Математический анализ
100 интегралов
ФИТиП ИС
1 курс 2 семестр

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1. $\int \frac{1}{\sqrt{2x-1}}\,\mathrm{d}x = (*)$

$$\text{Замена: } \begin{bmatrix} t = 2x-1 \\ x = \dfrac{t+1}{2} \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{2} \end{bmatrix} \quad => \quad (*) \quad = \quad \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} \quad = \quad \frac{1}{2}(2\sqrt{t} \quad + \quad C) \quad = \quad \sqrt{t} \quad + \quad C$$

2. $\int \frac{1}{(x+1)^2}\,\mathrm{d}x = (*)$

$$\text{Замена: } \begin{bmatrix} t = x+1 \\ x = t-1 \\ \mathrm{d}x = \mathrm{d}t \end{bmatrix} \quad => \quad (*) \quad = \quad \int \frac{\mathrm{d}t}{t^2} \quad = \quad -\frac{1}{t} \quad + \quad C$$

3. $\int 5^{2x-1}\,\mathrm{d}x = (*)$

$$\text{Замена: } \begin{bmatrix} t = 2x-1 \\ x = \dfrac{t+1}{2} \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{2} \end{bmatrix} \quad => \quad (*) \quad = \quad \frac{1}{2}\int 5^t\,\mathrm{d}t \quad = \quad \frac{1}{2}\left(\frac{5^t}{\ln 5} + C\right) \quad = \quad \frac{5^t}{2\ln 5} \; + \; C \quad = \quad \frac{5^{2x-1}}{2\ln 5} \; + \; C$$

4. $\int \sqrt{\frac{x+1}{x}}\,\mathrm{d}x = (*)$

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = \sqrt{\dfrac{x+1}{x}} & \mathrm{d}u = u'\,\mathrm{d}x = -\dfrac{1}{2x^2\sqrt{\dfrac{x+1}{x}}}\,\mathrm{d}x \\ v = x & \mathrm{d}v = \mathrm{d}x \end{array}\right] =>$$

$$(*) = \sqrt{\frac{x+1}{x}}x - \int x\left(-\frac{\mathrm{d}x}{2x^2\sqrt{\dfrac{x+1}{x}}}\right) = x\sqrt{\frac{x+1}{x}} + \frac{1}{2}\int \frac{\mathrm{d}x}{x\sqrt{\dfrac{x+1}{x}}} =$$

$$= \sqrt{x\,(x+1)} + \int \frac{\mathrm{d}x}{2\sqrt{x\,(x+1)}}$$

$$\int \frac{\mathrm{d}x}{2\sqrt{x\,(x+1)}} = (**)$$

$$\text{Замена: } \begin{bmatrix} t = \sqrt{x} \\ x = t^2 \\ \mathrm{d}t = \dfrac{\mathrm{d}x}{2\sqrt{x}} \end{bmatrix} =>$$

$$(**) = \int \frac{\mathrm{d}t}{\sqrt{1+t^2}} = \ln\left|t + \sqrt{t^2+1}\right| + C = \ln\left|\sqrt{x} + \sqrt{x+1}\right| + C =>$$

$$(*) = \sqrt{x\,(x+1)} + (**) =$$

$$= \sqrt{x\,(x+1)} + \ln\left|\sqrt{x} + \sqrt{x+1}\right| + C$$

5. $\int \dfrac{\mathrm{d}x}{\cos^2 x - \dfrac{\pi}{4}} = (*)$

Замена: $\left[\begin{array}{c} t = x - \dfrac{\pi}{4} \\ x = t + \dfrac{\pi}{4} \\ \mathrm{d}x = \mathrm{d}t \end{array}\right] \quad => \quad (*) \quad = \quad \int \dfrac{\mathrm{d}t}{\cos^2 t} \quad = \quad tg\,(t) \quad + \quad C \quad = \quad tg\left(x - \dfrac{\pi}{4}\right) \quad + \quad C$

6. $\int \frac{\ln x}{x}\,\mathrm{d}x = (*)$

Замена: $\left[\begin{array}{c} t = \ln x \\ \mathrm{d}t = \dfrac{\mathrm{d}x}{x} \end{array}\right] \quad => \quad (*) \quad = \quad \int t\,\mathrm{d}t \quad = \quad \dfrac{t^2}{2} \quad + \quad C \quad = \quad \dfrac{\ln x^2}{2} \quad + \quad C$

7. $\int 3^x\ e^{x+1}\,\mathrm{d}x = (*)$

Замена: $\left[\begin{array}{c} t = 3^x e^{x+1} \\ \mathrm{d}t = 3^x * \ln 3 * e^{x+1} + 3^x e^{x+1}\,\mathrm{d}x \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{3^x * \ln 3 * e^{x+1} + 3^x e^{x+1}} = \dfrac{\mathrm{d}t}{t * \ln 3 + t} = \dfrac{\mathrm{d}t}{t\,(\ln 3 + 1)} \end{array}\right] =>$

$$(*) = \int \frac{t}{t\,(\ln 3 + 1)}\,\mathrm{d}t = \int \frac{\mathrm{d}t}{\ln 3 + 1} = \frac{1}{\ln 3 + 1}\int \mathrm{d}t = \frac{t}{\ln 3 + 1} + C = \frac{3^x e^{x+1}}{\ln 3 + 1} + C$$

8. $\int \frac{1+\sqrt{x}^2}{x}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{1 + 2\sqrt{x} + x}{x}\,\mathrm{d}x = \int \left(\frac{1}{x} + \frac{2}{\sqrt{x}} + 1\right)\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x + \int \frac{2}{\sqrt{x}}\,\mathrm{d}x + \int \mathrm{d}x = \int \frac{\mathrm{d}x}{x} + 2\int \frac{\mathrm{d}x}{\sqrt{x}} + \int \mathrm{d}x =$$
$$= \ln|x| + 2 * 2\sqrt{x} + x + C = \ln|x| + 4\sqrt{x} + x + C$$

9. $\int \dfrac{\sqrt[3]{x^2} - x + 1}{\sqrt{x}}\,\mathrm{d}x = (*)$

$$(*) = \int \left(\frac{\sqrt[3]{x^2}}{\sqrt{x}} - \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)\mathrm{d}x = \int \left(x^{\frac{2}{3}-\frac{1}{2}} - x^{1-\frac{1}{2}} + \frac{1}{\sqrt{x}}\right)\mathrm{d}x = \int \left(x^{\frac{1}{6}} - \sqrt{x} + \frac{1}{\sqrt{x}}\right)\mathrm{d}x =$$
$$= \int x^{\frac{1}{6}}\,\mathrm{d}x - \int \sqrt{x}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{\sqrt{x}} = \frac{6}{7}\sqrt[6]{x^7} - \frac{2}{3}\sqrt{x}^3 + 2\sqrt{x} + C$$

10. $\int \left(\dfrac{3 * 2^x - 2 * 3^x}{5^{2x}}\right)\mathrm{d}x = (*)$

$$(*) \quad = \quad \int \left(\frac{3 * 2^x}{5^{2x}} - \frac{2 * 3^x}{5^{2x}}\right)\mathrm{d}x \quad = \quad \int \frac{3 * 2^x}{5^{2x}}\,\mathrm{d}x \ - \ \int \frac{2 * 3^x}{5^{2x}}\,\mathrm{d}x \quad = \quad 3\int \frac{2^x}{5^{2x}}\,\mathrm{d}x \ - \ 2\int \frac{3^x}{5^{2x}}\,\mathrm{d}x \quad =>$$

$$3\int \frac{2^x}{5^{2x}}\,\mathrm{d}x = (1)$$

$$\text{Замена:}\left[\begin{array}{c} t = \dfrac{2^x}{5^{2x}} \\ \mathrm{d}t = \dfrac{2^x * 5^{2x} * \ln 2 - 2^x * 5^{2x} * \ln 5}{5^{4x}}\,\mathrm{d}x = \left(\dfrac{2^x \ln 2}{5^{2x}} - \dfrac{2^x \ln 5}{5^{2x}}\right)\mathrm{d}x = \\ = \dfrac{2^x}{5^{2x}}\left(\ln 2 - 2\ln 5\right)\mathrm{d}x = t\left(\ln 2 - 2\ln 5\right)\mathrm{d}x \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{t\left(\ln 2 - 2\ln 5\right)} \end{array}\right] => (1) = 3\int \frac{t}{t\left(\ln 2 - 2\ln 5\right)}\,\mathrm{d}t =$$

$$= \frac{3}{\ln 2 - 2\ln 5}\int \mathrm{d}t = \frac{3t}{\ln 2 - 2\ln 5} + C = 3\frac{2^x}{5^{2x}}\left(\frac{1}{\ln 2 - 2\ln 5}\right) + C$$

Аналогично распишем вторую часть интеграла:

$$2\int \frac{3^x}{5^{2x}}\,\mathrm{d}x = (2)$$

$$\text{Замена:}\left[\begin{array}{c} t = \dfrac{3^x}{5^{2x}} \\ \mathrm{d}t = \dfrac{3^x * 5^{2x} * \ln 3 - 2^x * 5^{2x} * \ln 5}{5^{4x}}\,\mathrm{d}x = \left(\dfrac{3^x \ln 3}{5^{2x}} - \dfrac{3^x \ln 5}{5^{2x}}\right)\mathrm{d}x = \\ = \dfrac{3^x}{5^{2x}}\left(\ln 3 - 2\ln 5\right)\mathrm{d}x = t\left(\ln 3 - 2\ln 5\right)\mathrm{d}x \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{t\left(\ln 3 - 2\ln 5\right)} \end{array}\right] => (2) = 2\int \frac{t}{t\left(\ln 3 - 2\ln 5\right)}\,\mathrm{d}t =$$

$$= \frac{2}{\ln 3 - 2\ln 5}\int \mathrm{d}t = \frac{2t}{\ln 3 - 2\ln 5} + C = 2\frac{3^x}{5^{2x}}\left(\frac{1}{\ln 3 - 2\ln 5}\right) + C$$

Объединим полученные результаты:

$$(*) = (1) - (2) = \left(3\frac{2^x}{5^{2x}}\left(\frac{1}{\ln 2 - 2\ln 5}\right) + C\right) - \left(2\frac{3^x}{5^{2x}}\left(\frac{1}{\ln 3 - 2\ln 5}\right) + C\right) = \frac{6}{5^{2x}}\left(\frac{2^{x-1}}{\ln 2 - 2\ln 5} - \frac{3^{x-1}}{\ln 3 - 2\ln 5}\right) + C$$

11. $\int \frac{x^3}{x+1^2}\,\mathrm{d}x = (*)$

$$\text{Замена:}\left[\begin{array}{c} t = x + 1 \\ x = t - 1 \\ \mathrm{d}t = \mathrm{d}x \end{array}\right] => (*) = \int \frac{(t-1)^3}{t^2}\,\mathrm{d}x = \int \left(\frac{t^3 - 3t^2 + 3t - 1}{t^2}\right)\mathrm{d}t = \int \left(t - 3 + \frac{3}{t} - \frac{1}{t^2}\right)\mathrm{d}t =$$

$$= \int t\,\mathrm{d}t - 3\int \mathrm{d}t + 3\int \frac{\mathrm{d}t}{t} - \int \frac{\mathrm{d}t}{t^2} = \frac{1}{2}t^2 - 3t + 3\ln|t| + \frac{1}{t} + C = \frac{(x+1)^2}{2} - 3(x+1) + 3\ln|x+1| + \frac{1}{x+1} + C =$$

$$= \frac{x^2 + 2x + 1 - 6x - 6}{2} + 3\ln|x+1| + \frac{1}{x+1} + C = \frac{x^2 - 4x}{2} + 3\ln|x+1| + \frac{1}{x+1} + C =$$

$$= \frac{x(x-4)}{2} + 3\ln|x+1| + \frac{1}{x+1} + C$$

12. $\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x = (*)$

$$\text{Замена:}\left[\begin{array}{c} t = x^2 + 1 \\ \mathrm{d}t = 2x\,\mathrm{d}x \\ x\,\mathrm{d}x = \dfrac{\mathrm{d}t}{2} \end{array}\right] => (*) = \int \frac{1}{2\sqrt{t}}\,\mathrm{d}t = \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} = \sqrt{t} + C = \sqrt{x^2+1} + C$$

13. $\int \frac{1}{2x-1}\,\mathrm{d}x = (*)$

Замена: $\begin{bmatrix} t = 2x-1 \\ \mathrm{d}t = 2\,\mathrm{d}x \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{2} \end{bmatrix}$ $=> \quad (*) \;=\; \displaystyle\int \frac{1}{2t}\,\mathrm{d}t \;=\; \frac{1}{2}\int \frac{\mathrm{d}t}{t} \;=\; \frac{\ln|t|}{2} \;+\; C \;=\; \frac{\ln|2x-1|}{2} \;+\; C$

