DYNAMIC PROGRAMMING

Day 5

Youtube link:

Part 1: https://www.youtube.com/watch?v=95o1lWeV7fY

Part 2: https://www.youtube.com/watch?v=8HOlj0RgcCU

Contents:

1. Bitwise operations
2. Bitmasks
3. DP with bit masking

Bitwise Operations

Binary representation of numbers

int x=9;
Binary representation of 9 = 1001
Because 9 = 2^3 + 2^0
We know, int is 32-bit .
9 = ...000000001001

int x=7;
x = 0000000….00111

Logical Operations

&& - AND
|| - OR
! - NOT
These operations operate on boolean variables

```
if ( true && false ) {
```

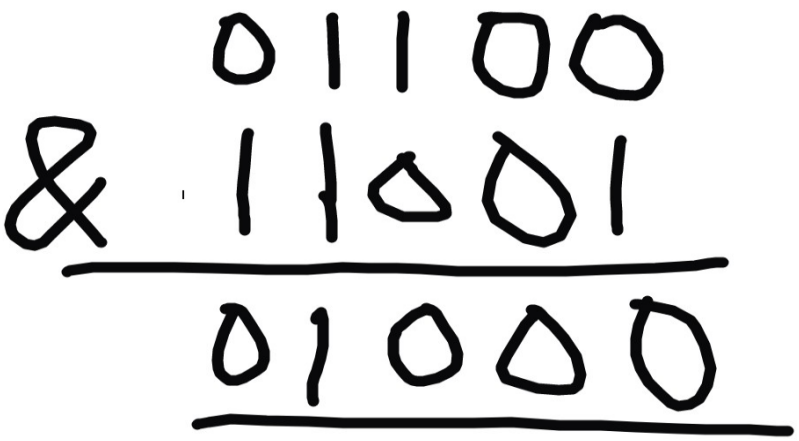
....
}
if(4 && 7) {
// Computer will first convert 4 & 7 to boolean (true/false) and then it will operate
// 4 - true (All non-zero values are treated as true)
// 7 - true (All non-zero values are treated as true)
}
Similar for || & NOT

Bitwise Operations

- Bit-per-bit calculations

& operator (Bitwise AND operation)

Eg
A = 12 = (1100) [in binary]
B = 25 = (11001) [in binary]

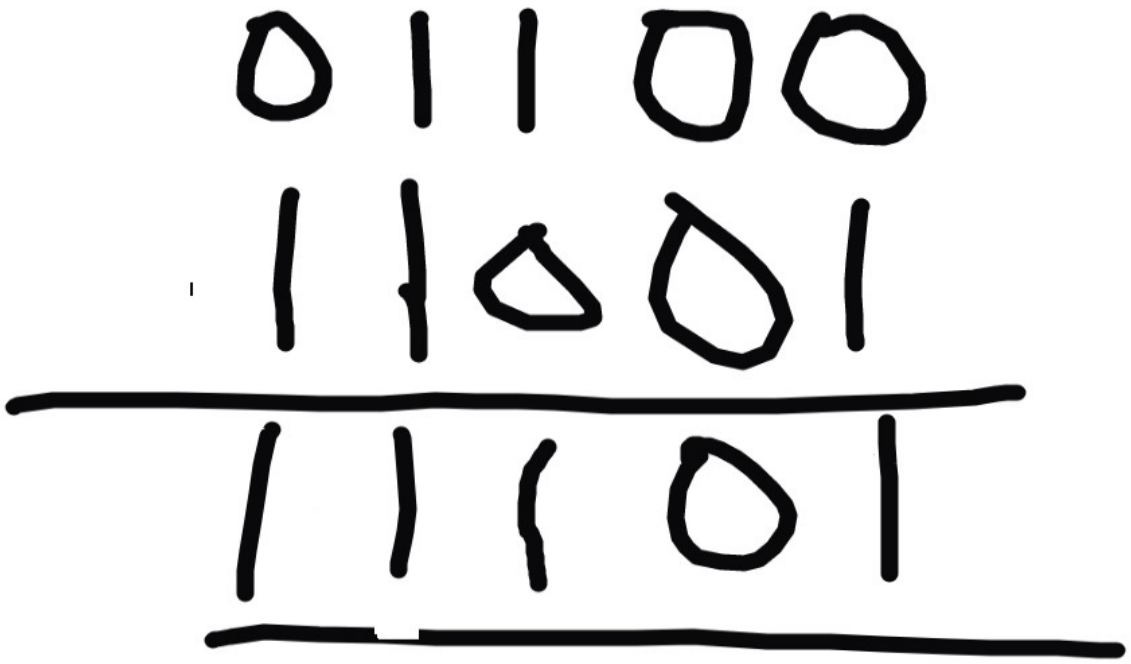


A&B = 8

| operator (Bitwise OR operation)

