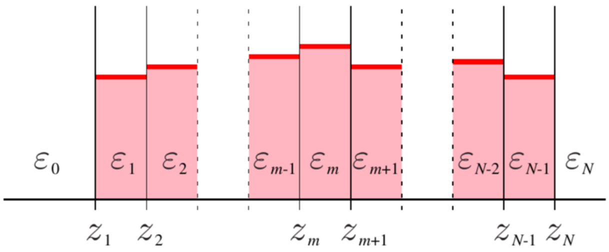

Figure 1: Problem formulation

1. Problem formulation

A unit-amplitude plane wave is incident normally from $z < 0$ onto a slab $z_1 < z < z_N$ (Fig. 1) with a piecewise-constant (and possibly discontinuous), complex-valued dielectric profile $\varepsilon(z)$ satisfying:

$$\varepsilon(z \to \pm\infty) = 1, \text{ i.e., } \varepsilon_0 = \varepsilon_N = 1. \tag{1}$$

The (generally complex) field $E(z)$ satisfies:

$$\frac{d^2E}{dz^2} + k_0^2\,\varepsilon(z)\,E = 0, \quad k_0 = \frac{\omega}{c}, \tag{2}$$

and matches the boundary conditions:

$$E(z) \;\to \begin{cases} e^{ik_0z} + Re^{-ik_0z}, & z \to -\infty, \\ Te^{ik_0z}, & z \to +\infty. \end{cases} \tag{3}$$

Find:

$$R, \quad T, \quad A = 1 - |R|^2 - |T|^2 \tag{4}$$

for an arbitrary distribution of $\varepsilon(z)$.
Extra: extend to oblique incidence angle θ and both s- and p-polarizations.

2. Solution

2.1. Normal incidence

In each m-th layer with dielectric constant ε_m, the general solution is:

$$E_m(z) = A_m e^{ik_m(z-z_m)} + B_m e^{-ik_m(z-z_m)} \tag{5}$$

where $k_m = k_0\sqrt{\varepsilon_m}$.

At m-th interface ($z = z_{m+1}$), tangential components of fields are continuous. Considering $\mathbf{E} \parallel \mathbf{e}_y$ and $\mathbf{H} \parallel \mathbf{e}_x$, the continuity conditions are:

$$E_m\Big|_{z_{m+1}} = E_{m+1}\Big|_{z_{m+1}} \tag{6a}$$

$$H_m\Big|_{z_{m+1}} = H_{m+1}\Big|_{z_{m+1}} \longleftrightarrow \frac{dE_m}{dz}\Big|_{z_{m+1}} = \frac{dE_{m+1}}{dz}\Big|_{z_{m+1}} \tag{6b}$$

Substitution of eq. (5) into eqs. (6) yields:

$$\begin{pmatrix} A_m \\ B_m \end{pmatrix} = \begin{pmatrix} e^{-ik_m d_m} & 0 \\ 0 & e^{ik_m d_m} \end{pmatrix} \begin{pmatrix} \frac{1}{t_{m+1}} & \frac{r_{m+1}}{t_{m+1}} \\ \frac{r_{m+1}}{t_{m+1}} & \frac{1}{t_{m+1}} \end{pmatrix} \begin{pmatrix} A_{m+1} \\ B_{m+1} \end{pmatrix} = P_m I_{m+1} \begin{pmatrix} A_{m+1} \\ B_{m+1} \end{pmatrix} \tag{7}$$

where $t_{m+1} = \dfrac{2k_m}{k_m + k_{m+1}}$ – local amplitude transmission coefficient, $r_{m+1} = \dfrac{k_m - k_{m+1}}{k_m + k_{m+1}}$ – local amplitude reflection coefficient, matrix P_m is the propagation matrix in the $m-$th layer, and matrix I_{m+1} is the interface matrix.

The wave propagates through $N-1$ layer and N interfaces. The subsequent application of transformation leads to a matrix expression of the following form:

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = M \begin{pmatrix} A_N \\ 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 \\ R \end{pmatrix} = M \begin{pmatrix} T \\ 0 \end{pmatrix} \tag{8}$$

where the transfer matrix M is expressed as:

$$M = \left(\prod_{m=1}^{N-1} I_m P_m \right) \times I_N = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{9}$$

Asymptotic expressions (12) and eq. (8) allow to obtain the required coefficients:

$$T = \frac{1}{m_{11}} \tag{10a}$$

$$R = \frac{m_{21}}{m_{11}} \tag{10b}$$

$$A = 1 - \left| \frac{m_{21}}{m_{11}} \right|^2 - \frac{1}{|m_{11}|^2} \tag{10c}$$

2.2. Oblique incidence

If the wave is incident at angle θ, then $\mathbf{k} = (\mathbf{k_x}, \mathbf{0}, \mathbf{k_z})$.

