On the Option Pricing Formula Based on the Bachelier Model

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This version: Sep. 2019

First version: Jul. 2019

Abstract

Under the recent negative interest rate situation, the Bachelier model has been attracting attention and adopted for evaluating the price of interest rate options. In this paper, we will derive an option pricing formula based on the Bachelier model and compare it with the prior researches. We will derive it by eight methods and clarify the property of the Bachelier model. Then we will confirm the validity of the Normal model that is actually used in the valuation of interest rate options under negative interest rate, while comparing it with the Bachelier model for stocks. We start from the natural setting of modeling the undiscounted stock price by the Ornstein=Uhlenbeck process, and derive the Bachelier formula in consideration of discount. On the other hand, since the major prior researches start from modeling the discounted stock price by the Brownian motion, their models of the undiscounted stock price has an unnatural setting that the price of the numeraire asset is included. Furthermore, It has been confirmed that their formulas are not consistent among them. During the derivation process, we have obtained various results concerning the Bachelier model. In particular, in the case of the Bachelier model, it has been confirmed that the utility function of a representative agent is the CARA utility function unlike the Black-Scholes model. The assumption of the exponential type utility function is quite natural setting. In addition, we have derived other expressions of the Bachelier's formula (the formula decomposed into the intrinsic value and the time value and the formula using a characteristic function and modified characteristic function). As for the Normal model used for pricing interest rate options, we have derived an original pricing formula (Modified Normal model) in which the unnatural points of the Normal model of the forward LIBOR and forward swap rate have been partially corrected.

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Keywords: Bachelier model, Ornstein=Uhlenbeck process, Bachelier partial diferential equation, Martingale approach, Intrinsic value and time value, Continuous-time CAPM, Representative agent, CARA utility function, Characteristic function, Interest rate option, Normal model

1 Introduction

Research on the theory of option pricing is well known in the Black-Scholes formula by Black & Scholes [4] and Merton [12] in 1973. Later it has been sophisticated and developed by Harisson & Kreps [7] and Harisson & Pliska [8] using abstract mathematics, with the fundamental theorem of asset pricing. But its history goes back to Bachelier [2] in 1900. Bachelier [2] is a pioneering paper on option pricing. he examine the property of the price of the stock, and found that it follows the normal distribution [1]. Under that condition, he derived the option pricing formula.

The Black-Scholes formula is mainly used to evaluate a price of option which underlying asset is stock or currency. After that, the Black formula which is the pricing formula of the futures option was published by Black [3] in 1976. In addition to bond options, it is used to evaluate interest rate options such as caps, floors, and swaptions that use interest rates as the underlying asset. When using stocks, currencies, and interest rates as underlying assets, it was generally considered that their value was a positive. So it was natural to use the Black-Scholes model or the Black model, which assumes that the underlying asset prices follow the log-normal distribution. However, with regard to interest rates, with the recent introduction of the negative interest rate policy, a situation has arisen in which option pricing by the Black model cannot be made. In order to respond to negative interest rates, the Bachelier model has been refocused, and adopted for the valuation of interest rate options. Since the Bachelier model assumes that the underlying asset prices follow the normal distribution, it can evaluate option prices regardless of whether the underlying asset prices are positive or negative.

This paper investigates the option pricing formula based on the Bachelier model in detail. First, after defining the Bachelier model for stocks, we derive the pricing formula of the option whose underlying asset is stock. We call it the Bachelier formula in this paper. After that we conduct comparative study with the Bachelier formula of prior research. We start from the natural setting of modeling the undiscounted stock price by the Ornstein=Uhlenbeck process, and derive the Bachelier formula in consideration of discount. On the other hand, since the major prior researches start from modeling the discounted stock price by the Brownian motion, their models of the undiscounted stock price has an unnatural setting that the price of the numeraire asset is included. Furthermore, It has been confirmed that their formulas are not consistent among them.

In the derivation of the Bachelier formula, we use various methods that have been developed and sophisticated in mathematical finance or financial engineering since the Black-Scholes model appeared. In addition, while obtaining various alternative expressions of the Bachelier formula, the properties of the Bachelier model are discussed in comparison with the Black-Scholes model. During the derivation process, we have obtained various results concerning the Bachelier model. In particular, in the case of

[1]The Black-Scholes model assumes that the underlying asset price follows the log-normal distribution.

the Bachelier model, it has been confirmed that the utility function of a representative agent is the CARA utility function unlike the Black-Scholes model. The assumption of the exponential type utility function is quite natural setting. In addition, we have derived other expressions of the Bachelier's formula (the formula decomposed into the intrinsic value and the time value and the formula using a characteristic function).

Furthermore, as a practical example of using the Bachelier model, we analyze the Normal model ,the market standard pricing model of interest rate option under a negarive interest rate situation, and confirm its validity. We have derived an original pricing formula (Modified Normal model) in which the unnatural points of the Normal model of the forward LIBOR and forward swap rate have been partially corrected.

On the other hand, as a developmental model of the Black-Scholes model, the stochastic volatility model, such as Heston model or SABR model, which stochastically models not only the underlying asset price but also its volatility are being used. These models assume that the underlying asset price is positive, like the Black-Scholes model. By changing these to models based on the Bachelier model, it is possible to use it to evaluate interest rate options under negative interest rate conditions. It is possible to use the Bachelier formula of this paper as a foothold for analyzing such more advanced models.

The outline of this paper is as follows. In Section 2, after confirming the setting of the Bachelier model and describing the basic concept, we present the Bachelier formula of this paper. In addition, we compare it with the prior research. In Section 3, the Bachelier formula is derived from the eight kinds of methods, referring to Andreasen [1] and Heston [10]. In Section 3.1, the partial differential equation (Bachelier partial differential equation) is derived by the discussion of the hedge by Black & Scholes [4] and Merton [12]. Section 3.2 shows how to derive using the martingale approach. In Section 3.3, we use the technique of local time to derive the Bachelier formula in the form of the option price divided into the intrinsic value and the time value. In Section 3.4, using forward partial differential equations, we evaluate call options as put options in dual economy, and indicate that the Bachelier formula is derived. In Section 3.5, we construct a discrete binomial model and derive the Bachelier formula by considering its limits. In Section 3.6, we show that Bachelier partial differential equations can be derived from continuous-time CAPM, which is a major achievement in financial economics. In Section 3.7, the Bachelier formula is derived by solving the utility maximization problem of a representative agency with an constant absolute risk aversion type utility function. In Section 3.8, we derive the Bachelier formula using the characteristic function and the modified characteristic function. In Section 4, we examine the problem of the pricing formula of the interest rate option considering the negative interest rate used in practice. The pricing formula is corrected by applying the results up to 3. In Section 5, we summarize this paper and describe future research issues.

2 The Bachelier formula and prior research

2.1 Setting of the Bachelier model and the Bachelier formula

We consider an economy in which there is a stock without dividend payments, a zero coupon bond (hereinafter called "bond") and an option whose underlying asset is this stock. In this paper, what models stock price and bond price as respectively (1) and (2), we call the Bachelier model. The price of stock is assumed to follow the stochastic differential equation(SDE)

$$dS_t = \mu S_t dt + \sigma dW_t^{\mathcal{P}}, \tag{1}$$

where μ and σ are constants and $(W_t^{\mathcal{P}})$ is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_t)$. That is, S_t follows Ornstein=Uhlenbeck process. The bond price is modeled by

$$\frac{dP(t,T)}{P(t,T)} = r dt, \ P(T,T) = 1, \tag{2}$$

where $P(t,T)$ represents the value of a bond at t with maturity T and r is a continuous compound interest rate, which is assumed to be constant. At this time, when we choose $P(t,T)$ as the numeraire, the SDE of S_t under the forward measure Q_T is

$$dS_t = r S_t dt + \sigma dW_t^{Q_T}. \tag{3}$$

We consider a European call option on the stock with maturity date T and strike price K . Thus, a payoff at T is

$$\max(S_T - K, 0) \equiv (S_T - K)^+.$$

There are no payments from contract point to maturity other than option premiums and maturity payoffs. In addition, there are no transaction costs and no restrictions on short-selling, and investors can continuously restructure their portfolio.

The following Bachelier formula can be derived from the above Bachelier model based on the same argument of Black & Scholes [4] and Merton [12].

