Problems

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Spivak Stuff and normal invariants

Question. When is a finite CW complex homotopic to a closed manifold?

Poincare Duality

Let X be a finite CW complex. The orientation character $\omega : \pi_1(X) \to \mathbb{Z}_2$. Using this we can define the chain complex twisted by the orientation character

$$\mathbb{Z}^\omega \otimes_{\mathbb{Z}\pi} C_*(\tilde{X})$$

$$H_n(\mathbb{Z}^\omega \otimes_{\mathbb{Z}\pi} C_*(\tilde{X})) \doteq H_n(X; \mathbb{Z}^\omega)$$

Definition. A connected m-dimensional geometric Poincare complex is a finite connected CW complex with orientation character $\omega : \pi_1(X) \to \mathbb{Z}_2$ and fundamental class $[X] \in H_n(X; \mathbb{Z}^\omega)$ that gives a $\mathbb{Z}\pi$ chain map

$$- \cap [x] : C^{m-*}(\tilde{X}) \to C_*(\tilde{X})$$

that is a $\mathbb{Z}\pi$ chain homotopy equivalence.

Remarks. This induces $H^{m-*}(X; \mathbb{Z}^\omega) = H_*(X; \mathbb{Z})$ and $H^{m-*}(X) = H_*(X; \mathbb{Z}^\omega)$.

This also works for pairs, $(X, \partial X)$, where the fundamental class is now $[X] \in H_m(X, \partial X; \mathbb{Z}^\omega)$. For example,

$$H^{m-*}(X, \partial X; \mathbb{Z}^\omega) \to H_*(X; \mathbb{Z})$$

Theorem. If M^m is a connected, closed manifold, then there exists a finite CW complex X homotopic to M such that X is a Poincare complex.

Remark. There exist Poincare spaces that are not homotopic to manifolds. For example, say $\pi_1(M^4) = \mathbb{Z}_p$, this then implies that $\sigma(\tilde{M}) = p\sigma(M)$. Then there do in fact exist Poincare spaces such that for all p prime M_p^4 is such that $\sigma(\tilde{M}_p) \neq p\sigma(M_p)$.

Thom Spaces and spherical fibrations

A vector bundle over a manifold $\xi : M \to BO(k)$ can be given a metric. With respect to this metric

$$D(\xi) = \{v \in E(\xi) \mid v \leq 1\}$$
$$S(\xi) = \{v \in E(\xi) \mid v = 1\}$$

Not that these are fiber bundles so we have

$$D^k \to D(\xi) = \{v \in E(\xi) \mid v \leq 1\} \to M$$
$$S^{k-1} \to S(\xi) = \{v \in E(\xi) \mid v = 1\} \to M$$

Then the Thom space is

$$Th(\xi) = D(\xi)/S(\xi)$$

Remarks.

$D(\xi)$ is the mapping cylinder of $(S(\xi) \to M)$.

$Th(\xi)$ is the mapping cone of $(S(\xi) \to M)$.

Theorem There exists $U_\xi \in \tilde{H}^k(Th(\xi); \mathbb{Z}^\omega) = H^k(D(\xi), S(\xi); \mathbb{Z}^\omega)$ such that we have isomorphims

$$U_\xi \cap - : \tilde{H}_*(Th(\xi)) = H_*(D(\xi), S(\xi)) \to H_{*-k}(D(\xi); \mathbb{Z}^\omega)$$

Then $H_{*-k}(D(\xi); \mathbb{Z}^\omega) \to H_{*-k}(M; \mathbb{Z}^\omega)$.

Definition. A spherical $k-1$-fibration

$$S^{k-1} \longrightarrow E \xrightarrow{p} X$$

has a disk bundle $DE = cyl(p)$

$$D^k \longrightarrow DE \longrightarrow X$$

and has a Thom space, which is just the cone of the projection.

Remarks.

