Unit-4

Numerical Solution of ordinary Differential Equation

Consider the ordinary diff^n eq^n

$$\frac{dy}{dx} = f(x,y)$$

with $y(x_0) = y_0$

Many analytical techniques exists for solving such eq^n. (2nd order only)
But, majority of d.e. in physical problems cannot be solved analytically. Thus it becomes necessity to discuss their sol^n by numerical method.

In numerical methods, we ~~don't~~ find the numerical values of the dependent variable for certain values of independent variable.

Solution

Single step

* This method which require only the numerical value y_i in order to compute next value y_{i+1}

→ Taylor's series

→ Picard's method

Multi-step

* This method which require not only the numerical value y_i but at least one of the past value $y_{i-1}, y_{i-2}, --$ to evaluate next value y_{i+1}

→ Euler

→ Modified Euler

→ R-K method

→ Milne's method.

form of sol^n

A series for y in terms of powers of x from which the value of y can be obtained by direct substitution

→ Picard's

A set of tabulated values of x and y.

→ (Rest Methods belong here)

Note:- ① In Euler's and R-K method, the interval range h should be kept small hence they can be applied for tabulating y only over a limited range.

② To get functional values over a wide range Milne's method may be used, which require starting values usually obtained by Picard's, Taylor's or RK-method.

① Picard's method

Consider the diff^n eq^n

$$\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0 \quad \text{—— (1)},$$

Integrating eq^n (1) b/w x_0 to x

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)\, dx$$

Now the first approximation is

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

second app. $$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

third app $$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\, dx$$

$\vdots$

n^{th} approx $$\boxed{y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx}$$ with $y(x_0) = y_0$.

... lution ... at a given argument,

Ex1. Use Picard's method to solve y at $x = 0.2$. 13

$$\frac{dy}{dx} = x - y \quad \text{with } y(0) = 1$$

Sol: $f(x,y) = (x-y)$, $x_0 = 0$, $y_0 = 1$

1st approx $\quad y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$

$$y^{(1)} = 1 + \int_0^x f(x,1)\, dx$$

$$= 1 + \int_0^x (x-1)\, dx = 1 - x + \frac{x^2}{2} \Rightarrow y^{(1)}(0.2) = 0.82$$

2nd appr. $\quad y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx$

$$= 1 + \int_0^x \left\{ x - 1 + x - \frac{x^2}{2} \right\} dx$$

$$y^{(2)} = 1 - x + x^2 - \frac{x^3}{6}, \quad y^{(2)}(0.2) = 0.83867$$

3rd appr. $\quad y^{(3)} = y_0 + \int_0^x f(x, y^{(2)})\, dx$

$$= 1 + \int_0^x \left\{ x - 1 + x - x^2 + \frac{x^3}{6} \right\} dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}, \quad y^{(3)}(0.2) = 0.83740$$

4th appr. $\quad y^{(4)} = y_0 + \int_0^x f(x, y^{(3)})\, dx$

$$= 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{24} \right) dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

$$y^{(4)}(0.2) = 0.83746.$$

Q2. If $\frac{dy}{dx} = \frac{y-x}{y+x}$, find $y(0.1)$

with $y(0)=1$.

Sol → $y^{(1)} = 1 + \int_0^x \left(\frac{1-x}{1+x}\right) dx = 1 + \int_0^x \left(\frac{2}{1+x} - 1\right) dx$

$= 1 - x + 2\log(1+x)$.

$y^{(2)} = 1 + x - 2\int_0^x \frac{x\,dx}{1+2\log(1+x)}$ which is difficult to integrate.

Hence only 1st approx. $\Rightarrow y^{(1)}(0.1) = 1.09062$.

Q3. If $\frac{dy}{dx} = y + x$ s.t. $y=1$ when $x=0$.

obtain a solⁿ up to fifth approx.

$\left\{ 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} \right\}$.

