjective, i.e. for every $[E]-[F] \in \widetilde{K}(X)$, there is an element $\{J\} \in Vect^s(X)$ such that $\varphi(\{J\}) = [J] - [\epsilon^{\mathrm{rank}(J)}]$. We know there exist $E'$ and $F'$ such that $E \oplus E' \cong F \oplus F' \cong \epsilon^n$. Thus,

$$\begin{aligned} &[E \oplus F'] - [\epsilon^n] \\ &= [E \oplus F'] - [F \oplus F'] \\ &= [E] - [F] \end{aligned}$$

and $\varphi(\{E \oplus F'\}) = [E \oplus F'] - [\epsilon^n] = [E] - [F]$.

We also want to show injectivity of $\varphi$. Mainly, if $[E] - [\epsilon^m] = [F] - [\epsilon^n] \in \widetilde{K}(X)$, then there exists an $Q$ such that

$$\begin{aligned} E \oplus \epsilon^n \oplus Q &\cong F \oplus \epsilon^m \oplus Q \\ E \oplus \epsilon^n \oplus Q \oplus Q' &\cong F \oplus \epsilon^m \oplus Q \oplus Q' \\ E \oplus \epsilon^n \oplus \epsilon^k &\cong F \oplus \epsilon^m \oplus \epsilon^k \\ E \oplus \epsilon^{n+k} &\cong F \oplus \epsilon^{m+k}, \end{aligned}$$

i.e. $E$ and $F$ are stably isomorphic. Thus $\varphi : Vect^s(X) \to \widetilde{K}(X)$ is an isomorphism.

*Remark.* It is also useful to note that when we pair the relationship $Vect_n(X) \cong [X : G_n(\mathbb{C}^\infty)] \cong [X : BU(n)]$ with the isomorphism above, we get

$$Vect^s(X) \cong \widetilde{K}(X) \cong [X, BU].$$

## Examples

1. A vector bundle over a point is just an $n$-dimensional plane attached to a point, thus $Vect(pt) \cong \mathbb{N}$. When we apply the Grothendick Construction to the natural numbers, we get the integers, thus $K(pt) \cong \mathbb{Z}$. Additionally $\widetilde{K}(pt) \cong 0$.
2. Using the equivalence $\widetilde{K}(S^n) \cong [S^n : BU] \cong \pi_n(BU) \cong \pi_{n-1}(U)$

   $$\cong \begin{cases} \mathbb{Z}, & \text{when } n \text{ even} \\ 0, & \text{when } n \text{ odd} \end{cases}$$

   by Bott periodicity.
3. Specifically, we can take $X = S^2 \cong \mathbb{CP}^1$, and we know that $\widetilde{K}(S^2) \cong \mathbb{Z}$ by the previous part. The tautological bundle $\gamma \to \mathbb{CP}^1$ has first chern class $c_1(\gamma) = -1$, and its dual has first chern class $c_1(\gamma^*) = 1$. The map $Vect_1(\mathbb{CP}^1) \to H^2(\mathbb{CP}^1; \mathbb{Z})$ given by $L \mapsto c_1(L)$ is actually a bijection.

   Suppose $c_1(L_1) = c_1(L_2) \in H^2(\mathbb{CP}^1; \mathbb{Z})$. Since $H^k(X; \mathbb{Z}) \cong [X; K(\mathbb{Z}, k)]$, we have an equivalence $H^2(\mathbb{CP}^1; \mathbb{Z}) \cong [\mathbb{CP}^1 : K(\mathbb{Z}, 2)] \cong \pi_2(\mathbb{CP}^\infty)$ and since $\mathbb{CP}^\infty \cong BU(1)$, then $c_1(L) = c_1(L_2) \in \pi_2(BU(1)$ implies $L_1 = L_2$ and thus this map is injective.

   For any integer $k$ in $H^2(\mathbb{CP}^1; \mathbb{Z}) \cong \mathbb{Z}$, there exists a line bundle $L$ such that $c_1(L) = k$ since $c_1(\gamma)^{\otimes n} = -n$ and $c_1(\gamma^*)^{\otimes n} = n$. Thus the map is bijective.