Eg
A = 12 = (1100) [in binary]

B = 25 = (11001) [in binary]



Decimal equivalent of A | B is 29

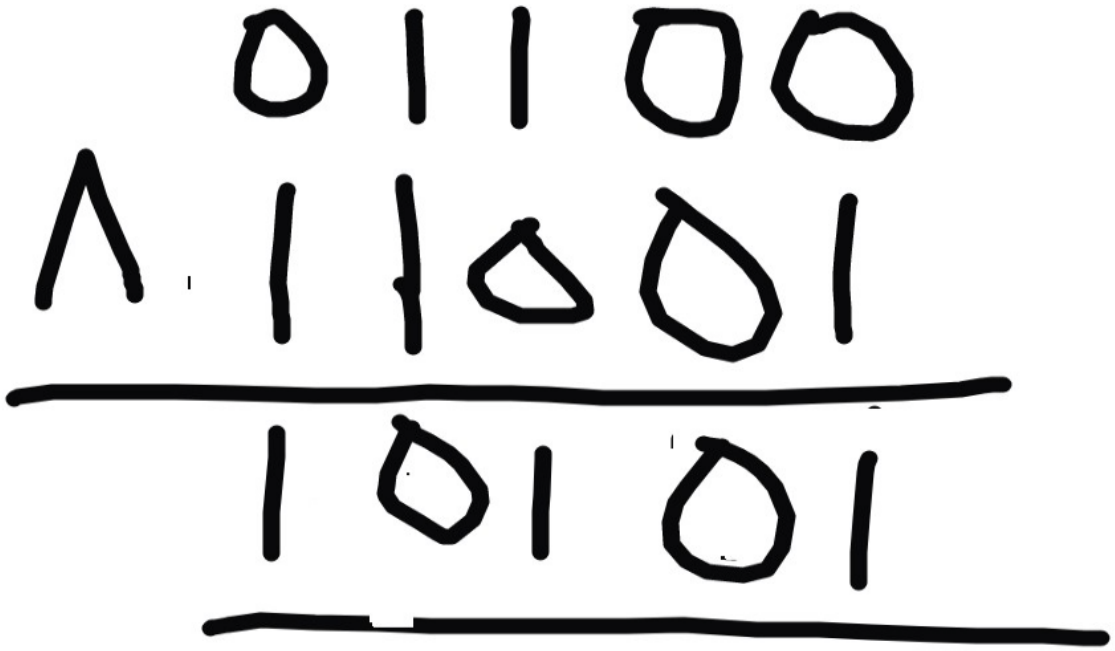
^ operator (Bitwise XOR operation)

- Same bit (both 0 or both 1), then it will result in a 0 bit
- Different bit will result in a 1 bit (0 ^ 1 = 1)

Eg

A = 12 = (1100) [in binary]

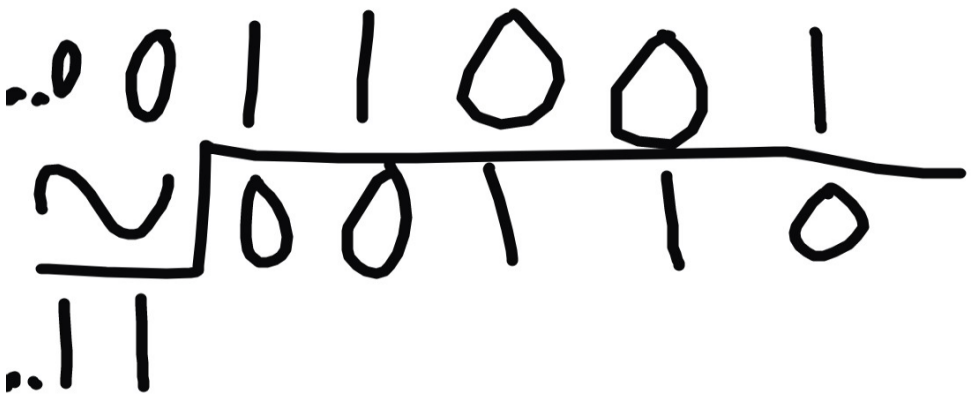
B = 25 = (11001) [in binary]



