The 62nd William Lowell Putnam Mathematical Competition
Saturday, December 1, 2001

A-1 Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

A-2 You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of fallings heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answers as a rational function of n.

A-3 For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

A-4 Triangle ABC has an area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.

A-5 Prove that there are unique positive integers a, n such that $a^{n+1} - (a+1)^n = 2001$.

A-6 Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

B-1 Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)(n)+1, (k-1)n+2, \ldots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

B-2 Find all pairs of real numbers (x, y) satisfying the system of equations

$$\frac{1}{x} + \frac{1}{2y} = (x^2+3y^2)(3x^2+y^2)$$
$$\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4).$$

B-3 For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

B-4 Let S denote the set of rational numbers different from $\{-1, 0, 1\}$. Define $f : S \to S$ by $f(x) = x - 1/x$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

B-5 Let a and b be real numbers in the interval $(0, 1/2)$, and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

B-6 Assume that $(a_n)_{n \geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

The 63rd William Lowell Putnam Mathematical Competition
Saturday, December 7, 2002

A1 Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

A2 Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

A3 Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \dots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

A4 In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

A5 Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots \right\}.$$

A6 Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = n f(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

B1 Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

B2 Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

> Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

B3 Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

B4 An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

B5 A palindrome in base b is a positive integer whose base-b digits read the same backwards and forwards; for example, 2002 is a 4-digit palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is the 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3-digit palindrome in base b for at least 2002 different values of b.

B6 Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients are congruent modulo p.)

The 64th William Lowell Putnam Mathematical Competition
Saturday, December 6, 2003

A1 Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

A2 Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$\begin{aligned}&(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \\ &\leq [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.\end{aligned}$$

A3 Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

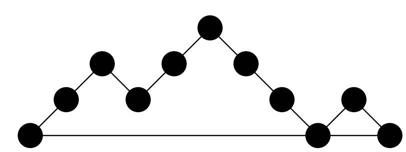
A4 Suppose that a, b, c, A, B, C are real numbers, $a \neq 0$ and $A \neq 0$, such that

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$

for all real numbers x. Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC|.$$

A5 A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

A6 For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n ?

B1 Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

B2 Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n < 2/n$.

B3 Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

B4 Let

$$\begin{aligned} f(z) &= az^4 + bz^3 + cz^2 + dz + e \\ &= a(z-r_1)(z-r_2)(z-r_3)(z-r_4)\end{aligned}$$

where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

B5 Let A, B, and C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let a, b, c be the distance from P to A, B, C, respectively. Show that there is a triangle with side lengths a, b, c, and that the area of this triangle depends only on the distance from P to O.

B6 Let $f(x)$ be a continuous real-valued function defined on the interval $[0,1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)|\, dx\, dy \geq \int_0^1 |f(x)|\, dx.$$

The 65th William Lowell Putnam Mathematical Competition
Saturday, December 4, 2004

A1 Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

A2 For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \le a_2, b_1 \le b_2, c_1 \le c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \le A_2$?

A3 Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \ge 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

A4 Show that for any positive integer n, there is an integer N such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1, 0, 1$.

A5 An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

A6 Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \le x \le 1, 0 \le y \le 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy \right)^2 dx$$
$$\le \left(\int_0^1 \int_0^1 f(x,y)dx\,dy \right)^2 + \int_0^1 \int_0^1 (f(x,y))^2\,dx\,dy.$$

B1 Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r,\ c_n r^2 + c_{n-1} r,\ c_n r^3 + c_{n-1} r^2 + c_{n-2} r,$$
$$\ldots,\ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

B2 Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

B3 Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y); 0 \le x \le a, 0 \le y \le f(x)\}$$

has perimeter k units and area k square units for some real number k.

B4 Let n be a positive integer, $n \ge 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then $R_2, \ldots,$ then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

B5 Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

B6 Let $\mathcal{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathcal{A}$ not exceeding x. Let $\mathcal{B}$ denote the set of positive integers b that can be written in the form $b = a - a'$ with $a \in \mathcal{A}$ and $a' \in \mathcal{A}$. Let $b_1 < b_2 < \cdots$ be the members of $\mathcal{B}$, listed in increasing order. Show that if the sequence $b_{i+1} - b_i$ is unbounded, then

$$\lim_{x \to \infty} N(x)/x = 0.$$

The 66th William Lowell Putnam Mathematical Competition
Saturday, December 3, 2005

A1 Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

A2 Let $\mathbf{S} = \{(a, b) | a = 1, 2, \ldots, n, b = 1, 2, 3\}$. A *rook tour* of $\mathbf{S}$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that

(i) $p_i \in \mathbf{S}$,

(ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$,

(iii) for each $p \in \mathbf{S}$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?

(An example of such a rook tour for $n = 5$ was depicted in the original.)

A3 Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

A4 Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

A5 Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\, dx.$$

A6 Let n be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

B1 Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.)

B2 Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

B3 Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

B4 For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

B5 Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) P(x_1, \ldots, x_n) = 0 \quad \text{(identically)}$$

and that

$$x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).$$

Show that $P = 0$ identically.

B6 Let S_n denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1}.$$

The 67th William Lowell Putnam Mathematical Competition
Saturday, December 2, 2006

A1 Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

A2 Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

A3 Let $1, 2, 3, \dots, 2005, 2006, 2007, 2009, 2012, 2016, \dots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \dots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

A4 Let $S = \{1, 2, \dots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

(i) $\pi(k) > \pi(k+1)$ for $k = 1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;

(iii) $\pi(k-1) < \pi(k)$ for $k = n$.

(For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

A5 Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \dots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer, and determine its value.

A6 Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

B1 Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

B2 Prove that, for every set $X = \{x_1, x_2, \dots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

B3 Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

B4 Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$.) [Editorial note: the proposers probably intended to write $Z(V)$ for "the number of points in $V \cap Z$", but this changes nothing.]

B5 For each continuous function $f : [0, 1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\, dx$ and $J(x) = \int_0^1 x \left(f(x)\right)^2\, dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

B6 Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$

The 68th William Lowell Putnam Mathematical Competition
Saturday, December 1, 2007

A1 Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

A2 Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$. (A set S in the plane is called *convex* if for any two points in S the line segment connecting them is contained in S.)

A3 Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

A4 A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.

A5 Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either $n = 0$ or p divides $n + 1$.

A6 A *triangulation* $\mathcal{T}$ of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side is a side of exactly one triangle in $\mathcal{T}$. Say that $\mathcal{T}$ is *admissible* if every internal vertex is shared by 6 or more triangles. For example, [figure omitted.] Prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.

B1 Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$. [Editor's note: one must assume f is nonconstant.]

B2 Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$\left| \int_0^\alpha f(x)\,dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$

B3 Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n \sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}. ($\lfloor a \rfloor$ means the largest integer $\leq a$.)

B4 Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and $\deg P > \deg Q$.

B5 Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n), \ldots, P_{k-1}(n)$ (which may depend on k) such that for any integer n,

$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}.$$

($\lfloor a \rfloor$ means the largest integer $\leq a$.)

B6 For each positive integer n, let $f(n)$ be the number of ways to make $n!$ cents using an unordered collection of coins, each worth $k!$ cents for some k, $1 \leq k \leq n$. Prove that for some constant C, independent of n,

$$n^{n^2/2 - Cn} e^{-n^2/4} \leq f(n) \leq n^{n^2/2 + Cn} e^{-n^2/4}.$$

The 69th William Lowell Putnam Mathematical Competition
Saturday, December 6, 2008

A1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y) + f(y,z) + f(z,x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) - g(y)$ for all real numbers x and y.

A2 Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

A3 Start with a finite sequence $a_1, a_2, \dots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\mathrm{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

A4 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

A5 Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \dots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n - 1$.

A6 Prove that there exists a constant $c > 0$ such that in every nontrivial finite group G there exists a sequence of length at most $c \ln |G|$ with the property that each element of G equals the product of some subsequence. (The elements of G in the sequence are not required to be distinct. A *subsequence* of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, $4, 4, 2$ is a subsequence of $2, 4, 6, 4, 2$, but $2, 2, 4$ is not.)

B1 What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

B2 Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. Evaluate

$$\lim_{n \to \infty} \frac{n! F_n(1)}{\ln n}.$$

B3 What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

B4 Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \dots, h(p^2 - 1)$ are distinct modulo p^2. Show that $h(0), h(1), \dots, h(p^3 - 1)$ are distinct modulo p^3.

B5 Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a, b) = 1$.) (Note: gcd means greatest common divisor.)

B6 Let n and k be positive integers. Say that a permutation σ of $\{1, 2, \dots, n\}$ is *k-limited* if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of k-limited permutations of $\{1, 2, \dots, n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.

Solutions to the 62nd William Lowell Putnam Mathematical Competition Saturday, December 1, 2001

Manjul Bhargava, Kiran Kedlaya, and Lenny Ng

A–1 The hypothesis implies $((b*a)*b)*(b*a) = b$ for all $a, b \in S$ (by replacing a by $b*a$), and hence $a*(b*a) = b$ for all $a, b \in S$ (using $(b*a)*b = a$).

A–2 Let P_n denote the desired probability. Then $P_1 = 1/3$, and, for $n > 1$,

$$\begin{aligned} P_n &= \left(\frac{2n}{2n+1}\right) P_{n-1} + \left(\frac{1}{2n+1}\right)(1 - P_{n-1}) \\ &= \left(\frac{2n-1}{2n+1}\right) P_{n-1} + \frac{1}{2n+1}. \end{aligned}$$

The recurrence yields $P_2 = 2/5$, $P_3 = 3/7$, and by a simple induction, one then checks that for general n one has $P_n = n/(2n+1)$.

Note: Richard Stanley points out the following noninductive argument. Put $f(x) = \prod_{k=1}^n (x+2k)/(2k+1)$; then the coefficient of x^i in $f(x)$ is the probability of getting exactly i heads. Thus the desired number is $(f(1) - f(-1))/2$, and both values of f can be computed directly: $f(1) = 1$, and

$$f(-1) = \frac{1}{3} \times \frac{3}{5} \times \cdots \times \frac{2n-1}{2n+1} = \frac{1}{2n+1}.$$

A–3 By the quadratic formula, if $P_m(x) = 0$, then $x^2 = m \pm 2\sqrt{2m} + 2$, and hence the four roots of P_m are given by $S = \{\pm\sqrt{m} \pm \sqrt{2}\}$. If P_m factors into two nonconstant polynomials over the integers, then some subset of S consisting of one or two elements form the roots of a polynomial with integer coefficients.

First suppose this subset has a single element, say $\sqrt{m} \pm \sqrt{2}$; this element must be a rational number. Then $(\sqrt{m} \pm \sqrt{2})^2 = 2 + m \pm 2\sqrt{2m}$ is an integer, so m is twice a perfect square, say $m = 2n^2$. But then $\sqrt{m} \pm \sqrt{2} = (n \pm 1)\sqrt{2}$ is only rational if $n = \pm 1$, i.e., if $m = 2$.

Next, suppose that the subset contains two elements; then we can take it to be one of $\{\sqrt{m} \pm \sqrt{2}\}$, $\{\sqrt{2} \pm \sqrt{m}\}$ or $\{\pm(\sqrt{m} + \sqrt{2})\}$. In all cases, the sum and the product of the elements of the subset must be a rational number. In the first case, this means $2\sqrt{m} \in \mathbb{Q}$, so m is a perfect square. In the second case, we have $2\sqrt{2} \in \mathbb{Q}$, contradiction. In the third case, we have $(\sqrt{m} + \sqrt{2})^2 \in \mathbb{Q}$, or $m + 2 + 2\sqrt{2m} \in \mathbb{Q}$, which means that m is twice a perfect square.

We conclude that $P_m(x)$ factors into two nonconstant polynomials over the integers if and only if m is either a square or twice a square.

Note: a more sophisticated interpretation of this argument can be given using Galois theory. Namely, if m is neither a square nor twice a square, then the number fields $\mathbb{Q}(\sqrt{m})$ and $\mathbb{Q}(\sqrt{2})$ are distinct quadratic fields, so their compositum is a number field of degree 4, whose Galois group acts transitively on $\{\pm\sqrt{m} \pm \sqrt{2}\}$. Thus P_m is irreducible.

A–4 Choose r, s, t so that $EC = rBC$, $FA = sCA$, $GB = tCB$, and let $[XYZ]$ denote the area of triangle XYZ. Then $[ABE] = [AFE]$ since the triangles have the same altitude and base. Also $[ABE] = (BE/BC)[ABC] = 1 - r$, and $[ECF] = (EC/BC)(CF/CA)[ABC] = r(1-s)$ (e.g., by the law of sines). Adding this all up yields

$$\begin{aligned} 1 &= [ABE] + [ABF] + [ECF] \\ &= 2(1-r) + r(1-s) = 2 - r - rs \end{aligned}$$

or $r(1+s) = 1$. Similarly $s(1+t) = t(1+r) = 1$.

Let $f : [0, \infty) \to [0, \infty)$ be the function given by $f(x) = 1/(1+x)$; then $f(f(f(r))) = r$. However, $f(x)$ is strictly decreasing in x, so $f(f(x))$ is increasing and $f(f(f(x)))$ is decreasing. Thus there is at most one x such that $f(f(f(x))) = x$; in fact, since the equation $f(z) = z$ has a positive root $z = (-1 + \sqrt{5})/2$, we must have $r = s = t = z$.

We now compute $[ABF] = (AF/AC)[ABC] = z$, $[ABR] = (BR/BF)[ABF] = z/2$, analogously $[BCS] = [CAT] = z/2$, and $[RST] = |[ABC] - [ABR] - [BCS] - [CAT]| = |1 - 3z/2| = \frac{7-3\sqrt{5}}{4}$.

Note: the key relation $r(1+s) = 1$ can also be derived by computing using homogeneous coordinates or vectors.

A–5 Suppose $a^{n+1} - (a+1)^n = 2001$. Notice that $a^{n+1} + [(a+1)^n - 1]$ is a multiple of a; thus a divides $2002 = 2 \times 7 \times 11 \times 13$.

Since 2001 is divisible by 3, we must have $a \equiv 1 \pmod 3$, otherwise one of a^{n+1} and $(a+1)^n$ is a multiple of 3 and the other is not, so their difference cannot be divisible by 3. Now $a^{n+1} \equiv 1 \pmod 3$, so we must have $(a+1)^n \equiv 1 \pmod 3$, which forces n to be even, and in particular at least 2.

If a is even, then $a^{n+1} - (a+1)^n \equiv -(a+1)^n \pmod 4$. Since n is even, $-(a+1)^n \equiv -1 \pmod 4$. Since

$2001 \equiv 1 \pmod 4$, this is impossible. Thus a is odd, and so must divide $1001 = 7 \times 11 \times 13$. Moreover, $a^{n+1} - (a+1)^n \equiv a \pmod 4$, so $a \equiv 1 \pmod 4$.

Of the divisors of $7 \times 11 \times 13$, those congruent to 1 mod 3 are precisely those not divisible by 11 (since 7 and 13 are both congruent to 1 mod 3). Thus a divides 7×13. Now $a \equiv 1 \pmod 4$ is only possible if a divides 13.

We cannot have $a = 1$, since $1 - 2^n \neq 2001$ for any n. Thus the only possibility is $a = 13$. One easily checks that $a = 13, n = 2$ is a solution; all that remains is to check that no other n works. In fact, if $n > 2$, then $13^{n+1} \equiv 2001 \equiv 1 \pmod 8$. But $13^{n+1} \equiv 13 \pmod 8$ since n is even, contradiction. Thus $a = 13, n = 2$ is the unique solution.

Note: once one has that n is even, one can use that $2002 = a^{n+1} + 1 - (a+1)^n$ is divisible by $a+1$ to rule out cases.

A–6 The answer is yes. Consider the arc of the parabola $y = Ax^2$ inside the circle $x^2 + (y-1)^2 = 1$, where we initially assume that $A > 1/2$. This intersects the circle in three points, $(0,0)$ and $(\pm\sqrt{2A-1}/A, (2A-1)/A)$. We claim that for A sufficiently large, the length L of the parabolic arc between $(0,0)$ and $(\sqrt{2A-1}/A, (2A-1)/A)$ is greater than 2, which implies the desired result by symmetry. We express L using the usual formula for arclength:

$$\begin{aligned} L &= \int_0^{\sqrt{2A-1}/A} \sqrt{1+(2Ax)^2}\,dx \\ &= \frac{1}{2A}\int_0^{2\sqrt{2A-1}} \sqrt{1+x^2}\,dx \\ &= 2 + \frac{1}{2A}\left(\int_0^{2\sqrt{2A-1}} (\sqrt{1+x^2}-x)\,dx - 2\right), \end{aligned}$$

where we have artificially introduced $-x$ into the integrand in the last step. Now, for $x \geq 0$,

$$\sqrt{1+x^2}-x = \frac{1}{\sqrt{1+x^2}+x} > \frac{1}{2\sqrt{1+x^2}} \geq \frac{1}{2(x+1)};$$

since $\int_0^\infty dx/(2(x+1))$ diverges, so does $\int_0^\infty (\sqrt{1+x^2}-x)\,dx$. Hence, for sufficiently large A, we have $\int_0^{2\sqrt{2A-1}}(\sqrt{1+x^2}-x)\,dx > 2$, and hence $L > 2$.

Note: a numerical computation shows that one must take $A > 34.7$ to obtain $L > 2$, and that the maximum value of L is about 4.0027, achieved for $A \approx 94.1$.

B–1 Let R (resp. B) denote the set of red (resp. black) squares in such a coloring, and for $s \in R \cup B$, let $f(s)n + g(s) + 1$ denote the number written in square s, where $0 \leq f(s), g(s) \leq n-1$. Then it is clear that the value of $f(s)$ depends only on the row of s, while the value of $g(s)$ depends only on the column of s. Since every row contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s\in R} f(s) = \sum_{s \in B} f(s).$$

Similarly, because every column contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s\in R} g(s) = \sum_{s \in B} g(s).$$

It follows that

$$\sum_{s\in R} f(s)n + g(s) + 1 = \sum_{s\in B} f(s)n + g(s) + 1,$$

as desired.

Note: Richard Stanley points out a theorem of Ryser (see Ryser, *Combinatorial Mathematics*, Theorem 3.1) that can also be applied. Namely, if A and B are $0-1$ matrices with the same row and column sums, then there is a sequence of operations on 2×2 matrices of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or vice versa, which transforms A into B. If we identify 0 and 1 with red and black, then the given coloring and the checkerboard coloring both satisfy the sum condition. Since the desired result is clearly true for the checkerboard coloring, and performing the matrix operations does not affect this, the desired result follows in general.

B–2 By adding and subtracting the two given equations, we obtain the equivalent pair of equations

$$\begin{aligned} 2/x &= x^4 + 10x^2y^2 + 5y^4 \\ 1/y &= 5x^4 + 10x^2y^2 + y^4. \end{aligned}$$

Multiplying the former by x and the latter by y, then adding and subtracting the two resulting equations, we obtain another pair of equations equivalent to the given ones,

$$3 = (x+y)^5, \qquad 1 = (x-y)^5.$$

It follows that $x = (3^{1/5}+1)/2$ and $y = (3^{1/5}-1)/2$ is the unique solution satisfying the given equations.

B–3 Since $(k-1/2)^2 = k^2 - k + 1/4$ and $(k+1/2)^2 = k^2 + k + 1/4$, we have that $\langle n \rangle = k$ if and only if

$k^2-k+1 \leq n \leq k^2+k$. Hence

$$\begin{aligned}
\sum_{n=1}^\infty \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} &= \sum_{k=1}^\infty \sum_{n, \langle n \rangle = k} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} \\
&= \sum_{k=1}^\infty \sum_{n=k^2-k+1}^{k^2+k} \frac{2^k+2^{-k}}{2^n} \\
&= \sum_{k=1}^\infty (2^k+2^{-k})(2^{-k^2+k} - 2^{-k^2-k}) \\
&= \sum_{k=1}^\infty (2^{-k(k-2)} - 2^{-k(k+2)}) \\
&= \sum_{k=1}^\infty 2^{-k(k-2)} - \sum_{k=3}^\infty 2^{-k(k-2)} \\
&= 3.
\end{aligned}$$

Alternate solution: rewrite the sum as $\sum_{n=1}^\infty 2^{-(n+\langle n \rangle)} + \sum_{n=1}^\infty 2^{-(n-\langle n \rangle)}$. Note that $\langle n \rangle \neq \langle n+1 \rangle$ if and only if $n = m^2+m$ for some m. Thus $n + \langle n \rangle$ and $n - \langle n \rangle$ each increase by 1 except at $n = m^2+m$, where the former skips from m^2+2m to m^2+2m+2 and the latter repeats the value m^2. Thus the sums are

$$\sum_{n=1}^\infty 2^{-n} - \sum_{m=1}^\infty 2^{-m^2} + \sum_{n=0}^\infty 2^{-n} + \sum_{m=1}^\infty 2^{-m^2} = 2+1 = 3.$$

B–4 For a rational number p/q expressed in lowest terms, define its *height* $H(p/q)$ to be $|p|+|q|$. Then for any $p/q \in S$ expressed in lowest terms, we have $H(f(p/q)) = |q^2-p^2| + |pq|$; since by assumption p and q are nonzero integers with $|p| \neq |q|$, we have

$$\begin{aligned}
H(f(p/q)) - H(p/q) &= |q^2-p^2| + |pq| - |p| - |q| \\
&\geq 3 + |pq| - |p| - |q| \\
&= (|p|-1)(|q|-1) + 2 \geq 2.
\end{aligned}$$

It follows that $f^{(n)}(S)$ consists solely of numbers of height strictly larger than $2n+2$, and hence

$$\cap_{n=1}^\infty f^{(n)}(S) = \emptyset.$$

Note: many choices for the height function are possible: one can take $H(p/q) = \max|p|, |q|$, or $H(p/q)$ equal to the total number of prime factors of p and q, and so on. The key properties of the height function are that on one hand, there are only finitely many rationals with height below any finite bound, and on the other hand, the height function is a sufficiently "algebraic" function of its argument that one can relate the heights of p/q and $f(p/q)$.

B–5 Note that $g(x) = g(y)$ implies that $g(g(x)) = g(g(y))$ and hence $x = y$ from the given equation. That is, g is injective. Since g is also continuous, g is either strictly increasing or strictly decreasing. Moreover, g cannot tend to a finite limit L as $x \to +\infty$, or else we'd have $g(g(x)) - ag(x) = bx$, with the left side bounded and the right side unbounded. Similarly, g cannot tend to a finite limit as $x \to -\infty$. Together with monotonicity, this yields that g is also surjective.

Pick x_0 arbitrary, and define x_n for all $n \in \mathbb{Z}$ recursively by $x_{n+1} = g(x_n)$ for $n > 0$, and $x_{n-1} = g^{-1}(x_n)$ for $n < 0$. Let $r_1 = (a + \sqrt{a^2+4b})/2$ and $r_2 = (a - \sqrt{a^2+4b})/2$ and r_2 be the roots of $x^2 - ax - b = 0$, so that $r_1 > 0 > r_2$ and $1 > |r_1| > |r_2|$. Then there exist $c_1, c_2 \in \mathbb{R}$ such that $x_n = c_1 r_1^n + c_2 r_2^n$ for all $n \in \mathbb{Z}$.

Suppose g is strictly increasing. If $c_2 \neq 0$ for some choice of x_0, then x_n is dominated by r_2^n for n sufficiently negative. But taking x_n and x_{n+2} for n sufficiently negative of the right parity, we get $0 < x_n < x_{n+2}$ but $g(x_n) > g(x_{n+2})$, contradiction. Thus $c_2 = 0$; since $x_0 = c_1$ and $x_1 = c_1 r_1$, we have $g(x) = r_1 x$ for all x. Analogously, if g is strictly decreasing, then $c_2 = 0$ or else x_n is dominated by r_1^n for n sufficiently positive. But taking x_n and x_{n+2} for n sufficiently positive of the right parity, we get $0 < x_{n+2} < x_n$ but $g(x_{n+2}) < g(x_n)$, contradiction. Thus in that case, $g(x) = r_2 x$ for all x.

B–6 Yes, there must exist infinitely many such n. Let S be the convex hull of the set of points (n, a_n) for $n \geq 0$. Geometrically, S is the intersection of all convex sets (or even all halfplanes) containing the points (n, a_n); algebraically, S is the set of points (x, y) which can be written as $c_1(n_1, a_{n_1}) + \cdots + c_k(n_k, a_{n_k})$ for some $c_1, \dots, c_k$ which are nonnegative of sum 1.

We prove that for infinitely many n, (n, a_n) is a vertex on the upper boundary of S, and that these n satisfy the given condition. The condition that (n, a_n) is a vertex on the upper boundary of S is equivalent to the existence of a line passing through (n, a_n) with all other points of S below it. That is, there should exist $m > 0$ such that

$$a_k < a_n + m(k-n) \qquad \forall k \geq 1. \qquad (1)$$

We first show that $n = 1$ satisfies (1). The condition $a_k/k \to 0$ as $k \to \infty$ implies that $(a_k - a_1)/(k-1) \to 0$ as well. Thus the set $\{(a_k-a_1)/(k-1)\}$ has an upper bound m, and now $a_k \leq a_1 + m(k-1)$, as desired.

Next, we show that given one n satisfying (1), there exists a larger one also satisfying (1). Again, the condition $a_k/k \to 0$ as $k \to \infty$ implies that $(a_k - a_n)/(k-n) \to 0$ as $k \to \infty$. Thus the sequence $\{(a_k-a_n)/(k-n)\}_{k>n}$ has a maximum element; suppose $k = r$ is the largest value of k that achieves this maximum, and put $m = (a_r - a_n)/(r-n)$. Then the line through (r, a_r) of slope m lies strictly above (k, a_k) for $k > r$ and passes through or lies above (k, a_k) for

$k < r$. Thus (1) holds for $n = r$ with m replaced by $m - \epsilon$ for suitably small $\epsilon > 0$.

By induction, we have that (1) holds for infinitely many n. For any such n there exists $m > 0$ such that for $i = 1, \ldots, n-1$, the points $(n-i, a_{n-i})$ and $(n+i, a_{n+i})$ lie below the line through (n, a_n) of slope m. That means $a_{n+i} < a_n + mi$ and $a_{n-i} < a_n - mi$; adding these together gives $a_{n-i} + a_{n+i} < 2a_n$, as desired.

Solutions to the 63rd William Lowell Putnam Mathematical Competition
Saturday, December 7, 2002

Kiran Kedlaya and Lenny Ng

A1 By differentiating $P_n(x)/(x^k-1)^{n+1}$, we find that $P_{n+1}(x) = (x^k-1)P_n'(x) - (n+1)kx^{k-1}P_n(x)$; substituting $x=1$ yields $P_{n+1}(1) = -(n+1)kP_n(1)$. Since $P_0(1) = 1$, an easy induction gives $P_n(1) = (-k)^n n!$ for all $n \geq 0$.

Note: one can also argue by expanding in Taylor series around 1. Namely, we have

$$\frac{1}{x^k-1} = \frac{1}{k(x-1)+\cdots} = \frac{1}{k}(x-1)^{-1} + \cdots,$$

so

$$\frac{d^n}{dx^n}\frac{1}{x^k-1} = \frac{(-1)^n n!}{k(x-1)^{-n-1}}$$

and

$$\begin{aligned} P_n(x) &= (x^k-1)^{n+1}\frac{d^n}{dx^n}\frac{1}{x^k-1} \\ &= (k(x-1)+\cdots)^{n+1} \\ &\quad \left(\frac{(-1)^n n!}{k}(x-1)^{-n-1}+\cdots\right) \\ &= (-k)^n n! + \cdots. \end{aligned}$$

A2 Draw a great circle through two of the points. There are two closed hemispheres with this great circle as boundary, and each of the other three points lies in one of them. By the pigeonhole principle, two of those three points lie in the same hemisphere, and that hemisphere thus contains four of the five given points.

Note: by a similar argument, one can prove that among any $n+3$ points on an n-dimensional sphere, some $n+2$ of them lie on a closed hemisphere. (One cannot get by with only $n+2$ points: put them at the vertices of a regular simplex.) Namely, any n of the points lie on a great sphere, which forms the boundary of two hemispheres; of the remaining three points, some two lie in the same hemisphere.

A3 Note that each of the sets $\{1\}, \{2\}, \dots, \{n\}$ has the desired property. Moreover, for each set S with integer average m that does not contain m, $S \cup \{m\}$ also has average m, while for each set T of more than one element with integer average m that contains m, $T \setminus \{m\}$ also has average m. Thus the subsets other than $\{1\}, \{2\}, \dots, \{n\}$ can be grouped in pairs, so $T_n - n$ is even.

A4 (partly due to David Savitt) Player 0 wins with optimal play. In fact, we prove that Player 1 cannot prevent Player 0 from creating a row of all zeroes, a column of all zeroes, or a 2×2 submatrix of all zeroes. Each of these forces the determinant of the matrix to be zero.

For $i,j = 1,2,3$, let A_{ij} denote the position in row i and column j. Without loss of generality, we may assume that Player 1's first move is at A_{11}. Player 0 then plays at A_{22}:

$$\begin{pmatrix} 1 & * & * \\ * & 0 & * \\ * & * & * \end{pmatrix}$$

After Player 1's second move, at least one of A_{23} and A_{32} remains vacant. Without loss of generality, assume A_{23} remains vacant; Player 0 then plays there.

After Player 1's third move, Player 0 wins by playing at A_{21} if that position is unoccupied. So assume instead that Player 1 has played there. Thus of Player 1's three moves so far, two are at A_{11} and A_{21}. Hence for i equal to one of 1 or 3, and for j equal to one of 2 or 3, the following are both true:

(a) The 2×2 submatrix formed by rows 2 and i and by columns 2 and 3 contains two zeroes and two empty positions.

(b) Column j contains one zero and two empty positions.

Player 0 next plays at A_{ij}. To prevent a zero column, Player 1 must play in column j, upon which Player 0 completes the 2×2 submatrix in (a) for the win.

Note: one can also solve this problem directly by making a tree of possible play sequences. This tree can be considerably collapsed using symmetries: the symmetry between rows and columns, the invariance of the outcome under reordering of rows or columns, and the fact that the scenario after a sequence of moves does not depend on the order of the moves (sometimes called "transposition invariance").

