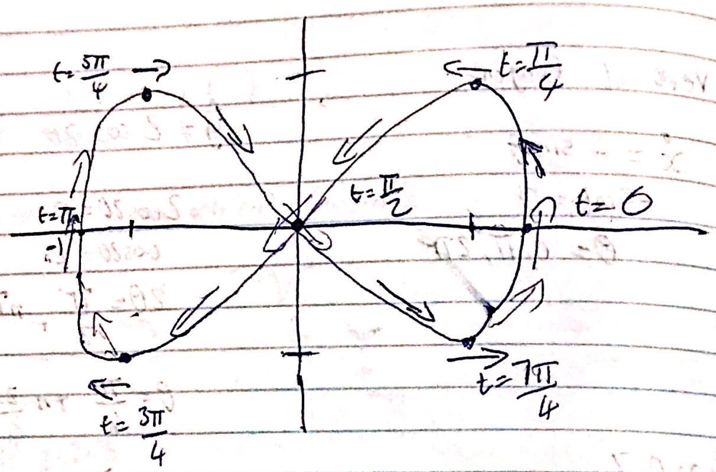


2.3 Another Application

$\underline{r} = (x(t), y(t))$

$\dot{\underline{r}} = (\dot{x}, \dot{y})$

$\underline{r} = (r\cos\theta, r\sin\theta)$

Th 2.3.1

$\tan a = \frac{r}{\dot{r}} \quad \dot{r} \neq 0$

Proof. Gradient of tangent

$$\frac{dy}{dx} = \tan(\theta + a) = \frac{\tan\theta + \tan a}{1 - \tan\theta\tan a} \quad \text{-- } ①$$

$\dot{\underline{r}} = (\dot{x}, \dot{y})$ $\quad x = r\cos\theta$, $y = r\sin\theta$

$$\frac{d\underline{r}}{d\theta} = (-r\sin\theta + \dot{r}\cos\theta,\ r\cos\theta + \dot{r}\sin\theta)$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{r\cos\theta + \dot{r}\sin\theta}{-r\sin\theta + \dot{r}\cos\theta} \times \frac{\frac{1}{\dot{r}\cos\theta}}{\frac{1}{r\sin\theta}}$$

$$= \frac{\frac{r}{\dot{r}} + \tan\theta}{-\frac{r}{\dot{r}}\tan\theta + 1} \quad \text{-- } ②$$

if $\dot{r} = 0$ then $\tan a$ is undefined here which $a = \frac{\pi}{2}$

tan ⊥ radius

curve is a circle

r is constant

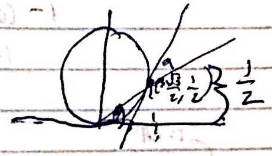
Compare eqn 1 and 2:

$$\frac{\tan\theta + \tan a}{1 - \tan a \cdot \tan\theta} = \frac{\tan\theta + \frac{r}{\dot{r}}}{1 - \frac{r}{\dot{r}}\tan\theta}$$

$\underline{r}(r\cos\theta, r\sin\theta) = (x, y)$

where $r = r(\theta)$

Example 2.3.2



$$\tan a = \frac{r}{\dot{r}} = \frac{2\sin\theta}{2\cos\theta} = \tan\theta$$

$\cancel{a = \theta}$

$\therefore a = \theta$

$M_{tangent}$ $\quad \tan\theta = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} = a$

$$M_{tangent} = \tan\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

$y = \sqrt{3}x + c$

$\frac{1}{2} = \sqrt{3} \cdot \frac{\sqrt{3}}{2} + c$

$c = \frac{1}{2} - \frac{3}{2} = -1$

eqn $\quad y = \sqrt{3}x - 1$

$\left(1+\frac{4}{9}\right)\int e^{2x}\sin 3x\,dx = \frac{-e^{2x}\cos 3x}{3} + \frac{2e^{2x}\sin 3x}{9} + C$

$= \frac{-3}{13}e^{2x}\cos 3x + \frac{2}{13}e^{2x}\sin 3x + C$

1.6.14

$\int \sin^n\theta\,d\theta = \int \sin^{n-1}\sin\theta\,d\theta$

let $u = \sin^{n-1}\theta$ and $dv = \sin\theta\,d\theta$

$du = (n-1)\sin^{n-2}\theta\,d\theta\cos\theta$ $\quad v = \cos\theta$

$= -\sin^{n-1}\theta\cos\theta + \int \cos^2\theta(n-1)\sin^{n-2}d\theta$

$= -\sin^{n-1}\theta\cos\theta + \int (1-\sin^2\theta)(n-1)\sin^{n-2}\theta\,d\theta$

