Mathematics - GATE

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Contents

Chapter 1

Linear Algebra

Chapter 2

Differential and Integral Calculus

Chapter 3

Vector Calculus

Chapter 4

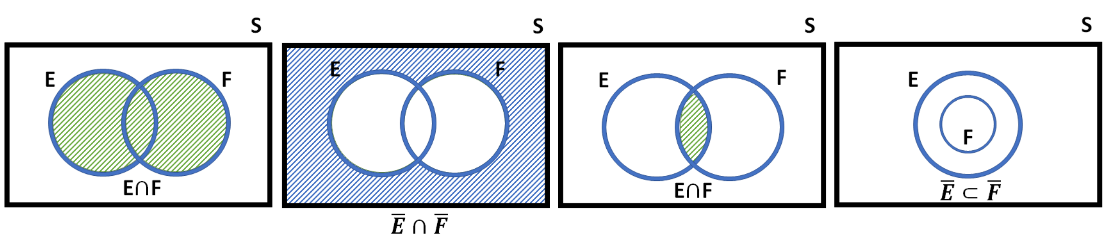
Differential Equations

Chapter 5

Probability

5.1 Introduction and Basics

- **Random Experiment** : An experiment in which the result is unpredictable, but the possibilities of the result is known. Eg. When you throw a coin, you don't know its gonna be head or tail. But you know it will be either one of them.
- **Sample Space(S)** : A set of all possible outcomes of a random experiment is called its sample space. For the above mentioned coin toss example the sample space is s:{H,T} $\boxed{P(s) = 1}$
- **Event** : A subset of the sample space is called as event. For the above mentioned example, the event can be in which the coin lands as H, or the event can be in which the coin lands as tail. Another example: Throwing a fair dice, S={1,2,3,4,5,6}. An event can be in which the number on the top of the dice will be even, So $\implies$ {2,4,6}. $\boxed{0 \leq P(E) \leq 1}$
  - **Simple event:** Only one element is possible in the event. Eg. coin toss $\implies$ {H} or {T}
  - **Compound event:** More than one element is possible in the event. Eg. Even number on the dice top.
- **Union($\cup$)** Let S={1,2,3,4,5,6,7,8,9,10} If three events are described,say:
  - A = should be a multiple of 2 = {2,4,6,8,10}
  - B = should be a multiple of 3 = {3,6,9}
  - C = Either a multiple of 2 or 3 = {2,3,4,6,8,9,10}
  - The event C can be written as $(\boldsymbol{A} \cup \boldsymbol{B})$ as it contains all the elements in A as well as all the elements in B
- **Intersection($\cap$)** For the same sample space mentioned above, let there be a fourth event D such that D = should be a multiple of both 2 and 3 = {6}. Then D can be written as $(\boldsymbol{A} \cap \boldsymbol{B})$



- **Permutations(^nP_r)**
  - Permutation means **Arrangement** $\boxed{^nP_r = \dfrac{n!}{(n-r)!}}$ $\Longleftarrow$ (Ways to arrange r things out of n things)
- **Combinations(^nC_r)**
  - Combination means **Selection** $\boxed{^nC_r = \dfrac{n!}{(n-r)!r!} = \dfrac{^nP_r}{r!}}$ $\Longleftarrow$ (Ways to select r things out of n things)

5.2 Types of Events

5.2.1 Complementary events

- If S={1,2,3,4,5} and A = even numbers - {2,4}, then the complimentary even of A is denoted by $\bar{A}$={1,3,5}

5.2.2 Equally likely events

- Two events are equally likely events when the probability for both the events are same. For Eg. Getting a head or a tail on a coin toss is 50/50

5.2.3 Mutually Exclusive events

- Two events are mutually exclusive if when one event occurs, the other cannot occur simultaneously. For eg. you can get only a head or a tail but not both on a coin toss on a single try.
- $\boxed{p(E_1 \cup E_2) = p(E_1) + p(E_2)} \Longleftarrow$ (E_1 and E_2 are mutually exclusive) $\Longrightarrow \boxed{p(E_1 \cap E_2) = 0}$

5.2.4 Collectively Exhaustive events

- If the union of two events give the sample space, then they are called collectively exhaustive events

