Solve the Inverse Kinematics Problem by Minimization

$$\vec{\theta}^* = \arg\min_{\vec{\theta}} \underbrace{\frac{1}{2} \left| \begin{bmatrix} \vec{g} \\ 1 \end{bmatrix} - \vec{F}(\vec{\theta}, \vec{t}) \right|^2}_{\equiv f(\theta)}$$

The Objective Function

$$f(\theta) \equiv \frac{1}{2}\left(\begin{bmatrix}\vec{g}\\1\end{bmatrix} - \vec{F}(\vec{\theta},\vec{t})\right)^T \left(\begin{bmatrix}\vec{g}\\1\end{bmatrix} - \vec{F}(\vec{\theta},\vec{t})\right)$$

A bit of Short hand notation to increase readability

$$\vec{g} = \begin{bmatrix} \vec{g} \\ 1 \end{bmatrix}$$

$$\vec{F}(\vec{\theta}, \vec{t}) = \vec{F}(\vec{\theta}) \begin{bmatrix} \vec{t} \\ 1 \end{bmatrix} = \vec{F}(\vec{\theta})$$

$$\vec{J}_k = \frac{\partial}{\partial \theta_k} \left(\vec{F}(\vec{\theta}) \right)$$

The gradient of the objective

$$
\begin{aligned}
\nabla f(\vec{\theta})^T &= \frac{\partial f(\vec{\theta})}{\partial \vec{\theta}} \\
&= \begin{bmatrix} \frac{\partial f(\vec{\theta})}{\partial \theta_0} & \dots & \frac{\partial f(\vec{\theta})}{\partial \theta_n} \end{bmatrix}
\end{aligned}
$$

The j'th component of the gradient

$$
\begin{aligned}
\frac{\partial f(\vec{\theta})}{\partial \theta_j} &= \frac{\partial}{\partial \theta_j} \left(\frac{1}{2} \left(\vec{g} - \vec{F}(\vec{\theta}) \right)^T \left(\vec{g} - \vec{F}(\vec{\theta}) \right) \right) \\
&= \left(\vec{g} - \vec{F}(\vec{\theta}) \right)^T \frac{\partial}{\partial \theta_j} \left(\vec{g} - \vec{F}(\vec{\theta}) \right) \\
&= - \left(\vec{g} - \vec{F}(\vec{\theta}) \right)^T \vec{J}_j
\end{aligned}
$$

The final equation for the gradient

$$\nabla f(\vec{\theta}) = -\mathbf{J} \left(\vec{g} - \vec{F}(\vec{\theta}) \right)$$

The Hessian of the objective function

$$\mathbf{H}_{ij} = \frac{\partial f(\vec{\theta})}{\partial \theta_i \partial \theta_j} \qquad i < j$$

$$= \frac{\partial}{\partial \theta_i} \left(- \left(\vec{g} - \vec{F}(\vec{\theta}) \right)^T \vec{J}_j \right)$$

$$= - \frac{\partial}{\partial \theta_i} \left(\left(\vec{g} - \vec{F}(\vec{\theta}) \right)^T \vec{J}_j \right)$$

The Product Rule

$$
\begin{aligned}
\mathbf{H}_{ij} &= -\frac{\partial}{\partial \theta_i}\left(\left(\vec{g} - \vec{F}(\vec{\theta})\right)^T \vec{J}_i\right) \\
&= -\left(\frac{\partial}{\partial \theta_i}\left(\vec{g} - \vec{F}(\vec{\theta})\right)^T\right)\vec{J}_j - \left(\vec{g} - \vec{F}(\vec{\theta})\right)^T\left(\frac{\partial}{\partial \theta_i}\vec{J}_j\right) \\
&= \vec{J}_i^T\vec{J}_j - \boxed{\left(\vec{g} - \vec{F}(\vec{\theta})\right)}^T \boxed{\left(\frac{\partial}{\partial \theta_i}\vec{J}_j\right)}
\end{aligned}
$$

From past knowledge we find

$$\frac{\partial}{\partial \theta_i}\vec{J}_j = \mathbf{T}_0 \cdots \mathbf{T}_{i-1}\frac{\partial \mathbf{T}_i}{\partial \theta_i}\mathbf{T}_{i+1} \cdots \mathbf{T}_{j-1}\frac{\partial \mathbf{T}_j}{\partial \theta_j}\mathbf{T}_{j+1} \cdots \mathbf{T}_n \begin{bmatrix} \vec{t} \\ 1 \end{bmatrix}$$

Using a geometric re-interpretation we have

$$\frac{\partial}{\partial \theta_i}\vec{J}_j = -\vec{J}_j$$

Finally we find

$$\begin{aligned}\mathbf{H}_{ij} &= \vec{J}_i^T \vec{J}_j - \left(\vec{g} - \vec{F}(\vec{\theta})\right)^T \left(\frac{\partial}{\partial \theta_i} \vec{J}_j\right) \\ &= \vec{J}_i^T \vec{J}_j + \left(\vec{g} - \vec{F}(\vec{\theta})\right)^T \vec{J}_j\end{aligned}$$