18.785 Analytic Number Theory Problem Set #8

Holden Lee

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Problem 1 *(Odd Maass forms)*

We assume $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \Gamma$.

Let $g(z) = \frac{1}{4\pi i}\frac{\partial f}{\partial x}(z)$. Note that if $z = x + yi$,

$$f(z) = f_0(y) + \sum_{n\neq 0} a(n) 2y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)e(nx)$$

then

$$g(z) = \frac{1}{4\pi i}\frac{\partial f}{\partial x}(z) = \frac{1}{4\pi i}\sum_{n\neq 0} a(n)2y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y)2\pi i n e(nx).$$

Using this and the identity

$$\int_0^\infty K_{s-\frac{1}{2}}(y)y^w\frac{dy}{y} = 2^{w-2}\Gamma\left(\frac{w+s+1/2}{2}\right)\Gamma\left(\frac{w-s+1/2}{2}\right),$$

we get

$$\begin{aligned}
\int_0^\infty g(iy)y^{w+\frac{1}{2}}\frac{dy}{y} &= \int_0^\infty \frac{1}{4\pi i}\sum_{n\neq 0} a(n)2y^{w+1}K_{s-\frac{1}{2}}(2\pi|n|y)2\pi i n e(nx)\frac{dy}{y} \\
&= \frac{1}{2\pi}\sum_{n\neq 0} a(n)\int_0^\infty K_{s-\frac{1}{2}}(2\pi|n|y)2\pi n y^{w+1}\frac{dy}{y} \\
&= \frac{1}{2\pi}\sum_{n\neq 0} a(n)\int_0^\infty K_{s-\frac{1}{2}}(y)2\pi n\frac{y^{w+1}}{2^{w+1}|n|^{w+1}\pi^{w+1}}\frac{dy}{y} && \left(y \leftarrow \frac{y}{2\pi|n|}\right) \\
&= \frac{1}{2\pi}\sum_{n\neq 0}\frac{a(n)}{|n|^w\operatorname{sign}(n)2^w\pi^w}\int_0^\infty K_{s-\frac{1}{2}}(y)y^{w+1}\frac{dy}{y} \\
&= \frac{1}{2\pi}\sum_{n\neq 0}\frac{a(n)}{|n|^w\operatorname{sign}(n)2^w\pi^w}2^{w-1}\Gamma\left(\frac{w+s+3/2}{2}\right)\Gamma\left(\frac{w-s+3/2}{2}\right) \\
&= \frac{1}{2\pi}\sum_{n=1}^\infty\frac{2a(n)}{2n^w\pi^w}\Gamma\left(\frac{w+s+3/2}{2}\right)\Gamma\left(\frac{w-s+3/2}{2}\right) && (a(-n) = -a(n))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\pi^{-w}\Gamma\left(\frac{w+s+3/2}{2}\right)\Gamma\left(\frac{w-s+3/2}{2}\right)\sum_{n=1}^{\infty}\frac{a(n)}{n^w} \\
&= \frac{1}{2\pi}L^*(w,f).
\end{aligned}
$$

Multiplying by 2π fives

$$L^*(w,f) = 2\pi\int_0^\infty g(iy)y^{w+\frac{1}{2}}\frac{dy}{y}.$$

Note the RHS defines an absolutely convergent function for all w, since $K_{s-\frac{1}{2}}(y) \sim (2\pi^{-1}y)^{-\frac{1}{2}}e^{-y}$ gives the convergence of the sum when $y \to \infty$, and the transformation $f(iy) = f(i/y)$ gives convergence when $y \to 0$. This gives the analytic continuation of $L^*(w,f)$.

