Some Elementary Fixed Point Theorems

Theorem. Every continuous map $f : D^2 \to D^2$ has a fixed point.

Proof. We can proceed by contradiction. Suppose that $f(x) \neq x, \forall x \in D^2$, then we can define a new function $g(x) : D^2 \to S^1$ to be the function that assigns a point on the boundary of D^2 to the ray starting at $f(x) \in D^2$ and passing through x. The map g is actually a retraction, since $g(x) = x$ if x is a point on the boundary of D^2. But no such retraction can exist since the map of fundamental groups induced by the inclusion of the boundary $\pi_1(S^1) \to \pi_1(D^2)$ is most definitely not invective. Thus every continuous map $f : D^2 \to D^2$ has a fixed point.

In the next section we will show how to extend these ideas to show which spaces have a similar fixed point property.

The Lefschetz Fixed Point Theorem

For any map $f : X \to X$ of a finite CW complex, define the Lefschetz number to be

$$\Lambda(f) = \sum_i (-1)^i \mathrm{tr}(f_* : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q})).$$

Note that if f is homotopic to the identity, we simply have $\Lambda(f) = \chi(X)$.

Theorem. If X is a finite simplicial complex or a retraction of a finite simplicial complex, then any continuous map $f : X \to X$ with $\Lambda(f) \neq 0$ has a fixed point.

This theorem is quite useful, because as it turns every compact, locally contractible space which can be embedded in $\mathbb{R}^n$ for some n is a retract of a finite simplicial complex. Examples of these spaces are finite CW complexes and compact manifolds.

Theorem. Every continuous map $f : D^n \to D^n$ has a fixed point.

Proof. The only non-zero homology group of the disk D^n is the 0th homology group $H_0(D^n) \cong \mathbb{Z}$. For a continuous map f between path-connected spaces, the induced map on H_0 is the identity, since any two points belong to the same class by path-connected because they are the boundary of some 1-simplex connecting them, and thus any continuous map will map the class of all points to the class of all points and be the identity. Since the trace of the identity map is 1, $\Lambda(f) = 1$ and so f must have a fixed point.

It immediately follows that for any continuous map $f : X \to X$ of a space X which has homology groups ismorphic to D^n mod torsion will have a fixed point. An example of such a space is $\mathbb{RP}^n$ for n even. Recall that the homology groups of $\mathbb{RP}^n$ for n even are:

$$H_k(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}_2 & \text{for } k \text{ odd, } 0 < k < n \\ 0 & \text{otherwise} \end{cases}$$

We can also prove a useful result about maps of spheres.

Theorem. A continuous map $f : S^n \to S^n$ has a fixed point unless its degree is equal to the degree of the antipodal map $x \mapsto -x$.

Proof. The only non-zero homology groups of S^n are $H_0(S^n) \cong H_n(S^n) \cong \mathbb{Z}$, so we only have to track the induced maps on these homology groups. As previously mentioned $f_* : H_0(S^n) \to H_0(S^n)$ is the identity, so this contributes $+1$ to $\Lambda(f)$. The contribution of $f_* : H_n(S^n) \to H_n(S^n)$ to $\Lambda(f)$ is given by the degree of f and note that the only time that $\Lambda(f) = 0$ is if $\deg(f) = -1$, i.e. when f is the antipodal map. By a previous remark we said that if $f : X \to X$ is homotopic to the identity, then $\Lambda(f) = \chi(X)$. Since $\chi(S^n) = 0$ when n is odd, the antipodal map is homotopic to the identity when n is odd.

We can also use the Lefschetz formula in other ways. We follow an example from Hatcher

Let X be the closed orientable surface of genus 3, and let $f : X \to X$ be the map which is a 180 degree rotation about a vertical axis passing through the central hole of X. Since f has no fixed points we should have $\Lambda(f) = 0$. As mentioned before, the induced map $f_* : H_0(X) \to H_0(X)$ is the identity and contributes 1 to $\Lambda(f)$. Now consider $H_1(X)$. There are six loops, which we will call $\alpha_{1,2,3}$ and $\beta_{1,2,3}$ that represent a basis for this group. Since the map f is a 180 degree rotation, it interchanges the homology classes of the loops which go around the left-side hole and the right-side hole. Let's say it interchanges loops α_1 and α_3, as well as loops β_1 and β_3. The homology classes of the two loops that go around the central whole gent sent to themselves, say β_2 is sent to itself and α_2 is sent to a loop α_2' which is clearly homologous to α_2. Therefore the induced map $f_* : H_1(X) \to H_1(X)$ contributes -2 to $\Lambda(f)$. Therefore, we can conclude that $f_* : H_2(X) \to H_2(X)$ is the identity map.

We can repeat the exercise from before for complex and quaternionic projective spaces, although this takes slightly more work.

Theorem. Every continuous map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ has a fixed point if n is even. When n is odd, there is a fixed point unless $f^*(\alpha) = -\alpha$ for α a generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$.

Proof. Note that $H_i(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}$ for i even and 0 otherwise. Therefore the alternating sum in the Lefschetz formula just becomes and ordinary sum and additionally by Poincare Duality we obtain

$$\Lambda(f) = \sum_{i=0}^{n/2} \mathrm{tr}(f^* : H^{2i}(\mathbb{CP}^n) \to H^{2i}(\mathbb{CP}^n))$$

We know the cup product structure on $\mathbb{CP}^n$.

$$H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$$

that is, if α is the generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$, then α^k is the generator of $H^{2k}(\mathbb{CP}^n; \mathbb{Z})$. Therefore if $f^*(\alpha) = j\alpha$ for some integer j, then

$$f^*(\alpha^k) = f^*(\alpha)^k = j^k \alpha^k$$

and moreover the Lefschetz number is now

$$\Lambda(f) = \sum_{i=0}^{n} j^i$$

It is not too difficult to prove that this sum is non-zero hence every such f has a fixed point. When n is odd and $j = -1$, that is $f^*(\alpha) = -\alpha$, then $\sum_{i=0}^{n} j^i = 0$.