Contents

1 Distributions

1.1 The spaces

Problem: Define the 3 spaces of functions $\mathcal{D}, \mathcal{S}, \mathcal{E}$ and of the corresponding distributions. What are the inclusions? Do they have the subspace topology?

1. The function spaces are
$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n).$$
In short,
   - $\mathcal{D}(\mathbb{R}^n)$: bounded derivatives+compact support
   - $\mathcal{S}(\mathbb{R}^n)$: bounded derivatives+decay
   - $\mathcal{E}(\mathbb{R}^n)$: bounded derivatives.

   Formally, $\mathcal{S}, \mathcal{E}$ are locally convex spaces with the following seminorms (each is defined as the space of functions where all the norms exist and are finite, so $\mathcal{D} = C_c^\infty$, the functions of compact support, while $\mathcal{S}, \mathcal{D}$ are subsets of C^∞). $\mathcal{D}$ is a little different; see below.

Space	Seminorms
$\mathcal{D}(\mathbb{R}^n)$	$(\sup \lvert\partial^\alpha \varphi\rvert)_\alpha$
$\mathcal{S}(\mathbb{R}^n)$	$(\sup \lvert x^\alpha \partial^\beta \varphi\rvert)_{\alpha,\beta}$
$\mathcal{E}(\mathbb{R}^n)$	$(\sup_K \lvert\partial^\alpha \varphi\rvert)_{\alpha,K}$

   NOTE: The topology we use on $\mathcal{D}(\mathbb{R}^n)$ is actually stronger than the one given by the seminorms: for $f_n \to 0$ we require $\bigcup \mathrm{Supp}(f_n)$ to be contained in a compact set. A function that keeps spreading out does NOT converge to 0. Hence $\mathcal{D} \to \mathcal{S}$ is a continuous map but not homeomorphic onto its image.

2. The spaces of distributions are the dual spaces,

$$\mathcal{D}'(\mathbb{R}^n) \supset \mathcal{S}'(\mathbb{R}^n) \supset \mathcal{E}'(\mathbb{R}^n).$$

They have the w^*-topology.

We use the following from functional analysis (Lemma 2.2.5).

Lemma 1.1: If X is a locally convex space, then X^* is the set of linear functionals u for which there exists C and a finite subset S of seminorms such that

$$|ux| \le C \max_{\|\cdot\| \in S} \|x\| .$$

Equivalently, it is the set of linear functionals such that $x_n \to 0$ implies $ux_n \to 0$ (sequential continuity).

Space	$\forall u \ldots$
$\mathcal{D}'(\mathbb{R}^n)$	$\forall K \exists C, N : \langle u, \varphi \rangle \le C \sum_{\lvert\alpha\rvert \le N} \sup \lvert\partial^\alpha \varphi\rvert$
$\mathcal{S}'(\mathbb{R}^n)$	$\exists C, N : \langle u, \varphi \rangle \le C \sum_{\lvert\alpha\rvert,\lvert\beta\rvert \le N} \sup \lvert x^\alpha \partial^\beta \varphi\rvert$
$\mathcal{E}'(\mathbb{R}^n)$	$\exists C, N, K : \langle u, \varphi \rangle \le C \sum_{\lvert\alpha\rvert \le N} \sup_K \lvert\partial^\alpha \varphi\rvert$

We also have the inclusions $\mathcal{D} \subset \mathcal{E}'$ (because functions in $\mathcal{D}$ have compact support), $\mathcal{S} \subset \mathcal{S}'$, $\mathcal{E} \subset \mathcal{D}'$ (just use Cauchy-Schwarz to show sequential continuity).

1.2 Derivatives and convolution

Problem: Define the translation, reflection, and derivative of a distribution. Relate the derivative to the typical notion of a derivative. Show that if $u' = 0$ then $u = 0$.

1. Define $\tau_h f(x) = f(x-h)$, $\check{f}(x) = f(-x)$. Define the derivative by $\langle \partial u, f \rangle = -\langle u, \partial f \rangle$. This is consistent with the definition for functions by integration by parts.

2. The derivative is continuous. (Easy.)

3. Lemma: $\langle \partial u, f \rangle = \lim_{h \to 0} \left\langle \frac{\tau_{-h} u - u}{h}, f \right\rangle$. Proof: Bounce the difference quotient to the other side, and then use the fact that the difference quotient converges uniformly (the remainder in the Taylor expansion is uniformly bounded).

