18.785 Analytic Number Theory Problem Set #10

Holden Lee

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Problem 1 *(f cuspidal iff $\tilde{f} \in L^2(\Gamma\backslash G)$)*

First suppose f is cuspidal. Then its norm with respect to the Petersson inner product is well defined, i.e.

$$\iint_{\Gamma\backslash H} |f(z)|^2 y^m \frac{dxdy}{y^2}$$

converges. Noting that $d\mu = \frac{dxdy}{y^2}$ and that $y = \Im[g(i)] = |j(g,i)|^{-2}$, this equals

$$\int_{\Gamma\backslash G} |j(g,i)^{-m} f(g(i))|^2 \, d\mu = \int_{\Gamma\backslash G} \tilde{f}(g) \, d\mu$$

so the latter converges.

Conversely suppose $\tilde{f} \in L^2(\Gamma\backslash G)$. By conjugation we may assume $P = P_0$ with the cusp at infinity. Now $\tilde{f} - \tilde{f}_P$ is rapidly decreasing so $\tilde{f} - \tilde{f}_P \in L^2$. By the triangle inequality,

$$\left\|\tilde{f}_P\right\| \le \left\|\tilde{f} - \tilde{f}_P\right\| + \left\|\tilde{f}\right\| < \infty.$$

Now $\tilde{f}_P = j(g,i)^{-m} a_0$ where a_0 is the constant term at the cusp. As above,

$$\left\|\tilde{f}_P\right\|^2 = \int_{\Gamma\backslash G} |\tilde{f}(g)|^2 \, d\mu = |a_0|^2 \int_{\Gamma\backslash H} y^m \frac{dxdy}{y^2}.$$

Taking a fundamental domain, it contains some region in the form $[a,b] \times [c,\infty)$, and the above integral diverges. Therefore $a_0 = 0$. This shows that f is actually a cusp form.

Problem 2 *($\mathcal{A}(\Gamma, J, \chi) = \bigoplus_{i=1}^{q} \mathcal{A}(\Gamma, J_i, \chi)$)*

First suppose $f \in \mathcal{A}(\Gamma, J, \chi)$. Let

$$P_j(x) = \prod_{\substack{1 \le i \le q \\ i \ne j}} (x - \lambda_i)^{n_i}.$$

Since the P_i are relatively prime there exist g_i such that

$$\sum_{i=1}^{q} g_i P_i = 1.$$

Then

$$f = \sum_{i=1}^{q} \underbrace{[(g_i P_i)(\mathcal{C})] f}_{\in \mathcal{A}(\Gamma, J_i, \chi)} \in \sum_{i=1}^{q} \mathcal{A}(\Gamma, J_i, \chi).$$

To see the inclusion, note that $(\mathcal{C} - \lambda_i)^{n_i}(g_i P_i)(\mathcal{C}) f = g_i P(\mathcal{C}) f = 0$.

It is clear that each $\mathcal{A}(\Gamma, J_i, \chi) \subseteq \mathcal{A}(\Gamma, J, \chi)$. Thus we are left to show the sum is actually a direct sum. Suppose

$$f_1 + \cdots + f_q = 0$$

where $f_i \in \mathcal{A}(\Gamma, J_i, \chi)$. Operate by $P_j(\mathcal{C})$, and note this annihilates every term except f_j. We get $P_j(\mathcal{C}) f_j = 0$. However, we also know $(\mathcal{C} - \lambda_j)^{n_j} f_j = 0$. Since $\gcd(P_j(x), (x - \lambda_j)^{n_j}) = 1$, we get $f_j = 0$. Hence $f_1 = \cdots = f_q = 0$, showing the sum is a direct sum.

Problem 3 *(Integral over N is 0)*

First we make several reductions.

1. f can be written as a sum of functions having specific K-type, so we may assume $f(gk) = f(g)\chi(k), g \in G, k \in K$ for some character χ.

2. By problem 2, f is a sum of $f_i \in \mathcal{A}(\Gamma, J_i, \chi)$, so it suffices to solve the problem for the f_i, i.e. we may assume $(\mathcal{C} - \lambda)^m$ annihilates f.

3. Next, we may assume N, A are the groups $\{\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] | x \in \mathbb{R}\}$ and $\left\{\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right] | t > 0\right\}$ since every other p-pair is obtained by conjugation.

4. Finally, note every $g \in G$ can be written as $g = n_g a_g k_g$ with $n_g \in N$, $a_g \in A$, and $k_g \in K$. Then
$$\int_N f(ng)\, dn = \int_N f(na_g)\chi(k_g)\, dn$$
so it suffices to show that $\int_N f(na)\, dn = 0$ for each $a \in A$.

