A guide for selecting water wave theories - Le Mehaute (1976)'s graph revisited

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1. Notation

a = 1^{st} order amplitude in Stokes wave or crest amplitude in cnoidal wave;
A_* = various order term of Stokes solution;
$A_n/A_{n,prec}$ = ratio of higher order to the preceding terms of Stokes equation;
b = trough amplitude in cnoidal wave
B_{**} = coefficients in the Stokes 5^{th} order equation from Fenton (1985);
c = phase speed;
C_* = various order term of cnoidal wave free surface;
d = trough depth;
g = gravitational acceleration, $9.81 m/s^2$ in this manuscript
H = crest to trough wave height (m);
H_B = wave breaking limit (m);
h = mean water depth in Stokes wave or still water depth in cnoidal wave (m);
i, n = an index to indicate the A, C term of various order;
k = wave number;
m = parameter of elliptic functions and integrals;
L = wave length (m);
t = time (s);
T = wave period (s);
Ur = $(H/L)/(h/L)^3$, Ursell number (-);
(x, z) = coordinates system;
α = H/d;
β = $\cosh 2kh$;
ε = a small quantity which expresses the relative variation slowness in x direction;
η = free surface displacement;
θ = a phase parameter;
ω = angular frequency;
σ = $\tanh kh$

2. Coefficients in the free surface solutions of the 5^{th} order Stokes wave theory

The list of coefficients, B_{**}, in the free surface solution of the 5^{th} order Stokes wave theory (Eq. 2 in the main manuscript) are shown here (Zhao and Liu, 2022),

$$
\begin{aligned}
B_{22} &= \frac{3-\sigma^2}{4\sigma^3}, \\
B_{31} &= \frac{3+8\sigma^2-9\sigma^4}{16\sigma^4}, \\
B_{33} &= \frac{27-9\sigma^2+9\sigma^4-3\sigma^6}{64\sigma^6}, \\
B_{42} &= \frac{60\beta^6+232\beta^5-118\beta^4-989\beta^3-607\beta^2+352\beta+260}{24(3\beta+2)(\beta-1)^4\sinh 2kh}, \\
B_{44} &= \frac{24\beta^6+116\beta^5+214\beta^4+188\beta^3+133\beta^2+101\beta+34}{24(3\beta+2)(\beta-1)^4\sinh 2kh}, \\
B_{51} &= \frac{121\beta^5+263\beta^4+376\beta^3-1999\beta^2+2509\beta-1108}{192(\beta-1)^5}, \\
B_{53} &= \frac{9(57\beta^7+204\beta^6-53\beta^5-782\beta^4-741\beta^3-52\beta^2+371\beta+186)}{128(\beta-1)^6(3\beta+2)}, \\
B_{55} &= \frac{5(300\beta^8+1579\beta^7+3176\beta^6+2949\beta^5+1188\beta^4+675\beta^3+1326\beta^2+827\beta+130)}{384(\beta-1)^6(12\beta^2+11\beta+2)}.
\end{aligned}
\tag{1}
$$

where $\sigma = \tanh kh$ and $\beta = \cosh 2kh$.

3. Coefficients in the free surface solutions of 3^{rd} and 5^{th}order cnoidal wave theories

In this section the free surface solutions of the 3^{rd} and 5^{th} orders cnoidal wave theories are provided for completeness. The coefficients, $C_i (i = 1, 2, 3)$, in the free surface elevation solution (η) of the 3^{rd} order cnoidal wave is first listed here (Fenton, 1999).

