1. The graph[1] of Europe $\mathcal{E} = \langle V, E \rangle$ is defined as follows: each vertex $v \in V$ is a Europe country; two vertices are adjacent ($\{u, v\} \in E$) if corresponding countries share a border. Let $\mathcal{E}^*$ be a maximum connected component of $\mathcal{E}$.
   (a) Draw the Europe graph $\mathcal{E}$ and prove that it is planar by showing that it is a plane map.
   (b) Find $|V|$, $|E|$, $\delta(\mathcal{E}^*)$, $\Delta(\mathcal{E}^*)$, $\mathrm{rad}(\mathcal{E}^*)$, $\mathrm{diam}(\mathcal{E}^*)$, $\mathrm{center}(\mathcal{E}^*)$.
   (c) Find the minimum vertex coloring[2] Z of $\mathcal{E}$.
   (d) Find the maximum stable set X of $\mathcal{E}$ and prove that it is maximal.
   (e) Find the maximum matching M of $\mathcal{E}$ and prove that it is maximal.
   (f) Find the minimum vertex cover R of $\mathcal{E}$ and prove that it is minimal.
   (g) Find the minimum edge cover F of $\mathcal{E}^*$ and prove that it is minimal.
   (h) Find the shortest closed path (or circuit) that visits every edge[3] of $\mathcal{E}^*$.
   (i) Add the weight function $w : E \to \mathbb{R}$ denoting the distance between the capitals. Find the minimum spanning tree T for the maximum connected component of the weighted Europe graph $\mathcal{E}_w = \langle V, E, w \rangle$.
2. Prove rigorously that every r-regular ($r > 0$) (n, m)-bipartite graph has $n = m$.
3. Prove rigorously that a connected graph $G = \langle V, E \rangle$ is a tree *iff*[4] $|E| = |V| - 1$.
4. Prove rigorously the TRIANGLE INEQUALITY for a connected graph $G = \langle V, E \rangle$:
$$\forall x, y, z \in V : \mathrm{dist}(x, y) + \mathrm{dist}(y, z) \geq \mathrm{dist}(x, z)$$

Minor Cheatsheet

* Graph G is a pair $\langle V, E \rangle$ of the set of vertices V and the set of edges E.
* $S^k = S \times \cdots \times S = \{\langle s_1, \ldots, s_k \rangle \mid s_1, \ldots, s_k \in S\}$ — set of k-tuples (Cartesian k-power of S).
* $S^{(k)} = \{\{s_1, \ldots, s_k\} \mid s_1 \neq \cdots \neq s_k \in S\}$ — set of all exactly-k-sized subsets of S.
* Simple **directed** graph has $E \subseteq V^2$. Simple **undirected** graph has $E \subseteq V^{(2)}$.
* $\delta(G) = \min_{v \in V} \deg(v)$ — **minimum degree**, $\Delta(G) = \max_{v \in V} \deg(v)$ — **maximum degree**.
* A graph is r-**regular** if all its vertices have the same degree: $\forall v \in V : \deg(v) = r$.
* The **distance** $\mathrm{dist}(v, w)$ between two vertices is the length or the shortest path $v \rightsquigarrow w$.
* A **stable set**[5] $X \subseteq V$ is a set of pairwise non-adjacent vertices.
* A **matching** $M \subseteq E$ is a set of pairwise non-adjacent edges.
* A **vertex cover** $R \subseteq V$ is a set of vertices that contains at least one endpoint of every edge: $\{u, v\} \in E \to u \in R \vee v \in R$.
* An **edge cover** $F \subseteq E$ is a set of edges such that every vertex of the graph is an endpoint of some element of F.
* Note the distinction between **maximum** ("globally") and **maximal** ("locally"):
  - Some thing A^* is **maximum** if there is no other thing A such that $|A| > |A^*|$.
  - Some thing A' is **maximal** is there is no other thing A such that $A \supset A'$ (or rather, $A \succ A'$).
  - Similarly, **minimum** / **minimal**.

[1] Hereinafter "graphs" are "simple undirected and unweighted", unless stated otherwise.
[2] Since $\mathcal{E}$ is planar, there exists a 4-coloring, but it may be not the minimum!
[3] This is a CHINESE POSTMAN PROBLEM.
[4] Recall that in order to prove $A \leftrightarrow B$ ("*iff*"), you must prove both $A \to B$ and $B \to A$.
[5] Also known as "independent set".