# CBNST – Computer based Numerical and Statistical Techniques

## Numerical Methods | Analysis

In Numerical Methods, we are dealing with numbers. So first term which is very important is

### Accuracy of Numbers

**Exact Number**

The number of the type $7, 4, 65, \frac{3}{2}, 6.43, \ldots$ will be in category of exact no.

(numbers are terminating after the decimal pts. or finite no. of no. after decimal pt.)

**Approximate number**

The numbers $\frac{1}{3} = 0.3333\ldots\infty$

$\pi = 3.141592\ldots$

$\sqrt{2} = 1.414213\ldots$

do not have finite decimal expansion i.e. infinite no. of digits after decimal point.

or numbers are not terminating after the decimal pt.

Imp Hence these numbers are approximate to some finite digits called significant digits for the purpose of calculation.

Significant digits :⇒ The digits used to express a number are called significant digits.

The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are significant digits.
'0' is also a significant digit except when it is used to fix the decimal point.

| | Number | significant digits | No. of S.D. |
|---|---|---|---|
| Example → | 7845 | 7, 8, 4, 5 | 4 |
| | 3.589 | 3, 5, 8, 9 | 4 |
| | 2.018 | 2, 0, 1, 8 | 4 |
| | 0.1234 | 1, 2, 3, 4 | 4 |
| | 0.0003090 | 3, 0, 9, 0 | 4 |
| | 3.0686 | 3, 0, 6, 8, 6 | 5 |

Rule for 0 is ⇒ ① lie between significant digits
② lie to the right of decimal point and at the same ~~point~~ time, to the right of a non-zero digit.

| | | | |
|---|---|---|---|
| | 3060 | 3, 0, 6 | 3 |
| | 3900 | 3, 9 | 2 |
| | 0.3969 | 3, 9, 6, 9 | 4 |
| Imp | 39.00 | 3, 9, 0, 0 | 4 |
| | 0.00390 | 3, 9, 0 | 3 |
| | $3.9 \times \underbrace{10^{k}}_{\text{Remove}}$ | 3, 9 | 2 |
| | $6 \times 10^{-2}$ | 6 | 1 |
| | $3.909 \times 10^{5}$ | 3, 9, 0, 9 | 4 |

**Rounding off** :- Let $\frac{22}{7} = 3.142857143 ----$

for calculation we take $\frac{22}{7} = 3.143$ or $3.14$.

This process of dropping unwanted digits is called rounding-off.

Rule → To round-off a no. to n significant digits, discard all digits to the right of $n^{th}$ digit and if this discarded ~~no~~ digit is

(i) less than 5 in $(n+1)^{th}$ place, leave the $n^{th}$ digit unaltered.

(ii) greater than 5 in $(n+1)^{th}$ place, increase the $n^{th}$ digit by unity.

(iii) exactly 5 in $(n+1)^{th}$ place, increase the $n^{th}$ digit by unity if it is odd otherwise leave it unchanged.

**Example**

| Number | Rounded-off to n significant figures. Three digits | four digits | Five digits |
|---|---|---|---|
| 00.543241 | 00.543 | 00.5432 | 00.54324 |
| 39.5255 | 39.5 | 39.52 | 39.526 |
| 69.4155 | 69.4 | 69.42 | 69.416 |
| 00.667676 | 00.667 | 00.6677 | 00.66768 |
| 75462 | 75500 | 75460 | 75462 |
| 865250 | 865000 | 865200 | 865250 |
| 37.46235 | 37.5 | 37.46 | 37.462 |
| 625.483 | 625 | 625.5 | 625.48 |

# Errors

[True value ~ Approximate value]

## Inherent Error

* Which are already present in the statements of a problem.
* Exist due to approximate given data or limitation to computing aids (tables, calculators etc.)
* Can be minimized by taking better data or by using high precision.

ex→ 4.3201 $10^{-4}$

4.32 $10^{-2}$

4.320106 $10^{-6}$

this no. has the greatest precision.

## Round-off Error

* Arise due to rounding-off the numbers while process of computation

Ex→ 7.5846712 round-off to 7.585

error = 7.5846712 ~ 7.585 = 0.0003288

## Truncation Error

* Arise on replacing an infinite process by a finite one.

Ex→ $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots \infty = X$

If it is truncated to $X' = 1 + x + \frac{x^2}{\lfloor 2}$, then

truncation error $= X - X'$

## Absolute Error

$E_a$ or $e_a$

Relative Error $E_r$ or $e_r$

Percentage Error $E_p$ or $e_p$

Absolute error $e_a = |X - X'|$

Relative error = $\frac{\text{Absolute error}}{\text{True Value}}$

$$e_r = \frac{|X - X'|}{X}$$

Percentage error = Relative Error × 100

$$e_p = \frac{|X - X'|}{X} \times 100$$

Note :- ① Relative & Percentage error are independent of units.

② If a no. is correct to n decimal places, then error $= \frac{1}{2} \times 10^{-n}$.

③ If the first significant digit of a no. is k and the no. is correct to n significant digits, then

relative error $< \frac{1}{k \times 10^{n-1}}$.

## Examples

1. Suppose 1.414 is used as an approximation to $\sqrt{2}$. find the absolute & relative errors.

Sol → True value $x$ of $\sqrt{2} = 1.41421356$

Approximate $x' = 1.414$

Error = True value − Approximate value

$= \sqrt{2} - 1.414$

$= 0.00021356$

Absolute error $= |x - x'| = 0.00021356$

Relative error $= \left|\frac{x-x'}{x}\right| = \frac{0.00021356}{1.41421356} = 0.0001510097$

2. If 0.333 is approximate value of $\frac{1}{3}$, find absolute, relative and percentage errors.

Sol → $\cancel{x = 1/0}$. $x = \frac{1}{3} = 0.33333333$, $x' = 0.333$

$e_a = |x - x'| = 0.00033333$

$e_r = \left|\frac{x-x'}{x}\right| = 0.00099999$

$\% \, e_p = e_r \times 100 = 0.099999\%$

3. Three approximate values of no. $\frac{1}{3}$ are given as 0.30, 0.33 and 0.34. which of these three is the best approximation.

Sol → $x = 0.33333333$

① $x' = 0.30$ ② $x' = 0.33$ ③ $x' = 0.34$

Absolute error =

$|x - x'|$

$= |0.3333333 - 0.30|$

$= 0.03$

(4) find the relative error of the no. 8.6 if both of its digits are correct. (6)

Sol → $X = 8.6$

$e_a = \frac{1}{2} \times 10^{-1} = (\ n$ decimal place$)$

$= 0.5 \times 10^{-1} = 0.05$

$e_r = \frac{0.05}{8.6} = 0.0058$

(5) Find the percentage error if 625.483 is approximated to three significant figures.

Sol → $X = 625.483$

$X' = 625$

$E_p = \left|\frac{X - X'}{X}\right| \times 100 = \frac{0.483}{625} \times 100 = 0.077\%$

(6) Round-off the no. 865250 and 37.46235 to four significant digits and compute $e_a$, $e_r$ & $e_p$.

Sol → $X = 865250$

$X' = 865200$

$e_a = \left|\frac{X - X'}{\cancel{X}}\right| = 50$

$e_r = \frac{50}{865250} = 0.00005779$

$e_p = e_r \times 100 = 0.005779$

$X = 37.46235$

$X' = 37.46$

$e_a = 0.00235$

$e_r = 6.2729 \times 10^{-5}$

$e_p = 6.2729 \times 10^{-3}$

(7) Evaluate the sum $S = \sqrt{3} + \sqrt{5} + \sqrt{7}$ to four significant digits and find its absolute & relative errors.

Sol → $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, $\sqrt{7} = 2.646$

$\frac{1}{2} \times 10^{-3}$

$0.5 \times 10^{-3}$

$0.0005$

Hence $S = 6.614$

$e_a = 0.0005 + 0.0005 + 0.0005$

$e_a = 0.0015$

$e_r = \frac{0.0015}{6.61} = 0.00026$

(8) It is required to obtain the roots of $x^2-2x+\log_{10}2=0$ to four decimal places. To what accuracy should $\log_{10}2$ be given?

Sol →. Roots of $x^2-2x+\log_{10}2=0$

$$x = \frac{2 \pm \sqrt{4-4\log_{10}2}}{2}, \quad x = 1 \pm \sqrt{1-\log_{10}2}$$

$$x = 1 \pm (1-\log_{10}2)^{1/2}$$

$$\Delta x = 0 \pm \frac{1}{2}(1-\log_{10}2)^{-1/2} \cdot (0-\Delta\log_{10}2)$$

$$\Delta x = \pm\frac{1}{2}\frac{\Delta\log_{10}2}{\sqrt{1-\log_{10}2}}$$

$$|\Delta x| = \frac{1}{2}\frac{\Delta\log_{10}2}{\sqrt{1-\log_{10}2}}$$

Now according to question absolute error $= \frac{1}{2}\times10^{-4}$
$= 0.5\times10^{-4}$

Thus $$|\Delta x| = \frac{1}{2}\frac{\Delta\log_{10}2}{\sqrt{1-\log_{10}2}} \leq 0.5\times10^{-4}$$

$$\Delta\log_{10}2 = 2\times0.5\times10^{-4}\left(\sqrt{1-\log_{10}2}\right)$$

$$\simeq 10^{-4}(8.3604)$$

i.e. $$\Delta\log_{10}2 \simeq 8.3604\times10^{-4}.$$

# Errors in Numerical Computations

## ① Errors in Addition of numbers :-

Let $X = x_1 + x_2 + \cdots + x_n$

and $X + \Delta X = (x_1 + \Delta x_1) + (x_2 + \Delta x_2) + \cdots + (x_n + \Delta x_n)$

Absolute error $|(X+\Delta X) - X|$

ie. $\Delta X = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n$

and Relative Error $= \frac{\Delta X}{X} = \frac{\Delta x_1}{X} + \frac{\Delta x_2}{X} + \cdots + \frac{\Delta x_n}{X}$

and Max. Relative Error is

$$\left|\frac{\Delta X}{X}\right| \leq \left|\frac{\Delta x_1}{X}\right| + \left|\frac{\Delta x_2}{X}\right| + \cdots + \left|\frac{\Delta x_n}{X}\right|$$

## ② Error in Subtraction of numbers :-

let $X = x_1 - x_2$

$\therefore \; X + \Delta X = (x_1 + \Delta x_1) - (x_2 + \Delta x_2)$

then Absolute error $= \Delta X = \Delta x_1 - \Delta x_2$

Relative error $= \frac{\Delta X}{X} = \frac{\Delta x_1}{X} - \frac{\Delta x_2}{X}$

and. Max. relative error $\left|\frac{\Delta X}{X}\right| \leq \left|\frac{\Delta x_1}{X}\right| + \left|\frac{\Delta x_2}{X}\right|$

## ③ Error in product of numbers :-

let $X = x_1 x_2 \cdots x_n$

as $X$ is a function of $x_1, x_2, \cdots x_n$, then

$$\Delta X = \frac{\partial X}{\partial x_1} \cdot \Delta x_1 + \frac{\partial X}{\partial x_2} \Delta x_2 + \frac{\partial X}{\partial x_3} \Delta x_3 + \cdots + \frac{\partial X}{\partial x_n} \Delta x_n$$

Now. $\frac{\Delta X}{X} = \frac{1}{X}\frac{\partial X}{\partial x_1} \Delta x_1 + \frac{1}{X}\frac{\partial X}{\partial x_2} \Delta x_2 + \cdots + \frac{1}{X}\frac{\partial X}{\partial x_n} \Delta x_n$

$\frac{1}{X}\frac{\partial X}{\partial x_1} = \frac{1}{X}(x_2 x_3 \cdots x_n) = \frac{x_2 x_3 \cdots x_n}{x_1 x_2 \cdots x_n} = \frac{1}{x_1}$

$\frac{1}{X}\frac{\partial X}{\partial x_2} = \frac{1}{x_2}$ , $\frac{1}{X}\frac{\partial X}{\partial x_3} = \frac{1}{x_3}$ - -

Now $\frac{\Delta X}{X} = \frac{\Delta x_1}{x_1} + \frac{\Delta x_2}{x_2} + \cdots + \frac{\Delta x_n}{x_n}$

Max. Relative error $= \left|\frac{\Delta X}{X}\right| \leq \left|\frac{\Delta x_1}{x_1}\right| + \left|\frac{\Delta x_2}{x_2}\right| + \cdots \left|\frac{\Delta x_n}{x_n}\right|$

Max. Absolute error $= \left|\frac{\Delta X}{X}\right| X = \left|\frac{\Delta X}{X}\right| \cdot (x_1 x_2 \cdots x_n)$

## Error in division of numbers :-

Let $X = \frac{x_1}{x_2}$

$$\Delta X = \frac{\partial X}{\partial x_1} \Delta x_1 + \frac{\partial X}{\partial x_2} \cdot \Delta x_2$$

$$\frac{\partial X}{\partial x_1} = \frac{1}{x_2}, \quad \frac{\partial X}{\partial x_2} = -\frac{x_1}{x_2^2}$$

$$\Delta X = \frac{1}{x_2} \Delta x_1 + \left(-\frac{x_1}{x_2^2}\right) \cdot \Delta x_2$$

$$\frac{\Delta X}{X} = \frac{1}{X}\left(\frac{\Delta x_1}{x_2}\right) + \frac{1}{X}\left(-\frac{x_1}{x_2^2}\right)\Delta x_2$$

$$= \frac{x_2}{x_1} \cdot \frac{\Delta x_1}{x_2} + \frac{x_2}{x_1}\left(-\frac{x_1}{x_2^2}\right)\Delta x_2$$

$$\frac{\Delta X}{X} = \frac{\Delta x_1}{x_1} - \frac{\Delta x_2}{x_2}$$

Max. Relative Error $= \left|\frac{\Delta X}{X}\right| \leq \left|\frac{\Delta x_1}{x_1}\right| + \left|\frac{\Delta x_2}{x_2}\right|$

Max. Absolute Error $= \left|\frac{\Delta X}{X}\right| \cdot X$

## 5 Error in evaluating $x^k$ :-

let $X = x^k$, $k$ = integer or fraction

$$\Delta X = k x^{k-1} \Delta x$$

$$\frac{\Delta X}{X} = k x^{k-1} \frac{\Delta x}{X} = k x^{k-1} \frac{\Delta x}{x^k}$$

$$\frac{\Delta X}{X} = k \frac{\Delta x}{x}$$

Relative Error $\left|\frac{\Delta X}{X}\right| \leq k \left|\frac{\Delta x}{x}\right|$

Ex1) If $u = \frac{4x^2y^3}{z^4}$ and errors in $x, y, z$ be 0.001, Compute the relative max. error in u when $x=y=z=1$.

Sol → u is the function of $x, y$ & $z$,

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z$$

$$\Delta u = \frac{8xy^3}{z^4}\Delta x + \frac{12x^2y^2}{z^4}\Delta y - \frac{16x^2y^3}{z^5}\Delta z$$

Maximum absolute error

$$|\Delta u| = \left|\frac{8xy^3}{z^4}\Delta x\right| + \left|\frac{12x^2y^2}{z^4}\Delta y\right| + \left|\frac{16x^2y^3}{z^5}\delta z\right|$$

$(x=y=z=1)$

$$= 8(0.001) + 12(0.001) + 16(0.001)$$

$$= 0.036$$

Max. Relative Error $= \left|\frac{\Delta u}{u}\right|_{\text{at } x=y=z=1} = \left|\frac{0.036}{4}\right| = 0.009.$

Ex2) Compute the percentage error in the time period $T = 2\pi\sqrt{\frac{l}{g}}$ for $l = 1\,m$ if the error in the measurement of $l$ is 0.01.

Sol → $T = 2\pi\sqrt{\frac{l}{g}}$

Now take log ⇒ $\log T = \log 2\pi + \frac{1}{2}(\log l - \log g)$

differentiate $\frac{1}{T}\delta T = 0 + \frac{1}{2}\cdot\frac{1}{l}\delta l - 0$

$$\frac{1}{T}\delta T = \frac{1}{2}\frac{\delta l}{l}$$

$$\frac{\delta T}{T} = \frac{1}{2}\,\frac{0.01}{1} = \frac{1}{2}\times 0.01$$

Percentage Error $= \frac{\delta T}{T}\times 100 = \frac{1}{2}\times 0.01\times 100$

$$= 0.5\%$$

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Ex3 If $u = 2v^6 - 5v$, find the percentage error in $u$ at $v = 1$ if error in $v$ is 0.05.

Sol → $u = 2v^6 - 5v$, $u$ is a func. of $v$

$\delta u = (12v^5 - 5)\delta v$ $\quad \partial u = \frac{\partial u}{\partial v} \cdot \delta v$

$$\frac{\delta u}{u} = \left(\frac{12v^5 - 5}{2v^6 - 5v}\right)\delta v$$

$$= \left(\frac{12-5}{2-5}\right)\delta v$$

$$= \frac{7}{-3}(0.05)$$

percentage error $= \frac{\delta u}{u} \times 100 = -\frac{7}{3} \times 0.05 \times 100$

$= -11.667\%$

Max. percentage error $= 11.667\%$

Ex4 If $r = 3h(h^6 - 2)$, find the percentage error in $r$ at $h = 1$, if the percentage error in $h$ is 5.

