4. 导热问题的数值解法

4.1 数值求解的基本思想

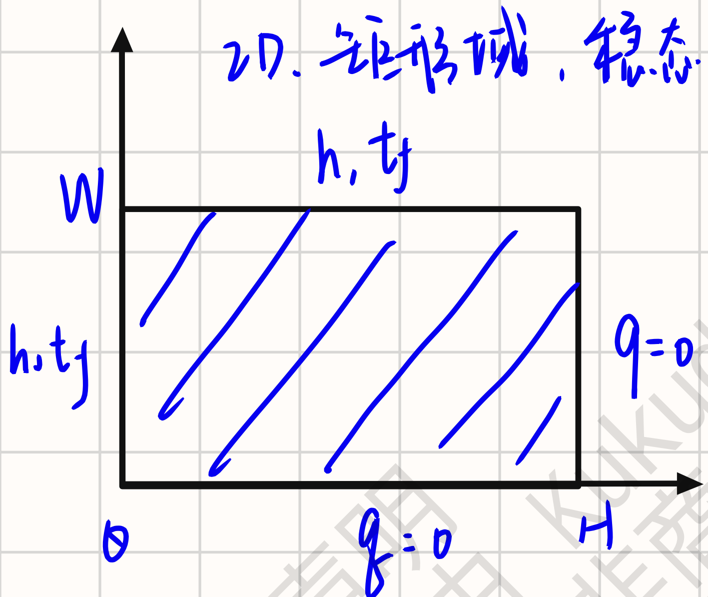
分析解：对导热微分方程在定解条件下的积分求解

数值解：在求解区域上空间、时间坐标系中的离散点的温度分布代替连续的温度场，用大量的代数方程代替微分方程

$$\begin{cases}\text{连续} \rightarrow \text{离散} \\ \text{微分方程} \rightarrow \text{代数方程}\end{cases}$$

4.2 数值求解的基本步骤

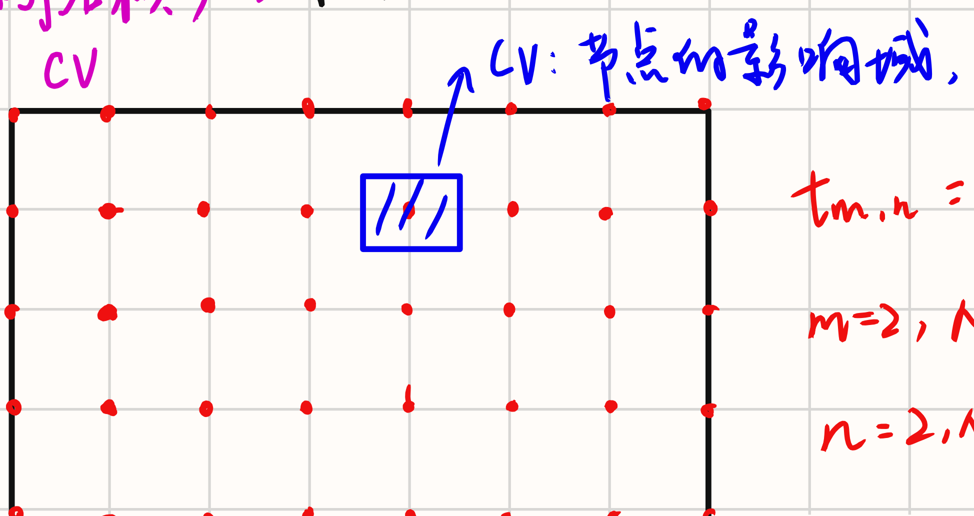
2D、矩形域，稳态，无内热源，常物性，ⅠBC & ⅢBC



$$\begin{cases}\dfrac{\partial^2 t}{\partial x^2} + \dfrac{\partial^2 t}{\partial y^2} = 0 \\ x=0,\ \lambda\dfrac{\partial t}{\partial x} = h(t-t_f) \\ x=H,\ \dfrac{\partial t}{\partial x} = 0 \\ y=0,\ \dfrac{\partial t}{\partial y} = 0 \\ y=W,\ -\lambda\dfrac{\partial t}{\partial y} = h(t-t_f)\end{cases}$$