$\int \sin^n\theta\,d\theta = -\sin^{n-1}\theta\cos\theta + (n-1)\int \sin^{n-2}\theta\,d\theta$

$-(n-1)\int \sin^n\theta$

symbolising original integral

$I_n + (n-1)I_n = -\sin^{n-1}\theta\cos\theta + (n-1).$

Repeated use of integration by parts

ex 1.6.~~7~~5

$\int x^3 e^{2x}\,dx$

let $u = x^3$ $\quad dv = e^{2x}\,dx$

$du = 3x^2\,dx$ $\quad v = \frac{e^{2x}}{2}$

$= x^3\left(\frac{e^{2x}}{2}\right) - \int (3x^2)\frac{e^{2x}}{2}\,dx$

let $u = 3x^2$ $\quad dv = \frac{e^{2x}}{2}\,dx$

$du = 6x\,dx$ $\quad v = \frac{e^{2x}}{4}$

$= x^3\left(\frac{e^{2x}}{2}\right) - (3x^2)\left(\frac{e^{2x}}{4}\right) + \left[\int (6x)\left(\frac{e^{2x}}{4}\right)dx\right]$

$= x^3\left(\frac{e^{2x}}{2}\right) - (3x^2)\left(\frac{e^{2x}}{4}\right) + \int (6x)\left(\frac{e^{2x}}{4}\right)dx$

let $u = 6x$ $\quad dv = \frac{e^{2x}}{4}\,dx$

$du = 6\,dx$ $\quad v = \frac{e^{2x}}{8}$

$= x^3\left(\frac{e^{2x}}{2}\right) - (3x^2)\left(\frac{e^{2x}}{4}\right) + (6x)\left(\frac{e^{2x}}{8}\right) - \int 6\frac{e^{2x}}{8}$

$= x^3\left(\frac{e^{2x}}{2}\right) - (3x^2)\left(\frac{e^{2x}}{4}\right) + (6x)\left(\frac{e^{2x}}{8}\right)$

$-(6)\left(\frac{e^{2x}}{16}\right) + C$

$\int u\,dv = uv - u'\int v + u''\iint v - u'''\iiint v + u''''\iiiint v$

let $u = x^5 + 7x^2$ $\quad dv = \sin 2x\,dx$

$du = (5x^4 + 14x)\,dx$ $\quad v = -\frac{\cos 2x}{2}$

$= (x^5+7x^2)\left(-\frac{\cos 2x}{2}\right) - (5x^4+14x)\left(-\frac{\sin 2x}{4}\right) + (20x^3+14)$

$\left(\frac{\cos 2x}{8}\right) - (60x^2)\left(\frac{\sin 2x}{16}\right) + (120x)\left(\frac{-\cos 2x}{32}\right) +$

Example 1.6.7

$\int x \arcsin x \, dx$ let $\theta = \arcsin x$

$\therefore \sin\theta = x$

$\cos\theta . d\theta = dx$

$\therefore \int \arcsin x \, dx = \sin\theta . \theta . \cos\theta \, d\theta$

$\therefore = \frac{1}{2}\int \theta . 2\sin\theta\cos\theta \, d\theta$

$= \frac{1}{2}\int \theta . \sin(2\theta) \, d\theta$

$u(\theta) = \theta$

$u'\theta = 1$

$v'(\theta) = \sin 2\theta$

$v(\theta) = \int v'(\theta) \, d\theta$

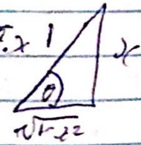
$= \int \sin 2\theta . d\theta$

$= -\frac{\cos(2\theta)}{2} + c$

$= \frac{1}{2}\left[-\frac{\theta}{2} . \cos(2\theta) + \int \frac{\cos(2\theta)}{2} d\theta\right]$

$= \frac{-\theta}{4}\cos(2\theta) + \frac{1}{8}\sin(2\theta) + C$

$= -\frac{1}{4}\arcsin x \left[-1 + 2\left(\sqrt{1-x^2}\right)^2\right] + \frac{1}{4} 2\sqrt{1-x^2} . x + C$



$= -\frac{1}{4}\arcsin x\left[-1 + 2 - 2x^2\right] + \frac{1}{4}\sqrt{1-x^2} . x + C$

