1 Algebra

1.1 Groups

1. A subgroup H of G is normal iff left cosets of H and right cosets of H coincide, i.e., for any $g \in G$, $Hg = gH$. If such is the case, then we can construct the quotient group G/H as the set of the cosets of H, with multiplication defined as $g_1H \cdot g_2H = g_1g_2H$. Then there is a natural group homomorphism from G to G/H sending g to gH, whose kernel is exactly H. The converse of the above statment is also true: if there is a group homomorphism from G to another group, then its kernel is a normal subgroup. Therefore, normal subgroups of G can be alternatively characterized as the kernels of group homomorphisms from G to another group.

   If we realize G as the fundamental group of a CW complex X, then normal subgroups of G correspond to regular covering spaces of X, and quotient groups of G correspond to their deck transformation groups.

2. Let $[G : H] = 2$. Then for any $g \in G$, $Hg = G\backslash H = gH$.

3. We use two facts. The first one is that the order of a subgroup (not necessarily normal) divides the order of the big group. The second one is that a normal subgroup is a union of conjugacy classes. Thus we can find the normal subgroups of A_4 by finding a sum of the order of its conjugacy classes that divides $|A_4| = 12$. Since the conjugacy classes of A_4 have orders 1 (the identity), 3 (two compositions of 2-cycles), 4 (3-cycles conjugate to (1 2 3)) and 4 (3 cycles cojugate to (1 2 4)) respectively, the only admissible sums are 1, 1 + 3 = 4 and 1 + 3 + 4 + 4 = 12. Therefore, A_4 has only three normal subgroups, $\{e\}$, A_4 itself, and the one containing the identity and 3 compositions of 2-cycles.

4. In S_3, the subgroup consisting of the identity and a 2-cycle is not normal. $SO_2(\mathbb{R})$ is not normal in $SL_2(\mathbb{R})$ as an orthonormal transformation in one basis may not be orthonormal in another basis.

5. No, as the following general construction shows. Let A be a normal group of B, and let C act on B by conjugation in a way that fixes B but does not fix A. Then B is a normal subgroup in the semidirect product

   $$G = B \rtimes C$$

   but A is not its normal subgroup.

   For example, let $A = \{0\} \times \mathbb{F}_2$ be a subgroup of $B = \mathbb{F}_2^2$, and $C = \mathbb{Z}/2 = \langle g \rangle$ act on B by conugation by letting $cbc^{-1} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)b$ for any $b \in B$, $c \in C$. Then $G = D_8$, as it is the only nonabelian group of order 8 that has a subgroup isomorphic to $\mathbb{F}_2^2$. B can be taken as the subgroup generated by horizontal and vertical reflections, and A can be taken as the subgroup generated by the horizontal reflection alone.

   (Note: if A is a *characteristic* subgroup of B (i.e., A is invariant under all automorphisms of B) and $B \trianglelefteq G$, then $A \trianglelefteq G$.)

6. A group G is solvable iff there is a composite sequence

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

such that all of the quotients G_i/G_{i-1} are abelian. S_3 is the (first) solvable nonabelian group, as it has a composite sequence

$$\{e\} \trianglelefteq A_3 \trianglelefteq S_3$$

whose successive quotients $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ are both abelian. A_4 is also solvable as it has a composite sequence

$$\{e\} \trianglelefteq (\mathbb{Z}/2\mathbb{Z})^2 \trianglelefteq A_4$$

whose successive quotients $(\mathbb{Z}/2\mathbb{Z})^2$ and $\mathbb{Z}/3\mathbb{Z}$ are both abelian.

By the Sylow theorems alone, we do not know the subgroup $(\mathbb{Z}/2\mathbb{Z})^2$ of A_4 is normal. This is because there are two other nonabelian groups of order 12: the dihedral group D_{12} (sometimes denoted by D_6) and the quaternion group Q_{12}, whose 2-Sylow groups are not normal. In fact both of them have three 2-Sylow groups conjugate to each other, which is also consistent with the Sylow theorems. However, if we use the "homomorphism trick", we can show the subgroup $(\mathbb{Z}/2\mathbb{Z})^2$ of A_4 is normal. Suppose not, then by the third Sylow theorem, it has three conjugate subgroups, on which A_4 acts by conjugation. Thus we have a homomorphism $\phi\colon A_4 \to S_3$. By the second Sylow theorem, the action is transitive, so the image of ϕ is either A_3 or S_3. If it is A_3, then $\ker\phi$ is a normal subgroup of A_4 of order 4, which contradicts our assumption. If it is S_3, then $\ker\phi$ is a normal subgroup of A_4 of order 2. However, there are 3 elements of order 2 in A_4, all conjugate to each other. Hence there is no normal subgroup of order 2 in A_4, which is also a contradiction.

7. The lower central sequence of a group G is a sequence

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n \cdots,$$

where $G_{n+1} = [G, G_n]$ for all $n \geq 0$. The upper central sequence of a group G is a sequence

$$\{e\} = Z_0 \trianglelefteq Z_1 \trianglelefteq \cdots \trianglelefteq Z_n \cdots,$$

where

$$Z_{n+1} = \{g \in G\colon \forall h \in G, [g,h] \in Z_n\}.$$

A group G is nilpotent iff its lower central sequence terminates at $\{1\}$, iff its upper central sequence terminates at G. Solvable groups are defined in 1.6.

8. Let g, $h \in G$. The commutator of g and h is defined as $g^{-1}h^{-1}gh$ (other conventions exist). The commutator subgroup of G is the subgroup generated by the commutators $[g,h]$ for all g, $h \in G$. It is a normal subgroup of G because for any $k \in G$, $k^{-1}[g,h]k = [k^{-1}gk, k^{-1}hk]$. The derived sequence of G is the composite sequence

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \cdots \trianglerighteq G^{(n)} \trianglerighteq \cdots,$$

where $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for all $n \in \mathbb{N}$.

A group G is solvable iff its derived series terminates at $\{e\}$. Want another "nontrivial" theorem about the derived series.

9.

10. The eigenvalues are quadratics integers, so if they are roots of unity their order divides 6. Answer: 1, 2, 3, 6, ∞.

11.

12.

13. We have $\sum_{C \text{ conjugacy class}} |C| = |G| = p^r$. Now $|C| \mid |G|$ because it is the orbit under the group action defined by conjugation. Thus if $|C| \neq 1$, then $p \mid |C|$. Since $|\{1\}| = 1$, there must be other conjugacy classes of 1 element, i.e., elements in the center.

14.

1.2 Representation theory

1. A representation is a homomorphism $\rho : G \to \mathrm{Aut}(V) \cong \mathrm{GL}_n(V)$. A subrepresentation is a G-invariant subspace $W \subseteq V$ (invariant means that $\rho(G)W \subseteq W$ (equality holds by invertibility)), along with $\rho : G \to \mathrm{Aut}(W)$. An irreducible representation is one with no proper subrepresentation.

   (Note there is an equivalence of categories:

   $$\left\{\begin{matrix} \text{Representatons } \rho \text{ of } G \text{ over } F \\ G\text{-invariant linear transformations } T \end{matrix}\right\} \xrightarrow{\cong} \left\{\begin{matrix} FG\text{-module} \\ \text{module homomorphism} \end{matrix}\right\}.)$$

   We can decompose finite-dimensional representations of finite groups into irreducible ones because of the existence of G-invariant inner product. Under this inner product, if W is a subrepresentation, so is $W^\perp$.

   Let $\langle \cdot, \cdot \rangle$ be any inner product. Then the following is G-invariant:

   $$\frac{1}{|G|} \sum_{g \in G} \langle gx, gy \rangle .$$

   (We need $|G| \perp \mathrm{char}(F)$ so that we can divide by $|G|$.)

2.

Theorem 1.1 (Maschke)**:** Suppose F has characteristic 0 or coprime to $|G|$, and ρ is a representation on V. Then V is a direct sum of G-invariant subspaces and ρ is a direct sum of irreducible representations corresponding to these subspaces.

See above for proof. What can go wrong when not coprime: $\mathbb{Z}/p \ni 1 \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/p)$. Over a real field it still works (?) but you may be able to reduce further in the algebraic closure (ex. rotation matrices).

3. See 1. The trace is $\mathrm{tr}(\rho(g))$.

4.

Lemma 1.2 (Schur)**:** Two versions.

(a) A homomorphism between simple modules $\varphi : A \to B$ is an isomorphism or 0.

(b) If ρ, ρ' are irreducible representations on V, V', and $T : V \to V'$ is G-invariant, then $T = 0$ or T is an isomorphism.

In particular, if $A = B$, then ρ is multiplication by a constant.

Proof. The kernel is 0 or A; the image is 0 or B. □

5. The following hold.

(a) Over $\mathbb{C}$ (or an algebraically closed field where $|G| \perp \mathrm{char}(F)$), the characters span the space of class functions: $\mathrm{span}\{\text{character } \chi\} = \mathrm{Fun}(\{\text{conjugacy class of } G\}, \mathbb{C})$.

(b) Define an inner product $\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_g \chi_1(g)\overline{\chi_2}(g)$. The irreducible representations form an orthnormal basis.

(c) More generally, $\langle \chi_U, \chi_V \rangle = \dim_{\mathbb{C}G} \mathrm{Hom}(U, V)$. Representations are isomorphic iff they have the same character.

Proof. (a) By Maschke, $A := \mathbb{C}G$ is semisimple, $A = \bigoplus n_i S_i$.

(b)

Theorem 1.3 (Wedderburn)**:** An algebra A is semisimple iff it is isomorphic with the direct sum of matrix algebras over division algebras. If F is algebraically closed, any semisimple algebra is isomorphic to a direct sum of matrix algebras over F.

