Linear Algebra

ZZ-1104 Essential Mathematics for Digital Science

Linear Algebra: Vectors & Matrices

- Geometric intuition for linear algebra
- Outline:
  - Matrices as linear transformations or as sets of constraints
  - Linear systems & vector spaces
  - Solving linear systems
  - Eigenvalues & eigenvectors

Scalar vs Vector

A vector is a quantity that has both magnitude and direction

force, velocity, and acceleration, which have both magnitude and direction

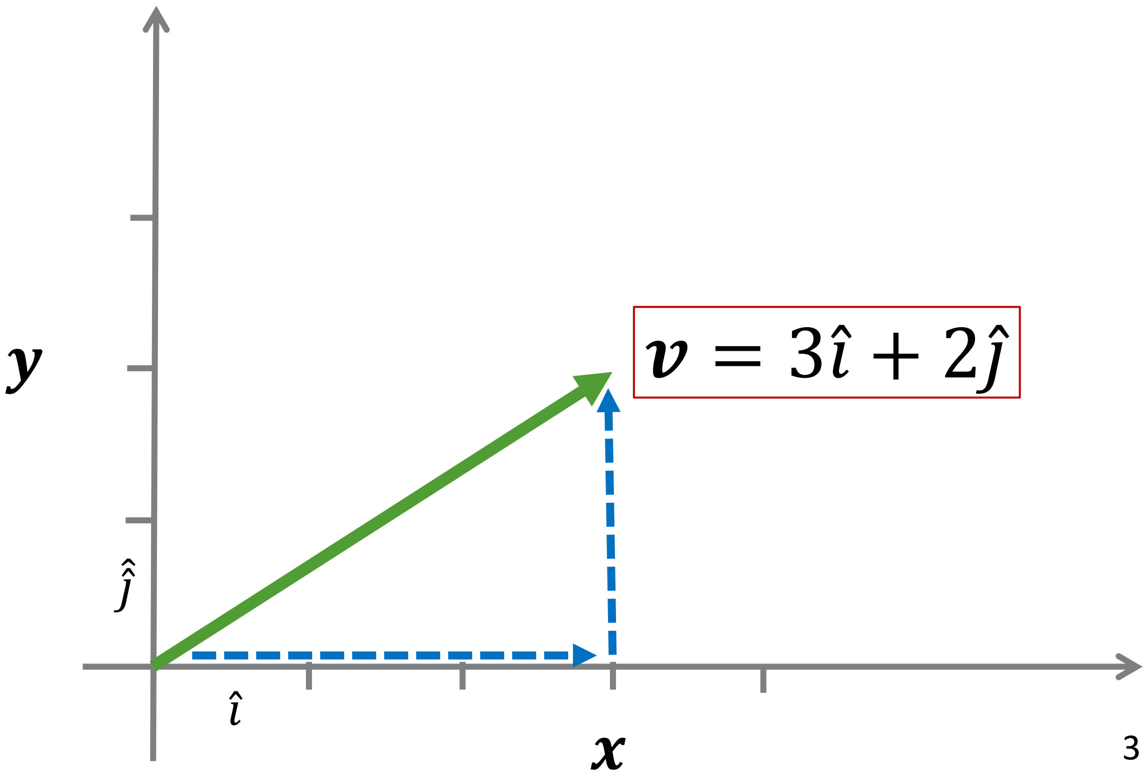
represented geometrically by directed line segments (arrows).

Vector Representation:

Vectors as summations/combinations of components

Different pieces of information put together

$$\boldsymbol{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



Vectors in Linear Algebra

- *Vector* in R^n is an ordered set of n real numbers.
  - e.g. v = (1,6,3,4) is in R^4
  - "(1,6,3,4)" is a column vector:
  - as opposed to a row vector:

$$\begin{pmatrix} 1 \\ 6 \\ 3 \\ 4 \end{pmatrix}$$

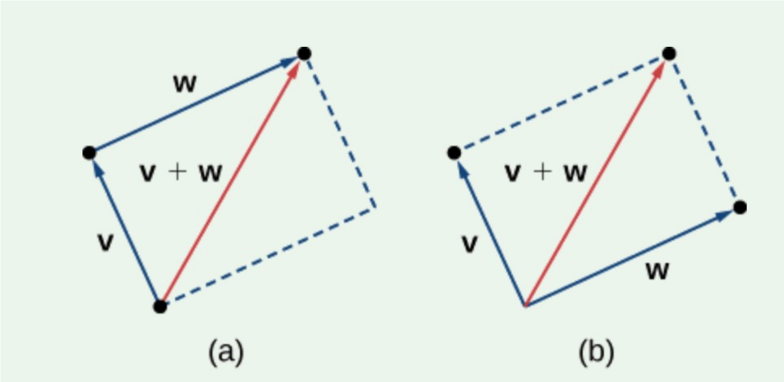
$$\begin{pmatrix} 1 & 6 & 3 & 4 \end{pmatrix}$$

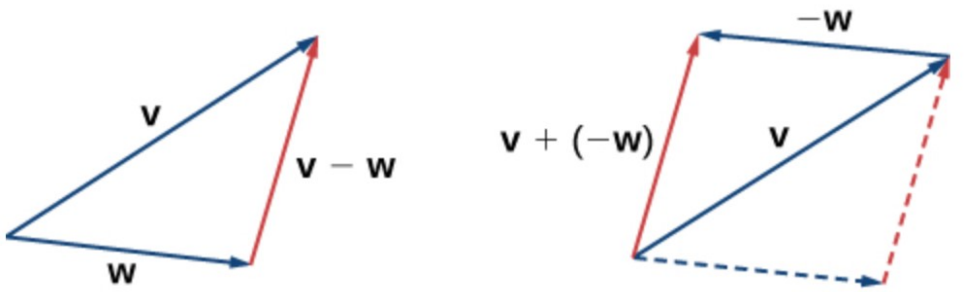
By a *vector* u, we mean a list of numbers, say, $a_1, a_2, \ldots, a_n$. Such a vector is denoted by

$$u = (a_1, a_2, \ldots, a_n)$$

Vectors

Parallelogram Method of Addition





Laws of vector addition

(*Commutative Law*)	$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
(*Associative Law*)	$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
(*Distributive Law*)	$k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$

Components of a Vector

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$$

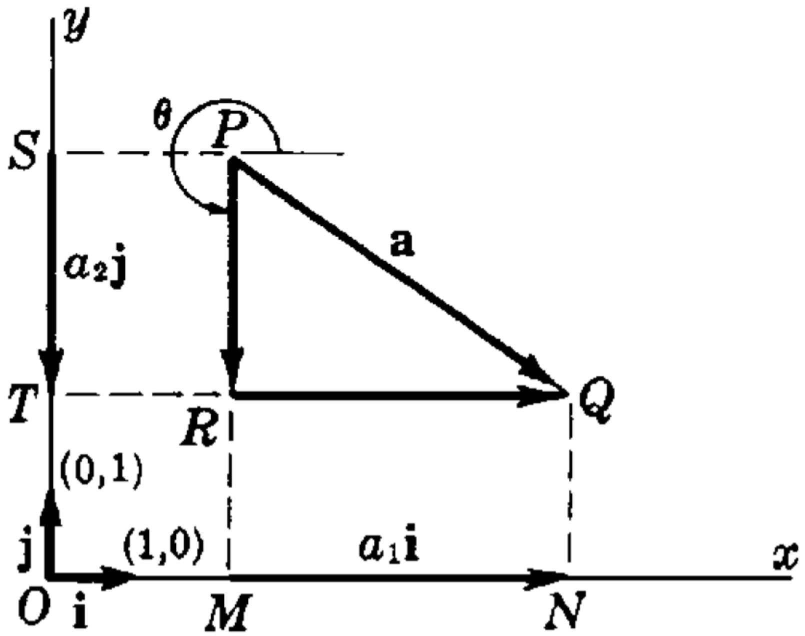
$a_1\mathbf{i}$ and $a_2\mathbf{j}$ the *vector components* of $\mathbf{a}$

- Magnitude of scalar components

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$$

- Unit vector in the direction of $\vec{a}$

$$= \frac{\vec{a}}{\|\vec{a}\|}$$



If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{j} + b_2\mathbf{j}$, then the following hold.

