How do we compute the gradient of the end-effector vector?
(this is called the Jacobian)

$$\frac{\partial}{\partial \vec{\theta}} \left(\vec{F}(\vec{\theta}) \begin{bmatrix} \vec{t} \\ 1 \end{bmatrix} \right) = \begin{bmatrix} \frac{\partial}{\partial \theta_0} \left(\vec{F}(\vec{\theta}) \begin{bmatrix} \vec{t} \\ 1 \end{bmatrix} \right) & \cdots & \frac{\partial}{\partial \theta_n} \left(\vec{F}(\vec{\theta}) \begin{bmatrix} \vec{t} \\ 1 \end{bmatrix} \right) \end{bmatrix}$$

Using our homogenous coordinates we have

$$\frac{\partial}{\partial \theta_j}\left(\vec{F}(\vec{\theta})\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right) = \frac{\partial}{\partial \theta_j}\left(\mathbf{T}_0 \cdots \mathbf{T}_n \begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)$$

From the product rule we have

$$
\begin{aligned}
\frac{\partial}{\partial \theta_j}\left(\vec{F}(\vec{\theta})\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right) &= \frac{\partial}{\partial \theta_j}\left(\mathbf{T}_0\cdots\mathbf{T}_j\cdots\mathbf{T}_n\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)\\
&= \left(\left(\frac{\partial}{\partial \theta_j}\mathbf{T}_0\right)\cdots\mathbf{T}_j\cdots\mathbf{T}_n\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)\\
&+\cdots\\
&+\left(\mathbf{T}_0\cdots\left(\frac{\partial}{\partial \theta_j}\mathbf{T}_j\right)\cdots\mathbf{T}_n\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)\\
&+\cdots\\
&+\left(\mathbf{T}_0\cdots\mathbf{T}_j\cdots\left(\frac{\partial}{\partial \theta_j}\mathbf{T}_n\right)\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)\\
&+\cdots\\
&+\left(\mathbf{T}_0\cdots\mathbf{T}_j\cdots\mathbf{T}_n\left(\frac{\partial}{\partial \theta_j}\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)\right)
\end{aligned}
$$

Using calculus we have

$$\frac{\partial}{\partial\theta_j}\begin{bmatrix}\cos\theta_k & -\sin\theta_k & l_{k,x}\\ \sin\theta_k & \cos\theta_k & l_{k,y}\\ 0 & 0 & 1\end{bmatrix} = \begin{cases}\begin{bmatrix}-\sin\theta_k & -\cos\theta_k & 0\\ \cos\theta_k & -\sin\theta_k & 0\\ 0 & 0 & 0\end{bmatrix} & j = k\\ \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} & j \neq k\end{cases}$$

Apply knowledge about terms that become zero

$$
\begin{aligned}
\frac{\partial}{\partial\theta_j}\left(\vec{F}(\vec{\theta})\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right) &= \frac{\partial}{\partial\theta_j}\left(\mathbf{T}_0\cdots\mathbf{T}_j\cdots\mathbf{T}_n\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)\\
&= \cancel{\left(\left(\frac{\partial}{\partial\theta_j}\mathbf{T}_0\right)\cdots\mathbf{T}_j\cdots\mathbf{T}_n\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)}\\
&+\cdots\\
&+\left(\mathbf{T}_0\cdots\left(\frac{\partial}{\partial\theta_j}\mathbf{T}_j\right)\cdots\mathbf{T}_n\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)\\
&+\cdots\\
&+\cancel{\left(\mathbf{T}_0\cdots\mathbf{T}_j\cdots\left(\frac{\partial}{\partial\theta_j}\mathbf{T}_n\right)\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)}\\
&+\cdots\\
&+\cancel{\left(\mathbf{T}_0\cdots\mathbf{T}_j\cdots\mathbf{T}_n\left(\frac{\partial}{\partial\theta_j}\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right)\right)}
\end{aligned}
$$