14. $\int (4x+1)^7\,\mathrm{d}x = (*)$

Замена: $\begin{bmatrix} t = 4x+1 \\ \mathrm{d}t = 4\,\mathrm{d}x \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{4} \end{bmatrix}$ $=> \quad (*) \;=\; \displaystyle\frac{1}{4}\int t^7\,\mathrm{d}t \;=\; \frac{1}{4} \,*\, \frac{1}{8}t^8 \;+\; C \;=\; \frac{t^8}{32} \;+\; C \;=\; \frac{(4x+1)^8}{32} \;+\; C$

15. $\int \sqrt[5]{(8-3x)^3}\,\mathrm{d}x = (*)$

Замена: $\begin{bmatrix} t = 8-3x \\ \mathrm{d}t = -3\,\mathrm{d}x \\ \mathrm{d}x = -\dfrac{\mathrm{d}t}{3} \end{bmatrix}$ $=> (*) = -\displaystyle\frac{1}{3}\int \sqrt[5]{t^3}\,\mathrm{d}t = -\frac{1}{3} * \frac{5}{8}\sqrt[5]{t^8} + C = -\frac{5\sqrt[5]{t^8}}{24} + C = -\frac{5}{24}\sqrt[5]{(8-3x)^8} + C$

16. $\int \sqrt[3]{x-5}\,\mathrm{d}x = (*)$

Замена: $\begin{bmatrix} t = x-5 \\ \mathrm{d}t = \mathrm{d}x \end{bmatrix}$ $=> \quad (*) \;=\; \displaystyle\int \sqrt[3]{t}\,\mathrm{d}t \;=\; \frac{3}{4}\sqrt[3]{t^4} \;+\; C \;=\; \frac{3}{4}\sqrt[3]{(x-5)^4} \;+\; C$

17. $\int e^{1-3x}\,\mathrm{d}x = (*)$

Замена: $\begin{bmatrix} t = e^{1-3x} \\ \mathrm{d}t = -3e^{1-3x}\,\mathrm{d}x \\ \mathrm{d}x = -\dfrac{\mathrm{d}t}{3t} \end{bmatrix}$ $=> \quad (*) \;=\; \displaystyle\int -\frac{t}{3t}\,\mathrm{d}t \;=\; -\frac{1}{3}\int \mathrm{d}t \;=\; -\frac{t}{3} \;+\; C \;=\; -\frac{1}{3}e^{1-3x} \;+\; C$

18. $\displaystyle\int \frac{\mathrm{d}x}{1+9x^2} = (*)$

Замена: $\begin{bmatrix} t = 3x \\ \mathrm{d}t = 3\,\mathrm{d}x \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{3} \end{bmatrix}$ $=> \quad (*) \;=\; \displaystyle\frac{1}{3}\int \frac{\mathrm{d}t}{1+t^2} \;=\; \frac{1}{3}arctg\,(t) \;+\; C \;=\; \frac{1}{3}arctg(3x) \;+\; C$

19. $\int xe^{-x^7}\,\mathrm{d}x = (*)$

Замена: $\begin{bmatrix} t = e^{-x^2} \\ \mathrm{d}t = -2xe^{-x^2}\,\mathrm{d}x \\ xe^{-x^2}\,\mathrm{d}x = -\dfrac{\mathrm{d}t}{2} \end{bmatrix}$ $=> (*) = -\displaystyle\int \frac{1}{2}e^{-x^5}\,\mathrm{d}t = -\frac{1}{2}\int \sqrt{t^5}\,\mathrm{d}t = -\frac{1}{7}\sqrt{t^7} + C = -\frac{1}{7}\sqrt{e^{-7x^2}} + C$

20. $\int e^{5x-1}\,\mathrm{d}x = (*)$

$$\text{Замена: } \left[\begin{array}{c} t = 5x-1 \\ \mathrm{d}t = 5\,\mathrm{d}x \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{5} \end{array}\right] \quad => \quad (*) \quad = \quad \frac{1}{5}\int e^t\,\mathrm{d}t \quad = \quad \frac{1}{5}e^t \; + \; C \quad = \quad \frac{1}{5}e^{5x-1} \; + \; C$$

21 $\int \dfrac{\mathrm{d}x}{\sin^4 x + \cos^4 x} = (*)$

$$\text{Замена: } \left[\begin{array}{c} \sin^2 x = \dfrac{1-\cos 2x}{2} \\ \cos^2 x = \dfrac{1+\cos 2x}{2} \end{array}\right] => (*) = \int \frac{\mathrm{d}x}{\left(\dfrac{1-\cos 2x}{2}\right)^2 + \left(\dfrac{1+\cos 2x}{2}\right)^2} =$$

$$= 4\int \frac{\mathrm{d}x}{1 - 2\cos 2x + \cos^2 2x + 1 + 2\cos 2x + \cos^2 x} = 4\int \frac{\mathrm{d}x}{2 + 2\cos^2 2x} = 2\int \frac{\mathrm{d}x}{1+\cos^2 2x} =$$

$$= 2\int \frac{\mathrm{d}x}{1 + \dfrac{1+\cos 4x}{2}} = 4\int \frac{\mathrm{d}x}{2 + 1 + \cos 4x} = 4\int \frac{\mathrm{d}x}{3+\cos 4x}$$

$$\text{Замена: } \left[\begin{array}{c} \cos 4x = \dfrac{1 - tg^2\,(2x)}{1 + tg^2\,(2x)} \\ t = tg\,(2x)\,,\;\; 2x = arctg\,(t)\,,\;\; x = \dfrac{arctg\,(t)}{2} \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{2(1+t^2)} \end{array}\right] =>$$

$$(*) = 4\int \frac{\mathrm{d}t}{2\,(1+t^2)\left(3 + \dfrac{1-t^2}{1+t^2}\right)} = 2\int \frac{\mathrm{d}t}{3 + 3t^2 + 1 - t^2} = 2\int \frac{\mathrm{d}t}{4 + 2t^2} = \int \frac{\mathrm{d}t}{2+t^2} = \int \frac{\mathrm{d}t}{\sqrt{2}^2 + t^2} =$$

$$= \frac{1}{\sqrt{2}}\; arctg\left(\frac{t}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}}\; arctg\left(\frac{tg\,(2x)}{\sqrt{2}}\right) + C$$

22. $\int (\arcsin x + \arccos x)\,\mathrm{d}x = (*)$

$$(*) = \int \arcsin x\,\mathrm{d}x + \int \arccos x\,\mathrm{d}x = (1) + (2)$$

Для первого интеграла распишем

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = \arcsin x & \mathrm{d}u = \dfrac{\mathrm{d}x}{\sqrt{1-x^2}} \\ v = x & \mathrm{d}v = \mathrm{d}x \end{array}\right] => (1) = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$$

Для второго интеграла распишем

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = \arccos x & \mathrm{d}u = -\dfrac{\mathrm{d}x}{\sqrt{1-x^2}} \\ v = x & \mathrm{d}v = \mathrm{d}x \end{array}\right] => (2) = x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x =>$$

$$(*) = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x + x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x =$$

$$= x\arcsin x + x\arccos x + C = x\,(\arcsin x + \arccos x) + C$$

23. $\int \dfrac{\mathrm{d}x}{1+\cos^2 x} = (*)$

$$(*) = \int \frac{\mathrm{d}x}{1+\dfrac{1+\cos 2x}{2}} = 2\int \frac{\mathrm{d}x}{2+1+\cos 2x} = 2\int \frac{\mathrm{d}x}{3+\cos 2x}$$

$$\text{Замена: } \left[\begin{array}{c} \cos 2x = \dfrac{1-tg^2(x)}{1+tg^2(x)} \\ t = tg\,(x)\,,\ x = arctg\,(t) \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{1+t^2} \end{array}\right] \Rightarrow (*) = 2\int \frac{\mathrm{d}x}{(1+t^2)\left(3+\dfrac{1-t^2}{1+t^2}\right)} = 2\int \frac{\mathrm{d}t}{3+3t^2+1-t^2} =$$

$$= 2\int \frac{\mathrm{d}t}{4+2t^2} = \int \frac{\mathrm{d}t}{2+t^2} = \int \frac{\mathrm{d}t}{\left(\sqrt{2}\right)^2+t^2} = \frac{1}{\sqrt{2}}\ arctg\left(\frac{t}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}}\ arctg\left(\frac{tg\,(x)}{\sqrt{2}}\right) + C$$

24. $\int \dfrac{\cos 2x}{\sin^2 x\ \cos^2 x}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x\ \cos^2 x}\,\mathrm{d}x$$

$$\text{Замена: } \left[\begin{array}{c} t = \sin x\ \cos x \\ \mathrm{d}t = \left(\cos^2 x\ - \sin^2 x\right)\mathrm{d}x \end{array}\right] \Rightarrow (*) = \int \frac{\mathrm{d}t}{t^2} = -\frac{1}{t} + C = -\frac{1}{\sin x\ \cos x} + C$$

25. $\int ctg^2\,(x)\ \ \mathrm{d}x = (*)$

$$(*) = \int \frac{\cos^2 x}{\sin^2 x}\,\mathrm{d}x$$

$$\text{Замена: } \left[\begin{array}{c} \cos^2 x = \dfrac{1+\cos 2x}{2} \\ \sin^2 x = \dfrac{1-\cos 2x}{2} \end{array}\right] \Rightarrow (*) = \int \frac{1+\cos 2x}{1-\cos 2x}\,\mathrm{d}x = \int \frac{1-\cos 2x+2\cos 2x}{1-\cos 2x}\,\mathrm{d}x =$$

$$= \int \mathrm{d}x + 2\int \frac{\cos 2x}{1-\cos 2x}\,\mathrm{d}x = x + 2\int \frac{\cos 2x}{1-\cos 2x}\,\mathrm{d}x$$

$$\text{Замена: } \left[\begin{array}{c} cos2x = \dfrac{1-tg^2(x)}{1+tg^2(x)} \\ t = tg^2\,(x)\,,\ x = arctg\,(t) \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{1+t^2} \end{array}\right] \Rightarrow (*) = x + 2\int \frac{\dfrac{1-t^2}{1+t^2}}{1-\dfrac{1-t^2}{1+t^2}}\,\frac{\mathrm{d}t}{1+t^2} = x + 2\int \frac{\left(1-t^2\right)\left(1+t^2\right)}{(1+t^2)^2\,(1+t^2-1+t^2)}\,\mathrm{d}t =$$

$$= x + 2\int \frac{1-t^2}{2t^2\ (1+t^2)}\,\mathrm{d}t = x + \int \frac{1-t^2}{t^2\ (1+t^2)}\,\mathrm{d}t$$

Разложим подынтегральное выражение на простые слагаемые:

$$\frac{1-t^2}{t^2\ (1+t^2)} = \frac{A}{t^2} + \frac{B}{t} + \frac{Ct+D}{1+t^2} = \frac{A\left(1+t^2\right) + Bt\left(1+t^2\right) + (Ct+D)\ t^2}{t^2\ (1+t^2)} \Rightarrow$$

$$A + At^2 + Bt + Bt^3 + Ct^3 + Dt^2 = 1 - t^2$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{c|l} t^0 & 1 = A \\ t^1 & 0 = B \\ t^2 & -1 = A + D \\ t^3 & 0 = B + C \end{array} \quad => A = 1,\ \ B = 0,\ \ C = 0,\ \ D = -2$$

$$\frac{1-t^2}{t^2\ \left(1+t^2\right)} = \frac{1}{t^2} - \frac{2}{1+t^2} => (*) = x + \int \frac{\mathrm{d}t}{t^2} - 2\int \frac{\mathrm{d}t}{1+t^2} = x - \frac{1}{t} - 2\int \frac{\mathrm{d}t}{t^2+1} = x - \frac{1}{t} - 2arctg\,(t) + C =$$
$$= x - \frac{1}{tg\,(x)} - 2arctg\,\big(tg\,(x)\big) + C = x - \frac{1}{tg\,(x)} - 2x + C = -x - ctg\,(x) + C$$

26. $\int \dfrac{\mathrm{d}x}{\cos^2 x - \cos 2x} = (*)$

$$(*) = \int \frac{\mathrm{d}x}{\cos^2 x - \cos^2 x + \sin^2 x} = \int \frac{\mathrm{d}x}{\sin^2 x} = -ctg\,(x) + C$$

