Strength of Materials - GATE

Kulasekaran

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Contents

Chapter 1

Properties of Materials

1.1 Basics

- **Longitudinal Axis:** The Line passing through the center of all the planes along the longest dimension of the member is called as Longitudinal Axis.
- **Cross Sectional Area (CS):** The plane normal to the longitudinal axis of the object
- **Prismatic Bar:** A member with constant cross-sectional area along its whole length

1.2 Stress(σ)

- **Stress(σ):** The internal resisting force which resists deformation in object when a force in acting on it. $\sigma = \dfrac{F}{A}$

    - Stress is developed only when the motion due to the force is restricted
    - Pressure and Stress are not the same. Pressure is external Normal force over a surface.
    - **Normal Stress(σ):** Stress acting perpendicular to CS area.
        Sign Convention:
        * Tensile stress ($\leftarrow \boxed{...} \rightarrow$) = Positive (+ve)
        * Compressive stress ($\rightarrow \boxed{...} \leftarrow$) = Negative (-ve)
    - **Shear Stress(τ):** Stress acting tangential(parallel) to CS area.

- **Engineering (or) Nominal Stress**

    $\rightarrow \boxed{\sigma_{Engg} = \dfrac{F}{A_0}}$

    $\rightarrow A_0$ = Original CS Area. Its called Original because, when an object is developing stress, there will be deformation to the object will change the cross sectional area. But, if we are using the initial cross sectional area to calculate stress, then its called Engineering stress (or) Nominal stress (or) **Average stress**

- **True stress (or) Actual stress**

    $\rightarrow \boxed{\sigma_{actual} = \dfrac{F}{A_a}}$

    $\rightarrow A_a = A_0 + \Delta A$
        * ΔA = +ve for Compression, as Area($\uparrow$) during compression
        * ΔA = -ve for Tension, as Area($\downarrow$) during Tension

$\implies$ In Tension $\boxed{\sigma_{True} > \sigma_{actual}}$ $\qquad \implies$ In Compression $\boxed{\sigma_{True} < \sigma_{actual}}$

1.3 Strain($\in$)

- $$\boxed{\in = \frac{Change\ in\ dimension}{Original\ dimension} = \frac{\Delta L}{L}} \implies \boxed{\in_{Engg} = \frac{\Delta L}{L_0}} \implies \boxed{\in_{actual} = \frac{\Delta L}{L_a}} \impliedby \boxed{L_a = L_0 \pm \Delta L}$$

- $L_a = L_0 + \Delta L$ for Tension $\qquad L_a = L_0 - \Delta L$ for Compression

- **Relationship between** σ_{Engg} **&** σ_{actual} **:** $\boxed{\sigma_{actual} = \sigma_{Engg}(1 \pm \in_{Engg})} \impliedby +ve(Tension)\ \&\ -ve(Compression)$

1.4 Stress-Strain curve

- The Mechanical properties of materials used in engineering are determined using experiments performed on small specimens
  - **ATSM** - American Society for Testing and Materials
  - **UTM** - Universal Testing Machine (for Tension test)
  - **Specimen spec** - Must be a cylindrical rod with L/D = 4
- **Stress-Strain curve for Tension**
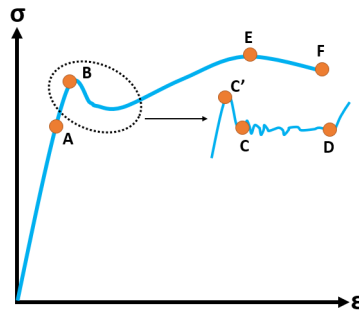
  - A = PROPORTIONAL LIMIT
    * **Hooke's Law:** stress $\propto$ strain
    * Hooke's law is valid upto this point i.e., Linear variation of stress and strain upto A
  - B = ELASTIC LIMIT
    * Maximum stress upto which material can retain its original dimension upon load removal
    * Material behaves perfectly elastic up until B
    * Only Elastic or Elastoplastic deformation (Elastoplastic = both plastic and elastic deformation)
  - C' = UPPER YIELD POINT
    * Depends on CS area, shape of specimen and type of the test equipment.
    * Has no practical significance
  - C = LOWER YIELD POINT
    * Also called as actual yield point and stress at C is the Yield stress(σ_y)
    * The yielding begins from this point
  - CD = PERFECTLY PLASTIC REGION
    * Strain occurring without any increase in stress
  - DE = STRAIN HARDENING REGION
    * strain increases with faster rate in this region
    * material undergoes change in the crystalline structure
  - E = ULTIMATE YIELD POINT
    * Stress corresponding to this point is called Ultimate stres (σ_U)
  - F = FRACTURE POINT
    * Stress corresponding to this point is called Ultimate (σ_F)
    * Region between EF is called the **Necking region**, where the CS area is drastically reduced.
- **Plastic Strain:** Strain before Yield point
- **Elastic Strain:** Strain After Yield point
- Fracture Strain($\in_F$) depends on Carbon content %. If carbon($\uparrow$), fracture strain decreases

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1.5   Properties of Materials

1.5.1   Ductility

- Large deformations are possible in ductile material before fracture
- These materials have post-elastic strain greater than 5%

1.5.2   Brittleness

- These materials have post-elastic strain less than 5%
- Fracture takes place immediately after elastic limit
- Fracture and ultimate points are the same

1.5.3   Malleability

- The property that tells if a metal can be converted into thin sheet by pressing it.
- This property is great use in operations like forging, hot rolling, stamping, etc.,

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1.5.4   Hardness

- Resistance to scratch or abrasion
- Two methods of Hardness measurement:
  - Mohr's test
  - Indentation hardness - Brinell, Rockwell, Vickers, Knoop

1.5.5   Toughness

- Property which enables material to absorb energy without fracture.
- If a material is tough it has ability to store large amount of strain energy before fracture.
- Ductile materials are tough and brittle materials are hard
- **Modulus of Toughness:** Total strain energy per unit volume up until fracture

- Modulus of Toughness $= \boxed{\dfrac{\sigma_y + \sigma_U}{2} * \in_F}$

1.6   Creep

- Permanent deformation in a material under constant loading after a long period of time
- Factors affecting creep: Load magnitude, type of loading, age of loading, Temperature
- **Homologous temperature:**   Half of melting point, creep becomes appreciable at this temperature

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1.7   Stress relaxation

- The reason why electric wires sag after a long period of time.
- The stress gradually diminishes and reaches a constant value after a period of time.

1.8 Elasticity

- The property by which original dimensions can be recovered upon unloading is called elasticity
- within elastic limit, the curve can be both linear and non-linear.

