1 Hardy-Littlewood Maximal Function

Throughout we work in $\mathbb{R}^n$.

Definition 1.1: Given a function f, define the **maximal function** as the maximal average of absolute value over balls centered at f.

$$Uf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy.$$

Theorem 1.2 (Maximal inequality)**:** 1. For all $\alpha > 0$,

$$m(\{x : Mf(x) > \alpha\}) \precsim \frac{1}{\alpha} \int |f|\, dx.$$

This essentially says that if $f \in L^1$, then $Mf \in L^{1,\infty}$.

2. If $f \in L^p$ and $1 < p \le \infty$, then $Mf \in L^p$, and

$$\|Mf\|_{L^p} \precsim \|f\|_{L^p}.$$

To prove this we need the covering lemma.

Lemma 1.3 (Vitali covering lemma)**:** Let E be a nonempty measurable set, covered by balls $B_1, \ldots, B_M$. There exists a subcollection $\{B_1, \ldots, B_N\}$ that is mutually disjoint and

$$\sum_{k=1}^{N} m(B_k) \ge 3^n m(E).$$

Suppose we have some weird set E. If we can cover it by a finite collection of balls, we can find a disjoint subcollection that still tells us something about the measure of E.

Proof. Use the greedy method ("analysis is being greedy"). Let $B_1 \in \{B_\alpha\}$ have the largest radius. Let B_2 be the ball disjoint from B_1 which has largest radius, and so forth.

They are mutually disjoint by construction. Suppose $B \in \{B_\alpha\}$ which is not one of $B_1, \ldots, B_N$. Then for some l, $B \cap B_\ell \neq \phi$. Suppose B_l is the largest (first) ball that it intersects. Then $r(B_l) > r(B)$ because otherwise B would have been chosen instead of B_l at that stage. Then the dilation of B_l by a factor of 3 covers B: $B \subseteq 3B_l$.

Then $3B_1, \ldots, 3B_N$ cover E, so

$$m(E) \le \sum_{k=1}^{N} m(3B_k) = 3^n \sum_{k=1}^{N} m(B_k).$$

□

Proof of maximal inequality 1.2. Define the **uncentered maximal function** by

$$\widetilde{M}f(x) = \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y)|\, dy.$$

It's clear that $Mf(x) \leq \widetilde{M}f(x)$.

It suffices to prove the theorem for $\widetilde{M}f$.

Set

$$E_\alpha = \left\{x : \widetilde{M}f(x) > \alpha\right\}.$$

For all $x \in E_\alpha$, there exists $B_x \ni x$ such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|\, dy > \alpha,$$

i.e. $m(B_x) \leq \frac{1}{\alpha} \int_{B_x} |f(y)|\, dy$. Let measure of E can be made arbitrarily close to that of E_α because of inner regularity.) Let $\{B_x\}_{x \in E_\alpha}$ be a collection of balls that cover E_α. By compactness we can choose a finite number $B_1, \ldots, B_M$ that cover E. By the Covering Lemma 1.3 we can choose $B_1, \ldots, B_N$ mutually disjoint and $\sum_{k=1}^N m(B_k) \geq 3^{-n} m(E)$. We have an upper bound on this already:

$$\begin{aligned} m(E) &\leq 3^n \sum_{k=1}^N m(B_k) \\ &\leq \frac{3^n}{\alpha} \sum_{k=1}^N \int_{B_f} |f(y)|\, dy \\ &\leq \frac{3^n}{\alpha} \int |f|\, dx \\ \implies m(E_\alpha) &\lesssim \frac{1}{\alpha} \int |f|\, dx. \end{aligned}$$

□

The norm is independent of the dimension.

Definition 1.4: Let f be measurable. The **distribution function** of f is a function $\lambda_f(\alpha) : [0, \infty) \to [0, \infty)$ given by

$$\lambda_f(\alpha) = m(\{|f(x)| > \alpha\}).$$

This gives us useful information. For example,

$$\int |f|^p\, dx = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, dx.$$

"Cut the cake" horizontally.