A^B = 21
A ^ 0 = A
A ^ A =0
A ^ (B ^ C) = (A ^ B) ^ C
Try this problem -
https://www.hackerrank.com/challenges/lonely-integer/problem
~ operator (Bitwise NOT operation or 1's complement)

- Replace 0 by 1 and 1 by 0

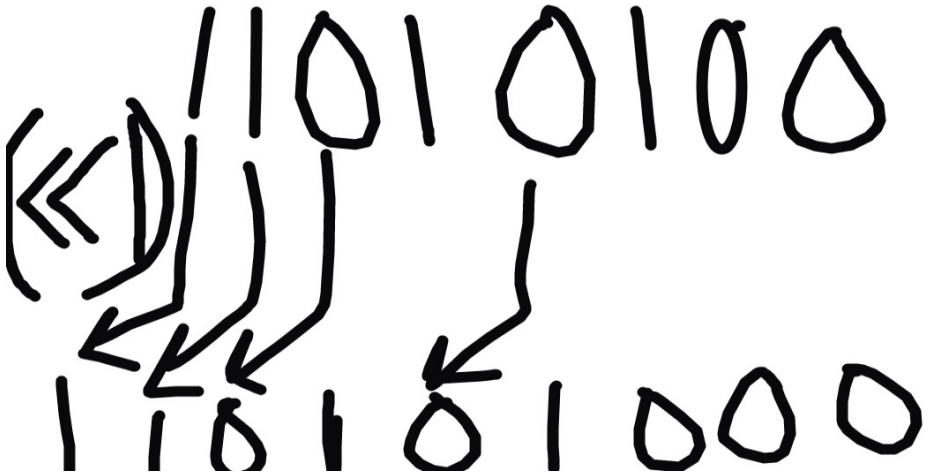
Eg
B = 25 = (11001) [in binary]



Note: Negative integers are represented in memory as their 2's complement form
i.e. -x = 2's complement of x
2's complement = 1's complement + 1
-x = ~x + 1 => ~x = - (x+1)
Eg. ~25 = -26 = -25 -1 = - (25+1)

<< operator (Left shift operation)

212 = (11010100)



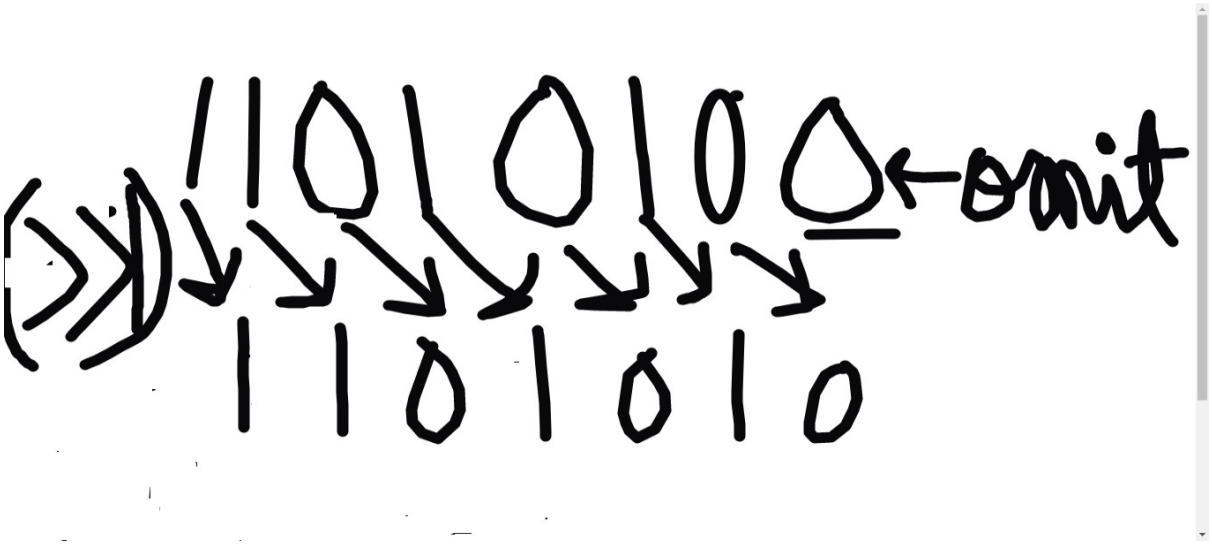
212 << 1 = 424 = 212 * 2
212 << 2 = 848 = 212 * 4
x << b = ? (Left-shift number x by b bits)
Important property:
$x << b = x * (2^b)$