We can define an orientation character for a spherical fibration and we get the same Thom isomorphism theorems.

$$U_p \in \tilde{H}^k(Th(p); \mathbb{Z}^\omega) = H^k(DE, E; \mathbb{Z}^\omega)$$

Whitney sum still works

$$S(\xi \oplus \xi') = S(\xi) * S(\xi')$$

Definition. Let $G(k)$ be the monoid of homotopy equivalences $S^{k-1} \to S^{k-1}$. Any monoid has an associated classifying space.

$$[X : BG(k)] = \text{spherical } k-1 \text{ fibrations}$$

$$BG \doteq \lim BG(k)$$

We have a map

$$BO(k) \xrightarrow{j_k} BG(k)$$

Pontryagin-Thom construction

Any closed M^m has a stable normal bundle $\nu_M : M \to BO$. Take a representative, $i : M \hookrightarrow \mathbb{R}^{m+k}$ so that

$$i^* T\mathbb{R}^{m+k} \cong TM \oplus \nu M$$

The tubular neighborhood theorem says that there is an diffeomorphism

$$f : (N(m), \partial N(m) \to (D(\nu M), S(\nu M))$$

The collapse map

$$c : \mathbb{R}^{m+k} \cup \{\infty\} = S^{m+k} \to Th(\nu M)$$

sends the interior $N(m)$ to the interior of $D(\nu M)$ and boundary to boundary, and all other points go to infinity.

Claim. $c_*[S^{m+k} =$ a generator.

Proof. c is a smooth degree 1 map as every point $x \in \text{int}D(\nu M)$ has 1 point in preimage.

$$H_{m+k}(S^{m+k}) \to \tilde{H}_{m+k}(Th(\nu M)) = H_{m+k}(D(\nu M), S(\nu M))$$

Comparing

	Spaces	Bundles	Characteristic classes	Classifying spaces
Topology/geometry	Manifolds	vector bundles	Pontryagin	BO
homotopy theory	CW/Poincare Complexes	spherical fibrations	Stiefel-Whitney classes	BG

Manifolds have a stable normal bundle $\nu_M : M \to BO$. For Poincare complexes the analog is a Spivak normal fibration $\nu_X : X \to BG$.

Definition. For an m-dimensional Poincare complex X with orientation character $\omega : \pi_1(X) \to \mathbb{Z}_2$, a $k-1$ Spivak normal structure on X is a $k-1$ spherical fibration $\nu_X : X \to BG(k)$ with the same orientation character.

$$S^{k-1} \longrightarrow E(\nu_X) \longrightarrow X$$

such that there exists a pointed map $c : S^{m+k} \to Th(\nu_X)$ which agrees with Thom iso:

$$[X] = U_{\nu_X} \cap c_*[S^{m+k}] \in H_m(X; \mathbb{Z}^\omega)$$

Example. Manifolds admit a Spivak normal fibration.

$$i : M \hookrightarrow \mathbb{R}^{m+k}$$

$$\nu_X : M \to BO(k) \to BG(k)$$

$$c : S^{m+k} \to Th(\nu_X)$$

and $c_*[S^{m+k}]$ generates $\tilde{H}_{m+k}(Th(\nu_X))$.

So for a choice of Thom class of ν_X,

$$U_{\nu_X} \in \tilde{H}^k(Th(\nu_X); \mathbb{Z}^\omega)$$

we have

$$\pm[X] = U_{\nu_X} \cap c_*[S^{m+k}]$$

Definition. Given a $(k-1)$ Spivak normal structure ν_X, the stable version $\nu_X : X \to BG(k) \to BG$ is the Spivak normal fibration.

Main claim. Let X be a finite CW complex. X is a Poincare complex iff X admits a Spivak normal fibration.

Idea of proof. X is homotopic to a finite simplicial complex K by the simplicial approximation theorem. K can be embbeded in $\mathbb{R}^{m+k}$. Take a regular neighborhood of X, $(N(X), \partial N(X)$. Note that $i : X \hookrightarrow N(X)$ induces isomorphims i^* and $i_a st$ because X is a strong deformation retract of $N(X)$. Let $[N(x)] \in H_{m+k}(N(x), \partial N(X))$. Let $u \in H^k(N(x), \partial N(X))$ then the following commutes.

$$\begin{tikzcd}
H^{m-*}(X) \arrow[rr] \arrow[dr, "\phi"'] & & H^{m+k-*}(N(x), \partial N(x)) \arrow[dl, "-\cap[N(X)]"] \\
& H_*(X) \cong H_*(N(X)) &
\end{tikzcd}$$

Set $u = U$ the Thom class, which implies that ϕ is an isomorphim which implies $-\cap[X] : H^{m-*} \to H_*(X)$ is an iso, where $[X] \doteq (i_*)^{-1}(U \cap [N(X)])$. So Spivak normal fibration implies Poincare Duality.