Result 1 (The Bachelier formula) *Let $C_t = C(S_t, t)$ denotes the price of a European call option at time t.*

- $r \neq 0$

$$C_t = \left(S_t - Ke^{-r(T-t)}\right)\Phi(z) + \sigma\sqrt{\frac{1-e^{-2r(T-t)}}{2r}}\phi(z) \tag{4}$$
$$z = \frac{S_t - Ke^{-r(T-t)}}{\sigma\sqrt{\frac{1-e^{-2r(T-t)}}{2r}}}$$

- $r = 0$

$$C_t = (S_t - K)\Phi(z) + \sigma\sqrt{T-t}\,\phi(z) \tag{5}$$
$$z = \frac{S_t - K}{\sigma\sqrt{T-t}}$$

where Φ denotes the cumulative distribution function of the standard normal distribution, and ϕ denotes the probability density function of the standard normal distribution.

Brooks & Brooks [5] presented more general Bashlier formula with interest rate and volatility as functions of time. In this paper, we will keep these parameters constant for comparison with prior researches, presentation of various derivation methods, and analysis of interest rate options.

2.2 Prior research

This section shows the Bachelier formulas presented in some main literatures.

2.2.1 Smith (1976) 's formula

Smith [13] derived the following formula in a form that makes the pricing formula of Bachelier [2] more general [2].

$$\begin{aligned} C_0 &= (S_0 - K)\Phi(z) + \sigma\sqrt{T}\phi(z) \\ z &= \frac{S_0 - K}{\sigma\sqrt{T}} \end{aligned}$$

This is the same result as the case of $r = 0$ of the result 1.

2.2.2 Haug(2007)'s formula

Haug [9] derived the same formula as Smith [13] and the following formula considering discount

$$\begin{aligned} C_0 &= \left(S_0 - Ke^{-rT}\right)\Phi(z) + \sigma\sqrt{T}\phi(z) \\ z &= \frac{S_0 - Ke^{-rT}}{\sigma\sqrt{T}}. \end{aligned}$$

The result does not match the case of $r = 0$ of the result 1, because the SDE of the stock price is different from this paper. Specifically, Haug [9] chooses the deposit B_t as the numeraire and sets the discounted stock price to follow the Brownian motion under the risk neutral measure Q. That is, the SDE of the discounted stock price is

$$d\left(\frac{S_t}{B_t}\right) = \sigma dW_t^Q,$$

and the deposit model is

$$\frac{dB_t}{B_t} = rdt,\ B_0 = 1.$$

[2] The pricing formula derived by Bachelier corresponds to the case of at-the-money ($S_0 = K$) in the Smith (1976)'s formula. That is, $C_0 = \sigma\sqrt{T}\phi(0)$.

Therefore, according to Ito's formula, the SDE of the undiscounted stock price is

$$\begin{aligned} dS_t &= rS_t dt + \sigma B_t dW_t^Q \\ &= rS_t dt + \sigma e^{rt} dW_t^Q. \end{aligned}$$

Compared to (3), the term with the Brownian motion includes the value of deposit that is the numeraire. We considere that it is an unnatural model setting that the price of the numeraire asset is included in the undiscounted stock price model. This problem stems from the fact that the model setting of the discounted stock price is the starting point.

2.2.3 Dawson et al.(2007)'s formula

Dawson et al. [6] derives the following formula,

$$\begin{aligned} C_0 &= e^{-rT}\left\{(F_0 - K)\,\Phi(z) + \sigma\sqrt{T}\phi(z)\right\} \\ z &= \frac{F_0 - K}{\sigma\sqrt{T}}. \end{aligned} \tag{6}$$

F_t is a forward price of the stock, and $F_t = S_t e^{r(T-t)}$. (6) is rewritten using this relation,

$$\begin{aligned} C_0 &= \left(S_0 - Ke^{-rT}\right)\Phi(z) + \sigma\sqrt{T}e^{-rT}\phi(z) \\ z &= \frac{S_0 - Ke^{-rT}}{\sigma\sqrt{T}e^{-rT}}. \end{aligned}$$

Dawson et al. [6] chooses the bond as the numeraire and sets the discounted stock price (forward price) to follow the Brownian motion under the forward measure Q_T. That is, the SDE of the discounted stock price is

$$dF_t = d\left(\frac{S_t}{P(t,T)}\right) = \sigma dW_t^{Q_T}$$

and the bond model is as mentioned above,

$$\frac{dP(t,T)}{P(t,T)} = rdt,\ \ P(T,T) = 1.$$

Therefore, according to Ito's formula, the SDE of the undiscounted stock price is

$$\begin{aligned} dS_t &= rS_t dt + \sigma P(t,T) dW_t^{Q_T} \\ &= rS_t dt + \sigma e^{-r(T-t)} dW_t^{Q_T}. \end{aligned}$$

Compared to (3), the term with the Brownian motion includes the price of bond that is the numeraire. Like Haug [9], it is an unnatural model setting. Again, this problem stems from the fact that the model setting of the discounted stock price is the starting point. However, Dawson(2007)'s formula (6) is generally used in the market (*market standard model*). For example, a formula in which F_t is replaced by forward LIBOR or forward swap rate is being used as a formula for pricing interest rate options under negative interest rate conditions. It is commonly called the Normal model. The pricing formula for interest rate options will be described in detail in Section 4.

2.3 Comparison with prior research

We compare the Bachelier formula of result 1 and those of the prior researches. Smith [13] does not consider the discount and corresponds to the case of $r = 0$ of the result 1. Haug [9] and Dawson et al. [6] both model the discounted stock price with a Brownian motion, and they choose different numeraire asset each other. As a result, their model of the undiscounted stock price include the price of the asset selected as the numeraire and their option pricing formulas are different each other. In this paper, we start from the natural setting of modeling the undiscounted stock price with the Ornstein=Uhlenbeck process, so we can obtain the Bachelier formula of result 1 using either deposit or bond as the numeraire.

In the Black-Scholes model, the undiscounted stock price is modeled by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathcal{P}}.$$

The discounted stock price also follows a geometric Brownian motion

$$d\left(\frac{S_t}{P(t,T)}\right) = (\mu - r)\left(\frac{S_t}{P(t,T)}\right)dt + \sigma\left(\frac{S_t}{P(t,T)}\right)dW_t^{\mathcal{P}} = \sigma\left(\frac{S_t}{P(t,T)}\right)dW_t^{Q_T}.$$

On the other hand, the Bachelier model does not work as well as the Black-Scholes model. If the discounted stock price is modeled by a Brownian motion, the undiscounted stock price does not follow a Brownian motion.

Section 3 describes how to derive the Bachelier formula, which uses the SDE of the undiscounted stock price

$$dS_t = \mu S_t dt + \sigma dW_t^{\mathcal{P}}.$$

At this time, the SDE of the discounted stock price is as follows.

$$d\left(\frac{S_t}{P(t,T)}\right) = (\mu - r)\left(\frac{S_t}{P(t,T)}\right)dt + \sigma\left(\frac{1}{P(t,T)}\right)dW_t^{\mathcal{P}} = \sigma\left(\frac{1}{P(t,T)}\right)dW_t^{Q_T}$$

3 Derivation of the Bachelier formula

Andreasen et al. [1] is based on the Black-Scholes model, and Heston [10] is based on the Heston model. So, we derive the Bachelier formula by changing the base model to the Bachelier model and applying various methods of above literatures.

3.1 Partial differential equation

In this section, we use the method first presented to derive the Black-Scholes formula by Black & Scholes [4] and Merton [12]. Brooks & Brooks [5] present the result of the following partial differential equation. This paper shows the detailed derivation process together with the resulting PDE.