$\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1$ and $a_2 = b_2$

$$k\mathbf{a} = ka_1\mathbf{i} + ka_2\mathbf{j}$$

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$$

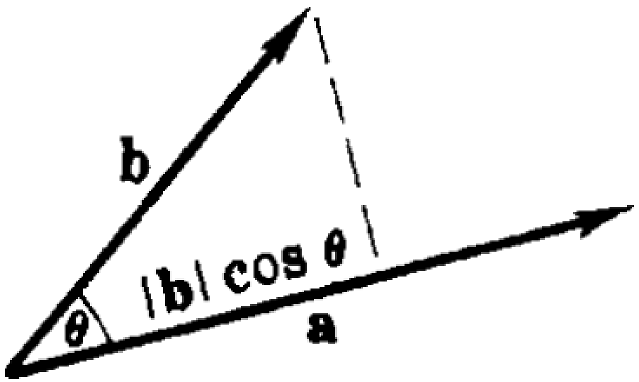
$$\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j}$$

Operations on Vectors

- If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$,
- **<u>Dot Product</u>** $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$
  - $\vec{a} \cdot \vec{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta$
  - If $\theta = 90°$ (and $\vec{a} \cdot \vec{b} = 0$)
    - Then vectors are orthogonal
  - If $\vec{a} = c\vec{b}$
    - Then vectors are parallel

Projections

a · **b** is the product of the length of **a** and the scalar projection of **b** on **a**.



$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$
$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \text{ and } |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$
$$\mathbf{a} \cdot \mathbf{b} = 0 \text{ if and only if } (\mathbf{a} = 0 \text{ or } \mathbf{b} = 0 \text{ or } \mathbf{a} \text{ is perpendicular to } \mathbf{b})$$
$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1 \text{ and } \mathbf{i} \cdot \mathbf{j} = 0$$
$$\mathbf{a} \cdot \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j}) \cdot (b_1\mathbf{i} + b_2\mathbf{j}) = a_1 b_1 + a_2 b_2$$
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$
$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$$

Vector Operations

EXAMPLE A.2 Let $u = (2, 3, -4)$ and $v = (1, -5, 8)$. Then

$$
\begin{aligned}
u + v &= (2 + 1,\ 3 - 5,\ -4 + 8) = (3, -2, 4) \\
5u &= (5 \cdot 2,\ 5 \cdot 3,\ 5 \cdot (-4)) = (10, 15, -20) \\
-v &= -1 \cdot (1, -5, 8) = (-1, 5, -8) \\
2u - 3v &= (4, 6, -8) + (-3, 15, -24) = (1, 21, -32) \\
u \cdot v &= 2 \cdot 1 + 3 \cdot (-5) + (-4) \cdot 8 = 2 - 15 - 32 = -45 \\
\|u\| &= \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}
\end{aligned}
$$

Vectors Algebra Scales Up

- Vectors in 2-D

$$\vec{v} = \langle v_1, v_2 \rangle$$

- Vectors in 3-D (just add *z*)

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

- To find a vector from the initial point (p_1, p_2, p_3) to the terminal point (q_1, q_2, q_3)

$$\vec{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$$

- If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{u} = \langle u_1, u_2, u_3 \rangle$,
- Addition
  - Add corresponding elements

$$\vec{v} + \vec{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

- Scalar multiplication
  - Distribute

$$c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle$$

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$, where θ is the smaller angle between **a** and **b**

$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a} = \mathbf{0}$, or $\mathbf{b} = \mathbf{0}$, or **a** and **b** are perpendicular

$|\mathbf{r}| = \sqrt{\mathbf{r} \,.\, \mathbf{r}} = \sqrt{x^2 + y^2 + z^2}$

Vectors in space

- Let $\vec{m} = \langle 1, 0, 3 \rangle$ and $\vec{n} = \langle -2, 1, -4 \rangle$
- Find $||\vec{m}||$

$$||\vec{m}|| = \sqrt{m_1^2 + m_2^2 + m_3^2}$$
$$= \sqrt{1^2 + 0^2 + 3^2}$$
$$= \sqrt{10}$$

- Find unit vector in direction of $\vec{m}$

$$\frac{\vec{m}}{||\vec{m}||} = \frac{\langle 1, 0, 3 \rangle}{\sqrt{10}}$$
$$= \left\langle \frac{1}{\sqrt{10}}, \frac{0}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$
$$= \left\langle \frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10} \right\rangle$$

- Find $\vec{m} + 2\vec{n}$

$$\langle 1, 0, 3 \rangle + 2\langle -2, 1, -4 \rangle$$
$$\langle 1, 0, 3 \rangle + \langle -4, 2, -8 \rangle$$
$$\langle -3, 2, -5 \rangle$$

Vectors in space

- Let $\vec{m} = \langle 1, 0, 3 \rangle$ and $\vec{n} = \langle -2, 1, -4 \rangle$
- Find $\vec{m} \cdot \vec{n}$

$$\langle 1, 0, 3 \rangle \cdot \langle -2, 1, -4 \rangle$$
$$1(-2) + 0(1) + 3(-4)$$
$$-14$$

- Find the angle between $\vec{m}$ and $\vec{n}$

$$\vec{m} \cdot \vec{n} = \|\vec{m}\|\|\vec{n}\| \cos\theta$$
$$-14 = \sqrt{1^2 + 0^2 + 3^2}\sqrt{(-2)^2 + 1^2 + (-4)^2} \cos\theta$$
$$-14 = \sqrt{10}\sqrt{21} \cos\theta$$
$$\frac{-14}{\sqrt{10}\sqrt{21}} = \cos\theta$$
$$\theta \approx 165.0°$$

Vectors in space

- Are $\vec{p} = \langle 1, 5, -2 \rangle$ and $\vec{q} = \left\langle -\frac{1}{5}, -1, \frac{2}{5} \right\rangle$ parallel, orthogonal, or neither?

Orthogonal if $\vec{p} \cdot \vec{q} = 0$

$$\langle 1, 5, -2 \rangle \cdot \left\langle -\frac{1}{5}, -1, \frac{2}{5} \right\rangle$$

$$1\left(-\frac{1}{5}\right) + 5(-1) + (-2)\left(\frac{2}{5}\right)$$

$$-\frac{1}{5} - 5 - \frac{4}{5} = -6$$

Not 0, so not orthogonal

- Parallel if $\vec{p} = c\vec{q}$

$$\langle 1, 5, -2 \rangle = c\left\langle -\frac{1}{5}, -1, \frac{2}{5} \right\rangle$$

Check x

$$1 = c\left(-\frac{1}{5}\right) \rightarrow c = -5$$

Check y

$$5 = c(-1) \rightarrow c = -5$$

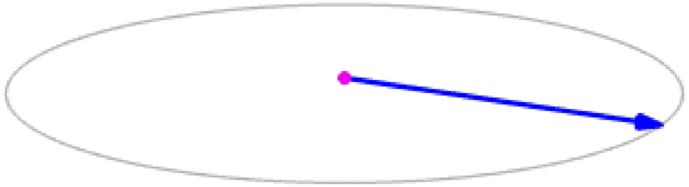
Check z

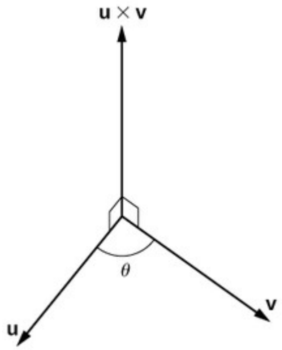
$$-2 = c\left(\frac{2}{5}\right) \rightarrow c = -5$$

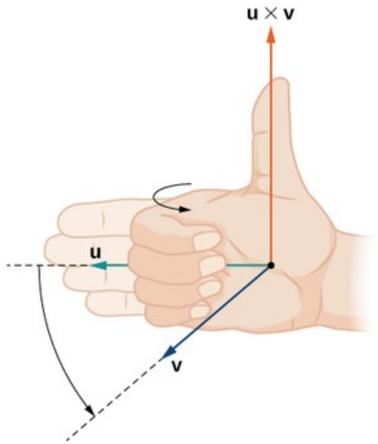
c is always the same, so they are parallel

Cross Products

- $\hat{\imath}$ is unit vector in x, $\hat{\jmath}$ is unit vector in y, and $\hat{k}$ is unit vector in z
- $\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$ and $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$
- $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- If $\vec{u} = \langle -2, 3, -3 \rangle$ and $\vec{v} = \langle 1, -2, 1 \rangle$, find $\vec{u} \times \vec{v}$







$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 3 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$
$$= 3\hat{\imath} - 6\hat{\imath} + (-3)\hat{\jmath} - (-2)\hat{\jmath} + 4\hat{k} - 3\hat{k}$$
$$= -3\hat{\imath} - \hat{\jmath} + \hat{k} = \langle -3, -1, 1 \rangle$$