Converse: $[X]$ exists and if we set ϕ to be Poincare duality map, this uniquely defines U up to sign. This is a candidate Thom class.

Consider $\partial \hookrightarrow N(X) \simeq X$.

Consequences.

Theorem. An m-dimensional simply connected Poincare space is homotopy equivalent to a manifolds M iff

(1) There exists a vector bundle $\xi : X \to BO(k)$ and $c : S^{m+k} \to Th(\xi)$ such that $U_\xi \cup c_*[S^{m+k}] = [X]$.

(2) If $m = 4k$, $\sigma(X) =< L_k(-\xi), [X] >$, if $m = 4k+2$, the $\mathbb{Z}_2$ valued Arf invariant of the self intersection form $\mu : \ker(f_* : H_{2k+1}(M; \mathbb{Z}_2 \to H_{2k+1}(M; \mathbb{Z}_2)\mathbb{Z}_2$ vanishes.

Definition. Given $J : BO \to BG$. We want a lift $\tilde{\nu_X}$

$$\begin{tikzcd} & BO \arrow[d] \\ X \arrow[ur, "\tilde{\nu_X}"] \arrow[r] & BG \end{tikzcd}$$

Define the mapping fiber G/O such that we have a fibration

$$G/O \longrightarrow BO \longrightarrow BG$$

We can extend this fibration to the right.

$$G/O \longrightarrow BO \longrightarrow BG \longrightarrow B(G/O)$$

The obstruction to $\tilde{\nu_X}$ is in $[X : B(G/O)]$ since the fibration above passes on to exact sequence

$$[X : BO] \longrightarrow [X : BG] \longrightarrow [X : B(G/O)]$$

So we want

$$X \xrightarrow{\nu_X} BG \longrightarrow B(G/O)$$

to be nullhomotopic.

Examples of Poincare complexes which are not manifolds. Consider the fibration

$$S^2 \longrightarrow E \longrightarrow S^3$$

, where E is the Poincare complex which is not homotopic to a PL-manifold.

Definition. The ith Stiefel-Whitney class of a spherical fibration $p : E \to X$ is ϕ- Thom iso, Sq^i-ith Steenrod square.

$$\begin{tikzcd} \tilde{H}^{m+k}(Th(p);\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^i"] & \tilde{H}^{m+k+i}(Th(p);\mathbb{Z}_2) \\ H^m(X;\mathbb{Z}_2) \arrow[u, "\phi"] \arrow[r] & H^{m+i}(X;\mathbb{Z}_2) \arrow[u, "\phi"] \end{tikzcd}$$

$$\omega_i(p) \doteq \phi^{-1}\mathrm{Sq}^i\phi(1)$$

where $1 \in H^M(X;\mathbb{Z}_2)$.

Browder Novikov theory

Definition. An n-dim normal map to (X, ν), where X is a Poincare complex and $\eta : X \to BO$ is a pair (f, b) such that $f : M^n \to X$ and $b : \nu_M \to \eta$.

$$\begin{tikzcd} \nu_M \cong f^*\eta \arrow[r] \arrow[d] & \eta \arrow[d] \\ M \arrow[r] & X \end{tikzcd}$$

A normal bordism of two normal maps $(f, b) : M \to X, (f', b') : M' \to X$ is a normal map

$$((F, B), (f, b), (f', b')) : (W^{m+1}, M, M') \to (X \times I, X \times \{0\}, X \times \{1\})$$

Denote by $\mathcal{N}_m(X)$ the set of bordism equivalence classes of normal maps to (X, η).

Proposition. $\mathcal{N}_m(X) = \pi^s_{m+k}(Th(\eta))$.

