18.997 Probabilistic Method Problem Set #6

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Problem 1 *(9.1)*

We first show the following.

Claim 1.1: There exists a $c > 1$ such that the following holds: For every $n \ge 1$, there exists a 3-regular bipartite graph with color classes A, B each containing n vertices, such that for every $k \le \frac{n}{2}$, every group of k vertices in either A or B is connected to at least ck vertices in B or A, respectively.

Consider three independent random matchings between the vertices of A and B, with each matching equally likely to be chosen. Let G be the bipartite graph with these edges. Given a set S of k vertices in A and a set T of $\lfloor ck \rfloor \le n$ vertices in B, the probability that S is only connected to vertices in T in a random matching is

$$\frac{\binom{\lfloor ck \rfloor}{k}}{\binom{n}{k}},$$

since any k-element set of B is equally likely to be the set of neighbors of A, and $\binom{\lfloor ck \rfloor}{k}$ of these sets lie in T. Hence the probability that S is only connected to vertices in T in G is

$$\left(\frac{\binom{\lfloor ck \rfloor}{k}}{\binom{n}{k}}\right)^3.$$

By the union bound the probability that *some* k-subset of A has all neighbors inside *some* $\lfloor ck \rfloor$-subset of B is at most

$$\left(\frac{\binom{\lfloor ck \rfloor}{k}}{\binom{n}{k}}\right)^3 \cdot \binom{n}{k}\binom{n}{\lfloor ck \rfloor} = \frac{\binom{\lfloor ck \rfloor}{k}^3 \binom{n}{\lfloor ck \rfloor}}{\binom{n}{k}^2}.$$

Thus letting p be the probability that for *some* $k \le \frac{n}{2}$, there exists a k-element subset of A with at most ck neighbors in B, we get

$$p \le \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{\lfloor ck \rfloor}{k}^3 \binom{n}{\lfloor ck \rfloor}}{\binom{n}{k}^2}.$$

We bound this sum in two steps.

Step 1: For sufficiently large n, sufficiently small $c > 1$,

$$\sum_{1 \le k \le \frac{n}{6}} \frac{\binom{\lfloor ck \rfloor}{k}^3 \binom{n}{\lfloor ck \rfloor}}{\binom{n}{k}^2} < \frac{1}{4}.$$

Using the approximation

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k,$$

and letting $d = \frac{\lfloor ck \rfloor}{k}$, for c close enough to 1,

$$\begin{aligned}
\frac{\binom{\lfloor ck \rfloor}{k}^3 \binom{n}{\lfloor ck \rfloor}}{\binom{n}{k}^2} &= \frac{\binom{dk}{(d-1)k}^3 \binom{n}{dk}}{\binom{n}{k}^2} \\
&\le \frac{\left(\frac{ed}{d-1}\right)^{3(d-1)k} \left(\frac{en}{dk}\right)^{dk}}{\left(\frac{n}{k}\right)^{2k}} \\
&\le \left(d^{2d-3} e^{4d-3} (d-1)^{-3(d-1)} \left(\frac{k}{n}\right)^{2-d} \right)^k \\
&\le \underbrace{\left(c^{2c-3} e^{4c-3} (c-1)^{-3(c-1)} \left(\frac{k}{n}\right)^{2-c} \right)^k}_{b(c,k,n)}
\end{aligned}$$

(since $(c-1)^{-(c-1)}$ is decreasing for $c > 1$ close to 1). We have $\lim_{c \to 1^+} b(c,k,n) = \frac{ek}{n}$. So for c close enough to 1 and $k \le \frac{n}{6}$, $b \le \frac{1}{2}$. Then $b^k \le \left(\frac{1}{2}\right)^k$. Hence

$$\sum_{1 \le k \le \frac{n}{6}} \frac{\binom{\lfloor ck \rfloor}{k}^3 \binom{n}{\lfloor ck \rfloor}}{\binom{n}{k}^2} \le \sum_{1 \le k \le 4} \frac{\binom{\lfloor ck \rfloor}{k}^3 \binom{n}{\lfloor ck \rfloor}}{\binom{n}{k}^2} + e^3 \sum_{5 \le k \le \frac{n}{6}} \left(\frac{1}{2}\right)^k = \sum_{1 \le k \le 4} \frac{\binom{\lfloor ck \rfloor}{k}^3 \binom{n}{\lfloor ck \rfloor}}{\binom{n}{k}^2} + \frac{e}{2^4}.$$

