Functional Analysis

Contents

Introduction

András Zsák (a.zsak@dpmms.cam.ac.uk) taught a course on Functional Analysis at the University of Cambridge in Michaelmas (Fall) 2013. The website is `http://www.dpmms.cam.ac.uk/~az10000`. These are my "live-TEXed" notes from the course. The template was created by Akhil Mathew.

We'll cover the main theorems that are useful in many other areas.

Here's an outline of the course.

1. Introduction
2. Hahn-Banach Theorem
3. Riesz Representation Theorem (dual of continuous functions in a compact Hausdorff space)
4. Weak topologies (weak and weak*, Banach–Alaoglu, etc.)
5. Convexity and Krein-Milman Theorem
6. Banach algebras
7. Holomorphic functional calculus
8. C^*-algebras (a type of Banach algebra)
9. Borel functional calculus and spectral theory

Here are some recommended books.

1. G. R. Allan: Introduction to Banach Spaces and Algebras, OUP
2. B. Bollobas: Linear Analysis.
3. G. J. Murphy: C^*-algebras and operator theory
4. W. Rudin: Real and Complex Analysis, Functional Analysis

Prerequisites are linear analysis, complex analysis (section 7), and measure theory (sections 3, 9).

Chapter 1

Introduction

1 Normed spaces

Definition 1.1.1: A **normed space** is a pair $(X, \|\cdot\|)$ where X is a real or complex vector space and $\|\cdot\|$ is a norm on X. Most of the time the choice of scalar field makes little difference; for convenience we'll use real scalars. A norm induces a metric: $d(x, y) = \|x - y\|$. This induces a topology on X, called the **norm topology**. A **Banach space** is a complete normed space.

Example 1.1.2:

1. (sequences) For $1 \leq p < \infty$, we have $\ell_p = \{(x_n) \text{ scalar sequence} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ with norm $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$. (Minkowski's inequality says that if $x, y \in \ell_p$ then $x + y \in \ell_p$, so $\|x + y\|_p \leq \|x\|_p + \|y\|_p$. Then ℓ_p is a Banach space.

2. (convergent sequences) $\ell_\infty = \{(x_n) \text{ scalar sequence} : (x_n) \text{ is bounded}\}$ with $\|x\|_\infty = \sup_{n \in N} |x_n|$. Then ℓ_∞ is a Banach space.

   $c_{00} = \{(x_n) \text{ scalar sequence} : \exists N \forall n > N, x_n = 0\}$. Let $e_n = (0, 0, \ldots, 0, \underbrace{1}_{n}, 0, \ldots)$; then $c_{00} = \operatorname{span}\{e_n : n \in \mathbb{N}\}$. Note that c_{00} is a subspace of ℓ_∞ but it's not closed: In $\ell_p, 1 \leq p < \infty$, $\ell_p = \overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\}$.

   $c_0 = \{(x_n) \in \ell_\infty : \lim_{n\to\infty} x_n = 0\}$ is a closed subspace of ℓ_∞, $c_0 = \overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\}$ in ℓ_∞.

   $c = \{(x_n) \in \ell_\infty : \lim_{n\to\infty} x_n \text{ exists}\}$ is a closed subspace of ℓ_∞. c_0 and c are Banach spaces.

3. (Euclidean space) $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p), 1 \leq p \leq \infty$.

4. K is any set, $\ell_\infty(K) = \{f : K \to \mathbb{R} : f \text{ is bounded}\}$ with norm $\|f\|_\infty = \sup_{x \in K} |f(x)|$. This is a Banach space, e.g. $\ell_\infty = \ell_\infty(\mathbb{N})$.

5. K compact topological space $C(K) = \{f \in \ell_\infty(K) : f \text{ continuous}\} = \{f : K \to \mathbb{R} : f \text{ is continuous}\}$. $C(K)$ is a closed subspace of $\ell_\infty(K)$ because any uniform limit of continuous functions is continuous, and hence it's a Banach space, e.g. $C[0, 1]$.

   We'll write $C^{\mathbb{R}}(K)$ and $C^{\mathbb{C}}(K)$ for the real and complex versions of $C(K)$, respectively.

6. Let (Ω, Σ, μ) be a measure space. Then for $1 \leq p < \infty$,
$$L_p(\mu) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable}, \int_\Omega |f|^p \, d\mu < \infty \right\}$$
with norm $\|f\|_p = (\int_\Omega |f|^p \, d\mu)^{\frac{1}{p}}$ is a Banach space (after identifying functions that are equal almost everywhere.

   When $p = \infty$, $L_\infty(\mu) = \{f : \Omega \to \mathbb{R} : f \text{ is measurable and essentially bounded}\}$. ("Essentially bounded" means that there exists a null-set N such that f is bounded on $\Omega \backslash N$.)
$$\|f\|_\infty = \operatorname{ess\,sup} |f| = \inf_N \sup_{\Omega \backslash N} |f|.$$

7. Hilbert spaces, e.g. ℓ_2, $L_2(\mu)$. All Hilbert spaces are isomorphic, but some different representation may be more natural.

Proposition 1.1.3: Let X, Y be normed spaces, $T : X \to Y$ linear. Then the following are equivalent.

1. T is continuous.
2. T is bounded: $\exists C \geq 0$, $\|Tx\| \leq C \|x\|$ for all $x \in X$.

Proof. To think about continuity at a, "translate" to 0 using linearity. □

Definition 1.1.4: Let $\mathcal{B}(X, Y) = \{T : X \to Y : T \text{ is linear and bounded}\}$. This is a normed space with the **operator norm**: $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$. T is an **isomorphism** if T is a linear bijection whose inverse is also continuous. (This is equivalent to T being a linear bijection and there existing $a > 0, b > 0$, with $a\|x\| \leq \|Tx\| \leq b\|x\|$ for all $x \in X$.)

If there exists such T, we say X, Y are isomorphic and we write $X \sim Y$.

If $T : X \to Y$ is a linear bijection such that $\|Tx\| = \|x\|$ for all $x \in X$ (i.e. $a = b = 1$), then T is an **isometric isomorphism** and we say X, Y are isometrically isomorphic and write $X \cong Y$[1].

$T : X \to Y$ is an **isomorphic embedding** if $T : X \to TX$ is an isomorphism. We write $X \hookrightarrow Y$.

Proposition 1.1.5: If Y is complete, then $\mathcal{B}(X, Y)$ is complete. In particular, $X^* = \mathcal{B}(X, \mathbb{R})$, the space of bounded linear functionals, called the **dual space** of X, is always complete.

Example 1.1.6:
1. For $1 < p < \infty$, then $\ell^* \cong \ell_q$ where $\frac{1}{p} + \frac{1}{q} = 1$. The proof uses Hölder's inequality: $x = (x_n) \in \ell_p$, $y = (y_n) \in \ell_q$ then $\sum |x_n y_n| \leq \|x\|_p \|y\|_q$. This isomorphism is $\varphi : \ell_q \to \ell_p^*, y \mapsto \varphi_y, \varphi_y(x) = \sum x_n y_n$.)
2. $c_0^* \cong \ell_1$, $\ell_1^* \cong \ell_\infty$. (Later we will see that c_0 cannot be a dual space.)

[1]Some people use $\cong$ for isomorphism. We use it to mean isometric isomorphism.

3. If H is a Hilbert space then $H^* \cong H$ (Riesz Representation Theorem).

4. If (Ω, Σ, μ) is a measure space, $1 < p < \infty$, then $L_p(\mu)^* \cong L_q(\mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$.
   If μ is σ-finite then $L_1(\mu)^* \cong L_\infty(\mu)$. (Else we only have $L_\infty(\mu) \hookrightarrow L_1(\mu)^*$.)

Lecture 2

Recall that if V is a finite-dimensional vector space, then any two norms on V are equivalent. Specifically, if $\|\cdot\|$ and $\|\cdot\|'$ are two norms on V, then there exist $a, b > 0$ such that

$$a\|x\| \leq \|x\|' \leq b\|x\| \, \forall x \in V.$$

In other words, $\mathrm{Id} : (V, \|\cdot\|) \to (V, \|\cdot\|')$ is an isomorphism.

Some consequences are the following.

Corollary 1.1.7:

1. If X, Y are normed spaces, $\dim X < \infty$, $T : X \to Y$ is linear, then T is bounded.

2. If $\dim X < \infty$ then X is complete.

3. If X is a normed space and E a subspace with $\dim E < \infty$, then E is closed.

Proof.

1. Set $\|x\|' = \|x\| + \|Tx\|$. This is a norm on X, so there exists $b > 0$, $\|x\|' \leq b\|x\|$ for all x, so $\|Tx\| \leq b\|x\|$ for all $x \in X$.

2. By (1), $X \sim \ell_2^n$ where $n = \dim X$.

□

2 Riesz's lemma and applications

The unit ball is compact and this characterizes finite-dimensionality. We use the following.

Lemma 1.2.1 (Riesz's Lemma)**:** lem:riesz Let Y be a proper closed subspace of a normed space X. Then for every $\varepsilon > 0$ there exists $x \in X$, $\|x\| = 1$, such that $d(x, Y) := \inf_{y \in Y} \|x - y\| > 1 - \varepsilon$.

We'd like to take the some sort of "perpendicular" vector to Y, or the vector which minimizes the distance from a point not on Y to Y. Note this is not in general possible since X may not be complete, and hence the "inf." However, we can come arbitrarily close to that inf, and get an "almost perpendicular" vector.

Proof. Pick $z \in X \backslash Y$ with Y proper. Since Y is closed, $d(z, Y) > 0$. There exists $y \in Y$ such that $\|z - y\| < \frac{d(z,Y)}{1-\varepsilon}$ (WLOG $\varepsilon < 1$).

Set $x = \frac{z-y}{\|z-y\|}$. Then

$$d(x, Y) = d\left(\frac{z-y}{\|z-y\|}, Y\right) = \frac{1}{\|z-y\|} d(z - y, Y) = \frac{d(z, Y)}{\|z-y\|} > 1 - \varepsilon.$$

□

We give two applications. First, some notation. In a metrix space (M, d), write

$$B(x, r) := \{y \in M : d(x, y) \leq r\}, x \in M, r \geq 0$$

for the closed ball of radius r at x. In a normed space,

$$B_X := B(0, 1) = \{x \in X : \|x\| \leq 1\}, \qquad B(x, r) = x + rB_X.$$

Also, $S_X = \{x \in X : \|x\| = 1\}$.

Theorem 1.2.2: Let X be a normed space. Then $\dim X < \infty$ iff B_X is compact.

Proof. "$\Rightarrow$" We have $X \sim \ell_2^n$ where $n = \dim X$.

"$\Leftarrow$" By compactness there exist $x_1, \ldots, x_n \in B_X$ such that $B_X \subseteq \bigcup_{i=1}^n B(x_i, \frac{1}{2})$. Let $Y = \operatorname{span}\{x_1, \ldots, x_n\}$. For all $x \in B_X$ there exists $y \in Y$ with $\|x - y\| \leq \frac{1}{2}$, so $d(x, Y) \leq \frac{1}{2}$. Thus there do not exist "almost orthogonal vectors" in the sense of Riesz's Lemma 1.2.1. This means Y is not a proper subspace of X, so X is finite-dimensional. □

Remark: We showed the following in the proof: If Y is a subspace of a normed space X and there exists $0 \leq \delta < 1$ such that for all $x \in B_X$ there exists $y \in Y$ with $\|x - y\| \leq \delta$, then Y is dense in X.

If we let $\delta = 1$ then this statement is trivial. The remark says that if we can do a little better than 1, the trivial estimate, then we can automatically approximate x with much smaller ε.

Theorem 1.2.3 (Stone-Weierstrass Theorem)**:** Let K be a compact topological space and A be a subalgebra of $C^{\mathbb{R}}(K)$. If A separates the points of K (i.e., for all $x \neq y$ in K, there exists $f \in A$, $f(x) \neq f(y)$), and A contains the constant functions, then A is dense in $C^{\mathbb{R}}(K)$.

In this case it does matter whether the field of scalars is $\mathbb{R}$ or $\mathbb{C}$.

The following proof is due to T. J. Ransford.

Proof. First we show that if E, F are disjoint closed subsets of K, then there exists $f \in A$ such that $-\frac{1}{2} \leq f \leq \frac{1}{2}$ on K and $f \leq -\frac{1}{4}$ on E and $f \geq \frac{1}{4}$ on F.

Fix $x \in E$. Then for all $y \in F$, there exists $h \in A$ such that $h(x) = 0$, $h(y) > 0$, $h \geq 0$ on K. (This is since A separates points, we can shift by a constant, and square the function.) Then there is an open neighborhood of y on which $h > 0$. An easy compactness argument gives that there exists $g = g_x \in A$ with $g(x) = 0$, $g > 0$ on F, $0 \leq g \leq 1$ on K. Pick $R = R_x \in \mathbb{N}$ such that $g > \frac{2}{R}$ on F, set $U = U_x = \left\{y \in K : g(y) < \frac{1}{2R}\right\}$.

Do this for all $x \in E$. Compactness gives a finite cover: there exist $x_1, \ldots, x_m$ such that $E \subseteq \bigcup_{i=1}^m U_{x_i}$. To simplify notation, set $g_i = g_{x_i}$, $R_i = R_{x_i}$, $U_i = U_{x_i}$, and $i = 1, \ldots, m$. For $n \in \mathbb{N}$, by Bernoulli's inequality,

$$\text{on } U_i \qquad (1 - g_i^n)^{R_i^n} \geq 1 - (g_i R_i)^n > 1 - 2^{-n} \to 1 \text{ as } n \to \infty$$

$$\text{on } F \qquad (1 - g_i^n)^{R_i^n} \leq \frac{1}{(1 + g_i^n)^{R_i^n}} \leq \frac{1}{(g_i R_i)^n} < \frac{1}{2^n} \to 0 \text{ as } n \to \infty$$

There exists $n_i \in \mathbb{N}$ such that $h_i = 1 - (1 - g_i^{n_i})^{R_i^{n_i}}$ satisties

- on U_i, $h_i \leq \frac{1}{4}$
- on F, $h_i \geq \left(\frac{3}{4}\right)^{\frac{1}{m}}$
- on K, $0 \leq h_i \leq 1$.

Set $h = h_1 h_2 \cdots h_m$. Then $h \leq \frac{1}{4}$ on E, $h \geq \frac{3}{4}$ on F, and $0 \leq h \leq 1$ on K. Set $f = h - \frac{1}{2}$. Given $g \in C^{\mathbb{R}}(K)$, $\|g\|_\infty \leq 1$, set

$$E = \left\{x \in K : g(x) \leq -\frac{1}{4}\right\},\ F = \left\{x \in K : g(x) \geq \frac{1}{4}\right\}.$$

Let $f \in A$ be as above. Then $\|f - g\| \leq \frac{3}{4}$, i.e., $d(g, A) \leq \frac{3}{4}$. By Riesz's Lemma 1.2.1, A is dense in $C^{\mathbb{R}}(K)$. □

Remark: The complex version says that if A is a subalgebra of $C^{\mathbb{C}}(K)$ that separates points of K contains the constant functions, and is closed under complex conjugation ($f \in A \implies \overline{f} \in A$), then A is dense in $C^{\mathbb{C}}(K)$.

3 Open mapping lemma

We'll assume the Baire category theorem and its consequences: principle of uniform boundedness, open mapping theorem (OMT), closed graph theorem (CGT).

Definition 1.3.1: Let A, B be subsets of a metric space (M, d) and let $\delta \geq 0$. Say A is **δ-dense** in B if for all $b \in B$ there exists $a \in A$ with $d(a, b) \leq \delta$.

Lemma 1.3.2 (Open mapping lemma)**:** lem:oml Let X, Y be normed spaces, X complete, $T \in \mathcal{B}(X, Y)$. Assume for some $M \geq 0$ and $0 \leq \delta < 1$ that $T(MB_X)$ is δ-dense in B_Y. Then T is surjective. More precisely, for all $y \in Y$ there exists $x \in X$ such that $y = Tx$ and

$$\|x\| \leq \frac{M}{1-\delta} \|y\|,$$

i.e.,

$$T\left(\frac{M}{1-\delta} B_X\right) \supseteq B_Y.$$

Moreover, Y is complete.