First, we consider the case of $r \neq 0$. Let Y_t denote the price of a European call option with strike price K and maturity T. We assume that it can be written as a twice continuously differentiable function of stock prices S_t and time t. That is,

$$Y_t = C(S_t, t).$$

Applying Ito formula to Y_t, we get

$$dY_t = \left(\mu S_t \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \frac{\partial C}{\partial S}\sigma dW_t^{\mathcal{P}}. \tag{7}$$

Let a_t denote the number of stocks held at time t, and b_t the bond holing amount. we assume that there is a self-financing trading strategy (a_t, b_t) that satisfies

$$a_t S_t + b_t P(t, T) = Y_t, \ \forall t \in [0, T]. \tag{8}$$

By linearity of the stochastic integral and the self-financing condition, we obtain

$$\begin{aligned} dY_t &= a_t dS_t + b_t dP(t, T) \\ &= (a_t \mu S_t + b_t r P(t, T))\, dt + a_t \sigma dW_t^{\mathcal{P}}. \end{aligned} \tag{9}$$

Therefore, two expressions are obtained for dY_t, and the drift and diffusion terms of (7) and (9) must be equal. First, comparing the diffusion terms, we get

$$a_t = \frac{\partial C}{\partial S}(S_t, t).$$

On the other hand, from (8) we get

$$b_t = \frac{1}{P(t, T)}\left(C(S_t, t) - S_t \frac{\partial C}{\partial S}(S_t, t)\right).$$

By comparing the drift terms from these results, we get

$$r S_t \frac{\partial C}{\partial S}(S_t, t) + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2}(S_t, t) + \frac{\partial C}{\partial t}(S_t, t) = rC(S_t, t),$$

and the following partial differential equation is obtained

$$rx \frac{\partial C}{\partial x}(x, t) + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2}(x, t) + \frac{\partial C}{\partial t}(x, t) = rC(x, t). \tag{10}$$

We call it the Bachelier partial differential equation. Given a European call option, C needs to meet the following boundary condition

$$C(x, T) = (x - K)^+. \tag{11}$$

To find the call option price, we need to solve (10)–(11). If we solve it using the Feynman-Kac formula, we can obtain the Bachelier formula (4)

$$C(S_t, t) = \left(S_t - Ke^{-r(T-t)}\right)\Phi\left(\frac{S_t - Ke^{-r(T-t)}}{\sigma\sqrt{\frac{1-e^{-2r(T-t)}}{2r}}}\right) + \sigma\sqrt{\frac{1 - e^{-2r(T-t)}}{2r}}\,\phi\left(\frac{S_t - Ke^{-r(T-t)}}{\sigma\sqrt{\frac{1-e^{-2r(T-t)}}{2r}}}\right).$$

We can also confirm that this solution satisfies the Bachelier partial differential equation.

In the case of $r = 0$, the Bachelier partial differential equation is

$$\frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2}(x, t) + \frac{\partial C}{\partial t}(x, t) = 0.$$

If we solve it combined with (11), we get Bachelier formula (5).

3.2 Martingale approach

In this section, we derive the Bachelier formula by the martingale approach. The martingale approach is a method to calculate the option price by calculating the expected value of the discounted option payoff under the measure under which the discounted stock price becomes martingale (equivalent martingale measure). The pricing formula of the European call option by martingale approach is as follows

$$C_t = e^{-r(T-t)} E_t^{Q_T}[(S_T - K)^+]. \tag{12}$$

Under the forward measure Q_T, the stock price follows the SDE (3). So the model of the undiscounted stock price is as follows.

$$\begin{aligned} dS_t &= rS_t dt + \sigma dW_t^{Q_T} \\ \Leftrightarrow S_T &= S_t e^{r(T-t)} + \sigma e^{r(T-t)} \int_0^{T-t} e^{-rs} dW_s^{Q_T} \end{aligned}$$

That is, under Q_T, S_T follows a normal distribution with the following mean and variance

$$m = E^{Q_T}[S_T] = S_0 e^{rT} \tag{13}$$

$$v^2 = Var^{Q_T}[S_T] = \sigma^2 \frac{e^{2rT} - 1}{2r}. \tag{14}$$

Then we obtain the Bachelier formula (4)

$$\begin{aligned} C_0 &= e^{-rT} \int_{\frac{K-m}{v}}^{\infty} (m + vx - K) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= (S_0 - Ke^{-rT}) \Phi\left(\frac{S_0 - Ke^{-rT}}{\sigma \sqrt{\frac{1-e^{-2rT}}{2r}}} \right) + \sigma \sqrt{\frac{1 - e^{-2rT}}{2r}} \phi \left(\frac{S_0 - Ke^{-rT}}{\sigma \sqrt{\frac{1-e^{-2rT}}{2r}}} \right). \end{aligned}$$

In the case of $r = 0$, the stock price follows the SDE

$$dS_t = \sigma dW_t^{Q_T}.$$

From this, under Q_T, the stock price becomes martingale and S_T follows a normal distribution with the following mean and variance

$$m = E^{Q_T}[S_T] = S_0 \tag{15}$$

$$v^2 = Var^{Q_T}[S_T] = \sigma^2 T \tag{16}$$

Then, we obtain the Bachelier formula (5)

$$\begin{aligned} C_0 &= \int_{\frac{K-S_0}{\sigma\sqrt{T}}}^{\infty} (S_0 + \sigma\sqrt{T}x - K) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= (S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}} \right) + \sigma\sqrt{T} \phi \left(\frac{S_0 - K}{\sigma\sqrt{T}} \right). \end{aligned}$$

3.3 The intrinsic value and the time value

In this section, we use the *local time* to derive another representation of the Bachelier formula. By analyzing a trading strategy known as the *stop-loss start-gain* strategy, a term with local time appears. We analyze the stop-loss start-gain strategy as follows. Let:

$$\begin{aligned} a_t &= \mathbf{1}_{\{S_t > KP(t,T)\}} \\ b_t &= -\mathbf{1}_{\{S_t > KP(t,T)\}} K, \ \forall t \in [0,T], \end{aligned}$$

where $\mathbf{1}_{\{A\}}$ is an indicator function. Next, let Y_t denote the value of the portfolio at time t

$$\begin{aligned} Y_t &= a_t S_t + b_t P(t,T) \\ &= \mathbf{1}_{\{S_t > KP(t,T)\}} S_t - \mathbf{1}_{\{S_t > KP(t,T)\}} KP(t,T) \\ &= (S_t - KP(t,T))^+ \,. \end{aligned} \tag{17}$$

It can be seen that Y_t duplicates the payoff of European call option and if $S_0 < KP(0,T)$, there is no cost in constructing the portfolio initially.

Now we apply the Tanaka-Meyer formula to $Y_t/P(t,T)$. We get

$$(F_t - K)^+ = (F_0 - K)^+ + \int_0^t \mathbf{1}_{\{F_u > K\}} dF_u + \Lambda_t(K), \ \forall t \in [0,T], \tag{18}$$

where $F_t = S_t/P(t,T)$ and $\Lambda_t(K)$ is the local time at K by time t. The term of local time in (18) can be interpreted as the external financing needed to trade by the stop-loss start-gain strategy. $\Lambda_t(K)$ is positive with a positive probability for any t, which indicates that the stop-loss start-gain strategy is not self-financing. The expected value under the forward measure Q_T of (18) is as follows

$$E_0^{Q_T}[(F_t - K)^+] = (F_0 - K)^+ + E_0^{Q_T}[\Lambda_t(K)], \ \forall t \in [0,T]. \tag{19}$$

Since F is Q_T-martingale, the integral term related to dF_u is also Q_T-martingale. So the expected value of the integral term is zero.

Now we evaluate (19) for $t = T$ and obtain the Bachelier formula. From (12),

$$\frac{C_0}{P(0,T)} = E_0^{Q_T}[(S_T - K)^+] = E_0^{Q_T}[(F_T - K)^+].$$

Substituting this to (19) gives

$$C_0 = \left(S_0 - Ke^{-rT}\right)^+ + e^{-rT} E_0^{Q_T}[\Lambda_T(K)]. \tag{20}$$

The first term on the right side is the intrinsic value of the option, and the second term is the time value of the option. Since F is a Q_T-Martingale, by Girsanov's theorem we get

$$dF_t = \frac{\sigma}{P(t,T)} dW_t^{Q_T}.$$

That is,

$$F_t = F_0 + \sigma e^{rT} \int_0^t e^{-rs} dW_s^{Q_T},$$

and the following transition density function for F is obtained

$$\psi(F_t, t; F_0, 0) = \frac{1}{\sigma e^{rT}\sqrt{\frac{1-e^{-2rt}}{2r}}}\phi\left(\frac{F_t - F_0}{\sigma e^{rT}\sqrt{\frac{1-e^{-2rt}}{2r}}}\right), \tag{21}$$

where $\phi(z) \equiv \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ is a probability density function of the standard normal distribution. When the calculation is performed using the theorem about local time, $E_0^{Q_T}[\Lambda_T(K)]$ becomes as follows

$$E_0^{Q_T}[\Lambda_T(K)] = \frac{\sigma^2}{2}\int_0^T e^{2r(T-t)}\frac{1}{\sigma e^{rT}\sqrt{\frac{1-e^{-2rt}}{2r}}}\phi\left(\frac{K - F_0}{\sigma e^{rT}\sqrt{\frac{1-e^{-2rt}}{2r}}}\right)dt. \tag{22}$$