Sol → $r$ is a func. of $h$. $\quad , r = (3h^7 - 6h)$

$$\partial r = \frac{\partial r}{\partial h} \cdot \delta h$$

$$= (21h^6 - 6)\,\delta h$$

$\frac{\delta h}{h} \times 100 = 5$

$$\frac{\partial r}{r} = \left(\frac{21h^6 - 6}{r}\right)\delta h$$

$$\frac{\partial r}{r} \times 100 = \left(\frac{21h^6 - 6}{3h^7 - 6h}\right)\left(\frac{\delta h}{h} \times 100\right)$$

$$= \left(\frac{21h^6 - 6}{3h^6 - 6}\right)(5)$$

$$= \left(\frac{21 - 6}{3 - 6}\right)(5)$$

$$= \left(\frac{15}{3-6}\right)(5)$$

$$= \left(-\frac{15}{3}\right)(5)$$

$$= -25\%$$

Max. Percentage error $= 25\%$

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(Ex5) The discharge Q over a notch for head H is calculated by the formula $Q = KH^{5/2}$, $K = \text{Const}$. If the head is 75 cm and an error of 0.15 cm is possible in its measurement, estimate the percentage error in computing the discharge.

Sol →

$$Q = KH^{5/2}$$

$$\log Q = \log K + \frac{5}{2}\log H$$

$$\frac{1}{Q}\Delta Q = 0 + \frac{5}{2}\cdot\frac{1}{H}\Delta H$$

$$\frac{\Delta Q}{Q} = \frac{5}{2}\frac{\Delta H}{H}$$

$$\frac{\Delta Q}{Q}\times 100 = \frac{5}{2}\cdot\frac{0.15}{75}\times 100$$

$$= \frac{1}{2} = 0.5\,\%$$

(Ex6) The error in the measurement to the area of a circle is not allowed to exceed 0.1 %. How accurately should the diameter be measured?

Sol →

$$A = \pi r^2 \qquad d = 2r$$
$$r = d/2$$

$$A = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi}{4}d^2$$

$$\log A = \log\frac{\pi}{4} + 2\log d$$

$$\frac{1}{A}\Delta A = 0 + 2\cdot\frac{1}{d}\Delta d$$

$$\frac{1}{d}\Delta d = \frac{1}{2}\cdot\frac{\Delta A}{A}$$

$$\frac{\Delta d}{d}\times 100 = \frac{1}{2}\cdot\underbrace{\frac{\Delta A}{A}\times 100}$$

$$= \frac{1}{2}(0.1)$$

$$= 0.05\,\%$$

**Ex 7** (i) Prove that the absolute error in common logarithm of a number is less than half the relative error of the given no.

**Sol →** let $x$ be any number and $\log_{10} x$ is the common log.

so let $N = \log_{10} x$

$$= \frac{\log_e x}{\log_e 10} = \log_{10} e \, \log_e x$$

$$N = 0.43429 \log_e x$$

Now $\Delta N = 0.43429 \left\{ \frac{1}{x} \Delta x \right\}$

$$\Rightarrow \Delta N < \frac{1}{2}\left(\frac{\Delta x}{x}\right)$$

absolute error — relative Error

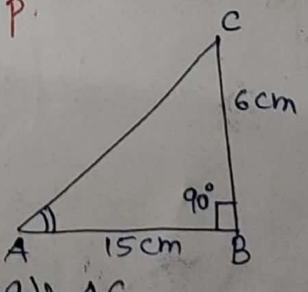
(ii) Prove that error in antilogarithm is many time the error in logarithm.

$$\left\{ \begin{array}{l} N = \log x \\ x = \text{Antilog } N \end{array} \right\}$$

**Sol →** $\Delta N = 0.43429 \left\{ \frac{\Delta x}{x} \right\}$

$$\Rightarrow \Delta x = \frac{x \, \Delta N}{0.43429} = 2.3026 \, x \, (\Delta N)$$

**Ex 8** In a $\Delta ABC$, $a = 6$ cm, $c = 15$ cm, $\angle B = 90°$. Find the possible error in computed value of $A$ if errors in measurements of $a$ & $c$ are 1 mm and 2 mm resp.




**Sol →** $\tan A = \frac{a}{c} \Rightarrow A = \tan^{-1} \frac{a}{c}$

Now $\Delta A = \frac{\partial A}{\partial a} \Delta a + \frac{\partial A}{\partial c} \Delta c$

$$= \frac{1}{1+\left(\frac{a}{c}\right)^2} \cdot \left(\frac{1}{c}\right) \Delta a + \frac{1}{1+\left(\frac{a}{c}\right)^2}\left(\frac{-(a)}{c^2}\right) \Delta c$$

$$\Delta A = \frac{c}{c^2+a^2} \Delta a + \frac{a}{c^2+a^2} \Delta c$$

Now $|\Delta A| \leq \left|\frac{c}{c^2+a^2} \Delta a\right| + \left|\frac{a}{c^2+a^2} \Delta c\right| = \frac{15}{261}(0.1) + \frac{6}{261}(0.2)$

Thus $\Delta A \leq 0.0103$ ~~radians~~ deg.

Ex 9 Approxim values of $\sqrt{29} = 5.385$ and $\sqrt{11} = 3.317$ are correct to four significant fig. Find the relative error in their sum and difference.

Sol →. No. $x_1 = 5.385$ and $x_2 = 3.317$ are correct to 4 sig. digits.

then absolute error $= \frac{1}{2} \times 10^{-3} = 0.0005$ , $X = x_1 + x_2 = 8.702$

$= 0.005$

Now $\Delta x_1 = 0.005$ and $\Delta x_2 = 0.005$

Now Relative error in their sum

$$\left|\frac{\Delta X}{X}\right| \leq \left|\frac{\Delta x_1}{X}\right| + \left|\frac{\Delta x_2}{X}\right| = \left|\frac{0.005}{\cancel{5.385}\ 8.702}\right| + \left|\frac{0.005}{\cancel{5.385}\ 8.702}\right|$$

$$= 1.149 \times 10^{-4}$$

Relative error $\leq 1.149 \times 10^{-4}$.

Relative error in their difference $\quad X = x_1 - x_2$, $X = 2.068$

$$\left|\frac{\Delta X}{X}\right| \leq \left|\frac{\Delta x_1}{X}\right| + \left|\frac{\Delta x_2}{X}\right|$$

$$= \left|\frac{0.0005}{2.068}\right| + \left|\frac{0.0005}{2.068}\right| = 4.835 \times 10^{-4}$$

Relative error $\leq 4.835 \times 10^{-4}$.

Ex 10 Sum the following numbers: 0.1532, 15.45, 0.000354, 305.1, 8.12, 143.3, 0.0212, 0.643 and 0.1734, where digits are correct.

Ex → Find the product of 346.1 and 865.2. State how many figures of the result are trustworthy? Given that the no. are correct to four significant figures.

Sol → $x_1 = 346.1, \quad x_2 = 865.2$

$X = x_1 x_2 \quad = 346.1 \times 865.2 = 299445.72 = 299446$

$\Delta X = \frac{\partial X}{\partial x_1}\Delta x_1 + \frac{\partial X}{\partial x_2}\Delta x_2 \qquad \Delta x_1 = 0.05 = \Delta x_2$

$\Delta X = x_2 \Delta x_1 + x_1 \Delta x_2$

$\frac{\Delta X}{X} = \frac{\Delta x_1}{x_1} + \frac{\Delta x_2}{x_2}$

$\left|\frac{\Delta X}{X}\right| \leq \left|\frac{\Delta x_1}{x_1}\right| + \left|\frac{\Delta x_2}{x_2}\right| = \frac{0.05}{346.1} + \frac{0.05}{865.2}$

$e_r \leq 0.000202$

$e_a = e_r \cdot X$

$= 0.000202 \times 299445.72$

$= 60.488035$

$= 60.49$ 4 sign

Thus true value will be $X \pm e_a$ ie value lies b/w $X - e_a$ and $X + e_a$

$299445.72 - 60.49$ & $299445.72 + 60.49$

$299385.23$ & $299506.21$.

3 figures are trustworthy.

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Ex) Given that $a = 10.00 \pm 0.05$

$b = 0.0356 \pm 0.0002$

$c = 15300 \pm 100$

$d = 62000 \pm 500.$

find the max. value of the absolute error in

(i) $a+b+c+d$ (ii) $a+5c-d$ (iii) $d^3$

(i) ↓

Max. absolute error

$|\Delta a+\Delta b+\Delta c+\Delta d|$

$\leq |\Delta a|+|\Delta b|+|\Delta c|+|\Delta d|$

$= 0.05+0.0002+100+500$

$= 600.0502$

(ii) ↓

Max. absol. error

$= |\Delta a+5\Delta c-\Delta d|$

$\leq |\Delta a|+5|\Delta c|+|\Delta d|$

$= 0.05+5(100)+500$

$= 1000.05$

(iii) ↓

absolute error

$3d^2\Delta d$

Max. absol. err

$= |3d^2\Delta d|$

$= 3(62000)^2(500)$

$= 5.766\times10^{12}.$

Ex→ If $R = \frac{1}{2}\left(\frac{r^2}{h}+h\right)$ — (1) and the error in R is at most 0.4%. find the percentage error allowable in r and h when $r = 5.1$ cm and $h = 5.8$ cm.

Sol→ Given $\frac{\Delta R}{R}\times100 = 0.4$ — (2)

(i) % error in $r = \frac{\Delta r}{r}\times100$

$= \frac{1}{r}\left(\Delta R.\frac{1}{\frac{\partial R}{\partial r}}\right)\times100$

$= \frac{1}{r}\left(\Delta R.\frac{h}{r}\right)\times100$

$= \frac{1}{r^2}\Delta R.h\times100$

$= \frac{100}{(5.1)^2}\times(0.0206)\times5.8$

$= 0.22968 \approx 2\%.$

$R = \frac{1}{2}\frac{r^2}{h}+\frac{h}{2}$

$\Delta R = \underbrace{\frac{\partial R}{\partial r}\times\Delta r}+\frac{\partial R}{\partial h}\times\Delta h$

Const

$\Delta r = \Delta R.\frac{1}{\frac{\partial R}{\partial r}}$

from (1) I

$\frac{\partial R}{\partial r} = \frac{1}{\cancel{2}}.\frac{\cancel{2}r}{h} = \frac{r}{h}$

from (2)

$\Delta R = \frac{0.4}{100}\times R$

$\Delta R = \frac{0.4}{100}\frac{1}{2}\left[\frac{(5.1)^2}{5.8}+5.8\right]$

$= 0.0206$

# Error in a Series Approximation

The error committed in a series approximation can be evaluated by using the remainder after n terms.

let Taylor's series for f(x) at $x=a$ is given by

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \cdots + \frac{(x-a)^{n-1}}{\lfloor n-1} f^{n-1}(a) + R_n(x)$$

where $R_n(x) = \frac{(x-a)^n}{\lfloor n} f^n(\theta)$ where $a < \theta < x$.
(error)

For a convergent series, $R_n(x) \to 0$ as $n \to \infty$. If we approximate f(x) by first n terms of series, then max. error committed is given by $R_n(x)$.

(Ex1) Find the no. of terms of the exponential series such that their sum given the value of $e^x$ correct to six decimal places at $x=1$.

Sol →

$$\underset{=f(x)}{e^x} = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots + \frac{x^{n-1}}{\lfloor n-1} + R_n(x).$$

where $R_n(x) = \frac{x^n}{\lfloor n} f(\theta)$

$$= \frac{x^n}{\lfloor n} e^\theta \quad , 0 < \theta < x$$

Max. absolute error $= \left| \frac{x^n}{\lfloor n} e^\theta \right|_{\theta = x} = \left| \frac{x^n}{\lfloor n} e^x \right|$

Max. Relative error $= \frac{\left| \frac{x^n}{\lfloor n} e^x \right|}{e^x} = \frac{x^n}{\lfloor n}$

Hence Max. Relative error at $x=1$, $(e_r)_{max} = \frac{1}{\lfloor n}$.

For a six decimal accuracy at $x=1$

$$\frac{1}{\lfloor n} < \frac{1}{2} \times 10^{-6} \quad \text{or} \quad \lfloor n > 2 \times 10^6$$

$$3628800 > 2000000$$

which is $\simeq 10$.

$\boxed{n \simeq 10.}$

$\lfloor 9 = 403,200$

Ex 2. Use the series $\log_e\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$ to compute the value of $\log(1.2)$ correct to seven decimal places and find the no. of terms retained.

Sol:

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1}\right] + R_n(x).$$

$$R_n(x) = \frac{2 \cdot x^{2n+1}}{2n+1} \log\left(\frac{1+\theta}{1-\theta}\right)$$

Now $\frac{1+x}{1-x} = 1.2$

$1 + x = 1.2 - 1.2x$

$2.2x = \cancel{2}1$

$1 - 1.2 = -(1+1.2)x$

$-0.2 = -2.2x$

$x = \frac{0.2}{2.2} = \frac{1}{11}$

$x = \frac{1}{11}$

Now max absolute error at $x = \frac{1}{11}$ $(\theta = x)$

$|R_n(x)|$ is

$$= \frac{2 \cdot \left(\frac{1}{11}\right)^{2n+1}}{2n+1} \log\left(\frac{1+x}{1-x}\right),$$

Max. absolute err $= \frac{1}{2(2n+1)}\left(\frac{1}{11}\right)^{2n+1}$.

For seven decimal accuracy,

$$\frac{2}{2n+1}\left(\frac{1}{11}\right)^{2n+1} \le \frac{1}{2} \times 10^{-7}$$

$$(2n+1)(11)^{2n+1} \ge 4 \times 10^{7}$$

40000000

$n=1$ $\quad 3(11)^3$

$n=2$ $\quad 5(11)^5$ $\quad$ 805255

$n=3$ $\quad 7(11)^7$ $\quad 7(19487171)$ $= \cancel{77948684}$ 136,410,197

which gives $n \le 3$

Hence retain the first three terms, we get

$$\log_e(1.2) = 2\left[\frac{1}{11} + \frac{\left(\frac{1}{11}\right)^3}{3} + \frac{\left(\frac{1}{11}\right)^5}{5}\right]$$

$= 0.1823215$

(Ex3) The func. $f(x) = \cos x$ can be expanded as

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots\cdots$$

Compute the no. of terms required to estimate $\cos\left(\frac{\pi}{4}\right)$ so that the result is correct to at least two significant digits.

Sol →

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots + R_n(x)$$

$$R_n(x) = \frac{(-1)^n x^{2n}}{\lfloor 2n} f(\theta) \qquad 0 < \theta < x$$

Max. absolute error at $\theta = x$

$$|R_n(x)| = \left| \frac{(-1)^n x^{2n}}{\lfloor 2n} \cos x \right| = \frac{x^{2n}}{\lfloor 2n} \cos x$$

Max. Relative error at $x = \pi/4$.

$$(e_r)_{max} = \left[ \frac{x^{2n}}{\lfloor 2n} \frac{\cancel{\cos x}}{\cancel{\cos x}} \right]_{x=\pi/4}$$

$$= \frac{\left(\frac{\pi}{4}\right)^{2n}}{\lfloor 2n}$$

Now two significant accuracy,

$$(e_r)_{max} \leq \frac{1}{2} \times 10^{-2}$$

$$\left(\frac{\pi}{4}\right)^{2n} \cdot \frac{1}{\lfloor 2n} \leq \frac{1}{2} \times 10^{-2}$$

$$\frac{\lfloor 2n}{(\pi/4)^{2n}} \geq 200 \Rightarrow \frac{\lfloor 2n}{(0.785)^{2n}} \geq 200$$

$$\boxed{n = 3}$$

# Solution of Algebraic and Trancedental Equation

Consider the equation of the form $f(x)=0$.

form of $f(x)$.

algebraic form of high degree

ex $(x^5+3x-1=0)$ - -

Trancedental form

(Contain, log, $e^x$, trignometric func.)

## ① Bisection Method

A numerical method in mathematics to find the sol$^n$ (root) [to find the sol$^n$ $f(x)=0$] of the given function.

* Root of a func. $f(x)$ is a value 'a' such that $f(a)=0$.

## Intermediate Value Theorem

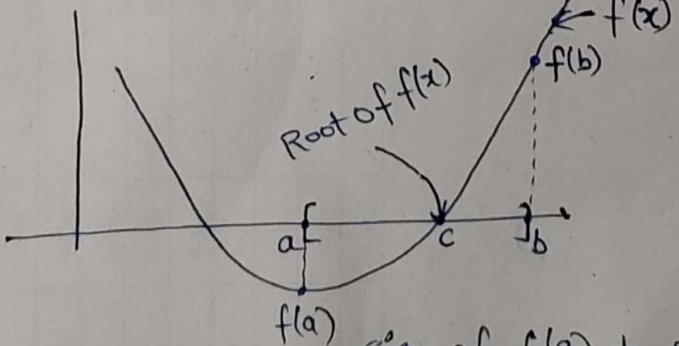
Imp

If a func. $f(x)$ is continous on the interval $[a,b]$ and sign of $f(a) \neq$ sign of $f(b)$, then

* There is a value $c \in [a,b]$ such that $f(c)=0$

i.e. $c$ is the root of $f(x)$ in the interval $[a,b]$.