Proof. The proof involves successive approximations. Let $y \in B_Y$. There exists $x_1 \in MB_X$ with $\|y - Tx_i\| \leq \delta$. Then $\frac{y - Tx_i}{\delta} \in B_Y$. There exists $x_2 \in MB_X$, with $\left\|\frac{y-Tx_i}{\delta} - Tx_2\right\| \leq \delta$, i.e., $\|y - Tx_1 - \delta Tx_2\| \leq \delta^2$, and so forth. Obtain (x_n) in MB_X such that

$$\left\|y - Tx_1 - \delta Tx_2 - \cdots - \delta^{n-1} Tx_n\right\| \leq \delta^n$$

for all n. Set $x = \sum_{n=1}^\infty \delta^{n-1} x_n$. This converges since $\sum_{n=1}^\infty \|\delta^{n-1} x_n\| \leq M \sum_{n=1}^\infty \delta^{n-1} = \frac{M}{1-\delta}$, and X is complete.[2] So $x \in \frac{M}{1-\delta} B_X$ and by continuity $Tx = \sum_{n=1}^\infty \delta^{n-1} Tx_n = y$. For the

[2] This kind of geometric sum argument comes up a lot in functional analysis!

"moreover" part, let $\hat{Y}$ be the completion of Y, and view T as a map $X \to \hat{Y}$. Since B_Y is dense in $B_{\hat{Y}}$, $T(MB_X)$ is δ'-dense in $B_{\hat{Y}}$ for $\delta < \delta' < 1$. By the first part, $T(X) = \hat{Y} = Y$, so Y is complete. □

Remark: Suppose $T \in \mathcal{B}(X, Y)$, X is complete, and the image of the ball is dense: $\overline{T(B_X)} \supseteq B_Y$. Suppose that for all $\varepsilon > 0$, $T((1+\varepsilon)B_X)$ is 1-dense in B_Y. Take $M > 1, 0 < \delta < 1$ so that $1+\varepsilon = \frac{M}{1-\delta}$; lemma 1.3.2 shows that $T\left((1+\varepsilon)B_X\right) \supseteq B_Y$. It follows that $T(B_X^\circ) \supseteq B_Y^\circ$. (For a subset A of a topological space, A° or $\mathrm{int}(A)$ denotes the interior of A.)

Lecture 3

3.1 Applications of the open mapping lemma

Theorem 1.3.3 (Open mapping theorem)**:** thm:omt Let X, Y be Banach spaces, $T \in \mathcal{B}(X, Y)$ be onto. Then T is an open map.

Proof. Let $Y = T(X) = \bigcup_{n=1}^{\infty} T(nB_X)$. The Baire category theorem tells us that there exists N with $\mathrm{int}(\overline{T(NB_X)}) \neq \phi$. Then there exists $r > 0$ with

$$\overline{T(NB_X)} \supseteq rB_Y.$$

By Lemma 1.3.2, for $M = \frac{2N}{r}$, we have $T(MB_X) \supseteq B_Y$. Therefore, U is open and $T(U)$ is open. □

Theorem 1.3.4 (Banach isomorphism theorem)**:** thm-bit If in addition T is injective, then T^{-1} is continuous.

Proof. An open map that is a bijection is a homeomorphism. □

Theorem 1.3.5 (Closed graph theorem)**:** Let X, Y be Banach spaces and $T : X \to Y$ be linear. Assume that whenever $x_n \to 0$ in X, $Tx_n \to y$ in Y, then $y = 0$. Then T is continuous.

This is a powerful result. Usually we have to show the sequence converges and show it converges to 0. This says that we only have to check the second part given the first part.

Proof. The assumption says that the graph of T

$$\Gamma(T) = \{(x, Tx) : x \in X\}$$

is closed in $X \oplus Y = \{(x, y) : x \in X, y \in Y\}$ with norm, e.g.

$$\|(x, y)\| = \|x\| + \|y\|.$$

So $\Gamma(T)$ is a Banach space. Consider $U : \Gamma(T) \to X$, $U(x, y) = x$. U is a linear bijection and $\|U\| \leq 1$. From the Banach isomorphism theorem 1.3.4, U^{-1} is continuous, i.e., $x \mapsto (x, Tx)$ is continuous. □

We'll give three more applications. The first one is to quotient spaces. Let X be a normed space and Y be a closed subspace. Then $X/Y = \{x + Y : x \in X\}$ is a normed space with

$$\|x + Y\| = d(x, Y) = d(0, x + Y) = \inf\{\|x + y\| : y \in Y\}.$$

We need Y closed to ensure that if $\|z\| = 0$ then $z = 0$ for $z \in X/Y$.

Proposition 1.3.6: pr:5Let X, Y be as above. If X is complete, then so is X/Y.

Proof. Consider the quotient map $q : X \to X/Y$, $q(x) = x + Y$. This is a bounded linear map, $q \in \mathcal{B}(X, X/Y)$, so

$$\|q(x)\| = d(x, Y) \le \|x\|$$

and $q(B_X^\circ) \subseteq B_{X/Y}^\circ$. If $\|x + Y\| < 1$ then there exists $y \in Y$ with $\|x + y\| < 1$ and $q(x + y) = q(x) = x + Y$ so $q(B_X^\circ) = B_{X/Y}^\circ$. So for any $M > 1$, $\overline{q(MB_X)} \supseteq B_{X/Y}$. By Lemma 1.3.2, X/Y is complete. $\square$

Proposition 1.3.7: Every separable Banach space X is (isometrically isomorphic to) a quotient of ℓ_1.

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be dense in B_X. Let (e_n) be the standard basis of ℓ_1. If $a = (a_n) \in \ell_1$, then $a = \sum_{n=1}^\infty a_n e_n$. Define

$$T : \ell_1 \to X, \qquad T\left(\sum_{n=1}^\infty a_n e_n\right) = \sum_{n=1}^\infty a_n x_n.$$

This is well defined because the sum converges: $\sum_{n=1}^\infty \|a_n x_n\| \le \sum_{n=1}^\infty |a_n| = \|a\|$. So $T \in \mathcal{B}(\ell_1, X)$, $\|T\| \le 1$. Also $T(B_{\ell_1}^\circ) \subseteq B_X^\circ$. We have $T(B_{\ell_1}) \supseteq \{x_n : n \in \mathbb{N}\}$, so $\overline{T(B_{\ell_1})} \supseteq B_X$.

So $T(B_{\ell_1}^\circ) = B_X^\circ$. We have a unique $\widetilde{T} : \ell_1/\ker T \to X$ such that

$$\begin{tikzcd}
\ell_1 \arrow[rr, "T"] \arrow[dr, "q"'] & & X \\
& \ell_1/\ker T \arrow[ur, "\widetilde{T}"'] &
\end{tikzcd}$$

commutes, i.e., $T = \widetilde{T}q$ where q is the quotient map. Moreover, $\widetilde{T}$ is a linear bijection

$$\widetilde{T}(B_{\ell_1/\ker T}^\circ) = \widetilde{T}(q(B_{\ell_1}^\circ)) = T(B_{\ell_1}^\circ) = B_X^\circ.$$

So $\widetilde{T}$ is an isometric isomorphism, and $X \cong \ell_1/\ker T$. $\square$

This suggests that to understand all Banach spaces we just have to understand ℓ_1. But ℓ_1 is quite complicated, not as innocent as it looks.

Recall the following definition.

Definition 1.3.8: A topological space K is **normal** if whenever E, F are disjoint closed sets in K, there are disjoint open sets U, V in K such that $E \subseteq U$, $F \subseteq V$.

Lemma 1.3.9 (Urysohn's lemma)**:** lem:urysohn If K is normal and E, F are disjoint closed subsets of K, then $\exists$ continuous $f : K \to [0,1]$ such that $f = 0$ on E and $f = 1$ on F.

This can be used to construct partitions of unity which we'll use in chapter 3.

Theorem 1.3.10 (Tietze's Extension Theorem)**:** If K is normal and L is a closed subset of K, $g : L \to \mathbb{R}$ is bounded and continuous, then there exists a bounded, continuous $f : K \to \mathbb{R}$ such that $f|_L = g$, $\|f\|_\infty = \|g\|_\infty$.

Proof. Let $C_b(K) = \{h : K \to \mathbb{R} : h \text{ bounded, continuous}\}$, a closed subspace of $\ell_\infty(K)$ in $\|\cdot\|_\infty$ (so it is a Banach space). Consider $R : C_b(K) \to C_b(L)$, $R(f) = f|_L$. We need $R(B_{C_b(K)}) = B_{C_b(L)}$ ("$\subseteq$" is clear, since $\|R\| \le 1$.) Let $g \in B_{C_b(L)}$ so $-1 \le g \le 1$. Let $E = \left\{y \in L : g(y) \le -\frac{1}{3}\right\}$, $F = \left\{y \in L : g(y) \ge \frac{1}{3}\right\}$. Urysohn's lemma gives $\exists f \in C_b(K)$ such that $-\frac{1}{3} \le f \le \frac{1}{3}$, $f = -\frac{1}{3}$ on E, and $f = \frac{1}{3}$ on F.

We have
$$\|R(f) - g\|_\infty \le \frac{2}{3}$$
and $\|f\| \le \frac{1}{3}$. So $R(\frac{1}{3}B_{C_b(K)})$ is $\frac{2}{3}$-dense in $B_{C_b(L)}$. By the Open Mapping Lemma 1.3.2, R is surjective. $\square$

Remark: The theorem holds in the complex case too.

Chapter 2

Hahn-Banach Theorem

1 Hahn-Banach Theorem

For a normed space X, the dual space X^* is the space of bounded linear functionals with the operator norm

$$\|f\| = \sup_{x \in B_X} |f(x)|, \qquad f \in X^*.$$

This gives

$$\forall f \in X^*, \forall x \in X, \qquad |f(x)| \leq \|f\| \, \|x\| \, .$$

We will use the notation $\langle x, f \rangle$ for $f(x)$ to remind us of inner products.

Example 2.1.1:

1. $\ell_p^* \cong \ell_q$ where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$
2. $c_0^* \cong \ell_1$
3. $\ell_1^* \cong \ell_\infty$
4. $H^* \cong H$ (conjugate linear in the complex case)

First we need to show this space is not trivial. We will give 3 versions of the Hahn-Banach Theorem. The first is purely algebraic, with no topology.

Definition 2.1.2: Let X be a real vector space. A map $p : X \to \mathbb{R}$ is

1. **positively homogeneous** if $p(tx) = tp(x)$ for all $x \in X$ and all $t \geq 0$,
2. **subadditive** if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

For example, a norm has these properties.

Theorem 2.1.3 (Hahn-Banach Theorem)**:** thm:hb1 Let X be a real vector space and $p : X \to \mathbb{R}$ positively homogeneous and subadditive. Let Y be a subspace of X and $g : Y \to \mathbb{R}$ a linear map dominated by p on Y: $g(y) \leq p(y)$ for all $y \in Y$.

Then g extends to a linear map $f : X \to \mathbb{R}$ which is dominated by p by $X \to \mathbb{R}$:

$$f|_Y = g, f(x) \leq p(x) \quad \forall x \in X.$$

We can always extend by 1 dimension. Then we use transfinite induction/axiom of choice/Zorn's lemma.

A **poset** is a pair $(P, \leq)$ where P is a set, $\leq$ is a partial order on P (reflexive, antisymmetric, transitive). A **chain** is a subset C of P linearly ordered by $\leq$, i.e., $\forall x, y \in C$, either $x \leq y$ or $y \leq x$. $x \in P$ is a maximal element if for all $y \in P$, $x \leq y \implies x = y$. A subset A of P is **bounded above** if there exists $x \in P$ such that for all $a \in A, a \leq x$.

Lemma 2.1.4 (Zorn's lemma)**:** lem:zorn A nonempty poset in which every chain is bounded above has a maximal element.

Proof of 2.1.3. Let

$$P = \{(Z, h) : Z \text{ is a subspace of } X, Z \supseteq Y, h : Z \to \mathbb{R} \text{ linear}, h|_Y = g, h(z) \leq p(z) \forall z \in Z\}.$$

For $(Z_1, h_1), (Z_2, h_2) \in P$, $(Z_1, h_1) \leq (Z_2, h_2)$ iff $Z_1 \subseteq Z_2, h_2|_{Z_1} = h_1$. This is a partial order on P. Note $P \neq \phi$ since $(Y, g) \in P$.

If $C \subseteq P$ is a nonempty chain, say $C = \{(Z_i, h_i) : i \in I\}$, then let $Z = \bigcup Z_i$. This is a subspace of X, $Y \subseteq Z$. Define $h : Z \to \mathbb{R}$ as follows: for $x \in Z$, there exists $i \in I$ such that $x \in Z_i$. Let $h(x) = h_i(x)$. The definition is independent of i by the way we defined a chain. So $(Z, h) \in P$ and $(Z_i, h_i) \leq (Z, h) \forall i \in I$. So by Zorn's lemma 2.1.4 P has a maximal element (Z, f). We need $Z = X$.

Assume not. Pick $x_0 \in X \backslash Z$. Let $Z_1 = \operatorname{span}\{Z \cup \{x_0\}\}$. We choose $\alpha \in \mathbb{R}$ and define $f_1 : Z_1 \to \mathbb{R}$ by $f_1(z + tx_0) = f(z) + \alpha t$ for $z \in Z, t \in \mathbb{R}$. We have f_1 is linear and $f_1|_{Z_1} = f$.

We want to choose α such that

$$\text{eq:fa1} f_1(z + tx_0) = f(z) + t\alpha \leq p(z + tx_0) \quad \forall z \in Z, t \in \mathbb{R}. \tag{2.1}$$

(Then $(Z_1, f_1) \in P$, $(Z, f) < (Z_1, f_1)$, contradiction.) Note (2.1) holds for $t = 0$. Considering $t > 0, t < 0$ and using positive homogeneity, 2.1 is equivalent to

$$\begin{aligned} f_1(z + x_0) = f(z) + \alpha \leq p(z + x_0) \quad \forall z \in Z \\ f_1(z - x_0) = f(z) - \alpha \leq p(z - x_0) \quad \forall z \in Z, \end{aligned}$$

i.e.

$$f(z_1) - p(z_1 - x_0) \leq \alpha \leq p(z_2 + x_0) - f(z_2) \forall z_1, z_2 \in Z.$$

Such an α exists iff

$$\text{eq:f2} \sup_{z_1 \in Z}(f(z_1) - p(z_1 - x_0)) \leq \inf_{z_2 \in Z}[p(z_2 + x_0) - f(z_2)]. \tag{2.2}$$

We have

$$f(z_1) + f(z_2) = f(z_1 + z_2) \leq p(z_1 + z_2) = p(z_1 - z_0 + z_2 + x_0) \leq p(z_1 - x_0) + p(z_2 + x_0).$$

So (2.2) holds. $\square$

This is a result over $\mathbb{R}$. To extend to $\mathbb{C}$, we need p to be a seminorm.

Definition 2.1.5: A **seminorm** on a real or complex vector space X is a function $p : X \to \mathbb{R}$ such that

1. $p(x) \geq 0 \quad \forall x \in X$.
2. $p(\lambda x) = |\lambda| p(x) \quad \forall x \in X$ and any scalar λ.
3. $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in X$.

Note that a norm is a seminorm, and a seminorm is postively homogeneous and subadditive. The additional condition is nonnegativity.

Theorem 2.1.6 (Hahn-Banach Theorem for seminorms)**:** thm:hb2 Let X be a real or complex vector space $p : X \to \mathbb{R}$ a seminorm on X, Y a subspace of X, g a linear functional on Y such that $|g(y)| \leq p(y) \forall y \in Y$. Then g extends to a linear functional f on X such that $|f(x)| \leq p(x) \forall x \in X$.

Proof. The real case follows from Theorem 2.1.3: $g(y) \leq |g(y)| \leq p(y)$ for all $y \in Y$. By Theorem 2.1.3, there exists $f : X \to \mathbb{R}$ linear such that $f(x) \leq p(x)$ for all $x \in X$. So $-f(x) = f(-x) \leq p(-x) = p(x)$. So $|f(x)| \leq p(x)$ for all $x \in X$.

For the complex case, we note when you have a complex space and a complex linear function, you can take the real part and apply the real case. When you have a real linear map on a complex space, it is the real part of a unique complex map.