$$\begin{aligned}
\frac{\eta}{d} =& 1 + \sum_{i=1}^{3} C_i; \\
C_1 &= \alpha \,\mathrm{cn}^2, \\
C_2 &= \frac{3}{4}\alpha^2 \,\mathrm{cn}^2 (\mathrm{cn}^2 - 1), \\
C_3 &= \frac{1}{80}\alpha^3 \,\mathrm{cn}^2 \left[-61 m^{-1} + 111 + (61 m^{-1} - 212)\,\mathrm{cn}^2 + 101\,\mathrm{cn}^4 \right],
\end{aligned} \tag{2}$$

where $\alpha = H/d$, H is the wave height, h is the still water depth, and d is the depth from wave trough to the seabed (see Fig. 3 in main manuscript). The notation "cn", short for $\mathrm{cn}\,(\sqrt{\varepsilon}(x - ct)/d|m)$, denotes Jacobi elliptic cosine function, c is phase speed, and m is the elliptical parameter. The small quantity which expresses the relative slowness of variation in the x direction, ε , normalized trough water depth, d/h, the normalized wavelength, L/h, and the normalized mean velocity ($\overline{U}/\sqrt{gd}$) are

$$\sqrt{\varepsilon} = \sqrt{\frac{3\alpha}{4m}} \left(1 + (\frac{\alpha}{m})(\frac{1}{4} - \frac{7}{8}m) + (\frac{\alpha}{m})^2(\frac{1}{32} - \frac{11}{32}m + \frac{111}{128}m^2) \right), \tag{3}$$

$$\begin{aligned}
\frac{d}{h} =& 1 + \left(\frac{H}{mh}\right)(1 - m - e) + \left(\frac{H}{mh}\right)^2 \left[-\frac{1}{2} + \frac{1}{2}m + \left(\frac{1}{2} - \frac{1}{4}m\right) e \right] \\
&+ \left(\frac{H}{mh}\right)^3 \left[\frac{133}{200} - \frac{399}{400}m + \frac{133}{400}m^2 + \left(-\frac{233}{200} + \frac{233}{200}m - \frac{1}{25}m^2 \right) e \right. \\
&\left. + \left(\frac{1}{2} - \frac{1}{4}m\right) e^2 \right],
\end{aligned} \tag{4}$$

$$\begin{aligned}
\frac{L}{h} =& 4K(m) \left\{ \left(\frac{3H}{mh}\right)^{-1/2} \left[1 + \left(\frac{H}{mh}\right)\left(\frac{5}{4} - \frac{5}{8}m - \frac{3e}{2}\right) \right] \right. \\
&\left. + \left(\frac{H}{mh}\right)^2 \left[-\frac{15}{32} + \frac{15}{32}m - \frac{21}{128}m^2 + \left(\frac{1}{8} - \frac{1}{16}m\right) e + \frac{3}{8}e^2 \right] \right\},
\end{aligned} \tag{5}$$

$$
\begin{aligned}
\frac{\overline{U}}{\sqrt{gd}} =&1 + \left(\frac{\alpha}{m}\right)\left(\frac{1}{2} - e\right) + \left(\frac{\alpha}{m}\right)^2 \left[-\frac{13}{120} - \frac{1}{60}m - \frac{1}{40}m^2 + \left(\frac{1}{3} + \frac{1}{12}m\right)e\right] \\
&+ \left(\frac{\alpha}{m}\right)^3 \left[-\frac{361}{2100} + \frac{1899}{5600}m - \frac{2689}{16800}m^2 + \frac{13}{280}m^3\right. \\
&\left.+ \left(\frac{7}{75} - \frac{103}{300}m + \frac{131}{600}m^2\right)e\right],
\end{aligned}
\tag{6}
$$

where $K(m)$ and $E(m)$ are the complete elliptic integral of the first and second kinds, respectively, and $e = E(m)/K(m)$.

The list of coefficients, C_i, in the free surface solutions of the 5^{th} order cnoidal wave theory ($m \geq 0.96$) is shown below

