18.997 Probabilistic Method Problem Set #2

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Problem 1 *(2.1, Hypergraph with no monochromatic edges)*

Independently color each edge with one of the four colors with probability $\frac{1}{4}$. Given an edge e, the probability that it is monochromatic is

$$P(e \text{ monochromatic}) = \frac{1}{4^{n-1}}$$

since the probability that all its vertices are a given color is $\left(\frac{1}{4}\right)^n$ and there are 4 choices for the color. Letting X_e be the indicator function for e being monochromatic and X be the number of monochromatic edges, we have by linearity of expectation

$$\mathbb{E}(X) = \sum_{e\in E} \mathbb{E}(X_e) = \sum_{e\in E} P(e \text{ monochromatic}) \le |E|\frac{1}{4^{n-1}} = 1.$$

Next note that if all vertices are colored the same color, $X = n \ge 2 > E(X)$. Hence there exists a coloring so that $X < E(X)$, i.e. $X = 0$, i.e. there is no monochromatic edge.

Problem 2 *(2.2, Subset avoiding an equation)*

We show the problem holds with $c = \frac{1}{7}$.

Step 1: Consider the case where $A \subseteq \mathbb{Z}\backslash\{0\}$.

Take $p > 2$ a prime so that $p > 2\max_{a\in A}|a|$ and p is in the form $7k + 2$. Then no two elements of A are equal modulo p (since they are between $-\frac{p}{2}$ and $\frac{p}{2}$). Let A' be A considered as a subset of $\mathbb{Z}/p\mathbb{Z}$.

Let $I = \left(\frac{3}{7}p, \frac{4}{7}p\right)$ as a subset of $\mathbb{Z}/p\mathbb{Z}$. We claim there exists m so that

$$|mA' \cap I| > \frac{1}{7}n.$$

Note that I consists of the $k + 1$ integers $3k + 1, \ldots, 4k + 1$. Choose the number m at random among $1, \ldots, p - 1$, each with probability $\frac{1}{p-1}$. Since p is prime, for $a \in A'$, ma ranges through all nonzero residues modulo p as m ranges through $1, \ldots, p - 1$. (Remember that $a \neq 0$.) The probability that $ma \in I$ is hence $\frac{k+1}{7k+1} > \frac{1}{7}$. Let X_a be the indicator function for $ma \in I$, and $X = |mA' \cap I|$. Then by linearity of expectation

$$\mathbb{E}(X) = \sum_{a \in A'} \mathbb{E}(X_a) = \sum_{a \in A'} P(ma \in I) > \frac{n}{7}.$$

Hence there exists m so that $mA' \cap I > \frac{1}{7}n$. Let $B = \{a \in A | ma \bmod p \in I\}$. If $b_1, b_2, b_3, b_4 \in B$ and $b_1 + 2b_2 = 2b_3 + 2b_4$ then this equation holds modulo p and multiplying by m gives

$$mb_1 + 2mb_2 \equiv 2mb_3 + 2mb_4 \pmod p.$$

However, $mb_1, mb_2, mb_3, mb_4 \bmod p$ are all in I. Thus the left hand side is in $3I = \left(\frac{2}{7}p, \frac{5}{7}p\right)$ while the right hand side is in $4I = \left(\frac{5}{7}p, p\right) \cup \left[0, \frac{2}{7}p\right)$. These are disjoint sets in $\mathbb{Z}/p\mathbb{Z}$, contradiction. So B is the desired set.

Step 2: Approximate reals with integers.

Theorem 2.1 (Dirichlet)**:** Let $\alpha_1, \ldots, \alpha_n$ be real numbers and $\varepsilon > 0$. There exists a positive integer N and integers m_k so that $|N\alpha_k - m_k| < \varepsilon$. Moreover, N can be chosen arbitrarily large.

