USA
AMC 12/AHSME
2010

A

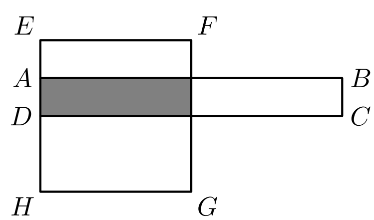
$\boxed{1}$ What is $(20-(2010-201))+(2010-(201-20))$?

(A) -4020 **(B)** 0 **(C)** 40 **(D)** 401 **(E)** 4020

$\boxed{2}$ A ferry boat shuttles tourists to an island every hour starting at 10 AM until its last trip, which starts at 3 PM. One day the boat captain notes that on the 10 AM trip there were 100 tourists on the ferry boat, and that on each successive trip, the number of tourists was 1 fewer than on the previous trip. How many tourists did the ferry take to the island that day?

(A) 585 **(B)** 594 **(C)** 672 **(D)** 679 **(E)** 694

$\boxed{3}$ Rectangle $ABCD$, pictured below, shares 50



(A) 4 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 10

$\boxed{4}$ If $x<0$, then which of the following must be positive?

(A) $\frac{x}{|x|}$ **(B)** $-x^2$ **(C)** -2^x **(D)** $-x^{-1}$ **(E)** $\sqrt[3]{x}$

$\boxed{5}$ Halfway through a 100-shot archery tournament, Chelsea leads by 50 points. For each shot a bullseye scores 10 points, with other possible scores being $8,4,2,0$ points. Chelsea always scores at least 4 points on each shot. If Chelsea's next n shots are bulleyes she will be guaranteed victory. What is the minimum value for n?

(A) 38 **(B)** 40 **(C)** 42 **(D)** 44 **(E)** 46

$\boxed{6}$ A *palindrome*, such as 83438, is a number that remains the same when its digits are reversed. The numbers x and $x+32$ are three-digit and four-digit palindromes, respectively. What is the sum of the digits of x?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

$\boxed{7}$ Logan is constructing a scaled model of his town. The city's water tower stands 40 meters high, and the top portion is a sphere that holes $100,000$ liters of water. Logan's miniature water tower holds 0.1 liters. How tall, in meters, should Logan make his tower?

(A) 0.04 **(B)** $\frac{0.4}{\pi}$ **(C)** 0.4 **(D)** $\frac{4}{\pi}$ **(E)** 4

$\boxed{8}$ Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?

(A) 60° **(B)** 75° **(C)** 90° **(D)** 105° **(E)** 120°

$\boxed{9}$ A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume, in cubic inches, of the remaining solid?

(A) 7 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 15

$\boxed{10}$ The first four terms of an arithmetic sequence are $p, 9, 3p - q$, and $3p + q$. What is the 2010^{th} term of the sequence?

(A) 8041 **(B)** 8043 **(C)** 8045 **(D)** 8047 **(E)** 8049

$\boxed{11}$ The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?

(A) $\frac{7}{15}$ **(B)** $\frac{7}{8}$ **(C)** $\frac{8}{7}$ **(D)** $\frac{15}{8}$ **(E)** $\frac{15}{7}$

$\boxed{12}$ In a magical swamp there are two species of talking amphibians: toads, whose statements are always true, and frogs, whose statements are always false. Four amphibians, Brian, Chris, LeRoy, and Mike live together in the swamp, and they make the following statements:

Brian: "Mike and I are different species." Chris: "LeRoy is a frog." LeRoy: "Chris is a frog." Mike: "Of the four of us, at least two are toads."

How many of these amphibians are frogs?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

$\boxed{13}$ For how many integer values of k do the graphs of $x^2 + y^2 = k^2$ and $xy = k$ <u>not</u> intersect?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 8

$\boxed{14}$ Nondegenerate $\triangle ABC$ has integer side lengths, BD is an angle bisector, $AD = 3$, and $DC = 8$. What is the smallest possible value of the perimeter?

(A) 30 **(B)** 33 **(C)** 35 **(D)** 36 **(E)** 37

15 A coin is altered so that the probability that it lands on heads is less than $\frac{1}{2}$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $\frac{1}{6}$. What is the probability that the coin lands on heads?

(A) $\frac{\sqrt{15}-3}{6}$ **(B)** $\frac{6-\sqrt{6\sqrt{6}+2}}{12}$ **(C)** $\frac{\sqrt{2}-1}{2}$ **(D)** $\frac{3-\sqrt{3}}{6}$ **(E)** $\frac{\sqrt{3}-1}{2}$

16 Bernardo randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

(A) $\frac{47}{72}$ **(B)** $\frac{37}{56}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{49}{72}$ **(E)** $\frac{39}{56}$

17 Equiangular hexagon $ABCDEF$ has side lengths $AB = CD = EF = 1$ and $BC = DE = FA = r$. The area of $\triangle ACE$ is 70

(A) $\frac{4\sqrt{3}}{3}$ **(B)** $\frac{10}{3}$ **(C)** 4 **(D)** $\frac{17}{4}$ **(E)** 6

18 A 16-step path is to go from $(-4, -4)$ to $(4, 4)$ with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \le x \le 2$, $-2 \le y \le 2$ at each step?

(A) 92 **(B)** 144 **(C)** 1568 **(D)** 1698 **(E)** 12,800

19 Each of 2010 boxes in a line contains a single red marble, and for $1 \le k \le 2010$, the box in the kth position also contains k white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let $P(n)$ be the probability that Isabella stops after drawing exactly n marbles. What is the smallest value of n for which $P(n) < \frac{1}{2010}$?

(A) 45 **(B)** 63 **(C)** 64 **(D)** 201 **(E)** 1005

20 Arithmetic sequences (a_n) and (b_n) have integer terms with $a_1 = b_1 = 1 < a_2 \le b_2$ and $a_n b_n = 2010$ for some n. What is the largest possible value of n?

(A) 2 **(B)** 3 **(C)** 8 **(D)** 288 **(E)** 2009

21 The graph of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ lies above the line $y = bx + c$ except at three values of x, where the graph and the line intersect. What is the largest of those values?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

22 What is the minimum value of $f(x) = |x-1| + |2x-1| + |3x-1| + \cdots + |119x-1|$?

(A) 49 **(B)** 50 **(C)** 51 **(D)** 52 **(E)** 53

23 The number obtained from the last two nonzero digits of $90!$ is equal to n. What is n?

(A) 12 **(B)** 32 **(C)** 48 **(D)** 52 **(E)** 68

$\boxed{24}$ Let $f(x) = \log_{10}(\sin(\pi x) \cdot \sin(2\pi x) \cdot \sin(3\pi x) \cdots \sin(8\pi x))$. The intersection of the domain of $f(x)$ with the interval $[0,1]$ is a union of n disjoint open intervals. What is n?

(A) 2 **(B)** 12 **(C)** 18 **(D)** 22 **(E)** 36

$\boxed{25}$ Two quadrilaterals are considered the same if one can be obtained from the other by a rotation and a translation. How many different convex cyclic quadrilaterals are there with integer sides and perimeter equal to 32?

(A) 560 **(B)** 564 **(C)** 568 **(D)** 1498 **(E)** 2255

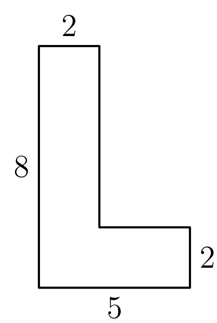
USA
AMC 12/AHSME
2010

B

$\boxed{1}$ Makayla attended two meetings during her 9-hour work day. The first meeting took 45 minutes and the second meeting took twice as long. What percent of her work day was spent attending meetings?

(A) 15 **(B)** 20 **(C)** 25 **(D)** 30 **(E)** 35

$\boxed{2}$ A big L is formed as shown. What is its area?



(A) 22 **(B)** 24 **(C)** 26 **(D)** 28 **(E)** 30

$\boxed{3}$ A ticket to a school play costs x dollars, where x is a whole number. A group of 9th graders buys tickets costing a total of \$48, and a group of 10th graders buys tickets costing a total of \$64. How many values of x are possible?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

$\boxed{4}$ A month with 31 days has the same number of Mondays and Wednesdays. How many of the seven days of the week could be the first day of this month?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

$\boxed{5}$ Lucky Larry's teacher asked him to substitute numbers for a, b, c, d, and e in the expression $a - (b - (c - (d + e)))$ and evaluate the result. Larry ignored the parentheses but added and subtracted correctly and obtained the correct result by coincidence. The numbers Larry substituted for a, b, c, and d were 1, 2, 3, and 4, respectively. What number did Larry substitute for e?

(A) -5 **(B)** -3 **(C)** 0 **(D)** 3 **(E)** 5

6 At the beginning of the school year, 50% of all students in Mr. Well's math class answered "Yes" to the question "Do you love math", and 50% answered "No." At the end of the school year, 70% answered "Yes" and 30% answered "No." Altogether, $x\%$ of the students gave a different answer at the beginning and end of the school year. What is the difference between the maximum and the minimum possible values of x?

(A) 0 **(B)** 20 **(C)** 40 **(D)** 60 **(E)** 80

7 Shelby drives her scooter at a speed of 30 miles per hour if it is not raining, and 20 miles per hour if it is raining. Today she drove in the sun in the morning and in the rain in the evening, for a total of 16 miles in 40 minutes. How many minutes did she drive in the rain?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

8 Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37th and 64th, respectively. How many schools are in the city?

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

9 Let n be the smallest positive integer such that n is divisible by 20, n^2 is a perfect cube, and n^3 is a perfect square. What is the number of digits of n?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

10 The average of the numbers $1, 2, 3, ..., 98, 99$, and x is $100x$. What is x?

(A) $\frac{49}{101}$ **(B)** $\frac{50}{101}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{51}{101}$ **(E)** $\frac{50}{99}$

11 A palindrome between 1000 and $10,000$ is chosen at random. What is the probability that it is divisible by 7?

(A) $\dfrac{1}{10}$ **(B)** $\dfrac{1}{9}$ **(C)** $\dfrac{1}{7}$ **(D)** $\dfrac{1}{6}$ **(E)** $\dfrac{1}{5}$

12 For what value of x does

$$\log_{\sqrt{2}}\sqrt{x} + \log_2 x + \log_4(x^2) + \log_8(x^3) + \log_{16}(x^4) = 40?$$

(A) 8 **(B)** 16 **(C)** 32 **(D)** 256 **(E)** 1024

13 In $\triangle ABC$, $\cos(2A - B) + \sin(A + B) = 2$ and $AB = 4$. What is BC?

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $2\sqrt{2}$ **(E)** $2\sqrt{3}$

14 Let a, b, c, d, and e be positive integers with $a + b + c + d + e = 2010$, and let M be the largest of the sums $a + b$, $b + c$, $c + d$, and $d + e$. What is the smallest possible value of M?

(A) 670 **(B)** 671 **(C)** 802 **(D)** 803 **(E)** 804

$\boxed{15}$ For how many ordered triples (x, y, z) of nonnegative integers less than 20 are there exactly two distinct elements in the set $\{i^x, (1+i)^y, z\}$, where $i = \sqrt{-1}$?

(A) 149 **(B)** 205 **(C)** 215 **(D)** 225 **(E)** 235

$\boxed{16}$ Positive integers a, b, and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \ldots, 2010\}$. What is the probability that $abc + ab + a$ is divisible by 3?

(A) $\dfrac{1}{3}$ **(B)** $\dfrac{29}{81}$ **(C)** $\dfrac{31}{81}$ **(D)** $\dfrac{11}{27}$ **(E)** $\dfrac{13}{27}$

$\boxed{17}$ The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

(A) 18 **(B)** 24 **(C)** 36 **(D)** 42 **(E)** 60

$\boxed{18}$ A frog makes 3 jumps, each exactly 1 meter long. The directions of the jumps are chosen independently and at random. What is the probability the the frog's final position is no more than 1 meter from its starting position?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

$\boxed{19}$ A high school basketball game between the Raiders and Wildcats was tied at the end of the first quarter. The number of points scored by the Raiders in each of the four quarters formed an increasing geometric sequence, and the number of points scored by the Wildcats in each of the four quarters formed an increasing arithmetic sequence. At the end of the fourth quarter, the Raiders had won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

(A) 30 **(B)** 31 **(C)** 32 **(D)** 33 **(E)** 34

$\boxed{20}$ A geometric sequence (a_n) has $a_1 = \sin x, a_2 = \cos x$, and $a_3 = \tan x$ for some real number x. For what value of n does $a_n = 1 + \cos x$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

$\boxed{21}$ Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$$P(1) = P(3) = P(5) = P(7) = a, \text{ and}$$
$$P(2) = P(4) = P(6) = P(8) = -a.$$

What is the smallest possible value of a?

(A) 105 **(B)** 315 **(C)** 945 **(D)** 7! **(E)** 8!

$\boxed{22}$ Let $ABCD$ be a cyclic quadrilateral. The side lengths of $ABCD$ are distinct integers less than 15 such that $BC \cdot CD = AB \cdot DA$. What is the largest possible value of BD?

(A) $\sqrt{\frac{325}{2}}$ **(B)** $\sqrt{185}$ **(C)** $\sqrt{\frac{389}{2}}$ **(D)** $\sqrt{\frac{425}{2}}$ **(E)** $\sqrt{\frac{533}{2}}$

$\boxed{23}$ Monic quadratic polynomials $P(x)$ and $Q(x)$ have the property that $P(Q(x))$ has zeroes at $x = -23, -21, -17, \text{and} -15$, and $Q(P(x))$ has zeroes at $x = -59, -57, -51, \text{and} -49$. What is the sum of the minimum values of $P(x)$ and $Q(x)$?

(A) -100 **(B)** -82 **(C)** -73 **(D)** -64 **(E)** 0

$\boxed{24}$ The set of real numbers x for which

$$\frac{1}{x-2009} + \frac{1}{x-2010} + \frac{1}{x-2011} \geq 1$$

is the union of intervals of the form $a < x \leq b$. What is the sum of the lengths of these intervals?

(A) $\frac{1003}{335}$ **(B)** $\frac{1004}{335}$ **(C)** 3 **(D)** $\frac{403}{134}$ **(E)** $\frac{202}{67}$

$\boxed{25}$ For every integer $n \geq 2$, let $\text{pow}(n)$ be the largest power of the largest prime that divides n. For example $\text{pow}(144) = \text{pow}(2^4 \cdot 3^2) = 3^2$. What is the largest integer m such that 2010^m divides

$$\prod_{n=2}^{5300} \text{pow}(n)?$$

(A) 74 **(B)** 75 **(C)** 76 **(D)** 77 **(E)** 78

Albania
BMO TST
2010

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1. **a)** Is the number $1111\cdots11$ (with 2010 ones) a prime number? **b)** Prove that every prime factor of $1111\cdots11$ (with 2011 ones) is of the form $4022j+1$ where j is a natural number.

2. Let $a \geq 2$ be a real number; with the roots x_1 and x_2 of the equation $x^2-ax+1=0$ we build the sequence with $S_n = x_1^n + x_2^n$. **a)**Prove that the sequence $\frac{S_n}{S_{n+1}}$, where n takes value from 1 up to infinity, is strictly non increasing. **b)**Find all value of a for the which this inequality hold for all natural values of n $\frac{S_1}{S_2} + \cdots + \frac{S_n}{S_{n+1}} > n-1$

3. Let K be the circumscribed circle of the trapezoid $ABCD$. In this trapezoid the diagonals AC and BD are perpendicular. The parallel sides $AB = a$ and $CD = c$ are diameters of the circles K_a and K_b respectively. Find the perimeter and the area of the part inside the circle K, that is outside circles K_a and K_b.

4. Let's consider the inequality $a^3+b^3+c^3 < k(a+b+c)(ab+bc+ca)$ where a, b, c are the sides of a triangle and k a real number. **a)** Prove the inequality for $k=1$. **b)** Find the smallest value of k such that the inequality holds for all triangles.

argentina
Team Selection Test
2010

Day 1 - *29 April 2010*

$\boxed{1}$ In a football tournament there are 8 teams, each of which plays exacly one match against every other team. If a team A defeats team B, then A is awarded 3 points and B gets 0 points. If they end up in a tie, they receive 1 point each. It turned out that in this tournament, whenever a match ended up in a tie, the two teams involved did not finish with the same final score. Find the maximum number of ties that could have happened in such a tournament.

$\boxed{2}$ Let ABC be a triangle with $AB = AC$. The incircle touches BC, AC and AB at D, E and F respectively. Let P be a point on the arc EF that does not contain D. Let Q be the second point of intersection of BP and the incircle of ABC. The lines EP and EQ meet the line BC at M and N, respectively. Prove that the four points P, F, B, M lie on a circle and $\frac{EM}{EN} = \frac{BF}{BP}$.

$\boxed{3}$ Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + xy + f(y)) = \left(f(x) + \frac{1}{2}\right)\left(f(y) + \frac{1}{2}\right)$ holds for all real numbers x, y.

argentina
Team Selection Test
2010

Day 2 - *30 April 2010*

$\boxed{4}$ Two players, A and B, play a game on a board which is a rhombus of side n and angles of 60° and 120°, divided into $2n^2$ equilateral triangles, as shown in the diagram for $n = 4$. A uses a red token and B uses a blue token, which are initially placed in cells containing opposite corners of the board (the 60° ones). In turns, players move their token to a neighboring cell (sharing a side with the previous one). To win the game, a player must either place his token on the cell containing the other player's token, or get to the opposite corner to the one where he started. If A starts the game, determine which player has a winning strategy.

$\boxed{5}$ Let p and q be prime numbers. The sequence (x_n) is defined by $x_1 = 1$, $x_2 = p$ and $x_{n+1} = px_n - qx_{n-1}$ for all $n \geq 2$. Given that there is some k such that $x_{3k} = -3$, find p and q.

$\boxed{6}$ Suppose $a_1, a_2, ..., a_r$ are integers with $a_i \geq 2$ for all i such that $a_1 + a_2 + ... + a_r = 2010$. Prove that the set $\{1, 2, 3, ..., 2010\}$ can be partitioned in r subsets $A_1, A_2, ..., A_r$ each with $a_1, a_2, ..., a_r$ elements respectively, such that the sum of the numbers on each subset is divisible by 2011. Decide whether this property still holds if we replace 2010 by 2011 and 2011 by 2012 (that is, if the set to be partitioned is $\{1, 2, 3, ..., 2011\}$).

brazil
National Olympiad
2010

Day 1 - *16 October 2010*

1. Find all functions f from the reals into the reals such that

$$f(ab) = f(a+b)$$

for all irrational a, b.