Cross Products

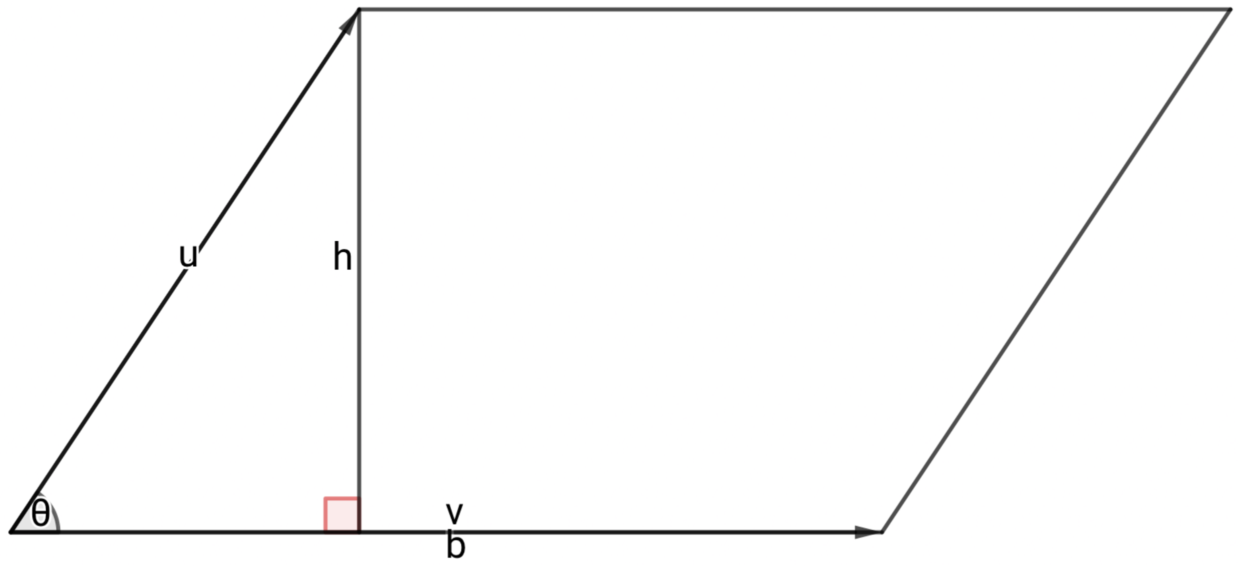
- Properties of Cross Products
- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
- $c(\vec{u} \times \vec{v}) = c\vec{u} \times \vec{v} = \vec{u} \times c\vec{v}$
- $\vec{u} \times \vec{u} = 0$
  - If $\vec{u} \times \vec{v} = 0$, then $\vec{u}$ and $\vec{v}$ are parallel
- $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
- $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Cross Products

- $A = bh$
- $h = ||\vec{u}|| \sin \theta$
- $A = ||\vec{v}||||\vec{u}|| \sin \theta$
- Area of a Parallelogram
  - $||\vec{u} \times \vec{v}||$ where $\vec{u}$ and $\vec{v}$ represent adjacent sides



Matrices in Linear Algebra

A *matrix* A is a rectangular array of numbers usually presented in the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- m-by-n *matrix* is an object with m rows and n columns, each entry fill with a real number:
- Each element is denoted by A[i,j] or a_{ij}
- row matrix and column matrix

$$\begin{pmatrix} 1 & 2 & 8 \\ 4 & 78 & 6 \\ 9 & 3 & 2 \end{pmatrix}$$

$A = \begin{bmatrix} 1 & -4 & 5 \\ 0 & 3 & -2 \end{bmatrix}$ is a 2×3 matrix

Operations on Matrices

$$A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

$$kA=\begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

The product AB is pictured in Fig. A-2.

$$\begin{bmatrix} a_{11} & \dots & a_{1p} \\ \cdot & \dots & \cdot \\ a_{i1} & \dots & a_{ip} \\ \cdot & \dots & \cdot \\ a_{m1} & \dots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ b_{p1} & \dots & b_{pj} & \dots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \cdot & \dots & \cdot \\ \cdot & c_{ij} & \cdot \\ \cdot & \dots & \cdot \\ c_{m1} & \dots & c_{mn} \end{bmatrix}$$

Matrix Multiplication

$$C * D = \begin{pmatrix} 3 & -9 & -8 \\ 2 & 4 & 3 \end{pmatrix} * \begin{pmatrix} 7 & -3 \\ -2 & 3 \\ 6 & 2 \end{pmatrix}$$

$$C * D = \begin{pmatrix} 3*7+(-9)*(-2)+(-8)*6 & 3*(-3)+(-9)*3+(-8)*2 \\ 2*7+4*(-2)+3*6 & 2*(-3)+4*3+3*2 \end{pmatrix}$$

$$C * D = \begin{pmatrix} 21+18-48 & -9-27-16 \\ 14-8+18 & -6+12+6 \end{pmatrix} = \begin{pmatrix} -9 & -52 \\ 24 & 12 \end{pmatrix}$$

$$A(BC) = (AB)C$$

$$A(B+C) = AB + AC$$

$$(A+B)C = AC + BC$$

However matrix multiplication is not commutative. That is to say *A***B* does not necessarily equal *B***A*. In fact, *B***A* often has no meaning since the dimensions rarely match up. However, you can take the transpose of matrix multiplication. In that case $(AB)^T = B^T A^T$.

Transpose of a Matrix

To take the **transpose** of a matrix, simply switch the rows and column of a matrix. The transpose of A can be denoted as A' or A^T.

For example

$$A = \begin{pmatrix} 1 & -5 & 4 \\ 2 & 5 & 3 \end{pmatrix}$$

$$A' = A^T = \begin{pmatrix} 1 & 2 \\ -5 & 5 \\ 4 & 3 \end{pmatrix}$$

If a matrix is its own transpose, then that matrix is said to be **symmetric**. Symmetric matrices must be square matrices, with the same number of rows and columns.

One example of a symmetric matrix is shown below:

$$A = \begin{pmatrix} 1 & -5 & 4 \\ -5 & 7 & 3 \\ 4 & 3 & 3 \end{pmatrix} = A' = A^T$$

Operations on Matrices

Find AB where $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$

$$AB = \begin{bmatrix} 17 & -6 & 14 \\ -1 & 2 & -14 \end{bmatrix}$$

Suppose $f(x) = 2x^2 - 3x + 5$. Then

$$f(A) = 2\begin{bmatrix} 7 & -6 \\ -9 & -22 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & -18 \\ -27 & 61 \end{bmatrix}$$

Suppose $g(x) = x^2 + 3x - 10$. Then

$$g(A) = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Special matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$ diagonal

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$ upper-triangular

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & e & 0 \\ 0 & f & g & h \\ 0 & 0 & i & j \end{pmatrix}$$ tri-diagonal

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$ lower-triangular

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ I (identity matrix)

System of Linear Equations and Matrices

Any system S of linear equations is equivalent to the matrix equation

$$AX = B$$

$$\begin{aligned} x + 2y - 3z &= 4 \\ 5x - 6y + 8z &= 9 \end{aligned} \quad \text{is equivalent to} \quad \begin{bmatrix} 1 & 2 & -3 \\ 5 & -6 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

Observe that the system is completely determined by the matrix

$$M = [A, B] = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 5 & -6 & 8 & 9 \end{bmatrix}$$

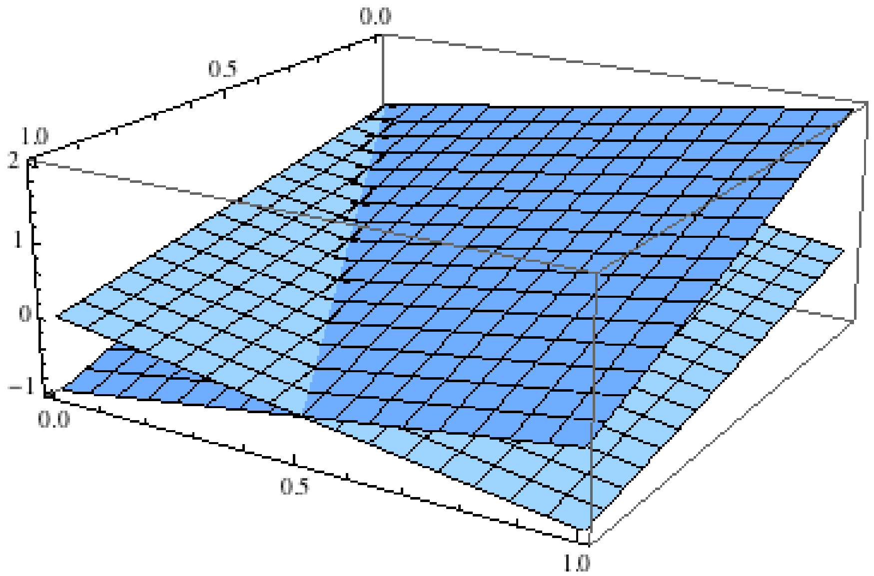
which is called the *augmented matrix* of the system.