4. See Lemma 1.4.3.

5. Computing: use ample use of FToC. Use intuition to guess, ex. $H' = \delta_0$.

Problem: 1. Define convolution, and extend its definition as far as you can. What are all the pairs of spaces that you can take convolution between? What are the continuity properties of $*$? How can we write $\langle\rangle$ as $*$?

2. How does differentiation act with $*$?

3. Prove that $*$ is associative and commutative. (What do you need to show?)

4. Prove that $\mathcal{D} \subset \mathcal{D}'$ is dense.

1. We have

$$\text{eq:dist-conv1}\,\mathcal{D} * \mathcal{E}' \subseteq \mathcal{D} \qquad \text{b/c both of compact support} \tag{1}$$
$$\text{eq:dist-conv2}\,\mathcal{D} * \mathcal{D}' \subseteq \mathcal{E} \tag{2}$$
$$\text{eq:dist-conv3}\,\mathcal{E} * \mathcal{E}' \subseteq \mathcal{E} \tag{3}$$
$$\text{eq:dist-conv4}\,\mathcal{D}' * \mathcal{E}' \subseteq \mathcal{D}'. \tag{4}$$

   (a) Original definition: $f * g = \int f(x-y)g(y)\,dy$.

   (b) Extend: $u * \varphi(x) = \langle u, \tau_x\check{\varphi}\rangle$.

   (c) Extend by $(u_1 * u_2) * \varphi = u_1 * (u_2 * \varphi)$ to (4).

   We have
$$\langle u, \varphi\rangle = u * \check{\varphi}(0)$$
   (this is useful because now we can use associativity).

2. 1.4.8. $\partial^\alpha(u * \varphi) = u * \partial^\alpha\varphi$ by 1.4.7; in particular, convolution is smooth.

3. 1.4.9. Schematic of argument:
$$(u * \varphi) * \psi = \int \langle u, \ldots\varphi\rangle\,\psi = \int \langle u, \varphi \ldots \psi\rangle = \lim \sum \langle u, \varphi \ldots \psi\rangle = \left\langle u, \lim \sum \varphi \ldots \psi\right\rangle.$$
   We want the integral to go inside the brackets, so we have to turn it into a Riemann sum.

   For commutativity, we need to show $u_1 * u_2 = u_2 * u_1$ if one is in $\mathcal{E}'$, the other in $\mathcal{D}'$. Note $(u_1 * u_2) * (\varphi * \psi) = (u_1 * \psi) * (u_2 * \varphi)$. Key: we can commute φ, ψ.

4. Convolute u with a "good kernel" and then cutoff: $\varphi_m = \chi_m(u * \phi_m) \to u$ where $\chi_m = \chi\left(\frac{x}{m}\right)$ is a bump function and $\phi_m(x) = m^n\psi(mx)$. Now use $\langle u, \varphi\rangle = u * \check{\varphi}(0)$ and associativity to get $u * (\phi_m * (\chi_m\theta)\check{})(x)$. Now $\phi_m * (\chi_m\theta)\check{} = \langle \chi_m\tau_{-x}\phi_m, \theta\rangle$, and $\chi_m\tau_{-x}\phi_m$ is like a good kernel at x, so $\langle\varphi_m, \theta\rangle \to \langle u, \theta\rangle$.

1.3 The Fourier transform

Problem: 1. How do you extend the definition of the Fourier transform? Give an alternate definition that resembles the original definition; when is it valid? What is the connection between the Fourier and Laplace transform?

2. Prove the Fourier inversion formula.

3. Fill in the table below. (Be careful with the normalizations.) What are the Fourier transforms of $e^{-\varepsilon x^2}, 1, \delta_0$?

1. Define the Fourier transform by $\widehat{f} = \int_{\lambda \in \mathbb{R}^n} e^{-i\lambda \cdot x} f(x)\, dx$. (See p. 136, SS.)

   We have
$$\hat{u} = \left\langle u, e^{-i\lambda \cdot x} \right\rangle$$
   by exchanging double integrals.

2.