2. Let $P(x)$ be a polynomial with real coefficients. Prove that there exist positive integers n and k such that k has n digits and more than $P(n)$ positive divisors.

3. What is the biggest shadow that a cube of side length 1 can have, with the sun at its peak? Note: "The biggest shadow of a figure with the sun at its peak" is understood to be the biggest possible area of the orthogonal projection of the figure on a plane.

brazil
National Olympiad
2010

Day 2 - *17 October 2010*

1. Let $ABCD$ be a convex quadrilateral, and M and N the midpoints of the sides CD and AD, respectively. The lines perpendicular to AB passing through M and to BC passing through N intersect at point P. Prove that P is on the diagonal BD if and only if the diagonals AC and BD are perpendicular.

2. Determine all values of n for which there is a set S with n points, with no 3 collinear, with the following property: it is possible to paint all points of S in such a way that all angles determined by three points in S, all of the same color or of three different colors, aren't obtuse. The number of colors available is unlimited.

3. Find all pairs (a, b) of positive integers such that

$$3^a = 2b^2 + 1.$$

china
China Girls Math Olympiad
2010

$\boxed{1}$ Let n be an integer greater than two, and let $A_1, A_2, \cdots, A_{2n}$ be pairwise distinct subsets of $\{1, 2, , n\}$. Determine the maximum value of

$$\sum_{i=1}^{2n} \frac{|A_i \cap A_{i+1}|}{|A_i| \cdot |A_{i+1}|}$$

Where $A_{2n+1} = A_1$ and $|X|$ denote the number of elements in X.

$\boxed{2}$ In triangle ABC, $AB = AC$. Point D is the midpoint of side BC. Point E lies outside the triangle ABC such that $CE \perp AB$ and $BE = BD$. Let M be the midpoint of segment BE. Point F lies on the minor arc $\widehat{AD}$ of the circumcircle of triangle ABD such that $MF \perp BE$. Prove that $ED \perp FD$.

$\boxed{3}$ Prove that for every given positive integer n, there exists a prime p and an integer m such that (a) $p \equiv 5 \pmod 6$ (b) $p \nmid n$ (c) $n \equiv m^3 \pmod p$

$\boxed{4}$ Let $x_1, x_2, \cdots, x_n$ be real numbers with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that

$$\sum_{k=1}^{n} \left(1 - \frac{k}{\sum\limits_{i=1}^{n} i x_i^2} \right)^2 \cdot \frac{x_k^2}{k} \leq \left(\frac{n-1}{n+1} \right)^2 \sum_{k=1}^{n} \frac{x_k^2}{k}$$

Determine when does the equality hold?

$\boxed{5}$ Let $f(x)$ and $g(x)$ be strictly increasing linear functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x)$ is an integer if and only if $g(x)$ is an integer. Prove that for any real number x, $f(x) - g(x)$ is an integer.

$\boxed{6}$ In acute triangle ABC, $AB > AC$. Let M be the midpoint of side BC. The exterior angle bisector of $\widehat{BAC}$ meet ray BC at P. Point K and F lie on line PA such that $MF \perp BC$ and $MK \perp PA$. Prove that $BC^2 = 4PF \cdot AK$.

$\boxed{7}$ For given integer $n \geq 3$, set $S = \{p_1, p_2, \cdots, p_m\}$ consists of permutations p_i of $(1, 2, \cdots, n)$. Suppose that among every three distinct numbers in $\{1, 2, \cdots, n\}$, one of these number does not lie in between the other two numbers in every permutations p_i $(1 \leq i \leq m)$. (For example, in the permutation $(1, 3, 2, 4)$, 3 lies in between 1 and 4, and 4 does not lie in between 1 and 2.) Determine the maximum value of m.

$\boxed{8}$ Determine the least odd number $a > 5$ satisfying the following conditions: There are positive integers m_1, m_2, n_1, n_2 such that $a = m_1^2 + n_1^2$, $a^2 = m_2^2 + n_2^2$, and $m_1 - n_1 = m_2 - n_2$.

china
Team Selection Test
2010

China TST

Day 1

1. Given acute triangle ABC with $AB > AC$, let M be the midpoint of BC. P is a point in triangle AMC such that $\angle MAB = \angle PAC$. Let O, O_1, O_2 be the circumcenters of $\triangle ABC, \triangle ABP, \triangle ACP$ respectively. Prove that line AO passes through the midpoint of O_1O_2.

2. Let $A = \{a_1, a_2, \cdots, a_{2010}\}$ and $B = \{b_1, b_2, \cdots, b_{2010}\}$ be two sets of complex numbers. Suppose
$$\sum_{1\le i<j\le 2010} (a_i + a_j)^k = \sum_{1\le i<j\le 2010} (b_i + b_j)^k$$
holds for every $k = 1, 2, \cdots, 2010$. Prove that $A = B$.

3. Let $n_1, n_2, \cdots, n_{26}$ be pairwise distinct positive integers satisfying (1) for each n_i, its digits belong to the set $\{1, 2\}$; (2) for each i, j, n_i can't be obtained from n_j by adding some digits on the right. Find the smallest possible value of $\sum_{i=1}^{26} S(n_i)$, where $S(m)$ denotes the sum of all digits of a positive integer m.

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Day 2

1. Let $G = G(V, E)$ be a simple graph with vertex set V and edge set E. Suppose $|V| = n$. A map $f : V \to \mathbb{Z}$ is called good, if f satisfies the followings: (1) $\sum_{v\in V} f(v) = |E|$; (2) color arbitarily some vertices into red, one can always find a red vertex v such that $f(v)$ is no more than the number of uncolored vertices adjacent to v. Let $m(G)$ be the number of good maps. Prove that if every vertex in G is adjacent to at least one another vertex, then $n \le m(G) \le n!$.

2. Given integer $a_1 \ge 2$. For integer $n \ge 2$, define a_n to be the smallest positive integer which is not coprime to a_{n-1} and not equal to $a_1, a_2, \cdots, a_{n-1}$. Prove that every positive integer except 1 appears in this sequence $\{a_n\}$.

3. Given integer $n \ge 2$ and real numbers $x_1, x_2, \cdots, x_n$ in the interval $[0, 1]$. Prove that there exist real numbers $a_0, a_1, \cdots, a_n$ satisfying the following conditions: (1) $a_0 + a_n = 0$; (2) $|a_i| \le 1$, for $i = 0, 1, \cdots, n$; (3) $|a_i - a_{i-1}| = x_i$, for $i = 1, 2, \cdots, n$.

Quiz 1

china
Western Mathematical Olympiad
2010

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1 Suppose that m and k are non-negative integers, and $p = 2^{2^m} + 1$ is a prime number. Prove that **(a)** $2^{2^{m+1}p^k} \equiv 1 \pmod{p^{k+1}}$; **(b)** $2^{m+1}p^k$ is the smallest positive integer n satisfying the congruence equation $2^n \equiv 1 \pmod{p^{k+1}}$.

2 AB is a diameter of a circle with center O. Let C and D be two different points on the circle on the same side of AB, and the lines tangent to the circle at points C and D meet at E. Segments AD and BC meet at F. Lines EF and AB meet at M. Prove that E, C, M and D are concyclic.

3 Determine all possible values of positive integer n, such that there are n different 3-element subsets $A_1, A_2, ..., A_n$ of the set $\{1, 2, ..., n\}$, with $|A_i \cap A_j| \neq 1$ for all $i \neq j$.

4 Let $a_1, a_2, .., a_n, b_1, b_2, ..., b_n$ be non-negative numbers satisfying the following conditions simultaneously:

(1) $\displaystyle\sum_{i=1}^{n}(a_i + b_i) = 1$;

(2) $\displaystyle\sum_{i=1}^{n} i(a_i - b_i) = 0$;

(3) $\displaystyle\sum_{i=1}^{n} i^2(a_i + b_i) = 10$.

Prove that $\max\{a_k, b_k\} \leq \dfrac{10}{10 + k^2}$ for all $1 \leq k \leq n$.

5 Let k be an integer and $k > 1$. Define a sequence $\{a_n\}$ as follows:

$a_0 = 0$,

$a_1 = 1$, and

$a_{n+1} = ka_n + a_{n-1}$ for $n = 1, 2, ...$

Determine, with proof, all possible k for which there exist non-negative integers $l, m (l \neq m)$ and positive integers p, q such that $a_l + ka_p = a_m + ka_q$.

6 ΔABC is a right-angled triangle, $\angle C = 90°$. Draw a circle centered at B with radius BC. Let D be a point on the side AC, and DE is tangent to the circle at E. The line through C perpendicular to AB meets line BE at F. Line AF meets DE at point G. The line through A parallel to BG meets DE at H. Prove that $GE = GH$.

7 There are n ($n \geq 3$) players in a table tennis tournament, in which any two players have a match. Player A is called not out-performed by player B, if at least one of player A's losers is not a B's loser.

Determine, with proof, all possible values of n, such that the following case could happen: after finishing all the matches, every player is not out-performed by any other player.

$\boxed{8}$ Determine all possible values of integer k for which there exist positive integers a and b such that $\dfrac{b+1}{a} + \dfrac{a+1}{b} = k$.

greece
National Olympiad
2010

$\boxed{1}$ Solve in the integers the diophantine equation :

$x^4 - 6x^2 + 1 = 7 \cdot 2^y$

Babis

$\boxed{2}$ If x, y are positive real numbers with sum $2a$, prove that :

$x^3y^3(x^2 + y^2)^2 \leq 4a^{10}$

When does equality hold ?

Babis

$\boxed{3}$ A triangle ABC is inscribed in a circle $C(O, R)$ and has incenter I. Lines AI, BI, CI meet the circumcircle (O) of triangle ABC at

points D, E, F respectively.

The circles with diameter ID, IE, IF meet the sides BC, CA, AB at pairs of points $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ respectively.

Prove that the six points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic.

Babis

$\boxed{4}$ On the plane are given $k+n$ distinct lines , where $k > 1$ is integer and n is integer as well.Any three of these lines do not pass through the

same point . Among these lines exactly k are parallel and all the other n lines intersect each other.All $k + n$ lines define on the plane a partition

of triangular , polygonic or not bounded regions. Two regions are colled different, if the have not common points

or if they have common points only on their boundary.A regions is called "good" if it contained in a zone between two parallel lines .

If in a such given configuration the minimum number of "good" regionrs is 176 and the maximum number of these regions is 221, find k and n.

Babis

india

International Mathematical Olympiad Training Camp

2010

$\boxed{1}$ Let ABC be a triangle in which $BC < AC$. Let M be the mid-point of AB; AP be the altitude from A on BC; and BQ be the altitude from B on to AC. Suppose QP produced meets AB (extended) in T. If H is the ortho-center of ABC, prove that TH is perpendicular to CM.

$\boxed{2}$ Two polynomials $P(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$ have real coefficients, and I is an interval on the real line of length greater than 2. Suppose $P(x)$ and $Q(x)$ take negative values on I, and they take non-negative values outside I. Prove that there exists a real number x_0 such that $P(x_0) < Q(x_0)$.

$\boxed{3}$ For any integer $n \geq 2$, let $N(n)$ be the maximum number of triples (a_j, b_j, c_j), $j = 1, 2, 3, \cdots, N(n)$, consisting of non-negative integers a_j, b_j, c_j (not necessarily distinct) such that the following two conditions are satisfied:

(a) $a_j + b_j + c_j = n$, for all $j = 1, 2, 3, \cdots N(n)$; (b) $j \neq k$, then $a_j \neq a_k$, $b_j \neq b_k$ and $c_j \neq c_k$. Determine $N(n)$ for all $n \geq 2$.

$\boxed{4}$ Let a, b, c be positive real numbers such that $ab + bc + ca \leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}} + \sqrt{\frac{b^2+c^2}{b+c}} + \sqrt{\frac{c^2+a^2}{c+a}} + 3 \leq \sqrt{2}(\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a})$$

$\boxed{5}$ Given an integer $k > 1$, show that there exist an integer an $n > 1$ and distinct positive integers $a_1, a_2, \cdots a_n$, all greater than 1, such that the sums $\sum_{j=1}^{n} a_j$ and $\sum_{j=1}^{n} \phi(a_j)$ are both k-th powers of some integers. (Here $\phi(m)$ denotes the number of positive integers less than m and relatively prime to m.)

$\boxed{6}$ Let $n \geq 2$ be a given integer. Show that the number of strings of length n consisting of $0'$s and $1'$s such that there are equal number of 00 and 11 blocks in each string is equal to

$$2\binom{n-2}{\lfloor \frac{n-2}{2} \rfloor}$$

$\boxed{7}$ Let $ABCD$ be a cyclic quadrilaterla and let E be the point of intersection of its diagonals AC and BD. Suppose AD and BC meet in F. Let the midpoints of AB and CD be G and H respectively. If Γ is the circumcircle of triangle EGH, prove that FE is tangent to Γ.

$\boxed{8}$ Call a positive integer **good** if either $N = 1$ or N can be written as product of *even* number of prime numbers, not necessarily distinct. Let $P(x) = (x - a)(x - b)$, where a, b are positive integers.

(a) Show that there exist distinct positive integers a, b such that $P(1), P(2), \cdots, P(2010)$ are all good numbers. (b) Suppose a, b are such that $P(n)$ is a good number for all positive integers n. Prove that $a = b$.

9 Let $A = (a_{jk})$ be a 10×10 array of positive real numbers such that the sum of numbers in row as well as in each column is 1. Show that there exists $j < k$ and $l < m$ such that

$$a_{jl}a_{km} + a_{jm}a_{kl} \geq \frac{1}{50}$$

10 Let ABC be a triangle. Let Ω be the brocard point. Prove that $\left(\frac{A\Omega}{BC}\right)^2 + \left(\frac{B\Omega}{AC}\right)^2 + \left(\frac{C\Omega}{AB}\right)^2 \geq 1$

11 Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x+y) + xy = f(x)f(y)$ for all reals x, y

12 Prove that there are infinitely many positive integers m for which there exists consecutive odd positive integers $p_m < q_m$ such that $p_m^2 + p_mq_m + q_m^2$ and $p_m^2 + m \cdot p_mq_m + q_m^2$ are both perfect squares. If m_1, m_2 are two positive integers satisfying this condition, then we have $p_{m_1} \neq p_{m_2}$

india
National Olympiad
2010

$\boxed{1}$ Let ABC be a triangle with circum-circle Γ. Let M be a point in the interior of triangle ABC which is also on the bisector of $\angle A$. Let AM, BM, CM meet Γ in A_1, B_1, C_1 respectively. Suppose P is the point of intersection of A_1C_1 with AB; and Q is the point of intersection of A_1B_1 with AC. Prove that PQ is parallel to BC.

$\boxed{2}$ Find all natural numbers $n > 1$ such that n^2 does not divide $(n-2)!$.

$\boxed{3}$ Find all non-zero real numbers x, y, z which satisfy the system of equations:

$$(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) = xyz$$

$$(x^4 + x^2y^2 + y^4)(y^4 + y^2z^2 + z^4)(z^4 + z^2x^2 + x^4) = x^3y^3z^3$$

$\boxed{4}$ How many 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ are there such that each of $a_1, a_2, a_3, a_4, a_5, a_6$ is from the set $\{1, 2, 3, 4\}$ and the six expressions

$$a_j^2 - a_ja_{j+1} + a_{j+1}^2$$

for $j = 1, 2, 3, 4, 5, 6$ (where a_7 is to be taken as a_1) are all equal to one another?

$\boxed{5}$ Let ABC be an acute-angled triangle with altitude AK. Let H be its ortho-centre and O be its circum-centre. Suppose KOH is an acute-angled triangle and P its circum-centre. Let Q be the reflection of P in the line HO. Show that Q lies on the line joining the mid-points of AB and AC.

$\boxed{6}$ Define a sequence $< a_n >_{n\geq 0}$ by $a_0 = 0$, $a_1 = 1$ and

$$a_n = 2a_{n-1} + a_{n-2},$$

for $n \geq 2$.

(a) For every $m > 0$ and $0 \leq j \leq m$, prove that $2a_m$ divides $a_{m+j} + (-1)^j a_{m-j}$.

(b) Suppose 2^k divides n for some natural numbers n and k. Prove that 2^k divides a_n.

indonesia

Indonesia TST

Jogjakarta, Indonesia 2010

Day 1

$\boxed{1}$ Let a, b, and c be non-negative real numbers and let x, y, and z be positive real numbers such that $a+b+c=x+y+z$. Prove that

$$\frac{a^3}{x^2}+\frac{b^3}{y^2}+\frac{c^3}{z^2}\geq a+b+c.$$

Hery Susanto, Malang

$\boxed{2}$ Let $A=\{n:1\leq n\leq 2009^{2009},n\in\mathbb{N}\}$ and let $S=\{n:n\in A,\gcd\left(n,2009^{2009}\right)=1\}$. Let P be the product of all elements of S. Prove that

$$P\equiv 1 \pmod{2009^{2009}}.$$

Nanang Susyanto, Jogjakarta

$\boxed{3}$ In a party, each person knew exactly 22 other persons. For each two persons X and Y, if X and Y knew each other, there is no other person who knew both of them, and if X and Y did not know each other, there are exactly 6 persons who knew both of them. Assume that X knew Y iff Y knew X. How many people did attend the party?

Yudi Satria, Jakarta

$\boxed{4}$ Let ABC be a non-obtuse triangle with CH and CM are the altitude and median, respectively. The angle bisector of $\angle BAC$ intersects CH and CM at P and Q, respectively. Assume that

$$\angle ABP=\angle PBQ=\angle QBC,$$

(a) prove that ABC is a right-angled triangle, and (b) calculate $\dfrac{BP}{CH}$.

Soewono, Bandung

indonesia
Indonesia TST
Jogjakarta, Indonesia 2010

Day 2

$\boxed{1}$ Let $ABCD$ be a trapezoid such that $AB \parallel CD$ and assume that there are points E on the line outside the segment BC and F on the segment AD such that $\angle DAE = \angle CBF$. Let I, J, K respectively be the intersection of line EF and line CD, the intersection of line EF and line AB, and the midpoint of segment EF. Prove that K is on the circumcircle of triangle CDJ if and only if I is on the circumcircle of triangle ABK.

