AAGT Homework #6

Jakub Grondziowski

In this homework I use the term $lg(a)$ as $lg_2(a)$.

1 Task a

In order to prove that (n, h) universal tree has at least the size equal to $\binom{\lfloor lg(n) \rfloor + h - 1}{h-1}$, I will first prove some lemmas, which then will be combined to form the complete proof.

Let's start with defining a function $size(n, h)$, which returns the least amount of leaves (size of the tree) of the (n, h) universal tree. I will argue that $size(n, h) = \sum_{i=1}^{n} size(\lfloor \frac{n}{i} \rfloor, h - 1)$.

From the properties of universal trees we can define the base case that will stop this recurrent equation: the size of $(1, h)$ universal tree is equal to 1, whilst the size of $(n, 1)$ universal tree is equal to n.

Let $T = (n, h)$ universal tree and let $i \in [1, n]$. Let's define T_i as a sub-tree of T of height $h - 1$, which we can build by removing all leaves from tree T and then removing all nodes at height $h - 1$ that had a degree smaller than i from before the deletion of leaves. If the parent of the deleted node has became a leaf (it didn't have any other children), we remove him from the tree and repeat the process of deleting the parents until we encounter a node which had 2 or more children from before the deletion process.

I will now try to proof that the tree T_i is actually a $(\lfloor \frac{n}{i} \rfloor, h - 1)$ universal tree. Let's define some example tree w of size $\lfloor \frac{n}{i} \rfloor$ and height $h - 1$. To every leaf of tree w we append i children, which creates a new tree W of height h and size $i * \lfloor \frac{n}{i} \rfloor$, which is less or equal to n. That means that the newly created tree W is actually a tree that embeds to T, since T is (n, h) universal. This naturally concludes another observation, that the tree w embeds to the tree T_i, because the leaves of w, which are also nodes at height $h-1$ in W, have a degree i in W, thus those nodes are also in T_i. This means that the number of nodes of T at height $h - 1$ of degree $\geq i$ is at least equal to $size(\lfloor \frac{n}{i} \rfloor, h - 1)$.

Let's now define n_i to be the exact number of nodes at height $(h-1)$ with degree equal to i. Using the previous conclusion we can now state that $\sum_{j=i}^{n} n_j \geq size(\lfloor \frac{n}{i} \rfloor, h - 1)$. That means that the total size of tree T is

$$\sum_{j=1}^{n} n_j * j = \sum_{i=1}^{n} \sum_{j=i}^{n} n_j \geq \sum_{i=1}^{n} size(\lfloor \frac{n}{i} \rfloor, h - 1)$$

That proves that indeed $size(n, h) = \sum_{i=1}^{n} size(\lfloor \frac{n}{i} \rfloor, h - 1)$.

For the second lemma I will prove a simple equation: $\binom{x+y}{x} = \binom{x+y}{y}$. It's quite easy, expanding those binomials yield the proof:

$$\binom{x+y}{x} = \frac{(x+y)!}{x! * (x+y-x)!} = \frac{(x+y)!}{x! * y!} = \frac{(x+y)!}{y! * (x+y-y)!} = \binom{x+y}{y}$$

It's not revolutionary in any sense, but it will help us in our next lemma, as we now know that

$$\binom{\lfloor lg(n) \rfloor + h - 1}{h-1} = \binom{\lfloor lg(n) \rfloor + h - 1}{\lfloor lg(n) \rfloor}$$

.

For the final lemma we just need to prove that $size(n,h) \geq \binom{\lfloor lg(n) \rfloor + h - 1}{\lfloor lg(n) \rfloor}$. Let $S(p,h) = size(2^p, h)$ for every $p \geq 0$ and $h > 0$. We can easily estimate the values of S using size:

$$S(0,h) = 1$$

$$S(p,1) \geq 1$$

$$S(p,h) \geq \sum_{r=0}^{p} S(p-r, h-1)$$

We can define another term which will work as a lower bound for our newly declared S. Let's call it $\bar{S}$.

$$\bar{S}(0,h) = 1$$

$$\bar{S}(p,1) = 1$$

$$\bar{S}(p,h) = \bar{S}(p,h-1) + \bar{S}(p-1,h)$$

We can now use the Pascal's identity to convert $\bar{S}(p,h)$ to a more friendly, binomial form:

$$\bar{S}(p,h) = \binom{p+h-1}{p} = \binom{p+h-2}{p} + \binom{p+h-2}{p-1}$$

We have now proven that $S(p,h) \geq \binom{p+h-1}{p}$ (Since $\bar{S}$ was the lower bound for S), which will now help us in defining the final inequality:

$$size(n,h) \geq S(\lfloor lg(n) \rfloor, h) \geq \bar{S}(\lfloor lg(n) \rfloor, h) = \binom{\lfloor lg(n) \rfloor + h - 1}{\lfloor lg(n) \rfloor}$$

This concludes the proof.