Proof. Thom's cobordism theorem. Theorem 6.10 Ranicki

Definition. For a n-dimensional Poincare complex X

1. A normal invariant is a pair (η, ρ) such that $\eta : X \to BO(k)$ with $\omega_1(\eta) = \omega_1(X)$ and $\rho : S^{m+k} \to Th(\eta)$ such that $\tau_\eta \cap h_*(\eta) = [X] \in H_n(X; \mathbb{Z}^\omega)$, where
$$h : \pi_*(Th(\eta)) \to H_*(Th(\eta))$$
2. $(\eta, \rho) \simeq (\eta', \rho')$ iff $c : \eta \oplus \epsilon^j \simeq \eta' \oplus \epsilon^l$ and
$$T(c)* : \pi^s_{m+k+j}(\Sigma^j Th(\eta)) \to \pi_{m+k+j}($$
bla bla, we just want the induced map on the thom spaces to send ρ to ρ'.
3. $\mathcal{I}(X)$ is the normal structure set is the set of equivalence classes of normal invariants using the relation from above.

Example.

$$\begin{tikzcd} M^m \arrow[r, "\nu_M"] \arrow[dr, "i"'] & S^{m+k} \arrow[d, "\rho"] \\ & Y/\partial Y = Th(\nu_M) \end{tikzcd}$$

Claim. (ν_M, ρ) is a normal invariant of M.

Proof. The Thom class is $\tau_e ta = Di_*[M]$ where $D = (- \cup [Y, \partial Y])^{-1}$. We have tho check that
$$Di_*[M] \cap h_*(\rho) = i_*[M]$$
$h_*(\rho) = p_*[S^{m+k}] = [Y, \partial Y]$ and then this follows by construction.

Theorem. Let X be an m-dimensional Poincare complex. Then the following are equivalent

1. $\mathcal{I}(X) \neq \emptyset$
2. $\exists$ degree 1 normal map to (X, η).
3. The Spivak normal fibration admits a vector bundle reduction
4. Then the composition $X \to BG \to B(G/O)$ is nullhomotopic.

Proof. 3 iff 4 has already been proved.

$1 \implies 2$, Given a normal invariant (η, ρ), make $\rho : S^{m+k} \to Th(\eta)$ transverse to the 0-section, where we have a mapping $X \to Th(\eta)$. Then if $f = \rho \mid : \rho^{-1}(X) = M \to X$.

$2 \implies 1$. Given (f, b), we can look at embedding $M \hookrightarrow S^{m+k}$, which gives us (ν_M, p), and then $\rho : S^{m+k} \to Th(\nu_M) \to Th(\eta)$, which gives (η, ρ).

$1 \implies 3$. Given (η, ρ), then $J\eta$ is the Spivak normal fibration of X.

$3 \implies 1$. If there is an η such that $J\eta \simeq \nu_X$. The Spivak normal fibration comes with a map satisfying the right conditions $S^{m+k} \to Th(\nu_X) \simeq Th(\eta)$.

Proposition. A geometric Poincare complex X^m is homotopy equivalent to a manifold M iff $t(\nu_X) : X \to B(G/O)$ and $[f, b]$ contains the homotopy equivalence.

Proposition.

1. $\mathcal{X} = \mathcal{N}_m(X)$.
2. $\mathcal{X} = [X : G/O]$.

Proof 2: Let (η, ρ) be a normal invariant and $[\alpha, \beta] \in [X, G/O]$, where $\alpha : X \to BO$ and $\beta = J\alpha \simeq * : X \to BG$ is nullhomotopic. We can to $(\alpha, \beta) \mapsto (\eta \oplus \alpha, \tilde{\rho})$.
$$\tilde{\rho} : S^{m+k+j} \to \Sigma^j Th(\eta) = Th(\eta \oplus \epsilon^j) \simeq Th(J\eta \oplus \epsilon^j)$$

Seifert Van Kampen Theorem.

Let X be topological space covered by two path connected open sets U_1, U_2 such that $U_1 \cap U_2$ is also path connected and non-empty. Then

$$\pi_1(X) = \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$$

Let X be the space described in the picture.