区域离散：将求解区域按照一定规划分为许多小区域，这个过程称作区域离散，

(控制容积)→ 每个小的区域的物理量值由一个点—节点来表示。



$$t_{m,n} = f(t_{m+1,n},\ t_{m-1,n},\ t_{m,n+1},\ t_{m,n-1})$$

$m=2, M-1$

$n=2, N-1$

每个节点与它相邻的节点存在一定关系，通过相应的物理定律，可以建立相应的关系式，又称为节点

的离散方程

把所有节点的离散方程联立起来，会组成一个封闭的方程组，对代数方程组，

求解代数方程组 $\begin{cases} \text{直接解法} & \text{速度快，对设备性能要求高。} \\ \text{迭代求解} & \text{速度慢，对设备性能要求低} \end{cases}$

迭代求解 → 最常用

Taylor级数展开法

$$\left.\frac{\partial^2 t}{\partial x^2}\right|_{m,n} + \left.\frac{\partial^2 t}{\partial y^2}\right|_{m,n} = 0$$

$$\begin{cases} t_{m+1,n} = t_{m,n} + \left.\frac{\partial t}{\partial x}\right|_{m,n}\Delta x + \left.\frac{\partial^2 t}{\partial x^2}\right|_{m,n}\frac{\Delta x^2}{2!} + \left.\frac{\partial^3 t}{\partial x^3}\right|_{m,n}\frac{\Delta x^3}{3!} + \cdots + O(\Delta x^4) \\ t_{m-1,n} = t_{m,n} - \left.\frac{\partial t}{\partial x}\right|_{m,n}\Delta x + \left.\frac{\partial^2 t}{\partial x^2}\right|_{m,n}\frac{\Delta x^2}{2} - \left.\frac{\partial^3 t}{\partial x^3}\right|_{m,n}\frac{\Delta x^3}{3!} + \cdots + O(\Delta x^4) \end{cases}$$

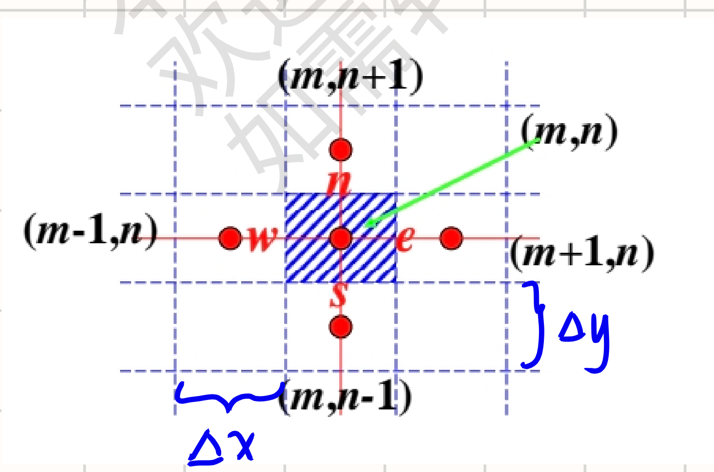
$$\Rightarrow \left.\frac{\partial^2 t}{\partial x^2}\right|_{m,n} = \frac{t_{m+1,n} - 2t_{m,n} + t_{m-1,n}}{\Delta x^2} + \underbrace{O(\Delta x^2)}_{\text{截断误差}}$$

如果 $\Delta x = \Delta y \Rightarrow t_{m,n} = \frac{1}{4}(t_{m+1,n} + t_{m-1,n} + t_{m,n+1} + t_{m,n-1})$

$\begin{cases} \text{优点：简单，精度好} \\ \text{缺点：物理含义不明，对非均分网格，变物性问题不方便} \end{cases}$

热平衡法 ☆重点

基本思想：能量守恒定律 "能量集中在一个点"



规定导入热量为正

$$\Phi_w = \lambda(1\cdot\Delta y)\frac{t_{m-1,n} - t_{m,n}}{\Delta x}$$

$$\Phi_e = \lambda(1\cdot\Delta y)\frac{t_{m+1,n} - t_{m,n}}{\Delta x}$$

$$\Phi_n = \lambda(1\cdot\Delta x)\frac{t_{m,n+1} - t_{m,n}}{\Delta y}$$

$$\Phi_s = \lambda(1\cdot\Delta x)\frac{t_{m,n-1} - t_{m,n}}{\Delta y}$$

$$\Phi_w + \Phi_e + \Phi_s + \Phi_n = 0$$

√ $\frac{t_{m+1,n}-2t_{m,n}+t_{m-1,n}}{\Delta x^2}+\frac{t_{m,n+1}-2t_{m,n}+t_{m,n-1}}{\Delta y^2}=0$