2. Taylor's method

Consider the first order eqⁿ $\frac{dy}{dx} = f(x,y)$.

with $y(x_0) = y_0$ or $y(x_p) = y_p$.

then Taylor's series expansion about the pt. x_p is given by

$$y(x) = y(x_p) + (x-x_p)\, y'(x_p) + \frac{(x-x_p)^2}{\lfloor 2} y''(x_p) + \cdots$$

find all the values of the right hand side and solve.

Q1. Find by Taylor's series method, the value of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from

$$\frac{dy}{dx} = x^2 y - 1 \; ; \; y(0) = 1.$$

Sol → $f(x,y) = x^2 y - 1$, $y(x_p) = y_p$

$x_p = 0$, $y_p = 1$

Now. $y' = f(x,y) = x^2 y - 1 = \frac{dy}{dx} = y'$

$f'(x,y)$ ie y'

$y' = x^2 y - 1$ at $x_p = 0 \Rightarrow y' = -1$

$y'' = 2xy + x^2 y' \Rightarrow y'' = 0$

$y''' = 2y + 2xy' + 2xy' + x^2 y'' \Rightarrow y''' = 2$

$= 2y + 4xy' + x^2 y''$

$y^{iv} = 2y' + 4y' + 4xy'' + 2xy'' + x^2 y'''$

$= 6y' + 6xy'' + x^2 y''' \Rightarrow y^{iv} = -6$

etc

Hence put in Taylor's series, we have

$$y(x) = y(x_p) + (x - x_p) \cdot y'(x_p) + \frac{(x-x_p)^2}{\underline{|2}} y''(x_p) + \cdots$$

$$= 1 + (x-0) \cdot (-1) + \frac{(x-0)^2}{\underline{|2}} (0) + \frac{(x-0)^3}{\underline{|3}} (2) + \frac{(x-0)^4}{\underline{|4}} (-6) + \cdots$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

at $x = 0.1$

$y(0.1) = 0.90033$

$y(0.2) = 0.80227$

Q2. Solve $y' = x + y$; $y(0) = 1$ by Taylor's series. find y at $x = 0.1$ and 0.2

$y(0.1) = 1.1103$

$y(0.2) = 1.2427$

Q3. Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for $\frac{dy}{dx} = 2y + 3e^x$; $y(0) = 0$. Compare the numerical sol^n obtained with the exact sol^n.

$y(0.2) = 0.8110$

exact

$$\frac{dy}{dx} - 2y = 3e^x$$

I.F. $= e^{-\int 2\,dx} = e^{-2x}$

sol^n

$$y e^{-2x} = \int 3e^x \cdot e^{-2x}\,dx + c$$

$$y = -3e^x + ce^{2x}$$

at $x = 0$, $y = 0 \Rightarrow c = 3 \Rightarrow y(x) = 3(e^{2x} - e^x)$

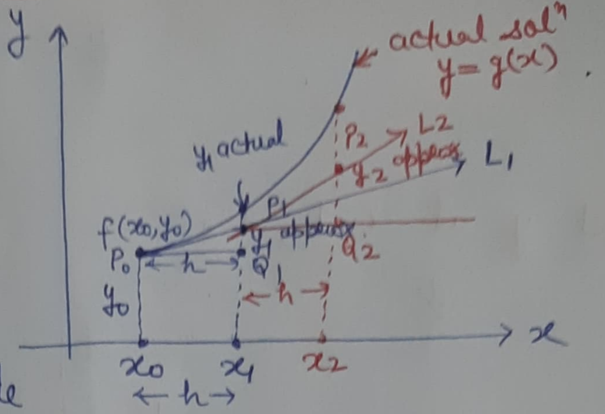
$y(0.2) = 0.8112$

3. Euler's method

Consider a diff^n eq^n as

$$\frac{dy}{dx} = f(x,y)$$

with $y(x_0) = y_0$.



