Poisson Statistics

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Abstract

In this experiment, we aim to observe the Poisson distribution of radioactive decay using the Geiger-Muller tube setup. We use two different sources which are ^{137}Cs and ^{133}Ba and count the number of gamma rays emitted by the radioactive source at two different time intervals, 1 sec and 10 sec. By using this data, we study the Poisson and Gaussian distribution and find that while the Poisson distribution approaches the Gaussian as the mean value λ becomes large, Poisson is more suitable to use for random and discrete events such as radioactive decay.

1 Theory

Radioactive decay is a condition and natural process in which subatomic particles (protons and neutrons) decay due to the instability of the atom. This produces ionized radiation in the form of alpha particles, beta particles or gamma rays. The rate of disintegration can be measured on a logarithmic scale. A material containing such unstable nuclei is considered radioactive.

Radioactive decay is a random process at the level of single atoms. According to quantum theory, it is impossible to predict when a particular atom will decay, regardless of how long the atom has existed. Since particle physics is based on quantum mechanics, and quantum mechanics is based on the paradigm of a probabilistic description of nature, so we can say that the radioactive decay of particles is probabilistic according to quantum mechanics.

In radioactive decay, the emission of radiation depends on the number of atoms that can decay and a probability function that is characteristic their natural lifetimes. The detection of particles is random and any two measurements of the particles over equal periods of time will most likely be different. For large numbers, the difference between the measurements will be a small percentage. The probability of detecting a specific number of events for a given measurement is given by the standard normal or Gaussian distribution . However, for the measurement of a small number of events, the probability distribution for detecting a specific number of events is different and is given by a Poisson distribution. For rare events, the average number detected might also be much less than one, The Poisson distribution applies to these measurements and is useful for determining the probability of detecting a single event or more than one event in the same period. The Poisson distribution is a special case of the binomial distribution, similar to the Gaussian distribution being a special case.

Binomial to Poisson

As previously stated, Poisson distribution is a special case of the binomial where the number of trials is large, and the probability of success in any given one is small. We can prove that the Poisson distribution is just the binomial with n approaching infinity and p approaching zero. The binomial distribution works when we have a fixed number of events n, each with a constant probability of success p.

We define a number where n is the number of trials and p is the probability of success;

$$\lambda = np \tag{1}$$

We recall that binomial distribution is

$$B(p,n) = P(X=k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k} \tag{2}$$

From the equation 1, we can say

$$p = \frac{\lambda}{n} \tag{3}$$

We substitute this expression for p into the binomial distribution and take the limit as n goes to infinity

$$\lim_{n\to\infty} P(X=k) = \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} \tag{4}$$

$$\left(\frac{\lambda^k}{k!}\right) \lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n^k}\right) \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k} \tag{5}$$

We take the limit of right-hand side separately

$$\lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n^k}\right) \tag{6}$$

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots(n-k)(n-k-1)\ldots(1)}{(n-k)(n-k-1)\ldots(1)} \left(\frac{1}{n^k}\right) \tag{7}$$

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k} \tag{8}$$

$$\lim_{n\to\infty} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \left(\frac{n-2}{n}\right) \ldots \left(\frac{n-k+1}{n}\right) = 1 \tag{9}$$

Secondly we find limit of the term in the middle of equation 4

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \tag{10}$$

$$e = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \tag{11}$$

We define a number x as

$$x = -\frac{n}{\lambda} \tag{12}$$

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{x}\right)^{x(-\lambda)} = e^{-\lambda} \tag{13}$$

We find the limit of the last term of the equation 4

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k} = 1^{-k} = 1 \tag{14}$$

$$\left(\frac{\lambda^k}{k!}\right)\lim_{n\to\infty}\frac{n!}{(n-k)!}\left(\frac{1}{n^k}\right)\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k} = \frac{\lambda^k}{k!}1e^{-\lambda}1 \tag{15}$$

$$\left(\frac{\lambda^k}{k!}\right)\lim_{n\to\infty}\frac{n!}{(n-k)!}\left(\frac{1}{n^k}\right)\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k} = \frac{\lambda^k}{k!}1e^{-\lambda}1 \tag{16}$$

