Twisted Coefficients

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July 11, 2024

CW structure of a sphere with two cells in each dimension.

Consider $S^2 \subset \mathbb{R}^3$.

2 cells.

$$e^2_+ \to S^2$$
$$(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$$

$$e^2_- \to S^2$$
$$(x, y) \mapsto (x, y - \sqrt{1 - x^2 - y^2})$$

1 cells.

$$e^1_+ \to S^2$$
$$(x) \mapsto (x, \sqrt{1 - x^2}, 0)$$

$$e^1_- \to S^2$$
$$(x) \mapsto (x, -\sqrt{1 - x^2}, 0)$$

0 cells.

e^0_+ just gets mapped to $(1, 0, 0)$, and e^0_- gets mapped to $(-1, 0, 0)$.

General case.

The pattern here is relatively nice. We can construct $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots x_{n+1}^2 = 1\}$ inductively by mapping in the 0,1,2 cells as described above. In general, when mapping the two k-cells e^k_- and e^l_+ in each dimension $k \leq n$, the maps into $\mathbb{R}^{n+1}$ are:

$$e^k_+ \to S^n$$
$$(x_1, \ldots x_k) \mapsto (x, \ldots, x_k, \sqrt{1 - x^2 - \cdots - x_k}, 0, \ldots, 0)$$

$$e^k_- \to S^n$$
$$(x_1, \ldots, x_k) \mapsto (x, \ldots, x_k, -\sqrt{1 - x^2 - \cdots - x^k}, 0 \ldots, 0)$$

Homology of S^n with two cells in each dimension.

When we consider the CW structure of the n-sphere, constructed in the above section, $S^n = e^0_+ \cup e^0_- \cup \cdots \cup e^n +$ $- \cup e^n_-$, the chain complex looks like this:

$$0 \xrightarrow{\partial_{n+1}=0} \mathbb{Z}^2 \xrightarrow{\partial_n} \mathbb{Z}^2 \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0=0} 0$$

Let's look at the boundary maps for the low dimensional case. When we consider the mapping $e^1_+ \to S^2$ given by $x \mapsto (x, \sqrt{1-x^2}, 0)$:

$$-1 \mapsto (-1,0,0) = e^0_-$$
$$1 \mapsto (1,0,0) = e^0_+$$

Therefore $\partial e^1_+ = e^0_+ - e^0_-$. The other 1-cell behaves similarly:

$$e^1_- \to S^2$$
$$(x) \mapsto (x, -\sqrt{1-x^2}, 0)$$
$$-1 \mapsto (-1,0,0) = e^0_-$$
$$1 \mapsto (1,0,0) = e^0_+$$
$$\partial e^1_+ = e^0_+ - e^0_-$$

Boundary map from 2-cells.

$$e^2_+ \to S^2$$
$$(x,y) \mapsto (x, y, \sqrt{1-x^2-y^2})$$

The boundary of the disk $e^2_+ = S^1$, so we only want to plug in points x, y such that $x^2 + y^2 = 1$, and moreover $\sqrt{1-x^2-y^2}$ disappears.

Hold on.

When we map a k-cell, using the mapping $(x_1, \ldots, x_k) \mapsto (x, \ldots, x_k, \pm\sqrt{1-x^2-\cdots-x^k}, 0\ldots, 0)$ and we map the boundary of the k-cell, i.e. all points $x_1, \ldots, x_k$ such that $x_1^2 + \cdots + x_k^2 = 1$, which is the same thing as $-x_1^2 - \cdots - x_k^2 = -1$, then the term $\pm\sqrt{1-x^2-\cdots-x^k}$ just goes to 0, and so $\partial e^k_+ = \partial e^k_- =$ something.

I mean it's kind of obvious that this something is $e^{k-1}_+ - e^{k-1}_-$. Which is good news, because this means the differential is just the matrix $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and indeed $\partial^2 = 0$.