Preventing Overflows in Left Shift (Very Important)

To prevent-overflows, use LL with constants
(to convert to long long)
Eg. (1LL << 40) is correct
But (1 << 40) is wrong (overflow)

>> operator (Right shift operation)

212 = (11010100)



212 >> 1 = (1101010) = 106 = 212/2

212 >> 2 = 53 = 212 / 4

x >> b = ? (Right-shift number x by b bits)

Important property:

$x >> b = floor(x / (2^b))$

Important property: All bitwise operations are performed in O(1)

Applications of bitwise operations (Very very important)

1. Check whether x^{th} bit (from the right) of a number n is set (1) or not (0) ?

```
if ( n & ( 1<<x ) ) {
    // x^th bit is set
} else {
// x^th bit is not set
}
```

n = 01110011

1 << x = 10000...0 (1 is at only the x-th position)

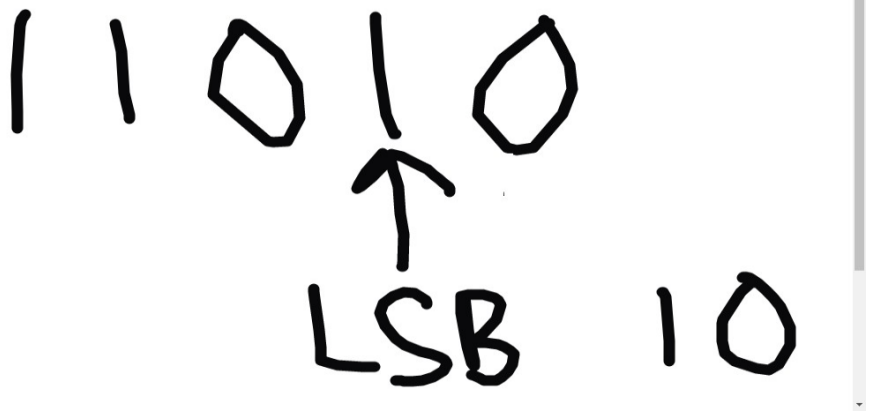
2. Set x^{th} bit (from the right) of a number n

   n = n | (1<<x)
   1 << x = 10000...0 (1 is at only the x-th position)

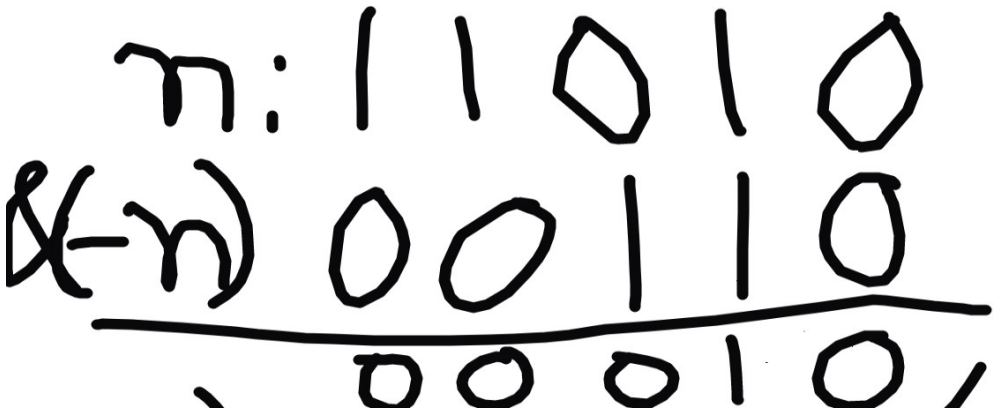
3. Toggle x^{th} bit (from the right) of a number n
   (Toggle means 0 becomes 1 and 1 becomes 0)

   n = n ^ (1<<x)
   1 << x = 10000...0 (1 is at only the x-th position)

4. Find LSB of n. Using this, also unset the LSB of n.
   (O(1))
   LSB = Lowest Set Bit



LSB = n & (-n)



To unset LSB , subtract LSB

Method 1:
n = n - (n &(-n))

Method 2:
n = n ^ (n &(-n))

5. Given number n, check whether it is a power of 2
   Eg. n = 6, not a power of 2
   n =8, is a power of 2 [$8 = 2^3$]

```
if ( n & (n-1) ) {
// n is not a power of 2
} else if (n>0){
 // n is a power of 2
}
```