Proof.

$$A \cong \bigoplus n_i S_i \tag{1}$$
$$A^{\mathrm{op}} \cong \mathrm{End}_A(A) \cong \bigoplus \mathcal{M}_{n_i}(\mathrm{End}_A(S_i)) \tag{2}$$
$$A \cong A^{\mathrm{opop}} \cong \bigoplus \mathcal{M}_{n_i}(\mathrm{End}_A(S_i)^{\mathrm{op}}). \tag{3}$$

Now $\mathrm{End}_A(S_i)^{\mathrm{op}}$ is a finite-dimensional division algebra, and the only such over an algebraically closed field is F (because otherwise, $\alpha \notin F$ is the root of a polynomial, since a division algebra is a domain and F is in the center, α coincides with the root already in F). □

(c) Write $\mathbb{C}G \cong \bigoplus_i^r \mathcal{M}_{n_i}(\mathbb{C})$. We calculate the center in 2 ways:

$$(\text{number of conjugacy classes}) = \dim_{\mathbb{C}} Z = r.$$

(d) Evaluate at I_{n_i} to show linear independence.

(e) We have

$$\dim_{\mathbb{C}G} U^G = \mathrm{Tr}\left(\frac{1}{|G|}\sum_{g\in G} g\right) = \frac{1}{|G|}\sum_{g\in G}\chi_U(g)$$

(f) Apply this to $\mathrm{Hom}_{\mathbb{C}G}(U,V) = \mathrm{Hom}(U,V)^G$ and use $\chi_{U^*\otimes V} = \chi_{U^*}\chi_V = \overline{\chi_U}\chi_V$.

□

In other words, the matrix $U = (\frac{|C_j|}{|G|}\chi_i(g_j))$ is unitary (g_j's are representatives of conjugacy classes C_j's). Considering U^*, we get $\frac{|G|}{|C|}$ (g,h in same cc C) $= \sum_\chi \chi(g)\overline{\chi}(h)$. An irreducible representation $\rho_1 \subseteq \rho_2$ iff $\langle \chi_1, \chi_2\rangle > 0$.

6. They are equal. See above.

7. The table $(\chi(g))$, χ irreducible, g representatives of conjugacy classes. Entries lie in $\mathbb{C}$.

8. Because the irreducible representations are a basis for the space of class functions. see above.

9. g has order 2. Otherwise, consider induced representations of $\langle g\rangle$, where g acts as any nth root of unity ($\mathrm{ord}\, g = n$).

10. The permutation representation of G on G, or the $\mathbb{C}G$-module $\mathbb{C}G$.

11. For A a FH-module, define the **(co)induced module** to be the FG-module

$$\mathrm{Ind}_H^G A = A \otimes_{FH} FG$$
$$\mathrm{Coind}_H^G A = \mathrm{Hom}_{FH}(FG, A).$$

For G finite, they are equivalent via $\varphi \mapsto \sum_{g\in G/H}\varphi(g^{-1})\otimes g$.

$$\begin{tikzcd}
\varphi \arrow[r] \arrow[d] & \sum \varphi(g^{-1}) \otimes g \arrow[d] \\
\varphi(\bullet h) \arrow[r] & \sum \varphi(g^{-1}h) \otimes g
\end{tikzcd}$$

12. See above.

13.

14.

15. We show something more general.

Theorem 1.4 (Adjoint associativity)**:** For an R-module M, (R, R')-bimodule N, and R'-module P, we have

$$\mathrm{Hom}_{R'}(M \otimes_R N, P) = \mathrm{Hom}_R(M, \mathrm{Hom}_{R'}(N, P)).$$

Hence, between the category of R-modules and R' modules, we have the adjoint functors

$$\text{if } R' \text{ is } R\text{-algebra}, \; \bullet \otimes_R R' \dashv \mathrm{Res}_R^{R'} \tag{4}$$

$$\text{if } R \text{ is } R'\text{-algebra}, \; \mathrm{Res}_{R'}^{R} \dashv \mathrm{Hom}_{R'}(R, \bullet). \tag{5}$$

Proof. The map is

$$\begin{aligned} \mathrm{Hom}_{R'}(M \otimes_R N, P) &= \mathrm{Hom}_R(M, \mathrm{Hom}_{R'}(N, P)) \\ f &\mapsto (m \mapsto (n \mapsto f(m \otimes n))) \\ (m \otimes n \mapsto (g(m))(n)) &\leftarrow g \end{aligned}$$

Now set $N = R'$ for the first adjunction and $N = R$ for the second. (Note $\mathrm{Hom}_{R'}(R', P)$ is P in the first because homomorphisms are multiplication by elements of P.) □

Letting R, R' be FG and FH we get the following.

Theorem 1.5 (Frobenius reciprocity)**:** Ind_H^G is the left adjoint functor to Res_H^G, and Coind_H^G is the right adjoint functor to Res_H^G. That is,

$$\begin{aligned} \mathrm{Hom}_{FG}(\mathrm{Ind}_H^G V, W) &\cong \mathrm{Hom}_{FH}(V, \mathrm{Res}_H^G W) \\ \mathrm{Hom}_{FG}(\mathrm{Res}_H^G W, V) &\cong \mathrm{Hom}_{FH}(W, \mathrm{Coind}_H^G W) \end{aligned}$$

are a natural isomorphisms. On characters, this means

$$\langle \chi, \psi|_H \rangle = \langle \chi^G, \psi \rangle .$$

16. No. Take $\{1\}$ and G. The induced representation is the regular representation.

17. kernels? Induction is right exact. (Tensoring is right exact.)

18. Yes, because their characters are equal.

19. $\mathrm{char}\, \mathbb{C} = 0$, and $\mathbb{C}$ is algebraically closed. Yes. If $\mathrm{char}(F) \mid |G|$, then Maschke fails.

20.

21.

22. Write the abelian group as $\prod \mathbb{Z}/n_i$. The reps are $\chi_{(a_i)} = \prod e\left(\frac{a_i x_i}{n_i}\right)$.

23. For $\mathbb{F}_q$, abstractly they are given by the above with $\mathbb{Z}/(q-1)$. Concretely they are

$$\chi_a = e^{2\pi i \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(ax)}$$

24.

1.3 Categories and Functors

1. The following universal property holds. For every R-module M, there is a bijection between bilinear maps $B : A \times B \to M$ and module homomorphisms $\varphi : A \otimes_R B \to M$. If B corresponds to φ, then $b(a,b) = \varphi(a \otimes b)$. What is this called?

2. If we're given an action of G on A, B, C then we can form the long exact sequences in homology and cohomology.

   If we're just given $1 \to A \to B \to C \to 1$ are abelian groups, we can tensor (over $\mathbb{Z}$) with G to and take the long exact sequence in homology $\mathrm{Tor}_n^{\mathbb{Z}}(G, A) H_n(G \otimes_{\mathbb{Z}} \bullet) = H_n(G \otimes_{\mathbb{Z}} P_\bullet)$ to get

$$\cdots \mathrm{Tor}_1^{\mathbb{Z}}(G,C) \to \mathrm{Tor}_0^{\mathbb{Z}}(G,C) \to \mathrm{Tor}_0^{\mathbb{Z}}(G,B) \to \mathrm{Tor}_1^{\mathbb{Z}}(G,A) \to 1.$$

   The sequence stops after $n = 1$ because every abelian group has a free resolution of length 2. Wikipedia says length 1. How do you define length anyway?

3. We have

$$\mathrm{Ext}_R^n(A,B) = H^n(\mathrm{Hom}_R(P_A, B)) = H^n(A, E^B).$$

   For abelian groups, we can take a free resolution of length 2

$$1 \to F_1 \xrightarrow{f} F_0 \to A \to 1,$$

   and take Hom,

$$1 \to \mathrm{Hom}(F_0, B) \xrightarrow{f^*} \mathrm{Hom}(F_1, B) \to 1.$$

   We have

$$\mathrm{Ext}_R^1(A,B) = \mathrm{Hom}(F_1, B)/\ker f^*.$$

   For $\mathrm{Ext}(\mathbb{Z}/m, G)$, $\mathbb{Z} = F_0 = F_1$ and f is multiplication by m, $\mathrm{Hom}(F_i, G)$ is G, so

$$\mathrm{Ext}(\mathbb{Z}/m, G) = G/mG$$

   Hence

$$\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/\gcd(m,n) \tag{6}$$
$$\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}) = \mathbb{Z}/m. \tag{7}$$

2 Real Analysis

2.1 Measure theory

1. Let $f : A \to B$ be a map between measure spaces. f is measurable if for every measurable set M, $f^{-1}(M)$ is measurable. This may look simple, but we must be careful with what σ-algebras we put on A and B. A function is *Borel measurable* if we put the σ-algebra of Borel sets on both A and B. It is *Lebesgue measurable* if we put the Borel algebra on B but the Lebesgue algebra on A. Thus, a Lebesgue measurable function may not be Borel measurable. For example, there are more Lebesgue measurable sets on $\mathbb{R}$ than Borel measurable sets. [1]

   This has subtle implications in terms of composition of measurable functions. The composition of two *Borel* measurable functions is again Borel measurable. However, the composition of two *Lebesgue* measurable functions need not be Lebesgue measurable. This is because the inverse preimage of an open set under the first function is only Lebesgue measurable, so we cannot say anything about its preimage under the second function.

   If we take $A = B = \mathbb{R}$, then there is a concrete example, [2] taken from `http://math.stackexchange.com/questions/283443/is-composition-of-measurable-functions-measurabl`

   Let F be the Cantor function

$$F(\sum_k a_k 3^{-k}) = \sum_k a_k 2^{-k}$$

   defined on the Cantor set C, and extended monotonically to $[0,1]$. Let $G(x) = x + F(x)$. Since G is continuous and strictly increasing, it is invertible, so G^{-1} is also continuous (hence measurable). Note $\mu(G(C)) = 2 - \mu(C) = 1$. By Vitali's theorem 2.1, there is a Lebesgue nonmeasurable set $D \subseteq G(C)$. Note $G^{-1}(D)$ is Lebesgue measurable because it is contained in C, which has measure 0. Now

$$(1_{G^{-1}(D)} \circ G^{-1})^{-1}(1) = G(G^{-1}(D)) = D$$

   is not measurable.

