Probability

Probability of an event denotes the likelihood of that event.

0 <= P(E) <= 1

P(E) = # fav outcomes/ # total outcomes

Expressions:

P(~E) = 1 - P(E)
P(a | b) = P(a) + P(b) - P(a&b)
If a and b are disjoint events, P(a&b) = 0

Conditional Probability:

P(a/b) = Probability of happening of event a given that event b has already occurred

P(a/b) = P(a&b)/P(b)

P(a&b) = P(a/b)P(b) = P(b/a)P(a)

Bayes thm: P(a/b) = P(b/a)P(a)/P(b)

Independent events:
So two events a and b are said to be independent iff
P(a/b) = P(a) or P(b/a) = P(b)

Condition for two events to be independent: P(a&b) = P(a)P(b)

Random Variables:

R V is a value that is generated by some random process.
Let's say we are throwing two dice,

X = sum of outcomes,
X = number of even outcomes,
X = (sum of outcomes)%5

Expected Value:

E(X) indicates the average value of a random value X.

E(X) = summation over v (P(X=v)*v)

For example:
When we throw a die, what is the expected value of the outcome?

E = (1/6)*1 + (1/6)*2 + (1/6)*3 + (1/6)*4 + (1/6)*5 + (1/6)*6 = 7/2

Here RV X = outcome of the throw

Q. Let's say you bought a lottery ticket for $2 and there is a 10% chance that you will win $10 and 2% chance that you will win $20. What is the expected value of the prize? Is it favorable to purchase the ticket?

E(prize) = 0.1*10 + 0.02*20 = 1.4
NO

Linearity of Expected Values:

E(x1 + x2 + x3 + … + xn) = E(x1) + E(x2) + … + E(xn)

It holds true even when the RV are dependent on each other.

E.g If we throw a die, EV(outcome) = 7/2
If we throw two die simultaneously, EV(sum of outcomes) = 7/2 + 7/2 = 7

But in the above case the random variables are independent.

Problem:
We have n boxes and we place n balls in them randomly. Calculate the expected number of empty boxes.

X1 = if box 1 is empty

E(X1) = P(X1)
= (n-1)^n/n^n
= ((n-1)/n)^n

Our final answer is E(X1 + X2 + X3 + … + Xn)
= E(X1) + E(X2) + … + E(Xn)
= n * E(X1)
= n*((n-1)/n)^n

For the above problem, our random variables are dependent.

Q. There is a necklace with n pearls and we want to color them with m colors. Beauty of a necklace is defined as the total number of distinct colors used. Calculate the expected value of the beauty.

X: # distinct color used
E(beauty) = P(X=1)*1 + P(X=2)*2 + … P(X=m)*m

Y: whether this color is used or not
E(beauty) = P1(y=1)*1 + P2(y=1)*1 + … + Pm(y=1)*1 (1)

Probability that ith color is used
Pi(Y=1) = (total ways - no of ways in which i is not used)/total ways
Total ways = m^n
no of ways in which i is not used = (m-1)^n

Pi(y=1) = (m^n - (m-1)^n)/m^n (2)

P1(y=1) = P2(y=1) = … = Pm(y=1) (3)

Using (1), (2) and (3)
E(beauty) = m*P(y=1) = m*(m^n - (m-1)^n)/m^n

Problem - 1543C - Codeforces
Solution: https://codeforces.com/contest/1543/submission/121602447

Homework:
Problem - 1525E - Codeforces

Kick Start - Merge Cards

K U's   k R's
1U
(n-k-1) U   (n-k) R
(k+1 n-k) U (k+ n-k-1) R
(n+1) U's (n-1) R's

(n+1+n-1)C(n+1)
2nC(n+1)=2nC(n-1)

y=x+k
a U's   (a-k)R's   —--> m U's n R's
1 U
(m-a-1) U's (n-(a-k))R's   (remaining)
Mirror image
(n-(a-k))U's (m-a-1) R's
(n-a+k+a+1) U's   (m-a-1+a-k)R's
(n+k+1) U's   (m-1-k)R's

(n+m)C(n+k+1)   (max prefix sum is >k)
(n+m)C(n+(k+1)+1) (max prefix sum is >(k+1))

(n+m)C(n+k+1) - (n+m)C(n+(k+1)+1) (max prefix sum = (k+1))

3, 7, 4, 1

(sum)*2/n   (a^2 +b^2 + c^2) != (a+b+c)^2

10, 4, 1

sum*2/(n-1)

14, 1
sum*2/(n-2)

sum*(2/n + 2/n-1 + 2/n-2 + 2/n-3 + ... + 2/2)

2*sum*(1/n + 1/n-1 +....1/1)

r>=l
dp[l][r]=(expected value for the subarray from l to r)

dp[l][r]=

1, 4, 2, 7, 5

1   4, 2, 7, 5   (dp[l][l]+dp[l+1][r]+sum)/(r-l)
1, 4   2, 7, 5   dp[l][l+1]+dp[l+2][r]+sum/r-l
1, 4, 2 7, 5   dp[l][l+2]+dp[l+3][r]+sum/r-l
1, 4, 2, 7 5   dp[l][l+3]+dp[l+4][r]+sum/r-l

Dp[l][l]+dp[l][l+1]+dp[l][l+2]+dp[l][l+3] =pref[l][r-1]

Dp[r][r]+dp[r-1][r]+dp[r-2][r]+dp[r-3][r] = suf[l+1][r]

[Problem - 1525E - Codeforces]

Discussion:

We have m points to conquer

E(points) = E(p1) + E(p2) + … + E(pm)

E(pi) = P(pi).1

Ans = E(points) = P(p1) + P(p2) + … + P(pm)

P(p1) = 1 - P(not p1)

P (not p1) ??
Say p1 is at a distance i from a city
Then what are the possible choices for the turn in which that city's monument is built?

Let say we build the monument in turn t
n - t < i
n - i < t

If p1 is at a distance 1 from a city, then P (not p1) = 0

```
int cnt[21];
int fact[N];
int n,m;

int perm(int n, int r){
     if(r>n) return 0;
     return (fact[n]*power(fact[n-r], mod-2, mod))%mod;
}

int ways(){
     int val=1;
     int done=0;
     for(int i=0;i<21;i++){
```

```
            val*=perm(i-done, cnt[i]);
            val%=mod;
            // cout<<val<<" ";
            done+=cnt[i];
        }
        val*=fact[n-done];
        val%=mod;
        // cout<<"\n";
        return val;
}

int32_t main()
{
    IOS;
    cin>>n>>m;
    int a[n][m];
    for(int i=0;i<n;i++){
        for(int j=0;j<m;j++){
            cin>>a[i][j];
        }
    }
    fact[0]=1;
    for(int i=1;i<N;i++){
        fact[i] = (fact[i-1]*i)%mod;
    }
    int x = power(fact[n], mod-2, mod);
    int ans=0;
    for(int j=0;j<m;j++){
        fill(cnt, 0);
        for(int i=0;i<n;i++){
            if(a[i][j]-1<n){
                cnt[a[i][j]-1]++;
            }
        }
        int curr = ways();
        curr = (curr*x)%mod;
        ans+=(1-curr+mod)%mod;
        ans%=mod;
```

```
    }
    cout<<ans;
}
```