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Also

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ -x-y \end{bmatrix} = \{(a,-a) \mid a \in \mathbb{Z}\} \cong \mathbb{Z}$$

which gives us the correct homology groups $H_n(S^n) = H_0(S^n) \cong \mathbb{Z}$.

Homology of $\mathbb{RP}^n$

We can give $\mathbb{RP}^n$ a cell structure with one cell in each dimension.

$$\mathbb{RP}^n = e^1 \cup e^2 \cup \cdots \cup e^n$$

Therefore we get the following chain complex,

$$\mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_n} \cdots \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

To compute d_k, we can use cellular boundary formula, so we have to compute the degree of the composition

$$\partial e^k \cong S^{k-1} \longrightarrow \mathbb{RP}^{n-1} \longrightarrow \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} = S^{k-1}$$

I think the idea is that the $S^{k-1} - S^{k-2} = D_-^{k-1} \cup D_+^{k-1}$ and $\mathbb{RP}^{k-1} - \mathbb{RP}^{k-2} = D^{k-1}$, so the composition from above restricts to a homeomorphism on each hemisphere D_-^{k-1}, D_+^{k-1} such that these homeomorphisms differ by precomposition with the antipodal map of the sphere S^{k-1}, which is $(-1)^l$. Then using some local degree argument, the degree of the composition from above, is $\deg(1) + \deg(a \circ 1) = 1 + (-1)^k$, so when k is even $d_k = 2$ and when k is odd $d_k = 0$.

Twisted homology of $\mathbb{RP}^n$

The chain complex we are considering is

$$\mathbb{Z}^2 \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathbb{Z}^\omega \longrightarrow \cdots \longrightarrow \mathbb{Z}^2 \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathbb{Z}^\omega \longrightarrow 0$$

The chain complex of the universal cover $C_k(\tilde{X}) = C_k(S^n) \cong \mathbb{Z}^2$ has two generators fore every dimension k given by the two k-cells e_-^k, e_+^k in the CW decomposition of S^n. By the boundary maps calculated in the previous section, we have

$$\partial_k e_+^k = \partial_k e_+^k = e_+^{k-1} - e_-^{k-1}$$

The trivial element in $\pi_1(\mathbb{RP}^n$ acts trivially on both sides of $\mathbb{Z}^2 \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathbb{Z}^\omega$. Let $\gamma \in \pi_1(\mathbb{RP}^n)$ be the non-trivial element. It acts on S^2 as the antipodal map, and on right side by flipping sign. Therefore for any $q, j, l \in \mathbb{Z}$, when we identify this action we get:

$$\gamma(q\langle e_+^k\rangle + j\langle e_-^k\rangle) \otimes l = (q\langle e_-^k\rangle + j\langle e_+^k\rangle) \otimes l = (q\langle e_+^k\rangle + j\langle e_-^k\rangle) \otimes \gamma l = (q\langle e_+^k\rangle + j\langle e_-^k\rangle) \otimes -l = (-q\langle e_+^k\rangle - j\langle e_-^k\rangle) \otimes l$$

$$(q\langle e_-^k\rangle + j\langle e_+^k\rangle) \otimes l = (-q\langle e_+^k\rangle - j\langle e_-^k\rangle) \otimes l$$

From this, we see that

$$\langle e_-^k\rangle \otimes 1 = -\langle e_+^k\rangle \otimes 1$$
$$\langle e_+^k\rangle \otimes 1 = -\langle e_-^k\rangle \otimes 1$$

Therefore the tensored chain complex has the following form

$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the differentials are given by $\partial_k \otimes 1$. Now

$$(\partial_k \otimes 1)(\langle e_+^k\rangle \otimes 1) = (e_+^{k-1} \otimes 1) + (e_-^{k-1} \otimes 1) = (e_+^{k-1} \otimes 1) - (e_+^{k-1} \otimes 1) = 0$$

Therefore all the differentials in this chain complex are just 0's and the homology is just $\mathbb{Z}$ in every dimension.