27. $\int \dfrac{1+\cos^2 x}{1+\cos 2x}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{1 + \frac{1+\cos 2x}{2}}{1+\cos 2x}\,\mathrm{d}x = \frac{1}{2}\int \frac{2+1+\cos 2x}{1+\cos 2x}\,\mathrm{d}x = \frac{1}{2}\int \frac{1+\cos 2x+2}{1+\cos 2x}\,\mathrm{d}x =$$
$$= \frac{1}{2}\int \mathrm{d}x + \int \frac{\mathrm{d}x}{1+\cos 2x} = \frac{1}{2}\left(x + \int \frac{\mathrm{d}x}{\cos^2 x}\right) = \frac{1}{2}\big(x + tg\,(x)\big) + C$$

28. $\int \dfrac{\sin x}{\cos^3 x}\,\mathrm{d}x = (*)$

$$\text{Замена: } \left[\begin{array}{c} t = \cos x \\ \mathrm{d}t = -\sin x\,\mathrm{d}x \end{array}\right] \quad => \quad (*) = -\int \frac{\mathrm{d}t}{t^3} = \frac{1}{2t^2} + C = \frac{1}{2\cos^2 x} + C$$

29. $\int tg\,(x)\,\mathrm{d}x = (*)$

$$(*) = \int \frac{\sin x}{\cos x}\,\mathrm{d}x$$

$$\text{Замена: } \left[\begin{array}{c} t = \cos x \\ \mathrm{d}t = -\sin x\,\mathrm{d}x \end{array}\right] => (*) = -\int \frac{\mathrm{d}t}{t} = -\ln|t| + C = -\ln|\cos x| + C$$

30. $\int \sin^2 3x\ \ \mathrm{d}x = (*)$

$$(*) = \int \frac{1-\cos 6x}{2}\,\mathrm{d}x = \frac{1}{2}\left(\int \mathrm{d}x - \int \cos 6x\,\mathrm{d}x\right) = \frac{1}{2}\left(x - \frac{1}{6}\int \cos 6x\,\mathrm{d}6x\right) = \frac{x}{2} - \frac{1}{1}2\sin 6x + C$$

31. $\int \frac{\sin 2x}{1+\cos^2 x}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{\sin 2x}{1+\frac{1+\cos 2x}{2}}\,\mathrm{d}x = 2\int \frac{\sin 2x}{2+1+\cos 2x}\,\mathrm{d}x = 2\int \frac{\sin 2x}{3+\cos 2x}\,\mathrm{d}x$$

$$\text{Замена: } \begin{bmatrix} t = 3+\cos 2x \\ \mathrm{d}t = -2\sin 2x\,\mathrm{d}x \\ \sin 2x\,\mathrm{d}x = -\frac{1}{2}\,\mathrm{d}t \end{bmatrix} \Rightarrow (*) = -\int \frac{\mathrm{d}t}{t} = -\ln|t| + C = -\ln|3+\cos 2x| + C$$

32. $\int \cos x\; e^{\sin x}\,\mathrm{d}x = (*)$

$$\text{Замена: } \begin{bmatrix} t = \sin x \\ \mathrm{d}t = \cos x\,\mathrm{d}x \end{bmatrix} \Rightarrow (*) = \int e^t\,\mathrm{d}t = e^t + C = e^{\sin x} + C$$

33. $\int \sin^4 x\,\mathrm{d}x = (*)$

$$(*) = \int \frac{3-4\cos 2x+\cos 4x}{8}\,\mathrm{d}x = \frac{3}{8}\int \mathrm{d}x - \frac{1}{2}\int \cos 2x\,\mathrm{d}x + \frac{1}{8}\int \cos 4x\,\mathrm{d}x = \frac{3}{8}x + \frac{1}{4}\sin 2x - \frac{1}{32}\sin 4x + C$$

34. $\int \sin x\;\; \sin 5x\,\mathrm{d}x = (*)$

$$(*) = \frac{1}{2}\int \Big(\cos(x-5x)+\cos(x+5x)\Big)\,\mathrm{d}x = \frac{1}{2}\int (\cos 4x + \cos 6x)\,\mathrm{d}x = \frac{1}{8}\int \cos 4x\,\mathrm{d}4x + \frac{1}{12}\int \cos 6x\,\mathrm{d}6x =$$
$$= \frac{\sin 4x}{8} + \frac{\sin 6x}{12} + C$$

35. $\int \cos^3 x\,\mathrm{d}x = (*)$

$$(*) = \int \cos x\;\Big(1-\sin^2 x\Big)\,\mathrm{d}x$$

$$\text{Замена: } \begin{bmatrix} t = \sin x \\ \mathrm{d}t = \cos x\,\mathrm{d}x \end{bmatrix} \Rightarrow (*) = \int \Big(1-t^2\Big)\,\mathrm{d}t = \int \mathrm{d}t - \int t^2\,\mathrm{d}t = t - \frac{t^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C =$$

36. $\int \cos^6 x \, \mathrm{d}x = (*)$

$$(*) = \int \left(\frac{1+\cos 2x}{2}\right)^3 \mathrm{d}x = \frac{1}{8}\int (1+\cos 2x)^3 \,\mathrm{d}x = \frac{1}{8}\int \left(1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x\right) \mathrm{d}x =$$
$$= \frac{1}{8}\int \left(1 + 3\cos 2x + \frac{3}{2}(1+\cos 4x) + \cos 2x \frac{1+\cos 4x}{2}\right) \mathrm{d}x =$$
$$= \frac{1}{8}\int \left(1 + 3\cos 2x + \frac{3+3\cos 4x}{2} + \frac{\cos 2x + \cos 2x \ \cos 4x}{2}\right) \mathrm{d}x =$$
$$= \frac{1}{16}\int (2 + 7\cos 2x + 3 + 3\cos 4x + \cos 2x \ \cos 4x) \,\mathrm{d}x =$$
$$= \frac{1}{16}\int \left(2 + 7\cos 2x + 3 + 3\cos 4x + \frac{1}{2}\left(\cos(2x+4x) + \cos(2x-4x)\right)\right) \mathrm{d}x =$$
$$= \frac{1}{32}\int (10 + 14\cos 2x + 6\cos 4x + \cos 6x + \cos 2x) \,\mathrm{d}x = \frac{1}{32}\int (10 + 15\cos 2x + 6\cos 4x + \cos 6x) \,\mathrm{d}x =$$
$$= \frac{10}{32}x + \frac{15}{64}\sin 2x + \frac{1}{32}\frac{6}{4}\sin 4x + \frac{1}{32}\frac{1}{6}\sin 6x + C = \frac{5}{16}x + \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{192}\sin 6x + C =$$
$$= \frac{1}{192}(60x + 45\sin 2x + 9\sin 4x + \sin 6x) + C$$

37. $\int \frac{\cos 2x}{1+\sin x \ \cos x} \,\mathrm{d}x = (*)$

$$(*) = \int \frac{2\cos 2x}{2+\sin 2x} \,\mathrm{d}x$$

Замена: $\left[\begin{array}{c} t = 2 + \sin 2x \\ \mathrm{d}t = 2\cos 2x \ \sin 2x \,\mathrm{d}x \end{array}\right] \Rightarrow (*) = \int \frac{\mathrm{d}t}{t} = \ln|t| + C = \ln|2+\sin 2x| + C$

38. $\int sin^5 x \,\mathrm{d}x = (*)$

Замена: $\left[\begin{array}{c} t = \cos^2 x \\ \mathrm{d}t = -2\cos x \sin x \,\mathrm{d}x \\ \sin x \,\mathrm{d}x = -\dfrac{\mathrm{d}t}{2\cos x} \end{array}\right] \Rightarrow (*) = -\int \sin^4 x \frac{\mathrm{d}t}{2\cos x} = -\int \left(1-\cos^2 x\right)^2 \frac{\mathrm{d}t}{2\cos x} = -\int \frac{(1-t)^2}{2\sqrt{t}} \,\mathrm{d}t =$

$$= -\frac{1}{2}\int \frac{1-2t+t^2}{\sqrt{t}} \,\mathrm{d}t = -\int \frac{\mathrm{d}t}{2\sqrt{t}} + \int \sqrt{t} \ \mathrm{d}t - \int \sqrt{t^3} \ \frac{\mathrm{d}t}{2} = -\sqrt{t} + \frac{2}{3}\sqrt{t^3} - \frac{1}{5}\sqrt{t^5} + C =$$
$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$

39. $\int \frac{\mathrm{d}x}{1+\sin^2 x} = (*)$

$$(*) = \int \frac{\mathrm{d}x}{1+\frac{1-\cos 2x}{2}} = 2\int \frac{\mathrm{d}x}{2+1-\cos 2x} = 2\int \frac{\mathrm{d}x}{3-\cos 2x}$$

$$\text{Замена: } \left[\begin{array}{c} \cos 2x = \dfrac{1-tg^2(x)}{1+tg^2(x)} \\ t = tg\,(x)\,,\ x = arctg\,(t) \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{1+t^2} \end{array}\right] => (*) = 2\int \frac{\mathrm{d}t}{(1+t^2)\left(3-\dfrac{1-t^2}{1+t^2}\right)} = 2\int \frac{\mathrm{d}t}{3+3t^2-1+t^2} =$$

$$= 2\int \frac{\mathrm{d}t}{2+4t^2} = \int \frac{\mathrm{d}t}{\left(\sqrt{2}t\right)^2+1} = \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}\sqrt{2}t}{\left(\sqrt{2}t\right)^2+1} = \frac{1}{\sqrt{2}}arctg\left(\sqrt{2}t\right) + C = \frac{1}{\sqrt{2}}arctg\left(\sqrt{2}tg\,(x)\right) + C$$

40. $\int \frac{\mathrm{d}x}{cosx} = (*)$

$$\text{Замена: } \left[\begin{array}{c} cosx = \dfrac{1-tg^2\left(\dfrac{x}{2}\right)}{1+tg^2\left(\dfrac{x}{2}\right)} \\ t = tg\left(\dfrac{x}{2}\right),\ \dfrac{x}{2} = arctg\,(t)\,,\ x = 2arctg\,(t) \\ \mathrm{d}x = \dfrac{2\,\mathrm{d}t}{1+t^2} \end{array}\right] => (*) = 2\int \frac{1+t^2}{1-t^2}\frac{\mathrm{d}t}{1+t^2} = 2\int \frac{\mathrm{d}t}{1-t^2} = 2\int \frac{\mathrm{d}t}{(1-t)(1+t)}$$

Разложим подынтегральное выражение на простые слагаемые:

$$\frac{2}{(1+t)\,(1-t)} = \frac{A}{1-t} + \frac{B}{1+t} = \frac{A\,(1+t) + B\,(1-t)}{(1-t)\,(1+t)} =>$$

$$A + At + B - Bt = 2$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{c|l} t^0 & 2 = A + B \\ t^1 & 0 = A - B \end{array} \quad => A = B = 1$$

$$\frac{1}{(1+t)\,(1-t)} = \frac{1}{1-t} + \frac{1}{1+t} => (*) = \int \frac{\mathrm{d}t}{1+t} + \int \frac{\mathrm{d}t}{1-t} = \int \frac{\mathrm{d}(1+t)}{1+t} - \int \frac{\mathrm{d}(1-t)}{1-t} =$$

$$= \ln|1+t| - \ln|1-t| + C = \ln\left|1+tg\left(\frac{x}{2}\right)\right| - \ln\left|1-tg\left(\frac{x}{2}\right)\right| + C = \ln\left|\frac{1+tg\left(\dfrac{x}{2}\right)}{1-tg\left(\dfrac{x}{2}\right)}\right| + C$$

41. $\int \dfrac{\mathrm{d}x}{\cos^4 x} = (*)$

$$(*) = \int \frac{\dfrac{\sin^2 x}{\cos^2 x}}{\dfrac{\sin^2 x}{\cos^2 x}\cos^4 x}\,\mathrm{d}x = \int \frac{tg^2\,(x)}{\sin^2 x\ \cos^2 x}\,\mathrm{d}x$$

$$\text{Замена: } \left[\begin{array}{c} t = tg\,(x) = \dfrac{\sin x}{\cos x} \Rightarrow\ \sin x = t\cos x,\ \sin x = t\sqrt{1-\sin^2 x}, \\ \sin^2 x = t^2(1-\sin^2 x),\ \sin^2 x = \dfrac{t^2}{1+t^2} \\ \mathrm{d}t = \dfrac{\mathrm{d}x}{\cos^2 x} \end{array}\right] \Rightarrow (*) = \int \frac{t^2}{\left(\dfrac{t^2}{1+t^2}\right)}\,\mathrm{d}t =$$

$$= \int \left(1+t^2\right)\mathrm{d}t = \int \mathrm{d}t + \int t^2\,\mathrm{d}t = t + \frac{1}{3}t^3 + C = tg\,(x) + \frac{1}{3}tg^3\,(x) + C$$