5.2.5 Independent events

- When the occurrence of one event doesn't affect the occurrence of another event. For eg. In a bag of 5 red balls and 3 black balls, if two balls are drawn at random one at a time **with replacement**, then the probability of getting say 2 red balls is not affected by whether you got red ball on the first pick or not.
- If it had been **without replacement**, then the probability will change
- If two events are independent, then conditional probability becomes marginal probability
- $\Longrightarrow p(A/B) = p(A), \;\; p(B/A) = p(B) \;\; \boxed{p(A \cap B) = p(A) * p(B)}$

5.3 De Morgan's Law

- $\boxed{\overline{(E_1 \cup E_2)} = (\bar{E_1} \cap \bar{E_2})}$ $\boxed{\overline{(E_1 \cap E_2)} = (\bar{E_1} \cup \bar{E_2})}$ $\boxed{p(\bar{E_1} \cap \bar{E_2}) \; i.e., (\; Neither\; E_1 \; nor \; E_2) = 1 - p(E_1 \cup E_2)}$

5.4 Approaches to Probability

5.4.1 Classical Approach

- $\boxed{p(E) = \frac{n(E)}{n(S)}} \Longleftarrow$ (Classical approach assumes that all outcomes are equally likely)

5.5 Rules of Probability

1. $\boxed{p(A \cup B) = p(A) + p(B) - p(A \cap B)} \Longleftarrow$ **(Inclusion-Exclusion rule)**
2. $\boxed{p(A \cup B) = p(A) * p(B/A) = p(B) * p(A/B) \;\; \Longleftarrow}$ **(Conditional Probability)**
   - Here, p(A) and p(B) are called Marginal probabilities
   - p(A/B) and p(B/A) are called Conditional probabilities
   - p(A/B) = Probability of occurrence of A when B has already occurred
   - p(B/A) = Probability of occurrence of B when A has already occurred

- Conditions for three events A,B,C to be independent:
  - $\boxed{\text{p(ABC) = p(A)p(B)p(C)}}$ $\boxed{\text{p(AB) = p(A)p(B)}}$ $\boxed{\text{p(BC) = p(B)p(C)}}$ $\boxed{\text{p(AC) = p(A)(C)}}$

3. $\boxed{p(A) = 1 - p(\bar{A})}$ $\boxed{p(A) + p(\bar{A}) = 1}$ $\Longleftarrow$ **(Complimentary probability)**

4. $\boxed{p(A/B) = \dfrac{p(A \cap B)}{p(B)}}$ $\boxed{p(B/A) = \dfrac{p(A \cup B)}{p(A)}}$ $\Longleftarrow$ **(Conditional probability - multiplication rule)**

5. $\boxed{p(E) = p(A \cap E) + p(B \cap E) = p(A) * (p(E/A) + p(B) * p(E/B))}$ $\Longleftarrow$ **(Total probability)**

   - If an event E can occur in two ways A and B, then the total probability for the occurrence of E is the sum the probability of it happening by A and the probability of E happening by B.

6. $\boxed{p\left(E_i/A\right) = \dfrac{p(E_i \cap A)}{A} = \dfrac{p(E_i) * p(E_i/A)}{\sum_{k=1}^{n} p(E_k).p(A/E_k)}}$ $\Longleftarrow$ **(Baye's Theorem)**

Chapter 6

Statistics

6.1 Basics and Introduction

- A branch of mathematics that gives us the means to work with large datasets and derive meaningful results from it
- Descriptive measures:
  - Measure of Central tendency: indicates the avg value of data
  - Measure of dispersion: denotes about the extent to which data items deviate from the central tendency value. In other words, "It quantifies the variation in data"

6.2 Arithmetic Mean($\bar{x}$)

- $\boxed{\bar{x} = \frac{\sum x}{n}}$ $\Longleftarrow$ (for raw data) $\boxed{\bar{x} = \frac{\sum (f_i x_i)}{\sum f_i}}$ $\Longleftarrow$ (for Grouped data)