Ambient Isotopy

Let N, M be two manifolds and $g, h : N \hookrightarrow M$ be two embeddings. A continuous map

$$F : M \times [0,1] \to M$$

is an ambient isotopy taking g to h if F_0 is the identity, each F_t is a homeomorphism from M to itself and $F_1 \circ g = h$.

What is $\pi_n(f)$

The mapping cylinder M_f for a function $f : X \to Y$ is defined as $X \times [0,1] \cup_f Y$. The homotopy group of a map $f : X \to Y$ is defined as $\pi_n(f) = \pi_n(M_f, X)$. Thus, elements of $\pi_n(f)$ can be represented by maps $(D^n, S^{n-1}) \to (M_f, X)$. Since M_f is homotopy equivalent to Y, it follows that elements in $\pi_n(f)$ can be represented as commutative diagrams:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\
D^n \arrow[r] & Y
\end{tikzcd}$$

Attaching cells

Lemma. A based map $f : (D^n, S^{n-1}) \to (X, A)$ represents zero in $\pi_n(X, A)$ iff it is homotopic rel S^{n-1} to a map $g : (D^n, S^{n-1}) \to (X, A)$ with image contained in A.

Proof. Given a maps f, g as above we know that there is a rel S^{n-1} homotopy $F : D^n \times [0,1] \to X$ between f and the constant map c. In other words, F is such that $F|_{D^n \times \{0\}} = f$ and $F|_{D^n \times \{1\}} = c$ and $F|_{S^{n-1} \times [0,1]} \subset A$. Set $f_1 = F| : D^n = d^n \times \{1\} \cup S^{n-1} \times [0,1] \to A$. Since the image of f_1 lies in A, it is enough to show that f is homotopic to g. The homotopy f_t is given by the restriction of F to $D^n = D^n \times \{t\} \cup S^{n-1} \times [0,t] \to X$: by construction we have, we have $f_1 = g$ and f_0 is equal to f.

Conversely, assume that f is homotopic rel S^{n-1} to a map g with image in A. Show that g represents zero in $\pi_n(X, A)$, i.e. that it is nullhomotopic rel S^{n-1}. As $g(D^n) \subset A$, we have a map $g : (D^n, S^{n-1}) \to (A, A)$. This map is nullhomotopic since D^n is contractible.

Claim. Let X be a CW complex with $f \in \pi_n(X)$. Let $Y = X \cup_f e^{n+1}$. Then $i : X \to Y$ is n-connected, or equivalently the pair (Y, X) is n-connected.

Proof. We must show that $\pi_i(Y, X) = 0$ for $i < n$. Let $f : (D^i, S^{i-1}) \to (Y, X)$ be a representative of $[f] \in \pi_i(Y, X)$. By the cellular approximation theorem $f(D^i) \subset X^i$.

If we have a space CW complex X with $[f] \in \pi_n(X)$, then there exists a space X' containing X such that $\pi_i(X) = \pi_I(X')$ for $i < n$ and $\pi_n(X') = \pi_n(X)/ < [f] >$. The space $X' = X \cup_f e^{n+1}$.

We can also kill relative homotopy groups. Let $f : X \to Y$ with $\omega \in \pi_{n+1}(f)$. Then there exists X' containing X such that $\pi_i(X') = \pi_i(X)$ for all $i < n+1$ and $\pi_{n+1}(f') = \pi_n(f)/ < \omega >$.

Proof. ω is represented by a pair of maps $(q, Q); (S^n, D^{n+1}) \to (X, Y)$. Let $X' = X \cup_q e^{n+1}$ and set the induced map $f' : X' \to Y$ to be $f' = f \cup Q : X \cup_q e^{n+1} \to Y$. Now consider the following commutative diagram:

$$\begin{tikzcd}
{} \arrow[r] & \pi_i(X) \arrow[r] \arrow[d] & \pi_i(Y) \arrow[r] \arrow[d] & \pi_i(f) \arrow[r] \arrow[d] & \pi_{i-1}(X) \arrow[r] \arrow[d] & \pi_i(Y) \arrow[r] & {} \\
{} \arrow[r] & \pi_i(X') \arrow[r] & \pi_i(Y) \arrow[r] & \pi_i(f') \arrow[r] & \pi_{i-1}(X') \arrow[r] & \pi_i(Y) \arrow[r] & {}
\end{tikzcd}$$

For $i < n-1$, the maps $i : \pi(X) \hookrightarrow \pi(X')$ are isomorphisms.

attaching cells again.