1.9 Resilience

- The total strain energy which can be stored in the given volume of the metal and can be released after unloading is called resilience.
- Resilience = Area under stress-strain curve within elastic limit
- **Modulus of resilience(U_r):** Maximum elastic energy per unit volume. This occurs when elastic limit coincides with yield point.
- $$\boxed{U_r = \frac{1}{2} * \sigma_y * \in_y = \frac{\sigma_y^2}{2E}} \impliedby \boxed{\in_y = \frac{\sigma_y}{E}}$$

1.10 Proof stress

- Some materials do not show clear yield point on the stress-strain curve.
- For such materials the yield point is calculated by offset method.
- A line parallel to the curve until the elastic limit is drawn starting from 0.2% of the strain. This line meets the curve at a point and the corresponding stress at this point is called Proof stress.

1.11 Elasto-Plastic behaviour

- during unloading if only part of the original dimension was recovered, then the remaining unrecoverable strain energy is called **Inelastic strain energy**
- Beyond elastic limit, if a material undergoes continuous loading and unloading, then yield limit of material increases continuously

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1.12 Types of Material Behaviour

Linear Elastic   NonLinear Elastic   Elastoplastic or Viscoplastic   Perfectly Plastic

Elastoplastic with strain hardening   Ideal fluid   Ideal Rigid
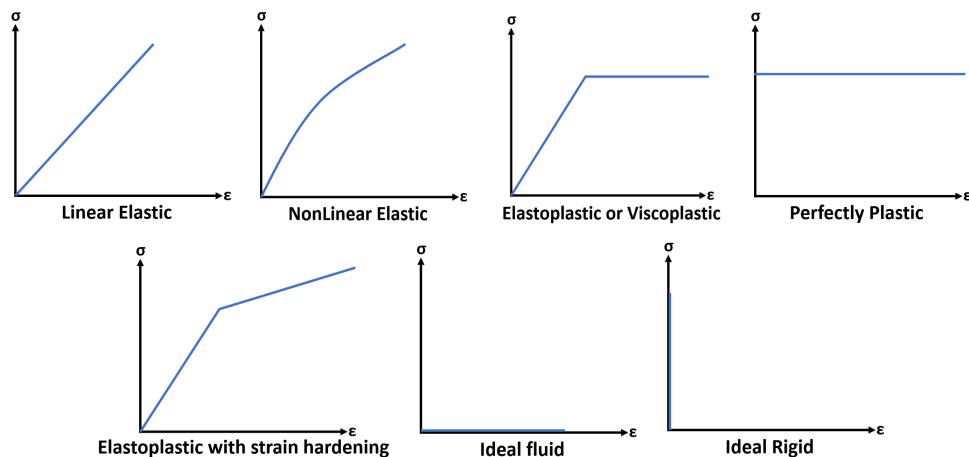
1.13   Fatigue

- Materials behave differently under static and dynamic loading

- Factors affecting fatigue: Loading, Temperature, Loading frequency, Corrosion, Stress concentration

- **Fatigue Initiation life:** The number of load cycles required to initiate a surface crack

- **Fatigue Propagation life:** The number additional load cycle required to propagate surface crack

- **Endurance limit:** The stress below which material has no probability of cracking even with infinite load cycles. Endurance limit exists between elastic limit and yield point.

1.14   Failure of Materials in Tension and Compression

1.14.1   Ductile Metals in Tension

- Ductile materials are weak in shear

- Cup and cone failure

- failure plane angle is 45˚

1.14.2   Brittle Metals in Tension

- Brittle materials are weak in tension

- Failure plane angle is 90˚to load

1.14.3   Ductile Metals in Compression

- Failure plane anlge is 90˚to load

1.14.4   Brittle Metals in Compression

- Brittle materials fail in shear

- Failure plane angle is 45˚

Chapter 2

Stress, Strain and Elastic Constants

2.1 Normal Stress

- Stress acting perpendicular to the CS area
- They are of two types: Direct axial stress, Bending stress

2.1.1 Direct axial stress

- These stress are produced when axial force is acting at CG of cross section

2.1.2 Bending stress

- Produced due to Bending moments. Bending stresses vary linearly from 0 at Neutral axis to Maximum at farthest fibre from Neutral Axis

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2.2 Shear (or) Tangential stress(τ)

- Shear stress(τ) = $\boxed{\dfrac{ShearForce}{Area} = \dfrac{S}{A}}$

- They are of two types: Direct shear, Torsional shear

2.2.1 Direct shear stress

- Due to direct shear force acting on the surface

2.2.2 Torsional shear stress

- Produced when a member is subjected to torsional moment (Twisting)

2.3 Matrix representation of stress and strain

- Stress and strain are called **tensor** quantities as are they are defined with respect to an area. They are 2nd order Tensors
- In a 3D body, stress or strain at a point has 9 components (3 Normal and 6 shear)
- In a 3D, there are 3 mutually perpendicular planes(xy,yz,xz). Each plane has 1 normal component and 2 shear component.

$$stress = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \qquad strain = \begin{bmatrix} \in_{xx} & \phi_{xy}/2 & \phi_{xz}/2 \\ \phi_{yx}/2 & \in_{yy} & \phi_{yz}/2 \\ \phi_{zx}/2 & \phi_{zy}/2 & \in_{zz} \end{bmatrix}$$

- Shear stresses in two mutually perpendicular directions are equal: $\tau_{yx} = \tau_{xy}$ $\tau_{zx} = \tau_{xz}$ $\tau_{yz} = \tau_{zy}$

- Total shear strain in x-y plane: $\dfrac{\phi_{xy}}{2} + \dfrac{\phi_{yx}}{2} = \phi_{xy}$

- Total shear strain in y-z plane: $\dfrac{\phi_{zy}}{2} + \dfrac{\phi_{yz}}{2} = \phi_{yz}$

- Total shear strain in x-z plane: $\dfrac{\phi_{xz}}{2} + \dfrac{\phi_{zx}}{2} = \phi_{xz}$

- $\in_{xx}, \in_{yy}, \in_{zz}$ are linear strains in x, y, z directions respectively

- Under pure normal stress = Volume changes, shape remains same

- Under pure shear stress = Volume remains same, shape changes.

2.3.1 Strain Types

Axial Strain($\in$)

- Strain the direction of the applied force. Also called as **Linear strain**

- $$\boxed{\in = \dfrac{Change\ in\ dimension}{original\ dimension} = \dfrac{\Delta L}{L}}$$

Lateral Strain($\in_L$)

- Strain in the direction perpendicular to the applied force.