Homology groups of Grassmannian

Gr(4,2)

We know that $\mathrm{Gr}_2(\mathbb{R}^4)$ is a closed orientable 4-manifold, therefore $H_0(\mathrm{Gr}_2(\mathbb{R}^4)) = H_4(\mathrm{Gr}_2(\mathbb{R}^4)) \cong \mathbb{Z}$. $\mathrm{Gr}_2(\mathbb{R}^4)$ can be given a cell structure with one 0-dimensional cell, one 1-dimensional cell, two 2-dimensional cells, one 3-dimensional cell and one 4-dimensional cell, therefore the CW chain complex is:

$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

Fundamental group of Gr(4,2)

As described below, there is a natural projection $q : V_2(\mathbb{R}^4) \to \mathrm{Gr}_2(\mathbb{R}^4)$, which has fiber $O(2)$. Therefore, we have a long exact sequence of homotopy groups;

$$\pi_3(\mathrm{Gr}_2(\mathbb{R}^4)) \longrightarrow \pi_2(O(2)) \longrightarrow \pi_2(V_2(\mathbb{R})^4) \longrightarrow \pi_2(Gr_2(\mathbb{R}^4)) \longrightarrow \pi_1(O(2))$$

$$\longrightarrow \pi_1(V_2(\mathbb{R})^4) \longrightarrow \pi_1(Gr_2(\mathbb{R}^4)) \longrightarrow \pi_0(O(2)) \longrightarrow \pi_0(V_2(\mathbb{R})^4) \longrightarrow \pi_0(Gr_2(\mathbb{R}^4)).$$

The Stiefel manifolds fit into fibrations

$$V_{k-1}(\mathbb{R}^{n-1}) \longrightarrow V_k(\mathbb{R}^n) \longrightarrow S^{n-1}$$

In our case we have a fibration,

$$V_1(\mathbb{R}^3) \longrightarrow V_2(\mathbb{R}^4) \longrightarrow S^3$$

But note that the set of orthonormal 1-frames in $\mathbb{R}^3$ is precisely the same thing as S^2, therefore $V_1(\mathbb{R}^3) \cong S^2$ and we have:

$$S^2 \longrightarrow V_2(\mathbb{R}^4) \longrightarrow S^3$$

The long exact sequence of homotopy groups gives us:

$$\pi_3(S^3) \longrightarrow \pi_2(S^2) \longrightarrow \pi_2(V_2(\mathbb{R}^4)) \longrightarrow \pi_2(S^3) \longrightarrow \pi_1(S^2) \longrightarrow \pi_1(V_2(\mathbb{R}^2)) \longrightarrow \pi_1(S^3)$$

which can be reduced to

$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \pi_2(V_2(\mathbb{R}^4)) \longrightarrow 0 \longrightarrow 0 \longrightarrow \pi_1(V_2(\mathbb{R}^2)) \longrightarrow 0$$

Therefore, $\pi_1(V_2(\mathbb{R}^4)) \cong 0$ and $\pi_2(V_2(\mathbb{R}^4))$ is a quotient of $\mathbb{Z}$. Therefore by this long exact sequence of homotopy groups, we see that $\pi_1(\mathrm{Gr}_2(\mathbb{R}^4)) \cong \mathbb{Z}_2$.

$$\pi_3(\mathrm{Gr}_2(\mathbb{R}^4)) \longrightarrow \pi_2(O(2) \longrightarrow \pi_2(V_2(\mathbb{R}^4)) \longrightarrow \pi_2(Gr_2(\mathbb{R}^4)) \longrightarrow \pi_1(O(2))$$

$$\longrightarrow \pi_1(V_2(\mathbb{R}^4)) \cong 0 \longrightarrow \pi_1(Gr_2(\mathbb{R}^4)) \longrightarrow \pi_0(O(2)) \cong \mathbb{Z}_2 \longrightarrow 0$$

Another way?

We can look at the way that we attach cells to Gr(1,3) or something else to obtain Gr(2,4) and show that the process of attaching cells leaves the fundamental group the same.