Matrices as sets of constraints

$$x + y + z = 1$$

$$2x - y + z = 2$$

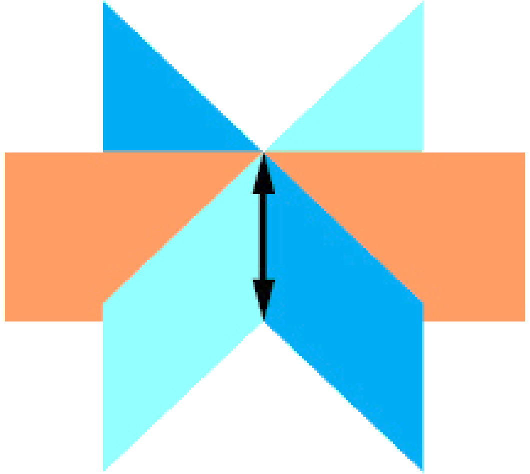
$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



Solutions to System of Linear Equations

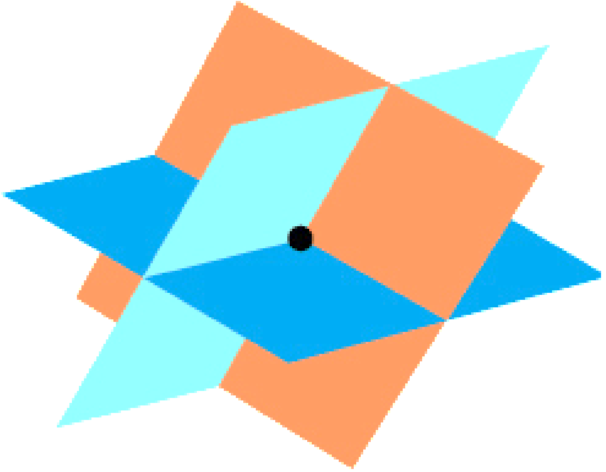
$$x + y + z = 2$$
$$6x - 4y + 5z = 31$$
$$5x + 2y + 2z = 13$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 6 & -4 & 5 & 31 \\ 5 & 2 & 2 & 13 \end{pmatrix}$$



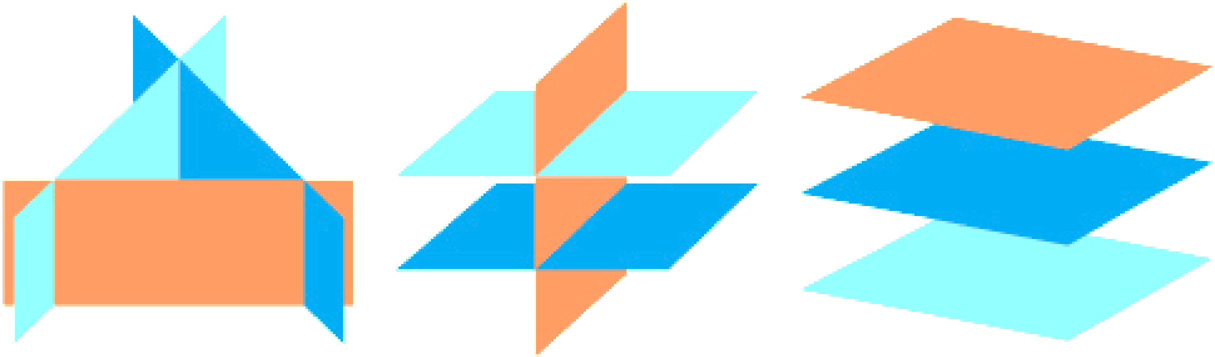
(b)

Infinite solutions



(a) Single solution

No Solutions



(a) (b) (c)

Gaussian Elimination Method

- Stage 1: Echelon form

$$\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 6 & -4 & 5 & 31 \\ 5 & 2 & 2 & 13 \end{array}\right)$$

A matrix A is called an *echelon matrix*, or is said to be in *echelon form*, if the following two conditions hold:

(i) All zero rows, if any, are on the bottom of the matrix.
(ii) Each leading nonzero entry is to the right of the leading nonzero entry in the preceding row.

- Stage 2: Row Canonical form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

The matrix is said to be in *row canonical form* if it has the following two additional properties:

(iii) Each leading nonzero entry is 1.
(vi) Each leading nonzero entry is the only nonzero entry in its column.

Gaussian Elimination Method

- Elementary Row Transformations

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix}$$

- (Row Swap) Exchange any two rows.
- (Scalar Multiplication) Multiply any row by a constant.
- (Row Sum) Add a multiple of one row to another row.

$$\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

Gaussian Elimination Example

- Elementary Row Transformations

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix}$$

apply the row operations "Add $-2R_1$ to R_2" and "Add $-3R_1$ to R_3."

"Add $-\frac{3}{2}R_2$ to R_3."

$$A \sim \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right)$$

multiply R_3 by $-1/2$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

Multiply R_2 by $\frac{1}{2}$

"Add $3R_1$ to R_1". $A \sim \begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 7 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Solving a System Using Gauss Elimination Method

Solve the system: $\begin{cases} x + 2y + z = 3 \\ 2x + 5y - z = -4 \\ 3x - 2y - z = 5 \end{cases}$

Reduce its augmented matrix M to echelon form and then to row canonical form as follows:

$$M = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus the system has the unique solution $x = 2$, $y = -1$, $z = 3$ or, equivalently, the vector $u = (2, -1, 3)$.

Solving Ax=b

x + 2y + z = 0

y - z = 2

x +2z= 1

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$$

Write system of equations in matrix form.

x + 2y + z = 0

y - z = 2

-2y + z= 1

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

Subtract first row from last row.

x + 2y + z = 0

y - z = 2

- z = 5

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 5 \end{pmatrix}$$

Add 2 copies of second row to last row.

Now solve by back-substitution: z = -5, y = 2-z = 7, x = -2y-z = -9

Solving a System Using Gauss Elimination Method

Reduce the matrix $A = \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{bmatrix}$ to row canonical form.

First reduce A to echelon form by applying the operations "Add $-R_1$ to R_2" and "Add $-2R_1$ to R_3," and then the operation "Add $-3R_2$ to R_3." These operations yield

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 9 & 3 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

Now use back-substitution on the echelon matrix to obtain the row canonical form of A. Specifically, first multiply R_3 by $\frac{1}{2}$ to obtain the pivot $a_{34} = 1$, and then apply the operations "Add $2R_3$ to R_2" and "Add $-R_3$ to R_t." These operations yield

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 & \frac{3}{2} \\ 0 & 3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Now multiply R_2 by $\frac{1}{3}$ making the pivot $a_{22} = 1$, and then apply the operation "Add $2R_2$ to R_1." We obtain

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{11}{3} & 0 & \frac{17}{6} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Matrix Inverse

A square matrix A is said to be *invertible* (or *nonsingular*) if there exists a matrix B such that

$$AB = BA = I, \qquad \text{(the identity matrix)}.$$

Such a matrix B is unique; it is called the *inverse* of A and is denoted by A^{-1}. Observe that B is the inverse of A if and only if A is the inverse of B. For example, suppose

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A and B are inverses.

Matrix Inverse (Gauss Elimination Method)

Find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$.

Form the matrix $M = (A, I)$ and reduce M to echelon form:

$$M = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

In echelon form, the left half of M is in triangular form; hence A is invertible. Further row reduce M to row canonical form:

$$M \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

The identity matrix is in the left half of the final matrix; hence the right half is A^{-1}. In other words,

$$A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

Matrix Inversion Summary

- To solve $Ax = B$, we can write a closed-form solution if we can find a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$ (identity matrix)

- Then $Ax = B$ iff $x = A^{-1}$ B:

$$x = Ix = A^{-1}Ax = A^{-1}B$$

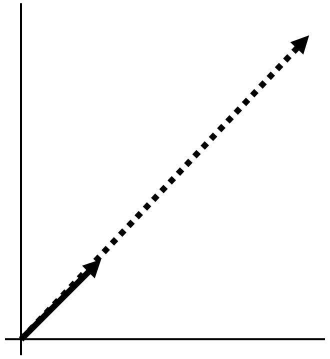
- A is *non-singular* iff A^{-1} exists iff $Ax = B$ has a unique solution.