- Eg. When an object is stretched, its length increases, but its width and height decreases. This decrease in width and height is called Lateral strain. The increase in length is called Linear strain

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Shear strain(ϕ)
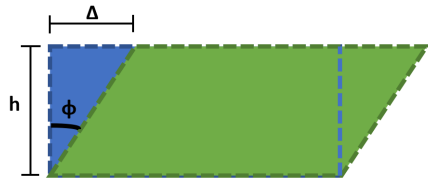
- Angular deformation caused by shearing force, $\boxed{\phi = \dfrac{\Delta}{h}}$

2.3.2 Differential form of Strains

- Consider a point P(x,y,z). A force acting on P, shifts its position to (u,v,w)

- Then linear strains and shear strains are given by:

$$\implies \boxed{\in_{xx} = \dfrac{\delta u}{\delta x}} \quad \boxed{\in_{yy} = \dfrac{\delta v}{\delta y}} \quad \boxed{\in_{zz} = \dfrac{\delta w}{\delta z}} \quad \boxed{(\phi_{xy} = \phi_{yx}) = \dfrac{\delta u}{\delta y} + \dfrac{\delta v}{\delta x}} \quad \boxed{(\phi_{xz} = \phi_{zx}) = \dfrac{\delta w}{\delta x} + \dfrac{\delta u}{\delta z}} \quad \boxed{(\phi_{yz} = \phi_{zy}) = \dfrac{\delta w}{\delta y} + \dfrac{\delta v}{\delta z}}$$

2.4 Allowable stresses

- **Strength:** The ability of a structure to resist loading is called strength. For safety reasons, materials should have higher strength than what is required due to loading.

- **Factor of Safety** $= \dfrac{Actual\ strength}{Strength\ required}$

- **Allowable Stress** $(\sigma_A) = \boxed{\sigma_{A(ductile)} \dfrac{Yield\ stress}{FOS}} = \boxed{\sigma_{A(brittle)} \dfrac{Ultimate\ stress}{FOS}}$

- A term called **Margin of safety** is used for aircrafts. Margin of safety = FOS - 1

2.5 Saint Venant's principle

- This principle states that the stress distribution in a prismatic bar is uniform except in the region of extreme ends.

- b = width of the prismatic bar

- Section(1-1): $\dfrac{b}{2}$ distance from the extreme ends

- Section(2-2): $\dfrac{b}{2} + \dfrac{b}{2}$ distance from the extreme ends

- $\boxed{\sigma_{1-1} = 1.387\sigma_{avg}} \quad \boxed{\sigma_{2-2} = 1.027\sigma_{avg}} \quad \boxed{\sigma_{3-3} = \sigma_{avg}}$

2.6 Hooke's law

- Assumptions: Homogeneous (made of same material), Isotropic (properties are same in all directions), elastic

- Stress $\propto$ Strain $\implies \boxed{\dfrac{\sigma}{\in} = E} \impliedby$ (Valid upto Proportional limit)

- E = Modulus of elasticity = slope of stress strain curve under proportional limit

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2.7 Elastic constants

2.7.1 Young's modulus(E)

- $\boxed{E = \dfrac{Direct\ stress}{Direct\ strain} = \dfrac{\sigma}{\in}}$

2.7.2 Shear modulus (or) Rigidity modulus(G)

- $\boxed{G = \dfrac{Shear\ Stress}{Shear\ strain} = \dfrac{\tau}{\phi}}$

2.7.3 Bulk modulus(k)

- $\boxed{k = \dfrac{Volumetric\ stress}{Volumetric\ strain} = \dfrac{\sigma_{vol}}{\in_V}} \quad \boxed{\in_V = \dfrac{\Delta Volume}{Volume}}$

2.7.4 Poisson's ratio(μ)

- $\boxed{\mu = \dfrac{-Lateral\ strain}{Linear\ strain}}$ $\Longleftarrow$ (defined in elastic region)

- $\mu = 0.05$ - 1 (glass), 0.1 - 0.2 (concrete),0.25 - 0.42 (metals), 0.5 (pure rubber, perfectly plastic)

- For non-elastic region, it is called **Contraction ratio**

2.7.5 Relationship between elastic constants

- $\boxed{E = 3K(1 - 2\mu)}$ $\boxed{E = 2G(1 + \mu)}$ $\boxed{E = \dfrac{9KG}{3K + G}}$ $\boxed{\mu = \dfrac{3K - 2G}{6K + 2G}}$

- **Orthotropic material** = 9 elastic constants, **Anisotropic material** = 21 elastic constants

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2.8 Applications of Hooke's law

2.8.1 Effect of Uniaxial Loading

- Consider a rectangular prismatic bar with tensile force acting along x-axis

$\boxed{\in_{xx} = \dfrac{\sigma_x}{E}}$ $\Longrightarrow$ $\boxed{\in_{yy} = -\mu\dfrac{\sigma_x}{E}}$ $\boxed{\in_{zz} = -\mu\dfrac{\sigma_x}{E}}$ $\Longleftarrow$ $\in_{yy}$ & $\in_{zz}$ are Lateral strains

2.8.2 Effect of Triaxial Loading

- Consider: σ_x acting along x-direction, σ_y acting along x-direction, σ_z acting along x-direction. (All are Tensile)

- $\boxed{\in_{xx} = \dfrac{\sigma_x}{E} - \mu\dfrac{\sigma_y}{E} - \mu\dfrac{\sigma_z}{E}}$ $\boxed{\in_{yy} = \dfrac{\sigma_y}{E} - \mu\dfrac{\sigma_x}{E} - \mu\dfrac{\sigma_z}{E}}$ $\boxed{\in_{zz} = \dfrac{\sigma_z}{E} - \mu\dfrac{\sigma_x}{E} - \mu\dfrac{\sigma_y}{E}}$

2.9 Volumetric Strain($\in_V$)

- How to derive?

- compute the volume of the member. take that as V.

- Now find ΔV, that is differentiate V wrt to each factor.

- Then use: $\boxed{\in_V = \in_x + \in_y + \in_z}$

2.9.1 For Rectangular prismatic member

- $\boxed{\dfrac{\sigma_x + \sigma_y + \sigma_z}{E} * (1 - 2\mu)}$

2.9.2 For Cylindrical rod

- $\boxed{\in_V = \in_L + 2\in_d}$ $\Longleftarrow$ ($\in_d$ = Diametrical strain)

2.9.3 For Spherical body

- $\boxed{\in_V = 3\in_d}$

2.10 Elongation in axially loaded members

- How to derive?

- consider an element and find its elongation using $\dfrac{PL}{AE}$, Then integrate that δL for the whole length

2.10.1 Axially loaded prismatic bar

- In this case Elemental elongation is $\dfrac{P_x d_x}{A_x E_x}$ $\Longleftarrow$ d_x = Length of the element.

- $\boxed{\Delta = \dfrac{PL}{AE}}$ (AE = Axial rigidity) and ($\dfrac{AE}{L}$ = Axial stiffness)

2.10.2 Axially loaded Circular Tapered bar

- Here Diamater of the element $= D_x = D_1 + \dfrac{D_2 - D_1}{L}x$

- $\boxed{\Delta = \dfrac{4PL}{\pi E D_1 D_2}}$ $\Longleftarrow$ D_1 = smaller diameter and D_2 = Larger diameter

2.10.3 Axially loaded Rectangular Tapered bar

- The thickness of the bar is uniform

- $\boxed{\Delta = \dfrac{PL \log_e \left(\dfrac{B_2}{B_1}\right)}{(B_2 - B_1)tE}}$ $\Longleftarrow$ (B_1 = Smaller height and B_2 = Larger height)

2.11 Principle of Superposition

- If a member is subjected to various loadings then the resultant deformation will be equal to the algebraic sum of the deformation caused by the individual forces acting on the member.