3.

f	$\widehat{f} = \int_{\lambda \in \mathbb{R}^n} e^{-i\lambda \cdot x} f(x)\, dx$	$\widehat{f} = \int_{\lambda \in \mathbb{R}^n} e^{-2\pi i\lambda \cdot x} f(x)\, dx$
Fourier inversion	$f = \frac{1}{2\pi} \int f(x) e^{i\lambda \cdot x}\, dx$	$f = \int f(x) e^{2\pi i\lambda \cdot x}\, dx$
Parseval	$\left\langle \widehat{f}, g \right\rangle = \langle f, \widehat{g} \rangle, \langle f, g \rangle = \frac{1}{2\pi} \left\langle \widehat{\check{f}}, \widehat{g} \right\rangle$	$\left\langle \widehat{f}, g \right\rangle = \langle f, \widehat{g} \rangle, \langle f, g \rangle = \left\langle \widehat{\check{f}}, \widehat{g} \right\rangle$
$f(x+h)$	$\widehat{f}(\lambda) e^{ih\lambda}$	$\widehat{f}(\lambda) e^{2\pi i h\lambda}$
$f(x)e^{-\varepsilon ih}$		$\widehat{f}$ when $\varepsilon = \frac{1}{2}$
$f(\delta x)$	$\delta^{-1} \widehat{f}(\delta^{-1}\lambda)$	$\delta^{-1} \widehat{f}(\delta^{-1}\lambda)$
f'	$i\lambda \widehat{f}$	$2\pi i\lambda \widehat{f}$
$Df = -if'$	$\lambda \widehat{f}$	$2\pi \widehat{f}$
xf	$-D\widehat{f}' = i\widehat{f}'$	$-\frac{1}{2\pi i} \widehat{f}'$
δ_0	1	1
1	$2\pi\delta_0$	δ_0
$e^{-\varepsilon x^2}$	$\sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{\lambda^2}{4\varepsilon}}$	$\sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{\pi}{\varepsilon}\lambda^2}$

Problem: 1. Define Sobolev space and local Sobolev space.

2. What are the inclusions?

3. "Fourier transform converts smoothness into decay." Make this statement precise and quantitative.

1. Definition **??** If $u \in S'(\mathbb{R}^n)$ such that $\hat{u}(\lambda)$ is a function and
$$\|u\|_{H^s}^2 := \int |\hat{u}(\lambda)|^2 (1 + |\lambda|^2)^s\, d\lambda < \infty$$

then we say $u \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. We say $u \in H^s_{\text{loc}}$ if $u\varphi \in H^s$ for every $\varphi \in \mathcal{D}$.

2. $H^{s>\frac{n}{2}}(\mathbb{R}^n) \subseteq C(\mathbb{R}^n)$ by Cauchy-Schwarz. (Careful: we need to show $\hat{u}$ as a function agrees with $\hat{u}$ as a distribution.)

   $\cap H^2(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n)$. (More generally, $H^{s>\frac{n}{2}+k}(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$. (I think so.) This shows $\hat{u}\,\langle\lambda\rangle^{-k} \in L^1$.)

   $D^\alpha : H^s \to H^{s-|\alpha|}$.

   $\mathcal{E}(\mathbb{R}^n) \subseteq \bigcup_s H^s(\mathbb{R}^n)$.

3. See above.

2 Applications of distributions

Problem: Define an elliptic operator.
Show the existence of a parametrix for an elliptic PDO. What space is it in?
State the elliptic regularity theorem.

1. (Definition **??**) $P(D), D = -i\partial$ corresponds to the polynomial $\sigma_P(\lambda)$. $P(D)$ is elliptic if for all $\sigma \neq 0$, $\sigma_P(\lambda) \neq 0$.

2. Define a parametrix to be $E \in \mathcal{D}'$ with $P(D)E = \delta_0 + \omega$, "fundamental solution up to $\mathcal{E}$." (Definition **??**.

   Lemma **??**:

   Lemma 2.1: Every elliptic PDO has a parametrix$\in \mathcal{S}'(\mathbb{R}^n)$.

   Proof: $\mathcal{F}^{-1}\left(\frac{1-\chi_R(\lambda)}{P(\lambda)}\right)$. Check $P(\lambda) \gtrsim \langle\lambda\rangle^N$ (Lemma **??**).

3. Note $E * f$ is more regular than f (compute the Sobolev norm). We get by writing $u = \delta_0 * u$ the following.

   Lemma 2.2: If $P(D)$ is elliptic of degree N; $P(D)u = f$ where $f \in H^s$; $u \in \mathcal{D}'$ with compact support (i.e., in $\mathcal{E}'$), then $u \in H^{N+s}$.

   Why is $\omega * u \in \mathcal{S}$? We don't have $\mathcal{E} * \mathcal{E}' \subseteq \mathcal{S}$!

