Counting

with Ajaz Bhat

Counting as a Tool

Counting is not just about "numbers"

but organizing, planning, and analysing different scenarios in structured ways.

Simple Examples:

How many outfits can you make with 3 shirts and 2 pairs of pants?

How many routes can you take to go from home to work, considering multiple transport options?

Basic Counting Principles

The sum rule: (Either/Or Rule)

Sum Rule Principle:

Suppose some event E can occur in m ways and a second event F can occur in n ways, and suppose both events cannot occur simultaneously. Then E or F can occur in $m + n$ ways.

Example:

SDS will award a free computer to either a DigSci student or a DigSci professor.
How many different choices are there, if there are 230 students and 15 professors?

There are 230 + 15 = 245 choices.

The Product rule:(And Rule)

Product Rule Principle:

Suppose there is an event E which can occur in m ways and, independent of this event, there is a second event F which can occur in n ways. Then combinations of E and F can occur in mn ways.

Example:

How many different license plates are there that contain exactly three English letters ?

Solution:

There are 26 possibilities to pick the first letter, then 26 possibilities for the second one, and 26 for the last one.

So there are $26 \cdot 26 \cdot 26 = 17576$ different license plates.

Basic Counting Principles

EXAMPLE 5.1 Suppose a college has 3 different history courses, 4 different literature courses, and 2 different sociology courses.

(a) The number m of ways a student can choose one of each kind of courses is:

Inclusion-Exclusion

How many bit strings of length 8 either start with a 1 or end with 00?

Task 1: Construct a string of length 8 that starts with a 1.

There is one way to pick the first bit (1),
two ways to pick the second bit (0 or 1),
two ways to pick the third bit (0 or 1),
.
.
.
two ways to pick the eighth bit (0 or 1).

Product rule: Task 1 can be done in $1 \cdot 2^7 = 128$ ways.

Inclusion-Exclusion

Task 2: Construct a string of length 8 that ends with 00.

There are two ways to pick the first bit (0 or 1),
two ways to pick the second bit (0 or 1),
.
.
.
two ways to pick the sixth bit (0 or 1),
one way to pick the seventh bit (0), and
one way to pick the eighth bit (0).

Product rule: Task 2 can be done in $2^6 = 64$ ways.

Inclusion-Exclusion

Since there are 128 ways to do Task 1 and 64 ways to do Task 2, does this mean that there are 192 bit strings either starting with 1 or ending with 00 ?

No, because here Task 1 and Task 2 can be done **at the same time**.

When we carry out Task 1 and create strings starting with 1, some of these strings end with 00.

Therefore, we sometimes do Tasks 1 and 2 at the same time, so **the sum rule does not apply**.

Inclusion-Exclusion

If we want to use the sum rule in such a case, we have to subtract the cases when Tasks 1 and 2 are done at the same time.

How many cases are there, that is, how many strings start with 1 **and** end with 00?

There is one way to pick the first bit (1),
two ways for the second, …, sixth bit (0 or 1),
one way for the seventh, eighth bit (0).

Product rule: In $2^5 = 32$ cases, Tasks 1 and 2 are carried out at the same time.

Inclusion-Exclusion

Since there are 128 ways to complete Task 1 and 64 ways to complete Task 2, and in 32 of these cases Tasks 1 and 2 are completed at the same time, there are

128 + 64 – 32 = 160 ways to do either task.

In set theory, this corresponds to sets A_1 and A_2 that are **not** disjoint. Then we have:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

This is called the **principle of inclusion-exclusion.**

Inclusion-Exclusion

In the below, Find the total number of students studying?

French, German, and Russian, given the following data:

65 study French,	20 study French and German,	
45 study German,	25 study French and Russian,	8 study all three languages.
42 study Russian,	15 study German and Russian,	

We want to find $n(F \cup G \cup R)$ where F, G, and R denote the sets of students studying French, German, and Russian, respectively.