Substituting (22) into (20), we get

$$C_0 = \left(S_0 - Ke^{-rT}\right)^+ + \frac{\sigma}{2}\int_0^T e^{-2rt}\frac{1}{\sqrt{\frac{1-e^{-2rt}}{2r}}}\phi\left(\frac{S_0 - Ke^{-rT}}{\sigma\sqrt{\frac{1-e^{-2rt}}{2r}}}\right)dt$$

Converting the variable t with $\nu \equiv \sigma\sqrt{\frac{1-e^{-2rt}}{2r}}\Big/\sqrt{\frac{1-e^{-2rT}}{2r}}$ gives another representation of the Bachelier formula [3].

$$C_0 = \left(S_0 - Ke^{-rT}\right)^+ + \sqrt{\frac{1-e^{-2rT}}{2r}}\int_0^\sigma \phi\left(\frac{S_0 - Ke^{-rT}}{\nu\sqrt{\frac{1-e^{-2rt}}{2r}}}\right)d\nu \tag{23}$$

(23) is equivalent to (4). It can be confirmed by the fact that the both result of the differentiation with respect to σ is

$$\sqrt{\frac{1-e^{-2rT}}{2r}}\phi\left(\frac{S_0 - Ke^{-rT}}{\sigma\sqrt{\frac{1-e^{-2rT}}{2r}}}\right), \tag{24}$$

and the boundary condition is $\left(S_0 - Ke^{-rT}\right)^+$ when $\sigma = 0$.

In the case of $r = 0$, the following formula is derived based on the same argument, and it can be confirmed that it is equivalent to (5).

$$C_0 = (S_0 - K)^+ + \sqrt{T}\int_0^\sigma \phi\left(\frac{S_0 - K}{\nu\sqrt{T}}\right)d\nu \tag{25}$$

[3] In the case of the Black-Scholes model, the Black-Scholes formula is as follows

$$C_0 = \left(S_0 - Ke^{-rT}\right)^+ + Ke^{-rT}\sqrt{T}\int_0^\sigma \phi\left(\frac{\ln\left(\frac{S_0}{Ke^{-rT}}\right) - \frac{1}{2}\nu^2 T}{\nu\sqrt{T}}\right)d\nu.$$

3.4 Forward partial differential equations and dual economy

In this section, we derive forward partial differential equations for the price of European call option. In this equation, by contrast of the backward partial differential equation obtained in Section 3.1, the strike price K and the maturity T are variables, and the current stock price S_0 and the current time are fixed. And the option pricing problem can be solved in a dual economy where the option is a put option, the underlying asset price is K, the strike price is equal to the spot price S_0, and the time goes to the opposite direction.

We start with the following pricing formula

$$\begin{aligned} C_0 &= e^{-rT} E^{Q_T}[(S_T - K)^+] \\ &= e^{-rT} \int_K^{\infty} (x - K)\psi(x, T) dx, \end{aligned} \tag{26}$$

where $\psi(x, T)$ is the Q_T-density function of S_T in the point x given S_0 at time 0. Due to the Markov property of the spot price, ψ satisfies the forward Fokker-Planck equation

$$0 = -\frac{\partial \psi}{\partial T} - \frac{\partial}{\partial x}[rx\psi] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2\psi],$$

subject to the initial boundary condition $\psi(x, 0) = \delta(x - S_0)$, where $\delta(\cdot)$ is the delta function. Then we obtain a forward PDE for the option price from this. Let $C(K, T)$ denote the initial price of the European call option with the strike price K and maturity T, then the following forward PDE for the price of the European call option is obtained

$$0 = -\frac{\partial C}{\partial T} - rK\frac{\partial C}{\partial K} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial K^2}, \tag{27}$$

and initial boundary condition is $C(K, 0) = (S_0 - K)^+$.

Now we evaluate

$$E^{\mathcal{R}}\left[(S_0 - K_0)^+ | K_T = K\right] \tag{28}$$

for the stochastic process

$$dK_t = (-r)K_t d(-t) + \sigma dW_t^{\mathcal{R}},$$

assuming that the time axis is reversed, S_0 is a fixed amount, and T is a initial time point, where $W^{\mathcal{R}}$ is the backward running Brownian motion under some probability measure $\mathcal{R}$. The forward equation (27) is the backward equation of this problem. Under $\mathcal{R}$, you get

$$K_0 = Ke^{-rT} + \sigma \int_T^0 e^{-rs} dW_s^{\mathcal{R}}.$$

Using this in the equation (28), the Bachelier formula is obtained by essentially the same calculations as those in Section 3.2. In the case of $r = 0$, the Bachelier formula can be obtained by the same argument.

3.5 Binomial model

In this section, Bachelier formula is derived by setting a binomial model as a discrete time model and converging it into a continuous-time model. We consider the case of $r = 0$.

First, consider the one period binomial model as follows. At current time, the stock has a price of S_0, and in the next period, it may go up to $S_0 + u$ or down to $S_0 + d$. This is assumed to occur with probabilities p and $1 - p$ respectively. In this economy, besides stocks, there is a risk-free zero coupon bond with an interest rate of 0 maturing in the next period, and a European call option of the stock with a strike price K maturing in the next period. Note that $u > 0 > d$ to avoid dominance.

We consider hedging the option by trading stocks in a units and bonds in b units. An exact replication of the option's payoff is achieved by

$$\begin{aligned} a &= \frac{C_u - C_d}{u - d} \\ b &= \frac{(S_0 + u)C_d - (S_0 + d)C_u}{u - d}. \end{aligned}$$

To prevent arbitrage opportunities, the value of the hedged portfolio must be equal to the present value of the call option. If this is described and arranged, we get

$$C_0 = qC_u + (1 - q)C_d, \tag{29}$$

where $q = -d/(u - d)$. Using the measure Q_T introduced by q, we can rewrite (29) as follows,

$$\frac{C_0}{P(0,T)} = E_0^{Q_T}\left[\frac{C_T}{P(T,T)}\right],\ P(0,T) = P(T,T) = 1.$$

Using the bond as numeraire, the price of the option discounted by bond is Q_T-martingale.

Now we devide the time-to-maturity T into n intervals and extend the one-period model to the n-period binomial model with n independent additive binomial movements. The call option price can be written as

$$C_0 = E_0^{Q_T}\left[\left(S_T^{(n)} - K\right)^+\right] = E_0^{Q_T}\left[S_T^{(n)}\mathbf{1}_{\{S_T^{(n)} > K\}}\right] - KQ_T\left(S_T^{(n)} > K\right), \tag{30}$$

where

$$\begin{aligned} S_T^{(n)} &= S_0 + ju + (n - j)d \\ j &\overset{Q_T}{\sim} bin(n, q), \end{aligned}$$

and *bin* represents the binomial distribution.

In the following, we will show that the call option price of the n period binomial model converges to the Bachelier formula by bringing n closer to infinity. First, the pricing formula in the continuous-time model is transformed as follows,

$$C_0 = E_0^{Q_T}\left[(S_T - K)^+\right] = E_0^{Q_T}\left[S_T\mathbf{1}_{\{S_T > K\}}\right] - KQ_T\left(S_T > K\right). \tag{31}$$

From (30), (31) and the distribution results (15)–(16), we choose the parameters of the binomial model so that

$$S_T^{(n)} \xrightarrow{Q_T} N\left(S_0, \sigma^2 T\right). \tag{32}$$

The important parameters are the sizes of u_n and d_n and we choose these as follows

$$u_n = \frac{\sigma}{\sqrt{n}}\sqrt{T} \tag{33}$$

$$d_n = -\frac{\sigma}{\sqrt{n}}\sqrt{T}. \tag{34}$$

Let M_n and V_n denote the mean and variance of $S_T^{(n)}$ respectively, then we have

$$\begin{aligned} M_n^{Q_T} &= S_0 + n(q_n u_n + (1-q_n)d_n) \\ V_n^{Q_T} &= nq_n(1-q_n)(u_n - d_n)^2 \end{aligned}$$

Noting that $q_n = -d_n/(u_n - d_n) = \frac{1}{2}$, we get

$$\begin{aligned} M_n^{Q_T} &\xrightarrow[n\to\infty]{} S_0 \\ V_n^{Q_T} &\xrightarrow[n\to\infty]{} \sigma^2 T, \end{aligned}$$

so the first and second moments converge. Therefore, the validity of (32) can be confirmed by the central limit theorem.

