Limits, Continuity & Differentiation

ZZ-1104 Essential Mathematics for Digital Science

Limits

Life as a shopkeeper's son:

1. Potatoes (a full sack), need to be sold as soon as possible.
    a. You weigh and you pay @ \$3 $per\ kg$
    b. Lucky weight not to be paid! i.e., \$0 $if\ weight\ = 2\ kg$

Let $y\ =\ f(x)$ represent the price of your potatoes in dollars as a function of its weight x in kilograms. Write an equation $y\ =\ f(x)$ as a piecewise defined function

$$f(x) = \begin{cases} 3x & if\ x \neq 2 \\ 0 & if\ x = 2 \end{cases}$$

Draw a graph of $y\ =\ f(x)$.

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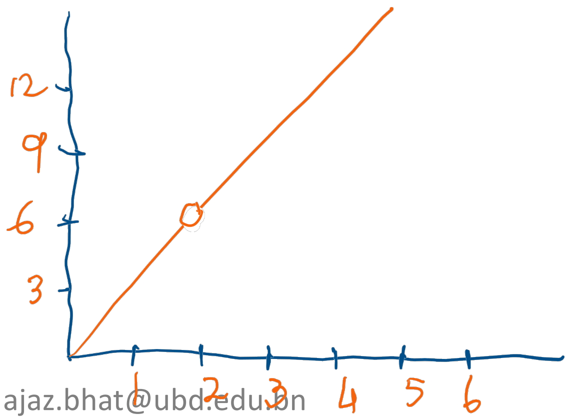

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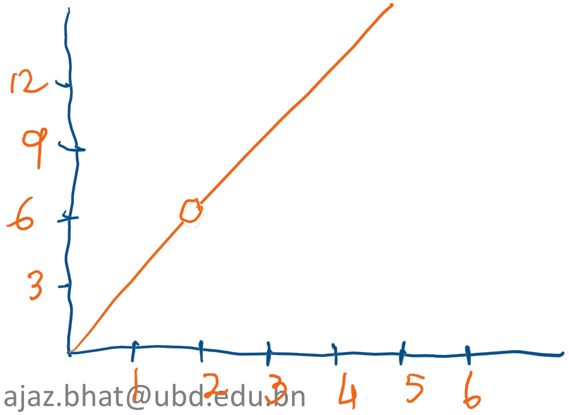


Describe the behaviour of $f(x)$ when x is near 2 but not equal to 2

$$\lim_{x \to 2} f(x) = 6$$

$$f(x=2) = 0$$

Limits Definition (Informal)

$$\lim_{x \to a} f(x) = L \qquad \text{means,}$$

that L is the limit of $f(x)$ as x approaches/tends to a

i.e., the value of $f(x)$ gets arbitrarily close to L as x gets arbitrarily close to a.

Notice the difference between $\lim_{x \to a} f(x) = L$ and $f(x) = L$

For example: (i) $\lim_{x \to 3} x^2 = 9$ *[case: limit and function have same values]*

since x^2 gets arbitrarily close to 9 as x approaches as close as one wishes to 3.

(ii) *[case: function not defined at x but the limit has a value]*

$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$, although $\frac{x^2 - 4}{x - 2}$ is not defined when $x = 2$. Since

$$\frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x - 2} = x + 2$$

Theorems on Limits

Theorem 7.1: If $f(x) = c$, a constant, then $\lim_{x\to a} f(x) = c$.

For the next five theorems, assume $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$.

Theorem 7.2: $\lim_{x\to a} c \cdot f(x) = c \lim_{x\to a} f(x) = cA.$

Theorem 7.3: $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B.$

Theorem 7.4: $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A \cdot B.$

Theorem 7.5: $\lim_{x\to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}$, if $B \neq 0$.

Theorem 7.6: $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{A}$, if $\sqrt[n]{A}$ is defined.

Solved Problems

(a) $\lim_{x \to 2} 5x = 5 \lim_{x \to 2} x = 5 \cdot 2 = 10$

(b) $\lim_{x \to 2} (2x + 3) = 2 \lim_{x \to 2} x + \lim_{x \to 2} 3 = 2 \cdot 2 + 3 = 7$

(d) $\lim_{x \to 3} \dfrac{x-2}{x+2} = \dfrac{\lim_{x \to 3}(x-2)}{\lim_{x \to 3}(x+2)} = \dfrac{1}{5}$

(e) $\lim_{x \to -2} \dfrac{x^2 - 4}{x^2 + 4} = \dfrac{4-4}{4+4} = 0$

(f) $\lim_{x \to 4} \sqrt{25 - x^2} = \sqrt{\lim_{x \to 4} (25 - x^2)} = \sqrt{9} = 3$