Let $g_1(y) = \Re g(y)$, $g_2(y) = \Im g(y), y \in Y$. Then g_1, g_2 are real linear functionals on Y. We have

$$g(iy) = g_1(iy) + i g_2(iy)$$
$$g(iy) = i g(y) = i g_1(y) - g_2(y)$$

so $g_2(y) = -g_1(iy)$. So $g(y) = g_1(y) - i g_1(iy)$. Have $|g_1(y)| \leq |g(y)| \leq p(y)$ for all $y \in Y$. By the real case there exists a real linear functional f_1 on X such that $f_1|_Y = g_1$ with $|f_1(x)| \leq p(x)$ for all $x \in X$. Define $f(x) = f_1(x) - i f_1(ix)$. Clearly f is $\mathbb{R}$-linear, and $f(ix) = f_1(ix) - i f_1(-x) = i f(x)$. So f is $\mathbb{C}$-linear. For all $y \in Y$,

$$f(y) = f_1(y) - i f_1(iy) = g_1(y) - i g_1(iy) = g(y).$$

So $f|_Y = g$. Given $x \in X$, choose $\theta \in \mathbb{R}$ such that $|f(x)| = e^{i\theta} f(x)$. Then

$$|f(x)| = f(e^{i\theta} x) = f_1(e^{i\theta} x) \leq p(e^{i\theta} x) = p(x).$$

□

Corollary 2.1.7: cor:hb Let X, p be as in Theorem 2.1.6. Let $x_0 \in X$. Then there exists a linear functional f on X such that $|f(x)| \leq p(x)$ for all $x \in X$, and $f(x_0) = p(x_0)$.

Proof. Let $Y = \operatorname{span}\{x_0\}$, define $g(\lambda x_0) = \lambda p(x_0)$ and apply Theorem 2.1.6. □

Theorem 2.1.8 (Hahn-Banach Theorem (existence of norming functionals))**:** thm:hb3 Let X be a normed space.

1. (Hahn-Banach extension theorem) If Y is a subspace of X, $g \in Y^*$, then $\exists f \in X^*$ such that $f|_Y = g$ and $\|f\| = \|g\|$.

2. For all $x_0 \in X$, $x_0 \neq 0$, $\exists f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.

Proof. 1. For all $y \in Y$, $|g(y)| \leq \|g\| \|y\|$. Apply Theorem 2.1.6 with $p(x) = \|g\| \|x\|$, $x \in X$.

2. Apply Corollary 2.1.7 with $p(x) = \|x\|$, $x \in X$.

$\square$

Remark:

By (2), $\forall x \neq y$ in X, $\exists f \in X^*$ such that $f(x) \neq f(y)$. In other words, the dual space "separates points"; it is rich.

(1) is a kind of linear version of Tietze's extension theorem. (Recall that Urysohn says that for a compact Hausdorff space, for any two points x, y there is a continuous function distinct at x and y; we can actually do this with disjoint closed sets.) Later we will see the Hahn-Banach separation theorem: given two convex sets we can separate them by a hyperplane.

f in (2) is called a **support functional at** x_0 or a **increasing functional at** x_0.

[A picture here]
$x_0 \in S_X, f \in S_{X^*}, f(x_0) = \|x_0\| = 1$. Note f is not unique.
We give some applications.

Definition 2.1.9: The **second dual** or **bidual** of a normed space X is $X^{**} := (X^*)^*$.

Theorem 2.1.10: The canonical map $X \to X^{**}$, $x \mapsto \hat{x}$ where $\hat{x}(f) = f(x)$ is an isometric embedding into X^{**}.

Proof. $\hat{x}$ is linear, $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\|$, so $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leq \|x\|$. Clearly, $x \mapsto \hat{x}$ is a linear map. By Theorem 2.1.8,

$$\|\hat{x}\| = \sup_{f \in B_{X^*}} |\hat{x}(f)| = \sup_{f \in B_{X^*}} |f(x)| = \|x\| .$$

$\square$

Remark: 1. For $x \in X$, $f \in X^*$, $\langle x, f \rangle = f(x) = \hat{x}(f) = \langle f, \hat{x} \rangle$. (Note $x \in X$, $f \in X^*$, $\hat{x} \in X^{**}$.)

2. $\hat{X} = \{\hat{x} : x \in X\}$ is closed in X^{**} iff X is complete. In general, the closure in X^{**} of $\hat{X}$ is a Banach space of which $\hat{X}$ is a dense subspace. (Why?) We have proved the existence of completions.

Definition 2.1.11: X is **reflexive** if $\hat{X} = X^{**}$.

Example 2.1.12: The following are reflexive.

1. For $\ell_p, 1 < p < \infty$ ($\ell_p^{**} \cong \ell_q^* \cong \ell_p$ for $\frac{1}{p} + \frac{1}{q} = 1$).
2. Any Hilbert space H ($H^* \cong H$)
3. $L_p(\mu), 1 < p < \infty$.
4. Any finite-dimensional space.

Warning: $X \cong X^{**}$ does not imply X is reflexive. The canonical map has to be an isomorphic isomorphism. For example, the James's space have codimension 1 in its second dual.

Thus in the examples, we have to check that the given isomorphism is in fact the embedding.

Note c_0 is not reflexive; $c_0^{**} = \ell_\infty$ and ℓ_∞ is non-separable. ℓ_0, ℓ_∞ are also not reflexive. Note X^* reflexive implies X reflexive.

Lecture 5

Definition 2.1.13: Let X, Y be normed spaces and $T \in \mathcal{B}(X, Y)$. The dual operator $T^* : Y^* \to X^*$ is defined by

$$T^*(g) = g \circ T,$$

i.e., $T^*(g)(x) = g(Tx)$ or $\langle x, T^*g \rangle = \langle Tx, g \rangle$ for $x \in X, g \in Y^*$.

One checks that T^* is well-defined, linear, and bounded. For boundedness, we have

$$\begin{aligned} \|T^*\| &= \sup_{g \in B_{Y^*}} \|T^*g\| \\ &= \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |T^*(g)(x)| \\ &= \sup_{x \in B_X} \underbrace{\sup_{g \in B_{Y^*}} |g(Tx)|}_{=\|Tx\| \text{ by Theorem 2.1.8(2)}} \\ &= \|T\|. \end{aligned}$$

Proposition 2.1.14: We have the following properties.

1. $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$.
2. If $S, T \in \mathcal{B}(X, Y)$ and λ, μ are scalars, then $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$. (Note there is no complex conjugation.)

   So $T \mapsto T^*$ is an isometric isomorphism of $\mathcal{B}(X, Y)$ into $\mathcal{B}(Y^*, X^*)$.
3. If $S \in \mathcal{B}(X, Y)$, $T \in \mathcal{B}(Y, Z)$, then $(TS)^* = S^*T^*$. So if $X \sim Y$ then $X^* \sim Y^*$. (The converse is false.)

4. For $T \in \mathcal{B}(X, Y)$ the following diagram commutes.[1]

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\hat{}"'] & Y \arrow[d, "\hat{}"] \\ X^{**} \arrow[r, "T^{**}"] & Y^{**} \end{tikzcd}$$

Proof. We just show the last statement. Let $x \in X, g \in Y^*$. Taking T to the other side and using the definition of $\hat{}$,

$$\left\langle g, \widehat{Tx} \right\rangle = \langle Tx, g \rangle = \langle x, T^* g \rangle = \langle T^* g, \hat{x} \rangle = \langle g, T^{**} \hat{x} \rangle .$$

$\square$

Example 2.1.15: For $X = \ell_p$, $1 \le p < \infty$ or c_0, $Y = X^* = \ell_q$, or ℓ_1, respectively, where $\frac{1}{p} + \frac{1}{q} = 1$, if $T : X \to X$ is the right shift $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$, then $T^* : Y \to Y$ is the left shift $T(x_1, x_2, \ldots) = (x_2, \ldots)$.

Theorem 2.1.16: If X^* is separable, then so is X.

Proof. Let $\{f_n : n \in \mathbb{N}\}$ be dense in S_{X^*}. For all $n \in \mathbb{N}$ there exists $x_n \in B_X$ such that $|f_n(x_n)| > \frac{1}{2}$. Let $Y = \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$.

We claim that $Y = X$; then we're done. Suppose not. Then $X/Y \ne 0$, so applying Hahn-Banach Theorem 2.1.8(2) to a nonzero vector in X/Y, there exists a $h \in S_{(Y/X)^*}$ with $\|h\| = 1$. Let $f = h \circ q$, where $q : X \to X/Y$ is the quotient map. We have

$$f(B_X^\circ) = h(q(B_X^\circ)) = h(B_{X/Y}^\circ).$$

So

$$\|f\| = \sup_{\|x\|<1, x \in X} |f(x)| = \sup_{\|z\|<1, z \in X/Y} |h(z)| = \|h\| = 1.$$

Note $Y \subseteq \ker f$ since $Y \subseteq \ker q$. There exists $n \in \mathbb{N}$ such that $\|f - f_n\| < \frac{1}{10}$. Then $\frac{1}{2} < |f_n(x_n)| = |f_n(x_n) - f(x_n)| < \frac{1}{10}$, contradiction. $\square$

Theorem 2.1.17: thm:embed-liy Every separable space embeds $X \hookrightarrow \ell_\infty$ isometrically.

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be dense in X. For all n there exists $f_n \in S_{X^*}$ such that $f_n(x_n) = \|x_n\|$ (Theorem 2.1.8(2)), WLOG $x_n \ne 0$.

For all $x \in X$, $(f_n(x))_{n=1}^\infty \in \ell_\infty$ and $|f_n(x)| < \|x\|$ for all n. We have a bounded linear map $T : X \to \ell_\infty$, $Tx = (f_n(x))_{n=1}^\infty$ and $\|Tx\| \le \|x\|$ for all $x \in X$. Given $x \in X, \varepsilon > 0$, there exists n such that $\|x - x_n\| < \varepsilon$. Then

$$\|Tx\| \ge |f_n(x)| \ge |f_n(x_n)| - |f_n(x_n - x)| = \|x_n\| - |f_n(x_n - x)|$$
$$\|x\| - \|x - x_n\| - \|x - x_n\| > \|x\| - 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\|Tx\| \ge \|x\|$. $\square$

[1] In other words, $\hat{}$ is a natural transformation from Id to **.

We have the following questions.

1. Does there exist separable Z such that every separable $X \hookrightarrow Z$ isomorphically? Isometrically? Yes, we'll do this later with $Z = C[0,1]$. (Note that ℓ^∞ is not separable.)

2. Does there exist a reflexive Z such that every separable reflexive $X \hookrightarrow Z$? Yes, trivially, take the direct sum of all reflexive subspaces of ℓ_∞. What if we require Z to be separable? The answer is no, but this is harder.

Theorem 2.1.18 (Vector-valued version of Liouville's Theorem)**:** thm:vv-liouville Let X be a complex Banach space and $f : \mathbb{C} \to X$ be a bounded analytic function. Then f is constant.

Proof. f is bounded, i.e., there exists $M \geq 0$ such that $\|f(z)\| \leq M$ for all $z \in \mathbb{C}$. f is analytic, i.e., $\lim_{z\to w} \frac{f(z)-f(w)}{z-w}$ exists for every $w \in \mathbb{C}$. (Convergence is in norm.)

Fix $\Lambda \in X^*$. Consider $\Lambda \circ f : \mathbb{C} \to \mathbb{C}$. This is bounded since

$$|\Lambda \circ f(z)| \leq \|\Lambda\| \, \|f(z)\| \leq M \, \|\Lambda\| \qquad \forall z \in \mathbb{C}$$

This is analytic (by linearity), so by scalar-valued Liouville, $\Lambda \circ f$ is constant. We get

$$\Lambda \circ f(0) = \Lambda \circ f(z) \qquad \forall z \in \mathbb{C}, \forall \Lambda \in X^*.$$

By Hahn-Banach (Theorem 2.1.8(2)), $f(0) = f(z)$ for all $z \in \mathbb{C}$. $\square$

We need to generalize from normed spaces to more general spaces. The wider context is topological vector spaces though we won't cover this.

2 Locally convex spaces

Definition 2.2.1: A **locally convex space** (LCS) is a pair $(X, \mathcal{P})$ where X is a real or complex vector space and $\mathcal{P}$ is a family of seminorms on X that separate points of X, i.e., $\forall x \in X, x \neq 0, \exists p \in \mathcal{P}$ such that $p(x) \neq 0$. X carries a topology defined as follows.

$U \subseteq X$ is open iff for all $x \in U$ there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$, $\varepsilon > 0$ such that

$$\{y \in X : p_i(y - x) < \varepsilon \text{ for } 1 \leq i \leq n\} \subseteq U.$$

(I.e., these sets form a neighborhood base.)

Remark: 1. It is easy to check addition and scalar multiplication are continuous.

2. If Y is a subspace of X then Y is a LCS with seminorms $\{p|_Y : p \in \mathcal{P}\}$. The topology on Y is the subspace topology.

3. The topology on X is Hausdorff. It's metrizable iff there exists a family Q of seminorms on X which is equivalent to $\mathcal{P}$ (yields the same topology) which is countable.

Definition 2.2.2: A complete metrizable LCS is a **Fréchet space**.

Example 2.2.3: 1. A normed space $(X, \|\cdot\|)$ is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.

2. Let $U \subset \mathbb{C}$ open and $\mathcal{O}(U) = \{f : U \to \mathbb{C} : f \text{ is analytic}\}$.

   For $K \subseteq U$, K compact, let $p_K(f) = \sup_{z \in K} |f(z)|$, $f \in \mathcal{O}(U)$, $\mathcal{P} = \{p_K : K \subseteq U, K \text{ compact}\}$. So $(\mathcal{O}(U), \mathcal{P})$ is a LCS. The topology is the topology of local uniform convergence.

   It's metrizable: there exists compact $K_n \subseteq U$, $n \in \mathbb{N}$ such that $U = \bigcup_{n=1}^{\infty} K_n$, and $K_n \subsetneq \operatorname{int}(K_{n+1})$ for all n. We have a countable family of seminorms $\{p_{K_n} : n \in \mathbb{N}\}$ equivalent to $\mathcal{P}$, so $(\mathcal{O}(U), P)$ is metrizable. It is in fact a Fréchet space.

   Note $\mathcal{O}(U)$ is not normable. There does not exist a single norm $\|\cdot\|$ equivalent to $\mathcal{P}$. (Hint: Use Montel's Theorem on normal families.)[2]

Lecture 6

Example 2.2.4: Let Ω be an open subset of $\mathbb{R}^n$. Let

$$C^\infty(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is infinitely differentiable}\}.$$

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ let $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. For compact $K \subseteq \Omega$, $\alpha \in \mathbb{N}_0^n$, define $\rho_{\alpha,K}(f) = \sup_{x \in K} |(D^\alpha f)(x)|$ and

$$\mathcal{P} = \{\rho_{\alpha,K} : \alpha \in \mathbb{N}_0^n, \text{ compact } K \subseteq \Omega\}.$$

Then $(C^\infty(\Omega), \mathcal{P})$ is a locally convex space. This is a Fréchet space, not normable. This arises in the theory of distributions.

Lemma 2.2.5: lem:f2-9 Let $(X, \mathcal{P})$, $(Y, \mathcal{Q})$ be locally convex spaces, and $T : X \to Y$ be linear. The following are equivalent.

1. T is continuous at 0.
2. T is continuous.
3. For all $q \in Q$, there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n$, $C \geq 0$ such that $q(Tx) \leq C \max_{1 \leq i \leq n} p_i(x)$.

Proof. (1) $\iff$ (2) follows from continuity of addition in a LCS.

(3) $\implies$ (1): given a neighborhood V of 0 in Y, there exists $m \in \mathbb{N}$, $q_1, \ldots, q_m \in Q$, $\varepsilon > 0$ such that

$$\{y \in Y : q_j(y) \leq \varepsilon \forall j\} \subseteq V.$$

For each $1 \leq j \leq m$, there exist $n_j \in \mathbb{N}$, $p_{j1}, \ldots, p_{jn_j}$ in $\mathcal{P}$, $c_j \geq 0$ such that $q_j(Tx) \leq C_j \max_{1 \leq i \leq n_j} p_{ji}(x)$. Then $U = \left\{x \in X : p_{j_i}(x) \leq \frac{\varepsilon}{C_j} \text{ for } 1 \leq j \leq m, 1 \leq i \leq n_j\right\}$ is a neighborhood of 0 in X, and $T(U) \subseteq V$.

