Bundles

Change of trivialization for bundles. Consider a bundle $\pi : E \to X$ with fiber F. Let x be a point in X and consider two trivializations of an open set around x

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times F$$

$$\varphi_j : \pi^{-1}(U_j) \to U_j \times F$$

The change of trivialization formula is then

$$\begin{gathered} \varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F \\ (x, f) \mapsto (x, g_{ij}(x)f), \text{where } g_{ij} : U_i \cap U_j \to G \end{gathered}$$

Principal Bundles

Let $\pi : E \to X$ be a fiber bundle where an open cover U_α of X gives rise to a local trivialization of the bundle

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times F.$$

A principal G bundle is a fiber bundle $\pi : P \to X$ equipped with a continuous right action $P \times G \to G$ that acts freely and transitively on the fibers and preserves them, i.e. for all $y \in E_x$, $gy \in E_x$ for all $g \in G$.

Example. Let M^m be a smooth manifold. Then let $x \in M$ and define $L_x M = \{(e_1, \ldots, e_m)_x :$ basis for $T_x M\} \cong GL_m(\mathbb{R})$. Then the frame bundle on M is

$$LM = \sqcup_{x \in M} L_x M.$$

There is a projection map $\pi : LM \to M$ given by $(e_1, \ldots, e_m)_x \mapsto x$ and there is a $GL(R)$ action on LM given by

$$((e_1, \ldots, e_m)_x \cdot g \mapsto (ge_1, \ldots, ge_m)_x.$$

Thus the frame bundle on M is a principle $GL(R)$ bundle.

How can we have a bundle map between a principal G bundle $E \to X$ and a principal H bundle $P \to M$?

Theorem. A principal bundle is trivial iff there is a smooth $\sigma : M \to P$.

Proof.

Associated Bundles

A vector bundle and its associated bundles come in pairs, i.e. they determine each other. Here is a brief way to construct the associated bundle given a vector bundle.

Let $\pi : E \to X$ be a fiber bundle with structure group G and fiber F, i.e. if . We can take the local trivialization of the bundle E by considering an open cover U_α of X and the maps

$$\phi_i$$

Let $P \to X$ be a principal G-bundle. Take another space F and define $P \times_\rho F$ by the equivalence relation

$$(p, f) \sim (pg^{-1}, \rho(g)f).$$

Examples

1. **Frame Bundle.** Let $E \to M$ be a vector bundle with fibers V.
2.

Geometry and Curvature

If M is a manifold, define a Riemannian metric g as a positive definite inner product:

$$g_p : T_pM \times T_pM \to \mathbb{R}.$$

or equivalently as a section

$$g \in \Gamma((TM \otimes TM)^*)$$

such that

$$g(v \otimes w) = g_p(v, w).$$

Consider an example of the polar metric $g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$.

A covariant derivative is a map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM).$$

A connection is a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$.

Fundamental Theorem of Riemannian Geometry. Let (M, g) be a Riemannian manifold (or pseudo-Riemannian manifold). Then there exists a unique connection ∇ which satisfies the following conditions:

1. For any vector fields X, Y, Z on M, we have

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

   where $X(g(Y, Z))$ denotes the derivative of the function $g(Y, Z)$ along the vector field X.
2. For any vector fields X, Y,

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

   where $[X, Y]$ denotes the Lie bracket of X and Y.

We can prove this by using the two identities to write down an explicit formula for the connection.

$$\begin{aligned} X\left(g(Y,Z)\right) + Y\left(g(X,Z)\right) - Z\left(g(X,Y)\right) &= \left(g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\right) + \left(g(\nabla_Y X, Z) + g(X, \nabla_Y Z)\right) \\ &\quad - \left(g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\right) \\ &= g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X) \\ &= g(2\nabla_X Y + [Y, X], Z) + g([X, Z], Y) + g([Y, Z], X). \end{aligned}$$

We can also express these ideas similarly using Christoffel symbols.

The Leibniz Formula is

$$\nabla_x(fY) = X(f)Y + f\nabla_X Y$$

where

$$X = \partial_i, Y = Y^j \partial_j$$

and we have

$$\begin{aligned}\nabla_{\partial_i} Y &= \nabla_{\partial_i}(Y^j \partial_j) \\ &= \partial_i Y^j \partial_j + Y^j \nabla_{\partial_i} \partial_j \\ &= \partial_i Y^j \partial_j + Y^j \Gamma_{ij}^k \partial_k\end{aligned}$$

where Γ_{ij}^k are the Christoffel symbols. Thus,

$$\nabla_X Y = X^i \left(\frac{\partial Y^j}{\partial x^i} + \Gamma_{ik}^j Y^k \right) \frac{\partial}{\partial x^j}$$

In Euclidean space or in a space with a flat connection $\Gamma_{ij}^k = 0$.

Torsion is given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

When $\Gamma_{ij}^k = \Gamma^k ji$, then $T = 0$.

Examples for polar and spherical ...