6.3 Median

6.3.1 Median for Raw data

- First arrange the elements in Ascending order
- $\boxed{\left(\frac{n+1}{2}\right)^{th} term}$ $\Longleftarrow$ (n=odd) $\boxed{\frac{\left(\frac{n}{2}\right)^{th} term + \left(\frac{n}{2}+1\right)^{th} term}{2}}$ $\Longleftarrow$ (n=even)

6.3.2 Median for Grouped data

$$\text{Median} = \boxed{L + \left[\frac{\left(\frac{N+1}{2}\right) - (F+1)}{f_m}\right] * h}$$

L = Lower limit of median class

N = Total number of data items = $\sum F$

F = cumulative frequency of the class immediately preceding the median class

f_m = Frequency of the median class

h = width off the median class

- Consider the following example for calculation of Median in grouped data:

Mark Range	f (No. of students)	Cumulative frequency
(00-20)	02	02
(21-40)	03	05
(41-60)	10	15
(61-80)	15	30
(81-100)	20	50

- In the above example, $\left(\frac{N+1}{2}\right) = \frac{(2+3+10+15+20)+1}{2} = 25.5$
- Class (61-80) is the median class as $30 > 25.5$
- Here, $\boxed{\text{L=61}}$ $\boxed{\text{h=20}}$ $\boxed{\text{F=15}}$ $\boxed{f_m = 15}$ So, Median $= 60 + \frac{25.5 - (15+1)}{15} * 20 = 69.66 \approx 69.7$

6.4 Mode

- Mode is the value that occurs most frequently in the data. If more than one data is occuring most frequently, then both of them are mode.

6.4.1 Mode of Raw data

- Found using observation

6.4.2 Mode of Grouped data

$$\boxed{Mode = L + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} * h}$$

L = Lower limit of the modal class
f_0 = Modal frequency i.e. Largest frequency
f_1=Frequency in the class preceding the modal class
f_2 = Frequency in the class succeeding the modal class
h = width of the class

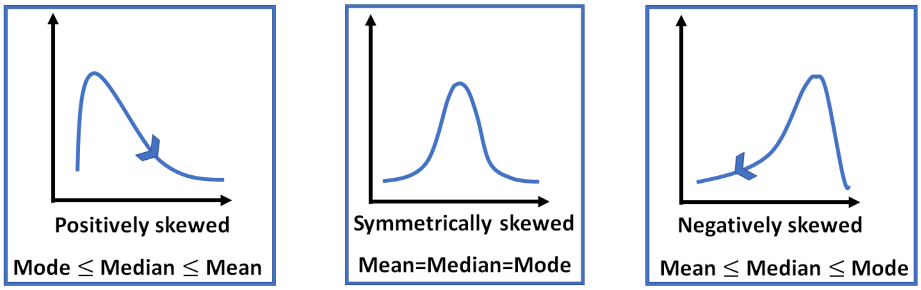
- Consider the following example for calculation of Mode in grouped data:

Height(in ft)	Number of students
3.0 - 3.5	12
3.6 - 4.0	37
4.1 - 4.5	79
4.6 - 5.0	152
5.1 - 5.5	65
5.6 - 6.0	7
Total	352

- Since, 152 is the largest frequency, the modal class is (4.6 - 5.0)
- Thus, $\boxed{\text{L=4.6}}$ $\boxed{f_0 = 152}$ $\boxed{f_1 = 79}$ $\boxed{f_2 = 65}$ $\boxed{\text{h = 0.5}}$ Mode $= 4.5 * \frac{152 - 79}{2(152) - 79 - 65} * 0.5 \approx 4.73$

6.5 Properties relating Mean, Median and Mode

- When an approx value of mode is required: $\boxed{Emprical\ mode = 3*(Median) - 2*(Mean)}$



6.6 Standard Deviation and Variance

6.6.1 Standard Deviation for RAW data

- It is a measure of variation among the data
- Consider there are five values: x_1, x_2, x_3, x_4, x_5 The mean $\bar{x} = \dfrac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$.
- The individual deviations are $(x_1 - \bar{x}),\ (x_2 - \bar{x}),\ (x_3 - \bar{x}),\ (x_4 - \bar{x}),\ (x_5 - \bar{x})$
- **Variance(σ^2)** = Mean of Square of each deviation of data = $\boxed{\sigma^2 = \dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$
- **Standard deviation(σ)** = $\boxed{\sqrt{Variance} = +\sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}}$

