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## 1 Kakeya problem

**Conjecture 1.1:** Let  $E \subseteq \mathbb{R}^n$  contain a line segment of length 1 in every direction (called a **Kakeya set**). Then

$$\dim_{\text{Haus}}(E) = n.$$

Note that Besicovich showed that a Kakeya set  $E$  can have  $\mu(E) = 0$ .

For  $n > 2$  this is a big open problem in harmonic analysis.

Tom Wolff proposed the discrete problem, over  $\mathbb{F}_q^n$ . Note here  $q$  is the parameter that varies rather than  $n$ . Think of it as a bigger and bigger discrete grid.

(Proof omitted here. See AMiT notes.)

Comments: In number theory we're inclined to use Lang-Weil. But here you get polynomials by interpolation so have no control over cohomology, etc. You only have Hilbert-polynomial like invariants like degree. All we really used is Bezout's Theorem: the intersection of a line and hypersurface either has  $\leq d$  points or it's a geometric containment. *What if you know more algebraic geometry; could you do more?*

A lot of people have used the "polynomial method" since: Tao, Guth, Katz, Quilodran, Dvir, Sharir, Bourgain, Zohl, Solymosi, Zhang, Iosevich,... They include combinatorialists, harmonic analysts, number theorists, etc.

## 2 Kakeya problem, analyst's version

Analysts like to think of operators on function spaces.

A subset corresponds to a characteristic function. Things that you can prove about subsets you should prove about all functions!

**Theorem 2.1** (Kakeya maximal function conjecture, Ellenberg, Oberlin, Tao 2010): Let  $f : \mathbb{F}_q^n \rightarrow \mathbb{R}$ . Define  $f^* : \mathbb{P}^{n-1}(\mathbb{F}_q) \rightarrow \mathbb{R}$  by

$$f^*(\omega) = \sup_{\ell \parallel \omega} \sum_{x \in \ell} |f(x)|.$$

Then

$$\|f^*\|_n < c_n q^{\frac{n-1}{n}} \|f\|_n$$

Note that the Kakeya condition on  $S$  tells us  $\chi_s^* = q$  for all  $\omega \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ , so this conjecture gives

$$q \cdot q^{\frac{n-1}{n}} < c_n q^{\frac{n-1}{n}} \underbrace{\|\chi_S\|_n}_{|S|^{\frac{1}{n}}}.$$

How to use the polynomial method? We use the multiplicity trick (Saraf-Sudan). Given a function  $f : \mathbb{F}_q^n \rightarrow \mathbb{Z}$ , interpolate a  $P$  of bounded degree vanishing to degree  $f(x)$  at  $x$ . This gives  $\sim f(x)^n$  linear conditions. Thus the  $L^n$  norm comes up naturally.

### 3 Open questions

Here's an open question.

**Problem 3.1** (Furstenberg-type problem): Let  $S \subseteq \mathbb{F}_q^n$  be a set such that for every  $K$ -plane  $V$ , there exists a  $k$ -plane  $W$  parallel to  $V$  such that  $|W \cap S| > q^n$ . Is  $|S| > q^{\frac{cn}{k}q}$ ?

Line segment intersects in set of Hausdorff dimension at least  $\beta$ .

**Theorem 3.2:** • (EOT) For  $c = k$ ,  $|S| > q^n(1 - q^{1-k})^{\binom{n}{2}}$ .

• (Zhang) For  $c = \frac{1}{2}$ ,  $\mathbb{F}_q = \mathbb{F}_p$ ,

$$|S| > q^{\frac{n}{2} + \frac{c}{n^2}}.$$

How does the finite field case differ from the Euclidean case? Dvir proves something too much!

- Besicovich: There exists a Kakeya set of measure 0.
- There is a Kakeya set in  $\mathbb{F}_q^n$  has measure  $> c_n \cdot (2^{-n})$ .

Scale is missing! We can't do multiscale analysis. (Terry Tao writes on graininess, planiness...) Lines can be clustered closely. In  $\mathbb{F}$ , two things are the same or not. There is no metric with multiple scales. In summary,

- $\mathbb{F}_q$  has only one scale.
- $\mathbb{R}$  has infinitely many scales.