Finally, we derive the Bachelier formula (5). Assuming that Z is a random variable following the standard normal distribution, we get

$$\begin{aligned} C_0 &= E_0^{Q_T}\left[S_T^{(n)}\mathbf{1}_{\{S_T^{(n)}>K\}}\right] - KQ_T\left(S_T^{(n)} > K\right) \\ &= E_0^{Q_T}\left[\left(\frac{S_T^{(n)} - S_0}{\sigma\sqrt{T}}\sigma\sqrt{T} + S_0\right)\mathbf{1}_{\left\{\frac{S_T^{(n)}-S_0}{\sigma\sqrt{T}} > \frac{K-S_0}{\sigma\sqrt{T}}\right\}}\right] - KQ_T\left(\frac{S_T^{(n)} - S_0}{\sigma\sqrt{T}} > \frac{K - S_0}{\sigma\sqrt{T}}\right) \\ &\xrightarrow[n\to\infty]{} E_0^{Q_T}\left[\left(Z\sigma\sqrt{T} + S_0\right)\mathbf{1}_{\left\{Z > \frac{K-S_0}{\sigma\sqrt{T}}\right\}}\right] - KQ_T\left(Z > \frac{K-S_0}{\sigma\sqrt{T}}\right) \\ &= (S_0 - K)\,Q_T\left(Z > \frac{K-S_0}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}E_0^{Q_T}\left[Z\mathbf{1}_{\left\{Z > \frac{K-S_0}{\sigma\sqrt{T}}\right\}}\right] \\ &= (S_0 - K)\,\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right). \end{aligned}$$

3.6 Continuous-time CAPM

In this section, we obtain the Bachelier partial differential equation in the continuous-time capital asset price model (continuous-time CAPM).

Assume that the market contains N risk assets (no dividend paying) and they follow the N-dimensional SDE

$$d\boldsymbol{S}_t = \boldsymbol{I}_{S_t}\boldsymbol{\mu}\,dt + \boldsymbol{\Sigma}\,d\boldsymbol{W}_t^{\mathcal{P}},$$

where $\mathbf{I}_{S_t}$ is a diagonal matrix with diagonal elements $(S_1, \ldots, S_N)$, $\boldsymbol{\mu}$ is a constant vector of N dimensions, $\boldsymbol{\Sigma}$ is a constant $N \times N$ matrix, $\mathbf{W}_t^{\mathcal{P}}$ is an N-dimensional Brownian motion under $\mathcal{P}$. For simplicity, we assume that $\boldsymbol{\Sigma}$ has full rank. Further, assume that there is a risk free asset that pays a certain continuously compound interest rate. We consider an investor that maximizes expected additive utility on $[0, \tau]$,

$$E^{\mathcal{P}}\left[\int_0^\tau u(x_t, t)dt\right],$$

subject to the self-financing constraint

$$\begin{aligned} dV_t &= \boldsymbol{a}_t' d\boldsymbol{S}_t + (V_t - \boldsymbol{a}_t'\boldsymbol{S}_t) r dt - x_t dt \\ &= \left(\boldsymbol{a}_t' \boldsymbol{I}_{S_t}(\boldsymbol{\mu} - r\mathbf{1}) + rV_t - x_t\right) dt + \boldsymbol{a}_t'\boldsymbol{\Sigma} d\boldsymbol{W}_t^{\mathcal{P}} \end{aligned}$$

where V_t is the wealth process, x_t is the consumption process, $\boldsymbol{a}_t$ is the holding amount vector of the risk assets, and $'$ denotes transposition of the vector or matrix. If we define the indirect utility function as follows

$$J(V_t, t) = \max_{(\boldsymbol{a}_s, x_s)_{s \geq t}} E_t^{\mathcal{P}}\left[\int_t^\tau u(x_s, s)ds\right],$$

we get the following Hamilton-Jacobi-Bellman (HJB) equation.

$$0 = \max_{(\boldsymbol{a}, x)}\left[u + \frac{\partial J}{\partial t} + (\boldsymbol{a}'\boldsymbol{I}_S(\boldsymbol{\mu} - r\mathbf{1}) + rV - x)\frac{\partial J}{\partial V} + \frac{1}{2}\boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{a}\frac{\partial^2 J}{\partial V^2}\right]$$

From the first order condition, the optimal solution is

$$\boldsymbol{a} = -\frac{\partial J/\partial V}{\partial^2 J/\partial V^2}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}\boldsymbol{I}_S(\boldsymbol{\mu} - r\mathbf{1}).$$

The optimal holding amount of the risk assets is that the vector $(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}\boldsymbol{I}_S(\boldsymbol{\mu} - r\mathbf{1})$ multiplied by the inverse of the Arrow-Pratt coefficient of absolute risk aversion with respect to the indirect utility function $J(V_t, t)$. Since the ratio a_i/a_j for all i and j is independent of wealth and utility, all investors hold the same portfolio of risk assets if they have additive separable utilitiy. This means that the market portfolio of risk assets is given by

$$\boldsymbol{a}_M = k(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}\boldsymbol{I}_S(\boldsymbol{\mu} - r\mathbf{1})$$

for some one-dimensional process k. Therefore, the expected instantaneous excess return of the market portfolio is

$$\mu_M^r = k(\boldsymbol{\mu} - r\mathbf{1})'\boldsymbol{I}_S(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}\boldsymbol{I}_S(\boldsymbol{\mu} - r\mathbf{1}),$$

and the local variance of the market returns is

$$v_M^2 = k^2(\boldsymbol{\mu} - r\mathbf{1})'\boldsymbol{I}_S(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}\boldsymbol{I}_S(\boldsymbol{\mu} - r\mathbf{1}).$$

Also, the local covariance vector of the instantaneous return between the market portfolio and the assets is

$$\boldsymbol{c} = \boldsymbol{I}_S(\boldsymbol{\mu} - r\mathbf{1}).$$

Combining these equations results in

$$\mu_i S_i = rS_i + \frac{c_i}{v_M^2}\mu_M^r. \tag{35}$$

This is the formula of CAPM in the Bachelier model.

Next, we assume that the option contract on S_i is introduced into the market so that the net supply is zero. Since the market is dynamically complete, the market equilibrium does not change and (35) is still valid. If the option price is a function of the price of the underlying stock and time, by Ito's formula we can write the local covariance of the market return and the option return as

$$\frac{\partial C}{\partial S}c_i.$$

Thus, the expected instantaneous return on option contracts is

$$rC + \frac{\partial C}{\partial S}\frac{c_i}{v_M^2}\mu_M^r. \tag{36}$$

On the other hand, applying Ito formula to the option price $C(S_i(t), t)$, the instantaneous return of the option contract is

$$\frac{\partial C}{\partial t} + \mu_i S_i \frac{\partial C}{\partial S_i} + \frac{1}{2}\|\Sigma_i\|^2 \frac{\partial^2 C}{\partial S_i^2}. \tag{37}$$

Equating (36) to (37), and substituting (35),we obtain the Bachelier partial differential equation

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial S^2} = rC,$$

where we omit the subscript on the stock and introduce the notation $\sigma^2 = \|\Sigma_i\|^2$. We have derived the Bachelier partial differential equation in the context of continuous-time CAPM. Thus, the method of Section 3.1 can be used to calculate the Bachelier formula. In the case of $r = 0$, the Bachelier partial differential equation can be derived by the same argument.