6.6.2 Standard Deviation for Grouped data

- $\boxed{Arithmetic\ Mean(\bar{x}) = \dfrac{\sum(f_i x_i)}{\sum f_i}}$ $\boxed{Variance(\sigma^2) = \sum_{i=1}^{n}\left(\dfrac{f_i(x_i - \bar{x})^2}{f_i}\right)}$

6.7 Coefficient of Variation

- The standard deviation is an absolute measure of variation and hence cannot be used for comparison of variation between 2 different data sets with different mean.
- For the above purpose a relative measure of variation called, **Coefficient of Variation(CV)** is used. $\boxed{CV = \dfrac{\sigma}{\bar{x}}}$.

  CV is often represented as a percentage

6.8 Probability Distribution

6.8.1 Random Variable

- It is a real valued function.

- For eg. Tossing 3 coins. x be the real valued function denoting the number of heads.

x	0	1	2	3
p(x)	1/8	3/8	3/8	1/8

- It is of two types: Discrete and Continuous

6.8.2 Discrete Random Variable

- eg. x = function representing the possible sum of values of two fair dice.
- x can have only discrete values of any one of {2,3,4,5,6,7,8,9,10,11,12}

6.8.3 Continuous Random Variable

- eg. x = function representing the volume of a water in a container
- In this case, x can have any value between 0 to the volume.

6.8.4 Probability Density Function(F(x))

- $\boxed{F(x) \geq 0}$ $\boxed{\int_{-\infty}^{\infty} F(x)dx = 1}$ $\boxed{p(a < x < b) = \int_a^b F(x)dx}$
- In simple terms, it tells us the likelihood of a continuous random variable having a value between any two points, rather than having a specific value as in the case of a discrete random variable.
- The pdf assigns a value between 0 and 1 to every possible value of the continuous random variable, such that the integral of the pdf over the entire range of the random variable is equal to 1.
- This means that the area under the curve of the pdf represents the total probability of the random variable being in that range of values.

6.8.5 Probability Mass Function(p(x))

- $\boxed{p(x) = P[X = x]}$ $\boxed{p(x) \geq 0}$ $\boxed{\sum p(x) = 1}$
- In simple terms, it gives the exact probability of a discrete random variable having a specific value. The PMF assigns a probability between 0 and 1 to each possible value of the discrete random variable such that the sum of the probabilities for all possible values is equal to 1.
- In other words, the PMF gives the probability distribution of a discrete random variable and allows us to see the likelihood of each possible outcome.

6.9 Distributions

- Distributions are also of two types based on the above:
  - Discrete distribution: Poisson, Hypergeometric
  - Continuous distribution: Uniform, Normal and Exponential

6.9.1 Properties of Discrete Distribution

- $\boxed{\sum P(x) = 1}$ $\boxed{E(x) = \sum xp(x)}$ $\boxed{V(x) = E(x^2) - (E(x))^2}$ (E(x) = Expected value, V(x) = Variance of RV)

6.9.2 Properties of Continuous Distribution

- $\boxed{\int_{-\infty}^{\infty} f(x)dx = 1}$ $\boxed{F(x) = CDF = \int_{-\infty}^{x} f(x)dx}$ $\boxed{E(x) = \int_{-\infty}^{\infty} xf(x)dx}$
- CDF = Cumulative Distribution Function.
- CDF gives us the cumulative probability of a random variable being less than or equal to a certain value, while the PDF gives us the likelihood of a random variable taking on a specific value within a certain range.
- $\boxed{p(a < x < b) = p(a \leq x < b) = p(a < x \leq b) = p(a \leq x \leq b) = \int_{a}^{b} f(x)dx}$