Note (due to Paul Cheng): one can reduce Determinant Tic-Tac-Toe to a variant of ordinary tic-tac-toe. Namely, consider a tic-tac-toe grid labeled as follows:

A_{11}	A_{22}	A_{33}
A_{23}	A_{31}	A_{12}
A_{32}	A_{13}	A_{21}

Then each term in the expansion of the determinant occurs in a row or column of the grid. Suppose Player 1 first plays in the top left. Player 0 wins by playing first in the top row, and second in the left column. Then there are only one row and column left for Player 1 to threaten, and Player 1 cannot already threaten both on the third move, so Player 0 has time to block both.

A5 It suffices to prove that for any relatively prime positive integers r, s, there exists an integer n with $a_n = r$ and $a_{n+1} = s$. We prove this by induction on $r+s$, the case $r + s = 2$ following from the fact that $a_0 = a_1 = 1$. Given r and s not both 1 with $\gcd(r, s) = 1$, we must have $r \neq s$. If $r > s$, then by the induction hypothesis we have $a_n = r - s$ and $a_{n+1} = s$ for some n; then $a_{2n+2} = r$ and $a_{2n+3} = s$. If $r < s$, then we have $a_n = r$ and $a_{n+1} = s - r$ for some n; then $a_{2n+1} = r$ and $a_{2n+2} = s$.

Note: a related problem is as follows. Starting with the sequence

$$\frac{0}{1}, \frac{1}{0},$$

repeat the following operation: insert between each pair $\frac{a}{b}$ and $\frac{c}{d}$ the pair $\frac{a+c}{b+d}$. Prove that each positive rational number eventually appears.

Observe that by induction, if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms in the sequence, then $bc - ad = 1$. The same holds for consecutive terms of the n-th *Farey sequence*, the sequence of rational numbers in $[0, 1]$ with denominator (in lowest terms) at most n.

A6 The sum converges for $b = 2$ and diverges for $b \geq 3$. We first consider $b \geq 3$. Suppose the sum converges; then the fact that $f(n) = nf(d)$ whenever $b^{d-1} \leq n \leq b^d - 1$ yields

$$\sum_{n=1}^{\infty} \frac{1}{f(n)} = \sum_{d=1}^{\infty} \frac{1}{f(d)} \sum_{n=b^{d-1}}^{b^d-1} \frac{1}{n}. \qquad (1)$$

However, by comparing the integral of $1/x$ with a Riemann sum, we see that

$$\begin{aligned}\sum_{n=b^{d-1}}^{b^d-1} \frac{1}{n} &> \int_{b^{d-1}}^{b^d} \frac{dx}{x} \\ &= \log(b^d) - \log(b^{d-1}) = \log b,\end{aligned}$$

where $\log$ denotes the natural logarithm. Thus (1) yields

$$\sum_{n=1}^{\infty} \frac{1}{f(n)} > (\log b) \sum_{n=1}^{\infty} \frac{1}{f(n)},$$

a contradiction since $\log b > 1$ for $b \geq 3$. Therefore the sum diverges.

For $b = 2$, we have a slightly different identity because $f(2) \neq 2f(2)$. Instead, for any positive integer i, we have

$$\sum_{n=1}^{2^i-1} \frac{1}{f(n)} = 1 + \frac{1}{2} + \frac{1}{6} + \sum_{d=3}^{i} \frac{1}{f(d)} \sum_{n=2^{d-1}}^{2^d-1} \frac{1}{n}. \quad (2)$$

Again comparing an integral to a Riemann sum, we see that for $d \geq 3$,

$$\begin{aligned}\sum_{n=2^{d-1}}^{2^d-1} \frac{1}{n} &< \frac{1}{2^{d-1}} - \frac{1}{2^d} + \int_{2^{d-1}}^{2^d} \frac{dx}{x} \\ &= \frac{1}{2^d} + \log 2 \\ &\leq \frac{1}{8} + \log 2 < 0.125 + 0.7 < 1.\end{aligned}$$

Put $c = \frac{1}{8} + \log 2$ and $L = 1 + \frac{1}{2} + \frac{1}{6(1-c)}$. Then we can prove that $\sum_{n=1}^{2^i-1} \frac{1}{f(n)} < L$ for all $i \geq 2$ by induction on i. The case $i = 2$ is clear. For the induction, note that by (2),

$$\begin{aligned}\sum_{n=1}^{2^i-1} \frac{1}{f(n)} &< 1 + \frac{1}{2} + \frac{1}{6} + c \sum_{d=3}^{i} \frac{1}{f(d)} \\ &< 1 + \frac{1}{2} + \frac{1}{6} + c \frac{1}{6(1-c)} \\ &= 1 + \frac{1}{2} + \frac{1}{6(1-c)} = L,\end{aligned}$$

as desired. We conclude that $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges to a limit less than or equal to L.

Note: the above argument proves that the sum for $b = 2$ is at most $L < 2.417$. One can also obtain a lower bound by the same technique, namely $1 + \frac{1}{2} + \frac{1}{6(1-c')}$ with $c' = \log 2$. This bound exceeds 2.043. (By contrast, summing the first 100000 terms of the series only yields a lower bound of 1.906.) Repeating the same arguments with $d \geq 4$ as the cutoff yields the upper bound 2.185 and the lower bound 2.079.

B1 The probability is $1/99$. In fact, we show by induction on n that after n shots, the probability of having made any number of shots from 1 to $n-1$ is equal to $1/(n-1)$. This is evident for $n = 2$. Given the result for n, we see that the probability of making i shots after $n+1$ attempts is

$$\begin{aligned}\frac{i-1}{n} \frac{1}{n-1} + \left(1 - \frac{i}{n}\right) \frac{1}{n-1} &= \frac{(i-1) + (n-i)}{n(n-1)} \\ &= \frac{1}{n},\end{aligned}$$

as claimed.

B2 (Note: the problem statement assumes that all polyhedra are connected and that no two edges share more than one face, so we will do likewise. In particular, these are true for all convex polyhedra.) We show that in fact the first player can win on the third move. Suppose the polyhedron has a face A with at least four edges. If the first player plays there first, after the second player's first move there will be three consecutive faces B, C, D adjacent to A which are all unoccupied. The first player wins by playing in C; after the second player's second move, at least one of B and D remains unoccupied, and either is a winning move for the first player.

It remains to show that the polyhedron has a face with at least four edges. (Thanks to Russ Mann for suggesting the following argument.) Suppose on the contrary that each face has only three edges. Starting with any face F_1 with vertices v_1, v_2, v_3, let v_4 be the other endpoint of the third edge out of v_1. Then the faces adjacent to F_1 must have vertices v_1, v_2, v_4; v_1, v_3, v_4; and v_2, v_3, v_4. Thus v_1, v_2, v_3, v_4 form a polyhedron by themselves, contradicting the fact that the given polyhedron is connected and has at least five vertices. (One can also deduce this using Euler's formula $V - E + F = 2 - 2g$, where V, E, F are the numbers of vertices, edges and faces, respectively, and g is the genus of the polyhedron. For a convex polyhedron, $g = 0$ and you get the "usual" Euler's formula.)

Note: Walter Stromquist points out the following counterexample if one relaxes the assumption that a pair of faces may not share multiple edges. Take a tetrahedron and remove a smaller tetrahedron from the center of an edge; this creates two small triangular faces and turns two of the original faces into hexagons. Then the second player can draw by signing one of the hexagons, one of the large triangles, and one of the small triangles. (He does this by "mirroring": wherever the first player signs, the second player signs the other face of the same type.)

B3 The desired inequalities can be rewritten as

$$1 - \frac{1}{n} < \exp\left(1 + n\log\left(1 - \frac{1}{n}\right)\right) < 1 - \frac{1}{2n}.$$

By taking logarithms, we can rewrite the desired inequalities as

$$-\log\left(1 - \frac{1}{2n}\right) < -1 - n\log\left(1 - \frac{1}{n}\right) < -\log\left(1 - \frac{1}{n}\right).$$

Rewriting these in terms of the Taylor expansion of $-\log(1-x)$, we see that the desired result is also equivalent to

$$\sum_{i=1}^{\infty} \frac{1}{i2^i n^i} < \sum_{i=1}^{\infty} \frac{1}{(i+1)n^i} < \sum_{i=1}^{\infty} \frac{1}{in^i},$$

which is evident because the inequalities hold term by term.

Note: David Savitt points out that the upper bound can be improved from $1/(ne)$ to $2/(3ne)$ with a slightly more complicated argument. (In fact, for any $c > 1/2$, one has an upper bound of $c/(ne)$, but only for n above a certain bound depending on c.)

B4 Use the following strategy: guess $1, 3, 4, 6, 7, 9, \dots$ until the target number n is revealed to be equal to or lower than one of these guesses. If $n \equiv 1 \pmod 3$, it will be guessed on an odd turn. If $n \equiv 0 \pmod 3$, it will be guessed on an even turn. If $n \equiv 2 \pmod 3$, then $n+1$ will be guessed on an even turn, forcing a guess of n on the next turn. Thus the probability of success with this strategy is $1335/2002 > 2/3$.

Note: for any positive integer m, this strategy wins when the number is being guessed from $[1, m]$ with probability $\frac{1}{m}\lfloor \frac{2m+1}{3} \rfloor$. We can prove that this is best possible as follows. Let a_m denote m times the probability of winning when playing optimally. Also, let b_m denote m times the corresponding probability of winning if the objective is to select the number in an even number of guesses instead. (For definiteness, extend the definitions to incorporate $a_0 = 0$ and $b_0 = 0$.)

We first claim that $a_m = 1 + \max_{1 \leq k \leq m}\{b_{k-1} + b_{m-k}\}$ and $b_m = \max_{1 \leq k \leq m}\{a_{k-1} + a_{m-k}\}$ for $m \geq 1$. To establish the first recursive identity, suppose that our first guess is some integer k. We automatically win if $n = k$, with probability $1/m$. If $n < k$, with probability $(k-1)/m$, then we wish to guess an integer in $[1, k-1]$ in an even number of guesses; the probability of success when playing optimally is $b_{k-1}/(k-1)$, by assumption. Similarly, if $n < k$, with probability $(m-k)/m$, then the subsequent probability of winning is $b_{m-k}/(m-k)$. In sum, the overall probability of winning if k is our first guess is $(1 + b_{k-1} + b_{m-k})/m$. For optimal strategy, we choose k such that this quantity is maximized. (Note that this argument still holds if $k = 1$ or $k = m$, by our definitions of a_0 and b_0.) The first recursion follows, and the second recursion is established similarly.

We now prove by induction that $a_m = \lfloor (2m+1)/3 \rfloor$ and $b_m = \lfloor 2m/3 \rfloor$ for $m \geq 0$. The inductive step relies on the inequality $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$, with equality when one of x, y is an integer. Now suppose that $a_i = \lfloor (2i+1)/3 \rfloor$ and $b_i = \lfloor 2i/3 \rfloor$ for $i < m$. Then

$$\begin{aligned} 1 + b_{k-1} + b_{m-k} &= 1 + \left\lfloor \frac{2(k-1)}{3} \right\rfloor + \left\lfloor \frac{2(m-k)}{3} \right\rfloor \\ &\leq \left\lfloor \frac{2m}{3} \right\rfloor \end{aligned}$$

and similarly $a_{k-1} + a_{m-k} \leq \lfloor (2m+1)/3 \rfloor$, with equality in both cases attained, e.g., when $k = 1$. The inductive formula for a_m and b_m follows.

B5 (due to Dan Bernstein) Put $N = 2002!$. Then for $d = 1, \dots, 2002$, the number N^2 written in base $b = N/d - 1$ has digits $d^2, 2d^2, d^2$. (Note that these really are digits because $2(2002)^2 < (2002!)^2/2002 - 1$.)

Note: one can also produce an integer N which has base b digits $1, *, 1$ for n different values of b, as follows. Choose c with $0 < c < 2^{1/n}$. For m a large positive integer, put $N = 1 + (m+1)\cdots(m+n)\lfloor cm \rfloor^{n-2}$. For m sufficiently large, the bases

$$b = \frac{N-1}{(m+i)m^{n-2}} = \prod_{j \neq i} (m+j)$$

for $i = 1, \dots, n$ will have the properties that $N \equiv 1 \pmod{b}$ and $b^2 < N < 2b^2$ for m sufficiently large.

Note (due to Russ Mann): one can also give a "nonconstructive" argument. Let N be a large positive integer. For $b \in (N^2, N^3)$, the number of 3-digit base-b palindromes in the range $[b^2, N^6 - 1]$ is at least

$$\left\lfloor \frac{N^6 - b^2}{b} \right\rfloor - 1 \geq \frac{N^6}{b^2} - b - 2,$$

since there is a palindrome in each interval $[kb, (k+1)b - 1]$ for $k = b, \dots, b^2 - 1$. Thus the average number of bases for which a number in $[1, N^6 - 1]$ is at least

$$\frac{1}{N^6} \sum_{b=N^2+1}^{N^3-1} \left(\frac{N^6}{b} - b - 2 \right) \geq \log(N) - c$$

for some constant $c > 0$. Take N so that the right side exceeds 2002; then at least one number in $[1, N^6 - 1]$ is a base-b palindrome for at least 2002 values of b.

B6 We prove that the determinant is congruent modulo p to

$$x \prod_{i=0}^{p-1} (y + ix) \prod_{i,j=0}^{p-1} (z + ix + jy). \qquad (3)$$

We first check that

$$\prod_{i=0}^{p-1} (y + ix) \equiv y^p - x^{p-1} y \pmod{p}. \qquad (4)$$

Since both sides are homogeneous as polynomials in x and y, it suffices to check (4) for $x = 1$, as a congruence between polynomials. Now note that the right side has $0, 1, \dots, p-1$ as roots modulo p, as does the left side. Moreover, both sides have the same leading coefficient. Since they both have degree only p, they must then coincide.

We thus have

$$\begin{aligned}
&x \prod_{i=0}^{p-1} (y + ix) \prod_{i,j=0}^{p-1} (z + ix + jy) \\
&\equiv x(y^p - x^{p-1} y) \prod_{j=0}^{p-1} ((z + jy)^p - x^{p-1}(z + jy)) \\
&\equiv (xy^p - x^p y) \prod_{j=0}^{p-1} (z^p - x^{p-1} z + jy^p - jx^{p-1} y) \\
&\equiv (xy^p - x^p y)((z^p - x^{p-1} z)^p \\
&\quad - (y^p - x^{p-1} y)^{p-1} (z^p - x^{p-1} z)) \\
&\equiv (xy^p - x^p y)(z^{p^2} - x^{p^2 - p} z^p) \\
&\quad - x(y^p - x^{p-1} y)^p (z^p - x^{p-1} z) \\
&\equiv xy^p z^{p^2} - x^p y z^{p^2} - x^{p^2-p+1} y^p z^p + x^{p^2} y z^p \\
&\quad - xy^{p^2} z^p + x^{p^2-p+1} y^p z^p + x^p y^{p^2} z - x^{p^2} y^p z \\
&\equiv xy^p z^{p^2} + yz^p x^{p^2} + zx^p y^{p^2} \\
&\quad - xz^p y^{p^2} - yx^p z^{p^2} - zy^p x^{p^2},
\end{aligned}$$

which is precisely the desired determinant.

Note: a simpler conceptual proof is as follows. (Everything in this paragraph will be modulo p.) Note that for any integers a, b, c, the column vector $[ax + by + cz, (ax + by + cz)^p, (ax + by + cz)^{p^2}]$ is a linear combination of the columns of the given matrix. Thus $ax + by + cz$ divides the determinant. In particular, all of the factors of (3) divide the determinant; since both (3) and the determinant have degree $p^2 + p + 1$, they agree up to a scalar multiple. Moreover, they have the same coefficient of $z^{p^2} y^p x$ (since this term only appears in the expansion of (3) when you choose the first term in each factor). Thus the determinant is congruent to (3), as desired.

Either argument can be used to generalize to a corresponding $n \times n$ determinant, called a Moore determinant; we leave the precise formulation to the reader. Note the similarity with the classical Vandermonde determinant: if A is the $n \times n$ matrix with $A_{ij} = x_i^j$ for $i, j = 0, \dots, n-1$, then

$$\det(A) = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Solutions to the 64th William Lowell Putnam Mathematical Competition
Saturday, December 6, 2003

Manjul Bhargava, Kiran Kedlaya, and Lenny Ng

A1 There are n such sums. More precisely, there is exactly one such sum with k terms for each of $k = 1, \dots, n$ (and clearly no others). To see this, note that if $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$, then

$$\begin{aligned} ka_1 &= a_1 + a_1 + \cdots + a_1 \\ &\leq n \leq a_1 + (a_1+1) + \cdots + (a_1+1) \\ &= ka_1 + k - 1. \end{aligned}$$

However, there is a unique integer a_1 satisfying these inequalities, namely $a_1 = \lfloor n/k \rfloor$. Moreover, once a_1 is fixed, there are k different possibilities for the sum $a_1 + a_2 + \cdots + a_k$: if i is the last integer such that $a_i = a_1$, then the sum equals $ka_1 + (i-1)$. The possible values of i are $1, \dots, k$, and exactly one of these sums comes out equal to n, proving our claim.

Note: In summary, there is a unique partition of n with k terms that is "as equally spaced as possible". One can also obtain essentially the same construction inductively: except for the all-ones sum, each partition of n is obtained by "augmenting" a unique partition of $n-1$.

A2 **First solution:** Assume without loss of generality that $a_i + b_i > 0$ for each i (otherwise both sides of the desired inequality are zero). Then the AM-GM inequality gives

$$\begin{aligned} &\left(\frac{a_1 \cdots a_n}{(a_1+b_1)\cdots(a_n+b_n)} \right)^{1/n} \\ &\leq \frac{1}{n} \left(\frac{a_1}{a_1+b_1} + \cdots + \frac{a_n}{a_n+b_n} \right), \end{aligned}$$

and likewise with the roles of a and b reversed. Adding these two inequalities and clearing denominators yields the desired result.

Second solution: Write the desired inequality in the form

$$(a_1+b_1)\cdots(a_n+b_n) \geq [(a_1\cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n}]^n,$$

expand both sides, and compare the terms on both sides in which k of the terms are among the a_i. On the left, one has the product of each k-element subset of $\{1, \dots, n\}$; on the right, one has $\binom{n}{k}(a_1\cdots a_n)^{k/n} \cdots (b_1 \dots b_n)^{(n-k)/n}$, which is precisely $\binom{n}{k}$ times the geometric mean of the terms on the left. Thus AM-GM shows that the terms under consideration on the left exceed those on the right; adding these inequalities over all k yields the desired result.

Third solution: Since both sides are continuous in each a_i, it is sufficient to prove the claim with $a_1, \dots, a_n$ all positive (the general case follows by taking limits as some of the a_i tend to zero). Put $r_i = b_i/a_i$; then the given inequality is equivalent to

$$(1+r_1)^{1/n} \cdots (1+r_n)^{1/n} \geq 1 + (r_1 \cdots r_n)^{1/n}.$$

In terms of the function

$$f(x) = \log(1+e^x)$$

and the quantities $s_i = \log r_i$, we can rewrite the desired inequality as

$$\frac{1}{n}(f(s_1) + \cdots + f(s_n)) \geq f\left(\frac{s_1 + \cdots + s_n}{n} \right).$$

This will follow from Jensen's inequality if we can verify that f is a convex function; it is enough to check that $f''(x) > 0$ for all x. In fact,

$$f'(x) = \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}$$

is an increasing function of x, so $f''(x) > 0$ and Jensen's inequality thus yields the desired result. (As long as the a_i are all positive, equality holds when $s_1 = \cdots = s_n$, i.e., when the vectors $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$. Of course other equality cases crop up if some of the a_i vanish, i.e., if $a_1 = b_1 = 0$.)

Fourth solution: We apply induction on n, the case $n = 1$ being evident. First we verify the auxiliary inequality

$$(a^n + b^n)(c^n + d^n)^{n-1} \geq (ac^{n-1} + bd^{n-1})^n$$

for $a, b, c, d \geq 0$. The left side can be written as

$$\begin{aligned} &a^n c^{n(n-1)} + b^n d^{n(n-1)} \\ &+ \sum_{i=1}^{n-1} \binom{n-1}{i} a^n c^{ni} d^{n(n-1-i)} \\ &+ \sum_{i=1}^{n-1} \binom{n-1}{i-1} b^n c^{n(n-i)} d^{n(i-1)}. \end{aligned}$$

Applying the weighted AM-GM inequality between matching terms in the two sums yields

$$\begin{aligned}&(a^n+b^n)(c^n+d^n)^{n-1}\\&\geq a^n c^{n(n-1)} + b^n d^{n(n-1)}\\&\quad+\sum_{i=1}^{n-1}\binom{n}{i} a^i b^{n-i} c^{(n-1)i} d^{(n-1)(n-i)},\end{aligned}$$

proving the auxiliary inequality.

Now given the auxiliary inequality and the $n-1$ case of the desired inequality, we apply the auxiliary inequality with $a = a_1^{1/n}$, $b = b_1^{1/n}$, $c = (a_2 \cdots a_n)^{1/n(n-1)}$, $d = (b_2 \dots b_n)^{1/n(n-1)}$. The right side will be the n-th power of the desired inequality. The left side comes out to

$$(a_1+b_1)((a_2\cdots a_n)^{1/(n-1)} + (b_2\cdots b_n)^{1/(n-1)})^{n-1},$$

and by the induction hypothesis, the second factor is less than $(a_2+b_2)\cdots(a_n+b_n)$. This yields the desired result.

Note: Equality holds if and only if $a_i = b_i = 0$ for some i or if the vectors $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ are proportional. As pointed out by Naoki Sato, the problem also appeared on the 1992 Irish Mathematical Olympiad. It is also a special case of a classical inequality, known as Hölder's inequality, which generalizes the Cauchy-Schwarz inequality (this is visible from the $n=2$ case); the first solution above is adapted from the standard proof of Hölder's inequality. We don't know whether the declaration "Apply Hölder's inequality" by itself is considered an acceptable solution to this problem.

A3 **First solution:** Write

$$\begin{aligned}f(x) &= \sin x + \cos x + \tan x + \cot x + \sec x + \csc x\\&= \sin x + \cos x + \frac{1}{\sin x \cos x} + \frac{\sin x + \cos x}{\sin x \cos x}.\end{aligned}$$

We can write $\sin x + \cos x = \sqrt{2}\cos(\pi/4 - x)$; this suggests making the substitution $y = \pi/4 - x$. In this new coordinate,

$$\sin x \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\cos 2y,$$

and writing $c = \sqrt{2}\cos y$, we have

$$\begin{aligned}f(y) &= (1+c)\left(1 + \frac{2}{c^2-1}\right) - 1\\&= c + \frac{2}{c-1}.\end{aligned}$$

We must analyze this function of c in the range $[-\sqrt{2},\sqrt{2}]$. Its value at $c=-\sqrt{2}$ is $2-3\sqrt{2} < -2.24$, and at $c=\sqrt{2}$ is $2+3\sqrt{2} > 6.24$. Its derivative is $1-2/(c-1)^2$, which vanishes when $(c-1)^2 = 2$, i.e., where $c = 1\pm\sqrt{2}$. Only the value $c = 1-\sqrt{2}$ is in bounds, at which the value of f is $1-2\sqrt{2} > -1.83$. As for the pole at $c=1$, we observe that f decreases as c approaches from below (so takes negative values for all $c<1$) and increases as c approaches from above (so takes positive values for all $c>1$); from the data collected so far, we see that f has no sign crossings, so the minimum of $|f|$ is achieved at a critical point of f. We conclude that the minimum of $|f|$ is $2\sqrt{2}-1$.

Alternate derivation (due to Zuming Feng): We can also minimize $|c + 2/(c-1)|$ without calculus (or worrying about boundary conditions). For $c > 1$, we have

$$1 + (c-1) + \frac{2}{c-1} \geq 1 + 2\sqrt{2}$$

by AM-GM on the last two terms, with equality for $c - 1 = \sqrt{2}$ (which is out of range). For $c < 1$, we similarly have

$$-1 + 1 - c + \frac{2}{1-c} \geq -1 + 2\sqrt{2},$$

here with equality for $1 - c = \sqrt{2}$.

Second solution: Write

$$f(a,b) = a + b + \frac{1}{ab} + \frac{a+b}{ab}.$$

Then the problem is to minimize $|f(a,b)|$ subject to the constraint $a^2+b^2-1=0$. Since the constraint region has no boundary, it is enough to check the value at each critical point and each potential discontinuity (i.e., where $ab=0$) and select the smallest value (after checking that f has no sign crossings).

We locate the critical points using the Lagrange multiplier condition: the gradient of f should be parallel to that of the constraint, which is to say, to the vector (a,b). Since

$$\frac{\partial f}{\partial a} = 1 - \frac{1}{a^2 b} - \frac{1}{a^2}$$

and similarly for b, the proportionality yields

$$a^2b^3 - a^3b^2 + a^3 - b^3 + a^2 - b^2 = 0.$$

The irreducible factors of the left side are $1+a$, $1+b$, $a-b$, and $ab-a-b$. So we must check what happens when any of those factors, or a or b, vanishes.

If $1+a=0$, then $b=0$, and the singularity of f becomes removable when restricted to the circle. Namely, we have

$$f = a + b + \frac{1}{a} + \frac{b+1}{ab}$$

and $a^2+b^2-1=0$ implies $(1+b)/a = a/(1-b)$. Thus we have $f=-2$; the same occurs when $1+b=0$.

If $a - b = 0$, then $a = b = \pm\sqrt{2}/2$ and either $f = 2 + 3\sqrt{2} > 6.24$, or $f = 2 - 3\sqrt{2} < -2.24$.

If $a = 0$, then either $b = -1$ as discussed above, or $b = 1$. In the latter case, f blows up as one approaches this point, so there cannot be a global minimum there.

Finally, if $ab - a - b = 0$, then

$$a^2b^2 = (a+b)^2 = 2ab + 1$$

and so $ab = 1 \pm \sqrt{2}$. The plus sign is impossible since $|ab| \leq 1$, so $ab = 1 - \sqrt{2}$ and

$$\begin{aligned} f(a,b) &= ab + \frac{1}{ab} + 1 \\ &= 1 - 2\sqrt{2} > -1.83. \end{aligned}$$

This yields the smallest value of $|f|$ in the list (and indeed no sign crossings are possible), so $2\sqrt{2} - 1$ is the desired minimum of $|f|$.

Note: Instead of using the geometry of the graph of f to rule out sign crossings, one can verify explicitly that f cannot take the value 0. In the first solution, note that $c + 2/(c-1) = 0$ implies $c^2 - c + 2 = 0$, which has no real roots. In the second solution, we would have

$$a^2b + ab^2 + a + b = -1.$$

Squaring both sides and simplifying yields

$$2a^3b^3 + 5a^2b^2 + 4ab = 0,$$

whose only real root is $ab = 0$. But the cases with $ab = 0$ do not yield $f = 0$, as verified above.

A4 We split into three cases. Note first that $|A| \geq |a|$, by applying the condition for large x.

Case 1: $B^2 - 4AC > 0$. In this case $Ax^2 + Bx + C$ has two distinct real roots r_1 and r_2. The condition implies that $ax^2 + bx + c$ also vanishes at r_1 and r_2, so $b^2 - 4ac > 0$. Now

$$\begin{aligned} B^2 - 4AC &= A^2(r_1 - r_2)^2 \\ &\geq a^2(r_1 - r_2)^2 \\ &= b^2 - 4ac. \end{aligned}$$

Case 2: $B^2 - 4AC \leq 0$ and $b^2 - 4ac \leq 0$. Assume without loss of generality that $A \geq a > 0$, and that $B = 0$ (by shifting x). Then $Ax^2 + Bx + C \geq ax^2 + bx + c \geq 0$ for all x; in particular, $C \geq c \geq 0$. Thus

$$\begin{aligned} 4AC - B^2 &= 4AC \\ &\geq 4ac \\ &\geq 4ac - b^2. \end{aligned}$$

Alternate derivation (due to Robin Chapman): the ellipse $Ax^2 + Bxy + Cy^2 = 1$ is contained within the ellipse $ax^2 + bxy + cy^2 = 1$, and their respective enclosed areas are $\pi/(4AC - B^2)$ and $\pi/(4ac - b^2)$.

Case 3: $B^2 - 4AC \leq 0$ and $b^2 - 4ac > 0$. Since $Ax^2 + Bx + C$ has a graph not crossing the x-axis, so do $(Ax^2 + Bx + C) \pm (ax^2 + bx + c)$. Thus

$$\begin{aligned} (B-b)^2 - 4(A-a)(C-c) \leq 0, \\ (B+b)^2 - 4(A+a)(C+c) \leq 0 \end{aligned}$$

and adding these together yields

$$2(B^2 - 4AC) + 2(b^2 - 4ac) \leq 0.$$

Hence $b^2 - 4ac \leq 4AC - B^2$, as desired.

A5 **First solution:** We represent a Dyck n-path by a sequence $a_1 \cdots a_{2n}$, where each a_i is either $(1,1)$ or $(1,-1)$.

Given an $(n-1)$-path $P = a_1 \cdots a_{2n-2}$, we distinguish two cases. If P has no returns of even-length, then let $f(P)$ denote the n-path $(1,1)(1,-1)P$. Otherwise, let $a_i a_{i+1} \cdots a_j$ denote the rightmost even-length return in P, and let $f(P) = (1,1)a_1 a_2 \cdots a_j (1,-1) a_{j+1} \cdots a_{2n-2}$. Then f clearly maps the set of Dyck $(n-1)$-paths to the set of Dyck n-paths having no even return.

We claim that f is bijective; to see this, we simply construct the inverse mapping. Given an n-path P, let $R = a_i a_{i+1} ... a_j$ denote the leftmost return in P, and let $g(P)$ denote the path obtained by removing a_1 and a_j from P. Then evidently $f \circ g$ and $g \circ f$ are identity maps, proving the claim.

Second solution: (by Dan Bernstein) Let C_n be the number of Dyck paths of length n, let O_n be the number of Dyck paths whose final return has odd length, and let X_n be the number of Dyck paths with no return of even length.