4. Idea: Use the lemma to bootstrap.

   By definition of H^m_{loc}: it suffices to show $u\varphi \in H^{s+N}$ for every $u \in \mathcal{D}$.

   Use lemma to get

$$P(D)(u\varphi_M) = \underbrace{(P(D)u)}_{f\in H^s}\varphi + \underbrace{[P(D),\varphi]}_{-(N-1)}\underbrace{(\varphi_{M-1}u)}_{t}.$$

$$t_M \geq \min\{s, t_{M-1} - N + 1\} + N.$$

where t_k is the regularity of $u\varphi_k$. (Warning: $P(D)(uv) \neq vP(D)u + uP(D)v$. For example, for $P(D) = D^2$ it is $(u''v) + (2u'v' + uv'')$.) We inserted φ_M because then we can iterate the process. At each stage,

$$t_k \geq \min\{s + N, t_{k-1} + 1\}$$

so pick $M = s + N - t$.

Problem: Define a fundamental solution. Find the fundamental solution of the Cauchy-Riemann and heat operator. Prove the Malgrange-Ehrenpries Theorem.

1. A fundamental solution for a PDO $P(D)$ is $E \in \mathcal{D}'$ such that $P(D)E = \delta_0$. Then for any f, $E * f$ solves $P(D)u = f$. (Warning: uniqueness is a different issue.)

2. The fundamental solution of $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$[1] is $\frac{1}{\pi z}$.

   (Proof: Use Green's Theorem $\oint F \cdot dv = \iint_A \nabla \times F \, dA$ on $\frac{\partial}{\partial \bar{z}}(\varphi E) = E\frac{\partial}{\partial \bar{z}}\varphi$. Note we get a $-$ since the region $|z| > \varepsilon$ is *outside*.)

   The fundamental solution of $P(D) = \frac{\partial}{\partial t} - \Delta_x$ is $(t > 0) \cdot (4\pi t)^{-\frac{n}{2}} \exp\left(\frac{-|x|^2}{4t}\right)$.

   (Pf. E basically solves the heat equation except at $t = 0$. To get over this, bounce to φ, write $\int = \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty$, and then bounce back to E. Note we get a boundary term in the t direction, $(E\varphi)_t$) in the last IbP.)

3.

   Theorem 2.3: Every constant coefficient PDO has a fundamental solution.

   Proof.

   - Guess the solution. $\widehat{E} = \frac{1}{P(\lambda)}$, so

   $$\langle E, \varphi \rangle = \frac{1}{(2\pi)^n} \left\langle \widehat{E}, \widetilde{\widehat{\varphi}} \right\rangle = \frac{1}{(2\pi)^n} \int \frac{\widehat{\varphi}(-\lambda)}{P(\lambda)} \, d\lambda.$$

   - Establish an estimaate for $\widehat{\varphi}$, when the last variable is thought of as complex. If $P(D)$ is elliptic and $\varphi \in \mathcal{D}$, then

   $$|\hat{\varphi}(\lambda', z)| \lesssim_m (1 + |z|)^{-m} e^{\delta|\Im z|}, \qquad m \in \mathbb{N}_0.$$

     Key: We don't care about the exponential term outside the (compact) support of φ, so it is bounded by $e^{\delta|\Im z|}$. Bound $|z^m \widehat{\varphi}|$, and use IbP to get the derivative onto φ.

[1] We have $(z, \bar{z}) = (x + iy, x - iy)$ so $\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial \bar{z}}{\partial x} \\ \frac{\partial z}{\partial y} & \frac{\partial \bar{z}}{\partial y} \end{pmatrix}$. The Jacobian is $\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ with inverse $\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} \frac{1}{-2i} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, so $\begin{pmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$.

- By Cauchy's Theorem,

$$\langle E, \varphi \rangle = \frac{1}{(2\pi)^n} \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \int_{\Im \lambda_n = c_i} d\lambda_n \, \frac{\hat{\varphi}(-\lambda_1' - \lambda_n)}{P(\lambda)}$$

  is a solution provided we can bound $P(\lambda)$ away from 0 and have an estimate on $\widehat{\varphi}$. Where do we use the last condition? Also check this is a valid distribution.

  Figure of Hörmander's staircase.

  Given a point μ', we can bound $P(\mu', \lambda_n)$ away from 0 on some line $\Im \lambda_n = c$ because P is not identically 0 by ellipticity so P has finitely many roots. We can extend to neigborhoods by continuity. Now choose get Δ_i by a compactness argument.