**Problem 3.3:** Is the Kakeya conjecture true for

- $\mathbb{F}_q[[z]]$
- $\mathbb{Z}_p$
- $\mathbb{F}_q[t]/t^2$  (This has only 2 different scales, but it's hard! Try it! This is an infinitesimal step towards understanding how to approach the general problem!)

**Theorem 3.4** (Dummit, Hoblicsch): There exists Kakeya sets in  $\mathbb{F}_q[[t]]$  and  $\mathbb{Z}_p^n$  of measure 0.

Their sets do have Minkowski dimension  $n$ . Thus these rings are rich enough for the Besicovich phenomenon. Their construction is not related to Besicovich's construction; it's very clever. It's iterative, and uses a martingale theorem. The limit of a stochastic process has to converge to 0; a little measure seeps away each step.

## 4 Guth-Katz

Guth and Katz showed:

**Theorem 4.1** (Guth, Katz, 2010, conjecture of Bourgain): Let  $L$  be a set of  $N^2$  lines in  $\mathbb{R}^3$  so that no more than  $N$  lie in any plane. Let  $S$  be a set of points such that each line in  $L$  has at least  $N$  points of  $S$ . Then

$$|S| > cN^3.$$

Asymptotically, the lines don't intersect and each line contributes  $\gtrsim N$  points.

**Theorem 4.2** (Guth-Katz, 2010, Erdős distance conjecture):  $N$  points in  $\mathbb{R}^2$  determine at least  $\frac{N}{\ln N}$  distinct distances.

This is almost sharp,  $\frac{N}{\sqrt{\ln N}}$  is attainable.

The Elekes-Sharir setup is as follows (they set up a program for these types of problems):

- few distinct distances mean that
- there are many pairs  $(x_1, y_1)$  and  $(x_2, y_2)$  such that  $d(x_1, y_1) = d(x_2, y_2)$ .
- What's a grown-up way of saying this? That there exist  $g \in G$ , the group of rigid motions, such that  $g(x_1) = x_2, g(y_1) = y_2$ .  $G(x_1, x_2) = \{g \in G : gx_1 = x_2\}$  a translate of  $\text{SO}_2 \in G$ .
- There are many points of intersection between  $G(x_1, x_2)$  in  $G$ , 1-dimensional lines in 3-dimensional space. That is the main theorem in their paper.

A lot of problems in extremal combinatorial geometry (distinct areas of simplices, etc.), as long as we can express the problem as orbits of groups. There's a desire for incidence theory not just for linear subspaces, but about varieties, using their degrees.

Guth-Katz, in solving the Erdős problem, uses polynomials to cut  $\mathbb{R}^n$  into cells. *This seriously uses  $K = \mathbb{R}$ .*

Guth-Katz in solving Bourgain's conjecture, uses a purely algebraic approach that works over  $\mathbb{C}$ .

The Bourgain conjecture on  $\mathbb{F}_q$  would be: Given  $q^2$  lines in  $\mathbb{F}_q$ , no  $q$  of which lie on the same plane, the union of the lines has  $cq^3$  points. This is false!

There is a piece that looks like differential geometry: there are three vectors (?) that are planar. This is not true in characteristic  $p$ ! Planes are the enemy, but in characteristic  $p$ . There is a non-plane curve in  $\mathbb{F}_p$  all of whose points are flex points! They have degree  $\geq p$ .

**Theorem 4.3** (Ellenberg, Hrbáček, 2014): Bourgain is true for  $\mathbb{F}_p$ ,  $p$  prime.

What I'm working on now. We're not satisfied about theorems about sets of points in space. (That's a zero-dimension set in affine space.) We want geometric Kakeya: extremal problems for 0-dimensional subschemes of  $\mathbb{A}^n$ . From Ravi Vakil, to prove stuff about smooth curves, degenerate them until they're as non-smooth as they can be. This looks scary but they're easier. Degenerate to fat points at the origin, then recover stuff about smooth schemes.

Painful combinatorial difficulties can be ameliorated by passing to the maximal degenerate case.

Bombieri-Pila: The theory of unlikely intersections. Real algebraic geometry, transcendental theory. Hardest to show polynomial is not 0.

Bounding number of rational points on curves.