Intro to Serre Spectral Sequence

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What is S^∞?

Exact couples

Start with the following pair of spaces, where every map is exact.

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

Then $jk : B \to B$ is a differential, because $(jk)^2 = (jkjk) = 0$. We then get a derived exact couple

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd}$$

where $B' = H(B, jk)$ and $A' = \mathrm{Im}(A)$. The maps are

$i' = i \mid A$

$j'(ia) = [ja] \in E'$. This is well-defined: $ja \in \ker d$ since $dja = jkja = 0$. Additionally if $ia_1 = ia_2$ then $a_1 - a_2 \in \ker i = \mathrm{Im}k$ so $ja_1 - ja_2 \in \mathrm{Im}jk = \mathrm{Im}d$.

$k'[e] = ke$, which lies in $A' = \mathrm{Im}i = \ker j$ since $e \in \ker d$ implies $jke = de = 0$. Furthermore, k' is well defined since $[e] = 0 \in E'$ implies $e \in \mathrm{Im}d \subset \mathrm{Im}j = \ker k$.

Fibrations

Definition. A continuous map $p : E \to B$ satisfies the homotopy lifting property for a space X if for every homotopy $h : X \times [0,1] \to B$ and every lift $\tilde{h}_0$, which is $p^{-1} \circ h \mid_{X\times\{0\}}$, there exists a homotop

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d, "i"] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"] & B \end{tikzcd}$$

Definition. A map $p : E \to B$ is a Serre fibration if it satisfies the homotopy lifting property for finite CW complexes.

Definition. A Hurewicz fibration is a Serre fibration which satisfies the homotopy lifting property for all spaces.

Definition. A fiber bundle is a continuou

Claim. A fiber bundle is a fibration given that the base space B is paracompact.

Proof. The homotopy lifting property for CW complexes is the same as the homotopy lifting property for disks or cubes. Let $p : E \to B$ be a fiber bundle with fiber F and $G : I^n \times I \to B$ a homotopy we want to lift. Choose an open cover $\{U_\alpha\}$ of B with local trivializtions $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$. This is explicitly written out in Hatcher 4.48. The idea is to subdivide I^n into smaller cubes C and I into smaller intervals $[i_t, i_j]$ such that each $C \times [i_t, i_j]$ is mapped into only one U_α by G and then somehow construct $\tilde{G}$ out of these smaller pieces.

Choose an open cover $\{U_\alpha\}$ of B with local trivializtions $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$.

Claim. Given a fibration, $\pi_1(B)$ acts on F.

Claim. $\pi_1(B)$ acts on $\pi_*(F)$, $H_*(F)$, $H^*(F)$.

The setup

For a Serre fibration $F \longrightarrow E \longrightarrow B$, we have a CW decomposition of the base.

$$B^{-1} \subset B^0 \subset \cdots B$$

Then $E^p \doteq \pi^{-1}(B^p)$.

$$C_n = C_n(E)$$
$$F_p C_n \doteq C_n(E^p)$$
$$E^0_{p,q} = C_{p+q}(E^p)/C_{p+q}(E^{p-1}) \Longrightarrow \frac{F_p H_{p+q} E}{F_{p-1} H_{p+q} E}$$
$$E^0_{p,q} = C_{p+q}(E^p)/C_{p+q}(E^{p-1}) = C_{p+q}(E^p, E^{p-1})$$
$$E^1 = H_{p+q}(E^p, E^{p-1})$$

The differential for the E_0 page is given in the following way, take $n = p + q$.

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} \cdots \\
\vdots & \vdots & & \vdots & \vdots \\
\cdots \xrightarrow{\partial_{n+1}|_{F_p}} & F_p C_n & \xrightarrow{\partial_n|_{F_p}} & F_p C_{n-1} & \xrightarrow{\partial_{n-1}|_{F_p}} \cdots \\
\cdots \xrightarrow{\partial_{n+1}|_{F_{p-1}}} & C_n & \xrightarrow{\partial_n|_{F_{p-1}}} & F_{p-1} C_{n-1} & \xrightarrow{\partial_{n-1}|_{F_{p-1}}} \cdots \\
\vdots & \vdots & & \vdots & \vdots
\end{array}$$

So $d_0^{p,q} = \partial_n \mid_{F_p}$ no this is wrong, but ok.

$$d^1_{p,q} : H_{p+q}(E_p, E_{p-1}; \mathbb{Z}) \to H_{p+q-1}(E_{p-1}, X_{E_{p-2}}$$

$$\begin{tikzcd}
 & H_{p+q-1}(E_{p-1}; \mathbb{Z}) \arrow[dr] & \\
H_{p+q}(E_p, E_{p-1}; \mathbb{Z}) \arrow[ur, "\partial"] \arrow[rr, "d^1_{p,q}"] & & H_{p+1-1}(E_{p-1}, E_{p-2}; \mathbb{Z})
\end{tikzcd}$$

E_2 page

Theorem. Let $F \longrightarrow E \longrightarrow B$ be a Serre fibration, with B path-connected and a trivial $\pi_1(B)$ action on $H_*(F,\mathbb{Z})$.