Proof of part 2 of Theorem 1.2. Let $f \in L^p$, $1 < p < \infty$. Then

$$\int |\widetilde{M}f(x)|^p\, dx = p \int_0^\infty \alpha^{p-1} m\left(\left\{x : \widetilde{M}f(x) > \alpha\right\}\right)\, dx \leq p \int_0^\infty \alpha^{p-1} m\left(\left\{x : \widetilde{M}g(x) > \frac{\alpha}{2}\right\}\right)\, dx.$$

where

$$g(x) := \begin{cases} |f(x)|, & |f(x)| > \frac{\alpha}{2} \\ 0, & \text{otherwise.} \end{cases}$$

We show the last inequality. The new function g contains all the information we need. We know

$$\begin{aligned} |f(x)| &\le \max\{g(x), \frac{\alpha}{2}\} \\ &\le |g(x)| + \frac{\alpha}{2}. \end{aligned}$$

This translates to the maximal function.

$$\begin{aligned} \widetilde{M}f(x) &\le \widetilde{M}g(x) + \frac{\alpha}{2} \\ \widetilde{M}f(x) > \alpha &\implies \widetilde{M}g(x) > \frac{\alpha}{2} \\ \{\widetilde{M}f(x) > \alpha\} &\subseteq \{\widetilde{M}g(x) > \frac{\alpha}{2}\}. \end{aligned}$$

We have

$$\begin{aligned} \int |\widetilde{M}f(x)|^p\,dx &\precsim \int_0^\infty \alpha^{p-1}\frac{1}{\alpha}\left(\int_{\mathbb{R}^n} |g(x)|\,dx\right)\,d\alpha \\ &= \int_0^\infty \alpha^{p-2}\left(\int_{\left\{x:|f(x)|>\frac{\alpha}{2}\right\}} |f(x)|\,dx\right)\,d\alpha \\ &= \int_{\mathbb{R}^n} |f(x)|\left(\int_0^{2|f(x)|}\alpha^{p-1}\,d\alpha\right)\,dx \precsim \int |f|^p\,dx. \end{aligned}$$

□

Let's build up what we need for the Calderon-Zygmund decomposition. First we need another (more annoying) covering lemma.

We can cover the complement of a closed set with cubes, where the diameter of the cube is proportion to the distance from the cube to the set.

Lemma 1.5 (Whitney decomposition)**:** Let F be a nonempty closed set. There exists a sequence of almost disjoint cubes (intersecting only on a set of measure 0, i.e., their boundary) $\{Q_k\}$ such that

$$F^c = \bigcup Q_k,$$

and there exists $C > 0$ such that

$$\operatorname{diam}(Q_k) \le d(Q_k, F) \le C\operatorname{diam}(Q_k).$$

Proof. Let M_0 be cubes of unit side length with vertices in $\mathbb{Z}^n$.

Define M_k by bisecting each cube in M_{k-1}, so that M_k consists of cubes with side length 2^{-k}, vertices $2^{-k}\mathbb{Z}^n$. (This is the "dyadic decomposition.")

Let $\Omega = F^c$; set

$$\Omega_k = \left\{x \in \mathbb{R}^n : C2^{-k} \le d(x, F) \le 2C2^{-k}\right\},$$

where C is a constant to be chosen. Think of them as bands around the original set.

For $Q \in M_k$, include $Q \in \mathcal{F}$ if $Q \cap \Omega_k \neq \phi$. We need a lower bound of the distance of the cube to the original set F. Let $x \in Q$. Then

$$\begin{aligned} d(x, F) &\geq C2^{-k} - \underbrace{\operatorname{diam}(Q)}_{\sqrt{n}2^{-k}} \\ &= (C - \sqrt{n})2^{-k} \geq \sqrt{n}2^{-k} = \operatorname{diam}(Q). \end{aligned}$$

where we take $C = 2\sqrt{n}$. This gives one direction.