We first exhibit a recursion for O_n; note that $O_0 = 0$. Given a Dyck n-path whose final return has odd length, split it just after its next-to-last return. For some k (possibly zero), this yields a Dyck k-path, an upstep, a Dyck $(n-k-1)$-path whose odd return has even length, and a downstep. Thus for $n \geq 1$,

$$O_n = \sum_{k=0}^{n-1} C_k (C_{n-k-1} - O_{n-k-1}).$$

We next exhibit a similar recursion for X_n; note that $X_0 = 1$. Given a Dyck n-path with no even return, splitting as above yields for some k a Dyck k-path with no even return, an upstep, a Dyck $(n-k-1)$-path whose final return has even length, then a downstep. Thus for $n \geq 1$,

$$X_n = \sum_{k=0}^{n-1} X_k (C_{n-k-1} - O_{n-k-1}).$$

To conclude, we verify that $X_n = C_{n-1}$ for $n \geq 1$, by induction on n. This is clear for $n = 1$ since $X_1 = C_0 = 1$. Given $X_k = C_{k-1}$ for $k < n$, we have

$$\begin{aligned}
X_n &= \sum_{k=0}^{n-1} X_k (C_{n-k-1} - O_{n-k-1}) \\
&= C_{n-1} - O_{n-1} + \sum_{k=1}^{n-1} C_{k-1}(C_{n-k-1} - O_{n-k-1}) \\
&= C_{n-1} - O_{n-1} + O_{n-1} \\
&= C_{n-1},
\end{aligned}$$

as desired.

Note: Since the problem only asked about the *existence* of a one-to-one correspondence, we believe that any proof, bijective or not, that the two sets have the same cardinality is an acceptable solution. (Indeed, it would be highly unusual to insist on using or not using a specific proof technique!) The second solution above can also be phrased in terms of generating functions. Also, the C_n are well-known to equal the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$; the problem at hand is part of a famous exercise in Richard Stanley's *Enumerative Combinatorics, Volume 1* giving 66 combinatorial interpretations of the Catalan numbers.

A6 **First solution:** Yes, such a partition is possible. To achieve it, place each integer into A if it has an even number of 1s in its binary representation, and into B if it has an odd number. (One discovers this by simply attempting to place the first few numbers by hand and noticing the resulting pattern.)

To show that $r_A(n) = r_B(n)$, we exhibit a bijection between the pairs (a_1, a_2) of distinct elements of A with $a_1 + a_2 = n$ and the pairs (b_1, b_2) of distinct elements of B with $b_1 + b_2 = n$. Namely, given a pair (a_1, a_2) with $a_1 + a_2 = n$, write both numbers in binary and find the lowest-order place in which they differ (such a place exists because $a_1 \neq a_2$). Change both numbers in that place and call the resulting numbers b_1, b_2. Then $a_1 + a_2 = b_1 + b_2 = n$, but the parity of the number of 1s in b_1 is opposite that of a_1, and likewise between b_2 and a_2. This yields the desired bijection.

Second solution: (by Micah Smukler) Write $b(n)$ for the number of 1s in the base 2 expansion of n, and $f(n) = (-1)^{b(n)}$. Then the desired partition can be described as $A = f^{-1}(1)$ and $B = f^{-1}(-1)$. Since $f(2n) + f(2n+1) = 0$, we have

$$\sum_{i=0}^{n} f(n) = \begin{cases} 0 & n \text{ odd} \\ f(n) & n \text{ even.} \end{cases}$$

If p, q are both in A, then $f(p) + f(q) = 2$; if p, q are both in B, then $f(p) + f(q) = -2$; if p, q are in different sets, then $f(p) + f(q) = 0$. In other words,

$$2(r_A(n) - r_B(n)) = \sum_{p+q=n, p<q} (f(p) + f(q))$$

and it suffices to show that the sum on the right is always zero. If n is odd, that sum is visibly $\sum_{i=0}^{n} f(i) = 0$. If n is even, the sum equals

$$\left(\sum_{i=0}^{n} f(i)\right) - f(n/2) = f(n) - f(n/2) = 0.$$

This yields the desired result.

Third solution: (by Dan Bernstein) Put $f(x) = \sum_{n \in A} x^n$ and $g(x) = \sum_{n \in B} x^n$; then the value of $r_A(n)$ (resp. $r_B(n)$) is the coefficient of x^n in $f(x)^2 - f(x^2)$ (resp. $g(x)^2 - g(x^2)$). From the evident identities

$$\begin{aligned}
\frac{1}{1-x} &= f(x) + g(x) \\
f(x) &= f(x^2) + xg(x^2) \\
g(x) &= g(x^2) + xf(x^2),
\end{aligned}$$

we have

$$\begin{aligned}
f(x) - g(x) &= f(x^2) - g(x^2) + xg(x^2) - xf(x^2) \\
&= (1-x)(f(x^2) - g(x^2)) \\
&= \frac{f(x^2) - g(x^2)}{f(x) + g(x)}.
\end{aligned}$$

We deduce that $f(x)^2 - g(x)^2 = f(x^2) - g(x^2)$, yielding the desired equality.

Note: This partition is actually unique, up to interchanging A and B. More precisely, the condition that $0 \in A$ and $r_A(n) = r_B(n)$ for $n = 1, \dots, m$ uniquely determines the positions of $0, \dots, m$. We see this by induction on m: given the result for $m - 1$, switching the location of m changes $r_A(m)$ by one and does not change $r_B(m)$, so it is not possible for both positions to work. Robin Chapman points out this problem is solved in D.J. Newman's *Analytic Number Theory* (Springer, 1998); in that solution, one uses generating functions to find the partition and establish its uniqueness, not just verify it.

B1 No, there do not.

First solution: Suppose the contrary. By setting $y = -1, 0, 1$ in succession, we see that the polynomials $1 - x + x^2, 1, 1 + x + x^2$ are linear combinations of $a(x)$ and $b(x)$. But these three polynomials are linearly independent, so cannot all be written as linear combinations of two other polynomials, contradiction.

Alternate formulation: the given equation expresses a diagonal matrix with $1, 1, 1$ and zeroes on the diagonal, which has rank 3, as the sum of two matrices of rank 1. But the rank of a sum of matrices is at most the sum of the ranks of the individual matrices.

Second solution: It is equivalent (by relabeling and rescaling) to show that $1+xy+x^2y^2$ cannot be written as $a(x)d(y)-b(x)c(y)$. Write $a(x)=\sum a_ix^i$, $b(x)=\sum b_ix^i$, $c(y)=\sum c_jy^j$, $d(y)=\sum d_jy^j$. We now start comparing coefficients of $1+xy+x^2y^2$. By comparing coefficients of $1+xy+x^2y^2$ and $a(x)d(y)-b(x)c(y)$, we get

$$\begin{aligned} 1 &= a_id_i - b_ic_i \qquad (i=0,1,2) \\ 0 &= a_id_j - b_ic_j \qquad (i \neq j). \end{aligned}$$

The first equation says that a_i and b_i cannot both vanish, and c_i and d_i cannot both vanish. The second equation says that $a_i/b_i = c_j/d_j$ when $i \neq j$, where both sides should be viewed in $\mathbb{R} \cup \{\infty\}$ (and neither is undetermined if $i,j \in \{0,1,2\}$). But then

$$a_0/b_0 = c_1/d_1 = a_2/b_2 = c_0/d_0$$

contradicting the equation $a_0d_0 - b_0c_0 = 1$.

Third solution: We work over the complex numbers, in which we have a primitive cube root ω of 1. We also use without further comment unique factorization for polynomials in two variables over a field. And we keep the relabeling of the second solution.

Suppose the contrary. Since $1+xy+x^2y^2 = (1-xy/\omega)(1-xy/\omega^2)$, the rational function $a(\omega/y)d(y)-b(\omega/y)c(y)$ must vanish identically (that is, coefficient by coefficient). If one of the polynomials, say a, vanished identically, then one of b or c would also, and the desired inequality could not hold. So none of them vanish identically, and we can write

$$\frac{c(y)}{d(y)} = \frac{a(\omega/y)}{b(\omega/y)}.$$

Likewise,

$$\frac{c(y)}{d(y)} = \frac{a(\omega^2/y)}{b(\omega^2/y)}.$$

Put $f(x) = a(x)/b(x)$; then we have $f(\omega x) = f(x)$ identically. That is, $a(x)b(\omega x) = b(x)a(\omega x)$. Since a and b have no common factor (otherwise $1+xy+x^2y^2$ would have a factor divisible only by x, which it doesn't since it doesn't vanish identically for any particular x), $a(x)$ divides $a(\omega x)$. Since they have the same degree, they are equal up to scalars. It follows that one of $a(x), xa(x), x^2a(x)$ is a polynomial in x^3 alone, and likewise for b (with the same power of x).

If $xa(x)$ and $xb(x)$, or $x^2a(x)$ and $x^2b(x)$, are polynomials in x^3, then a and b are divisible by x, but we know a and b have no common factor. Hence $a(x)$ and $b(x)$ are polynomials in x^3. Likewise, $c(y)$ and $d(y)$ are polynomials in y^3. But then $1+xy+x^2y^2 = a(x)d(y)-b(x)c(y)$ is a polynomial in x^3 and y^3, contradiction.

Note: The third solution only works over fields of characteristic not equal to 3, whereas the other two work over arbitrary fields. (In the first solution, one must replace -1 by another value if working in characteristic 2.)

B2 It is easy to see by induction that the j-th entry of the k-th sequence (where the original sequence is $k=1$) is $\sum_{i=1}^k \binom{k-1}{i-1}/(2^{k-1}(i+j-1))$, and so $x_n = \frac{1}{2^{n-1}}\sum_{i=1}^n \binom{n-1}{i-1}/i$. Now $\binom{n-1}{i-1}/i = \binom{n}{i}/n$; hence

$$x_n = \frac{1}{n2^{n-1}} \sum_{i=1}^n \binom{n}{i} = \frac{2^n-1}{n2^{n-1}} < 2/n,$$

as desired.

B3 **First solution:** It is enough to show that for each prime p, the exponent of p in the prime factorization of both sides is the same. On the left side, it is well-known that the exponent of p in the prime factorization of $n!$ is

$$\sum_{i=1}^n \left\lfloor \frac{n}{p^i} \right\rfloor.$$

(To see this, note that the i-th term counts the multiples of p^i among $1,\dots,n$, so that a number divisible exactly by p^i gets counted exactly i times.) This number can be reinterpreted as the cardinality of the set S of points in the plane with positive integer coordinates lying on or under the curve $y = np^{-x}$: namely, each summand is the number of points of S with $x = i$.

On the right side, the exponent of p in the prime factorization of $\mathrm{lcm}(1,\dots,\lfloor n/i \rfloor)$ is $\lfloor \log_p \lfloor n/i \rfloor \rfloor = \lfloor \log_p(n/i) \rfloor$. However, this is precisely the number of points of S with $y = i$. Thus

$$\sum_{i=1}^n \lfloor \log_p \lfloor n/i \rfloor \rfloor = \sum_{i=1}^n \left\lfloor \frac{n}{p^i} \right\rfloor,$$

and the desired result follows.

Second solution: We prove the result by induction on n, the case $n=1$ being obvious. What we actually show is that going from $n-1$ to n changes both sides by the same multiplicative factor, that is,

$$n = \prod_{i=1}^{n-1} \frac{\mathrm{lcm}\{1,2,\dots,\lfloor n/i \rfloor\}}{\mathrm{lcm}\{1,2,\dots,\lfloor (n-1)/i \rfloor\}}.$$

Note that the i-th term in the product is equal to 1 if n/i is not an integer, i.e., if n/i is not a divisor of n. It is also equal to 1 if n/i is a divisor of n but not a prime power, since any composite number divides the lcm of all smaller numbers. However, if n/i is a power of p, then the i-th term is equal to p.

Since n/i runs over all proper divisors of n, the product on the right side includes one factor of the prime p for each factor of p in the prime factorization of n. Thus the whole product is indeed equal to n, completing the induction.

B4 **First solution:** Put $g = r_1 + r_2$, $h = r_3 + r_4$, $u = r_1 r_2$, $v = r_3 r_4$. We are given that g is rational. The following are also rational:

$$\begin{aligned}\frac{-b}{a} &= g + h \\ \frac{c}{a} &= gh + u + v \\ \frac{-d}{a} &= gv + hu\end{aligned}$$

From the first line, h is rational. From the second line, $u + v$ is rational. From the third line, $g(u+v) - (gv + hu) = (g-h)u$ is rational. Since $g \neq h$, u is rational, as desired.

Second solution: This solution uses some basic Galois theory. We may assume $r_1 \neq r_2$, since otherwise they are both rational and so then is $r_1 r_2$.

Let τ be an automorphism of the field of algebraic numbers; then τ maps each r_i to another one, and fixes the rational number $r_1 + r_2$. If $\tau(r_1)$ equals one of r_1 or r_2, then $\tau(r_2)$ must equal the other one, and vice versa. Thus τ either fixes the set $\{r_1, r_2\}$ or moves it to $\{r_3, r_4\}$. But if the latter happened, we would have $r_1 + r_2 = r_3 + r_4$, contrary to hypothesis. Thus τ fixes the set $\{r_1, r_2\}$ and in particular the number $r_1 r_2$. Since this is true for any τ, $r_1 r_2$ must be rational.

Note: The conclusion fails if we allow $r_1 + r_2 = r_3 + r_4$. For instance, take the polynomial $x^4 - 2$ and label its roots so that $(x - r_1)(x - r_2) = x^2 - \sqrt{2}$ and $(x - r_3)(x - r_4) = x^2 + \sqrt{2}$.

B5 Place the unit circle on the complex plane so that A, B, C correspond to the complex numbers $1, \omega, \omega^2$, where $\omega = e^{2\pi i/3}$, and let P correspond to the complex number x. The distances a, b, c are then $|x - 1|, |x - \omega|, |x - \omega^2|$. Now the identity

$$(x - 1) + \omega(x - \omega) + \omega^2(x - \omega^2) = 0$$

implies that there is a triangle whose sides, as vectors, correspond to the complex numbers $x - 1, \omega(x - \omega), \omega^2(x - \omega^2)$; this triangle has sides of length a, b, c.

To calculate the area of this triangle, we first note a more general formula. If a triangle in the plane has vertices at 0, $v_1 = s_1 + it_1$, $v_2 = s_2 + it_2$, then it is well known that the area of the triangle is $|s_1 t_2 - s_2 t_1|/2 = |v_1 \overline{v_2} - v_2 \overline{v_1}|/4$. In our case, we have $v_1 = x - 1$ and $v_2 = \omega(x - \omega)$; then

$$v_1 \overline{v_2} - v_2 \overline{v_1} = (\omega^2 - \omega)(x\overline{x} - 1) = i\sqrt{3}(|x|^2 - 1).$$

Hence the area of the triangle is $\sqrt{3}(1 - |x|^2)/4$, which depends only on the distance $|x|$ from P to O.

B6 **First solution:** (composite of solutions by Feng Xie and David Pritchard) Let μ denote Lebesgue measure on $[0,1]$. Define

$$\begin{aligned}E_+ &= \{x \in [0,1] : f(x) \geq 0\} \\ E_- &= \{x \in [0,1] : f(x) < 0\};\end{aligned}$$

then E_+, E_- are measurable and $\mu(E_+) + \mu(E_-) = 1$. Write μ_+ and μ_- for $\mu(E_+)$ and $\mu(E_-)$. Also define

$$\begin{aligned}I_+ &= \int_{E_+} |f(x)|\,dx \\ I_- &= \int_{E_-} |f(x)|\,dx,\end{aligned}$$

so that $\int_0^1 |f(x)|\,dx = I_+ + I_-$.

From the triangle inequality $|a + b| \geq \pm(|a| - |b|)$, we have the inequality

$$\begin{aligned}&\iint_{E_+ \times E_-} |f(x) + f(y)|\,dx\,dy \\ &\geq \pm \iint_{E_+ \times E_-} (|f(x)| - |f(y)|)\,dx\,dy \\ &= \pm(\mu_- I_+ - \mu_+ I_-),\end{aligned}$$

and likewise with $+$ and $-$ switched. Adding these inequalities together and allowing all possible choices of the signs, we get

$$\begin{aligned}&\iint_{(E_+ \times E_-) \cup (E_- \times E_+)} |f(x) + f(y)|\,dx\,dy \\ &\geq \max\left\{0, 2(\mu_- I_+ - \mu_+ I_-), 2(\mu_+ I_- - \mu_- I_+)\right\}.\end{aligned}$$

To this inequality, we add the equalities

$$\begin{aligned}\iint_{E_+ \times E_+} |f(x) + f(y)|\,dx\,dy &= 2\mu_+ I_+ \\ \iint_{E_- \times E_-} |f(x) + f(y)|\,dx\,dy &= 2\mu_- I_- \\ -\int_0^1 |f(x)|\,dx &= -(\mu_+ + \mu_-)(I_+ + I_-)\end{aligned}$$

to obtain

$$\begin{aligned}&\int_0^1 \int_0^1 |f(x) + f(y)|\,dx\,dy - \int_0^1 |f(x)|\,dx \\ &\geq \max\{(\mu_+ - \mu_-)(I_+ + I_-) + 2\mu_-(I_- - I_+), \\ &\qquad (\mu_+ - \mu_-)(I_+ - I_-), \\ &\qquad (\mu_- - \mu_+)(I_+ + I_-) + 2\mu_+(I_+ - I_-)\}.\end{aligned}$$

Now simply note that for each of the possible comparisons between μ_+ and μ_-, and between I_+ and I_-, one of the three terms above is manifestly nonnegative. This yields the desired result.

Second solution: We will show at the end that it is enough to prove a discrete analogue: if $x_1, \dots, x_n$ are real numbers, then

$$\frac{1}{n^2} \sum_{i,j=1}^{n} |x_i + x_j| \geq \frac{1}{n} \sum_{i=1}^{n} |x_i|.$$

In the meantime, we concentrate on this assertion.

Let $f(x_1,\dots,x_n)$ denote the difference between the two sides. We induct on the number of nonzero values of $|x_i|$. We leave for later the base case, where there is at most one such value. Suppose instead for now that there are two or more. Let s be the smallest, and suppose without loss of generality that $x_1=\cdots=x_a=s$, $x_{a+1}=\cdots=x_{a+b}=-s$, and for $i>a+b$, either $x_i=0$ or $|x_i|>s$. (One of a,b might be zero.)

Now consider

$$f(\overbrace{t,\cdots,t}^{a\text{ terms}},\overbrace{-t,\cdots,-t}^{b\text{ terms}},x_{a+b+1},\cdots,x_n)$$

as a function of t. It is piecewise linear near s; in fact, it is linear between 0 and the smallest nonzero value among $|x_{a+b+1}|,\dots,|x_n|$ (which exists by hypothesis). Thus its minimum is achieved by one (or both) of those two endpoints. In other words, we can reduce the number of distinct nonzero absolute values among the x_i without increasing f. This yields the induction, pending verification of the base case.

As for the base case, suppose that $x_1=\cdots=x_a=s>0$, $x_{a+1}=\cdots=x_{a+b}=-s$, and $x_{a+b+1}=\cdots=x_n=0$. (Here one or even both of a,b could be zero, though the latter case is trivial.) Then

$$\begin{aligned}f(x_1,\dots,x_n)&=\frac{s}{n^2}(2a^2+2b^2+(a+b)(n-a-b))\\&\quad-\frac{s}{n}(a+b)\\&=\frac{s}{n^2}(a^2-2ab+b^2)\\&\geq 0.\end{aligned}$$

This proves the base case of the induction, completing the solution of the discrete analogue.

To deduce the original statement from the discrete analogue, approximate both integrals by equally-spaced Riemann sums and take limits. This works because given a continuous function on a product of closed intervals, any sequence of Riemann sums with mesh size tending to zero converges to the integral. (The domain is compact, so the function is uniformly continuous. Hence for any $\epsilon>0$ there is a cutoff below which any mesh size forces the discrepancy between the Riemann sum and the integral to be less than ϵ.)

Alternate derivation (based on a solution by Dan Bernstein): from the discrete analogue, we have

$$\sum_{1\leq i<j\leq n}|f(x_i)+f(x_j)|\geq\frac{n-2}{2}\sum_{i=1}^n|f(x_i)|,$$

for all $x_1,\dots,x_n\in[0,1]$. Integrating both sides as $(x_1,\dots,x_n)$ runs over $[0,1]^n$ yields

$$\begin{aligned}&\frac{n(n-1)}{2}\int_0^1\int_0^1|f(x)+f(y)|\,dy\,dx\\&\geq\frac{n(n-2)}{2}\int_0^1|f(x)|\,dx,\end{aligned}$$

or

$$\int_0^1\int_0^1|f(x)+f(y)|\,dy\,dx\geq\frac{n-2}{n-1}\int_0^1|f(x)|\,dx.$$

Taking the limit as $n\to\infty$ now yields the desired result.

Third solution: (by David Savitt) We give an argument which yields the following improved result. Let μ_p and μ_n be the measure of the sets $\{x\ :\ f(x)>0\}$ and $\{x\ :\ f(x)<0\}$ respectively, and let $\mu\leq 1/2$ be $\min(\mu_p,\mu_n)$. Then

$$\begin{aligned}&\int_0^1\int_0^1|f(x)+f(y)|\,dx\,dy\\&\geq(1+(1-2\mu)^2)\int_0^1|f(x)|\,dx.\end{aligned}$$

Note that the constant can be seen to be best possible by considering a sequence of functions tending towards the step function which is 1 on $[0,\mu]$ and -1 on $(\mu,1]$.

Suppose without loss of generality that $\mu=\mu_p$. As in the second solution, it suffices to prove a strengthened discrete analogue, namely

$$\frac{1}{n^2}\sum_{i,j}|a_i+a_j|\geq\left(1+\left(1-\frac{2p}{n}\right)^2\right)\left(\frac{1}{n}\sum_{i=1}^n|a_i|\right),$$

where $p\leq n/2$ is the number of $a_1,\dots,a_n$ which are positive. (We need only make sure to choose meshes so that $p/n\to\mu$ as $n\to\infty$.) An equivalent inequality is

$$\sum_{1\leq i<j\leq n}|a_i+a_j|\geq\left(n-1-2p+\frac{2p^2}{n}\right)\sum_{i=1}^n|a_i|.$$

Write $r_i=|a_i|$, and assume without loss of generality that $r_i\geq r_{i+1}$ for each i. Then for $i<j$, $|a_i+a_j|=r_i+r_j$ if a_i and a_j have the same sign, and is r_i-r_j if they have opposite signs. The left-hand side is therefore equal to

$$\sum_{i=1}^n(n-i)r_i+\sum_{j=1}^n r_jC_j,$$

where

$$\begin{aligned}C_j&=\#\{i<j\ :\mathrm{sgn}(a_i)=\mathrm{sgn}(a_j)\}\\&\quad-\#\{i<j:\mathrm{sgn}(a_i)\neq\mathrm{sgn}(a_j)\}.\end{aligned}$$

Consider the partial sum $P_k=\sum_{j=1}^k C_j$. If exactly p_k of $a_1,\dots,a_k$ are positive, then this sum is equal to

$$\binom{p_k}{2}+\binom{k-p_k}{2}-\left[\binom{k}{2}-\binom{p_k}{2}-\binom{k-p_k}{2}\right],$$

which expands and simplifies to

$$-2p_k(k-p_k)+\binom{k}{2}.$$

For $k \leq 2p$ even, this partial sum would be minimized with $p_k = \frac{k}{2}$, and would then equal $-\frac{k}{2}$; for $k < 2p$ odd, this partial sum would be minimized with $p_k = \frac{k\pm 1}{2}$, and would then equal $-\frac{k-1}{2}$. Either way, $P_k \geq -\lfloor \frac{k}{2} \rfloor$. On the other hand, if $k > 2p$, then

$$-2p_k(k-p_k)+\binom{k}{2} \geq -2p(k-p)+\binom{k}{2}$$

since p_k is at most p. Define Q_k to be $-\lfloor \frac{k}{2} \rfloor$ if $k \leq 2p$ and $-2p(k-p)+\binom{k}{2}$ if $k \geq 2p$, so that $P_k \geq Q_k$. Note that $Q_1 = 0$.

Partial summation gives

$$\begin{aligned}\sum_{j=1}^{n} r_j C_j &= r_n P_n + \sum_{j=2}^{n}(r_{j-1}-r_j)P_{j-1} \\ &\geq r_n Q_n + \sum_{j=2}^{n}(r_{j-1}-r_j)Q_{j-1} \\ &= \sum_{j=2}^{n} r_j(Q_j - Q_{j-1}) \\ &= -r_2 - r_4 - \cdots - r_{2p} \\ &\quad + \sum_{j=2p+1}^{n}(j-1-2p)r_j.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{1\leq i<j\leq n}|a_i+a_j| &= \sum_{i=1}^{n}(n-i)r_i + \sum_{j=1}^{n} r_j C_j \\ &\geq \sum_{i=1}^{2p}(n-i-[i \text{ even}])r_i \\ &\quad + \sum_{i=2p+1}^{n}(n-1-2p)r_i \\ &= (n-1-2p)\sum_{i=1}^{n} r_i + \\ &\quad \sum_{i=1}^{2p}(2p+1-i-[i \text{ even}])r_i \\ &\geq (n-1-2p)\sum_{i=1}^{n} r_i + p\sum_{i=1}^{2p} r_i \\ &\geq (n-1-2p)\sum_{i=1}^{n} r_i + p\frac{2p}{n}\sum_{i=1}^{n} r_i\,,\end{aligned}$$

as desired. The next-to-last and last inequalities each follow from the monotonicity of the r_i's, the former by pairing the i^{th} term with the $(2p+1-i)^{\text{th}}$.

Note: Compare the closely related Problem 6 from the 2000 USA Mathematical Olympiad: prove that for any nonnegative real numbers $a_1,\ldots,a_n,b_1,\ldots,b_n$, one has

$$\sum_{i,j=1}^{n}\min\{a_ia_j, b_ib_j\} \leq \sum_{i,j=1}^{n}\min\{a_ib_j, a_jb_i\}.$$

Solutions to the 65th William Lowell Putnam Mathematical Competition
Saturday, December 4, 2004

Kiran Kedlaya and Lenny Ng

A1 Yes. Suppose otherwise. Then there would be an N such that $S(N) < 80\%$ and $S(N+1) > 80\%$; that is, O'Keal's free throw percentage is under 80% at some point, and after one subsequent free throw (necessarily made), her percentage is over 80%. If she makes m of her first N free throws, then $m/N < 4/5$ and $(m+1)/(N+1) > 4/5$. This means that $5m < 4n < 5m+1$, which is impossible since then $4n$ is an integer between the consecutive integers $5m$ and $5m+1$.

Remark: This same argument works for any fraction of the form $(n-1)/n$ for some integer $n > 1$, but not for any other real number between 0 and 1.

A2 **First solution:** (partly due to Ravi Vakil) Yes, it does follow. For $i = 1, 2$, let P_i, Q_i, R_i be the vertices of T_i opposide the sides of length a_i, b_i, c_i, respectively.

We first check the case where $a_1 = a_2$ (or $b_1 = b_2$ or $c_1 = c_2$, by the same argument after relabeling). Imagine T_2 as being drawn with the base Q_2R_2 horizontal and the point P_2 above the line Q_2R_2. We may then position T_1 so that $Q_1 = Q_2$, $R_1 = R_2$, and P_1 lies above the line $Q_1R_1 = Q_2R_2$. Then P_1 also lies inside the region bounded by the circles through P_2 centered at Q_2 and R_2. Since $\angle Q_2$ and $\angle R_2$ are acute, the part of this region above the line Q_2R_2 lies within T_2. In particular, the distance from P_1 to the line Q_2R_2 is less than or equal to the distance from P_2 to the line Q_2R_2; hence $A_1 \leq A_2$.

To deduce the general case, put

$$r = \max\{a_1/a_2, b_1/b_2, c_1/c_2\}.$$

Let T_3 be the triangle with sides ra_2, rb_2, rc_2, which has area r^2A_2. Applying the special case to T_1 and T_3, we deduce that $A_1 \leq r^2A_2$; since $r \leq 1$ by hypothesis, we have $A_1 \leq A_2$ as desired.

Remark: Another geometric argument in the case $a_1 = a_2$ is that since angles $\angle Q_2$ and $\angle R_2$ are acute, the perpendicular to Q_2R_2 through P_2 separates Q_2 from R_2. If $A_1 > A_2$, then P_1 lies above the parallel to Q_2R_2 through P_2; if then it lies on or to the left of the vertical line through P_2, we have $c_1 > c_2$ because the inequality holds for both horizontal and vertical components (possibly with equality for one, but not both). Similarly, if P_1 lies to the right of the vertical, then $b_1 > b_2$.

Second solution: (attribution unknown) Retain notation as in the first paragraph of the first solution. Since the angle measures in any triangle add up to π, some angle of T_1 must have measure less than or equal to its counterpart in T_2. Without loss of generality assume that $\angle P_1 \leq \angle P_2$. Since the latter is acute (because T_2 is acute), we have $\sin \angle P_1 \leq \sin \angle P_2$. By the Law of Sines,

$$A_1 = \frac{1}{2} b_1 c_1 \sin \angle P_1 \leq \frac{1}{2} b_2 c_2 \sin \angle P_2 = A_2.$$

Remark: Many other solutions are possible; for instance, one uses Heron's formula for the area of a triangle in terms of its side lengths.

A3 Define a sequence v_n by $v_n = (n-1)(n-3)\cdots(4)(2)$ if n is odd and $v_n = (n-1)(n-3)\cdots(3)(1)$ if n is even; it suffices to prove that $u_n = v_n$ for all $n \geq 2$. Now $v_{n+3}v_n = (n+2)(n)(n-1)!$ and $v_{n+2}v_{n+1} = (n+1)!$, and so $v_{n+3}v_n - v_{n+2}v_{n+1} = n!$. Since we can check that $u_n = v_n$ for $n = 2, 3, 4$, and u_n and v_n satisfy the same recurrence, it follows by induction that $u_n = v_n$ for all $n \geq 2$, as desired.

A4 It suffices to verify that

$$\begin{aligned} &x_1 \cdots x_n \\ &= \frac{1}{2^n n!} \sum_{e_i \in \{-1,1\}} (e_1 \cdots e_n)(e_1x_1 + \cdots + e_nx_n)^n. \end{aligned}$$

To check this, first note that the right side vanishes identically for $x_1 = 0$, because each term cancels the corresponding term with e_1 flipped. Hence the right side, as a polynomial, is divisible by x_1; similarly it is divisible by $x_2, \dots, x_n$. Thus the right side is equal to $x_1 \cdots x_n$ times a scalar. (Another way to see this: the right side is clearly odd as a polynomial in each individual variable, but the only degree n monomial in $x_1, \dots, x_n$ with that property is $x_1 \cdots x_n$.) Since each summand contributes $\frac{1}{2^n} x_1 \cdots x_n$ to the sum, the scalar factor is 1 and we are done.