---

[2]

Theorem (Montel's Theorem): (`http://en.wikipedia.org/wiki/Montel's_theorem`) A uniformly bounded family of holomorphic functions defined on an open subset of $\mathbb{C}$ is normal. A family is **normal** if every sequence has a subsequence which converges uniformly on every compact subset.

(1) $\implies$ (3) Given $q \in Q$, there exists a neighborhood U of 0 in X such that $T(U) \subseteq \{y \in Y : q(y) \leq 1\}$

There exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$, $\varepsilon > 0$ such that $\{x \in X : p_i(x) \leq \varepsilon \forall i\} \subseteq U$. Given $x \in X$, if $\max_i p_i(x) > 0$, then $\frac{\varepsilon x}{\max_i p_i(x)} \in U$ so $q\left(\frac{\varepsilon T(x)}{\max_i p_i(x)}\right) \leq 1$ and therefore we have $q(Tx) \leq \frac{1}{\varepsilon} \max_i p_i(x)$.

If $\max_i p_i(x) = 0$, then $\lambda x \in U$ for all scalars λ, so $|\lambda| q(Tx) \leq 1$ for all scalars λ, so $q(Tx) = 0$, so $q(Tx) \leq \frac{1}{\varepsilon} \max_i p_i(x)$ holds. □

Definition 2.2.6: For a LCS X, we let X^* be the space of all continuous linear functionals on X.

Theorem 2.2.7 (Hahn-Banach Theorem for LCSs)**:** thm:hb-lcs 4-10 Let $(X, \mathcal{P})$ be a LCS. Then

1. If Y is a subspace of X, $g \in Y^*$ then there exists $f \in X^*$ such that $f|_Y = g$.
2. (Separation of points) For every $x_0 \in X, x_0 \neq 0, \exists f \in X^*$ such that $f(x_0) \neq 0$.

The proof will be fairly easy since we have done most of the work. This is the analogue for LCS of Theorem 2.1.8.

Proof. 1. By Lemma 2.2.5, $\exists n \in \mathbb{N}, p_1, \ldots, p_n \in \mathcal{P}, C \geq 0$ such that

$$|g(y)| \leq C \max_{1 \leq i \leq n} p_i(y) \forall y \in Y.$$

Let $p(x) = C \max_{1 \leq i \leq n} p_i(x)$, $x \in X$. Then p is a seminorm on X, $|g(y)| \leq p(y) \forall y \in Y$. By Theorem 2.1.6, g extends to a linear functional f on X^* such that $|f(x)| \leq p(x)$ for all $x \in X$. By Lemma 2.2.5, $f \in X^*$.

2. If $x_0 \neq 0$, then there exists $p \in \mathcal{P}$ such that $p(x_0) \neq 0$. By Corollary 2.1.7 there exist linear functional f on X such that $|f(x)| \leq p(x)$ for all $x \in X$ and $f(x_0) = p(x_0)$. By Lemma 9, $f \in X^*$.

□

Theorem 2.2.8: thm:f2-11 Let $(X, \mathcal{P})$ be a LCS, f be a linear functional on X. Then $f \in X^*$ iff $\ker f$ is closed.

Proof. $\implies$: Clear.

$\Leftarrow$: Let $Y = \ker f$, WLOG $Y \neq X$. Pick $x_0 \in X \backslash Y$. Since Y is closed, there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}, \varepsilon > 0$ such that setting

$$U = \{x \in X : p_i(x) \leq \varepsilon \forall i\},$$

we have $(x_0 + U) \cap Y = \phi$.

For all $x \in U$ and all scalars $\lambda, |\lambda| \leq 1$, $\lambda x \in U$. Since f is linear, $\forall z \in f(U)$ and all scalars $\lambda, |\lambda| \leq 1$, $\lambda z \in f(U)$. So if $f(U)$ is unbounded, then it's the entire scalar field, and hence so is $f(x_0 + U)$, contradiction, as $0 \notin f(x_0 + U)$. So there exists $|f(x)| \leq C$ for all $x \in U$. By usual argument, $|f(x)| \leq \frac{C}{\varepsilon} \max_{1 \leq i \leq n} p_i(x)$ for all $x \in X$.

By Lemma 2.2.5, f is continuous. □

Chapter 3

Riesz Representation Theorem

Our aim is to describe the dual space of $C(K)$ where K is a compact Hausdorff space.

1 Functions $K \to \mathbb{C}$ or $\mathbb{R}$

Definition 3.1.1: df:f3-1 . Let $C(K), C^{\mathbb{R}}(K)$ be the complex, real Banach space of all continuous functions $f : K \to \mathbb{C}, K \to \mathbb{R}$ with the sup norm, respectively. Write $\|f\| = \sup_{x\in K}(f(x))$ and let

$$C^+(K) = \{f : K \to \mathbb{R} : f \text{ continuous}, f(x) \geq 0 \quad \forall x \in K\}.$$

Let $M(K) = C(K)^*$,

$$M^{\mathbb{R}}(K) = \left\{\varphi \in M(K) : \varphi(f) \in \mathbb{R} \quad \forall f \in C^{\mathbb{R}}(K)\right\} \subset M(K).$$

It is a real subspace of $M(K)$. (Note its definition is not $C^{\mathbb{R}}(K)^*$, but we will see it is isomorphic below.) Let

$$M^+(K) = \left\{\varphi : C(K) \to \mathbb{C} : \varphi \text{ linear}, \varphi(f) \geq 0 \quad \forall f \in C^+(K)\right\}.$$

I.e., functions take nonnegative real numbers on nonnegative functions. The elements of $M^+(K)$ are called **positive linear functionals**.

The relationship between these spaces is given by the following lemma.

Lemma 3.1.2: lem:f3-1

1. For all $\varphi \in M(K)$ there exists a unique $\varphi_1, \varphi_2 \in M^{\mathbb{R}}(K)$ such that $\varphi = \varphi_1 + i\varphi_2$.
2. For $\varphi \in M^{\mathbb{R}}(K)$, $\|\varphi\| = \sup\left\{|\varphi(f)| : f \in C^{\mathbb{R}}(K), \|f\| \leq 1\right\}$. So $\varphi \mapsto \varphi|_{C^{\mathbb{R}}(K)}$ is an isometric real-linear isomorphism $M^{\mathbb{R}}(K) \to C^{\mathbb{R}}(K)^*$.
3. $M^+(K) \subseteq M^{\mathbb{R}}(K)$ and moreover

$$M^+(K) = \{\varphi \in M(K) : \|\varphi\| = \varphi(1)\}.$$

4. For all $\varphi \in M^{\mathbb{R}}(K)$ there exist unique $\varphi^+, \varphi^- \in M^+(K)$ such that

$$\varphi = \varphi^+ - \varphi^-, \qquad \|\varphi\| = \left\|\varphi^+\right\| + \left\|\varphi^-\right\|.$$

Proof. 1. For $\varphi \in M(K)$, define $\varphi^*(f) = \overline{\varphi(\overline{f})}$ for any $f \in C(K)$. Then $\varphi^* \in M(K)$, $\|\varphi^*\| = \|\varphi\|$. We have $\varphi^* = \varphi$ iff $\varphi \in M^{\mathbb{R}}(K)$. The map $\varphi \mapsto \varphi^*$ is conjugate linear.

For uniqueness, if $\varphi = \varphi_1 + i\varphi_2$ for $\varphi_1, \varphi_2 \in M^{\mathbb{R}}(K)$, then $\varphi^* = \varphi_1 - i\varphi_2$. Then

$$\text{eq:f3-1}\varphi_1 = \frac{1}{2}(\varphi + \varphi^*), \qquad \varphi_2 = \frac{1}{2i}(\varphi - \varphi^*) \tag{3.1}$$

For existence, define φ_1, φ_2 using (3.1) and check it works.

2. Let $\varphi \in M^{\mathbb{R}}(K)$. Clearly, $\|\varphi\| \geq \sup\left\{|\varphi(f)| : f \in C^{\mathbb{R}}(K), \|f\| \leq 1\right\} =: \|\varphi\|_r$. Given $f \in C(K)$ with $\|f\| \leq 1$, choose $\theta \in \mathbb{R}$ such that $|\varphi(f)| = e^{i\theta}\varphi(f)$. Let $f_1 = \Re(e^{i\theta} f)$, $f_2 = \Im(e^{i\theta} f)$. Then
$$|\varphi(f)| = \varphi(f_1) + i\varphi(f_2) \in \mathbb{R}$$
and $\varphi(f_1), \varphi(f_2) \in \mathbb{R}$. So $\varphi(f_2) = 0$, and $|\varphi(f)| = \varphi(f_1) \leq \|\varphi\|_r \|f_1\| \leq \|\varphi\|_r$. Taking sup over all such f gives $\|\varphi\| \leq \|\varphi\|_r$.

   Finally, given $\psi \in C^{\mathbb{R}}(K)^*$, define $\varphi(f) = \psi(\Re f) + i\psi(\Im f)$, $f \in C(K)$. Then $\varphi \in M^{\mathbb{R}}(K)$ and $\varphi|_{C^{\mathbb{R}}(K)} = \psi$.

3. Given $\varphi \in M^+(K)$, for $f \in C^{\mathbb{R}}(K)$ we can write $f = f_1 - f_2$, $f_1, f_2 \in C^+(K)$, for instance, $f_1 = f \vee 0$ and $f_2 = (-f) \vee 0$, and $\varphi(f) = \varphi(f_1) - \varphi(f_2) \in \mathbb{R}$.

   Given $f \in C^{\mathbb{R}}(K)$ with $\|f\| \leq 1$ (so $-1 \leq f \leq 1$), we have $1 \pm f \geq 0$, so $\varphi(1 \pm f) \geq 0$. It follows that $|\varphi(f)| \leq \varphi(1)$. So $\|\varphi\| = \varphi(1)$. Suppose $\varphi \in M(K)$, $\|\varphi\| = \varphi(1)$. WLOG $\varphi(1) = 1$. Given $f \in C^+(K)$, $0 \leq f \leq 1$, let $\varphi(f) = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. Consider for $t \in \mathbb{R}$, (using the fact f is real valued)

$$\|f + i\beta t\|^2 = \|f\| + \beta^2 t^2 \leq 1 + \beta^2 t^2$$

   and
$$|\varphi(f + i\beta t)| = \alpha^2 + \beta^2(1+t)^2.$$
   We have $2\beta t \leq 1 - \alpha^2 - \beta^2$ for all $t \in \mathbb{R}$, so $\beta = 0$. We have

$$\varphi(1 - f) = 1 - \varphi(f) \leq \|1 - f\| \leq 1,$$

   so $\varphi(f) \geq 0$.

4. For existence, define for $f \in C^+(K)$, $\varphi^+(f) = \sup\{\varphi(g) : 0 \leq g \leq f, g \in C^+(K)\}$. Note $\varphi^+(f) \geq 0 \vee \varphi(f)$. Let $f_1, f_2 \in C^+(K)$. Then whenever $0 \leq g_i \leq f_i$, $i = 1, 2$, we have $0 \leq g_1 + g_2 \leq f_1 + f_2$, so $\varphi^+(f_1 + f_2) \geq \varphi(g_1 + g_2) = \varphi(g_1) + \varphi(g_2)$. Taking sup over all such g_1, g_2,
$$\varphi^+(f_1 + f_2) \geq \varphi^+(f_1) + \varphi^+(f_2).$$

Conversely, given $0 \leq g \leq f_1 + f_2$, we have $0 \leq g \wedge f_1 \leq f_1$ and $0 \leq g - g \wedge f_1 \leq f_2$. So $\varphi^+(f_1) + \varphi^+(f_2) \geq \varphi(g \wedge f_1) + \varphi(g - g \wedge f_1) = \varphi(g)$. Taking the sup over g,

$$\varphi^+(f_1) + \varphi^+(f_2) \geq \varphi^+(f_1 + f_2).$$

So φ^+ is additive on $C^+(K)$. Also $\varphi^+(tf) = t\varphi^+(f)$ for all $f \in C^+(K)$ and $t \geq 0$.

We now define $\varphi^+(f) = \varphi^+(f_1) - \varphi^+(f_2)$ where $f \in C^{\mathbb{R}}(K)$, $f = f_1 - f_2$, $f_1, f_2 \in C^+(K)$. This is well-defined, extends φ^+, and it's real linear. Finally, for $f \in C(K)$, define $\varphi^+(f) = \varphi^+(\Re f) + i\varphi^+(\Im f)$. This extends the definition of φ^+ and $\varphi^+ \in M^+(K)$. Set $\varphi^- = \varphi^+ - \varphi \in M^+(K)$ ($\varphi^+(f) \geq \varphi(f)$ for all $f \in C^+(K)$) and $\varphi = \varphi^+ - \varphi^-$. Of course, $\|\varphi\| \leq \|\varphi^+\| + \|\varphi^-\|$. By (3), $\|\varphi^+\| = \varphi^+(1)$. Since $\varepsilon > 0$, $\exists g \in C^+(K), 0 \leq g \leq 1$ such that $\varphi^+(1) \leq \varphi(g) + \varepsilon$. We have

$$\begin{aligned}\left\|\varphi^+\right\| + \left\|\varphi^-\right\| &= \varphi^+(1) + \varphi^-(1) \leq 2\varphi(g) + 2\varepsilon - \varphi(1) \\ &= \varphi(2g - 1) + 2\varepsilon \leq \|\varphi\| + 2\varepsilon\end{aligned}$$

since $-1 \leq 2g - 1 \leq 1$. Here $\varepsilon > 0$ was arbitrary, so we're done.

For uniqueness, assume $\varphi = \psi^+ - \psi^-$, and $\|\varphi\| = \|\psi^+\| + \|\psi^-\|$ for some $\psi^+, \psi^- \in M^+(K)$. If $f, g \in C^+(K)$, $0 \leq g \leq f$, then $\varphi(g) \leq \psi^+(g) \leq \psi^+(f)$. It follows that $\varphi^+(f) \leq \psi^+(f)$, and hence $\varphi^-(f) \leq \psi^-(f)$, and so $\psi^+ - \varphi^+, \psi^- - \varphi^- \in M^+(K)$. (Thus φ^+ is the smallest possible thing you can try when you have such a decomposition; this motivates our definition of φ^+.)

We have

$$\|\varphi\| = \left\|\psi^+\right\| + \left\|\psi^-\right\| = \psi^+(1) + \psi^-(1) \geq \varphi^+(1) + \varphi^-(1) = \left\|\varphi^+\right\| + \left\|\varphi^-\right\| = \|\varphi\|.$$

So $\psi^+(1) = \varphi^+(1)$ and $\psi^-(1) = \varphi^-(1)$. So $\|\psi^+ - \varphi^+\| = (\psi^+ - \varphi^+)(1) = 0$, and then $\psi^+ = \varphi^+$ and $\psi^- = \varphi^-$.

□

We will see that positive linear functionals are given by a measure.
First we need some topological and measure theoretic preliminaries.

1. A compact Hausdorff space K is normal: if E and F are disjoint closed subsets of K, then there exist disjoint open sets U, V such that $E \subseteq U, F \subseteq V$. Equivalently, if $E \subseteq U$, E is closed and U is open then there exists open V such that $E \subseteq V \subseteq \overline{V} \subseteq U$.

2. Urysohn's Lemma: if E, F are disjoint closed subsets of a normal space K, then there exists continuous $f : K \to [0, 1]$ such that $f = 0$ on E and $f = 1$ on F.

3. Notation: $E \prec f$ will mean that E is a closed subset of K, $f : K \to [0, 1]$ continuous, and $f \equiv 1$ on E.

   $f \prec U$ means that U is an open subset of K, $f : K \to [0, 1]$ is continuous, and $\mathrm{Supp}(f) = \overline{\{x \in K : f(x) \neq 0\}} \subseteq U$.

   Thus, think of $E \prec f \prec U$ as saying, "f moves from being 1 on E, decreasing in the middle $U \backslash E$, until it's 0 outside of U."

