Probability Assignment 5

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Question : There are 5% defective items in a bulk of items. What is the probability that a sample of 10 items will include not more than one defective item ?

Solution : Let X denote the number of defective items out of 10 items. Clearly, X has the binomial distribution with n = 10 and p = 0.05, being the probability of item being defective.

$$n = 10 \tag{1}$$
$$p = 0.05 \tag{2}$$
$$q = 1 - p = 0.95 \tag{3}$$

Since X has binomial distribution,

$$\implies P_X(r) = {}^{n}C_r\,(p)^r\,(q)^{n-r} \tag{4}$$
$$\implies F_X(r) = \sum_{r=0}^{r} {}^{n}C_r\,(p)^r\,(q)^{n-r} \tag{5}$$

Therefore,

$$\Pr(X <= 1) = F_X(1) \tag{6}$$
$$i = \sum_{r=0}^{1} {}^{10}C_r\,(0.05)^r\,(0.95)^{n-r} \tag{7}$$
$$= 0.914 \tag{8}$$

Now for Gaussian distribution,

$$\mu(mean) = np \tag{9}$$
$$= 0.5 \tag{10}$$
$$\sigma(standardDeviation) = \sqrt{npq} \tag{11}$$
$$= 0.6982 \tag{12}$$

Since binomial is for discrete random variable, we need to add correction factor in bounds. For this case,

$$upperbound = 1 + 0.5 \tag{13}$$
$$= 1.5 \tag{14}$$

Where 0.5 is the correction factor .Therefore,

$$P(x) = \int_{-\infty}^{1.5} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) e^{\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)} dx \tag{15}$$
$$= 0.926 \tag{16}$$

The results obtained from both binomial and gaussian distribution are quite close.

Showing binomial can be approximated by Gaussian :

Let,

$$W = {}^{N}C_n\,(p)^n\,(q)^{N-n} \tag{17}$$
$$= \frac{N!}{(N-n)! \times n!}\,(p)^n\,(q)^{N-n} \tag{18}$$

Using the approximation that,

$$ln\,(x!) = xln\,(x) - x \tag{19}$$

Therefore,

$$lnW = Nln\,(N) - ((N-n)ln\,(N-n) - lnq) - n\,(ln\,(n) - lnp) \tag{20}$$

For finding most probable value of n,

$$\frac{\partial W}{\partial n} = ln\,(N-n) - ln\,(n) + ln\,(p) - ln\,(q) = 0 \tag{21}$$
$$\implies \frac{(N-n)\,p}{n\,(1-p)} = 1 \tag{22}$$
$$\implies n^* = Np \tag{23}$$

Where n^* denotes most probable value of n. Similairly,

$$\frac{\partial^2 W(n^*)}{\partial n^2} = -\frac{1}{Npq} \tag{24}$$

Now with the help of Taylor's series,

$$f\,(x) = f\,(x_o) + (x - x_o)\,f'\,(x_o) + \frac{(x - x_o)^2}{2!} f''(x_o).... \tag{25}$$

Where ln(W) denotes f(x)

$$lnW = lnW\,(n^*) - \frac{(n-n^*)^2}{2!} \times \frac{1}{Npq} \tag{26}$$
$$W = W\,(n^*)\,e^{\left(\frac{-(n-n^*)^2}{2\times Npq}\right)} \tag{27}$$
$$W = W\,(n^*)\,e^{\left(\frac{-(n-\mu)^2}{2\sigma^2}\right)} \tag{28}$$

Expression (28) is similar to expression (15).