// Corner case: When n=0, treat separately

Try this problem:
https://www.hackerrank.com/challenges/counter-game/problem

7. Given 2 equations on a & b. Given values of n & m. Find values of a & b that satisfy it: [Linkedin coding round 2020]

a + b = m

a ^ b = n

1 <= a, b, n, m <=10^9

a <= b

(**Link to a similar problem (Only constraints are different):** https://codeism.contest.codeforces.com/group/qXv2tukHZE/contest/309847/problem/A)

Solution

From half-adder (or binary addition)

S = a^b

C = a&b

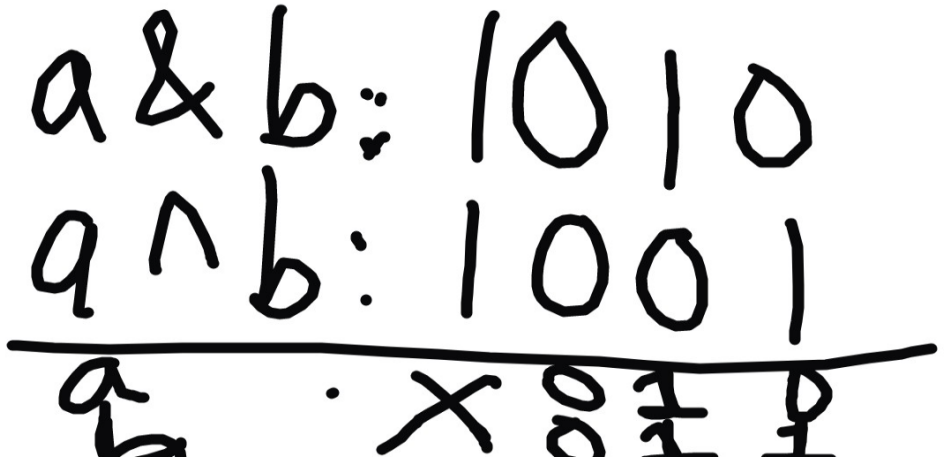
a + b = a^b + ((a&b) << 1)

(a + b) - (a^b) = ((a&b) * 2)

a & b = floor (((a + b) - (a^b)) / 2)

Let k = a & b = floor ((m- n) / 2)

Iterate through all bits and check the bit value of n & k,



HW questions

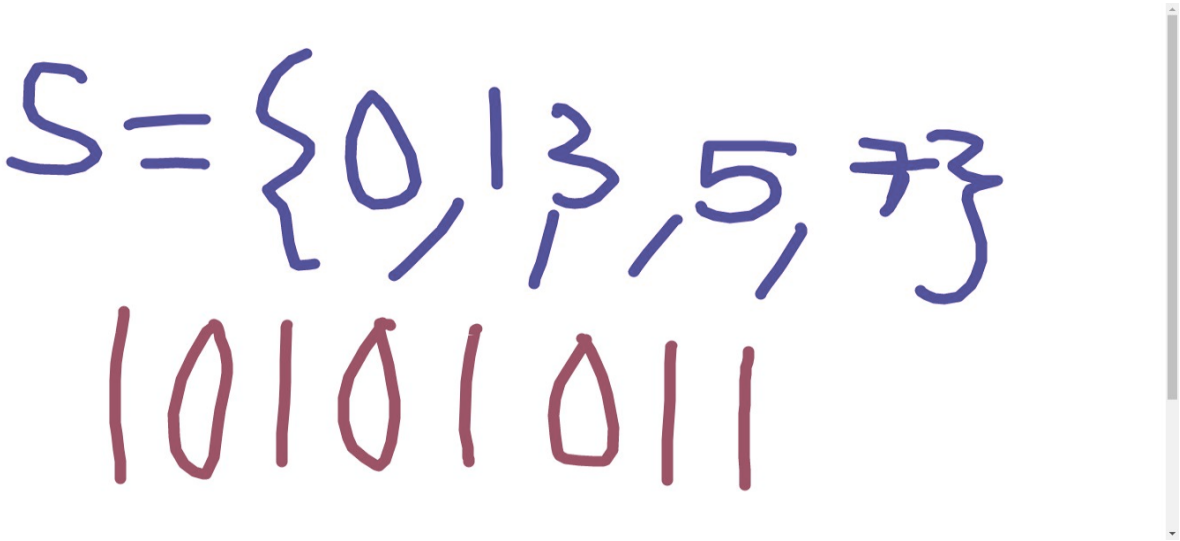
8. Find addition of 2 numbers using only bit-wise operations.
https://leetcode.com/problems/sum-of-two-integers/