Note Urysohn gives that if $E \subseteq U$, E closed, and U is open then there exists $f, E \prec f \prec U$.

Lemma 3.1.3 (Partitions of unity)**:** lem:f3-2 Let E be a closed set in K and assume $E \subseteq \bigcup_{i=1}^n U_i$, U_i is open for all i.

1. (Shrinkage lemma) There exist open sets V_i such that $E \subseteq \bigcup_{i=1}^n V_i$, and $\overline{V_i} \subseteq U_i$ for all i.
2. (Existence of partition of unity) There exists f_i such that $f_i \prec U_i$ for all i, $\sum_{i=1}^n f_i = 1$ on E, and $0 \leq \sum f_i \leq 1$ on K.

Proof. 1. Induct on n. For $n = 1$, see remark 1 above. ($E \subseteq V_1 \subseteq \overline{V_1} \subseteq U_i$.)

For $n > 1$, $E \backslash U_n \subseteq \bigcup_{i=1}^{n-1} U_i$, so there exist open sets V_i, $1 \leq i \leq n-1$ such that $E \backslash U_n \subseteq \bigcup_{i=1}^{n-1} V_i$, $\overline{V_i} \subseteq U_i$ for $1 \leq i \leq n-1$.

$E \backslash \bigcup_{i=1}^{n-1} V_i \subseteq U_n$ so there exists an open V_n such that $E \backslash \bigcup_{i=1}^{n-1} V_i \subseteq V_n \subseteq \overline{V_n} \subseteq U_n$.

2. Use (1) to get open $V_i, 1 \leq i \leq n$, $E \subseteq \bigcup_{i=1}^n V_i$, $\overline{V_i} \subseteq U_i$ for all i. Let $U_0 = K \backslash \bigcup_{i=1}^n \overline{V_i}$. This is an open subset of K. Note $K = \bigcup_{i=0}^n U_i$.

Urysohn gives there exists g_0 such that $K \backslash \bigcup_{i=1}^n V_i \prec g_0 \prec K \backslash E$, and $\overline{V_i} \prec g_i \prec U_i$, $1 \leq i \leq n$. Then $g = \sum_{i=0}^n g_i > 0$ on K and $g_0 \equiv 0$ on E. Setting $f_i = \frac{g_i}{g}$, $1 \leq i \leq n$, we get $\sum_{i=1}^n f_i \equiv 1$ on E, $0 \leq \sum_{i=1}^n f_i \leq 1$ on K, as needed.

□

2 Review of measure theory

We give some measure theoretic preliminaries.

Definition 3.2.1: A **measure space** is a triple $(X, \mathcal{F}, \mu)$ where

1. X is a set
2. $\mathcal{F}$ is a σ**-field (or** σ**-algebra)** on X, i.e., $\mathcal{F} \subseteq \mathcal{P}X$, $\phi \in \mathcal{F}$, and where $A \in \mathcal{F} \implies X \backslash A \in \mathcal{F}$, and $A_n \in \mathcal{F}$ for $n \in \mathbb{N}$ implies $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$.
3. μ is a measure on $(X, \mathcal{F})$: $\mu : \mathcal{F} \to [0, \infty]$ with $\mu(\phi) = 0$ and $\mu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mu(A_n)$ for $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, pairwise disjiont.

Example 3.2.2: If X is a topological space, the **Borel** σ**-field** $\mathcal{B}$ on X is the σ-field generated by the family $\mathcal{G}$ of open subsets of X (i.e., $\mathcal{B}$ is the smallest σ-field in X that contains $\mathcal{G}$). Elements of $\mathcal{B}$ are **Borel sets**.

A **Borel measure on** X is a measure μ on $(X, \mathcal{B})$. μ is regular if $\mu(E) < \infty$ for every compact set $E \subseteq X$ (X is Hausdorff), and

$$\mu(A) = \inf \{\mu(U) : U \in \mathcal{G}, U \supseteq A\}, A \in \mathcal{B}$$
$$\mu(U) = \sup \{\mu(E) : E \subseteq U, E \text{ is compact}\}, U \in \mathcal{G}.$$

The nice thing about a regular Borel measure is that knowing the measure on compact sets, or the measure on open sets, determines it completely. If X is compact Hausdorff, then μ is regular iff

$$\mu(A) = \inf\{\mu(U) : U \text{ open } U \supseteq A\} = \sup\{\mu(E) : E \text{ closed}, E \subseteq A\}, A \in \mathcal{B}.$$

For example, on $\mathbb{R}$, λ is the Lebesgue measure, $\lambda([a,b]) = b - a$.

The following will be helpful in constructing measures.

Definition 3.2.3: X is now any set. An **outer measure** on X is function $\mu^* : \mathcal{P}X \to [0,\infty]$ such that

$$\begin{aligned} \mu^*(\phi) &= 0 \\ \mu^*(A) &\le \mu^*(B) \text{ if } A \subseteq B \\ \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) &\le \sum_{i=1}^{\infty} \mu^*(A_i). \end{aligned}$$

(The last condition is called **countable subadditivity**.)

Call $A \subseteq X$ **μ^*-measurable** if

$$\mu^*(B) - \mu^*(B \cap A) = \mu^*(B \backslash A) \text{ for all } B \subseteq X.$$

Here, $\le$ is clear from subadditivity; one only needs to check $\ge$.

We have the following.

Proposition 3.2.4: Let μ^* be an outer measure. $\mathcal{M} = \{A \subseteq X : A\, \mu^*\text{-measurable}\}$ is a σ-field on X and $\mu = \mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$.

Lecture 8 We now review integration. Suppose we have a measure space $(X, \mathcal{F}, \mu)$. Say that $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is **measurable** if $f^{-1}(B) \in \mathcal{F}$ for every Borel set B. This is the analogue in measure theory of what continuous functions are in topology.

Example 3.2.5:

1. Simple functions, i.e., functions of the form $\sum_{i=1}^n a_i 1_{A_i}$ where a_i are scalars, the $A_i \in \mathcal{F}$. Here $1_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}$.

2. If X is a topological space and $\mathcal{F}$ is the Borel σ-field on X, then every continuous function $X \to \mathbb{R}$ (or $\mathbb{C}$) is measurable.

If $f \ge 0$ is a simple function, i.e., $f = \sum_{i=1}^n a_i 1_{A_i}$ where $a_i \in [0,\infty)$ for all i and $A_i \in \mathcal{F}$ for all i, then define

$$\int_X f\, d\mu = \sum_{i=1}^n a_i \mu(A_i).$$

(The convention is that $0 \cdot \infty = 0 = \infty \cdot 0$. If $f \ge 0$ is measurable, then $\int_X f\, d\mu = \sup\{\int_X g\, d\mu : 0 \le g \le f, g \text{ simple}\}$.)

If $f : X \to \mathbb{R}$ is measurable, we say f is integrable if $\int_X |f| \, d\mu < \infty$. Set

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu.$$

(Here $f_+ = f \vee 0$ and $f_- = (-f) \vee 0$.) If $f : X \to \mathbb{C}$ is measurable, set $\int_X f \, d\mu = \int_X \Re f \, d\mu + i \int_X \Im f \, d\mu$.

Proposition 3.2.6:

1. (Linearity) If $f \geq 0$, $g \geq 0$ is measurable, $\alpha, \beta \geq 0$ scalars, then $\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu$. If $f : X \to \mathbb{C}, g : X \to \mathbb{C}$ are integrable, $\alpha, \beta \in \mathbb{C}$, then $\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu$.
2. If $f : X \to \mathbb{C}$ is integrable, then $|\int_X f \, d\mu| \leq \int_X |f| \, d\mu$.
3. (Monotone convergence) If $0 \leq f_n$ are measurable, $n \in \mathbb{N}$, and $f_n \nearrow f$, then $\int_X f_n \, d\mu \nearrow \int_X f \, d\mu$.
4. (Dominated convergence) If $f_n, n \in \mathbb{N}$ are measurable, g is integrable, $|f_n| \leq g$ for all n and if $f_n \to f$ pointwise, then f is integrable and $\int_X f_n \, d\mu \to \int_X f \, d\mu$.

Definition 3.2.7: df:c-meas Let X be any set and $\mathcal{F}$ be a σ-field on X. A **complex measure** on $\mathcal{F}$ is a function $\mu : g \to \mathbb{C}$ such that $\mu(\phi) = 0$, $\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i)$ whenever $A_i \in \mathcal{F}$ for all i, $A_i \cap A_j = \phi$ for all $i \neq j$. The total variation measure $|\mu|$ of μ is defined as follows.

$$|\mu|(A) = \sup \left\{ \sum_{i=1}^n |\mu(A_i)| : n \in \mathbb{N}, A = \bigsqcup_{i=1}^n A_i, A_i \in \mathcal{F} \forall i \right\} \text{ for } A \in \mathcal{F}.$$

Then $|\mu|$ is a measure on $\mathcal{F}$. The **total variation** $\|\mu\|_1$ of μ is $\|\mu\|_1 = |\mu|(X)$.

Theorem 3.2.8: The total variation of a complex measure $|\mu|$ is a positive finite measure: for every set X, $|\mu|(X) < \infty$.

Proof. Rudin [1, Theorem 6.2, 6.4].[1] □

A **signed measure** on $\mathcal{F}$ is a complex measure $\mu : \mathcal{F} \to \mathbb{R}$.

Theorem 3.2.9 (Hahn-Jordan decomposition of μ)**:** thm:hahn-jordan There exists a unique measure μ^+, μ^- such that $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$.

---

[1] Summary:

1. $|\mu|$ is a measure by a "refining partitions" argument.
2. That $|\mu|$ is finite rests on the lemma that for S finite,

$$\max_{A \subseteq S} \sum_{z \in A} |z| \geq \frac{1}{\pi} \sum_{z \in S} |z|.$$

Prove by taking those z that are within $90°$ of some θ; average over θ to find there's one θ for which the sum is large.

Now if $|\mu|(E) = \infty$, then there's some subset where $|\mu|(A)$ is much larger than $\mu(E)$, and use the lemma to show we can write $E = A \sqcup B$ with $|\mu(A)| > 1$ and $\mu(B) = \infty$; iterate; we get divergence.

Proof sketch. Show $\sup\{\mu(A) : A \in \mathcal{F}\} < \infty$ is attained by some $P \in \mathcal{F}$.[2] Let $\mu^+(A) = \mu(A \cap P), \mu^-(A) = \mu(A \cap N)$, $N = X \backslash P$. $\square$

Let $\mu : \mathcal{F} \to \mathbb{C}$ be a complex measure. A measurable function $f : X \to \mathbb{C}$ is integrable if $\int_X |f|\, d|\mu| < \infty$. Then define

$$\int_X f\, d\mu = \int_X f\, d\mu_1^+ - \int_X f\, d\mu_1^- + i \int_X f\, d\mu_1^+ - i \int_X f\, d\mu_2^-$$

where $\mu_1 = \Re\mu$, $\mu_2 = \Im\mu$, and $\mu_1 = \mu_1^+ - \mu_1^-, \mu_2 = \mu_2^+ - \mu_2^-$ are the Hahn-Jordan decompositions of μ_1, μ_2, respectively. The previous properties (linearity, dominated convergence, etc.) hold in this more general setting. In the special case when X is a topological space, $\mathcal{F}$ is the Borel σ-field on X, then if $\mu : \mathcal{F} \to \mathbb{C}$ is a complex measure (called a complex Borel measure on X), then any bounded continuous function $f : X \to \mathbb{C}$ is integrable, $|\int_X f\, d\mu| \le \int_X |f|\, d\mu$.

It follows that $C_b(X) = \{f : X \to \mathbb{C} : f \text{ bounded, continuous}\} \to \mathbb{C}$, $f \mapsto \int_X f\, d\mu$ is a bounded linear functional with norm is at most $\|\mu\|_1 := |\mu|(X)$. (We are using the norm $\|\cdot\|_\infty$ on $C_b(X)$.) The complex measure μ is **regular** if $|\mu|$ is regular.

3 Riesz Representation

Theorem 3.3.1 (Riesz Representation Theorem)**:** Let K be a compact Hausdorff space, and let $\varphi \in M^+(K)$. Then there exists a unique regular, finite Borel measure μ on K such that

$$\varphi(f) = \int_K f\, d\mu \text{ for all } f \in C(K).$$

(In Lemma 3.1.2 we showed it's sufficient to describe $M^+(K)$, and we'll understand all functionals.)

The idea is we'd like to define $\mu(E) = \varphi(1_E)$. We can't do this because φ is only defined on continuous functions. We'll use Urysohn to approximate 1_E with continuous fnctions.

Proof. Uniqueness: Suppose μ_1, μ_2 both represent φ in the sense of the theorem. To show $\mu_1 = \mu_2$, it's enough to show that they agree on closed sets, by regularity. Let E be a closed set. Then by Urysohn's Lemma, for any open $U \supseteq E$, there exists f, $E \prec f \prec U$, and

$$\mu_1(E) \le \int_K f\, d\mu_1 = \int_K f\, d\mu_2 \le \mu_2(U).$$

μ_2 is regular, so taking inf over all open $U \supseteq E$, $\mu_1(E) \le \mu_2(E)$. Reversing the roles of μ_1, μ_2 gives $\mu_1(E) = \mu_2(E)$.

Existence: We define $\mu^* : \mathcal{P}K \to [0, \infty)$ as follows. For $U \in \mathcal{G}$ (the family of all open subsets of K), define

$$\mu^*(U) = \sup\{\varphi(f) : f \prec U\}.$$

[2]Let s be the sup. Take $A_1 \subseteq A_2 \subseteq \cdots$ with $\mu(A_i) > (1 - \frac{1}{2^i})s$. Now consider $\bigcap_{i \ge n} A_i$. Use the formula $\mu(A) + \mu(B) - \mu(A \cap B) = \mu(A \cup B) \le s$ to show their measures converge.

For arbitrary $A \subseteq K$, define

$$\mu^*(A) = \inf \{\mu^*(U) : U \supseteq A, U \in \mathcal{G}\}.$$

Note $\mu^*(\phi) = 0$, $\mu^*(K) = \varphi(1) = \|\varphi\|$, the two definitions of μ^* agree on $\mathcal{G}$, and $\mu^*(A) \leq \mu^*(B)$ whenever $A \subseteq B \subseteq K$. We need to check:

1. *μ^* is countably subadditive on $\mathcal{G}$.* Let $U_n \in \mathcal{G}$, $n \in \mathbb{N}$, $U = \bigcup_{n=1}^{\infty} U_n$. To get subadditivity we need to get a function f on U and "break it up."

   Given $f \prec U$, by compactness there exists $n \in \mathbb{N}$, $\operatorname{Supp} f \subseteq \bigcup_{i=1}^{n} U_i$. By Lemma 3.1.3(2), there exists $h_i \prec U_i$ such that $\sum_{i=1}^{n} h_i = 1$ on $\operatorname{Supp} f$, and $\sum_{i=1}^{n} h_i \in [0,1]$ on K. Then $f = \sum_{i=1}^{n} f h_i$ and $f h_i \prec U_i$ for all i. So

   $$\varphi(f) = \sum_{i=1}^{n} \varphi(f h_1) \leq \sum_{i=1}^{n} \mu^*(U_i) \leq \sum_{i=1}^{n} \mu^+(U_i).$$

   Take the sup over all $f \prec U$, to get $\mu^*(U) \leq \sum_{n=1}^{\infty} \mu^*(U_n)$.

2. *μ^* is countably subadditive on $\mathcal{P}K$.* Given $A_i \subseteq K$, $i \in \mathbb{N}$, given $\varepsilon > 0$, choose open $U_i \supseteq A_i$ such that $\mu^*(U_i) < \mu^*(A_i) + \frac{\varepsilon}{2^i}$. Then

   $$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \mu^*\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{n=1}^{\infty} \mu^*(U_i) < \sum_{n=1}^{\infty} \mu^*(A_i) + \varepsilon.$$

   Now

   $$\mathcal{M} = \{A \subseteq K : A \text{ is } \mu^*\text{-measurable}\}$$

   is a σ-field on K, and $\mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$.

3. *$\mathcal{M}$ is large enough to contain all Borel sets.* Since the Borel sets are generated by the open sets, it suffices to show $\mathcal{G} \subseteq \mathcal{M}$. Let $U \in \mathcal{G}$. We need to show

   $$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \backslash U) \quad \forall A \subseteq K.$$

   To show subadditivity we had to take f on U and "break it up"; to show superadditivity here we have to do the reverse: take functions on the subsets and add them up.