6.9.3 Properties of Expectation(E(x)) and Variance(V(x))

- $\boxed{E(c) = c}$ $\boxed{V(c) = 0}$ $\boxed{E(cx) = cE(x) \implies E(-x) = -E(x)}$ $\Longleftarrow$ (c = constant)
- $\boxed{V(cx) = c^2V(x) \implies V(-x) = (-1)^2V(x) = V(x)}$ $\boxed{V(c_1x_1 + c_2x_2) = c_1^2V(x_1) + c_2^2V(x_2) + 2c_1c_2cov(x_1, x_2)}$
- If (x_1, x_2) are independent variables, then $cov(x_1, x_2) = 0$
- $\boxed{V(x_1 + x_2) = V(x_1 - x_2) = V(x_1) + V(x_2)}$
- $\boxed{E(X + Y) = E(X) + E(Y)}$ $\Longleftarrow$ (X,Y are two random variables)
- $\boxed{E(XY) = E(x)E(Y)}$ $\boxed{V(X + Y) = V(X) + V(Y)}$ $\Longleftarrow$ (X,Y are two independent Random Variables)

6.10 Discrete Distributions

6.10.1 General Discrete Distribution

- Let X be the number which comes on a single throw of dice. The probability distribution table is given by

x	1	2	3	4	5	6
p(x)	1/6	1/6	1/6	1/6	1/6	1/6

- For this case, p(x) for all x is same. But this will not be the case always.
- But, $\sum p(x) = 1$ always
- From the above table, we can easily calculate E(x), V(x).
- E(x) is the expected value of x and is similar to the average value of x after infinite trials. So E(x) is sometimes written as μ_x
- V(x) represents the variability of X. So it is written sometimes as σ_x^2
- $\boxed{cov(x, y) = E(xy) - E(x)E(y)}$ $\boxed{cov(x, y) = 0}$ $\Longleftarrow$ (If x,y are independent Random variables)

6.10.2 Binomial Distribution (n,p)

- Consider a trial experiment of n independent trials
- each of which will result in success with probability p or failure with probability 1-p
- If X represents the number of successes that occur in the n trials, then X is the Binomial Random variable with parameters (n,p)
- Conditions to be satisfied for Binomial distribution:
  - Only 2 outcomes are possible: success or failure
  - The probability of outcome should remain same from trial to trial
  - The outcome of one trial should not affect the outcome of another.
- Then the probability of obtaining x successes from n trials is given by: $\boxed{p(X=x) = ({}^{n}C_{x})p^{x}(1-p)^{n-x}}$
- Here, $\boxed{E(X) = np}$ $\boxed{V(x) = np(1-p)}$
- By **Recurrence relation:** $\boxed{p(X=(x+1)) = \frac{(n-x)p}{(n+x)(1-p)}p(x)}$

6.10.3 Hypergeometric Distribution

- If probability changes from trial to trial, then one condition of the binomial distribution is violated. In such a case, Hypergeometric distribution is used. **Mainly used in case of sampling without replacement from a finite population**
- For eg. Consider the situation:
  - There are 10 markers on a table - 6 are defective(D), 4 are NonDefective(ND)
  - 3 are randomly taken without replacement
  - Find the probability of exactly one marker being defective
- Since it is without replacement, after each draw(trial), the probability will change.
- Solution: Let X denote the num of defective marker. $\implies$ $\boxed{p(X=x) = \frac{({}^{6}C_{x})({}^{4}C_{3-x})}{{}^{1}0C_{3}}}$
- With the above formula for the above problem, we can easily now find the solution as well find other solutions like: At least one marker being defective $\implies$ $(p(X=(x \geq 1)) = p(X=0) + p(X=1))$

  Generalized formula for Hypergeometric distribution:

  - Consider n objects of with r are of type 1 and (n-r) are of type 2. From this w objects are drawn.

$$\boxed{p(X=x) = \frac{({}^{r}C_{x})({}^{n-r}C_{w-x})}{{}^{n}C_{w}}} \quad \boxed{E(x) = n * \left(\frac{r}{n}\right)}$$

6.10.4 Geometric Distribution (p)

- **Consider an experiment to be performed until successes is obtained**
- let the probability of success be p and probability of failure be (1-p)=q
- let x denote the number of times the experiment has to be repeated to obtain success
- The distribution of random variable x is given by:

k	1	2	3	4	...
p(k)	q^0p	q^1p	q^2p	q^3p	....