Aim :- With the help of initial condition and diff^n eq^n, find the approximate value which is nearer to the actual value or sol^n.

hence $y_1 = y_0 + P_1Q_1$

$$P_1Q_1 = h\, f(x_0,y_0)$$

by $\tan\theta = \frac{P_1Q_1}{h}$

$$\left(\frac{dy}{dx}\right)_{(x_0,y_0)} = \frac{P_1Q_1}{h}$$

first approx $\quad y_1 = y_0 + h\, f(x_0,y_0)$

$y(x_0+h)$

second approx $\quad y_2 = y_1 + P_2Q_2$

$$= y_1 + h\, f(x_1,y_1)$$

$y(x_0+2h)$

third approx $\quad y_3 = y_2 + P_3Q_3$

$$= y_2 + h\, f(x_2,y_2)$$

General or $(n+1)$th approx

$$\boxed{y_{n+1} = y_n + h\, f(x_n,y_n)}$$

Ex1. Using Euler's method, find an approximate value of y corresponding to $x = 0.2$; given that $\frac{dy}{dx} = x+y$ and $y=1$ when $x=0$. $[h = 0.1]$ / ~~h=0.1~~

Sol → $f(x,y) = x+y$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $y(0.2) = ?$

Make table

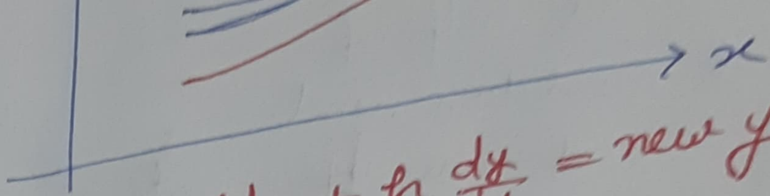
new y = old $y + h\left(\frac{dy}{dx}\right)$

	x	y	$\frac{dy}{dx} = x+y$
x_0	0	1	1
x_1	0.1	1.1	1.2
x_2	0.2	1.22	$\boxed{1.42}$

$1 + 0.1(1) = 1.1$

$1.1 + 0.1(1.2) = 1.22$

Q) find y at $x=1$, then



x	y	$x+y=\frac{dy}{dx}$	$\text{old } y + h\frac{dy}{dx} = \text{new } y$
0	1	1	$1 + 0.1(1) = 1.1$
0.1	1.10	1.20	$1.10 + 0.1(1.20) = 1.22$
0.2	1.22	1.42	$1.22 + 0.1(1.42) = 1.36$
0.3	1.36	1.66	$1.36 + 0.1(1.66) = 1.53$
0.4	1.53	1.93	$1.53 + 0.1(1.93) = 1.72$
0.5	1.72	2.22	1.94
0.6	1.94	2.54	2.19
0.7	2.19	2.89	2.48
0.8	2.48	3.29	2.81
0.9	2.81	3.71	3.18
1.0	$\boxed{3.18}$		

Q2. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with $y(0)=1$. find $y(0.1)$.
$(h = 0.02)$.

Sol →

x	y	dy/dx
0	1	1
0.02	1.02	0.9615
0.04	1.0392	0.926
0.06	1.0577	0.893
0.08	1.0756	0.862
0.10	1.0928	

old $y + h\frac{dy}{dx}$ = new y

$1 + 0.02(1) = 1.02$

$1.02 + 0.02(0.9615) = 1.0392$

$1.0392 + 0.02(0.926) = 1.0577$

1.0756

1.0928

4. Modified Euler's Method

In Euler's method;

$$y_1 = y_0 + h f(x_0, y_0)$$

Now we find a better approximation $y_1^{(1)}$ as

1st approx $\quad y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(\underbrace{x_0 + h}_{x_1}, y_1)\right]$

or

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(E)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

till two consecutive value of y agree. This is repeated to be taken as the value of y_1.

2nd approx $\quad y_2 = y_1 + h f(x_1, y_1)$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(E)})\right]$$

$$y_2^{(2)} = y_2 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

repeat till similar values.

Then we proceed to calculate y_3 and so on.

Ex1:- Use Modified Euler's method, find an app. value of y when $x = 0.3$, given that $\frac{dy}{dx} = x + y$ & $y = 1$ when $x = 0$.