For the s-polarized wave, $\mathbf{E} = (\mathbf{0}, \mathbf{E_y}, \mathbf{0})$. The y-component of the field satisfies:

$$\frac{d^2 E_y}{dz^2} + \left(k_0^2\, \varepsilon(z) - k_x^2\right) E_y = 0 \tag{11}$$

where $k_x = k_0 \sin\theta = k_m \sqrt{\varepsilon_m} \sin\theta_m$.

The electric field matches the asymptotic conditions:

$$E_y(z) \rightarrow \begin{cases} e^{ik_{z,0}z} + Re^{-ik_{z,0}z}, & z \rightarrow -\infty, \\ Te^{ik_{z,N}z}, & z \rightarrow +\infty. \end{cases} \tag{12}$$

Eq. (11) can be reduced to the form of eq. (2) by introducing $k_z = \sqrt{k_0^2 \varepsilon(z) - k_x^2}$. Therefore, the expressions for T, R, and A can be obtained using eqs. (10) and substituting $k_{z,m}$ instead of k_m in matrix expressions.

For the p-polarized wave, $\mathbf{E} = (\mathbf{E_x}, \mathbf{0}, \mathbf{E_z})$ and $\mathbf{H} = (\mathbf{0}, \mathbf{H_y}, \mathbf{0})$. From Maxwell's equations one can obtain the following expression for the H_y-component of the magnetic field:

$$\frac{d}{dz}\left(\frac{1}{\varepsilon(z)} \frac{dH_y}{dz}\right) + \left(k_0^2 - \frac{k_x^2}{\varepsilon(z)}\right) H_y = 0 \tag{13}$$

The magneric field matches the asymptotic conditions:

$$H_y(z) \ \rightarrow \begin{cases} e^{ik_{z,0}z} + R_H e^{-ik_{z,0}z}, & z \rightarrow -\infty, \\ T_H e^{ik_{z,N}z}, & z \rightarrow +\infty. \end{cases} \tag{14}$$

For each layer, the equation can be simplified to:

$$\frac{d^2 H_y}{dz^2} + k_{m,z} H_y = 0 \tag{15}$$

where $k_{m,z} = \sqrt{k_0^2 \varepsilon_m - k_x^2}$.

The general solution is:

$$H_m(z) = A_m e^{ik_{m,z}(z-z_m)} + B_m e^{-ik_{m,z}(z-z_m)} \tag{16}$$

At m-th interface ($z = z_{m+1}$), tangential components of fields are continuous. Thus:

$$H_{y,m}\bigg|_{z_{m+1}} = H_{y,m+1}\bigg|_{z_{m+1}} \tag{17a}$$

$$E_{x,m}\bigg|_{z_{m+1}} = E_{x,m+1}\bigg|_{z_{m+1}} \longleftrightarrow \frac{1}{\varepsilon_m}\frac{dH_{y,m}}{dz}\bigg|_{z_{m+1}} = \frac{1}{\varepsilon_{m+1}}\frac{dH_{y,m+1}}{dz}\bigg|_{z_{m+1}} \tag{17b}$$

Substitution of eq. (16) into eqs. (17) results in:

$$\begin{pmatrix} A_m \\ B_m \end{pmatrix} = P_m I_{m+1} \begin{pmatrix} A_{m+1} \\ B_{m+1} \end{pmatrix} \tag{18}$$

where matrices P_m and I_{m+1} have the same representation as previously, $t_{m+1} = \dfrac{2p_m}{p_m + p_{m+1}}$, $r_{m+1} = \dfrac{p_m - p_{m+1}}{p_m + p_{m+1}}$, $p_m = k_{m,z}/\varepsilon_m$.

Following the previous steps, one can obtain T_H and R_H:

$$T_H = \frac{1}{m_{11}} \tag{19a}$$

$$R_H = \frac{m_{21}}{m_{11}} \tag{19b}$$

Since $E_x \propto \partial_z H_y/\varepsilon$ with $\varepsilon_0 = \varepsilon_N = 1$, the following relation between amplitude reflection/transmission coefficients arise:

$$R_E = -R_H \tag{20}$$

$$T_E = T_H \tag{21}$$

Therefore, the final expressions are as follows:

$$T_E = \frac{1}{m_{11}} \tag{22a}$$

$$R_E = -\frac{m_{21}}{m_{11}} \tag{22b}$$

$$A_E = 1 - \left|\frac{m_{21}}{m_{11}}\right|^2 - \frac{1}{|m_{11}|^2} \tag{22c}$$