优点：物理含义清晰，推导简单，不均匀网格、变物性均适用

缺点：计算精度不明确

4.3 边界节点离散方程建立及代数方程求解

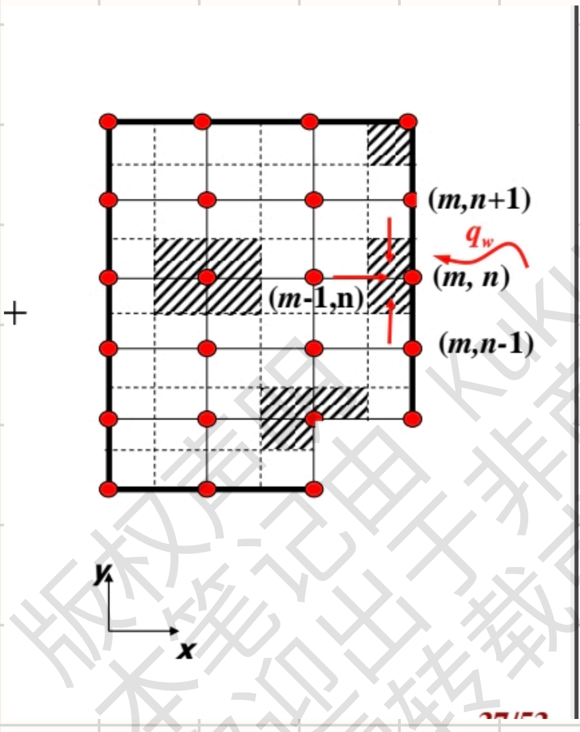
第一类边界条件：边界温度已知，方程封闭

第二、三类边界条件：边界温度未知，代数方程组不封闭

建立ⅡBC、ⅢBC边界节点离散方程

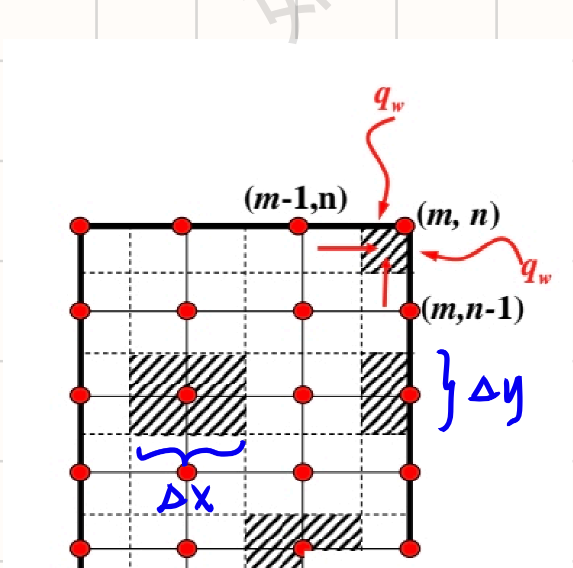
假设物体有内热源，网格均匀（Δx不一定等于Δy，x方向、y方向各自均匀）

① 平直边界

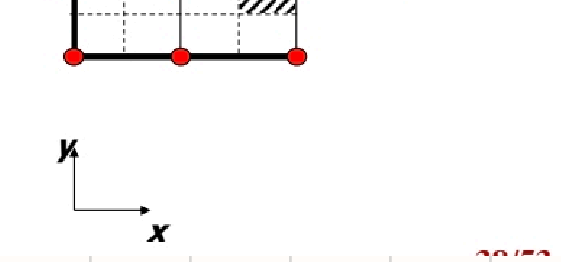


$$\lambda\frac{t_{m-1,n}-t_{m,n}}{\Delta x}\Delta y+q_w\Delta y+\lambda\frac{t_{m,n-1}-t_{m,n}}{\Delta y}\cdot\frac{\Delta x}{2}+\lambda\frac{t_{m,n+1}-t_{m,n}}{\Delta y}\frac{\Delta x}{2}+\dot{\Phi}_{m,n}\frac{\Delta x}{2}\Delta y=0$$