Example 1.6.8

$\int (\ln x)^2 \, dx =$

$u = \ln$

$du = \frac{1}{x}$

$v'(x) = \ln x$

$\therefore = \int (\ln x)^2 \, dx$

$= x(\ln x)^2 - x\ln x - \int \ln x - 1 \, dx$

$V(x) = \int v'(x) \, dx = \int \ln x \, dx = x\ln x - x + c$ → Ex 1.6.4

$= x(\ln x)^2 - x\ln x - x\ln x + 2x + c$

$= x(\ln x)^2 - 2x\ln x + 2x + C$

Option 2:

$u = \ln(x)$ $du = \frac{1}{x} dx$

$e^u = x$ $e^u du = dx$ ← do this to find dx

$\int (\ln x)^2 \, dx = \int u^2 e^u - 2\int u e^u . du$

$= u^2 e^u - 2\left[u e^u - \int e^u du\right]$

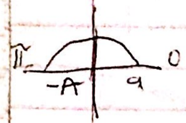
$= u^2 e^u - 2ue^u + 2e^u + C$

$= (\ln x)^2 . x - 2\ln x . x + 2x + C$

1.6.1

$\int_a^b u(x) v'(x) \, dx = \left[u(x) \cdot v(x)\right]_a^b - \int_a^b u'(x) v(x)$

Example: 2.6.3



$$CSA = \int_{\theta=a}^{\theta=\beta} 2\pi |y(\theta)| \sqrt{r^2+\dot{r}^2}\, d\theta$$

where $x = a\cos\theta$, $y = r\sin\theta$ $\therefore r(\theta) = a$

$$= 2\pi \int_0^{\pi} |a\sin\theta| \sqrt{a^2+0}\,.d\theta \qquad \dot{r} = \frac{dr}{d\theta} = 0$$

$$= 2\pi \int_0^{\pi} a^2 \sin\theta\, d\theta$$

$$= -2\pi a^2 \cos\theta \Big]_0^{\pi}$$

$$= -2\pi a^2[-1-1]$$

$$= 4\pi a^2$$

Example 2.6.4 $\quad x = t - \sin t$ or

$$CSA = \int_{t_1}^{t_2} 2\pi |y(t)| \sqrt{\dot{x}^2+\dot{y}^2}\, dt \qquad \frac{dx}{dt} = 1-\cos t$$

$$= \int_0^{2\pi} 2\pi |1-\cos t| \sqrt{(1-\cos t)^2+\sin^2 t}\, dt \qquad \frac{dy}{dt} = \sin t$$

$$= 2\pi \int_0^{2\pi} (1-\cos t)\sqrt{2-2\cos t}\, dt$$

$$= 2\pi\sqrt{2} \int_0^{2\pi} (1-\cos t)\sqrt{1-\cos t}$$

$$2\sqrt{2}\pi \int_0^{2\pi} (1-\cos t)^{\frac{3}{2}}\, dt$$

forming the double angle

$$= \cancel{2\pi}\, 2\sqrt{2}\pi \int_0^{2\pi} \left(1 - \left[1-2\sin^2\frac{t}{2}\right]\right)^{\frac{3}{2}} dt$$

$$= 2\sqrt{2}\pi \int_0^{2\pi} \left(2\sin^2\frac{t}{2}\right)^{\frac{3}{2}} dt$$

$$= 2\sqrt{2}\cdot\sqrt{2^3}\,\pi \int_0^{2\pi} \sin^3\frac{t}{2}\, dt$$

$$= 2^3\pi \int_0^{2\pi} \sin\frac{t}{2}\left[1-\cos^2\frac{t}{2}\right] dt \qquad \text{u.s.}$$

$$= 8\pi \int_0^{2\pi} \sin\frac{t}{2} + \left[-\sin\frac{t}{2}\right]\left[\cos^2\frac{t}{2}\right] dt \qquad \text{u substitution}$$

$$= 8\pi\left[-2\cos\frac{t}{2} + \frac{1}{3}\cdot 2\cos^3\frac{t}{2}\right]_0^{2\pi}$$

$$= 8\pi\left[-2(-1) - \frac{2}{3} - \left[-2+\frac{2}{3}\right]\right]$$

Δ

Example 2.6.5

$y = \sqrt{x} \quad x = 0 \quad x = 3$

$$CSA = \int_{x_1}^{x_2} 2\pi |y| \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$= \int_0^3 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,.dx$$

$$= \int_0^3 2\pi\sqrt{x}\sqrt{\frac{4x+1}{4x}}\, dx$$