- Consider the following example:
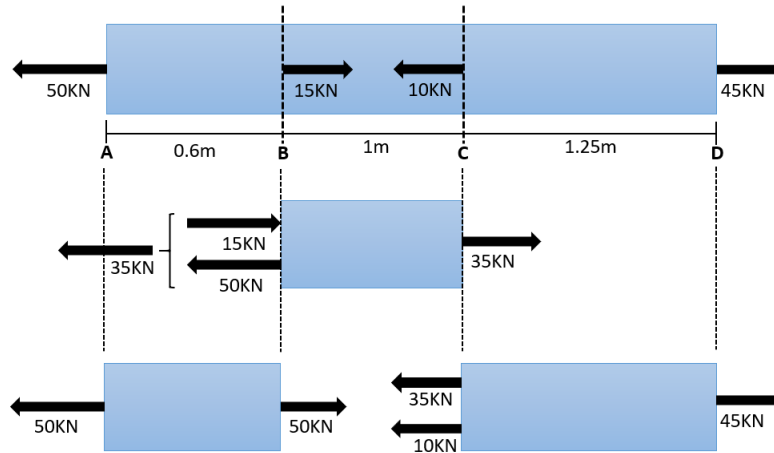
- Total elongation of the element(Δ) $= \Delta_{AB} + \Delta_{BC} + \Delta_{CD}$

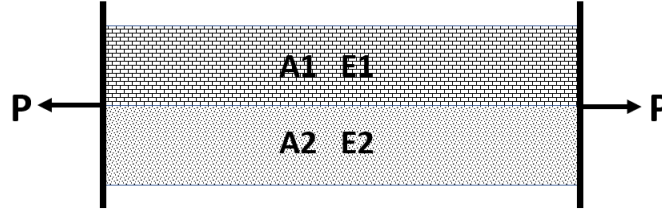## 2.12   Elongation in Composite members

- Composite structures are those that are made of more than one material

2.12.1   Elongation in composite rectangular member

- Condition of Equilibrium: $P = P_1 + P_2$

- As two materials are joined firmly: $\Delta_1 = \Delta_2 \implies \dfrac{P_1 L_1}{A_1 E_1} = \dfrac{P_2 L_2}{A_2 E_2}$

- $\boxed{\Delta = \dfrac{PL}{A_1 E_1 + A_2 E_2}}$ and $E_{eq} = \dfrac{A_1 E_1 + A_2 E_2}{A_1 + A_2} \implies \Delta = \dfrac{PL}{(A_1 + A_2)E_{eq}}$

2.13   Elongation due to Self-Weight

- How to derive?

- Find the elemental deflection due to the self weight and then Integrate for the whole length.

2.13.1   Rectangular Prismatic bar

- $\gamma = Unit\ weight = \dfrac{Weight}{unit\ volume}$

- $W_x = \gamma(A * x) \impliedby W_x = $ Self weight acting at an element x distance away from the bottom of the bar

- $\boxed{\Delta = \dfrac{\gamma L^2}{2E}} \implies \boxed{\dfrac{WL}{2AE}}$

2.13.2   Conical Bar

- $\boxed{W_x = \gamma * \left(\dfrac{1}{3}\right) \pi \left(\dfrac{D_x}{2}\right)^2 x}$ $\boxed{\Delta = \dfrac{\gamma L^2}{6E}}$

2.13.3   Bar of Uniform strength

- It is possible to maintain uniform stress at all the sections by increasing the area from the lower level to Higher level.

- $\boxed{A_1 = A_2 e^{\left(\dfrac{\rho g L}{\sigma}\right)}}$

2.14 Statically Indeterminate Axial Loaded structures

- Those structures which cannot be solved using static equilibrium equations alone are called Statically Indeterminate structures
- They can be solved using flexibility approach or stiffness approach
- Flexibility approach involves equations using the relations in Slope, deflection and rotation and are called **compatibility equation**

2.15 Thermal stresses

- **NOTE:** Thermal stresses do not depend on the area of cross section
- Thermal Strain $\boxed{\in_{Th} = \alpha T}$ $\impliedby$ (α = Coefficient of thermal expansion)
- Consider a rectangular prismatic bar of length L between fixed supports. If the temperature of the bar is raised, then $\Delta = 0$ (As the bar is fixed)
- But due to rise in temperature of T°C, the bar will try to expand and so will be under compression
- $\Delta = 0 = L\alpha T - \dfrac{\sigma_{Th} L}{E}$ $\implies$ $\boxed{\sigma_{Th} = E\alpha T}$ $\impliedby$ (Thermal stresses are independent of member dimensions)

2.15.1 Thermal stresses in Composite bars

- Mostly we use three sets of relations
- $P_1 = P_2$ (under no other external load) $\implies \sigma_1 A_1 = \sigma_2 A_2$
- $\Delta_1 = \Delta_2$
- The material with larger α value, will tend to deform more under thermal load
- Consider a composite cantilever bar made of copper and steel and both the materials are rigidly fixed to each other. Copper has higher coefficient of thermal expansion and so upon temperature rise will tend to expand more than steel.
- As a result, copper part will be under compression as steel is not letting it to expand and steel part will be under tension as copper is pulling it.
- As they are affixed to each other rigidly, both their end deformation value will be the same. Using that relation we

  can write: Copper free expansion + contraction = steel free expansion + tension $\implies$ $\boxed{L\alpha_c T - \dfrac{\sigma_c L}{E_c} = L\alpha_s T + \dfrac{\sigma_s L}{E_s}}$

2.16 Stresses in Nuts and Bolts

- **Effect of Tightening of Nut**
- P = pitch of screw on the bolt, D_b = Bolt diameter
- A_s = Area of steel bolt = $\dfrac{\pi D_b^2}{4}$
- D_i = Inner diameter of copper tube, D_O = Outer diameter of copper tube
- θ = Nut is rotated by θ°, n = Number of turns = $\dfrac{\theta}{360}$
- np = Axial movement of nut
- Total tensile for in bolt = total compressive force in copper tube $\implies \sigma_s A_s = \sigma_c A_c$
- $\boxed{np = \left(\dfrac{\sigma_s L}{E_s}\right) + \left(\dfrac{\sigma_c L}{E_c}\right)}$ $\impliedby$ Using the above relation, solve for σ_s and σ_c

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2.17   Strain Energy(U)

- The strain energy is equal to the work done by the load provided no energy is added or subtracted in the form of heat.