Gr(1,3) is homeomorphic to $\mathbb{RP}^2$ and has the same cell structure, a cell in each dimension. We can obtain Gr(4,2) from this by attaching a 2-cell, 3-cell, 4-cell. The 3-cell and 4-cell, I think are not attached to the 1-cell, so will not have any effect on π_1. How is the 2-cell attached? I have no bloody idea.

Homology of Gr(4,2)

Since $\pi_1(\mathrm{Gr}_2(\mathbb{R}^4)) \cong \mathbb{Z}_2$, then $H_1(\mathrm{Gr}_2(\mathbb{R}^4)) \cong \mathbb{Z}_2$. By Poincare duality $H^3(\mathrm{Gr}_2(\mathbb{R}^4)) \cong \mathbb{Z}_2$. By the universal coefficient theorem, there is a short exact sequence that splits:

$$0 \longrightarrow Ext(H_2(\mathrm{Gr}_2(\mathbb{R}^4);\mathbb{Z}),\mathbb{Z}) \longrightarrow H^3(\mathrm{Gr}_2(\mathbb{R}^4);\mathbb{Z}) \longrightarrow Hom(H_3(\mathrm{Gr}_2(\mathbb{R}^4);\mathbb{Z}),\mathbb{Z}) \longrightarrow 0$$

Therefore,

$$\mathbb{Z}_2 \cong Hom(H_3(\mathrm{Gr}_2(\mathbb{R}^4);\mathbb{Z}),\mathbb{Z}) \oplus Ext(H_2(\mathrm{Gr}_2(\mathbb{R}^4);\mathbb{Z}),\mathbb{Z})$$

We can now use the fact that $Ext(H;\mathbb{Z})$ is isomorphic to the torsion subgroup of H if H is finitely generated and $Hom(H;\mathbb{Z})$ is isomorphic to the free part of H. Therefore the torsion subgroup of H_2 is isomorphic to $\mathbb{Z}_2$ and the free part of H_3 is isomorphic to 0. Furthermore notice that $\chi(\mathrm{Gr}_2(\mathbb{R}^4)) = 1 - 1 + 2 - 1 + 1 = 2$. The Euler Characteristic of a manifold M is also

$$\chi(\mathrm{Gr}_2(\mathbb{R}^4)) = \sum (-1)^n \ \mathrm{rank}(H_n) = 2$$

Using the UCT, we just determined that $\mathrm{rank}(H_3) = 0$, so using the Euler Characteristic

$$\chi(\mathrm{Gr}_2(\mathbb{R}^4)) = 1 - (0) + (x) - (0) + 1 = 2$$

wee see that $\mathrm{rank}(H_2) = 0$ and the homology groups of $\mathrm{Gr}_2(\mathbb{R}^4))$ are.

$$H_0(\mathrm{Gr}_2(\mathbb{R}^4)) = H_4(\mathrm{Gr}_2(\mathbb{R}^4)) \cong \mathbb{Z}$$
$$H_1(\mathrm{Gr}_2(\mathbb{R}^4)) = H_2(\mathrm{Gr}_2(\mathbb{R}^4)) \cong \mathbb{Z}_2$$
$$H_3(\mathrm{Gr}_2(\mathbb{R}^4)) \cong 0$$

The last statement follows from the fact that

$$H^4 \cong H_0 \cong Ext(H_3,\mathbb{Z}) \oplus Hom(H_4,\mathbb{Z}) \cong \mathbb{Z}$$

Which implies that H_3 has no torsion and that in fact $H_3 \cong 0$.

Oriented Grassmannian.

Definition. The oriented Grassmannian $\mathrm{Gr}^+(n,k)$ is the set of oriented k-dimensional subspaces through the origin of $\mathbb{R}^n$.

Claim. The Grassmannian admits a double cover

$$\mathrm{Gr}^+(n,k) \to \mathrm{Gr}(n,k)$$

Proof. No idea.

Claim. The oriented Grassmannian is simply connected, given that n, k...

Proof. No idea.

Corollary $\pi_1(\mathrm{Gr}(n,k)) \cong \mathbb{Z}_2$.

