Gaussian Elimination and Gauss-Jordan Method

Learning Outcomes

At the end of this session, the students should be able to:

1. discuss the difference between the Gaussian and Gauss-Jordan Elimination methods;
2. discuss the importance of pivoting;
3. perform the Gaussian Elimination and Gauss-Jordan Elimination algorithms in R; and
4. interpret the results of the said methods.

Content

I. Gaussian and Gauss Jordan Elimination
II. Partial Pivoting
III. Importing R Code
IV. Swapping Matrix Rows

Gaussian Elimination and Gauss-Jordan Elimination

These two methods are for solving a system of linear equations. A system of linear equations can be expressed into an augmented coefficient matrix.

$n \times n$ System of Linear Equations:	$n \times n+1$ Augmented Coefficient Matrix
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$	$$\left[\begin{array}{cccc\|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{array}\right]$$

Given an augmented coefficient matrix, one can use either Gaussian Elimination or Gauss-Jordan Elimination.

Gaussian Elimination

Gaussian Elimination will produce an upper-triangular matrix. An upper triangular matrix is of the form:

$$\underset{i \times j}{A} = \begin{cases} A_{ij}, & i \le j \\ 0, & i > j \end{cases}$$

To perform Gaussian Elimination, one must do *forward elimination* and *backward substitution*. In finding the multiplier, there might be times where the pivot element might be zero. Thus, to prevent a division by zero, pivoting is needed to be employed.

Pivoting

Pivoting is an elementary matrix operation where the rows and/or columns of the matrix are being swapped. *Partial pivoting* is done when rows are being swapped. This is valid because each row is treated as a separate equation.

A complete pivoting is done when columns are also being swapped. It should be taken note that when columns are swapped, the ordering of the variables are also being swapped in the augmented coefficient matrix. It should be taken note of that pivoting is completely optional. The advantage of employing pivoting is that it reduces the round-off errors in representing floating-point numbers and the division by zero errors that may occur.

ALGORITHM 1: Forward Elimination

```
for i in 1 to n-1:
    find PIVOT ROW: row with max(abs(a[i:n,i]))
    if a[PIVOT ROW, i] == 0:
        no unique solution exists, STOP
    do PARTIAL PIVOTING: swap(PIVOT ROW, a[i,])
    for j in i + 1 to n:
        find PIVOT ELEMENT: a[i,i]
        find MULTIPLIER: a[j,i] / PIVOT ELEMENT
        find NORMALIZED ROW: MULTIPLIER * a[i,]
        find a[j,]: a[j,] - NORMALIZED ROW
```

ALGORITHM 2: Backward Substitution

```
for i in n - 1 to 1:
    x[i] = (b[i] - sum(a[i, i+1:n] * x[i+1:n])) / a[i,i]
```

Gauss-Jordan Elimination

For Gauss-Jordan Elimination, it will produce an identity matrix on the $n \times n$ portion of the augmented coefficient matrix, and the results will be present on its last column. Pivoting may still be employed.

ALGORITHM 3. Gauss-Jordan Elimination

```
for i in 1 to n:
    if i != n:
        find PIVOT ROW: row with max(abs(a[i:n,i]))
        if a[PIVOT ROW, i] == 0: no unique solution exists, STOP
        do PARTIAL PIVOTING: swap(PIVOT ROW, a[i,])
    find a[i,]: a[i,] / a[i,i]
    for j in 1 to n:
        if i == j: continue
        find NORMALIZED ROW: a[j,i] * a[i,]
```

Importing R Code

R code made in another file can be made available into the current file by the `source()` function. One can enter an absolute file path or a relative file path.

```
> source("sample.r") #Given a file named sample.r in the same directory
> source("../sample.r") #Given a file named sample.r up one level
```

Swapping Matrix Rows

To swap matrix rows (and columns), one can use the `order()` function. The `order()` function accepts a vector on what to sort. It returns a vector of the positioning of the items when it is sorted. It also accepts an additional parameter to sort in descending order. It sorts in ascending order by default.

```
> x = c(40, 20, 10, 30)
> x
[1] 40 20 10 30
> order(x)
[1] 3 2 4 1
> order(x, decreasing = TRUE)
[1] 1 4 2 3
```

To obtain a sorted version of such items, one must use the result of the `order()` function in indexing.

```
> x[order(x)]
[1] 10 20 30 40
> y = matrix(1:9, nrow=3, dimnames=list(c("p1", "p2", "p3"), c("q1", "q2", "q3")))
> y
   q1 q2 q3
p1  1  4  7
p2  2  5  8
p3  3  6  9
> y[order(c(2, 1, 3)), ]
   q1 q2 q3
p2  2  5  8
p1  1  4  7
p3  3  6  9
> y[ ,order(c("q3", "q1", "q2"))]
   q2 q3 q1
p1  4  7  1
p2  5  8  2
p3  6  9  3
```

Learning Experiences

1. Students will have hands-on experience in importing previously constructed codes in R.
2. Students will create a labeled list in R.
3. Students will create a modification of a matrix by swapping the rows present in it.
4. Students will create their own R script that will solve for the solution of a system of linear equations, using both the Gaussian and Gauss-Jordan Elimination.
5. Students will use their R script to solve a given problem that uses a system of linear equations.
6. Students will accomplish ***sample exercises for self-learning*** provided below.