Utari Wijayanti, Bandung

$\boxed{2}$ Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^3 + y^3) = xf(x^2) + yf(y^2)$$

for all real numbers x and y.

Hery Susanto, Malang

$\boxed{3}$ Let x, y, and z be integers satisfying the equation

$$\frac{2008}{41y^2} = \frac{2z}{2009} + \frac{2007}{2x^2}.$$

Determine the greatest value that z can take.

Budi Surodjo, Jogjakarta

$\boxed{4}$ For each positive integer n, define $f(n)$ as the number of digits 0 in its decimal representation. For example, $f(2) = 0$, $f(2009) = 2$, etc. Please, calculate

$$S = \sum_{k=1}^{n} 2^{f(k)},$$

for $n = 9,999,999,999$.

Yudi Satria, Jakarta

Day 3

$\boxed{1}$ Let f be a polynomial with integer coefficients. Assume that there exists integers a and b such that $f(a) = 41$ and $f(b) = 49$. Prove that there exists an integer c such that 2009 divides $f(c)$.

Nanang Susyanto, Jogjakarta

$\boxed{2}$ Given an equilateral triangle, all points on its sides are colored in one of two given colors. Prove that the is a right-angled triangle such that its three vertices are in the same color and on the sides of the equilateral triangle.

Alhaji Akbar, Jakarta

$\boxed{3}$ Let $a_1, a_2, \dots$ be sequence of real numbers such that $a_1 = 1$, $a_2 = \frac{4}{3}$, and

$$a_{n+1} = \sqrt{1 + a_n a_{n-1}}, \quad \forall n \geq 2.$$

Prove that for all $n \geq 2$,

$$a_n^2 > a_{n-1}^2 + \frac{1}{2}$$

and

$$1 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} > 2a_n.$$

Fajar Yuliawan, Bandung

$\boxed{4}$ Let ABC be an acute-angled triangle such that there exist points D, E, F on side BC, CA, AB, respectively such that the inradii of triangle AEF, BDF, CDE are all equal to r_0. If the inradii of triangle DEF and ABC are r and R, respectively, prove that

$$r + r_0 = R.$$

Soewono, Bandung

Day 4

1 The integers $1, 2, \ldots, 20$ are written on the blackboard. Consider the following operation as one step: *choose two integers a and b such that $a - b \geq 2$ and replace them with $a-1$ and $b+1$*. Please, determine the maximum number of steps that can be done.

Yudi Satria, Jakarta

2 Circles Γ_1 and Γ_2 are internally tangent to circle Γ at P and Q, respectively. Let P_1 and Q_1 are on Γ_1 and Γ_2 respectively such that P_1Q_1 is the common tangent of P_1 and Q_1. Assume that Γ_1 and Γ_2 intersect at R and R_1. Define O_1, O_2, O_3 as the intersection of PQ and P_1Q_1, the intersection of PR and P_1R_1, and the intersection QR and Q_1R_1. Prove that the points O_1, O_2, O_3 are collinear.

Rudi Adha Prihandoko, Bandung

3 Determine all real numbers a such that there is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$x + f(y) = af(y + f(x))$$

for all real numbers x and y.

Hery Susanto, Malang

4 Prove that for all integers m and n, the inequality

$$\frac{\phi(\gcd(2^m+1, 2^n+1))}{\gcd(\phi(2^m+1), \phi(2^n+1))} \geq \frac{2\gcd(m,n)}{2^{\gcd(m,n)}}$$

holds.

Nanang Susyanto, Jogjakarta

Day 5

$\boxed{1}$ Is there a triangle with angles in ratio of $1:2:4$ and the length of its sides are integers with at least one of them is a prime number?

Nanang Susyanto, Jogjakarta

$\boxed{2}$ Consider a polynomial with coefficients of real numbers $\phi(x) = ax^3 + bx^2 + cx + d$ with three positive real roots. Assume that $\phi(0) < 0$, prove that

$$2b^3 + 9a^2d - 7abc \leq 0.$$

Hery Susanto, Malang

$\boxed{3}$ Let $\mathbb{Z}$ be the set of all integers. Define the set $\mathbb{H}$ as follows: (1). $\dfrac{1}{2} \in \mathbb{H}$, (2). if $x \in \mathbb{H}$, then $\dfrac{1}{1+x} \in \mathbb{H}$ and also $\dfrac{x}{1+x} \in \mathbb{H}$. Prove that there exists a bijective function $f : \mathbb{Z} \to \mathbb{H}$.

$\boxed{4}$ Prove that the number $(\underbrace{9999\ldots99}_{2005})^{2009}$ can be obtained by erasing some digits of $(\underbrace{9999\ldots99}_{2008})^{2009}$ (both in decimal representation).

Yudi Satria, Jakarta

APMO 2010

$\boxed{1}$ Let ABC be a triangle with $\angle BAC \neq 90^\circ$. Let O be the circumcircle of the triangle ABC and Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be the diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

$\boxed{2}$ For a positive integer k, call an integer a *pure* $k-th$ *power* if it can be represented as m^k for some integer m. Show that for every positive integer n, there exists n distinct positive integers such that their sum is a pure $2009-$th power and their product is a pure $2010-$th power.

$\boxed{3}$ Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

$\boxed{4}$ Let ABC be an acute angled triangle satisfying the conditions $AB > BC$ and $AC > BC$. Denote by O and H the circumcentre and orthocentre, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcentre of the triangle MNH lies on the line OH.

$\boxed{5}$ Find all functions f from the set $\mathbb{R}$ of real numbers into $\mathbb{R}$ which satisfy for all $x, y, z \in \mathbb{R}$ the identity $f(f(x)+f(y)+f(z)) = f(f(x)-f(y)) + f(2xy+f(z)) + 2f(xz-yz)$

Balkan MO 2010

$\boxed{1}$ Let a, b and c be positive real numbers. Prove that

$$\frac{a^2b(b-c)}{a+b} + \frac{b^2c(c-a)}{b+c} + \frac{c^2a(a-b)}{c+a} \geq 0$$

$\boxed{2}$ Let ABC be an acute triangle with orthocentre H, and let M be the midpoint of AC. The point C_1 on AB is such that CC_1 is an altitude of the triangle ABC. Let H_1 be the reflection of H in AB. The orthogonal projections of C_1 onto the lines AH_1, AC and BC are P, Q and R, respectively. Let M_1 be the point such that the circumcentre of triangle PQR is the midpoint of the segment MM_1. Prove that M_1 lies on the segment BH_1.

$\boxed{3}$ A strip of width w is the set of all points which lie on, or between, two parallel lines distance w apart. Let S be a set of n ($n \geq 3$) points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2.

$\boxed{4}$ For each integer n ($n \geq 2$), let $f(n)$ denote the sum of all positive integers that are at most n and not relatively prime to n. Prove that $f(n+p) \neq f(n)$ for each such n and every prime p.

Baltic Way 2002

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1 Solve the system of simultaneous equations

$$\begin{cases} a^3 + 3ab^2 + 3ac^2 - 6abc = 1 \\ b^3 + 3ba^2 + 3bc^2 - 6abc = 1 \\ c^3 + 3ca^2 + 3cb^2 - 6abc = 1 \end{cases}$$

in real numbers.

2 Let a, b, c, d be real numbers such that

$$a + b + c + d = -2$$

$$ab + ac + ad + bc + bd + cd = 0$$

Prove that at least one of the numbers a, b, c, d is not greater than -1.

3 Find all sequences $0 \le a_0 \le a_1 \le a_2 \le \ldots$ of real numbers such that

$$a_{m^2+n^2} = a_m^2 + a_n^2$$

for all integers $m, n \ge 0$.

4 Let n be a positive integer. Prove that

$$\sum_{i=1}^{n} x_i(1-x_i)^2 \le \left(1 - \frac{1}{n}\right)^2$$

for all nonnegative real numbers $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \ldots x_n = 1$.

5 Find all pairs (a, b) of positive rational numbers such that

$$\sqrt{a} + \sqrt{b} = \sqrt{2 + \sqrt{3}}.$$

6 The following solitaire game is played on an $m \times n$ rectangular board, $m, n \ge 2$, divided into unit squares. First, a rook is placed on some square. At each move, the rook can be moved an arbitrary number of squares horizontally or vertically, with the extra condition that each move has to be made in the $90°$ clockwise direction compared to the previous one (e.g. after a move to the left, the next one has to be done upwards, the next one to the right etc). For which values of m and n is it possible that the rook visits every square of the board exactly once and returns to the first square? (The rook is considered to visit only those squares it stops on, and not the ones it steps over.)

$\boxed{7}$ We draw n convex quadrilaterals in the plane. They divide the plane into regions (one of the regions is infinite). Determine the maximal possible number of these regions.

$\boxed{8}$ Let P be a set of $n \geq 3$ points in the plane, no three of which are on a line. How many possibilities are there to choose a set T of $\binom{n-1}{2}$ triangles, whose vertices are all in P, such that each triangle in T has a side that is not a side of any other triangle in T?

$\boxed{9}$ Two magicians show the following trick. The first magician goes out of the room. The second magician takes a deck of 100 cards labelled by numbers $1, 2, \ldots, 100$ and asks three spectators to choose in turn one card each. The second magician sees what card each spectator has taken. Then he adds one more card from the rest of the deck. Spectators shuffle these 4 cards, call the first magician and give him these 4 cards. The first magician looks at the 4 cards and guesses what card was chosen by the first spectator, what card by the second and what card by the third. Prove that the magicians can perform this trick.

$\boxed{10}$ Let N be a positive integer. Two persons play the following game. The first player writes a list of positive integers not greater than 25, not necessarily different, such that their sum is at least 200. The second player wins if he can select some of these numbers so that their sum S satisfies the condition $200 - N \leq S \leq 200 + N$. What is the smallest value of N for which the second player has a winning strategy?

$\boxed{11}$ Let n be a positive integer. Consider n points in the plane such that no three of them are collinear and no two of the distances between them are equal. One by one, we connect each point to the two points nearest to it by line segments (if there are already other line segments drawn to this point, we do not erase these). Prove that there is no point from which line segments will be drawn to more than 11 points.

$\boxed{12}$ A set S of four distinct points is given in the plane. It is known that for any point $X \in S$ the remaining points can be denoted by Y, Z and W so that $|XY| = |XZ| + |XW|$ Prove that all four points lie on a line.

$\boxed{13}$ Let ABC be an acute triangle with $\angle BAC > \angle BCA$, and let D be a point on side AC such that $|AB| = |BD|$. Furthermore, let F be a point on the circumcircle of triangle ABC such that line FD is perpendicular to side BC and points F, B lie on different sides of line AC. Prove that line FB is perpendicular to side AC .

$\boxed{14}$ Let L, M and N be points on sides AC, AB and BC of triangle ABC, respectively, such that BL is the bisector of angle ABC and segments AN, BL and CM have a common point. Prove that if $\angle ALB = \angle MNB$ then $\angle LNM = 90°$.

$\boxed{15}$ A spider and a fly are sitting on a cube. The fly wants to maximize the shortest path to the spider along the surface of the cube. Is it necessarily best for the fly to be at the point opposite to the spider? (Opposite means symmetric with respect to the centre of the cube.)

$\boxed{16}$ Find all nonnegative integers m such that

$$a_m = (2^{2m+1})^2 + 1$$

is divisible by at most two different primes.

$\boxed{17}$ Show that the sequence

$$\binom{2002}{2002}, \binom{2003}{2002}, \binom{2004}{2002}, \ldots$$

considred modulo 2002, is periodic.

$\boxed{18}$ Find all integers $n > 1$ such that any prime divisor of $n^6 - 1$ is a divisor of $(n^3 - 1)(n^2 - 1)$.

$\boxed{19}$ Let n be a positive integer. Prove that the equation

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

$\boxed{20}$ Does there exist an infinite non-constant arithmetic progression, each term of which is of the form a^b, where a and b are positive integers with $b \geq 2$?

Benelux 2010

Amsterdam/Netherlands

$\boxed{1}$ A finite set of integers is called *bad* if its elements add up to 2010. A finite set of integers is a *Benelux-set* if none of its subsets is bad. Determine the smallest positive integer n such that the set $\{502, 503, 504, ..., 2009\}$ can be partitioned into n Benelux-sets. (A partition of a set S into n subsets is a collection of n pairwise disjoint subsets of S, the union of which equals S.)

(2nd Benelux Mathematical Olympiad 2010, Problem 1)

$\boxed{2}$ Find all polynomials $p(x)$ with real coeffcients such that

$$p(a+b-2c)+p(b+c-2a)+p(c+a-2b)=3p(a-b)+3p(b-c)+3p(c-a)$$

for all $a, b, c \in \mathbb{R}$.

(2nd Benelux Mathematical Olympiad 2010, Problem 2)

$\boxed{3}$ On a line l there are three different points A, B and P in that order. Let a be the line through A perpendicular to l, and let b be the line through B perpendicular to l. A line through P, not coinciding with l, intersects a in Q and b in R. The line through A perpendicular to BQ intersects BQ in L and BR in T. The line through B perpendicular to AR intersects AR in K and AQ in S. (a) Prove that P, T, S are collinear. (b) Prove that P, K, L are collinear.

(2nd Benelux Mathematical Olympiad 2010, Problem 3)

$\boxed{4}$ Find all quadruples (a, b, p, n) of positive integers, such that p is a prime and

$$a^3+b^3=p^n.$$

(2nd Benelux Mathematical Olympiad 2010, Problem 4)

CentroAmerican 2010

$\boxed{1}$ Denote by $S(n)$ the sum of the digits of the positive integer n. Find all the solutions of the equation

$n(S(n)-1)=2010.$

$\boxed{2}$ Let ABC be a triangle and L, M, N be the midpoints of BC, CA and AB, respectively. The tangent to the circumcircle of ABC at A intersects LM and LN at P and Q, respectively. Show that CP is parallel to BQ.

$\boxed{3}$ A token is placed in one square of a $m\times n$ board, and is moved according to the following rules:

In each turn, the token can be moved to a square sharing a side with the one currently occupied. [/*:m] The token cannot be placed in a square that has already been occupied. [/*:m] Any two consecutive moves cannot have the same direction.[/*:m]

The game ends when the token cannot be moved. Determine the values of m and n for which, by placing the token in some square, all the squares of the board will have been occupied in the end of the game.

$\boxed{4}$ Find all positive integers N such that an $N\times N$ board can be tiled using tiles of size 5×5 or 1×3.

Note: The tiles must completely cover all the board, with no overlappings.

$\boxed{5}$ If p, q and r are nonzero rational numbers such that $\sqrt[3]{pq^2}+\sqrt[3]{qr^2}+\sqrt[3]{rp^2}$ is a nonzero rational number, prove that

$\frac{1}{\sqrt[3]{pq^2}}+\frac{1}{\sqrt[3]{qr^2}}+\frac{1}{\sqrt[3]{rp^2}}$

is also a rational number.

$\boxed{6}$ Let Γ and Γ_1 be two circles internally tangent at A, with centers O and O_1 and radii r and r_1, respectively ($r>r_1$). B is a point diametrically opposed to A in Γ, and C is a point on Γ such that BC is tangent to Γ_1 at P. Let A' the midpoint of BC. Given that O_1A' is parallel to AP, find the ratio r/r_1.

International Zhautykov Olympiad 2010

Day 1

1. Find all primes p, q such that $p^3 - q^7 = p - q$.

2. In a cyclic quadrilateral $ABCD$ with $AB = AD$ points M,N lie on the sides BC and CD respectively so that $MN = BM + DN$. Lines AM and AN meet the circumcircle of $ABCD$ again at points P and Q respectively. Prove that the orthocenter of the triangle APQ lies on the segment MN .

3. A rectangle formed by the lines of checkered paper is divided into figures of three kinds: isosceles right triangles (1) with base of two units, squares (2) with unit side, and parallelograms (3) formed by two sides and two diagonals of unit squares (figures may be oriented in any way). Prove that the number of figures of the third kind is even.

   [img]http://up.iranblog.com/Files7/dda310bab8b6455f90ce.jpg[/img]

Day 2

1. Positive integers $1, 2, ..., n$ are written on blackboard ($n > 2$). Every minute two numbers are erased and the least prime divisor of their sum is written. In the end only the number 97 remains. Find the least n for which it is possible.

2. In every vertex of a regular n -gon exactly one chip is placed. At each *step* one can exchange any two neighbouring chips. Find the least number of steps necessary to reach the arrangement where every chip is moved by $[\frac{n}{2}]$ positions clockwise from its initial position.

3. Let ABC arbitrary triangle ($AB \neq BC \neq AC \neq AB$) And O,I,H it's circum-center, incenter and ortocenter (point of intersection altitudes). Prove, that 1) $\angle OIH > 90^0$(2 points) 2)$\angle OIH > 135^0$(7 points)

   balls for 1) and 2) not additive.

Junior Balkan MO 2010

$\boxed{1}$ The real numbers a, b, c, d satisfy simultaneously the equations

$$abc - d = 1, \quad bcd - a = 2, \quad cda - b = 3, \quad dab - c = -6.$$

Prove that $a + b + c + d \neq 0$.

$\boxed{2}$ Find all integers n, $n \geq 1$, such that $n \cdot 2^{n+1} + 1$ is a perfect square.

$\boxed{3}$ Let AL and BK be angle bisectors in the non-isosceles triangle ABC (L lies on the side BC, K lies on the side AC). The perpendicular bisector of BK intersects the line AL at point M. Point N lies on the line BK such that LN is parallel to MK. Prove that $LN = NA$.

$\boxed{4}$ A 9×7 rectangle is tiled with tiles of the two types: L-shaped tiles composed by three unit squares (can be rotated repeatedly with $90°$) and square tiles composed by four unit squares. Let $n \geq 0$ be the number of the 2×2 tiles which can be used in such a tiling. Find all the values of n.

Middle European Mathematical Olympiad 2010

$\boxed{1}$ Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$, we have

$$f(x+y) + f(x)f(y) = f(xy) + (y+1)f(x) + (x+1)f(y).$$

(4th Middle European Mathematical Olympiad, Individual Competition, Problem 1)

$\boxed{2}$ All positive divisors of a positive integer N are written on a blackboard. Two players A and B play the following game taking alternate moves. In the firt move, the player A erases N. If the last erased number is d, then the next player erases either a divisor of d or a multiple of d. The player who cannot make a move loses. Determine all numbers N for which A can win independently of the moves of B.

