18.997 Probabilistic Method Problem Set #5

Holden Lee

4/18/11

Problem 1 *(6.1)*

Note Q equals the probability that in a random subgraph H of G obtained by picking each edge of G with probability $\frac{1}{2}$, that both H and $G\backslash H$ are connected, where $G\backslash H$ consists of the vertices of G and the edges of G not in H. (Just associate one color with "being in H" and the other color with "not being in H.")

Note "H connected" is a monotone increasing graph property and "$G\backslash H$ connected" is a monotone decreasing graph property (with respect to H), because deleting an edge in H corresponds to adding an edge in $G\backslash H$. Thus by Theorem 6.3.2 applied to the set of edges of G, we get that

$$\begin{aligned} P(H \text{ connected and } G\backslash H \text{ connected}) &\le P(H \text{ connected})P(G\backslash H \text{ connected}) \\ &= P(H \text{ connected})^2. \end{aligned}$$

The last follows from the fact that the distribution for H and $G\backslash H$ is the same, since H is equally likely to be any subgraph H_0 of G, and in particular, the probability that $H = H_0$ is the same as the probability that $H = G\backslash H_0$.

Thus $Q \le P^2$.

Problem 2 *(6.3)*

Note for any vertex v, "v has degree at most $k-1$" is a monotone decreasing graph property.

Label the vertices $v_1, \ldots, v_{2k}$. By repeated application of 6.3.3, (basically an induction; in the induction step noting that if P_1 and P_2 are monotone decreasing then so it $P_1 \wedge P_2$),

$$P(v_1, \ldots, v_{2k} \text{ all have degree } \le k-1) \ge \prod_{i=1}^{2k} P(v_1 \text{ has degree } \le k-1) = \left(\frac{1}{2}\right)^{2k} = \frac{1}{4^k}.$$

The equality come from the fact that there are $2k-1$ edges coming out of v, each chosen independently with probability $\frac{1}{2}$, so the degree of v gives a binomial distribution symmetric around $\frac{2k-1}{2}$. In particular, it is as likely to have degree at most $k-1$ as degree at least k, i.e. both probabilities are $\frac{1}{2}$.

Problem 3 *(7.2)*

Lemma 3.1:

$$\mathbb{E}[\chi(H)] \geq 500.$$

Proof. We show that given $U \subseteq V$, $\chi(G[U])+\chi(G[U^c]) \geq 1000$. Indeed, take proper colorings of $G[U]$ and $G[U^c]$ with $\chi(G[U])$ and $\chi(G[U^c])$ different colors, such that the colors used in U are different from any color used in U^c. This gives a proper coloring of G, since the only edges in G neither in $G[U]$ nor $G[U^c]$ are those between U and U^c, which will be between different colors. Since $\chi(G) = 1000$, there must be at least 1000 colors used.

Now summing $\chi(G[U]) + \chi(G[U^c]) \geq 1000$ over all $2^{|V|}$ subsets $U \subseteq V$ and dividing by $2^{|V|+1}$ gives $\mathbb{E}[\chi(H)] \geq 500$. □

Color G properly with 1000 colors, and let $A_1, \ldots, A_{1000}$ be the color classes. Note that each A_i is an independent set.

Let $B_j = \bigcup_{i=1}^{j} A_j$; consider the gradation $B_0 \subset B_1 \subset \cdots \subset B_{1000} = V$. Let $L : \mathcal{P}(V) \to \mathbb{Z}$ be the function $L(W) = \chi(G[W])$. Let

$$X_j(U) = \mathbb{E}[L(W)|B_j \cap W = B_j \cap U].$$

In other words, $X_j(U)$ is the expected value of the chromatic number of $G[W]$, where W is a random subset of V that matches U on $A_1, \ldots, A_j$. Note $X_0(U) = \mathbb{E}[\chi(H)]$, while $X_{1000}(U) = \chi(H)$. (Here $H = G[U]$.)