We want to provide the following chain isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & H_{p+q}(E_p, E_{p-1}) \arrow[r, "\partial"] \arrow[d] & H_{p+q-1}(E_{p-1}, E_{p-2}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_p(B_p, B_{p-1}) \otimes H_q(F) \arrow[r, "\partial\otimes 1"] & H_{p-1}(E_{p-1}, E_{p-2}) \otimes H_q(F) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Let B^p be the p-skeleton of B. WE can filter the integral singular chain complex $C_*(E)$ by defining $F_pC_*(E) = C_*(\pi^{-1}(B^p))$. The graded pieces are $G_pC_*(E) = C_*(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$. The E^1 page is then

$$E^1_{p,q} = H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$$

The d^1 differential is defined as the composition

$$H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})) \to H_{p+q-1}(\pi^{-1}(B^{p-1})) \to H_{p+q-1}(\pi^{-1}(B^{p-1}), \pi^{-1}(B^{p-2}))$$

Since $H_p(B^p, B^{p-1})$ is a free group over $\mathbb{Z}$ generated by the p-cells of B this is isomorphic to $\oplus_\alpha H_q(F)$. Let ϕ_α be a characteristic map for a p-cell D_α. We can construct a pullback square

$$\begin{tikzcd}
\tilde{D}_\alpha \arrow[r] & E \arrow[d] \\
D_\alpha \arrow[r] & B
\end{tikzcd}$$

and set $\tilde{S}_\alpha$ to be the preimage of the boundary S_α under .

Explicit example for Hopf fibration.

Consider $S^3 \subset \mathbb{C}^2$ and $S^2 \subset \mathbb{C} \times \mathbb{R}$. The Hopf fibration is then a map $p : S^3 \to S^2$ defined by

$$(z_0, z_1) \mapsto (2z_0z_1^*, N(z_0) - N(z_1))$$

The reason that this is S^2 is that

$$2z_0z_1^*2z_0^*z_1 + (N(z_0) - N(z_1))^2 = (N(z_0) + N(z_2))^2 = 1$$

Any two points which differ by a complex number λ such that $N(\lambda) = 1$ get mapped to the same point on S^2, therefore the fiber of this fibration is S^1.

$$B^2 = S^2$$
$$B^1 = *$$
$$B^0 = *$$
$$E^2 = S^3$$
$$E^1 = S^1$$
$$E^0 = S^1$$

The E_0 page, $E_0^{p,q} = C_{p+q}(E^p)/C_{p+q}(E^{p-1})$ looks like this

$$C_1(E^0) = C_1(S^1) \qquad C_2(E^1)/C_2(E^0) = 0 \qquad C_3(E^2)/C_3(E^1) = C_3(S^3)$$

$$C_0(E^0) = C_0(S^1) \qquad C_1(E^1)/C_1(E^0) = 0 \qquad C_2(E^2)/C_2(E^1) = C_2(S^3)$$

Additionally, we know that $E_1^{p,q} = H_p(B^p, B^{p-1}) \otimes H_q(F) = C_p^{CW}(B; H_q(F))$, so the E_1 page is

$$\begin{array}{cccc}
C_0(S^2, H_2(S^1)) & C_1(S^2, H_2(S^1)) & C_2(S^2, H_2(S_1)) & C_3(S^3, H_2(S^1)) \\
C_0(S^2, H_1(S^1)) & C_1(S^2, H_1(S^1)) & C_2(S^2, H_1(S_1)) & C_3(S^3, H_1(S^1)) \\
C_0(S^2, H_0(S^1)) & C_1(S^2, H_0(S^1)) & C_2(S^2, H_0(S_1)) & C_3(S^3, H_0(S^1))
\end{array}$$

and this is equal to

$$\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 \\
\mathbb{Z} & 0 & \mathbb{Z} & 0
\end{array}$$

all the differentials are 0, so the E^2 page is the same. The only non-zero differential on the E^2 page is $d_2 : E^2_{2,0} \to E^2_{0,2}$ and this is zero because.

Generalization of the Hopf fibration.

We can define the Hopf fibration more elegantly as a map $p : S^3 \to \mathbb{CP}^1$, where $S^3 \subset \mathbb{C}^2$ and

$$p(z_1, z_2) \mapsto [z_1 : z_2]$$

, where $[z_1 : z_2]$ is the line that passes through the two points z_1, z_2. We can now identify $\mathbb{CP}^1$ with $\mathbb{C} \cup \{\infty\}$ with an explicit homeomorphism.

$$\phi : \mathbb{CP}^1 \to \mathbb{C} \cup \{\infty\}$$
$$\phi([z_1 : z_2]) \mapsto z_1/z_2$$

We can define $S^{2n+1} = \{x \in \mathbb{C}^{n+1} \mid |x| = 1\}$. We can also define $\mathbb{CP}^n = \mathbb{C}^{n+1}/\sim$, where $(z_1, \ldots, z_{n+1}) \sim (\omega)_1, \ldots, \omega_{n+1})$ if $\alpha(z_1, \ldots, z_{n+1}) = (\omega_1, \ldots, \omega_{n+1})$ for some $\alpha \in \mathbb{C}$.

Claim. The map $p : S^{2n+1} \to \mathbb{CP}^n$ given by $(z_1, \ldots, z_{n+1}) \mapsto [z_1, \ldots, z_{n+1}]$ is a fiber bundle with fiber S^1.

Proof. Let $U_i = \{[z_1, \ldots, z_{n+1}] \mid z_i \neq 0\}$ be an open set in $\mathbb{CP}^n$.

$$p^{-1}(U_i) = \{z_1, \ldots z_{n+1} \mid z_i \neq 0\}$$

Define a map $h_i : p^{-1}(U_i) \to U_i \times S^1$ by

$$(z_0, \ldots, z_n) \mapsto ([z_0, \ldots, z_n], z_i/|z_i|)$$