It is clear that the first term goes to 0 as $n \to \infty$, so for sufficiently large n and c sufficiently close to 1, this is at most $\frac{1}{4}$.

Step 2: For sufficiently large n, sufficiently small $c > 1$,

$$\sum_{\frac{n}{6} < k \le \frac{n}{2}} \frac{\binom{\lfloor ck \rfloor}{k}^3 \binom{n}{\lfloor ck \rfloor}}{\binom{n}{k}^2} < \frac{1}{4}.$$

We use the formula

$$\binom{n}{k} = 2^{n\left(-\frac{k}{n}\log_2\left(\frac{k}{n}\right) - \left(1-\frac{k}{n}\right)\log_2\left(1-\frac{k}{n}\right) + o(1)\right)} = \left(\frac{k}{n}\right)^{-k} \left(1 - \frac{k}{n}\right)^{-(n-k)} 2^{no(1)}, \quad 0 < k < n.$$

(Note the $o(1)$ can be bounded independently of k, n.) Then letting $d = \frac{\lfloor ck \rfloor}{k}$ and $r = \frac{k}{n}$,

$$\begin{aligned}
\frac{\binom{\lfloor ck \rfloor}{k}^3 \binom{n}{\lfloor ck \rfloor}}{\binom{n}{k}^2} &= \frac{\binom{dk}{k}^3 \binom{n}{dk}}{\binom{n}{k}^2} \\
&= \frac{\left(\frac{1}{d}\right)^{-3rn} \left(1 - \frac{1}{d}\right)^{-3(d-1)rn} (dr)^{-drn} (1-dr)^{-n(1-dr)}}{r^{-2nr}(1-r)^{-2n(1-r)}} \\
&= \left(\left(d^{d-3} \left(1 - \frac{1}{d}\right)^{3d-3} \right)^{-r} r^{r(2-d)} (1-r)^{2(1-r)} (1-dr)^{-(1-dr)} \right)^n .
\end{aligned}$$

Note $d^{d-3}\left(1-\frac{1}{d}\right)^{3d-3} \to 1$ as $d \to 1^+$. Note d can be assumed arbitrarily close to c, by letting n be sufficiently large. As $r < 1$, this means we for sufficiently large n and c close to 1, $\left(d^{d-3}\left(1-\frac{1}{d}\right)^{3d-3}\right)^{-r}$ can be made as close to 1 as needed. For the rest of the expression $r^{r(2-d)}(1-r)^{2(1-r)}(1-dr)^{-(1-dr)}$, note that it is continuous in d and $r \in [\frac{1}{6}, \frac{1}{2}]$, so as $d \to 1^+$, it converges uniformly to the function $r^r(1-r)^{1-r}$ on $r \in [\frac{1}{6}, \frac{1}{2}]$. This is clearly bounded away from 1, less than 1, for r in this interval. Therefore there exists $q < 1$ so that for n sufficiently large and c sufficiently close to 1, $\frac{\binom{\lfloor ck \rfloor}{k}^3 \binom{n}{\lfloor ck \rfloor}}{\binom{n}{k}^2} \le q^n$. Then the desired sum is at most $\left(\frac{n}{2} - \frac{n}{6} + 1\right) q^n$, which is less than $\frac{1}{4}$ for sufficiently large n.