(4th Middle European Mathematical Olympiad, Individual Competition, Problem 2)

$\boxed{3}$ We are given a cyclic quadrilateral $ABCD$ with a point E on the diagonal AC such that $AD = AE$ and $CB = CE$. Let M be the center of the circumcircle k of the triangle BDE. The circle k intersects the line AC in the points E and F. Prove that the lines FM, AD and BC meet at one point.

(4th Middle European Mathematical Olympiad, Individual Competition, Problem 3)

$\boxed{4}$ Find all positive integers n which satisfy the following tow conditions: (a) n has at least four different positive divisors; (b) for any divisors a and b of n satisfying $1 < a < b < n$, the number $b - a$ divides n.

(4th Middle European Mathematical Olympiad, Individual Competition, Problem 4)

$\boxed{5}$ Three strictly increasing sequences

$$a_1, a_2, a_3, \ldots, \qquad b_1, b_2, b_3, \ldots, \qquad c_1, c_2, c_3, \ldots$$

of positive integers are given. Every positive integer belongs to exactly one of the three sequences. For every positive integer n, the following conditions hold: (a) $c_{a_n} = b_n + 1$; (b) $a_{n+1} > b_n$; (c) the number $c_{n+1}c_n - (n+1)c_{n+1} - nc_n$ is even. Find a_{2010}, b_{2010} and c_{2010}.

(4th Middle European Mathematical Olympiad, Team Competition, Problem 1)

$\boxed{6}$ For each integer $n \geqslant 2$, determine the largest real constant C_n such that for all positive real numbers $a_1, \ldots, a_n$ we have

$$\frac{a_1^2 + \ldots + a_n^2}{n} \geqslant \left(\frac{a_1 + \ldots + a_n}{n}\right)^2 + C_n \cdot (a_1 - a_n)^2.$$

(4th Middle European Mathematical Olympiad, Team Competition, Problem 2)

$\boxed{7}$ In each vertex of a regular n-gon, there is a fortress. At the same moment, each fortress shoots one of the two nearest fortresses and hits it. The *result of the shooting* is the set of the hit fortresses; we do not distinguish whether a fortress was hit once or twice. Let $P(n)$ be the number of possible results of the shooting. Prove that for every positive integer $k \geqslant 3$, $P(k)$ and $P(k+1)$ are relatively prime.

(4th Middle European Mathematical Olympiad, Team Competition, Problem 3)

$\boxed{8}$ Let n be a positive integer. A square $ABCD$ is partitioned into n^2 unit squares. Each of them is divided into two triangles by the diagonal parallel to BD. Some of the vertices of the unit squares are colored red in such a way that each of these $2n^2$ triangles contains at least one red vertex. Find the least number of red vertices.

(4th Middle European Mathematical Olympiad, Team Competition, Problem 4)

$\boxed{9}$ The incircle of the triangle ABC touches the sides BC, CA, and AB in the points D, E and F, respectively. Let K be the point symmetric to D with respect to the incenter. The lines DE and FK intersect at S. Prove that AS is parallel to BC.

(4th Middle European Mathematical Olympiad, Team Competition, Problem 5)

$\boxed{10}$ Let A, B, C, D, E be points such that $ABCD$ is a cyclic quadrilateral and $ABDE$ is a parallelogram. The diagonals AC and BD intersect at S and the rays AB and DC intersect at F. Prove that $\sphericalangle AFS = \sphericalangle ECD$.

(4th Middle European Mathematical Olympiad, Team Competition, Problem 6)

$\boxed{11}$ For a nonnegative integer n, define a_n to be the positive integer with decimal representation

$$1\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}1.$$

Prove that $\frac{a_n}{3}$ is always the sum of two positive perfect cubes but never the sum of two perfect squares.

(4th Middle European Mathematical Olympiad, Team Competition, Problem 7)

$\boxed{12}$ We are given a positive integer n which is not a power of two. Show that ther exists a positive integer m with the following two properties: (a) m is the product of two consecutive positive integers; (b) the decimal representation of m consists of two identical blocks with n digits.

(4th Middle European Mathematical Olympiad, Team Competition, Problem 8)

Romanian Masters In Mathematics 2010

$\boxed{1}$ For a finite non empty set of primes P, let $m(P)$ denote the largest possible number of consecutive positive integers, each of which is divisible by at least one member of P

(i) Show that $|P| \leq m(P)$, with equality if and only if $min(P) > |P|$

(ii) Show that $m(P) < (|P|+1)(2^{|P|}-1)$

(The number $|P|$ is the size of set P)

$\boxed{2}$ For each positive integer n, find the largest integer C_n with the following property. Given any $n-$real valued functions $f_1(x), f_2(x), \cdots, f_n(x)$ defined on the closed interval $0 \leq x \leq 1$, one can find numbers $x_1, x_2, \cdots x_n$, such that $0 \leq x_i \leq 1$ satisfying $|f_1(x_1) + f_2(x_2) + \cdots f_n(x_n) - x_1 x_2 \cdots x_n| \geq C_n$

$\boxed{3}$ Let $A_1A_2A_3A_4$ be a quadrilateral with no pair of parallel sides. For each $i = 1, 2, 3, 4$, define ω_1 to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i, A_iA_{i+1}$ and $A_{i+1}A_{i+2}$ (indices are considered modulo 4 so $A_0 = A_4, A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.

$\boxed{4}$ Determine whether there exists a polynomial $f(x_1, x_2)$ with two variables, with integer coefficients, and two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in the plane, satisfying the following conditions.

(i) A is an integer point (i.e a_1 and a_2 are integers);

(ii) $|a_1 - b_1| + |a_2 - b_2| = 2010$;

(iii) $f(n_1, n_2) > f(a_1, a_2)$ for all integer points (n_1, n_2) in the plane other than A;

(iv) $f(x_1, x_2) > f(b_1, b_2)$ for all integer points (x_1, x_2) in the plane other than B

$\boxed{5}$ Let n be a given positive integer. Say that a set K of points with integer coordinates in the plane is connected if for every pair of points $R, S \in K$, if there exists a positive integer l and a sequence $R = T_0.T_1, T_2, \cdots T_l = S$ of points in K, where each T_i is distance 1 away from T_{i+1}. For such a set$K$, we define the set of vectors $\Delta(K) = \{\overrightarrow{RS} | R, S \in K\}$. What is the maximum value of $|\Delta(K)|$ over all connected sets K of $2n+1$ points with integer coordinates in the plane?

$\boxed{6}$ Given a polynomial $f(x)$ with rational coefficients, with degree $d \geq 2$, we define a sequence of sets $f^0(\mathbb{Q}), f^1(\mathbb{Q}), \cdots$ as $f^0(\mathbb{Q}) = 0, f^{n+1}(\mathbb{Q}) = f(f^{n-1}(\mathbb{Q}))$ for $n \geq 0$. (Given a set S, we write $f(S)$ for the set $\{f(x), x \in S\}$)

Let $f^\omega(\mathbb{Q}) = \bigcap_{n=0}^{\infty} f^n(\mathbb{Q})$ be the set of numbers that are in all of the sets $f^n(\mathbb{Q})$. Prove that $f^\omega(\mathbb{Q})$ is a finite set.

iran
National Math Olympiad (3rd Round)
2010

Day 1

$\boxed{1}$ suppose that polynomial $p(x) = x^{2010} \pm x^{2009} \pm ... \pm x \pm 1$ does not have a real root. what is the maximum number of coefficients to be -1?(14 points)

$\boxed{2}$ a, b, c are positive real numbers. prove the following inequality:
$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{(a+b+c)^2} \geq \frac{7}{25}(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{a+b+c})^2$
(20 points)

$\boxed{3}$ prove that for each natural number n there exist a polynomial with degree $2n+1$ with coefficients in $\mathbb{Q}[x]$ such that it has exactly 2 complex zeros and it's irreducible in $\mathbb{Q}[x]$.(20 points)

$\boxed{4}$ For each polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ we define it's derivative as this and we show it by $p'(x)$:

$$p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + ... + 2 a_2 x + a_1$$

a) For each two polynomials $p(x)$ and $q(x)$ prove that:(3 points)

$$(p(x)q(x))' = p'(x)q(x) + p(x)q'(x)$$

b) Suppose that $p(x)$ is a polynomial with degree n and $x_1, x_2, ..., x_n$ are it's zeros. prove that:(3 points)

$$\frac{p'(x)}{p(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}$$

c) $p(x)$ is a monic polynomial with degree n and $z_1, z_2, ..., z_n$ are it's zeros such that:

$$|z_1| = 1, \quad \forall i \in \{2, .., n\} : |z_i| \leq 1$$

Prove that $p'(x)$ has at least one zero in the disc with length one with the center z_1 in complex plane. (disc with length one with the center z_1 in complex plane: $D = \{z \in \mathbb{C} : |z - z_1| \leq 1\}$)(20 points)

$\boxed{5}$ x, y, z are positive real numbers such that $xy + yz + zx = 1$. prove that: $3 - \sqrt{3} + \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq (x+y+z)^2$ (20 points)

the exam time was 6 hours.

Day 2

1. suppose that $a = 3^{100}$ and $b = 5454$. how many zs in $[1, 3^{99})$ exist such that for every c that $gcd(c, 3) = 1$, two equations $x^z \equiv c$ and $x^b \equiv c \pmod a$ have the same number of answers?($\frac{100}{6}$ points)

2. R is a ring such that $xy = yx$ for every $x, y \in R$ and if $ab = 0$ then $a = 0$ or $b = 0$. if for every Ideal $I \subset R$ there exist $x_1, x_2, .., x_n$ in R (n is not constant) such that $I = (x_1, x_2, ..., x_n)$, prove that every element in R that is not 0 and it's not a unit, is the product of finite irreducible elements.($\frac{100}{6}$ points)

3. If p is a prime number, what is the product of elements like g such that $1 \leq g \leq p^2$ and g is a primitive root modulo p but it's not a primitive root modulo p^2, modulo p^2?($\frac{100}{6}$ points)

4. sppose that $\sigma_k : \mathbb{N} \longrightarrow \mathbb{R}$ is a function such that $\sigma_k(n) = \sum_{d|n} d^k$. $\rho_k : \mathbb{N} \longrightarrow \mathbb{R}$ is a function such that $\rho_k * \sigma_k = \delta$. find a formula for ρ_k.($\frac{100}{6}$ points)

5. prove that if p is a prime number such that $p = 12k + \{2, 3, 5, 7, 8, 11\}(k \in \mathbb{N} \cup \{0\})$, there exist a field with p^2 elements.($\frac{100}{6}$ points)

6. g and n are natural numbers such that $gcd(g^2 - g, n) = 1$ and $A = \{g^i | i \in \mathbb{N}\}$ and $B = \{x \equiv (n) | x \in A\}$(by $x \equiv (n)$ we mean a number from the set $\{0, 1, ..., n-1\}$ which is congruent with x modulo n). if for $0 \leq i \leq g-1$ $a_i = |[\frac{ni}{g}, \frac{n(i+1)}{g}) \cap B|$ prove that $g - 1 | \sum_{i=0}^{g-1} i a_i$.(the symbol $|\ |$ means the number of elements of the set)($\frac{100}{6}$ points)

   the exam time was 4 hours

iran
National Math Olympiad (3rd Round)
2010

Day 3

$\boxed{1}$ 1. In a triangle ABC, O is the circumcenter and I is the incenter. X is the reflection of I to O. A_1 is foot of the perpendicular from X to BC. B_1 and C_1 are defined similarly. prove that AA_1,BB_1 and CC_1 are concurrent.(12 points)

$\boxed{2}$ in a quadrilateral $ABCD$, E and F are on BC and AD respectively such that the area of triangles AED and BCF is $\frac{4}{7}$ of the area of $ABCD$. R is the intersection point of digonals of $ABCD$. $\frac{AR}{RC} = \frac{3}{5}$ and $\frac{BR}{RD} = \frac{5}{6}$. a) in what ratio does EF cut the digonals?(13 points) b) find $\frac{AF}{FD}$.(5 points)

$\boxed{3}$ in a quadrilateral $ABCD$ digonals are perpendicular to each other. let S be the intersection of digonals. K,L,M and N are reflections of S to AB,BC,CD and DA. BN cuts the circumcircle of SKN in E and BM cuts the circumcircle of SLM in F. prove that $EFLK$ is concyclic.(20 points)

$\boxed{4}$ in a triangle ABC, I is the incenter. BI and CI cut the circumcircle of ABC at E and F respectively. M is the midpoint of EF. C is a circle with diameter EF. IM cuts C at two points L and K and the arc BC of circumcircle of ABC (not containing A) at D. prove that $\frac{DL}{IL} = \frac{DK}{IK}$.(25 points)

$\boxed{5}$ In a triangle ABC, I is the incenter. D is the reflection of A to I. the incircle is tangent to BC at point E. DE cuts IG at P (G is centroid). M is the midpoint of BC. prove that a) $AP||DM$.(15 points) b) $AP = 2DM$. (10 points)

$\boxed{6}$ In a triangle ABC, $\angle C = 45$. AD is the altitude of the triangle. X is on AD such that $\angle XBC = 90 - \angle B$ (X is in the triangle). AD and CX cut the circumcircle of ABC in M and N respectively. if tangent to circumcircle of ABC at M cuts AN at P, prove that P,B and O are collinear.(25 points)

the exam time was 4 hours and 30 minutes.

iran
National Math Olympiad (3rd Round)
2010

Day 4

$\boxed{1}$ suppose that $\mathcal{F} \subseteq X^{(k)}$ and $|X| = n$. we know that for every three distinct elements of $\mathcal{F}$ like A, B, C, at most one of $A \cap B$,$B \cap C$ and $C \cap A$ is ϕ. for $k \leq \frac{n}{2}$ prove that: a) $|\mathcal{F}| \leq max(1, 4 - \frac{n}{k}) \times \binom{n-1}{k-1}$.(15 points) b) find all cases of equality in a) for $k \leq \frac{n}{3}$.(5 points)

$\boxed{2}$ suppose that $\mathcal{F} \subseteq \bigcup_{j=k+1}^{n} X^{(j)}$ and $|X| = n$. we know that $\mathcal{F}$ is a sperner family and it's also H_k. prove that: $\sum_{B \in \mathcal{F}} \frac{1}{\binom{n-1}{|B|-1}} \leq 1$ (15 points)

$\boxed{3}$ suppose that $\mathcal{F} \subseteq p(X)$ and $|X| = n$. we know that for every $A_i, A_j \in \mathcal{F}$ that $A_i \supseteq A_j$ we have $3 \leq |A_i| - |A_j|$. prove that: $|\mathcal{F}| \leq \lfloor \frac{2^n}{3} + \frac{1}{2} \binom{n}{\lfloor \frac{n}{2} \rfloor} \rfloor$ (20 points)

$\boxed{4}$ suppose that $\mathcal{F} \subseteq X^{(K)}$ and $|X| = n$. we know that for every three distinct elements of $\mathcal{F}$ like A, B and C we have $A \cap B \not\subset C$.

a)(10 points) Prove that :

$$|\mathcal{F}| \leq \binom{k}{\lfloor \frac{k}{2} \rfloor} + 1$$

b)(15 points) if elements of $\mathcal{F}$ do not necessarily have k elements, with the above conditions show that:

$$|\mathcal{F}| \leq \binom{n}{\lceil \frac{n-2}{3} \rceil} + 2$$

$\boxed{5}$ suppose that $\mathcal{F} \subseteq p(X)$ and $|X| = n$. prove that if $|\mathcal{F}| > \sum_{i=0}^{k-1} \binom{n}{i}$ then there exist $Y \subseteq X$ with $|Y| = k$ such that $p(Y) = \mathcal{F} \cap Y$ that $\mathcal{F} \cap Y = \{F \cap Y : F \in \mathcal{F}\}$(20 points) you can see this problem also here: COMBINATORIAL PROBLEMS AND EXERCISES-SECOND EDITION-by LASZLO LOVASZ-AMS CHELSEA PUBLISHING- chapter 13- problem 10(c)!!!

$\boxed{6}$ Suppose that X is a set with n elements and $\mathcal{F} \subseteq X^{(k)}$ and $X_1, X_2, ..., X_s$ is a partition of X. we know that for every $A, B \in \mathcal{F}$ and every $1 \leq j \leq s$, $E = B \cap (\bigcup_{i=1}^{j} X_i) \neq A \cap (\bigcup_{i=1}^{j} X_i) = F$ shows that non of E, F have the other one. prove that:

$$|\mathcal{F}| \leq max_{\sum_{i=1}^{s} w_i = k} \prod_{i=1}^{s} \binom{|X_i|}{w_i}$$

(15 points)

the exam time was 5 hours and 20 minutes.

iran

National Math Olympiad (3rd Round)

2010

Day 5

1 **two variable ploynomial**

$P(x,y)$ is a two variable polynomial with real coefficients. degree of a monomial means sum of the powers of x and y in it. we denote by $Q(x,y)$ sum of monomials with the most degree in $P(x,y)$. (for example if $P(x,y)=3x^4y-2x^2y^3+5xy^2+x-5$ then $Q(x,y)=3x^4y-2x^2y^3$.) suppose that there are real numbers x_1,y_1,x_2 and y_2 such that $Q(x_1,y_1)>0$, $Q(x_2,y_2)<0$ prove that the set $\{(x,y)|P(x,y)=0\}$ is not bounded. (we call a set S of plane bounded if there exist positive number M such that the distance of elements of S from the origin is less than M.)

time allowed for this question was 1 hour.

2 **rolling cube**

a,b and c are natural numbers. we have a $(2a+1)\times(2b+1)\times(2c+1)$ cube. this cube is on an infinite plane with unit squares. you call roll the cube to every side you want. faces of the cube are divided to unit squares and the square in the middle of each face is coloured (it means that if this square goes on a square of the plane, then that square will be coloured.) prove that if any two of lengths of sides of the cube are relatively prime, then we can colour every square in plane.

time allowed for this question was 1 hour.