Proof. Choose a positive integer r so that $\frac{1}{r} < \varepsilon$. Consider the n-tuple $S_N := (\{N\alpha_1\}, \ldots, \{N\alpha_n\})$. They all fall in one of the rectangles

$$\left[\frac{t_1}{r}, \frac{t_1+1}{r}\right) \times \cdots \times \left[\frac{t_n}{r}, \frac{t_n+1}{r}\right)$$

where $t_i = 0, 1, \ldots$ or $r-1$. Hence by the Box Principle, there exist M and M' so that S_M and $S_{M'}$ fall in the same rectangle. Without loss of generality $M > M'$. Then we can take $N = M - M'$, $m_k = \lfloor M\alpha_k \rfloor - \lfloor M'\alpha_k \rfloor$ and find that $|N\alpha_k - m_k| < \frac{1}{r} < \varepsilon$.

To see we can choose N arbitrarily large, let $N_0 \in \mathbb{N}$ be given, Find $N' > 0$ and m'_k so that $|N'\alpha_k - m'_k| < \frac{\varepsilon}{N_0}$. Then let $N = N_0\alpha_k \geq N_0$ and $m_k = N_0 m'_k$. □

Now given $A = \{a_1, \ldots, a_n\} \subseteq \mathbb{R}\backslash\{0\}$, let $\varepsilon = \frac{1}{7}$ in the lemma and choose N large enough so that $N \min_{a \in A} |a| > 1$ and $N \min_{a,b \in A,\, a \neq b} |a - b| > 1$; then we will have $m_k \neq 0$ in the lemma and $m_i \neq m_j$ for $i \neq j$. We may replace A with NA as scaling doesn't change whether $b_1 + 2b_2 = 2b_3 + 2b_4$ holds, so we can assume $|a_k - m_k| < \frac{1}{7}$.

Now apply Step 1 to $\{m_1, \ldots, m_n\}$ to find $B' = \{m_{i_1}, \ldots, m_{i_j}\}$ so that $|B'| > \frac{n}{7}$ and so that

$$b_1 + 2b_2 \neq 2b_3 + 2b_4 \tag{1}$$

for any $b_1, b_2, b_3, b_4 \in B'$. Since this is an inequality in integers, the two sides must differ by at least 1. Now take $B = \{a_{i_1}, \ldots, a_{i_n}\}$. Replacing the b_i in (1) with their corresponding elements in B, we get that the new LHS differs from the old LHS by less than $\frac{3}{7}$, and the new RHS differs from the old RHS by less than $\frac{4}{7}$. Thus equality still cannot hold, and B is the desired set.

Problem 3 *(2.5, No monochromatic copy of H)*

Let G be the graph with n vertices and t edges containing no copy of H. We show that k copies of G suffice to cover K_n. Labeling the vertices of G and K_n with $1, \dots, n$, each permutation σ of $\{1, \dots, n\}$ gives a way of embedding G into K_n. Call the imbedded graph $\sigma(G)$. For e an edge in G, let $\sigma(e)$ denote the corresponding edge in $\sigma(G)$.

Take k independent random permutations $\sigma_1, \dots, \sigma_k$, each permutation chosen with probability $\frac{1}{n!}$. Given an edge $e \in K_n$ and an index i,

$$P(e \in \sigma(G)) = \frac{t}{\binom{n}{2}}$$

since there are t edges in G and $\binom{n}{2}$ edges in K_n, and for $e' \in G$, $\sigma(e')$ has equal probability of being any edge in K_n, and $\sigma(e') \neq \sigma(e'')$ for $e' \neq e''$. Then using the the independence of the σ_i and linearity of expectation,

$$\begin{aligned}
P(e \notin \sigma_i(G)) &= 1 - \frac{t}{\binom{n}{2}} \\
P(e \notin \sigma_i(G) \text{ for any } i, 1 \le i \le k) &= \left(1 - \frac{t}{\binom{n}{2}}\right)^k \\
\mathbb{E}(\text{number of edges of } K_n \text{ not in any } \sigma_i(G)) &= \sum_{e \in K_n} P(e \notin \sigma_i(G) \text{ for any } i, 1 \le i \le k) \\
&\le |E(K_n)| P(e \notin \sigma_i(G) \text{ for any } i, 1 \le i \le k) \\
&\le \binom{n}{2} \left(1 - \frac{t}{\binom{n}{2}}\right)^k.
\end{aligned}$$