3.7 Utility function of a representative agent

In this section, we show that the Bachelier formula can be derived in a model in which the assumption of continuous trading is replaced with the assumption of a representative agent with exponential utility. First, consider a one-period model that can trade at time 0 and T. Assume that a representative agent maximizes the expected utility of consumption at maturity T

$$E^{\mathcal{P}}[u(x_T)]$$

subject to the budget constraint

$$\begin{aligned} x_T &= \boldsymbol{a}'\boldsymbol{S}_T \\ V_0 &= \boldsymbol{a}'\boldsymbol{S}_0, \end{aligned}$$

where u is the utility function of a representative agent, $\boldsymbol{a}$ is the portfolio holding vector, $\boldsymbol{S}$ is the asset price vector, V_0 is the initial wealth, and $'$ denotes transposition of the vector. In order to solve the maximization problem, we form the Lagrange function

$$L = E^{\mathcal{P}}[u(x_T)] - \lambda(\boldsymbol{a}'\boldsymbol{S}_0 - V_0),$$

and the first order condition is

$$\boldsymbol{S}_0 = \lambda^{-1} E^{\mathcal{P}}[u'(x_T)\boldsymbol{S}_T], \tag{38}$$

where $'$ denotes the first derivative and λ is the Lagrange multiplier. Especially for the risk free asset, we get

$$e^{-rT} = \lambda^{-1} E^{\mathcal{P}}[u'(x_T)]. \tag{39}$$

Combining (38) and (39) yields the asset pricing formula

$$\boldsymbol{S}_0 = e^{-rT} E^{\mathcal{P}} \left[\frac{u'(x_T)}{E^{\mathcal{P}}[u'(x_T)]} \boldsymbol{S}_T \right]. \tag{40}$$

Next, we assume that in the market there is a representative agent with constant absolute risk aversion utility function (CARA utility function)

$$u(x) = \frac{1}{\alpha} e^{\alpha x} \tag{41}$$

where $\alpha < 0$, i.e. $-\alpha$ is a constant absolute risk aversion of the representative agent [4]. Now let S be the price of one paticular stock, and V_0 be the initial total wealth. Assuming that the stock price at time T and the total consumption at time T follow the simultaneous normal distribution, we model as follows,

$$S_T = S_0 e^{\mu T} + \sigma e^{\mu T} \int_0^T e^{-\mu s} dW_s^{\mathcal{P}} \tag{42}$$

$$x_T = V_0 e^{\mu_x T} + \sigma_x e^{\mu_x T} \int_0^T e^{-\mu_x s} dW_{x,s}^{\mathcal{P}}, \tag{43}$$

where constant correlation of $\int_0^T e^{-\mu s} dW_s^{\mathcal{P}}$ and $\int_0^T e^{-\mu_x s} dW_{x,s}^{\mathcal{P}}$ is ρ. From (41) and (43), we get

$$\frac{u'(x_T)}{E^{\mathcal{P}}[u'(x_T)]} = e^{-\frac{1}{2}\alpha^2\sigma_x^2 \frac{e^{2\mu_x T}-1}{2\mu_x} + \alpha\sigma_x e^{\mu_x T} \int_0^T e^{-\mu_x s} dW_{x,s}^{\mathcal{P}}}. \tag{44}$$

In order for the market to be in equilibrium, the asset pricing formula (40) needs to hold for the stock. Substituting (42) and (44) into (40), we obtain

$$S_0 = S_0 e^{(\mu-r)T} + e^{(\mu-r)T} \alpha\rho\sigma\sigma_x \sqrt{\frac{e^{2\mu T}-1}{2\mu}} \sqrt{\frac{e^{2\mu_x T}-1}{2\mu_x}}.$$

[4] In the Black-Scholes model, the existence of a representative agent with constant relative risk aversion utility function (CRRA utility function) is assumed.

Therefore,

$$\mu = r \; and \; \rho = 0. \tag{45}$$

Finally, we evaluate the European call option price on S_T with a strike K and a maturity T. By using (40), (42) and (44) and introducing the simultaneous probability density function of $\int_0^T e^{-\mu s} dW_s^{\mathcal{P}}$ and $\int_0^T e^{-\mu_x s} dW_{x,s}^{\mathcal{P}}$, the expected value of the option price can be calculated as follows,

$$\begin{aligned}
C_0 &= e^{-rT} E^{\mathcal{P}} \left[\frac{u'(x_T)}{E^{\mathcal{P}}[u'(x_T)]} (S_T - K)^+ \right] \\
&= \left(S_0 e^{(\mu - r)T} - K e^{-rT} \right) \Phi(d) + \sigma e^{(\mu - r)T} \sqrt{\frac{e^{2\mu T} - 1}{2\mu}} \phi(d) \\
&\quad + e^{(\mu - r)T} \alpha \rho \sigma \sigma_x \sqrt{\frac{e^{2\mu T} - 1}{2\mu}} \sqrt{\frac{e^{2\mu_x T} - 1}{2\mu_x}} \phi(d),
\end{aligned}$$

where

$$d = \frac{S_0 e^{\mu T} - K + \alpha \rho \sigma \sigma_x \sqrt{\frac{e^{2\mu T} - 1}{2\mu}} \sqrt{\frac{e^{2\mu_x T} - 1}{2\mu_x}}}{\sigma \sqrt{\frac{e^{2\mu T} - 1}{2\mu}}}.$$

By substituting the condition (45), the Bachelier formula can be obtained. In the case of $r = 0$, the Bachelier formula can be derived by the same argument.

The Black-Scholes model assumes that a representative agent in the economy has CRRA utility function. On the other hand, in the Bachelier model, it is assumed that a representative agent has CARA utility function. It can be seen that the assumptions of representative agent differ between the Black-Scholes model and the Bachelier model.

3.8 The Bachelier formula using characteristic function

In this section, we rewrite the Bachelier formula into an expression using a characteristic function. An expression using a characteristic function necessary, for example, when deriving an analytical solution of the Heston model in Heston [10]. In the following, based on the method derived by Takada [14], we show the Bachelier formula using characteristic functions.

Consider a forward and a European call option on the same underlying stock. Let C_T be the value at time T of the European call option with strike K and maturity T, and let F be the current price of the forward. Let $X = S_T - F$ and let $\varphi_X(u)$ be a characteristic function under forward measure Q_T $(u \in \mathbb{R})$,

$$\varphi_X(u) = E^{Q_T} \left[e^{iuX} \right] = \int_{-\infty}^{+\infty} e^{iux} f_X(x) dx, \tag{46}$$

where $f_X(x)$ is a probability density function of X. In addition, we define the following function that extends this characteristic function (modified characteristic function).

$$\tilde{\varphi}_X(u) = E^{Q_T} \left[e^{-\alpha X} e^{iuX} \right] = \int_{-\infty}^{+\infty} e^{-\alpha x} e^{iux} f_X(x) dx, \tag{47}$$

where $\alpha \in \mathbb{R}$ and $\alpha > 0$.

Let $k = K - F$, the option pricing formula is as follows,

$$\begin{aligned} C_T &= E^{Q_T}\left[(S_T - K)^+\right] = E^{Q_T}\left[(X-k)^+\right] \\ &= E^{Q_T}[X] - k + E^{Q_T}\left[(k-X)^+\right] \\ &= -i\varphi_X'(0) - k + \int_{-\infty}^{+\infty}(k-x)\mathbf{1}_{\{k-x>0\}} f_X(x)dx. \end{aligned}$$

Then we introduce $\alpha \in \mathbb{R}(\alpha > 0)$ and apply the convolution theorem for Fourier transforms, we get

$$\begin{aligned} C_T &= -i\varphi_X'(0) - k + e^{\alpha k}\int_{-\infty}^{+\infty} e^{-\alpha(k-x)}(k-x)\mathbf{1}_{\{k-x>0\}}e^{-\alpha x} f_X(x)dx \\ &= -i\varphi_X'(0) - k + e^{\alpha k}\mathcal{F}^{-1}\left[\mathcal{F}\left[\int_{-\infty}^{+\infty} e^{-\alpha(k-x)}(k-x)\mathbf{1}_{\{k-x>0\}}e^{-\alpha x} f_X(x)dx\right]\right] \\ &= -i\varphi_X'(0) - k + e^{\alpha k}\mathcal{F}^{-1}\left[\mathcal{F}\left[\int_{-\infty}^{+\infty} e^{-\alpha x} x\mathbf{1}_{\{x>0\}}dx\right]\mathcal{F}\left[\int_{-\infty}^{+\infty} e^{-\alpha x} f_X(x)dx\right]\right] \\ &= -i\varphi_X'(0) - k + e^{\alpha k}\mathcal{F}^{-1}\left[\frac{\tilde{\varphi}_X(u)}{(-\alpha+iu)^2}\right], \end{aligned}$$

where $\mathcal{F}$ indicates the Fourier transform, and $\mathcal{F}^{-1}$ indicates the Fourier inverse transform. The final solution is as follows [5],

$$C_T = F - K - i\varphi_{S_T-F}'(0) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{\varphi}_{S_T-F}(u)e^{-(-\alpha+iu)(K-F)}}{(-\alpha+iu)^2}du \tag{48}$$

$$= F - K - i\varphi_{S_T-F}'(0) + \frac{1}{\pi}\int_{-\infty}^{+\infty}\mathrm{Re}\left[\frac{\tilde{\varphi}_{S_T-F}(u)e^{-(-\alpha+iu)(K-F)}}{(-\alpha+iu)^2}\right]du \tag{49}$$