3 **points in plane**

set A containing n points in plane is given. a *copy* of A is a set of points that is made by using transformation, rotation, homogeneity or their combination on elements of A. we want to put n *copies* of A in plane, such that every two copies have exactly one point in common and every three of them have no common elements. a) prove that if no 4 points of A make a parallelogram, you can do this only using transformation. (A doesn't have a parallelogram with angle 0 and a parallelogram that it's two non-adjacent vertices are one!) b) prove that you can always do this by using a combination of all these things.

time allowed for this question was 1 hour and 30 minutes

4 **carpeting**

suppose that S is a figure in the plane such that it's border doesn't contain any lattice points. suppose that x,y are two lattice points with the distance 1 (we call a point lattice point if it's coordinates are integers). suppose that we can cover the plane with copies of S such that x,y always go on lattice points (you can rotate or reverse copies of S). prove that the area of S is equal to lattice points inside it.

time allowed for this question was 1 hour.

5 **interesting sequence**

n is a natural number and $x_1, x_2, ...$ is a sequence of numbers 1 and -1 with these properties:

it is periodic and its least period number is $2^n - 1$. (it means that for every natural number j we have $x_{j+2^n-1} = x_j$ and $2^n - 1$ is the least number with this property.)

There exist distinct integers $0 \le t_1 < t_2 < ... < t_k < n$ such that for every natural number j we have

$$x_{j+n} = x_{j+t_1} \times x_{j+t_2} \times ... \times x_{j+t_k}$$

Prove that for every natural number s that $s < 2^n - 1$ we have

$$\sum_{i=1}^{2^n-1} x_i x_{i+s} = -1$$

Time allowed for this question was 1 hours and 15 minutes.

6 **polyhedral**

we call a 12-gon in plane good whenever: first, it should be regular, second, it's inner plane must be filled!!, third, it's center must be the origin of the coordinates, forth, it's vertices must have points $(0,1)$,$(1,0)$,$(-1,0)$ and $(0,-1)$. find the faces of the massivest polyhedral that it's image on every three plane xy,yz and zx is a good 12-gon. (it's obvios that centers of these three 12-gons are the origin of coordinates for three dimensions.)

time allowed for this question is 1 hour.

7 **interesting function**

S is a set with n elements and $P(S)$ is the set of all subsets of S and $f : P(S) \to \mathbb{N}$ is a function with these properties: for every subset A of S we have $f(A) = f(S - A)$. for every two subsets of S like A and B we have $max(f(A), f(B)) \ge f(A \cup B)$ prove that number of natural numbers like x such that there exists $A \subseteq S$ and $f(A) = x$ is less than n.

time allowed for this question was 1 hours and 30 minutes.

8 **numbers $n^2 + 1$**

prove that there infinitly many natural numbers in the form $n^2 + 1$ such that they don't have any divider in the form of $k^2 + 1$ except 1 and itself.

time allowed for this question was 45 minutes.

iran
National Math Olympiad (3rd Round)
2010

Day 6

1 prove that the group of oriention-preserving symmetries of a cube is isomorph to S_4(group of permutations of $\{1, 2, 3, 4\}$).(20 points)

2 prove the third sylow theorem: suppose that G is a group and $|G| = p^e m$ which p is a prime number and $(p, m) = 1$. suppose that a is the number of p-sylow subgroups of G ($H < G$ that $|H| = p^e$). prove that $a|m$ and $p|a - 1$.(Hint: you can use this: every two p-sylow subgroups are conjugate.)(20 points)

3 suppose that $G < S_n$ is a subgroup of permutations of $\{1, ..., n\}$ with this property that for every $e \neq g \in G$ there exist exactly one $k \in \{1, ..., n\}$ such that $g.k = k$. prove that there exist one $k \in \{1, ..., n\}$ such that for every $g \in G$ we have $g.k = k$.(20 points)

4 a) prove that every discrete subgroup of $(\mathbb{R}^2, +)$ is in one of these forms: i-$\{0\}$. ii-$\{mv|m \in \mathbb{Z}\}$ for a vector v in $\mathbb{R}^2$. iii-$\{mv + nw|m, n \in \mathbb{Z}\}$ for tho linearly independent vectors v and w in $\mathbb{R}^2$.(lattice L) b) prove that every finite group of symmetries that fixes the origin and the lattice L is in one of these forms: $\mathcal{C}_i$ or $\mathcal{D}_i$ that $i = 1, 2, 3, 4, 6$ ($\mathcal{C}_i$ is the cyclic group of order i and $\mathcal{D}_i$ is the dyhedral group of order i).(20 points)

5 suppose that p is a prime number. find that smallest n such that there exists a non-abelian group G with $|G| = p^n$.

SL is an acronym for Special Lesson. this year our special lesson was Groups and Symmetries.

the exam time was 5 hours.

iran

National Math Olympiad (Second Round)

2010

$\boxed{1}$ Let a, b be two positive integers and $a > b$.We know that $\gcd(a-b, ab+1) = 1$ and $\gcd(a+b, ab-1) = 1$. Prove that $(a-b)^2 + (ab+1)^2$ is not a perfect square.

$\boxed{2}$ There are n points in the page such that no three of them are collinear.Prove that number of triangles that vertices of them are chosen from these n points and area of them is 1,is not greater than $\frac{2}{3}(n^2 - n)$.

$\boxed{3}$ Circles W_1, W_2 meet at $D$and P.A and B are on W_1, W_2 respectively,such that AB is tangent to W_1 and W_2.Suppose D is closer than P to the line AB. AD meet circle W_2 for second time at C.If M be the midpoint of BC,prove that

$$D\hat{P}M = B\hat{D}C$$

$\boxed{4}$ Let $P(x) = ax^3 + bx^2 + cx + d$ be a polynomial with real coefficients such that

$$\min\{d, b+d\} > \max\{|c|, |a+c|\}$$

Prove that $P(x)$ do not have a real root in $[-1, 1]$.

$\boxed{5}$ In triangle ABC,$\hat{A} = \frac{\pi}{3}$.Construct E and F on continue of AB and AC respectively such that $BE = CF = BC$.EF meet circumcircle of $\triangle ACE$ in K.($K \not\equiv E$).Prove that K is on the bisector of $\hat{A}$.

$\boxed{6}$ A school has n students and some super classes are provided for them. Each student can participate in any number of classes that he/she wants. Every class has at least two students participating in it. We know that if two different classes have at least two common students, then the number of the students in the first of these two classes is different from the number of the students in the second one. Prove that the number of classes is not greater that $(n-1)^2$.

japan

Kyoto University Entry Examination

2010

$\boxed{\text{1A}}$ In the coordinate plane, denote by $S(a)$ the area of the region bounded by the line passing through the point $(1,\ 2)$ with the slope a and the parabola $y = x^2$. When a varies in the range of $0 \leq a \leq 6$, find the value of a such that $S(a)$ is minimized.

$\boxed{\text{1B}}$ Given a $\triangle ABC$ such that $AB = 2,\ AC = 1$. A bisector of $\angle BAC$ intersects with BC at D. If $AD = BD$, then find the area of $\triangle ABC$.

$\boxed{2}$ In the coordinate palne, when the point $P(x,\ y)$ moves in the domain of $4x + y \leq 9,\ x + 2y \geq 4,\ 2x - 3y \geq -6$, find the maximum and minimum value of $2x + y,\ x^2 + y^2$ respectively.

$\boxed{3}$ Arrange numbers $1,\ 2,\ 3,\ 4,\ 5$ in a line. Any arrangements are equiprobable. Find the probability such that the sum of the numbers for the first, second and third equal to the sum of that of the third, fourth and fifth. Note that in each arrangement each number are used one time without overlapping.

$\boxed{4}$ Given a regular decagon with the center O and two neibouring vertices $A,\ B$. Take a point P on the line segmemt OB such that $OP^2 = OB \cdot PB$. Prove that $OP = AB$.

$\boxed{5}$ In the coordinate space, consider the cubic with vertices $O(0,\ 0,\ 0),\ A(1,\ 0,\ 0),\ B(1,\ 1,\ 0),\ C(0,\ 1,\ 0),\ D(0,\ 0$
Find the volume of the solid generated by revolution of the cubic around the diagonal OF as the axis of rotation.

japan
Tokio University Entry Examination
2010

$\boxed{1}$ Let the lengths of the sides of a cuboid be denoted a, b and c. Rotate the cuboid in 90° the side with length b as the axis of the cuboid. Denote by V the solid generated by sweeping the cuboid.

(1) Express the volume of V in terms of $a,\ b,\ c$.

(2) Find the range of the volume of V with $a + b + c = 1$.

$\boxed{2}$ (1) Show the following inequality for every natural number k.

$$\frac{1}{2(k+1)} < \int_0^1 \frac{1-x}{k+x}dx < \frac{1}{2k}$$

(2) Show the following inequality for every natural number $m,\ n$ such that $m > n$.

$$\frac{m-n}{2(m+1)(n+1)} < \log\frac{m}{n} - \sum_{k=n+1}^{m}\frac{1}{k} < \frac{m-n}{2mn}$$

$\boxed{3}$ There are two boxes L and R such that L contains x balls and R contains $30 - x$ balls for $0 \leq x \leq 30$. Repeat the following operation ():

() Let z be the number of ball in L. Flip a coin with equal probability of coming up Head and Tails, if Head comes up, we move $K(z)$' s balls from box R to box L and if Tails comes up, we move $K(z)$'s balls from box L to box R . Note that $K(z) = z$ when $0 \leq z \leq 15,\ K(z) = 30 - z$ when $16 \leq z \leq 30$.

After m^{th} operation, denote by $P_m(x)$ the probability such that the number of balls in box L is 30. For example, $P_1(15) = P_2(15) = \frac{1}{2}$. Answer the following questions (1), (2), (3).

(1) For $m \geq 2$, by selecting y for x successfully, express $P_m(x)$ in terms of $P_{m-1}(y)$.

(2) For each natural number n, find $P_{2n}(10)$.

in addition to, for applicants for Science,

(3) For each natural number n, find $P_{4n}(6)$.

$\boxed{4}$ In the coordinate plane with $O(0,\ 0)$, consider the function $C:\ y = \frac{1}{2}x + \sqrt{\frac{1}{4}x^2 + 2}$ and two distinct points $P_1(x_1,\ y_1),\ P_2(x_2,\ y_2)$ on C.

(1) Let H_i $(i = 1,\ 2)$ be the intersection points of the line passing through P_i $(i = 1,\ 2)$, parallel to x axis and the line $y = x$. Show that the area of $\triangle OP_1H_1$ and $\triangle OP_2H_2$ are equal.

(2) Let $x_1 < x_2$. Express the area of the figure bounded by the part of $x_1 \leq x \leq x_2$ for C and line segments $P_1O,\ P_2O$ in terms of $y_1,\ y_2$.

$\boxed{5}$ Let C be the circumference of a circle with radius 1. Take a point A on C. Three points $P,\ Q,\ R$ start from A at the time $t = 0$ on C at fixed speed respectively, $P,\ Q$ move in counterclockwise, R moves in clockwise during $0 \leq t \leq 2\pi$ with speed of $P,\ Q$ and R, $m,\ 1$ and 2 respectively,or Q moves around C one time. Note that m is integer such that $1 \leq m \leq 10$. Determine all possible pairs of $(m,\ t)$ such that $\triangle PQR$ is a Rectangular Isoceles Triangle with hypotenuse PR.

$\boxed{6}$ Given a tetrahedron with four congruent faces such that $OA = 3,\ OB = \sqrt{7},\ AB = 2$. Denote by L a plane which contains three points $O,\ A,\ B$.

(1) Let H be the foot of the perpendicular drawn fron the point C to the plane L. Express $\overrightarrow{OH}$ in terms of $\overrightarrow{OA},\ \overrightarrow{OB}$.

(2) For a real number t with $0 < t < 1$, let $P_t,\ Q_t$ be the points which divide internally the line segments $OA,\ OB$ respectively. Denote by M a plane which is perpendicular to the plane L. Find the sectional area $S(t)$ of the tetrahedron $OABC$ cut by the plane M.

(3) When t moves in the range of $0 < t < 1$, find the maximum value of $S(t)$.

japan

Tokyo Institute Of Technology Admission Office Entrance Examination

2010

$\boxed{1}$ Find all positive integers n such that $n!$ is a multiple of n^2.

2010 Tokyo Institute of Technology Admission Office entrance exam, Problem I-1/Science

$\boxed{2}$ Given a triangle with side lengths $a,\ b,\ c$. Let $a,\ b,\ c$ vary, find the range of $\frac{a^2+b^2+c^2}{ab+bc+ca}$.

2010 Tokyo Institute of Technology Admission Office entrance exam, Problem I-2/Science

$\boxed{3}$ Find all positive integers n such that there exists the polynomial with degree n satisfying $f(x^2+1)=f(x)^2+1$.

2010 Tokyo Institute of Technology Admission Office entrance exam, Problem -1/Science

$\boxed{4}$ Given a regular nonagon inscribed in a circle with radius 1. Find five numbers of the edges of a regular polygon which has all vertices in the perimeter of the nonagon, then for each n, give an example of such regular n polygon to find the side length.

2010 Tokyo Institute of Technology Admission Office entrance exam, Problem -2/Science

kazakhstan
National Olympiad
2010

Day 9

$\boxed{1}$ Triangle ABC is given. Circle ω passes through B, touch AC in D and intersect sides AB and BC at P and Q respectively. Line PQ intersect BD and AC at M and N respectively. Prove that ω, circumcircle of DMN and circle, touching PQ in M and passes through B, intersects in one point.

$\boxed{2}$ Exactly $4n$ numbers in set $A = \{1, 2, 3, ..., 6n\}$ of natural numbers painted in red, all other in blue. Proved that exist $3n$ consecutive natural numbers from A, exactly $2n$ of which numbers is red.

$\boxed{3}$ Positive real A is given. Find maximum value of M for which inequality
$\frac{1}{x} + \frac{1}{y} + \frac{A}{x+y} \geq \frac{M}{\sqrt{xy}}$
holds for all $x, y > 0$

$\boxed{4}$ Let x- minimal root of equation $x^2 - 4x + 2 = 0$. Find two first digits of number $\{x + x^2 + ... + x^{20}\}$ after 0, where $\{a\}$- fractional part of a.

$\boxed{5}$ Arbitrary triangle ABC is given (with $AB < BC$). Let M - midpoint of AC, N- midpoint of arc AC of circumcircle ABC, which is contains point B. Let I - in-center of ABC. Proved, that $\angle IMA = \angle INB$

$\boxed{6}$ Let numbers $1, 2, 3, ..., 2010$ stand in a row at random. Consider row, obtain by next rule: For any number we sum it and it's number in a row (For example for row $(2, 7, 4)$ we consider a row $(2 + 1; 7 + 2; 4 + 3) = (3; 9; 7)$); Proved, that in resulting row we can found two equals numbers, or two numbers, which is differ by 2010

kazakhstan
National Olympiad
2010

Day 10

$\boxed{1}$ Triangle ABC is given. Consider ellipse Ω_1, passes through C with focuses in A and B. Similarly define ellipses Ω_2, Ω_3 with focuses B, C and C, A respectively. Prove, that if all ellipses have common point D then A, B, C, D lies on the circle.

Ellipse with focuses X, Y, passes through Z- locus of point T, such that $XT+YT = XZ+YZ$

$\boxed{2}$ On sides of convex quadrilateral $ABCD$ on external side constructed equilateral triangles ABK, BCL, CDM, DAN. Let P, Q- midpoints of BL, AN respectively and X- circumcenter of CMD. Prove, that PQ perpendicular to KX

$\boxed{3}$ Call $A \in \mathbb{N}^0$ be $numberofyear$ if all digits of A equals 0, 1 or 2 (in decimal representation). Prove that exist infinity $N \in \mathbb{N}$, such that N can't presented as $A^2 + B$ where $A \in \mathbb{N}^0$; B-$numberofyear$.

$\boxed{4}$ Given, that for any $n \in \mathbb{N}$ exist natural a, such that $a^{n-1} \equiv 1(modn)$ and for any prime divisor p of $(n-1)$, $a^{\frac{n-1}{p}} - 1$ don't divided by n.

Prove that n is prime.

$\boxed{5}$ Let $n \geq 2$ be an integer. Define $x_i = 1$ or -1 for every $i = 1, 2, 3, \cdots, n$.

Call an operation *adhesion*, if it changes the string $(x_1, x_2, \cdots, x_n)$ to $(x_1x_2, x_2x_3, \cdots, x_{n-1}x_n, x_nx_1)$.

Find all integers $n \geq 2$ such that the string $(x_1, x_2, \cdots, x_n)$ changes to $(1, 1, \cdots, 1)$ after finitely *adhesion* operations.

$\boxed{6}$ Let $ABCD$ be convex quadrilateral, such that exist M, N inside $ABCD$ for which $\angle NAD = \angle MAB; \angle NBC = \angle MBA; \angle MCB = \angle NCD; \angle NDA = \angle MDC$ Prove, that $S_{ABM} + S_{ABN} + S_{CDM} + S_{CDN} = S_{BCM} + S_{BCN} + S_{ADM} + S_{ADN}$, where S_{XYZ}-area of triangle XYZ

kazakhstan
National Olympiad
2010

Day 11

$\boxed{1}$ Given, that for any $n \in \mathbb{N}$ exist natural a, such that $a^{n-1} \equiv 1(modn)$ and for any prime divisor p of $(n-1)$, $a^{\frac{n-1}{p}} - 1$ don't divided by n.

Prove that n is prime.

$\boxed{2}$ Let $n \geq 2$ be an integer. Define $x_i = 1$ or -1 for every $i = 1, 2, 3, \cdots, n$.

Call an operation *adhesion*, if it changes the string $(x_1, x_2, \cdots, x_n)$ to $(x_1x_2, x_2x_3, \cdots, x_{n-1}x_n, x_nx_1)$.

Find all integers $n \geq 2$ such that the string $(x_1, x_2, \cdots, x_n)$ changes to $(1, 1, \cdots, 1)$ after finitely *adhesion* operations.