Next note that $f(iy) = f\left(\frac{i}{y}\right)$ gives $g(iy) = g\left(\frac{i}{y}\right)$ so

$$
\begin{aligned}
L^*(-1-w,f) &= 2\pi\int_0^\infty g(iy)y^{-w-\frac{1}{2}}\frac{dy}{y} \\
&= 2\pi\int_0^\infty g\left(\frac{i}{y}\right)y^{-w-\frac{1}{2}}\frac{dy}{y} \\
&= 2\pi\int_\infty^0 g(iu)u^{w+\frac{1}{2}}u\cdot -\frac{du}{u^2} && \left(y \leftarrow \frac{1}{u}\right) \\
&= 2\pi\int_0^\infty g(iu)u^{w+\frac{1}{2}}\frac{du}{u} \\
&= L^*(w,f).
\end{aligned}
$$

Problem 2 *(Properties of convolution)*

If $Y \in \mathfrak{g}$ and $e^{tY}z = u(t) + iv(t)$, then Y acts on functions by

$$\left.\frac{du(t)}{dt}\right|_{t=0}\frac{\partial}{\partial x} + i\left.\frac{dv(t)}{dt}\right|_{t=0}\frac{\partial}{\partial y}.$$

Since we want derivatives at $t=0$, it suffices to calculate the x and y derivatives of $(I+tY)z$; the higher order terms in the power series expansion of e^{tY} have derivative 0 at $t=0$. For $Y = F$,

$$(I+tF)z = \begin{bmatrix}1 & 0\\ t & 1\end{bmatrix}(x+iy) = \frac{x+iy}{t(x+iy)+1} = \frac{tx^2+ty^2+x}{(tx+1)^2+(ty)^2} + \frac{y}{(tx+1)^2+(ty)^2}i.$$

Taking the derivative with respect to t and setting $t=0$ gives

$$\frac{(x^2+y^2)-(tx^2+ty^2+x)(2(tx+1)x+2ty^2)}{((tx+1)^2+(ty)^2)^2} - \frac{y(2(tx+1)x+2ty^2)}{((tx+1)^2+(ty)^2)^2}i = (y^2-x^2)-2xyi.$$

Hence
$$F = (y^2 - x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}.$$
Simlarly,
$$(I + tH)z = \begin{bmatrix} 1+t & 0 \\ 0 & 1-t \end{bmatrix}(x + iy) = \frac{1+t}{1-t}(x + yi).$$
Taking the derivative with respect to t and setting $t = 0$ gives
$$\frac{2}{(1-t)^2}(x + iy) = 2x + 2yi.$$
Hence
$$H = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}.$$
Then
$$\begin{aligned}
\mathcal{C} &= \frac{1}{2}H^2 + EF + FE \\
&= \frac{1}{2}\cdot 2\left(x\frac{\partial}{\partial x}2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + y\frac{\partial}{\partial y}2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\right) + \frac{\partial}{\partial x}\left((y^2 - x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}\right) \\
&\quad + \left(2(y^2 - x^2)\frac{\partial^2}{\partial x^2} - 4xy\frac{\partial^2}{\partial y\partial x}\right) \\
&= 2\left(x\left(\frac{\partial}{\partial x} + x\frac{\partial^2}{\partial x^2}\right) + \cancel{2xy\frac{\partial^2}{\partial x\partial y}} + y\left(\frac{\partial}{\partial y} + y\frac{\partial^2}{\partial y^2}\right)\right) + \left(-2x\frac{\partial}{\partial x} + (y^2 - x^2)\frac{\partial^2}{\partial x^2} - 2y\left(\frac{\partial}{\partial y} + \cancel{x\frac{\partial^2}{\partial x\partial y}}\right)\right) \\
&\quad + \left((y^2 - x^2)\frac{\partial^2}{\partial x^2} - \cancel{2xy\frac{\partial^2}{\partial y\partial x}}\right) \\
&= 2y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -\frac{1}{2}\Delta.
\end{aligned}$$

Problem 3 *(Properties of convolution)*

(i)
Given that f is continuous and $g \in C_0^\infty(G)$,
$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\,dy$$
is in $C^\infty(G)$ because the integrand $f(y)g(y^{-1}x)$ is a C^∞ function in y that vanishes off a compact set; to take derivatives of $f * g$ we may take them inside the integral.