   The way this example works is that $1^{-1}_{G^{-1}(D)}(D) = G^{-1}(D)$, which is Lebesgue measurable but not Borel measurable, so its preimage under the continuous function G^{-1} fails to be Lebesgue measurable.

   Theorem 2.1 (Vitali)**:** Given any set $S \subset \mathbb{R}$ of positive measure, there exists a nonmeasurable subset of S.

   Proof. (See also p. 28 here: `http://web.mit.edu/~holden1/www/coursework/math/18125/notes.pdf`.) Since

$$\sum_{n \in \mathbb{Z}} m(S \cap [n, n+1)) = m(S) > 0,$$

[1] Assuming the Axiom of Choice, of course.

[2] Assuming the Axiom of Choice, of course.

there is some $n_0 \in \mathbb{Z}$ such that $m(S \cap [n_0, n_0+1)) > 0$. We may WLOG assume $n_0 = 0$, so that $S \subset [0, 1]$.

Consider the equivalence relation $x \sim y \iff x - y \in \mathbb{Q}$. Let $\mathcal{N}$ be a set with 1 representative from every class, as in Problem 1.3. We claim that one of the following sets in the union is not measurable.

$$\bigcup_{q\in\mathbb{Q}} S \cap (\mathcal{N} + q) = S.$$

Indeed, assume all sets in the union are measurable. Then

$$\sum_{q\in\mathbb{Q}} m(S \cap (\mathcal{N} + q)) = m(S) > 0.$$

Thus there is some q_0 such that $m(S \cap (\mathcal{N} + q_0)) > 0$. Since $S \subset [0, 1]$, $m([0, 1] \cap (\mathcal{N} + q_0)) > 0$. Since $x \to x + q \bmod 1$ is a measure preserving map on $[0, 1]$, for any $q \in \mathbb{Q}$, $m([0, 1] \cap (\mathcal{N} + q)) = m([0, 1] \cap (\mathcal{N} + q_0))$. (Take the addition mod 1?) Since $\mathcal{N} + q$ are disjoint for distinct q,

$$m([0, 1]) \geq \sum_{q\in\mathbb{Q}} m([0, 1] \cap (\mathcal{N} + q)) = \infty,$$

which is a contradiction. $\square$

2. Here is a sketch of how measure is defined on $\mathbb{R}$.
   (a) Define it for intervals $\mu((a, b)) = b - a$.
   (b) Extend it to the σ-algebra generated by open sets (note every open set is a countable union of open intervals). Recall

   Definition 2.2: A σ-algebra of E is a collection of subsets of E such that
   i. $E \in \mathcal{B}$.
   ii. $A \in \mathcal{B} \implies A^c \in \mathcal{B}$.
   iii. $A_i \in \mathcal{B} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$.

   This also implies a countable union is in the σ-algebra.
   (c) Extend to *Lebesgue* measurable sets. There are three ways to do this:
   i. Define sets of measure 0.

   Definition 2.3: A set of measure 0 is a set S such that

   $$\inf_{B\supseteq S \text{ Borel}} \mu(B) = 0.$$

   We may as well restrict to B being unions of disjoint intervals (so as to not require the previous definition of measure).

(A cardinality argument shows that not all sets of measure 0 are Borel.) Now define $\mu(A) = m$ if A differs from a Borel set of measure m by a set of measure 0.

ii. Let the Lebesgue measurable sets be such that the inner and outer measures are equal:

$$\mu^*(S) := \sup_{B \subseteq S} \mu(B) = \inf_{B \supseteq S} \mu(B) =: \mu_*(S)$$

We may as well restrict to compact sets on the inside and open sets on the outside (?).

iii. Let the Lebesgue measurable sets be such that

$$\mu^*(B) = \mu^*(B \backslash A) + \mu^*(B \cap A) \text{ for all Borel sets } A.$$

3. No, assuming the axiom of choice. The construction of a nonmeasurable set $\mathcal{N}$ uses the axiom of choice and rests on a simple equivalence relation among real numbers in $[0, 1]$. For $x, y \in [0, 1]$, the equivalence relation is defined as $x \sim y$ if and only if $x - y$ is rational. Then the set $[0, 1]$ can be written as disjoint union of all equivalence classes

$$[0, 1] = \bigcup_{\alpha} \mathcal{E}_\alpha.$$

With the axiom of choice, we can construct the set $\mathcal{N}$ by choosing exactly one element x_α from each $\mathcal{E}_\alpha$ and setting $\mathcal{N} = \{x_\alpha\}$. We will prove $\mathcal{N}$ is nonmeasurable. We prove by contradiction. Suppose that $\mathcal{N}$ is measurable. Let $\{r_k\}$ be the enumeration of rational numbers in $[0, 1]$ and construct $\mathcal{N}_k = \mathcal{N} + \{r_k\}$ by translation. Then $\{\mathcal{N}_k\}$ are disjoint sets and

$$[0, 1] \subset \bigcup_{k} \mathcal{N}_k \subset [-1, 2].$$

Indeed, if there exists an element in $\mathcal{N}_{k_1} \bigcap \mathcal{N}_{k_2}$, then we have $x_\alpha, x_\beta \in \mathcal{N}$ and rational numbers $r_{k_1} \neq r_{k_2}$ such that

$$x_\alpha - x_\beta = r_{k_2} - r_{k_1} \neq 0.$$

Thus, $\alpha \neq \beta$ and $x_\alpha - x_\beta$ is rational which contradicts the construction of $\mathcal{N}$. Thus $\{\mathcal{N}_k\}$ are disjoint. The two inclusions are verified by the construction. Then if $\mathcal{N}$ is measurable, we have

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3,$$

which is impossible.

4. By countable additivity, the measure of a countable set has 0 measure. An uncountable set can have 0 measure. For example, consider a Cantor set where we start from $[0, 1]$ and remove the middle half each time.

5. Yes. Recall:

Definition 2.4: A F_σ set is a countable union of closed sets, and a G_δ set is a countable intersection of open sets.

By Problem 1.6, $\mathbb{Q}$ is not G_δ. Taking the complement we know $\mathbb{R}\backslash\mathbb{Q}$ is not F_σ. To construct a set that is neither G_δ nor F_σ, take $A = ((-2,-1)\cap\mathbb{Q}) \cup ([1,2]\backslash\mathbb{Q})$. Indeed, if $A = \cap_{n\in\mathbb{N}} G_n$ is G_δ, then $(-2,-1)\cap\mathbb{Q}) = \cap_{n\in\mathbb{N}} G_n \cap (-2,-1)$ would also be G_δ. If $A = \cup_{n\in\mathbb{N}} F_n$ is F_σ, then $[1,2]\backslash\mathbb{Q} = \cup_{n\in\mathbb{N}} F_n \cap [1,2]$ would also be F_σ. These contradict Problem 1.6.

6. No. Assume the contrary that $\mathbb{Q}$ is a G_δ set. Since $\mathbb{R}\backslash\mathbb{Q} = \cap_{q\in\mathbb{Q}}\mathbb{R}\backslash\{q\}$ is G_δ, $\varnothing$ is also G_δ. This contradicts the Baire category theorem which says that any G_δ set in $\mathbb{R}$ is dense.

7. We show the existence of a Lebesgue measurable set that is not Borel measurable by a cardinality argument.

   Since the Cantor set (here we start from $[0,1]$ and remove the middle half each time) is a set of measure zero, any of its subset is Lebesgue measurable of measure zero. Since the Cantor set has cardinality c (the continuum), we have found 2^c Lebesgue measurable set.

   Next we show that there are at most c Borel measurable sets, using transfinite induction up to the first uncountable ordinal ω_1. This proof comes from `http://math.stackexchange.com/questions/70880/cardinality-of-borel-sigma-algebra`.

   We define a family $B_\alpha \subset \mathcal{P}(\mathbb{R})$ for all ordinals $\alpha \le \omega_1$ as follows. Let B_0 be the set of open intervals in $\mathbb{R}$. For each ordinal $\alpha < \omega_1$, let

   $$B_{\alpha+1} = \{\bigcup_{i\in I} A_i \bigcup \bigcup_{j\in J} A_j^c : |I|, |J| \le \aleph_0, A_i \in B_\alpha\}.$$

   (Not that we need it here, but how do we show the chain is strict inclusions?) If β is a limit ordinal, let

   $$B_\beta = \bigcup_{\alpha<\beta} B_\alpha.$$

   By transfinite induction, B_α is well defined for all $\alpha \le \omega_1$.

   Next we show $|B_\alpha| \le c$ for every ordinal $\alpha \le \omega_1$. Clearly $|B_0| = c$. Since each element of $B_{\alpha+1}$ is built out of countably many elements in B_α, $|B_{\alpha+1}| \le \aleph_0 \cdot |B_\alpha|^{\aleph_0}$. Then from $|B_\alpha| \le c$ we can deduce $|B_{\alpha+1}| \le c$. Finally, for any $\beta \le \omega_1$, $|\{\alpha : \alpha < \beta\}| \le |\omega_1| \le c$. So if $|B_\alpha| \le c$ for all $\alpha < \beta$, then $|B_\beta| \le c^2 = c$. By transfinite induction, $|B_{\omega_1}| \le c$.