Only one term survive

$$\frac{\partial}{\partial\theta_j}\left(\vec{F}(\vec{\theta})\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right) = \mathbf{T}_0\cdots\mathbf{T}_{j-1}\frac{\partial\mathbf{T}_j}{\partial\theta_j}\mathbf{T}_{j+1}\cdots\mathbf{T}_n\begin{bmatrix}\vec{t}\\1\end{bmatrix}$$

A little mathematical exercise
The permutation is the “hat” operation in 2D

$$\frac{\partial}{\partial \theta_j} \begin{bmatrix} \cos\theta_j & -\sin\theta_j & l_{j,x} \\ \sin\theta_j & \cos\theta_j & l_{j,x} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta_j & -\cos\theta_j & 0 \\ \cos\theta_j & -\sin\theta_j & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\sin\theta_j & -\cos\theta_j & 0 \\ \cos\theta_j & -\sin\theta_j & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_j & -\sin\theta_j & l_{j,x} \\ \sin\theta_j & \cos\theta_j & l_{j,y} \\ 0 & 0 & 1 \end{bmatrix}$$

Using the re-write we have

$$\frac{\partial}{\partial \theta_j}\left(\vec{F}(\vec{\theta})\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right) = \mathbf{T}_0 \cdots \mathbf{T}_{j-1}\frac{\partial \mathbf{T}_j}{\partial \theta_j}\mathbf{T}_{j+1} \cdots \mathbf{T}_n \begin{bmatrix}\vec{t}\\1\end{bmatrix}$$

$$\frac{\partial}{\partial \theta_j}\left(\vec{F}(\vec{\theta})\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right) = \mathbf{T}_0 \cdots \mathbf{T}_{j-1}\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}\mathbf{T}_j \cdots \mathbf{T}_n \begin{bmatrix}\vec{t}\\1\end{bmatrix}$$

This allows us to easily make a geometric re-interpretation

$$\frac{\partial}{\partial \theta_j}\left(\vec{F}(\vec{\theta})\begin{bmatrix}\vec{t}\\ 1\end{bmatrix}\right) = \underbrace{\mathbf{T}_0 \cdots \mathbf{T}_{j-1}}_{\vec{F}(\theta_0,\ldots,\theta_{j-1})} \underbrace{\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\widehat{\cdot}} \underbrace{\mathbf{T}_j \cdots \mathbf{T}_n \begin{bmatrix}\vec{t}\\ 1\end{bmatrix}}_{\left[\Delta\right]_j}$$

The hat of the difference between tool position and link i origin as seen from world coordinate frame…

