Solutions:

1. Since $\alpha'(t) = \left(1, \sqrt{2}t, t^2\right)$, $\alpha''(t) = \left(0, \sqrt{2}, 2t\right)$, $\alpha'''(t) = (0,0,2)$,

$\|\alpha'(t)\| = t^2 + 1$, $\alpha' \times \alpha'' = \begin{vmatrix} U_1 & U_2 & U_3 \\ 1 & \sqrt{2}t & t^2 \\ 0 & \sqrt{2} & 2t \end{vmatrix} = \left(\sqrt{2}t^2, -2t, \sqrt{2}\right)$,

$\|\alpha' \times \alpha''\| = \sqrt{2}(t^2+1)$, $(\alpha' \times \alpha'') \cdot \alpha''' = \left(\sqrt{2}t^2, -2t, \sqrt{2}\right) \cdot (0,0,2) = 2\sqrt{2}$, then

$\kappa = \|\alpha' \times \alpha''\| / \|\alpha'\|^3 = \sqrt{2}/(t^2+1)^2$,

$\tau = (\alpha' \times \alpha'') \cdot \alpha''' / \|\alpha' \times \alpha''\|^2 = \sqrt{2}/(t^2+1)^2$,

then we have $\frac{\tau}{\kappa} = 1$. **[10]**

Hence α is a cylindrical helix. If we write $1 = \tau/\kappa = \cot\vartheta$, then $\vartheta = \pi/4$ **[3]**. Hence we have $T = \frac{\alpha'}{\|\alpha'\|} = \frac{(1,\sqrt{2}t,t^2)}{t^2+1}$, $B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} = \frac{(t^2,-\sqrt{2}t,1)}{t^2+1}$ **[6]**so that

$$\boldsymbol{u} = \cos\vartheta\, T + \sin\vartheta\, B = \frac{\left(1, \sqrt{2}t, t^2\right)}{\sqrt{2}(t^2+1)} + \frac{\left(t^2, -\sqrt{2}t, 1\right)}{\sqrt{2}(t^2+1)} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right). \textbf{[6]}$$

2. First note that

$$\begin{aligned}\left[U, [V, W]\right] &= [U, \nabla_V W - \nabla_W V] = \nabla_U(\nabla_V W - \nabla_W V) - \nabla_{(\nabla_V W - \nabla_W V)} U \\ &= \nabla_U \nabla_V W - \nabla_U \nabla_W V - \nabla_{\nabla_V W} U + \nabla_{\nabla_W V} U. \textbf{[5]}\end{aligned}$$

So we have

$$\begin{aligned}\left[U, [V, W]\right] + \left[V, [W, U]\right] + \left[W, [U, V]\right] &= \left(\nabla_U \nabla_V W - \nabla_U \nabla_W V - \nabla_{\nabla_V W} U + \nabla_{\nabla_W V} U\right) \\ &\quad + \left(\nabla_V \nabla_W U - \nabla_V \nabla_U W - \nabla_{\nabla_W U} V + \nabla_{\nabla_U W} V\right) \\ &\quad + \left(\nabla_W \nabla_U V - \nabla_W \nabla_V U - \nabla_{\nabla_U V} W + \nabla_{\nabla_V U} W\right) \\ &= (UVW - UWV - VWU + WVU) \\ &\quad + (VWU - VUW - WUV + UWV) \\ &\quad + (WUV - WVU - UVW + VUW) \\ &= 0. \qquad \textbf{[15]}\end{aligned}$$

3. a) The toroidal frame field is is the same as the spherical frame field, since the directions of increasing ρ, ϑ, φ are the same as in the spherical case. So

$E_1 = F_1 = \cos\vartheta \cos\varphi\, U_1 + \sin\vartheta \cos\varphi\, U_2 + \sin\varphi\, U_3$

$E_2 = F_2 = -\sin\vartheta\, U_1 + \cos\vartheta\, U_2$

$E_3 = F_3 = -\cos\vartheta \sin\varphi \, U_1 - \sin\vartheta \sin\varphi \, U_2 + \cos\varphi \, U_3$. **[10]**

Since

$$\|E_1\| = \|E_2\| = \|E_3\| = 1, \quad E_1 \cdot E_2 = E_1 \cdot E_3 = E_2 \cdot E_3 = 0$$

then E_1, E_2, E_3 constitute a frame field. **[5]**

b) Since the attitude matrix of the toroidal frame field is

$$A = \begin{pmatrix} \cos\vartheta\cos\varphi & \sin\vartheta\cos\varphi & \sin\varphi \\ -\sin\vartheta & \cos\vartheta & 0 \\ -\cos\vartheta\sin\varphi & -\sin\vartheta\sin\varphi & \cos\varphi \end{pmatrix}, \quad [\mathbf{5}]$$

then we have

$$\begin{aligned} \omega &= dA \cdot A^t \\ &= \begin{pmatrix} -\sin\vartheta\cos\varphi\, d\vartheta - \cos\vartheta\sin\varphi\, d\varphi & \cos\vartheta\cos\varphi\, d\vartheta - \sin\vartheta\sin\varphi\, d\varphi & \cos\varphi\, d\varphi \\ -\cos\vartheta\, d\vartheta & -\sin\vartheta\, d\vartheta & 0 \\ \sin\vartheta\sin\varphi\, d\vartheta - \cos\vartheta\cos\varphi\, d\varphi & -\cos\vartheta\sin\varphi\, d\vartheta - \sin\vartheta\cos\varphi\, d\varphi & -\sin\varphi\, d\varphi \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} \cos\vartheta\cos\varphi & -\sin\vartheta & -\cos\vartheta\sin\varphi \\ \sin\vartheta\cos\varphi & \cos\vartheta & -\sin\vartheta\sin\varphi \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix} \\ &= \begin{pmatrix} 0 & \cos\varphi\, d\vartheta & d\varphi \\ -\cos\varphi\, d\vartheta & 0 & \sin\varphi\, d\vartheta \\ -d\varphi & -\sin\varphi\, d\vartheta & 0 \end{pmatrix}. \end{aligned}$$

[10]

4. (a) If β is a cylindrical helix, then τ/κ is constant **[5]**. So we have

$$\frac{\bar{\tau}}{\bar{\kappa}} = \frac{(sgnF)\tau}{\kappa} = constant.$$

Hence $F(\beta)$ is a cylindrical helix. **[5]**

(b) If β has spherical image σ, then $\sigma = \beta' = T$. If we write $F(\beta) = \bar{\beta}$, then the spherical image of $F(\beta)$ is $\bar{\beta}' = F(\beta)'$ **[5]**. So we have

$$\bar{\beta}' = F_*(\beta') = C(\beta') = C(\sigma). \quad [\mathbf{5}]$$

Since $\|C(\sigma)\| = \|\bar{\beta}'\| = \|F_*(\beta')\| = \|\beta'\| = 1$, then $C(\sigma)$ is the spherical image of $F(\beta)$. **[5]**