- **NOTE:** Strain energy of an elastic body due to more than one load cannot be found by simply adding the strain energy obtained from individual loads. In other words, **The principle of superposition is not applicable on strain energy**. This is because, Unlike deformation which is linear function, strain energy is a quadratic function.

- $\boxed{U = \dfrac{1}{2}P\Delta}$ $\impliedby$ Δ = Axial deflection

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2.17.1   Strain energy in Prismatic bar

- $\boxed{U = \dfrac{PL^2}{2AE}}$

2.17.2   Strain energy in Prismatic bar of varying cross section

- $\boxed{U = \dfrac{P^2}{2E}\left[\dfrac{L_1}{A_1} + \dfrac{L_2}{A_2} + ...\right]}$

2.17.3   Strain energy due to shear force

- $\boxed{U = \displaystyle\int \dfrac{S_x^2 dx}{2A_r G}}$ $\implies$ $(A_r$ = Reduced area$)$

2.17.4   Strain energy in terms of principal stresses

- $\boxed{U = \dfrac{1}{2E}(\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2)}$ $\impliedby$ (2D) $\boxed{U = \dfrac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]}$ $\impliedby$ (3D)

2.17.5   Strain energy due to Bending moment

- $\boxed{U = \displaystyle\int \dfrac{M_x^2 ds}{2EI}}$ $\impliedby$ (I = MOI about NA)

2.17.6   Strain energy due to Torque

- $\boxed{U = \displaystyle\int \dfrac{T_x^2 ds}{2GI_P}}$ $\impliedby$ $(I_P$ = Polar MOI$)$

Chapter 3

Shear force and Bending Moment

3.1 Types of Loading

- Point load
- Uniformly Distributed Load
- Uniformly Varying Load
- Couple - Concentrated moment at any point is called couple

3.2 Types of supports

3.2.1 Types of 2D Supports

- Fixed support - R_x, R_y, M_z
- Hinged Support - R_x, R_y
- Roller support - R_y
- Double roller support - R_y, M_z

3.2.2 Types of 3D Supports

- Fixed support - $R_x, R_y, R_z, M_x, M_y.M_z$
- Hinged support - R_x, R_y, R_z
- Roller support - R

3.3 Types of Beams

- Simply supported beam
- Cantilever beam
- Propped cantilever beam
- Fixed beam
- Continuous beam
- Overhanging beam
- **NOTE: If an internal hinge is present in a beam, then Bending moment at hinge is Zero. Hinges carry only Axial load.**

3.4 Statically determinate structure

- A structure is said to be statically determinate if all the reaction forces can be found using the static equilibrium equations (2D): $\boxed{\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_z = 0}$

- If R denotes the number of reactions and E denotes the number of equilibrium equations available, then if (R = E) the structure is determinate, if (R > E) then the structure is indeterminate

3.5 Shear Force

- Clockwise shear = +ve and Anticlockwise shear = -ve

3.6 Bending Moment

- Sagging Bending moment = +ve and Hogging Bending moment = -ve

3.7 Important points about SFD and BMD

- SFD is one degree higher than Loading diagram and BMD is one degree higher than SFD
- If a point load is present, then the SFD will change by that load's magnitude
- If a concentrated moment is present, then the BMD will change by that moment's magnitude
- Bending will be maximum/minimum at the point where Shear force is zero, but the inverse is not true
- **Point of Contraflexure** - The point where Bending moment changes sign
- **Focal length** - The distance between two adjacent points of contraflexure
- **Shear span** - the portion of beam where shear force is constant.
- Relation between Loading rate and Shear force: $\boxed{\dfrac{dS}{dx} = -W}$

- Relation between Shear force and bending moment: $\boxed{\dfrac{dM}{dx} = S_x} \implies$ Slope of bending moment = shear force.

- Relation between Bending moment and Loading rate: $\boxed{\dfrac{d^2 M}{dx^2} = -w}$

3.8 Curve Tracing for SFD and BMD
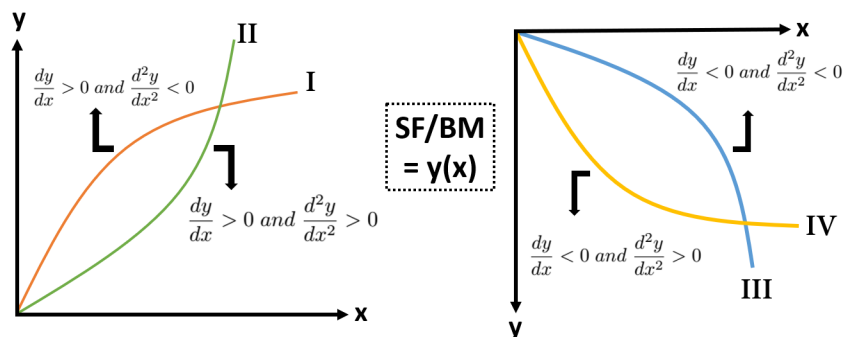
3.9 SFD and BMD by Integration

- Use the relations between Loading rate, shear force and Bending moment:

$$\implies \boxed{\frac{d^2M}{dx^2} = -w} \ ...(i) \quad \boxed{\frac{dM}{dx} = S_x = -Wx + C_1} \ ...(ii) \quad \boxed{M_x = \frac{-Wx^2}{2} + C_1x + C_2} ...(iii)$$

  - **NOTE:** Eqn(i)'s RHS will change according the loading. For eg. in case of Uniformly variying load,

$$\boxed{\frac{d^2M}{dx^2} = -W_x = \frac{-Wx}{L}}$$

- The constants C_1 and C_2 can be found by boundary conditions
- From (ii) SFD can be plotted, from (iii) BMD can be plotted

3.10 Effect of Concentrated moment on SFD and BMD

- At the point of concentrated moment, discontinuity occurs in BMD. The BMD ordinate changes by the magnitude of the moment. For SFD, concentrated moment induces support reactions only.
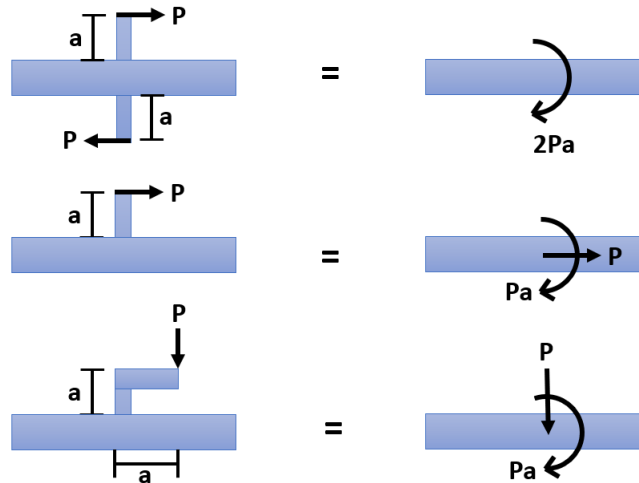
3.11 Loading diagram and BMD from SFD

- In SFD, the portion where SF is constant, it means no load is acting in that portion
- In SFD, Vertical lines represent loads (Generally upward loads denote reactions)
- Inclined line in SFD denotes Uniformly distributed load. Intensity of UDL = slope of line in SFD
- Parabolic curve in SFD denotes Uniformly varying load.
- If $\sum M$ at supports is zero, then the support is hinged or roller. If its not zero, then it is a fixed support.