Spin Structure. Let M be a spin manifold. Let S be a spinor bundle.

Then, the covariant derivative lifts to:

$$\nabla : \Gamma(S) \to \Gamma(T^*M \otimes S)$$

Dirac Operator. Define the Dirac operator D by:

$$D = \sum_i e_i \cdot \nabla_{e_i}$$

where e_i is a local orthonormal frame and $\cdot$ denotes Clifford multiplication.

Why Clifford Multiplication? Clifford multiplication satisfies:

- It uniquely encodes the spin representation.
- It is first-order and coordinate invariant.
- It reproduces the behavior of spinors under rotation and parallel transport.

Flat vs Curved. In flat space:

$$D^2 = -\Delta$$

In curved space:

$$D^2 = \nabla^* \nabla + \frac{1}{4} \text{Scal}$$

where Scal denotes the scalar curvature. This is the Lichnerowicz formula.

Curvature Tensors.

- Riemann tensor: $R(X, Y)Z$
- Ricci tensor: $\text{Ric}(X, Y) = \text{tr}(Z \mapsto R(Z, X)Y)$
- Scalar curvature: $R = \text{tr}_g(\text{Ric})$

Bochner Laplacian. The Bochner Laplacian on spinors is:

$$\Delta_B = \nabla^* \nabla$$

Positivity. In general, the Bochner Laplacian is not necessarily positive-definite.

Covariant Derivatives

A covariant derivative on a vector bundle $\pi : E \to B$ is a $\mathbb{R}$-linear map

$$\nabla : \Omega^*(B; E) \to \Omega^{*+1}(B; E)$$

where $\Omega^q(B; E) \doteq \Gamma(\Lambda^q T^*B \otimes E)$ is the space of E-valued q-forms on B.

It turns out that to construct a covariant derivative we only need to work at the $* = 0$ level. Thus given the map

$$\nabla_0 : \Omega^0(B; E) \to \Omega^1(B; E)$$
$$\nabla_0 : \Gamma(\Lambda^0 T^*B \otimes E) \to \Gamma(\Lambda^1 T^*B \otimes E)$$
$$\nabla_0 : \Gamma(E) \to \Gamma(T^*B \otimes E)$$

there exixts a unique $\nabla : \Omega^*(B; E) \to \Omega^{*+1}(B; E)$ such that ∇ and ∇_0 agree at the $* = 0$ level.

0.1 Connections

There is an intimate relationship between covariant derivatives and connections.

A connection on a vector bundle $\pi : E \to M$ is a map

$$\nabla^* : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$
$$(X, \sigma) \mapsto \nabla^*_X \sigma$$

Given a connection ∇^*, we have an induced covariant derivative

$$\nabla : \Gamma(E) \to \mathrm{Hom}(\Gamma(TM), \Gamma(E))$$
$$\sigma \mapsto (X \mapsto \nabla^*_X \sigma).$$

Alternatively, given a covariant derivative $\nabla : \Gamma(E) \to \Gamma(T^*M)$ we can form a connection

$$\nabla^* : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$
$$(X, \sigma) \mapsto (\nabla\sigma)(X).$$

Dirac Bundles and Index Theory

Dirac Bundles. Let $E \to M$ be a (typically complex, sometimes real) vector bundle over a smooth manifold M equipped with:

- a connection ∇,
- a Clifford multiplication $c : T^*M \to \mathrm{End}(E)$,

satisfying the Leibniz rule:

$$\nabla_X(c(\omega)s) = c(\nabla_X \omega)s + c(\omega)\nabla_X s$$

for any vector field X, 1-form ω, and section s of E.

Spin Manifolds and Spinor Bundles. If $\dim M = n$, then a spin manifold has a spinor bundle S, which is a bundle of left modules over the Clifford algebra $\mathrm{Cl}(T^*M)$.

If M is parallelizable, then the spinor bundle S is trivial:

$$S \cong M \times \mathbb{C}^k.$$

Clifford Multiplication. Let $\{e_i\}$ be an orthonormal basis (ONB) of T_pM. Clifford multiplication satisfies:

$$e_i \cdot e_j + e_j \cdot e_i = -2g_{ij}$$

in Riemannian signature.

Dirac Operator. The Dirac operator D is a first-order differential operator defined as:

$$D = \sum_i c(e_i)\nabla_{e_i}$$

where $c(e_i)$ denotes Clifford multiplication.

Properties of D.

- D is elliptic.
- D is self-adjoint if the metric and connection are compatible.
- D is a first-order, coordinate-invariant differential operator.
- D has discrete spectrum on compact manifolds.

Ellipticity and Index. Let $D : \Gamma(E) \to \Gamma(F)$ be a Dirac-type operator. Its principal symbol $\sigma_D(\xi)$ is invertible for all $\xi \neq 0$, hence D is elliptic.

The analytic index of D is defined as:

$$\mathrm{ind}(D) = \dim \ker D - \dim \operatorname{coker} D.$$