   Finally we show that B_{ω_1} contains the Borel algebra. Since it contains B_0, the set of all open intervals, we only need to show that it is closed under countable unions and complements. Pick $A \in B_{\omega_1}$. Then $A \in B_\alpha$ for some $\alpha < \omega_1$, from which it follows that $A^c \in B_{\alpha+1}$, with $\alpha + 1 < \omega_1$, so $A^c \in B_{\omega_1}$. Pick $A_n \in B_{\omega_1}$ for $n \in \mathbb{N}$. Then $A_n \in B_{\alpha_n}$ for some $\alpha_n < \omega_1$. Since ω_1 is an uncountable ordinal, there is $\alpha < \omega_1$ such that $\alpha > \alpha_n$ for all $n \in \mathbb{N}$. Then $A_n \in B_\alpha$, and we know that $\cup_{n\in\mathbb{N}} A_n \in B_{\alpha+1} \subset B_{\omega_1}$.

   To conclude, the Borel algebra on $\mathbb{R}$ is contained in a countable set B_{ω_1}, so it is strictly contained in the Lebesgue algebra.

8. Let $(X, \mathcal{B}, \mu)$ be a measure space. The completion of $\mathcal{B}$ is the σ-algebra $\bar{\mathcal{B}}$ generated by $\mathcal{B}$ and all sets that are contained in a set of measure zero. Then μ extends uniquely to a measure $\bar{\mu}$ on $\bar{\mathcal{B}}$, which is called the completion of μ. On $\mathbb{R}$, for example, if $\mathcal{B}$ is the Borel algebra, then $\bar{\mathcal{B}}$ is the Lebesgue algebra.

9.

   Theorem 2.5: Let K be compact Hausdorff. Let $C(K)$ be the Banach space of continuous functions $K \to \mathbb{C}$ on K with the sup norm. Then there is a bijection between $M(K) = C(K)^*$ and regular complex Borel measures on K, given by

$$C(K)^* \xrightarrow{\cong} \{\text{regular complex Borel measures}\} \tag{8}$$

$$\varphi(f) = \int_K f \, d\mu \leftarrow\!\shortmid \mu. \tag{9}$$

   The associated φ and μ satisfy $\|\varphi\| = \|\mu\|_1$.

   Here is a summary of the proof (taken from notes here: `https://www.sharelatex.com/project/52d37dcd582ebcfe2f00169e`)

   (a) Pf. Consider $\varphi \in M^+(K)$ first. We'd like to define $\mu(E) = \varphi(1_E)$ but 1_E is not continuous; approximate it using Urysohn: $\mu^*(U) = \sup\{\varphi(f) : f \ll U\}$ and $\mu^*(A) = \inf\{\mu^*(U) : U \supseteq A\}$.

   (b) Countable subadditivity for open sets: use partitions of unity. For arbitrary sets: approximate with opens (the $\sum_n \varepsilon_n = \varepsilon$ trick).

   (c) We show superadditivity: $\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \backslash U)$ for all A and open U, to get a measure on all Borel sets.

   (d) Show $\varphi(f) = \int_K f \, d\mu$ by taking a partition of unity h_i on sets U_i where f is approximately constant, and taking φ of $f = \sum f h_i$.

   (e) For arbitrary φ: Break up $\varphi = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4$. To show $\|\varphi\| = \|\mu\|_1$, given $\bigsqcup E_i$, use PoU to get functions on U_i close to E_i, and take a linear combination of them.

2.2 Lebesgue integration

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8. Yes: define for $N \in \mathbb{Z}_+$,

$$f_N(x) = \begin{cases} f(x), & \text{if } |f(x)| \le N, \\ 0, & \text{otherwise.} \end{cases}$$

We easily check that the functions f_N are measurable, and by the monotone convergence theorem, $||f - f_N||_{L^1(X)} \xrightarrow{N \to \infty} 0$. So for any measurable set $E \subset X$,

$$\begin{aligned} \int_E |f| &= \int_E |f - f_N| + \int_E f_N, \\ &\le ||f - f_N||_{L^1(X)} + Nm(E). \end{aligned}$$

So given $\epsilon > 0$, choose N so $||f - f_N||_{L^1(X)} < \epsilon/2$ and then choose $\delta > 0$ so $N\delta < \epsilon/2$.

9.

10.

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12.

13. It suffices to consider $x \ge 0$. Divide the integral into 2 pieces.

    (a) $\int_0^1$: for $s \le 1$, for $1 < s < 2$, the integral converges (use $x \sim \sin x$), for $s \ge 2$ the integral diverges.

    (b) $\int_1^\infty$: it is Lebesgue integrable when $s > 1$ but not otherwise. It is Riemann integrable for any $s > 0$ by this criterion: if $f(x)$ is positive and decreasing and $\int_a^b g(x) < C$ for any a, b, then $\int_0^\infty f(x)g(x)\,dx < \infty$.

    Conclusion: for $0 < s \le 1$ it converges for Riemann and and diverges for Lebesgue; for $1 < s < 2$ it converges, for $s \ge 2$ it diverges. (For $s = 0$ the integral is not defined.)

14. This question has likely been incorrectly stated. It appears in the real section here: `http://web.math.princeton.edu/generals/fung_francis`. Based on the examinee responses there, the condition was instead probably

$$\int_1^\infty f(x)x^{-n}dx = 0 \text{ for all } n \ge 2.$$

Change variables, $x \mapsto 1/t$, to obtain

$$\int_0^1 g(t)t^n dt = 0 \text{ for all } n \ge 0,$$

where $g(t) = f(1/t)$. Setting $n = 0$ we see that $g \in L^1([0,1])$. Now suppose that $g(t_0) \ne 0$, say $g(t_0) > 0$ for a $t_0 \in (0,1]$. By the Stone-Weierstrass theorem we may choose a sequence of polynomials P_N converging uniformly in $[0,1]$ to ρ, a smooth bump function supported in a neighborhood B about t_0 in which g is positive. So $\int_0^1 g(t)P_N(t)dt = 0$ by assumption, but by the dominated convergence theorem,

$$\int_0^1 g(t)P_N(t)dt \xrightarrow{N\to\infty} \int_0^1 g(t)\rho(t)dt > 0,$$

a contradiction.

2.3 Fourier transform and Fourier series

1. If we consider $\mathbb{T}$ as

   (a) $[0,1]$, then the Fourier series is

   $$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}, \qquad a_n = \int_0^1 f(x)e^{-2\pi i n x}\, dx.$$

   (b) the circle in the complex plane, then the Fourier series is (make the substitution $z = e^{2\pi i x}, dz = 2\pi i z\, dz$ from the above)

   $$\sum_{n=\infty}^{\infty} a_n z^n, \qquad a_n = \int_0^1 f(z)z^{-n-1}.$$

   Note this is just the Laurent series around the origin.

   The Riemann-Lebesgue lemma says: if $f \in L^1(\mathbb{R})$ then $\lim_{t\to\pm\infty} \widehat{f}(t) = 0$. Proof sketch: first prove this for compactly supported C^1 functions using the idea in the next problem (but for $\mathbb{R}$). Now use the fact that every function in L^1 is a limit of C^1 functions in the L^1 norm (why?), and $\widehat{\bullet}$ has norm 1 on L^1 functions. For the Fourier series version, think of f as a function on $[0,1]$ and let $t \in \mathbb{Z}$.

2. The Fourier transform switches smoothness and decay. By integration by parts,

   $$\int_0^1 f^{(k)}(x)e(-nx)\, dx = \frac{1}{-2\pi i n}\int_0^1 f^{(k+1)}(x)e(-nx).$$

   Thus by induction, if f is C^k, then $\widehat{f}(n)$ decays as $O\left(\frac{1}{n^k}\right)$.

3.

4.

5. From the second question in this section, the Fourier series of a smooth function f has coefficients that are $O(\frac{1}{n^2})$, for example. So its Fourier series converges uniformly to a smooth function with the same Fourier coefficients as f (switch the sum and integral when computing the Fourier coefficients of the series using uniform convergence of the series), and the convergence of the Fourier series to f follows by the uniqueness of Fourier coefficients for continuous functions.

To see this uniqueness, suppose f is a nonzero continuous function on $[-\pi,\pi]$ with $\hat{f}(n)=0$ for all n. We may assume $f(0)>2$, and integrate against an "approximation to the identity" constructed with trigonometric polynomials to derive a contradiction. Explicitly, there exists a $\delta>0$ so that $f(x)>1$ if $|x|<\delta$; define $P(x)=\cos x+\epsilon$ for so that $P(x)>1$ if and only if $|x|<\delta$. Then, we obtain the following contradiction:

$$\begin{aligned} 0 &= \int_{-\pi}^{\pi} f(x)P(x)^N dx, \\ &= \underbrace{\int_{|x|<\delta} f(x)P(x)^N dx}_{\xrightarrow{N\to\infty}\infty} + \underbrace{\int_{\delta\le|x|\le\pi} f(x)P(x)^N dx}_{\xrightarrow{N\to\infty} 0 \text{ by dom. conv.}}. \end{aligned}$$

6.

7. We will use the Banach-Steinhaus theorem:

   Let X be a Banach space, Y a normed vector space, and let $\{T_a\}_{a\in A}$ be a family of bounded linear operators from X to Y. Then either $\sup_a ||T_a|| < \infty$ or there exists $x\in X$ such that $\sup_a ||T_a x||_Y = \infty$.