Using the geometric re-interpretation allow for a simpler computation

$$\vec{\Delta} = \vec{e} - \vec{p}_j$$

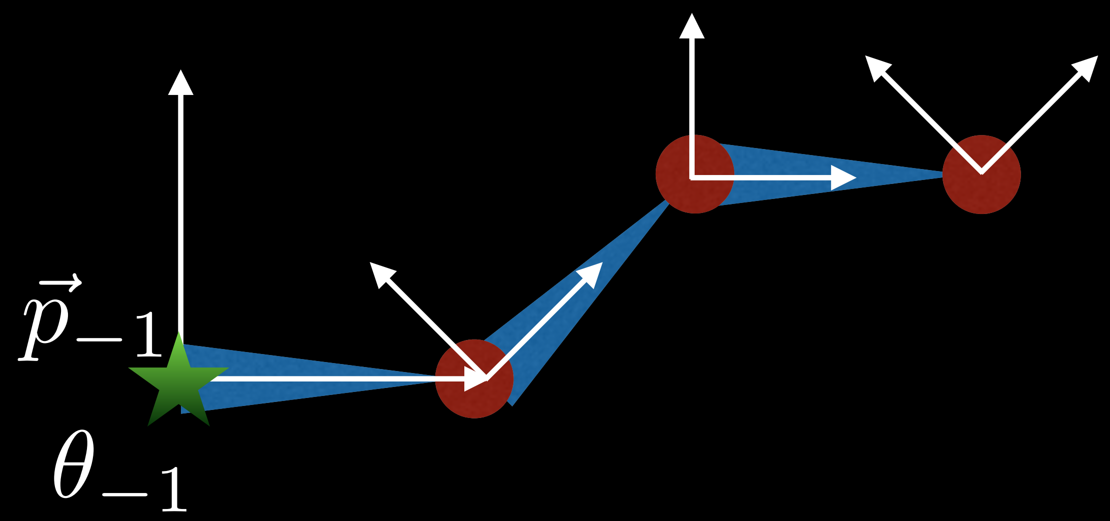
Origin of link j frame in WCS

$$\frac{\partial}{\partial \theta_j}\left(\vec{F}(\vec{\theta})\begin{bmatrix}\vec{t}\\1\end{bmatrix}\right) = \begin{bmatrix}-\Delta_y\\ \Delta_x\\ 0\end{bmatrix}$$

We need to compute origin of link frames to apply this

How do we compute link frame origins?

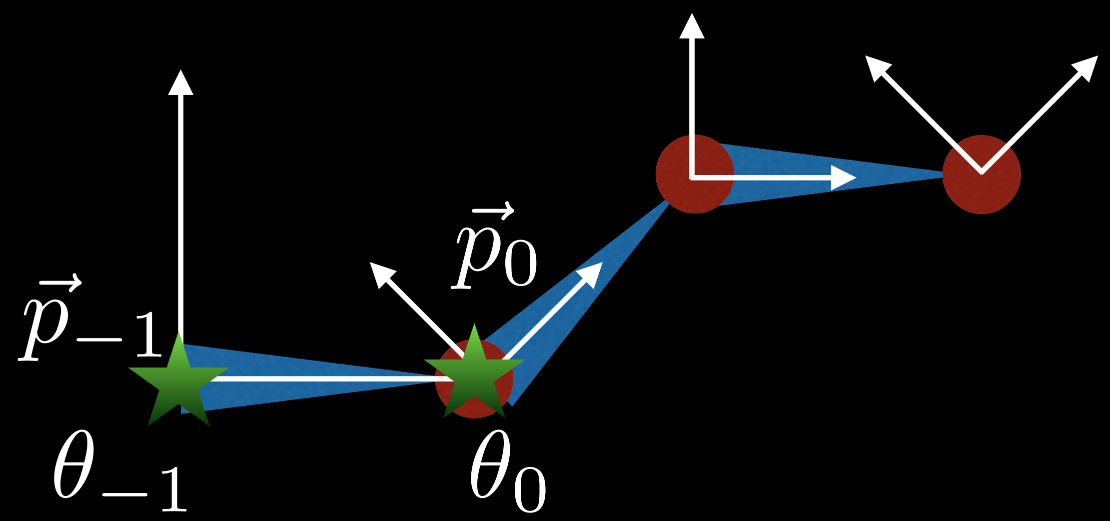
Let us use a forward sweep approach



$$\vec{p}_{-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\theta_{-1} = 0$$