Equation 15 simplifies to the following which is equal to probability density function for the Poisson distribution

$$P(\lambda,k) = \left(\frac{\lambda^k e^{-\lambda}}{k!}\right) \tag{17}$$

Poisson to Gaussian

A limiting form of the Poisson distribution is the Gaussian distribution as the mean value λ is very large. Firstly we write Poisson as

$$P(\lambda,n) = \left(\frac{\lambda^n e^{-\lambda}}{n!}\right) \tag{18}$$

Let

$$x = n = \lambda(1+\delta) \tag{19}$$

where

$$\lambda \gg 1 \quad \text{and} \quad \delta \ll 1 \tag{20}$$

Since $\langle n\rangle = \lambda$, this means that we will also be concerned with large values of n, in which case the discrete P_n goes over to a continuous probability density function in the variable x. Using Stirling's formula for n!:

$$x! \to \sqrt{2\pi x}e^{-x}x^x \quad \text{as} \quad x\to\infty \tag{21}$$

$$p(x) = \frac{\lambda^{\lambda(1+\delta)}e^{-\lambda}}{\sqrt{2\pi}e^{-\lambda(1+\delta)}[\lambda(1+\delta)]^{\lambda(1+\delta)+1/2}} \tag{22}$$

$$= \frac{e^{\lambda\delta}(1+\delta)^{-\lambda(1+\delta)-1/2}}{\sqrt{2\pi\lambda}} \tag{23}$$

$$= \frac{e^{-\lambda\delta^2/2}}{\sqrt{2\pi\lambda}} \tag{24}$$

$$\delta = (x-\lambda)/\lambda \tag{25}$$

We find the following equation which is the Gaussian with the mean value or expectation as λ

$$p(x) = \frac{e^{-(x-\lambda)^2/(2\lambda)}}{\sqrt{2\pi\lambda}} \tag{26}$$

We can write it as the following where the σ is the standard deviation and equals to $\sqrt{\lambda}$

$$p(x) = \frac{1}{(2\pi)^{1/2}\sigma} e^{-(x-\lambda)^2/2\sigma^2} \tag{27}$$

2 Experiment

Apparatus

The Poisson Experiment setup consists of Geiger- Müller tube, Geiger-Müller counter, sample holder, various Gamma-Ray sources and Potentiometric recorder.

Figure 1: The Experiment Setup

The Geiger- Muller Tube:

A Geiger-Müller tube consists of a chamber filled with a gas mixture at a low pressure of about 0.1 atmosphere. The chamber contains two electrodes, between which there is a potential difference of several hundred volts. The walls of the tube are either metal or have their inside surface coated with a conducting material or a spiral wire to form the cathode, while the anode is a wire mounted axially in the centre of the chamber.

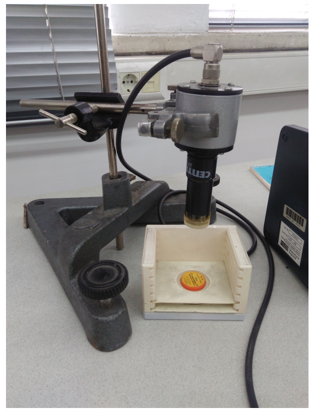

Figure 2: The Geiger-Müller Tube

It is the sensing element of the Geiger counter instrument used for the detection of ionizing radiation which is radiation that carries enough energy to detach electrons from atoms or molecules, thereby ionizing them. Ionizing radiation is made up of energetic subatomic particles, ions or atoms moving at high speeds, and electromagnetic waves on the high-energy end of the electromagnetic spectrum.

G-M tube is a gaseous ionization detector and uses the Townsend avalanche phenomenon, which is a gas ionisation process where free electrons are accelerated by an electric field, collide with gas molecules, and consequently free additional electrons, to produce an easily detectable electronic pulse from as little as a single ionising event due to a radiation particle. It is used for the detection of gamma radiation, X-rays, and alpha and beta particles. There are three basic types of gaseous ionization detectors; ionization chambers, proportional counters, Geiger-Müller tubes.