   Now let $X=C(\mathbb{T})$ with the norm $||\cdot||_\infty$ and let $Y=\mathbb{C}$. Define $T_N : X\to Y$ by

$$T_N f = S_N f(0) = \int_{-1/2}^{1/2} f(t) D_N(t) dt,$$

   where

$$D_N(t) = \sum_{k=-N}^{N} e^{2\pi i k t} = \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}$$

   is the Dirichlet kernel. And we can easily prove that

$$||T_N|| = L_N = \int_{-1/2}^{1/2} |D_N(t)| dt.$$

   Now it suffices to prove

$$L_N \to \infty \quad as \quad N\to\infty.$$

   This result is derived from a direct computation:

$$\begin{aligned} L_N &= 2\int_0^{1/2} \left|\frac{\sin(\pi(2N+1)t)}{\pi t}\right| dt + O(1) \\ &= 2\int_0^{N+1/2} \left|\frac{\sin(\pi t)}{\pi t}\right| dt + O(1) \\ &= 2\sum_{k=0}^{N-1}\int_k^{k+1} \left|\frac{\sin(\pi t)}{\pi t}\right| dt + O(1) \\ &= \frac{2}{\pi}\sum_{k=0}^{N-1}\int_0^1 \frac{|\sin(\pi t)|}{t+k} dt + O(1) \\ &= \frac{2}{\pi}\int_0^1 |\sin(\pi t)| \sum_{k=1}^{N-1}\frac{1}{t+k} dt + O(1) \\ &= \frac{4}{\pi^2}\log N + O(1). \end{aligned}$$

Hence, we prove desired result.

8.

9.

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11.

12. Given a function $f \in L^1(\mathbb{R})$, define its Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi} dx.$$

13.

14. The Riemann-Lebesgue lemma is

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$$

With direct computation, we derive

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x} dx \\ &= -\int_{\mathbb{R}} f(x)e^{-2\pi i\xi(x+1/2\xi)} dx \\ &= -\int_{\mathbb{R}} f(x-1/2\xi)e^{-2\pi i\xi x} dx.\end{aligned}$$

Thus,

$$|\hat{f}(\xi)| \le \int_{\mathbb{R}} |f(x) - f(x-1/2\xi)| dx.$$

Now, we easily derive

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$$

2.4 Functional Analysis

1.

2. Any two separable Hilbert spaces are isometric. Thus, for example, the spaces

$$\ell_2, \quad L_2[0,1], \quad L_2(\mathbb{T}^n), \quad L^2(\mathbb{R}^n), \quad W^{m,2}(\mathbb{R}^n)$$

are all isometric.

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10. A Banach algebra homomorphism $\phi : B \to \mathbb{C}$ is necessarily bounded, with norm less or equal to 1. Suppose, for contradiction, that there is an element $b \in B$ such that $|\phi(b)| > \|b\|$. Set $b_1 = \frac{b}{|\phi(b)|}$. Then $\|b_1\| < 1$ but $|\phi(b_1)| = 1$. However,

$$s_n = b_1 + b_1^2 + ... + b_1^n$$

converges to an element $b_0 \in B$ (here we use the completeness of B). However,

$$b_1 s_n = s_{n+1} - b_1,$$

thus, sending n to infinity, we get

$$b_1 b_0 = b_0 - b_1.$$

Thus,

$$\phi(b_1)\phi(b_0) = \phi(b_0) - \phi(b_1)$$

which is of course impossible, since $\phi(b_1) = 1$.

2.5 L^p spaces

1.

2. Hölder's inequality: if $f \in L^p(X)$ and $g \in L^q(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p \in [1, \infty]$ then $fg \in L^1(X)$ and $||fg||_{L^1(X)} \leq ||f||_{L^p(X)} ||g||_{L^q(X)}$.

It is immediate that the inequality is true for $p = 1$, and that equality holds if and only if $g(x) = c$ for some constant c for a.e. x such that $f(x) \neq 0$.

Hölder's inequality can be proved for $p \in (1, \infty)$ using Young's inequality, which states for a, b positive numbers and p, q as before:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Young's inequality can be shown using the convexity of the exponential function:

$$ab = \exp(\frac{1}{p} \log a^p + \frac{1}{q} \log b^q) \leq \frac{1}{p} \exp(\log a^p) + \frac{1}{q} \exp(\log b^q).$$

To then show Hölder's inequality, let $a = |f|/||f||_{L^p(X)}$ and $b = |g|/||g||_{L^q(X)}$, and integrate over X to see that $||(f/||f||_p)(g/||g||_q)||_1 \leq 1$. From the proof, we see that equality is obtained in Young's inequality exactly when $a^p = b^q$. So equality in Hölder's inequality, $p \in (1, \infty)$ occurs exactly when one of $|f|^p$ and $|g|^q$ is a scalar multiple of the other a.e.

3.

4.

5.

6. The inclusions of ℓ^p is $\ell^p \subset \ell^q$ for any $1 \leq p < q \leq \infty$. To see this, suppose $\{a_n\} \in \ell^p$, or $\sum_n |a_n|^p < \infty$. Then in particular, $a_n \to 0$ as $n \to \infty$. Thus there is $N \in \mathbb{N}$ such that $a_n < 1$ for all $n > N$. As such, we have

$$\sum_n |a_n|^q \leq \sum_{n \leq N} |a_n|^q + \sum_{n > N} |a_n|^p < \infty,$$

so $\{a_n\} \in \ell^q$.

The reverse inclusion need not hold. For example, let $a_n = n^{-2/(p+q)}$. Then $\{a_n\} \in \ell^q \backslash \ell^p$.

7.

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10. Consider the functions

$$f_n(x) = (-1)^{\left\lfloor \frac{x}{2^n} \right\rfloor}.$$

Then $\|f_i(x) - f_j(x)\|_{L^1} \geq \frac{1}{2}$. They cannot converge to any function, so $L^1[0,1]$ is not sequentially compact, and hence not compact.

We can show the more general theorem.

Theorem 2.6: Let X be a normed space. If the closed unit ball $B_1(X)$ is compact, then X is finite-dimensional.

Proof. Inductively find a sequence of points whose distance is bounded from below. Suppose you've found $x_1, \ldots, x_n$. Let $V_n = \text{span}(x_1, \ldots, x_n)$. There exists y such that $\inf_{x \in V_n} \|y - x\| = 1$. There exists $x \in V_n$, $\|y - x\| < 1 + \varepsilon$. Let $x_{n+1} = \frac{y-x}{\|y-x\|}$. We get

$$d\left(\frac{y-x}{\|y-x\|}, V_n\right) > 1 - \varepsilon.$$

Choose ε small. (This is the argument of Riesz's lemma.) $\square$

2.6 Different kinds of convergence

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5. It is not true. Consider the sequence of functions $\{f_n\}$, where f_n is zero outside $[1/2n, 1/n]$ and its graph on this interval is exactly the triangle with vertices $(1/2n, 0)$, $(1/n, 0)$ and $(3/4n, n)$. One can easily check that $f_n \to 0$ pointwise, whereas $\int_0^1 f_n = 1/4$ for every n.

6. The answer is positive in the finite case and negative in the infinite case. If $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$ and $f_n \to f$ uniformly on X, then

$$\left\| \int_X f_n - \int_X f \right\| \leq \mu(X) \|f_n - f\|_\infty \longrightarrow 0.$$

Thus,

$$\int_X f_n \longrightarrow \int_X f.$$

For the infinite case, consider just the sequence of constant functions $f_n(x) = \frac{1}{n}$ defined on $\mathbb{R}$. Then $f_n \to 0$ uniformly on $\mathbb{R}$, but

$$\int_{\mathbb{R}} f_n = \infty \neq 0.$$

7. Yes, the convergence is indeed uniform. Suppose $f_n \to f$ pointwise in the interval $[a, b]$ and $|f_n'(x)| \leq M$ for every n and $x \in [a, b]$. We will need the Arzela-Ascoli Theorem. ' **Theorem.** (Arzela-Ascoli) Let K a compact metric space and a subset $A \subseteq C(K)$.

A is compact if and only if it is closed, bounded and equicontinuous.

From the mean-value theorem, all the functions f_n are $M-$Lipschitz, thus the set $\{f_n : n = 1, 2, ...\}$ is equicontinuous. It is also bounded, since for any n and $x \in [a, b]$ there exists an $\xi_n \in (a, x)$ such that

$$\begin{aligned} |f_n(x)| &\leq |f_n(x) - f_n(a)| + |f_n(a)| \\ &= |f_n'(\xi_n)||x - a| + |f_n(a)| \\ &\leq M|b - a| + |f_n(a)|, \end{aligned}$$

which is of course (uniformly) bounded, since $\{f_n(a)\}$ is also convergent. Thus, the set

$$A = \overline{\{f_n : n = 1, 2, ...\}}$$

is closed, bounded and equicontinuous, thus compact. So, there exists a subsequence $\{f_{k_n}\}$ of $\{f_n\}$ such that $f_{k_n} \to f$ uniformly.

We are now done, since, if we suppose that f_n does not converge uniformly to f, then there exists an $\varepsilon > 0$ and a subsequence $\{f_{k_n}\}$ of $\{f_n\}$ such that $\|f_{k_n} - f\|_\infty \geq \varepsilon$. But then, you can apply the previous result to the sequence $\{f_{k_n}\}$ and this is a contradiction.

3 Complex Analysis

3.1 Basic complex analysis

1. Write a complex function as $f(x+iy) = F(x,y) + iG(x,y)$. The fact that h is holomorphic gives that
$$\frac{F_y + iG_y}{i} = F_x + iG_y \tag{10}$$
$$\implies -F_y = G_x \tag{11}$$
$$G_y = F_x. \tag{12}$$
We can also write this as $\frac{\partial f}{\partial \bar{z}} = 0$ where $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

2. The radius of convergence of the power series of meromorphic f at z_0 is $\sup\{r : f$ has no pole with $|z -$
Here the poles are ± 1. Answer: $\sqrt{10001}$.

3. $\int_0^1 f(e^{2\pi it}) 2\pi i e^{2\pi it}\, dt$ exists.

4. Counterclockwise outside, clockwise inside.

5.

