1 Jensen's Theorem

Theorem 1.1 (Jensen's formula)**:** thm:jensenC Let f be holomorphic in $B_R(0)$ with zeros $(z_k)_{k=1}^n$ (with multiplicity) inside. Then

$$\ln|f(0)| = \sum \ln\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})\, d\theta.$$

Corollary 1.2 (Jensen's inequality)**:** cor:jensenC Keep the setup. Then

$$|f(0)| \geq \frac{R^n}{\prod_k |z_k|} \max_{|z|=R} f(z).$$

Proof.

1. Check that if f, g satisfy the theorem then so does fg.
2. For f nonvanishing, apply Cauchy on $\ln f$ (and take the real part).
3. For $f = z - a_k$, we're reduced to showing $\oint \ln|e^{i\theta} - a_k|\, d\theta = 0$. Interpret this as the mean value of a harmonic function, so it's 0.

□

2 Hadamard's Theorem

Definition 2.1: The **order** of an entire function f is

$$\inf\left\{\alpha : |f(z)| \precsim e^{|z|^\alpha}\right\}.$$

Let z_k be the zeros of f with multiplicity. The **genus** is

$$\inf\left\{s : \sum_k \frac{1}{|z_k|^s} < \infty\right\}.$$

Theorem 2.2: product-development Let z_n be a sequence with $\lim_{n\to\infty}|z_n| = \infty$ (or a finite sequence). If f is entire with order $\alpha < \infty$ with zeros $z_1, z_2, \ldots$ (with multiplicity, not including 0), then it has a product formula

$$\text{product-formula}\; f(z) = z^r e^{g(z)} \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \cdots + \frac{1}{k}\left(\frac{z}{z_k}\right)^k}, \tag{1}$$

where

- $k = \lfloor \alpha \rfloor$,
- r is the order of vanishing of f at 0, and
- g is a polynomial of degree at most a.

The product converges uniformly locally. Moreover,

$$\text{num-zeros}|\{k : z_k < R\}| \precsim_\varepsilon R^{\alpha+\varepsilon}. \tag{2}$$

Conversely, if $a = \lfloor \alpha \rfloor$ and z_k is a sequence satisfying (2), then the RHS of (1) defines an entire function of order at most α.

Proof. (Following Stein-Shakarchi. Ahlfors seems shorter but I haven't read it.)

1. Massage Jensen to give us info about the count of zeros (cf. Abel summation).

   (1.2) Let $n(r)$ be the number of zeros in $B_r(0)$. Then $\int_0^R n(r)\frac{dr}{r} = \sum \ln\left|\frac{R}{z_k}\right|$.

   Proof. Just consier 1 zero, $z - z_k$. Now multiply.

2. (2) The relationship between order and genus. Let ρ be the order. Then

   (a) $n(r) \le C_\varepsilon r^{\rho_\varepsilon}$ for some C,

   (b) genus$\le \rho$.

   Proof. Plugging 1.2 into Jensen 1.1, we get $\int_0^R n(r)\frac{dr}{r} \le C' R^{\rho'}$. Noting $n(r)$ is increasing, if it's $> C'' R^{\rho'}$ infinitely often, summing $\int_R^{2R} C'' R^{\rho-1}\,dr$ makes it $\succ R^{\rho'}$.

3. The product is well-defined. (It is a natural guess; the exponential term is to make it converge.)

   (a) (3.2) If $|F_n - 1| \le c_n$, $\sum c_n < \infty$, then $\prod_{n=1}^\infty = F$ uniformly, $\frac{F'}{F} = \sum \frac{F_n'}{F_n}$.

   (b) (4.1) Let $P_k = z + \frac{z^2}{2} + \cdots + \frac{z^k}{k!}$. Let $E_k = (1-z)e^{P_k(z)}$. Then for $|z| \le \frac{1}{2}$, $|1 - E_k| \le C|z|^{k+1}$, C independent of k.
   Proof.
   $$|1 - E_m| = |1 - (1-z)e^{-\ln(1-z)+O(z^{k+1})}| = O(z^{k+1}).$$

   (c) The Hadamard product converges and has order $\le \alpha$.
   Proof. Use the above criterion and
   $$\sum\left(1 - E_m\left(\frac{z}{z_n}\right)\right) \le \sum c\left|\frac{z}{|z_n|/|z|}\right|^{k+1}$$
   finite if $\sum \frac{1}{|z_n|^{k+1}} < \infty$.
   To see it's order α: terms for large $|z_n|$ don't contribute much; terms for small $|z_n|$ can contribute $\prod_{\left|\frac{z}{z_n}\right| \le \frac{1}{2}}(C|z|e^{C\left(\frac{z}{z_n}\right)^k})$. The $|z|$ contributes an ε. In the exponent we have by Abel summation
   $$\sum_n C\left|\frac{z}{z_n}\right|^k = \int^R z^k|z|^{\alpha-k+\varepsilon}\underbrace{\left(\sum_{|z_n|<R}\frac{1}{|z_n|^{s+\varepsilon}}\right)}_{\le C}.$$
   Exponentiating gives the bound.

4. Given arbitrary f of order $\rho_0 \in [k, k+1)$, look at $\frac{f}{\text{Hadamard product}}$. It's entire and nonvanishing by construction; we need it to have order $\leq \rho_0$ as well, so it equals e^g. Thus we need a lower bound on the growth of the Hadamard product.

   (a) (5.2)
   $$E_k \geq \begin{cases} e^{-c|z|^{k+1}}, & |z| \leq \frac{1}{2} \\ |1-z|e^{-c'|z|^k}, & |z| \geq \frac{1}{2}. \end{cases}$$
   Proof. For $|z| \leq \frac{1}{2}$, $E_k = e^{-\frac{z^{k+1}}{k+1}+\cdots}$; that first term dominates. For $|z| \geq \frac{1}{2}$, $|1-z|e^{\bullet} = e^{-\cdots-\frac{z^k}{k}}$; the last term dominates.

   (b) (5.3) For any $\rho_0 < s < k+1$, $\left|\prod_{n=1}^{\infty} E_k\left(\frac{z}{z_n}\right)\right| > e^{-c|z|^s}$ when $z \notin \bigcup_n B_{|a_n|^{-k-1}}(a_n)$.
   Proof. Split into 3 products.
   $$\prod_{|z_n| \leq 2|z|} \left|1 - \frac{z}{z_n}\right| e^{-c'\left|\frac{z}{z_n}\right|^k} \prod_{|z_n| > 2|z|} e^{-c\left|\frac{z}{z_n}\right|^{k+1}}.$$

      i. First: use the fact that z is not in the balls.
      ii. Make $\prod e \to e^{\sum}$, separate out a factor $|z_n|^{-s}$.
      iii. Same idea.

   (c) (5.4–5.5) We have $\bigcup B \subseteq \{z : |z| \in I\}$ where I has finite measure. We can apply Cauchy to bound $\ln\left(\frac{f}{\text{Hadamard}}\right)$ on the valid radii, and get it's a poly of degree $\leq s$.

□