18.785 Analytic Number Theory Problem Set #11

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Problem 1 *($\mathcal{C}$ has real eigenvalues)*

It suffices to show that $\mathcal{C}$ is self-adjoint, that is,

$$\langle f, \mathcal{C}h\rangle = \langle \mathcal{C}f, h\rangle$$

for all automorphic forms $f, h \in L^2$. Then it will follow that all eigenvalues of $\mathcal{C}$ on this space are real.

By writing f, h as the sum of functions with specific K-type, we can reduce to the case where f has K-type m_1 and h has K-type m_2. Then

$$\langle f, h\rangle = \int_{\Gamma\backslash G} f\overline{h}\, dg = \int_{\Gamma\backslash G/K}\int_K f(gk)\overline{h(gk)}\, d\mu dk = \int_{\Gamma\backslash G/K}\int_K f(g)\overline{h(g)}\chi_{m_1}(k_1)\overline{\chi_{m_2}(k_1)}\, d\mu dk$$

If $m_1 \neq m_2$, integrating over K gives 0, and the assertion is obvious. If $m_1 = m_2$, then $\chi_{m_1}(k_1)\overline{\chi_{m_2}(k_1)} = 1$ and

$$\langle f, h\rangle = \int_{\Gamma\backslash G/K} f(g)\overline{h(g)}\, dg$$

$\mathcal{C} \in \mathcal{Z}$ so $\mathcal{C}$ commutes with right translation, and if f has K-type m then so does $\mathcal{C}f$. Using the fact that $\mathcal{C} = -2\Delta$ and Green's identity, noting that f, g vanish at ∞, we get, with appropriate normalization,

$$\begin{aligned}
\langle f, \mathcal{C}g\rangle &= \int_{\Gamma\backslash G/K} f(\overline{\mathcal{C}g})\, d\mu \\
&= -2\int_{\Gamma\backslash \mathcal{H}} f(\overline{\Delta g})\, d\mu \\
&= -2\int_{\Gamma\backslash \mathcal{H}} (\Delta f)\overline{g}\, d\mu \\
&= \int_{\Gamma\backslash G/K} (Cf)\overline{g}\, d\mu \\
&= \langle Cf, g\rangle
\end{aligned}$$

as needed.

Problem 2

Lemma 2.1: For $f \in L^2(\Gamma\backslash G)$ and $\varphi \in C_c^1(G)$, there exists a constant C depending on φ so that

$$|(f * \varphi)(x)| \le Ca(x)^\rho \|f\|_2 .$$

(The L^2 norm is with respect to $\Gamma\backslash G$.)

Proof. Since $f, f*\varphi$ are left Γ-invariant and a fundamental domain for $\Gamma\backslash G$ can be covered by finitely many Siegel sets, it suffices to prove this for x in some Siegel set $\mathfrak{S}_{\omega,t} = \omega A_t K$ relative to a cuspidal parabolic P.

Let U be a relatively compact, symmetric neighborhood containing $\operatorname{Supp}\varphi$. We have, by Cauchy-Schwarz,

$$\begin{aligned} |f * \alpha(x)| &= \left|\int_G f(y)\varphi(y^{-1}x)\,dy\right| \\ &= \left|\int_{xU} f(y)\varphi(y^{-1}x)\,dy\right| \\ &\le \left(\int_G |\varphi(y)|^2\,dy\right)^{\frac12} \left(\int_{xU} |f(y)|^2\,dy\right)^{\frac12}. \end{aligned} \tag{1}$$

Since U is relatively compact, so is KU, and we can find compact subsets $C_A \subseteq A$ and $C_N \subseteq N$ such that $KU \subseteq C_N C_A K$. Then

$$xU = n(x)a(x)k(x)U \subseteq \omega a(x) C_N C_A K = \left(\omega a(x) C_N a(x)^{-1}\right) a(x) C_A K.$$

Note conjugation by $a(x)$ corresponds to dilation by $a(x)^{2\rho}$ on N, that $\omega a(x) C_N a(x)^{-1} \subseteq N$, and $a(x)C_A \subseteq A_{t'}$ for some fixed t' (since C_A is compact; therefore $a(y)^\rho$ has a minimum for $y \in C_A$). Hence this set is contained in at most $ka(x)^{2\rho}$ fundamental domains for some constant k. Therefore,

$$\left(\int_{xU} |f(y)|^2\,dy\right)^{\frac12} \le k^{\frac12} a(x)^\rho \left(\int_{\Gamma\backslash G} |f(y)|^2\,dy\right)^{\frac12},$$

which together with (1) gives the desired estimate. □

Lemma 2.2: For $f \in {}^\circ L^2(\Gamma\backslash G)$ and $\varphi \in C_c^1(G)$, there exists a constant C depending on φ so that

$$|(f * \varphi)(x)| \le C \|f\|_2 .$$

Proof. Let P be a cuspidal parabolic subgroup for G and $\mathfrak{S}$ a Siegel set relative to P. Then by Borel (5.7), there exists c_1 depending on φ so that