$$
\begin{aligned}
\frac{\eta}{d} =&1 + \sum_{i=1}^{5} C_i; \\
C_1 &= \alpha\,\mathrm{cn}^2, \\
C_2 &= \alpha^2 \left(-\frac{3}{4}\,\mathrm{cn}^2 + \frac{3}{4}\,\mathrm{cn}^4\right), \\
C_3 &= \alpha^3 \left(\frac{5}{8}\,\mathrm{cn}^2 - \frac{151}{80}\,\mathrm{cn}^4 + \frac{101}{80}\,\mathrm{cn}^6\right), \\
C_4 &= \alpha^4 \left(-\frac{8209}{6000}\,\mathrm{cn}^2 + \frac{11641}{3000}\,\mathrm{cn}^4 + \frac{112393}{24000}\,\mathrm{cn}^6 + \frac{17367}{8000}\,\mathrm{cn}^8\right), \\
C_5 &= \alpha^5 \left(\frac{364671}{196000}\,\mathrm{cn}^2 - \frac{2920931}{392000}\,\mathrm{cn}^4 + \frac{2001361}{21568000}\,\mathrm{cn}^6\right. \\
&\left.- \frac{17906339}{1568000}\,\mathrm{cn}^8 + \frac{1331817}{313600}\,\mathrm{cn}^{10}\right),
\end{aligned}
\tag{7}
$$

and small quantity (ε), the trough water depth (d), wavelength (L), and mean velocity ($\overline{U}$) are

$$
\sqrt{\varepsilon} = \sqrt{\frac{3\alpha}{4}}\left(1 - \frac{5}{8}\alpha + \frac{71}{128}\alpha^2 - \frac{100627}{179200}\alpha^3 + \frac{16259737}{28672000}\alpha^4\right),
\tag{8}
$$

$$
\begin{aligned}
\frac{d}{h} =&1 + \frac{H}{h}(-e) + \left(\frac{H}{h}\right)^2 \frac{e}{4} + \left(\frac{H}{h}\right)^3 \left(-\frac{e}{25} + \frac{e^2}{4}\right) + \left(\frac{H}{h}\right)^4 \left(\frac{573e}{2000} - \frac{57e^2}{400} + \frac{e^3}{4}\right) \\
&+ \left(\frac{H}{h}\right)^5 \left(-\frac{302159e}{1470000} + \frac{1779e^2}{2000} - \frac{123e^3}{400} + \frac{e^4}{4}\right),
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\frac{\overline{U}}{\sqrt{gd}} =& 1+\alpha\left(\frac{1}{2}-e\right)+\alpha^2\left(-\frac{3}{20}+\frac{5}{12}e\right)+\alpha^3\left(\frac{3}{56}-\frac{19}{600}e\right) \\
&+\alpha^4\left(-\frac{309}{5600}+\frac{3719}{21000}e\right)+\alpha^5\left(\frac{12237}{616000}-\frac{997699}{8820000}e\right),
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\frac{L}{h} =& 4K(m)\left(\frac{3H}{h}\right)^{-1/2}\left[1+\left(\frac{H}{h}\right)\left(\frac{5}{8}-\frac{3e}{2}\right)+\left(\frac{H}{h}\right)^2\left(-\frac{21}{128}+\frac{1}{16}e+\frac{3}{8}e^2\right)\right] \\
&+\left(\frac{H}{h}\right)^3\left(\frac{20127}{179200}-\frac{409}{6400}e+\frac{7}{64}e^2+\frac{1}{16}e^3\right) \\
&+\left(\frac{H}{h}\right)^4\left(-\frac{1575087}{28672000}+\frac{1086367}{1792000}e-\frac{2679}{25600}e^2+\frac{13}{128}e^3+\frac{3}{128}e^4\right).
\end{aligned} \tag{11}
$$

4. Comparisons between Stokes and cnoidal wave theories for $Ur = 26$

In this appendix, comparisons between Stokes wave solutions and cnoidal wave solutions for $U_r = 26$ are discussed. For a fixed Ursell number, say $U_r = 26$, and a fixed water depth, $h = 10$m, different H/h values can be obtained by varying the wavelength L. In this appendix, results for $H/h = 0.065$ and 0.26 are examined. The corresponding h/L values are 0.05 and 0.1, respectively. Using the cnoidal 3^{rd} order theory and the Stokes 5^{th} order theory, Figure 4.1 shows that the free surface profiles of both theories agree well with each other. The differences in the crest elevation are less than 0.2% along the whole curve with $U_r = 26$.