(b) $\lim_{x \to \pm\infty} \dfrac{6x^2 + 2x + 1}{5x^2 - 3x + 4} = \lim_{x \to \pm\infty} \dfrac{6 + 2/x + 1/x^2}{5 - 3/x + 4/x^2} = \dfrac{6+0+0}{5-0+0} = \dfrac{6}{5}$

4. Given $f(x) = x^2 - 3x$, find $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Since $f(x) = x^2 - 3x$, we have $f(x+h) = (x+h)^2 - 3(x+h)$ and

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x^2 + 2hx + h^2 - 3x - 3h) - (x^2 - 3x)}{h} = \lim_{h \to 0} \frac{2hx + h^2 - 3h}{h}$$

$$= \lim_{h \to 0} (2x + h - 3) = 2x - 3.$$

Continuity

A function f is defined to be continuous at x_0 if the following three conditions hold:

(i) $f(x_0)$ is defined;

(ii) $\lim_{x \to x_0} f(x)$ exists;

(iii) $\lim_{x \to x_0} f(x) = f(x_0)$.

Use above conditions to test below questions

$f(x) = x^2 + 1$ is continuous at 2

$f(x) = \sqrt{4 - x^2}$ is continuous at 3?

Plot & Check Continuity at 2

$$f(x) = \frac{1}{x-2}$$

Continuity

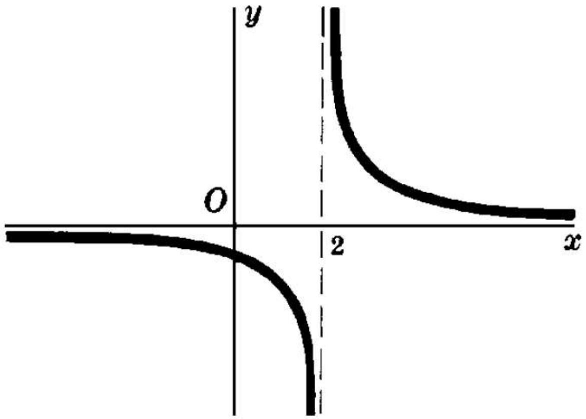
A function f is defined to be continuous at x_0 if the following three conditions hold:

(i) $f(x_0)$ is defined;

(ii) $\lim_{x \to x_0} f(x)$ exists;

(iii) $\lim_{x \to x_0} f(x) = f(x_0)$.

$f(x) = \dfrac{1}{x-2}$ is discontinuous at 2 because $f(2)$ is not defined and also because $\lim_{x \to 2} f(x)$ does not exist (since $\lim_{x \to 2} f(x) = \infty$). See Fig. 8-1.



$f(x) = \dfrac{x^2-4}{x-2}$ is discontinuous at 2 because $f(2)$ is not defined. However, $\lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{(x+2)(x-2)}{x-2} = \lim_{x \to 2}(x+2) = 4$ so that condition (ii) holds.

Removable Discontinuity, by defining $f(2) = 4$

Continuity: Solved Problems

(a) $f(x) = \frac{2}{x}$. Nonremovable discontinuity at $x = 0$.

(c) $f(x) = \frac{(x+2)(x-1)}{(x-3)^2}$. Nonremovable discontinuity at $x = 3$.

(k) $f(x) = \begin{cases} x & \text{if } x \le 0. \\ x^2 & \text{if } 0 < x < 1 \\ 2 - x & \text{if } x \ge 1. \end{cases}$ No discontinuities.

Rate of Change of Functions

$y = f(x)$. Consider any number x_0 in the domain of f. Let Δx (read "delta x") represent a small change in the value of x, from x_0 to $x_0 + \Delta x$, and then let Δy (read "delta y") denote the corresponding change in the value of y. So, $\Delta y = f(x_0 + \Delta x) - f(x_0)$. Then the ratio

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called the *average rate of change* of the function f on the interval between x_0 and $x_0 + \Delta x$.

EXAMPLE 9.1: Let $y = f(x) = x^2 + 2x$. Starting at $x_0 = 1$, change x to 1.5. Then $\Delta x = 0.5$. The corresponding change in y is $\Delta y = f(1.5) - f(1) = 5.25 - 3 = 2.25$. Hence, the average rate of change of y on the interval between $x = 1$ and $x = 1.5$ is $\frac{\Delta y}{\Delta x} = \frac{2.25}{0.5} = 4.5$.