$\boxed{3}$ Let $ABCD$ be convex quadrilateral, such that exist M, N inside $ABCD$ for which $\angle NAD = \angle MAB; \angle NBC = \angle MBA; \angle MCB = \angle NCD; \angle NDA = \angle MDC$ Prove, that $S_{ABM} + S_{ABN} + S_{CDM} + S_{CDN} = S_{BCM} + S_{BCN} + S_{ADM} + S_{ADN}$, where S_{XYZ}-area of triangle XYZ

$\boxed{4}$ For $x; y \geq 0$ prove the inequality:

$\sqrt{x^2 - x + 1}\sqrt{y^2 - y + 1} + \sqrt{x^2 + x + 1}\sqrt{y^2 + y + 1} \geq 2(x + y)$

$\boxed{5}$ Let O be the circumcircle of acute triangle ABC, AD-altitude of ABC ($D \in BC$), $AD \cap CO = E$, M-midpoint of AE, F-feet of perpendicular from C to AO. Proved that point of intersection OM and BC lies on circumcircle of triangle BOF

$\boxed{6}$ Call $A \in \mathbb{N}^0$ be $numberofyear$ if all digits of A equals 0, 1 or 2 (in decimal representation). Prove that exist infinity $N \in \mathbb{N}$, such that N can't presented as $A^2 + B$ where $A \in \mathbb{N}^0; B$-$numberofyear$.

Korea
National Olympiad
2010

Day 1 - *27 March 2010*

$\boxed{1}$ Given an arbitrary triangle ABC, denote by P, Q, R the intersections of the incircle with sides BC, CA, AB respectively. Let the area of triangle ABC be T, and its perimeter L. Prove that the inequality $(\frac{AB}{PQ})^3 + (\frac{BC}{QR})^3 + (\frac{CA}{RP})^3 \geq \frac{2}{\sqrt{3}} \cdot \frac{L^2}{T}$ holds.

*Sorry, I made a typo: I wrote D, E, F where P, Q, R should be.

$\boxed{2}$ Let I be the incenter and O the circumcenter of a given acute triangle ABC. The incircle is tangent to BC at D. Assume that $\angle B < \angle C$ and the segments AO and HD are parallel. Let the intersection of the line OD and AH be E. If the midpoint of CI is F, prove that E, F, I, O are concyclic.

[Edit] H is the orthocenter of ABC.

$\boxed{3}$ There are n websites $1, 2, ..., n (n \geq 2)$. If there is a link website i to j, we can use this link so we can move website i to j. All for $i \in 1, 2, ..., n-1$, There is a link website $i-1$ to i. Prove that we can add less or equal than $3(n-1)\log_2(\log_2 n)$ links so that all integer $1 \leq i < j \leq n$, starting with website i, and using at most three links to website j.(If we use a link, website's number should increase. For example, No.7 to 4 is impossible.)

Sorry for my bad English.

Day 2 - *28 March 2010*

$\boxed{4}$ Given is a trapezoid $ABCD$ where AB and CD are parallel, and A, B, C, D are clockwise in this order. Let Γ_1 be the circle with center A passing through B, Γ_2 be the circle with center C passing through D. The intersection of line BD and Γ_1 is P $(\neq B, D)$. Denote by Γ the circle with diameter PD, and let Γ and Γ_1 meet at $X(\neq P)$. Γ and Γ_2 meet at Y. If the circumcircle of triangle XBY and Γ_2 meet at Q, prove that B, D, Q are collinear.

$\boxed{5}$ On a circular table are sitting $2n$ people, equally spaced in between. m cookies are given to these people, and they give cookies to their neighbors according to the following rule.

(i) One may give cookies only to people adjacent to himself. (ii) In order to give a cookie to one's neighbor, one must eat a cookie.

Select arbitrarily a person A sitting on the table. Find the minimum value m such that there is a strategy in which A can eventually receive a cookie, independent of the distribution of cookies at the beginning.

$\boxed{6}$ An arbitrary prime p is given. If an integer sequence $(n_1, n_2, \cdots, n_k)$ satisfying the conditions - For all $i = 1, 2, \cdots, k$, $n_i \geq \frac{p+1}{2}$ - For all $i = 1, 2, \cdots, k$, $p^{n_i} - 1$ is divisible by n_{i+1}, and $\frac{p^{n_i}-1}{n_{i+1}}$ is coprime to n_{i+1}. Let $n_{k+1} = n_1$. exists not for $k = 1$, but exists for some $k \geq 2$, then call the prime a good prime. Prove that a prime is good iff it is not 2.

moldova
Team Selection Test
2010

Day 1

$\boxed{1}$ Find all 3-digit numbers such that placing to the right side of the number its successor we get a 6-digit number which is a perfect square.

$\boxed{2}$ Prove that for any real number x the following inequality is true: $\max\{|\sin x|, |\sin(x+2010)|\} > \frac{1}{\sqrt{17}}$

$\boxed{3}$ Let $ABCD$ be a convex quadrilateral. We have that $\angle BAC = 3\angle CAD$, $AB = CD$, $\angle ACD = \angle CBD$. Find angle $\angle ACD$

$\boxed{4}$ Let $n \geq 6$ be a even natural number. Prove that any cube can be divided in $\frac{3n(n-2)}{4} + 2$ cubes.

moldova

Team Selection Test

2010

Day 2

1. Let $p \in \mathbb{R}_+$ and $k \in \mathbb{R}_+$. The polynomial $F(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + k^4$ with real coefficients has 4 negative roots. Prove that $F(p) \geq (p+k)^4$

2. Let $x_1, x_2, \ldots, x_n$ be positive real numbers with sum 1. Find the integer part of: $E = x_1 + \dfrac{x_2}{\sqrt{1-x_1^2}} + \dfrac{x_3}{\sqrt{1-(x_1+x_2)^2}} + \cdots + \dfrac{x_n}{\sqrt{1-(x_1+x_2+\cdots+x_{n-1})^2}}$

3. Let ABC be an acute triangle. H is the orthocenter and M is the middle of the side BC. A line passing through H and perpendicular to HM intersect the segment AB and AC in P and Q. Prove that $MP = MQ$

4. In a chess tournament $2n+3$ players take part. Every two play exactly one match. The schedule is such that no two matches are played at the same time, and each player, after taking part in a match, is free in at least n next (consecutive) matches. Prove that one of the players who play in the opening match will also play in the closing match.

portugal
NMO
2010

Day 1

1. There are several candles of the same size on the Chapel of Bones. On the first day a candle is lit for a hour. On the second day two candles are lit for a hour, on the third day three candles are lit for a hour, and successively, until the last day, when all the candles are lit for a hour. On the end of that day, all the candles were completely consumed. Find all the possibilities for the number of candles.

2. On a circumference, points A and B are on opposite arcs of diameter CD. Line segments CE and DF are perpendicular to AB such that $A-E-F-B$ (i.e., A, E, F and B are collinear on this order). Knowing $AE=1$, find the length of BF.

3. On each day, more than half of the inhabitants of vora eats *sericaia* as dessert. Show that there is a group of 10 inhabitants of vora such that, on each of the last 2010 days, at least one of the inhabitants ate *sericaia* as dessert.

portugal
NMO
2010

Day 2

1. Giraldo wrote five distinct natural numbers on the vertices of a pentagon. And next he wrote on each side of the pentagon the least common multiple of the numbers written of the two vertices who were on that side and noticed that the five numbers written on the sides were equal. What is the smallest number Giraldo could have written on the sides?

2. Show that any triangle has two sides whose lengths a and b satisfy $\frac{\sqrt{5}-1}{2} < \frac{a}{b} < \frac{\sqrt{5}+1}{2}$.

3. Consider a square $(p-1) \times (p-1)$, where p is a prime number, which is divided by squares 1×1 whose sides are parallel to the initial square's sides. Show that it is possible to select p vertices such that there are no three collinear vertices.

russia

All-Russian Olympiad

2010

Grade 9

Day 1

1. There are 24 different pencils, 4 different colors, and 6 pencils of each color. They were given to 6 children in such a way that each got 4 pencils. What is the least number of children that you can randomly choose so that you can guarantee that you have pencils of all colors.

   P.S. for 10 grade gives same problem with 40 pencils, 10 of each color and 10 children.

2. There are 100 random, distinct real numbers corresponding to 100 points on a circle. Prove that you can always choose 4 consecutive points in such a way that the sum of the two numbers corresponding to the points on the outside is always greater than the sum of the two numbers corresponding to the two points on the inside.

   It's problem was in 9 and 10 grade.

3. Lines tangent to circle O in points A and B, intersect in point P. Point Z is the center of O. On the minor arc AB, point C is chosen not on the midpoint of the arc. Lines AC and PB intersect at point D. Lines BC and AP intersect at point E. Prove that the circumcentres of triangles ACE, BCD, and PCZ are collinear.

4. There are 100 apples on the table with total weight of 10 kg. Each apple weighs no less than 25 grams. The apples need to be cut for 100 children so that each of the children gets 100 grams. Prove that you can do it in such a way that each piece weighs no less than 25 grams.

space*0.4cm

Day 2

1. Let $a \neq ba, b \in \mathbb{R}$ such that $(x^2 + 20ax + 10b)(x^2 + 20bx + 10a) = 0$ has no roots for x. Prove that $20(b - a)$ is not an integer.

   It problem was in 9 and 10 grade.

2. Each of 1000 elves has a hat, red on the inside and blue on the outside or vise versa. An elf with a hat that is red outside can only lie, and an elf with a hat that is blue outside can only tell the truth. One day every elf tells every other elf, Your hat is red on the outside. During that day, some of the elves turn their hats inside out at any time during the day. (An elf can do that more than once per day.) Find the smallest possible number of times any hat is turned inside out.

$\boxed{3}$ Let us call a natural number *unlucky* if it cannot be expressed as $\frac{x^2-1}{y^2-1}$ with natural numbers $x, y > 1$. Is the number of *unlucky* numbers finite or infinite?

$\boxed{4}$ In a acute triangle ABC, the median, AM, is longer than side AB. Prove that you can cut triangle ABC into 3 parts out of which you can construct a rhombus.

Grade 10

Grade 11

russia

Sharygin Geometry Olympiad

2010

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1. Does there exist a triangle, whose side is equal to some of its altitudes, another side is equal to some of its bisectors, and the third is equal to some of its medians?

2. Bisectors AA_1 and BB_1 of a right triangle ABC ($\angle C = 90°$) meet at a point I. Let O be the circumcenter of triangle CA_1B_1. Prove that $OI \perp AB$.

3. Points A', B', C' lie on sides BC, CA, AB of triangle ABC. for a point X one has $\angle AXB = \angle A'C'B' + \angle ACB$ and $\angle BXC = \angle B'A'C' + \angle BAC$. Prove that the quadrilateral $XA'BC'$ is cyclic.

4. The diagonals of a cyclic quadrilateral $ABCD$ meet in a point N. The circumcircles of triangles ANB and CND intersect the sidelines BC and AD for the second time in points A_1, B_1, C_1, D_1. Prove that the quadrilateral $A_1B_1C_1D_1$ is inscribed in a circle centered at N.

5. A point E lies on the altitude BD of triangle ABC, and $\angle AEC = 90°$. Points O_1 and O_2 are the circumcenters of triangles AEB and CEB; points F, L are the midpoints of the segments AC and O_1O_2. Prove that the points L, E, F are collinear.

6. Points M and N lie on the side BC of the regular triangle ABC (M is between B and N), and $\angle MAN = 30°$. The circumcircles of triangles AMC and ANB meet at a point K. Prove that the line AK passes through the circumcenter of triangle AMN.

7. The line passing through the vertex B of a triangle ABC and perpendicular to itst median BM intersects the altitudes dropped from A and C (or their extensions) in points K and N. Points O_1 and O_2 are the circumcenters of the triangles ABK and CBN respectively. Prove that $O_1M = O_2M$.

8. Let AH be the altitude of a given triangle ABC. The points I_b and I_c are the incenters of the triangles ABH and ACH respectively. BC touches the incircle of the triangle ABC at a point L. Find $\angle LI_bI_c$.

9. A point inside a triangle is called "*good*" if three cevians passing through it are equal. Assume for an isosceles triangle ABC ($AB = BC$) the total number of "*good*" points is odd. Find all possible values of this number.

10. Let three lines forming a triangle ABC be given. Using a two-sided ruler and drawing at most eight lines construct a point D on the side AB such that $\frac{AD}{BD} = \frac{BC}{AC}$.

11. A convex $n-$gon is split into three convex polygons. One of them has n sides, the second one has more than n sides, the third one has less than n sides. Find all possible values of n.

12. Let AC be the greatest leg of a right triangle ABC, and CH be the altitude to its hypotenuse. The circle of radius CH centered at H intersects AC in point M. Let a point B' be the

reflection of B with respect to the point H. The perpendicular to AB erected at B' meets the circle in a point K. Prove that

a) $B'M \parallel BC$

b) AK is tangent to the circle.

13 Let us have a convex quadrilateral $ABCD$ such that $AB = BC$. A point K lies on the diagonal BD, and $\angle AKB + \angle BKC = \angle A + \angle C$. Prove that $AK \cdot CD = KC \cdot AD$.

14 We have a convex quadrilateral $ABCD$ and a point M on its side AD such that CM and BM are parallel to AB and CD respectively. Prove that $S_{ABCD} \geq 3S_{BCM}$.

Remark. S denotes the area function.

15 Let AA_1, BB_1 and CC_1 be the altitudes of an acute-angled triangle ABC. AA_1 meets B_1C in a point K. The circumcircles of triangles A_1KC_1 and A_1KB_1 intersect the lines AB and AC for the second time at points N and L respectively. Prove that

a) The sum of diameters of these two circles is equal to BC,

b) $\frac{A_1N}{BB_1} + \frac{A_1L}{CC_1} = 1$.

16 A circle touches the sides of an angle with vertex A at points B and C. A line passing through A intersects this circle in points D and E. A chord BX is parallel to DE. Prove that XC passes through the midpoint of the segment DE.

17 Construct a triangle, if the lengths of the bisectrix and of the altitude from one vertex, and of the median from another vertex are given.

18 A point B lies on a chord AC of circle ω. Segments AB and BC are diameters of circles ω_1 and ω_2 centered at O_1 and O_2 respectively. These circles intersect ω for the second time in points D and E respectively. The rays O_1D and O_2E meet in a point F, and the rays AD and CE do in a point G. Prove that the line FG passes through the midpoint of the segment AC.

19 A quadrilateral $ABCD$ is inscribed into a circle with center O. Points P and Q are opposite to C and D respectively. Two tangents drawn to that circle at these points meet the line AB in points E and F. (A is between E and B, B is between A and F). The line EO meets AC and BC in points X and Y respectively, and the line FO meets AD and BD in points U and V respectively. Prove that $XV = YU$.

20 The incircle of an acute-angled triangle ABC touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, B_2 are the midpoints of the segments B_1C_1, A_1C_1 respectively. Let P be a common point of the incircle and the line CO, where O is the circumcenter of triangle ABC. Let also A' and B' be the second common points of PA_2 and PB_2 with the incircle.

Prove that a common point of AA' and BB' lies on the altitude of the triangle dropped from the vertex C.

21 A given convex quadrilateral $ABCD$ is such that $\angle ABD + \angle ACD > \angle BAC + \angle BDC$. Prove that

$$S_{ABD} + S_{ACD} > S_{BAC} + S_{BDC}.$$

22 A circle centered at a point F and a parabola with focus F have two common points. Prove that there exist four points A, B, C, D on the circle such that the lines AB, BC, CD and DA touch the parabola.

23 A cyclic hexagon $ABCDEF$ is such that $AB \cdot CF = 2BC \cdot FA, CD \cdot EB = 2DE \cdot BC$ and $EF \cdot AD = 2FA \cdot DE$. Prove that the lines AD, BE and CF are concurrent.

24 Let us have a line ℓ in the space and a point A not lying on ℓ. For an arbitrary line ℓ' passing through A, XY (Y is on ℓ') is a common perpendicular to the lines ℓ and ℓ'. Find the locus of points Y.

25 For two different regular icosahedrons it is known that some six of their vertices are vertices of a regular octahedron. Find the ratio of the edges of these icosahedrons.

switzerland

Schweizer Mathematik-Olympiade

2010

$\boxed{1}$ Three coins lie on integer points on the number line. A move consists of choosing and moving two coins, the first one 1 unit to the right and the second one 1 unit to the left. Under which initial conditions is it possible to move all coins to one single point?

$\boxed{2}$ Let $\triangle ABC$ be a triangle with $AB \neq AC$. The incircle with centre I touches BC, CA, AB at D, E, F, respectively. Furthermore let M the midpoint of EF and AD intersect the incircle at $P \neq D$. Show that $PMID$ ist cyclic.

$\boxed{3}$ For $n \in \mathbb{N}$, determine the number of natural solutions (a, b) such that

$$(4a - b)(4b - a) = 2010^n$$

holds.

$\boxed{4}$ Let x, y, $z \in \mathbb{R}^+$ satisfying $xyz = 1$. Prove that

$$\frac{(x+y-1)^2}{z} + \frac{(y+z-1)^2}{x} + \frac{(z+x-1)^2}{y} \geqslant x + y + z.$$

$\boxed{5}$ Some sides and diagonals of a regular n-gon form a connected path that visits each vertex exactly once. A *parallel pair* of edges is a pair of two different parallel edges of the path. Prove that (a) if n is even, there is at least one *parallel pair.* (b) if n is odd, there can't be one single *parallel pair.*

$\boxed{6}$ Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that for all x, $y \in \mathbb{R}$,

$$f(f(x)) + f(f(y)) = 2y + f(x - y)$$

holds.

$\boxed{7}$ Let m, n be natural numbers such that $m + n + 1$ is prime and divides $2(m^2 + n^2) - 1$. Prove that $m = n$.

$\boxed{8}$ In a village with at least one inhabitant, there are several associations. Each inhabitant is a member of at least k associations, and any two associations have at most one common member. Prove that at least k associations have the same number of members.

$\boxed{9}$ Let k and k' two concentric circles centered at O, with k' being larger than k. A line through O intersects k at A and k' at B such that O seperates A and B. Another line through O intersects k at E and k' at F such that E separates O and F. Show that the circumcircle of $\triangle OAE$ and the circles with diametres AB and EF have a common point.

$\boxed{10}$ Let $n \geqslant 3$ and P a convex n-gon. Show that P can be, by $n - 3$ non-intersecting diagonals, partitioned in triangles such that the circumcircle of each triangle contains the whole area of P. Under which conditions is there exactly one such triangulation?

turkey
Team Selection Tests
2010

Day 1 - *27 March 2010*

1. D, E, F are points on the sides AB, BC, CA, respectively, of a triangle ABC such that $AD = AF$, $BD = BE$, and $DE = DF$. Let I be the incenter of the triangle ABC, and let K be the point of intersection of the line BI and the tangent line through A to the circumcircle of the triangle ABI. Show that $AK = EK$ if $AK = AD$.