Theorem 3.1 (Stokes)**:**
$$\oint \vec{f} \cdot d\vec{r} = \iint_A (\vec{\nabla} \times \vec{f}) \cdot d\vec{n}.$$
When A is in the xy-plane, we get Green's Theorem.
$$\int f_x\, dx + f_y\, dy = \iint_A \left(-\frac{\partial}{\partial y} f_x + \frac{\partial}{\partial x} f_y\right) dA.$$
For Cauchy's Theorem, see the next problem.

6.

Theorem 3.2 (Goursat)**:** Let R be a rectangle, and suppose f is holomorphic in R. Then
$$\int_{\partial R} f(z)\, dz = 0.$$
(Note: We can only use the fact that f is once-differentiable in the proof.)

Proof. Two steps.

(a) Taylor expand f. For any z_0, we have (the last function depending on z_0)
$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \underbrace{h(z - z_0)}_{o(z-z_0)}.$$
Verify that the integral of $f(z_0) + f'(z_0)(z - z_0)$ (linear term) around the rectangle is 0.

(b) Suppose the integral is not 0. Subdivide the rectangle into 2^{2n} rectangles, and split the integral to be over these rectangles. At each stage take the rectangle which gives the largest integral. Let z_0 be the intersection of the rectangles. As $n \to \infty$, the integral is $\geq c\frac{1}{2^{2n}}$, but because $h = o(z)$, it is also $o\left(\frac{1}{2^n}\right)\frac{1}{2^n}$, contradiction.

$\square$

7. After proving Goursat:

   (a) Cauchy's theorem for circles: Can define integral $g(z) = \int_{z_0}^{z} f(u)\, du$ along a rectangle. By Goursat, which rectangle doesn't matter.

   (b) It holds for the union of circles. Now "cover the homotopy with circles."

8. $f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{z-z_0}\, dz$.

9. Use the above with z_0 equal to the point, and the integral on the boundary.

10. Assuming its $(n+1)$th derivative is nonzero, it's in the form $a_{n+1}z^{n+1} + \cdots$. This looks like z^{n+1}, which is $(n+1)$-to-1 except at the origin. (In fact we can choose two charts so it is z^{n+1}.

11.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta} \frac{1}{1-\frac{z}{\zeta}}\, d\zeta \\ &= \frac{1}{2\pi i} \int_{C_r} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \zeta^n\, d\zeta \\ &= \sum_{n=0}^{\infty} \left(\left(\frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \right) z^n \right) \end{aligned}$$

12. Real-analytic means that the function is equal to its power series everywhere locally. (In $\mathbb{R}^2 \to \mathbb{R}^2 \cong \mathbb{C}$, this means that both the x, y coordinates of $f(x, y)$ are power series in $x^j y^k$.) Just substitute $z = x + iy$.

13. Laurent series are $f(z) = \sum_{n \geq L} a_n z^n$. (Can replace z by $z - z_0$ on the right.) We have

$$a_n = \frac{1}{2\pi i} \int \frac{f(z)}{z^{n+1}}\, dz.$$

14. Laurent series converge in $(\limsup_{n\to\infty} a_{-n}^{\frac{1}{n}}, \limsup_{n\to\infty} a_n^{-\frac{1}{n}})$. They converge uniformly in any compact subset in this annulus. (Proof: geometric series formula.) They fail to converge outside the closure. They may or many not converge on the boundary.

15. Given this information, it's $\frac{A}{z-1} + \frac{B}{z-2i} + g(z)$, g entire. (Assume simple poles. For nth degree poles replace $\frac{1}{z-c}$ by $P\left(\frac{1}{z-c}\right)$ with P of degree n.) The power series converges for $|z| < 1$.

16. $e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k! z^k}$.

17. It is a_{-1} in the Laurent expansion around z_0. It relates poles with integrals around closed curves.

18. $\oint_{\partial R} f(z)\, dz = 2\pi i \sum_{z_0} W(\gamma, z_0) \operatorname{Res}_{z_0} f$.

19. There are 4 poles. Let the residues of $e^{\frac{2\pi i(2k-1)}{8}}$ be r_k. Let U_r be the upper semicircle of radius r. By Cauchy

$$\int_{-r}^{r} (1+x^4)^{-1}\, dx + \int_{U_r} (1+x^4)^{-1}\, dx = \frac{1}{2\pi i}(r_1 + r_2).$$

The residue is $r_k = \lim_{x \to r_k} \frac{x - r_k}{x^4+1} = \frac{1}{(x^4+1)'}|_{x=r_k}$. Let $r \to \infty$. Then the second term is 0 so the answer is

$$\frac{1}{2\pi i}(r_1 + r_2) = \frac{1}{2\pi i}\left(\frac{1}{4r_1^3} + \frac{1}{4r_2^3}\right) = -\frac{1}{16\pi}$$

20. Same idea. The multiplicity doesn't make a difference. (Warning: having a double pole does not mean the residue is 0.)

21. Same idea!

22. Note the map $i\frac{1-z}{1+z}$ (inverse $\frac{i-w}{i+w}$) takes $\mathbb{D}$ to $\mathcal{H}$ and $(-i, 1, i, -1)$ to $(-1, 0, 1, \infty)$. We have

$$\int_1^{z'} \frac{1}{\sqrt{(1-z^2)(1+z^2)}}\, dz = \int_0^{\frac{i-w'}{i+w'}} \frac{1}{\sqrt{\left(1 - \left(\frac{i-w}{i+w}\right)^2\right)\left(1 + \left(\frac{i-w}{i+w}\right)^2\right)}} \frac{-2i}{(i+z)^2} dw$$
$$= \int_0^{\frac{i-w'}{i+w'}} \frac{-i}{\sqrt{iz(z^2-1)}}\, dw$$

which is now in standard form and can be found to be the inverse of an elliptic function.

23. Another question: $\int_{-\infty}^{\infty} \frac{\sin z}{z}\, dz$.

It might be easier to integrate along a rectangular contour.

We try to integrate $\frac{e^{iz}}{z}$. If $z = x + iy, \frac{e^{iz}}{z} = \frac{e^{-y}e^{ix}}{z}$. Let/find

$$I(r, \theta_1, \theta_2) := \int_{\gamma = re^{i\theta}, \theta \in [\theta_1, \theta_2]} \frac{e^{iz}}{z}\, dz = \int_{\theta_1}^{\theta_2} ie^{-r\sin\theta} e^{ir\cos\theta}\, d\theta.$$

Choose $\frac{1}{r} \prec \theta(r) \prec 1$ as $r \to \infty$. Integrate over the upper half-annulus with inner radius $r_1 \to 0$ and outer radius $r_2 \to \infty$. The total integral is 0 by Cauchy.

(a) Outer integral. Split into 3 parts.

i.
$$I(r,\theta(r),\pi-\theta(r)) \le \pi \max |e^{r\sin\theta}| = 2\pi e^{-r\sin\theta(r)} \to 0.$$

ii.
$$I(r,0,\theta(r)) + I(r,\pi-\theta(r),\pi) \le 2\theta(r)\cdot 1 \to 0.$$

(b) Inner integral.
$$I(r,\pi,0) = \int_\pi^0 ie^{-r\sin\theta}e^{ir\cos\theta}\,d\theta \to -\pi i$$
because $e^{-r\sin\theta+ir\cos\theta} \to 1$ uniformly in θ as $r\to 0$.

(c) Thus $\lim_{r_2\to\infty, r_1\to 0^+} \int_{(-r_2,r_1)\cup(r_1,r_2)} \frac{e^{iz}}{z}\,dz = 0$. Taking the imaginary part gives $\int_{-\infty}^\infty \frac{\sin z}{z} = \pi$.

Our question is easier. Integrate $\int_{-\infty}^\infty \frac{e^{2iz}-1}{z^2}\,dz$. Here $I(r,\theta_1,\theta_2) = \int_{\theta_1}^{\theta_2} \frac{ie^{-2r\sin\theta}e^{2ir\cos\theta}}{re^{i\theta}}\,d\theta$. The outer circle gives 0 because we have r in the denominator. The inner circle gives, using $e^{2iz} = 1+2iz+O(z^2)$, $-(2i)i\pi = 2\pi$. We get the same answer for $e^{-2iz}-1$ integrating below the x-axis. Adding and dividing by 4, we get π. factor of i off?

24. (Following [SS] Exer. 2.1) Idea: we know $\int_0^\infty e^{-x^2}\,dx$, so integrate along a boundary where one of the sides gives this, and another gives $\int e^{-ix^2}$.

Integrate e^{-z^2} along the wedge from 0 to $\frac{\pi}{4}$. By Cauchy, the integral is
$$\begin{aligned} 0 &= \lim_{r\to\infty}\left(\int_0^r e^{-z^2}\,dz - \int_0^{re^{\frac{\pi i}{4}}} e^{-z^2}\,dz + \int_0^{\frac{\pi}{4}} e^{i(re^{i\theta})^2}\, ire^{i\theta}\,d\theta\right)\\ &= \frac{\sqrt{\pi}}{2} - e^{-\frac{i\pi}{4}}\int_0^\infty e^{iz^2}\,dz + \lim_{r\to\infty} L_r \end{aligned}$$

We have
$$|L_r| \le \int_0^{\frac{\pi}{4}} e^{-r^2\sin(2\theta)} r\,d\theta \le \int_0^{\frac{\pi}{4}} e^{-cr^2\theta} r\,d\theta \le \frac{1}{cr}(1-e^{-cr\frac{\pi}{4}}) \to 0.$$

Thus
$$\int_0^\infty e^{-iz^2}\,dz = \frac{\sqrt{\pi}}{2\sqrt{2}} + \frac{\sqrt{\pi}}{2\sqrt{2}}i = \left(\int_0^\infty \cos(x^2)\,dx\right) + i\left(\int_0^\infty \sin(x^2)\,dx\right)$$

25. $W(\gamma,z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz$. Locally the integral is $\ln(z-z_0)$, so it's an integer.