Remark: Several variants on the above construction are possible; for instance,

$$\begin{aligned} &x_1 \cdots x_n \\ &= \frac{1}{n!} \sum_{e_i \in \{0,1\}} (-1)^{n-e_1-\cdots-e_n} (e_1x_1 + \cdots + e_nx_n)^n \end{aligned}$$

by the same argument as above.

Remark: These construction work over any field of characteristic greater than n (at least for $n > 1$). On the other hand, no construction is possible over a field of characteristic $p \leq n$, since the coefficient of $x_1 \cdots x_n$ in the expansion of $(e_1x_1 + \cdots + e_nx_n)^n$ is zero for any e_i.

Remark: Richard Stanley asks whether one can use fewer than 2^n terms, and what the smallest possible number is.

A5 **First solution:** First recall that any graph with n vertices and e edges has at least $n - e$ connected components (add each edge one at a time, and note that it reduces the number of components by at most 1). Now imagine the squares of the checkerboard as a graph, whose vertices are connected if the corresponding squares share a side and are the same color. Let A be the number of edges in the graph, and let B be the number of 4-cycles (formed by monochromatic 2×2 squares). If we remove the bottom edge of each 4-cycle, the resulting graph has the same number of connected components as the original one; hence this number is at least

$$mn - A + B.$$

By the linearity of expectation, the expected number of connected components is at least

$$mn - E(A) + E(B).$$

Moreover, we may compute $E(A)$ by summing over the individual pairs of adjacent squares, and we may compute $E(B)$ by summing over the individual 2×2 squares. Thus

$$E(A) = \frac{1}{2}(m(n-1) + (m-1)n),$$
$$E(B) = \frac{1}{8}(m-1)(n-1),$$

and so the expected number of components is at least

$$mn - \frac{1}{2}(m(n-1) + (m-1)n) + \frac{1}{8}(m-1)(n-1)$$
$$= \frac{mn + 3m + 3n + 1}{8} > \frac{mn}{8}.$$

Remark: A "dual" approach is to consider the graph whose vertices are the corners of the squares of the checkerboard, with two vertices joined if they are adjacent and the edge between then does not separate two squares of the same color. In this approach, the 4-cycles become isolated vertices, and the bound on components is replaced by a call to Euler's formula relating the vertices, edges and faces of a planar figure. (One must be careful, however, to correctly handle faces which are not simply connected.)

Second solution: (by Noam Elkies) Number the squares of the checkerboard $1, \dots, mn$ by numbering the first row from left to right, then the second row, and so on. We prove by induction on i that if we just consider the figure formed by the first i squares, its expected number of monochromatic components is at least $i/8$. For $i = 1$, this is clear.

Suppose the i-th square does not abut the left edge or the top row of the board. Then we may divide into three cases.

– With probability $1/4$, the i-th square is opposite in color from the adjacent squares directly above and to the left of it. In this case adding the i-th square adds one component.

– With probability $1/8$, the i-th square is the same in color as the adjacent squares directly above and to the left of it, but opposite in color from its diagonal neighbor above and to the left. In this case, adding the i-th square either removes a component or leaves the number unchanged.

– In all other cases, the number of components remains unchanged upon adding the i-th square.

Hence adding the i-th square increases the expected number of components by $1/4 - 1/8 = 1/8$.

If the i-th square does abut the left edge of the board, the situation is even simpler: if the i-th square differs in color from the square above it, one component is added, otherwise the number does not change. Hence adding the i-th square increases the expected number of components by $1/2$; likewise if the i-th square abuts the top edge of the board. Thus the expected number of components is at least $i/8$ by induction, as desired.

Remark: Some solvers attempted to consider adding one row at a time, rather than one square; this must be handled with great care, as it is possible that the number of components can drop rather precipitously upon adding an entire row.

A6 By approximating each integral with a Riemann sum, we may reduce to proving the discrete analogue: for $x_{ij} \in \mathbb{R}$ for $i, j = 1, \dots, n$,

$$n \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x_{ij} \right)^2 + n \sum_{j=1}^{n} \left(\sum_{i=1}^{n} x_{ij} \right)^2$$
$$\leq \left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \right)^2 + n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^2.$$

The difference between the right side and the left side is

$$\frac{1}{4} \sum_{i,j,k,l=1}^{n} (x_{ij} + x_{kl} - x_{il} - x_{kj})^2,$$

which is evidently nonnegative. If you prefer not to discretize, you may rewrite the original inequality as

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 F(x,y,z,w)^2 \, dx \, dy \, dz \, dw \geq 0$$

for

$$F(x,y,z,w) = f(x,y) + f(z,w) - f(x,w) - f(z,y).$$

Remark: (by Po-Ning Chen) The discrete inequality can be arrived at more systematically by repeatedly applying the following identity: for any real $a_1, \dots, a_n$,

$$\sum_{1 \leq i < j \leq n} (x_i - x_j)^2 = n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i \right)^2.$$

Remark: (by David Savitt) The discrete inequality can also be interpreted as follows. For $c, d \in \{1, \dots, n-1\}$ and $\zeta_n = e^{2\pi i/n}$, put

$$z_{c,d} = \sum_{i,j} \zeta_n^{ci+dj} x_{ij}.$$

Then the given inequality is equivalent to

$$\sum_{c,d=1}^{n-1} |z_{c,d}|^2 \geq 0.$$

B1 Let k be an integer, $0 \leq k \leq n-1$. Since $P(r)/r^k = 0$, we have

$$\begin{aligned} &c_n r^{n-k} + c_{n-1} r^{n-k+1} + \cdots + c_{k+1} r \\ &\quad = -(c_k + c_{k-1} r^{-1} + \cdots + c_0 r^{-k}). \end{aligned}$$

Write $r = p/q$ where p and q are relatively prime. Then the left hand side of the above equation can be written as a fraction with denominator q^{n-k}, while the right hand side is a fraction with denominator p^k. Since p and q are relatively prime, both sides of the equation must be an integer, and the result follows.

Remark: If we write $r = a/b$ in lowest terms, then $P(x)$ factors as $(bx - a)Q(x)$, where the polynomial Q has integer coefficients because you can either do the long division from the left and get denominators divisible only by primes dividing b, or do it from the right and get denominators divisible only by primes dividing a. The numbers given in the problem are none other than a times the coefficients of Q. More generally, if $P(x)$ is divisible, as a polynomial over the rationals, by a polynomial $R(x)$ with integer coefficients, then P/R also has integer coefficients; this is known as "Gauss's lemma" and holds in any unique factorization domain.

B2 **First solution:** We have

$$(m+n)^{m+n} > \binom{m+n}{m} m^m n^n$$

because the binomial expansion of $(m+n)^{m+n}$ includes the term on the right as well as some others. Rearranging this inequality yields the claim.

Remark: One can also interpret this argument combinatorially. Suppose that we choose $m+n$ times (with replacement) uniformly randomly from a set of $m+n$ balls, of which m are red and n are blue. Then the probability of picking each ball exactly once is $(m+n)!/(m+n)^{m+n}$. On the other hand, if p is the probability of picking exactly m red balls, then $p < 1$ and the probability of picking each ball exactly once is $p(m^m/m!)(n^n/n!)$.

Second solution: (by David Savitt) Define

$$S_k = \{i/k : i = 1, \dots, k\}$$

and rewrite the desired inequality as

$$\prod_{x \in S_m} x \prod_{y \in S_n} y > \prod_{z \in S_{m+n}} z.$$

To prove this, it suffices to check that if we sort the multiplicands on both sides into increasing order, the i-th term on the left side is greater than or equal to the i-th term on the right side. (The equality is strict already for $i = 1$, so you do get a strict inequality above.)

Another way to say this is that for any i, the number of factors on the left side which are less than $i/(m+n)$ is less than i. But since $j/m < i/(m+n)$ is equivalent to $j < im/(m+n)$, that number is

$$\begin{aligned} &\left\lceil \frac{im}{m+n} \right\rceil - 1 + \left\lceil \frac{in}{m+n} \right\rceil - 1 \\ &\leq \frac{im}{m+n} + \frac{in}{m+n} - 1 = i - 1. \end{aligned}$$

Third solution: Put $f(x) = x(\log(x+1) - \log x)$; then for $x > 0$,

$$\begin{aligned} f'(x) &= \log(1 + 1/x) - \frac{1}{x+1} \\ f''(x) &= -\frac{1}{x(x+1)^2}. \end{aligned}$$

Hence $f''(x) < 0$ for all x; since $f'(x) \to 0$ as $x \to \infty$, we have $f'(x) > 0$ for $x > 0$, so f is strictly increasing.

Put $g(m) = m \log m - \log(m!)$; then $g(m+1) - g(m) = f(m)$, so $g(m+1) - g(m)$ increases with m. By induction, $g(m+n) - g(m)$ increases with n for any positive integer n, so in particular

$$\begin{aligned} g(m+n) - g(m) &> g(n) - g(1) + f(m) \\ &\geq g(n) \end{aligned}$$

since $g(1) = 0$. Exponentiating yields the desired inequality.

B3 The answer is $\{a \,|\, a > 2\}$. If $a > 2$, then the function $f(x) = 2a/(a-2)$ has the desired property; both perimeter and area of R in this case are $2a^2/(a-2)$. Now suppose that $a \leq 2$, and let $f(x)$ be a nonnegative continuous function on $[0, a]$. Let $P = (x_0, y_0)$ be a point on the graph of $f(x)$ with maximal y-coordinate; then the area of R is at most ay_0 since it lies below the line $y = y_0$. On the other hand, the points $(0,0)$, $(a,0)$, and P divide the boundary of R into three sections. The length of the section between $(0,0)$ and P is at least the distance between $(0,0)$ and P, which is at least y_0; the length of the section between P and $(a,0)$ is similarly at least y_0; and the length of the section between $(0,0)$ and $(a,0)$ is a. Since $a \leq 2$, we have $2y_0 + a > ay_0$ and hence the perimeter of R is strictly greater than the area of R.

B4 **First solution:** Identify the xy-plane with the complex plane $\mathbb{C}$, so that P_k is the real number k. If z is sent to z' by a counterclockwise rotation by θ about P_k, then $z' - k = e^{i\theta}(z-k)$; hence the rotation R_k sends z to $\zeta z + k(1-\zeta)$, where $\zeta = e^{2\pi i/n}$. It follows that R_1 followed by R_2 sends z to $\zeta(\zeta z + (1-\zeta)) + 2(1-\zeta) = \zeta^2 z + (1-\zeta)(\zeta+2)$, and so forth; an easy induction shows that R sends z to

$$\zeta^n z + (1-\zeta)(\zeta^{n-1} + 2\zeta^{n-2} + \cdots + (n-1)\zeta + n).$$

Expanding the product $(1-\zeta)(\zeta^{n-1} + 2\zeta^{n-2} + \cdots + (n-1)\zeta + n)$ yields $-\zeta^n - \zeta^{n-1} - \cdots - \zeta + n = n$. Thus R sends z to $z+n$; in cartesian coordinates, $R(x,y) = (x+n, y)$.

Second solution: (by Andy Lutomirski, via Ravi Vakil) Imagine a regular n-gon of side length 1 placed with its top edge on the x-axis and the left endpoint of that edge at the origin. Then the rotations correspond to rolling this n-gon along the x-axis; after the n rotations, it clearly ends up in its original rotation and translated n units to the right. Hence the whole plane must do so as well.

Third solution: (attribution unknown) Viewing each R_k as a function of a complex number z as in the first solution, the function $R_n \circ R_{n-1} \circ \cdots \circ R_1(z)$ is linear in z with slope $\zeta^n = 1$. It thus equals $z + T$ for some $T \in \mathbb{C}$. Since $f_1(1) = 1$, we can write $1 + T = R_n \circ \cdots \circ R_2(1)$. However, we also have

$$R_n \circ \cdots \circ R_2(1) = R_{n-1} \circ R_1(0) + 1$$

by the symmetry in how the R_i are defined. Hence

$$R_n(1+T) = R_n \circ R_1(0) + R_n(1) = T + R_n(1);$$

that is, $R_n(T) = T$. Hence $T = n$, as desired.

B5 **First solution:** By taking logarithms, we see that the desired limit is $\exp(L)$, where $L = \lim_{x\to 1^-} \sum_{n=0}^\infty x^n \left(\ln(1+x^{n+1}) - \ln(1+x^n)\right)$. Now

$$\begin{aligned}
&\sum_{n=0}^N x^n \left(\ln(1+x^{n+1}) - \ln(1+x^n)\right) \\
&= 1/x \sum_{n=0}^N x^{n+1}\ln(1+x^{n+1}) - \sum_{n=0}^N x^n \ln(1+x^n) \\
&= x^N \ln(1+x^{N+1}) - \ln 2 + (1/x - 1)\sum_{n=1}^N x^n \ln(1+x^n);
\end{aligned}$$

since $\lim_{N\to\infty}(x^N \ln(1+x^{N+1})) = 0$ for $0 < x < 1$, we conclude that $L = -\ln 2 + \lim_{x\to 1^-} f(x)$, where

$$\begin{aligned}
f(x) &= (1/x - 1)\sum_{n=1}^\infty x^n \ln(1+x^n) \\
&= (1/x-1)\sum_{n=1}^\infty \sum_{m=1}^\infty (-1)^{m+1} x^{n+mn}/m.
\end{aligned}$$

This final double sum converges absolutely when $0 < x < 1$, since

$$\begin{aligned}
\sum_{n=1}^\infty \sum_{m=1}^\infty x^{n+mn}/m &= \sum_{n=1}^\infty x^n(-\ln(1-x^n)) \\
&< \sum_{n=1}^\infty x^n(-\ln(1-x)),
\end{aligned}$$

which converges. (Note that $-\ln(1-x)$ and $-\ln(1-x^n)$ are positive.) Hence we may interchange the summations in $f(x)$ to obtain

$$\begin{aligned}
f(x) &= (1/x-1)\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{(-1)^{m+1} x^{(m+1)n}}{m} \\
&= (1/x-1)\sum_{m=1}^\infty \frac{(-1)^{m+1}}{m}\left(\frac{x^m(1-x)}{1-x^{m+1}}\right).
\end{aligned}$$

This last sum converges absolutely uniformly in x, so it is legitimate to take limits term by term. Since $\lim_{x\to 1^-} \frac{x^m 1-x}{1-x^{m+1}} = \frac{1}{m+1}$ for fixed m, we have

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \sum_{m=1}^\infty \frac{(-1)^{m+1}}{m(m+1)} \\
&= \sum_{m=1}^\infty (-1)^{m+1}\left(\frac{1}{m} - \frac{1}{m+1}\right) \\
&= 2\left(\sum_{m=1}^\infty \frac{(-1)^{m+1}}{m}\right) - 1 \\
&= 2\ln 2 - 1,
\end{aligned}$$

and hence $L = \ln 2 - 1$ and the desired limit is $2/e$.

Remark: Note that the last series is not absolutely convergent, so the recombination must be done without rearranging terms.

Second solution: (by Greg Price, via Tony Zhang and Anders Kaseorg) Put $t_n(x) = \ln(1+x^n)$; we can then write $x^n = \exp(t_n(x)) - 1$, and

$$L = \lim_{x\to 1^-} \sum_{n=0}^\infty (t_n(x) - t_{n+1}(x))(1 - \exp(t_n(x))).$$

The expression on the right is a Riemann sum approximating the integral $\int_0^{\ln 2}(1-e^t)\,dt$, over the subdivision of $[0, \ln(2))$ given by the $t_n(x)$. As $x \to 1^-$, the maximum difference between consecutive $t_n(x)$ tends to 0, so the Riemann sum tends to the value of the integral. Hence $L = \int_0^{\ln 2}(1-e^t)\,dt = \ln 2 - 1$, as desired.

B6 **First solution:** (based on a solution of Dan Bernstein) Note that for any b, the condition that $b \notin \mathcal{B}$ already forces $\limsup N(x)/x$ to be at most 1/2: pair off $2mb+n$ with $(2m+1)b+n$ for $n = 1, \dots, b$, and note that at most one member of each pair may belong to $\mathcal{A}$. The idea of the proof is to do something similar with pairs replaced by larger clumps, using long runs of excluded elements of B.

Suppose we have positive integers $b_0 = 1, b_1, \dots, b_n$ with the following properties:

(a) For $i = 1, \dots, n$, $c_i = b_i/(2b_{i-1})$ is an integer.

(b) For $e_i \in \{-1, 0, 1\}$, $|e_1 b_1 + \cdots + e_n b_n| \notin \mathcal{B}$.

Each nonnegative integer a has a unique "base expansion"

$$a = a_0 b_0 + \cdots + a_{n-1} b_{n-1} + m b_n \qquad (0 \leq a_i < 2c_i);$$

if two integers have expansions with the same value of m, and values of a_i differing by at most 1 for $i = 0, \dots, n-1$, then their difference is not in $\mathcal{B}$, so at most one of them lies in $\mathcal{A}$. In particular, for any $d_i \in \{0, \dots, c_i - 1\}$, any $m_0 \in \{0, 2c_0 - 1\}$ and any m_n, the set

$$\{m_0 b_0 + (2d_1 + e_1) b_0 + \cdots \\ + (2d_{n-1} + e_{n-1}) b_{n-1} + (2m_n + e_n) b_n\},$$

where each e_i runs over $\{0, 1\}$, contains at most one element of $\mathcal{A}$; consequently, $\limsup N(x)/x \leq 1/2^n$.

We now produce such b_i recursively, starting with $b_0 = 1$ (and both (a) and (b) holding vacuously). Given $b_0, \dots, b_n$ satisfying (a) and (b), note that $b_0 + \cdots + b_{n-1} < b_n$ by induction on n. By the hypotheses of the problem, we can find a set S_n of $6b_n$ consecutive integers, none of which belongs to $\mathcal{B}$. Let b_{n+1} be the second-smallest multiple of $2b_n$ in S_n; then $b_{n+1} + x \in S_n$ for $-2b_n \leq x \leq 0$ clearly, and also for $0 \leq x \leq 2b_n$ because there are most $4b_n - 1$ elements of S_n preceding b_{n+1}. In particular, the analogue of (b) with n replaced by $n+1$ holds for $e_{n+1} \neq 0$; of course it holds for $e_{n+1} = 0$ because (b) was already known. Since the analogue of (a) holds by construction, we have completed this step of the construction and the recursion may continue.

Since we can construct $b_0, \dots, b_n$ satisfying (a) and (b) for any n, we have $\limsup N(x)/x \leq 1/2^n$ for any n, yielding $\lim N(x)/x = 0$ as desired.

Second solution: (by Paul Pollack) Let S be the set of possible values of $\limsup N(x)/x$; since $S \subseteq [0,1]$ is bounded, it has a least upper bound L. Suppose by way of contradiction that $L > 0$; we can then choose $\mathcal{A}, \mathcal{B}$ satisfying the conditions of the problem such that $\limsup N(x)/x > 3L/4$.

To begin with, we can certainly find some positive integer $m \notin \mathcal{B}$, so that $\mathcal{A}$ is disjoint from $\mathcal{A}+m = \{a+m : a \in \mathcal{A}\}$. Put $\mathcal{A}' = \mathcal{A} \cup (\mathcal{A}+m)$ and let $N'(x)$ be the size of $\mathcal{A}' \cap \{1, \dots, x\}$; then $\limsup N'(x)/x = 3L/2 > L$, so $\mathcal{A}'$ cannot obey the conditions of the problem statement. That is, if we let $\mathcal{B}'$ be the set of positive integers that occur as differences between elements of $\mathcal{A}'$, then there exists an integer n such that among any n consecutive integers, at least one lies in $\mathcal{B}'$. But

$$\mathcal{B}' \subseteq \{b + em : b \in \mathcal{B}, e \in \{-1, 0, 1\}\},$$

so among any $n + 2m$ consecutive integers, at least one lies in $\mathcal{B}$. This contradicts the condition of the problem statement.

We conclude that it is impossible to have $L > 0$, so $L = 0$ and $\lim N(x)/x = 0$ as desired.

Remark: A hybrid between these two arguments is to note that if we can produce $c_1, \dots, c_n$ such that $|c_i - c_j| \notin \mathcal{B}$ for $i, j = 1, \dots, n$, then the translates $\mathcal{A} + c_1, \dots, \mathcal{A} + c_n$ are disjoint and so $\limsup N(x)/x \leq 1/n$. Given $c_1 \leq \cdots \leq c_n$ as above, we can then choose c_{n+1} to be the largest element of a run of $c_n + 1$ consecutive integers, none of which lie in $\mathcal{B}$.

Solutions to the 66th William Lowell Putnam Mathematical Competition
Saturday, December 3, 2005

Manjul Bhargava, Kiran Kedlaya, and Lenny Ng

A1 We proceed by induction, with base case $1 = 2^0 3^0$. Suppose all integers less than $n-1$ can be represented. If n is even, then we can take a representation of $n/2$ and multiply each term by 2 to obtain a representation of n. If n is odd, put $m = \lfloor \log_3 n \rfloor$, so that $3^m \leq n < 3^{m+1}$. If $3^m = n$, we are done. Otherwise, choose a representation $(n-3^m)/2 = s_1 + \cdots + s_k$ in the desired form. Then

$$n = 3^m + 2s_1 + \cdots + 2s_k,$$

and clearly none of the $2s_i$ divide each other or 3^m. Moreover, since $2s_i \leq n - 3^m < 3^{m+1} - 3^m$, we have $s_i < 3^m$, so 3^m cannot divide $2s_i$ either. Thus n has a representation of the desired form in all cases, completing the induction.

Remarks: This problem is originally due to Paul Erdős. Note that the representations need not be unique: for instance,

$$11 = 2 + 9 = 3 + 8.$$

A2 We will assume $n \geq 2$ hereafter, since the answer is 0 for $n = 1$.

First solution: We show that the set of rook tours from $(1,1)$ to $(n,1)$ is in bijection with the set of subsets of $\{1, 2, ..., n\}$ that include n and contain an even number of elements in total. Since the latter set evidently contains 2^{n-2} elements, so does the former.

We now construct the bijection. Given a rook tour P from $(1,1)$ to $(n,1)$, let $S = S(P)$ denote the set of all $i \in \{1, 2, \dots, n\}$ for which there is either a directed edge from $(i,1)$ to $(i,2)$ or from $(i,3)$ to $(i,2)$. It is clear that this set S includes n and must contain an even number of elements. Conversely, given a subset $S = \{a_1, a_2, \dots, a_{2r} = n\} \subset \{1, 2, \dots, n\}$ of this type with $a_1 < a_2 < \cdots < a_{2r}$, we notice that there is a unique path P containing $(a_i, 2 + (-1)^i), (a_1, 2)$ for $i = 1, 2, \dots, 2r$. This establishes the desired bijection.

Second solution: Let A_n denote the set of rook tours beginning at $(1,1)$ and ending at $(n,1)$, and let B_n denote the set of rook tours beginning at $(1,1)$ and ending at $(n,3)$.

For $n \geq 2$, we construct a bijection between A_n and $A_{n-1} \cup B_{n-1}$. Any path P in A_n contains either the line segment P_1 between $(n-1,1)$ and $(n,1)$, or the line segment P_2 between $(n,2)$ and $(n,1)$. In the former case, P must also contain the subpath P_1' which joins $(n-1,3)$, $(n,3)$, $(n,2)$, and $(n-1,2)$ consecutively; then deleting P_1 and P_1' from P and adding the line segment joining $(n-1,3)$ to $(n-1,2)$ results in a path in A_{n-1}. (This construction is reversible, lengthening any path in A_{n-1} to a path in A_n.) In the latter case, P contains the subpath P_2' which joins $(n-1,3)$, $(n,3)$, $(n,2)$, $(n,1)$ consecutively; deleting P_2' results in a path in B_{n-1}, and this construction is also reversible. The desired bijection follows.

Similarly, there is a bijection between B_n and $A_{n-1} \cup B_{n-1}$ for $n \geq 2$. It follows by induction that for $n \geq 2$, $|A_n| = |B_n| = 2^{n-2}(|A_1| + |B_1|)$. But $|A_1| = 0$ and $|B_1| = 1$, and hence the desired answer is $|A_n| = 2^{n-2}$.

Remarks: Other bijective arguments are possible: for instance, Noam Elkies points out that each element of $A_n \cup B_n$ contains a different one of the possible sets of segments of the form $(i,2), (i+1,2)$ for $i = 1, \dots, n-1$. Richard Stanley provides the reference: K.L. Collins and L.B. Krompart, The number of Hamiltonian paths in a rectangular grid, *Discrete Math.* **169** (1997), 29–38. This problem is Theorem 1 of that paper; the cases of $4 \times n$ and $5 \times n$ grids are also treated. The paper can also be found online at the URL `kcollins.web.wesleyan.edu/vita.htm`.

A3 Note that it is implicit in the problem that p is nonconstant, one may take any branch of the square root, and that $z = 0$ should be ignored.

First solution: Write $p(z) = c\prod_{j=1}^n (z - r_j)$, so that

$$\frac{g'(z)}{g(z)} = \frac{1}{2z}\sum_{j=1}^n \frac{z + r_j}{z - r_j}.$$

Now if $z \neq r_j$ for all j,then

$$\frac{z + r_j}{z - r_j} = \frac{(z + r_j)(\overline{z} - \overline{r_j})}{|z - r_j|^2} = \frac{|z|^2 - 1 + 2\mathrm{Im}(\overline{z} r_j)}{|z - r_j|^2},$$

and so

$$\mathrm{Re}\,\frac{z g'(z)}{g(z)} = \frac{|z|^2 - 1}{2}\left(\sum_j \frac{1}{|z - r_j|^2}\right).$$

Since the quantity in parentheses is positive, $g'(z)/g(z)$ can be 0 only if $|z| = 1$. If on the other hand $z = r_j$ for some j, then $|z| = 1$ anyway.

Second solution: Write $p(z) = c\prod_{j=1}^n (z - r_j)$, so that

$$\frac{g'(z)}{g(z)} = \sum_{j=1}^n \left(\frac{1}{z - r_j} - \frac{1}{2z}\right).$$

We first check that $g'(z) \neq 0$ whenever z is real and $z > 1$. In this case, for $r_j = e^{i\theta_j}$, we have $z - r_j = (z - \cos(\theta_j)) + \sin(\theta_j)i$, so the real part of $\frac{1}{z-r_j} - \frac{1}{2z}$ is

$$\begin{aligned} &\frac{z - \cos(\theta_j)}{z^2 - 2z\cos(\theta_j) + 1} - \frac{1}{2z} \\ &= \frac{z^2 - 1}{2z(z^2 - 2z\cos(\theta_j) + 1)} > 0. \end{aligned}$$

Hence $g'(z)/g(z)$ has positive real part, so $g'(z)/g(z)$ and hence $g(z)$ are nonzero.

Applying the same argument after replacing $p(z)$ by $p(e^{i\theta}z)$, we deduce that g' cannot have any roots outside the unit circle. Applying the same argument after replacing $p(z)$ by $z^n p(1/z)$, we also deduce that g' cannot have any roots inside the unit circle. Hence all roots of g' have absolute value 1, as desired.

Third solution: Write $p(z) = c\prod_{j=1}^n (z - r_j)$ and put $r_j = e^{2i\theta_j}$. Note that $g(e^{2i\theta})$ is equal to a nonzero constant times

$$\begin{aligned} h(\theta) &= \prod_{j=1}^n \frac{e^{i(\theta+\theta_j)} - e^{-i(\theta+\theta_j)}}{2i} \\ &= \prod_{j=1}^n \sin(\theta + \theta_j). \end{aligned}$$

Since h has at least $2n$ roots (counting multiplicity) in the interval $[0, 2\pi)$, h' does also by repeated application of Rolle's theorem. Since $g'(e^{2i\theta}) = 2ie^{2i\theta}h'(\theta)$, $g'(z^2)$ has at least $2n$ roots on the unit circle. Since $g'(z^2)$ is equal to z^{-n-1} times a polynomial of degree $2n$, $g'(z^2)$ has all roots on the unit circle, as then does $g'(z)$.

Remarks: The second solution imitates the proof of the Gauss-Lucas theorem: the roots of the derivative of a complex polynomial lie in the convex hull of the roots of the original polynomial. The second solution is close to problem B3 from the 2000 Putnam. A hybrid between the first and third solutions is to check that on the unit circle, $\mathrm{Re}(zg'(z)/g(z)) = 0$ while between any two roots of p, $\mathrm{Im}(zg'(z)/g(z))$ runs from $+\infty$ to $-\infty$ and so must have a zero crossing. (This only works when p has distinct roots, but the general case follows by the continuity of the roots of a polynomial as functions of the coefficients.) One can also construct a solution using Rouché's theorem.

A4 **First solution:** Choose a set of a rows $r_1, \dots, r_a$ containing an $a \times b$ submatrix whose entries are all 1. Then for $i, j \in \{1, \dots, a\}$, we have $r_i \cdot r_j = n$ if $i = j$ and 0 otherwise. Hence

$$\sum_{i,j=1}^a r_i \cdot r_j = an.$$

On the other hand, the term on the left is the dot product of $r_1 + \cdots + r_a$ with itself, i.e., its squared length. Since this vector has a in each of its first b coordinates, the dot product is at least $a^2 b$. Hence $an \geq a^2 b$, whence $n \geq ab$ as desired.

Second solution: (by Richard Stanley) Suppose without loss of generality that the $a \times b$ submatrix occupies the first a rows and the first b columns. Let M be the submatrix occupying the first a rows and the last $n - b$ columns. Then the hypothesis implies that the matrix MM^T has $n-b$'s on the main diagonal and $-b$'s elsewhere. Hence the column vector v of length a consisting of all 1's satisfies $MM^T v = (n-ab)v$, so $n - ab$ is an eigenvalue of MM^T. But MM^T is semidefinite, so its eigenvalues are all nonnegative real numbers. Hence $n - ab \geq 0$.