- Note: If A^{-1}, B^{-1} exist, then $(AB)^{-1} = B^{-1}A^{-1}$, and $(A^T)^{-1} = (A^{-1})^T$

Matrices as linear transformations

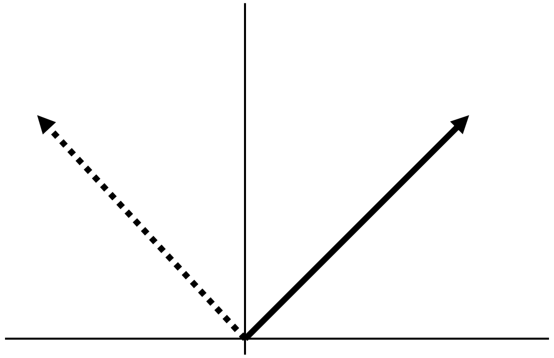
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$



(stretching)

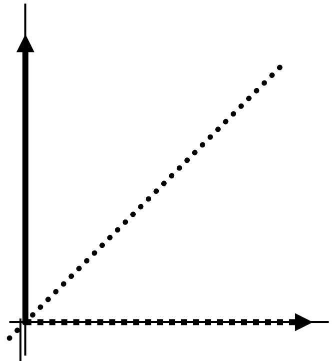
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



(rotation)

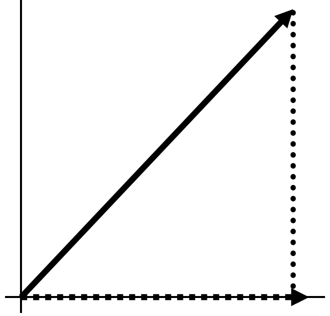
Matrices as linear transformations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



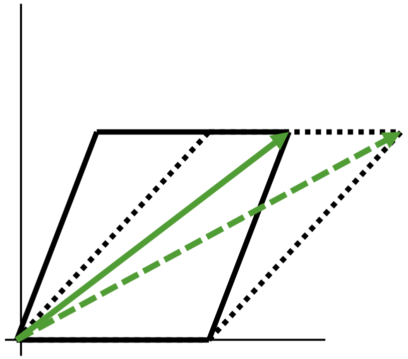
(reflection)

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



(projection)

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + cy \\ y \end{pmatrix}$$



(shearing)

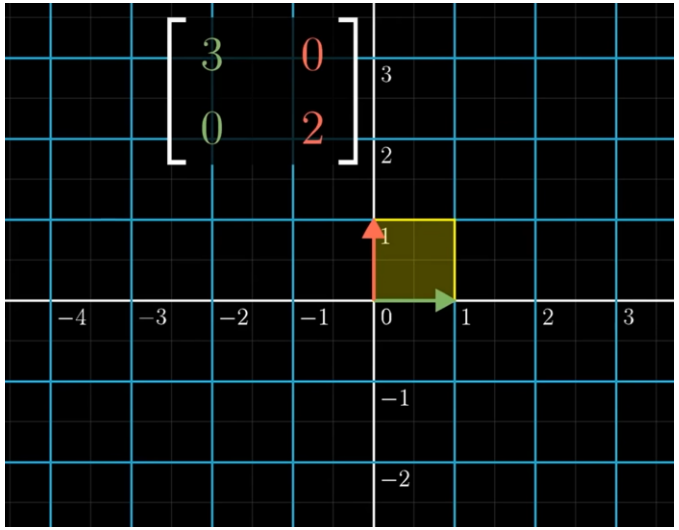
$\boxed{\text{function view of matrix} \quad \Rightarrow \quad \boldsymbol{y} = f(x) = \boldsymbol{A}\boldsymbol{x}}$

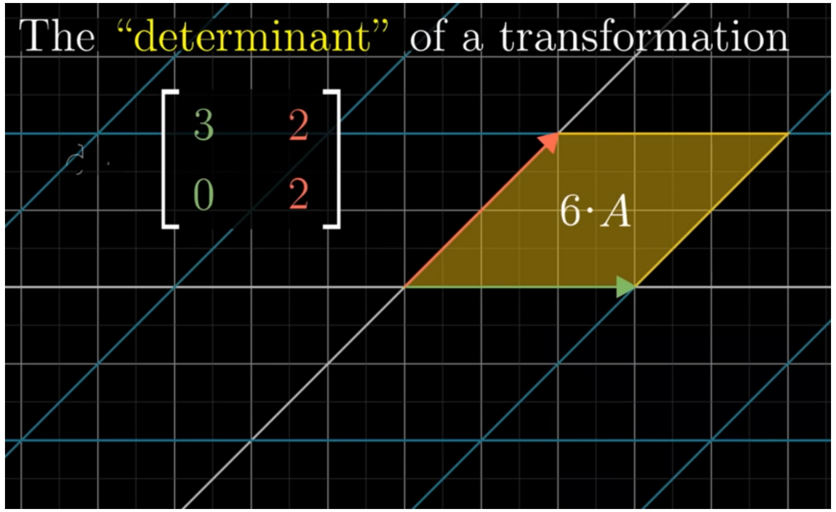
Determinants

Take a unit matrix, think of its area.

Apply shear and scale the basis vectors and see the area.

The scaling factor by which a matrix transformation changes any area is called the **determinant** of the matrix





Determinants

The **determinate** of a square, 2 × 2 matrix *A* is

$$det(A) = |A| = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1} * a_{2,2} - a_{1,2} * a_{2,1}$$

For example

$$det(A) = |A| = \begin{vmatrix} 5 & 2 \\ 7 & 2 \end{vmatrix} = 5 * 2 - 2 * 7 = -4$$

For a 3 × 3 matrix *B*, the determinate is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

For example:

$$det(B) = |B| = \begin{vmatrix} 4 & 0 & -1 \\ 2 & -2 & 3 \\ 7 & 5 & 0 \end{vmatrix} = -1 \begin{vmatrix} 2 & -2 \\ 7 & 5 \end{vmatrix} - 3 \begin{vmatrix} 4 & 0 \\ 7 & 5 \end{vmatrix} + 0 \begin{vmatrix} 4 & 0 \\ 2 & -2 \end{vmatrix}$$

Determinants

- If det(A) = 0, then A is singular.
- If det(A) ≠ 0, then A is invertible.
- To compute:
  - Simple example:
  - Python: function

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

The general definition of a determinant of order n is as follows:

$$\det(A) = \sum \operatorname{sgn}(\sigma) a_{1j_1} a_{2j_2} \ldots a_{nj_n}$$

where the sum is taken over all permutations $\sigma = \{j_1, j_2, \ldots, j_n\}$ of $\{1, 2, \ldots, n\}$. Here $\operatorname{sgn}(\sigma)$ equals $+1$ or -1

Norms in Linear Algebra

The **norm** of a vector or matrix is a measure of the "length" of said vector or matrix. For a vector *x*, the most common norm is the $\mathbf{L_2}$ **norm**, or **Euclidean norm**. It is defined as

$$\|x\| = \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Other common vector norms include the $\mathbf{L_1}$ **norm**, also called the **Manhattan norm** and **Taxicab norm**.

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

The most commonly used matrix norm is the **Frobenius norm**. For a $m \times n$ matrix A, the Frobenius norm is defined as:

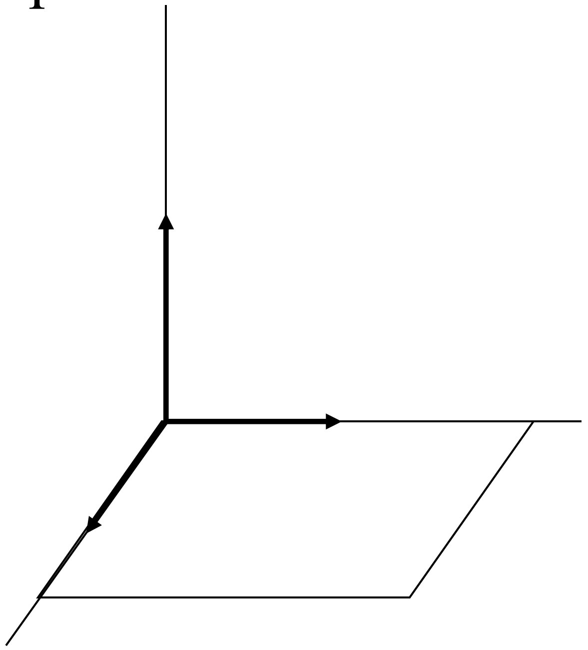
$$\|A\| = \|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j}^2}$$

Vector spaces

- Formally, a *vector space* is a set of vectors which is closed under addition and multiplication by real numbers.
- A *subspace* is a subset of a vector space which is a vector space itself, e.g. the plane $z=0$ is a subspace of R^3 (It is essentially R^2.).
- We'll be looking at R^n and subspaces of R^n

Our notion of planes in R^3 may be extended to *hyperplanes* in R^n (of dimension n-1)

Note: subspaces must include the origin (zero vector).