Sol→ $\quad y(0.3) = ?$

$$h = 0.1$$

$$x_0 = 0$$

$$x_1 = x_0 + h = 0.1$$

$$x_2 = x_0 + 2h = 0.2$$

$$x_3 = x_0 + 3h = 0.3$$

x	$x+y=\frac{dy}{dx}$	Mean slope	old y + (0.1)[Mean slope] = new y
0	0+1 = 1 (old y)	–	1+(0.1)(1) = 1.10
0.1	0.1+1.1 = 1.2	$\frac{1}{2}$(1+1.2) = 1.1	1+(0.1)(1.1) = 1.11
0.1	0.1+1.11 = 1.21	$\frac{1}{2}$(1+1.21) = 1.105	1+(0.1)(1.105) = 1.1105
0.1	0.1+1.1105 = 1.2105	$\frac{1}{2}$(1+1.2105) = 1.1052	1+(0.1)(1.1052) = 1.1105

Since last two values are same, (equal.) hence $y_1 = 1.1105$

0.1	1.2105 slope	–	1.1105+(0.1)(1.2105) = 1.2316
0.2	1.4316	$\frac{1}{2}$(1.2105+1.4316) = 1.3211	1.1105+(0.1)(1.3211) = 1.2426
0.2	0.2+1.2426 = 1.4426	$\frac{1}{2}$(1.2105+1.4426) = 1.3266	1.1105+(0.1)(1.3266) = 1.2432
0.2	0.2+1.2432 = 1.4432	$\frac{1}{2}$(1.2105+1.4432) = 1.3268	1.1105+(0.1)(1.3268) = 1.2432

y_2 or $y(0.2) = 1.2432$

0.2	1.4432	–	1.2432+(0.1)(1.4432) = 1.3875
0.3	0.3+1.3875 = 1.6875	$\frac{1}{2}$[1.4432+1.6875) = 1.5654	1.2432+(0.1)(1.5654) = 1.3997
0.3	0.3+1.3997 = 1.6997	$\frac{1}{2}$(1.4432+1.6997) = 1.5715	1.2432+(0.1)(1.5715) = 1.4003
0.3	0.3+(1.4003) = 1.7003	$\frac{1}{2}$(1.4432+1.7003) = 1.5718	1.2432+(0.1)(1.5718) = 1.4004
0.3	0.3+1.4004 = 1.7004	$\frac{1}{2}$(1.4432)+(1.7004) = 1.5718	1.2432+(0.1)(1.5718) = 1.4004

hence y_2 or $y(0.2) = 1.4004$ approx

1.32905

Q2. Use Modified Euler method & find $y(0.2)$ & $y(0.4)$ for $\frac{dy}{dx} = y + e^x$; $y(0) = 0$ $\quad h = 0.2$

Sol →

x	$\frac{dy}{dx} = y + e^x$	Meanslope	old y + h(Meanslope) = new y
0	1 slope	–	$0 + (0.2)(1) = 0.2$
0.2	$0.2 + e^{0.2} = 1.4214$	$\frac{1}{2}(1 + 1.4214) = 1.2107$	$0 + (0.2)(1.2107) = 0.2421$
0.2	$0.2421 + e^{0.2} = 1.4635$	$\frac{1}{2}(1 + 1.4635) = 1.2317$	$0 + 0.2(1.2317) = 0.2463$
0.2	$0.2463 + e^{0.2} = 1.4677$	$\frac{1}{2}(1 + 1.4677) = 1.2338$	$0 + 0.2(1.2338) = 0.2468$ ✓
0.2	$0.2468 + e^{0.2} = 1.4682$	$\frac{1}{2}(1 + 1.4682) = 1.2341$	$0 + 0.2(1.2341) = 0.2468$ ✓ old y