Ex 1




$c \in [a,b]$.

sign of $f(a) \neq$ sign of $f(b)$

## Procedure of Bisection Method

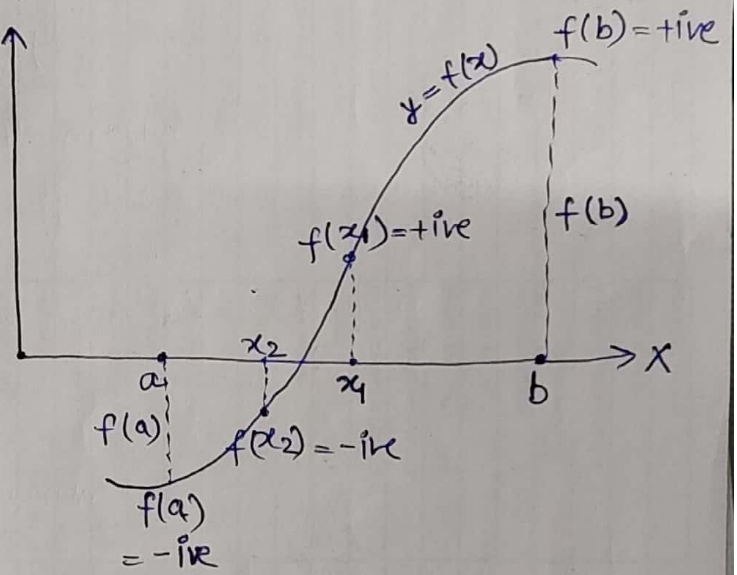
This method is based on the repeated application of inter-mediate value property.

Let $f(x)$ be continous b/w $a$ & $b$.

and $f(a) = +ive$
$f(b) = +ive$

then first approximation to the root is $x_1 = \frac{1}{2}(a+b)$.

Now check $f(x_1)$




If $f(x_1) = +ive$, so now root will lie in the interval $[a, x_1]$.

Then we again bisect this interval as

$$x_2 = \frac{a+x_1}{2} \quad \text{(second approx.)}$$

& $f(x_2) = -ive$

so now root will lie in $[x_2, x_1]$, so third approximation to the root is

$$x_3 = \frac{x_2+x_1}{2}$$ and so on.

Note :- ① No. of iterations required may be determined by

$$\frac{|b-a|}{2^n} \leq \epsilon.$$

or $$n \geq \frac{\log|b-a| - \log \epsilon}{\log 2}.$$

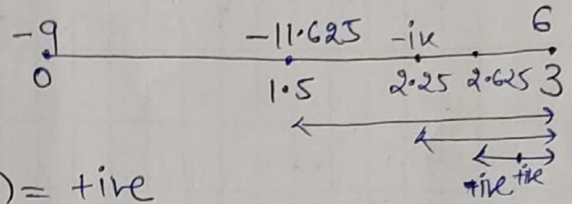
② Requires a large no. of iterations to achieve a certain accuracy to the root.

**Ex1** find a root of the eqⁿ $x^3-4x-9=0$ using the bisection method correct to four decimal places.

**Sol →** $f(x) = x^3 - 4x - 9$

Now $f(0) = -9$ $\quad f(1) = -12, \; f(2) = -9, \; f(3) = 6$

Interval $[\underset{a}{0} \;\; \underset{b}{3}]$




Since

$f(0) = -\text{ive}$, $f(3) = +\text{ive}$

So the root lies between 0 and 3.

First approximation :– $x_1 = \frac{a+b}{2} = \frac{3}{2} = 1.5$

$f(x_1) = (1.5)^3 - 4(1.5) - 9$

$= -11.625$

ie. $f(x_1) = -\text{ive}$ and $f(3) = +\text{ive}$.

So root lies b/w $x_1$ and 3.

Second approximation :– $x_2 = \frac{x_1+3}{2} = \frac{1.5+3}{2} = \frac{4.5}{2} = 2.25$

$f(2.25) = (2.25)^3 - 4(2.25) - 9$

$= -6.609375$

$f(x_2) = -\text{ive}$ and $f(3) = +\text{ive}$

So root lies b/w $x_2$ and 3.

Third Approximation :– $x_3 = \frac{x_2+3}{2} = \frac{2.25+3}{2} = 2.625$

$f(2.625) = -1.412109$

$f(x_3) = -\text{ive}$, $f(3) = +\text{ive}$

So root lies b/w $x_3$ and 3.

fouth Approximation :– $x_4 = \frac{x_3+3}{2} = \frac{2.625+3}{2} = 2.8125$

$f(2.8125) = 1.997314$

$f(2.625) = -\text{ive}$, $f(2.8125) = +\text{ive}$

root lies b/w $[2.625, 2.8125]$

**fifth Approximation :-**

$$x_5 = \frac{x_3 + x_4}{2}$$

$$= \frac{2.625 + 2.8125}{2}$$

$$= 2.71875$$

$$f(2.71875) = (2.71875)^3 - 4(2.71875) - 9$$

$$= 0.22091$$

$= +ive$

-ive +ive +ive

2.625 2.71875 2.8125 3

**Sixth approximation :-**

$$x_6 = \frac{2.625 + 2.71875}{2}$$

$$= 2.671875$$

$$f(2.671875) = -0.613208$$

$= -ive$

+ive -ive +ive

2.625 2.671875 2.71875

**Seventh approx. :-**

$$x_7 = \frac{2.671875 + 2.71875}{2}$$

$$= 2.6953125.$$

$$f(2.6953125) = -0.20058$$

$= -ive$

-ive -ive +ive

2.67187 2.6953 2.71875

**eighth approx. :-**

$$x_8 = \frac{2.6953125 + 2.71875}{2}$$

$$= 2.70703125$$

$f(x_8) = 0.009049$ +ive.

-ive +ive +ive

2.69 2.70 2.71

**Ninth approx :-**

$$x_9 = \frac{2.6953125 + 2.70703125}{2}$$

$$= 2.701171875$$

$$f(x_9) = -0.0960$$

-ive +ive

2.7011 2.7070

**Tenth approx :-**

$$x_{10} = \frac{2.701171875 + 2.70703125}{2}$$

$$= 2.7041015625$$

$f(x_{10}) = -0.04356$
$= -ive$

$-ive$ 2.7041 $+ive$ 2.7070

Eleventh approx :- $x_{11} = \frac{2.7070325 + 2.7041015625}{2}$

$= 2.705566$

$f(x_{11}) = -0.0172$
$= -ive$

$-ive$ 2.7055 $+ive$ 2.7070

12th approx. :→ $x_{12} = \frac{2.705566 + 2.707031}{2}$

$= 2.7062985$

$f(x_{12}) = -0.0041242192$

$-ive$ 2.7062 $+ive$ 2.7070

13th approx. :- $x_{13} = \frac{2.7062985 + 2.707031}{2}$

$= 2.70666475$

$f(x_{13}) = +0.0024$

$-ive$ 2.7062 $+ive$ 2.7066

14th approx :- $x_{14} = \frac{2.70666475 + 2.7062985}{2}$

$= 2.706481625$

$f(x_{14}) = -0.0083$
$= -ive$

$-ive$ 2.7064 $+ive$ 2.7066

15th approx $x_{15} = \frac{2.70666475 + 2.706481625}{2}$

$= 2.706573188$

$f(x_{15}) = 0.0008131287$
$= +ive$

$-ive$ 2.7064 $+ive$ 2.7065

16th approx $x_{16} = \frac{2.706573188 + 2.706481625}{2} = 2.706527$

$\boxed{x_{16} = 2.706527}$

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Now $x_{15} = 2.706573188$

$x_{16} = 2.706527$

$\therefore$ $\underbrace{2.7065}$ is correct to 4 decimal places.

7

**Ex2** Perform five iterations of bisection method to obtain the smallest +ive root of the eq$^n$.

$$f(x) \equiv x^3 - 5x + 1 = 0$$

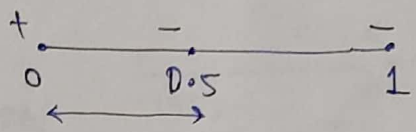
Sol → $f(0) = 1$, $f(1) = 1-5+1 = -3$, $a=0, b=1$

root lies b/w 0 & 1.

**first Approx.** :- $x_1 = \frac{a+b}{2} = \frac{1}{2} = 0.5$

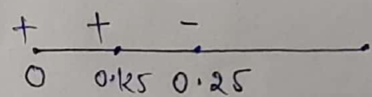
$f(0.5) = (0.5)^3 - 5(0.5) + 1$

$= -1.375$




**2$^{nd}$ Approx** $x_2 = \frac{0+0.5}{2} = 0.25$

$f(0.25) = -0.2343$

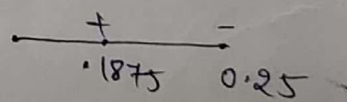



**3$^{rd}$ Approx** $x_3 = \frac{0+0.25}{2} = 0.125$

$f(0.125) = 0.3769$

**4$^{th}$ Approx** $x_4 = \frac{0.125+0.25}{2} = 0.1875$

$f(0.1875) =$ +ive $(0.6909)$

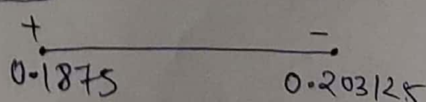



**5$^{th}$ Approx** $x_5 = \frac{.1875+0.25}{2} = 0.21875$

$f(0.21875) =$ −ve.

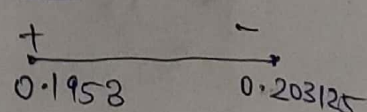

**6$^{th}$ Approx** $x_6 = \frac{.1875+.21875}{2} = 0.203125$

$f(0.203125) =$ −ve.




**7$^{th}$ Approx.** $x_7 = \frac{0.1875+0.203125}{2} = 0.1953125$

$f(0.1953125) =$ +ive

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0.2

8th Approx

$$x_8 = \frac{0.1953125 + 0.203125}{2} = 0.19921875$$

$f(x_8) =$ -ve.

+ -

·1953 ·19921

9th Approx

$x_9 =$ ~~0.19726~~

$$\frac{0.1953125 + 0.19921875}{2}$$

+ +

·1953 ·19726

$= 0.197265625$

$f(x_9) =$ 0.0213 +ive.

+ +

·1953 ·1972

10th Approx

$$x_{10} = \frac{.197265625 + 0.19921875}{2}$$

$= \underbrace{0.198}2421875$

$f(x_{10}) =$ +ive

+ -

0.198 ·19

11th Approx

$x_{11} = \underbrace{0.198}73$

$f(x_{11}) =$ +.

Ex 4) Use bisection method to find out the positive square root of 30 correct to 4 decimal places.

Sol →

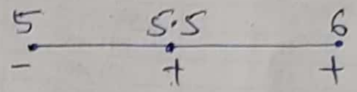
$f(x) = x^2 - 30 = 0$    let $x = \sqrt{30}$

$x^2 - 30 = 0$

$f(5) = -5, \quad f(6) = 6$

Root lies b/w 5 and 6.

first Approx. :- $x_1 = \frac{5+6}{2} = 5.5$

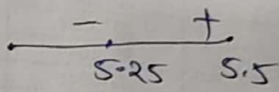



$f(x_1) = +ve$

root lie b/w 5 and 5.5

Second Approx :- $x_2 = \frac{5+5.5}{2} = 5.25$

$f(x_2) = -ve$




root lie b/w 5.25 & 5.5

3rd Approx :- $x_3 = \frac{5.25+5.5}{2} = 5.375$

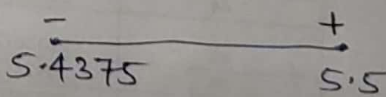
$f(x_3) = -ve$

root lie b/w 5.375 & 5.5


4th Approx :- $x_4 = \frac{5.375+5.5}{2}$

$= 5.4375$

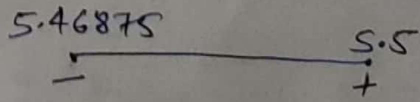
$f(x_4) = -ve$




root lie b/w 5.4375 & 5.5

5th Approx :- $x_5 = \frac{5.4375+5.5}{2} = 5.46875$

$f(x_5) = -ve$




6th Approx :- $x_6 = 5.484375$

$f(x_6) = +ve$

7th Approx. :- $x_7 = \frac{5.464375 + 5.484375}{2}$

$= 5.47438$

$f(x_7) = -ve$

8th Approx :- $x_8 = \frac{5.47438 + 5.48438}{2}$

-ve 5.47438 + 5.48438

$= 5.47938$

$f(x_8) = +ve.$

− 5.47438 + 5.47938

9th Approx. :- $x_9 = \frac{5.47438 + 5.47938}{2}$

$= 5.47688$

$f(x_9) = -ve.$

− 5.47688 + 5.47938

10th Approx :- $x_{10} = 5.47813$

$f(x_{10}) = +ve$

− 5.47688 + 5.47813

11th Approx :- $x_{11} = \frac{5.47688 + 5.47813}{2}$

$= 5.47751$

$f(x_{11}) = +ve$

− 5.47688 + 5.47751

12th Approx :- $x_{12} = \frac{5.47688 + 5.47751}{2}$

$= 5.47720$

$f(x_{12}) = -ve$

-ve 5.47720 + 5.47751

13th Approx :- $x_{13} = 5.477355$

$f(x_{13}) = +ve$

− 5.47720 + 5.47736

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$14^{th}$ Approx :- $x_{14} = \dfrac{5.47720 + 5.47736}{2}$

$= 5.47728$

$f(x_{14}) =$ +ve

$-$ 5.47720 —— $+$ 5.47728

15th Approx :- $x_{15} = \dfrac{5.47728 + 5.47720}{2}$

$x_{15} = 5.47724$

$f(x_{15}) =$ +ve

$-$ 5.47720 —— $+$ 5.47728

$$\boxed{\begin{aligned} x_{14} &= 5.47720 \\ x_{15} &= 5.47728 \end{aligned}}$$

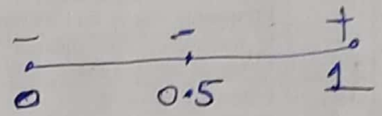
Ans. $= 5.4772$

Q→ Find a positive real root of $x - \cos x = 0$ by bisection method, correct to 3 decimal places.

Sol→ $f(x) = x - \cos x$

$f(0) = -1$, $f(1) = +ve$ $= 0.4596$

Root lie b/w $= \frac{0+1}{2} = 0$ & 1.

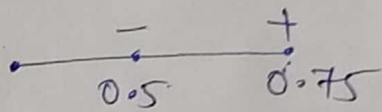



1st approx :- $x_1 = \frac{0+1}{2} = 0.5$

$f(x_1) = -ve$

root lie b/w 0.5 & 1

2nd approx :- $x_2 = \frac{0.5+1}{2} = 0.75$



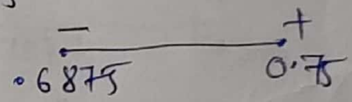

$f(x_2) = +ve$

root lies b/w 0.5 & 0.75.

3rd approx. :- $x_3 = \frac{0.5+0.75}{2} = 0.625$

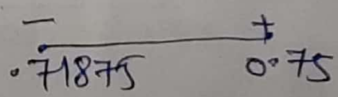

$f(x_3) = -ve$

4th approx $x_4 = \frac{0.625+0.75}{2} = 0.6875$



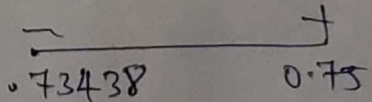

$f(x_4) = -ve$

5th Approx $x_5 = \frac{0.6875+0.75}{2} = 0.71875$




$f(x_5) = -ve$

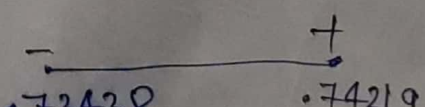
6th Approx $x_6 = \frac{0.71875+0.75}{2} = 0.734375$
$= 0.73438$




$f(x_6) = -ve$

7th Approx $x_7 = \frac{0.73438+0.75}{2} = 0.74219$

$f(x_7) = +ve$



8th Approx. :-

$$x_8 = \frac{0.73438 + 0.74219}{2}$$

$$= 0.73828$$

$f(x_3) = -ve$

$-$ .73828 $+$ .74219

9th Approx

$$x_9 = \frac{0.73828 + 0.74219}{2}$$

$$= 0.74024$$

$f(x_9) = +ve$

$-$ .73828 $+$ .74024

10th Approx

$$x_{10} = \frac{0.73828 + 0.74024}{2}$$

$$= 0.73926$$

$f(x_{10}) = +ve$

$-$ .73828 $+$ .73926

11th Approx.

$$x_{11} = \frac{0.73828 + 0.73926}{2}$$

$$= 0.73877$$

$f(x_{11}) = -ve$

12th Approx.