$$|(f * \varphi)(x)| \le c_1 \|f\|_2 a(x)^\rho$$

for all $f \in L^2(\Gamma\backslash G)$. Since $D(f*\varphi) = f * (D\varphi)$, by Borel (5.7) there exists c_2 depending on φ, D such that

$$|(D(f * \varphi))(x)| \le c_2 \|f\|_2 a(x)^\rho$$

Integrate $D(f*\varphi)(nx)$ over $\Gamma_N\backslash N$, noting that $a(nx)=a(x)$ for $n\in N$, to get the constant term satisfies the inequality

$$|(D(f*\varphi)_P)(x)|\le c_2\left\|f\right\|_2 a(x)^\rho$$

Since f is cuspidal, $f_P=0$ and $(f*\varphi)_P=0$. Then by Lemma 9.1.1 in the notes,

$$\begin{aligned}|(f*\varphi)(x) &= |((f*\varphi)-(f*\varphi)_P)(x)|\\ &\le c_3a(x)^{-\alpha}\left(\sum_{i=1}^3 |X_i(f*\varphi)|_P(x)\right)\\ &\le C'\left\|f\right\|_2 a(x)^{\rho-\alpha}\end{aligned}$$

for some C'. Since $\alpha=2\rho$,

$$|(D(f*\varphi)_P)(x)|\ll \left\|f\right\|_p \max(a(x)^\rho,a(x)^{-\rho}).$$

Either $\rho\le 0$ or $-\rho\le 0$, giving the desired bound on a Siegel set corresponding to P. The result follows since $\Gamma\backslash G$ is covered by finitely many Siegel sets. □

From the lemma, since $D(f*\varphi)=f*(D\varphi)$, we also get

$$|D(f*\varphi)(x)|\le C\left\|f\right\|_2$$

for C depending on D,φ.

Consider a subset U of $\Gamma\backslash G$ that is the homeomorphic image of a neighborhood of G with coordinates x_1,x_2,x_3. Consider a bounded subset S of ${}^\circ L^2(\Gamma\backslash G)$. Then by the above, the image T of S under $*\varphi$ is bounded; and inside U, its derivatives with respect to x_1,x_2,x_3 are also bounded. Hence the functions in T, restricted to U, are equicontinuous. By Arzela-Ascoli, any sequence $f_n*\varphi$ in ${}^\circ L^2(\Gamma\backslash G)$ has a uniformly convergent subsequence $f_{n_i}*\varphi$, when we restrict the domain to U. Covering $\Gamma\backslash G$ with countably many such subsets U and using a diagonalization argument, there exist $f_{n_i}*\varphi$ that converge locally uniformly to a continuous bounded function f. Hence $\overline{T}$ is sequentially compact, and T is relatively compact.

Problem 3

Write $P=P_0$ and $N=N_0$. We know that the constant term of $E_{P,s}(g)$ is $\varphi_{P,s}(g)+c(s)\varphi_{P,-s}(g)$ for some meromorphic $c(s)$. By Bruhat decomposition (4.7.1),

$$\Gamma=\mathrm{SL}_2(\mathbb{Z})=\Gamma_P\sqcup\bigcup_{c>0}\bigcup_{d\,(\mathrm{mod}^\times c)}\Gamma_P\begin{bmatrix}* & *\\ c & d\end{bmatrix}\Gamma_P.$$

Calling the second part Γ_w, this gives

$$\Gamma_P\backslash\Gamma_w/\Gamma_N=\left\{\begin{bmatrix}* & *\\ c & d\end{bmatrix}: 0\le d<c,\ \gcd(c,d)=1\right\}.$$

The constant term of $E_{P,s}$ is

$$(E_{P,s})_P = \int_{\Gamma_N\backslash N} \left(\varphi_{P,s}(ng) + \sum_{\gamma\in\Gamma_P\backslash\Gamma_w} \varphi_{P,s}(\gamma ng) \right) dn \tag{2}$$

The first term gives $\varphi_{P,s}(g)$, so we focus on the second term. As in the notes (Lemma 10.2.3), we unfold the integral over $\Gamma_N\backslash N$ to one over N.

$$\begin{aligned}
\sum_{\gamma\in\Gamma_N\backslash\Gamma_w} \int_{\Gamma_N\backslash N} \varphi_{P,s}(\gamma ng)\,dn &= \sum_{\gamma\in\Gamma_N\backslash\Gamma_w/\Gamma_N} \sum_{\delta\in\Gamma_N} \int_{\Gamma_N\backslash N} \varphi_{P,s}(\gamma\delta ng)\,dn \\
&= \sum_{\gamma\in\Gamma_P\backslash\Gamma_w/\Gamma_N} \int_N \varphi_{P,s}(\gamma nx)\,dn \\
&= \sum_{0\le d<c,\,\gcd(c,d)=1} \int_{\mathbb{R}} \varphi_{P,s}\left(\begin{bmatrix} * & * \\ c & d \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right) dx
\end{aligned}$$