42. $\int \dfrac{\mathrm{d}x}{3\sin^2 x + 4\cos^2 x} = (*)$

$$(*) = \int \frac{\mathrm{d}x}{\cos^2 c\left(3tg^2\,(x)+4\right)} = \frac{1}{\sqrt{3}}\int \frac{\mathrm{d}(\sqrt{3}tg\,(x))}{\left(\sqrt{3}tg\,(x)\right)^2 + 2^2} = \frac{1}{2\sqrt{3}}arctg\left(\frac{\sqrt{3}tg\,(x)}{2}\right) + C$$

43. $\int ctg\left(x - \dfrac{\pi}{8}\right)\mathrm{d}x = (*)$

$$\text{Замена: } \left[\begin{array}{c} \alpha = x - \dfrac{\pi}{8} \\ \mathrm{d}\alpha = \mathrm{d}x \end{array}\right] \Rightarrow (*) = \int ctg\,(\alpha)\,\mathrm{d}\alpha = \int \frac{\cos\alpha}{\sin\alpha}\,\mathrm{d}\alpha$$

$$\text{Замена: } \left[\begin{array}{c} t = \sin\alpha \\ \mathrm{d}t = \cos\alpha\,\mathrm{d}\alpha \end{array}\right] \Rightarrow (*) = \int \frac{\mathrm{d}t}{t} = \ln|t| + C = \ln\left|\sin\left(x - \frac{\pi}{8}\right)\right| + C$$

44. $\int \dfrac{\cos x}{\sqrt[3]{\sin^2 x}}\,\mathrm{d}x = (*)$

$$\text{Замена: } \left[\begin{array}{c} t = \sin x \\ \mathrm{d}t = \cos x\,\mathrm{d}x \end{array}\right] \Rightarrow (*) = \int \frac{\mathrm{d}t}{\sqrt[3]{t^2}} = 3\sqrt[3]{t} + C = 3\sqrt[3]{\sin x} + C$$

45. $\int \sin 2x\ e^{\cos 2x}\,\mathrm{d}x = (*)$

$$\text{Замена: } \left[\begin{array}{c} t = \cos 2x \\ \mathrm{d}t = -2\sin 2x\,\mathrm{d}x \\ \sin 2x\,\mathrm{d}x = -\dfrac{\mathrm{d}t}{2} \end{array}\right] \Rightarrow (*) = -\frac{1}{2}\int e^t\,\mathrm{d}t = -\frac{e^t}{2} + C = -\frac{1}{2}\int e^{\cos 2x}\,\mathrm{d}t = -\frac{e^t}{2} + C$$

46. $\int \dfrac{\sqrt{x}+\sqrt[3]{x}}{x}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{\mathrm{d}x}{\sqrt{x}} + \int \frac{\mathrm{d}x}{\sqrt[3]{x^2}} = 2\sqrt{x} + 3\sqrt[3]{x} + C$$

47. $\int x\,\sqrt{1-x^2}\,\mathrm{d}x = (*)$

$$\text{Замена: } \begin{bmatrix} t = 1-x^2 \\ \mathrm{d}t = -2x\,\mathrm{d}x \\ x\,\mathrm{d}x = -\dfrac{\mathrm{d}t}{2} \end{bmatrix} \quad => \quad (*) = -\frac{1}{2}\int \sqrt{t}\,\mathrm{d}t = -\frac{1}{3}\sqrt{t^3} + C = -\frac{1}{3}\sqrt{(1-x^2)^3} + C$$

48. $\int 3^{-x}\,\mathrm{d}x = (*)$

$$(*) = -\int 3^{-x}\,\mathrm{d}{-x} = -\frac{3^{-x}}{\ln 3} + C$$

49. $\int \dfrac{x}{x^4+1}\,\mathrm{d}x = (*)$

$$\text{Замена: } \begin{bmatrix} t = x^2 \\ \mathrm{d}t = 2x\,\mathrm{d}x \\ x\,\mathrm{d}x = \dfrac{\mathrm{d}t}{2} \end{bmatrix} \quad => \quad (*) = \frac{1}{2}\int \frac{\mathrm{d}t}{1+t^2} = \frac{1}{2}arctg\,(t) + C = \frac{1}{2}arctg\left(x^2\right) + C$$

50. $\int \sqrt{\dfrac{1-x}{1+x}}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{1-x}{\sqrt{(1+x)\,(1-x)}\,\mathrm{d}x} = \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} - \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x - \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}$$

$$\text{Замена: } \begin{bmatrix} t = 1-x^2 \\ \mathrm{d}t = -2x\,\mathrm{d}x \\ x\,\mathrm{d}x = -\dfrac{\mathrm{d}t}{2} \end{bmatrix} => (*) = \arcsin x + \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} = \arcsin x + \sqrt{t} + C = \arcsin x + \sqrt{1-x^2} + C$$

51. $\int \dfrac{e^x}{e^{2x}+9}\,\mathrm{d}x = (*)$

$$\text{Замена: } \begin{bmatrix} t = e^x \\ \mathrm{d}t = e^x\,\mathrm{d}x \end{bmatrix} \quad => \quad (*) = \int \frac{\mathrm{d}t}{9+t^2} = \int \frac{\mathrm{d}t}{3^2+t^2} = \frac{1}{3}arctg\left(\frac{t}{3}\right) + C = \frac{1}{3}arctg\left(\frac{e^x}{3}\right) + C$$

52. $\displaystyle\int \frac{\mathrm{d}x}{x\ \ln x} = (*)$

Замена: $\left[\begin{array}{c} t = \ln x \\ \mathrm{d}t = \dfrac{\mathrm{d}x}{x} \end{array}\right] \quad => \quad (*) \quad = \quad \displaystyle\int \frac{\mathrm{d}t}{t} \quad = \quad \ln|t| \quad + \quad C \quad = \quad \ln|\ln x| \quad + \quad C$

53. $\displaystyle\int \frac{x\,\mathrm{d}x}{x - \sqrt{x^2-1}} = (*)$

$$(*) = \int \frac{x\left(x+\sqrt{x^2-1}\right)}{\left(x+\sqrt{x^2-1}\right)\left(x-\sqrt{x^2-1}\right)}\,\mathrm{d}x = \int \frac{x\left(x+\sqrt{x^2-1}\right)}{x^2-(x^2-1)}\,\mathrm{d}x =$$
$$= \int x\left(x+\sqrt{x^2-1}\right)\mathrm{d}x = \int x^2 + x\sqrt{x^2-1}\,\mathrm{d}x = \int x^2\,\mathrm{d}x + \int x\sqrt{x^2-1}\,\mathrm{d}x = \frac{x^3}{3} + \int x\sqrt{x^2-1}\,\mathrm{d}x$$

Замена: $\left[\begin{array}{c} t = x^2-1 \\ \mathrm{d}t = 2x\,\mathrm{d}x \\ x\,\mathrm{d}x = \dfrac{\mathrm{d}t}{2} \end{array}\right] => (*) = \dfrac{x^3}{3} + \dfrac{1}{2}\displaystyle\int \sqrt{t}\,\mathrm{d}t = \dfrac{x^3}{3} + \dfrac{\sqrt{t^3}}{3} + C = \dfrac{x^3}{3} + \dfrac{\sqrt{(x^2-1)^3}}{3} + C$

54. $\displaystyle\int x\ \sqrt[3]{1+x}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{x\,(1+x)}{\sqrt[3]{(1+x)^2}}\,\mathrm{d}x$$

Замена: $\left[\begin{array}{c} t = (1+x)^2 \\ \mathrm{d}t = 2\,(1+x)\,\mathrm{d}x \\ (1+x)\,\mathrm{d}x = \dfrac{\mathrm{d}t}{2} \end{array}\right] => (*) = \dfrac{1}{2}\displaystyle\int \frac{\sqrt{t}-1}{\sqrt[3]{t}}\,\mathrm{d}t = \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt[6]{t}} - \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt[3]{t}} = \frac{3}{7}\sqrt[6]{t^7} - \frac{3}{4}\sqrt[3]{t^2} + C =$

$$= \frac{3}{7}\sqrt[3]{(1+x)^7} - \frac{3}{4}\sqrt[3]{(1+x)^4} + C$$

55. $\displaystyle\int \frac{\mathrm{d}x}{1-4x^2} = (*)$

$$(*) = \int \frac{\mathrm{d}x}{1-(2x)^2} = \frac{1}{2}\int \frac{\mathrm{d}x}{1-2x} + \frac{1}{2}\int \frac{\mathrm{d}t}{1+2x} =$$
$$= -\frac{1}{4}\int \frac{\mathrm{d}(1-2x)}{1-2x} + \frac{1}{4}\int \frac{\mathrm{d}(1+2x)}{1+2x} = -\frac{1}{4}\ln|1-2x| + \frac{1}{4}\ln|1+2x| + C = \frac{1}{4}\ln\left|\frac{1+2x}{1-2x}\right| + C$$

56. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{4-9x^2}} = (*)$

Замена: $\left[\begin{array}{c} t = 3x \\ \mathrm{d}t = 3\,\mathrm{d}x \\ \mathrm{d}x = \dfrac{\mathrm{d}t}{3} \end{array}\right] \quad => \quad (*) \quad = \quad \dfrac{1}{3}\displaystyle\int \frac{\mathrm{d}t}{\sqrt{2^2-t^2}} \quad = \quad \frac{1}{3}\arcsin\left(\frac{t}{2}\right) \ + \ C \quad = \quad \frac{1}{3}\arcsin\left(\frac{3}{2}x\right) \ + \ C$

57. $\int \dfrac{x\sqrt{1+x}}{\sqrt{1-x}}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{x\,(1+x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} + \int \frac{x^2\,\mathrm{d}x}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{\mathrm{d}\big(1-x^2\big)}{\sqrt{1-x^2}} - \int \frac{-x^2}{\sqrt{1-x^2}}\,\mathrm{d}x =$$
$$= -\sqrt{1-x^2} - \int \sqrt{1-x^2}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + \arcsin x - \int \sqrt{1-x^2}\,\mathrm{d}x$$
$$I = \int \sqrt{1-x^2}\,\mathrm{d}x$$
$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = \sqrt{1-x^2} & \mathrm{d}u = -\dfrac{x\,\mathrm{d}x}{\sqrt{1-x^2}} \\ v = x & \mathrm{d}v = \mathrm{d}x \end{array}\right] => I = x\sqrt{1-x^2} + \int \frac{x^2\,\mathrm{d}x}{\sqrt{1-x^2}} =$$
$$= x\sqrt{1-x^2} - \int \frac{1-x^2\,\mathrm{d}x}{\sqrt{1-x^2}} + \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} =>$$
$$2I = x\sqrt{1-x^2} + \arcsin x => I = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$$
$$(*) = -\sqrt{1-x^2} + \arcsin x - I =$$
$$= -\sqrt{1-x^2} + \arcsin x - \left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x\right) + C = -\sqrt{1-x^2}\left(1 + \frac{x}{2}\right) + \frac{1}{2}\arcsin x + C$$

58. $\int \dfrac{1+x}{\sqrt{1-x^2}}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} + \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x \ + \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}$$
$$\text{Замена: } \left[\begin{array}{c} t = 1-x^2 \\ \mathrm{d}t = -2x\,\mathrm{d}x \\ x\,\mathrm{d}x = -\dfrac{\mathrm{d}t}{2} \end{array}\right] => (*) = \arcsin x \ - \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} = \arcsin x \ - \sqrt{t} + C = \arcsin x \ - \sqrt{1-x^2} + C$$

59. $\int \dfrac{1+x-x^2}{\sqrt{\left(1-x^2\right)^3}}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{x}{\sqrt{\left(1-x^2\right)^3}}\,\mathrm{d}x + \int \frac{1-x^2}{\sqrt{\left(1-x^2\right)^3}}\,\mathrm{d}x = \int \frac{x\,\mathrm{d}x}{\sqrt{\left(1-x^2\right)^3}} + \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int \frac{x\,\mathrm{d}x}{\sqrt{\left(1-x^2\right)^3}} + \arcsin x$$
$$\text{Замена: } \left[\begin{array}{c} t = 1-x^2 \\ \mathrm{d}t = -2x\,\mathrm{d}x \\ x\,\mathrm{d}x = -\dfrac{\mathrm{d}t}{2} \end{array}\right] => (*) = \arcsin x - \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t^3}} = \frac{1}{\sqrt{t}} + \arcsin x + C = \frac{1}{\sqrt{1-x^2}} + \arcsin x + C$$

60. $\int x \ \cos x\,\mathrm{d}x = (*)$

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = x & \mathrm{d}u = \mathrm{d}x \\ v = \sin x & \mathrm{d}v = \cos x\,\mathrm{d}x \end{array}\right] => (*) = x \ \sin x - \int \sin x\,\mathrm{d}x = x \ \sin x + \cos x + C$$