Remarks: A matrix as in the problem is called a *Hadamard matrix*, because it meets the equality condition of Hadamard's inequality: any $n \times n$ matrix with ± 1 entries has absolute determinant at most $n^{n/2}$, with equality if and only if the rows are mutually orthogonal (from the interpretation of the determinant as the volume of a paralellepiped whose edges are parallel to the row vectors). Note that this implies that the columns are also mutually orthogonal. A generalization of this problem, with a similar proof, is known as *Lindsey's lemma*: the sum of the entries in any $a \times b$ submatrix of a Hadamard matrix is at most $\sqrt{abn}$. Stanley notes that Ryser (1981) asked for the smallest size of a Hadamard matrix containing an $r \times s$ submatrix of all 1's, and refers to the URL `www3.interscience.wiley.com/cgi-bin/abstract/110550861/ABSTRACT` for more information.

A5 **First solution:** We make the substitution $x = \tan\theta$, rewriting the desired integral as

$$\int_0^{\pi/4} \log(\tan(\theta) + 1)\, d\theta.$$

Write

$$\begin{aligned} &\log(\tan(\theta) + 1) \\ &\quad = \log(\sin(\theta) + \cos(\theta)) - \log(\cos(\theta)) \end{aligned}$$

and then note that $\sin(\theta) + \cos(\theta) = \sqrt{2}\cos(\pi/4 - \theta)$. We may thus rewrite the integrand as

$$\frac{1}{2}\log(2) + \log(\cos(\pi/4 - \theta)) - \log(\cos(\theta)).$$

But over the interval $[0, \pi/4]$, the integrals of $\log(\cos(\theta))$ and $\log(\cos(\pi/4 - \theta))$ are equal, so their contributions cancel out. The desired integral is then just the integral of $\frac{1}{2}\log(2)$ over the interval $[0, \pi/4]$, which is $\pi\log(2)/8$.

Second solution: (by Roger Nelsen) Let I denote the desired integral. We make the substitution $x = (1 -$

$u)/(1+u)$ to obtain

$$\begin{aligned} I &= \int_0^1 \frac{(1+u)^2 \log(2/(1+u))}{2(1+u^2)} \frac{2\,du}{(1+u)^2} \\ &= \int_0^1 \frac{\log(2) - \log(1+u)}{1+u^2}\,du \\ &= \log(2) \int_0^1 \frac{du}{1+u^2} - I, \end{aligned}$$

yielding

$$I = \frac{1}{2}\log(2) \int_0^1 \frac{du}{1+u^2} = \frac{\pi \log(2)}{8}.$$

Third solution: (attributed to Steven Sivek) Define the function

$$f(t) = \int_0^1 \frac{\log(xt+1)}{x^2+1}\,dx$$

so that $f(0) = 0$ and the desired integral is $f(1)$. Then by differentiation under the integral,

$$f'(t) = \int_0^1 \frac{x}{(xt+1)(x^2+1)}\,dx.$$

By partial fractions, we obtain

$$\begin{aligned} f'(t) &= \left.\frac{2t\arctan(x) - 2\log(tx+1) + \log(x^2+1)}{2(t^2+1)}\right|_{x=0}^{x=1} \\ &= \frac{\pi t + 2\log(2) - 4\log(t+1)}{4(t^2+1)}, \end{aligned}$$

whence

$$f(t) = \frac{\log(2)\arctan(t)}{2} + \frac{\pi\log(t^2+1)}{8} - \int_0^t \frac{\log(t+1)}{t^2+1}\,dt$$

and hence

$$f(1) = \frac{\pi\log(2)}{4} - \int_0^1 \frac{\log(t+1)}{t^2+1}\,dt.$$

But the integral on the right is again the desired integral $f(1)$, so we may move it to the left to obtain

$$2f(1) = \frac{\pi\log(2)}{4}$$

and hence $f(1) = \pi\log(2)/8$ as desired.

Fourth solution: (by David Rusin) We have

$$\int_0^1 \frac{\log(x+1)}{x^2+1}\,dx = \int_0^1 \left(\sum_{n=1}^\infty \frac{(-1)^{n-1}x^n}{n(x^2+1)} \right) dx.$$

We next justify moving the sum through the integral sign. Note that

$$\sum_{n=1}^\infty \int_0^1 \frac{(-1)^{n-1}x^n\,dx}{n(x^2+1)}$$

is an alternating series whose terms strictly decrease to zero, so it converges. Moreover, its partial sums alternately bound the previous integral above and below, so the sum of the series coincides with the integral.

Put

$$J_n = \int_0^1 \frac{x^n\,dx}{x^2+1};$$

then $J_0 = \arctan(1) = \frac{\pi}{4}$ and $J_1 = \frac{1}{2}\log(2)$. Moreover,

$$J_n + J_{n+2} = \int_0^1 x^n\,dx = \frac{1}{n+1}.$$

Write

$$\begin{aligned} A_m &= \sum_{i=1}^m \frac{(-1)^{i-1}}{2i-1} \\ B_m &= \sum_{i=1}^m \frac{(-1)^{i-1}}{2i}; \end{aligned}$$

then

$$\begin{aligned} J_{2n} &= (-1)^n(J_0 - A_n) \\ J_{2n+1} &= (-1)^n(J_1 - B_n). \end{aligned}$$

Now the $2N$-th partial sum of our series equals

$$\begin{aligned} &\sum_{n=1}^N \frac{J_{2n-1}}{2n-1} - \frac{J_{2n}}{2n} \\ &= \sum_{n=1}^N \frac{(-1)^{n-1}}{2n-1}(J_1 - B_{n-1}) - \frac{(-1)^n}{2n}(J_0 - A_n) \\ &= A_N(J_1 - B_{N-1}) + B_N(J_0 - A_N) + A_N B_N. \end{aligned}$$

As $N \to \infty$, $A_N \to J_0$ and $B_N \to J_1$, so the sum tends to $J_0 J_1 = \pi\log(2)/8$.

Remarks: The first two solutions are related by the fact that if $x = \tan(\theta)$, then $1 - x/(1+x) = \tan(\pi/4 - \theta)$. The strategy of the third solution (introducing a parameter then differentiating it) was a favorite of physics Nobelist (and Putnam Fellow) Richard Feynman. Noam Elkies notes that this integral is number 2.491#8 in Gradshteyn and Ryzhik, *Table of integrals, series, and products*. The *Mathematica* computer algebra system (version 5.2) successfully computes this integral, but we do not know how.

A6 **First solution:** The angle at a vertex P is acute if and only if all of the other points lie on an open semicircle. We first deduce from this that if there are any two acute angles at all, they must occur consecutively. Suppose the contrary; label the vertices $Q_1, \dots, Q_n$ in counterclockwise order (starting anywhere), and suppose that the angles at Q_1 and Q_i are acute for some i with $3 \leq i \leq n-1$. Then the open semicircle starting at Q_2 and proceeding counterclockwise must contain all of $Q_3, \dots, Q_n$, while the open semicircle starting at Q_i and proceeding counterclockwise must contain $Q_{i+1}, \dots, Q_n, Q_1, \dots, Q_{i-1}$. Thus two open semicircles cover the entire circle, contradiction.

It follows that if the polygon has at least one acute angle, then it has either one acute angle or two acute angles occurring consecutively. In particular, there is a unique pair of consecutive vertices Q_1, Q_2 in counterclockwise order for which $\angle Q_2$ is acute and $\angle Q_1$ is not acute. Then the remaining points all lie in the arc from the antipode of Q_1 to Q_1, but Q_2 cannot lie in the arc, and the remaining points cannot all lie in the arc from the antipode of Q_1 to the antipode of Q_2. Given the choice of Q_1, Q_2, let x be the measure of the counterclockwise arc from Q_1 to Q_2; then the probability that the other points fall into position is $2^{-n+2} - x^{n-2}$ if $x \leq 1/2$ and 0 otherwise.

Hence the probability that the polygon has at least one acute angle with a *given* choice of which two points will act as Q_1 and Q_2 is

$$\int_0^{1/2} (2^{-n+2} - x^{n-2})\,dx = \frac{n-2}{n-1} 2^{-n+1}.$$

Since there are $n(n-1)$ choices for which two points act as Q_1 and Q_2, the probability of at least one acute angle is $n(n-2)2^{-n+1}$.

Second solution: (by Calvin Lin) As in the first solution, we may compute the probability that for a particular one of the points Q_1, the angle at Q_1 is not acute but the following angle is, and then multiply by n. Imagine picking the points by first choosing Q_1, then picking $n-1$ *pairs* of antipodal points and then picking one member of each pair. Let $R_2, \dots, R_n$ be the points of the pairs which lie in the semicircle, taken in order away from Q_1, and let $S_2, \dots, S_n$ be the antipodes of these. Then to get the desired situation, we must choose from the pairs to end up with all but one of the S_i, and we cannot take R_n and the other S_i or else $\angle Q_1$ will be acute. That gives us $(n-2)$ good choices out of 2^{n-1}; since we could have chosen Q_1 to be any of the n points, the probability is again $n(n-2)2^{-n+1}$.

B1 Take $P(x,y) = (y-2x)(y-2x-1)$. To see that this works, first note that if $m = \lfloor a \rfloor$, then $2m$ is an integer less than or equal to $2a$, so $2m \leq \lfloor 2a \rfloor$. On the other hand, $m+1$ is an integer strictly greater than a, so $2m+2$ is an integer strictly greater than $2a$, so $\lfloor 2a \rfloor \leq 2m+1$.

B2 By the arithmetic-harmonic mean inequality or the Cauchy-Schwarz inequality,

$$(k_1 + \cdots + k_n)\left(\frac{1}{k_1} + \cdots + \frac{1}{k_n}\right) \geq n^2.$$

We must thus have $5n - 4 \geq n^2$, so $n \leq 4$. Without loss of generality, we may suppose that $k_1 \leq \cdots \leq k_n$.

If $n = 1$, we must have $k_1 = 1$, which works. Note that hereafter we cannot have $k_1 = 1$.

If $n = 2$, we have $(k_1, k_2) \in \{(2,4), (3,3)\}$, neither of which work.

If $n = 3$, we have $k_1 + k_2 + k_3 = 11$, so $2 \leq k_1 \leq 3$. Hence $(k_1, k_2, k_3) \in \{(2,2,7), (2,3,6), (2,4,5), (3,3,5), (3,4,4)\}$, and only $(2,3,6)$ works.

If $n = 4$, we must have equality in the AM-HM inequality, which only happens when $k_1 = k_2 = k_3 = k_4 = 4$.

Hence the solutions are $n = 1$ and $k_1 = 1$, $n = 3$ and (k_1, k_2, k_3) is a permutation of $(2,3,6)$, and $n = 4$ and $(k_1, k_2, k_3, k_4) = (4,4,4,4)$.

Remark: In the cases $n = 2, 3$, Greg Kuperberg suggests the alternate approach of enumerating the solutions of $1/k_1 + \cdots + 1/k_n = 1$ with $k_1 \leq \cdots \leq k_n$. This is easily done by proceeding in lexicographic order: one obtains $(2,2)$ for $n = 2$, and $(2,3,6), (2,4,4), (3,3,3)$ for $n = 3$, and only $(2,3,6)$ contributes to the final answer.

B3 **First solution:** The functions are precisely $f(x) = cx^d$ for $c, d > 0$ arbitrary except that we must take $c = 1$ in case $d = 1$. To see that these work, note that $f'(a/x) = dc(a/x)^{d-1}$ and $x/f(x) = 1/(cx^{d-1})$, so the given equation holds if and only if $dc^2a^{d-1} = 1$. If $d \neq 1$, we may solve for a no matter what c is; if $d = 1$, we must have $c = 1$. (Thanks to Brad Rodgers for pointing out the $d = 1$ restriction.)

To check that these are all solutions, put $b = \log(a)$ and $y = \log(a/x)$; rewrite the given equation as

$$f(e^{b-y})f'(e^y) = e^{b-y}.$$

Put

$$g(y) = \log f(e^y);$$

then the given equation rewrites as

$$g(b-y) + \log g'(y) + g(y) - y = b - y,$$

or

$$\log g'(y) = b - g(y) - g(b-y).$$

By the symmetry of the right side, we have $g'(b-y) = g'(y)$. Hence the function $g(y) + g(b-y)$ has zero derivative and so is constant, as then is $g'(y)$. From this we deduce that $f(x) = cx^d$ for some c, d, both necessarily positive since $f'(x) > 0$ for all x.

Second solution: (suggested by several people) Substitute a/x for x in the given equation:

$$f'(x) = \frac{a}{xf(a/x)}.$$

Differentiate:

$$f''(x) = -\frac{a}{x^2 f(a/x)} + \frac{a^2 f'(a/x)}{x^3 f(a/x)^2}.$$

Now substitute to eliminate evaluations at a/x:

$$f''(x) = -\frac{f'(x)}{x} + \frac{f'(x)^2}{f(x)}.$$

Clear denominators:

$$xf(x)f''(x) + f(x)f'(x) = xf'(x)^2.$$

Divide through by $f(x)^2$ and rearrange:

$$0 = \frac{f'(x)}{f(x)} + \frac{xf''(x)}{f(x)} - \frac{xf'(x)^2}{f(x)^2}.$$

The right side is the derivative of $xf'(x)/f(x)$, so that quantity is constant. That is, for some d,

$$\frac{f'(x)}{f(x)} = \frac{d}{x}.$$

Integrating yields $f(x) = cx^d$, as desired.

B4 **First solution:** Define $f(m,n,k)$ as the number of n-tuples $(x_1, x_2, \dots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \leq m$ and exactly k of $x_1, \dots, x_n$ are nonzero. To choose such a tuple, we may choose the k nonzero positions, the signs of those k numbers, and then an ordered k-tuple of positive integers with sum $\leq m$. There are $\binom{n}{k}$ options for the first choice, and 2^k for the second. As for the third, we have $\binom{m}{k}$ options by a "stars and bars" argument: depict the k-tuple by drawing a number of stars for each term, separated by bars, and adding stars at the end to get a total of m stars. Then each tuple corresponds to placing k bars, each in a different position behind one of the m fixed stars.

We conclude that

$$f(m,n,k) = 2^k \binom{m}{k} \binom{n}{k} = f(n,m,k);$$

summing over k gives $f(m,n) = f(n,m)$. (One may also extract easily a bijective interpretation of the equality.)

Second solution: (by Greg Kuperberg) It will be convenient to extend the definition of $f(m,n)$ to $m,n \geq 0$, in which case we have $f(0,m) = f(n,0) = 1$.

Let $S_{m,n}$ be the set of n-tuples $(x_1, \dots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \leq m$. Then elements of $S_{m,n}$ can be classified into three types. Tuples with $|x_1| + \cdots + |x_n| < m$ also belong to $S_{m-1,n}$. Tuples with $|x_1| + \cdots + |x_n| = m$ and $x_n \geq 0$ correspond to elements of $S_{m,n-1}$ by dropping x_n. Tuples with $|x_1| + \cdots + |x_n| = m$ and $x_n < 0$ correspond to elements of $S_{m-1,n-1}$ by dropping x_n. It follows that

$$\begin{aligned} &f(m,n) \\ &= f(m-1,n) + f(m,n-1) + f(m-1,n-1), \end{aligned}$$

so f satisfies a symmetric recurrence with symmetric boundary conditions $f(0,m) = f(n,0) = 1$. Hence f is symmetric.

Third solution: (by Greg Martin) As in the second solution, it is convenient to allow $f(m,0) = f(0,n) = 1$. Define the generating function

$$G(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m,n) x^m y^n.$$

As equalities of formal power series (or convergent series on, say, the region $|x|, |y| < \frac{1}{3}$), we have

$$\begin{aligned} G(x,y) &= \sum_{m \geq 0} \sum_{n \geq 0} x^m y^n \sum_{\substack{k_1, \dots, k_n \in \mathbb{Z} \\ |k_1| + \cdots + |k_n| \leq m}} 1 \\ &= \sum_{n \geq 0} y^n \sum_{k_1, \dots, k_n \in \mathbb{Z}} \sum_{m \geq |k_1| + \cdots + |k_n|} x^m \\ &= \sum_{n \geq 0} y^n \sum_{k_1, \dots, k_n \in \mathbb{Z}} \frac{x^{|k_1| + \cdots + |k_n|}}{1-x} \\ &= \frac{1}{1-x} \sum_{n \geq 0} y^n \left(\sum_{k \in \mathbb{Z}} x^{|k|} \right)^n \\ &= \frac{1}{1-x} \sum_{n \geq 0} y^n \left(\frac{1+x}{1-x} \right)^n \\ &= \frac{1}{1-x} \cdot \frac{1}{1 - y(1+x)/(1-x)} \\ &= \frac{1}{1-x-y-xy}. \end{aligned}$$

Since $G(x,y) = G(y,x)$, it follows that $f(m,n) = f(n,m)$ for all $m,n \geq 0$.

B5 **First solution:** Put $Q = x_1^2 + \cdots + x_n^2$. Since Q is homogeneous, P is divisible by Q if and only if each of the homogeneous components of P is divisible by Q. It is thus sufficient to solve the problem in case P itself is homogeneous, say of degree d.

Suppose that we have a factorization $P = Q^m R$ for some $m > 0$, where R is homogeneous of degree d and not divisible by Q; note that the homogeneity implies that

$$\sum_{i=1}^{n} x_i \frac{\partial R}{\partial x_i} = dR.$$

Write ∇^2 as shorthand for $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$; then

$$\begin{aligned} 0 &= \nabla^2 P \\ &= 2mnQ^{m-1}R + Q^m \nabla^2 R + 2 \sum_{i=1}^{n} 2mx_i Q^{m-1} \frac{\partial R}{\partial x_i} \\ &= Q^m \nabla^2 R + (2mn + 4md) Q^{m-1} R. \end{aligned}$$

Since $m > 0$, this forces R to be divisible by Q, contradiction.

Second solution: (by Noam Elkies) Retain notation as in the first solution. Let P_d be the set of homogeneous polynomials of degree d, and let H_d be the subset of P_d of polynomials killed by ∇^2, which has dimension $\geq \dim(P_d) - \dim(P_{d-2})$; the given problem amounts to showing that this inequality is actually an equality.

Consider the operator $Q\nabla^2$ (i.e., apply ∇^2 then multiply by Q) on P_d; its zero eigenspace is precisely H_d. By the calculation from the first solution, if $R \in P_d$, then

$$\nabla^2(QR) - Q\nabla^2 R = (2n+4d)R.$$

Consequently, $Q^j H_{d-2j}$ is contained in the eigenspace of $Q\nabla^2$ on P_d of eigenvalue

$$(2n+4(d-2j)) + \cdots + (2n+4(d-2)).$$

In particular, the $Q^j H^{d-2j}$ lie in distinct eigenspaces, so are linearly independent within P_d. But by dimension counting, their total dimension is at least that of P_d. Hence they exhaust P_d, and the zero eigenspace cannot have dimension greater than $\dim(P_d) - \dim(P_{d-2})$, as desired.

Third solution: (by Richard Stanley) Write $x = (x_1, \dots, x_n)$ and $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. Suppose that $P(x) = Q(x)(x_1^2 + \cdots + x_n^2)$. Then

$$P(\nabla)P(x) = Q(\nabla)(\nabla^2)P(x) = 0.$$

On the other hand, if $P(x) = \sum_\alpha c_\alpha x^\alpha$ (where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$), then the constant term of $P(\nabla)P(x)$ is seen to be $\sum_\alpha c_\alpha^2$. Hence $c_\alpha = 0$ for all α.

Remarks: The first two solutions apply directly over any field of characteristic zero. (The result fails in characteristic $p > 0$ because we may take $P = (x_1^2 + \cdots + x_n^2)^p = x_1^{2p} + \cdots + x_n^{2p}$.) The third solution can be extended to complex coefficients by replacing $P(\nabla)$ by its complex conjugate, and again the result may be deduced for any field of characteristic zero. Stanley also suggests Section 5 of the arXiv e-print `math.CO/0502363` for some algebraic background for this problem.

B6 **First solution:** Let I be the identity matrix, and let J_x be the matrix with x's on the diagonal and 1's elsewhere. Note that $J_x - (x-1)I$, being the all 1's matrix, has rank 1 and trace n, so has $n-1$ eigenvalues equal to 0 and one equal to n. Hence J_x has $n-1$ eigenvalues equal to $x-1$ and one equal to $x+n-1$, implying

$$\det J_x = (x+n-1)(x-1)^{n-1}.$$

On the other hand, we may expand the determinant as a sum indexed by permutations, in which case we get

$$\det J_x = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) x^{\nu(\pi)}.$$

Integrating both sides from 0 to 1 (and substituting $y = 1-x$) yields

$$\begin{aligned}\sum_{\pi \in S_n} \frac{\operatorname{sgn}(\pi)}{\nu(\pi)+1} &= \int_0^1 (x+n-1)(x-1)^{n-1}\,dx \\ &= \int_0^1 (-1)^{n+1}(n-y)y^{n-1}\,dy \\ &= (-1)^{n+1}\frac{n}{n+1},\end{aligned}$$

as desired.

Second solution: We start by recalling a form of the principle of inclusion-exclusion: if f is a function on the power set of $\{1, \dots, n\}$, then

$$f(S) = \sum_{T \supseteq S} (-1)^{|T|-|S|} \sum_{U \supseteq T} f(U).$$

In this case we take $f(S)$ to be the sum of $\sigma(\pi)$ over all permutations π whose fixed points are exactly S. Then $\sum_{U \supseteq T} f(U) = 1$ if $|T| \geq n-1$ and 0 otherwise (since a permutation group on 2 or more symbols has as many even and odd permutations), so

$$f(S) = (-1)^{n-|S|}(1-n+|S|).$$

The desired sum can thus be written, by grouping over fixed point sets, as

$$\begin{aligned}&\sum_{i=0}^n \binom{n}{i}(-1)^{n-i}\frac{1-n+i}{i+1} \\ &= \sum_{i=0}^n (-1)^{n-i}\binom{n}{i} - \sum_{i=0}^n (-1)^{n-i}\frac{n}{i+1}\binom{n}{i} \\ &= 0 - \sum_{i=0}^n (-1)^{n-i}\frac{n}{n+1}\binom{n+1}{i+1} \\ &= (-1)^{n+1}\frac{n}{n+1}.\end{aligned}$$

Third solution: (by Richard Stanley) The *cycle indicator* of the symmetric group S_n is defined by

$$Z_n(x_1, \dots, x_n) = \sum_{\pi \in S_n} x_1^{c_1(\pi)} \cdots x_n^{c_n(\pi)},$$

where $c_i(\pi)$ is the number of cycles of π of length i. Put

$$F_n = \sum_{\pi \in S_n} \sigma(\pi) x^{\nu(\pi)} = Z_n(x, -1, 1, -1, 1, \dots)$$

and

$$f(n) = \sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi)+1} = \int_0^1 F_n(x)\,dx.$$

A standard argument in enumerative combinatorics (the Exponential Formula) gives

$$\sum_{n=0}^{\infty} Z_n(x_1,\dots,x_n)\frac{t^n}{n!} = \exp\sum_{k=1}^{\infty} x_k \frac{t^k}{k},$$

yielding

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n)\frac{t^n}{n!} &= \int_0^1 \exp\left(xt - \frac{t^2}{2} + \frac{t^3}{3} - \cdots\right)\,dx \\
&= \int_0^1 e^{(x-1)t+\log(1+t)}\,dx \\
&= \int_0^1 (1+t)e^{(x-1)t}\,dx \\
&= \frac{1}{t}(1-e^{-t})(1+t).
\end{aligned}$$

Expanding the right side as a Taylor series and comparing coefficients yields the desired result.

Fourth solution (sketch): (by David Savitt) We prove the identity of rational functions

$$\sum_{\pi\in S_n} \frac{\sigma(\pi)}{\nu(\pi)+x} = \frac{(-1)^{n+1}n!(x+n-1)}{x(x+1)\cdots(x+n)}$$

by induction on n, which for $x=1$ implies the desired result. (This can also be deduced as in the other solutions, but in this argument it is necessary to formulate the strong induction hypothesis.)

Let $R(n,x)$ be the right hand side of the above equation. It is easy to verify that

$$\begin{aligned}
R(x,n) &= R(x+1,n-1) + (n-1)!\frac{(-1)^{n+1}}{x} \\
&\quad + \sum_{l=2}^{n-1}(-1)^{l-1}\frac{(n-1)!}{(n-l)!}R(x,n-l),
\end{aligned}$$

since the sum telescopes. To prove the desired equality, it suffices to show that the left hand side satisfies the same recurrence. This follows because we can classify each $\pi\in S_n$ as either fixing n, being an n-cycle, or having n in an l-cycle for one of $l=2,\dots,n-1$; writing the sum over these classes gives the desired recurrence.

Solutions to the 67th William Lowell Putnam Mathematical Competition
Saturday, December 2, 2006

Kiran Kedlaya and Lenny Ng

A1 We change to cylindrical coordinates, i.e., we put $r = \sqrt{x^2+y^2}$. Then the given inequality is equivalent to

$$r^2 + z^2 + 8 \leq 6r,$$

or

$$(r-3)^2 + z^2 \leq 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x-3)^2 + z^2 \leq 1$ in the xz-plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π, times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

A2 Suppose on the contrary that the set B of values of n for which Bob has a winning strategy is finite; for convenience, we include $n = 0$ in B, and write $B = \{b_1, \dots, b_m\}$. Then for every nonnegative integer n not in B, Alice must have some move on a heap of n stones leading to a position in which the second player wins. That is, every nonnegative integer not in B can be written as $b+p-1$ for some $b \in B$ and some prime p. However, there are numerous ways to show that this cannot happen.

First solution: Let t be any integer bigger than all of the $b \in B$. Then it is easy to write down t consecutive composite integers, e.g., $(t+1)!+2, \dots, (t+1)!+t+1$. Take $n = (t+1)!+t$; then for each $b \in B$, $n-b+1$ is one of the composite integers we just wrote down.

Second solution: Let $p_1, \dots, p_{2m}$ be any prime numbers; then by the Chinese remainder theorem, there exists a positive integer x such that

$$\begin{aligned} x - b_1 &\equiv -1 \pmod{p_1 p_{m+1}} \\ &\dots \\ x - b_n &\equiv -1 \pmod{p_m p_{2m}}. \end{aligned}$$

For each $b \in B$, the unique integer p such that $x = b+p-1$ is divisible by at least two primes, and so cannot itself be prime.

Third solution: (by Catalin Zara) Put $b_1 = 0$, and take $n = (b_2-1)\cdots(b_m-1)$; then n is composite because $3, 8 \in B$, and for any nonzero $b \in B$, $n - b_i + 1$ is divisible by but not equal to $b_i - 1$. (One could also take $n = b_2 \cdots b_m - 1$, so that $n - b_i + 1$ is divisible by b_i.)

A3 We first observe that given any sequence of integers $x_1, x_2, \dots$ satisfying a recursion

$$x_k = f(x_{k-1}, \dots, x_{k-n}) \qquad (k > n),$$

where n is fixed and f is a fixed polynomial of n variables with integer coefficients, for any positive integer N, the sequence modulo N is eventually periodic. This is simply because there are only finitely many possible sequences of n consecutive values modulo N, and once such a sequence is repeated, every subsequent value is repeated as well.

We next observe that if one can rewrite the same recursion as

$$x_{k-n} = g(x_{k-n+1}, \dots, x_k) \qquad (k > n),$$

where g is also a polynomial with integer coefficients, then the sequence extends uniquely to a doubly infinite sequence $\dots, x_{-1}, x_0, x_1, \dots$ which is fully periodic modulo any N. That is the case in the situation at hand, because we can rewrite the given recursion as

$$x_{k-2005} = x_{k+1} - x_k.$$

It thus suffices to find 2005 consecutive terms divisible by N in the doubly infinite sequence, for any fixed N (so in particular for $N = 2006$). Running the recursion backwards, we easily find

$$\begin{gathered} x_1 = x_0 = \cdots = x_{-2004} = 1 \\ x_{-2005} = \cdots = x_{-4009} = 0, \end{gathered}$$

yielding the desired result.

A4 **First solution:** By the linearity of expectation, the average number of local maxima is equal to the sum of the probability of having a local maximum at k over $k = 1, \dots, n$. For $k = 1$, this probability is 1/2: given the pair $\{\pi(1), \pi(2)\}$, it is equally likely that $\pi(1)$ or $\pi(2)$ is bigger. Similarly, for $k = n$, the probability is 1/2. For $1 < k < n$, the probability is 1/3: given the pair $\{\pi(k-1), \pi(k), \pi(k+1)\}$, it is equally likely that any of the three is the largest. Thus the average number of local maxima is

$$2 \cdot \frac{1}{2} + (n-2) \cdot \frac{1}{3} = \frac{n+1}{3}.$$

Second solution: Another way to apply the linearity of expectation is to compute the probability that $i \in \{1, \dots, n\}$ occurs as a local maximum. The most efficient way to do this is to imagine the permutation as consisting of the symbols $1, \dots, n, *$ written in a circle in some order. The number i occurs as a local maximum if the two symbols it is adjacent to both belong to the set $\{*, 1, \dots, i-1\}$. There are $i(i-1)$ pairs of such symbols and $n(n-1)$ pairs in total, so the probability of

i occurring as a local maximum is $i(i-1)/(n(n-1))$, and the average number of local maxima is

$$\begin{aligned}\sum_{i=1}^{n} \frac{i(i-1)}{n(n-1)} &= \frac{2}{n(n-1)} \sum_{i=1}^{n} \binom{i}{2} \\ &= \frac{2}{n(n-1)} \binom{n+1}{3} \\ &= \frac{n+1}{3}.\end{aligned}$$

One can obtain a similar (if slightly more intricate) solution inductively, by removing the known local maximum n and splitting into two shorter sequences.

Remark: The usual term for a local maximum in this sense is a *peak*. The complete distribution for the number of peaks is known; Richard Stanley suggests the reference: F. N. David and D. E. Barton, *Combinatorial Chance*, Hafner, New York, 1962, p. 162 and subsequent.

A5 Since the desired expression involves symmetric functions of $a_1, \dots, a_n$, we start by finding a polynomial with $a_1, \dots, a_n$ as roots. Note that

$$1 \pm i \tan\theta = e^{\pm i\theta} \sec\theta$$

so that

$$1 + i\tan\theta = e^{2i\theta}(1 - i\tan\theta).$$

Consequently, if we put $\omega = e^{2in\theta}$, then the polynomial

$$Q_n(x) = (1+ix)^n - \omega(1-ix)^n$$

has among its roots $a_1, \dots, a_n$. Since these are distinct and Q_n has degree n, these must be exactly the roots.