Using the estimate $1 - x < e^{-x}$ for $x \neq 0$,

$$\begin{aligned}
\mathbb{E}(\text{number of edges not in any } \sigma_i(G)) &\le \binom{n}{2} e^{-\frac{tk}{\binom{n}{2}}} \\
&< \binom{n}{2} e^{-\frac{n^2 \ln n}{\binom{n}{2}}} \\
&< \binom{n}{2} n^{-2} \\
&< 1.
\end{aligned}$$

Hence there exists $\sigma_1, \dots, \sigma_k$ so that every edge of K_n is in one of the $\sigma_i(G)$. Let E_i be the set of edges in $\sigma_i(G)$ not in $\sigma_j(G)$ for $j < i$. Then the E_i form a partition of the edges of K_n. Color E_i with color i. Since E_i is contained in $\sigma_i(G)$, it does not contain a copy of H. The resulting coloring does not give rise to a monochromatic copy of H.

Problem 4 *(2.7, Sperner's Lemma)*

Note $X \leq 1$ always, since if $i, j \in \{i : \{\sigma(1), \ldots, \sigma(i)\} \in \mathcal{F}\}$ and $i < j$, then

$$\{\sigma(1), \ldots, \sigma(i)\} \subset \{\sigma(1), \ldots, \sigma(j)\}$$

would be an inclusion of sets contained in $\mathcal{F}$. Hence

$$\mathbb{E}(X) \leq 1. \tag{2}$$

On the other hand, for each set $A \in \mathcal{F}$, let X_A be the indicator function for the event that

$$\{\sigma(1), \ldots, \sigma(|A|)\} = A.$$

Then $X = \sum_{A \in \mathcal{F}} X_A$ so by linearity of expectation,

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{A \in \mathcal{F}} \mathbb{E}(X_A) \\
&= \sum_{A \in \mathcal{F}} P(\{\sigma(1), \ldots, \sigma(|A|)\} = A) \\
&= \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}
\end{aligned}$$

since there are $\binom{n}{|A|}$ subsets of size $|A|$ and $\{\sigma(1), \ldots, \sigma(|A|)\}$ is equally likely to be any of those. But the maximum of $\binom{n}{k}$ is attained when $k = \lfloor \frac{n}{2} \rfloor$. Hence

$$\mathbb{E}(X) \geq |\mathcal{F}| \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}. \tag{3}$$

Putting (2) and (3) together give

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Problem 5 *(2.9, List coloring of bipartite graph)*

Let A and B be the two classes in the bipartite graph. For each color that appears in some list, either cross it out from all vertices in A, or cross it out from all vertices in B, with probability $\frac{1}{2}$. For a vertex v, let $S'(v)$ be the list of colors remaining after this operation.

Note that

$$P(S'(v) = \phi) = \left(\frac{1}{2}\right)^{|S(v)|} \leq \left(\frac{1}{2}\right)^{\log_2 n} \leq \frac{1}{n}$$

because each color in $S(v)$ has probability $\frac{1}{2}$ of being crossed out from the list. Let X_v be the indicator function for $S'(v) = \phi$ and X be the number of v such that $S'(v) = \phi$. By linearity of expectation,

$$\mathbb{E}(X) = \sum_{v \in V} \mathbb{E}(X_v) = \sum_{v \in V} P(S'(v) = \phi) \leq |V| \frac{1}{n} = 1.$$

However, if $n > 2$, then without loss of generality A has more than 1 vertex. Crossing out each color from all vertices in A, we have that $S'(v) = \phi$ for all $v \in A$, and hence $X > 1 \geq \mathbb{E}(X)$ in this case. Therefore there must exist X so that $X < \mathbb{E}(X)$, i.e. $X = 0$,i.e. there exists a method of deletion so that every vertex still has a nonempty list.

Now color each vertex v with any color from $S'(v)$. For every color, it can only appear in B or only appear in A, since it was either crossed out from all lists in A or all lists in B. Since all edges are between A and B, this is a proper coloring.