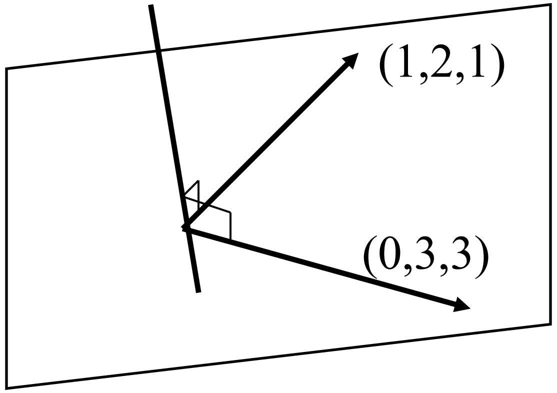


Linear system & subspaces

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$u \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + v \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- Linear systems ($Ax = B$) define certain subspaces
- $Ax = B$ is solvable iff B may be written as a linear combination of the columns of A
- The set of possible B vectors forms a subspace, called the *column space* of A



Linear system & subspaces

The set of solutions to Ax = 0 forms a subspace called the *null space* of A.

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

→ Null space: {(0,0)}

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

→ Null space: {(c,c,-c)}

Linear Combination of Vectors

Let $x_1, x_2, \ldots, x_n$ be *m*-dimensional vectors. Then a **linear combination** of $x_1, x_2, \ldots, x_n$ is any *m*-dimensional vector that can be expressed as

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

where $c_1, \ldots, c_n$ are all scalars. For example: [A result of vector additions and scalar multiplications only.]

$$x_1 = \begin{pmatrix} 3 \\ 8 \\ -2 \end{pmatrix}, x_2 = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$$

$$y = \begin{pmatrix} -5 \\ 12 \\ -8 \end{pmatrix} = 1 * \begin{pmatrix} 3 \\ 8 \\ -2 \end{pmatrix} + (-2) * \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} = 1 * x_1 + (-2) * x_2$$

So *y* is a linear combination of x_1 and x_2. The set of all linear combinations of $x_1, x_2, \ldots, x_n$ is called the **span** of $x_1, x_2, \ldots, x_n$. In other words

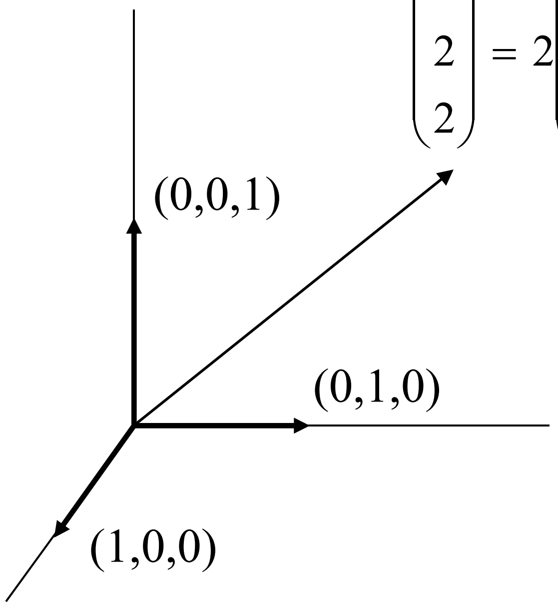
$$span(\{x_1, x_2, \ldots, x_n\}) = \{v | v = \sum_{i=1}^{n} c_i x_i, c_i \in \mathbb{R}\}$$

Linear Combinations and Span

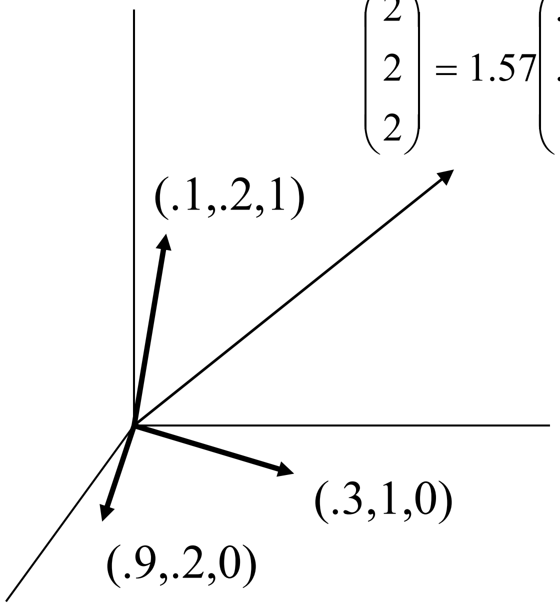
- If all vectors in a vector space may be expressed as linear combinations of $v_1,\ldots,v_k$,

then $v_1,\ldots,v_k$ *span* the space.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$



$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



Linear Independence

A set of vectors $x_1, x_2, \ldots, x_n$ is **linearly independent** if none of the vectors in the set can be expressed as a linear combination of the other vectors. Another way to think of this is a set of vectors $x_1, x_2, \ldots, x_n$ are linearly independent if the only solution to the below equation is to have $c_1 = c_2 = \ldots = c_n = 0$, where $c_1, c_2, \ldots, c_n$ are scalars, and 0 is the zero vector (the vector where every entry is 0).

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = 0$$

If a set of vectors is not linearly independent, then they are called **linearly dependent**.

Linear Independence

$$x_1 = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}, x_2 = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}, x_3 = \begin{pmatrix} 6 \\ 8 \\ -2 \end{pmatrix}$$

Does there exist a vector *c*, such that,

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$$

To answer the question above, let:

$$\begin{aligned} 3c_1 + 4c_2 + 6c_3 &= 0, \\ 4c_1 - 2c_2 + 8c_3 &= 0, \\ -2c_1 + 2c_2 - 2c_3 &= 0 \end{aligned}$$

Thus $\{x_1, x_2, x_3\}$ is linearly independent.

Solving the above system of equations shows that the only possible solution is $c_1 = c_2 = c_3 = 0$.

$$x_1 = \begin{pmatrix} 1 \\ -8 \\ 8 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

In this case $\{x_1, x_2, x_3\}$ are linearly dependent, because if $c = (-1,1,-3)^T$, then

$$\mathrm{c}X = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} (x_1 \; x_2 \; x_3) = -1 \begin{pmatrix} 1 \\ -8 \\ 8 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

Linear Independence

- Vectors $v_1, \dots, v_k$ are linearly independent if

$$c_1 v_1 + \cdots + c_k v_k \;=\; 0 \text{ implies } c_1 = \cdots = ck = 0$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Recall nullspace contained only (u,v)=(0,0).
i.e., the columns are linearly independent.

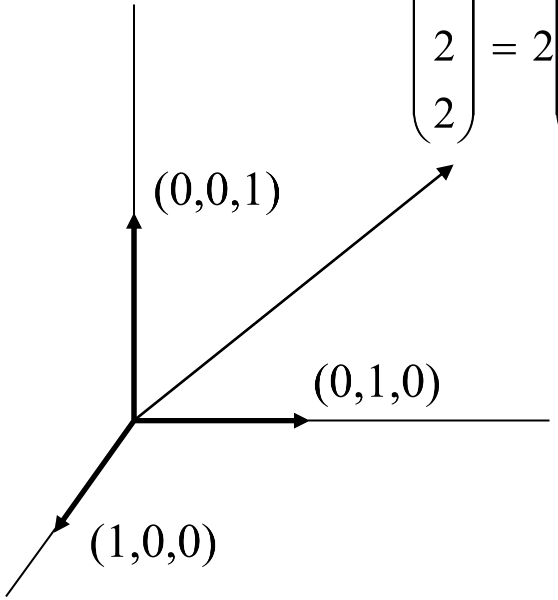
i.e., the nullspace is the origin

$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

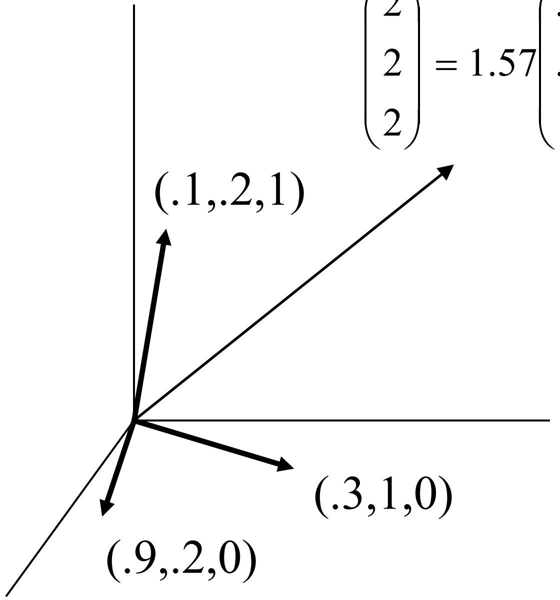
Linear independence and basis

- A *basis* is a set of linearly independent vectors which span the space.
- The *dimension* of a space is the # of "degrees of freedom" of the space; it is the number of vectors in any basis for the space.
- A basis is a maximal set of linearly independent vectors and a minimal set of spanning vectors.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$



$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



Orthogonal Vectors

Two vectors, *x* and *y*, are **orthogonal** if their dot product is zero.