Gr(2,5)

Cell structure of Gr(2,5):

$$Gr(2,5) = e^0 \cup e^1 \cup e^2_a \cup e^2_b \cup e^3_a \cup e^3_b \cup e^4_a \cup e^4_b \cup e^5 \cup e^6$$

Therefore the Euler characteristic of Gr(2,5) is

$$\chi(\mathrm{Gr}_2(\mathbb{R}^5)) = 1 - 1 + 2 - 2 + 2 - 1 + 1 = 2$$

Using a similar argument to last time we know that $H_0(\mathrm{Gr}_2(\mathbb{R}^5)) \cong 0, H_0(\mathrm{Gr}_2(\mathbb{R}^5)) \cong \mathbb{Z}_2$.

Since Gr(2,5) is non-orientable, we cannot use Poincare duality like last time, let's pass on to the universal cover and work there, so that we can compute the homology with twisted coefficients.

The universal cover is the oriented Grassmannian, which is simply connected as shown above. So $H_0(\mathrm{Gr}_2^+(\mathbb{R}^5)) = H_6(\mathrm{Gr}_2^+(\mathbb{R}^5)) \cong \mathbb{Z}$ and $H_1(\mathrm{Gr}_2^+(\mathbb{R}^5)) \cong 0$. It also follows from the section above that

$$\pi_2(V_2(\mathbb{R}^5)) \cong 0 \Longrightarrow \pi_2(\mathrm{Gr}_2^+(\mathbb{R}^5)) \cong \pi_1(SO(2)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Therefore by the Hurewicz Theorem $H_2(\mathrm{Gr}_2^+(\mathbb{R}^5)) \cong \mathbb{Z}$.

We also know what the Euler Characterstic is:

$$\chi(\mathrm{Gr}_2^+(\mathbb{R}^5)) = 4$$

$$4 = \sum(-1)^n \mathrm{rank}(H_n)$$
$$4 = (1) - (0) + (1) - (a) + (b) - (c) + (1)$$

Using Poincare duality and UCT we know that

$$H_1 \cong H^5 \cong Ext(H_4) \oplus Hom(H_5) \cong 0$$

So, H_4 has no torsion and $\mathrm{rank}(H_5) = 0 \Longrightarrow c = 0$.

Using Poincare duality and UCT again, we have:

$$H_2 \cong H^4 \cong Ext(H_3) \oplus Hom(H_4) \cong \mathbb{Z}$$

So, H_3 has no torsion and $\mathrm{rank}(H_4) = 1 \Longrightarrow H_4 \cong \mathbb{Z} \Longrightarrow b = 1$.

Moreover, since

$$4 = (1) - (0) + (1) - (a) + (b = 1) - (c = 0) + (1)$$
$$a = 0$$
$$\mathrm{rank}(H_3) = 0$$

and since $Ext(H_2, \mathbb{Z})$, this implies that $H_3 \cong 0$.

Additionally $H_5 \cong 0$, because

$$H^6 \cong H_0 \cong \mathbb{Z} \cong Ext(H_5, \mathbb{Z}) \oplus Hom(H_6, \mathbb{Z})$$

So H_5 has no torsion and $H_5 \cong 0$. The final homology groups of $\mathrm{Gr}_2^+(\mathbb{R}^5)$ are:

$$H_0 \cong H_2 \cong H_4 \cong H_6 \cong \mathbb{Z}$$
$$H_1 \cong H_3 \cong H_5 \cong 0$$

Back to the homology of Gr(5,2)

Since $\mathrm{Gr}_2(\mathbb{R}^5)$ is non-orientable and path-connected, $H_6 \cong 0$, and $H_0 \cong 1$. Additionally, for the same reasons as before $H_1 \cong \mathbb{Z}_2$. The Euler Characteristic of $\mathrm{Gr}_2(\mathbb{R}^5)$ is 2, therefore