If we write

$$Q_n(x) = c_n x^n + \cdots + c_1 x + c_0,$$

then $a_1 + \cdots + a_n = -c_{n-1}/c_n$ and $a_1 \cdots a_n = -c_0/c_n$, so the ratio we are seeking is c_{n-1}/c_0. By inspection,

$$\begin{aligned}c_{n-1} &= ni^{n-1} - \omega n(-i)^{n-1} = ni^{n-1}(1-\omega) \\ c_0 &= 1 - \omega\end{aligned}$$

so

$$\frac{a_1 + \cdots + a_n}{a_1 \cdots a_n} = \begin{cases} n & n \equiv 1 \pmod 4 \\ -n & n \equiv 3 \pmod 4. \end{cases}$$

Remark: The same argument shows that the ratio between any two *odd* elementary symmetric functions of $a_1, \dots, a_n$ is independent of θ.

A6 **First solution:** (by Daniel Kane) The probability is $1 - \frac{35}{12\pi^2}$. We start with some notation and simplifications. For simplicity, we assume without loss of generality that the circle has radius 1. Let E denote the expected value of a random variable over all choices of P, Q, R. Write $[XYZ]$ for the area of triangle XYZ.

If P, Q, R, S are the four points, we may ignore the case where three of them are collinear, as this occurs with probability zero. Then the only way they can fail to form the vertices of a convex quadrilateral is if one of them lies inside the triangle formed by the other three. There are four such configurations, depending on which point lies inside the triangle, and they are mutually exclusive. Hence the desired probability is 1 minus four times the probability that S lies inside triangle PQR. That latter probability is simply $E([PQR])$ divided by the area of the disc.

Let O denote the center of the circle, and let P', Q', R' be the projections of P, Q, R onto the circle from O. We can write

$$[PQR] = \pm[OPQ] \pm [OQR] \pm [ORP]$$

for a suitable choice of signs, determined as follows. If the points P', Q', R' lie on no semicircle, then all of the signs are positive. If P', Q', R' lie on a semicircle in that order and Q lies inside the triangle OPR, then the sign on $[OPR]$ is positive and the others are negative. If P', Q', R' lie on a semicircle in that order and Q lies outside the triangle OPR, then the sign on $[OPR]$ is negative and the others are positive.

We first calculate

$$E([OPQ] + [OQR] + [ORP]) = 3E([OPQ]).$$

Write $r_1 = OP, r_2 = OQ, \theta = \angle POQ$, so that

$$[OPQ] = \frac{1}{2} r_1 r_2 (\sin\theta).$$

The distribution of r_1 is given by $2r_1$ on $[0,1]$ (e.g., by the change of variable formula to polar coordinates), and similarly for r_2. The distribution of θ is uniform on $[0, \pi]$. These three distributions are independent; hence

$$\begin{aligned}&E([OPQ]) \\ &= \frac{1}{2} \left(\int_0^1 2r^2\,dr \right)^2 \left(\frac{1}{\pi} \int_0^\pi \sin(\theta)\,d\theta \right) \\ &= \frac{4}{9\pi},\end{aligned}$$

and

$$E([OPQ] + [OQR] + [ORP]) = \frac{4}{3\pi}.$$

We now treat the case where P', Q', R' lie on a semicircle in that order. Put $\theta_1 = \angle POQ$ and $\theta_2 = \angle QOR$; then the distribution of θ_1, θ_2 is uniform on the region

$$0 \le \theta_1, \quad 0 \le \theta_2, \quad \theta_1 + \theta_2 \le \pi.$$

In particular, the distribution on $\theta = \theta_1 + \theta_2$ is $\frac{2\theta}{\pi^2}$ on $[0, \pi]$. Put $r_P = OP, r_Q = OQ, r_R = OR$. Again, the distribution on r_P is given by $2r_P$ on $[0,1]$, and similarly for r_Q, r_R; these are independent from each other and from the joint distribution of θ_1, θ_2. Write $E'(X)$ for the expectation of a random variable X restricted to this part of the domain.

Let χ be the random variable with value 1 if Q is inside triangle OPR and 0 otherwise. We now compute

$$\begin{aligned}
&E'([OPR]) \\
&= \frac{1}{2}\left(\int_0^1 2r^2\,dr\right)^2 \left(\int_0^\pi \frac{2\theta}{\pi^2}\sin(\theta)\,d\theta\right) \\
&= \frac{4}{9\pi} \\
&E'(\chi[OPR]) \\
&= E'(2[OPR]^2/\theta) \\
&= \frac{1}{2}\left(\int_0^1 2r^3\,dr\right)^2 \left(\int_0^\pi \frac{2\theta}{\pi^2}\theta^{-1}\sin^2(\theta)\,d\theta\right) \\
&= \frac{1}{8\pi}.
\end{aligned}$$

Also recall that given any triangle XYZ, if T is chosen uniformly at random inside XYZ, the expectation of $[TXY]$ is the area of triangle bounded by XY and the centroid of XYZ, namely $\frac{1}{3}[XYZ]$.

Let χ be the random variable with value 1 if Q is inside triangle OPR and 0 otherwise. Then

$$\begin{aligned}
&E'([OPQ]+[OQR]+[ORP]-[PQR]) \\
&= 2E'(\chi([OPQ]+[OQR]) + 2E'((1-\chi)[OPR]) \\
&= 2E'(\frac{2}{3}\chi[OPR]) + 2E'([OPR]) - 2E'(\chi[OPR]) \\
&= 2E'([OPR]) - \frac{2}{3}E'(\chi[OPR]) = \frac{29}{36\pi}.
\end{aligned}$$

Finally, note that the case when P', Q', R' lie on a semicircle in some order occurs with probability $3/4$. (The case where they lie on a semicircle proceeding clockwise from P' to its antipode has probability 1/4; this case and its two analogues are exclusive and exhaustive.) Hence

$$\begin{aligned}
&E([PQR]) \\
&= E([OPQ]+[OQR]+[ORP]) \\
&\quad - \frac{3}{4}E'([OPQ]+[OQR]+[ORP]-[PQR]) \\
&= \frac{4}{3\pi} - \frac{29}{48\pi} = \frac{35}{48\pi},
\end{aligned}$$

so the original probability is

$$1 - \frac{4E([PQR])}{\pi} = 1 - \frac{35}{12\pi^2}.$$

Second solution: (by David Savitt) As in the first solution, it suffices to check that for P, Q, R chosen uniformly at random in the disc, $E([PQR]) = \frac{35}{48\pi}$. Draw the lines PQ, QR, RP, which with probability 1 divide the interior of the circle into seven regions. Put $a = [PQR]$, let b_1, b_2, b_3 denote the areas of the three other regions sharing a side with the triangle, and let c_1, c_2, c_3 denote the areas of the other three regions. Put $A = E(a)$, $B = E(b_1)$, $C = E(c_1)$, so that $A + 3B + 3C = \pi$.

Note that $c_1 + c_2 + c_3 + a$ is the area of the region in which we can choose a fourth point S so that the quadrilateral $PQRS$ fails to be convex. By comparing expectations, we have $3C + A = 4A$, so $A = C$ and $4A + 3B = \pi$.

We will compute $B + 2A = B + 2C$, which is the expected area of the part of the circle cut off by a chord through two random points D, E, on the side of the chord not containing a third random point F. Let h be the distance from the center O of the circle to the line DE. We now determine the distribution of h.

Put $r = OD$; the distribution of r is $2r$ on $[0,1]$. Without loss of generality, suppose O is the origin and D lies on the positive x-axis. For fixed r, the distribution of h runs over $[0, r]$, and can be computed as the area of the infinitesimal region in which E can be chosen so the chord through DE has distance to O between h and $h + dh$, divided by π. This region splits into two symmetric pieces, one of which lies between chords making angles of $\arcsin(h/r)$ and $\arcsin((h+dh)/r)$ with the x-axis. The angle between these is $d\theta = dh/(r^2 - h^2)$. Draw the chord through D at distance h to O, and let L_1, L_2 be the lengths of the parts on opposite sides of D; then the area we are looking for is $\frac{1}{2}(L_1^2 + L_2^2)d\theta$. Since

$$\{L_1, L_2\} = \sqrt{1-h^2} \pm \sqrt{r^2-h^2},$$

the area we are seeking (after doubling) is

$$2\frac{1 + r^2 - 2h^2}{\sqrt{r^2-h^2}}.$$

Dividing by π, then integrating over r, we compute the distribution of h to be

$$\begin{aligned}
&\frac{1}{\pi}\int_h^1 2\frac{1+r^2-2h^2}{\sqrt{r^2-h^2}}2r\,dr \\
&= \frac{16}{3\pi}(1-h^2)^{3/2}.
\end{aligned}$$

We now return to computing $B + 2A$. Let $A(h)$ denote the smaller of the two areas of the disc cut off by a chord at distance h. The chance that the third point is in the smaller (resp. larger) portion is $A(h)/\pi$ (resp. $1 - A(h)/\pi$), and then the area we are trying to compute is $\pi - A(h)$ (resp. $A(h)$). Using the distribution on h, and the fact that

$$\begin{aligned}
A(h) &= 2\int_h^1 \sqrt{1-h^2}\,dh \\
&= \frac{\pi}{2} - \arcsin(h) - h\sqrt{1-h^2},
\end{aligned}$$

we find

$$\begin{aligned}
&B + 2A \\
&= \frac{2}{\pi}\int_0^1 A(h)(\pi - A(h))\,\frac{16}{3\pi}(1-h^2)^{3/2}\,dh \\
&= \frac{35+24\pi^2}{72\pi}.
\end{aligned}$$

Since $4A + 3B = \pi$, we solve to obtain $A = \frac{35}{48\pi}$ as in the first solution.

Third solution: (by Noam Elkies) Again, we reduce to computing the average area of a triangle formed by three random points A, B, C inside a unit circle. Let O be the center of the circle, and put $c = \max\{OA, OB, OC\}$; then the probability that $c \leq r$ is $(r^2)^3$, so the distribution of c is $6c^5\,dc$ on $[0,1]$.

Given c, the expectation of $[ABC]$ is equal to c^2 times X, the expected area of a triangle formed by two random points P, Q in a circle and a fixed point R on the boundary. We introduce polar coordinates centered at R, in which the circle is given by $r = 2\sin\theta$ for $\theta \in [0,\pi]$. The distribution of a random point in that circle is $\frac{1}{\pi} r\,dr\,d\theta$ over $\theta \in [0,\pi]$ and $r \in [0, 2\sin\theta]$. If (r,θ) and (r',θ') are the two random points, then the area is $\frac{1}{2} rr' \sin|\theta - \theta'|$.

Performing the integrals over r and r' first, we find

$$\begin{aligned}
X &= \frac{32}{9\pi^2}\int_0^\pi \int_0^\pi \sin^3\theta \sin^3\theta' \sin|\theta-\theta'|\,d\theta'\,d\theta \\
&= \frac{64}{9\pi^2}\int_0^\pi \int_0^\theta \sin^3\theta \sin^3\theta' \sin(\theta-\theta')\,d\theta'\,d\theta.
\end{aligned}$$

This integral is unpleasant but straightforward; it yields $X = 35/(36\pi)$, and $E([PQR]) = \int_0^1 6c^7 X\,dc = 35/(48\pi)$, giving the desired result.

Remark: This is one of the oldest problems in geometric probability; it is an instance of Sylvester's four-point problem, which nowadays is usually solved using a device known as Crofton's formula. We defer to `http://mathworld.wolfram.com/` for further discussion.

B1 The "curve" $x^3 + 3xy + y^3 - 1 = 0$ is actually reducible, because the left side factors as

$$(x + y - 1)(x^2 - xy + y^2 + x + y + 1).$$

Moreover, the second factor is

$$\frac{1}{2}((x+1)^2 + (y+1)^2 + (x-y)^2),$$

so it only vanishes at $(-1,-1)$. Thus the curve in question consists of the single point $(-1,-1)$ together with the line $x + y = 1$. To form a triangle with three points on this curve, one of its vertices must be $(-1,-1)$. The other two vertices lie on the line $x + y = 1$, so the length of the altitude from $(-1,-1)$ is the distance from $(-1,-1)$ to $(1/2, 1/2)$, or $3\sqrt{2}/2$. The area of an equilateral triangle of height h is $h^2\sqrt{3}/3$, so the desired area is $3\sqrt{3}/2$.

Remark: The factorization used above is a special case of the fact that

$$\begin{aligned}
&x^3 + y^3 + z^3 - 3xyz \\
&= (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z),
\end{aligned}$$

where ω denotes a primitive cube root of unity. That fact in turn follows from the evaluation of the determinant of the *circulant matrix*

$$\begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}$$

by reading off the eigenvalues of the eigenvectors $(1, \omega^i, \omega^{2i})$ for $i = 0, 1, 2$.

B2 Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. For $i = 0, \dots, n$, put $s_i = x_1 + \cdots + x_i$ (so that $s_0 = 0$). Sort the numbers $\{s_0\}, \dots, \{s_n\}$ into ascending order, and call the result $t_0, \dots, t_n$. Since $0 = t_0 \leq \cdots \leq t_n < 1$, the differences

$$t_1 - t_0, \dots, t_n - t_{n-1}, 1 - t_n$$

are nonnegative and add up to 1. Hence (as in the pigeonhole principle) one of these differences is no more than $1/(n+1)$; if it is anything other than $1 - t_n$, it equals $\pm(\{s_i\} - \{s_j\})$ for some $0 \leq i < j \leq n$. Put $S = \{x_{i+1}, \dots, x_j\}$ and $m = \lfloor s_i \rfloor - \lfloor s_j \rfloor$; then

$$\begin{aligned}
\left| m + \sum_{s \in S} s \right| &= |m + s_j - s_i| \\
&= |\{s_j\} - \{s_i\}| \\
&\leq \frac{1}{n+1},
\end{aligned}$$

as desired. In case $1 - t_n \leq 1/(n+1)$, we take $S = \{x_1, \dots, x_n\}$ and $m = -\lceil s_n \rceil$, and again obtain the desired conclusion.

B3 The maximum is $\binom{n}{2} + 1$, achieved for instance by a convex n-gon: besides the trivial partition (in which all of the points are in one part), each linear partition occurs by drawing a line crossing a unique pair of edges.

First solution: We will prove that $L_S = \binom{n}{2} + 1$ in any configuration in which no two of the lines joining points of S are parallel. This suffices to imply the maximum in all configurations: given a maximal configuration, we may vary the points slightly to get another maximal configuration in which our hypothesis is satisfied. For convenience, we assume $n \geq 3$, as the cases $n = 1, 2$ are easy.

Let P be the line at infinity in the real projective plane; i.e., P is the set of possible directions of lines in the plane, viewed as a circle. Remove the directions corresponding to lines through two points of S; this leaves behind $\binom{n}{2}$ intervals.

Given a direction in one of the intervals, consider the set of linear partitions achieved by lines parallel to that direction. Note that the resulting collection of partitions depends only on the interval. Then note that the collections associated to adjacent intervals differ in only one element.

The trivial partition that puts all of S on one side is in every such collection. We now observe that for any other linear partition $\{A, B\}$, the set of intervals to which $\{A, B\}$ is:

(a) a consecutive block of intervals, but

(b) not all of them.

For (a), note that if ℓ_1, ℓ_2 are nonparallel lines achieving the same partition, then we can rotate around their point of intersection to achieve all of the intermediate directions on one side or the other. For (b), the case $n = 3$ is evident; to reduce the general case to this case, take points P, Q, R such that P lies on the opposite side of the partition from Q and R.

It follows now that that each linear partition, except for the trivial one, occurs in exactly one place as the partition associated to some interval but not to its immediate counterclockwise neighbor. In other words, the number of linear partitions is one more than the number of intervals, or $\binom{n}{2} + 1$ as desired.

Second solution: We prove the upper bound by induction on n. Choose a point P in the convex hull of S. Put $S' = S \setminus \{P\}$; by the induction hypothesis, there are at most $\binom{n-1}{2} + 1$ linear partitions of S'. Note that each linear partition of S restricts to a linear partition of S'. Moreover, if two linear partitions of S restrict to the same linear partition of S', then that partition of S' is achieved by a line through P.

By rotating a line through P, we see that there are at most $n - 1$ partitions of S' achieved by lines through P: namely, the partition only changes when the rotating line passes through one of the points of S. This yields the desired result.

Third solution: (by Noam Elkies) We enlarge the plane to a projective plane by adding a line at infinity, then apply the polar duality map centered at one of the points $O \in S$. This turns the rest of S into a set S' of $n - 1$ lines in the dual projective plane. Let O' be the point in the dual plane corresponding to the original line at infinity; it does not lie on any of the lines in S'.

Let ℓ be a line in the original plane, corresponding to a point P in the dual plane. If we form the linear partition induced by ℓ, then the points of $S \setminus \{O\}$ lying in the same part as O correspond to the lines of S' which cross the segment $O'P$. If we consider the dual affine plane as being divided into regions by the lines of S', then the lines of S' crossing the segment $O'P$ are determined by which region P lies in.

Thus our original maximum is equal to the maximum number of regions into which $n-1$ lines divide an affine plane. By induction on n, this number is easily seen to be $1 + \binom{n}{2}$.

Remark: Given a finite set S of points in $\mathbb{R}^n$, a *non-Radon partition* of S is a pair (A, B) of complementary subsets that can be separated by a hyperplane. *Radon's theorem* states that if $\#S \geq n+2$, then not every (A, B) is a non-Radon partition. The result of this problem has been greatly extended, especially within the context of matroid theory and oriented matroid theory. Richard Stanley suggests the following references: T. H. Brylawski, A combinatorial perspective on the Radon convexity theorem, *Geom. Ded.* **5** (1976), 459-466; and T. Zaslavsky, Extremal arrangements of hyperplanes, *Ann. N. Y. Acad. Sci.* **440** (1985), 69-87.

B4 The maximum is 2^k, achieved for instance by the subspace

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 = \cdots = x_{n-k} = 0\}.$$

First solution: More generally, we show that any affine k-dimensional plane in $\mathbb{R}^n$ can contain at most 2^k points in Z. The proof is by induction on $k + n$; the case $k = n = 0$ is clearly true.

Suppose that V is a k-plane in $\mathbb{R}^n$. Denote the hyperplanes $\{x_n = 0\}$ and $\{x_n = 1\}$ by V_0 and V_1, respectively. If $V \cap V_0$ and $V \cap V_1$ are each at most $(k-1)$-dimensional, then $V \cap V_0 \cap Z$ and $V \cap V_1 \cap Z$ each have cardinality at most 2^{k-1} by the induction assumption, and hence $V \cap Z$ has at most 2^k elements. Otherwise, if $V \cap V_0$ or $V \cap V_1$ is k-dimensional, then $V \subset V_0$ or $V \subset V_1$; now apply the induction hypothesis on V, viewed as a subset of $\mathbb{R}^{n-1}$ by dropping the last coordinate.

Second solution: Let S be a subset of Z contained in a k-dimensional subspace of V. This is equivalent to asking that any $t_1, \dots, t_{k+1} \in S$ satisfy a nontrivial linear dependence $c_1 t_1 + \cdots + c_{k+1} t_{k+1} = 0$ with $c_1, \dots, c_{k+1} \in \mathbb{R}$. Since $t_1, \dots, t_{k+1} \in \mathbb{Q}^n$, given such a dependence we can always find another one with $c_1, \dots, c_{k+1} \in \mathbb{Q}$; then by clearing denominators, we can find one with $c_1, \dots, c_{k+1} \in \mathbb{Z}$ and not all having a common factor.

Let $\mathbb{F}_2$ denote the field of two elements, and let $\overline{S} \subseteq \mathbb{F}_2^n$ be the reductions modulo 2 of the points of S. Then any $t_1, \dots, t_{k+1} \in \overline{S}$ satisfy a nontrivial linear dependence, because we can take the dependence from the end of the previous paragraph and reduce modulo 2. Hence $\overline{S}$ is contained in a k-dimensional subspace of $\mathbb{F}_{2^n}$, and the latter has cardinality exactly 2^k. Thus $\overline{S}$ has at most 2^k elements, as does S.

Variant (suggested by David Savitt): if $\overline{S}$ contained $k+1$ linearly independent elements, the $(k+1)\times n$ matrix formed by these would have a nonvanishing maximal minor. The lift of that minor back to $\mathbb{R}$ would also not vanish, so S would contain $k+1$ linearly independent elements.

Third solution: (by Catalin Zara) Let V be a k-dimensional subspace. Form the matrix whose rows are the elements of $V \cap Z$; by construction, it has row rank at most k. It thus also has column rank at most k; in particular, we can choose k coordinates such that each point of $V \cap Z$ is determined by those k of its coordinates. Since each coordinate of a point in Z can only take two values, $V \cap Z$ can have at most 2^k elements.

Remark: The proposers probably did not realize that this problem appeared online about three months before the exam, at `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=105991`. (It may very well have also appeared even earlier.)

B5 The answer is $1/16$. We have

$$\begin{aligned}&\int_0^1 x^2 f(x)\,dx - \int_0^1 x f(x)^2\,dx\\ &= \int_0^1 (x^3/4 - x(f(x)-x/2)^2)\,dx\\ &\leq \int_0^1 x^3/4\,dx = 1/16,\end{aligned}$$

with equality when $f(x) = x/2$.

B6 **First solution:** We start with some easy upper and lower bounds on a_n. We write $O(f(n))$ and $\Omega(f(n))$ for functions $g(n)$ such that $f(n)/g(n)$ and $g(n)/f(n)$, respectively, are bounded above. Since a_n is a nondecreasing sequence, $a_{n+1}-a_n$ is bounded above, so $a_n = O(n)$. That means $a_n^{-1/k} = \Omega(n^{-1/k})$, so

$$a_n = \Omega\left(\sum_{i=1}^n i^{-1/k}\right) = \Omega(n^{(k-1)/k}).$$

In fact, all we will need is that $a_n \to \infty$ as $n \to \infty$.

By Taylor's theorem with remainder, for $1 < m < 2$ and $x > 0$,

$$|(1+x)^m - 1 - mx| \leq \frac{m(m-1)}{2}x^2.$$

Taking $m = (k+1)/k$ and $x = a_{n+1}/a_n = 1 + a_n^{-(k+1)/k}$, we obtain

$$\left| a_{n+1}^{(k+1)/k} - a_n^{(k+1)/k} - \frac{k+1}{k} \right| \leq \frac{k+1}{2k^2} a_n^{-(k+1)/k}.$$

In particular,

$$\lim_{n\to\infty} a_{n+1}^{(k+1)/k} - a_n^{(k+1)/k} = \frac{k+1}{k}.$$

In general, if x_n is a sequence with $\lim_{n\to\infty} x_n = c$, then also

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n x_i = c$$

by Cesaro's lemma. Explicitly, for any $\epsilon > 0$, we can find N such that $|x_n - c| \leq \epsilon/2$ for $n \geq N$, and then

$$\left| c - \frac{1}{n}\sum_{i=1}^n x_i \right| \leq \frac{n-N}{n}\frac{\epsilon}{2} + \frac{N}{n}\left|\sum_{i=1}^N (c - x_i)\right|;$$

for n large, the right side is smaller than ϵ.

In our case, we deduce that

$$\lim_{n\to\infty} \frac{a_n^{(k+1)/k}}{n} = \frac{k+1}{k}$$

and so

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k} = \left(\frac{k+1}{k}\right)^k,$$

as desired.

Remark: The use of Cesaro's lemma above is the special case $b_n = n$ of the *Cesaro-Stolz theorem*: if a_n, b_n are sequences such that b_n is positive, strictly increasing, and unbounded, and

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = L,$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Second solution: In this solution, rather than applying Taylor's theorem with remainder to $(1+x)^m$ for $1 < m < 2$ and $x > 0$, we only apply convexity to deduce that $(1+x)^m \geq 1 + mx$. This gives

$$a_{n+1}^{(k+1)/k} - a_n^{(k+1)/k} \geq \frac{k+1}{k},$$

and so

$$a_n^{(k+1)/k} \geq \frac{k+1}{k}n + c$$

for some $c \in \mathbb{R}$. In particular,

$$\liminf_{n\to\infty} \frac{a_n^{(k+1)/k}}{n} \geq \frac{k+1}{k}$$

and so

$$\liminf_{n\to\infty} \frac{a_n}{n^{k/(k+1)}} \geq \left(\frac{k+1}{k}\right)^{k/(k+1)}.$$

But turning this around, the fact that

$$\begin{aligned}
&a_{n+1} - a_n \\
&= a_n^{-1/k} \\
&\leq \left(\frac{k+1}{k}\right)^{-1/(k+1)} n^{-1/(k+1)}(1+o(1)),
\end{aligned}$$

where $o(1)$ denotes a function tending to 0 as $n \to \infty$, yields

$$\begin{aligned}
&a_n \\
&\leq \left(\frac{k+1}{k}\right)^{-1/(k+1)} \sum_{i=1}^n i^{-1/(k+1)}(1+o(1)) \\
&= \frac{k+1}{k}\left(\frac{k+1}{k}\right)^{-1/(k+1)} n^{k/(k+1)}(1+o(1)) \\
&= \left(\frac{k+1}{k}\right)^{k/(k+1)} n^{k/(k+1)}(1+o(1)),
\end{aligned}$$

so

$$\limsup_{n\to\infty} \frac{a_n}{n^{k/(k+1)}} \leq \left(\frac{k+1}{k}\right)^{k/(k+1)}$$

and this completes the proof.

Third solution: We argue that $a_n \to \infty$ as in the first solution. Write $b_n = a_n - Ln^{k/(k+1)}$, for a value of L to be determined later. We have

$$\begin{aligned}
&b_{n+1} \\
&= b_n + a_n^{-1/k} - L((n+1)^{k/(k+1)} - n^{k/(k+1)}) \\
&= e_1 + e_2,
\end{aligned}$$

where

$$\begin{aligned}
e_1 &= b_n + a_n^{-1/k} - L^{-1/k} n^{-1/(k+1)} \\
e_2 &= L((n+1)^{k/(k+1)} - n^{k/(k+1)}) \\
&\quad - L^{-1/k} n^{-1/(k+1)}.
\end{aligned}$$

We first estimate e_1. For $-1 < m < 0$, by the convexity of $(1+x)^m$ and $(1+x)^{1-m}$, we have

$$\begin{aligned}
1 + mx &\leq (1+x)^m \\
&\leq 1 + mx(1+x)^{m-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
-\frac{1}{k} L^{-(k+1)/k} n^{-1} b_n &\leq e_1 - b_n \\
&\leq -\frac{1}{k} b_n a_n^{-(k+1)/k}.
\end{aligned}$$

Note that both bounds have sign opposite to b_n; moreover, by the bound $a_n = \Omega(n^{(k-1)/k})$, both bounds have absolutely value strictly less than that of b_n for n sufficiently large. Consequently, for n large,

$$|e_1| \leq |b_n|.$$

We now work on e_2. By Taylor's theorem with remainder applied to $(1+x)^m$ for $x > 0$ and $0 < m < 1$,

$$\begin{aligned}
1 + mx &\geq (1+x)^m \\
&\geq 1 + mx + \frac{m(m-1)}{2} x^2.
\end{aligned}$$

The "main term" of $L((n+1)^{k/(k+1)} - n^{k/(k+1)})$ is $L\frac{k}{k+1} n^{-1/(k+1)}$. To make this coincide with $L^{-1/k} n^{-1/(k+1)}$, we take

$$L = \left(\frac{k+1}{k}\right)^{k/(k+1)}.$$

We then find that

$$|e_2| = O(n^{-2}),$$

and because $b_{n+1} = e_1 + e_2$, we have $|b_{n+1}| \leq |b_n| + |e_2|$. Hence

$$|b_n| = O\left(\sum_{i=1}^n i^{-2}\right) = O(1),$$

and so

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k} = L^{k+1} = \left(\frac{k+1}{k}\right)^k.$$

Remark: The case $k = 2$ appeared on the 2004 Romanian Olympiad (district level).

Remark: One can make a similar argument for any sequence given by $a_{n+1} = a_n + f(a_n)$, when f is a *decreasing* function.

Remark: Richard Stanley suggests a heuristic for determining the asymptotic behavior of sequences of this type: replace the given recursion

$$a_{n+1} - a_n = a_n^{-1/k}$$

by the differential equation

$$y' = y^{-1/k}$$

and determine the asymptotics of the latter.

Solutions to the 68th William Lowell Putnam Mathematical Competition
Saturday, December 1, 2007

Manjul Bhargava, Kiran Kedlaya, and Lenny Ng

A1 The only such α are $2/3, 3/2, (13 \pm \sqrt{601})/12$.

First solution: Let C_1 and C_2 be the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$, respectively, and let L be the line $y = x$. We consider three cases.

If C_1 is tangent to L, then the point of tangency (x, x) satisfies

$$2\alpha x + \alpha = 1, \qquad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

by symmetry, C_2 is tangent to L there, so C_1 and C_2 are tangent. Writing $\alpha = 1/(2x+1)$ in the first equation and substituting into the second, we must have

$$x = \frac{x^2 + x}{2x+1} + \frac{1}{24},$$

which simplifies to $0 = 24x^2 - 2x - 1 = (6x+1)(4x-1)$, or $x \in \{1/4, -1/6\}$. This yields $\alpha = 1/(2x+1) \in \{2/3, 3/2\}$.

If C_1 does not intersect L, then C_1 and C_2 are separated by L and so cannot be tangent.

If C_1 intersects L in two distinct points P_1, P_2, then it is not tangent to L at either point. Suppose at one of these points, say P_1, the tangent to C_1 is perpendicular to L; then by symmetry, the same will be true of C_2, so C_1 and C_2 will be tangent at P_1. In this case, the point $P_1 = (x, x)$ satisfies

$$2\alpha x + \alpha = -1, \qquad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

writing $\alpha = -1/(2x+1)$ in the first equation and substituting into the second, we have

$$x = -\frac{x^2 + x}{2x+1} + \frac{1}{24},$$

or $x = (-23 \pm \sqrt{601})/72$. This yields $\alpha = -1/(2x+1) = (13 \pm \sqrt{601})/12$.