61. $\int \arccos x \,\mathrm{d}x = (*)$

Интегрируем по частям: $\left[\begin{array}{r|l} u = \arccos x & \mathrm{d}u = -\dfrac{\mathrm{d}x}{\sqrt{1-x^2}} \\ v = x & \mathrm{d}v = \mathrm{d}x \end{array}\right] \Rightarrow (*) = x\ \arccos x + \displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}$

Замена: $\left[\begin{array}{c} t = 1-x^2 \\ \mathrm{d}t = -2x\,\mathrm{d}x \\ x\,\mathrm{d}x = -\dfrac{\mathrm{d}t}{2} \end{array}\right] \Rightarrow (*) = x\ \arccos x - \dfrac{1}{2}\displaystyle\int \frac{\mathrm{d}t}{\sqrt{t}} = x\ \arccos x - \sqrt{t} + C = x\ \arccos x - \sqrt{1-x^2} + C$

62. $\int x\ arctgx \,\mathrm{d}x = (*)$

Интегрируем по частям: $\left[\begin{array}{r|l} u = arctg(x) & \mathrm{d}u = \dfrac{\mathrm{d}x}{1+x^2} \\ v = \dfrac{x^2}{2} & \mathrm{d}v = x\,\mathrm{d}x \end{array}\right] \Rightarrow (*) = \dfrac{x^2}{2} arctg(x) - \dfrac{1}{2}\displaystyle\int \frac{x^2}{1+x^2}\,\mathrm{d}x =$

$$= \frac{x^2}{2} arctg(x) - \frac{1}{2}\int \mathrm{d}x + \frac{1}{2}\int \frac{\mathrm{d}x}{1+x^2} = \frac{x^2}{2} arctg(x) - \frac{x}{2} + \frac{1}{2} arctg(x) + C = \frac{1}{2}\left(x^2+1\right) arctg(x) - \frac{x}{2} + C$$

63. $\int x^2\ e^x \,\mathrm{d}x = (*)$

Интегрируем по частям: $\left[\begin{array}{c|c} u = x^2 & \mathrm{d}u = 2x\,\mathrm{d}x \\ v = e^x & \mathrm{d}v = e^x\,\mathrm{d}x \end{array}\right] \Rightarrow (*) = x^2\ e^x - 2\displaystyle\int x\ e^x\,\mathrm{d}x$

Интегрируем по частям: $\left[\begin{array}{c|c} u = x & \mathrm{d}u = \mathrm{d}x \\ v = e^x & \mathrm{d}v = e^x\,\mathrm{d}x \end{array}\right] \Rightarrow$

$$(*) = x^2\ e^x - 2x\ e^x + 2\int e^x\,\mathrm{d}x = x^2\ e^x - 2x\ e^x + 2e^x + C = e^x\ \left(x^2 - 2x + 2\right) + C$$

64. $\int \ln x \,\mathrm{d}x = (*)$

Интегрируем по частям: $\left[\begin{array}{r|l} u = \ln x & \mathrm{d}u = \dfrac{\mathrm{d}x}{x} \\ v = x & \mathrm{d}v = \mathrm{d}x \end{array}\right] \Rightarrow (*) = x\ \ln x - \displaystyle\int \mathrm{d}x = x\ \ln x - x + C = x\ (\ln x - 1) + C$

65. $\int \sqrt{x^2+1}\,\mathrm{d}x = (*)$

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = \sqrt{x^2+1} & \mathrm{d}u = \dfrac{x\,\mathrm{d}x}{\sqrt{x^2+1}} \\ v = x & \mathrm{d}v = \mathrm{d}x \end{array}\right] => (*) = x\ \sqrt{x^2+1} - \int \frac{x^2\,\mathrm{d}x}{\sqrt{x^2+1}} =$$

$$= x\ \sqrt{x^2+1} - \int \frac{x^2+1}{\sqrt{x^2+1}}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = x\ \sqrt{x^2+1} - \int \sqrt{x^2+1}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{\sqrt{x^2+1}}$$

$$\int \sqrt{x^2+1}\,\mathrm{d}x = I => 2I = x\ \sqrt{x^2+1} + \int \frac{\mathrm{d}x}{\sqrt{x^2+1}}$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = (1)$$

$$\text{Замена: } \left[\begin{array}{c} t = x + \sqrt{x^2+1} \\ \mathrm{d}t = 1 + \dfrac{x\,\mathrm{d}x}{\sqrt{x^2+1}} = \dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\,\mathrm{d}x = \dfrac{t\,\mathrm{d}x}{\sqrt{x^2+1}} \\ \dfrac{\mathrm{d}x}{\sqrt{x^2+1}} = \dfrac{\mathrm{d}t}{t} \end{array}\right] => (1) = \int \frac{\mathrm{d}t}{t} = \ln|t| + C = \ln\left|x + \sqrt{x^2+1}\right| + C =>$$

$$2I = x\ \sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right| + C => (*) = I = \frac{x}{2}\ \sqrt{x^2+1} + \frac{1}{2}\ln\left|x + \sqrt{x^2+1}\right| + C$$

66. $\int e^{2x}\ \sin 3x\,\mathrm{d}x = (*)$

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = e^{2x} & \mathrm{d}u = 2e^{2x}\,\mathrm{d}x \\ v = -\dfrac{1}{3}\cos 3x & \mathrm{d}v = \sin 3x\,\mathrm{d}x \end{array}\right] => (*) = -\frac{1}{3}e^{2x}\ \cos 3x - \int -\frac{2}{3}e^{2x}\ \cos 3x\,\mathrm{d}x =$$

$$= -\frac{1}{3}e^{2x}\ \cos 3x + \frac{2}{3}\int e^{2x}\ \cos 3x\,\mathrm{d}x$$

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = e^{2x} & \mathrm{d}u = 2e^{2x}\,\mathrm{d}x \\ v = \dfrac{1}{3}\sin 3x & \mathrm{d}v = \cos 3x\,\mathrm{d}x \end{array}\right] =>$$

$$(*) = -\frac{1}{3}e^{2x}\ \cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\ \sin 3x - \frac{2}{3}\int e^{2x}\ \sin 3x\,\mathrm{d}x\right) =$$

$$= -\frac{1}{3}e^{2x}\ \cos 3x + \frac{2}{9}e^{2x}\ \sin 3x - \frac{4}{9}\int e^{2x}\ \sin 3x\,\mathrm{d}x$$

$$\int e^{2x}\ \sin 3x\,\mathrm{d}x = I =>$$

$$I = -\frac{1}{3}e^{2x}\ \cos 3x + \frac{2}{9}e^{2x}\ \sin 3x - \frac{4}{9}I$$

$$\frac{13}{9}I = -\frac{1}{3}e^{2x}\left(\frac{2}{3}\sin 3x - cos3x\right) + C$$

$$(*) = I = \frac{3}{13}e^{2x}\left(\frac{2}{3}\sin 3x - cos3x\right) + C$$

67. $\int \dfrac{3x-1}{x^2+9}\,\mathrm{d}x = (*)$

$$(*) = 3\frac{x\,\mathrm{d}x}{x^2+9} - \int \frac{\mathrm{d}x}{x^2+9} = \frac{3}{2}\int \frac{\mathrm{d}\left(x^2+9\right)}{x^2+9} - \int \frac{\mathrm{d}x}{x^2+3^2} = \frac{3}{2}\ln\left|x^2+9\right| - \frac{1}{3}arctg\left(\frac{x}{3}\right) + C$$

68. $\displaystyle\int \frac{\mathrm{d}x}{x^4+2x^2+1} = (*)$

$$(*) = \int \frac{4x^3+4x-4x^3-4x+1}{\left(x^2+1\right)^2}\,\mathrm{d}x = \int \frac{\mathrm{d}\left(x^4+2x^2+1\right)}{x^4+2x^2+1} - \int \frac{4x^3+4x-1}{x^4+2x^2+1}\,\mathrm{d}x = \ln\left|x^4+2x^2+1\right| - \int \frac{4x^3+4x-1}{\left(x^2+1\right)^2}\,\mathrm{d}x$$

Разложим подынтегральное выражение на простые слагаемые:

$$\int \frac{4x^3+4x-1}{\left(x^2+1\right)^2}dx = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

$$(Ax+B)\left(x^2+1\right) + Cx + D = 4x^3+4x-1$$

$$Ax^3+Ax+Bx^2+B+Cx+D = 4x^3+4x-1$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{r|l} t^0 & -1 = B+D \\ t^1 & 4 = A+C \\ t^2 & 0 = B \\ t^3 & 4 = A \end{array} \quad => A=4,\ B=0,\ C=0,\ D=-1$$

$$(*) = \ln\left|x^4+2x^2+1\right| - \int \frac{4x\,\mathrm{d}x}{x^2+1} + \int \frac{\mathrm{d}x}{\left(x^2+1\right)^2}$$

$$\int \frac{4x\,\mathrm{d}x}{x^2+1} = (1)$$

$$\text{Замена: } \begin{bmatrix} t = x^2+1 \\ \mathrm{d}t = 2x\,\mathrm{d}x \end{bmatrix} => (1) = 2\int \frac{\mathrm{d}t}{t} = 2\ln|t| + C = 2\ln\left|x^2+1\right| + C$$

$$\int \frac{\mathrm{d}x}{\left(x^2+1\right)^2} = (2)$$

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = \dfrac{1}{x} & \mathrm{d}u = -\dfrac{\mathrm{d}x}{x^2} \\ \mathrm{d}v = \dfrac{x\,\mathrm{d}x}{\left(x^2+1\right)^2} & v = -\dfrac{1}{2\left(x^2+1\right)} \end{array}\right] =>$$

$$(2) = -\frac{1}{2x\left(x^2+1\right)} - \frac{1}{2}\int \frac{\mathrm{d}x}{x^2\left(x^2+1\right)}$$

$$\frac{1}{x^2\left(x^2+1\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} =>$$

$$A(x^3+x) + B(x^2+1) + Cx^3 + Dx^2 = 1 => A=0,\ B=1,\ C=0,\ D=-1 =>$$

$$(2) = -\frac{1}{2x\left(x^2+1\right)} - \frac{1}{2}\int \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right)\mathrm{d}x = -\frac{1}{2x\left(x^2+1\right)} + \frac{1}{2x} + \frac{1}{2}arctgx + C =>$$

$$(*) = \ln\left|x^4+2x^2+1\right| - (1) + (2) = \ln\left|x^4+2x^2+1\right| - 2\ln\left|x^2+1\right| - \frac{1}{2x\left(x^2+1\right)} + \frac{1}{2x} + \frac{1}{2}arctgx + C =$$

$$= \frac{x}{2} + \frac{1}{2}arctgx + C$$

69. $\int \ln^2 x \,\mathrm{d}x = (*)$

Интегрируем по частям: $\left[\begin{array}{r|l} u = \ln^2 x & \mathrm{d}u = \dfrac{2\ \ln x}{x}\,\mathrm{d}x \\ v = x & \mathrm{d}v = \mathrm{d}x \end{array}\right] \Rightarrow (*) = x\ \ln^2 x - 2\displaystyle\int \ln x \,\mathrm{d}x$

Интегрируем по частям: $\left[\begin{array}{r|l} u = \ln x & \mathrm{d}u = \dfrac{\mathrm{d}x}{x} \\ v = x & \mathrm{d}v = \mathrm{d}x \end{array}\right] \Rightarrow (*) = x\ \ln^2 x - 2\left(x\ \ln x - \displaystyle\int dx\right) = x\ \ln^2 x - 2x\ \ln x + 2x + C$

70. $\displaystyle\int \frac{x^2}{\left(x^2+1\right)^2}\,\mathrm{d}x = (*)$

Интегрируем по частям: $\left[\begin{array}{r|l} u = x & \mathrm{d}u = \mathrm{d}x \\ v = -\dfrac{1}{2}\left(\dfrac{1}{x^2+1}\right) & \mathrm{d}v = \dfrac{x}{\left(x^2+1\right)^2}\,\mathrm{d}x \end{array}\right] \Rightarrow$

$$(*) = -\frac{1}{2}\left(\frac{x}{x^2+1}\right) - \frac{1}{2}\int \frac{\mathrm{d}x}{x^2+1} = -\frac{1}{2}\left(\frac{x}{x^2+1} + arctgx\right) + C$$