For $n \geq 2$, $\pi_1(X') = \pi_1(X)$ so the universal covers $\tilde{X}$, $\tilde{X}'$

Let W^{m+1} be the trace of a k-surgery performed on a manifold M^m along the φ_0 with resultant M'. Notice that W has the homotopy type of $M \cup_\varphi e^{k+1}$, where e^{k+1} is just a $k+1$ dimensional cell attached along φ. This is true, because there is a deformation retract of $W = M \times [0,1] \cup_\varphi D^{k+1} \times D^{n-k}$ onto $M \times [0,1] \cup_{\varphi_0} D^{k+1} \times \{0\}$, so attaching a $(k+1)$ handle has the same homotopy theoretic effect as attaching a $k+1$ cell. Note that this implies that the inclusion $M \hookrightarrow W$ is k-connected and $\pi_k(W) = \pi_k(M)/[\varphi_0]$.

0 surgery is connected sum.

If we take two disjoint manifolds M, N of the same dimension and perform a 0 surgery on them, where the embedding $S^0 \times D^n \hookrightarrow M \coprod N$ the resulting manifolds is a connected sum of them.

If we take a single manifold M^n and perform 0 surgery, the output is $M\#(S^1 \times S^{n-1})$.

Framing and uniqueness

A framing of an embedding $\varphi_0 : S^k \hookrightarrow M^n$ is an identification of the normal bundle $\nu(\varphi_0)$ with $S^k \times \mathbb{R}^{n-k}$.

Claim. Let M^n be an n-dimensional manifold. A framed embedding $\varphi_0 : S^k \hookrightarrow M^n$ gives rise to an embedding $\varphi : S^k \times D^{n-k} \hookrightarrow M^n$ such that $\varphi \times 0 = \varphi_0$.

Proof. By the Tubular neighborhood theorem φ_0 extends to a codimension 0 embedding $\nu(\varphi_0) \hookrightarrow M^n$. Along with the framing of $\nu(\varphi_0)$ this becomes an embedding $S^k \times \mathbb{R}^{n-k} \hookrightarrow M^n$ and taking the unit disc bundle we get the desired embedding.

$$\varphi : S^k \times D^{n-k} \hookrightarrow M^n$$

I am slightly confused, Ranicki says that the framings of an embedding $\varphi_0 : S^k \hookrightarrow M^n$ are in one to one correspondence with extensions of φ to an embedding $\varphi : S^k \times D^{n-k} \hookrightarrow M^n$ by the tubular neighborhood theorem. But all the tubular neighborhood theorem says is that the φ_0 embedding extends to φ, but says nothing about the uniqueness of φ.

Let $\varphi, \varphi' : S^k \times D^{n-k} \hookrightarrow M^n$ determined by the same φ_0 and the same framing of $\nu(\varphi_0)$.

Attaching cells

If we have a space CW complex X with $[f] \in \pi_n(X)$, then there exists a space X' containing X such that $\pi_i(X) = \pi_I(X')$ for $i < n$ and $\pi_n(X') = \pi_n(X)/ < [f] >$. The space $X' = X \cup_f e^{n+1}$.

We can also kill relative homotopy groups. Let $f : X \to Y$ with $\omega \in \pi_{n+1}(f)$. Then there exists X' containing X such that $\pi_i(X') = \pi_i(X)$ for all $i < n+1$ and $\pi_{n+1}(f') = \pi_n(f)/ < \omega >$.

Proof. ω is represented by a pair of maps $(q, Q); (S^n, D^{n+1}) \to (X, Y)$. Let $X' = X \cup_q e^{n+1}$ and set the induced map $f' : X' \to Y$ to be $f' = f \cup Q : X \cup_q e^{n+1} \to Y$. Now consider the following commutative diagram:

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{} \arrow[r] & \pi_i(X') \arrow[r] & \pi_i(Y) \arrow[r] & \pi_i(f') \arrow[r] & \pi_{i-1}(X') \arrow[r] & \pi_i(Y) \arrow[r] & {}
\end{tikzcd}$$

For $i < n-1$, the maps $i : \pi(X) \hookrightarrow \pi(X')$ are isomorphisms.

attaching cells again.