Procedure

At the beginning of the experiment we try to determine the operator voltage of the Geiger tube. We select a source and place it on the sample holder beneath the Geiger tube assembly. We set the counter to 100 sec, single mode and take the data at 20 V intervals starting from 300 V until 480 V.

We plot the number of the counts and select a voltage value somewhere on the plateau (flat region), beyond the point where the counts reach the approximately constant value.

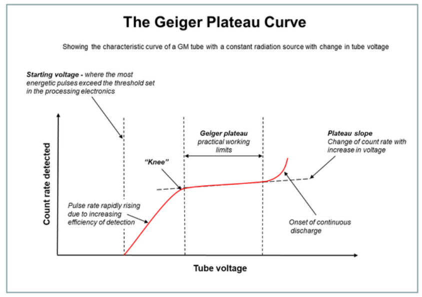

Figure 3

After that, we set the Geiger counter to the voltage determined as operator voltage and to continues mode with 10 sec interval. Using the same gamma- ray source as previous, we adjust the tray position to get about 100 counts and then record the number of counts 100 times for 10 sec intervals. Without moving the tray, we do the same steps for 1 sec intervals and take 100 data. We change the source and adjust the tray position and repeat the procedure for 10 sec and 1 sec intervals. In the end we have four sets of data.

In the second part of experiment, we try to obtain distribution of successive counts. To achieve this, we select a source then set the Potentiometric recorder and turn it on. We run it about 1 minutes and remove the portion of paper from the recorder. We measure the time interval between adjacent pulses.

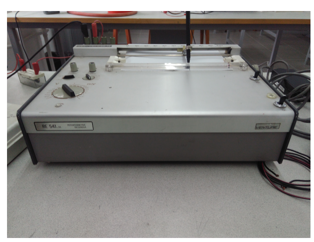

Figure 4: Potentiometric recorder

Data

We use ^{137}Cs source to determine the operator voltage. According to plot, 400 V is the starting point of plateau, so we determine 400 V as our operator voltage. Following is our voltage vs count graph with error bars

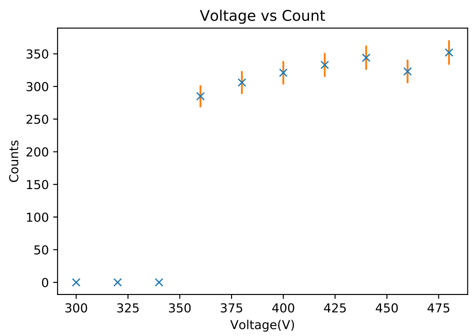

Figure 5: Voltage vs Count Graph with Error Bars

We use ^{137}Cs and ^{133}Ba sources to take two different time interval data. Following table is the part of our data

Cs-137 (10sec)	Cs-137 (1sec)	Ba-133 (10sec)	Ba-133 (1sec)
92	7	69	3
91	14	69	4
94	6	61	6
103	3	64	6
79	10	70	8

Table 1: Four Sets of Data with Two Different Sources

Following is the part of our data we take by measuring the time interval between adjacent pulses. And using it we calculate the time intervals for n =0 and n=1

	n=0	n=1
0	1.2	2.0
1.2	0.8	2.2
2.0	1.4	4.1
3.4	2.7	2.9
6.1	0.3	1.0

Table 2: Time interval between adjacent pulses, n=0 and n=1

3 Analysis

^{137}Cs for 1 sec interval:

