



Figure 1: Problem formulation

1. Problem formulation

A unit-amplitude plane wave is incident normally from $z < 0$ onto a slab $z_1 < z < z_N$ (Fig. 1) with a piecewise-constant (and possibly discontinuous), complex-valued dielectric profile $\varepsilon(z)$ satisfying:

$$\varepsilon(z \rightarrow \pm\infty) = 1, \text{ i.e., } \varepsilon_0 = \varepsilon_N = 1. \quad (1)$$

The (generally complex) field $E(z)$ satisfies:

$$\frac{d^2 E}{dz^2} + k_0^2 \varepsilon(z) E = 0, \quad k_0 = \frac{\omega}{c}, \quad (2)$$

and matches the boundary conditions:

$$E(z) \rightarrow \begin{cases} e^{ik_0 z} + R e^{-ik_0 z}, & z \rightarrow -\infty, \\ T e^{ik_0 z}, & z \rightarrow +\infty. \end{cases} \quad (3)$$

Find:

$$R, \quad T, \quad A = 1 - |R|^2 - |T|^2 \quad (4)$$

for an arbitrary distribution of $\varepsilon(z)$.

Extra: extend to oblique incidence angle θ and both s - and p -polarizations.

2. Solution

2.1. Normal incidence

In each m -th layer with dielectric constant ε_m , the general solution is:

$$E_m(z) = A_m e^{ik_m z} + B_m e^{-ik_m z} = E_m^+(z) + E_m^-(z) \quad (5)$$

where $k_m = k_0 \sqrt{\varepsilon_m}$.

At m -th interface ($z = z_{m+1}$), tangential components of fields are continuous. Considering $\mathbf{E} \parallel \mathbf{e}_y$ and $\mathbf{H} \parallel \mathbf{e}_x$, the continuity conditions are:

$$E_m \Big|_{z_{m+1}} = E_{m+1} \Big|_{z_{m+1}} \quad (6a)$$

$$H_m \Big|_{z_{m+1}} = H_{m+1} \Big|_{z_{m+1}} \iff \frac{dE_m}{dz} \Big|_{z_{m+1}} = \frac{dE_{m+1}}{dz} \Big|_{z_{m+1}} \quad (6b)$$

Substitution of eq. (5) into eqs. (6) yields:

$$\begin{pmatrix} E_m^+ \\ E_m^- \end{pmatrix} = \frac{1}{t_{m+1}} \begin{pmatrix} 1 & r_{m+1} \\ r_{m+1} & 1 \end{pmatrix} \begin{pmatrix} E_{m+1}^+ \\ E_{m+1}^- \end{pmatrix} = I_{m+1} \begin{pmatrix} E_{m+1}^+ \\ E_{m+1}^- \end{pmatrix} \quad (7)$$

where $t_{m+1} = \frac{2k_m}{k_m + k_{m+1}}$ – local amplitude transmission coefficient, $r_{m+1} = \frac{k_m - k_{m+1}}{k_m + k_{m+1}}$ – local amplitude reflection coefficient.

Considering the continuity of the wave in a layer, the amplitudes at the left and right sides of the m -th layer can be linked by the propagation matrix P_m :

$$P_m = \begin{pmatrix} e^{-ik_m d_m} & 0 \\ 0 & e^{ik_m d_m} \end{pmatrix} \quad (8)$$

where $d_m = z_{m+1} - z_m$ is the thickness of the m -th layer, since:

$$E_m(z_{m+1}) = A_m e^{ik_m(z_m + d_m)} + B_m e^{-ik_m(z_m + d_m)} = E_m^+ e^{ik_m d_m} + E_m^- e^{-ik_m d_m} \quad (9)$$

The wave propagates through $N-1$ layer and N interfaces. The subsequent application of transformation leads to a matrix expression of the following form:

$$\begin{pmatrix} E_0^+ \\ E_0^- \end{pmatrix} = M \begin{pmatrix} E_N^+ \\ 0 \end{pmatrix} \quad (10)$$

where the transfer matrix M is expressed as:

$$M = \left(\prod_{m=1}^{N-1} I_m P_m \right) \times I_N = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad (11)$$

Asymptotic expressions (14) and eq. (10) allow to obtain the required coefficients:

$$T = \frac{e^{ik_0(z_1 - z_N)}}{m_{11}} \quad (12a)$$

$$R = \frac{m_{21}}{m_{11}} e^{2ik_0 z_1} \quad (12b)$$

$$A = 1 - \left| \frac{m_{21}}{m_{11}} \right|^2 - \frac{1}{|m_{11}|^2} \quad (12c)$$

2.2. Oblique incidence

If the wave is incident at angle θ , then $\mathbf{k} = (\mathbf{k}_x, \mathbf{0}, \mathbf{k}_z)$.