Example Problems on Rate of Change

2. If a body (that is, a material object) starts out at rest and then falls a distance of s feet in t seconds, then physical laws imply that $s = 16t^2$. Find $\Delta s/\Delta t$ as t changes from t_0 to $t_0 + \Delta t$. Use the result to find $\Delta s/\Delta t$ as t changes: (a) from 3 to 3.5, (b) from 3 to 3.2, and (c) from 3 to 3.1.

$$\frac{\Delta s}{\Delta t} = \frac{16(t_0 + \Delta t)^2 - 16t_0^2}{\Delta t} = \frac{32t_0 \Delta t + 16(\Delta t)^2}{\Delta t} = 32t_0 + 16\,\Delta t$$

(a) Here $t_0 = 3$, $\Delta t = 0.5$, and $\Delta s/\Delta t = 32(3) + 16(0.5) = 104$ ft/sec.
(b) Here $t_0 = 3$, $\Delta t = 0.2$, and $\Delta s/\Delta t = 32(3) + 16(0.2) = 99.2$ ft/sec.
(c) Here $t_0 = 3$, $\Delta t = 0.1$, and $\Delta s/\Delta t = 97.6$ ft/sec.

Since Δs is the displacement of the body from time $t = t_0$ to $t = t_0 + \Delta t$,

$$\frac{\Delta s}{\Delta t} = \frac{\text{displacement}}{\text{time}} = \text{average velocity of the body over the time interval}$$

Instantaneous Rate of Change = Derivative

If $y = f(x)$ and x_0 is in the domain of f, then by the *instantaneous rate of change* of f at x_0 we mean the limit of the average rate of change between x_0 and $x_0 + \Delta x$ as Δx approaches 0:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

provided that this limit exists. This limit is also called the *derivative* of f at x_0.

Notations: $D_x y = \dfrac{dy}{dx} = y' = f'(x) = \dfrac{d}{dx} y = \dfrac{d}{dx} f(x) = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

Solved Problems on Derivatives

3. Find dy/dx, given $y = x^3 - x^2 - 4$. Find also the value of dy/dx when (a) $x = 4$, (b) $x = 0$, (c) $x = -1$.

$$y + \Delta y = (x + \Delta x)^3 - (x + \Delta x)^2 - 4$$

$$= x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^2 - 2x(\Delta x) - (\Delta x)^2 - 4$$

$$\Delta y = (3x^2 - 2x)\Delta x + (3x - 1)(\Delta x)^2 + (\Delta x)^3$$

$$\frac{\Delta y}{\Delta x} = 3x^2 - 2x + (3x - 1)\Delta x + (\Delta x)^2$$

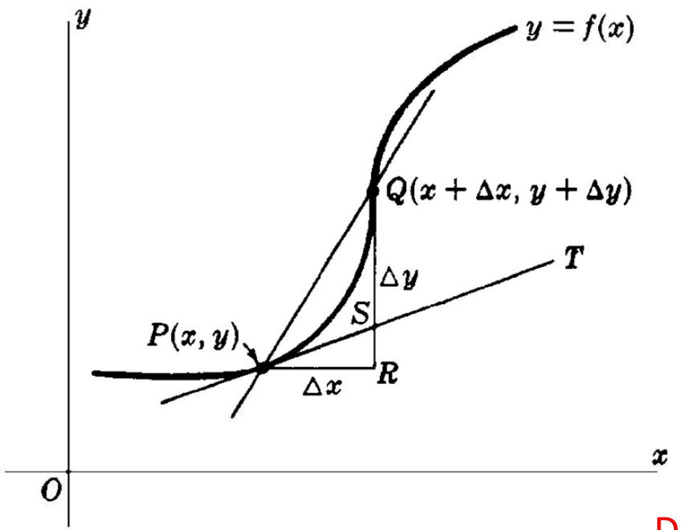
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} [3x^2 - 2x + (3x - 1)\Delta x + (\Delta x)^2] = 3x^2 - 2x$$

(a) $\left.\dfrac{dy}{dx}\right|_{x=4} = 3(4)^2 - 2(4) = 40;$ (b) $\left.\dfrac{dy}{dx}\right|_{x=0} = 3(0)^2 - 2(0) = 0;$ (c) $\left.\dfrac{dy}{dx}\right|_{x=-1} = 3(-1)^2 - 2(-1) = 5$

Derivative: Slope of the Tangent to Curve

9. Interpret dy/dx geometrically.

From Fig. 9-1 we see that $\Delta y/\Delta x$ is the slope of the secant line joining an arbitrary but fixed point $P(x, y)$ and a nearby point $Q(x + \Delta x, y + \Delta y)$ of the curve. As $\Delta x \to 0$, P remains fixed while Q moves along the curve toward P, and the line PQ revolves about P toward its limiting position, the tangent line PT moves to the curve at P. Thus, dy/dx gives the slope of the tangent line at P to the curve $y = f(x)$.



Derivative may change along the function

Fig. 9-1

For example, from Problem 3, the slope of the cubic $y = x^3 - x^2 - 4$ is $m = 40$ at the point $x = 4$; it is $m = 0$ at the point $x = 0$; and it is $m = 5$ at the point $x = -1$.