18.785 Analytic Number Theory Problem Set #9

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Problem 1

For $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ let H_θ denote the parabolic subgroup fixing the line in $\mathbb{R}^2$ that forms an angle of θ with the $+x$ axis. Let $k_\theta = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$, rotation by θ. Then

$$k_\theta^{-1} H_\theta k_\theta = H_0.$$

However, the matrices fixing the x axis have lower left hand corner 0 so

$$H_0 = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) \right\}.$$

Hence $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in H_\theta$ iff

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in H_0,$$

i.e. iff the lower left hand corner of the above matrix is 0. By calculation:

$$-a\sin\theta\cos\theta + c\cos^2\theta - b\sin^2\theta + d\sin\theta\cos\theta = 0$$

First consider the case $\theta \neq 0$. Then dividing by $-\sin^2\theta$ gives

$$a\cot\theta + b - c\cot^2\theta - d\cot\theta = 0.$$

Putting $z = \cot\theta$, this is equivalent to

$$\begin{aligned} az + b - cz^2 - dz &= 0 \\ \iff \frac{az+b}{cz+d} &= z, \end{aligned}$$

i.e. this is equivalent to $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ fixing $z = \cot\theta$. Since cot is bijective, we've shown that all parabolic subgroups are isotropy subgroups and vice versa, except for H_0 and the isotropy subgroup of ∞. But in this case, $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2(\mathbb{Z})$ is in H_0 iff $c \neq 0$, iff $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ fixes ∞, as needed.

Problem 2

Note

$$\begin{aligned}\|f\|_p^p = \int_G |f|^p \, d\mu &= \int_{\{x:|f|\geq 1\}} |f|^p \, d\mu + \int_{\{x:|f|<1\}} |f|^p \, d\mu \\ &\geq \mu(\{x : |f| \geq 1\}).\end{aligned}$$

Hence $f \in L^p(G)$ implies $\mu(\{x : |f| \geq 1\}) < \infty$. Additionally noting that $\max |f|^r < \infty$ since f is bounded, we get

$$\begin{aligned}\|f\|_r^r = \int_G |f|^r \, d\mu &= \int_{\{x:|f|\geq 1\}} |f|^r \, d\mu + \int_{\{x:|f|<1\}} |f|^r \, d\mu \\ &\leq \mu(\{x : |f| \geq 1\}) \max |f|^r + \int_{\{x:|f|<1\}} |f|^p \, d\mu \\ &\leq \mu(\{x : |f| \geq 1\}) \max |f|^r + \int_G |f|^p \, d\mu \\ &< \infty.\end{aligned}$$

Hence $f \in L^r(G)$.

Problem 3

By Theorem 8.5.2, f is continuous. By Harish-Chandra (Theorem 8.6.1), there exists a smooth function α with compact support such that $f * \alpha = f$. Young's inequality says that for $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s} + 1$,

$$\|f * \alpha\|_r \leq \|f\|_p \|\alpha\|_s .$$

Putting in $q = s = \infty$ gives

$$\|f * \alpha\|_\infty \leq \|f\|_p \|\alpha\|_\infty .$$

The L^∞ norm is just the maximum of the function (provided that the function is continuous). Using $f * \alpha = f$ and the fact that $\max(\alpha) < \infty$ (α is continuous on a compact set), this becomes

$$\max(f) \leq \|f\|_p \max(\alpha) < \infty.$$

Hence f is bounded. By Problem 2, $f \in L^r(G)$ for $r \geq p$.