For the s-polarized wave, $\mathbf{E} = (\mathbf{0}, \mathbf{E}_y, \mathbf{0})$. The y-component of the field satisfies:

$$\frac{d^2 E_y}{dz^2} + (k_0^2 \varepsilon(z) - k_x^2) E_y = 0 \quad (13)$$

where $k_x = k_0 \sin \theta = k_m \sqrt{\varepsilon_m} \sin \theta_m$.

The electric field matches the asymptotic conditions:

$$E_y(z) \rightarrow \begin{cases} e^{ik_{z,0}z} + R e^{-ik_{z,0}z}, & z \rightarrow -\infty, \\ T e^{ik_{z,N}z}, & z \rightarrow +\infty. \end{cases} \quad (14)$$

Eq. (13) can be reduced to the form of eq. (2) by introducing $k_z = \sqrt{k_0^2 \varepsilon(z) - k_x^2}$. Therefore, the expressions for T , R , and A can be obtained using eqs. (12) and substituting $k_{z,m}$ instead of k_m in matrix expressions.

For the p-polarized wave, $\mathbf{E} = (\mathbf{E}_x, \mathbf{0}, \mathbf{E}_z)$ and $\mathbf{H} = (\mathbf{0}, \mathbf{H}_y, \mathbf{0})$. From Maxwell's equations one can obtain the following expression for the H_y -component of the magnetic field:

$$\frac{d}{dz} \left(\frac{1}{\varepsilon(z)} \frac{dH_y}{dz} \right) + \left(k_0^2 - \frac{k_x^2}{\varepsilon(z)} \right) H_y = 0 \quad (15)$$

The magnetic field matches the asymptotic conditions:

$$H_y(z) \rightarrow \begin{cases} e^{ik_{z,0}z} + R_H e^{-ik_{z,0}z}, & z \rightarrow -\infty, \\ T_H e^{ik_{z,N}z}, & z \rightarrow +\infty. \end{cases} \quad (16)$$

For each layer, the equation can be simplified to:

$$\frac{d^2 H_y}{dz^2} + k_{m,z} H_y = 0 \quad (17)$$

where $k_{m,z} = \sqrt{k_0^2 \varepsilon_m - k_x^2}$.

The general solution is:

$$H_m(z) = A_m e^{ik_{m,z}z} + B_m e^{-ik_{m,z}z} = H_m^+(z) + H_m^-(z) \quad (18)$$

At m -th interface ($z = z_{m+1}$), tangential components of fields are continuous. Thus:

$$H_{y,m} \Big|_{z_{m+1}} = H_{y,m+1} \Big|_{z_{m+1}} \quad (19a)$$

$$E_{x,m} \Big|_{z_{m+1}} = E_{x,m+1} \Big|_{z_{m+1}} \longleftrightarrow \frac{1}{\varepsilon_m} \frac{dH_{y,m}}{dz} \Big|_{z_{m+1}} = \frac{1}{\varepsilon_{m+1}} \frac{dH_{y,m+1}}{dz} \Big|_{z_{m+1}} \quad (19b)$$

Substitution of eq. (18) into eqs. (19) results in:

$$\begin{pmatrix} H_m^+ \\ H_m^- \end{pmatrix} = I_{m+1} \begin{pmatrix} H_{m+1}^+ \\ H_{m+1}^- \end{pmatrix} \quad (20)$$

where matrix I_{m+1} has the same representation as previously, $t_{m+1} = \frac{2p_m}{p_m + p_{m+1}}$, $r_{m+1} = \frac{p_m - p_{m+1}}{p_m + p_{m+1}}$, $p_m = k_{m,z}/\varepsilon_m$. The propagation matrix P_m is expressed similarly to eq. (8) using $k_{m,z}$ for diagonal elements.

Following the previous steps, one can obtain T_H and R_H :

$$T_H = \frac{e^{ik_{z,0}(z_1 - z_N)}}{m_{11}} \quad (21a)$$

$$R_H = \frac{m_{21}}{m_{11}} e^{2ik_{z,0}z_1} \quad (21b)$$

Since $E_x \propto \partial_z H_y / \varepsilon$ with $\varepsilon_0 = \varepsilon_N = 1$, the following relation between amplitude reflection/transmission coefficients arise:

$$R_E = -R_H \quad (22)$$

$$T_E = T_H \quad (23)$$

Therefore, the final expressions are as follows:

$$T_E = \frac{e^{ik_{z,0}(z_N - z_1)}}{m_{11}} \quad (24a)$$

$$R_E = -\frac{m_{21}}{m_{11}} e^{2ik_{z,0}z_1} \quad (24b)$$

$$A_E = 1 - \left| \frac{m_{21}}{m_{11}} \right|^2 - \frac{1}{|m_{11}|^2} \quad (24c)$$