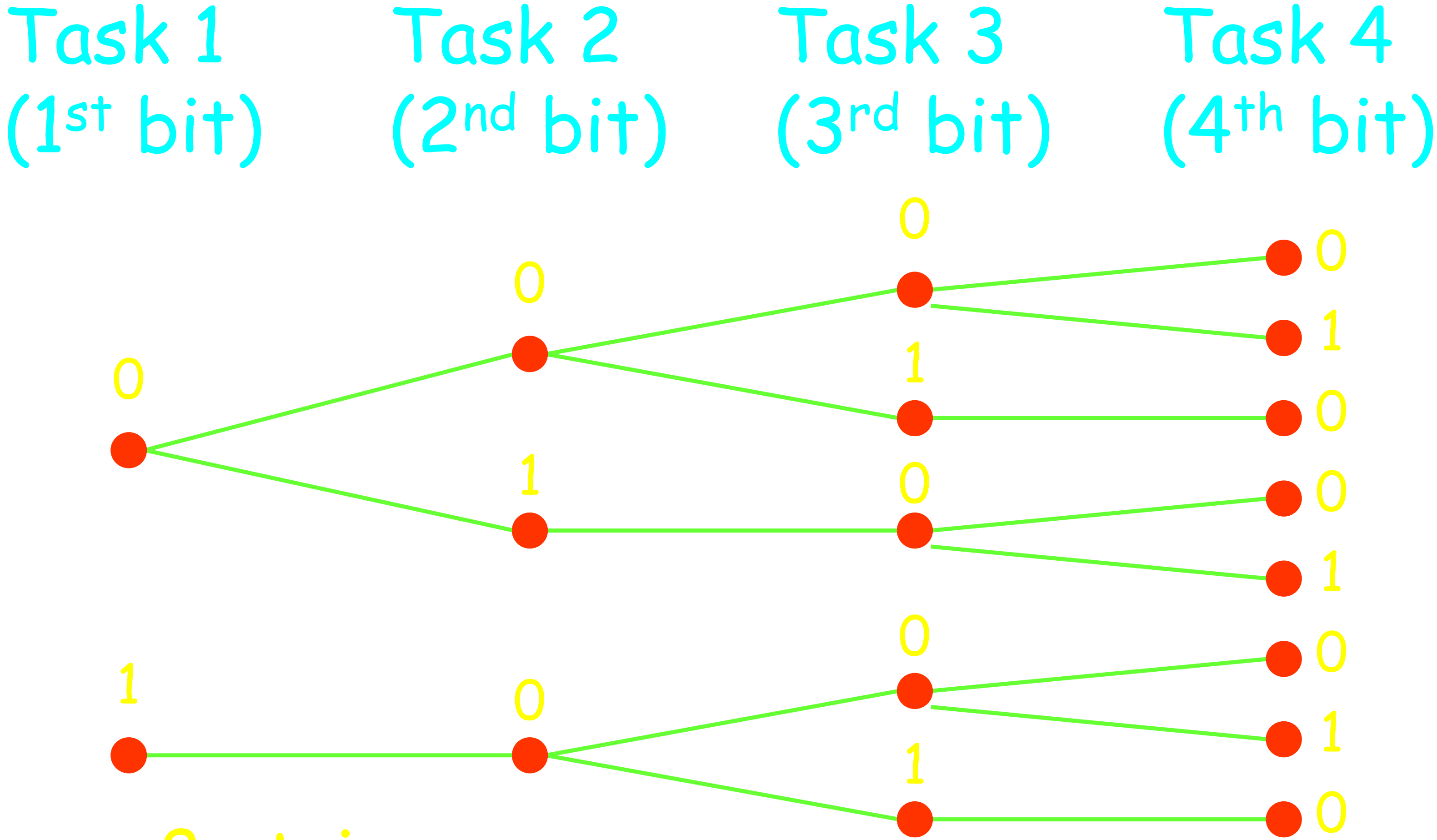
By the Inclusion–Exclusion Principle,

$$\begin{aligned} n(F \cup G \cup R) &= n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

Namely, 100 students study at least one of the three languages.

Tree Diagrams

How many bit strings of length four do not have two consecutive 1s?



There are 8 strings.

The Pigeonhole Principle

The pigeonhole principle: If N [>(k + 1)] or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example 1: If there are 11 players in a soccer team that wins 12-0, there must be at least one player in the team who scored at least twice.

Example 2: If you have 6 classes from Monday to Friday, there must be at least one day on which you have at least two classes.

The Pigeonhole Principle

The generalized pigeonhole principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ of the objects.

If k pigeonholes are occupied by xk+1 or more pigeons, (x>0), then at least one pigeonhole has x+1 or more pigeons

Example 1: In our 100-student class, at least 20 students will get the same letter grade (A, B, C, D, or F).

Example 2: Suppose a class contains 13 students, then two of the students (pigeons N) were born in the same month (pigeonholes k).

The Pigeonhole Principle

Example 3: Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

There are two types of socks, so if you pick at least 3 socks, there must be either at least two brown socks or at least two black socks.

Generalized pigeonhole principle: $\lceil 3/2 \rceil = 2$.

The Pigeonhole Principle

Example 4: Find the minimum number of students in a class to be sure that three of them are born in the same month.

Here the $n = 12$ months are the pigeonholes, and $x + 1 = 3$ so $x = 2$.

Hence among any $xk + 1 = 25$ students (pigeons), three of them are born in the same month

Generalized pigeonhole principle: $\lceil 25/12 \rceil = 3$.

Summary so far

<u>Sum/Product rules</u> handle the basic structure.