We know this is a multiple of $\varphi_s(g)$. To find the coefficient, we simply need to evaluate at $g = I$:

$$\sum_{0\le d<c,\,\gcd(c,d)=1} \int_{\mathbb{R}} \varphi_{P,s}\left(\begin{bmatrix} * & * \\ c & d \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right) dx = \sum_{0\le d<c,\,\gcd(c,d)=1} \int_{\mathbb{R}} \frac{y^{\frac{s+1}{2}} \varphi_P\left(\begin{bmatrix} * & * \\ c & d \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right)}{|cz+d|^{s+1}}\,dx$$

where $z = \left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} i \right) = i + x$ and $y = \Im\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} i \right) = 1$. Since $cz + d = (cx + d) + ci$, assuming the K-type is 0, this equals

$$\begin{aligned}
\sum_{0\le d<c,\,\gcd(c,d)=1} \int_{\mathbb{R}} \frac{\varphi_P\left(\begin{bmatrix} * & * \\ c & d \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right)}{((cx+d)^2+c^2)^{\frac{s+1}{2}}}\,dx &= \sum_{0\le d<c,\,\gcd(c,d)=1} c^{-(s+1)} \int_{\mathbb{R}} \frac{1}{\left(\left(x+\left(\frac{d}{c}\right)\right)^2+1\right)^{\frac{s+1}{2}}}\,dx \\
&= \sum_{c=1}^{\infty} \varphi(c) c^{-(s+1)} \int_{\mathbb{R}} \frac{1}{\left(\left(x+\left(\frac{d}{c}\right)\right)^2+1\right)^{\frac{s+1}{2}}}\,dx \\
&= \sum_{c=1}^{\infty} \varphi(c) c^{-(s+1)} \int_{\mathbb{R}} \frac{1}{(x^2+1)^{\frac{s+1}{2}}}\,dx \\
&= \frac{\zeta(s+1)}{\zeta(s)} \frac{\sqrt{\pi}\,\Gamma\left(\frac{s}{2}\right)}{\left(\frac{s+1}{2}\right)},
\end{aligned}$$

where for the last step we use

$$\sum_{c=1}^{\infty} \varphi(c) c^{-(s+1)} = \frac{\zeta(s+1)}{\zeta(s)} \tag{3}$$

$$\int_{\mathbb{R}} \frac{1}{(x^2+1)^{\frac{s+1}{2}}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}. \tag{4}$$

To show (3), note that

$$\begin{aligned}
\left(\sum_{c=1}^{\infty} \varphi(s) c^{-(s+1)}\, dx\right) \zeta(s+1) &= \left(\sum_{c=1}^{\infty} \frac{\varphi(s)}{c} c^{-s)}\, dx\right) \left(\sum_{c=1}^{\infty} \frac{1}{c} c^{-s}\, dx\right) \\
&= \sum_{c=1}^{\infty} \sum_{ab=c} \frac{\varphi(a)}{a} \frac{1}{b} c^{-s} \\
&= \sum_{c=1}^{\infty} \frac{1}{c} \sum_{a|c} \varphi(a) c^{-s} \\
&= \sum_{c=1}^{\infty} c^{-s} \\
&= \zeta(s),
\end{aligned}$$

where we used $\sum_{a|c} \varphi(a) = c$. For equation (4), see Theory of Integration, D. Stroock, 5.2.20(ii).

So

$$(E_{P,s})_P = \varphi_{P,s}(g) + \frac{\zeta(s+1)}{\zeta(s)} \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)}{\left(\frac{s+1}{2}\right)} \varphi_{P,-s}.$$

Note that P_0 is the sole parabolic subgroup of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ up to conjugation. The system of equations is, for $F_\mu(s, g)$ a linear combination of μ_i's,

1. For any $\varphi \in C_c^\infty(G)$,

$$\int_G F_\mu(s, g) \left(\mathcal{C} - \frac{s^2 - 1}{2}\right) \varphi(g)\, dg = 0.$$

2. Letting ψ be the characteristic function on a Siegel set for P and $\Lambda^t(f) = f - \psi f_P$,

$$\begin{aligned}
F_\mu(s) &= \Psi_\mu(s) + g(s) \\
g(s) &= -(\Lambda^t \circ (*\alpha) - \lambda_\alpha(s))^{-1} (\Lambda^t(\Psi_\mu(s) * \alpha)) \\
\Psi_\mu(s) &= \mu_+ \psi \varphi_{P,s} + \mu_- \psi \varphi_{P,-s}
\end{aligned}$$

3. $\Lambda^t(F_\mu(s) * \alpha) = \lambda_\alpha(s) \Lambda^t(F_\mu(s))$.

4. $\mu_+ = 1$

Uniqueness follows from Lemma 10.3.8 in the notes.