Exercises (4.3):

① $x(u,v) = (\underbrace{r\cos u\cos v}_{p_1}, \underbrace{r\sin u\cos v}_{p_2}, r\sin v)$

a) $f(p) = p_1^2 + p_2^2$

$$f(x(u,v)) = (r\cos u\cos v)^2 + (r\sin u\cos v)^2 = r^2\cos^2 v$$

b) $f(p) = (p_1 - p_2)^2 + p_3^2$

$$f(x(u,v)) = (r\cos u\cos v - r\sin u\cos v)^2 + (r\sin v)^2 = r^2\cos^2 v\underbrace{(\cos u - \sin u)^2}_{1-\sin 2u} + r^2\sin^2 v = r^2 - r^2\cos^2 v\sin 2u$$

$$= r^2(1 - \cos^2 v\sin 2u)$$

② $x(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$

a) $\alpha(t) = x(t,t) = (\underbrace{(R + r\cos t)\cos t}_{\alpha_1}, \underbrace{(R + r\cos t)\sin t}_{\alpha_2}, \underbrace{r\sin t}_{\alpha_3})$

b) $\alpha(t+p) = \alpha(t) \Rightarrow \cancel{r}\sin(t+p) = \cancel{r}\sin t \Rightarrow p = 2\pi$. So the period of α is 2π.

③ a) Since y is a mapping then its coordinate expression with respect to the patch x is $x^{-1}y : E \to D$. If $\bar{u}, \bar{v}$ are the Euclidean coord. funct. of $x^{-1}y$, then $y = x\underbrace{x^{-1}y}_{(\bar{u},\bar{v})} = x(\bar{u},\bar{v})$.

These are the only such functions, for if $y = x(\bar{w},\bar{z})$, then $(\bar{u},\bar{v}) = x^{-1}y = x^{-1}x(\bar{w},\bar{z}) = (\bar{w},\bar{z})$.

b) Since $y(u,v) = x(\overset{x_1}{\overline{\bar{u}(u,v)}}, \overset{x_2}{\overline{\bar{v}(u,v)}})$, then we have

$$\frac{\partial y}{\partial u}(u,v) = \frac{\partial x}{\partial x_1}(u,v)\frac{\partial \bar{u}}{\partial u}(u,v) + \frac{\partial x}{\partial x_2}(u,v)\frac{\partial \bar{v}}{\partial u}(u,v) \overset{\text{funct. not.}}{\Longrightarrow} y_u = x_u \frac{\partial \bar{u}}{\partial u} + x_v \frac{\partial \bar{v}}{\partial u}$$

c) $$y_u \times y_v = \left(\frac{\partial \bar{u}}{\partial u} x_u + \frac{\partial \bar{v}}{\partial u} x_v\right) \times \left(\frac{\partial \bar{u}}{\partial v} x_u + \frac{\partial \bar{v}}{\partial v} x_v\right) = \frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial v} x_u \times x_v + \frac{\partial \bar{v}}{\partial u}\frac{\partial \bar{u}}{\partial v} x_v \times x_u = \underbrace{\left(\frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial v} - \frac{\partial \bar{u}}{\partial v}\frac{\partial \bar{v}}{\partial u}\right)}_{J = \begin{vmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{u}}{\partial v} \\ \frac{\partial \bar{v}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \end{vmatrix}} x_u \times x_v$$

$=0$

4 a) If $x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$, then we have

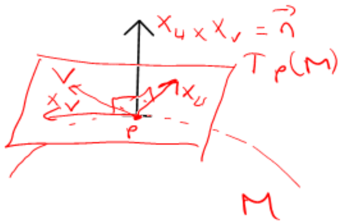
$U_1(f) = \partial f / \partial u_1$

$$x_*(U_1) = (U_1[x_1], U_1[x_2], U_1[x_3]) = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right) = x_u$$

b) If f is a diff. funct. on M, then we have $x_u(f) = \sum \frac{\partial x_i}{\partial u}(u,v)\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial u}(f(x))$.

5 a) We can write $M: g = 0$, where $g(x,y,z) = z - f(x,y)$. Since M is a surface, then $\nabla g = \sum \frac{\partial g}{\partial x_i} U_i$ $= (-f_x, -f_y, 1)$, is a nonvanishing normal vector field on the entire surface. Hence $v = (v_1, v_2, v_3)$ is tangent to M at a point p if and only if $v \cdot \nabla g = 0$ or $(v_1, v_2, v_3) \cdot (-f_x, -f_y, 1) = 0 \Rightarrow -\frac{\partial f}{\partial x} v_1 - \frac{\partial f}{\partial y} v_2 + v_3 = 0$.

b) Since $x_u(u,v)$ and $x_v(u,v)$ constitute a basis for the tangent plane of M at $x(u,v)$, then $x_u(u,v) \times x_v(u,v)$ is normal to M at $x(u,v)$. Hence v is tangent to M at $x(u,v)$ iff $v \cdot x_u(u,v) \times x_v(u,v) = 0$.