hence y_1 or $y(x_1)$ or $y(x_0+h)$, $y(0.2) = 0.2468$

x	$\frac{dy}{dx} = y + e^x$	Meanslope	old y + h(Meanslope) = new y
0.2	1.4682	–	$0.2468 + (0.2)(1.4682) = 0.5404$
0.4	$0.5404 + e^{0.4} = 2.0322$	$\frac{1}{2}(1.4682 + 2.0322) = 1.7502$	$0.2468 + (0.2)(1.7502) = 0.5968$
0.4	$0.5968 + e^{0.4} = 2.0887$	$\frac{1}{2}(1.4682 + 2.0887) = 1.7784$	$0.2468 + (0.2)(1.7784) = 0.6025$
0.4	$0.6025 + e^{0.4} = 2.0943$	$\frac{1}{2}(1.4682 + 2.0943) = 1.78125$	$0.2468 + (0.2)(1.78125) = 0.6030$
0.4	$0.6030 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7815$	$0.2468 + (0.2)(1.7815) = 0.6031$
0.4	$0.6031 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7816$	$0.2468 + (0.2)(1.7816) = 0.6031$

hence y_2 or $y(x_0 + 2h) = y(0.4) = 0.6031$ approx.

5. Runge-Kutta Method (RK-method)

These methods agree with Taylor's series sol^n upto the term in h^r where r differs from method to method and is called the order of that method.

first order RK method :- Euler's method

second order RK method :- Modified Euler's method.

Third order RK method :- Runge's method.

Fourth order RK method - This method is most commonly used and is often referred to as RK method only.

Working Rule :- $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$.

To find the increment k of y corresponding to increment h of x is

$$K_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$K_4 = hf(x_0 + h, y_0 + k_3).$$

then compute $K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$.

which gives the required approx. value as $y_1 = y_0 + K$.

Q1. Apply RK method of 4th order to find approx. value of y when $x = 0.2$, $\frac{dy}{dx} = x + y$, $y(0) = 1$.

Sol^n. $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x,y) = x + y$

$$k_1 = hf(x_0, y_0)$$
$$= (0.2)[0 + 1] = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= h\left\{\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.2}{2}\right)\right\}$$
$$= (0.2)\{0.1 + 1.1\} = 1.2 \times .2$$
$$= .24$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= (0.2)f\left(0.1, 1 + \frac{.24}{2}\right)$$
$$= (0.2)f(0.1, 1.12)$$
$$= 0.2(1.22)$$
$$= .2440$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= (0.2)f(0.2, 1 + .244)$$
$$= (0.2)f(0.2, 1.244)$$
$$= 0.2(1.444)$$
$$= .2888$$

Now $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 0.2428.$$

hence $y_1 = y_0 + k$

$$= 1 + 0.2428$$
$$= 1.2428$$

Q2. Use RK method, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2$ & 0.4

Sol-1. $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0=0$, $y_0=1$, $h=0.1$

$k_1 = 0.2 f(0,1) = 0.2$

$k_2 = 0.2 f(0.1, 1.1) = .19672$

$k_3 = 0.2 f(0.1, 1.09836) = .1967$

$k_4 = 0.2 f(0.2, 1.1967) = .1891$

$k = \frac{1}{6}(0.2 + 2(.19672) + 2(.1967) + .1891)$

$= .19599$

$y_1 = y_0 + k = 1.196.$

To find $y(0.4)$ $\qquad h=0.2$

$x_1 = 0.2$, $y_1 = 1.196$, $h=0.2$

$k_1 = hf(x_1, y_1) = .1891$

$k_2 = hf(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}) = .1795$

$k_3 = hf(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}) = .1793$

$k_4 = hf(x_1+h, y_1+k_3) = .1688$

$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$= 0.1792$

$y_2 = y_1 + k$

$= 1.196 + .1792$

$= 1.3752$

Q-1 $\frac{dy}{dx} = x + y^2$, $y(0) = 1$. find $y(0.2)$ in steps 0.1

(1.2736)

Q-1 $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$, $x_0 = 1$, $y_0 = 1$ $y(1.2)$ & $y(1.4)$

.1402 0.2705

6. Milne's Method

$$\frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0.$$

first we get the approximate value of y_{n+1} by predictor formula and then improve ~~the~~ this using a corrector formula.