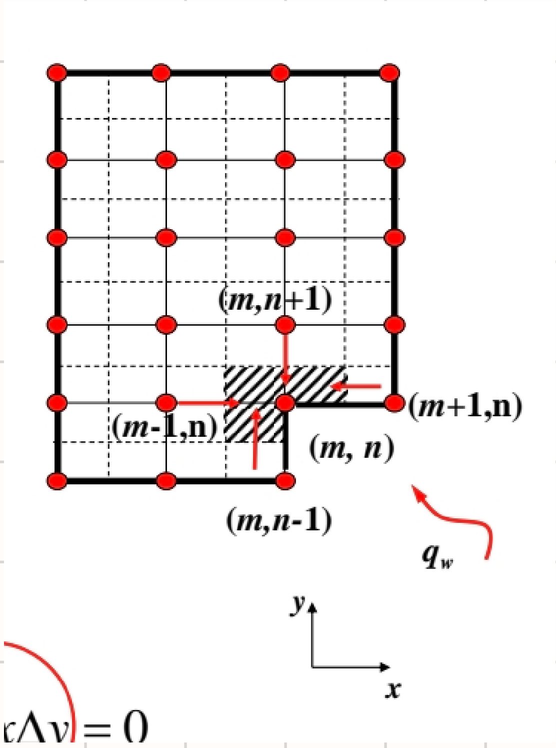
② 外角点



$$\lambda\frac{t_{m-1,n}-t_{m,n}}{\Delta x}\frac{\Delta y}{2}+q_w\cdot\frac{\Delta y}{2}+\lambda\frac{t_{m,n-1}-t_{m,n}}{\Delta y}\cdot\frac{\Delta x}{2}+q_w\frac{\Delta x}{2}+\dot{\Phi}_{m,n}\frac{\Delta x}{2}\cdot\frac{\Delta y}{2}=0.$$



③ 内角点



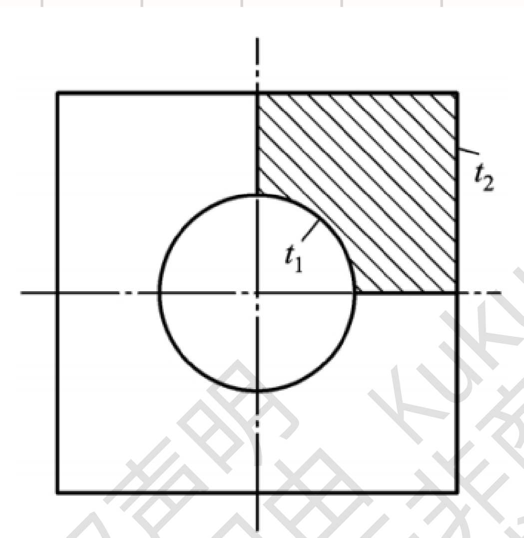
$$\lambda\frac{t_{m-1,n}-t_{m,n}}{\Delta x}\Delta y+q_w\cdot\frac{\Delta y}{2}+\lambda\frac{t_{m+1,n}-t_{m,n}}{\Delta x}\cdot\frac{\Delta y}{2}$$

$$+\lambda\frac{t_{m,n-1}-t_{m,n}}{\Delta y}\cdot\frac{\Delta x}{2}+q_w\frac{\Delta x}{2}+\lambda\frac{t_{m,n+1}-t_{m,n}}{\Delta y}\Delta x$$

$$+\Phi_{m,n}\frac{3}{4}\Delta x\Delta y=0.$$

④ 不规则边界

阶梯逼近法



方程的求解：① 直接求解：Cramer 法则

② Gauss 消元法

③ Jacobi 迭代法

④ Gauss-Seidel 迭代

迭代收敛：$\left|\frac{t_{m,n}^{(i+1)}-t_{m,n}^{(i)}}{t_{m,n}^{(i)}}\right|\le\varepsilon$

分母加一个小量

$\left|\frac{t_{m,n}^{(i+1)}-t_{m,n}^{(i)}}{t_{max}}\right|\le\varepsilon$

解决分母为0的问题

主对角占优原则:

迭代公式的选择应使每一个迭代变量的系数总是大于或等于该式中其它变量系数绝对值的代数和，此时，用迭代法求解代数方程，一定收敛

4.4 非稳态导热问题的数值解法

① 向前差分

② 向后差分

③ 中心差分

对称 ⟺ 温度梯度为0

① 向前差分

$$t_n^{(i+1)} = t_n^{(i)} + \left.\frac{\partial t}{\partial \tau}\right|_{(n,i)} \Delta\tau + \frac{1}{2!}\left.\frac{\partial^2 t}{\partial \tau^2}\right|_{(n,i)} (\Delta\tau)^2 + \cdots$$

$$\left.\frac{\partial t}{\partial \tau}\right|_{(n,i)} = \frac{t_n^{(i+1)} - t_n^{(i)}}{\Delta\tau} + O(\Delta\tau)$$