We show L satisfies the Lipschitz condition. Suppose W, W' differ only on $B_{j+1} - B_j = A_{j+1}$. Color $G[W]$ as follows: color the vertices in $W \cap A_{j+1}^c$ the same as in $W' \cap A_{j+1}^c$, and then color the vertices in $W \cap A_{j+1}$ another color (which is okay since A_{j+1} is an independent set). Then we have a proper coloring of $G[W]$ with at most $\chi(G[W']) + 1$ colors. So $L(W) \leq L(W') + 1$. The other inequality similarly holds, so $|L(W) - L(W')| \leq 1$.

Now apply Theorem 7.4.1 to X_i (but stated in terms of subsets, rather than functions), to conclude $|X_{i+1} - X_i| \leq 1$ for $0 \leq i < 1000$.

Let $\mu = \mathbb{E}[\chi(H)]$; by the lemma $\mu \geq 500$. By Azuma's inequality with $m = 1000$ and $\lambda = \sqrt{10}$,

$$\begin{aligned}
P[\chi(H)) \leq 400] &\leq P[\chi(H) - \mu \leq -100] \\
&= P[X_{1000} - X_0 < -\sqrt{10}\sqrt{1000}] \\
&= e^{-\frac{\sqrt{10}^2}{2}} \\
&= e^{-5} < \frac{1}{100}.
\end{aligned}$$

Problem 4 *(7.3)*

Let $\varepsilon = \frac{1}{300}$. Let $u = u(n, \varepsilon)$ be the least integer so that $P(\chi(G) \leq u) > \varepsilon$. Define $Z(G)$ to be the maximal size of a set of vertices for which the induced graph can be u-colored, and let $Y = n - Z$. Note Z (and hence Y) satisfies the vertex Lipschitz condition since if we add a vertex, Z either stays the same or increases by 1. Let $\mu = \mathbb{E}[Y]$ and use Azuma's inequality on the vertex exposure martingale to get

$$P(Y \leq \mu - \lambda\sqrt{n-1}) < e^{-\frac{\lambda^2}{2}}$$
$$P(Y \geq \mu - \lambda\sqrt{n-1}) < e^{-\frac{\lambda^2}{2}}.$$

Let $\lambda = \sqrt{-2\ln\varepsilon}$ so this becomes

$$P(Y \leq \mu - \lambda\sqrt{n-1}) < \varepsilon$$
$$P(Y \geq \mu - \lambda\sqrt{n-1}) < \varepsilon.$$

Now $P(Y = 0) = P(Z = n) = P(\chi(G) \leq u) > \varepsilon$ so the first inequality forces $\mu \leq \lambda\sqrt{n-1}$. The second inequality then gives

$$P(Y \geq 2\lambda\sqrt{n}) \leq P(Y \geq \mu + \lambda\sqrt{n-1}) < \varepsilon.$$

In other words, there is probability at least $1-\varepsilon$ that there is a u-coloring of all but at most $2\lambda\sqrt{n}$ vertices. Call these set of uncolored vertices U.

Since $\chi(G) \sim \frac{n}{2\log_2 n}$ almost surely, there exists c so that $P\left(\chi(G) \leq \frac{cn}{\log n}\right) \geq 1 - \varepsilon$ for all $n > 1$. Assuming $|U| \leq 2\lambda\sqrt{n}$, applying this to $G[U]$ we get that

$$\begin{aligned} 1 - \varepsilon &\leq P\left(\chi(G[U]) \leq \frac{c(2\lambda\sqrt{n})}{\log(2\lambda\sqrt{n})}\right) \\ &= P\left(\chi(G[U]) \leq \frac{c2\lambda\sqrt{n}}{\log(2\lambda) + \frac{1}{2}\log n}\right) \\ &\leq P\left(\chi(G[U]) \leq \frac{c'\sqrt{n}}{\log n}\right) \end{aligned}$$

for some appropriate constant c'.

Given $|U| \leq 2\lambda\sqrt{n}$, with probability at least $1-\varepsilon$, $G[U]$ can be colored with at most $\frac{c'\sqrt{n}}{\log n}$ further colors, giving a coloring of G with at most $u + \frac{c'\sqrt{n}}{\log n}$ colors. By minimality of u, there is probability at least $1-\varepsilon$ that u colors are needed for G. Hence

$$P\left(u \leq \chi(G) \leq u + \frac{c'\sqrt{n}}{\log n}\right) \geq 1 - 3\varepsilon = .99$$