Newton's forward Interpolation formula in terms of y' and p is

$$y' = y_0' + p\Delta y_0' + \frac{p(p-1)}{2}\Delta^2 y_0' + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0' + \frac{p(p-1)(p-2)(p-3)}{24}\Delta^4 y_0' \quad \text{—— (1)}$$

where $p = \frac{x-x_0}{h}$

or $x = x_0 + ph$

Integrate (1) over x_0 to (x_0+4h)

$$\int_{x_0}^{x_0+4h} y'\,dx = h\int_0^4 y'\,dp \qquad \left\{ \because \begin{array}{l} x = x_0 + ph \\ dx = h\,dp \end{array} \right.$$

$$(y_4 - y_0) = h\int_0^4 \{\text{put from (1)}\}\,dp$$

$$= h\left\{4y_0' + 8\Delta y_0' + \frac{20}{3}\Delta^2 y_0' + \frac{8}{3}\Delta^3 y_0' + \frac{28}{90}\Delta^4 y_0'\right)$$

substitute the values of I, II, III differences, we get

$$y_4 - y_0 = h\left\{4y_0' + 8(E-1)y_0' + \frac{20}{3}(E-1)^2 y_0' + \frac{8}{3}(E-1)^3 y_0' + \cdots\right\}$$

$$= \frac{4h}{3}\{2y_1' - y_2' + 2y_3'\} + \cdots$$

$$\boxed{y_4 = y_0 + \frac{4h}{3}\{2y_1' - y_2' + 2y_3'\}} + \cdots \quad \text{—— (2)}$$

This is Milne's Predictor formula

or

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$f_1 = y_1'$

$f_2 = y_2'$

$f_3 = y_3'$

It is used to predict the values of y_4 when the value of y_0, y_1, y_2 and y_3 are known.

To obtain the corrector formula, we integrate (1) over the interval x_0 to (x_0+2h) (or $p=0$ to 2) and we get

$$y_2 - y_0 = h\left(2y_0' + 2\Delta y_0' + \frac{1}{3}\Delta^2 y_0' - \cdots\right)$$

Express in I, II, ~~III~~ differences in terms of (E-1).

$$y_2 - y_0 = \frac{h}{3}\left(y_0' + 4y_1' + y_2'\right)$$

$$\boxed{y_2 = y_0 + \frac{h}{3}\left(y_0' + 4y_1' + y_2'\right)} \quad \text{— (3)}$$

This is milne's corrector formula

or

~~$y_2 = y_0$~~.

Eq^n (2) and (3) Can be expressed as

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \quad \text{Predictor}$$

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \quad \text{Corrector}$$

An improved of f_4 is then computed again & again by Milne's corrector method, find a still better value of y_4. we repeat this step untill y_4 remains unchanged.

Use it

Milne's Predictor-Corrector Method

Consider the d.e. $\frac{dy}{dx} = f(x,y)$

with I.C. $y(x_0) = y_0$

Aim :- find $y(x_n)$ (n must 4 atleast).

find $y(x_1), y(x_2), y(x_3)$ using any one of the following method

1. Picard's method
2. Euler
3. Modified Euler method
4. Taylor's series
5. R-k method

Then Calculate

$f_0 = f(x_0, y_0)$ $\quad f_2 = f(x_2, y_2)$

$f_1 = f(x_1, y_1)$ $\quad f_3 = f(x_3, y_3)$

By Milne's Predictor method,

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

then find $f_4 = f(x_4, y_4)$

By Milne's corrector method,

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

An impeoved of f_4 is then computed and again Milne's corrector method is applied to find a still better value of y_4, we repeat this step untill y_4 remains unchanged.

Q→ Using R-K method of 4^{th} order, to find y for $x=0.1$, 0.2, 0.3 given that $\frac{dy}{dx} = xy + y^2$, $y(0)=1$.
Continue the solⁿ at $x=0.4$ using Milne's method.