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Extension of an embedding

We can kill elements $x \in \pi_k(M)$ provided that they are represented by a framed embedding $\varphi : S^k \hookrightarrow M$. But what we really want is to kill homotopy groups of a map. Let $f : X \to Y$ be a smooth map and do surgery on the embedding $\varphi : S^n \times D^{n-k} \hookrightarrow M$. Note that f induces a map $f' : M' \to X$ if φ extends to an embedding $\Phi : D^{k+1} \times D^{n-k} \hookrightarrow X$. **Isn't there a dimension mismatch, how can this be an embedding?**

Define $f' : M' \to X$ as $f \cup \Phi$

Claim. Let $f : M^n \to X$ be a map and assume that $k < 2n$ or $k < 2n+1$. Let $\varphi : S^k \times D^{n-k} \hookrightarrow M$ be an embedding with an extension to a map $\Phi : D^{k+1} \times D^{n-k} \to X$. If $x \in \pi_{k+1}(f)$ denotes the homotopy class defined by (ϕ, Φ) and f' is the induced map resulting from surgery on φ, then

$$\pi_{k+1}(f') = \pi_{k+1}(f)/ < x > \text{ and } \pi_j(f') = \pi_j(f) \text{ for } j \leq k$$

Proof. Note that up to homotopy, the trace of the surgery W along φ is obtained by adding both a $(k+1)$ cell to $M \times [0,1]$ and adding a $(n-k)$ cell to $M' \times [0,1]$. f extends to a map $F = f \cup \Phi : W \to X$. This implies that $\pi_{k+1}(F) = \pi_{k+1}(f)/ < x >$ as well as $\pi_j(F) = \pi_j(f)$ for $j \leq k$ and $\pi_j(f') = \pi_j(F)$ for $j \leq n-k-1$. Thus $\pi_{j+1}(f) = \pi_j(f')$ for $2j \leq n-1$.

Hatcher stuff

The algebraic effect of a geometric surgery on a manifold M is determined by Poincare duality isomorphims

$$H^{m-*}(M) \simeq H_*(M)$$

This is a global expression of the local property of being a manifold, namely

$$H^{m-*}(\{x\}) \simeq H_*(M, M/\{x\})$$

To piece these local isomorphisms together we require orientability.

Orientations and Poincare duality.

Definition. A local orientation of M at x is a choice of generator $\mu_x \in H_m(M, M/\{x\}; \mathbb{Z}) \simeq H_m(\mathbb{R}^m, \mathbb{R}^m - \{0\}; \mathbb{Z}) = \mathbb{Z}$.

A global orientation of M is a consistent choice of generator μ_x for all $x \in M$, satisfying local consistency condition:

For all $x \in M$, there exists an open ball B containing X of finite radius such that, there exits $\mu_B \in H_m(M, M - B) = \mathbb{Z}$ mapping to μ_y for all $y \in B$ under the map $H_n(M, M-B) \to H_n(M, M - \{y\})$.

M is orientable if an orientation exists. $M_{\mathbb{Z}_2} \to M$.

Every manifold M has an orientable 2-sheeted cover

$$M_{\mathbb{Z}_2} = \{\mu_x \mid x \in M\}$$

This is a double cover because over each point x we have ± 1. We can topologize $M_{\mathbb{Z}_2}$: Given B containing x and μ_B generating $H_m(M, M-B; \mathbb{Z})$, we have a corresponding open set in $M_{\mathbb{Z}_2}$, $U(M_{\mathbb{Z}_2})$.

$$U(\mu_B) = \{\mu_x \in M_{\mathbb{Z}_2} \mid x \in B, \mu_B \to \mu_x\}$$

Definition. If M is connected, the orientation character or the 1st Stiefel-Whitney class is

$$\omega : \pi(M) \to \{\pm 1\} = Aut(\mathbb{Z})$$

Define this by taking $\gamma \in \pi_1(M)$. If γ lifts to a loop in $M_{\mathbb{Z}_2}$, then $\omega(\gamma) = 1$ and if it lifts to a path in $M_{\mathbb{Z}_2}$, then $\omega(\gamma) = -1$.