   Using this we can figure out the ring structure of $\widetilde{K}(S^2)$. If we take the element $\{\gamma^*\} \in Vect^s(S^2)$ and map it to $[\gamma^*] - [\epsilon] := H \in \widetilde{K}(S^2)$, we can show that $H^2 = 0 \in \widetilde{K}(S^2)$. (Note that $H \neq 0 \in \widetilde{K}(S^2)$, because if it did, then there would exist an $n, m$ such that $\gamma^* \oplus \epsilon^m \cong \epsilon^n$, but this would imply $c_1(\gamma^* \oplus \epsilon^m) = c_1(\epsilon^n)$, which is wrong because $1 \neq 0$.)

   Since $\gamma^*$ is a complex line bundle, we have:

   $$[\gamma^*]^2 = [\gamma^* \otimes \gamma^*] = [(\gamma^*)^{\otimes 2}].$$

   The first Chern class $c_1(\gamma^*)^{\otimes 2} = 2$ so its class in $K^0(S^2)$ is:

   $$[(\gamma^*)^{\otimes 2}] = 1 + 2H.$$

   Substituting into the earlier expression:

   $$\begin{aligned} H^2 &= [\gamma^*]^2 - 2[\gamma^*] + 1 \\ &= (1 + 2H) - 2(1 + H) + 1 \\ &= 1 + 2H - 2 - 2H + 1 \\ &= 0. \end{aligned}$$

Thus, $H^2 = 0$ in $\widetilde{K}(S^2)$, and ring structure is

$$\widetilde{K}(S^2) \cong \mathbb{Z}[H]/(H^2),$$

where $H = [\gamma^*] - 1$ is a generator.

## K-theory as a generalized Cohomology theory

## Spectral Sequences.

We can use the Atiyah-Hirzebruch Spectral Sequence (AHSS) to relate the $K$-theory of $X$ with ordinary cohomology of $X$.

**Questions**

1. Can bundles of different ranks be isomorphic in $Vect_n(X)$? Give example
2. Show explicit isomorphism $E \cong E \oplus X \times \{0\}$
3. Show explicit isomorphism $E \cong E \otimes (E \times \mathbb{C})$
4. Is the above isomorphism only valid for a trivial line bundle or also for $X \times \mathbb{C}^n$?
5. Check that this is a valid equivalence relation.

## Combinatorics Note

$$\frac{P(m,k)}{P(n,k)} = \frac{m!}{(m-k)!} \times \frac{(n-k)!}{n!} = \frac{m!(n-k)!}{n!(m-k)!}.$$

When we multiply this by $\frac{n!}{m!(n-m)!} = \binom{n}{m}$, we get

$$\frac{m!(n-k)!}{n!(m-k)!} \times \frac{n!}{m!(n-m)!} = \binom{n-k}{n-m}.$$

Thus,

$$\sum_{k=0}^{m} \frac{P(m,k)}{P(n,k)} = \frac{1}{\binom{n}{m}} \sum_{k=0}^{m} \binom{n-k}{n-m}.$$

We need find an expression for

$$\sum_{k=0}^{m} \binom{n-k}{n-m}.$$

We can index this using $i = n - k$ instead to obtain

$$\sum_{i=n-m}^{n} \binom{i}{n-m}.$$

Now,

$$\begin{aligned}
\binom{i}{n-m} &= \frac{i!}{(n-m)!(i+m-n)!} \\
&= \frac{i!(n-m+1)}{(n-m+1)!(i+m-n)!} \\
&= \frac{i!((i+1)-(i+m-n))}{(n-m+1)!(i+m-n)!}
\end{aligned}$$

$$= \frac{(i+1)! - i!(i+m-n)}{(n-m+1)!(i+m-n)!}$$

$$= \frac{(i+1)!}{(n-m+1)!(i+m-n)!} - \frac{i!(i+m-n)}{(n-m+1)!(i+m-n)!}$$

$$= \binom{i+1}{n-m+1} - \binom{i}{n-m+1},$$

summing over $i$,

$$\sum_{i=n-m}^{n} \binom{i+1}{n-m+1} - \sum_{i=n-m}^{n} \binom{i}{n-m+1}$$

$$= \binom{n+1}{n-m+1}$$

$$= \binom{n+1}{(n+1)-(n-m+1)}$$

$$= \binom{n+1}{m}.$$

Finally, when we divide by $\binom{n}{m}$ we obtain

$$\frac{\binom{n+1}{m}}{\binom{n}{m}}$$

$$= \frac{(n+1)!}{m!(n-m+1)!} \times \frac{m!(n-m)!}{n!}$$

$$= \frac{n+1}{n-m+1}$$