$-$ .73877 $+$ .73926

$$x_{12} = \frac{0.73877 + 0.73926}{2}$$

$$= 0.73902$$

$f(x_{12}) = -ve$

$-$ .73902 $+$ .73926

13th Approx

$$x_{13} = \frac{0.73902 + 0.73926}{2}$$

$$= 0.73914$$

$x_{12} = \underline{0.73902}$

$x_{13} = \underline{0.73914}$

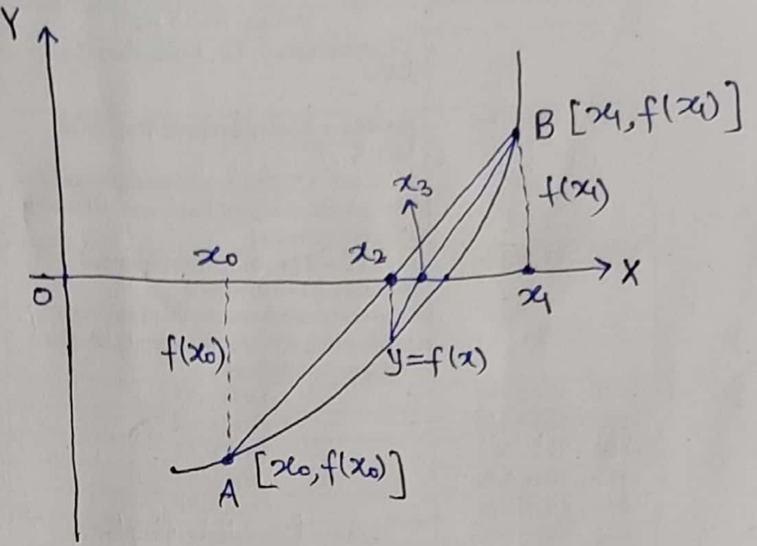
three decimal place accuracy.

# ② Regula-Falsi Method
## (Method of False position)

The bisectio method is very slow ( however it guaranteed converge).

In this method, we take two points $x_0$ and $x_1$ s.t. $f(x_0)$ and $f(x_1)$ are of opposite signs.

~~Since~~ So the root must lie in between these points.




Eq$^n$ of the chord joining pts. $\{x_0, f(x_0)\}$ and $\{x_1, f(x_1)\}$

$$y - f(x_0) = \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right](x_1 - x_0)$$

The method consists in replacing the curve AB by means of the chord AB and taking the pt. of intersection of the chord with X-axis as an approximation to the root.

Now. at $y=0$ (x-axis), the abscissa of the pt. where chord cuts $y=0$ is given by

$$0 - f(x_0) = \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right][x_2 - x_0]$$

$$\Rightarrow \quad x_2 = x_0 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_0).$$

which is the approximation to the root.

$$x_5 = 0.49402 - \left[\frac{1 - 0.49402}{-2.17798 - 0.07079}\right](0.07079)$$

$$= 0.50995$$

$f(x_5) = 0.02360$

$+$ ——— $-$
$.50995$ $1$

Now $x_0 = 0.50995$
$f(x_0) = 0.02360$

$$x_6 = 0.50995 - \left[\frac{1 - 0.50995}{-2.17798 - 0.02360}\right](0.02360)$$

$$= 0.51520$$

$f(x_6) = 0.00776$

$+$ ——— $-$
$0.51520$ $1$

Now $x_0 = 0.51520$
$f(x_0) = 0.00776$

$$x_7 = 0.51520 - \left[\frac{1 - 0.51520}{-2.17798 - 0.00776}\right](0.00776)$$

$$= 0.51692$$

Now $x_6 = 0.51520$
$x_7 = \underline{0.51}692$

2 decimal place accuracy.

Hence root is $\boxed{0.51}$ $A_2$.

(Ex1) Find the root of the eq$^n$ $xe^x = \cos x$ in the interval $(0,1)$ using Regula-Falsi method correct to 2 decimal places.

Sol → Let $f(x) \equiv \cos x - xe^x = 0$

$f(0) = 1$, $f(1) = \cos 1 - e = -2.17798$

root lies b/w 0 & 1.

By Regula-Falsi Method

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

$x_0 = 0$, $f(x_0) = 1$, $x_1 = 1$, $f(1) = -2.17798$

$$x_2 = 0 - \left[\frac{(1-0)}{-2.17798 - 1}\right] 1 = 0.31467$$

+    -ve
0   0.31467   1
$x_0$   → $x_0$

$f(x_2) = 0.51987$,

root lies b/w 0.31467 and 1.

Now $x_0 = 0.31467$, $f(x_0) = 0.51987$

Now $x_3 = .31467 - \left(\dfrac{1 - 0.31467}{-2.17798 - 0.51987}\right)(0.51987)$

$= 0.44673$

$f(x_3) = 0.20356$
$= +ive$

+    −
0.44673   1
$x_0$   $x_1$

Now $x_0 = .44673$, $f(x_0) = 0.20356$

$$x_4 = 0.44673 - \left[\frac{1 - 0.44673}{-2.17798 - 0.20356}\right](0.20356)$$

$= 0.49402$

$f(x_4) = 0.07079$

+    −
.49402   1

Now $x_0 = 0.49402$
$f(x_0) = 0.07079$

# Newton's - Raphson Method

Expanding $f(x_0+h)$ by Taylor's Series,

$$f(x_0) + h f'(x_0) + \frac{h^2}{\lfloor 2} f''(x_0) + \cdots = 0$$

take first two steps

$$f(x_0) + h f'(x_0) = 0$$

$$h = -\frac{f(x_0)}{f'(x_0)}$$

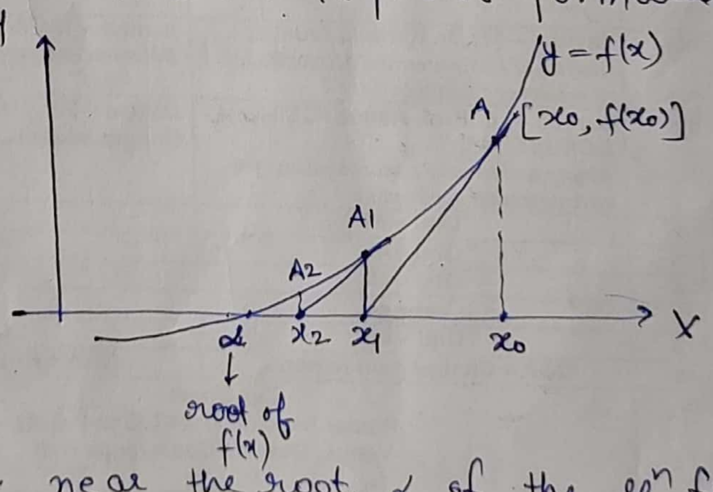
A better approximation than $x_0$ is therefore given by $x_1$, where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximations are given by $x_2, x_3, \cdots, x_{n+1}$,

where
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \cdots$$

which is the Newton-Raphson formula.




Let $x_0$ be a pt. near the root $\alpha$ of the eq$^n$ $f(x) = 0$, then tangent at $A\,[x_0, f(x_0)]$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

It cuts x-axis ($y = 0$ & $x = x_1$)

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

which is the $1^{st}$ approximation to the root $\alpha$.

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Ex(2) find the real root of $x = e^{-x}$ using Newton-Raphson method.

sol→ $f(x) = x - e^{-x}$ , $f'(x) = 1 + e^{-x}$

$f(0) = 0 - e^{-0} = -1$

$f(1) = 1 - e^{-1} = +ive.$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n - e^{-x_n})}{(1 + e^{-x_n})}$$

$$= \frac{\cancel{x_n} + x_n e^{-x_n} - \cancel{x_n} + e^{-x_n}}{1 + e^{-x_n}} = \frac{(x_n + 1) e^{-x_n}}{1 + e^{-x_n}}$$

put $n = 0$.

$$x_1 = \frac{(x_0 + 1) e^{-x_0}}{1 + e^{-x_0}}$$ take $x_0 = 0.5$.

$$x_1 = \frac{(1 + 0.5) e^{-0.5}}{1 + e^{-0.5}} = \frac{0.9098}{1.6065} = 0.5663$$

$$x_2 = \frac{(1 + 0.5663) e^{-0.5663}}{1 + e^{-0.5663}} = \frac{0.8892}{1.5676} = 0.5672$$

$$x_3 = \frac{(1 + 0.5672) e^{-0.5672}}{1 + e^{-0.5672}} = \frac{0.8888}{1.5671} = 0.5671$$

$$x_4 = \frac{(1 + 0.5671) e^{-0.5671}}{1 + e^{-0.5671}} = \frac{0.8888}{1.5671} = 0.5671$$

$$\boxed{x_3 = x_4}$$

root of the eqn is 0.5671.

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Q→ Using Newton-Raphson method, find the real root of the eqn $3x = \cos x + 1$ correct to 4 decimal places.

Sol →
$$f(x) = 3x - \cos x - 1$$
$$f(0) = -2$$
$$f(1) = 3 - \cos 1 - 1 = 1.4597 = +ive.$$

root of $f(x)$ lies b/w 0 & 1.

0 ——— 0.5 or 0.6 ——— 1

take $x_0 = 0.6$ as a root.

$$f(x_n) = 3x_n - \cos x_n - 1$$
$$f'(x_n) = 3 + \sin x_n$$

Newton's iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{(3x_n - \cos x_n - 1)}{(3 + \sin x_n)}$$
$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}$$

put $n = 0$
$$x_1 = \frac{x_0 \sin(x_0) + \cos x_0 + 1}{3 + \sin x_0}$$
$$= \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)} = 0.6071$$

put $n = 1$
$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1}$$
$$= \frac{(0.6071)\sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$
$$= 0.6071.$$

Here $x_1 = x_2$

So 0.6071 is the root correct to 4 dec places.

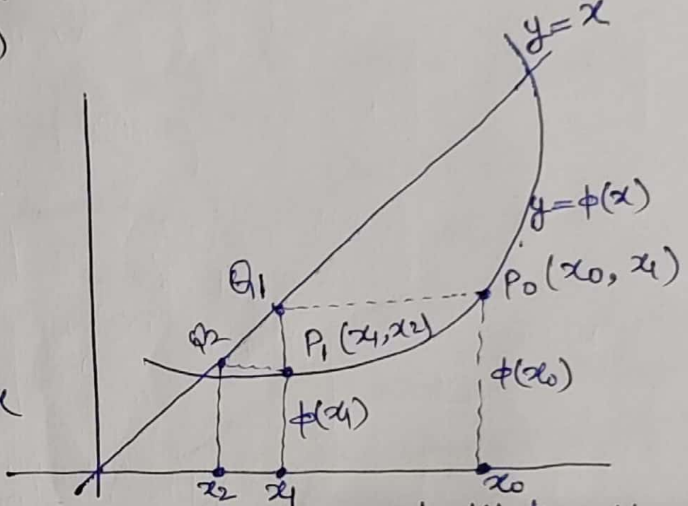
# Fixed pt. interative Method
or
# Successive approximation Method

**Fixed pt.** :⇒ fixed pt. is a point at which input is equal to output.

$$x = \phi(x).$$

$$x = \phi(x)$$

In this method $f(x)=0$ is written as $x=\phi(x)$.




**Method** :⇒

① choose the initial value $x_0$ which lies in the range of roots.

[ we find the interval of range first by substituting the value $x=a$ & $x=b$ for which $f(a)<0$ and $f(b)>0$. Take $x_0$ as average of a & b].

② Express the given eq$^n$, in the form $x=\phi(x)$ such that $|\phi'(x)|<1$ at $x=x_0$.

{ If there are more than one possibility of $\phi(x)$, choose the $\phi(x)$ which has the minimum value of $\phi'(x)$ at $x=x_0$ }.

③ Now $1^{st}$ approx. $x_1 = \phi(x_0)$

$2^{nd}$ appr $x_2 = \phi(x_1)$

thus we get the iteration form

$$\boxed{x_{n+1} = \phi(x_n)}, \quad n = 0,1,2 \dots$$

Q→ $f(x) = x^3 + x^2 - 1 = 0 \qquad (0,1)$

$x^3 + x^2 - 1 = 0$

$x = (1-x^2)^{1/3}$
$\phi(x)$

$\phi'(x) = \frac{1}{3}(1-x^2)^{-2/3}(-2x^2)$

$|\phi'(x)| = \left|\frac{2}{3}\frac{x^2}{(1-x^2)^{2/3}}\right|$

at $x_0 = 0.5$

$= \left|\frac{2}{3}\frac{(0.5)^2}{(0.5)^{2/3}}\right|$

$< 1$

$x^2 = 1 - x^3$

$x = (1-x^3)^{1/2}$
$\phi(x)$

$\phi'(x) = \frac{1}{2}(1-x^3)^{-1/2}(-3x^2)$

$|\phi'(x)| = \left|\frac{3}{2}\frac{x^2}{(1-x^3)^{1/2}}\right|$

at $x_0 = 0.5$

$< 1$

$x^2(1+x) - 1 = 0$

$x^2 = \frac{1}{1+x}$

$x = \sqrt{\frac{1}{1+x}}$
$\phi(x)$

$\phi'(x) = \left|\frac{1}{2\sqrt{1+x}}\right|$

$|\phi'(x)| < 1$

at $x_0 = 0.5$

(which gives minimum choose that range)

Q→ $f(x) \equiv x^3 + 6x^2 + 10x - 20 = 0$ near $x = 1$.

$x = (20 - 6x^2 - 10x)^{1/3}$

$\phi'(x) = \frac{1}{3}(20 - 6x^2 - 10x)^{-2/3}(-12x - 10)$

$|\phi'(x)| = \frac{12x + 10}{3(20 - 6x^2 - 10x)^{2/3}}$

at $x = 1$

$|\phi'(x)| = \frac{22}{3(4)^{2/3}} > 1$

$= 2.91$

No value satisfies $|\phi'(x)| < 1$

$x = (20 - 10x - x^3)^{1/2}$

$\phi'(x) = \frac{1}{2}(20 - 10x - x^3)^{-1/2}(-10 - 3x^2)$

$|\phi'(x)| = \frac{(10 + 3x^2)}{2(20 - 10x - x^3)^{1/2}}$

at $x = 1$

$|\phi'(x)| = \frac{13}{2(11)^{1/2}}$

$= 1.95 > 1$

Fitting
Exact
straling

19-7L

Ex1 find a real root of the $eq^n$ $\cos x = 3x-1$ correct to 3 decimal places using iteration method.

Sol → $f(x) = \cos x - 3x + 1 = 0$

$f(0) = 2 = +ve$, $f(1) = -ve$.

root lie in the Interval $(0,1)$.

Now rewrite the $eq^n$ as $\cos x = 3x-1$

$$3x = \cos x + 1$$

$$x = \frac{1}{3}[\cos x + 1] = \phi(x)$$

Now $\phi'(x) = -\frac{1}{3}\sin x$

$$|\phi'(x)| = \left|-\frac{1}{3}\sin x\right| = \frac{1}{3}|\sin x| < 1 \quad \text{in } (0,1).$$

Hence iteration method can be applied and start with $x_0 = 0$.

$$x_n = \phi(x_{n-1})$$

$n = 1, 2, 3, 4, 5 - - - - -$, $x_0 = 0$

$x_1 = \phi(x_0)$

$$x_1 = \frac{1}{3}[\cos x_0 + 1] = \frac{1}{3}[\cos 0 + 1] = \frac{2}{3} = 0.6667$$

$$x_2 = \frac{1}{3}[\cos x_1 + 1] = \frac{1}{3}[\cos(0.6667) + 1] = 0.5953$$

$$x_3 = \frac{1}{3}[\cos(0.5953) + 1] = 0.6093$$

$$x_4 = \frac{1}{3}[\cos(0.6093) + 1] = 0.6067$$

$$x_5 = \frac{1}{3}[\cos(0.6067) + 1] = 0.6072$$

$$x_6 = \frac{1}{3}[\cos(0.6072) + 1] = 0.6071$$

root is $0.607$

Ex2) find a real root of $2x - \log_{10} x = 7$ correct to four decimal places using iteration method.

Sol →

$f(x) = 2x - \log_{10} x - 7$

$f(1) = -5$ , $f(2) = -3.3010$ , $f(3) = -1.4771$

$f(4) = 0.3979$ (+ve)

$2x - \log_{10} x - 7 = 0$

$2x = \log_{10} x + 7 \Rightarrow x = \frac{1}{2}[\log_{10} x + 7]$

$\phi(x) = \frac{1}{2}[\log_{10} x + 7]$

$\phi'(x) = \frac{1}{2}\left[\frac{1}{x}\log_{10} e\right]$

$|\phi'(x)| = \left|\frac{1}{2}\frac{0.4343}{x}\right|$

$3 < x < 4$

$|\phi'(x)| < 1$

Hence iteration method can be applied.

take $x_0 = 3.6$ ,

$x_1 = \phi(x_0)$

$= \frac{1}{2}[\log_{10} 3.6 + 7] = 3.77815$.