   (a) First consider $A = V \in \mathcal{G}$. Given $\varepsilon > 0$, choose $f \prec V \cap U$ such that $\mu^*(V \cap U) < \varphi(f) + \varepsilon$. Let $g \prec V \backslash \operatorname{Supp}(f)$. Then $f + g \prec V$. So

   $$\mu^*(V) \geq \varphi(f+g) = \varphi(f) + \varphi(g) > \mu^*(V \cap U) - \varepsilon + \varphi(g).$$

   The sup over g is

   $$\mu^*(V) \geq \mu^*(V \cap U) - \varepsilon + \mu^*(V \backslash \operatorname{Supp}(f)) \geq \mu^*(V \cap U) + \mu^*(V \backslash U) - \varepsilon$$

   since $V \backslash \operatorname{Supp}(f) \supseteq V \backslash U$. Lecture 9

(b) For arbitrary $A \subseteq K$, we take an open set $V \supseteq A$, then

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \backslash U) \geq \mu^*(A \cap U) + \mu^*(A \backslash U)$$

since $V \cap U \supseteq A \cap U$ and $V \backslash U \supseteq A \backslash U$. Taking the inf over all such V gives $\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \backslash U)$.

We have $\mathcal{M}$, the μ^*-measurable sets, contains $\mathcal{G}$ and hence $\mathcal{B}$, the Borel σ-field. Moreover, $\mu = \mu^*|_{\mathcal{B}}$ is a Borel measure on K. It's finite since $\mu(K) = \varphi(1)$, and μ is regular.

4. *$\varphi(f) = \int_K f\, d\mu$ for all $f \in C(K)$.*

It's sufficient to consider $f \in C^{\mathbb{R}}(K)$ and then it's enough to show that $\varphi(f) \leq \int_K f\, d\mu$. (Applying this to $-f$ gives the reverse inequality.) To do this, we might think to approximate f with a simple function $\sum y_i 1_{A_i}$, and get that the integral of f is roughly a linear combination of measure of sets. The problem with this is that, as before, is that φ is not defined on indicator sets, so we break f up using a partition of unity with respect to $U_i \supseteq A_i$.

Choose $a < b$ in $\mathbb{R}$ such that $(a, b] \supseteq f(K)$. Given $\varepsilon > 0$, choose $a = y_0 < y_1 < \cdots < y_n = b$ such that

$$\text{eq:rr1}\, y_i - y_{i-1} < \varepsilon \text{ for all } i = 1, \ldots, n. \tag{3.2}$$

Let $A_i = f^{-1}((y_{i-1}, y_i])$, $i = 1, \ldots, n$, $A_i \in \mathcal{B}$, $A_1, \ldots, A_n$ is a partition of K, $\sum_{i=1}^n \mu(A_i) = \mu(K) = \varphi(1)$. On A_i, $f \geq y_{i-1}$. Next choose open sets $U_i \supseteq A_i$ such that

$$\text{eq:rr2}\, \mu(U_i \backslash A_i) < \frac{\varepsilon}{n} \tag{3.3}$$

and

$$\text{eq:rr3}\, f < y_i + \varepsilon \text{ on } U_i. \tag{3.4}$$

By Lemma 3.1.3(2), there exist $h_i \prec U_i$ such that $\sum_{i=1}^n h_i \equiv 1$ on K. Note $\varphi(\sum h_i) = \sum \varphi(h_i) = \varphi(1) = \mu(K)$, $fh_i \leq (y_i + \varepsilon)h_i$ for $i = 1, \ldots, n$ by (3.4). Hence

$$\varphi(f) = \sum_{i=1}^n \varphi(fh_i) \leq \sum_{i=1}^n (y_i + \varepsilon)\varphi(h_i) \text{ for } \varphi \in M^+(K).$$

As a technicality we need to consider when y_i could be negative, so choose $|a|$ such that $|a| + y_i + \varepsilon > 0$ for each i. Then

$$\begin{aligned}
\varphi(f) &= \sum_{i=1}^{n} \varphi(fh_i) \le \sum_{i=1}^{n} (y_i + \varepsilon)\varphi(h_i) \\
&= \sum_{i=1}^{n} (|a| + y_i + \varepsilon)\varphi(h_i) - |a|\mu(K) \\
&\le \sum_{i=1}^{n} (|a| + y_{i-1} + 2\varepsilon)\left(\mu(A_i) + \frac{\varepsilon}{n}\right) - |a|\mu(K) \quad (\mu(U_i) \ge \varphi(h_i) \text{ by definition of } \mu) \\
&= \underbrace{\sum_{i=1}^{n} y_{i-1}\mu(A_i)}_{\int_K \sum_{i=1}^{n} y_{i-1} 1_{A_i}\, d\mu} + \varepsilon(|a| + |b| + 2\mu(K) + 2\varepsilon) \\
&\le \int_K f\, d\mu + \varepsilon(|a| + |b| + 2\mu(K) + 2\varepsilon).
\end{aligned}$$

Note ε was arbitrary, so we're done.

□

Corollary 3.3.2: cor:rr Given $\varphi \in M(K) = C(K)^*$, there exists a unique regular complex Borel measure μ on K such that

$$\varphi(f) = \int_K f\, d\mu$$

for all $f \in C(K)$. Moreover, $\|\varphi\| = \|\mu\|_1$.

Proof. By lemma 3.1.2(1) and (4), $\varphi = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4$ where $\varphi_j \in M^+(K), j = 1, 2, 3, 4$. By Lemma 3.1.2 there exists finite regular measures μ_j, $1 \le j \le 4$, that represent φ_j. Set $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$, and get $\varphi(f) = \int_K f\, d\mu$ for all $f \in C(K)$. Note that $|\mu| \le \mu_1 + \mu_2 + \mu_3 + \mu_4$, so $|\mu|$ is K-regular.

Uniqueness will follow from the "moreover" part, which we now show.

We have

$$|\varphi(f)| = \left|\int_K f\, d\mu\right| \le \int_K |f|\, d|\mu| \le \|f\|\, |\mu|(K),$$

so $\|\varphi\| \le \|\mu\|_1$. For the reverse inequality, we need to show: for any Borel subsets $A_1, \dots, A_n$ of K that partition K,

$$\sum_{i=1}^{n} |\mu(A_i)| \le \|\varphi\|.$$

Given $\varepsilon > 0$, by regularity of μ, there exist closed sets $E_i \subseteq A_i$ such that $|\mu|(A_i \backslash E_i) < \frac{\varepsilon}{n}$. The E_i's are pairwise disjoint because the A_i's are. By regularity, there exist open sets U_i such that $E_i \subseteq U_i \subseteq K \backslash \bigcup_{j \ne i} E_j$.and $|\mu|(U_i \backslash E_i) < \frac{\varepsilon}{n}$.

By Lemma 3.1.3(2), there exist $h_i \prec U_i$ such that $\sum_{i=1}^{n} h_i \equiv 1$ on $\bigcup_{i=1}^{n} E_i$ and $0 \le \sum_{i=1}^{n} h_i \le 1$ on K.

This implies that $E_i \prec h_i \prec U_i$.

Choose $\lambda_j \in \mathbb{C}$, $|\lambda_j| \le 1$ such that $|\mu(E_j)| = \lambda_j \mu(E_j)$, $1 \le j \le n$. Set $f = \sum_{j=1}^{n} \lambda_j h_j$. Note that by the triangle inequality, $\|f\| \le 1$.

We have

$$\left| |\mu(E_j)| - \int_K \lambda_j h_j \, d\mu \right| = \left| \mu(E_j) - \int_K h_j \, d\mu \right| \leq |\mu|(U_j \backslash E_j) < \frac{\varepsilon}{n}.$$

So

$$\sum_{j=1}^{n} |\mu(A_j)| \leq \sum_{j=1}^{n} |\mu(E_j)| + \varepsilon \leq \underbrace{\left| \int_K \sum_{j=1}^{n} \lambda_j h_j \, d\mu \right|}_{\varphi(f)} + 2\varepsilon \leq \|\varphi\| + 2\varepsilon.$$

□

Remark: If $\varphi \in M^{\mathbb{R}}(K) \cong C^{\mathbb{R}}(K)^*$, the corresponding μ is a regular Borel signed measure. The space of regular, Borel complex measures on K with $\|\cdot\|_1$ is a complex Banach space isometrically isomorphic to $M(K)$. Similarly the space of regular, Borel signed measures on K with $\|\cdot\|_1$ is a real Banach space isometrically isomorphic to $M^{\mathbb{R}}(K)$.

Chapter 4

Weak Topologies

1 Weak Topologies

Definition 4.1.1: df:weak-topology Let X be a set. Let $\mathcal{F}$ be a family of functions, where each $f \in \mathcal{F}$ is a function $f : X \to Y_f$ and Y_f is a topological space. The **weak topology** on X generated by $\mathcal{F}$ denoted by $\sigma(X, \mathcal{F})$, is the smallest topology on X such that every $f \in \mathcal{F}$ is continuous.

Remark:

1. In other words, $\{f^{-1}(U) : f \in \mathcal{F}, U \text{ open subset of } Y_f\}$ is a subbase of $\sigma(X, \mathcal{F})$.

   More generally, if S_f is a subbase of the topology at Y_f, $f \in \mathcal{F}$, then

   $$\left\{f^{-1}(U) : f \in \mathcal{F}, U \in S_f\right\}$$

   is also a subbase of $\sigma(X, \mathcal{F})$.

2. For $V \subseteq X$, we have $V \in \sigma(X, \mathcal{F})$ iff for all $x \in V$, there exists $n \in \mathbb{N}$ and there exists $f_1, \ldots, f_n \in \mathcal{F}$ and open sets U_i in Y_{f_i} such that $x \in \bigcap_{i=1}^n f_i^{-1}(U_i) \subseteq V$.

3. (Universality Property) (This is a simple but very useful fact.) For any topological space Z and any function $g : Z \to X$, g is continuous iff $f \circ g : Z \to Y_f$ is continuous for all $f \in \mathcal{F}$. ($\Longrightarrow$ is clear. For "$\Leftarrow$," it's enough to check members of the subbase. We have $g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U)$ is open in Z for all $f \in \mathcal{F}$ and all open $U \subseteq Y_f$.)

   As an exercise, if τ is a topology on X with this property, then $\tau = \sigma(X, \mathcal{F})$.

4. If τ is a topology on X with this property, then $\tau = \sigma(X, \mathcal{F})$.

5. If Y_f is Hausdorff for all $f \in \mathcal{F}$, and $\mathcal{F}$ separates points of X, then $\sigma(X, \mathcal{F})$ is Hausdorff. For all $x \neq y$ in X there exists $f \in \mathcal{F}$, $f(x) \neq f(y)$.

Example 4.1.2:

1. (Subspace topology) Let X be a topological space, $Y \subseteq X$, and $i : Y \to X$ be the inclusion map. Then $\sigma(Y, \{i\})$ is the subspace topology.

   Given a set X, $\mathcal{F}$ as above, given $Y \subseteq X$, then $\sigma(X, \mathcal{F})|_Y$ (the subspace topology on Y induced by $\sigma(X, \mathcal{F})$) is the same as $\sigma(Y, \{f|_Y : f \in \mathcal{F}\})$. This is routine to verify and useful.

2. (Product topology) Given topological spaces $X_\gamma, \gamma \in \Gamma$, let $X = \prod_{\gamma\in\Gamma} X_\gamma$, i.e., $X = \{x : x \text{ is a function on } \Gamma\}$ with $x(\gamma) \in X_\gamma$ for all $\gamma \in \Gamma$.

   Have $\pi_\gamma : X \to X_\gamma, x \mapsto x(\gamma), \gamma \in \Gamma$. The product topology on X is $\sigma(X, \{\pi_\gamma : \gamma \in \Gamma\})$.

Proposition 4.1.3: pr:f4-1 Let X be a set. For each $n \in \mathbb{N}$, suppose we are given a metric space (Y_n, d_n) and a function $f_n : X \to Y_n$. Assume the f_n separate points of X, i.e., for all $x \neq y$ in X, there exists $n \in \mathbb{N}$, $f_n(x) \neq f_n(y)$. Then $\sigma(X, \{f_n : n \in \mathbb{N}\})$ is metrizable.

Proof. WLOG $d_n \leq 1$ (replace d_n with the equivalent metric $\min(1, d_n)$). We define

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(f_n(x), f_n(y)), \qquad x, y \in X.$$

It is easy to verify that d is a metric on X. Then $\mathrm{id} : (X, d) \to (X, \sigma)$ is continuous by the universality property (here $\sigma = \sigma(X, \{f_n : n \in \mathbb{N}\})$). (A map into a weak topology is continuous if, when composing with any of the defining maps it is continuous.)

We have $\mathrm{id} : (X, \sigma) \to (X, d)$ is continuous: check for all $x \in X$, $r > 0$, $B(x, r)$ is a σ-neighborhood of x. This is an easy exercise. □

The separating condition is needed to get positivity in the metric. Without the separation property d is a pseudo-metric.

Theorem 4.1.4 (Tychonov's Theorem)**:** thm:f4-2 The product of compact topological spaces is compact.

Proof. Let $X_\gamma, \gamma \in \Gamma$ be compact spaces, and $X = \prod_{\gamma\in\Gamma} X_\gamma$ with the product topology. Let $\mathcal{F}$ be a nonempty family of closed subsets of X with the FIP (finite intersection property): for all $n \in \mathbb{N}$ and all $F_1, \ldots, F_n \in \mathcal{F}$, $\bigcap_{i=1}^n F_i \neq \phi$. We need $\bigcap \mathcal{F} \neq \phi$.

Using Zorn, there exists a maximal (with respect to inclusion) family of (not necessarily closed) subsets of X, $\mathcal{A}$, such that $\mathcal{A} \supseteq F$ and $\mathcal{A}$ has FIP. It's enough to show that $\bigcap_{A\in\mathcal{A}} \overline{A} \neq \phi$.

1. For every $A, B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$: If $B_1, \ldots, B_n \in \mathcal{A}$ then $A \cap B \cap B_1 \cap \cdots \cap B_n \neq \phi$, so $\mathcal{A} \bigcup \{A \cap B\}$ has FIP, so it equals $\mathcal{A}$.

2. For all $A \subseteq X$, if $A \cap B \neq \phi$ for all $B \in \mathcal{A}$¡ then $A \in \mathcal{A}$: given $B_1, \ldots, B_n \in \mathcal{A}$, by (1), $\bigcap_{i=1}^n B_i \in \mathcal{A}$, so $A \cap \bigcap_{i=1}^n B_i \neq \phi$. So $\mathcal{A} \cup \{A\}$ has FIP, and $A \in \mathcal{A}$.

3. Given $A \in \mathcal{A}, B \subseteq X$, if $A \subseteq B$ then $B \in \mathcal{A}$: for any $X \in \mathcal{A}$, $A \cap C \neq \phi$, so $B \cap C \neq \phi$. By (2), $B \in \mathcal{A}$.

Let $\pi_y : X \to X_\gamma$ be the projection $\pi_\gamma(x) = x(\gamma)$. For $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathcal{A}$, $\overline{\bigcap_{i=1}^n \pi_\gamma(A_i)} \supseteq \underline{\pi_\gamma\left(\bigcap_{i=1}^n A_i\right) \neq \phi}$. So $\{\pi_\gamma(A) : A \in \mathcal{A}\}$ has FIP. X_γ is compact so $\bigcap_{A\in\mathcal{A}} \overline{\pi_\gamma(A)} \neq \phi$. Pick $x_\gamma \in \bigcap_{A\in\mathcal{A}} \overline{\pi_\gamma(A)}$. Set $x = (x_\gamma)_{\gamma\in\Gamma}$. We claim $x \in \bigcap_{A\in\mathcal{A}} \overline{A}$. Let V be a neighborhood of x. Then $\exists n \in \mathbb{N}, \gamma_1, \ldots, \gamma_n \in \Gamma$, open sets U_i on X_{γ_i}, $1 \leq i \leq n$, such that $x \in \bigcap_{i=1}^n \pi_{\gamma_i}^{-1}(U_i) \subseteq V$. For each i, $x_{\gamma_i} \in U_i$, so $U_i \cap \pi_{\gamma_i}(A) \neq \phi$ for all $A \in \mathcal{A}$.