<u>Inclusion-Exclusion</u> handles complex overlaps.

<u>Pigeonhole</u> applies to impossible situations.

<u>Permutations/Combinations</u> handle arranging and selecting items.

Permutations and Combinations

How many ways are there to pick a set of 3 people from a group of 6?

There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so there are $6 \cdot 5 \cdot 4 = 120$ ways to do this.

This is not the correct result!

For example, picking person *C*, then person *A*, and then person E leads to the **same group** as first picking E, then *C*, and then *A*.

However, these cases are counted **separately** in the above equation.

Permutations and Combinations

So how can we compute how many different subsets of people can be picked (that is, we want to disregard the order of picking)?

To find out about this, we need to look at **permutations and combinations.**

A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an **r-permutation**.

Permutations and Combinations

Example: Let $S = \{1, 2, 3\}$.
The arrangement 3, 1, 2 is a permutation of S.
The arrangement 3, 2 is a 2-permutation of S.

The number of r-permutations of a set with n distinct elements is denoted by **$P(n, r)$**.

We can calculate $P(n, r)$ with the product rule:

$P(n, r) = n\cdot(n - 1)\cdot(n - 2) \cdot\ldots\cdot(n - r + 1)$.

(n choices for the first element, (n – 1) for the second one, (n – 2) for the third one…)

Permutations and Combinations

Example:

$P(8, 3) = 8 \cdot 7 \cdot 6 = 336$
$= (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

General formula:

$P(n, r) = n!/(n - r)!$

Knowing this, we can return to our initial question:

How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

Permutations and Combinations

An **r-combination** of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

Example: Let $S = \{1, 2, 3, 4\}$.
Then $\{1, 3, 4\}$ is a 3-combination from S.

The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

Example: $C(4, 2) = 6$, since, for example, the 2-combinations of a set $\{1, 2, 3, 4\}$ are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$.

Permutations and Combinations

How can we calculate $C(n, r)$?

Consider that we can obtain the r-permutation of a set in the following way:

First, we form all the r-combinations of the set (there are $C(n, r)$ such r-combinations).

Then, we generate all possible orderings in each of these r-combinations (there are $P(r, r)$ such orderings in each case).

Therefore, we have:

$$P(n, r) = C(n, r) \cdot P(r, r)$$

Permutations and Combinations

$$C(n, r) = P(n, r)/P(r, r)$$
$$= n!/(n - r)!/(r!/(r - r)!) = P(n, r)/r!$$
$$= n!/(r!(n - r)!)$$

Now we can answer our initial question:

How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

$C(6, 3) = 6!/(3!\cdot 3!) = 720/(6\cdot 6) = 720/36 = 20$

There are 20 different ways, that is, 20 different groups to be picked.

Permutations and Combinations

Corollary:

Let n and r be nonnegative integers with $r \leq n$.
Then $C(n, r) = C(n, n - r)$.

Note that **"picking a group of r people from a group of n people"** is the same as **"splitting a group of n people into a group of r people and another group of (n – r) people".**

Combinations

We also saw the following:

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = C(n,r)$$

This symmetry is intuitively plausible. For example, let us consider a set containing six elements (n = 6).

Picking two elements and **leaving four** is essentially the same as **picking four** elements and **leaving two**.

In either case, our number of choices is the number of possibilities to **divide** the set into one set containing two elements and another set containing four elements.

Permutations and Combinations

Example:

A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

$$C(8, 6) \cdot C(7, 5) = 8!/(6!\cdot 2!) \cdot 7!/(5!\cdot 2!)$$
$$= 28\cdot 21$$
$$= 588$$

Sequences & Series

Sequences

Sequences represent **ordered lists** of elements.

A **sequence** is defined as a function from a subset of **N** to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

Example:

subset of **N**:	1	2	3	4	5	…
S:	2	4	6	8	10	…

Sequences

We use the notation $\{a_n\}$ to describe a sequence.

Important: Do not confuse this with the {} used in set notation.

It is convenient to describe a sequence with a **formula**.

For example, the sequence on the previous slide

2 4 6 8 10 … can be specified as

$\{a_n\}$, where $a_n = 2n$.

The Formula Game

What are the formulas that describe the following sequences $a_1, a_2, a_3, \ldots$?

1, 3, 5, 7, 9, …	$a_n = 2n - 1$
-1, 1, -1, 1, -1, …	$a_n = (-1)^n$
2, 5, 10, 17, 26, …	$a_n = n^2 + 1$
0.25, 0.5, 0.75, 1, 1.25 …	$a_n = 0.25n$
3, 9, 27, 81, 243, …	$a_n = 3^n$

Summations

What does $\sum_{j=m}^{n} a_j$ stand for?