Taking n sufficiently large to work for steps 1 and 2, we have $p < \frac{1}{2}$, but by symmetry p also equals the probability that for some $k \le \frac{n}{2}$, there exists a k-element subset of B with at most ck neighbors in A, and the probability of either one of these events happening is less than 1. This proves the claim 1.1 for large n, say $n > N$.

For $n \le N$, take any three matchings of the bipartite graph so that they form a connected graph. (In one matching match the ith vertex to the ith vertex; in another match the ith with $(i+1)$th (modulo n).) For every group of $k \le \frac{n}{2}$ vertices in A or B, we claim its set of neighbors has at least $k+1$ elements. Since the graph is 3-regular, the number e of edges between A and $N(A)$ is $3|A|$, and is also at most $3|N(A)|$. Hence $|N(A)| \ge |A|$, with equality only if there are no edges from $N(A)$ to outside A. But this is impossible as we chose a connected graph. Then the constant $\frac{\frac{N}{2}+1}{\frac{N}{2}}$ works in this case, and we can take the minimum of the constants for the $n < N$ and $n \ge N$ cases.

Problem 2 *(9.2)*

Theorem 2.1 (Corollary 9.2.5)**:** Given $G = (V, E)$ a (n, d, λ)-graph, for every two sets of vertices B and C of G, where $|B| = bn$ and $|C| = cn$, we have

$$|e(B, C) - cbdn| \le \lambda \sqrt{bc}n.$$

If equality holds, then $|N_B(v)| = bd$ for every $v \in V \backslash C$ and $|N_C(v)| = cd$ for every $v \in V \backslash B$.

Proof. This is Corollary 9.2.5. The second statement holds from symmetry and the fact that one of the inequalities used in the proof is

$$\sum_{v \in C} (|N_B(v)| - bd)^2 \leq \sum_{v \in V} (|N_B(v)| - bd)^2.$$

$\square$

For a color y, let B_y denote the set of vertices not adjacent to any vertex of color y. Suppose by way of contradiction that no vertex of G has a neighbor of each of the k colors. Then every vertex is in some B_y. Take z to be the color such that $|B_z|$ is largest. Since there are n vertices and k colors, $|B_z| \geq \frac{n}{k}$. Let B be a subset of B_z vertices with $\frac{n}{k}$ elements, and C be the set of vertices of color z. We have

$$e(B, C) = 0.$$

By assumption, $|C| = \frac{n}{k}$. Hence by the theorem above,

$$\left| e(B, C) - \frac{1}{k^2} dn \right| \leq \lambda \frac{1}{k} n$$
$$e(B, C) \geq \frac{1}{k^2} dn - \lambda \frac{1}{k} n = \frac{n}{k} \left(\frac{d}{k} - \lambda \right) \geq 0,$$

where we used the assumption $k\lambda \leq d$. Equality must hold everwhere above, so by the theorem, $N_C(v) = \frac{d}{k}$ for every $v \notin V \backslash B$. In particular, every vertex outside of B has a neighbor of color z. Hence $|B_z| = |B| = \frac{n}{k}$; since B_z was assumed largest among the B_y and $\bigcup B_y = V$, we conclude $|B_y| = \frac{n}{k}$ for each y. By the argument above applied to each B_y, we get if a vertex v is in B_z, then it is adjacent to exactly $\frac{d}{k}$ vertices of each color $y \neq z$, but not adjacent to any element of color z. Thus v has degree $(k-1)\frac{d}{k}$, contradicting the fact that G is d-regular.

Problem 3 *(Ramsey numbers)*

(i)

We use the following (proved in class):

Theorem 3.1: There exists a constant c depending on H such that if $H = (A \cup B, E)$ is a bipartite graph such that all vertices in B have degree at most r, then $\text{ex}(n, H) \leq cn^{2-\frac{1}{r}}$.