Let

$$\varphi(g) = \int_N f(ng)\, dn$$

and

$$\phi(t) = \varphi \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Since $A = \left\{ \left[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right] | t \in \mathbb{R} \right\}$, it suffices to show that $\phi(t) \equiv 0$. The fact that f is integrable over $G = NAK$ gives that φ is integrable over AK, so the following is finite:

$$\begin{aligned} \int_K \int_A |\varphi(ak)| a^{-2} \, da \, dk &= \int_K \int_A |\varphi(a)\chi(k)| a^{-2} \, da \, dk \\ &= \int_A |\varphi(a)| a^{-2} \, da \int_K |\chi(k)| \, dk. \end{aligned}$$

Since the second integral is positive, $\int_A |\varphi(a)| a^{-2} < \infty$. But this equals $\int_t \phi(t) e^{-2t} \, dt$. So we have

$$\int_t \phi(t) e^{-2t} \, dt < \infty. \tag{1}$$

Write $Y\phi(t)$ as shorthand for $(Y\varphi)(g)\big|_{g = \left[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right]}$. We claim that

$$\mathcal{C}\phi(t) = \left(\frac{1}{2} \frac{d^2}{dt^2} - \frac{d}{dt} \right) \phi(t). \tag{2}$$

First we show $H\phi(t) = \frac{d}{dt}\phi(t)$.Indeed,

$$\begin{aligned} H\phi(t) &= H\varphi(g)\big|_{g = \left[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right]} \\ &= \frac{d}{dt_1} \varphi\left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} e^{t_1 H} \right)\Bigg|_{t_1 = 0} \\ &= \frac{d}{dt_1} \varphi\left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{bmatrix} \right)\Bigg|_{t_1 = 0} \\ &= \frac{d}{dt_1} \varphi\left(\begin{bmatrix} e^{t+t_1} & 0 \\ 0 & e^{-(t+t_1)} \end{bmatrix} \right)\Bigg|_{t_1 = 0} \\ &= \frac{d}{dt_1} \varphi\left(\begin{bmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{bmatrix} \right)\Bigg|_{t_1 = t} \\ &= \frac{d}{dt}\phi(t). \end{aligned}$$

Next we show $EF\varphi(t) = 0$. First note $\varphi(g)$ is left N-invariant since the integral is over $n \in N$ and n appears on the left in the argument of f. Note

$$\begin{aligned} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} e^{t_1 E} &= \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t & t_1 e^t \\ 0 & e^{-t} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 1 & t_1 e^{2t} \\ 0 & 1 \end{bmatrix}}_{\in N} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}. \end{aligned}$$

Now

$$
\begin{aligned}
EF\varphi(g)\big|_{g=\left[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t}\end{smallmatrix}\right]} &= \frac{\partial^2}{\partial t_1 \partial t_2}\varphi\left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t}\end{bmatrix} e^{t_1E}e^{t_2F}\right)\bigg|_{t_1=t_2=0} \\
&= \frac{\partial^2}{\partial t_1 \partial t_2}\varphi\left(\begin{bmatrix} 1 & e^{2t}t_1 \\ 0 & 1\end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{-t}\end{bmatrix} e^{t_2F}\right)\bigg|_{t_1=t_2=0} \\
&= \frac{\partial^2}{\partial t_1 \partial t_2}\varphi\left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t}\end{bmatrix} e^{t_2F}\right)\bigg|_{t_1=t_2=0} && \text{left } N\text{-invariance} \\
&= 0.
\end{aligned}
$$

Putting the above two results together and using $\mathcal{C} = \frac{1}{2}H^2 - H + EF$ gives (2).

Now $(\mathcal{C}-\lambda)^m f = 0$ gives $(\mathcal{C}-\lambda)^m\phi(t) = 0$, which turns into the differential equation

$$\left(\frac{1}{2}\frac{d^2}{dt^2} - \frac{d}{dt} + \lambda\right)^m \phi = 0.$$

Since the quadratic $\frac{1}{2}x^2 - x + \lambda$ has vertex at 1, its zeros are $1 \pm s$ for some s. Then

$$\phi(t) = p(t)e^{t(1+s)} + q(t)e^{t(1-s)} = e^t(p(t)e^{ts} + q(t)e^{-ts})$$

for some polynomials p, q. Since f is $\mathcal{Z}$-finite, K-finite, and integrable so is φ; hence φ, and hence ϕ is bounded by PSet 9 question 3.

We claim one of $p(t)$ or $q(t)$ is 0. Else $\phi(t)$ grows at least like e^t: Indeed, if $|p(t)e^{ts}| \nsim |q(t)e^{-ts}|$ are of different orders then the larger one is at least constant. If $|p(t)e^{ts}| \sim |q(t)e^{-ts}|$, then $s = is'$ is pure imaginary, the leading terms of p and q are the same, and the expression in parenthesis is

$$2p(t)\cos s'\theta + [q(t) - p(t)]e^{ts} = p(t)\left(\cos s'\theta + \frac{q(t)-p(t)}{p(t)}e^{ts}\right)$$

which does not approach 0 as $\cos s'\theta$ is infinitely often 1 and $\frac{q(t)-p(t)}{p(t)} \to 0$.

WLOG, $q(t) = 0$. BWOC, $p(t) \neq 0$. Then for ϕ to be bounded, $\Re s = -1$ and $p(t)$ is constant. We've shown that $\phi(t)e^{-2t} = p(t)e^{t(-1+s)}$ is integrable (1). But if $\Re s = -1$ and $p(t) \neq 0$, this blows up as $t \to -\infty$. Hence $p(t) = 0$ and $\phi(t) = 0$, exactly what we need.