9. Using the above, find multiplication of 2 numbers using only bit-wise operations
(Similar thing was discussed in number theory - think of something like iterative version of binary exponentation)

10. Unset x^{th} bit of a number n,
n = n & ~(1<<x)

Bitmasks

- Representation of sets (sets in Set theory) using an integer
- If x^{th} bit is set (1) in the integer , that element x is present in the set
- If x^{th} bit is unset (0) in the integer , that element x is not present in the set
- There are 2^n subsets of a set of n elements



- **Finding the number of set bits (1-bits) in any integer x**
  (in O(1))

cout<<__builtin_popcount(x);
Eg.

```
    cout<<__builtin_popcount(9);
// “1001” has 2 set bits
```

Now, try to solve this problem:
https://codeforces.com/problemset/problem/1097/B

Observations:

- Order of rotations doesn’t matter
- We need to consider all the possible combinations of clockwise and anti-clockwise rotations
- We need to iterate through all the subset of a set that has n elements (n rotations)

Let us represent this set of rotations as bitmask

```
int vec[n];
```

```
for (int mask = 0; mask < (1<<n); mask++)
{
    // 1 - clockwise
    // 0 - anticlockwise
    // iterate through all bits of this number
mask
   int sum=0; // final value of rotation
  (Assume clockwise - +ve)
   for( int i =0; i<n; i++)
   {
     if ( mask & (1<<i) )
         {
sum=sum+vec[i];
    }
else {
   sum = sum - vec[i];
}
  }
 if (sum%360 == 0)
 {
   cout<<"YES";
   return 0;
  }
}

cout<<"NO";
return 0;
```

Q. Given x, y, find whether x is a submask (subset) of y. (in O(1))

Eg.

If x = 1101

y = 1111
Then x is a submask (subset) of y

If x = 01101
y = 01110
Then x is not a submask(subset) of y

Method 1

```
if (x & y == x)
{
// x is a subset of y
}
```

Method 2

```
if (x | y == y )
{
// x is a subset of y
}
```

Q. Given any number n, iterate through all the submasks of n. (in decreasing order)

Eg. n=1010
1010, 1000, 0010, 0000
(1010 - 1 = 1001)
(1001 & 1010 = 1000)
(1000 -1 = 0111)
(0111 & 1010 = 0010)
n = 11011
11011, 11010, 11000, 10011, 10010, 10001…….

- Subtract 1 from each sub mask
- And then take bitwise and (&) with the original mask

```
for( int x=n; x>0; x = (x-1) & n)
{
```

```
  cout<<x<<'\n';
}
cout<<0<<'\n'; // treat 0 separately
```

Time complexity : O(2^set_bits)

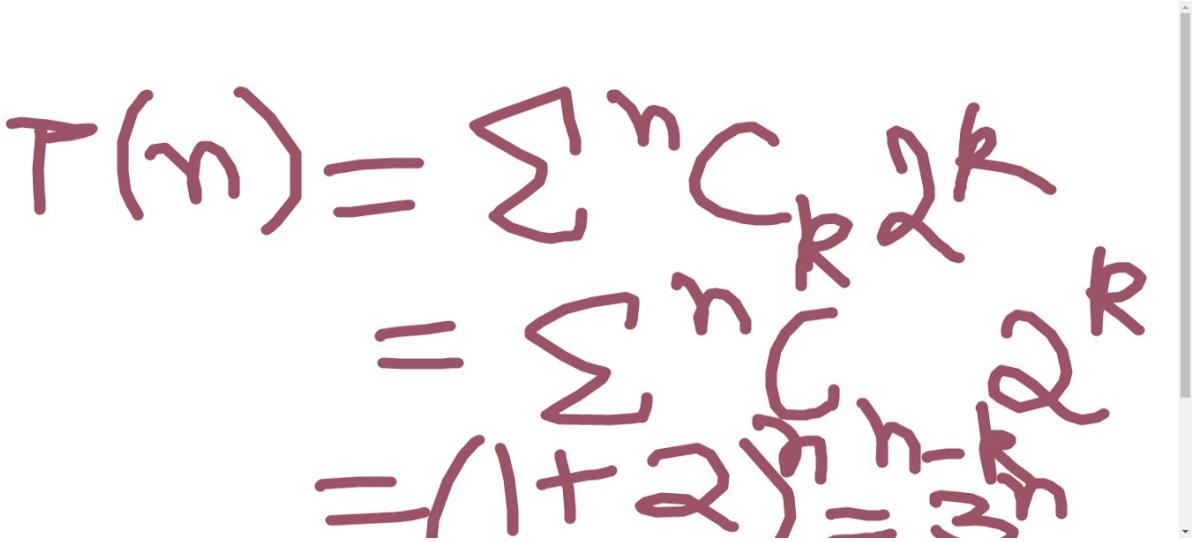
Q. Given any set of n elements, iterate through all the subsets (submasks) of all the subsets (submasks) of size n