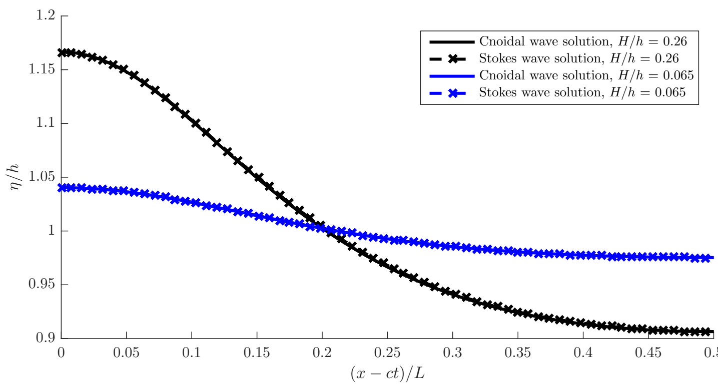

Figure 4.1: Free surface profiles for $h = 10\text{m}$, $Ur = 26$, and $H/h = 0.065, and 0.26$. The blue color corresponds to the solutions with $H/h = 0.065$, and the black color denotes $H/h = 0.26$. The solid lines are the 3^{rd} order cnoidal theory solution, and lines with markers are the 5^{th} order Stokes wave theory result. Note that the solutions of Stokes wave theory were shifted to align the $z = 0$ at the seabed for this figure.

The comparisons of velocity profiles under the crest and trough are shown in Figure 4.2. With a very small $H/h = 0.065$ (or $h/L = 0.05$, the differences in velocity at the water column under crest and trough are negligible ($<$ 1.5%). For $H/h = 0.26$ (or $h/L = 0.1$), although the free surface profiles appear identical, the maximum difference in the u profiles becomes $\approx 11.10\%$, occurring at the wave crest. The differences between the two solutions are even greater if H/h and h/L increase. Readers are advised to be cautious about the magnitude of velocity due to different wave theories at higher H/h values.

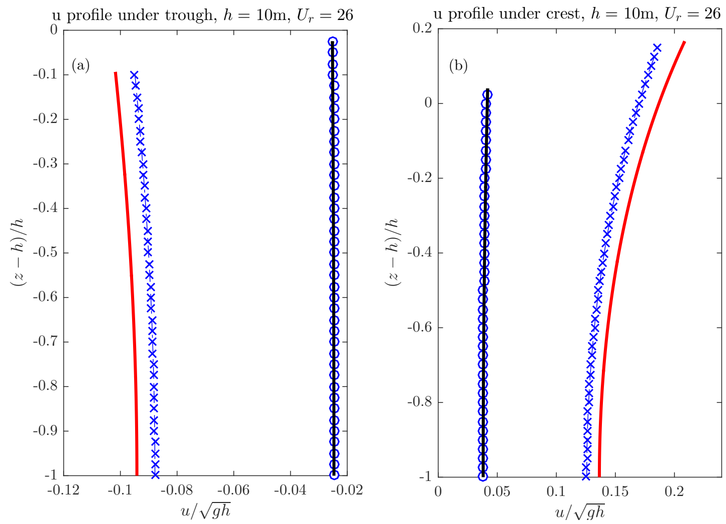

Figure 4.2: Horizontal velocity component u under wave trough (left panel - a) and crest (right panel - b) along the water column. Black and red solid lines represent solutions from cnoidal wave theory, and dashed lines with $\circ$ and $\times$ represent results from Stokes wave theory. The $\circ$ and black lines correspond to results for $H/h = 0.065$; the $\times$ and red lines correspond to results for $H/h = 0.26$.

5. Discussion on various order of cnoidal wave theory

As shown in Figure 5.1, the curve representing $U_r = 10$ is plotted in the parameter space and is almost identical with the curve denoting $m = 0.5$. On the other hand, the curve for $U_r = 26$ tracks closely the curve corresponding to $m = 0.855$. Moreover, the curve for $m = 0.96$ closely coincides with the curve of $U_r = 50$. Note that the $m = 0.96$ curve was obtained using the wavelength expression of the 5^{th} order cnoidal wave theory. According to Fenton (1999), the 5^{th} order cnoidal wave theory is applicable if $m \geq 0.96$, and the 3^{rd} order theory is applicable in the parameter space between $m = 0.5$ and $m = 0.96$.