26. Argument principle:
$$W(f\circ\gamma,0) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = 2\pi i \sum_{z_0} W(\gamma,z_0)\,\mathrm{ord}_{z_0} f.$$

The winding number of $f\circ\gamma$ around 0 is the same as the total number of roots of f in γ (minus poles).

27. Expand in power series and use Cauchy.

28. See 26.

29.

Theorem 3.3 (Rouché)**:** If $|g| < |f|$ on the boundary of K, then $f, f+g$ have the same number of zeros in K.

Proof. Use the argument principle on $\frac{g}{f}$ which we want to have equal number of poles and roots. $\left|\frac{g}{f} - 1\right| < 1$ on γ so it does not wind around the origin. $\square$

30.

31.

Theorem 3.4: A continuous map from the unit disc to itself has a fixed point.

Proof. Else there is a deformation retract by projecting z to the boundary in the direction of $f(z)$. $\square$

32. Use Rouché's theorem. (Is there a simpler way to do this?) Consider a small circle around a zero. Changing f slightly, it still has the same number of zeros in the circle.

33. If U is a connected open subset of $\mathbb{C}$ and f is nonconstant holomorphic on U, then f is an open map on U.

34.

Theorem 3.5 (Jensen's formula)**:** Suppose f is holomorphic in $\overline{B_r}$. Let ρ_k be the absolute values of the zeros with $|\cdot| \le r$. Then

$$|f(0)| \le \frac{\rho_1 \cdots \rho_n}{r^n} \max_{|z|=r} |f(z)|.$$

By Schwarz, $\max_{|z|=1} f \le 1$. Thus $|f(0)| \le \left(\frac{1}{3}\right)^{50}$.

35. If f, g are equal on a set with an accumulation point in Ω, then $f \equiv g$. Reduce to the case where $g = 0$. Proof: if $f(z) - f(z_0)$ has a 0 of order $k < \infty$ at z_0, then it is k-to-1 locally. But it's ∞-to-1, so it's 0.

36. Infinitely many, for example, the zeros of $\sin\left(\frac{1}{1-x}\right)$ have limit point 1. (An entire function can only have finitely many though, because zeros form a discrete set.)

37. It's 0. Everything in its power series is 0.

38. If f is holomorphic, $f \equiv 5$ by uniqueness.

    For f harmonic: A linear function works. (It's $\Re(5 + ce^{i\frac{\pi}{4}}z)$.) How to find all such functions?

39.

    Theorem 3.6 (Morera)**:** If f is continuous in Ω and $\int_\gamma f = 0$ for every closed curve γ in Ω, then f is analytic.

    Proof. It's the derivative of a function F. Since F has a derivative, it's analytic; so is F'. □

40. See 1.6. A triangle can be treated similarly as a rectangle.

41. Yes. We have $f = \oint \frac{f_n}{\zeta - z}\, d\zeta$ whose derivative converges uniformly. This is false for C^∞ real functions (as uniform convergence does not imply uniform convergence of derivatives). Take functions converging to a continuous function with cusps.

42. Nothing. Take $\frac{1}{n}\sin(nx)$. We need convergence on an open set in order to use the integral formula.

43. This is
    $$L(1, \chi_4) = \sum_{n \ge 0} (-1)^n (2n+1)^{-s}.$$
    It can be continued to an entire function using a functional equation. Note $\mathbb{Q}[i]$ has discriminant -4. We have
    $$L(1, \chi_4) = \mathrm{Res}_{z=1}\, \zeta_{\mathbb{Q}(i)} = \frac{2\pi(h_K = 1)\cancel{\mathrm{Reg}_K}}{\sqrt{|\Delta_K| = 4}(w_K = 4)} = \frac{\pi}{4}.$$
    See Ch. 35 of number theory notes. `https://github.com/holdenlee/number-theory`

44. If it's compact-locally uniformly convergent. (Counterexample seems kind of difficult. See `http://mathoverflow.net/questions/117633/a-question-about-the-limit-of-a-sequen`

45. Vitali convergence?

46. We have $\iint_{z=x+iy,|z|<1} |f(x+iy)|\, dx\, dy \ge \iint_{B_{\frac{1}{2}}(z_0)} |f|\, dx\, dy \ge \frac{\pi}{4}|f(z_0)|$ by Cauchy when $|z_0| \le \frac{1}{2}$ so $|f| \le \frac{4}{\pi}$ on $B_{\frac{1}{2}}$ (same argument works inside any radius bounded away from 1). Thus the family is uniformly bounded in $B_{r/2}$. Now apply Montel.

47. Nothing:(

48. Sketch: We apply the below to $g = f_m - f_n$ to show convergence.

$$\begin{aligned} g(a) &= \frac{1}{\varepsilon_1} \int_0^{\varepsilon_1} \int_0^{2\pi} g(a + re^{i\theta})\, d\theta\, dr \\ &= \frac{1}{\varepsilon_1} \iint r^{\frac{1}{2}} (r^{\frac{1}{2}} g(z))\, d\theta\, dr \\ &= \frac{1}{\varepsilon_1} \left(\iint r \right)^{\frac{1}{2}} \left(\iint |g(x)|^2 r \right)^{\frac{1}{2}} \\ &\le \frac{1}{\varepsilon_1} \sqrt{\pi \varepsilon_1^2}\, \|g\|_{L^2}\,. \end{aligned}$$

Thus the f_m converge uniformly on the compact subset.

49. If U is compact, then yes by 1.48. If U is open, then no. Ex. for D, consider $\frac{2n+2}{2\pi} z^n$ which converges to 0 on compact subsets but has L^2 norm 1.

50. It can be continuous: just take any Fourier series with decay on the order of $\frac{1}{n^\alpha}$, $1 < \alpha < \infty$, replace $e^{2\pi ikx}$ with z^k. If it were analytic, the radius of convergence would be > 1.

51. Yes, $\sin\left(\frac{\pi}{1-x}\right)$.

52. ?

53. $\to 0$ by Weyl equidistribution, `https://www.sharelatex.com/project/52724f0756f405b1710041a`

3.2 Entire functions

1. f' exists everywhere.

2. Bounded entire function is constant. Use Cauchy with f' with radii $\to \infty$.

3. See above.

4. Use Cauchy with $f^{(d)}$.

5. If f has no zero $\frac{1}{f}$ is bounded entire, so constant.

6. Yes, because $\ln|z| < |z|$. harmonic?

7. It's either constant, or not entire.

8. Poly degree ≤ 100.

9. Yes. Compose with a biholomorphic function from a strip to D. An entire function is constant.

10. Then $|f|$ is also bounded, so it's constant.

11. It's constant. It's bounded inside the circle; use Schwarz reflection; it's bounded everywhere; use Liouville.

12. As many as you want, as long as they don't have a limit point. Proof of Jensen sketch: see hadamard.pdf here `https://github.com/holdenlee/mathnotes/tree/master/Analysis/complex`

13. We have $\left\lfloor \frac{1}{2} \right\rfloor = 0$, so $\sum \frac{1}{|a_k|^0} < \infty$, so it has finitely many zeros.

14. $\ln f$ is at most linear, so $f = e^{az+b}$.

15. It's order is ≤ 100. Let a_k be zeros. $\sum \frac{1}{|a_k|^{100}} < \infty$. It has a Hadamard factorization with 100.

16. See hadamard.pdf here `https://github.com/holdenlee/mathnotes/tree/master/Analysis/complex` See 33.2.6 here: `https://github.com/holdenlee/number-theory` Do last part.

17. Take ln and use Cauchy.

18. It's not bounded. $\sin^2 + \cos^2 = 1$ eveywhere.

19. e^x. Not for 2 points by Picard Little.

20. It's constant: we can map those regions to bounded regions. Picard Little is generalization.

21. This is the proof using modular functions. Suppose f is an entire function omitting two values y_1, y_2. Note $\mu_2 = 6$ so

$$g(X(2)) = 1 + \mu_N \cdot \left. \frac{N-6}{12N} \right|_{N=2} = 0.$$

The number of cusps is

$$\frac{\mu_N}{N} = 3.$$

There is one cusp at ∞ and two inequivalent cusps on $\mathbb{R}$. Note that $\Gamma(2)\backslash\mathcal{H}$ is analytically isomorphic to $\mathbb{C} - \{y_1, y_2\}$ (say, via φ) since they both have genus 0 and two finite points omitted.(From [?], any compact Riemann surface of genus 0 and no cusps is analytically isomorphic to the Riemann sphere.) Thus f induces a holomorphic map $g : \mathbb{C} \to \Gamma(2)\backslash\mathcal{H}$. Now $\mathcal{H}$ is a covering space of $\Gamma(2)\backslash\mathcal{H}$ so g induces a analytic map h so that the following diagram commutes. (Here π is the projection map.)

$$\begin{tikzcd}
& \mathcal{H} \arrow[d, "\pi"] \\
\mathbb{C} \arrow[ur, "h"] \arrow[r, "g"] \arrow[dr, "f"'] & \Gamma(2)\backslash\mathcal{H} \arrow[d, "\varphi", "\cong"'] \\
& \mathbb{C} - \{y_1, y_2\}.
\end{tikzcd}$$

Now $u(z) = e^{iz}$ is an analytic map from $\mathcal{H}$ to $D - \{0\}$ (D being the unit disc centered at 0). Hence $u(h(z))$ is an entire function with image contained in D. Then $u(h(z))$ is bounded so constant by Liouville's Theorem. But the inverse image of any point under u is discrete, so this means that $h(z)$ is constant, and $f(z) = \varphi(\pi(h(z))$ is constant.