We're given that H is bipartite with maximum degree at most $a \ln n$ for some a. Choose c as above, so $\text{ex}(N, H) \leq cN^{2-\frac{1}{a \ln n}}$. Suppose

$$cN^{2-\frac{1}{a \ln n}} < \frac{1}{2}\binom{N}{2}. \tag{1}$$

Then no matter how K_N is colored with two colors, one of the colors contains at least $\frac{1}{2}\binom{N}{2}$ edges. By (1) and the theorem applied to the subgraph of that color, there is a copy of H in that color.

Thus it suffices to show that there is $N = n^{O(1)}$ such that (1) holds for all $n \geq 2$. (For $n = 1$, trivially $r(H) = 1$.) Put in $N = n^k$ and rewrite 1 as follows:

$$\begin{aligned} cN^{2-\frac{1}{a\ln n}} &< \frac{N(N-1)}{4} \\ \iff cn^{k\left(2-\frac{1}{a\ln n}\right)} &< \frac{n^k(n^k-1)}{4} \\ \iff 4cn^{2k-\frac{k}{a\ln n}} + n^k &< n^{2k} \\ \iff 4cn^{-\frac{k}{a\ln n}} + n^{-k} &< 1. \end{aligned}$$

Note

$$\lim_{n\to\infty} n^{\frac{1}{\ln n}} = e^{\lim_{n\to\infty}\ln\left(n^{\frac{1}{\ln n}}\right)} = e^{\lim_{n\to\infty}\frac{1}{\ln n}\ln(n)} = e,$$

so if k is fixed, then

$$\lim_{n\to\infty} 4cn^{-\frac{k}{a\ln n}} + n^{-k} = 4ce^{-\frac{k}{a}}.$$

Choose $k' \in \mathbb{N}$ such that $4ce^{-\frac{k'}{a}} < 1$. Then there exists L so that $4cn^{-\frac{k}{a\ln n}} + n^{-k} < 1$ for all $n \geq L$. Now if $n \geq 2$ is fixed, $4cn^{-\frac{k}{a\ln n}} + n^{-k}$ is decreasing in k and

$$\lim_{k\to\infty} 4cn^{-\frac{k}{a\ln n}} + n^{-k} = 0.$$

Thus there exists $k \geq k'$ such that $4cn^{-\frac{k}{a\ln n}} + n^{-k} < 1$ for all $n < L$. For $n \geq L$ we also have $4cn^{-\frac{k}{a\ln n}} + n^{-k} \leq 4cn^{-\frac{k'}{a\ln n}} + n^{-k'} < 1$. Then $N = n^k$ satisfies (1) for all n, as needed.

(ii)

Suppose H has n vertices and average degree $f(n)\log_2 n$ where $f(n) = \omega(1)$, i.e. $\lim_{n\to\infty} f(n) = \infty$. Then H has $\frac{nf(n)\log_2 n}{2}$ edges. Label the vertices of H from 1 to n.

Color each edge of K_N red or blue with probability $\frac{1}{2}$. Given an ordered set of n vertices in K_N, the probability that the imbedding of H into those n vertices, following the order, is monochromatic is $2\left(\frac{1}{2}\right)^{\frac{nf(n)\log_2 n}{2}}$: there are 2 colors to choose from, and each of the $\frac{nf(n)\log_2 n}{2}$ imbedded edges has $\frac{1}{2}$ chance of being that color.

Since there are $N^{\underline{n}} = N(N-1)\cdots(N-n+1)$ ordered sets of n vertices, the probability that there is some monochromatic copy of H in K_N is at most

$$2\left(\frac{1}{2}\right)^{\frac{nf(n)\log_2 n}{2}} N^{\underline{n}} \leq 2n^{-\frac{nf(n)}{2}}N^n.$$

If $N \leq n^k$, then the RHS is at most

$$2n^{-\frac{nf(n)}{2}+kn} = 2n^{n\left(k-\frac{f(n)}{2}\right)}.$$

Since $f(n) \to \infty$, this is less than 1 for sufficiently large n. Thus if $N \leq n^k$, then for sufficiently large n there exists a coloring of K_N with no monochromatic copy of H, showing $r(H) = n^{\omega(1)}$.