For example

$$\boxed{e \cdot f = \begin{pmatrix} 2 & 5 & 4 \end{pmatrix} * \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = 2 * 4 + (5) * (-2) + 4 * 5 = 8 - 10 + 20 = 18}$$

Vectors *e* and *f* are not orthogonal.

$$g \cdot h = \begin{pmatrix} 2 & 3 & -2 \end{pmatrix} * \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = 2 * 4 + (3) * (-2) + (-2) * 1 = 8 - 6 - 2 = 0$$

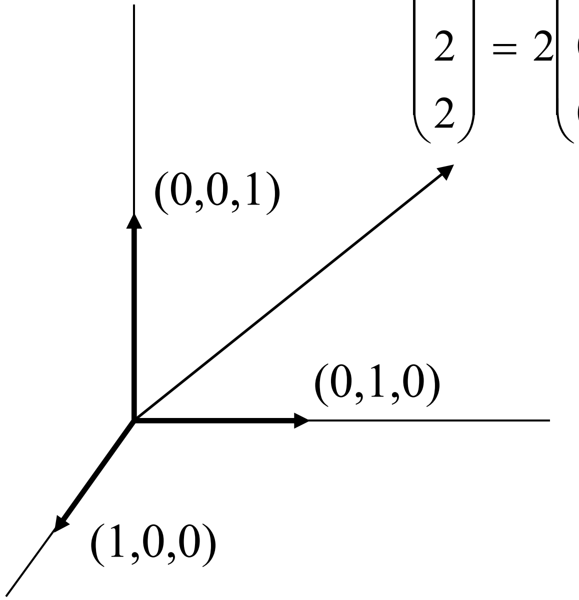
However, vectors *g* and *h* are orthogonal.

Orthogonal can be thought of as an expansion of perpendicular for higher dimensions.

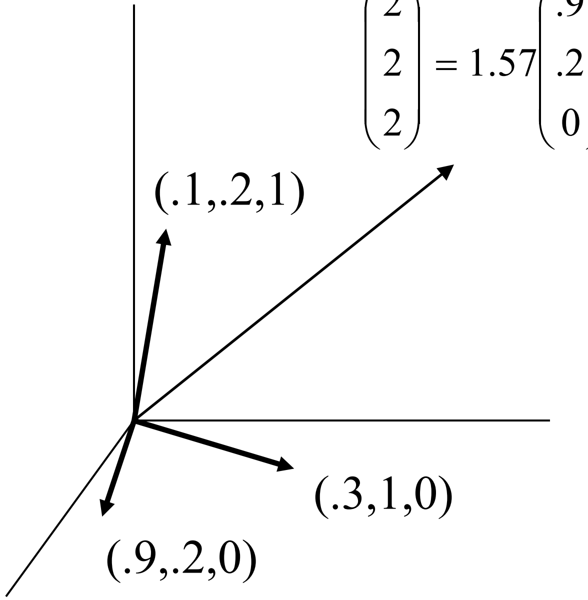
Linear independence and basis

- An *orthogonal basis* consists of orthogonal vectors.
- An *orthonormal basis* consists of orthogonal vectors of unit length.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$



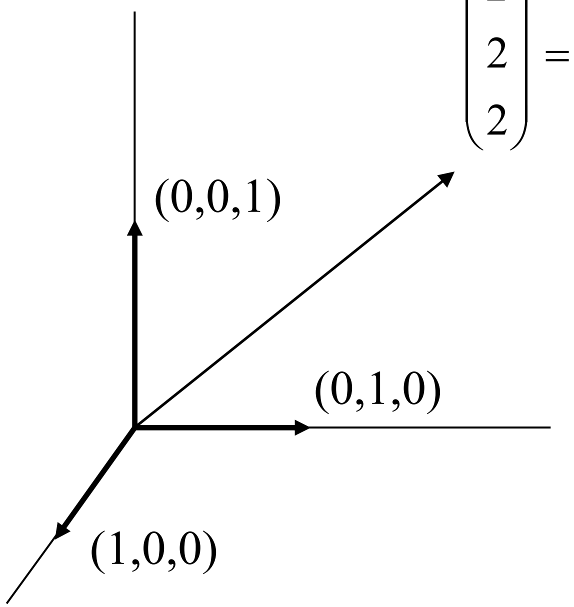
$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



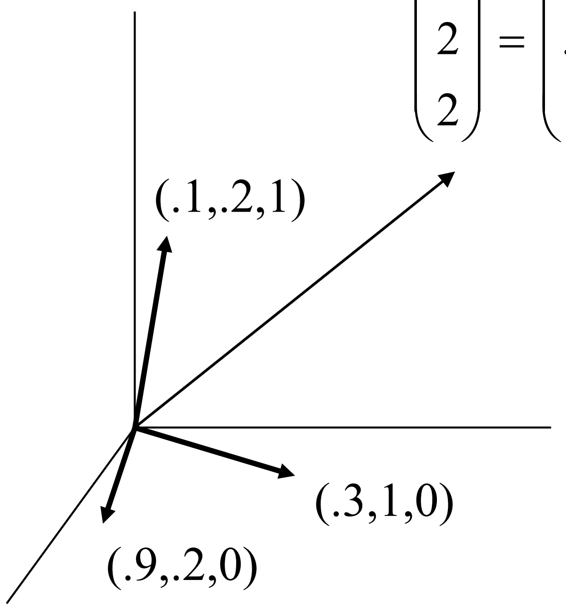
Basis transformations

We may write v=(2,2,2) in terms of an alternate basis:

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}2\\2\\2\end{pmatrix}$$



$$\begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}.9&.3&.1\\.2&1&.2\\0&0&1\end{pmatrix}\begin{pmatrix}1.57\\1.29\\2\end{pmatrix}$$



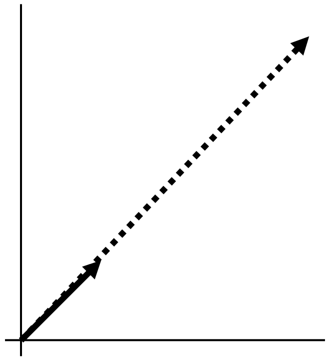
Components of (1.57,1.29,2) are projections of v onto new basis vectors, normalized so new v still has same length.

Matrices as linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}}_{} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

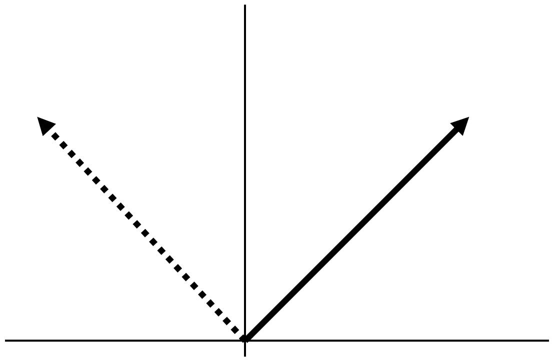
Linear combinations of new (basis) vectors

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$



(stretching)

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



(rotation)

Eigenvalues & eigenvectors

An **eigenvector** of a matrix A is a vector whose product when multiplied by the matrix is a scalar multiple of itself. The corresponding multiplier is often denoted as $lambda$ and referred to as an **eigenvalue**. In other words, if A is a matrix, v is a eigenvector of A, and λ is the corresponding eigenvalue, then $Av = \lambda v$.

$$A = \begin{pmatrix} 4 & 0 & -1 \\ 2 & -2 & 3 \\ 7 & 5 & 0 \end{pmatrix}$$

$$v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$\boxed{\text{function view of matrix} \quad \Rightarrow \quad \boldsymbol{y} = f(x) = \boldsymbol{Ax}}$

$$Av = \begin{pmatrix} 4 & 0 & 1 \\ 2 & -2 & 3 \\ 5 & 7 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4*1+0*1+1*2 \\ 2*1+-2*1+3*2 \\ 5*1+7*1+0*2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = 6v$$

v is an eigenvector of A, and the corresponding eigenvalue is 6.