71. $\displaystyle\int \frac{x^3}{\sqrt{x-1}}\,\mathrm{d}x = (*)$

Замена: $\left[\begin{array}{c} t = \sqrt{x-1},\ x = t^2+1 \\ \mathrm{d}t = \dfrac{\mathrm{d}x}{2\sqrt{x-1}} \\ \dfrac{\mathrm{d}x}{\sqrt{x-1}} = 2\,\mathrm{d}t \end{array}\right] \Rightarrow (*) = \dfrac{1}{2}\displaystyle\int \left(t^2+1\right)^3 \mathrm{d}t = \dfrac{1}{2}\displaystyle\int \left(t^6+3t^4+3t^2+1\right)\mathrm{d}t =$

$$= \frac{1}{2}\left(\int t^6\,\mathrm{d}t + 3\int t^4\,\mathrm{d}t + 3\int t^2\,\mathrm{d}t + \int \mathrm{d}t\right) = \frac{t^7}{14} + \frac{3t^5}{10} + \frac{t^3}{2} + \frac{t}{2} + C =$$

$$= \frac{1}{14}\sqrt{(x-1)^7} + \frac{3}{10}\sqrt{(x-1)^5} + \frac{1}{2}\sqrt{(x-1)^3} + \frac{1}{2}\sqrt{x-1} + C$$

72. $\displaystyle\int \frac{\mathrm{d}x}{x\ \sqrt{1+x}}$

Замена: $\left[\begin{array}{c} t = \sqrt{1+x},\ x = t^2-1 \\ \mathrm{d}t = \dfrac{\mathrm{d}x}{2\sqrt{1+x}} \\ \dfrac{\mathrm{d}x}{\sqrt{1+x}} = 2\,\mathrm{d}t \end{array}\right] \Rightarrow (*) = \dfrac{1}{2}\displaystyle\int \frac{\mathrm{d}t}{t^2-1} = \dfrac{1}{2}\displaystyle\int \frac{\mathrm{d}t}{(t+1)(t-1)} =$

$$= \frac{1}{2}\int \left(-\frac{1}{2(t+1)} + \frac{1}{2(t-1)}\right)\mathrm{d}t = \frac{1}{4}\int \frac{\mathrm{d}t}{t-1} - \frac{1}{4}\int \frac{\mathrm{d}t}{t+1} =$$

$$= \frac{1}{4}\ln|t-1| - \frac{1}{4}\ln|t+1| + C = \frac{1}{4}\ln\left|\sqrt{1+x}-1\right| - \frac{1}{4}\ln\left|\sqrt{1+x}+1\right| + C = \frac{1}{4}\ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right| + C$$

73. $\displaystyle\int \frac{\mathrm{d}x}{1+\sqrt{x}} = (*)$

$$\text{Замена: } \begin{bmatrix} t = 1+\sqrt{x},\ \sqrt{x} = t-1 \\ \mathrm{d}t = \dfrac{\mathrm{d}x}{2\sqrt{x}} \\ \mathrm{d}x = 2\sqrt{x}\,\mathrm{d}x \end{bmatrix} => (*) = \int \frac{2\,(t-1)}{t}\,\mathrm{d}t = 2\int \mathrm{d}t - 2\int \frac{\mathrm{d}t}{t} = 2t - 2\ln|t| + C =$$

$$= 2 + 2\sqrt{x} - 2\ln\left|1+\sqrt{x}\right| + C$$

74. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}} = (*)$

$$\text{Замена: } \begin{bmatrix} x = t^6,\ t = \sqrt[6]{x} \\ \mathrm{d}x = 6t^5\,\mathrm{d}t \end{bmatrix} => (*) = 6\int \frac{t^5}{t^3+t^2}\,\mathrm{d}t = 6\int \frac{t^5}{t^2\,(t+1)}\,\mathrm{d}t = 6\int \frac{t^3}{t+1}\,\mathrm{d}t = 6\int \left(t^2 - t + 1 - \frac{1}{t+1}\right)\mathrm{d}t =$$

$$= 6\left(\int t^2\,\mathrm{d}t - \int t\,\mathrm{d}t + \int \mathrm{d}t - \int \frac{\mathrm{d}t}{t+1}\right) = 2t^3 - 3t^2 + 6t - 6\ln|t+1| + C =$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left|\sqrt[6]{x}+1\right| + C$$

75. $\displaystyle\int \frac{x^2}{x+1}\,\mathrm{d}x = (*)$

$$\text{Замена: } \begin{bmatrix} t = x+1,\ x^2 = (t-1)^2 \\ \mathrm{d}t = \mathrm{d}x \end{bmatrix} => (*) = \frac{(t-1)^2}{t}\,\mathrm{d}t = \int \frac{t^2-2t+1}{t}\,\mathrm{d}t = \int \left(t - 2 + \frac{1}{t}\right)\mathrm{d}t =$$

$$= \int t\,\mathrm{d}t - 2\int \mathrm{d}t + \int \frac{\mathrm{d}t}{t} = \frac{1}{2}t^2 - 2t + \ln|t| + C = \frac{1}{2}(x+1)^2 - 2\,(x+1) + \ln|x+1| + C$$

76. $\displaystyle\int \frac{3-4x}{2x^2-3x+1}\,\mathrm{d}x = (*)$

$$\text{Замена: } \begin{bmatrix} t = 2x^2-3x+1 \\ \mathrm{d}t = (4x-3)\,\mathrm{d}x \end{bmatrix} \quad => \quad (*) \quad = \quad -\int \frac{\mathrm{d}t}{t} \quad = \quad -\ln|t| \ + \ C \quad = \quad -\ln\left|2x^2-3x+1\right| \ + \ C$$

77. $\displaystyle\int \frac{x-3}{\sqrt{3-2x-x^2}}\,\mathrm{d}x = (*)$

$$(*) = -\frac{1}{2}\int \frac{-2-2x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x + \int \frac{-4}{\sqrt{3-2x-x^2}}\,\mathrm{d}x = \int \frac{1+x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x - 4\int \frac{\mathrm{d}x}{\sqrt{4-(x+1)^2}} =$$

$$= \int \frac{1+x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x + 4\arccos\left(\frac{x+1}{2}\right)$$

$$\text{Замена: } \begin{bmatrix} t = 3-2x-x^2 \\ dt = (-2-2x)\,\mathrm{d}x \end{bmatrix} => (*) = 4\arccos\left(\frac{x+1}{2}\right) + \int -\frac{\mathrm{d}t}{2\sqrt{t}} =$$

$$= 4\arccos\left(\frac{x+1}{2}\right) - \sqrt{t} + C = 4\arccos\left(\frac{x+1}{2}\right) - \sqrt{3-2x-x^2} + C$$

78. $\int \dfrac{\mathrm{d}x}{\sqrt{9x^2 - 6x + 2}} = (*)$

$$(*) \quad = \quad \int \frac{\mathrm{d}x}{\sqrt{(3x)^2 - 2*3x + 1 + 1}} \quad = \quad \int \frac{\mathrm{d}x}{\sqrt{(3x+1)^2 + 1}} \quad = \quad \frac{1}{3}\ln\left|(3x+1) + \sqrt{(3x+1)^2+1}\right| \;+\; C$$

79. $\int \dfrac{\sqrt{x}\,\mathrm{d}x}{\sqrt{2x+3}} = (*)$

$$\text{Замена: } \left[\begin{array}{c} t = \sqrt{x},\ x = t^2 \\ \mathrm{d}x = 2t\,\mathrm{d}t \end{array}\right] \Rightarrow (*) = \int \frac{2t^2\,\mathrm{d}t}{\sqrt{2t^2+3}} = \int \frac{2t^2+3}{\sqrt{2t^2+3}}\,\mathrm{d}t - 3\int \frac{\mathrm{d}t}{\sqrt{2t^2+3}} =$$

$$= \int \sqrt{2t^2+3}\,\mathrm{d}t - 3\ln\left|\sqrt{2}t + \sqrt{2t^2+3}\right|$$

$$I = \int \sqrt{2t^2+3}\,\mathrm{d}t$$

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = \sqrt{2t^2+3} & \mathrm{d}u = \dfrac{2t\,\mathrm{d}t}{\sqrt{2t^2+3}} \\ v = t & \mathrm{d}v = \mathrm{d}t \end{array}\right] \Rightarrow I = t\sqrt{2t^2+3} - \int \frac{2t^2\,\mathrm{d}t}{\sqrt{2t^2+3}} =$$

$$= t\sqrt{2t^2+3} - \int \frac{2t^2+3}{\sqrt{2t^2+3}}\,\mathrm{d}t + 3\int \frac{\mathrm{d}t}{\sqrt{2t^2+3}} = t\sqrt{2t^2+3} - \int \sqrt{2t^2+3}\,\mathrm{d}t + \ln\left|\sqrt{2}t + \sqrt{2t^2+3}\right| + C \Rightarrow$$

$$2I = t\sqrt{2t^2+3} + \ln\left|\sqrt{2}t + \sqrt{2t^2+3}\right| + C$$

$$I = \frac{1}{2}\left(t\sqrt{2t^2+3} + \ln\left|\sqrt{2}t + \sqrt{2t^2+3}\right|\right) + C \Rightarrow$$

$$(*) = I - 3\ln\left|\sqrt{2}t + \sqrt{2t^2+3}\right| =$$

$$= \frac{1}{2}\left(t\sqrt{2t^2+3} + \ln\left|\sqrt{2}t + \sqrt{2t^2+3}\right|\right) - 3\ln\left|\sqrt{2}t + \sqrt{2t^2+3}\right| + C = \frac{1}{2}\left(t\sqrt{2t^2+3} - 5\ln\left|\sqrt{2}t + \sqrt{2t^2+3}\right|\right) + C$$

80. $\int \dfrac{\mathrm{d}x}{x^4+x^2+1} = (*)$

Разложим подынтегральное выражение на простые слагаемые:

$$\frac{1}{x^4+x^2+1} = \frac{1}{(x^2-x+1)(x^2+x+1)} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+x+1} =>$$

$$A\left(x^3+x^2+x\right) + B\left(x^2+x+1\right) + C\left(x^3-x^2+x\right) + D\left(x^2-x+1\right) = 1$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{r|l} t^0 & 1 = B+D \\ t^1 & 0 = A+B+C-D \\ t^2 & 0 = A+B-C+D \\ t^3 & 0 = A+C \end{array} \quad => A = -\frac{1}{2},\ B = \frac{1}{2},\ C = \frac{1}{2},\ D = -\frac{1}{2}$$

$$\frac{1}{x^4+x^2+1} = -\frac{1}{2}\frac{x-1}{x^2-x+1}\,\mathrm{d}x + \frac{1}{2}\frac{x+1}{x^2+x+1}\,\mathrm{d}x => (*) = -\frac{1}{2}\int\frac{x-1}{x^2-x+1}\,\mathrm{d}x + \frac{1}{2}\int\frac{x+1}{x^2+x+1}\,\mathrm{d}x =$$

$$= -\frac{1}{4}\int\frac{2x-2}{x^2-x+1}\,\mathrm{d}x + \frac{1}{4}\int\frac{2x+2}{x^2+x+1}\,\mathrm{d}x =$$

$$= -\frac{1}{4}\left(\int\frac{\mathrm{d}\left(x^2-x+1\right)}{x^2-x+1} - \int\frac{\mathrm{d}x}{x^2-x+1}\right) + \frac{1}{4}\left(\int\frac{\mathrm{d}\left(x^2+x+1\right)}{x^2+x+1} + \int\frac{\mathrm{d}x}{x^2+x+1}\right) =$$

$$= -\frac{1}{4}\ln\left|x^2-x+1\right| + \frac{1}{4}\ln\left|x^2+x+1\right| + \frac{1}{4}\left(\int\frac{\mathrm{d}x}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \int\frac{\mathrm{d}x}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right) =$$

$$= \frac{1}{4}\ln\left|\frac{x^2+x+1}{x^2-x+1}\right| + \frac{1}{4}\frac{2}{\sqrt{3}}\left(arctg\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) + arctg\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right) + C =$$

$$= \frac{1}{4}\ln\left|\frac{x^2+x+1}{x^2-x+1}\right| + \frac{1}{2\sqrt{3}}\left(arctg\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) + arctg\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right) + C$$

81. $\int \dfrac{\sqrt{x}}{1-\sqrt{x}}\,\mathrm{d}x = (*)$

$$(*) = \int\frac{\sqrt{x}\left(1+\sqrt{x}\right)}{1-x}\,\mathrm{d}x$$

$$\text{Замена: } \left[\begin{array}{c} t = 1+\sqrt{x},\ \sqrt{x} = t-1 \\ \mathrm{d}t = \dfrac{\mathrm{d}x}{2\sqrt{x}} \\ \mathrm{d}x = 2\sqrt{x}\,\mathrm{d}t = 2\left(t-1\right)\mathrm{d}t \end{array}\right] => (*) = 2\int\frac{t\left(t-1\right)^2\mathrm{d}t}{1-\left(t-1\right)^2} =$$

$$= 2\int\frac{t\left(t-1\right)^2\mathrm{d}t}{1-t^2+2t-1} = 2\int\frac{t\left(t-1\right)^2\mathrm{d}t}{t\left(2-t\right)} = -2\int\frac{\left(t-1\right)^2\mathrm{d}t}{t-2} = -2\int\frac{t^2-2t+1}{t-2} =$$

$$= -2\int t\,\mathrm{d}t - 2\int\frac{\mathrm{d}t}{t-2} = -t^2 - 2\ln\left|t-2\right| + C = -\left(1+\sqrt{x}\right)^2 - 2\ln\left|\sqrt{x}-1\right| + C$$