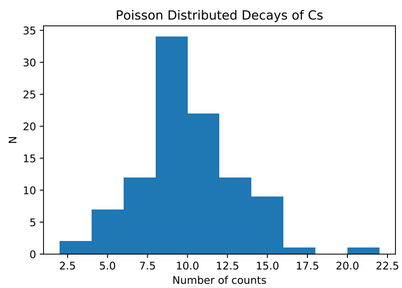

Figure 6

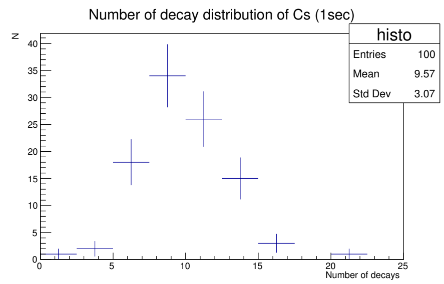

Figure 7

$\mu = 9.57$
$\sigma = 3.07$

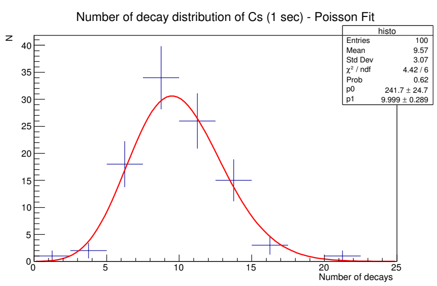

Figure 8

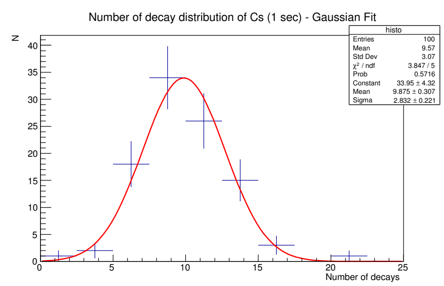

Figure 9

For Poisson fit:
$\chi^2/\text{ndf} = 4.42/6 = 0.7367$
$\mu \pm \sigma_\mu = 9.999 \pm 0.289$

For Gaussian fit:
$\chi^2/\text{ndf} = 3.847/5 = 0.7694$
$\mu \pm \sigma_\mu = 9.875 \pm 0.307$
$\sigma \pm \sigma_\sigma = 2.832 \pm 0.221$

^{137}Cs for 10 sec interval:

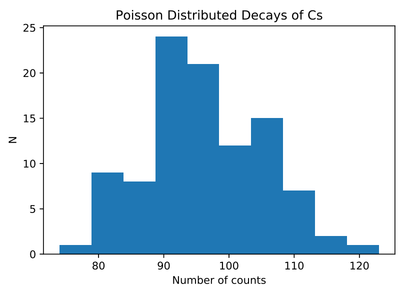

Figure 10

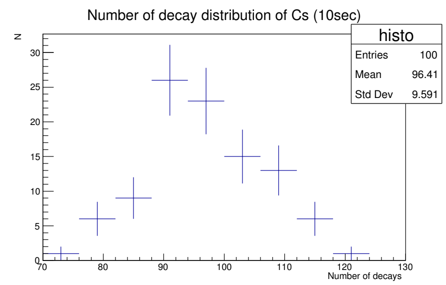

Figure 11

$\mu = 96.41$
$\sigma = 9.591$

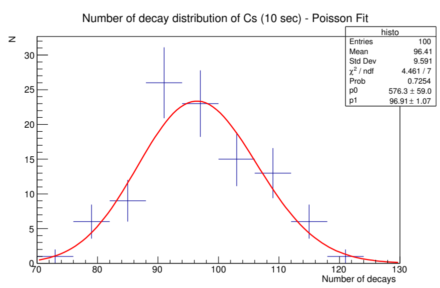

Figure 12

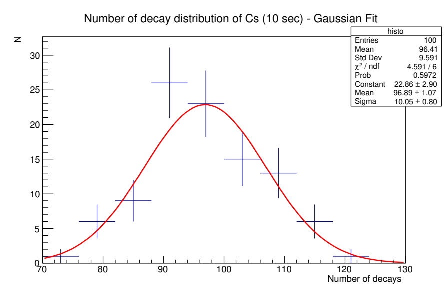

Figure 13

For Poisson fit:
$\chi^2/\text{ndf} = 4.461/7 = 0.6373$
$\mu \pm \sigma_\mu = 96.91 \pm 1.07$

For Gaussian fit:
$\chi^2/\text{ndf} = 4.591/6 = 0.7652$
$\mu \pm \sigma_\mu = 96.89 \pm 1.07$
$\sigma \pm \sigma_\sigma = 10.05 \pm 0.80$