The first Stiefel-Whitney class is the obstruction to being able to orient M.

$M_{\mathbb{Z}_2} \to M$ can be embedded into a larger covering space $M_{\mathbb{Z}} \to M$, called the orientation sheaf of M, where

$$M_{\mathbb{Z}} = \{\alpha_x \in H_n(M, M - \{x\}; \mathbb{Z}) \mid x \in M\}$$

We can topologize this. At $\alpha_x \cong 0$, we get a copy of M, and a copy of $M_{\mathbb{Z}_2}$ for each $n \in \mathbb{Z}_{>0}$ given by $\pm n\mu_x$ for each μ_x a generator of $H_n(M, M - \{x\})$.

Example

$$M = S^1$$
$$M_{\mathbb{Z}_2} = S^1 \times \mathbb{Z}_2$$
$$M_{\mathbb{Z}} = S^1 \times \mathbb{Z}$$

Example M is the Mobius band.

Definition. A continuous map $M \to M_{\mathbb{Z}}$,

$$x \mapsto \alpha x \in H_n(M, M - \{x\}; \mathbb{Z})$$

is called a section of the covering space.

An orientation of M is a section $x \mapsto \mu_x$ such that μ_x is a generator for all x.

Replace $\mathbb{Z}$ with any commutative ring R with identity.

Definition. An R-orientation is a section from $M \to M_R$ such that $x \mapsto \mu_x$ a generator, i.e. μ_x is a unit in R, where we again require local consistency.

$$H_n(M, M - \{x\}; R) \cong H_n(M, M\{x\}; \mathbb{Z}) \otimes R$$

$$M_R = \bigcup_{r \in R} M_r$$

where $M_r = \{\pm\mu_x \otimes r\}$.

$$M_r = M \text{ if } r \text{ is of order 2}$$
$$M_r = M_{\mathbb{Z}_2}\text{otherwise}$$

Corollary. If M is $\mathbb{Z}$-orientable $\implies$ M is R-orientable $\forall R$, because $1 \in \mathbb{R}$, and take $\mathbb{Z}$ orientation tensor 1.

If M is not $\mathbb{Z}$-orientable, but has an element of order 2, then M is R-orientable.

M is always $\mathbb{Z}_2$-orientable.

Theorem. If M is closed and R-orientable, then $H_n(M; R) \to H_n(M, M - \{x\}; R)$ is an isomorphism for all $x \in M$.

Definition. An element $[M] \in H_n(M; R)$ mapping to μ_x for all $x \in M$ is called a fundamental class of M with R-coefficients.

Remark. M closed, R-orientable $\implies \exists[M]$.

Poincare duality Theorem. For any closed R-orientable m-dimensional manifold, if $[M] \in H_m(M; R)$ is a fundamental class then

$$[M] \cap - : H^*(M; R) \to H_{m-*}(M; R)$$

is an isomorphism.

Local Coefficients

Local coefficients are a tool to organize information about the action of the fundamental group on various other abelian groups.

Local coefficients give a fundamental class for non-orientable manifolds and thus this gives a way to extend $\mathbb{Z}$ Poincare duality.

2 points of view:

(1) Main chain complex of interest for M is that of the universal cover $\tilde{M}$, viewed as $\mathbb{Z}\pi(M)$-module chain complex. Local coefficients are modules over $\mathbb{Z}\pi(M)$.

(2) Think of fiber bundles over M with abelian group G fibers, of transition functions in the automorphisms of G. This gives a chain complex of formal sums of singular simplices σ with coefficients over σ.

Remark. π_1 need not be commutative so left modules and right modules don't have to be the same.

First note that $C_*(\tilde{M})$ is a right $\mathbb{Z}\pi$-module chain complex.

Example. $M = S^1$, $\tilde{M} = \mathbb{R}$.