6 b) The function $y^{-1}x$ is defined only for those points (u,v) in D such that $x(u,v)$ lies in the image $y(D)$ of y.

Namely, $y^{-1}x$ is defined at the points $(u,v) = (r\cos t, r\sin t)$, where $0 < r < 1$ and $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

$$x(u,v) = (u, v, f(u,v)) = (r\cos t, r\sin t, \sqrt{1-r^2}) \overset{?}{\in} y(D)$$

If we write $y(u,v) = (v, f(u,v), u) = (p_1, p_2, p_3)$ then $y^{-1}(p_1,p_2,p_3) = (p_3, p_1)$. Hence

$$y^{-1}x(u,v) = y^{-1}(\underbrace{u, v, f(u,v)}_{x(u,v)}) = (f(u,v), u) = (\sqrt{1-u^2-v^2}, u).$$

c) The function $x^{-1}y$ is defined only for those points (u,v) in D such that $y(u,v)$ lies in the image $x(D)$ of x.

Namely, $x^{-1}y$ is defined at the points $(u,v) = (r\cos t, r\sin t)$, where $0 < r < 1$ and $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

If we write $x(u,v) = (u, v, f(u,v)) = (p_1, p_2, p_3)$ then $x^{-1}(p_1,p_2,p_3) = (p_1, p_2)$. Hence

$$x^{-1}y(u,v) = x^{-1}(v, f(u,v), u) = (v, f(u,v)) = (v, \sqrt{1-u^2-v^2}).$$

7) If we write $M: g=0$, where $g(x,y,z) = z - xy$, then $\nabla g = (-y,-x,1)$ is a nonvanishing normal vector field on the entire surface. Hence $V = (v_1,v_2,v_3)$ is a tangent vector field iff $V \cdot \nabla g = 0$ or

$$0 = (v_1,v_2,v_3)\cdot(-y,-x,1) \Rightarrow v_3 = y v_1 + x v_2 .$$

So $V_1 = (0,1,x)$ and $V_2 = (1,0,y)$ are two tangent vector fields that are linearly independent at each point.

8) a) Since $\alpha = x(a_1,a_2)$, then by the chain rule

$$\alpha' = x_u(a_1,a_2)\frac{da_1}{dt} + x_v(u,v)\frac{da_2}{dt}$$

Now, since $x_u(u,v) = \underline{(-\sin u, \cos u, 0)}$, $x_v(u,v) = \underline{(\cos u, \sin u, 1)}$, $\frac{da_1}{dt} = \sqrt{2}$, $\frac{da_2}{dt} = e^t$, then we have

$$\underline{\alpha' = \sqrt{2}\, x_u(\sqrt{2}t, e^t) + e^t x_v(\sqrt{2}t, e^t)}.$$

b)

$$\cos\theta = \frac{\alpha'\cdot x_u}{\|\alpha'\|\cdot\|x_u\|} = \frac{\alpha'\cdot x_v}{\|\alpha'\|\cdot\|x_v\|} \Longrightarrow \frac{\alpha'\cdot x_u}{\|x_u\|} = \frac{\alpha'\cdot x_v}{\|x_v\|}.$$

$$\alpha'\cdot\frac{x_u}{\|x_u\|} = (\sqrt{2}x_u + e^t x_v)\cdot\frac{x_u}{\|x_u\|} = \sqrt{2}\,\frac{\|x_u\|^2}{\|x_u\|} = \sqrt{2}\|x_u\| = \sqrt{2}e^t \quad \text{and}$$

$$\alpha'\cdot\frac{x_v}{\|x_v\|} = (\sqrt{2}x_u + e^t x_v)\cdot\frac{x_v}{\|x_v\|} = e^t\,\frac{\|x_v\|^2}{\|x_v\|} = e^t\|x_v\| = \sqrt{2}e^t.$$

(10) a) $M: g(x,y,z) = x^2+y^2+(z-1)^2 = 1 \Rightarrow \nabla g(x,y,z) = (2x, 2y, 2(z-1))$. $T_p(M)$ consists of all points $r=(x,y,z)$ in $\mathbb{R}^3$ such that $(r-p)\cdot \nabla g(p) = 0$. So