- $\boxed{p(k) = p(x-k) = q^{k-1}p}$ $\boxed{E(x) = 1/p}$ $\boxed{V(x) = \frac{q}{p^2}}$ $\boxed{CD = F(k) = -1 - q^k}$ $\boxed{p(x > r) = q^r}$

6.10.5 Poisson Distribution(λ)

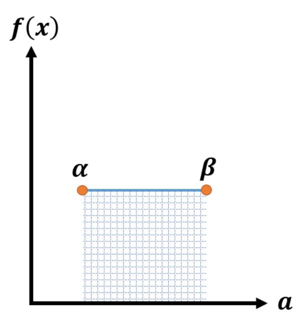
- **$\boldsymbol{\lambda}$ = average number of occurrences of event in an observation period $\boldsymbol{\Delta t}$** $\implies$ $\boxed{\lambda = \alpha \Delta t}$
- $\alpha \implies$ number of occurrence of event per unit time
- $\boxed{p(x=x) = \frac{e^{-\lambda}\lambda^x}{x!}}$ $\boxed{p(x=(x+1)) = \frac{\lambda}{x+1}p(x)}$ $\Longleftarrow$ (Recurrence relation) (obtained by p(x)/p(x+1))
- **NOTE:** Poisson distribution is used to approximate Binomial distribution when n is very large and p is very small for binomial distribution. So we'll use $\boxed{\boldsymbol{\lambda = np}}$
- For Poisson distribution: $E(x) = V(x) = \lambda$

6.11 Continuous Distribution

6.11.1 General Continuous Distribution

- Let X be a Continuous Random Variable. A continuous distribution of X can be defined by a PDF f(x): $p(-\infty \leq x \leq \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$
- We know, E(x) = $\mu_x = \int_{-\infty}^{\infty} xf(x)dx$ and V(x) = $E(x^2) - [E(x)]^2$
- The cumulative distribution function : $F(x) = p(X \leq x) = \int_{-\infty}^{x} f(x)dx$
- **NOTE:** Differentiation of the cumulative distribution function (CDF) does not necessarily give the probability density function (PDF). However, if the CDF is differentiable, then it's derivative is the PDF. This is known as the **"Probability Density Function Theorem"**

6.11.2 Uniform Distribution(α, β)



X is a Uniform Random Variable on the interval (α, β) if its PDF:

$\boxed{f(x) = \frac{1}{\beta - \alpha}}$ $\Longleftarrow$ (if $\alpha < x < \beta$) $\boxed{f(x) = 0}$ $\Longleftarrow$ (Otherwise)

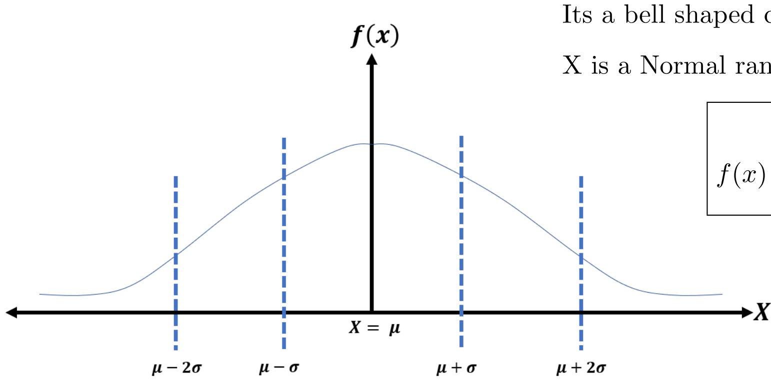
For Discrete Uniform Distribution:

Mean = $\boxed{E(x) = \frac{\beta + \alpha}{2}}$ Variance = $\boxed{V(x) = \frac{(\beta - \alpha)^2}{12}}$

6.11.3 Exponential Distribution (λ)

- A Continuous random variable whose PDF is given by:
- $\boxed{f(x) = \lambda e^{-\lambda x}}$ $\Longleftarrow$ (if $x \geq 0$) $\boxed{f(x) = 0}$ $\Longleftarrow$ (if $x < 0$)
- The CDF is given by: $F(a) = p(X \leq a) = \int_0^a \lambda e^{-\lambda x}dx = \left(-e^{-\lambda x}\right)_0^a = \boxed{1 - e^{-\lambda a}}, a \geq 0$
- In the above expression, if its for $p(a \leq x \leq b)$ then the CDF = F(b) - F(a)
- $\boxed{Mean = E(x) = 1/\lambda}$ $\boxed{Variance = V(x) = 1/\lambda^2}$