If instead the tangents to C_1 at P_1, P_2 are not perpendicular to L, then we claim there cannot be any point where C_1 and C_2 are tangent. Indeed, if we count intersections of C_1 and C_2 (by using C_1 to substitute for y in C_2, then solving for y), we get at most four solutions counting multiplicity. Two of these are P_1 and P_2, and any point of tangency counts for two more. However, off of L, any point of tangency would have a mirror image which is also a point of tangency, and there cannot be six solutions. Hence we have now found all possible α.

Second solution: For any nonzero value of α, the two conics will intersect in four points in the complex projective plane $\mathbb{P}^2(\mathbb{C})$. To determine the y-coordinates of these intersection points, subtract the two equations to obtain

$$(y - x) = \alpha(x - y)(x + y) + \alpha(x - y).$$

Therefore, at a point of intersection we have either $x = y$, or $x = -1/\alpha - (y + 1)$. Substituting these two possible linear conditions into the second equation shows that the y-coordinate of a point of intersection is a root of either $Q_1(y) = \alpha y^2 + (\alpha - 1)y + 1/24$ or $Q_2(y) = \alpha y^2 + (\alpha + 1)y + 25/24 + 1/\alpha$.

If two curves are tangent, then the y-coordinates of at least two of the intersection points will coincide; the converse is also true because one of the curves is the graph of a function in x. The coincidence occurs precisely when either the discriminant of at least one of Q_1 or Q_2 is zero, or there is a common root of Q_1 and Q_2. Computing the discriminants of Q_1 and Q_2 yields (up to constant factors) $f_1(\alpha) = 6\alpha^2 - 13\alpha + 6$ and $f_2(\alpha) = 6\alpha^2 - 13\alpha - 18$, respectively. If on the other hand Q_1 and Q_2 have a common root, it must be also a root of $Q_2(y) - Q_1(y) = 2y + 1 + 1/\alpha$, yielding $y = -(1+\alpha)/(2\alpha)$ and $0 = Q_1(y) = -f_2(\alpha)/(24\alpha)$.

Thus the values of α for which the two curves are tangent must be contained in the set of zeros of f_1 and f_2, namely $2/3$, $3/2$, and $(13 \pm \sqrt{601})/12$.

Remark: The fact that the two conics in $\mathbb{P}^2(\mathbb{C})$ meet in four points, counted with multiplicities, is a special case of *Bézout's theorem*: two curves in $\mathbb{P}^2(\mathbb{C})$ of degrees m, n and not sharing any common component meet in exactly mn points when counted with multiplicity.

Many solvers were surprised that the proposers chose the parameter $1/24$ to give two rational roots and two nonrational roots. In fact, they had no choice in the matter: attempting to make all four roots rational by replacing $1/24$ by β amounts to asking for $\beta^2 + \beta$ and $\beta^2 + \beta + 1$ to be perfect squares. This cannot happen outside of trivial cases ($\beta = 0, -1$) ultimately because the elliptic curve 24A1 (in Cremona's notation) over $\mathbb{Q}$ has rank 0. (Thanks to Noam Elkies for providing this computation.)

However, there are choices that make the radical milder, e.g., $\beta = 1/3$ gives $\beta^2 + \beta = 4/9$ and $\beta^2 + \beta + 1 = 13/9$, while $\beta = 3/5$ gives $\beta^2 + \beta = 24/25$ and $\beta^2 + \beta + 1 = 49/25$.

A2 The minimum is 4, achieved by the square with vertices $(\pm 1, \pm 1)$.

First solution: To prove that 4 is a lower bound, let S be a convex set of the desired form. Choose $A, B, C, D \in S$ lying on the branches of the two hyperbolas, with A in the upper right quadrant, B in the upper left, C in the lower left, D in the lower right. Then the area of the quadrilateral $ABCD$ is a lower bound for the area of S.

Write $A = (a, 1/a)$, $B = (b, -1/b)$, $C = (-c, -1/c)$, $D = (-d, 1/d)$ with $a, b, c, d > 0$. Then the area of the quadrilateral $ABCD$ is

$$\frac{1}{2}(a/b + b/c + c/d + d/a + b/a + c/b + d/c + a/d),$$

which by the arithmetic-geometric mean inequality is at least 4.

Second solution: Choose A, B, C, D as in the first solution. Note that both the hyperbolas and the area of the convex hull of $ABCD$ are invariant under the transformation $(x, y) \mapsto (xm, y/m)$ for any $m > 0$. For m small, the counterclockwise angle from the line AC to the line BD approaches 0; for m large, this angle approaches π. By continuity, for some m this angle becomes $\pi/2$, that is, AC and BD become perpendicular. The area of $ABCD$ is then $AC \cdot BD$.

It thus suffices to note that $AC \geq 2\sqrt{2}$ (and similarly for BD). This holds because if we draw the tangent lines to the hyperbola $xy = 1$ at the points $(1, 1)$ and $(-1, -1)$, then A and C lie outside the region between these lines. If we project the segment AC orthogonally onto the line $x = y = 1$, the resulting projection has length at least $2\sqrt{2}$, so AC must as well.

Third solution: (by Richard Stanley) Choose A, B, C, D as in the first solution. Now fixing A and C, move B and D to the points at which the tangents to the curve are parallel to the line AC. This does not increase the area of the quadrilateral $ABCD$ (even if this quadrilateral is not convex).

Note that B and D are now diametrically opposite; write $B = (-x, 1/x)$ and $D = (x, -1/x)$. If we thus repeat the procedure, fixing B and D and moving A and C to the points where the tangents are parallel to BD, then A and C must move to $(x, 1/x)$ and $(-x, -1/x)$, respectively, forming a rectangle of area 4.

Remark: Many geometric solutions are possible. An example suggested by David Savitt (due to Chris Brewer): note that AD and BC cross the positive and negative x-axes, respectively, so the convex hull of $ABCD$ contains O. Then check that the area of triangle OAB is at least 1, et cetera.

A3 Assume that we have an ordering of $1, 2, \ldots, 3k + 1$ such that no initial subsequence sums to 0 mod 3. If we omit the multiples of 3 from this ordering, then the remaining sequence mod 3 must look like $1, 1, -1, 1, -1, \ldots$ or $-1, -1, 1, -1, 1, \ldots$. Since there is one more integer in the ordering congruent to 1 mod 3 than to -1, the sequence mod 3 must look like $1, 1, -1, 1, -1, \ldots$.

It follows that the ordering satisfies the given condition if and only if the following two conditions hold: the first element in the ordering is not divisible by 3, and the sequence mod 3 (ignoring zeroes) is of the form $1, 1, -1, 1, -1, \ldots$. The two conditions are independent, and the probability of the first is $(2k+1)/(3k+1)$ while the probability of the second is $1/\binom{2k+1}{k}$, since there are $\binom{2k+1}{k}$ ways to order $(k + 1)$ 1's and k -1's. Hence the desired probability is the product of these two, or $\frac{k!(k+1)!}{(3k+1)(2k)!}$.

A4 Note that n is a repunit if and only if $9n + 1 = 10^m$ for some power of 10 greater than 1. Consequently, if we put

$$g(n) = 9f\left(\frac{n-1}{9}\right) + 1,$$

then f takes repunits to repunits if and only if g takes powers of 10 greater than 1 to powers of 10 greater than 1. We will show that the only such functions g are those of the form $g(n) = 10^c n^d$ for $d \geq 0$, $c \geq 1 - d$ (all of which clearly work), which will mean that the desired polynomials f are those of the form

$$f(n) = \frac{1}{9}(10^c(9n + 1)^d - 1)$$

for the same c, d.

It is convenient to allow "powers of 10" to be of the form 10^k for any integer k. With this convention, it suffices to check that the polynomials g taking powers of 10 greater than 1 to powers of 10 are of the form $10^c n^d$ for any integers c, d with $d \geq 0$.

First solution: Suppose that the leading term of $g(x)$ is ax^d, and note that $a > 0$. As $x \to \infty$, we have $g(x)/x^d \to a$; however, for x a power of 10 greater than 1, $g(x)/x^d$ is a power of 10. The set of powers of 10 has no positive limit point, so $g(x)/x^d$ must be equal to a for $x = 10^k$ with k sufficiently large, and we must have $a = 10^c$ for some c. The polynomial $g(x) - 10^c x^d$ has infinitely many roots, so must be identically zero.

Second solution: We proceed by induction on $d = \deg(g)$. If $d = 0$, we have $g(n) = 10^c$ for some c. Otherwise, g has rational coefficients by Lagrange's interpolation formula (this applies to any polynomial of degree d taking at least $d + 1$ different rational numbers to rational numbers), so $g(0) = t$ is rational. Moreover, g takes each value only finitely many times, so the sequence $g(10^0), g(10^1), \ldots$ includes arbitrarily large powers of 10. Suppose that $t \neq 0$; then we can choose a positive integer h such that the numerator of t is not divisible by 10^h. But for c large enough, $g(10^c) - t$ has numerator divisible by 10^b for some $b > h$, contradiction.

Consequently, $t = 0$, and we may apply the induction hypothesis to $g(n)/n$ to deduce the claim.

Remark: The second solution amounts to the fact that g, being a polynomial with rational coefficients, is continuous for the 2-adic and 5-adic topologies on $\mathbb{Q}$. By contrast, the first solution uses the "∞-adic" topology, i.e., the usual real topology.

A5 In all solutions, let G be a finite group of order m.

First solution: By Lagrange's theorem, if m is not divisible by p, then $n = 0$. Otherwise, let S be the set of p-tuples $(a_0, \dots, a_{p-1}) \in G^p$ such that $a_0 \cdots a_{p-1} = e$; then S has cardinality m^{p-1}, which is divisible by p. Note that this set is invariant under cyclic permutation, that is, if $(a_0, \dots, a_{p-1}) \in S$, then $(a_1, \dots, a_{p-1}, a_0) \in S$ also. The fixed points under this operation are the tuples $(a, \dots, a)$ with $a^p = e$; all other tuples can be grouped into orbits under cyclic permutation, each of which has size p. Consequently, the number of $a \in G$ with $a^p = e$ is divisible by p; since that number is $n + 1$ (only e has order 1), this proves the claim.

Second solution: (by Anand Deopurkar) Assume that $n > 0$, and let H be any subgroup of G of order p. Let S be the set of all elements of $G \setminus H$ of order dividing p, and let H act on G by conjugation. Each orbit has size p except for those which consist of individual elements g which commute with H. For each such g, g and H generate an elementary abelian subgroup of G of order p^2. However, we can group these g into sets of size $p^2 - p$ based on which subgroup they generate together with H. Hence the cardinality of S is divisible by p; adding the $p-1$ nontrivial elements of H gives $n \equiv -1 \pmod p$ as desired.

Third solution: Let S be the set of elements in G having order dividing p, and let H be an elementary abelian p-group of maximal order in G. If $|H| = 1$, then we are done. So assume $|H| = p^k$ for some $k \geq 1$, and let H act on S by conjugation. Let $T \subset S$ denote the set of fixed points of this action. Then the size of every H-orbit on S divides p^k, and so $|S| \equiv |T| \pmod p$. On the other hand, $H \subset T$, and if T contained an element not in H, then that would contradict the maximality of H. It follows that $H = T$, and so $|S| \equiv |T| = |H| = p^k \equiv 0 \pmod p$, i.e., $|S| = n + 1$ is a multiple of p.

Remark: This result is a theorem of Cauchy; the first solution above is due to McKay. A more general (and more difficult) result was proved by Frobenius: for any positive integer m, if G is a finite group of order divisible by m, then the number of elements of G of order dividing m is a multiple of m.

A6 For an admissible triangulation $\mathcal{T}$, number the vertices of P consecutively $v_1, \dots, v_n$, and let a_i be the number of edges in $\mathcal{T}$ emanating from v_i; note that $a_i \geq 2$ for all i.

We first claim that $a_1 + \cdots + a_n \leq 4n - 6$. Let V, E, F denote the number of vertices, edges, and faces in $\mathcal{T}$. By Euler's Formula, $(F+1) - E + V = 2$ (one must add 1 to the face count for the region exterior to P). Each face has three edges, and each edge but the n outside edges belongs to two faces; hence $F = 2E - n$. On the other hand, each edge has two endpoints, and each of the $V - n$ internal vertices is an endpoint of at least 6 edges; hence $a_1 + \cdots + a_n + 6(V - n) \leq 2E$. Combining this inequality with the previous two equations gives

$$\begin{aligned} a_1 + \cdots + a_n &\leq 2E + 6n - 6(1 - F + E) \\ &= 4n - 6, \end{aligned}$$

as claimed.

Now set $A_3 = 1$ and $A_n = A_{n-1} + 2n - 3$ for $n \geq 4$; we will prove by induction on n that $\mathcal{T}$ has at most A_n triangles. For $n = 3$, since $a_1 + a_2 + a_3 = 6$, $a_1 = a_2 = a_3 = 2$ and hence $\mathcal{T}$ consists of just one triangle.

Next assume that an admissible triangulation of an $(n-1)$-gon has at most A_{n-1} triangles, and let $\mathcal{T}$ be an admissible triangulation of an n-gon. If any $a_i = 2$, then we can remove the triangle of $\mathcal{T}$ containing vertex v_i to obtain an admissible triangulation of an $(n-1)$-gon; then the number of triangles in $\mathcal{T}$ is at most $A_{n-1} + 1 < A_n$ by induction. Otherwise, all $a_i \geq 3$. Now the average of $a_1, \dots, a_n$ is less than 4, and thus there are more $a_i = 3$ than $a_i \geq 5$. It follows that there is a sequence of k consecutive vertices in P whose degrees are $3, 4, 4, \dots, 4, 3$ in order, for some k with $2 \leq k \leq n - 1$ (possibly $k = 2$, in which case there are no degree 4 vertices separating the degree 3 vertices). If we remove from $\mathcal{T}$ the $2k - 1$ triangles which contain at least one of these vertices, then we are left with an admissible triangulation of an $(n-1)$-gon. It follows that there are at most $A_{n-1} + 2k - 1 \leq A_{n-1} + 2n - 3 = A_n$ triangles in $\mathcal{T}$. This completes the induction step and the proof.

Remark: We can refine the bound A_n somewhat. Supposing that $a_i \geq 3$ for all i, the fact that $a_1 + \cdots + a_n \leq 4n - 6$ implies that there are at least six more indices i with $a_i = 3$ than with $a_i \geq 5$. Thus there exist six sequences with degrees $3, 4, \dots, 4, 3$, of total length at most $n + 6$. We may thus choose a sequence of length $k \leq \lfloor \frac{n}{6} \rfloor + 1$, so we may improve the upper bound to $A_n = A_{n-1} + 2\lfloor \frac{n}{6} \rfloor + 1$, or asymptotically $\frac{1}{6}n^2$.

However (as noted by Noam Elkies), a hexagonal swatch of a triangular lattice, with the boundary as close to regular as possible, achieves asymptotically $\frac{1}{6}n^2$ triangles.

B1 The problem fails if f is allowed to be constant, e.g., take $f(n) = 1$. We thus assume that f is nonconstant.

Write $f(n) = \sum_{i=0}^d a_i n^i$ with $a_i > 0$. Then

$$\begin{aligned} f(f(n)+1) &= \sum_{i=0}^d a_i (f(n)+1)^i \\ &\equiv f(1) \pmod{f(n)}. \end{aligned}$$

If $n=1$, then this implies that $f(f(n)+1)$ is divisible by $f(n)$. Otherwise, $0 < f(1) < f(n)$ since f is nonconstant and has positive coefficients, so $f(f(n)+1)$ cannot be divisible by $f(n)$.

B2 Put $B = \max_{0 \leq x \leq 1} |f'(x)|$ and $g(x) = \int_0^x f(y)\,dy$. Since $g(0) = g(1) = 0$, the maximum value of $|g(x)|$ must occur at a critical point $y \in (0,1)$ satisfying $g'(y) = f(y) = 0$. We may thus take $\alpha = y$ hereafter.

Since $\int_0^\alpha f(x)\,dx = -\int_0^{1-\alpha} f(1-x)\,dx$, we may assume that $\alpha \leq 1/2$. By then substituting $-f(x)$ for $f(x)$ if needed, we may assume that $\int_0^\alpha f(x)\,dx \geq 0$. From the inequality $f'(x) \geq -B$, we deduce $f(x) \leq B(\alpha - x)$ for $0 \leq x \leq \alpha$, so

$$\begin{aligned} \int_0^\alpha f(x)\,dx &\leq \int_0^\alpha B(\alpha-x)\,dx \\ &= -\left.\frac{1}{2}B(\alpha-x)^2\right|_0^\alpha \\ &= \frac{\alpha^2}{2}B \leq \frac{1}{8}B \end{aligned}$$

as desired.

B3 **First solution:** Observing that $x_2/2 = 13$, $x_3/4 = 34$, $x_4/8 = 89$, we guess that $x_n = 2^{n-1}F_{2n+3}$, where F_k is the k-th Fibonacci number. Thus we claim that $x_n = \frac{2^{n-1}}{\sqrt{5}}(\alpha^{2n+3} - \alpha^{-(2n+3)})$, where $\alpha = \frac{1+\sqrt{5}}{2}$, to make the answer $x_{2007} = \frac{2^{2006}}{\sqrt{5}}(\alpha^{3997} - \alpha^{-3997})$.

We prove the claim by induction; the base case $x_0 = 1$ is true, and so it suffices to show that the recursion $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5}\rfloor$ is satisfied for our formula for x_n. Indeed, since $\alpha^2 = \frac{3+\sqrt{5}}{2}$, we have

$$\begin{aligned} x_{n+1} - (3+\sqrt{5})x_n &= \frac{2^{n-1}}{\sqrt{5}}(2(\alpha^{2n+5} - \alpha^{-(2n+5)}) \\ &\quad - (3+\sqrt{5})(\alpha^{2n+3} - \alpha^{-(2n+3)})) \\ &= 2^n\alpha^{-(2n+3)}. \end{aligned}$$

Now $2^n\alpha^{-(2n+3)} = (\frac{1-\sqrt{5}}{2})^3(3-\sqrt{5})^n$ is between -1 and 0; the recursion follows since x_n, x_{n+1} are integers.

Second solution: (by Catalin Zara) Since x_n is rational, we have $0 < x_n\sqrt{5} - \lfloor x_n\sqrt{5}\rfloor < 1$. We now have the inequalities

$$\begin{aligned} x_{n+1} - 3x_n < x_n\sqrt{5} &< x_{n+1} - 3x_n + 1 \\ (3+\sqrt{5})x_n - 1 < x_{n+1} &< (3+\sqrt{5})x_n \\ 4x_n - (3-\sqrt{5}) < (3-\sqrt{5})x_{n+1} &< 4x_n \\ 3x_{n+1} - 4x_n < x_{n+1}\sqrt{5} &< 3x_{n+1} - 4x_n + (3-\sqrt{5}). \end{aligned}$$

Since $0 < 3-\sqrt{5} < 1$, this yields $\lfloor x_{n+1}\sqrt{5}\rfloor = 3x_{n+1} - 4x_n$, so we can rewrite the recursion as $x_{n+1} = 6x_n - 4x_{n-1}$ for $n \geq 2$. It is routine to solve this recursion to obtain the same solution as above.

Remark: With an initial 1 prepended, this becomes sequence A018903 in Sloane's On-Line Encyclopedia of Integer Sequences: (`http://www.research.att.com/~njas/sequences/`). Therein, the sequence is described as the case $S(1,5)$ of the sequence $S(a_0,a_1)$ in which a_{n+2} is the least integer for which $a_{n+2}/a_{n+1} > a_{n+1}/a_n$. Sloane cites D. W. Boyd, Linear recurrence relations for some generalized Pisot sequences, *Advances in Number Theory (Kingston, ON, 1991)*, Oxford Univ. Press, New York, 1993, p. 333–340.

B4 The number of pairs is 2^{n+1}. The degree condition forces P to have degree n and leading coefficient ± 1; we may count pairs in which P has leading coefficient 1 as long as we multiply by 2 afterward.

Factor both sides:

$$\begin{aligned} &(P(X)+Q(X)i)(P(X)-Q(X)i) \\ &= \prod_{j=0}^{n-1}(X - \exp(2\pi i(2j+1)/(4n))) \\ &\quad \cdot \prod_{j=0}^{n-1}(X + \exp(2\pi i(2j+1)/(4n))). \end{aligned}$$

Then each choice of P,Q corresponds to equating $P(X)+Q(X)i$ with the product of some n factors on the right, in which we choose exactly of the two factors for each $j = 0,\dots,n-1$. (We must take exactly n factors because as a polynomial in X with complex coefficients, $P(X)+Q(X)i$ has degree exactly n. We must choose one for each j to ensure that $P(X)+Q(X)i$ and $P(X)-Q(X)i$ are complex conjugates, so that P,Q have real coefficients.) Thus there are 2^n such pairs; multiplying by 2 to allow P to have leading coefficient -1 yields the desired result.

Remark: If we allow P and Q to have complex coefficients but still require $\deg(P) > \deg(Q)$, then the number of pairs increases to $2\binom{2n}{n}$, as we may choose any n of the $2n$ factors of $X^{2n}+1$ to use to form $P(X)+Q(X)i$.

B5 For n an integer, we have $\left\lfloor \frac{n}{k} \right\rfloor = \frac{n-j}{k}$ for j the unique integer in $\{0,\dots,k-1\}$ congruent to n modulo k; hence

$$\prod_{j=0}^{k-1}\left(\left\lfloor \frac{n}{k}\right\rfloor - \frac{n-j}{k}\right) = 0.$$

By expanding this out, we obtain the desired polynomials $P_0(n), \dots, P_{k-1}(n)$.

Remark: Variants of this solution are possible that construct the P_i less explicitly, using Lagrange interpolation or Vandermonde determinants.

B6 (Suggested by Oleg Golberg) Assume $n \geq 2$, or else the problem is trivially false. Throughout this proof, any C_i will be a positive constant whose exact value is immaterial. As in the proof of Stirling's approximation, we estimate for any fixed $c \in \mathbb{R}$,

$$\sum_{i=1}^{n} (i+c)\log i = \frac{1}{2}n^2 \log n - \frac{1}{4}n^2 + O(n \log n)$$

by comparing the sum to an integral. This gives

$$\begin{aligned} n^{n^2/2 - C_1 n} e^{-n^2/4} &\leq 1^{1+c} 2^{2+c} \cdots n^{n+c} \\ &\leq n^{n^2/2 + C_2 n} e^{-n^2/4}. \end{aligned}$$

We now interpret $f(n)$ as counting the number of n-tuples $(a_1, \dots, a_n)$ of nonnegative integers such that

$$a_1 1! + \cdots + a_n n! = n!.$$

For an upper bound on $f(n)$, we use the inequalities $0 \leq a_i \leq n!/i!$ to deduce that there are at most $n!/i! + 1 \leq 2(n!/i!)$ choices for a_i. Hence

$$\begin{aligned} f(n) &\leq 2^n \frac{n!}{1!} \cdots \frac{n!}{n!} \\ &= 2^n 2^1 3^2 \cdots n^{n-1} \\ &\leq n^{n^2/2 + C_3 n} e^{-n^2/4}. \end{aligned}$$

For a lower bound on $f(n)$, we note that if $0 \leq a_i < (n-1)!/i!$ for $i = 2, \dots, n-1$ and $a_n = 0$, then $0 \leq a_2 2! + \cdots + a_n n! \leq n!$, so there is a unique choice of a_1 to complete this to a solution of $a_1 1! + \cdots + a_n n! = n!$. Hence

$$\begin{aligned} f(n) &\geq \frac{(n-1)!}{2!} \cdots \frac{(n-1)!}{(n-1)!} \\ &= 3^1 4^2 \cdots (n-1)^{n-3} \\ &\geq n^{n^2/2 + C_4 n} e^{-n^2/4}. \end{aligned}$$

Solutions to the 69th William Lowell Putnam Mathematical Competition
Saturday, December 6, 2008

Kiran Kedlaya and Lenny Ng

A1 The function $g(x) = f(x,0)$ works. Substituting $(x,y,z) = (0,0,0)$ into the given functional equation yields $f(0,0) = 0$, whence substituting $(x,y,z) = (x,0,0)$ yields $f(x,0)+f(0,x) = 0$. Finally, substituting $(x,y,z) = (x,y,0)$ yields $f(x,y) = -f(y,0) - f(0,x) = g(x) - g(y)$.

Remark: A similar argument shows that the possible functions g are precisely those of the form $f(x,0)+c$ for some c.

Reinterpretation: The first singular cohomology of the space $\mathbb{R}^2$ with coefficients in $\mathbb{R}$ is trivial.

A2 Barbara wins using one of the following strategies.

First solution: Pair each entry of the first row with the entry directly below it in the second row. If Alan ever writes a number in one of the first two rows, Barbara writes the same number in the other entry in the pair. If Alan writes a number anywhere other than the first two rows, Barbara does likewise. At the end, the resulting matrix will have two identical rows, so its determinant will be zero.

Second solution: (by Manjul Bhargava) Whenever Alan writes a number x in an entry in some row, Barbara writes $-x$ in some other entry in the same row. At the end, the resulting matrix will have all rows summing to zero, so it cannot have full rank.

A3 We first prove that the process stops. Note first that the product $a_1 \cdots a_n$ remains constant, because $a_j a_k = \gcd(a_j,a_k)\operatorname{lcm}(a_j,a_k)$. Moreover, the last number in the sequence can never decrease, because it is always replaced by its least common multiple with another number. Since it is bounded above (by the product of all of the numbers), the last number must eventually reach its maximum value, after which it remains constant throughout. After this happens, the next-to-last number will never decrease, so it eventually becomes constant, and so on. After finitely many steps, all of the numbers will achieve their final values, so no more steps will be possible. This only happens when a_j divides a_k for all pairs $j < k$.

We next check that there is only one possible final sequence. For p a prime and m a nonnegative integer, we claim that the number of integers in the list divisible by p^m never changes. To see this, suppose we replace a_j, a_k by $\gcd(a_j,a_k), \operatorname{lcm}(a_j,a_k)$. If neither of a_j,a_k is divisible by p^m, then neither of $\gcd(a_j,a_k), \operatorname{lcm}(a_j,a_k)$ is either. If exactly one a_j,a_k is divisible by p^m, then $\operatorname{lcm}(a_j,a_k)$ is divisible by p^m but $\gcd(a_j,a_k)$ is not. If both are, then $\gcd(a_j,a_k), \operatorname{lcm}(a_j,a_k)$ are as well.

If we started out with exactly h numbers not divisible by p^m, then in the final sequence $a'_1,\dots,a'_n$, the numbers $a'_{h+1},\dots,a'_n$ are divisible by p^m while the numbers $a'_1,\dots,a'_h$ are not. Repeating this argument for each pair (p,m) such that p^m divides the initial product $a_1,\dots,a_n$, we can determine the exact prime factorization of each of $a'_1,\dots,a'_n$. This proves that the final sequence is unique.

Remark: (by David Savitt and Noam Elkies) Here are two other ways to prove the termination. One is to observe that $\prod_j a_j^j$ is *strictly* increasing at each step, and bounded above by $(a_1\cdots a_n)^n$. The other is to notice that a_1 is nonincreasing but always positive, so eventually becomes constant; then a_2 is nonincreasing but always positive, and so on.

Reinterpretation: For each p, consider the sequence consisting of the exponents of p in the prime factorizations of $a_1,\dots,a_n$. At each step, we pick two positions i and j such that the exponents of some prime p are in the wrong order at positions i and j. We then sort these two position into the correct order for every prime p simultaneously.

It is clear that this can only terminate with all sequences being sorted into the correct order. We must still check that the process terminates; however, since all but finitely many of the exponent sequences consist of all zeroes, and each step makes a nontrivial switch in at least one of the other exponent sequences, it is enough to check the case of a single exponent sequence. This can be done as in the first solution.

Remark: Abhinav Kumar suggests the following proof that the process always terminates in at most $\binom{n}{2}$ steps. (This is a variant of the worst-case analysis of the *bubble sort* algorithm.)

Consider the number of pairs (k,l) with $1 \leq k < l \leq n$ such that a_k does not divide a_l (call these *bad pairs*). At each step, we find one bad pair (i,j) and eliminate it, and we do not touch any pairs that do not involve either i or j. If $i < k < j$, then neither of the pairs (i,k) and (k,j) can become bad, because a_i is replaced by a divisor of itself, while a_j is replaced by a multiple of itself. If $k < i$, then (k,i) can only become a bad pair if a_k divided a_i but not a_j, in which case (k,j) stops being bad. Similarly, if $k > j$, then (i,k) and (j,k) either stay the same or switch status. Hence the number of bad pairs goes down by at least 1 each time; since it is at most $\binom{n}{2}$ to begin with, this is an upper bound for the number of steps.

Remark: This problem is closely related to the classification theorem for finite abelian groups. Namely, if $a_1, \dots, a_n$ and $a'_1, \dots, a'_n$ are the sequences obtained at two different steps in the process, then the abelian groups $\mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_n\mathbb{Z}$ and $\mathbb{Z}/a'_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a'_n\mathbb{Z}$ are isomorphic. The final sequence gives a canonical presentation of this group; the terms of this sequence are called the *elementary divisors* or *invariant factors* of the group.

Remark: (by Tom Belulovich) A *lattice* is a partially ordered set L in which for any two $x, y \in L$, there is a unique minimal element z with $z \geq x$ and $z \geq y$, called the *join* and denoted $x \wedge y$, and there is a unique maximal element z with $z \leq x$ and $z \leq y$, called the *meet* and denoted $x \vee y$. In terms of a lattice L, one can pose the following generalization of the given problem. Start with $a_1, \dots, a_n \in L$. If $i < j$ but $a_i \not\leq a_j$, it is permitted to replace a_i, a_j by $a_i \vee a_j, a_i \wedge a_j$, respectively. The same argument as above shows that this always terminates in at most $\binom{n}{2}$ steps. The question is, under what conditions on the lattice L is the final sequence uniquely determined by the initial sequence?

It turns out that this holds if and only if L is *distributive*, i.e., for any $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

(This is equivalent to the same axiom with the operations interchanged.) For example, if L is a *Boolean algebra*, i.e., the set of subsets of a given set S under inclusion, then $\wedge$ is union, $\vee$ is intersection, and the distributive law holds. Conversely, any finite distributive lattice is contained in a Boolean algebra by a theorem of Birkhoff. The correspondence takes each $x \in L$ to the set of $y \in L$ such that $x \geq y$ and y cannot be written as a join of two elements of $L \setminus \{y\}$. (See for instance Birkhoff, *Lattice Theory*, Amer. Math. Soc., 1967.)

