Probability Assignment 5

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Question : There are 5% defective items in a bulk of items. What is the probability that a sample of 10 items will include not more than one defective item ?

Solution : Let X denote the number of defective items out of 10 items. Clearly, X has the binomial distribution with n = 10 and p = 0.05, being the probability of item being defective.

$$n = 10 \tag{1}$$
$$p = 0.05 \tag{2}$$
$$q = 1 - p = 0.95 \tag{3}$$

Since X has binomial distribution,

$$\implies P_X(r) = {}^nC_r\,(p)^r\,(q)^{n-r} \tag{4}$$
$$\implies F_X(r) = \sum_{r=0}^{r} {}^nC_r\,(p)^r\,(q)^{n-r} \tag{5}$$

Therefore,

$$\Pr(X <= 1) = F_X(1) \tag{6}$$
$$i = \sum_{r=0}^{1} {}^{10}C_r\,(0.05)^r\,(0.95)^{n-r} \tag{7}$$
$$= 0.914 \tag{8}$$

Now for Gaussian distribution,

$$\mu(mean) = np \tag{9}$$
$$= 0.5 \tag{10}$$
$$\sigma(standardDeviation) = \sqrt{npq} \tag{11}$$
$$= 0.6982 \tag{12}$$

Since binomial is for discrete random variable, we need to add correction factor in bounds. For this case,

$$upperbound = 1 + 0.5 \tag{13}$$
$$= 1.5 \tag{14}$$

Where 0.5 is the correction factor .Therefore,

$$P(x) = \int_{-\infty}^{1.5} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) e^{\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)} dx \tag{15}$$
$$= 0.926 \tag{16}$$

The results obtained from both binomial and gaussian distribution are quite close.

Showing binomial can be approximated by Gaussian :

The Binomial distribution can be approximated by Gaussian distribution when the number of trials is large and probability of success is not too close to 0 or 1.

Binomial distribution to obtain probability of X success in n trials with probability p is given by:

$$\Pr(X = k) = {}^nC_k\,(p)^k\,(1-p)^{n-k} \tag{17}$$

Using Stirilings's approximation for factorial,

$${}^nC_k \approx \sqrt{2\pi n} \times \left(\frac{n}{e}\right)^n \times \left(\frac{1}{k}\right)^k \times \left(\frac{1}{n-k}\right)^{n-k} \tag{18}$$

Subtituting this approximation into the binomial distribution formula, we get:

$$\Pr(X = k) \approx \left(\frac{1}{\sqrt{2\pi np(1-p)}} \times \left(\frac{n}{e}\right)^n \times \left(\frac{p}{e}\right)^k \times \left(\frac{1-p}{e}\right)^{n-k}\right) \tag{19}$$

This expression can be simplified further using the Q function, which is definied as:

$$Q(x) = \left(\frac{1}{\sqrt{2\pi}}\right) \times \int_x^{\infty} e^{\frac{-t^2}{2}} dt \tag{20}$$

The probability of X being less than or equal to k can be expressed as:

$$\Pr(X \le k) \approx Q\left(\frac{k + 0.5 - \mu}{\sigma}\right) \tag{21}$$

Where,

$$\mu(Mean) = np \tag{22}$$
$$\sigma(Standard\ Deviation) = \sqrt{np(1-p)} \tag{23}$$

Therefore, Binomial can be approximated by the Gaussian distribution in terms of the Q function.