Sol→ To find $y(0.1)$

$K_1 = hf(x_0, y_0) = (0.1)f(0,1) = 0.1$

$K_2 = hf(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}) = 0.1155$

$K_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}) = 0.1172$

$K_4 = hf(x_0+h, y_0+k_3) = 0.1360$

$K = \frac{1}{6}(K_1+2K_2+2k_3+K_4) = 0.1169$

$y_1 = y_0+k = 1+0.1169 = 1.1169$

To find $y(0.2)$ $\quad x_1 = 0.1,\ y_1 = 1.1169,\ h=0.1$

$K_1 = hf(x_1, y_1) = 0.1359$

$K_2 = hf(x_1+\frac{h}{2}, y_1+\frac{K_1}{2}) = 0.1581$

$K_3 = 0.1609$

$K_4 = 0.1888$

$K = \frac{1}{6}(K_1+2K_2+2K_3+K_4) = 0.1605$

$y_2 = y_1+k = 1.1169+0.1605 = 1.2774$.

To find $y(0.3)$ $\quad x_2 = 0.2,\ y_2 = 1.2774,\ h=0.1$

$K_1 = 0.1887,\ K_2 = 0.2224,\ K_3 = 0.2275,\ K_4 = 0.2716$

$K = 0.2267$

$y_3 = 1.5041$

So we have $\quad h=0.1 \quad f(x,y) = xy+y^2$

$x_0 = 0 \quad y_0 = 1 \quad f_0 = f(x_0,y_0) = 1$

$x_1 = 0.1 \quad y_1 = 1.1169 \quad f_1 = f(x_1,y_1) = 1.3591$

$x_2 = 0.2 \quad y_2 = 1.2774 \quad f_2 = f(x_2,y_2) = 1.8869$

$x_3 = 0.3 \quad y_3 = 1.5041 \quad f_3 = f(x_3,y_3) = 2.7132$

By Milne's predictor method

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1.8344$$

$\Rightarrow x_4 = 0.4,\ y_4 = 1.8344,\ f_4 = f(x_4, y_4) = 4.0988.$

By Corrector method

$$y_4' = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$y_4^{(1)} = 1.8387$$

$$f_4 = f(x_4, y_4) = 4.1162$$

Again $y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

$$\boxed{y_4^{(2)} = 1.8393}$$

$$f_4 = 4.1186$$

Again $y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

$$\boxed{y_4^{(3)} = 1.8393}$$

Q2. find $y(2)$ if $\frac{dy}{dx} = \frac{1}{2}(x+y)$.

$y(0) = 2$

Given $y(0.5) = 2.636,\ y(1) = 3.595,\ y(1.5) = 4.968$

Sol → $x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5$

$y_0 = 2,\ y_1 = 2.636,\ y_2 = 3.595,\ y_3 = 4.968$

$f(x, y) = \frac{1}{2}(x+y).$

Solⁿ → 6.873.

- unequally

Q3. Use Milne's predictor-corrector method, obtain $y(0.4)$ from the given set of tabulated value & $\frac{dy}{dx} = y^2 - x^2$.

x	x_0 0	x_1 0.1	x_2 0.2	x_3 0.3
y	1 y_0	0.11 y_1	1.25 y_2	1.42 y_3
f	1 f_0	1.22 f_1	1.52 f_2	1.92 f_3

$h = 0.1$

Sol: → Use predictor formula

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1 + \frac{4(0.1)}{3}(2\times1.22 - 1.52 + 2(1.92))$$

$$= 1.63466$$

then, $f_4 = f(x_4, y_4)$

$$= (y_4^2 - x_4^2) = 2.51211$$

Now apply corrector formula,

$$y_4^{(1)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.25 + \frac{(0.1)}{3}\left[1.52 + 4(1.92) + 2.51211\right]$$

$$y_4^{(1)} = 1.64040 \Rightarrow f_4^{(1)} = y_4^{(1)2} - x_4^2 = (1.64040)^2 - (0.4)^2$$

Again; $y_4^{(2)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{(1)})$

$$= 1.64103 \Rightarrow f_4^{(2)} = (1.64103)^2 - (0.4)^2$$

$\Rightarrow$

Again; $y_4^{(2)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{(2)})$

$$y_4^{(2)} = 1.64109 \simeq 1.6411 \Rightarrow f_4^{(3)} = \checkmark$$

Again $y_4^{(3)} \simeq 1.6411$

$\boxed{\text{hence } f(0.4) \text{ or } y_4 = 1.6411}$