4. Example of use: see example at the end of §4.2.

Problem: Prove the structure theorem for $\mathcal{E}'(X)$.

Theorem 2.4 (Structure Theorem for $\mathcal{E}'(X)$, **??**)**:** Every $u \in \mathcal{E}$ can be written as a linear combination of derivatives of continuous functions (supported in X). ("$\mathcal{E} \subseteq \operatorname{span}_{f \in C(X), \alpha} \partial^\alpha f$.")

Proof. Idea: The FT takes differential operators to polynomials. First show $u \in H^s$, it does not grow too fast. Transfer to a problem in the Fourier domain. Considering $\langle u, \varphi \rangle$, take a polynomial term away from $\widehat{u}$; this results in derivatives on φ back in the original domain.

1. We know $\widehat{u} \in H^s$ for some s. WHY? Thus $\widehat{u} \langle \lambda \rangle^{-2m} \in L^1 \cap C$ for $m > \frac{s-n}{4}$ by C-S, and it has a Fourier transform/inverse. (The Fourier inversion holds on continuous L^1 functions whose transform is also L^1 (the FT is always continuous).)

2. Note $\varphi \in \mathcal{E}(X)$ may not be the FT of a function. But all functions in $\mathcal{D} \subseteq \mathcal{S}$ are FT's of functions. First step: because u has compact support, for a bump funtion ρ, $\langle u, \varphi \rangle = \langle u, \rho\varphi \rangle$.

3. Now

$$\begin{aligned}
\langle u, \rho\varphi \rangle &= \frac{1}{(2\pi)^n} \langle (\widehat{u})\check{}, \widehat{\rho\varphi} \rangle \\
&= \frac{1}{(2\pi)^n} \left\langle \underbrace{\langle \lambda \rangle^{-2m} \widehat{u}}_{\in L^1}\check{}, \widehat{\rho\varphi} \langle \lambda \rangle^{2m} \right\rangle \\
&= \langle \cdots, P(D)(\rho\varphi) \rangle = \cdots .
\end{aligned}$$

□

Problem: State and prove the Paley-Wiener-Schwartz Theorem.

Theorem 2.5 (??): Write $\mathcal{D}_X(\mathbb{R}^n)$ for the functions in $\mathcal{D}(\mathbb{R}^n)$ supported on X[2]. The Fourier-Laplace transform is a bijection

$$\mathcal{D}_{\overline{B_\delta}} \xrightarrow{\widehat{\bullet}} \left\{\text{entire } u : \forall N, \frac{\widehat{u}(z)}{e^{\delta|\Im z|}} \prec \langle\lambda\rangle^N\right\}$$
$$\mathcal{E}'_{\overline{B_\delta}}(\mathbb{R}^n) \xrightarrow{\widehat{\bullet}} \left\{\text{entire } u : \exists N, \frac{\widehat{u}(z)}{e^{\delta|\Im z|}} \precsim \langle\lambda\rangle^N\right\}.$$

Proof. 1. Bound the growth of $\hat{u}$ as a complex analytic function (Lemma **??**): If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\mathrm{Supp}(u) \subseteq \overline{B_\delta} = \{x \in \mathbb{R}^n : |x| \le \delta\}$ then there exists $N \ge 0$ such that

$$|\hat{u}(z)| \precsim (1+|z|)^N e^{\delta|\Im z|}$$

for each $z \in \mathbb{C}^n$.

Proof: If u were a function, we would write out the integral and note that it vanishes outside the support of u. Here we have to treat u as a distribution, and use

$$\left\langle u, e^{-ix\cdot z}\right\rangle \le C \sum_{|\alpha|\le N} \sup_K |\partial^\alpha e^{-ix\cdot z}|.$$

Can we take $K = \overline{B_\delta}$? In general, we can't always take $K = \mathrm{Supp}(u)$, but in this case we almost can. To do this, replace u by $u\rho_\varepsilon$ where $\rho_\varepsilon = \psi(\frac{1}{\varepsilon}(|x|-\delta))$ where $\psi = \begin{cases} 1, & x \le 0 \\ 0, & x \ge 1. \end{cases}$. We get

$$\left\langle u, e^{-ix\cdot z}\right\rangle \le \sum_{|\alpha_1+\alpha_2|\le N} C_{\alpha_1,\alpha_2} \sup_K \partial^{\alpha_1}\rho_\varepsilon \partial^{\alpha_2}\varphi \precsim \left(\frac{1}{\varepsilon}\right)^{|\alpha_1|} e^{(\delta+\varepsilon)|\Im z|}(|\sup_K \partial^{\alpha_2}\varphi|).$$

Take $\varepsilon \to 0$? Seems like an $o(1)$ in the exponent.