4 Interest rate options and the Bachelier model

In this section, we will consider the Bachelier model that is used to price interest rate options in practice. Typical examples of interest rate options are cap (caplet), floor (floorlet) and swaption. Since interest rate such as LIBOR and swap rate have been assumed to have positive values, the Black model have traditionally been adopted for the valuation of these interest rate options. However, under the recent negative interest rate conditions, it has become impossible to evaluate the price of interest rate options using the Black model. As a model for dealing with this negative interest rate, the

[5] In the Black-Scholes model, the Black-Scholes formula can be rewritten as follows using the characteristic function,

$$\begin{aligned} C_T &= F\left(\frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{iu}\varphi_{\ln(S_T/F)}(u-i)e^{-iu\ln(K/F)}du\right) - K\left(\frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{iu}\varphi_{\ln(S_T/F)}(u)e^{-iu\ln(K/F)}du\right) \\ &= F\left(\frac{1}{2} + \frac{1}{\pi}\int_{0}^{+\infty}\mathrm{Re}\left[\frac{1}{iu}\varphi_{\ln(S_T/F)}(u-i)e^{-iu\ln(K/F)}du\right]\right) - K\left(\frac{1}{2} + \frac{1}{\pi}\int_{0}^{+\infty}\mathrm{Re}\left[\frac{1}{iu}\varphi_{\ln(S_T/F)}(u)e^{-iu\ln(K/F)}du\right]\right). \end{aligned}$$

Bachelier model has attracted attention and is being used. In this section, we will call the Bachelier model used in practice *Normal model* in accordance with the common name.

4.1 The Normal model

In this section, theoretical analysis is performed on the Normal model for forward LIBOR, using a caplet as an example. If we let T denote the start time of interest calculation and $T+\Delta$ denote the end time, then forward LIBOR($L(t,T,T+\Delta)$) at time t is defined as follows.

Definition 1 (Forward LIBOR)

$$
\begin{aligned}
L(t,T,T+\Delta) &:= \frac{1}{\Delta}\left(\frac{P(t,T)-P(t,T+\Delta)}{P(t,T+\Delta)}\right) \\
P(t,T) &: Price\ of\ a\ bond\ at\ time\ t\ with\ maturity\ T \\
\Delta &: Interest\ calculation\ period\ of\ \text{LIBOR}
\end{aligned}
$$

From this definition, forward LIBOR becomes a martingale under the forward measure $Q_{T+\Delta}$, when taking a bond with maturity $T+\Delta(T+\Delta\text{bond})$ as numeraire. Therefore, in the Black model, from the assumption that interest rates generally do not become negative, forward LIBOR is modeled as follows,

$$
\begin{aligned}
dL(t,T,T+\Delta) &= \sigma_B L(t,T,T+\Delta)dW_t^{Q_{T+\Delta}} \\
\Leftrightarrow L(t,T,T+\Delta) &= L(0,T,T+\Delta)e^{-\frac{1}{2}\sigma_B^2 t+\sigma W_t^{Q_{T+\Delta}}},
\end{aligned} \tag{50}
$$

where σ_B represents Black volatility. On the other hand, in the Normal model, forward LIBOR is modeled as follows in order to allow negative interest rates,

$$
\begin{aligned}
dL(t,T,T+\Delta) &= \sigma_N dW_t^{Q_{T+\Delta}} \\
\Leftrightarrow L(t,T,T+\Delta) &= L(0,T,T+\Delta)+\sigma_N W_t^{Q_{T+\Delta}},
\end{aligned} \tag{51}
$$

where σ_N represents Normal volatility. Now we consider a caplet with option maturity T, interest delivery date $T+\Delta$, strike rate K, and notional amount 1. The pricing formula of this caplet based on the Normal model is as follows

$$
\begin{aligned}
Caplet_N(0,T,T+\Delta) &= P(0,T+\Delta)E^{Q_{T+\Delta}}\left[\frac{\Delta(L(T,T,T+\Delta)-K)^+}{P(T+\Delta,T+\Delta)}\right] \\
&= \Delta P(0,T+\Delta)\left\{(L(0,T,T+\Delta)-K)\Phi(z)+\sigma_N\sqrt{T}\phi(z)\right\} \\
z &= \frac{L(0,T,T+\Delta)-K}{\sigma_N\sqrt{T}}.
\end{aligned} \tag{52}
$$

This is equivalent to the formula which replaces F_0 with $L(0,T,T+\Delta)$ in the Dawson et al's formula (6) and is multiplied by the interest calculation period Δ. This is a market standard option pricing formula of interest rate options.

In modellng the forward LIBOR with the Normal model, it is modeled by the Brownian motion based on the fact that the forward LIBOR is martingale under $Q_{T+\Delta}$ from its definition. That is, the definition of the forward LIBOR is used only to derive the property that it is $Q_{T+\Delta}$-martingale. And the model of bond prices, which are components of the forward LIBOR, is out of consideration. Now if the bond portfolio value $(P(t,T) - P(t,T+\Delta))$ and the $(T+\Delta)$-bond price in the definition of the forward LIBOR are modeled respectively as

$$\begin{aligned} d(P(t,T) - P(t,T+\Delta)) &= r(P(t,T) - P(t,T+\Delta))dt + \sigma_N \Delta P(t,T+\Delta) dW_t^{Q_{T+\Delta}} \quad (53) \\ dP(t,T+\Delta) &= rP(t,T+\Delta)dt, \end{aligned}$$

then (51) can be obtained by Ito formula. It is similar to the model of S_t and $P(t,T)$ in Dawson et al. [6]. In this modeling, the following two problems can be considered.

Problem 1 *The bond portfolio $P(t,T) - P(t,T+\Delta)$ is modeled by stochastic differential equation and $(T+\Delta)$-bond $P(t,T+\Delta)$ is modeled by ordinary differential equation, although they are both the bonds.*

This is a problem that applies to both the Black model and the Normal model. In this paper, we accepts this problem by simply considering the bond portfolio and the $(T+\Delta)$-bond as independent assets.

Problem 2 *The diffusion term of the stochastic differential equation of the bond portfolio $(P(t,T) - P(t,T+\Delta))$ includes the numeraire asset $P(t,T+\Delta)$ and Δ.*

In the Black model, (53) is rewritten by the geometric Brownian motion as

$$\begin{aligned} d(P(t,T) - P(t,T+\Delta)) &= r(P(t,T) - P(t,T+\Delta))dt + \sigma_B (P(t,T) - P(t,T+\Delta)) dW_t^{Q_{T+\Delta}} \\ dP(t,T+\Delta) &= rP(t,T+\Delta)dt. \end{aligned}$$

Therefore by Ito formula (50) can be derived. But in the Normal model, (53) should not be tolerated. At this time, the SDE of the bond portfolio (53) should be modified using the SDE of the stock price (3) in Section 2.1.

4.2 The Modified Normal model

In this section, we modify the SDE of the bond portfolio based on the Bachelier model in this paper. This modified model will be called the Modified Normal model. The modified SDE of the bond portfolio and the $(T+\Delta)$-bond are as follows,

$$\begin{aligned} d(P(t,T) - P(t,T+\Delta)) &= r(P(t,T) - P(t,T+\Delta))dt + \sigma_{MN} dW_t^{Q_{T+\Delta}} \\ dP(t,T+\Delta) &= rP(t,T+\Delta)dt, \end{aligned}$$

where σ_{MN} is the volatility of the Modified Normal model. Then, by Ito formula, the SDE of the forward LIBOR is as follows,

$$\begin{aligned} dL(t,T,T+\Delta) &= \frac{\sigma_{MN}}{\Delta P(t,T+\Delta)} dW_t^{Q_{T+\Delta}} \\ \Leftrightarrow L(t,T,T+\Delta) &= L(0,T,T+\Delta) + \frac{\sigma_{MN}}{\Delta P(0,T+\Delta)} \int_0^t e^{-rs} dW_s^{Q_{T+\Delta}}. \end{aligned}$$

Then the forward LIBOR follows the following normal distribution,

$$L(t, T, T + \Delta) \sim N\left(L(0, T, T + \Delta), \frac{\sigma_{MN}^2}{(\Delta P(0, T + \Delta))^2} \frac{1 - e^{-2rt}}{2r}\right).$$

Finally, we modify the caplet pricing formula,

$$\begin{aligned}
&Caplet_{MN}(0, T, T + \Delta) \\
&= \Delta P(0, T + \Delta) \left\{ (L(0, T, T + \Delta) - K)\Phi(z) + \frac{\sigma_{MN}}{\Delta P(0, T + \Delta)} \sqrt{\frac{1 - e^{-2rT}}{2r}} \phi(z) \right\} \quad (54) \\
&z = \Delta P(0, T + \Delta) \left(\frac{L(0, T, T + \Delta) - K}{\sigma_{MN} \sqrt{\frac{1 - e^{-2rT}}{2r}}} \right).
\end{aligned}$$

Comparing the pricing formula of the Modified Normal model (54) with the one of the Normal model (52), the following relation of the caplet volatility can be derived

$$\sigma_{MN} = \Delta P(0, T + \Delta) \frac{\sqrt{T}}{\sqrt{\frac{1 - e^{-2rT}}{2r}}} \sigma_N.$$

4.3 Consideration of swaption

In this section, we consider the swaption pricing formula of the Normal model. Like the cap and the floor, the Black model has been generally used for the valuation of a swaption and the proof of its validity was made by Jamshidian [11]. In addition, the Normal model is used to deal with negative forward swap rate.