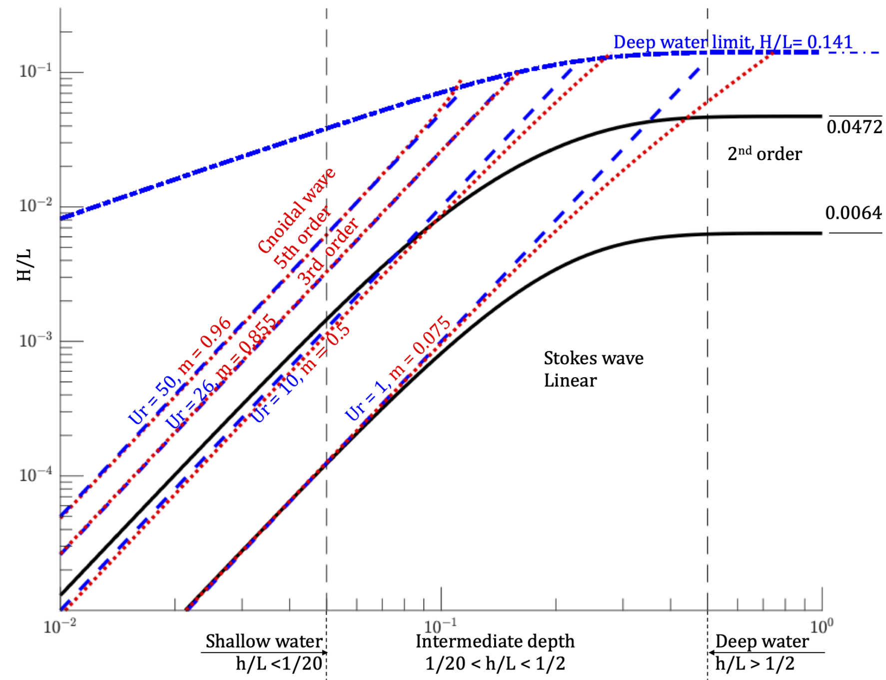

Figure 5.1: Discussion on various orders of cnoidal wave theory in the parameter space of h/L and H/L. Blue dashed lines were plotted using the equation for the corresponding U_r values as labelled in blue text, and red dotted lines were plotted based on various m values as labelled in red text.

6. Comparison of wave theory selection using updated and original Le Mehaute's chart

To evaluate the differences systematically, the authors selected typical points along the demarcation line of the updated chart (see Figure 6.1b below), calculated the corresponding h/gT^2, H/gT^2, and superimposed on the original Le Mehaute's chart (see Figure 6.1a below). The values of h/gT^2, H/gT^2, h/L, and H/L can be found from Table 1.

For example, from the demarcation line of $R_5 = 1\%$, we can see that these points fall in the region of Stokes 3rd order wave in original Le Mehaute's chart. It means if users refer to original chart, they would find third order Stokes wave theory were adequate. However, if they use the updated chart, it is recommended to use fifth order Stokes wave theory as the crest contribution from 5th order is 1% already. In general, the updated chart is more conservative in selecting higher order Stokes waves.