2. Show that

$$\sum_{cyc} \sqrt[4]{\frac{(a^2+b^2)(a^2-ab+b^2)}{2}} \le \frac{2}{3}(a^2+b^2+c^2)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

for all positive real numbers a, b, c.

3. A teacher wants to divide the 2010 questions she asked in the exams during the school year into three folders of 670 questions and give each folder to a student who solved all 670 questions in that folder. Determine the minimum number of students in the class that makes this possible for all possible situations in which there are at most two students who did not solve any given question.

turkey
Team Selection Tests
2010

Day 2 - *28 March 2010*

1. Let $0 \leq k < n$ be integers and $A = \{a \,:\, a \equiv k \pmod n\}$. Find the smallest value of n for which the expression
$$\frac{a^m + 3^m}{a^2 - 3a + 1}$$
does not take any integer values for $(a, m) \in A \times \mathbb{Z}^+$.

2. For an interior point D of a triangle ABC, let Γ_D denote the circle passing through the points A, E, D, F if these points are concyclic where $BD \cap AC = \{E\}$ and $CD \cap AB = \{F\}$. Show that all circles Γ_D pass through a second common point different from A as D varies.

3. Let Λ be the set of points in the plane whose coordinates are integers and let F be the collection of all functions from Λ to $\{1, -1\}$. We call a function f in F *perfect* if every function g in F that differs from f at finitely many points satisfies the condition
$$\sum_{0<d(P,Q)<2010} \frac{f(P)f(Q) - g(P)g(Q)}{d(P,Q)} \geq 0$$
where $d(P, Q)$ denotes the distance between P and Q. Show that there exist infinitely many *perfect* functions that are not translates of each other.

ukraine
Kyiv Mathematical Festival
2010

Grade 8

$\boxed{1}$ Bob has picked positive integer $1 < N < 100$. Alice tells him some integer, and Bob replies with the remainder of division of this integer by N. What is the smallest number of integers which Alice should tell Bob to determine N for sure?

$\boxed{2}$ Denote by $S(n)$ the sum of digits of integer n. Find 1) $S(3) + S(6) + S(9) + \ldots + S(300)$; 2) $S(3) + S(6) + S(9) + \ldots + S(3000)$.

$\boxed{3}$ Let O be the circumcenter and I be the incenter of triangle ABC. Prove that if $AI \perp OB$ and $BI \perp OC$ then $CI \parallel OA$.

$\boxed{4}$ 1) The numbers $1, 2, 3, \ldots, 2010$ are written on the blackboard. Two players in turn erase some two numbers and replace them with one number. The first player replaces numbers a and b with $ab-a-b$ while the second player replaces them with $ab+a+b$. The game ends when a single number remains on the blackboard. If this number is smaller than $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2010$ then the first player wins. Otherwise the second player wins. Which of the players has a winning strategy?

2) The numbers $1, 2, 3, \ldots, 2010$ are written on the blackboard. Two players in turn erase some two numbers and replace them with one number. The first player replaces numbers a and b with $ab - a - b + 2$ while the second player replaces them with $ab + a + b$. The game ends when a single number remains on the blackboard. If this number is smaller than $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2010$ then the first player wins. Otherwise the second player wins. Which of the players has a winning strategy?

$\boxed{5}$ 1) Cells of 8×8 table contain pairwise distinct positive integers. Each integer is prime or a product of two primes. It is known that for any integer a from the table there exists integer written in the same row or in the same column such that it is not relatively prime with a. Find maximum possible number of prime integers in the table.

2) Cells of $2n \times 2n$ table, $n \geq 2$, contain pairwise distinct positive integers. Each integer is prime or a product of two primes. It is known that for any integer a from the table there exist integers written in the same row and in the same column such that they are not relatively prime with a. Find maximum possible number of prime integers in the table.

Grade 9

Grade 10

Undergraduate Competitions

IMC

2010

Day 1 - *26 July 2010*

1. Let $0 < a < b$. Prove that $\int_a^b (x^2+1)e^{-x^2}dx \geq e^{-a^2} - e^{-b^2}$.

2. Compute the sum of the series $\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + ...$

3. Define the sequence $x_1, x_2, ...$ inductively by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Compute $\lim_{n\to\infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_n}{x_{n+1}}$.

4. Let a, b be two integers and suppose that n is a positive integer for which the set $\mathbb{Z} \backslash \{ax^n + by^n \mid x, y \in \mathbb{Z}\}$ is finite. Prove that $n = 1$.

5. Suppose that a, b, c are real numbers in the interval $[-1, 1]$ such that $1 + 2abc \geq a^2 + b^2 + c^2$. Prove that $1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$ for all positive integers n.

Day 2 - *27 July 2010*

1 [url=http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1961253p1961253]**IMC 2010 , Day 2**[/url]

Problem 1. (a) A sequence $x_1, x_2, \dots$ of real numbers satisfies

$$x_{n+1} = x_n \cos x_n \text{ for all } n \geq 1.$$

Does it follows that this sequence converges for all initial values x_1? (5 points)

(b) A sequence $y_1, y_2, \dots$ of real numbers satisfies

$$y_{n+1} = y_n \sin y_n \text{ for all } n \geq 1.$$

Does it follows that this sequence converges for all initial values y_1? (5 points)

2 [url=http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1961253p1961253]IMC 2010, Day 2[/url]

Let $a_0, a_1, \dots, a_n$ be positive real numbers such that $a_{k+1} - a_k \geq 1$ for all $k = 0, 1, \dots, n-1$. Prove that

$$1 + \frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right)\cdots\left(1 + \frac{1}{a_n - a_0}\right) \leq \left(1 + \frac{1}{a_0}\right)\left(1 + \frac{1}{a_1}\right)\cdots\left(1 + \frac{1}{a_n}\right).$$

3 **[url=http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1961253p1961253]IMO 2010, Day 2[/url]**

Denote by S_n the group of permutations of the sequence $(1, 2, \dots, n)$. Suppose that G is a subgroup of S_n, such that for every $\pi \in G \setminus \{e\}$ there exists a unique $k \in \{1, 2, \dots, n\}$ for which $\pi(k) = k$. (Here e is the unit element of the group S_n.) Show that this k is the same for all $\pi \in G \setminus \{e\}$.

4 [url=http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1961253p1961253]**IMC 2010, Day 2**[/url] Let A be a symmetric $m \times m$ matrix over the two-element field all of whose diagonal entries are zero. Prove that for every positive integer n each column of the matrix A^n has a zero entry.

5 [url=http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1961253p1961253]**IMC 2010, Day 2**[/url]

Suppose that for a function $f : \mathbb{R} \to \mathbb{R}$ and real numbers $a < b$ one has $f(x) = 0$ for all $x \in (a, b)$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ if

$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0$$

for every prime number p and every real number y.

USA
AIME
2010

I

$\boxed{1}$ Maya lists all the positive divisors of 2010^2. She then randomly selects two distinct divisors from this list. Let p be the probability that exactly one of the selected divisors is a perfect square. The probability p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

$\boxed{2}$ Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}}$ is divided by 1000.

$\boxed{3}$ Suppose that $y = \frac{3}{4}x$ and $x^y = y^x$. The quantity $x+y$ can be expressed as a rational number $\frac{r}{s}$, where r and s are relatively prime positive integers. Find $r+s$.

$\boxed{4}$ Jackie and Phil have two fair coins and a third coin that comes up heads with probability $\frac{4}{7}$. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where m and n are relatively prime positive integers. Find $m+n$.

$\boxed{5}$ Positive integers a, b, c, and d satisfy $a > b > c > d$, $a+b+c+d = 2010$, and $a^2-b^2+c^2-d^2 = 2010$. Find the number of possible values of a.

$\boxed{6}$ Let $P(x)$ be a quadratic polynomial with real coefficients satisfying $x^2 - 2x + 2 \le P(x) \le 2x^2 - 4x + 3$ for all real numbers x, and suppose $P(11) = 181$. Find $P(16)$.

$\boxed{7}$ Define an ordered triple (A, B, C) of sets to be minimally intersecting if $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $A \cap B \cap C = \emptyset$. For example, $(\{1,2\},\{2,3\},\{1,3,4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1,2,3,4,5,6,7\}$. Find the remainder when N is divided by 1000.

Note: $|S|$ represents the number of elements in the set S.

$\boxed{8}$ For a real number a, let $\lfloor a \rfloor$ denominate the greatest integer less than or equal to a. Let $\mathcal{R}$ denote the region in the coordinate plane consisting of points (x, y) such that $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25$. The region $\mathcal{R}$ is completely contained in a disk of radius r (a disk is the union of a circle and its interior). The minimum value of r can be written as $\frac{\sqrt{m}}{n}$, where m and n are integers and m is not divisible by the square of any prime. Find $m+n$.

$\boxed{9}$ Let (a, b, c) be the real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

10 Let N be the number of ways to write 2010 in the form $2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$, where the a_i's are integers, and $0 \leq a_i \leq 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N.

11 Let $\mathcal{R}$ be the region consisting of the set of points in the coordinate plane that satisfy both $|8 - x| + y \leq 10$ and $3y - x \geq 15$. When $\mathcal{R}$ is revolved around the line whose equation is $3y - x = 15$, the volume of the resulting solid is $\frac{m\pi}{n\sqrt{p}}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m+n+p$.

12 Let $M \geq 3$ be an integer and let $S = \{3, 4, 5, \ldots, m\}$. Find the smallest value of m such that for every partition of S into two subsets, at least one of the subsets contains integers a, b, and c (not necessarily distinct) such that $ab = c$.

Note: a partition of S is a pair of sets A, B such that $A \cap B = \emptyset$, $A \cup B = S$.

13 Rectangle $ABCD$ and a semicircle with diameter AB are coplanar and have nonoverlapping interiors. Let $\mathcal{R}$ denote the region enclosed by the semicircle and the rectangle. Line ℓ meets the semicircle, segment AB, and segment CD at distinct points N, U, and T, respectively. Line ℓ divides region $\mathcal{R}$ into two regions with areas in the ratio $1 : 2$. Suppose that $AU = 84$, $AN = 126$, and $UB = 168$. Then DA can be represented as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

14 For each positive integer n, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \leq 300$.

Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

15 In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that $\frac{AM}{CM} = \frac{p}{q}$. Find $p + q$.

USA
AIME
2010

II

1. Let N be the greatest integer multiple of 36 all of whose digits are even and no two of whose digits are the same. Find the remainder when N is divided by 1000.

2. A point P is chosen at random in the interior of a unit square S. Let $d(P)$ denote the distance from P to the closest side of S. The probability that $\frac{1}{5} \leq d(P) \leq \frac{1}{3}$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

3. Let K be the product of all factors $(b-a)$ (not necessarily distinct) where a and b are integers satisfying $1 \leq a < b \leq 20$. Find the greatest positive integer n such that 2^n divides K.

4. Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After waiting at that gate, Dave is told the departure gate has been changed to a different gate, again at random. Let the probability that Dave walks 400 feet or less to the new gate be a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

5. Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

6. Find the smallest positive integer n with the property that the polynomial $x^4 - nx + 63$ can be written as a product of two nonconstant polynomials with integer coefficients.

7. Let $P(z) = z^3 + az^2 + bz + c$, where a, b, and c are real. There exists a complex number w such that the three roots of $P(z)$ are $w+3i$, $w+9i$, and $2w-4$, where $i^2 = -1$. Find $|a+b+c|$.

8. Let N be the number of ordered pairs of nonempty sets $\mathcal{A}$ and $\mathcal{B}$ that have the following properties:

   $\mathcal{A} \cup \mathcal{B} = \{1,2,3,4,5,6,7,8,9,10,11,12\}$, $\mathcal{A} \cap \mathcal{B} = \emptyset$, The number of elements of $\mathcal{A}$ is not an element of $\mathcal{A}$, The number of elements of $\mathcal{B}$ is not an element of $\mathcal{B}$.

   Find N.

9. Let $ABCDEF$ be a regular hexagon. Let G, H, I, J, K, and L be the midpoints of sides AB, BC, CD, DE, EF, and AF, respectively. The segments AH, BI, CJ, DK, EL, and FG bound a smaller regular hexagon. Let the ratio of the area of the smaller hexagon to the area of $ABCDEF$ be expressed as a fraction $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

$\boxed{10}$ Find the number of second-degree polynomials $f(x)$ with integer coefficients and integer zeros for which $f(0) = 2010$.

$\boxed{11}$ Define a *T-grid* to be a 3×3 matrix which satisfies the following two properties:

(1) Exactly five of the entries are 1's, and the remaining four entries are 0's. (2) Among the eight rows, columns, and long diagonals (the long diagonals are $\{a_{13}, a_{22}, a_{31}\}$ and $\{a_{11}, a_{22}, a_{33}\}$, no more than one of the eight has all three entries equal.

Find the number of distinct T-grids.

$\boxed{12}$ Two noncongruent integer-sided isosceles triangles have the same perimeter and the same area. The ratio of the lengths of the bases of the two triangles is $8 : 7$. Find the minimum possible value of their common perimeter.

$\boxed{13}$ The 52 cards in a deck are numbered $1, 2, \ldots, 52$. Alex, Blair, Corey, and Dylan each picks a card from the deck without replacement and with each card being equally likely to be picked, The two persons with lower numbered cards from a team, and the two persons with higher numbered cards form another team. Let $p(a)$ be the probability that Alex and Dylan are on the same team, given that Alex picks one of the cards a and $a + 9$, and Dylan picks the other of these two cards. The minimum value of $p(a)$ for which $p(a) \geq \frac{1}{2}$ can be written as $\frac{m}{n}$. where m and n are relatively prime positive integers. Find $m + n$.

$\boxed{14}$ In right triangle ABC with right angle at C, $\angle BAC < 45$ degrees and $AB = 4$. Point P on AB is chosen such that $\angle APC = 2\angle ACP$ and $CP = 1$. The ratio $\frac{AP}{BP}$ can be represented in the form $p + q\sqrt{r}$, where p, q, r are positive integers and r is not divisible by the square of any prime. Find $p + q + r$.

$\boxed{15}$ In triangle ABC, $AC = 13, BC = 14$, and $AB = 15$. Points M and D lie on AC with $AM = MC$ and $\angle ABD = \angle DBC$. Points N and E lie on AB with $AN = NB$ and $\angle ACE = \angle ECB$. Let P be the point, other than A, of intersection of the circumcircles of $\triangle AMN$ and $\triangle ADE$. Ray AP meets BC at Q. The ratio $\frac{BQ}{CQ}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m - n$.

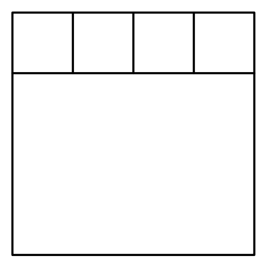
USA
AMC 10
2010

A

$\boxed{1}$ Mary's top book shelf holds five books with the following widths, in centimeters: 6, $\frac{1}{2}$, 1, 2.5, and 10. What is the average book width, in centimeters?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

$\boxed{2}$ Four identical squares and one rectangle are placed together to form one large square as shown. The length of the rectangle is how many times as large as its width?



(A) $\frac{5}{4}$ **(B)** $\frac{4}{3}$ **(C)** $\frac{3}{2}$ **(D)** 2 **(E)** 3

$\boxed{3}$ Tyrone had 97 marbles and Eric had 11 marbles. Tyrone then gave some of his marbles ot Eric so that Tyrone ended with twice as many marbles as Eric. How many marbles did Tyrone give to Eric?

(A) 3 **(B)** 13 **(C)** 18 **(D)** 25 **(E)** 29

$\boxed{4}$ A book that is to be recorded onto compact discs takes 412 minutes to read aloud. Each disc can hold up to 56 minutes of reading. Assume that the smallest possible number of discs is used and that each disc contains the same length of reading. How many minutes of reading will each disc contain?

(A) 50.2 **(B)** 51.5 **(C)** 52.4 **(D)** 53.8 **(E)** 55.2

$\boxed{5}$ The area of a circle whose circumference is 24π is $k\pi$. What is the value of k?

(A) 6 **(B)** 12 **(C)** 24 **(D)** 36 **(E)** 144

$\boxed{6}$ For positive numbers x and y the operation $\spadesuit(x, y)$ is defined as

$$\spadesuit(x, y) = x - \frac{1}{y}$$

What is $\spadesuit(2,\spadesuit(2,2))$?

(A) $\frac{2}{3}$ **(B)** 1 **(C)** $\frac{4}{3}$ **(D)** $\frac{5}{3}$ **(E)** 2

7 Crystal has a running course marked out for her daily run. She starts this run by heading due north for one mile. She then runs northeast for one mile, then southeast for one mile. The last portion of her run takes her on a straight line back to where she started. How far, in miles is this last portion of her run?

(A) 1 **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $2\sqrt{2}$

8 Tony works 2 hours a day and is paid \$0.50 per hour for each full year of his age. During a six month period Tony worked 50 days and earned \$630. How old was Tony at the end of the six month period?

(A) 9 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

9 A *palindrome*, such as 83438, is a number that remains the same when its digits are reversed. The numbers x and $x+32$ are three-digit and four-digit palindromes, respectively. What is the sum of the digits of x?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

10 Marvin had a birthday on Tuesday, May 27 in the leap year 2008. In what year will his birthday next fall on a Saturday?

(A) 2011 **(B)** 2012 **(C)** 2013 **(D)** 2015 **(E)** 2017

11 The length of the interval of solutions of the inequality $a \le 2x+3 \le b$ is 10. What is $b-a$?

(A) 6 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 30

12 Logan is constructing a scaled model of his town. The city's water tower stands 40 meters high, and the top portion is a sphere that holes $100,000$ liters of water. Logan's miniature water tower holds 0.1 liters. How tall, in meters, should Logan make his tower?

(A) 0.04 **(B)** $\frac{0.4}{\pi}$ **(C)** 0.4 **(D)** $\frac{4}{\pi}$ **(E)** 4

13 Angelina drove at an average rate of 80 kph and then stopped 20 minutes for gas. After the stop, she drove at an average rate of 100 kph. Altogether she drove 250 km in a total trip time of 3 hours including the stop. Which equation could be used to solve for the time t in hours that she drove before her stop?

(A) $80t+100(8/3-t)=250$ **(B)** $80t=250$ **(C)** $100t=250$
(D) $90t=250$ **(E)** $80(8/3-t)+100t=250$

14 Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?