^{133}Ba for 1 sec interval:

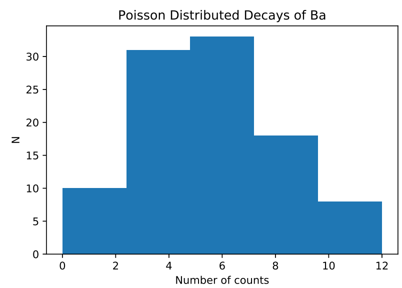

Figure 14

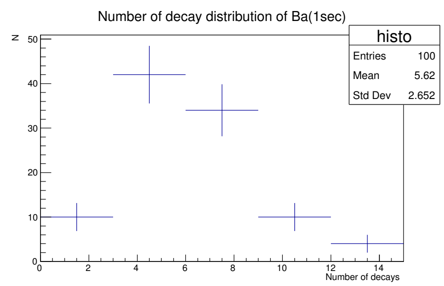

Figure 15

$\mu = 5.62$
$\sigma = 2.652$

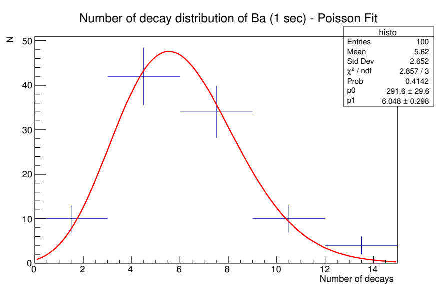

Figure 16

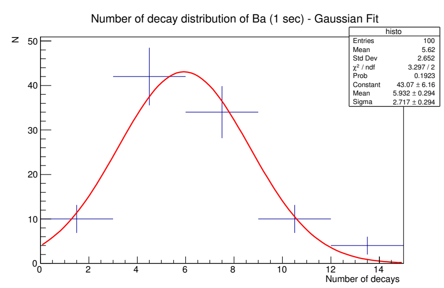

Figure 17

For Poisson fit:
$\chi^2/\text{ndf} = 2.857/3 = 0.9523$
$\mu \pm \sigma_\mu = 6.048 \pm 0.298$

For Gaussian fit:
$\chi^2/\text{ndf} = 3.297/2 = 1.648$
$\mu \pm \sigma_\mu = 5.932 \pm 0.294$
$\sigma \pm \sigma_\sigma = 2.717 \pm 0.294$

^{133}Ba for 10 sec interval:

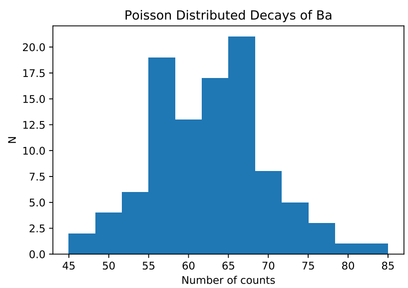

Figure 18

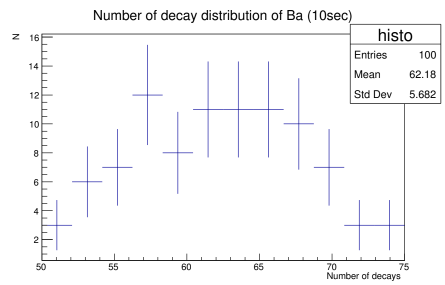

Figure 19

$\mu = 62.18$
$\sigma = 5.682$

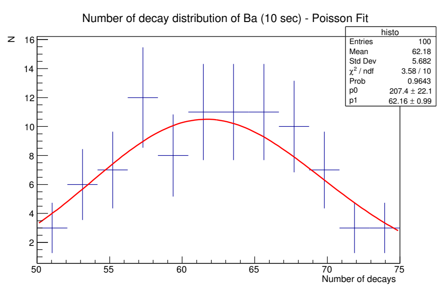

Figure 20

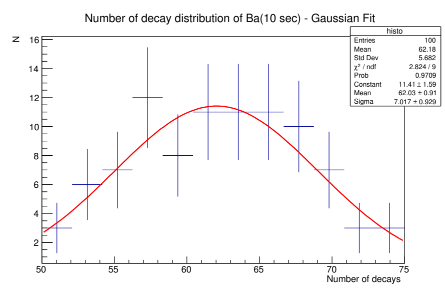

Figure 21

For Poisson fit:
$\chi^2/\text{ndf} = 3.58/10 = 0.358$
$\mu \pm \sigma_\mu = 62.16 \pm 0.99$