Sample Exercises for Self-learning

1. Required Competencies:
   Gaussian Elimination, Gauss-Jordan Elimination, Importing R Code, Labelled Lists, Swapping Matrix Rows

   Create an R code that imports the previous exercise on the augmented coefficient matrix. On the same code, create two functions named `GaussianElimination` and `GaussJordanElimination`, which contains statements to perform the Gaussian and Gauss-Jordan Elimination algorithms.

   The functions should return a labeled list which has the following labels:

   [a] `solutionSet`
   A vector of numeric values which contains the solution of the system of linear equations. In the event that the system has no solution, or has infinitely many solutions, return the value `NA`.
   [b] `variables`
   A vector of character values which contains the variable names used in the system of linear equations.
   [c] matrix
   The resulting matrix after the Forward Elimination of the Gaussian Elimination, or the resulting matrix after Gauss-Jordan Elimination, depending on which function was called.

2. Required Competencies:
   Gaussian Elimination, Gauss-Jordan Elimination

   Using the Gaussian and Gauss-Jordan Elimination functions made in the previous item, solve the following items in the section marked as Exercises, as assigned by your laboratory instructor. In a sheet of paper, write:

   [a] the resulting augmented coefficient matrix
   [b] the resulting matrix after the algorithm was deployed, and
   [c] the interpretation of the answer to the question as posed in each item.

Assessment Tool

1. A ***programming exercise*** on the implementation of Gaussian and Gauss-Jordan Elimination algorithms, written in an R script.
2. A ***written exercise*** on the interpretation of the results of Gaussian and Gauss-Jordan Elimination algorithms, as applied to real-life problems.

References

[1] Base. (n.d.). Retrieved September 3, 2018, from https://www.rdocumentation.org/packages/base/versions/3.5.1/topics/order

[2] Gauss Jordan Elimination Through Pivoting. (n.d.). Retrieved September 3, 2018, from https://people.richland.edu/james/lecture/m116/matrices/pivot.html

[3] McKinney, D. C. (n.d.). Numerical Methods for Civil Engineers. Retrieved September 3, 2018, from http://www.ce.utexas.edu/prof/mckinney/ce311k/handouts/Linear_Equations.pdf

Exercises

1. A forester wants to know how to source wood from different planting sites. These are to be used by a construction company. There are six sites, and six species of wood that is needed. Some species are not applicable to some sites because of the soil composition.

   The table below shows the timber density for each site.

Planting Site	Volume per hectare (m^3 / ha)	Yakal (%)	Falcata (%)	Gimelina (%)	Apitong (%)	Ipil (%)	Nato (%)
Abra	350	70	10	5	5	0	10
Zambales	270	20	60	10	0	5	5
Biliran	300	10	0	75	5	5	5
Leyte	290	10	10	10	60	10	0
Bukidnon	325	5	5	0	5	75	10
Davao City	360	0	10	5	5	20	60

The construction company needs to be provided with the following wood types, in cubic meters.

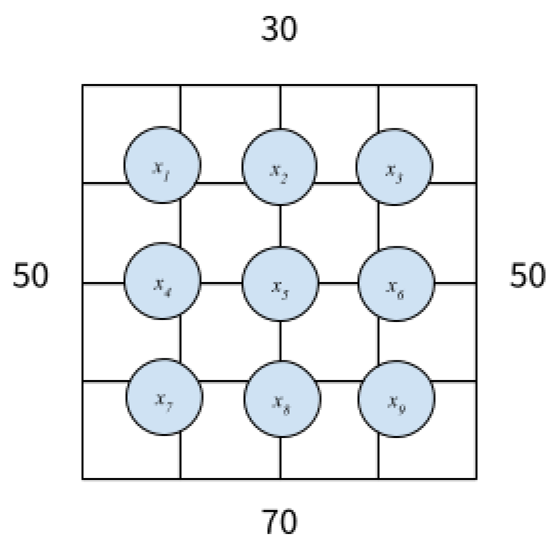
Wood	Volume (m^3)	Wood	Volume (m^3)	Wood	Volume (m^3)
Yakal	1000	Gimelina	1300	Ipil	1500
Falcata	700	Apitong	900	Yato	1200

How many hectares should be logged in each planting site, to deliver the required volume of logs?

2. A square plate of uniform material has been heated across the sides into different temperatures. There are nine points in which temperatures are needed to be recorded. These nine points, as well as the temperatures of the boundaries, are being laid out in the figure below.

   At a certain period, the temperature of the internal points will reach an equilibrium. This means that all of the temperatures will be equal. To gather the temperatures at each internal point, one must know the mean-value property:

   If a plate has reached a thermal equilibrium, and x_i is a grid point not on the boundary of the plate, then the temperature at x_i is the average of the temperatures of the four closest grid points to x_i. (This should find answering this question instructive.)



   What are the temperatures in each of the internal points?

3. Five reagents are needed to produce Chemical X. Given different amounts of such reagents, it produces a different yield for Chemical X. The table below shows the different amounts that are needed, as well as the yield.

   The precipitate which is a gaseous byproduct of the reaction is also important. The amount of precipitate is also recorded.

Reagent 1	Reagent 2	Reagent 3	Reagent 4	Reagent 5	Yield	Precipitate
200	90.3	70.3	130.4	200	295.4	140.5
127.2	200	77.5	133.5	200	293.5	150.7
133.5	99.3	200	135.9	200	300.6	142.6
153.7	94.2	79.5	200	200	298.3	143.9
144.2	91.5	78.2	137.3	200	288.7	140.1

   What are the volumes needed to produce the yields and the amounts of the precipitate?