Given $D \in \mathcal{U}(\mathfrak{g})$,

$$\begin{aligned}
D(f * g) &= \frac{d}{dt}(f * g)(xe^{tD})|_{t=0} \\
&= \frac{d}{dt}\int_G f(y)g(y^{-1}xe^{tD})\,dy|_{t=0} \\
&= \int_G f(y)\frac{d}{dt}g(y^{-1}xe^{tD})|_{t=0}\,dy \\
&= \int_G f(y)Dg(y^{-1}x)\,dy.
\end{aligned}$$

(ii)
Given f, g, h locally integrable with g, h having compact support,

$$\begin{aligned}
(f * (g * h))(x) &= \int_G f(xy)(g * h)(y^{-1})\,dy \\
&= \int_G f(xy)\int_G g(y^{-1}z)h(z^{-1})\,dz\,dy \\
&= \int_G\int_G f(xy)g(y^{-1}z)h(z^{-1})\,dy\,dz && \text{(Fubini)} \\
&= \int_G\int_G f(xy)g(y^{-1}z)\,dy\,h(z^{-1})\,dz \\
&= \int_G\int_G f(xzy)g(y^{-1})\,dy\,h(z^{-1})\,dz && (y \hookleftarrow z^{-1}y) \\
&= \int_G (f * g)(xz)h(z^{-1})\,dz \\
&= ((f * g) * h)(x).
\end{aligned}$$

Note the integrand in the double integral is integrable because if g has support K_1 and h has support K_2 then $g(y^{-1}z)h(z)$, as a function of (y, z), has support contained in $K_2K_1^{-1} \times K_2$, which is compact.

Problem 4 *(Converging harmonic functions)*

For a set S let $B_r(S) = \{x | \exists y \in S, d(x, y) < r\}$.

Let K be a compact subset of U. Choose R so that $B_R(K) \subseteq U$. Let $\varphi \in C^\infty(B_R(0))$ be radially symmetric with integral 1. Let $\varphi_y(x) = \varphi(x - y)$. Note φ_y has support contained in U. By the Mean Value Property for harmonic functions,

$$\begin{aligned}
f_i(y) &= f_i(y)\int_{B_R(0)} \varphi(x)\,dx \\
&= f_i(y)2\pi\int_0^R r\varphi(r)\,dr \\
&= \int_0^R r\varphi(r)2\pi f_i(y)\,dr
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^R \int_0^{2\pi} r\varphi(r(\cos\theta, \sin\theta)) f_i(y + r(\cos\theta, \sin\theta))\, d\theta\, dr \quad \text{MVP and radial symmetry of } \varphi \\
&= \int_{B_R(y)} \varphi(x-y) f_i(x)\, dx \\
&= (f_i * \varphi)(y) = \int_{B_R(y)} \varphi_y(x) f_i(x)\, dx.
\end{aligned}
$$

By assumption $(f_i * \varphi)(y)$ converges as $i \to \infty$, so $f_i(y)$ converges pointwise on K.

Let $B = \{\varphi_y | y \in K\}$. The T_{f_i} are linear operators on the test functions with $\sup_{i\in\mathbb{N}} |T_{f_i}(g)| < \infty$ for each $g \in B$, since $T_{f_i}(g) = \int_{B_R(y)} \varphi_y(x) f_i(x)\, dx$ converges to a finite limit. Hence by the uniform boundedness principle for Fréchet spaces, the T_{f_i} are equicontinuous, giving that the f_i are equicontinuous. (Equicontinuity of the T_{f_i} and the fact that $\varphi_{y'} \to \varphi_y$ when $y' \to y$ give that, for a given $\varepsilon > 0$, there exists $\delta > 0$ so that $d(y, y') < \delta$ implies $\sup_y |T_{f_i}\varphi_{y'} - T_{f_i}\varphi_y| < \varepsilon$. But this equals $\sup_y |f_i(y') - f_i(y)|$ by our calculations above so we get $\sup_y |f_i(y') - f_i(y)| < \varepsilon$ for all y, y' with $d(y, y') < \delta$.) Since the f_i are equicontinuous and converge pointwise, they converge uniformly.

To see $f = \lim_{i\to\infty} f_i$ is harmonic, take the limit of

$$
f_i(y) = \frac{1}{\pi R^2} \int_{B_R(y)} f_i(x)\, dx
$$

as $i \to \infty$, now legal since the f_i converge uniformly, to conclude that the mean value property holds for f, and hence that f is harmonic.