For Gaussian fit:
$\chi^2/\text{ndf} = 2.824/9 = 0.3138$
$\mu \pm \sigma_\mu = 62.03 \pm 0.91$
$\sigma \pm \sigma_\sigma = 7.017 \pm 0.929$

Distribution of Successive Counts

In the second part of experiment we measure the time interval between adjacent pulses. Then we calculate it for n=0 which means no event, and n=1 which means 1 event in an interval. Then we histogram our data and fit according to Poisson function

$$P(\lambda t, n) = \left(\frac{(\lambda t)^n e^{-\lambda t}}{n!}\right) \tag{28}$$

For n = 0 ;

$$P(\lambda t,0) = \left(\frac{(\lambda t)^0 e^{-\lambda t}}{0!}\right) = \lambda e^{-\lambda t} \tag{29}$$

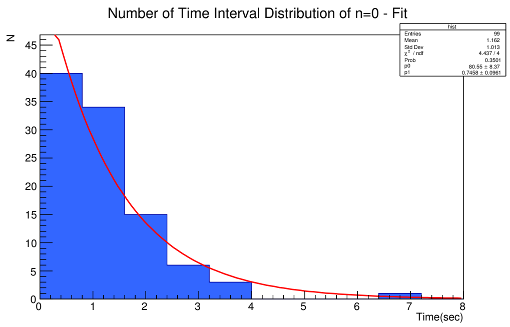

Figure 22: n=0 Histogram and Fit Graph

$$\lambda \pm \sigma_\lambda = 0.7458 \pm 0.0961 \tag{30}$$

$$\chi^2/\text{ndf} = 4.437/4 = 1.109 \tag{31}$$

For n= 1

$$P(\lambda t,1) = \left(\frac{(\lambda t)^1 e^{-\lambda t}}{1!}\right) = \lambda^2 t e^{-\lambda t} \tag{32}$$

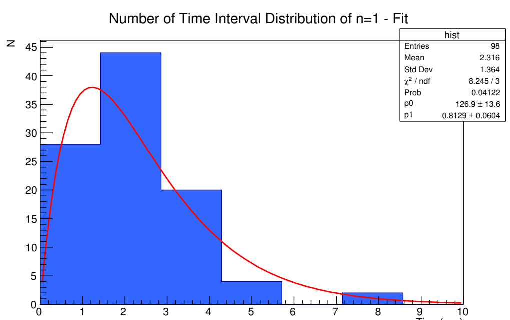

Figure 23: n=1 Histogram and Fit Graph

$$\lambda \pm \sigma_\lambda = 0.8129 \pm 0.0604 \tag{33}$$

$$\chi^2/\text{ndf} = 8.245/3 = 2.748 \tag{34}$$

4 Conclusion

At the end of the experiment, we were able to fit a Poisson distribution to the radioactive decays of ^{137}Cs and ^{133}Ba sources. We further verified the Poisson fit by comparing the experimental and theoretical results and found them closely related. Additionally, we observed that the result values of Poisson and Gaussian fits show that as n becomes large, Gaussian distribution and Poisson distribution behaves quite similarly. However, Poisson distribution is more suitable for determining the probability of detecting a single event or more than one event in the same period.

References

[1] Advanced Physics Experiments Gulmez, E.(1999), Istanbul : BogaziciUniversitesi

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[9] http://www.phys.utk.edu/labs/modphys/PoissonStatistics.pdf

Appendix

Codes are in my GitHub account: https://github.com/KubraCakir/poisson_experiment