3.12 Loading diagram from BMD

- In BMD, the portion where there is an inclined line, that means no load is acting in that region
- Parabolic curve in BMD denotes Unifromly distributed load.
- Cubic curve in BMD denotes Uniformly varying load.
- The point where BMD changes its ordinate suddenly, a concentrated moment is acting at that point.
- BMD is one degree higher than SFD and two degree higher than Loading diagram

Chapter 4

Centroid and Moment of Inertia

4.1   Centroid($\bar{x}$,$\bar{y}$)

- $$\bar{x} = \frac{A_1\bar{x_1} + A_2\bar{x_2} + ... + A_n\bar{x_n}}{A_1 + A_2 + ... + A_n} = \sum_{i=1}^{n} \frac{A_i\bar{x_i}}{A_i} = \frac{\int x dA}{A} \qquad \bar{y} = \frac{A_1\bar{y_1} + A_2\bar{y_2} + ... + A_n\bar{y_n}}{A_1 + A_2 + ... + A_n} = \sum_{i=1}^{n} \frac{A_i\bar{y_i}}{A_i} = \frac{\int y dA}{A}$$

- It is the center of a shape

    - If the shape is Bi-Symmetrical, then the centroid will lie on the point of intersection of the two symmetrical axis. Eg. Rectangle, Square, Circle, etc.
    - If the shape is symmetrical about only one axis, then centroid will be anywhere on that symmetrical axis. Eg. Triangle

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4.1.1   Finding centroid for a composite shape

- A composite shape is a shape that is made of many regular shapes like: semicircle, circle, rectangle, square etc,

- How to find Centroid for a composite shape? An example of composite shape is shown below
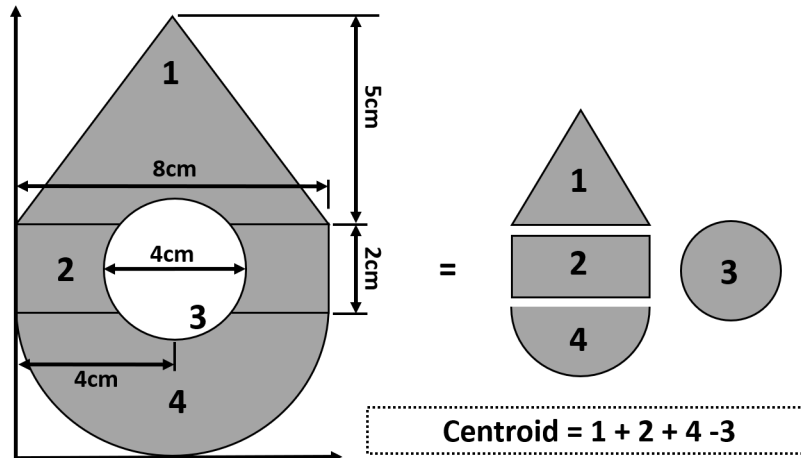
- Break down the composite shapes into its regular shapes. Then form a table with the following columns:

$$shape, \quad Area(A), \quad \bar{x}, \quad \bar{y}, \quad A\bar{x}, \quad A\bar{y}$$

- Then find: $\sum A$, $\sum A\bar{x}$, $\sum A\bar{y}$ and then use, $\bar{x} = \dfrac{\sum A\bar{x}}{\sum A}$ and $\bar{y} = \dfrac{\sum A\bar{y}}{\sum A}$

- For example, the above shape can be broken down into 4 parts as shown above: Triangle, Rectangle, Circle and Semi-circle

- Now, we need to know 3 things about each of the shape above to compute the centroid of the whole shape.

1. Area of the shape   2. $\bar{x}$ 3. $\bar{y}$

- For triangle:

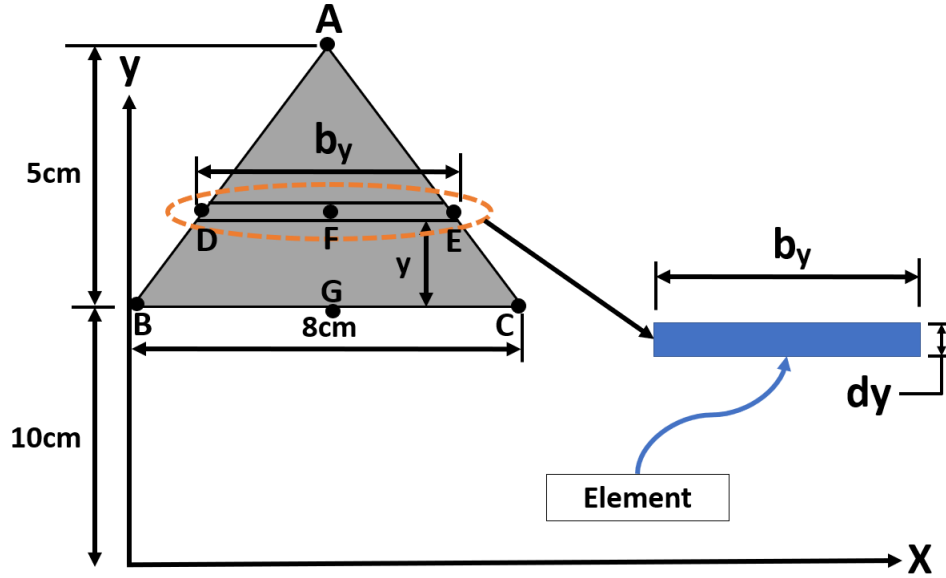$\rightarrow$ Area: $\left(\dfrac{1}{2} * Base * Height\right) = \dfrac{1}{2} * 8 * 5 = 20cm^2$

$\rightarrow \bar{x} = $ Base/2 (as the triangle is symmetrical about the y-axis) = 4cm

$\rightarrow \bar{y} = 10 + h/3 = (10+1.6667cm)$

    * 10 is added because, the triangle is at a height of 10cm from x-axis

    * Why $\bar{y} = $ h/3 ? Derived below:

4.1.2   Finding $\bar{y}$ for a triangle

- We know that $\boxed{\bar{y} = \dfrac{A_1\bar{y_1} + A_2\bar{y_2} + ... + A_n\bar{y_n}}{A_1 + A_2 + ... + A_n} = \sum\limits_{i=1}^{n}\dfrac{A_i\bar{y_i}}{A_i} = \dfrac{\sum A_i\bar{y_i}}{A}}$

- Here, Area $= 20cm^2$

- Consider a horizontal element of thickness (dy) at a height of y from origin.