Given set can be written as 111111...1 (1's n times)

```
for( int i=0; i< (1<<n) ; i++)
{
   for( int x=i; x>0; x = (x-1) & i)
   {
    cout<<x<<'\n';
   }
cout<<0<<'\n'; // treat 0 separately
}
```

Time complexity : O(3^n)

Proof: Let k=number of set bits



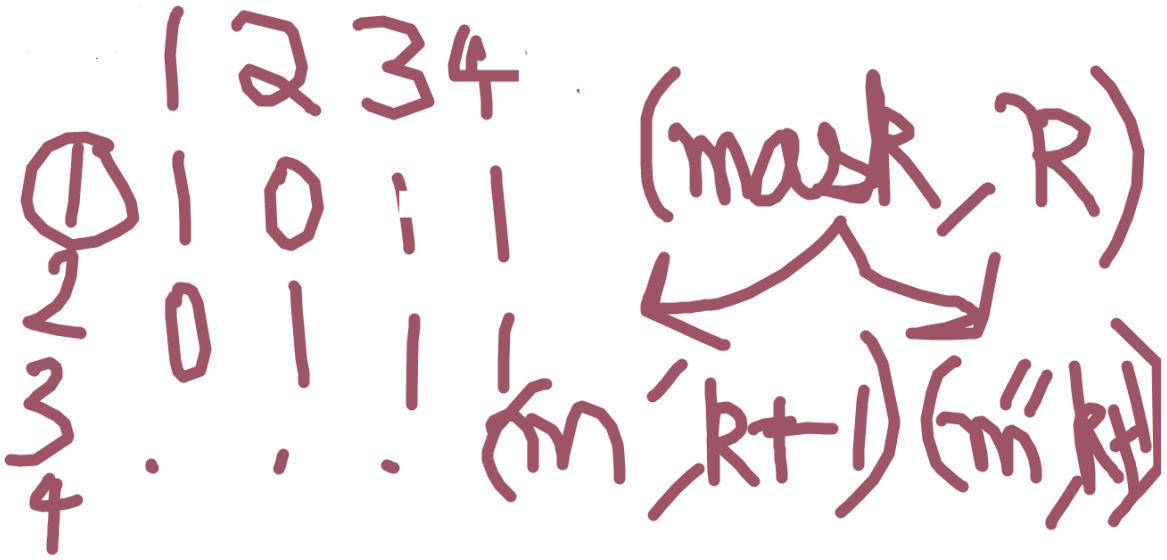
DP with bitmasking

- **If some state of DP problem can be represented as a subet or a set, then we use DP with bitmasking**

  Problem 1: https://www.spoj.com/problems/ASSIGN/

  - **In questions involving bitmasks, the constraints are generally very low. (n<=20 or 25)**
  - **Consider a set of n topics .**

Let us consider in a ,mask, if i^{th} bit is 1 represents that i^th student has been given a subject.
If i^th bit is 0, then that student has not taken any subject yet.
Let k = smallest index of topic left to be distributed




```
int dp [ 1<<n ][ n ] // initially each value -1
int solve(int mask, int k)
{
```

```
int ans=0;
if (k == n) {
  return 1;
}
 if (dp[mask][k]!=-1)
      return dp[mask][k];
    for(i=0; i<n; i++)
    {
         if(a[i][k]==1 && !(mask & (1<<i)) )
          {
             ans = ans + solve (mask | (1<<i), k+1 )
          }
     }
return dp[mask][k]=ans;
}

 // in main
 solve(0, 0)
```