② 向后差分

$$t_n^{(i-1)} = t_n^{i} - \left.\frac{\partial t}{\partial \tau}\right|_{(n,i)} \Delta\tau + \frac{1}{2!}\left.\frac{\partial^2 t}{\partial \tau^2}\right|_{(n,i)} (\Delta\tau)^2 - \cdots$$

$$\left.\frac{\partial t}{\partial \tau}\right|_{(n,i)} = \frac{t_n^{(i)} - t_n^{(i-1)}}{\Delta\tau} + O(\Delta\tau)$$

③ 中心差分

$$t_n^{(i+1)} = t_n^{(i)} + \left.\frac{\partial t}{\partial \tau}\right|_{(n,i)} \Delta\tau + \frac{1}{2!}\left.\frac{\partial^2 t}{\partial \tau^2}\right|_{(n,i)} (\Delta\tau)^2 + \cdots$$

$$t_n^{(i-1)} = t_n^{(i)} - \left.\frac{\partial t}{\partial \tau}\right|_{(n,i)} \Delta\tau + \frac{1}{2!}\left.\frac{\partial^2 t}{\partial \tau^2}\right|_{(n,i)} (\Delta\tau)^2 - \cdots$$

$$\left.\frac{\partial t}{\partial \tau}\right|_{(n,i)} = \frac{t_n^{(i+1)} - t_n^{(i-1)}}{2\Delta\tau} + O(\Delta\tau^2)$$

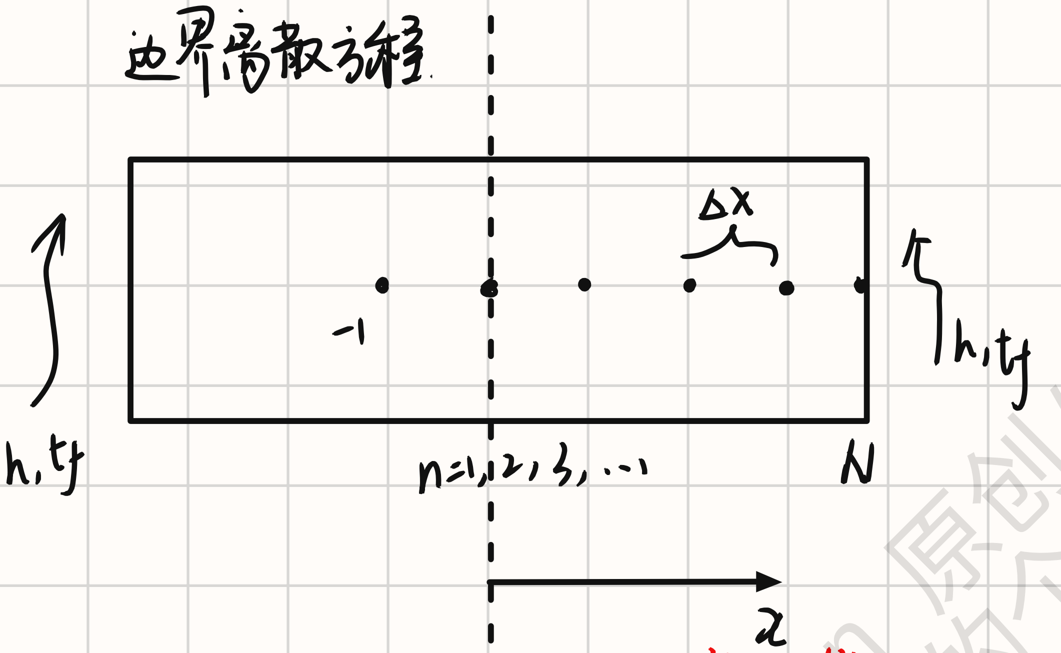
内节点离散方程

$$t_n^{(i+1)} = \left(1-2\frac{a(\Delta\tau)}{\Delta x^2}\right)t_n^{(i)} + \frac{a(\Delta\tau)}{\Delta x^2}\left(t_{n+1}^{(i)} - t_n^{(i)}\right)$$

$$\frac{t_n^{(i+1)} - t_n^{(i)}}{\Delta\tau} = \begin{cases} a\dfrac{t_{n+1}^{(i)} - 2t_n^{(i)} + t_{n-1}^{(i)}}{\Delta x^2} & \text{(explicit)} \\ a\dfrac{t_{n+1}^{(i+1)} - 2t_n^{(i+1)} + t_{n-1}^{(i+1)}}{\Delta x^2} & \text{(implicit)} \end{cases}$$

$$\left(1+2\frac{a(\Delta\tau)}{\Delta x^2}\right)t_n^{(i+1)} = t_n^{(i)} + \frac{a(\Delta\tau)}{\Delta x^2}\left(t_{n+1}^{(i+1)} + t_{n-1}^{(i+1)}\right)$$

