CW Complex

We often want to restrict our attention to nice topological spaces which have properties that yield more fruitful results. One such example of a topological space is a CW complex. A CW complex is made up of k-cells e^k, which are each homeomorphic to an open disk B^k. Constructing a cell complex is then done in the following inductive manner:

- Start with a discrete set of points, which are the 0-cells and call this the zero skeleton X^0.
- Create inductively the k skeleton X^k by attaching cells of dimensions k to the $k-1$ skeleton X^{k-1} such that we have $X^k = X^{k-1} \sqcup_\alpha e^k_\alpha$. These cells are attached via attaching maps $\phi_\alpha : \partial D^k_\alpha \to X^{k-1}$. Thus, the k-skeleton X^k is the quotient space $X^{k-1} \sqcup_\alpha e^k_\alpha / \sim$, where $x \sim \phi_\alpha(x)$ for $x \in \partial e^n_\alpha$.
- These attaching maps ϕ_α extend to so called characteristic maps $\Phi_\alpha : D^k_\alpha \to X$ such that the interior of D^k_α gets mapped homeomorphically to e^k_α.
- Finally, if X is a finite CW complex then $X^n = X$ for some finite n. If X is an infinite CW complex then $X = \cup_n X^n$ and we give it the weak topology, that is, a subset $A \subset X$ is open or closed iff $A \cap X^n$ is open closed for each n.

For example the space which consists of a single 0-cell e^0 and a single n-cell e^n attached via the constant map $S^{n-1} \to e^0$ is a cell structure for S^n. An explicit homeomorphism exists by considering S^n as $\mathbb{R}^n \cup \{\infty\}$ and defining a map

$$\varphi : D^n/\partial D^n \to \mathbb{R}^n \cup \{\infty\}$$

sending the interior of D^n to $\mathbb{R}^n$ and the boundary to the point at infinity.

Multiplicativity of the Euler Characteristic with respect to covers.

One way of defining the Euler Characteristic of a finite CW complex X is

$$\chi(X) = \text{number of even dimensional cells of } X \; - \text{number of odd dimensional cells of } X.$$

If $\tilde{X}$ is an n-sheeted covering of X, then one can lift to a CW structure on $\tilde{X}$ by lifting each characteristic map $\phi_\alpha : D^k \to X$ for each k-cell of X. Since $\tilde{X}$ is a n-sheeted covering, there exist n lifts of the characteristic map and for each k-cell in X, there exist n number of k-cells in $\tilde{X}$, so

$$\chi(\tilde{X}) = n\,(\text{number of even dimensional cells of } X \; - \text{number of odd dimensional cells of } X) = n{\cdot}\chi(X).$$

Proposition. The Euler Characteristic of a CW complex X is

$$\chi(X) = \sum_i (-1)^i \operatorname{rank}(H_i(X))$$

Proof. Consider the cellular chain complex of X.

$$0 \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

We have short exact sequences.

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n \longrightarrow 0$$

where $B_n = \text{Im } d_{n+1}$, $Z_n = \ker d_n$, $H_n = Z_n/B_n$.

By the rank nullity theorem

$$\text{rank } C_n = \text{rank } Z_n + \text{rank } B_{n-1}$$

$$\text{rank } Z_n = \text{rank } B_n + \text{rank } H_n$$

Therefore,

$$\text{rank } C_n = \text{rank } B_n + \text{rank } H_n + \text{rank } B_{n-1}$$

and $\sum_n (-1)^n \text{rank } C_n = (-1)^n \text{rank } B_n + \sum_n (-1)^n \text{rank } H_n$, and since $d_{n+1} = 0$ we have

$$\chi(X) = \sum_n (-1)^n \text{rank } H_n(X)$$

One can also arrive at a similar result using this fact we just proved. If we once again consider an n-sheeted covering $\tilde{X}$ of X, we get a fibration with discrete fiber F

$$F \longrightarrow \tilde{X} \longrightarrow X \ .$$

The Serre Spectral sequence says that we have the following first quadrant homological spectral sequence

$$E^2_{p,q} \cong H_p(X; (\mathcal{H}_q(F))) \Longrightarrow H_{p+q}(\tilde{X})$$

In the case of a discrete fiber the only non-zero row of the E^2 page is the 0th row, so all the differentials on the E_2 page and on every other page are 0 and we have

$$H_p(\tilde{X}) \cong H_p(X; \mathcal{H}_0(F)).$$

Since $\mathcal{H}_0$ is locally $\mathbb{Z}^n$ and this local system corresponds to a sheaf on X we can compute sheaf homology of X and get something along the lines of

$$H_p(X; \mathcal{H}_0) \cong nH_p(X; \mathbb{Z})$$