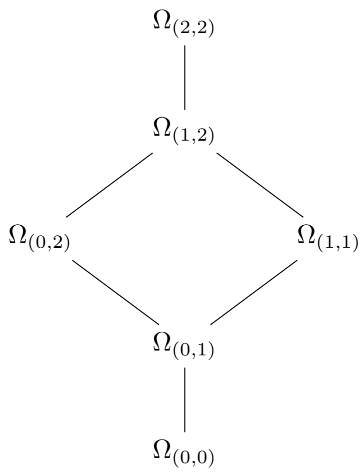
$$2 = (1) - (0) + (b_2) - (b_3) + (b_4) - (b_5) + (0)$$

Homology of Gr(5,2)

$$H_0 \cong \mathbb{Z}$$
$$H_1 \cong \mathbb{Z}_2$$
$$H_2 \cong \mathbb{Z}_2$$
$$H_3 \cong \mathbb{Z}_2$$

$$H_4 \cong \mathbb{Z}$$
$$H_5 \cong \mathbb{Z}_@$$
$$H_6 \cong 0$$

I want to work out the homology though. I'll include these Hasse diagrams to see how the cells sit inside of each other. The first one is the Hasse diagram for Gr(4,2) and the second one is the Hasse diagram for Gr(5,2). Note that the indices might be flipped, because I'm pretty sure I messed up dimension and codimension when writing that paper.



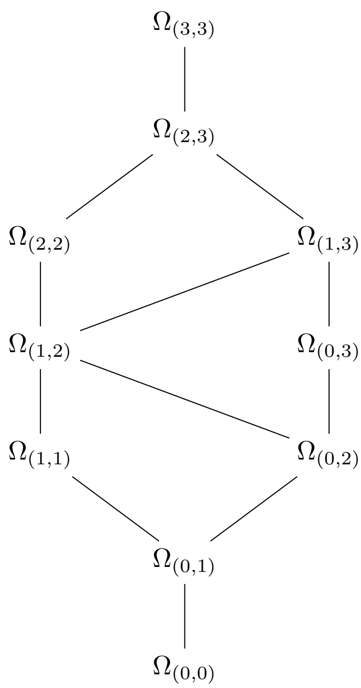
The chain complex was

$$\mathbb{Z} \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

We want the homology to be $H_0 = H_4 \cong \mathbb{Z}$ and $H_1 = H_2 = \mathbb{Z}_2$ and $H_3 = 0$. Therefore

$$\ker \partial_4 = \mathbb{Z} \Longrightarrow \partial_4 = 0$$
$$\ker \partial_3 = 0 \Longrightarrow \text{Im } \partial_3 = \mathbb{Z} \Longrightarrow$$
$$\partial_0 = 0 \Longrightarrow \partial_1 = 0 \Longrightarrow \text{Im } \partial_2 = 2\mathbb{Z}$$

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z}^2 \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$



Another way?

We can look at the cells (I'll denote cells using Schubert Symbols) in a few Grassmannians and try to relate their attaching maps.

$$\mathrm{Gr}_2(\mathbb{R}^3) = [(1,2)]_0 \cup [(1,3)]_1 \cup [(2,3)]_2$$
$$\mathrm{Gr}_1(\mathbb{R}^3) = [(1)]_0 \cup [(2)]_1 \cup [(3)]_2$$
$$\mathrm{Gr}_2(\mathbb{R}^4) = [(1,2)]_0 \cup [(1,3)]_1 \cup [(2,3) \cup (1,4)]_2 \cup [(2,4)]_3 \cup [(3,4)]_4$$

The idea is that I know that $\mathrm{Gr}_1(\mathbb{R}^3) \cong \mathrm{Gr}_2(\mathbb{R}^3) \cong \mathbb{RP}^2$, so I know how the 2-dimensional cells are attached. Therefore, if I identify these cases with $\mathrm{Gr}_2(\mathbb{R}^4)$, then I should be able to figure out how the cells there are attached.

Claim. $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is isomorphic to $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ as CW complexes.