22. Yes. See hadamard.pdf.

23. Some subring of $\mathbb{C}[[x]]$. more info?

24. do this

25. It's $O(\ln(\Im s))$.

3.3 Singularities

1. Removable singularities, poles, and essential singularities.

2.

Theorem 3.7 (Riemann's theorem on removable singularities)**:** If f is bounded on $\Omega - \{z_0\}$ then f extends analytically to z_0.

Proof. Try to define $f(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$; use this to extend to z_0. PROBLEM: we want γ to contain z_0 and z. This is OK because we can replace γ by 2 small circles around z and z_0; the integral around z_0 goes to 0 because it's bounded. This shows $f(z) = \int_\gamma \cdots$ for $z \neq z_0$.

This is a continuous extension, and you can differentiate it at z_0. □

3. Defined everywhere except poles. For $\frac{1}{\sin z}$, $\pi\mathbb{Z}$ with residues 1. (Calculate $\frac{2i}{e^{ix} - e^{-ix}} = \frac{2i}{(ix) - (-ix) + \cdots} = \frac{1}{x} + \cdots$.

4. It goes to ∞ in every direction: the leading term $a_{-k} z^{-k}$ swamps everything else.

5. The holomorphic ones, because the integrals around every pole must be 0, so they can't be poles.

6. See above.

7. No. f has finitely many zeros. So $\frac{1}{f}$ has finitely many poles, and has zeros in a dense set, so is 0.

8.

9. It hits every value. See 3.10.

10.

Theorem 3.8 (Casorati-Weierstrass)**:** Let U be an open set around an essential singularity z_0 of f. Then $f(U)$ is dense in $\mathbb{C}$.

Proof. Else there's a ball of radius δ around a that it misses. Then $\frac{1}{f(x)-a}$ is bounded, so extendable to z_0, contradiction. □

11. Big Picard!!!

12. Yes; otherwise it has a regular pole or is defined.

13. How does this make sense? $z = 0$ is not inside an open set where $\ln z$ is defined.

14. It blows up.

15. Some subring of $\mathbb{C}((x))$. It's not algebraically closed (it doesn't have $\sqrt{x}$ for instance).

3.4 Infinite products

See hadamard.pdf `https://github.com/holdenlee/mathnotes/tree/master/Analysis/complex`

1. Hadamard product theorem

2. ...

3. $z\prod_n(1-\frac{z}{n})$ (take symmetrically, or multiply each by $e^{-z/n}$)

4. ... $\prod_{n\in\mathbb{N}}(1+\frac{z}{n})$ won't converge.

5.

6.

7.

8.

9. Factoring ζ (or ξ), PNT reduces to getting information on zeros.

3.5 Analytic continuation

1. See number theory notes.

2. See number theory notes.

3.

4. If it's real on the interval...

5. See 1.1 in conformal.pdf: `https://github.com/holdenlee/mathnotes/tree/master/Analysis/complex`.

6.

7. Calculate the power series; check the annuli of convergence from this.

8. ?

9. $\oint = 2\pi i \neq 0$.

10.

11. ?

12. For D, $\sum z^{2^n}$ works. What bout others?

3.6

3.7 Maximum principles

1. On a closed set Ω, analytic f attains $\max|f|$ on the boundary. [It suffices to be continuous on the boundary.] Proof. If maximum is in interior at z_0, then $|f| = |f(z_0)|$ on any circle around z_0 inside Ω.

   (Note this would force f to be constant, as $f' \neq 0$ means f is locally open.)

2. By maximum modulus $|f'| \le 1$ inside; by integrating, $|f(z) - f(0)| \le |z|$.

3.

   Theorem 3.9 (Hadamard 3 lines)**:** Let f be analytic on $\{0 \le \Re f \le 1\}$. Let $M(x) = \sup_{y\in\mathbb{R}} |f(x+iy)|$. Then
   $$M(x) \le M(0)^{1-x} M(1)^x.$$
   In other words, M is log-concave.

   (It's easy to scale to (a,b).)

   Proof. Consider $f_n(x) = e^{-\frac{1}{n}} e^{\frac{x^2}{n}} M(0)^{x-1} M(1)^{-x}$. It's bounded by 1. (This is the point of the $\frac{1}{n}$.) For any z, $\varepsilon > 0$, there is n such that $|f_n(z)| > |f(z)| - \varepsilon$. Note $f_n \to 0$ as $\Im(z) \to 0$ (this was the point of $e^{-x^2/n}$). □

   So here $|f| \le 1$.

4. $|f(x+iy)| \le 5^x$. harmonic?

5.

6.

7.

8.

9. http://en.wikipedia.org/wiki/Borel%E2%80%93Carath%C3%A9odory_theorem

3.8

3.9 Conformal mappings

1. It preserves angles.

2. Conformal maps $U \to V$ are exactly holomorphic functions $U \to V$ with $f' \neq 0$.

3. No, it's not if $f' = 0$. If $f'(z) = 0$ it looks like z^k locally, k-to-1 except at the point. A conformal map $\mathbb{C} \to \mathbb{C}$ is always holomorphic because the condition tells us the derivative exists.

4. See above.

5. They are circles and lines. Möbius transformations are compositions of translations, rotation/homotheties, and inversions. Inversions send circles/lines to circles/lines (just calculate).

6. It suffices to show that any finite subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is simultaneously diagonalizable (then acting on $\mathcal{H}$, ∞ is the fixed point). It suffices to show it's abelian. How to do that? http://mathoverflow.net/questions/107902/finite-subgroups-of-sl-2r?rq=1

7. Consider S^2 as $\mathbb{C}_\infty$. If a, b are the zero, pole, respectively, then $f \cdot \frac{z-b}{z-a}$ is constant (any function between compact Riemann surfaces is constant or surjective). Thus the conformal maps are the Möbius transformations.

8. $\mathcal{H} \to D$ given by $\frac{i-z}{i+z}$. Just calculate.

9. $e^x : \{0 < \Im z < \pi\} \to \mathcal{H}$ followed by $\frac{i-z}{i+z}$.

10. $\ln x : \mathcal{H} \to \{0 < \Im z < \pi\}$.

11. Map $z \to iz$ and then use 9.9.

12. Square and use 9.9. $\frac{i-z^2}{i+z^2}$.

13. Square. You can't use a Möbius transformation, because a Möbius transformation extends to a map $\mathbb{C}_\infty \to \mathbb{C}_\infty$.

14. ln takes it to a rectangle. Now use the inverse of Schwarz-Christoffel.

15. You can choose the images of 3 vertices (ex. 0, 1, λ); the other is determined. (Note by changing the order of vertices, if 3 points are $0, 1, \lambda$, the fourth point can be $\frac{\lambda-1}{\lambda}, \frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, \frac{-1}{\lambda-1}, \lambda, -(\lambda-1)$.) In the case of a square, one obvious possibility is $\lambda = -1$ (since this makes $\int_{-1}^{0} \frac{1}{\sqrt{z(z-1)(z-\lambda)}}\,dz = \int_0^1 \frac{1}{\sqrt{z(z-1)(z-\lambda)}}\,dz$) so the other possibilities are 2, $\frac{1}{2}$.

16. See above.

17. First map $D \to \mathcal{H}$. Now use Schwarz-Christoffel. (See §7 here: `https://github.com/holdenlee/mathnotes/tree/master/Analysis/complex`.)

18. Let external angles be β_i. Then it's $\int_0^z \prod_k (\zeta - A_k)^{\beta_k}\,d\zeta$. (Or just include 2 angles.)

19. See above.

20. Use Schwarz's lemma 2.2 here: `https://github.com/holdenlee/mathnotes/tree/master/Analysis/complex`. It's $\mathrm{SL}_2(\mathbb{R})$.

21.

22. Compose S-C with $\mathcal{H} \to D$. It extends continuously; see 6.1 here: `https://github.com/holdenlee/mathnotes/tree/master/Analysis/complex`.

3.10 Riemann mapping theorem

1. It contradicts Liouville.

2. Proper simply connected regions are all isomorphic.

3. $\mathbb{C}$.

4. See 10.2. We have continuity up to boundary if the intersection of a small enough ball around $x \in \partial S$ with ∂S is a curve. 2nd part?

5. See 4.2 `https://github.com/holdenlee/mathnotes/tree/master/Analysis/complex`.

6. See above.

7. See 2.1 `https://github.com/holdenlee/mathnotes/tree/master/Analysis/complex`.

8.

9.

10. If f is nonvanishing on a simply connected domain, $\ln f$ is defined there (as $\int \frac{f'}{f}\,dz$). Define $\sqrt{f} = e^{\frac{1}{2}\ln f}$.

11. ?

12. ?

13. When the radial-ratios are equal. That's the only parameter.

    Suppose A, B are annuli with radii $(1, r_1)$, $(1, r_2)$, $r_1 \neq r_2$. By Schwarz reflection (across the boundary of the annuli), we can define a function $f : \mathbb{C} \to \mathbb{C}$ holomorphic except possibly at 0, taking $z = |r|$ approximately to $z = |r^{\log_{r_1} r_2}|$. (Details: apply Schwarz reflection after using the map $z \mapsto -i \ln z + \ln r_j \in \mathbb{C}/2\pi i\mathbb{Z}$ on the annuli. I.e., the function extended across the outer radius is $e^{i(g(\overline{-i \ln z + \ln r_1)}) - \ln r_2}$.)

    Assume $r_1 < r_2$ so that f is differentiable (holomorphic) at 0. 2 ways to finish: (1) But $f'(0) = 0$, and f is injective near 0, contradiction. (2) By Cauchy's formula for $f^{(d)}$, any function growing slower than a polynomial is a polynomial. So f is actually a polynomial, and is not 1-to-1 unless it's linear.

14.

15.

16.

17. Minus 1 point: just translate. Minus > 1 point: fundamental groups are different, so no.

18.

3.11