Time complexity: $O(n^2*2^n)$ or $O(n*2^n)$

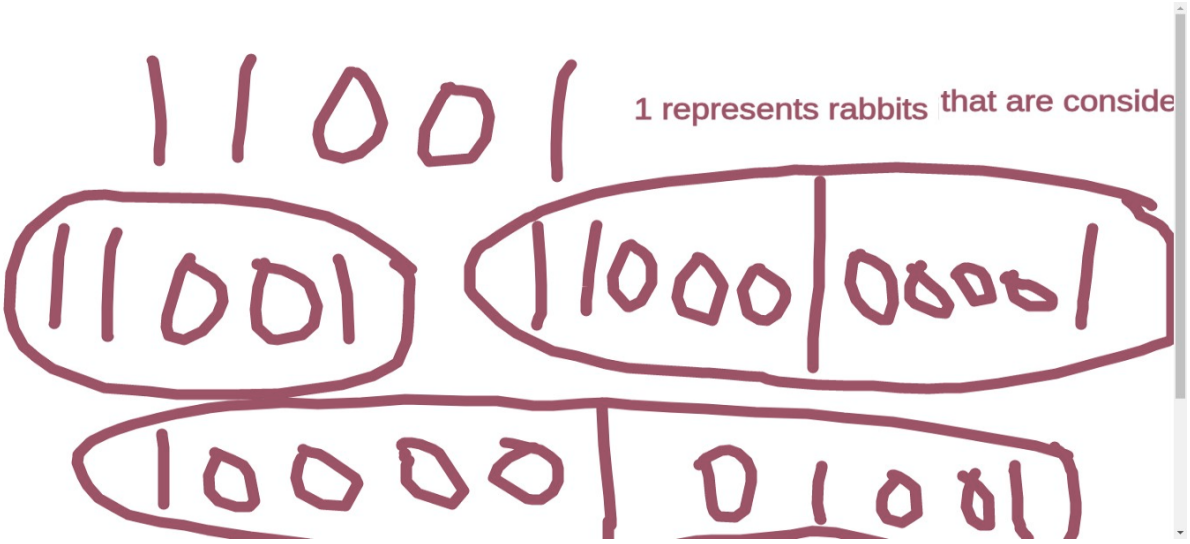
This solution will give MLE / TLE
We can reduce the state for k as follows:

```
int dp [ 1<<n ]
int solve(int mask)
{
int k = __builtin_popcount(mask);
// Now, k = set bits in mask calculated in O(1)
// Effectively , k = index of topic to be assigned
int ans=0;
```

```
if (k == n) {
  return 1;
 }
 if (dp[mask]!=-1)
      return dp[mask];
    for(i=0; i<n; i++)
    {
         if(a[i][k]==1 && !(mask & (1<<i)) )
          {
             ans = ans + solve (mask | (1<<i))
          }
     }
return dp[mask]=ans;
}
// In main,
solve(0)
```

Time complexity: O(2^n * n)

PROBLEM Atcoder DP contest - U Grouping
https://atcoder.jp/contests/dp/tasks/dp_u



```
dp[1<<n] ; // initalize with -INF or -1e18

int solve (int mask)
{
  if ( dp[mask]!= -INF)
  {
       return dp[mask];
   }
  // Case 1: All the set bits are in same group
int ans=0;
for(int i=0; i<n; i++)
  {
     for(int j=i+1; j<n; j++)
      {
         if ( (mask & (1<<i)) && (mask & (1<<j)) )
                ans += a[i][j];
       }
   }
// Case 2: iterate through all submasks of given mask
for( int i = mask&(mask-1); i>0; i=(i-1) & mask)
{
ans  = max(ans, solve(i) + solve( mask ^ i ) );
}
return dp[mask]=ans;
}
```

// dp[mask] = maximum cost to build any possible grouping of rabbits which have 1 (set bit) in mask
Eg. dp[1001] = maximum cost to build any grouping of rabbits 0 and rabbit 3

```
// in int main()
cout<<solve( (1<<n) - 1 );
// because (1<<n) - 1 = 1111111....111 (  n times 1 )
```

HW questions

1. https://www.hackerearth.com/practice/algorithms/dynamic-programming/bit-masking/practice-problems/algorithm/mehta-and-the-tricky-triplets/
2. https://leetcode.com/problems/number-of-ways-to-wear-different-hats-to-each-other/
3. (For those who have studied graphs) https://www.hackerrank.com/challenges/synchronous-shopping

Also, try long challenge problems here:
https://codeism.contest.codeforces.com/

Another trick for some number theory + bitmask problems
Any number n can be represented as a mask of its prime divisors.
Prime numbers = {2,3,5,7,11}
Eg. n=6, 6=2*3, n = 001100
Sometimes, this helps to calculate GCD, LCM of 2 numbers effectively using bitwise & and bitwise | operations respectively.
If you want to iterate through all the prime divisors of n, just iterate through all the set bits of n
If you want to iterate through all the divisors of n, just iterate through all the submasks of n