So $\pi_{\gamma_i}^{-1}(U_i) \cap A \neq \phi$ for all $A \in \mathcal{A}$. By (2) above $\pi_{\gamma_i}^{-1}(U_i) \in \mathcal{A}$. Then by (1) and (3), $V \in \mathcal{A}$. So $V \cap A \neq \phi$ for all $A \in \mathcal{A}$. Since this holds for every neighborhood V of x we get $x \in \overline{A}$ for all $A \in \mathcal{A}$. □

Let E be a (real or complex) vector space, let F be a subspace of the space of all linear functionals on E. Assume F separates the points of E, i.e., for all $x \neq 0$ in E there exists $f \in F$ such that $f(x) \neq 0$. We'll consider the weak topology $\sigma(E, F)$ on E. A subset V of E is open iff for all $x \in V$ there exists $n \in \mathbb{N}$, $f_1, \dots, f_n \in F$, $\varepsilon > 0$ such that $\{y \in E : |f_i(y) - f_i(x)| < \varepsilon \forall 1 \leq i \leq n\} \subseteq V$.

This is precisely a LCS given by the seminorms $x \mapsto |f(x)| : E \to \mathbb{R}$, $f \in F$.

In a moment we'll look at 2 particular examples, but first we'll give some basic results.

$\sigma(E, F)$ is Hausdorff, and addition and scalar multiplication are continuous.

Lemma 4.1.5: lem:f4-3 Let E be a vector space, and let $f, g_1, \dots, g_n$ be linear functionals on E such that $\bigcap_{i=1}^n \ker g_i \subseteq \ker f$. Then $f \in \operatorname{span}\{g_1, \dots, g_n\}$.

(There is a nice quantitative version of this; see the second example sheet.)

Proof. Let K be the scalar field. We define $q : E \to K^n$ by taking $q(x) = (g_1(x), \dots, g_n(x))$. Then $\ker q = \bigcap_{i=1}^n \ker g_i \subseteq \ker f$. Hence f factors through q: there exists a linear map $\widetilde{f} : K^n \to K$ such that

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "q"'] & & K \\ & K^n \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

commutes: $f = \widetilde{f} \circ q$. (Define $\widetilde{f}(q(x)) = f(x)$, this is well-defined and linear on im$q$, now just extend to the whole of K^n. There exist $a_1, \dots, a_n \in K$ such tht $\widetilde{f}(y_1, \dots, y_n) = \sum_{i=1}^n a_i y_i$, so $f(x) = \widetilde{f}(q(x)) = \sum_{i=1}^n a_i g_i(x)$, $x \in E$. Thus $f = \sum_{i=1}^n a_i g_i$. $\square$

Proposition 4.1.6: pr:f4-4 Let E be a (real or complex) vector space, F be a subspace of all linear functionals on E separating the points of E. let f be a linear functional on E. Then f is $\sigma(E, F)$-continuous iff $f \in F$. So $(E, \sigma(E, F))^* = F$.

The weak continuous was the smallest topology that makes all members of F continuous, but the point was that there was nothing else.

Proof. "$\Leftarrow$" is clear by definition.

"$\Rightarrow$": Note $\{x \in E : |f(x)| < 1\}$ is a neighborhood of 0, so it contains a set of the form $\{x \in E : |g_i(x)| < \varepsilon \text{ for all } 1 \leq i \leq n\}$ where $n \in \mathbb{N}$, $g_1, \dots, g_n \in \mathcal{F}$, $\varepsilon > 0$. If $x \in \bigcap_{i=1}^\infty \ker g_i$ then so is λx for all scalar λ, so $|f(\lambda x)| < 1$ for all λ. So $f(x) = 0$.

By Lemma 4.1.5, $f \in \operatorname{span}\{g_1, \dots, g_n\} \subseteq F$. $\square$

We have now made all the preparations to define the weak and weak star topologies.

2 w and w^* topologies

Recall from Chapter 2 the canonical map $X \hookrightarrow X^{**}$, $x \mapsto \hat{x}$ where $\langle f, \hat{x} \rangle = \langle x, f \rangle$ for all $f \in X^*$ is an isometric isomorphism of X into X^{**}. The image $\hat{X} = \{x : x \in X\}$ is closed in X^{**} iff X is complete. X is reflexive if $\hat{X} = X^{**}$.

Definition 4.2.1: Let X be a normed space. The **weak topology (w-topology)** on X is $\sigma(X, X^*)$ (i.e., $E = X$, $F = X^*$, F separates the points of E by Hahn-Banach).

Note that $V \subseteq X$ is in $\sigma(X, X^*)$ (say V is **weak open (w-open)** iff for all $x \in V$, there exists $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in X^*, \varepsilon > 0$ such that $\{y \in X : |x_i^*(x - y)| < \varepsilon, 1 \leq i \leq n\} \subseteq V$.

Definition 4.2.2: The **weak-star topology (w^*-topology)** on X^* is the weak topology $\sigma(X^*, X)$, i.e., $E = X^*$, $F = X = \hat{X} \subseteq X^{**}$. So $V \subseteq X^*$ is in $\sigma(X^*, X)$ (say V is w^***-open**) iff for all $x^* \in V$, there exists $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, $\varepsilon > 0$ such that

$$x^* \in \{y^* : |(x^* - y^*)(x_i)| < \varepsilon \text{ for } 1 \leq i \leq n\} \subseteq V.$$

These topologies look similar but we will see they are very different. Proposition 4.1.6 immediately gives the following—no more functionals are continuous than the ones we force to be.

Proposition 4.2.3: pr:f4-5 Let X be a normed space, let f be a linear functional on X, and φ a linear functional on X^*. Then we say

1. f is continuous in $\sigma(X, X^*)$ (say f is **weakly continuous (w-continuous**) iff $f \in X^*$.
2. φ is continuous in $\sigma(X^*, X)$ (say ϕ is w^***-continuous**) iff $\varphi \in X$, i.e., iff ϕ is an evalution: there exists $x \in X, \varphi = \hat{x}$.

It follows that $\sigma(X^*, X^{**}) = \sigma(X^*, X)$ iff X is reflexive.

We have the following properties.

Proposition 4.2.4:

1. X with the w-topology and X^* with the w^*-topology are LCSs. So they are Hausdorff, and addition and scalar multiplication are continuous.
2. $\sigma(X, X^*) \subseteq \|\cdot\|$-topology. This is because in the $\|\cdot\|$-topology, all the linear functionals are continuous, and the weak topology is the weakest such topology. $\sigma(X^*, X) \subseteq \sigma(X^*, X^{**})$ because in the LHS, we are making less functionals continuous (equality iff X is reflexive).
3. If $\dim X < \infty$, then these topologies coincide. (Exercise. Show that anything open in norm topology will be open in the weak topology.)
4. If $\dim X = \infty$, $U \neq \phi$ is an open set in $\sigma(X, X^*)$, then U is unbounded. (Given a finite number of functionals, the intersection of their kernels is a nontrivial subspace.) So it follows that $\sigma(X, X^*) \subsetneq \|\cdot\|$-topology. So $\sigma(X^*, X) \subset \sigma(X^*, X^{**}) \subsetneq \|\cdot\|$-topology.
5. If $\dim X = \infty$, then $\sigma(X, X^*)$ is not metrizable (and not even first countable, i.e., does not have countable neigborhood base).
6. In particular, if $\dim X$ is uncountable (e.g. $\dim X = \infty$, X complete, since the Baire category theorem gives that any complete normed space cannot have a countable Hamel basis), then $\sigma(X^*, X)$ is not metrizable, and not even first countable.

7. $\sigma(X^{**}, X^*)|_X = \sigma(X, X^*)$. (We use the fact that given a set X and a family $\mathcal{F}$, if we restrict to subspace Y, we get exactly the weak topology induced by the restrictions of $\mathcal{F}$.)

We introduce some notation. Let X be a normed space. Let $x_n \in X$, $n \in N$, $x \in X$. Say x_n **converges weakly** to x, and write $x_n \xrightarrow{w} x$ if x_n converges to x in the w-topology. This happens iff it converges pointwise at any functional.

$$\langle x_n, x^* \rangle \to \langle x, x^* \rangle \qquad \forall x^* \in X^*.$$

Given x_n^*, $n \in \mathbb{N}$, $x^* \in X^*$, we say x_n^* **converges** w^* **to** x^* if x_n^* converges to x^* in the w^*-topology, and write $x_n^* \xrightarrow{w^*} x^*$. This happens iff

$$\langle x, x_n^* \rangle \to \langle x, x^* \rangle \qquad \forall x \in X.$$

Note that the same applies to nets.

Recall the following.

Theorem 4.2.5 (Principle of Uniform Boundedness)**:** thm:pub Let X be Banach and Y be a normed space, and $\mathcal{T} \subseteq \mathcal{B}(X, Y)$. If $\mathcal{T}$ is pointwise bounded, i.e. $\sup_{T \in \mathcal{T}} \|Tx\| < \infty$ for all $x \in X$, then $\sup_{T \in \mathcal{T}} \|T\| < \infty$.

A subset A of a normed space X is **weakly bounded (**w**-bounded)** if $\{\langle x, x^* \rangle : x \in A\}$ is bounded for all $x^* \in X^*$, i.e., $\{\hat{x} : x \in A\} \subseteq X^{**} = \mathcal{B}(X^*, \mathbb{R} \text{ or } \mathbb{C})$ is pointwise bounded. A subset $B \subseteq X^*$ is w^***-bounded** if $\{\langle x, x^* \rangle : x^* \in B\}$ is bounded for all $x \in X$. We immediately get the following.

Proposition 4.2.6: pr:f4-6 Let X be a normed space, $A \subseteq X$, $B \subseteq X^*$. Then

1. A is w-bounded iff A is $\|\cdot\|$-bounded.

2. If X is a Banach space and B is w^*-bounded, then B is $\|\cdot\|$-bounded.

We use some facts of sequences. Since weak convergence of a sequence is defined as pointwise convergence, a weakly convergent sequence is pointwise bounded. Recall the following.

Theorem 4.2.7 (Banach-Steinhaus)**:** thm:bs Let X be Banach, Y be a normed space, and $T_n \in \mathcal{B}(X, Y)$, $n \in \mathbb{N}$. Assume T_n converges pointwise to some map $T : X \to Y$. Then $T \in \mathcal{B}(X, Y)$ and $\|T\| \le \liminf \|T_n\|$.

Proposition 4.2.8: pr:norm-bounded Let X be a normed space, $x_n \xrightarrow{w} x$ in X, and $x_n^* \xrightarrow{w^*} x^*$ in X^*. Then

1. $\sup \|x_n\| < \infty$, $\|x\| \le \liminf \|x_n\|$, and

2. if X is complete, then $\sup \|x_n^*\| < \infty$, and $\|x^*\| \le \liminf \|x_n^*\|$.

3 Hahn-Banach separation theorems

Definition 4.3.1: Let $(X, \mathcal{P})$ be a LCS. Let C be a convex subset of X such that $0 \in \operatorname{int}(C)$. We define the **Minkowski functional** associated to C by

$$\mu_C(x) = \inf \{t > 0 : x \in tC\}.$$

For any $x \in X$, $0x \in C$, so since scalar multiplication is continuous, $\exists \delta > 0$ such that $\delta x \in C$, so $x \in \frac{1}{\delta} C$ (so μ_C is well-defined).

For example, if X is a normed space, $C = B_X$, then $\mu_C = \|\cdot\|$.

Lemma 4.3.2: lem:f4-8 Let $(X, \mathcal{P})$ be a LCS, C a convex subset of X with $0 \in \operatorname{int}(C)$. Then μ_C is a positive homogeneous subadditive function. Moreover, $\{x \in X : \mu_C(x) < 1\} \subseteq C \subseteq \{x \in X : \mu_C(x) \leq 1\}$. Moreover, if C is open, we have equality in the first case: $C = \{x \in X : \mu_C(x) < 1\}$.

Proof. Positive homogeneity: Clearly for $x \in X$, $\alpha > 0$, we have $x \in tC \iff \alpha x \in \alpha t C$, so $\mu_C(\alpha x) = \alpha \mu_C(x)$ and $\mu_C(0) = 0$.

First observe that if $\mu_C(x) < t$, then $\exists s < t$ such that $x \in sC$. So $\frac{x}{s} \in C$ and hence $\frac{x}{t} = \frac{s}{t} + \frac{x}{s} + \left(1 - \frac{s}{t}\right) 0 \in C$ by convexity. So $x \in tC$. Now assume $x, y \in X$, pick $s > \mu_C(x)$, $t > \mu_C(y)$. Then $x \in sC$, $y \in tC$. So by convexity, $\frac{x+y}{s+t} = \frac{s}{s+t}\frac{x}{s} + \frac{t}{s+t}\frac{y}{t} \in C$. Thus $\mu_C(x+y) \leq s + t$. Taking inf over s, t we get $\mu_C(x+y) \leq \mu_C(x) + \mu_C(y)$.

We observed that $\mu_C(x) < 1$ then $x \in C$. Also if $x \in C$, then $\mu_C(x) \leq 1$. If C is open, then for $x \in C$, $1x = x \in C$ implies by continuity of scalar multiplication that there exists $\delta > 0$ such that $(1 + \delta)x \in C$, so $\mu_C(x) < 1$. □

6th Nov

Remark:

1. The only place where we used the fact that X is a LCS is to show μ_C is well-defined; we used continuity of scalar multiplication. All this makes sense in a vector space except that μ_C may take the value ∞.

2. If in the real case C is symmetric ($x \in C \implies -x \in C$) or in the complex case, C is balanced ($x \in C, |\lambda| = 1 \implies \lambda x \in C$), then μ_C is a seminorm.

The following is the most basic version of the separation theorem.

Theorem 4.3.3 (Hahn-Banach Separation Theorem)**:** thm:f4-9 Let $(X, \mathcal{P})$ be a LCS, C an open convex subset of X, $0 \in C$, and $x_0 \notin C$. Then for all $x \in C$, there exists $f \in X^*$ such that $f(x) < f(x_0)$ for all $x \in C$. (In the complex case, $\Re f(x) < \Re f(x_0)$ for all $x \in C$.)

Proof. Let μ_C be the Minkowski functional of C. C is open, so $\{x \in X : \mu_C(x) < 1\} = C$. Let $Y = \operatorname{span}\{x_0\}$. Define $g : Y \to \mathbb{R}$ by $g(\lambda x_0) = \lambda$ for all $\lambda \in \mathbb{R}$. So

- if $\lambda \geq 0$ then since $x_0 \notin C$, $g(\lambda x_0) = \lambda \leq \lambda \mu_C(x_0) = \mu_C(\lambda x_0)$, and
- if $\lambda < 0$ then $g(\lambda x_0) = \lambda \leq 0 \leq \mu_C(\lambda x_0)$.

So g is dominated by μ_C on Y. By the Hahn Banach Theorem 2.1.3 g extends to a linear map $f : X \to \mathbb{R}$ such that $f(x) \leq \mu_C(x)$ for all $x \in X$. For $x \in C$, $f(x) \leq \mu_C(x) < 1 = f(x_0)$. C is open and $0 \in C$, so there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$, $\varepsilon > 0$ such that $\{x \in X : p_i(x) \leq \varepsilon \forall 1 \leq i \leq n\} \subseteq C$. So if $\max_{1\leq i\leq n} p_i(x) \leq \varepsilon$ then $f(x) < 1$. It follows that $|f(x)| \leq \frac{1}{\varepsilon} \max_{1\leq i\leq n} p_i(x)$ for all $x \in X$. By Lemma 2.2.5, f is continuous.

In the complex case, we take the real part: View X as a real space; we get a real-linear $f_1 : X \to \mathbb{R}$ such that f_1 is continuous, $f_1(x) < f_1(x_0)$ for all $x \in C$. Define $f(x) = f_1(x) - if_1(ix)$ for $x \in X$. Then f is complex, linear, continuous functional with $\Re f = f_1$. □

In the next result the statement is for the real case only. The complex case follows as above.