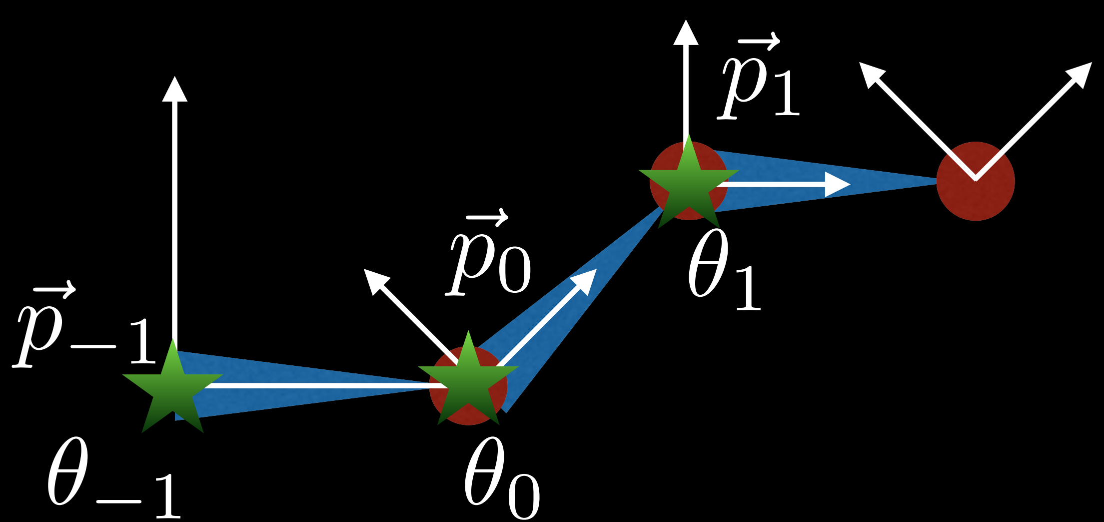
First link



$$\beta_0 = \theta_{-1}$$

$$\vec{p}_0 = \vec{p}_{-1} + \begin{bmatrix} \cos\beta_0 & -\sin\beta_0 \\ \sin\beta_0 & \cos\beta_0 \end{bmatrix} \vec{l}_0$$

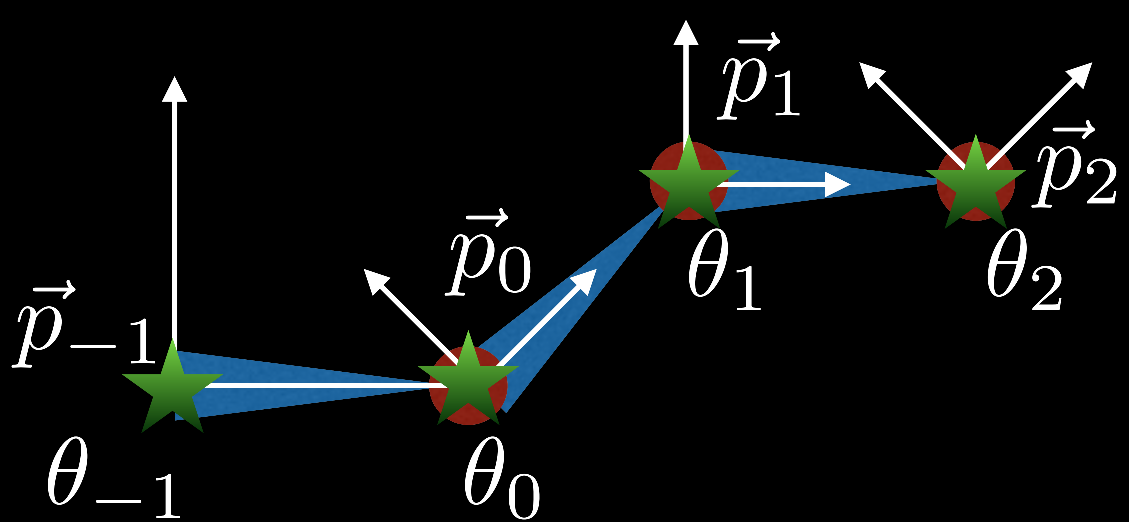
Second link



$$\beta_1 = \beta_0 + \theta_0$$

$$\vec{p}_1 = \vec{p}_0 + \begin{bmatrix} \cos\beta_1 & -\sin\beta_1 \\ \sin\beta_1 & \cos\beta_1 \end{bmatrix} \vec{l}_1$$