边界离散方程



更新边界温度

$$t_1^{(i)} = t_2^{(i)}$$

把边节看作内节点

$$t_{-1}^{(i)} = t_2^{(i)}$$

右端边界点

$$\lambda \cdot 1 \cdot \frac{t_{N-1}^{(i)} - t_N^{(i)}}{\Delta x} + h \cdot 1 \cdot \left(t_f - t_N^{(i)}\right)$$

$$= \rho c\left(\frac{\Delta x}{2}\cdot 1\right)\frac{t_N^{(i+1)} - t_N^{(i)}}{\Delta\tau}$$

$$t_N^{(i+1)} = t_N^{(i)}\left(1-2\frac{a\Delta\tau}{\Delta x^2} - \frac{2h\Delta\tau}{\rho c\Delta x}\right) + 2\frac{a\Delta\tau}{\Delta x^2}t_{N-1}^{(i)} + \frac{2h\Delta\tau}{\rho c\Delta x}t_f$$

显式代数方程

$$t_n^{(i+1)} = \left(1-2\frac{a(\Delta\tau)}{\Delta x^2}\right)t_n^{(i)} + \frac{a\Delta\tau}{\Delta x^2}\left(t_{n+1}^{(i)} + t_{n-1}^{(i)}\right) \quad n = 2, 3, \dots, N-1$$

$$t_n^{(0)} = t_0 \quad (n = 1, 2, 3, \dots, N)$$

$$t_{-1}^{(i)} = t_2^{(i)}$$

$$t_N^{(i+1)} = t_N^{(i)}\left(1-2\frac{a\Delta\tau}{\Delta x^2} - \frac{2h\Delta\tau}{\rho c\Delta x}\right) + 2\frac{a\Delta\tau}{\Delta x^2}t_{N-1}^{(i)} + \frac{2h\Delta\tau}{\rho c\Delta x}t_f$$

$$t_n^{(i+1)} = (1-2Fo_\Delta)t_n^{(i)} + Fo_\Delta\left(t_{n+1}^{(i)} + t_{n-1}^{(i)}\right) \quad n = 2, 3, \dots, N-1$$

$$t_n^{(0)} = t_0 \quad (n = 1, 2, \dots, N)$$

$$t_{-1}^{(i)} = t_2^{(i)}$$

$$t_N^{(i+1)} = (1-2Fo_\Delta - 2Fo_\Delta \cdot Bi_\Delta)\, t_N^{(i)} + 2Fo_\Delta\, t_{N-1}^{(i)} + 2Fo_\Delta \cdot Bi_\Delta \cdot t_f$$

稳定性分析

$$t_n^{(i+1)} = (1-2Fo_\Delta)\, t_n^{(i)} + Fo_\Delta \left(t_{n+1}^{(i)} + t_{n-1}^{(i)}\right) \quad n=2,3,\cdots,N-1$$

$$t_N^{(i+1)} = (1-2Fo_\Delta - 2Fo_\Delta \cdot Bi_\Delta)\, t_N^{(i)} + 2Fo_\Delta\, t_{N-1}^{(i)} + 2Fo_\Delta \cdot Bi_\Delta \cdot t_f$$

系数必须反映"正"影响

$$1-2Fo_\Delta \ge 0 \quad\Downarrow\quad Fo_\Delta \le \frac{1}{2}$$

$$1-2Fo_\Delta - 2Fo_\Delta \cdot Bi_\Delta \ge 0 \quad\Downarrow\quad Fo_\Delta \le \frac{1}{2(1+Bi_\Delta)}$$

隐式格式的数值稳定性

$$(1+2Fo_\Delta)\, t_n^{(i+1)} = t_n^{(i)} + Fo_\Delta \left(t_{n+1}^{(i+1)} + t_{n-1}^{(i+1)}\right)$$

$$(1+2Fo_\Delta + 2Fo_\Delta \cdot Bi_\Delta)\, t_N^{(i+1)} = t_N^{(i)} + 2Fo_\Delta \cdot t_{N-1}^{(i+1)} + 2Fo_\Delta \cdot Bi_\Delta \cdot t_f$$

绝对稳定

计算复杂，工作量大.