On one hand, if L is distributive, it can be shown that the j-th term of the final sequence is equal to the meet of $a_{i_1} \wedge \cdots \wedge a_{i_j}$ over all sequences $1 \leq i_1 < \cdots < i_j \leq n$. For instance, this can be checked by forming the smallest subset L' of L containing $a_1, \dots, a_n$ and closed under meet and join, then embedding L' into a Boolean algebra using Birkhoff's theorem, then checking the claim for all Boolean algebras. It can also be checked directly (as suggested by Nghi Nguyen) by showing that for $j = 1, \dots, n$, the meet of all joins of j-element subsets of $a_1, \dots, a_n$ is invariant at each step.

On the other hand, a lattice fails to be distributive if and only if it contains five elements $a, b, c, 0, 1$ such that either the only relations among them are implied by

$$1 \geq a, b, c \geq 0$$

(this lattice is sometimes called the *diamond*), or the only relations among them are implied by

$$1 \geq a \geq b \geq 0, \qquad 1 \geq c \geq 0$$

(this lattice is sometimes called the *pentagon*). (For a proof, see the Birkhoff reference given above.) For each of these examples, the initial sequence a, b, c fails to determine the final sequence; for the diamond, we can end up with $0, *, 1$ for any of $* = a, b, c$, whereas for the pentagon we can end up with $0, *, 1$ for any of $* = a, b$.

Consequently, the final sequence is determined by the initial sequence if and only if L is distributive.

A4 The sum diverges. From the definition, $f(x) = x$ on $[1, e]$, $x \ln x$ on $(e, e^e]$, $x \ln x \ln \ln x$ on $(e^e, e^{e^e}]$, and so forth. It follows that on $[1, \infty)$, f is positive, continuous, and increasing. Thus $\sum_{n=1}^\infty \frac{1}{f(n)}$, if it converges, is bounded below by $\int_1^\infty \frac{dx}{f(x)}$; it suffices to prove that the integral diverges.

Write $\ln^1 x = \ln x$ and $\ln^k x = \ln(\ln^{k-1} x)$ for $k \geq 2$; similarly write $\exp^1 x = e^x$ and $\exp^k x = e^{\exp^{k-1} x}$. If we write $y = \ln^k x$, then $x = \exp^k y$ and $dx = (\exp^k y)(\exp^{k-1} y) \cdots (\exp^1 y) dy = x(\ln^1 x) \cdots (\ln^{k-1} x) dy$. Now on $[\exp^{k-1} 1, \exp^k 1]$, we have $f(x) = x(\ln^1 x) \cdots (\ln^{k-1} x)$, and thus substituting $y = \ln^k x$ yields

$$\int_{\exp^{k-1} 1}^{\exp^k 1} \frac{dx}{f(x)} = \int_0^1 dy = 1.$$

It follows that $\int_1^\infty \frac{dx}{f(x)} = \sum_{k=1}^\infty \int_{\exp^{k-1} 1}^{\exp^k 1} \frac{dx}{f(x)}$ diverges, as desired.

A5 Form the polynomial $P(z) = f(z) + ig(z)$ with complex coefficients. It suffices to prove that P has degree at least $n-1$, as then one of f, g must have degree at least $n-1$.

By replacing $P(z)$ with $aP(z) + b$ for suitable $a, b \in \mathbb{C}$, we can force the regular n-gon to have vertices $\zeta_n, \zeta_n^2, \dots, \zeta_n^n$ for $\zeta_n = \exp(2\pi i/n)$. It thus suffices to check that there cannot exist a polynomial $P(z)$ of degree at most $n-2$ such that $P(i) = \zeta_n^i$ for $i = 1, \dots, n$.

We will prove more generally that for any complex number $t \notin \{0, 1\}$, and any integer $m \geq 1$, any polynomial $Q(z)$ for which $Q(i) = t^i$ for $i = 1, \dots, m$ has degree at least $m-1$. There are several ways to do this.

First solution: If $Q(z)$ has degree d and leading coefficient c, then $R(z) = Q(z+1) - tQ(z)$ has degree d and leading coefficient $(1-t)c$. However, by hypothesis, $R(z)$ has the distinct roots $1, 2, \dots, m-1$, so we must have $d \geq m-1$.

Second solution: We proceed by induction on m. For the base case $m = 1$, we have $Q(1) = t^1 \neq 0$, so Q must be nonzero, and so its degree is at least 0. Given the assertion for $m-1$, if $Q(i) = t^i$ for $i = 1, \dots, m$, then the polynomial $R(z) = (t-1)^{-1}(Q(z+1) - Q(z))$ has degree one less than that of Q, and satisfies $R(i) = t^i$ for $i = 1, \dots, m-1$. Since R must have degree at least $m-2$ by the induction hypothesis, Q must have degree at least $m-1$.

Third solution: We use the method of *finite differences* (as in the second solution) but without induction. Namely, the $(m-1)$-st finite difference of P evaluated at 1 equals

$$\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}Q(m-j) = t(1-t)^{m-1} \neq 0,$$

which is impossible if Q has degree less than $m-1$.

Remark: One can also establish the claim by computing a Vandermonde-type determinant, or by using the Lagrange interpolation formula to compute the leading coefficient of Q.

A6 For notational convenience, we will interpret the problem as allowing the empty subsequence, whose product is the identity element of the group. To solve the problem in the interpretation where the empty subsequence is not allowed, simply append the identity element to the sequence given by one of the following solutions.

First solution: Put $n = |G|$. We will say that a sequence S *produces* an element $g \in G$ if g occurs as the product of some subsequence of S. Let H be the set of elements produced by the sequence S.

Start with S equal to the empty sequence. If at any point the set $H^{-1}H = \{h_1h_2 : h_1^{-1}, h_2 \in H\}$ fails to be all of G, extend S by appending an element g of G not in $H^{-1}H$. Then $Hg \cap H$ must be empty, otherwise there would be an equation of the form $h_1 g = h_2$ with $h_1, h_2 \in G$, or $g = h_1^{-1}h_2$, a contradiction. Thus we can extend S by one element and double the size of H.

After $k \leq \log_2 n$ steps, we must obtain a sequence $S = a_1, \dots, a_k$ for which $H^{-1}H = G$. Then the sequence $a_k^{-1}, \dots, a_1^{-1}, a_1, \dots, a_k$ produces all of G and has length at most $(2/\ln 2)\ln n$.

Second solution:

Put $m = |H|$. We will show that we can append one element g to S so that the resulting sequence of $k+1$ elements will produce at least $2m - m^2/n$ elements of G. To see this, we compute

$$\begin{aligned}
\sum_{g\in G}|H \cup Hg| &= \sum_{g\in G}(|H| + |Hg| - |H \cap Hg|) \\
&= 2mn - \sum_{g\in G}|H \cap Hg| \\
&= 2mn - |\{(g,h) \in G^2 : h \in H \cap Hg\}| \\
&= 2mn - \sum_{h\in H}|\{g \in G : h \in Hg\}| \\
&= 2mn - \sum_{h\in H}|H^{-1}h| \\
&= 2mn - m^2.
\end{aligned}$$

By the pigeonhole principle, we have $|H \cup Hg| \geq 2m - m^2/n$ for some choice of g, as claimed.

In other words, by extending the sequence by one element, we can replace the ratio $s = 1 - m/n$ (i.e., the fraction of elements of G not generated by S) by a quantity no greater than

$$1 - (2m - m^2/n)/n = s^2.$$

We start out with $k = 0$ and $s = 1 - 1/n$; after k steps, we have $s \leq (1 - 1/n)^{2^k}$. It is enough to prove that for some $c > 0$, we can always find an integer $k \leq c\ln n$ such that

$$\left(1 - \frac{1}{n}\right)^{2^k} < \frac{1}{n},$$

as then we have $n - m < 1$ and hence $H = G$.

To obtain this last inequality, put

$$k = \lfloor 2\log_2 n \rfloor < (2/\ln 2)\ln n,$$

so that $2^{k+1} \geq n^2$. From the facts that $\ln n \leq \ln 2 + (n-2)/2 \leq n/2$ and $\ln(1 - 1/n) < -1/n$ for all $n \geq 2$, we have

$$2^k \ln\left(1 - \frac{1}{n}\right) < -\frac{n^2}{2n} = -\frac{n}{2} < -\ln n,$$

yielding the desired inequality.

Remark: An alternate approach in the second solution is to distinguish betwen the cases of H small (i.e., $m < n^{1/2}$, in which case m can be replaced by a value no less than $2m - 1$) and H large. This strategy is used in a number of recent results of Bourgain, Tao, Helfgott, and others on *small doubling* or *small tripling* of subsets of finite groups.

In the second solution, if we avoid the rather weak inequality $\ln n \leq n/2$, we instead get sequences of length $\log_2(n \ln n) = \log_2(n) + \log_2(\ln n)$. This is close to optimal: one cannot use fewer than $\log_2 n$ terms because the number of subsequences must be at least n.

B1 There are at most two such points. For example, the points $(0,0)$ and $(1,0)$ lie on a circle with center $(1/2, x)$ for any real number x, not necessarily rational.

On the other hand, suppose $P = (a,b), Q = (c,d), R = (e,f)$ are three rational points that lie on a circle. The midpoint M of the side PQ is $((a+c)/2, (b+d)/2)$, which is again rational. Moreover, the slope of the line PQ is $(d-b)/(c-a)$, so the slope of the line through M perpendicular to PQ is $(a-c)/(b-d)$, which is rational or infinite.

Similarly, if N is the midpoint of QR, then N is a rational point and the line through N perpendicular to QR has rational slope. The center of the circle lies on both of these lines, so its coordinates (g,h) satisfy two linear equations with rational coefficients, say $Ag + Bh = C$

and $Dg + Eh = F$. Moreover, these equations have a unique solution. That solution must then be

$$g = (CE - BD)/(AE - BD)$$
$$h = (AF - BC)/(AE - BD)$$

(by elementary algebra, or Cramer's rule), so the center of the circle is rational. This proves the desired result.

Remark: The above solution is deliberately more verbose than is really necessary. A shorter way to say this is that any two distinct rational points determine a *rational line* (a line of the form $ax + by + c = 0$ with a, b, c rational), while any two nonparallel rational lines intersect at a rational point. A similar statement holds with the rational numbers replaced by any field.

Remark: A more explicit argument is to show that the equation of the circle through the rational points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is

$$0 = \det \begin{pmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x^2 + y^2 & x & y & 1 \end{pmatrix}$$

which has the form $a(x^2 + y^2) + dx + ey + f = 0$ for a, d, e, f rational. The center of this circle is $(-d/(2a), -e/(2a))$, which is again a rational point.

B2 We claim that $F_n(x) = (\ln x - a_n)x^n/n!$, where $a_n = \sum_{k=1}^n 1/k$. Indeed, temporarily write $G_n(x) = (\ln x - a_n)x^n/n!$ for $x > 0$ and $n \geq 1$; then $\lim_{x\to 0} G_n(x) = 0$ and $G_n'(x) = (\ln x - a_n + 1/n)x^{n-1}/(n-1)! = G_{n-1}(x)$, and the claim follows by the Fundamental Theorem of Calculus and induction on n.

Given the claim, we have $F_n(1) = -a_n/n!$ and so we need to evaluate $-\lim_{n\to\infty} \frac{a_n}{\ln n}$. But since the function $1/x$ is strictly decreasing for x positive, $\sum_{k=2}^n 1/k = a_n - 1$ is bounded below by $\int_2^n dx/x = \ln n - \ln 2$ and above by $\int_1^n dx/x = \ln n$. It follows that $\lim_{n\to\infty} \frac{a_n}{\ln n} = 1$, and the desired limit is -1.

B3 The largest possible radius is $\frac{\sqrt{2}}{2}$. It will be convenient to solve the problem for a hypercube of side length 2 instead, in which case we are trying to show that the largest radius is $\sqrt{2}$.

Choose coordinates so that the interior of the hypercube is the set $H = [-1, 1]^4$ in $\mathbb{R}^4$. Let C be a circle centered at the point P. Then C is contained both in H and its reflection across P; these intersect in a rectangular paralellepiped each of whose pairs of opposite faces are at most 2 unit apart. Consequently, if we translate C so that its center moves to the point $O = (0, 0, 0, 0)$ at the center of H, then it remains entirely inside H.

This means that the answer we seek equals the largest possible radius of a circle C contained in H *and centered at O*. Let $v_1 = (v_{11}, \dots, v_{14})$ and $v_2 = (v_{21}, \dots, v_{24})$ be two points on C lying on radii perpendicular to each other. Then the points of the circle can be expressed as $v_1 \cos\theta + v_2 \sin\theta$ for $0 \leq \theta < 2\pi$. Then C lies in H if and only if for each i, we have

$$|v_{1i} \cos\theta + v_{2i} \sin\theta| \leq 1 \qquad (0 \leq \theta < 2\pi).$$

In geometric terms, the vector (v_{1i}, v_{2i}) in $\mathbb{R}^2$ has dot product at most 1 with every unit vector. Since this holds for the unit vector in the same direction as (v_{1i}, v_{2i}), we must have

$$v_{1i}^2 + v_{2i}^2 \leq 1 \qquad (i = 1, \dots, 4).$$

Conversely, if this holds, then the Cauchy-Schwarz inequality and the above analysis imply that C lies in H.

If r is the radius of C, then

$$\begin{aligned} 2r^2 &= \sum_{i=1}^4 v_{1i}^2 + \sum_{i=1}^4 v_{2i}^2 \\ &= \sum_{i=1}^4 (v_{1i}^2 + v_{2i}^2) \\ &\leq 4, \end{aligned}$$

so $r \leq \sqrt{2}$. Since this is achieved by the circle through $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$, it is the desired maximum.

Remark: One may similarly ask for the radius of the largest k-dimensional ball inside an n-dimensional unit hypercube; the given problem is the case $(n, k) = (4, 2)$. Daniel Kane gives the following argument to show that the maximum radius in this case is $\frac{1}{2}\sqrt{\frac{n}{k}}$. (Thanks for Noam Elkies for passing this along.)

We again scale up by a factor of 2, so that we are trying to show that the maximum radius r of a k-dimensional ball contained in the hypercube $[-1, 1]^n$ is $\sqrt{\frac{n}{k}}$. Again, there is no loss of generality in centering the ball at the origin. Let $T : \mathbb{R}^k \to \mathbb{R}^n$ be a similitude carrying the unit ball to this embedded k-ball. Then there exists a vector $v_i \in \mathbb{R}^k$ such that for $e_1, \dots, e_n$ the standard basis of $\mathbb{R}^n$, $x \cdot v_i = T(x) \cdot e_i$ for all $x \in \mathbb{R}^k$. The condition of the problem is equivalent to requiring $|v_i| \leq 1$ for all i, while the radius r of the embedded ball is determined by the fact that for all $x \in \mathbb{R}^k$,

$$r^2(x \cdot x) = T(x) \cdot T(x) = \sum_{i=1}^n x \cdot v_i.$$

Let M be the matrix with columns $v_1, \dots, v_k$; then $MM^T = r^2 I_k$, for I_k the $k \times k$ identity matrix. We then have

$$\begin{aligned} kr^2 &= \mathrm{Trace}(r^2 I_k) = \mathrm{Trace}(MM^T) \\ &= \mathrm{Trace}(M^T M) = \sum_{i=1}^n |v_i|^2 \\ &\leq n, \end{aligned}$$

yielding the upper bound $r \leq \sqrt{\frac{n}{k}}$.

To show that this bound is optimal, it is enough to show that one can find an orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^k$ so that the projections of the e_i all have the same norm (one can then rescale to get the desired configuration of $v_1, \dots, v_n$). We construct such a configuration by a "smoothing" argument. Startw with any projection. Let $w_1, \dots, w_n$ be the projections of $e_1, \dots, e_n$. If the desired condition is not achieved, we can choose i, j such that

$$|w_i|^2 < \frac{1}{n}(|w_1|^2 + \cdots + |w_n|^2) < |w_j|^2.$$

By precomposing with a suitable rotation that fixes e_h for $h \neq i, j$, we can vary $|w_i|, |w_j|$ without varying $|w_i|^2 + |w_j|^2$ or $|w_h|$ for $h \neq i, j$. We can thus choose such a rotation to force one of $|w_i|^2, |w_j|^2$ to become equal to $\frac{1}{n}(|w_1|^2 + \cdots + |w_n|^2)$. Repeating at most $n-1$ times gives the desired configuration.

B4 We use the identity given by Taylor's theorem:

$$h(x+y) = \sum_{i=0}^{\deg(h)} \frac{h^{(i)}(x)}{i!} y^i.$$

In this expression, $h^{(i)}(x)/i!$ is a polynomial in x with integer coefficients, so its value at an integer x is an integer.

For $x = 0, \dots, p-1$, we deduce that

$$h(x+p) \equiv h(x) + ph'(x) \pmod{p^2}.$$

(This can also be deduced more directly using the binomial theorem.) Since we assumed $h(x)$ and $h(x+p)$ are distinct modulo p^2, we conclude that $h'(x) \not\equiv 0 \pmod{p}$. Since h' is a polynomial with integer coefficients, we have $h'(x) \equiv h'(x+mp) \pmod{p}$ for any integer m, and so $h'(x) \not\equiv 0 \pmod{p}$ for *all* integers x.

Now for $x = 0, \dots, p^2-1$ and $y = 0, \dots, p-1$, we write

$$h(x+yp^2) \equiv h(x) + p^2 y h'(x) \pmod{p^3}.$$

Thus $h(x), h(x+p^2), \dots, h(x+(p-1)p^2)$ run over all of the residue classes modulo p^3 congruent to $h(x)$ modulo p^2. Since the $h(x)$ themselves cover all the residue classes modulo p^2, this proves that $h(0), \dots, h(p^3-1)$ are distinct modulo p^3.

Remark: More generally, the same proof shows that for any integers $d, e > 1$, h permutes the residue classes modulo p^d if and only if it permutes the residue classes modulo p^e. The argument used in the proof is related to a general result in number theory known as *Hensel's lemma*.

B5 The functions $f(x) = x+n$ and $f(x) = -x+n$ for any integer n clearly satisfy the condition of the problem; we claim that these are the only possible f.

Let $q = a/b$ be any rational number with $\gcd(a,b) = 1$ and $b > 0$. For n any positive integer, we have

$$\frac{f(\frac{an+1}{bn}) - f(\frac{a}{b})}{\frac{1}{bn}} = bnf\left(\frac{an+1}{bn}\right) - nbf\left(\frac{a}{b}\right)$$

is an integer by the property of f. Since f is differentiable at a/b, the left hand side has a limit. It follows that for sufficiently large n, both sides must be equal to some integer $c = f'(\frac{a}{b})$: $f(\frac{an+1}{bn}) = f(\frac{a}{b}) + \frac{c}{bn}$. Now c cannot be 0, since otherwise $f(\frac{an+1}{bn}) = f(\frac{a}{b})$ for sufficiently large n has denominator b rather than bn. Similarly, $|c|$ cannot be greater than 1: otherwise if we take $n = k|c|$ for k a sufficiently large positive integer, then $f(\frac{a}{b}) + \frac{c}{bn}$ has denominator bk, contradicting the fact that $f(\frac{an+1}{bn})$ has denominator bn. It follows that $c = f'(\frac{a}{b}) = \pm 1$.

Thus the derivative of f at any rational number is ± 1. Since f is continuously differentiable, we conclude that $f'(x) = 1$ for all real x or $f'(x) = -1$ for all real x. Since $f(0)$ must be an integer (a rational number with denominator 1), $f(x) = x+n$ or $f(x) = -x+n$ for some integer n.

Remark: After showing that $f'(q)$ is an integer for each q, one can instead argue that f' is a continuous function from the rationals to the integers, so must be constant. One can then write $f(x) = ax+b$ and check that $b \in \mathbb{Z}$ by evaluation at $a = 0$, and that $a = \pm 1$ by evaluation at $x = 1/a$.

B6 In all solutions, let $F_{n,k}$ be the number of k-limited permutations of $\{1, \dots, n\}$.

First solution: (by Jacob Tsimerman) Note that any permutation is k-limited if and only if its inverse is k-limited. Consequently, the number of k-limited permutations of $\{1, \dots, n\}$ is the same as the number of k-limited involutions (permutations equal to their inverses) of $\{1, \dots, n\}$.

We use the following fact several times: the number of involutions of $\{1, \dots, n\}$ is odd if $n = 0, 1$ and even otherwise. This follows from the fact that non-involutions come in pairs, so the number of involutions has the same parity as the number of permutations, namely $n!$.

For $n \leq k+1$, all involutions are k-limited. By the previous paragraph, $F_{n,k}$ is odd for $n = 0, 1$ and even for $n = 2, \dots, k+1$.

For $n > k+1$, group the k-limited involutions into classes based on their actions on $k+2, \dots, n$. Note that for C a class and $\sigma \in C$, the set of elements of $A = \{1, \dots, k+1\}$ which map into A under σ depends only on C, not on σ. Call this set $S(C)$; then the size of C is exactly the number of involutions of $S(C)$. Consequently, $|C|$ is even unless $S(C)$ has at most one element. However, the element 1 cannot map out of A because we are looking at k-limited involutions. Hence if $S(C)$ has one element and $\sigma \in C$, we must have

$\sigma(1) = 1$. Since σ is k-limited and $\sigma(2)$ cannot belong to A, we must have $\sigma(2) = k + 2$. By induction, for $i = 3, \ldots, k+1$, we must have $\sigma(i) = k + i$.

If $n < 2k + 1$, this shows that no class C of odd cardinality can exist, so $F_{n,k}$ must be even. If $n \geq 2k+1$, the classes of odd cardinality are in bijection with k-limited involutions of $\{2k+2, \ldots, n\}$, so $F_{n,k}$ has the same parity as $F_{n-2k-1,k}$. By induction on n, we deduce the desired result.

Second solution: (by Yufei Zhao) Let $M_{n,k}$ be the $n \times n$ matrix with

$$(M_{n,k})_{ij} = \begin{cases} 1 & |i-j| \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Write $\det(M_{n,k})$ as the sum over permutations σ of $\{1, \ldots, n\}$ of $(M_{n,k})_{1\sigma(1)} \cdots (M_{n,k})_{n\sigma(n)}$ times the signature of σ. Then σ contributes ± 1 to $\det(M_{n,k})$ if σ is k-limited and 0 otherwise. We conclude that

$$\det(M_{n,k}) \equiv F_{n,k} \pmod 2.$$

For the rest of the solution, we interpret $M_{n,k}$ as a matrix over the field of two elements. We compute its determinant using linear algebra modulo 2.

We first show that for $n \geq 2k+1$,

$$F_{n,k} \equiv F_{n-2k-1,k} \pmod 2,$$

provided that we interpret $F_{0,k} = 1$. We do this by computing $\det(M_{n,k})$ using row and column operations. We will verbally describe these operations for general k, while illustrating with the example $k = 3$.

To begin with, $M_{n,k}$ has the following form.

$$\left(\begin{array}{ccccccc|c}
1&1&1&1&0&0&0&\emptyset\\
1&1&1&1&1&0&0&\emptyset\\
1&1&1&1&1&1&0&\emptyset\\
1&1&1&1&1&1&1&\emptyset\\
0&1&1&1&1&1&1&?\\
0&0&1&1&1&1&1&?\\
0&0&0&1&1&1&1&?\\
\hline
\emptyset&\emptyset&\emptyset&\emptyset&?&?&?&*
\end{array}\right)$$

In this presentation, the first $2k+1$ rows and columns are shown explicitly; the remaining rows and columns are shown in a compressed format. The symbol $\emptyset$ indicates that the unseen entries are all zeroes, while the symbol $?$ indicates that they are not. The symbol $*$ in the lower right corner represents the matrix $F_{n-2k-1,k}$. We will preserve the unseen structure of the matrix by only adding the first $k+1$ rows or columns to any of the others.

We first add row 1 to each of rows $2, \ldots, k+1$.

$$\left(\begin{array}{ccccccc|c}
1&1&1&1&0&0&0&\emptyset\\
0&0&0&0&1&0&0&\emptyset\\
0&0&0&0&1&1&0&\emptyset\\
0&0&0&0&1&1&1&\emptyset\\
0&1&1&1&1&1&1&?\\
0&0&1&1&1&1&1&?\\
0&0&0&1&1&1&1&?\\
\hline
\emptyset&\emptyset&\emptyset&\emptyset&?&?&?&*
\end{array}\right)$$

We next add column 1 to each of columns $2, \ldots, k+1$.

$$\left(\begin{array}{ccccccc|c}
1&0&0&0&0&0&0&\emptyset\\
0&0&0&0&1&0&0&\emptyset\\
0&0&0&0&1&1&0&\emptyset\\
0&0&0&0&1&1&1&\emptyset\\
0&1&1&1&1&1&1&?\\
0&0&1&1&1&1&1&?\\
0&0&0&1&1&1&1&?\\
\hline
\emptyset&\emptyset&\emptyset&\emptyset&?&?&?&*
\end{array}\right)$$

For $i = 2$, for each of $j = i+1, \ldots, 2k+1$ for which the $(j, k+i)$-entry is nonzero, add row i to row j.

$$\left(\begin{array}{ccccccc|c}
1&0&0&0&0&0&0&\emptyset\\
0&0&0&0&1&0&0&\emptyset\\
0&0&0&0&0&1&0&\emptyset\\
0&0&0&0&0&1&1&\emptyset\\
0&1&1&1&0&1&1&?\\
0&0&1&1&0&1&1&?\\
0&0&0&1&0&1&1&?\\
\hline
\emptyset&\emptyset&\emptyset&\emptyset&\emptyset&?&?&*
\end{array}\right)$$

Repeat the previous step for $i = 3, \ldots, k+1$ in succession.

$$\left(\begin{array}{ccccccc|c}
1&0&0&0&0&0&0&\emptyset\\
0&0&0&0&1&0&0&\emptyset\\
0&0&0&0&0&1&0&\emptyset\\
0&0&0&0&0&0&1&\emptyset\\
0&1&1&1&0&0&0&?\\
0&0&1&1&0&0&0&?\\
0&0&0&1&0&0&0&?\\
\hline
\emptyset&\emptyset&\emptyset&\emptyset&\emptyset&\emptyset&\emptyset&*
\end{array}\right)$$

Repeat the two previous steps with the roles of the rows and columns reversed. That is, for $i = 2, \ldots, k+1$, for each of $j = i+1, \ldots, 2k+1$ for which the $(j, k+i)$-

entry is nonzero, add row i to row j.

$$\left(\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \emptyset \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \emptyset \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \emptyset \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \emptyset \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \emptyset \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \emptyset \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \emptyset \\ \hline \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & * \end{array}\right)$$

We now have a block diagonal matrix in which the top left block is a $(2k+1) \times (2k+1)$ matrix with nonzero determinant (it results from reordering the rows of the identity matrix), the bottom right block is $M_{n-2k-1,k}$, and the other two blocks are zero. We conclude that

$$\det(M_{n,k}) \equiv \det(M_{n-2k-1,k}) \pmod{2},$$

proving the desired congruence.

To prove the desired result, we must now check that $F_{0,k}, F_{1,k}$ are odd and $F_{2,k}, \dots, F_{2k,k}$ are even. For $n = 0, \dots, k+1$, the matrix $M_{n,k}$ consists of all ones, so its determinant is 1 if $n = 0, 1$ and 0 otherwise. (Alternatively, we have $F_{n,k} = n!$ for $n = 0, \dots, k+1$, since every permutation of $\{1, \dots, n\}$ is k-limited.) For $n = k+2, \dots, 2k$, observe that rows k and $k+1$ of $M_{n,k}$ both consist of all ones, so $\det(M_{n,k}) = 0$ as desired.

Third solution: (by Tom Belulovich) Define $M_{n,k}$ as in the second solution. We prove $\det(M_{n,k})$ is odd for $n \equiv 0, 1 \pmod{2k+1}$ and even otherwise, by directly determining whether or not $M_{n,k}$ is invertible as a matrix over the field of two elements.

Let r_i denote row i of $M_{n,k}$. We first check that if $n \equiv 2, \dots, 2k \pmod{2k+1}$, then $M_{n,k}$ is not invertible. In this case, we can find integers $0 \leq a < b \leq k$ such that $n + a + b \equiv 0 \pmod{2k+1}$. Put $j = (n+a+b)/(2k+1)$. We can then write the all-ones vector both as

$$\sum_{i=0}^{j-1} r_{k+1-a+(2k+1)i}$$

and as

$$\sum_{i=0}^{j-1} r_{k+1-b+(2k+1)i}.$$

Hence $M_{n,k}$ is not invertible.

We next check that if $n \equiv 0, 1 \pmod{2k+1}$, then $M_{n,k}$ is invertible. Suppose that $a_1, \dots, a_n$ are scalars such that $a_1 r_1 + \cdots + a_n r_n$ is the zero vector. The m-th coordinate of this vector equals $a_{m-k} + \cdots + a_{m+k}$, where we regard a_i as zero if $i \notin \{1, \dots, n\}$. By comparing consecutive coordinates, we obtain

$$a_{m-k} = a_{m+k+1} \qquad (1 \leq m < n).$$

In particular, the a_i repeat with period $2k+1$. Taking $m = 1, \dots, k$ further yields that

$$a_{k+2} = \cdots = a_{2k+1} = 0$$

while taking $m = n-k, \dots, n-1$ yields

$$a_{n-2k} = \cdots = a_{n-1-k} = 0.$$

For $n \equiv 0 \pmod{2k+1}$, the latter can be rewritten as

$$a_1 = \cdots = a_k = 0$$

whereas for $n \equiv 1 \pmod{2k+1}$, it can be rewritten as

$$a_2 = \cdots = a_{k+1} = 0.$$

In either case, since we also have

$$a_1 + \cdots + a_{2k+1} = 0$$

from the $(k+1)$-st coordinate, we deduce that all of the a_i must be zero, and so $M_{n,k}$ must be invertible.

Remark: The matrices $M_{n,k}$ are examples of *banded matrices*, which occur frequently in numerical applications of linear algebra. They are also examples of *Toeplitz matrices*.