82. $\displaystyle\int \frac{\mathrm{d}x}{x\left(x^2+1\right)} = (*)$

Разложим подынтегральное выражение на простые слагаемые:

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \Rightarrow$$

$$A\left(x^2+1\right) + Bx^2 + Cx = 1$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{r|l} t^0 & 1 = A \\ t^1 & 0 = C \\ t^2 & 0 = A + B \end{array} \Rightarrow A = 1,\ B = -1,\ C = 0$$

$$\frac{1}{x\left(x^2+1\right)} = \frac{1}{x} - \frac{x}{x^2+1} \Rightarrow (*) = \int \frac{\mathrm{d}x}{x} - \int \frac{x}{x^2+1}\,\mathrm{d}x =$$

$$= \int \frac{\mathrm{d}x}{x} - \frac{1}{2}\int \frac{\mathrm{d}\left(x^2+1\right)}{x^2+1} = \ln|x| - \frac{1}{2}\ln\left|x^2+1\right| + C = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$$

83. $\displaystyle\int \sqrt{\frac{2x-1}{2x+3}}\,\mathrm{d}x = (*)$

Замена: $\left[\begin{array}{c} t = 2x+3,\ 2x-1 = t-4 \\ \mathrm{d}t = 2\,\mathrm{d}x \end{array}\right] \Rightarrow (*) = \dfrac{1}{2}\displaystyle\int \sqrt{\frac{t-4}{t}}\,\mathrm{d}t$

Интегрируем по частям: $\left[\begin{array}{r|l} u = \sqrt{\dfrac{t-4}{t}} & \mathrm{d}u = 2\dfrac{\mathrm{d}t}{t^2\sqrt{\dfrac{t-4}{t}}} \\ v = t & \mathrm{d}v = \mathrm{d}t \end{array}\right] \Rightarrow (*) = t\sqrt{\dfrac{t-4}{t}} - 2\displaystyle\int t\frac{\mathrm{d}t}{t^2\sqrt{\dfrac{t-4}{t}}} =$

$$= t\sqrt{\frac{t-4}{t}} - 2\int \frac{\mathrm{d}t}{t\sqrt{\dfrac{t-4}{t}}} = \sqrt{t\,(t-4)} - 2\int \frac{\mathrm{d}t}{\sqrt{t\,(t-4)}}$$

Замена: $\left[\begin{array}{c} k = \sqrt{t} \\ \mathrm{d}k = \dfrac{\mathrm{d}t}{2\sqrt{t}} \end{array}\right] \Rightarrow (*) = \sqrt{t\,(t-4)} - \displaystyle\int \frac{\mathrm{d}k}{\sqrt{k^2-4}} = \sqrt{t\,(t-4)} - \arcsin\frac{k}{2} + C = \sqrt{t\,(t-4)} - \arcsin\frac{\sqrt{t}}{2} + C =$

$$= \sqrt{(2x+3)\,(2x-1)} - \arcsin\frac{\sqrt{2x+3}}{2} + C$$

84. $\displaystyle\int \frac{\mathrm{d}x}{x^3-1} = (*)$

$$(*) = \int \frac{\mathrm{d}x}{(x-1)\left(x^2+x+1\right)} = \int \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\right)\mathrm{d}x =>$$

$$1 = A\left(x^2+x+1\right) + B\left(x^2-x\right) + C\,(x-1)$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{c|l} t^0 & 1 = A + C \\ t^1 & 0 = A - B + C \\ t^2 & 0 = A + B \end{array} \quad => A = \frac{1}{3},\ B = -\frac{1}{3},\ C = -\frac{2}{3}$$

$$(*) = \frac{1}{3}\int \frac{\mathrm{d}x}{x-1} - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,\mathrm{d}x = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x-2}{x^2+x+1}\,\mathrm{d}x =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{\mathrm{d}\left(x^2+x+1\right)}{x^2+x+1} + \frac{5}{6}\int \frac{\mathrm{d}x}{x^2+x+1} =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| + \frac{5}{6}\int \frac{\mathrm{d}x}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| + \frac{5}{6}\frac{2}{\sqrt{3}} arctg \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) + C =$$

$$= \frac{1}{3}\ln\left|\frac{x-1}{\sqrt{x^2+x+1}}\right| + \frac{5}{3\sqrt{3}} arctg \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) + C =$$

$$= \frac{1}{6}\ln\left|1 + \frac{x}{x^2+x+1}\right| + \frac{5}{3\sqrt{3}} arctg \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) + C$$

85. $\displaystyle\int \frac{\mathrm{d}x}{x^2\left(x^2+1\right)} = (*)$

$$(*) = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}\right)\mathrm{d}x => \quad 1 = A\left(x^3+x\right) + B\left(x^2+1\right) + Cx^3 + Dx^2$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{c|l} t^0 & 1 = B \\ t^1 & 0 = A \\ t^2 & 0 = B + D \\ t^3 & 0 = A + C \end{array} \quad => A = 0,\ B = 1,\ C = 0,\ D = -1$$

$$(*) = \int \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right)\mathrm{d}x = \int \frac{\mathrm{d}x}{x^2} - \int \frac{\mathrm{d}x}{x^2+1} = -\frac{1}{x} - arctg(x) + C$$

86. $\displaystyle\int \frac{\mathrm{d}x}{(x+1)^2(x^2+1)} = (*)$

$$(*) = \int \left(\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} \right) =>$$

$$1 = A(x+1)\left(x^2+1\right) + B\left(x^2+1\right) + Cx(x+1)^2 + D(x+1)^2 =$$

$$= A\left(x^3+x^2+x+1\right) + B\left(x^2+1\right) + C\left(x^3+2x^2+x\right) + D\left(x^2+2x+1\right)$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{r|l} t^0 & 1 = A+B+D \\ t^1 & 0 = A+C+2D \\ t^2 & 0 = A+B+2C+D \\ t^3 & 0 = A \end{array} \quad => A = 0,\ B = \frac{3}{4},\ C = -\frac{1}{2},\ D = \frac{1}{4}$$

$$(*) = \int \left(\frac{3}{4(x+1)^2} - \frac{1}{4}\left(\frac{2x-1}{x^2+1}\right) \right) \mathrm{d}x = \frac{3}{4}\int \frac{\mathrm{d}(x+1)}{(x+1)^2} - \frac{1}{4}\left(\int \frac{\mathrm{d}\left(x^2+1\right)}{x^2+1} - \int \frac{\mathrm{d}x}{x^2+1} \right) =$$

$$= \frac{1}{4}\left(-\frac{3}{x+1} - \ln\left(x^2+1\right) + arctg(x) \right) + C$$

87. $\displaystyle\int \frac{x^3-6}{x^4+6x^2+8}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{x^3-6}{(x^2+2)(x^2+4)}\,\mathrm{d}x = \int \left(\frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+4} \right) \mathrm{d}x =>$$

$$x^3 - 6 = (Ax+B)\left(x^2+4\right) + (Cx+D)\left(x^2+2\right) =$$

$$= A\left(x^3+4x\right) + B\left(x^2+4\right) + C\left(x^3+2x\right) + D\left(x^2+2\right)$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{r|l} t^0 & -6 = 4B+2D \\ t^1 & 0 = 4A+2C \\ t^2 & 0 = B+D \\ t^3 & 1 = A+C \end{array} \quad => A = -1,\ B = -3,\ C = 2,\ D = 3$$

$$(*) = \int \frac{-x-3}{x^2+2}\,\mathrm{d}x + \int \frac{2x+3}{x^2+4}\,\mathrm{d}x = -\frac{1}{2}\int \frac{2x+6}{x^2+2}\,\mathrm{d}x + \int \frac{2x+3}{x^2+4}\,\mathrm{d}x =$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}\left(x^2+2\right)}{x^2+2} - 3\int \frac{\mathrm{d}x}{x^2+\left(\sqrt{2}\right)^2} + \int \frac{\mathrm{d}\left(x^2+4\right)}{x^2+4} + 3\int \frac{\mathrm{d}x}{x^2+2^2} =$$

$$= -\frac{1}{2}\ln\left(x^2+2\right) + \ln\left(x^2+4\right) - \frac{3}{\sqrt{2}}arctg(x) + \frac{3}{2}arctg(x) + C =$$

$$= \ln\left(\frac{x^2+4}{\sqrt{x^2+2}}\right) + \frac{3}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{2}}\right) arctg(x) + C$$

88. $\displaystyle\int \frac{x^3}{\left(x^2-1\right)^2}\,\mathrm{d}x = (*)$

Интегрируем по частям: $\left[\begin{array}{r|l} u = x^2 & \mathrm{d}u = 2x\,\mathrm{d}x \\ v = -\dfrac{1}{2\left(x^2-1\right)} & \mathrm{d}v = \dfrac{x\,\mathrm{d}x}{\left(x^2-1\right)^2} \end{array}\right] => (*) = -\dfrac{x^2}{2\left(x^2-1\right)} + \displaystyle\int \frac{x\,\mathrm{d}x}{x^2-1} =$

$$= -\frac{x^2}{2\left(x^2-1\right)} + \frac{1}{2}\int \frac{\mathrm{d}\left(x^2-1\right)}{x^2-1} = -\frac{x^2}{2\left(x^2-1\right)} + \frac{1}{2}\ln\left|x^2-1\right| + C = \frac{1}{2}\left(\ln\left|x^2-1\right| + \frac{x^2}{1-x^2}\right) + C$$

89. $\displaystyle\int \frac{x^5\,\mathrm{d}x}{\left(x^2-1\right)(x-1)^2} = (*)$

$$(*) = \int \frac{x^5\,\mathrm{d}x}{(x+1)(x-1)^3} => (*) = \int (x+2)\,\mathrm{d}x + \int \frac{4x^3-2x^2-3x+2}{(x+1)(x-1)^3}\,\mathrm{d}x = \frac{(x+2)^2}{2} + \int \frac{4x^3-2x^2-3x+2}{(x+1)(x-1)^3}\,\mathrm{d}x$$

$$\int \frac{4x^3-2x^2-3x+2}{(x+1)(x-1)^3}\,\mathrm{d}x = (1)$$

$$(1) = \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}\right) =>$$

$$A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3 = 4x^3-2x^2-3x+2$$

$$x+1=0:$$

$$D\left(x^3-3x^2+3x-1\right) = 4x^3-2x^2-3x+2$$

$$D(-1-3-3-1) = -4-2+3+2$$

$$-8D = -1 => D = \frac{1}{8}$$

$$A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) = 4x^3-2x^2-3x+2-\frac{1}{8}(x-1)^3$$

$$A\left(8x^3-8x^2-8x+8\right) + B\left(8x^2-8\right) + C(8x+8) = 31x^3-13x^2-27x+17$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{r|l} t^0 & 17 = 8A-8B+8C \\ t^1 & -27 = -8A+8C \\ t^2 & -13 = -8A+8B \\ t^3 & 31 = 8A \end{array} \quad => A = \frac{31}{8},\ B = \frac{9}{4},\ C = \frac{1}{2},\ D = \frac{1}{8}$$

$$(1) = \frac{31}{8}\int \frac{\mathrm{d}x}{x-1} + \frac{9}{4}\int \frac{\mathrm{d}x}{(x-1)^2} + \frac{1}{2}\int \frac{\mathrm{d}x}{(x-1)^3} + \frac{1}{8}\int \frac{\mathrm{d}x}{x+1} =$$

$$= \frac{31}{8}\ln|x-1| - \frac{9}{4(x-1)} - \frac{1}{(x-1)^2} + \frac{1}{8}\ln|x+1| + C = \frac{1}{8}\ln\left|(x-1)^{30}\left(x^2-1\right)\right| - \frac{x+3}{4(x-1)^2} + C =>$$

$$(*) = \frac{1}{8}\ln\left|(x-1)^{30}\left(x^2-1\right)\right| - \frac{x+3}{4(x-1)^2} + \frac{(x+2)^2}{2} + C$$