For the other,

$$\begin{aligned} d(Q, F) &\leq 2C2^{-k} && \text{since it intersects the band} \\ &= 2 \cdot 2\sqrt{n}2^{-k} \\ &= 4d(Q). \end{aligned}$$

For all $Q \in \mathcal{F}$, we have

$$\operatorname{diam}(Q) \leq d(Q, F) \leq 4 \operatorname{diam}(Q).$$

We also have to remove redundant cubes. Proof omitted.

□

Theorem 1.6 (Calderon-Zygmund decomposition)**:** Let $f \in L^1$. Fix $\alpha > 0$. Then there exists a decomposition $f = g + \sum_k b_k$ and a sequence of almost disjoint cubes $\{Q_k\}$ such that

1. (the good part is bounded) $|g(x)| \leq \alpha$,
2. $\operatorname{Supp}(b_k) \subseteq Q_k$, $\int_{Q_k} b_k = 0$, and $\int |b_k(x)| \precsim \alpha \cdot m(Q_k)$,
3. $\sum m(Q_k) \precsim \frac{1}{\alpha} \int |f| \, dx$.

Bounds ($\precsim$) depend only on the dimension.

This is very useful. To construct g, b_k, it's tempting to just cut off f at α. But the correct thing to do is to cut off the maximal function at α.

Idea: The bad set will be covered by a bunch of cubes; if you dilate the cubes by a fixed factor it intersects the good set, so you have a bound on the maximal function.

Proof. We cutoff where $\widetilde{M}f(x) > \alpha$. Let $E_\alpha = \left\{x : \widetilde{M}f(x) > \alpha\right\}$. E_α^c is closed (WLOG nonempty and not all of $\mathbb{R}^n$). Apply Whitney decomposition 1.5 to cover E_α by $\{Q_k\}$. Set

$$g(x) = \begin{cases} |f(x)|, & x \in E_\alpha^c \\ \frac{1}{m(Q_k)} \int_{Q_k} f(y) \, dy, & x \in Q_k \\ 0, & \text{elsewhere.} \end{cases}$$

On E_α^c we have $|f(x)| \precsim |\widetilde{M}f(x)| \leq \alpha$.

Claim: We have $|g(x)| \precsim \alpha$ almost everywhere.

If $x \in Q_k$ (This is a set completely contained in the bad region, but we can blow it up so that it overlaps the good region.),

$$\begin{aligned} |g(x)| &\leq \frac{1}{m(Q_k)} \int_{Q_k} |f(y)| \, dy \\ &\leq \frac{4^n}{m(Q_k^*)} \int_{Q_k^*} |f(y)| \, dy && Q_k^* := 4Q_k \\ &\precsim \alpha. \end{aligned}$$

Let

$$b_k(x) = \chi_{Q_k}(x) \left(f(x) - \frac{1}{m(Q_k)} \int_{Q_k} f(y) \, dy \right);$$

note that $\int b_k(x) = 0$; we have $f = g + \sum b_k$.

Now we need to estimate the L^1 norm of each. We find (the "$\leq \alpha$" comes from taking $x \in Q_k^* \cap E_\alpha$ and noting $\frac{1}{m(Q_k^*)} \int_{Q_k^*} |f(y)| \, dy \leq \widetilde{M} f(x)$ by definition of $\widetilde{M} f$)

$$\begin{aligned} \int |b_k(x)| \, dx &\leq 2 \int_{Q_k} |f(y)| \, dy \\ &\leq \int_{Q_k^*} |f(y)| \, dy \\ &= m(Q_k^*) \cdot \underbrace{\frac{1}{m(Q_k^*)} \int_{Q_k^*} |f(y)| \, dy}_{\leq \alpha} \\ &\precsim \alpha m(Q_k^*) \precsim \alpha m(Q_k) \\ \sum m(Q_k) &= m(\left\{ x : \widetilde{M} f(x) > \alpha \right\}) \\ &\precsim \frac{1}{\alpha} \int |f| \, dx. \end{aligned}$$

□

What can you do with this decomposition? You can estimate in singular integrals. This comes up in L^p elliptic regularity results for Laplace's equation. There's an integral kernel that can be estimated using Calderon-Zygmund.