$x_2 = \phi(x_1) = 3.78863$

$x_3 = 3.78924$

$x_4 = 3.78927$

Ans $3.7892$ //.

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# Order of Convergence

Measure of effectiveness with which each iteration reduces the approximation error.
or simply by which rate you are approaching to next approximation from the previous one. [ $1^{st} \to 2^{nd} \to 3^{rd} \to$ ]

Let $f(x)=0$ be an $eq^n$; we are solving this equation by iterative methods and ~~get~~ end up with $x_0, x_1, x_2, ---$
and let $\alpha$ be the exact root of the above $eq^n$, so our sequence $\{x_n\}$ is trying to converge to root $\alpha$.
So the error in each of this iteration is

$$e_n = -x_n + \alpha \quad \text{i.e.} \quad e_n = \alpha - x_n$$

or $\boxed{x_n = e_n + \alpha}$

The order of convergence is evaluated by

$$\boxed{|e_{n+1}| \cancel{\leq} A\,|e_n|^k}$$

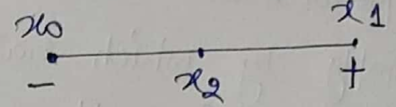
OR

If $|e_{n+1}| = A\,|e_n|^k$, then method is of order $k$.

$A =$ Asymptotic err constant and depend on the derivative of $f(x)$ at $x=\alpha$.

# Order of convergence of Bisection Method

Let $f(x)=0$ be an eqⁿ and $\alpha$ be the exact root of this eqⁿ so $f(\alpha)=0$




$x_2, x_3, \dots x_n$ are the approx.

Let $x_n$ be differ from $\alpha$ by error $e_n$.

ie. $x_n = e_n + \alpha$ , $x_{n+1} = e_{n+1} + \alpha$

$x_2 = \dfrac{x_0 + x_1}{2}$

Generalize it

$$x_{n+1} = \frac{x_{n-1} + x_n}{2}$$

$$e_{n+1} + \alpha = \frac{(e_{n-1} + \alpha) + (e_n + \alpha)}{2}$$

$$e_{n+1} + \cancel{\alpha} = \cancel{\alpha} + \left(\frac{e_{n-1} + e_n}{2}\right)$$

$$e_{n+1} = \frac{e_n + e_{n-1}}{2}$$

$$= \frac{e_n}{2}\left[1 + \frac{e_{n-1}}{e_n}\right]$$

$\left\{ \dfrac{e_{n-1}}{e_n} \text{ is very small. so neglect it} \right\}$

$$e_{n+1} \simeq \frac{e_n}{2}$$

or

$$e_{n+1} \simeq \frac{1}{2} e_n$$

Now

$$e_{n+1} = A\, e_n^{K}$$

once compare we get

$A = \frac{1}{2}$ and $\boxed{K=1}$

means linear convergence.

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## Order of Convergence of N-R ~~Bisection~~ Method

$f(x)=0$ be an eqn

$\alpha$ is the exact root so $f(\alpha)=0$

$e_n = x_n + \alpha$ , $e_{n+1} = x_{n+1} + \alpha$

N-R Method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$e_{n+1} + \cancel{\alpha} = (e_n + \cancel{\alpha}) - \frac{f(e_n+\alpha)}{f'(x_n+\alpha)}$$

$$e_{n+1} = e_n - \left[\frac{\cancelto{0}{f(\alpha)} + e_n f'(\alpha) + \frac{e_n^2}{\lfloor 2} f''(\alpha) + \cdots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{\lfloor 2} f'''(\alpha) + \cdots}\right].$$

$$e_{n+1} = e_n - \left[\frac{e_n f'(\alpha)}{f'(\alpha) + e_n f''(\alpha)}\right]$$

$e_n^2, e_n^3 \dots$ small so neglect those terms.

$$= \frac{\cancel{e_n f'(\alpha)} + e_n^2 f''(\alpha) - \cancel{e_n f'(\alpha)}}{f'(\alpha) + e_n f''(\alpha)}$$

$$\frac{e_{n+1}}{1} = \frac{e_n^2 f''(\alpha)}{f'(\alpha)\left[1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\right]}$$

$$\frac{e_{n+1}}{e_n^2} = \frac{f''(\alpha)}{f'(\alpha)} \left[1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

$$= \frac{f''(\alpha)}{f'(\alpha)} \left[1 - e_n \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]$$

$$\frac{e_{n+1}}{e_n^2} \simeq \frac{f''(\alpha)}{f'(\alpha)}$$

$$e_{n+1} \simeq \frac{f''(\alpha)}{f'(\alpha)} e_n^2$$

here $A = \frac{f''(\alpha)}{f'(\alpha)}$

$K=2$

order of convergence $= 2$.

# Order of Convergence of Regula-Falsi

Let $f(x)=0$ be a eq$^n$ and $\alpha$ be the exact root i.e.

$f(\alpha)=0$, $x_n$ is differ from $\alpha$ by $e_n$.

$$x_n = \alpha + e_n, \quad x_{n+1} = e_{n+1} + \alpha$$

Method $\quad x_{n+2} = \dfrac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}$

$$e_{n+2} + \alpha = \frac{(\alpha+e_n) f(\alpha+e_{n+1}) - (e_{n+1}+\alpha) f(e_n+\alpha)}{f(e_{n+1}+\alpha) - f(e_n+\alpha)}$$

$$e_{n+2} + \cancel{\alpha} = \cancel{\alpha} + \frac{e_n f(\alpha+e_{n+1}) - e_{n+1} f(\alpha+e_n)}{f(e_{n+1}+\alpha) - f(e_n+\alpha)}$$

$$e_{n+2} = \frac{e_n\left[f(\alpha) + e_{n+1} f'(\alpha) + \frac{(e_{n+1})^2}{\lfloor 2} f''(\alpha) + \cdots\right] - e_{n+1}\left[f(\alpha) + e_n f'(\alpha) + \frac{(e_n)^2}{\lfloor 2} f''(\alpha) + \cdots\right]}{\left[f(\alpha) + (e_{n+1}) f'(\alpha) + \cdots\right] - \left[f(\alpha) + e_n f'(\alpha) + \cdots\right]}$$

$\boxed{f(\alpha)=0}$

$$= \frac{\left[\cancel{e_n e_{n+1} f'(\alpha)} + \frac{e_n (e_{n+1})^2}{\lfloor 2} f''(\alpha)\right] - \left[\cancel{e_n e_{n+1} f'(\alpha)} - \frac{e_{n+1}}{2} e_n^2 f''(\alpha)\right]}{e_{n+1} f'(\alpha) + \frac{e_{n+1}^2}{2} f''(\alpha) - e_n f'(\alpha) - \frac{e_n^2}{\lfloor 2} f''(\alpha)}$$

$$= \frac{e_n e_{n+1} f''(\alpha)\left[e_{n+1} - e_n\right]}{f'(\alpha)\left[e_{n+1} - e_n\right] + \frac{f''(\alpha)}{2}\left(e_{n+1}^2 - e_n^2\right)}$$

$$= \frac{e_n e_{n+1} f''(\alpha)\left[\cancel{e_{n+1} - e_n}\right]}{2\left(\cancel{e_{n+1} - e_n}\right)\left[f'(\alpha) + \frac{f''(\alpha)}{2}(e_n + e_{n+1})\right]}$$

$$= \frac{e_n e_{n+1}}{2} \frac{f''(\alpha)}{f'(\alpha)}\left[1 + \frac{f''(\alpha)}{2f'(\alpha)}(e_n + e_{n+1})\right]^{-1}$$

$$e_{n+2} = \frac{e_n e_{n+1}}{2} \frac{f''(\alpha)}{f'(\alpha)}\left[1 - \frac{f''(\alpha)}{2f'(\alpha)}(e_n + e_{n+1})\right]$$

$$e_{n+2} \simeq e_n e_{n+1} \frac{f''(\alpha)}{2f'(\alpha)}$$

$$e_{n+2} \simeq M \cdot e_n e_{n+1} \quad \text{—} (1)$$

Now $e_{n+1} = A e_n^k$ — (2)

$$e_n = \left(\frac{e_{n+1}}{A}\right)^{1/k} \quad \text{—} (3)$$

~~put (2) in (1)~~

$$\cancel{e_{n+2} = M \left(\frac{e_{n+1}}{A}\right)^{1/k} e_{n+1}}$$

$$\cancel{e_{n+2} = e_n e_{n+1} \left(\frac{e_{n+1}}{A}\right)^{1/k} e_{n+1}}$$

$$\cancel{e_{n+2} =}$$

Now from (2) $\quad e_{n+2} = A e_{n+1}^k$ — (4)

put the values from (1), (3) in (4)

$$M \cdot e_n e_{n+1} = A e_{n+1}^k$$

$$M \cdot \left(\frac{e_{n+1}}{A}\right)^{1/k} e_{n+1} = A e_{n+1}^k$$

$$M A^{-\frac{1}{k}} e_{n+1}^{\frac{1}{k}+1} = A e_{n+1}^k$$

Compare the coeff. of $e_{n+1}$

$$\frac{1}{k} + 1 = k$$

$$k^2 = k + 1$$

$$k^2 - k - 1 = 0$$

$$k = \frac{1 \pm \sqrt{1+4}}{2}$$

$$k = \frac{1+\sqrt{5}}{2} \simeq 1.618$$

$$\boxed{k = 1.618}$$

## Order of Convergence of fixed-pt. Iteration Method

The eqⁿ is (formula)

$$y = f(x) = x - \phi(x)$$

or

$$x_{n+1} = \phi(x_n)$$

$$e_{n+1} + \alpha = \phi(e_n + \alpha)$$

$$e_{n+1} + \alpha = \phi(\alpha) + e_n \phi'(\alpha) + \frac{e_n^2}{\lfloor 2} \phi''(\alpha) + \cdots$$

Now $\Rightarrow$ $\phi(\alpha) = \alpha$

$$e_{n+1} + \cancel{\alpha} = \cancel{\phi(\alpha)} + e_n \phi'(\alpha) + \frac{e_n^2}{\lfloor 2} \phi''(\alpha) + \cdots$$

$$e_{n+1} \simeq e_n \phi'(\alpha) \quad \text{if } \phi'(\alpha) \neq 0$$

then $K = 1$, $A = \phi'(\alpha)$

If $\phi'(\alpha) = 0$ then

$$e_{n+1} \simeq \frac{e_n^2}{\lfloor 2} \phi''(\alpha)$$

$$A = \frac{\phi''(\alpha)}{2}, \quad K = 2$$

and so - on.

# Solution of Linear Equations

Consider a system of equation as

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\}_{3\times 3} \quad \text{—} (1)$$

Types of Methods to solve

**Direct Method**

1. Cramer's Rule
2. Matrix Inversion Method.
3. Gauss elimination method
4. Gauss Jordan method
5. factorization method
   or
   L-U decomposition method

**Iterative Method**

1. Jacobi's Iteration method
2. Gauss-Seidal Iter. method
3. S-O-R (Successive-over-relaxation method)

## ① Gauss-elimination Method

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution.

\# 1 On writing the given system of $eq^n$ as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or

$$AX = D$$

this method consists in transforming the coefficient matrix A to upper triangular matrix by elementary row trans-formations only.

\# 2 **Partial pivoting**

1) numerically largest coefficient of $x$ is selected and brought as the first pivot by interchanging the first $eq^n$ with the $eq^n$ having largest cofficient of $x$.

2) Now same step for $y$, but from the remaining $eq^n$.

3) Repeat for variables till we arrive at the $eq^n$ with single variable.

**Complete pivoting**

1) We can choose at each stage, the numerically largest coefficient of the entire matrix.

2) It requires not only an interchange of $eq^n$ but also an interchange of the position of the variables.

\# It is more complicated and doesn't improve the accuracy.

Ex1 Apply Gauss-Elimination method to solve the eqn

$x+4y-z=-5$
$x+y-6z=-12$
$3x-y-z=4$

Sol → with partial pivoting

$$A=\begin{bmatrix}1&4&-1\\1&1&-6\\3&-1&-1\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad D=\begin{bmatrix}-5\\-12\\4\end{bmatrix}$$

$$[A:D]=\begin{array}{c}\left[\begin{array}{ccc|c}1&4&-1&-5\\1&1&-6&-12\\3&-1&-1&4\end{array}\right]\end{array}\begin{array}{l}R_1\\R_2\\R_3\end{array}$$

numerically largest x ←

So $R_1 \leftrightarrow R_3$

$$=\left[\begin{array}{ccc|c}3&-1&-1&4\\1&1&-6&-12\\1&4&-1&-5\end{array}\right]\begin{array}{l}R_1\\R_2\\R_3\end{array}$$

↳ Numei largest coeff of y

so $R_2 \leftrightarrow R_3$

$$=\left[\begin{array}{ccc|c}3&-1&-1&4\\1&4&-1&-5\\1&1&-6&-12\end{array}\right]$$

Now $\left[R_2-\frac{1}{3}R_1\right]$, $\left[R_3-\frac{1}{3}R_1\right]$

$$=\left[\begin{array}{ccc|c}3&-1&-1&4\\0&\frac{13}{3}&-\frac{2}{3}&-\frac{19}{3}\\0&\frac{4}{3}&-\frac{17}{3}&-\frac{40}{3}\end{array}\right]=\left[\begin{array}{ccc|c}3&-1&-1&4\\0&\frac{13}{3}&\frac{-2}{3}&\frac{-19}{3}\\0&0&-\frac{71}{13}&\frac{-148}{13}\end{array}\right]$$

Now $R_2-\frac{12}{39}R_2$

Now

$$3x - y - z = 4 \quad \text{——} (1)$$

$$\frac{13}{3}y - \frac{2}{3}z = \frac{-19}{3} \quad \text{——} (2)$$

$$-\frac{71}{13}z = \frac{-148}{13} \Rightarrow -71z = -148$$

Back substitution

$$\boxed{z = 2.0845}$$

put z in (2)

$$\frac{13}{\cancel{3}}y - \frac{2}{\cancel{3}}(2.0845) = -\frac{19}{\cancel{3}}$$

$$13y = -14.8310$$

$$\boxed{y = -1.1408}$$

put y & z in (1).

$$3x + 1.1408 - 2.0845 = 4$$

$$3x = 4.9437$$

$$\boxed{x = 1.6479}$$

Hence $x = 1.6479$, $y = -1.1408$, $z = 2.0845$

Ans.

with complete pivoting

$$[A:D] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{array}\right]$$

Numerically large

So

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & -6 & -12 \\ 1 & 4 & -1 & -5 \\ 3 & -1 & -1 & 4 \end{array}\right]$$

Now

$$\begin{array}{cccc} z & y & x & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} -6 & 1 & 1 & -12 \\ -1 & 4 & +1 & -5 \\ -1 & -1 & 3 & 4 \end{array}\right]$$

$$R_2 \longleftrightarrow R_2 - \frac{1}{6}R_1$$

$$\left[\begin{array}{ccc|c} -6 & 1 & 1 & -12 \\ -1+1 & 4-\frac{1}{6} & 1-\frac{1}{6} & -5+\frac{12}{6} \\ -1+1 & -1-\frac{1}{6} & 3-\frac{1}{6} & 4+\frac{12}{6} \end{array}\right] = \left[\begin{array}{ccc|c} -6 & 1 & 1 & -12 \\ 0 & 23/6 & 5/6 & -18/6 \\ 0 & -7/6 & 17/6 & 36/6 \end{array}\right]$$

$$\left[\begin{array}{cccc} -6 & 1 & 1 & -12 \\ 0 & \frac{23}{6} & \frac{5}{6} & -\frac{18}{6} \\ 0 & \cancel{-\frac{7}{6}+\frac{23}{6}\times\frac{6}{23}\times\frac{7}{6}} \to 0 & \frac{17}{6}+\frac{5}{6}\times\frac{6\times 7}{23\times 6} & \frac{36}{6}+\frac{18}{6}\times\frac{6}{23}\times\frac{7}{6} \end{array}\right]$$

138

$$\begin{array}{cccc} z & y & x & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} -6 & 1 & 1 & -12 \\ 0 & 23/6 & 5/6 & 1/6 \\ 0 & 0 & \frac{71}{23} & \cancel{\frac{835}{138}}\ \frac{117}{23} \end{array}\right]$$

Now $-6z + y + x = -12$ — (1)

$\frac{23}{6}y + \frac{5}{6}x = \frac{-18}{6}$ — (2)

$\frac{71}{23}x = \cancel{\frac{835}{1386}}\ \frac{117}{23}$ — (3)

$\boxed{\cancel{x = 1.960}}$

put in (2)

$\cancel{23y + 5x = 1}$

$\cancel{23y = 1 - 5 \times 1.960}$

$\boxed{\cancel{y = -0.3826}}$

in (1)

$\cancel{-6z - 0.3826 + 1.960 = -12}$

$\boxed{\cancel{z = 2.2629}}$

$x = 1.6479$

$y = -1.11408$

$z = 2.0845$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

$$R_2 \leftrightarrow R_2 - \frac{1}{6} R_3$$

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

$$R_1 \leftrightarrow R_1 - R_2$$

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

$$R_3 \rightarrow \frac{R_3}{12}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$R_1 \rightarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} x = 1 \\ y = 3 \\ z = 5 \end{array}} \text{ Ans}$$

Q→

$$x+2y+3z-u=10$$
$$2x+3y-3z-u=1$$
$$2x-y+2z+3u=7$$
$$3x+2y-4z+3u=2$$

Q→

$$x+y+z=9$$
$$2x-3y+4z=13$$
$$3x+4y+5z=40$$

Q→

$$2x-y+3z=9$$
$$x+y+z=6$$
$$x-y+z=2.$$

Q→

$$2x+4y+z=3$$
$$3x+2y-2z=-2$$
$$x-y+z=2.$$

## Gauss–Jordan Method

This is a modification of the Gauss-elimination method. In this method, elimination of unknowns is performed in such a way that ultimately you get the system as a diagonal matrix form.