Eigenvalues & eigenvectors

- How can we characterize matrices? Using Eigenvectors
  - If we apply a matrix transformation to a space, most vectors in space will rotate off the current position
  - Except Eigenvectors
    - The vector that stays on the same span after the matrix transformation
    - If there is a common vector around which the matrices can be rotated [axis of rotation]
    - May just scale up or down

- The solutions to $Av = \lambda v$ in the form of eigenpairs

(λ, v) = (eigenvalue,eigenvector) where v is non-zero

- To find eigenvalues λ, we must solve the characteristic equation

$$\mathbf{(A - \lambda I)v = 0}$$

- λ is an eigenvalue iff $\det(A - \lambda I) = 0$

Eigenvalues & eigenvectors

(A – λI)x = 0

λ is an eigenvalue iff det(A – λI) = 0

Example:

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 3/4 & 6 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 4 & 5 \\ 0 & 3/4-\lambda & 6 \\ 0 & 0 & 1/2-\lambda \end{pmatrix} = (1-\lambda)(3/4-\lambda)(1/2-\lambda)$$

$$\lambda = 1, \lambda = 3/4, \lambda = 1/2$$

Eigenvalues

$$A = \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$$

The characteristic equation of *A* is listed below.

$$det(A - \lambda I) = det(\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = det \begin{pmatrix} 4-\lambda & 3 \\ 2 & -1-\lambda \end{pmatrix} = 0$$

$$det(A - \lambda I) = (4-\lambda)(-1-\lambda) - 3 * 2 = \lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5) = 0$$

eigenvalues of A must be -2 and 5.

Eigenvectors?

$$A = \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$$

$\boxed{\text{eigenvalues of A must be -2 and 5.}}$

$$\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} * \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 5 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 4v_1 + 3v_2 \\ 2v_1 - 1v_2 \end{pmatrix} = \begin{pmatrix} 5v_1 \\ 5v_2 \end{pmatrix}$$

And then solve the resulting system of linear equations to get

$$v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

For $\lambda = -2$, simply set up the equation as below, where the unknown eigenvector is $w = (w_1, w_2)$.

$$\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} * \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\begin{pmatrix} 4w_1 + 3w_2 \\ 2w_1 - 1w_2 \end{pmatrix} = \begin{pmatrix} -2w_1 \\ -2w_2 \end{pmatrix}$$

And then solve the resulting system of linear equations to get

$$w = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Eigenvalues & eigenvectors

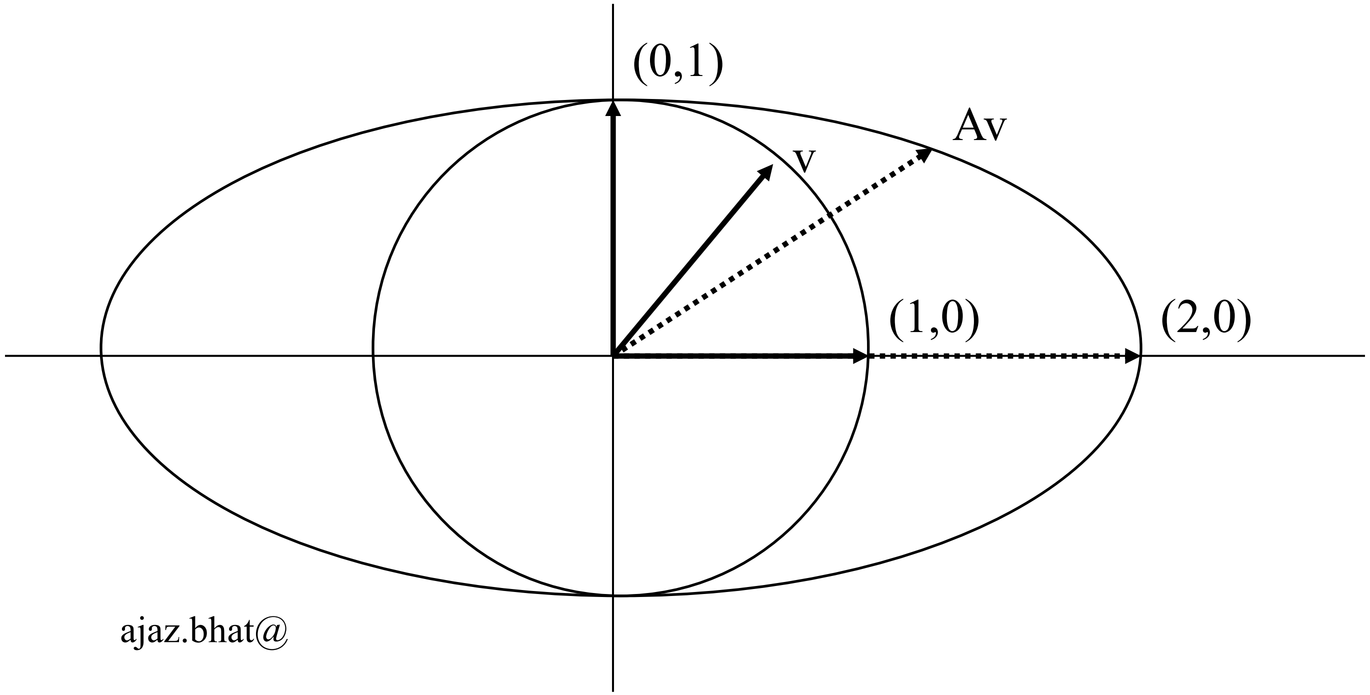
Eigenvectors of a linear transformation A are not rotated (but will be scaled by the corresponding eigenvalue) when A is applied.

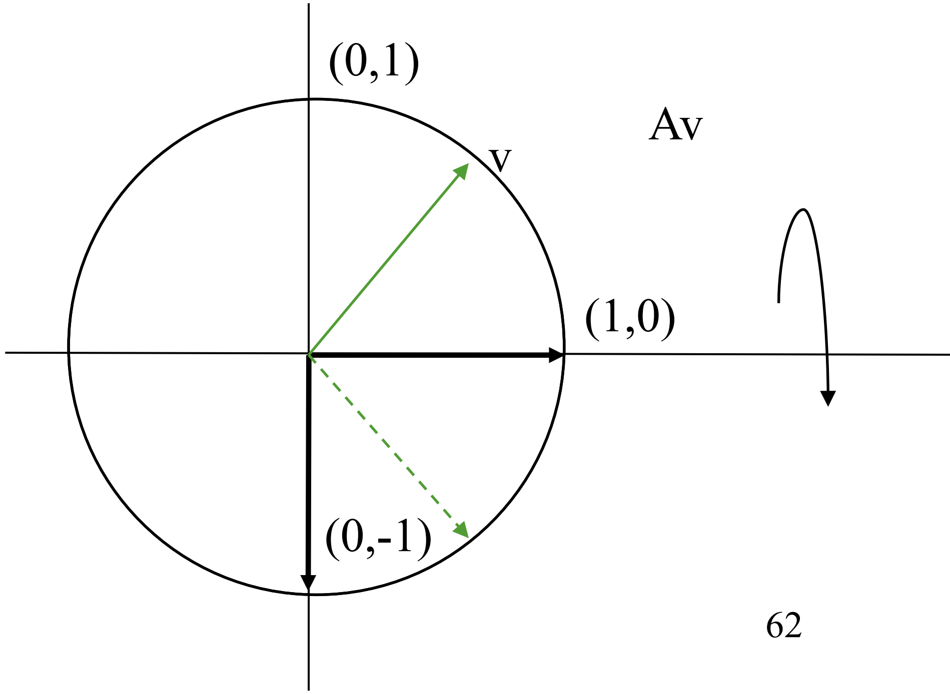
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues λ = 2, 1 with eigenvectors (1,0), (0,1)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenvalue λ = 1 with eigenvector (1,0)





References

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- 3Blue1Brown Video series on the Essence of Linear Algebra https://youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&si=U3AYjl98X3dBY6ML