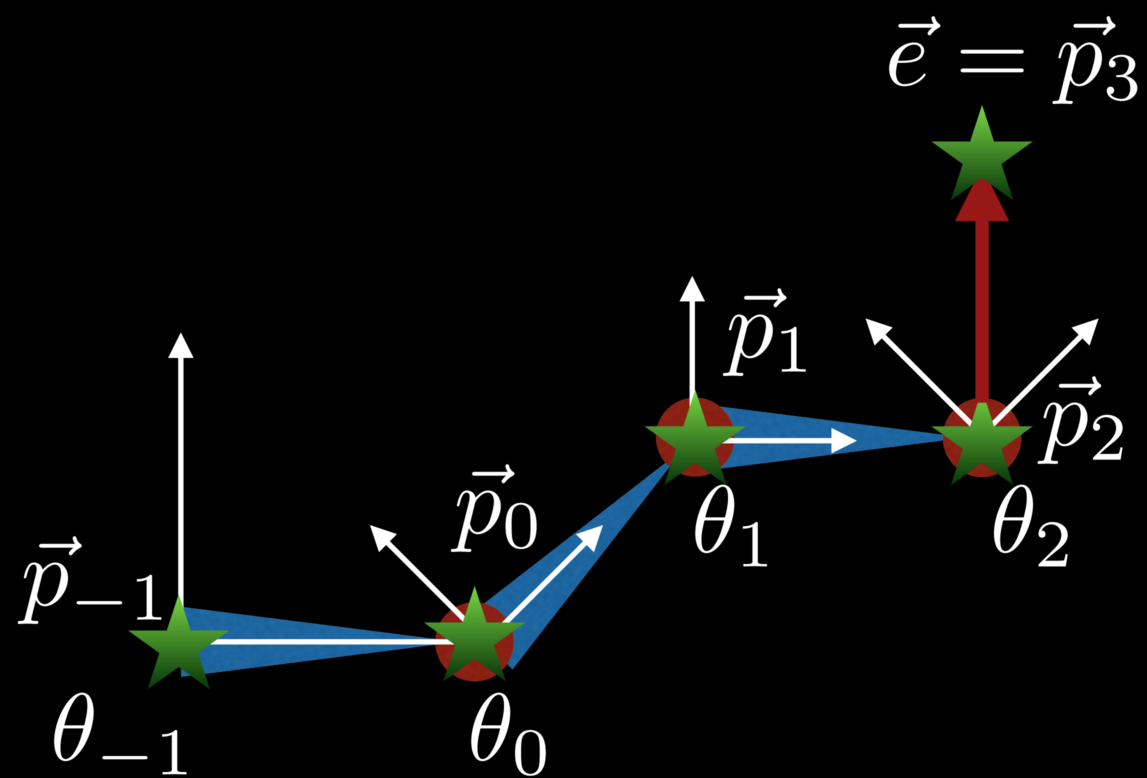
Third Link



$$\beta_2 = \beta_1 + \theta_1$$

$$\vec{p}_2 = \vec{p}_1 + \begin{bmatrix} \cos\beta_2 & -\sin\beta_2 \\ \sin\beta_2 & \cos\beta_2 \end{bmatrix} \vec{l}_2$$

End-Effector if non-zero tool position



$$\beta_3 = \beta_2 + \theta_2$$

$$\vec{e} = \vec{p}_3 = \vec{p}_2 + \begin{bmatrix} \cos\beta_3 & -\sin\beta_3 \\ \sin\beta_3 & \cos\beta_3 \end{bmatrix} \vec{t}$$

By construction we have found the incremental update rule for a forward sweep

$$\beta_i = \beta_{i-1} + \theta_{i-1}$$

$$\vec{p}_i = \vec{p}_{i-1} + \begin{bmatrix} \cos\beta_i & -\sin\beta_i \\ \sin\beta_i & \cos\beta_i \end{bmatrix} \vec{l}_i$$

The Final Algorithm

- Step 1 Compute end-effector vector and link frame origins
- Step 2 Fill in the Jacobian column-wise