2. (1) To show rapid decay, consider $z^\alpha \widehat{u}$ for all α.

$$|z^\alpha \widehat{u}(z)| \precsim \int (D^\alpha u)e^{-i\lambda\cdot z}\,dx = \int u(-ix)^\alpha \underbrace{e^{-i\lambda\cdot z}}_{\text{compact support}} \le e^{\delta|\Im z|}$$

Conversely, define u by the inverse Fourier transform

$$u = \frac{1}{(2\pi)^n}\int_{\mathbb{R}} \underbrace{\widehat{u}(z)}_{|\cdot|\le(1+|z|^{-N})e^{\delta 0}} e^{ix\cdot z}\,dz.$$

convergent by rapid decay. To use the bound, $e^{\delta\Im z}$ we have to shift the contour of integration to $\mathbb{R}+i\eta$, so that we get the integrand ($\Im z = \eta$)

$$|\cdot| \le e^{-x\cdot\eta+\delta\eta}(1+|z|)^{-N}.$$

[2] we may as well write $\mathcal{D}(\mathbb{R}^n)$, right?

Given $|x| > \delta$, choose η in the direction of x, $\to \infty$. (N.B. η is a vector.)

(2) We showed the forward direction. For the reverse, regularize by $\varphi_\varepsilon = \varphi\left(\frac{x}{\varepsilon}\right)$.

$$|\widehat{u * \varphi_\varepsilon}| \precsim |U(\lambda)\widehat{(}\varphi_\varepsilon)| \le U(\lambda)\varepsilon e^{\varepsilon\lambda}|$$

so $\mathrm{Supp}(u_\varepsilon) \subseteq \overline{B_{\delta+\varepsilon}}$. Now continuity of $*$ gives $\varphi_\varepsilon \xrightarrow{\mathcal{S}'} \delta_0 \implies u * \varphi_\varepsilon \xrightarrow{\mathcal{E}'} u$.

□

Problem: Can we take $K = \mathrm{Supp}(u)$ in the definition of $\mathcal{E}'$?

3 Oscillatory integrals

Problem:

1. Define an oscillatory integral. (Define a phase function and space of symbols first.)
2. State and prove the lemma of stationary phase.
3. Write $I_\Phi(a)$ as the limit of a sequence of functions in $\mathcal{D}$.
4. Define singular support and find a set that it's included in.
5. Find the singular support for the solution E to the wave equation $\frac{1}{c^2}E_{tt} - \Delta_x E = 0$, $E(0,x) = 0$, $E_t(0,x) = \delta_0(x)$.

"Prerequisite": Recall the IbP argument for: if all the derivatives of f up to degree k are L^1, then $\widehat{f} = O\left(\frac{1}{|f|^k}\right)$.

1. An **oscillatory integral** is denoted

$$I_{\Phi,a} = \int e^{i\Phi(x,\theta)} a(x,\theta)\, d\theta.$$

and is the distribution $\in \mathcal{D}'$ defined by

$$\langle I_{\Phi,a}, \varphi\rangle = \iint e^{i\Phi(x,\theta)} a(x,\theta)\varphi(x)\, dx\, d\theta$$

where

- $\Phi : X \times \mathbb{R}^k \to \mathbb{R}$ is a **phase function** (i.e., homogeneous of degree 1 in θ, smooth on $X \times \mathbb{R}^k \backslash 0$, continuous on $X \times \mathbb{R}^k$, $d\Phi \neq 0$). Example: $x \cdot \theta$.
- $a \in \mathrm{Sym}(X, \mathbb{R}^n; N)$ is a **symbol** (i.e., a function $X \times \mathbb{R}^n \to \mathbb{R}$ with the growth condition

$$|\partial_x^\alpha \partial_\theta^\beta a| \le \langle\theta\rangle^{N-\beta})$$

Example: a polynomial in θ of degree N. (It is easy to see $\mathrm{Sym}(X, \mathbb{R}^n)$ is a graded algebra, D_x^α doesn't change the order, D_θ^β reduces it by $|\beta|$.)

Notes for remembering:

(a) The domain for x does not have to be $\mathbb{R}^m$, it can be some $X \subseteq \mathbb{R}^m$; the domain for θ has to be $\mathbb{R}^n$.