Proof. I think the idea is to use the fact that $\rho : \mathrm{Gr}_k(\mathbb{R}^{n+k}) \to \mathrm{Gr}_n(\mathbb{R}^{n+k})$ which maps a k-plane to its perpendicular n-plane is a diffeomorphism, but to construct a cell map. Let $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \cdots \subset \mathbb{R}^{n+k}$ be the sequence of spaces which we use to obtain a cell structure on $\mathrm{Gr}_k(\mathbb{R}^{n+k})$. To obtain a cell structure on $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ use the sequence of spaces $(\mathbb{R}^{n+k})^\perp \subset (\mathbb{R}^{n+k-1})^\perp \subset \cdots \subset (\mathbb{R})^\perp$. We want to show that ρ sends a cell from one Grassmannian to another.

If $\dim(X \cap \mathbb{R}^m) = j$, then $\dim(X^\perp \cap (\mathbb{R}^m)^\perp) = k + j - m$.

Attaching maps for the Grassmannian

Definition. The upper half plane $\mathbb{H}^k \subset \mathbb{R}^n$ is the set of points $(x_1, \ldots, x_k)$ such that $x_1, \ldots x_k > 0$.

Claim. An n-plane X belongs to $e(\lambda)$ iff it possess a basis $x_1, \ldots x_n$ such that $x_1 \in \mathbb{H}^{\lambda_1}, \ldots \mathbb{H}^{\lambda_n}$.

Claim. Each n-plane $X \in e(\lambda)$ possesses a unique orthonormal basis $(x_1, \ldots x_n)$ which belongs to $\mathbb{H}^{\lambda_1} \times \cdots \times \mathbb{H}^{\lambda_n}$.

Proof. The basis vector x_1 has to lie in the 1-dimensional vector space $(X \cap \mathbb{R}^{\lambda_1})$ and be a unit vector. This gives us two possibilities for x_1, but the fact that the λ_1-th coordinate has to be positive, enforces this decision on us. Again, x_2 lies in the 2-dimensional vector space $(X \cap \mathbb{R}^{\lambda_2})$, and the fact that it has to be positive and orthogonal to x_1 leaves us with only one candidate. Repeating this procedure for all other x_i, we have obtained a unique basis $(x_1, \ldots x_n)$.

Definition. $V_n^0(\mathbb{R}^m)$ denotes the subset of $V_n(\mathbb{R}^m)$ of all orthonormal n-frames.

Definition. Let $e'(\lambda) = V_n^0(\mathbb{R}^m) \cap (\mathbb{H}^{\lambda_i} \times \cdots \times \mathbb{H}^{\lambda_n})$ denote the set of all orthonormal n-frames $(x_1, \ldots x_n)$ such that each $x_i \in \mathbb{H}^{\lambda_i}$. Let $\overline{e'}(\lambda)$ denote the set of orthonormal n-frames $(x_1, \ldots x_n)$ such that each $x_i \in \overline{\mathbb{H}}^{\lambda_i}$.

Claim. $\overline{e'}(\lambda)$ is a closed cell of dimension $d(\lambda) = (\lambda_1 - 1) + \cdots + (\lambda_n - n)$ with interior $e'(\lambda)$. Furthermore q maps the interior $e'(\lambda)$ homeomorphically onto $e(\lambda)$, therefore $e(\lambda)$ is an open cell of dimension $d(\lambda)$.

$$q\mid_{\overline{e'}(\lambda)} : \overline{e'}(\lambda) \to \mathrm{Gr}_n(\mathbb{R}^m)$$

will serve as a characteristic map.

Example.

Let's take the the two dimensional cell of $\mathrm{Gr}_2(\mathbb{R}^4)$ indexed by the Schubert symbol $\lambda = (2,3)$. By the previous claims $e(2,3)$ posses a unique orthonormal bases $(x_1, x_2) \in \mathbb{H}^2 \times \mathbb{H}^3$.