6.11.4 Normal Distribution (μ, σ^2)



Its a bell shaped curve

X is a Normal random variable if the PDF is given by:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}} \Longleftarrow (-\infty < x < \infty)$$

$$\boxed{E(x) = \mu} \quad \boxed{V(x) = \sigma^2}$$

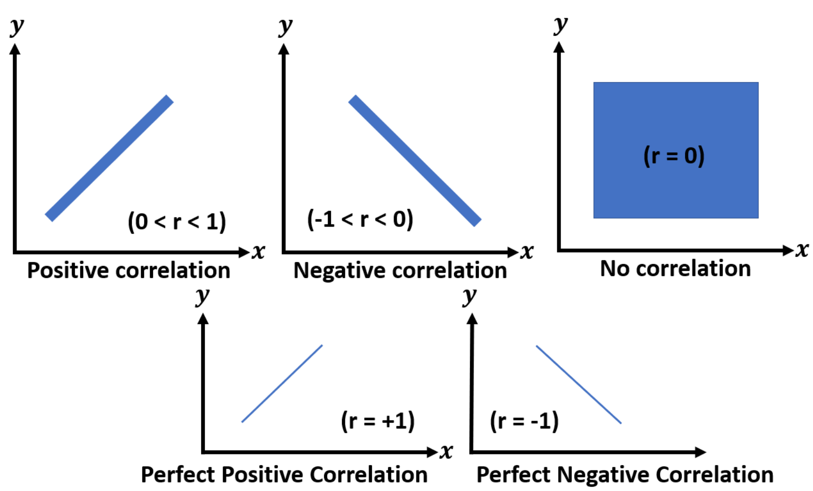
- Approximate areas: $\boxed{A_x(\mu - \sigma, \mu + \sigma) = 0.68A}$ $\boxed{A_x(\mu - 2\sigma, \mu + 2\sigma) = 0.95A}$ $\boxed{A_x(\mu - 3\sigma, \mu + 3\sigma) = 0.977A}$

6.11.5 Standard Normal Distribution

- Since in the Normal distribution, the N(μ, σ^2) varies with μ and σ^2, the integral can be evaluated only numerically
- So, its converted to Standard normal distribution N(0,1) for which, the shape, hence the integral remains constant.
- The conversion of N(μ, σ^2) to N(0,1) is effected by: $\boxed{Z = \frac{X - \mu}{\sigma}}$ $\Longleftarrow$ (Z = Standard normal variate)
- Mean E(x) = 0, Variance V(x) = 1

6.11.6 Correlation

- If there are two variables x and y, such that change in x changes y, then the two variables are said to in correlation.
- If the ratio of two variables is always constant no matter the change, then they are said to be perfect
- The correlation is represented using **Scatter or dot diagram**
- Coefficient of Correlation (r) = $\boxed{\frac{Cov(x,y)}{(\sigma_x) * (\sigma_y)} = \frac{\sum xy}{\sqrt{(\sum x^2)(\sum y^2)}}}$ $\Longleftarrow$ $(-1 \le r \le 1)$



6.11.7 Regression

- Regression analysis is the method for estimating the unknown values of one variable corresponding to the known value of another variable
- It can be a curve of regression or a line of regression
- A line of regression is the straight line which gives the best fit in the least square sense to the given frequency
- Let line of regression: y = a +bx ...(Eqn(1))
- Since it is the line of best fit, the value of a and b are given by: $\sum y = na + b \sum x$
- n = number of pairs of (x,y) $\implies \frac{\sum}{n} = a + b\frac{\sum x}{n} \implies \boxed{\bar{y} = a + b\bar{x}}_{Eqn(2)} \impliedby$ ($\bar{x}, \bar{y}$ are means of x and y respectively)
- The line of regression passes through $(\bar{x}, \bar{y})$
- Eqn(1)-Eqn(2) $= y - \bar{y} = b(x - \bar{x})$

Chapter 7

Numerical Methods

Chapter 8

Complex Numbers

Chapter 9

Transform Theory