$(x,y,z)\cdot(0,0,-2) = 0 \Rightarrow -2z = 0$ or $\boxed{z=0}$.

c) $x(u,v) = (u\cos v, u\sin v, 2v) \Rightarrow x_u = (\cos v, \sin v, 0)$, $x_v = (-u\sin v, u\cos v, 2)$

$$x_u \times x_v = \begin{vmatrix} u_1 & u_2 & u_3 \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 2 \end{vmatrix} = (2\sin v, -2\cos v, u).$$

$x(2,\pi/4) = (\sqrt{2}, \sqrt{2}, \frac{\pi}{2})$, $x_u(2,\pi/4)\times x_v(2,\pi/4) = (\sqrt{2}, -\sqrt{2}, 2)$.

$[\underbrace{r}_{(x,y,z)} - x(2,\frac{\pi}{4})]\cdot x_u(2,\frac{\pi}{4})\times x_v(2,\frac{\pi}{4}) = 0 \Rightarrow (x-\sqrt{2}, y-\sqrt{2}, z-\frac{\pi}{2})\cdot(\sqrt{2},-\sqrt{2},2) = 0 \Rightarrow \sqrt{2}(x-\sqrt{2}) - \sqrt{2}(y-\sqrt{2}) + 2(z-\frac{\pi}{2}) = 0$

$\Rightarrow \boxed{\sqrt{2}x - \sqrt{2}y + 2z = \pi}$

$$\frac{\partial}{\partial u}\left(\phi(y_v)\right) = \frac{\partial}{\partial u}\left(\phi(x_u)\frac{\partial \bar{u}}{\partial v} + \phi(x_v)\frac{\partial \bar{v}}{\partial v}\right) = \frac{\partial}{\partial u}\Big(\underset{x = x(\bar{u},\bar{v})}{\phi(x_u)}\Big)\frac{\partial \bar{u}}{\partial v} + \phi(x_u)\frac{\partial^2 \bar{u}}{\partial u \partial v} + \frac{\partial}{\partial u}\left(\phi(x_v)\right)\frac{\partial \bar{v}}{\partial v} + \phi(x_v)\frac{\partial^2 \bar{v}}{\partial v \partial u}$$

$$= \left[\underbrace{\frac{\partial}{\partial \bar{u}}\left(\phi(x_u)\right)\frac{\partial \bar{u}}{\partial u}}_{1} + \underline{\underline{\frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\frac{\partial \bar{v}}{\partial u}}}\right]\frac{\partial \bar{u}}{\partial v} + \cancel{\phi(x_u)\frac{\partial^2 \bar{u}}{\partial u \partial v}}$$

$$+ \left[\underbrace{\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right)}\frac{\partial \bar{u}}{\partial u} + \underbrace{\frac{\partial}{\partial \bar{v}}\left(\phi(x_v)\right)\frac{\partial \bar{v}}{\partial u}}_{2}\right]\frac{\partial \bar{v}}{\partial v} + \cancel{\phi(x_v)\frac{\partial^2 \bar{v}}{\partial v \partial u}}$$

$$\frac{\partial}{\partial v}\left(\phi(y_u)\right) = \left[\underbrace{\frac{\partial}{\partial \bar{v}}\phi(x_v)\frac{\partial \bar{v}}{\partial v}}_{2} + \underbrace{\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right)}\frac{\partial \bar{u}}{\partial v}\right]\frac{\partial \bar{v}}{\partial u} + \cancel{\phi(x_v)\frac{\partial^2 \bar{v}}{\partial v \partial u}}$$

$$+ \left[\underline{\underline{\frac{\partial}{\partial \bar{v}}\phi(x_u)\frac{\partial \bar{v}}{\partial v}}} + \underbrace{\frac{\partial}{\partial \bar{u}}\left(\phi(x_u)\right)\frac{\partial \bar{u}}{\partial v}}_{1}\right]\frac{\partial \bar{u}}{\partial u} + \cancel{\phi(x_u)\frac{\partial^2 \bar{u}}{\partial u \partial v}}$$

$$= \frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right)\underbrace{\left[\frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial v} - \frac{\partial \bar{u}}{\partial v}\frac{\partial \bar{v}}{\partial u}\right]}_{J} - \frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\underbrace{\left[\frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial v} - \frac{\partial \bar{u}}{\partial v}\frac{\partial \bar{v}}{\partial u}\right]}_{J}$$