**Ex →** Apply Gauss-Jordan method to solve the equations

$$x+y+z=9$$
$$2x-3y+4z=13$$
$$3x+4y+5z=40.$$

40
27
13

**Sol →**

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

$R_3 - 3R_1,\ R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & -5 & 2 & -5 \end{array}\right]$$

$R_3 \leftrightarrow R_3 + 5R_2$

$$\begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

$C_2 - C_1,\ C_3 - C_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

$C_3 - 2C_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

$R_3 \rightarrow R_3/12$

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$x=9,\ y=13,\ z=5$

## Examples

(1) $2x + 5y + 7z = 52$; $2x + y - z = 0$; $x + y + z = 9$

(2) $2x - 3y + z = -1$; $x + 4y + 5z = 25$; $3x - 4y + z = 2$

(3) $x + 3y + 3z = 16$; $x + 4y + 3z = 18$; $x + 3y + 4z = 19$.

| | 9-10 | 1-2 | 2-3 | 3-4 |
|---|---|---|---|---|
| Mon | CE, AI,DS | B.A Psy | CST All | G,H |
| Tue | | B.A Psy | CST All | G,H |
| Wed | | CE, AIDS | CST All | G, H |
| Thur | CST All | B.A Psy | | G, H  CE AI,DS ← clashing |
| Fri. | | B.A. Psy | | CE, AI,DS |

Penalties

G: 53, 34, 68, 51, 31

H: 33, 52, 37, 9, 20, 51, 52, 3, 62, 15, 60, 47, 66, 67, 68, 70

## (3) Gauss-Seidal Iteration Method

The system of equations :

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad -(1)$$

System (1) is rewritten as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad -(2)$$

The initial approximations are $x_0, y_0, z_0$.

The iteration process for the Gauss-Seidal is

$$x^{(K+1)} = \frac{1}{a_1}\left[d_1 - b_1y^{(K)} - c_1z^{(K)}\right]$$

$$y^{(K+1)} = \frac{1}{b_2}\left[d_2 - a_2x^{(K+1)} - c_2z^{(K)}\right]$$

$$z^{(K+1)} = \frac{1}{c_3}\left[d_3 - a_3x^{(K+1)} - b_3y^{(K+1)}\right]$$

Note:- (1) The absolute value of the largest coefficient in atleast one eqn is almost equal to or greater than the sum of the absolute values of all the remaining coefficients. {diagonally dominated}.

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Ex)

$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 36$$

Solve by G-Seidal Method.

$|8| \geq |3+2|$
$|11| \geq |4+1|$ diag. dominated.
$|12| \geq |6+3|$

Sol →

$$x^{(k+1)} = \frac{1}{8}\left[20 + 3y^{(k)} - 2z^{(k)}\right]$$

$$y^{(k+1)} = \frac{1}{11}\left[33 - 4x^{(k+1)} + z^{(k)}\right]$$

$$z^{(k+1)} = \frac{1}{12}\left[36 - 6x^{(k+1)} - 3y^{(k+1)}\right]$$

Start with $x^{(0)} = 0,\ y^{(0)} = 0$ ;·

1st iteration :- $\boxed{K=0}$

$$x^{(1)} = \frac{1}{8}\left[20 + 3y^{(0)} - 2z^{(0)}\right]$$

$$x^{(1)} = \frac{20}{8} = 2.5$$

$$y^{(1)} = \frac{1}{11}\left[33 - 4x^{(1)} + z^{(0)}\right] = \frac{1}{11}\left[33 - 4(2.5) + 0\right]$$
$$= 2.0909$$

$$z^{(1)} = \frac{1}{12}\left[36 - 6x^{(1)} - 3y^{(1)}\right] = \frac{1}{12}\left[36 - 6(2.5) - 3(2.0909)\right]$$
$$= 1.2273$$

| Iterations K | $x^{(k)}$ | $y^{(k)}$ | $z^{(k)}$ |
|---|---|---|---|
| 0 | 2.500 | 2.0909 | 1.2273 |
| 1 | 2.9773 | 2.0289 | 1.0041 |
| 2 | 3.0098 | 1.9968 | 0.9959 |
| 3 | 2.9998 | 1.9997 | 1.0002 |
| 4 | 2.9998 | 2.0001 | 1.0001 |
| 5 | 3.0000 | 2.0000 | 1.0000 |

$x = 3,\ y = 2,\ z = 1$.

Ans

Ex1 $8x - 3y + 2z = 20$
$4x + 11y - z = 33$
$6x + 3y + 12z = 36$

Solve by G-Seidal Method.

$|8| \geq |3+2|$ diag. dominated.
$|11| \geq |4+1|$
$|12| \geq |6+3|$

Sol →

$$x^{(K+1)} = \frac{1}{8}\left[20 + 3y^{(K)} - 2z^{(K)}\right]$$

$$y^{(K+1)} = \frac{1}{11}\left[33 - 4x^{(K+1)} + z^{(K)}\right]$$

$$z^{(K+1)} = \frac{1}{12}\left[36 - 6x^{(K+1)} - 3y^{(K+1)}\right]$$

Start with $x^{(0)} = 0$, $y^{(0)} = 0$;

1st iteration :- $\boxed{K=0}$ $x^{(1)} = \frac{1}{8}\left[20 + 3y^{(0)} - 2z^{(0)}\right]$

$$x^{(1)} = \frac{20}{8} = 2.5$$

$$y^{(1)} = \frac{1}{11}\left[33 - 4x^{(1)} + z^{(0)}\right] = \frac{1}{11}\left[33 - 4(2.5) + 0\right]$$

$$= 2.0909$$

$$z^{(1)} = \frac{1}{12}\left[36 - 6x^{(1)} - 3y^{(1)}\right] = \frac{1}{12}\left[36 - 6(2.5) - 3(2.0909)\right]$$

$$= 1.2273$$

| Iterations K | $x^{(K)}$ | $y^{(K)}$ | $z^{(K)}$ |
|---|---|---|---|
| 0 | 2.500 | 2.0909 | 1.2273 |
| 1 | 2.9773 | 2.0289 | 1.0041 |
| 2 | 3.0098 | 1.9968 | 0.9959 |
| 3 | 2.9998 | 1.9997 | 1.0002 |
| 4 | 2.9998 | 2.0001 | 1.0001 |
| 5 | 3.0000 | 2.0000 | 1.0000 |

Ans $x = 3$, $y = 2$, $z = 1$

Ex →

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

$$\left\{ \begin{array}{l} 20 > 1+2 \\ 20 > 3+1 \\ 20 > 2+3 \end{array} \right.$$

Sol →

$$x^{(K+1)} = \frac{1}{20}\left[17 - y^{(K)} + 2z^{(K)}\right]$$

$$y^{(K+1)} = \frac{1}{20}\left[-18 - 3x^{(K+1)} + z^{(K)}\right]$$

$$z^{(K+1)} = \frac{1}{20}\left[25 - 2x^{(K+1)} + 3y^{(K+1)}\right]$$

| Iteration | $x^{(K)}$ | $y^{(K)}$ | $z^{(K)}$ |
|---|---|---|---|
| 1 | 0.8500 | −1.0275 | 1.0109 |
| 2 | 1.0025 | −0.9998 | 0.9998 |
| 3 | 1.0000 | −1.0000 | 1.0000 |

Q → a)
$$2x + y + 6z = 9$$
$$8x + 3y + 2z = 13$$
$$x + 5y + z = 7$$

b)
$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$2x + 2y + 10z = 14$$

# Interpolation

"Interpolation is the art of reading between the lines of the table."

Suppose, we are given the following values of $y=f(x)$ for a set of values of $x$ :

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | $x_0$ | $x_1$ | $x_2$ | ------- | $x_n$ |
| $y$ : | $y_0$ | $y_1$ | $y_2$ | ------ | $y_n$ |

Thus the process of finding the value of $y$ corresponding to any value of $x=x_i$ between $x_0$ and $x_n$ is called interpolation.

## Methods of Interpolation

The method of graph

The method of Curve-fitting

Use of Calculus of finite-difference formulae.

## Finite Differences

The calculus of finite differences deals with the changes that ~~the~~ take place in the value of the function due to finite changes in the independent variable.

Suppose, we are given a set of values $(x_i, y_i)$; $i=1,2,--n$ of any func. $y_i = f(x_i)$, called arguments
entry

suppose that the func. $y=f(x)$ is tabulated for the equally spaced values $x=x_0, x_0+h, x_0+2h, \cdot ——, x_0+nh$ giving $y=y_0, y_1, y_2, ——y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate value of $x$, given three types of differences are useful :-

## 1. Forward differences

The differences $y_1-y_0, y_2-y_1, y_3-y_2, \ldots, y_n-y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}$ resp. are called the first forward differences where $\Delta$ is the forward difference operator.

Thus, the first foward differences are defined by

$$\Delta y_r = y_{r+1} - y_r \qquad r=0,1,2 \ldots$$

Secom forward differences are given by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

or
$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$= (y_2-y_1)-(y_1-y_0)$$
$$= y_2 - 2y_1 + y_0.$$

Similarly $\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$

Forward differ Table

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|
| $x_0$ | $y_0$ leading term | | | | | |
| | | $\Delta y_0$ | | | | |
| | | | $\Delta^2 y_0$ | | | |
| $x_1 = x_0+h$ | $y_1$ | | | leading differences. | | |
| | | $\Delta y_1$ | | $\Delta^3 y_0$ | | |
| | | | $\Delta^2 y_1$ | | $\Delta^4 y_0$ | |
| $x_2 = x_0+2h$ | $y_2$ | | | | | |
| | | $\Delta y_2$ | | $\Delta^3 y_1$ | | $\Delta^5 y_0$ |
| $x_3 = x_0+3h$ | $y_3$ | | $\Delta^2 y_2$ | | $\Delta^4 y_1$ | |
| | | $\Delta y_3$ | | $\Delta^3 y_2$ | | |
| | | | $\Delta^2 y_3$ | | | |
| $x_4 = x_0+4h$ | $y_4$ | | | | | |
| | | $\Delta y_4$ | | | | |
| $x_5 = x_0+5h$ | $y_5$ | | | | | |

## 2. Backward difference

The differences $y_1 - y_0, y_2 - y_1, \cdots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \cdots, \nabla y_n$ resp. are called first backward differences where $\nabla$ is the backward difference operator.

We define higher order backward differences as

$$\nabla y_r = y_r - y_{r-1}$$

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$

or $$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

$$= y_2 - 2y_1 + y_0$$

## Backward Difference table

| $x$ | $y$ | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ | $\nabla^5 y$ |
|---|---|---|---|---|---|---|
| $x_0$ | $y_0$ | | | | | |
| | | $\nabla y_1$ | | | | |
| $x_1 = x_0+h$ | $y_1$ | | $\nabla^2 y_2$ | | | |
| | | $\nabla y_2$ | | $\nabla^3 y_3$ | | |
| $x_2 = x_0+2h$ | $y_2$ | | | | $\nabla^4 y_4$ | |
| | | | $\nabla^2 y_3$ | | | $\nabla^5 y_5$ |
| | | $\nabla y_3$ | | $\nabla^3 y_4$ | | |
| $x_3 = x_0+3h$ | $y_3$ | | $\nabla^2 y_4$ | | $\nabla^4 y_5$ | |
| | | $\nabla y_4$ | | $\nabla^3 y_5$ | | |
| $x_4 = x_0+4h$ | $y_4$ | | $\nabla^2 y_5$ | | | |
| | | $\nabla y_5$ | | | | |
| $x_5 = x_0+5h$ | $y_5$ | | | | | |

3. Central differences :- The central difference operator $\delta$ is defined by the relation

$$y_1 - y_0 = \delta y_{1/2}, \quad y_2 - y_1 = \delta y_{3/2}, \; \cdots, \; y_n - y_{n-1} = \delta y_{n-1/2}$$

### central difference table

| $x$ | $y$ | $\delta y$ | $\delta^2 y$ | $\delta^3 y$ | $\delta^4 y$ |
|---|---|---|---|---|---|
| $x_0$ | $y_0$ | | | | |
| | | $\delta y_{1/2}$ | | | |
| $x_1$ | $y_1$ | | $\delta^2 y_{2/2}$ | | |
| | | $\delta y_{3/2}$ | | $\delta^3{}_{3/2}$ | |
| $x_2$ | $y_2$ | | | | $\delta^4 y_2$ |
| | | | $\delta^2 y_{4/2}$ | | |
| | | $\delta_{5/2}$ | | $\delta^3{}_{5/2}$ | |
| $x_3$ | $y_3$ | | $\delta^2 y_{6/2}$ | | |
| | | $\delta_{7/2}$ | | | |
| $x_4$ | $y_4$ | | | | |

## other difference operators

1. **Shift operator E :-** It is the operation of increasing the argument $x$ by $h$ so that

$$Ef(x) = f(x+h)$$
$$E^2 f(x) = f(x+2h)$$ or $E^n f(x) = f(x+nh)$

The inverse operator $E^{-1}$ is defined by

$$E^{-1} f(x) = f(x-h)$$

Also $E^n y_x = y_{x+nh}$.

2. **Average operator $\mu$ :-** It is defined by

$$\mu y_x = \frac{1}{2}\left[ y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}} \right].$$

In differential calculus, E is the fundamental operator and $\nabla, \Delta, \delta, \mu$ can be expressed in terms of E.

## Relation b/w operators

1. $\Delta \equiv E-1$ or $E \equiv \Delta+1$

proof :-
$$\Delta f(x) = f(x+h) - f(x)$$
$$= Ef(x) - f(x)$$
$$= (E-1)f(x)$$
$$\Rightarrow \Delta \equiv E-1$$

2. $\nabla \equiv 1-E^{-1}$

proof :-
$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1}f(x)$$
$$= (1-E^{-1})f(x)$$
$$\Rightarrow \nabla \equiv 1-E^{-1}$$

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3. $\delta \equiv E^{1/2} - E^{-1/2}$

proof :- $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

$= E^{1/2} f(x) - E^{-1/2} f(x)$

$= (E^{1/2} - E^{-1/2}) f(x)$

$\Rightarrow \delta \equiv E^{1/2} - E^{-1/2}$

4. $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

proof :- $\mu f(x) = \frac{1}{2}[f(x+\frac{h}{2}) + f(x-\frac{h}{2})]$

$= \frac{1}{2}[E^{1/2} f(x) + E^{-1/2} f(x)]$

$= \frac{1}{2}(E^{1/2} + E^{-1/2}) f(x)$

$\mu \equiv \frac{(E^{1/2} + E^{-1/2})}{2}$

from 3 ~~&~~ 4

5. $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

proof :- $E(\nabla f(x)) = E(f(x) - f(x-h))$

$= Ef(x) - Ef(x-h)$

$= f(x+h) - f(x)$

$= \Delta f(x)$

$\Rightarrow \boxed{E\nabla \equiv \Delta}$

# Now $\nabla(E f(x)) = \nabla(f(x+h))$

$= f(x+h) - f(x)$

$= \Delta f(x)$

$\Rightarrow \boxed{\nabla E \equiv \Delta}$

# $\delta E^{1/2} f(x) = \delta(f(x+\frac{h}{2}))$

$= f(x+\frac{h}{2}+\frac{h}{2}) - f(x+\cancel{\frac{h}{2}}-\cancel{\frac{h}{2}})$

$= f(x+h) - f(x) = \Delta f(x)$

$\boxed{\delta E^{1/2} \equiv \Delta}$

6. $E \equiv e^{hD}$

proof :- $Ef(x) = f(x+h)$

$$= f(x) + h.f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \cdots$$

$$= f(x) + hDf(x) + \frac{h^2}{\lfloor 2} D^2 f(x) + \cdots$$

$$= \left[1 + hD + \frac{h^2}{\lfloor 2} D^2 + \cdots\right] f(x)$$

$$= e^{hD} f(x)$$

$\Rightarrow$ $\boxed{E \equiv e^{hD}}$

Rules :-
1. $\Delta f(x) = f(x+h) - f(x)$
2. $\nabla f(x) = f(x) - f(x-h)$
3. $\mu f(x) = \frac{1}{2}\left[f(x+\frac{h}{2}) + f(x-\frac{h}{2})\right]$
4. $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$
5. $Ef(x) = f(x+h)$

# Differences of a polynomial

# The $n^{th}$ differences of a polynomial of $n^{th}$ degree are constant. &
all the higher order differences are 0, when the values of ind. variable $x$ are equally spaced. } Imp

# First difference of a polynomial of $n^{th}$ degree is a polynomial of degree $(n-1)$.

# second difference is of degree $(n-2)$.

- - - so on.