- The width of the element can be found using **Law of similar triangles** concept. Based on that:

$$\frac{AG}{BC} = \frac{AF}{DE} \implies \frac{5}{8} = \frac{5-y}{b_y} \implies b_y = \frac{8(5-y)}{5}$$

- Now, Area of this element is $A_i = b_y dy = \dfrac{8(5-y)}{5}dy$ and $\bar{y_i} = 10 + y + \dfrac{dy}{2} \impliedby$ But already dy is very small. So dy/2 is negligble $\implies \bar{y_i} = 10 + y$

- Applying the formula of $\bar{y}$ we get,

$$\bar{y} = \frac{\sum \dfrac{8(5-y)}{5}dy(10+y)}{20} = \frac{\displaystyle\int_0^5 \dfrac{8(5-y)}{5}(10+y)dy}{20}$$

$$= \frac{1}{100}\int_0^5 (40-8y)(10+y)dy = \frac{1}{100}\int_0^5 400 + 40y - 80y - 8y^2 = \frac{1}{100}\int_0^5 400 - 40y - 8y^2$$

$$= \frac{8}{100}\int_0^5 50 - 5y - y^2 = 0.08 * \left[50y - 5\frac{y^2}{2} - \frac{y^3}{3}\right]_0^5 = 0.08 * \left[50(5) - 5(12.5) - \frac{125}{3}\right]$$

$$= 0.08 * (145.833) = 11.667 \implies 10 + 1.667 \implies 10 + 5/3 \implies 10 + h/3$$

- Hence, $\bar{y}$ for a triangle is h/3

4.1.3 Finding centroid for area under a given curve

- Consider a vertical strip of thickness(dx) at a distance of x from origin and height(y)=f(x)

- The area of this element will be dA = ydx = f(x)dx

- Integrate this small area over the entire width of the curve and find Total area(A).

- Use the formula, $\bar{x} = \dfrac{\int x dA}{A}$ and $\bar{y} = \dfrac{\int y dA}{A}$

4.2 Moment of Inertia (I)

- Normally Inertia means, from Newton's first law, it is the resistance to change of state.

- Moment of Inertia are of various types.
    - Area Moment of Inertia - Resistance to Bending
    - Mass Moment of Inertia - Resistance to Rotation (based on mass distribution)
    - Polar Moment of Inertia - Resistance to Twisting (based on Perpendicular axis theorem)
    - Product of Inertia - Resistance to Rotation (based on two perpendicular axes)

- Here we are concerned with Area MOI and Polar MOI

- Area MOI is also called as 2nd moment of Area.

- Area MOI is always Non-zero and Positive

- Area MOI about x-axis = $\boxed{I_x = \int y^2 dA}$

- Area MOI about y-axis = $\boxed{I_y = \int x^2 dA}$

- The further a material is spread from its bending axis, the more resistant it is towards bending. See the figure below for better understanding.
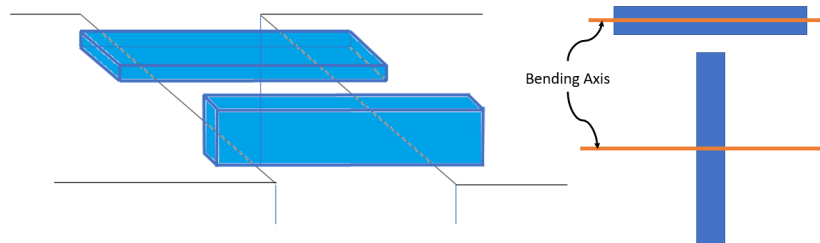

Bending Axis

- In the figure above, the first bar is flatly laid. When a load is acting at its center, its more prone to bending and hence breaking compared to the same bar that is laid in a different position and subjected to the same load.

- Because, in the first bar, the material is less spread from the bending axis, compared to the second bar

- This demonstrates why computing MOI is very important to consider the safety of structures.

4.2.1 Steps to find Area MOI for any area

- Consider elementary strip (horizontal strip for I_x and vertical strip for I_y) and find it's area (dA)

- Use the above two formula with appropriate limits

4.2.2 Finding Area MOI for a rectangle about its centroidal axis

- Remember: ($\int_{-a}^{a}(even\ function) = 2\int_{0}^{a}(even\ function)$)
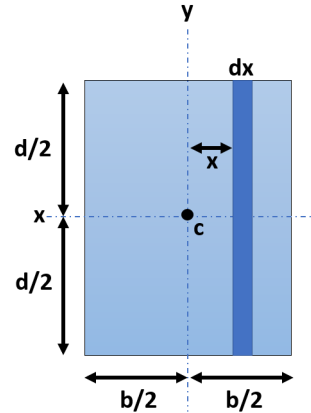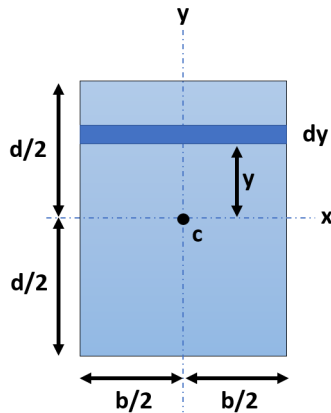
$$dA = b(dy)$$
$$We\ know\ I_x = \int y^2 dA$$
$$I_x = \int_{-d/2}^{d/2} y^2(bdy) = 2b\int_{0}^{d/2} y^2 dy$$
$$I_x = 2b\left(\frac{y^3}{3}\right)_0^{d/2} = 2b\left(\frac{d^3}{8*3}\right)$$
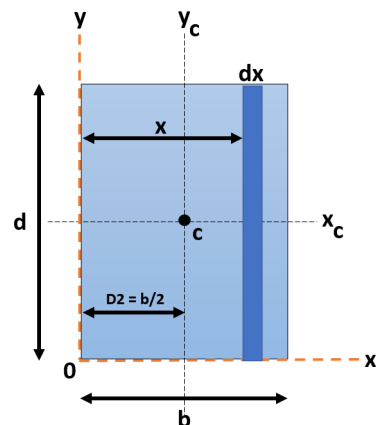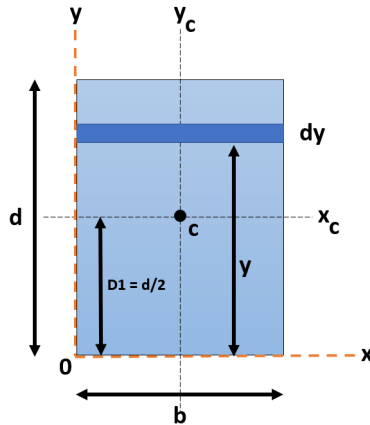$$\boxed{I_x = \frac{bd^3}{12}}$$

$$dA = d(dx)$$
$$We\ know\ I_y = \int x^2 dA$$
$$I_y = \int_{-b/2}^{b/2} x^2(d(dx)) = 2d\int_{0}^{b/2} x^2 dx$$
$$I_y = 2d\left(\frac{x^3}{3}\right)_0^{b/2} = 2d\left(\frac{b^3}{8*3}\right)$$
$$\boxed{I_y = \frac{db^3}{12}}$$