90. $\int \dfrac{x^2\,\mathrm{d}x}{1-x^4} = (*)$

$$(*) = \int \frac{x^2\,\mathrm{d}x}{(1-x^2)\,(1+x^2)} = \int \frac{x^2\,\mathrm{d}x}{(1-x)\,(1+x)\,(1+x^2)} = \int \left(\frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}\right)\mathrm{d}x =>$$

$$x^2 = A\,(1+x)\left(1+x^2\right) + B\,(1-x)\left(1+x^2\right) + (Cx+D)\left(1-x^2\right) =$$

$$= A\left(1+x+x^2+x^3\right) + B\left(1-x+x^2-x^3\right) + C\left(x-x^3\right) + D\left(1-x^2\right)$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$\begin{array}{r|l} t^0 & 0 = A+B+D \\ t^1 & 0 = A-B+C \\ t^2 & 1 = A+B-D \\ t^3 & 0 = A-B-C \end{array} \quad => A = \frac{1}{4},\ B = \frac{1}{4},\ C = 0,\ D = -\frac{1}{2}$$

$$(*) = \int \frac{1}{4}\left(\frac{1}{1-x} + \frac{1}{1+x} - \frac{2}{1+x^2}\right)\mathrm{d}x = -\frac{1}{4}\ln|1-x| + \frac{1}{4}\ln|1+x| - \frac{1}{2}\int \frac{\mathrm{d}x}{1+x^2} = \frac{1}{4}\ln\left|\frac{1+x}{1-x}\right| - \frac{1}{2}arctg\,(x) + C$$

91. $\int \sqrt{1-4x-x^2}\,\mathrm{d}x = (*)$

$$I = (*) = \int \sqrt{\left(\sqrt{5}\right)^2 - (x+2)^2}\,\mathrm{d}x$$

$$\text{Замена: } \left[\begin{array}{c} t = x+2 \\ dt = dx \end{array}\right] => (*) = \int \sqrt{\left(\sqrt{5}\right)^2 - t^2}\,\mathrm{d}x$$

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = \sqrt{\left(\sqrt{5}\right)^2 - t^2} & \mathrm{d}u = -\dfrac{t\,\mathrm{d}t}{\sqrt{\left(\sqrt{5}\right)^2 - t^2}} \\ v = t & \mathrm{d}v = \mathrm{d}t \end{array}\right] => (*) = t\sqrt{\left(\sqrt{5}\right)^2 - t^2} + \int \frac{t^2\,\mathrm{d}t}{\sqrt{\left(\sqrt{5}\right)^2 - t^2}} =$$

$$= t\sqrt{\left(\sqrt{5}\right)^2 - t^2} - \int \sqrt{\left(\sqrt{5}\right)^2 - t^2}\,\mathrm{d}x + 5\int \frac{\mathrm{d}t}{\sqrt{\left(\sqrt{5}\right)^2 - t^2}}$$

$$2I = t\sqrt{\left(\sqrt{5}\right)^2 - t^2} + 5\int \frac{\mathrm{d}t}{\sqrt{\left(\sqrt{5}\right)^2 - t^2}} =>$$

$$(*) = \frac{1}{2}\left(t\sqrt{\left(\sqrt{5}\right)^2 - t^2} + 5\int \frac{\mathrm{d}t}{\sqrt{\left(\sqrt{5}\right)^2 - t^2}}\right) + C = \frac{1}{2}\left(t\sqrt{\left(\sqrt{5}\right)^2 - t^2} + 5\arcsin\frac{t}{\sqrt{5}}\right) + C =$$

$$= \frac{1}{2}\left((x+2)\ \sqrt{1-4x-x^2} + 5\arcsin\frac{x+2}{\sqrt{5}}\right) + C$$

92. $\int \sqrt{x^2-2x-1}\,\mathrm{d}x = (*)$

$$(*) = \int \sqrt{x^2-2x+1-2}\,\mathrm{d}x = \int \sqrt{(x-1)^2-2}\,\mathrm{d}x$$

$$\text{Замена: } \begin{bmatrix} t = x-1 \\ \mathrm{d}t = \mathrm{d}x \end{bmatrix} => (*) = \int \sqrt{t^2-2}\,\mathrm{d}x$$

$$\text{Интегрируем по частям: } \left[\begin{array}{r|l} u = \sqrt{t^2-2} & \mathrm{d}u = \dfrac{t\,\mathrm{d}t}{\sqrt{t^2-2}} \\ v = t & \mathrm{d}v = \mathrm{d}t \end{array}\right] => (*) = t\,\sqrt{t^2-2} - \int \frac{t^2\,\mathrm{d}t}{\sqrt{t^2-2}} =$$

$$= t\,\sqrt{t^2-2} - \int \sqrt{t^2-2}\,\mathrm{d}x - 2\int \frac{\mathrm{d}t}{\sqrt{t^2-2}} => (*) = I = t\,\sqrt{t^2-2} - I - 2\int \frac{\mathrm{d}t}{\sqrt{t^2-2}}$$

$$2I = t\,\sqrt{t^2-2} - 2\int \frac{\mathrm{d}t}{\sqrt{t^2-2}} => I = \frac{1}{2}t\,\sqrt{t^2-2} - \ln\left|t+\sqrt{t^2-2}\right| + C =$$

$$= \frac{1}{2}(x-1)\sqrt{x^2-2x-1} - \ln\left|x-1+\sqrt{x^2-2x-1}\right| + C$$

93. $\int \dfrac{\mathrm{d}x}{2\sin x + 3\cos x} = (*)$

$$\text{Замена: } \begin{bmatrix} tg\left(\dfrac{x}{2}\right) = t \\ dx = \dfrac{2\,\mathrm{d}t}{1+t^2} \end{bmatrix} => (*) = \int \frac{\dfrac{2\,\mathrm{d}t}{1+t^2}}{2\dfrac{2t}{1+t^2} + 3\dfrac{1-t^2}{1+t^2}} = \int \frac{2\,\mathrm{d}t}{4t+3(1-t^2)} =$$

$$= 2\int \frac{\mathrm{d}t}{4t+3-3t^2} = -2\int \frac{\mathrm{d}t}{3t^2-4t-3} = -2\int \frac{\mathrm{d}t}{\left(t-\dfrac{2-\sqrt{13}}{3}\right)\left(t-\dfrac{2+\sqrt{13}}{3}\right)} =$$

$$= -2\int \left(\frac{A}{\left(t-\dfrac{2-\sqrt{13}}{3}\right)} + \frac{B}{\left(t-\dfrac{2+\sqrt{13}}{3}\right)}\right)\mathrm{d}t => A\left(t-\frac{2+\sqrt{13}}{3}\right) + B\left(t-\frac{2-\sqrt{13}}{3}\right) = 1$$

Коэффициенты перед степенями t удовлетворяют следующим уравнениям:

$$A = -\frac{3}{\sqrt{13}},\ B = \frac{3}{\sqrt{13}}$$

$$(*) = \frac{6}{\sqrt{13}}\int \left(\frac{1}{\left(t-\dfrac{2-\sqrt{13}}{3}\right)} - \frac{1}{\left(t-\dfrac{2+\sqrt{13}}{3}\right)}\right)\mathrm{d}t = \frac{6}{\sqrt{13}}\left(\int \frac{\mathrm{d}t}{\left(t-\dfrac{2-\sqrt{13}}{3}\right)} - \int \frac{\mathrm{d}t}{\left(t-\dfrac{2+\sqrt{13}}{3}\right)}\right) =$$

$$= \frac{6}{\sqrt{13}}\left(\ln\left|t-\frac{2-\sqrt{13}}{3}\right| - \ln\left|t-\frac{2+\sqrt{13}}{3}\right|\right) + C = \frac{6}{\sqrt{13}}\ln\left|\frac{3t-2+\sqrt{13}}{3t-2-\sqrt{13}}\right| + C$$

94. $\displaystyle\int \frac{\mathrm{d}x}{x^2\left(x+\sqrt{x^2+1}\right)} = (*)$

$$(*) = \int \frac{x-\sqrt{x^2+1}}{x^2x^2\left(x^2-x^2-1\right)\mathrm{d}x} = -\int \frac{x-\sqrt{x^2+1}}{x^2}\,\mathrm{d}x = -\int \frac{\mathrm{d}x}{x} + \int \frac{\sqrt{x^2+1}}{x^2}\,\mathrm{d}x = -\ln|x| + \int \frac{\sqrt{x^2+1}}{x^2}\,\mathrm{d}x$$

Интегрируем по частям: $\left[\begin{array}{r|l} u=\sqrt{x^2+1} & \mathrm{d}u = \dfrac{x\,\mathrm{d}x}{\sqrt{x^2+1}} \\ v=-\dfrac{1}{x} & dx = \dfrac{\mathrm{d}x}{x^2} \end{array}\right] => (*) = -\ln|x| - \dfrac{\sqrt{x^2+1}}{x} + \displaystyle\int \frac{x\,\mathrm{d}x}{x\sqrt{x^2+1}} =$

$$= -\ln|x| - \frac{\sqrt{x^2+1}}{x} + \int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = -\ln|x| - \frac{\sqrt{x^2+1}}{x} + \ln\left|x+\sqrt{x^2+1}\right| + C = -\frac{\sqrt{x^2+1}}{x} + \ln\left|\frac{x+\sqrt{x^2+1}}{x}\right| + C =$$

$$= \ln\left|1+\sqrt{1+\frac{1}{x^2}}\right| - \sqrt{1+\frac{1}{x^2}} + C$$

95. $\displaystyle\int \frac{x - arctg^3 x}{1+x^2}\,\mathrm{d}x = (*)$

$$(*) = \int \frac{x\,\mathrm{d}x}{1+x^2} - \int \frac{arctg^3 x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}\left(1+x^2\right)}{1+x^2} - \int arctg^3 x\,\mathrm{d}(arctgx) = \frac{1}{2}\ln\left|1+x^2\right| - \frac{1}{4}arctg^4 x + C$$

97. $\displaystyle\int \frac{\mathrm{d}x}{x-\sqrt{x^2-1}} = (*)$

$$(*) = \int \frac{x+\sqrt{x^2-1}}{x^2-x^2+1}\,\mathrm{d}x = \int \left(x+\sqrt{x^2-1}\right)\mathrm{d}x = \int x\,\mathrm{d}x + \int \sqrt{x^2-1}\,\mathrm{d}x = \frac{x^2}{2} + \int \sqrt{x^2-1}\,\mathrm{d}x$$

Интегрируем по частям: $\left[\begin{array}{r|l} u=\sqrt{x^2-1} & \mathrm{d}u = \dfrac{x\,\mathrm{d}x}{\sqrt{x^2-1}} \\ v=x & \mathrm{d}v = \mathrm{d}x \end{array}\right] => (*) = \dfrac{x^2}{2} + x\sqrt{x^2-1} - \displaystyle\int \frac{x^2}{\sqrt{x^2-1}}\,\mathrm{d}x =$

$$= \frac{x^2}{2} + x\sqrt{x^2-1} - \int \frac{x^2-1}{\sqrt{x^2-1}}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \frac{x^2}{2} + x\sqrt{x^2-1} - \int \sqrt{x^2-1}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{\sqrt{x^2-1}}$$

$$I = \int \sqrt{x^2-1} => 2I = x\sqrt{x^2-1} + \int \frac{\mathrm{d}x}{\sqrt{x^2-1}}$$

$$I = \frac{x}{2}\sqrt{x^2-1} + \ln\left|x+\sqrt{x^2-1}\right| + C =>$$

$$(*) = \frac{x^2}{2} + I = \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2-1} + \ln\left|x+\sqrt{x^2-1}\right| + C$$

98. $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2} = (*)$

Интегрируем по частям: $\left[\begin{array}{r|l} u=\sqrt{1-x^2} & \mathrm{d}u = -\dfrac{x\,\mathrm{d}x}{\sqrt{1-x^2}} \\ v=-\dfrac{1}{x} & \mathrm{d}v = \dfrac{\mathrm{d}x}{x^2} \end{array}\right] => (*) = -\dfrac{\sqrt{1-x^2}}{x} + \displaystyle\int \frac{x\,\mathrm{d}x}{x\sqrt{1-x^2}} =$

$$= -\frac{\sqrt{1-x^2}}{x} + \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = -\sqrt{\frac{1}{x^2}-1} - \arcsin x + C$$

99. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{(1-x^2)\arccos x}} = (*)$

Замена: $\left[\begin{array}{c} t = \arccos x \\ \mathrm{d}t = -\dfrac{\mathrm{d}x}{\sqrt{1-x^2}} \end{array}\right]$ => $(*) = -\displaystyle\int \frac{\mathrm{d}t}{\sqrt{t}} = -2\sqrt{t} + C = -2\sqrt{\arccos x} + C$

100. $\displaystyle\int \frac{1+x}{x\,(x+\ln x)} = (*)$

$$(*) = \int \frac{\frac{1}{x}+1}{x+\ln x}\,\mathrm{d}x = \int \frac{\mathrm{d}(x+\ln x)}{x+\ln x} = \ln|x+\ln x| + C$$