(b) Note that the integral HAS to be taken over x first because compact support of φ makes the integral converge. You cannot use Fubini to interchange limits because the integrand does not converge absolutely—it oscillates in θ.

2. Lemma of stationary phase (Lemma **??**): Let $\Phi \in C^\infty(\mathbb{R})$ such that

- $\Phi(0) = \Phi'(0) = 0$,
- $\Phi'(\theta) \neq 0$ on $\mathbb{R}\backslash\{0\}$, and
- $\Phi''(\theta) \neq 0$ for all θ.

Then for $\chi \in D(\mathbb{R})$ we have

$$\left|\int e^{i\lambda\Phi(\theta)}\chi(\theta)\,d\theta\right| \precsim |\lambda|^{-\frac{1}{2}}, \quad |\lambda| \to \infty.$$

Relation to oscillatory integrals: This is NOT an OI because Φ is not homogeneous of degree 1 in θ (ex. $\Phi = \theta^2$). This shows what goes wrong: if we then integrate over λ now the integral won't converge. "We expect major contributions from points satisfying $\partial_\theta\Phi = 0$," so?

Proof: (1) Use a bump function to kill the bad point 0. Let ρ be bump at 0; shrink it to 0. Write

$$\chi(\theta) = \rho\left(\frac{\theta}{\delta}\right)\chi + \underbrace{\left(\rho\left(\frac{\theta}{\delta}\right)\right)\chi}_{\varphi}.$$

The integral of the first part is bounded by $\boxed{\delta}$. (2) Invent L that is constant on the first term, and then IbP N times. Let

$$L = \frac{1}{i\lambda\Phi'(\theta)}\frac{d}{d\theta} \implies L* = \frac{\Phi''}{i\lambda\Phi'^2} - \frac{1}{i\lambda\Phi'}\frac{d}{d\theta} = -\frac{1}{i\lambda\Phi'}\frac{d}{d\theta} + O(\lambda^{-1}|\theta|^{-2}).$$

Iterate N times to get terms of the form

$$\sum_{\alpha+\beta=N} \lambda^{-N}\Phi'^{-N-\alpha}\frac{d^\beta}{d\theta^\beta}P_{\alpha,\beta}(\Phi'', \Phi''', \ldots),$$

$P_{\alpha,\beta}$ polynomials (induct). Now use

(a) $|\Phi'| \succsim |\theta|$, by $\Phi'' \neq 0$ (hence it doesn't change sign and has a minimum $|\cdot|$).

(b) ∂_θ throws off $\frac{1}{\delta}$.

(c) ψ is only nonzero when $|\theta| > \delta$.

Taking $(\alpha,\beta)=(N,0),(0,N)$, get the integrand

$$\lesssim \max(\lambda^{-N}|\theta|^{-2N},\delta^{-N}\lambda^{-N}|\theta|^{-N}) \lesssim \boxed{\lambda^{-N}\delta^{-2N+1}}.$$

Balancing with δ, let $\delta=|\lambda|^{-\frac{1}{2}}$.

3. (Theorem **??**) If Φ is a phase function and $a\in \mathrm{Sym}(X,\mathbb{R}^k;N)$ then $I_\Phi(a):=\lim_{\varepsilon\searrow 0} I_{\Phi,\varepsilon}(a)$ belongs to $D'(X)$ and has order no greater than $N+k+1$. Here

$$I_{\Phi,\varepsilon}(a)=\int e^{i\Phi(x,\theta)}a(x,\theta)\chi(\varepsilon\theta)\,d\theta$$

where $\chi\in D(\mathbb{R}^n)$ is fixed with $\chi=1$ on $|\theta|<1$.

(a) Find a differential operator with $L^*e^{i\Phi}=e^{i\Phi}$ and iterate it. Baby example:

$$\iint e^{i\theta x}f\,dx\,d\theta = -\iint e^{i\theta x}\frac{1}{i\theta}f'\,dx\,d\theta = -\int \frac{1}{i\theta}\widehat{f'}(\theta)\,d\theta = \int \widehat{f}(\theta)\,d\theta = 2\pi f(0)$$

so $I_{\theta x}(1)=2\pi\delta_0$.
(Lemma **??**) Find a differential operator L of the form

$$L=\sum_{j=1}^k a_j(x,\theta)\frac{\partial}{\partial\theta_j}+\sum_{j=1}^n b_j(x,\theta)\frac{\partial}{\partial x_j}+c(x,\theta)$$