It represents the sum $a_m + a_{m+1} + a_{m+2} + \ldots + a_n$.

The variable j is called the **index of summation**, running from its **lower limit** m to its **upper limit** n. We could as well have used any other letter to denote this index.

Summations

How can we express the sum of the first 1000 terms of the sequence $\{a_n\}$ with $a_n=n^2$ for $n = 1, 2, 3, \dots$?

We write it as $\sum_{j=1}^{1000} j^2$.

What is the value of $\sum_{j=1}^{6} j$?

It is 1 + 2 + 3 + 4 + 5 + 6 = 21.

What is the value of $\sum_{j=1}^{100} j$?

It is so much work to calculate this…

Summations

It is said that Friedrich Gauss came up with the following formula:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

When you have such a formula, the result of any summation can be calculated much more easily, for example:

$$\sum_{j=1}^{100} j = \frac{100(100+1)}{2} = \frac{10100}{2} = 5050$$

Arithemetic Series

How does: $\boxed{\sum_{j=1}^{n} j = \frac{n(n+1)}{2}}$???

Observe that:

$1 + 2 + 3 + \ldots + n/2 + (n/2 + 1) + \ldots + (n - 2) + (n - 1) + n$

$= [1 + n] + [2 + (n - 1)] + [3 + (n - 2)] + \ldots + [n/2 + (n/2 + 1)]$

$= (n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1)$ *(with n/2 terms)*

$= n(n + 1)/2.$

Geometric Series

How does: $$\boxed{\sum_{j=0}^{n} a^j = \frac{a^{(n+1)} - 1}{(a-1)}}$$???

Observe that:

$$S = 1 + a + a^2 + a^3 + \ldots + a^n$$

$$aS = a + a^2 + a^3 + \ldots + a^n + a^{(n+1)}$$

so, $(aS - S) = (a - 1)S = a^{(n+1)} - 1$

Therefore, $1 + a + a^2 + \ldots + a^n = (a^{(n+1)} - 1) / (a - 1)$.

For example: $1 + 2 + 4 + 8 + \ldots + 1024 = 2047$.

Follow me for a walk through...

Mathematical Induction

Induction

The **principle of mathematical induction** is a useful tool for proving that a certain proposition is true for **all natural numbers**.

It cannot be used to discover theorems, but only to prove them.

Induction

If we have a propositional function $P(n)$, and we want to prove that $P(n)$ is true for any natural number n, we do the following:

- Show that $P(0)$ is true.
  (basis step)
- Show that if $P(k)$ then $P(k + 1)$ for any $n \in N$.
  (inductive step)
- Then $P(n)$ must be true for any $n \in N$.
  (conclusion)

Induction

Example:

Show that $n < 2^n$ for all positive integers n.

Let P(n) be the proposition "$n < 2^n$."

1. Show that P(1) is true.
   (basis step)

P(1) is true, because $1 < 2^1 = 2$.

Induction

2. Show that if $P(k)$ is true, then $P(k + 1)$ is true.
   (inductive step)

Assume that $k < 2^k$ is true.

We need to show that $P(k + 1)$ is true, i.e.

$k + 1 < 2^{k+1}$

We start from $k < 2^k$:

$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2^{k+1}$

Therefore, if $k < 2^k$ then $k + 1 < 2^{k+1}$

Induction

- Then P(n) must be true for any positive integer.
  (conclusion)

$n < 2^n$ is true for any positive integer.

End of proof.

Induction

Another Example ("Gauss"):

$1 + 2 + \ldots + n = n\,(n + 1)/2$

- Show that P(0) is true.
  (basis step)

For $n = 0$ we get $0 = 0$. True.

Induction

- Show that if $P(k)$ then $P(k + 1)$ for any $n \in \mathbb{N}$. (inductive step)

$$1 + 2 + \ldots + k = k\,(k + 1)/2$$

$$1 + 2 + \ldots + k + (k + 1) = k\,(k + 1)/2 + (k + 1)$$

$$= (2k + 2 + k\,(k + 1))/2$$

$$= (2k + 2 + k^2 + k)/2$$

$$= (2 + 3k + k^2)/2$$

$$= (k + 1)\,(k + 2)/2$$

$$= (k + 1)\,((k + 1) + 1)/2$$

Induction

- Then P(n) must be true for any $n \in N$. (conclusion)

$1 + 2 + \ldots + n = n(n + 1)/2$ is true for all $n \in N$.

End of proof.

Induction

Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n) : 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

$$P(1) = 1^2$$

Assuming $P(k)$ is true, we add $2k + 1$ to both sides of $P(k)$, obtaining

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) - k^2 + (2k + 1) = (k + 1)^2$$

which is $P(k + 1)$. In other words, $P(k + 1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction, P is true for all n.