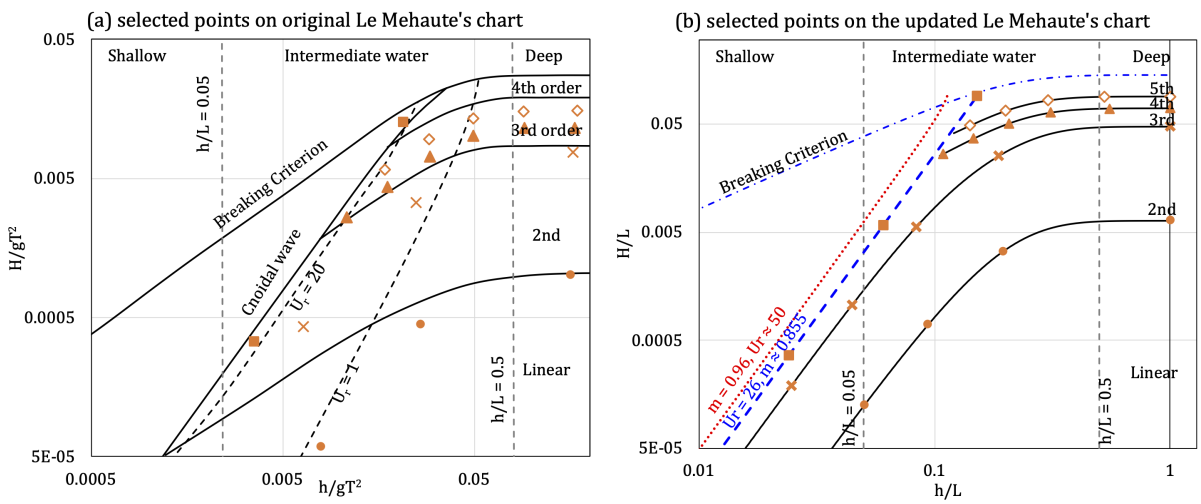

Figure 6.1: Test wave theories for selected points based on (a) original Le Mehaute's chart, and (b) updated chart as in the present paper. Points on $R_2 = 1\%$ are marked as filled circles ('•'), Points on $R_3 = 1\%$ are marked as cross ('×'), Points on $R_4 = 1\%$ is marked as filled triangle('▲'), Points on $R_5 = 1\%$ are marked as orange empty diamond('◇'), and points on $Ur = 26$ are marked as filled square ('■').

Table 1: List of wave conditions for comparison of wave theory selection

h, m	H, m	T, s	h/gT^2	H/gT^2	h/L	H/L
100.420	0.637	8.00	1.60E-01	1.01E-03	1.00E+00	6.37E-03
19.622	0.333	8.71	2.63E-02	4.47E-04	1.96E-01	3.33E-03
9.398	0.070	10.99	7.93E-03	5.88E-05	9.40E-02	6.96E-04
5.049	0.013	14.44	2.47E-03	6.19E-06	5.05E-02	1.27E-04
100.420	4.717	7.92	1.63E-01	7.67E-03	1.00E+00	4.72E-02
18.759	2.528	8.75	2.50E-02	3.36E-03	1.88E-01	2.53E-02
8.353	0.557	11.51	6.43E-03	4.29E-04	8.35E-02	5.57E-03
4.459	0.105	15.30	1.94E-03	4.59E-05	4.46E-02	1.05E-03
2.477	0.019	20.36	6.09E-04	4.72E-06	2.48E-02	1.92E-04
100.420	6.970	7.82	1.68E-01	1.16E-02	1.00E+00	6.97E-02
55.185	6.937	7.82	9.19E-02	1.16E-02	5.52E-01	6.94E-02
31.086	6.351	7.99	4.96E-02	1.01E-02	3.11E-01	6.35E-02
20.713	5.045	8.46	2.95E-02	7.18E-03	2.07E-01	5.04E-02
14.658	3.655	9.22	1.76E-02	4.38E-03	1.47E-01	3.65E-02
10.920	2.648	10.16	1.08E-02	2.62E-03	1.09E-01	2.65E-02
100.420	8.956	7.70	1.73E-01	1.54E-02	1.00E+00	8.96E-02
52.701	8.910	7.71	9.03E-02	1.53E-02	5.27E-01	8.91E-02
30.326	8.223	7.89	4.96E-02	1.35E-02	3.03E-01	8.22E-02
20.036	6.590	8.40	2.89E-02	9.52E-03	2.00E-01	6.59E-02
14.156	4.818	9.19	1.71E-02	5.82E-03	1.42E-01	4.82E-02
2.409	0.036	20.62	5.77E-04	8.71E-06	2.41E-02	3.63E-04
6.051	0.576	13.17	3.55E-03	3.38E-04	6.05E-02	5.76E-03
15.199	9.129	8.54	2.13E-02	1.28E-02	1.52E-01	9.13E-02