# $n^{th}$ difference is of degree 0 (constant).

# $(n+1)^{th}$ difference will be 0. } Imp

⇒ Converse of this theorem is also true.

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l$$

$$\Delta f(x) = f(x+h) - f(x)$$

$$= a\left[(x+h)^n - x^n\right] + b\left[(x+h)^{n-1} - x^{n-1}\right] + \cdots + k\left[(x+h) - x\right] + [l-l]$$

$$= anhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \cdots + k'h + l'$$

degree $= n-1$

$$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$$

$$= anh\left[(x+h)^{n-1} - x^{n-1}\right] + b'\left[(x+h)^{n-2} - x^{n-2}\right] + \cdots + k'h$$

$$= an(n-1)h^2 x^{n-2} + b'' x^{n-3} + \cdots k''$$

degree $= n-2$

$$\Delta^n f(x) = (an(n-1)(n-2) \cdots 1)h^n$$

$$= a\,\underline{|n}\,h^n$$

Examples

1. Construct the forward difference table, given that :- and find $\Delta^2 y_{10}$, $\Delta^4 y_5$

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|
| 5 | 9962 | | | | | |
| | | -114 | | | | |
| 10 | 9848 | | -75 | | | |
| | | -189 | | 2 | | |
| 15 | 9659 | | $\boxed{-73}$ | | $\boxed{-1}$ | |
| | | -262 | | 1 | | 3 |
| 20 | 9397 | | -72 | | 2 | |
| | | -334 | | 3 | | |
| 25 | 9063 | | -69 | | | |
| | | -403 | | | | |
| 30 | 8660 | | | | | |

$\Delta^2 y_{10} = -73$

$\Delta^2 y_5 = -1$

2. Construct a backward difference table for $y = \log x$, given & find $\nabla^3 \log 40$ & $\nabla^4 \log 50$.

| $x$ | $y$ | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ |
|---|---|---|---|---|---|
| 10 | 1 | | | | |
| | | 0.3010 | | | |
| 20 | 1.3010 | | -0.1249 | | |
| | | 0.1761 | | $\boxed{0.0738}$ | |
| 30 | 1.4771 | | -0.0511 | | $\boxed{-0.0508}$ |
| | | 0.1250 | | 0.0230 | |
| 40 | 1.6021 | | -0.0281 | | |
| | | 0.0969 | | | |
| 50 | 1.6990 | | | | |

$\nabla^3 \log 40 = 0.0738$

$\nabla^4 \log 40 = -0.0508$

Ex 3. find $f(6)$, given $f(0)=-3$, $f(1)=6$, $f(2)=8$, $f(3)=12$; third difference being constant.

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|---|---|---|---|---|
| 0 | -3 | | | |
| | | 9 | | |
| 1 | 6 | | -7 | |
| | | 2 | | 9 |
| 2 | 8 | | 2 | |
| | | 4 | | |
| 3 | 12 | | | |

$$f(6) = f(0+6) = E^6 f(0) = (1+\Delta)^6 f(0)$$

$$= \left[1 + 6\Delta + \frac{6(6-1)}{\underline{|2}} \Delta^2 + \frac{6(6-1)(6-2)}{\underline{|3}} \Delta^3\right] f(0)$$

$$= \left[1 + 6\Delta + 15\Delta^2 + 20\Delta^3\right] f(0)$$

$$= 1 + 6\Delta f(0) + 15\Delta^2 f(0) + 20\Delta^3 f(0)$$

$$= 1 + 6(9) + 15(-7) + 20(9)$$

$$\boxed{f(6) = 126}$$

Ex4. Evaluate

(i) $\Delta \tan^{-1} x$

$= \tan^{-1}(x+h) - \tan^{-1} x$

$= \tan^{-1}\left[\frac{(x+h-x)}{1+x(x+h)}\right]$

$= \tan^{-1}\left[\frac{h}{1+hx+x^2}\right]$

(ii) $\Delta^2 \cos 2x$

$= \Delta\left[\cos 2(x+h) - \cos 2x\right]$

$= \cos 2(x+2h) - \cos 2(x+h) - [\cos 2(x+h) - \cos 2x]$

$= \{\cos 2(x+2h) - \cos 2(x+h)\} - \{\cos 2(x+h) - \cos 2x\}$

$= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h$

$= -2\sin h\left[2\cos(2x+2h)\sin h\right]$

$= -4\sin^2 h \cos(2x+2h)$.

Ex5. If $f(x) = e^{2x}$, evaluate $\Delta^n f(x)$

Sol →. $\Delta f(x) = \Delta e^{2x} = e^{2(x+h)} - e^{2(x)} = e^{2x}e^{2h} - e^{2x}$

$= e^{2x}\left[e^{2h} - 1\right]$

$\Delta^2 f(x) = \Delta^2 e^{2x} = \Delta(\Delta e^{2x}) = \Delta\left(e^{2x}(e^{2h}-1)\right)$

$= (e^{2h}-1)\,\Delta e^{2x} = (e^{2h}-1)\left[e^{2(x+h)} - e^{2x}\right]$

$= (e^{2h}-1)(e^{2h}-1)(e^{2x})$

$= (e^{2h}-1)^2 e^{2x}$

$\Delta^3 f(x) = (e^{2h}-1)^3 e^{2x}$

⋮

$\Delta^n f(x) = (e^{2h}-1)^n e^{2x}$.

Ex-6 Prove that

(i) $(E^{1/2}+E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta$

$[E = 1+\Delta]$

proof ⇒ L.H.S. $(E^{1/2}+E^{-1/2})(1+\Delta)^{1/2}$

$= (E^{1/2}+E^{-1/2})\, E^{1/2}$

$= (E+1)$

$= (1+\Delta)+1 = 2+\Delta.$

(ii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\left(\frac{\delta^2}{4}\right)}$

~~proof~~ ⇒ R.H.S $\frac{1}{2}(E^{1/2}-E^{-1/2})^2 + (E^{1/2}-E^{-1/2})\sqrt{1+\frac{(E^{1/2}-E^{-1/2})^2}{4}}$

$= \frac{1}{2}(E+E^{-1}-2E^{1/2}\cdot E^{-1/2}) + (E^{1/2}-E^{-1/2})\left[\frac{E^{1/2}+E^{-1/2}}{2}\right]$

$= \frac{1}{2}(E+E^{-1}-2) + \frac{1}{2}(E-E^{-1})$

$= E-1$

$= \Delta$

(iii) $\Delta^3 y_2 = \nabla^3 y_5$

proof ⇒ L.H.S. $\Delta^3 y_2 = (E-1)^3 y_2$

$= -(1-E)^3 y_2$

$= -\left[1-3E+\frac{3(3-1)}{\lfloor 2}E^2 - \frac{3(3-1)(3-2)}{\lfloor 3}E^3\right] y_2$

$= -[1-3E+3E^2-E^3]\, y_2$

$= (E^3-3E^2+3E-1)\, y_2$

$= y_5 - 3y_4 + 3y_3 - y_2$

R.H.S. $= \nabla^3 y_5 = (1-E^{-1})^3 y_5$

$= (1-3E^{-1}+3E^{-2}-E^{-3})\, y_5$

$= y_5 - 3y_4 + y_3 - y_2$

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(iv) $hD = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

proof :- $\cancel{hD} \; E = e^{hD}$

$\log E = \log e^{hD}$

$\log E = hD$

$hD = \log E = -\log(E^{-1})$

$hD = -\log(1-\nabla)$

Now, $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$, $\delta = E^{1/2} - E^{-1/2}$

$\mu\delta = \frac{1}{2}(E^{1} - E^{-1})$

$= \frac{1}{2}[e^{hD} - e^{-hD}]$

$\mu\delta = \sinh(hD)$

$hD = \sinh^{-1}(\mu\delta)$.

\# Note :→ 1) $\Delta^n x^n = \lfloor n$

2) $\Delta^{n+1} x^n = 0$

## Missing term Technique

Suppose $n$ values out of $(n+1)$ values of $y = f(x)$ are given, the values of $x$ being equidistant. We construct the difference table.

\# equating to zero the $n^{th}$ difference, we can get the value of missing term.

Ex1 :- find the missing value in the table.

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 45 | 3 | $y_1-3$ | $5-2y_1$ | $3y_1+y_3-9$ |
| 50 | $\boxed{y_1}$ | $2-y_1$ | $y_1+y_3-4$ | $3.6-y_1-3y_3$ |
| 55 | 2 | $y_3-2$ | $-0.4-2y_3$ | |
| 60 | $\boxed{y_3}$ | $-2.4-y_3$ | | |
| 65 | $-2.4$ | | | |

As only three enteries $y_0, y_2, y_4$ are given, the func $y$ can be represented by a second degree polynomial.

so $\Delta^3 y_0 = 0$ ; $\Delta^3 y_1 = 0$

$3y_1 + y_3 - 9 = 0$ ; $3.6 - y_1 - 3y_3 = 0$

$y_1 + 3y_3 = 3.6$ ;

$3y_1 + y_3 = 9$ ;

$\Rightarrow y_1 = 2.925$ ; $y_2 = 0.225$

Ex2 :- Estimate the missing term in the following table:

(i)

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y(x)$ | 1 | 3 | 9 | ? $y_3$ | 81 |

(ii)

| $x$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $y$ | 45 | 49.2 | 54.1 | ? | 67.4 |

Sol →

(i) We are given four values

so $\Delta^4 y(x) = 0$

$\Rightarrow (E-1)^4 f(a) = 0$

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 4 | $y_3-19$ | $124-4y_3$ |
| 1 | 3 | 6 | $y_3-15$ | $105-3y_3$ | |
| 2 | 9 | $y_3-9$ | $90-2y_3$ | | |
| 3 | $y_3$ | $81-y_3$ | | | |
| 4 | 81 | | | | |

$\Rightarrow 4y_3 - 124 = 0$

$y_3 = \frac{124}{4}$

$\boxed{y_3 = 31}$

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Important

# The study of interpolation is based on the calculus of finite differences.

We begin with two imp. interpolation formulae by means of forward and backward differences of a func.

These are employed in engineering and scientific investigation.

1. Newton's Interpolation formulae:

Let $y = f(x)$ be a func. which takes the values $y_0, y_1, \ldots y_n$ corresponding to the values $x_0, x_1, \ldots x_n$ of $x$. and let $x$ be equi-spaced s.t

$$x_i = x_0 + ih \;(i = 0, 1, \ldots)$$

(1) Forward formula :-

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \ldots + \frac{p(p-1)(p-2)\ldots(p-\overline{n-1})}{\lfloor n}\Delta^n y_0$$

Imp $\boxed{x_p = x_0 + ph}$

(2) Backward formula :-

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3}\nabla^3 y_n + \ldots$$

# Use of finite difference calculus for the purpose of interpolation.

# We will do three cases

→ Interpolation with equal interval

Newton's formulae → forward, backward

Gauss formulae

Bessel's formula

Stirling formula

Everett's formula

→ Interpolation with unequal interval

Lagrange's interpolation

Newton's divided difference

$-5+65$

$13\times5$

$65$

Ex. The table gives the distance in miles of the visible horizon for the given heights in feet above the earth's surface:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | 100 | 150 | 200 | 250 | 300 | 350 | 400 |
| $y$ : | 10.63 | 13.03 | 15.04 | 16.81 | 18.42 | 19.90 | 21.27. |

find the values of $y$ when (i) $x = 160$ feet (ii) $x = 410$ ft

| | $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ | $\Delta^6 y$ |
|---|---|---|---|---|---|---|---|---|
| $x_0$ | 100 | 10.63 | | | | | | |
| | | | 2.40 | | | | | |
| | $\boxed{150}$ | $\boxed{13.03}$ | | −0.39 | | | | |
| | | | 2.01 | | 0.15 | | | |
| | 200 | 15.04 | | −0.24 | | −0.07 | | |
| | | | 1.77 | | 0.08 | | 0.02 | |
| | 250 | 16.81 | | −0.16 | | −0.05 | | 0.02 |
| | | | 1.61 | | 0.03 | | 0.04 | |
| | 300 | 18.42 | | −0.13 | | −0.01 | | |
| | | | 1.48 | | 0.02 | | | |
| | 350 | 19.90 | | −0.11 | | | | |
| | | | 1.37 | | | | | |
| | 400 | 21.27 | | | | | | |

(i) $x_p = 160$ feet [in starting] $\cancel{\Delta^2 y_{160} = (-0.24) \ldots}$

$\cancel{x_0 = 160}$, $\cancel{y_{160} = 13.03}$, $\cancel{\Delta y_{160} = 2.01}$

Now $\cancel{x = x_0 + ph} \Rightarrow$

$$y_p = y_{160} = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \cdots$$

Now $x_p = x_0 + ph \Rightarrow 160 = 100 + p(50) \Rightarrow p = \frac{60}{50} = 1.2$

$$y_{160} = 10.63 + 1.2(2.40) + \frac{(1.2)(1.2-1)}{2}(-0.39) + \frac{(1.2)(1.2-1)(1.2-2)}{\lfloor 3}(0.15)$$
$$+ \frac{(1.2)(1.2-1)(1.2-2)(1.2-3)}{\lfloor 4}(-0.07) + (1.2)(.2)(-0.8)(-0.7)(-0.6)$$
$$\simeq 13.46$$

(ii) $x = 410$ feet [at the end]

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3}\nabla^3 y_n + \cdots$$

$x_n = 400$, $410 = 400 + p(50)$

$p = \frac{10}{50} = 0.2$

$$= 21.27 + (0.2)(1.37) + \frac{(0.2)(0.2+1)}{\lfloor 2}(-0.11) + \cdots$$
$$\simeq 21.53 \text{ (feet) Miles}$$

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(14)

Ex 2. from the table, estimate the no. of students who obtained marks b/w 40 and 45.

| Marks | 30-40 | 40-50 | 50-60 | 60-70 | 70-80 |
|---|---|---|---|---|---|
| No of stud. | 31 | 42 | 51 | 35 | 31 |

| Marks less than $x$ | No of stu. $(y)$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|---|
| $x_0$ 40 | 31 | 31 | | | | |
| | | | 42 | | | |
| | | | | 9 | | |
| 50 | 42+31 | 73 | | | -25 | |
| | | | 51 | | | |
| 60 | 51+42+31 | 124 | | -16 | | 37 |
| | | | 35 | | 12 | |
| 70 | 35+51+42+31 | 159 | | -4 | | |
| | | | 31 | | | |
| 80 | 31+35+51+42 +31 | 190 | | | | |

$x_p = x_0 + ph$

$45 = 40 + p(10)$

$p = 5/10 = 0.5$

B/w 40 and 45

ie find less than 45 = a

and less than 40 (given) - b

then B/w 40-45 = a-b

hence

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{\lfloor 2} \Delta^2 y_{40} + \cdots$$

$$y_{45} = 31 + (0.5)(42) + \cdots$$

$$y_{45} = 47.87 \simeq 48.$$

Now $a = 48$, $b = 31$

No. of students b/w 40-45

$= 48-31$

$= 17$

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Ex → find a cubic polynomial which takes the values:-

$x$ : 0 1 2 3

$f(x)$ : 1 2 1 10

Hence Evaluate $f(4)$.

Sol→

| $x$ | $y/f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 0 | 1 | | | |
| | | 1 | | |
| 1 | 2 | | -2 | |
| | | -1 | | 12 |
| 2 | 1 | | 10 | |
| | | 9 | | |
| 3 | 10 | | | |

By N-forward formula :-

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3} \Delta^3 y_0$$

$x_p = x_0 + ph$ $\qquad p = \frac{x_p - x_0}{h}$

$x = 0 + p(1)$ $\qquad$ take $x_p = x$

$x = p$ $\qquad h = 1$

or $p = x$

$$y_x = y_0 + x\Delta y_0 + \frac{x(x-1)}{2} \Delta^2 y_0 + \frac{x(x-1)(x-2)}{\lfloor 3} \Delta^3 y_0$$

$$= 1 + x(1) + \frac{x^2 - x}{\cancel{2}}(-\cancel{2}) + \frac{x(x^2-3x+2)}{\cancel{6}} \times \cancel{12}^{2}$$

$$= 1 + x + x - x^2 + 2x^3 - 6x^2 + 4x$$

$$y_x / f(x) = 2x^3 - 7x^2 + 6x + 1$$

$$f(4) = 2(4)^3 - 7(4)^2 + 6(4) + 1$$

$$= 41.$$

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# Central Difference Interpolation formulae

Best suited for interpolation near the middle of the table :-

## Gauss's Forward Interpolation formula :-

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$p$ b/w 0 and 1.

## Gauss's backward Interpolation formula :-

$$y_p = y_0 + p\,\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2}$$

$p$ b/w $-1$ & 0.