4.2.3 Finding Area MOI for a rectangle with origin at bottom left corner

$$I_x = \int_{0}^{d} y^2(bdy) = b\int_{0}^{d} y^2 dy$$
$$I_x = b\left[\frac{y^3}{3}\right]_0^d = \frac{bd^3}{3}$$

$$I_y = \int_{0}^{b} x^2(d(dx)) = d\int_{0}^{b} x^2 dx$$
$$I_y = d\left[\frac{x^3}{3}\right]_0^b = \frac{db^3}{3}$$

- The above can also be solved using Parallel axis theorem, since the axis I_x is parallel to the centroidal axis I_{x_c} and the axis I_y is parallel to the centroidal axis I_{y_c}

4.3 Parallel axis theorem

- $\boxed{I_x = I_{xc} + Ad_1^2}$ $\boxed{I_y = I_{yc} + Ad_2^2}$ $\boxed{I_{xy} = I_{x_c y_c} + Ad_1 d_2}$

- d_1 = Distance of between x-axis and centroidal x-axis

- d_2 = Distance between y-axis and centroidal y-axis

4.3.1 Finding Area MOI of rectangle with origin at bottom right corner using Parallel axis theorem

- We have derived that Area MOI of the rectangle(b*d) about its centroidal axis is:

- $\boxed{I_{x_c} = \dfrac{bd^3}{12}}$ $\boxed{I_{y_c} = \dfrac{db^3}{12}}$

- Now using the above result, we can find I_x and I_y that is parallel to their respective centroidal axis as follows:

$$I_x = I_{x_c} + Ad_1^2 \impliedby (Here\ d_1 = d/2)$$
$$I_x = \frac{bd^3}{12} + (b*d)\left(\frac{d}{2}\right)^2$$
$$I_x = \frac{bd^3}{12} + \frac{bd^3}{4}$$
$$\implies \boxed{I_x = \frac{bd^3}{3}}$$

$$I_y = I_{y_c} + Ad_2^2 \impliedby (Here\ d_2 = b/2)$$
$$I_y = \frac{db^3}{12} + (b*d)\left(\frac{b}{2}\right)^2$$
$$I_y = \frac{db^3}{12} + \frac{db^3}{4}$$
$$\implies \boxed{I_y = \frac{db^3}{3}}$$

- From the above, it is evident how necessary and useful it is to remember the Area MOI of regular shapes about their centroidal axis as it can greatly help in deriving their Area MOI about any axis that is parallel to their centroidal axis.

4.4 Product of Inertia (I_{xy})

- If $I_{xy} = 0$, then that axis is called **Principal axis**

- Unlike Area MOI, Product of Inertia can be +ve, 0, -ve

- Product of Inertia about any symmetrical axis will be zero
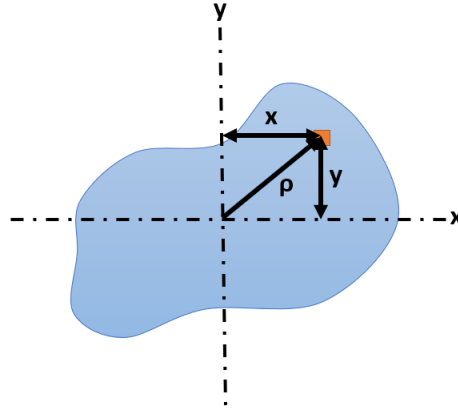
- $\boxed{I_{xy} = \displaystyle\int xy\,dA}$
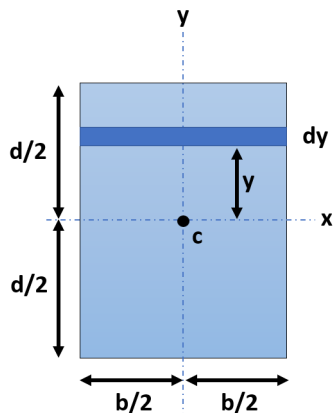
$$I_{x_c y_c} = \int \bar{x}\bar{y}\,dA = \int_{-d/2}^{d/2} \left(\frac{b}{2}\right) y(b\,dy)$$

$$I_{x_c y_c} = b^2 \left[\frac{y^2}{2}\right]_{-d/2}^{d/2}$$

$$I_{x_c y_c} = b^2(0) = 0 \quad (As\ x_c\ and\ y_c\ are\ Symmetrical\ axis)$$

4.5 Perpendicular axis theorem

- The Polar moment of Inertia (I_z) is based on the perpendicular axis theorem which defines an Area MOI wrt to an axis that is perpendicular to the cross-sectional area

- $\boxed{I_z = I_x + I_y} \implies I_z = \int \rho^2 dA = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA = I_y + I_x$

- It is based on Pythagorean theorem

- This can be used to calculate indirectly the Area MOI of a circle of radius R about its centroidal axes

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4.5.1 Finding Area MOI of a circle about its centroidal axes using I_z
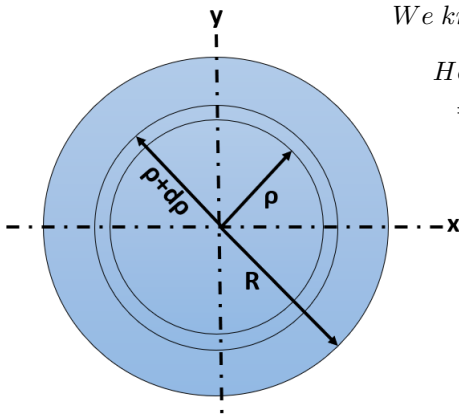
$$We\ know,\ I_z = \int \rho^2 dA$$

$$Here,\ dA = \pi(\rho + d\rho)^2 - \pi\rho^2 = \pi(\rho^2 + (d\rho)^2 + 2\rho d\rho - \rho^2)$$

$$\implies dA = \pi 2\rho d\rho$$

$$I_z = \int_0^R \rho^2(\pi 2\rho d\rho) = 2\pi \int_0^R \rho^3 d\rho$$

$$I_z = 2\pi \left[\frac{\rho^4}{4}\right]_0^R = \frac{\pi R^4}{2}$$

$$I_z = I_x + I_y \implies \frac{\pi R^4}{2} = \frac{\pi R^4}{4} + \frac{\pi R^4}{4}$$

$$So,\ \boxed{I_x = I_y = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}}$$

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