(A) 60° **(B)** 75° **(C)** 90° **(D)** 105° **(E)** 120°

15 In a magical swamp there are two species of talking amphibians: toads, whose statements are always true, and frogs, whose statements are always false. Four amphibians, Brian, Chris, LeRoy, and Mike live together in the swamp, and they make the following statements:

Brian: "Mike and I are different species." Chris: "LeRoy is a frog." LeRoy: "Chris is a frog." Mike: "Of the four of us, at least two are toads."

How many of these amphibians are frogs?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

16 Nondegenerate $\triangle ABC$ has integer side lengths, BD is an angle bisector, $AD = 3$, and $DC = 8$. What is the smallest possible value of the perimeter?

(A) 30 **(B)** 33 **(C)** 35 **(D)** 36 **(E)** 37

17 A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume, in cubic inches, of the remaining solid?

(A) 7 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 15

18 Bernardo randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

(A) $\frac{47}{72}$ **(B)** $\frac{37}{56}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{49}{72}$ **(E)** $\frac{39}{56}$

19 Equiangular hexagon $ABCDEF$ has side lengths $AB = CD = EF = 1$ and $BC = DE = FA = r$. The area of $\triangle ACE$ is 70

(A) $\frac{4\sqrt{3}}{3}$ **(B)** $\frac{10}{3}$ **(C)** 4 **(D)** $\frac{17}{4}$ **(E)** 6

20 A fly trapped inside a cubical box with side length 1 meter decides to relieve its boredom by visiting each corner of the box. It will begin and end in the same corner and visit each of the other corners exactly once. To get from a corner to any other corner, it will either fly or crawl in a straight line. What is the maximum possible length, in meters, of its path?

(A) $4 + 4\sqrt{2}$ **(B)** $2 + 4\sqrt{2} + 2\sqrt{3}$ **(C)** $2 + 3\sqrt{2} + 3\sqrt{3}$ **(D)** $4\sqrt{2} + 4\sqrt{3}$
(E) $3\sqrt{2} + 5\sqrt{3}$

$\boxed{21}$ The polynomial $x^3 - ax^2 + bx - 2010$ has three positive integer zeros. What is the smallest possible value of a?

(A) 78 **(B)** 88 **(C)** 98 **(D)** 108 **(E)** 118

$\boxed{22}$ Eight points are chosen on a circle, and chords are drawn connecting every pair of points. No three chords intersect in a single point inside the circle. How many triangles with all three vertices in the interior of the circle are created?

(A) 28 **(B)** 56 **(C)** 70 **(D)** 84 **(E)** 140

$\boxed{23}$ Each of 2010 boxes in a line contains a single red marble, and for $1 \le k \le 2010$, the box in the kth position also contains k white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let $P(n)$ be the probability that Isabella stops after drawing exactly n marbles. What is the smallest value of n for which $P(n) < \frac{1}{2010}$?

(A) 45 **(B)** 63 **(C)** 64 **(D)** 201 **(E)** 1005

$\boxed{24}$ The number obtained from the last two nonzero digits of $90!$ is equal to n. What is n?

(A) 12 **(B)** 32 **(C)** 48 **(D)** 52 **(E)** 68

$\boxed{25}$ Jim starts with a positive integer n and creates a sequence of numbers. Each successive number is obtained by subtracting the largest possible integer square less than or equal to the current number until zero is reached. For example, if Jim starts with $n = 55$, then his sequence contains 5 numbers:

$$\begin{aligned} & 55 \\ 55 - 7^2 = {} & 6 \\ 6 - 2^2 = {} & 2 \\ 2 - 1^2 = {} & 1 \\ 1 - 1^2 = {} & 0 \end{aligned}$$

Let N be the smallest number for which Jim's sequence has 8 numbers. What is the units digit of N?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

USA
AMC 10
2010

B

$\boxed{1}$ What is $100(100-3)-(100\cdot 100-3)$?

(A) $-20,000$ **(B)** $-10,000$ **(C)** -297 **(D)** -6 **(E)** 0

$\boxed{2}$ Makayla attended two meetings during her 9-hour work day. The first meeting took 45 minutes and the second meeting took twice as long. What percent of her work day was spent attending meetings?

(A) 15 **(B)** 20 **(C)** 25 **(D)** 30 **(E)** 35

$\boxed{3}$ A drawer contains red, green, blue, and white socks with at least 2 of each color. What is the minimum number of socks that must be pulled from the drawer to guarantee a matching pair?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 8 **(E)** 9

$\boxed{4}$ For a real number x, define $\heartsuit(x)$ to be the average of x and x^2. What is $\heartsuit(1)+\heartsuit(2)+\heartsuit(3)$?

(A) 3 **(B)** 6 **(C)** 10 **(D)** 12 **(E)** 20

[Thanks PowerOfPi, that's exactly how the heart looks like.]

$\boxed{5}$ A month with 31 days has the same number of Mondays and Wednesdays. How many of the seven days of the week could be the first day of this month?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

$\boxed{6}$ A circle is centered at O, $\overline{AB}$ is a diameter and C is a point on the circle with $\angle COB = 50^\circ$. What is the degree measure of $\angle CAB$?

(A) 20 **(B)** 25 **(C)** 45 **(D)** 50 **(E)** 65

$\boxed{7}$ A triangle has side lengths 10, 10, and 12. A rectangle has width 4 and area equal to the area of the rectangle. What is the perimeter of this rectangle?

(A) 16 **(B)** 24 **(C)** 28 **(D)** 32 **(E)** 36

$\boxed{8}$ A ticket to a school play costs x dollars, where x is a whole number. A group of 9th graders buys tickets costing a total of \$48, and a group of 10th graders buys tickets costing a total of \$64. How many values of x are possible?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

$\boxed{9}$ Lucky Larry's teacher asked him to substitute numbers for a, b, c, d, and e in the expression $a - (b - (c - (d + e)))$ and evaluate the result. Larry ignored the parentheses but added and subtracted correctly and obtained the correct result by coincedence. The numbers Larry substituted for a, b, c, and d were 1, 2, 3, and 4, respectively. What number did Larry substitute for e?

(A) -5 **(B)** -3 **(C)** 0 **(D)** 3 **(E)** 5

$\boxed{10}$ Shelby drives her scooter at a speed of 30 miles per hour if it is not raining, and 20 miles per hour if it is raining. Today she drove in the sun in the morning and in the rain in the evening, for a total of 16 miles in 40 minutes. How many minutes did she drive in the rain?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

$\boxed{11}$ A shopper plans to purchase an item that has a listed price greater than \$100 and can use any one of the three coupns. Coupon A gives 15% off the listed price, Coupon B gives \$30 the listed price, and Coupon C gives 25% off the amount by which the listed price exceeds \$100.

Let x and y be the smallest and largest prices, respectively, for which Coupon A saves at least as many dollars as Coupon B or C. What is $y - x$?

(A) 50 **(B)** 60 **(C)** 75 **(D)** 80 **(E)** 100

$\boxed{12}$ At the beginning of the school year, 50% of all students in Mr. Well's math class answered "Yes" to the question "Do you love math", and 50% answered "No." At the end of the school year, 70% answered "Yes" and 30% answered "No." Altogether, x% of the students gave a different answer at the beginning and end of the school year. What is the difference between the maximum and the minimum possible values of x?

(A) 0 **(B)** 20 **(C)** 40 **(D)** 60 **(E)** 80

$\boxed{13}$ What is the sum of all the solutions of $x = |2x - |60 - 2x||$?

(A) 32 **(B)** 60 **(C)** 92 **(D)** 120 **(E)** 124

$\boxed{14}$ The average of the numbers $1, 2, 3, ..., 98, 99$, and x is $100x$. What is x?

(A) $\frac{49}{101}$ **(B)** $\frac{50}{101}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{51}{101}$ **(E)** $\frac{50}{99}$

$\boxed{15}$ On a 50-question multiple choice math contest, students receive 4 points for a correct answer, 0 points for an answer left blank, and -1 point for an incorrect answer. Jesse's total score on the contest was 99. What is the maximum number of questions that Jesse could have answered correctly?

(A) 25 **(B)** 27 **(C)** 29 **(D)** 31 **(E)** 33

$\boxed{16}$ A square of side length 1 and a circle of radius $\sqrt{3}/3$ share the same center. What is the area inside the circle, but outside the square?

(A) $\frac{\pi}{3} - 1$ **(B)** $\frac{2\pi}{9} - \frac{\sqrt{3}}{3}$ **(C)** $\frac{\pi}{18}$ **(D)** $\frac{1}{4}$ **(E)** $2\pi/9$

$\boxed{17}$ Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37th and 64th, respectively. How many schools are in the city?

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

$\boxed{18}$ Positive integers a, b, and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \ldots, 2010\}$. What is the probability that $abc + ab + a$ is divisible by 3?

(A) $\dfrac{1}{3}$ **(B)** $\dfrac{29}{81}$ **(C)** $\dfrac{31}{81}$ **(D)** $\dfrac{11}{27}$ **(E)** $\dfrac{13}{27}$

$\boxed{19}$ A circle with center O has area 156π. Triangle ABC is equilateral, $\overline{BC}$ is a chord on the circle, $OA = 4\sqrt{3}$, and point O is outside $\triangle ABC$. What is the side length of $\triangle ABC$?

(A) $2\sqrt{3}$ **(B)** 6 **(C)** $4\sqrt{3}$ **(D)** 12 **(E)** 18

$\boxed{20}$ Two circles lie outside regular hexagon $ABCDEF$. The first is tangent to $\overline{AB}$, and the second is tangent to $\overline{DE}$. Both are tangent to lines BC and FA. What is the ratio of the area of the second circle to that of the first circle?

(A) 18 **(B)** 27 **(C)** 36 **(D)** 81 **(E)** 108

$\boxed{21}$ A palindrome between 1000 and $10,000$ is chosen at random. What is the probability that it is divisible by 7?

(A) $\dfrac{1}{10}$ **(B)** $\dfrac{1}{9}$ **(C)** $\dfrac{1}{7}$ **(D)** $\dfrac{1}{6}$ **(E)** $\dfrac{1}{5}$

$\boxed{22}$ Seven distinct pieces of candy are to be distributed among three bags. The red bag and the blue bag must each receive at least one piece of candy; the white bag may remain empty. How many arrangements are possible?

(A) 1930 **(B)** 1931 **(C)** 1932 **(D)** 1933 **(E)** 1934

$\boxed{23}$ The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

(A) 18 **(B)** 24 **(C)** 36 **(D)** 42 **(E)** 60

$\boxed{24}$ A high school basketball game between the Raiders and Wildcats was tied at the end of the first quarter. The number of points scored by the Raiders in each of the four quarters formed an increasing geometric sequence, and the number of points scored by the Wildcats in each of the four quarters formed an increasing arithmetic sequence. At the end of the fourth quarter, the Raiders had won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

(A) 30 **(B)** 31 **(C)** 32 **(D)** 33 **(E)** 34

25 Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$$P(1) = P(3) = P(5) = P(7) = a, \text{ and}$$

$$P(2) = P(4) = P(6) = P(8) = -a.$$

What is the smallest possible value of a?

(A) 105 **(B)** 315 **(C)** 945 **(D)** 7! **(E)** 8!

USA
Team Selection Test
2010

Day 1

1 Let P be a polynomial with integer coefficients such that $P(0) = 0$ and

$$\gcd(P(0), P(1), P(2), \ldots) = 1.$$

Show there are infinitely many n such that

$$\gcd(P(n) - P(0), P(n+1) - P(1), P(n+2) - P(2), \ldots) = n.$$

2 Let a, b, c be positive reals such that $abc = 1$. Show that

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

3 Let h_a, h_b, h_c be the lengths of the altitudes of a triangle ABC from A, B, C respectively. Let P be any point inside the triangle. Show that

$$\frac{PA}{h_b + h_c} + \frac{PB}{h_a + h_c} + \frac{PC}{h_a + h_b} \geq 1.$$

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2010

Day 2

$\boxed{1}$ Let ABC be a triangle. Point M and N lie on sides AC and BC respectively such that $MN||AB$. Points P and Q lie on sides AB and CB respectively such that $PQ||AC$. The incircle of triangle CMN touches segment AC at E. The incircle of triangle BPQ touches segment AB at F. Line EN and AB meet at R, and lines FQ and AC meet at S. Given that $AE = AF$, prove that the incenter of triangle AEF lies on the incircle of triangle ARS.

$\boxed{2}$ Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and, for $n > 1$,

$$a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/3 \rfloor} + \ldots + a_{\lfloor n/n \rfloor} + 1.$$

Prove that there are infinitely many n such that $a_n \equiv n \pmod{2^{2010}}$.

$\boxed{3}$ Let T be a finite set of positive integers greater than 1. A subset S of T is called *good* if for every $t \in T$ there exists some $s \in S$ with $\gcd(s, t) > 1$. Prove that the number of good subsets of T is odd.

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Day 3

$\boxed{1}$ In triangle ABC, let P and Q be two interior points such that $\angle ABP = \angle QBC$ and $\angle ACP = \angle QCB$. Point D lies on segment BC. Prove that $\angle APB + \angle DPC = 180^\circ$ if and only if $\angle AQC + \angle DQB = 180^\circ$.

$\boxed{2}$ Let m, n be positive integers with $m \geq n$, and let S be the set of all n-term sequences of positive integers $(a_1, a_2, \ldots a_n)$ such that $a_1 + a_2 + \cdots + a_n = m$. Show that

$$\sum_S 1^{a_1} 2^{a_2} \cdots n^{a_n} = \binom{n}{n} n^m - \binom{n}{n-1} (n-1)^m + \cdots + (-1)^{n-2} \binom{n}{2} 2^m + (-1)^{n-1} \binom{n}{1}.$$

$\boxed{3}$ Determine whether or not there exists a positive integer k such that $p = 6k + 1$ is a prime and

$$\binom{3k}{k} \equiv 1 \pmod{p}.$$

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2010

Day 1 - *27 April 2010*

$\boxed{1}$ A *permutation* of the set of positive integers $[n] = 1, 2, ..., n$ is a sequence $(a_1, a_2, \ldots, a_n)$ such that each element of $[n]$ appears precisely one time as a term of the sequence. For example, $(3, 5, 1, 2, 4)$ is a permutation of $[5]$. Let $P(n)$ be the number of permutations of $[n]$ for which ka_k is a perfect square for all $1 \leq k \leq n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

$\boxed{2}$ Let $n > 1$ be an integer. Find, with proof, all sequences $x_1, x_2, \ldots, x_{n-1}$ of positive integers with the following three properties: (a). $x_1 < x_2 < \cdots < x_{n-1}$; (b). $x_i + x_{n-i} = 2n$ for all $i = 1, 2, \ldots, n-1$; (c). given any two indices i and j (not necessarily distinct) for which $x_i + x_j < 2n$, there is an index k such that $x_i + x_j = x_k$.

$\boxed{3}$ Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Day 2 - *28 April 2010*

$\boxed{4}$ A triangle is called a parabolic triangle if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

$\boxed{5}$ Two permutations $a_1, a_2, \ldots, a_{2010}$ and $b_1, b_2, \ldots, b_{2010}$ of the numbers $1, 2, \ldots, 2010$ are said to *intersect* if $a_k = b_k$ for some value of k in the range $1 \le k \le 2010$. Show that there exist 1006 permutations of the numbers $1, 2, \ldots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

$\boxed{6}$ Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

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USAMO
2010

$\boxed{1}$ Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

$\boxed{2}$ There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

The most progress I made on this problem is that at the end, student n must be behind student $n-1$, student $n-1$ must be behind student $n-2$, etc. all the way around until we get that student 1 must be behind student n.

$\boxed{3}$ The 2010 positive numbers $a_1, a_2, \ldots, a_{2010}$ satisfy the inequality $a_i a_j \le i + j$ for all distinct indices i, j. Determine, with proof, the largest possible value of the product $a_1 a_2 \ldots a_{2010}$.

$\boxed{4}$ Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

$\boxed{5}$ Let $q = \frac{3p-5}{2}$ where p is an odd prime, and let

$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}$$

Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m - n$ is divisible by p.

$\boxed{6}$ A blackboard contains 68 pairs of nonzero integers. Suppose that for each positive integer k at most one of the pairs (k, k) and $(-k, -k)$ is written on the blackboard. A student erases some of the 136 integers, subject to the condition that no two erased integers may add to 0. The student then scores one point for each of the 68 pairs in which at least one integer is erased. Determine, with proof, the largest number N of points that the student can guarantee to score regardless of which 68 pairs have been written on the board.

vietnam
Team Selection Tests
2010

Day 1

1. Let n be a positive integer. Let T_n be a set of positive integers such that:

$$T_n = \{11(k+h) + 10(n^k + n^h) | (1 \leq k, h \leq 10)\}$$

Find all n for which there don't exist two distinct positive integers $a, b \in T_n$ such that $a \equiv b \pmod{10}$

2. Let ABC be a triangle with $\widehat{BAC} \neq 90^\circ$. Let M be the midpoint of BC. We choose a variable point D on AM. Let (O_1) and (O_2) be two circle pass through D and tangent to BC at B and C. The line BA and CA intersect $(O_1), (O_2)$ at P, Q respectively.

a) Prove that tangent line at P on (O_1) and Q on (O_2) must intersect at S.

b) Prove that S lies on a fix line.

3. We call a rectangle of the size 1×2 a domino. Rectangle of the 2×3 removing two opposite (under center of rectangle) corners we call tetramino. These figures can be rotated.

It requires to tile rectangle of size 2008×2010 by using dominoes and tetraminoes. What is the minimal number of dominoes should be used?

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Team Selection Tests

2010

Day 2

$\boxed{1}$ Let a, b, c be positive integers which satisfy the condition: $16(a+b+c) \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that

$$\sum_{cyc} \left(\frac{1}{a+b+\sqrt{2a+2c}} \right)^3 \leq \frac{8}{9}$$

$\boxed{2}$ We have n countries. Each country have m persons who live in that country ($n > m > 1$). We divide $m \cdot n$ persons into n groups each with m members such that there don't exist two persons in any groups who come from one country. Prove that one can choose n people into one class such that they come from different groups and different countries.

$\boxed{3}$ Let S_n be sum of squares of the coefficient of the polynomial $(1+x)^n$. Prove that $S_{2n} + 1$ is not divisible by 3.