**Ex1.** find $f(30)$ from the table

backward ↑ ऊपर

forward ↓

| | $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|---|
| $-2$ | 21 | 18.4708 | | | | |
| | | | $-0.6564$ | | | |
| | | | | $-0.0510$ | | |
| $-1$ | 25 | 17.8144 | | | $-0.0074$ | |
| | | | $-0.7074$ | | | $-0.0022$ |
| | | | | $-0.0564$ | | |
| $0$ | 29 | 17.1070 | | | $-0.0076$ | |
| | 30 → | | $-0.7638$ | | | |
| | | | | $-0.0640$ | | |
| $1$ | 33 | 16.3432 | | | | |
| | | | $-0.8278$ | | | |
| $2$ | 37 | 15.5154 | | | | |

$x_p = x_0 + ph$

$30 = 29 + p(4)$

$1 = p(4)$

$p = \frac{1}{4} = 0.25$

(forward formula)

$$y_{30} = 17.1070 + (0.25)(-0.7638)$$

$$+ \frac{p(p-1)}{2}(-0.0564) + \frac{(p+1)p(p-1)}{6}(-0.0076)$$

$$+ \frac{(p+1)p(p-1)(p-2)}{24}(-0.0022)$$

$$y_{30} = 16.9216$$

Ex2. Using the table, find y(8). [Use backward]

| | x | y | Δy | Δ²y | Δ³y | Δ⁴y | Δ⁵y |
|---|---|---|---|---|---|---|---|
| −2 | 0 | 7 | | | | | |
| | | | 4 | | | | |
| −1 | 5 | 11 | | −1 | | | |
| | | | 3 | | 2 | | |
| 0 | $\boxed{10}$ | 14 | | 1 | | −1 | 0 |
| | | | 4 | | 1 | | |
| 1 | 15 | 18 | | 2 | | −1 | |
| | | | 6 | | 0 | | |
| 2 | 20 | 24 | | 2 | | | |
| | | | 8 | | | | |
| 3 | 25 | 32 | | | | | |

add, subtr, add

$x_p = x_0 + ph$
$8 = 5 + p(5)$
$3 = 5p$ X
$p = \frac{3}{5}$

$8 = 10 + 5p$
$-2 = 5p$
$p = -0.4$ ✓

$$y_8 = 14 + p(3) + \frac{p(p+1)}{\lfloor 2}(1) + \frac{(p+1)p(p+1)}{\lfloor 3}(2) + \frac{(p-1)p(p+1)(p+2)}{\lfloor 4}(-1)$$

$y_8 = 12.826.$

Ex3. find in 1974.

| | Year x | Population y | Δy | Δ²y | Δ³y | Δ⁴y | Δ⁵y |
|---|---|---|---|---|---|---|---|
| −3 | 1939 | 12 | | | | | |
| | | | 3 | | | | |
| | | | | 2 | | | |
| −2 | 1949 | 15 | | | 0 | | |
| | | | 5 | | | 3 | |
| | | | | 2 | | | |
| −1 | 1959 | 20 | | | 3 | | −10 |
| | | | 7 B | | | | |
| | | | | 5 | | −7 | |
| 0 | 1969 | $\boxed{27}$ | forward | | | | |
| | | | 12 F | | −4 | | |
| 1 | 1979 | 39 | | 1 | | | |
| | | | 13 | | | | |
| 2 | 1989 | 52 | | | | | |

$x_p = x_0 + ph$
(F) $1974 = 1969 + p(10)$
$5 = p10 \Rightarrow p = 0.5$
$1974 = 1979 + p(10)$
$-5 = p(10)$
(B) $p = -0.5$

32.532

# Stirling's formula

G.F. =

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{\lfloor 3}\Delta^3 y_{-1} + \dots \quad \text{— (1)}$$

G.B =

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{\lfloor 2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{\lfloor 3}\Delta^3 y_{-2} + \dots \quad \text{— (2)}$$

Take mean of (1) & (2)

$$y_p = \frac{1}{2}\left[2y_0 + p\{\Delta y_0 + \Delta y_{-1}\} + \left(\frac{p^2-1+p^2+1}{\lfloor 2}\right)\Delta^2 y_{-1} + \frac{p(p^2-1)}{\lfloor 3}\{\Delta^3 y_{-1} + \Delta^3 y_{-2}\} + \dots\right]$$

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{\lfloor 2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{\lfloor 3}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \dots$$

p lying b/w $-\frac{1}{4}$ and $\frac{1}{4}$.

$-0.25 \leftrightarrow 0.25$

Ex → find $y_{12.2}$

| | x | y | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|---|
| −2 | 10 | 23967 | | | | |
| | | | 4093 | | | |
| | | | | −365 | | |
| −1 | 11 | 28060 | | | 58 | |
| | | | 3728 | | | |
| | | | | −307 | | −13 |
| 0 | $\boxed{12}$ | 31788 | | | | |
| | | | 3421 | | 45 | |
| 1 | 13 | 35209 | | −262 | | |
| | | | 3159 | | | |
| 2 | 14 | 38368 | | | | |

$x_p = x_0 + ph$

$12.2 = 12 + p1$

$p = 0.2$

$$y_{1.2} = 31788 + 0.2\left\{\frac{3728+3421}{2}\right\} + \dots$$

Ex→ $\boxed{\tan 16^\circ}$

| $\theta(x)$ | $\tan\theta\,(y)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ | $\Delta^6 y$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | |
| | | 0.0875 | | | | | |
| 5 | 0.0875 | | 0.0013 | | | | |
| | | 0.0888 | | 0.0015 | 0.0002 | | |
| 10 | 0.1763 | | 0.0028 | | | −0.0002 | |
| | | 0.0916 | | 0.0017 | | | 0.0011 leave |
| 15 | $\boxed{0.2679}$ | | 0.0045 | | 0.0000 | | |
| | | 0.961 | | 0.0017 | | 0.0009 leave | |
| 20 | 0.3640 | | 0.0062 | | 0.0009 | | |
| | | 0.1023 | | 0.0026 | | | |
| 25 | 0.4663 | | 0.0088 | | | | |
| | | 0.1111 | | | | | |
| 30 | 0.5774 | | | | | | |

$x_p = x_0 + ph$

$16 = 15 + p(5)$

$1 = 5p \Rightarrow p = \frac{1}{5} = 0.2$

Stirling ↗

$$y_p = y_0 + p\left\{\frac{0.0916 + 0.961}{2}\right\} + \frac{p^2}{\lfloor 2}(0.0045) + \frac{p(p^2-1)}{\lfloor 3}\left\{\frac{0.0017+0.0017}{2}\right\} + 0$$
$$= 0.28676$$

Ex→

| | $x$ | $y(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|---|
| −2 | 25 | 4 | | | | | |
| | | | −0.154 | | | | |
| | | | | 0.012 | | | |
| −1 | 26 | 3.846 | | | −0.003 | | |
| | | | −0.142 | | | | |
| | | | | 0.009 | | 0.004 | |
| 0 $x_0$ | 27 | $\boxed{3.704}$ | | | −0.001 | | −0.005 |
| | | | −0.133 | | | | |
| | | | | 0.010 | | −0.001 | |
| 1 | 28 | 3.571 | | | −0.002 | | |
| | | | −0.123 | | | | |
| | | | | 0.008 | | | |
| 2 | 29 | 3.448 | | | | | |
| | | | −0.115 | | | | |
| 3 | 30 | 3.333 | | | | | |

find $y(27.5)$

$27.5 = 27 + p(1)$

$0.5 = p$

$p = \frac{1}{2}$ $\quad \frac{1}{4}\left(\frac{-1}{2}\right)\frac{3}{4}$

$$y_p = 3.704 + p(-0.133) + \frac{p(p-1)}{\lfloor 2}\left\{\frac{0.009+0.010}{2}\right\} + 0 + \frac{(p+1)p(p+1)(p-2)}{\lfloor 4}\left\{\frac{0.004-0.001}{2}\right\} + \dots$$

$$= 3.585$$

# Bessel's Interpolation formula

$$y_p = \left\{\frac{y_0+y_1}{2}\right\} + \left(p-\tfrac{1}{2}\right)\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\left\{\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}\right\} +$$

$$\frac{(p-1)\left(p-\frac{1}{2}\right)p}{\lfloor 3}\,\Delta^3 y_{-1} + \frac{(p+1)(p-1)p(p-2)}{\lfloor 4}\left\{\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}\right\}$$

$+ \cdots$

best $\exists$ $p$ lies b/w $\frac{1}{4}$ and $\frac{3}{4}$, ($.25$, $.75$) can give for b/w 0 & 1.

Ex1. Given ; find tan 16°

| $\theta$ | $\tan\theta$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|
| 0 | 0 | | | | | |
| | | 0.0875 | | | | |
| 5 | 0.0875 | | 0.0013 | | | |
| | | 0.0888 | | 0.0015 | | |
| 10 | 0.1763 | | 0.0028 | | 0.0002 | |
| | | 0.0916 | | 0.0017 | | −0.0002 |
| $\boxed{15}$ | $\boxed{0.2679}$ $y_0$ | | 0.0045 | | 0.0000 | |
| | | 0.0961 | | 0.0017 | | 0.0009 |
| 20 | 0.3640 | | 0.0062 | | 0.0009 | |
| | | 0.1023 | | 0.0026 | | |
| 25 | 0.4663 | | 0.0088 | | | |
| | | 0.1111 | | | | |
| 30 | 0.5774 | | | | | |

at $\theta = 16°$

$x_0 = 15$

$16 = 15 + p5$

$1 = 5p$

$p = \frac{1}{5} = 0.2$

$y_{16} =$

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\left\{\frac{\Delta^2 y_0+\Delta^2 y_{-1}}{2}\right\} + \frac{p(p-1)(p-1/2)}{\lfloor 3}\,\Delta^3 y_{-1} +$$

$$\frac{p(p-1)(p+1)(p-2)}{\lfloor 4}\left\{\frac{\Delta^4 y_{-1}+\Delta^4 y_{-2}}{2}\right\} + \cdots$$

$$+ \frac{p\left(p-\frac{1}{2}\right)(p^2-1^2)\cdots\cdots(p-n-1)(p+n-2)\,\Delta^{2n-1} y_{-n+1}}{\lfloor 2n+1}$$

$$+ \frac{(p-2)(p-1)\left(p-\frac{1}{2}\right)p(p+1)}{\lfloor 5}\,\Delta^5 y_{-2}$$

$$+ \frac{(p-3)(p-2)(p-1)\,p\,(p+1)(p+2)}{\lfloor 6}\left\{\frac{\Delta^6 y_{-2}+\Delta^6 y_{-3}}{2}\right\}$$

## ① Lagrange's Interpolation formula

If $y=f(x)$ takes the value $y_0, y_1, \ldots, y_n$ corresponding to $x=x_0, x_1, x_2, \ldots, x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$
$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

Ex1.

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|---|
| $x$ : | 5 | 7 | 11 | 13 | 17 |
| $f(x)$ : | 150 | 392 | 1452 | 2366 | 5202 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |

find $f(9)$.

Sol→ $$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}(150) + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}(392)$$
$$+ \cdots + \cdots + \cdots$$
$$= 810$$

Ex2.

| $x$ | 0 | 1 | 2 | 5 |
|---|---|---|---|---|
| $f(x)$ | 2 | 3 | 12 | 147 |

find $f(3) = 35$.

Ex3. find Lagrange's interpolating polynomial from

| $x$ | 0 | 1 | 2 | 5 |
|---|---|---|---|---|
| $f(x)$ | 2 | 3 | 12 | 147 |

Sol→ $$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} 3 + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} 12$$
$$+ \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$
$$= x^3 + x^2 - x + 2$$

Ex4. A curve passes through $(0,18), (1,10), (3,-18), (6,90)$. find the slope of the curve at $x=2$.

Ex →

| time $t$ : | 0 | 1 | 3 | 4 |
|---|---|---|---|---|
| velocity $v$ : | 21 | 15 | 12 | 10 |

find distance at 4 seconds. & acceleration at the end of 4 seconds.

distance $= \int_0^4 v\,dt = 54.9$ | acce $= \frac{dv}{dt} = -3.4$

## Divided difference

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$ be the given pts, then the first divided difference for the arguments $x_0, x_1$ is defined by

$$[x_0, x_1] \text{ or } \underset{x_1}{\Delta} y_0 = \frac{y_1 - y_0}{x_1 - x_0}$$

$$[x_1, x_2] \text{ or } \underset{x_2}{\Delta} y_1 = \frac{y_2 - y_1}{x_2 - x_1} \text{ and so on.}$$

Second divided difference for $x_0, x_1, x_2$ is

$$\underset{x_1, x_2}{\Delta^2} y_0 = [x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

Third divided difference

$$\underset{x_1, x_2, x_3}{\Delta^3} y_0 = [x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

P :- (1) Symmetric ie. $[x_0, x_1] = [x_1, x_0]$

(2) $n^{th}$ divided difference of a polynomial of degree n is constant.

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2, \ldots, x_n] = \frac{1}{\lfloor n} \frac{1}{h^n} \underbrace{\Delta^n y_0}_{\text{Const}}$$

$$[x_0, x_1, x_2] = \frac{1}{\lfloor 2} \frac{1}{h^2} \Delta^2 y_0$$

$$= \frac{1}{\lfloor n} \frac{1}{h^n}$$

Newton's formula $\Rightarrow$

$$y = y_0 + (x - x_0) \Delta y_0 + (x - x_0)(x - x_1) \Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2) \Delta^3 y_0$$
$$+ \cdots + (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1}) \Delta^n y_0$$

Ex1. Construct the divided difference table

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 1 | 22 | | | | |
| | | $\frac{30-22}{2-1}=8$ | | | |
| 2 | 30 | | $\frac{26-8}{4-1}=6$ | | |
| | | $\frac{82-30}{4-2}=26$ | | $\frac{-3.6-6}{7-1}=-1.6$ | |
| 4 | 82 | | $\frac{8-26}{7-2}=-3.6$ | | $\frac{0.535+1.6}{12-1}=0.194$ |
| | | $\frac{106-82}{7-4}=8$ | | $\frac{1.75+3.6}{12-2}=0.535$ | |
| 7 | 106 | | $\frac{22-8}{12-4}=1.75$ | | |
| | | $\frac{216-106}{12-7}=22$ | | | |
| 12 | 216 | | | | |

Ex2. find f(8) and f(15).

| $x$ | $f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|
| $x_0$ 4 | $y_0$ 48 | | | | | |
| | | $\Delta y_0$ 52 | | | | |
| $x_1$ 5 | 100 | | $\Delta^2 y_0$ 15 | | | |
| | | 97 | | $\Delta^3 y_0$ 1 | | |
| $x_2$ 7 | 294 | | 21 | | $\Delta^4 y_0$ 0 | |
| | | 202 | | 1 | | 0 |
| $x_3$ 10 | 900 | | 27 | | 0 | |
| | | 310 | | 1 | | |
| $x_4$ 11 | 1210 | | 33 | | | |
| | | 409 | | | | |
| 13 | 2028 | | | | | |

$f(8) = 48 + (8-4)52 + (8-4)(8-5)15 + (8-4)(8-5)(8-7)1$
$+ (8-4)(8-5)(8-7)(8-10)\,0 + 0$
$= 448$

$f(15) = 48 + (15-4)52 + (15-4)(15-5)15 + (15-4)(15-5)(15-7)1$
$= 3150$

Scanned with CamScanner

Ex3. Determine $f(x)$ for

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| -4 | 1245 | | | | |
| | | -404 | | | |
| | | | 94 | | |
| | | | | -14 | |
| -1 | 33 | | | | 3 |
| | | -28 | | | |
| | | | 10 | | |
| | | | | 13 | |
| 0 | 5 | | | | |
| | | 2 | | | |
| | | | 88 | | |
| 2 | 9 | | | | |
| | | 442 | | | |
| 5 | 1335 | | | | |

$$y = 1245 + (x+4)(-404) + (x+4)(x+1)94 + (x+4)(x+1)(x-0)(-14)$$
$$+ (x+4)(x+1)x(x-2)3$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

Ex4. find missing term

| $x$ | 1 | 2 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $f(x)$ | 14 | 15 | 5 | ? | 9 |

left

| $x$ | $f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 1 | 14 | | | |
| | | $\frac{15-14}{2-1} = 1$ | | |
| | | | $\frac{-5-1}{4-1} = -2$ | |
| | | | | $\frac{\frac{7}{4}+2}{6-1} = \frac{3}{4}$ |
| 2 | 15 | | | |
| | | $\frac{5-15}{4-2} = -5$ | | |
| | | | $\frac{2+5}{6-2} = \frac{7}{4}$ | |
| 4 | 5 | | | |
| | | $\frac{9-5}{6-4} = 2$ | | |
| 6 | 9 | | | |

$$y = 14 + (x-1)1 + (x-1)(x-2)(-2) + (x-1)(x-2)(x-4)\frac{3}{4}$$

$\Rightarrow$ at $x = 5$

$$y(5) = 14 + 4\cdot 1 + 4\cdot 3\cdot(-2) + 4\cdot 3\cdot 1\cdot \frac{3}{4}$$
$$= 3.$$