such that $L^*e^{i\Phi}=e^{i\Phi}$. We want it to be a "differential operator on $\mathrm{Sym}(X,\mathbb{R}^k)$ of some order." (We can think of differential operators on $\mathrm{Sym}(X,\mathbb{R}^k)$ as $\mathrm{Sym}(X,\mathbb{R}^k)[\partial_{x_j},\partial_{\theta_j}]$ where ∂_x are order 0 and ∂_θ are order -1.)
We are given the condition $d\Phi\neq 0$ which we use. Note that $\Phi_{\theta_j}\in \mathrm{Sym}(X,\mathbb{R}^k;0)$, $\Phi_{x_j}\in\mathrm{Sym}(X,\mathbb{R}^k;1)$. In the below, the purple is what we get from $\partial_{\theta_j},\partial_{x_j}$. The rest is the a_j and b_j. The underbrace is the order of the differential operator; the ones underneath the purple represent the order of $\partial_{\theta_j},\partial_{x_j}$.

$$\widetilde{L}\frac{1}{\underbrace{|\theta|^2}_{2}\underbrace{|\nabla_\theta\Phi|^2}_{0}+\underbrace{|\nabla_x\Phi|^2}_{2}}\left(\underbrace{|\theta|^2}_{2}\underbrace{\sum(-i\Phi_{\theta_j})}_{-1}\underbrace{(i\Phi_{\theta_j})e^{i\Phi}}_{\text{(from diff.) }-1}+\underbrace{\sum(-i\Phi_{x_j})}_{0}\underbrace{(i\Phi_{x_j})}_{\text{(from diff.) }0}\right)$$

To fix blowup of the denominator near 0, use instead $L=(1-\rho)\widetilde{L}+\rho$ for a bump function ρ. (Behavior near 0 doesn't affect the order.)

(b) We have

$$I_{\Phi,\varepsilon}(a)=\iint e^{i\Phi(x,\theta)}a(x,\theta)\chi(\varepsilon\theta)\varphi(x)\,dx\,d\theta$$

. Put in L^M, integrate by parts, noting that since L decrease order by 1, taking $M=N+k+1$ suffices (as a has order N, $\chi(\varepsilon\theta)\varphi(x)$ has order 0 uniformly in ε (easy), and $\int_{\mathbb{R}^k}\langle\theta\rangle^{-k-1}\,d\theta$ converges).

4. (Definition **??**) The **singular support** of a distribution is defined to be the complement of the union of all the open sets on which the distribution is smooth:

$$\operatorname{sing\,supp}(u) = \left(\bigcup \{U \text{ open} : u \text{ smooth on } U\}\right)^c.$$

(Theorem **??**) If Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then $\operatorname{sing\,supp} I_\Phi(a) \subseteq M(\Phi)$ where

$$M(\Phi) = \left\{x : \nabla_\theta \Phi(x, \theta) = 0 \text{ for some } \theta \in (\mathbb{R}^k \backslash \{0\}) \cap \mathrm{Supp}[a(x, \theta)]\right\}$$

(a) Lemma **??**: If Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then the function

$$x \mapsto \int e^{i\Phi(x,\theta)} a(x, \theta)\rho(\theta)\, d\theta$$

is smooth for any $\rho \in \mathcal{D}(\mathbb{R}^k)$.

(b) Write $a = a\rho + a(1 - \rho)$, so we may assume $a = 0$ in a neighborhood of $x = 0$.

(c) Take x_0 such that $|\nabla_\theta \Phi(x_0, \theta)| \neq 0$, and a neighborhood N where it's bounded below. We need to show that for $\mathrm{Supp}(\psi) \in N$ that $\psi I_\Phi(a)$ is smooth.

(d) Find $Le^{i\Phi} = e^{i\Phi}$. We can take $L = -i\frac{1}{|\nabla_\theta \Phi|^2}\Phi_\theta \cdot \nabla_\theta$ (We're using that $|\nabla_\theta \Phi(x_0, \theta)| \gtrsim 1$ here, so L can be chosen to not involve x.) Write $I = \lim_{\varepsilon \to 0^+} I_{\Phi,\varepsilon}(a)$ and IbP M times. Choose M large to make the integral absolutely convergent.

5. Take a Fourier transform in x to obtain an ODE in t. Take the Fourier inverse and separate out behavior near 0 to get