The forward swap rate at time t ($S(t, T_0, T_{n+1})$), where the start time of the interest rate swap is T_0 and the end time is T_{n+1}, is defined as follows.

Definition 2 (Forward swap rate)

$$\begin{aligned}
S(t, T_0, T_{n+1}) &:= \frac{P(t, T_0) - P(t, T_{n+1})}{\sum_{i=1}^{n+1} \Delta_{i-1} P(t, T_i)} = \frac{P(t, T_0) - P(t, T_{n+1})}{BPV(t, T_0, T_{n+1})} \\
P(t, T) &: Price\ of\ a\ bond\ at\ time\ t\ with\ maturity\ T \\
\Delta_i &: Interest\ calculation\ period\ for\ the\ interest\ rate\ swap \\
&\quad that\ is\ the\ underlying\ asset \\
BPV(t, T_0, T_{n+1}) &: Sensitivity\ of\ the\ interest\ rate\ swap\ at\ time\ t \\
&\quad whose\ start\ time\ is\ T_0\ and\ maturity\ is\ T_{n+1}\ (basis\ point\ value)
\end{aligned}$$

The basis point value (BPV) is the amount of change in the value of the interest rate swap with respect to the change in the interest rate 1 basis point. From this definition, assuming $BPV(t, T_0, T_{n+1})$ as a numeraire, the forward swap rate is a martingale under the *forward swap measure* Q_{BPV}. So in the Normal model, the forward swap rate is modeled as follows in order to allow negative interest rates,

$$\begin{aligned} dS(t, T_0, T_{n+1}) &= \sigma_N dW_t^{Q_{BPV}} \\ \Leftrightarrow S(t, T_0, T_{n+1}) &= S(0, T_0, T_{n+1}) + \sigma_N W_t^{Q_{BPV}}. \end{aligned} \tag{55}$$

Therefore, by the same argument as the case of a caplet, the pricing formula based on the Normal model of a payers swaption with the option maturity T_0, the maturity of underlying interest rate swap T_{n+1}, strike rate K, and notional amount 1, is as follows

$$\begin{aligned} Swptn_N(0, T_0, T_{n+1}) &= BPV(0, T_0, T_{n+1}) E^{Q_{BPV}} \left[\frac{BPV(T_0, T_0, T_{n+1})(S(T_0, T_0, T_{n+1}) - K)^+}{BPV(T_0, T_0, T_{n+1})} \right] \\ &= BPV(0, T_0, T_{n+1}) \left\{ (S(0, T_0, T_{n+1}) - K)\Phi(z) + \sigma_N \sqrt{T_0}\phi(z) \right\} \\ z &= \frac{S(0, T_0, T_{n+1}) - K}{\sigma_N \sqrt{T_0}} \end{aligned}$$

Similar to the forward LIBOR, (55) shows that the following is assumed,

$$\begin{aligned} d(P(t, T_0) - P(t, T_{n+1})) &= r(P(t, T_0) - P(t, T_{n+1}))dt + \sigma_N BPV(t, T_0, T_{n+1}) dW_t^{Q_{BPV}} \\ dBPV(t, T_0, T_{n+1}) &= rBPV(t, T_0, T_{n+1})dt. \end{aligned} \tag{56}$$

and the same problems occur as with the forward LIBOR. So we modify (56) as follows

$$\begin{aligned} d(P(t, T_0) - P(t, T_{n+1})) &= r(P(t, T_0) - P(t, T_{n+1}))dt + \sigma_{MN} dW_t^{Q_{BPV}} \\ dBPV(t, T_0, T_{n+1}) &= rBPV(t, T_0, T_{n+1})dt. \end{aligned}$$

Then, by Ito formula, the SDE of the forward swap rate is as follows,

$$\begin{aligned} dS(t, T_0, T_{n+1}) &= \frac{\sigma_{MN}}{BPV(t, T_0, T_{n+1})} dW_t^{Q_{BPV}} \\ \Leftrightarrow S(t, T_0, T_{n+1}) &= S(0, T_0, T_{n+1}) + \frac{\sigma_{MN}}{BPV(0, T_0, T_{n+1})} \int_0^t e^{-rs} dW_s^{Q_{BPV}}, \end{aligned}$$

and the forward swap rate follows the following normal distribution,

$$S(t, T_0, T_{n+1}) \sim N\left(S(0, T_0, T_{n+1}), \frac{\sigma_{MN}^2}{BPV(0, T_0, T_{n+1})^2} \frac{1 - e^{-2rt}}{2r} \right).$$

Finally, the pricing formula of the payers swaption based on the Modified Normal model is

$$\begin{aligned} &Swptn_{MN}(0, T_0, T_{n+1}) \\ &= BPV(0, T_0, T_{n+1}) \left\{ (S(0, T_0, T_{n+1}) - K)\Phi(z) + \frac{\sigma_{MN}}{BPV(0, T_0, T_{n+1})} \sqrt{\frac{1 - e^{-2rT_0}}{2r}} \phi(z) \right\} \\ &z = BPV(0, T_0, T_{n+1}) \left(\frac{S(0, T_0, T_{n+1}) - K}{\sigma_{MN} \sqrt{\frac{1 - e^{-2rT_0}}{2r}}} \right). \end{aligned}$$

We can also get the following relation of the swaption volatility

$$\sigma_{MN} = BPV(0, T_0, T_{n+1}) \frac{\sqrt{T_0}}{\sqrt{\frac{1-e^{-2rT_0}}{2r}}} \sigma_N.$$

5 Conclusion

In this paper, the following three themes were investigated in detail about the option pricing formula based on the Bachelier model.

1. Analysis of the Bachelier formula in prior researches and comparison with this formula
2. Derivation of the Bachelier formula using eight methods and investigation of the property of the Bachelier model
3. Analysis and examination of the Normal model used to evaluate the interest rate options in practice

In the following, we will summarize this paper on these three themes and summarize future research prospects.

For Theme 1, we pointed out the deficiencies in model settings of prior researches. Specifically, they started from modeling the discounted stock price by the Brownian motion. So we started from the natural setting of modeling the undiscounted underlying asset price using the Ornstein=Uhlenbeck process and derived the Bachelier formula in consideration of discount.

For Theme 2, we derived the Bachelier formula by applying the eight methods for deriving the Black-Scholes formula to the Bachelier model. During the derivation process, various following results were also obtained.

(1) The Bachelier partial differential equation

(2) Representation of the intrinsic value and the time value using the local time

(3) Forward partial differential equation

(4) Continuous-time CAPM in the Bachelier model

(5) The option pricing formula using the characteristic function (modified characteristic function)

Furthermore, we derived the Bachelier formula given the existence of a representative agent with the CARA utility function (exponential type utility function). On the other hand, in the Black-Scholes model, it is assumed to have CRRA utility function (power type utility function). Therefore, although the exsistence of a representative agent is assumed in both the Bachelier model and the Black-Scholes model, it can be seen that its utility function is different from each other.

For theme 3, it was confirmed that the model setting of the Normal model that is actually used to evaluate interest rate options was unnatural compared to the Black model. The unnatural point is that the forward LIBOR and the forward swap rate are modeled by Brownian motion. It is essentially equivalent to starting from the modeling of discounted stock price in the pricing the option on the stock. Therefore, based on the setting of the Bachelier model in this paper, we corrected the above unnatural points in the model of the forward LIBOR and the forward swap rate and presented an original pricing formula (the Modified Normal model).

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