Now, $\overline{e'}(2,3)$ is the set of orthonormal 2-frames $(x_1, x_2) \in \overline{\mathbb{H}}^2 \times \overline{\mathbb{H}}^3$. More specifically,

$$x_1 = (0, 0, a, b), \text{ where } a, b \geq 0, \text{ and where } a^2 + b^2 = 1$$
$$x_2 = (0, c, d, e), \text{ where } c, d, e \geq 0, \text{ and where } c^2 + d^2 + e^2 = 1$$
$$\text{Additionally we want } ad + be = 0$$

Stiefel Manifolds.

This section is devoted to understanding the homotopy groups of Stiefel Manifolds.

Orientability of Grassmannian.

Definition. The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is the set of k-dimensional subspaces through the origin of $\mathbb{R}^{n+k}$.

Claim. The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ can be topologized.

Proof. The collection of all k-tuples of linearly independent vectors in $\mathbb{R}^{n+k}$ is an open subset of the k-fold Cartesian product $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$, which is a manifold called the Stiefel Manifold $V_k(\mathbb{R}^{n+k})$. We can give $\mathrm{Gr}_k(\mathbb{R}^{n+k})$, the quotient topology, by considering the map $q : V_k(\mathbb{R}^{n+k}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k})$, which maps each n-tuple of linearly independent vectors, to the subspace that it spans. So, a subset $U \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ is open iff $q^{-1}(U)$ is open.

Claim. The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is a compact topological manifold of dimension nk.

Proof.

Definition. The canonical bundle $\gamma(\mathbb{R}^{n+k})$ over $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ has total space E consisting of pairs all pairs (V, v), where $V \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ and v a vector in this n-plane. This can be topologized as a subset of $\mathrm{Gr}_k(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$. The projection map $\pi : \gamma(\mathbb{R}^{n+k}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k})$ is given by $\pi(V, v) = V$. The structure of the fiber over a point p, is given by the vector space structure of p.

Definition. Let V be a subspace of $\mathbb{R}^n$. Then the orthogonal complement $V^\perp$ is the set of all vectors in $\mathbb{R}^n$ which are orthogonal to every vector in V.

Claim. When V is a subspace of $\mathbb{R}^n$ then $V \oplus V^\perp = \mathbb{R}^n$.

Proof.

Claim. Let $V \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$, then the subset $U \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ consisting of all k-planes W such $V^\perp \cap W = \{0\}$ is an open subset of $\mathrm{Gr}_k(\mathbb{R}^{n+k})$.

Proof. We just need to show that $q^{-1}(U)$ is open in $V_k(\mathbb{R}^{n+k})$.

Claim. If a vector ξ has a Euclidean metric then it is isomorphic to its dual bundle.

then a neighborhood U of x,

Claim. $\gamma(\mathbb{R}^{n+k}$ is a vector bundle.

Axioms for Stiefel-Whitney classes.

1. To every vector bundle ξ with base space B, there corresponds cohomology classes

$$w_i(\xi) \in H^i(B; \mathbb{Z}_2)$$

   called the Stiefel-Whitney classes of ξ.

$$w_0(\xi) = 1 \in H^0(B; \mathbb{Z}_2)$$

2. For $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then

$$w_i(\xi) = f^* w_i(\eta).$$

3. If ξ and η are vector bundles over the same base space, then
4.

The tangent bundle of $Gr(k, n)$ is $TGr(k, n) = \mathrm{Hom}(\gamma, \gamma^\perp) \simeq \gamma^* \otimes \gamma^\perp \simeq \gamma \otimes \gamma^\perp$.

Since $\gamma \oplus \gamma^\perp = \epsilon$, $\omega_1(\gamma) = \omega_1(\gamma^\perp)$. Therefore,

$$\begin{aligned}\omega_1(\omega \otimes \omega^\perp) &= \mathrm{rank}(\gamma)\omega_1(\gamma) + \mathrm{rank}(\gamma^\perp)\omega_1(\gamma) \\ &= k\omega_1(\gamma) + (n-k)\omega_1(\gamma) = n\omega_1(\gamma)\end{aligned}$$

I'm pretty sure that $\omega_1(\gamma)$ is non-zero, so $n\omega_1(\gamma) = 0$ iff n is even.