Theorem 4.3.4 (Hahn-Banach separation theorem for convex sets)**:** thm:f4-10 Let $(X, \mathcal{P})$ be a LCS. Let A, B be non-empty, disjoint convex sets.

1. If A is open, then there exists $f \in X^*$, $\alpha \in \mathbb{R}$ such that $f(a) < a \leq f(b)$ for all $a \in A, b \in B$.

2. If A is compact, B is closed, then there exists $f \in X^*$ such that $\sup_{a\in A} f(a) < \inf_{b\in B} f(b)$.

In the first case, the hyperplane where $f = \alpha$ is allowed to touch B, and in the second, there is a hyperplane strictly separating the sets.

Proof. 1. Pick $a_0 \in A, b_0 \in B$. Set $C = A - B + b_0 - a_0$, $x_0 = b_0 - a_0$. Now $C = \bigcup_{b\in B}(A - b + b_0 - a_0)$, so C is an open convex set with $0 \in C$. $x_0 \notin C$ since $A \cap B = \phi$. By Theorem 4.3.3, there exists $f \in X^*$ such that $f(x) < f(x_0)$ for all $x \in C$. So $f(a) < f(b)$ for all $a \in A, b \in B$. Set $\alpha = \inf_{b\in B} f(b)$. Clearly, $f(b) \geq \alpha$ for all $b \in B$. Note that $f(x_0) > f(0) = 0$; given $a \in A$ there exists $n \in \mathbb{N}$ such that $a + \frac{1}{n}x_0 \in A$ (A is open). Then $f(a) < f(a) + \frac{1}{n}f(x_0) = f\left(a + \frac{1}{n}x_0\right) \leq \alpha$.

2. Let U be a neighborhood of 0: there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}, \varepsilon > 0$ such that $U \supseteq \{x \in X : p_i(x) < \varepsilon \forall 1 \leq i \leq n\}$. Set $V = \left\{x \in X : p_i(x) < \frac{\varepsilon}{2} \forall 1 \leq i \leq n\right\}$. Then V is open, convex, and $V + V \subseteq U$. For all $a \in A$ there exists an open neighborhood U_a of 0 such that $a + U_a \cap B = \phi$. (B is closed and $a \notin B$.) There exists an open convex neighborhood V_a of 0 such that $V_a + V_a \subseteq U_a$. (This is a standard trick.) Now $\{a + V_a : a \in A\}$ is an open cover for A, so there exist $a_1, \ldots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^n a_i + V_{a_i}$. Let $V = \bigcap_{i=1}^n V_{a_i}$. Then V is an open convex neighborhood of 0, and $(A + V) \cap B = \phi$. Given $a \in A$, there exists i such that $a \in \alpha + V_{a_i}$ so $a + V \subseteq a + V_{a_i} + V_{a_i} \subseteq a + Ua_i$ which is disjoint from B. Apply (1) to $A + V$, B to get $f \in X^*$, $\alpha \in \mathbb{R}$ such that $f(a + v) < \alpha \leq f(b)$ for all $a \in A, v \in V, b \in B$. Hence

$$\sup_A f < \inf_B f.$$

Note the sup is attained as A is compact.

□

These are all the separation theorems we will need. Now we return to weak topologies.

4 Results on weak topologies

4.1 Closure and compactness, Banach-Alaoglu

Theorem 4.4.1 (Mazur)**:** thm:f4-11 Let X be a normed space and C a convex subset of X. Then C is w-closed iff C is $\|\cdot\|$-closed. (In general, $\|\cdot\|$-closed may not imply w-closed.) In particular, the closure in the w-topology and $\|\cdot\|$-topology are the same, $\overline{C}^w = \overline{C}^{\|\cdot\|}$.

Proof. "$\Leftarrow$": Assume C is $\|\cdot\|$-closed. Let $x_0 \notin C$. By Hahn-Banach Separation Theorem 4.3.4 there exists $f \in X^*$ such that $f(x_0) < \inf_{x\in C} f(x)$ (take $A = \{x_0\}, B = C$ in Theorem 4.3.4(2)). Then $\{x \in X : f(x) < \alpha\}$ is a w-neighborhood of x_0 disjoint from C. So $X \backslash C$ is w-open. □

Corollary 4.4.2: cor:f4-12 Suppose $x_n \xrightarrow{w} 0$ in a normed space X. Then for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0, \infty)$ such that $\sum_{i=1}^n t_i = 1$ and $\|\sum_{i=1}^n t_i x_i\| < \varepsilon$.

Proof. Let $C = \operatorname{conv}\{x_n : n \in \mathbb{N}\} = \{\sum_{i=1}^n t_i x_i : n \in \mathbb{N}, t_i \geq 0 \forall i, \sum_{i=1}^n t_i = 1\}$. C is convex. By assumption $0 \in \overline{C}^w = \overline{C}^{\|\cdot\|}$ by Mazur. □

Remark: In particular we can find $p_1 < q_1 < p_2 < q_2 < \cdots$ and convex combinations $\sum_{i=p_n}^{q_n} t_i x_i \to 0$ as $n \to \infty$.

The following is the most important theorem in this section.

Theorem 4.4.3 (Banach-Alaoglu Theorem)**:** thm:ba Let X be a normed space. Then (B_{X^*}, w^*) is compact.

Problem with X^* with usual topology: it's an infinite dimensional space that is not compact. So this result is very good.

Proof. For $x \in X$, let $F_x = \{\lambda \text{ scalar} : |\lambda| \leq \|x\|\}$. This is a bounded closed subset of scalars so by Theorem 4.1.4 (Tychonov), $F = \prod_{x\in X} F_x$ is compact in the product topology. We will realize B_{X^*} as a closed subset of F.

Let $\pi_x : F \to F_x$ be the coordinate projection $\pi_x(f) = f(x)$. Define $\theta : (B_{X^*}, w^*) \to F$ by $\theta(x^*) = (x^*(x))_{x\in X}$.

It is clear θ is injective; we show θ is continuous. This comes from the universality property of the property: $\pi_x \circ \theta(x^*) = x^*(x)$ is continuous in x^*, so θ is continuous.

We have $\theta^{-1} : \operatorname{im}(\theta) \to (B_{X^*}, w^*)$ is continuous by the university property in the remark after Definition 4.1.1: $\hat{x} \circ \theta^{-1}(f) = f(x) = \pi_*(f)$. So $\hat{x} \circ \theta^{-1} = \pi_x|_{\operatorname{im}(\theta)}$ is continuous on $\operatorname{im}(\theta)$. So (B_{X^*}, w^*) is homeomorphic to $\operatorname{im}(\theta)$.

It suffices to show $\operatorname{im}(\theta)$ is closed, because a closed subset of a compact space is compact. We have

$$\operatorname{im}(\theta) = \{f \in F : f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) \forall x, y \in X, \lambda, \mu \text{ scalars}\}.$$

This is the inverse image of a closed set:

$$\bigcap_{x,y\in X,\lambda,\mu \text{ scalars}} \{f \in F : (\pi_{\lambda x+\mu y} - \lambda\pi_x - \mu\pi_y)(f) = 0\}.$$

This is closed in F and hence compact. □

Lecture 8-11

4.2 Separability and metrizability

Proposition 4.4.4: pr:f4-14 Let X be a normed space, and K a compact Hausdorff space. Then

1. X is separable iff (B_{X^*}, w^*) is metrizable.
2. $C(K)$ is separable iff K is metrizable.

Proof.

1. $\implies$: Let $\{x_n : x \in \mathbb{N}\}$ be dense in X. Consider $\mathcal{F} = \{\widehat{x}_n|_{B_{X^*}} : n \in \mathbb{N}\}$. $(\hat{x}(x^*) = x^*(x), x \in X, x^* \in X^*)$ Let σ be the weak topology $\sigma(B_{X^*}, \mathcal{F})$ on B_{X^*}. Clearly σ is in the w^*-topology. Also, since $\{x_n : n \in \mathbb{N}\}$ is dense in X, $\mathcal{F}$ separates points of B_{X^*} (if $x^*(x_n)$ for all n, then $x^* = 0$), so by Proposition 4.1.3, σ is metrizable. The formal id : $(B_{X^*}, w^*) \to (B_{X^*}, \sigma)$ is a continuous bijection from a compact space (Theorem 4.4.3) to a Hausdorff space, so it is a homeomorphism.

2. $\Leftarrow$: K is a compact metric space, so it's separable; let $\{t_n : n \in \mathbb{N}\}$ be dense in K. Let $f_n(t) = d(t, t_n)$, $t \in K$, $n \in \mathbb{N}$, where d is the distance on K. Since $\{t_n : n \in \mathbb{N}\}$ is dense in K, the f_n's separate the points of K. Let A be the algebra generated by $\{f_n : n \in \mathbb{N}\} \cup \{1\}$. Then A is a unital subalgebra of $C(K)$, separates the points of K and, in the complex case, is closed under complex conjugation. So by Stone-Weierstrass, $\overline{A} = C(K)$. Since A is countably generated, A and hence $C(K)$ is separable.

1. $\Leftarrow$: Consider $X \to C(K)$ where $K = (B_{X^*}, w^*)$ is a compact metric space given by $x \to \widehat{x}|_K$. By definition,

$$\|\hat{x}|_K\|_\infty = \sup_{x^*\in B_{X^*}} \|x^*(x)\| = \|x\|$$

so the embedding is isometric. By (ii)$\Leftarrow$, $C(K)$ is separable so X is separable.

2. $\Rightarrow$: Note $\|\cdot\|_\infty$ is a norm on $C(K)$ since X is compact. $X = (C(K), \|\cdot\|_\infty)$ is separable, so (B_{X^*}, w^*) is metrizable by (i) $\implies$. Consider $\varphi : K \to (B_{X^*}, w^*)$, $\varphi(k) = \delta_k$, where $\delta_k(f) = f(k)$. If $k \neq k'$ in K, then by Urysohn there exists $f \in C(K)$, $f(k) \neq f(k')$. So $\delta_k(f) \neq \delta_{k'}(f)$. So φ is injective.

   φ is continuous: $(\hat{x} \circ \varphi)(k) = \hat{x}(\delta_k) = \delta_k(x) = x(k)$, $x \in C(K)$. So $\hat{x} \circ \varphi \in C(K)$ for all $x \in C(K)$, so φ is continuous. So φ is a continuous bijection from compact K to Hausdorff $\varphi(K)$, so K is homeomorphic to $\varphi(K) \subseteq (B_{X^*}, w^*)$.

□

Remark: 1. A compact metrizable set is sequentially compact.

2. X is separable implies X^* is w^*-separable, as $X^* = \bigcup_{n=1}^{\infty} nB_{X^*}$. For all n, (nB_{X^*}, w^*) is a compact metric space, so separable. The converse is false in general, for example, for example, $X = \ell_\infty$.

3. X is separable iff X is w-separable. For $\implies$, note the norm topology is stronger, so it is harder to be separable in the norm topology. For $\Leftarrow$, if $X = \overline{A}^w$, and A is countable, then $\overline{\operatorname{span}}^{\|\cdot\|} A = \overline{\operatorname{span}}^w(A) \supseteq \overline{A}^w = X$.

Proposition 4.4.5: pr:f4-15 X^* is separable iff (B_X, w) is metrizable.

Proof. $\implies$: By Proposition 4.4.4, $(B_{X^{**}}, w^*)$ is metrizable. Since the w-topology on B_X is the restriction to B_X of the w^* topology on $B_{X^{**}}$, (B_X, w) is metrizable.

It's not clear why metrizability goes the other way; we have to work a bit.

$\Leftarrow$: Let d be a metric on B_X inducing the w-topology. Then every open ball with respect to d must contain a basis element of the w-topology: For all $n \in \mathbb{N}$, there exist $x^*_{n1}, \dots, x^*_{nk_n}$ and $\varepsilon_n > 0$ such that

$$U_n = \{y \in B_X : |x^*_{ni}(y)| < \varepsilon_n \forall 1 \le i \le k_n\} \subseteq B\left(0, \frac{1}{n}\right) = \left\{y \in B_X : d(y,0) < \frac{1}{n}\right\}.$$

Let $Y = \overline{\operatorname{span}}\{x^*_{ni} : n \in \mathbb{N}, 1 \le i \le k_n\}$. Let $x^* \in B_{X^*}$, there exists $n \in \mathbb{N}$ with $U_n \subseteq \left\{y \in B_X : |x^*(y)| < \frac{1}{2}\right\}$. If $y \in \bigcap_{i=1}^{k_n} \ker x^*_{ni} \cap B_X$ then $y \in U_n$, and hence $|x^*(y)| < \frac{1}{2}$. So

$$\left\| x^*|_{\bigcap_{i=1}^{k_n} \ker x^*_{ni}} \right\| \le \frac{1}{2}.$$

Let z^* be an extension of $x^*| \bigcap_{i=1}^{k_i} \ker x^*_{ni}$ to X with $\|z^*\| \le \frac{1}{2}$ (Hahn-Banach).

We have $\ker(x^* - z^*) \supseteq \bigcap_{i=1}^{k_n} \ker x^*_{ni}$. So by Lemma 4.1.5, $x^* - z^* \in \operatorname{span}\{x^*_{ni} : 1 \le i \le k_n\}$. So $d(x^*, Y) \le \frac{1}{2}$. By Riesz's lemma 1.2.1, $Y = X^*$. $\square$

We show weakly compact subsets are metrizable under certain conditions.

Proposition 4.4.6: pr:f4-16 Let K be a w-compact subset of a Banach space X. If X^* is w^*-separable (e.g. if X is separable), then (K, w) is metrizable.

Proof. Let $\{x^*_n : n \in \mathbb{N}\}$ be w^*-dense in X^*. Then $\mathcal{F} = \{x^*_n : n \in \mathbb{N}\}$ separates the points of X, and hence the points of K. So $\sigma = \sigma(K, \mathcal{F})$ is metrizable and weaker than the w-topology (it's only making the x^*_n continuous). So $\mathrm{id} : (K, w) \to (K, \sigma)$ is a continuous bijection from a compact to a Hausdorff space, so it's a homeomorphism. $\square$

4.3 Reflexivity, Goldstine's Theorem

We aim to give a complete characterization of reflexivity. We'll give two proofs, the first relying on the following lemma.

Lemma 4.4.7 (Local reflexivity)**:** lem:f4-17 Let X be a normed space, F be a finite-dimensional subspace of X^*, $x^{**} \in X^{**}$ and $\varepsilon > 0$. Then $\exists x \in X$, $\hat{x}|_F = x^{**}|_F$ and $\|x\| \le (1+\varepsilon)\|x^{**}\|$.

Proof. The proof basically uses Hahn-Banach separation in finite dimensions. Let $x_1^*, \ldots, x_n^*$ be a basis of F. WLOG, $\|x^{**}\| \le 1$. Consider $T : X \to \mathbb{R}^n$, $Tx = (x_1^*(x), \ldots, x_n^*(x))$. T is linear, and onto: if it is not onto, then there exists $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ orthogonal to $T(X)$, so $\sum \alpha_i x_i^*(x) = 0$ for all x, i.e., $\sum \alpha_i x_i^* = 0$.

So T is an open map by the Open Mapping Theorem 1.3.3[1]. It follows that $A = \{Tx : \|x\| < 1 + \varepsilon\}$ is an open convex set in $\mathbb{R}^n$ and $0 \in A$. We need $(x^{**}(x_1^*), \ldots, x^{**}(x_n^*)) \in A$. If not, then by the Hahn-Banach Theorem 4.3.3, there exist $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$\sum_{i=1}^n \alpha_i x_i^*(x) < \sum \alpha_i x^{**}(x_i^*)$$

for all $x \in X$, $\|x\| < 1 + \varepsilon$.

Taking the sup over x,

$$\left\| \sum \alpha_i x_i^* \right\| (1 + \varepsilon) \le x^{**} \left(\sum \alpha_i x_i^* \right) \le \left\| \sum \alpha_i x_i^* \right\|.$$

This is a contradiction as the x_i^* are linearly independent and $(\alpha_1, \ldots, \alpha_n) \neq 0$ (because of $<$ above). $\square$

Theorem 4.4.8 (Goldstine's Theorem)**:** thm:goldstine $\overline{B_X}^{w^*} = B_{X^{**}}$ where X is a normed space, $\overline{B_X}^{w^*}$ is the w^*-closure in X^{**} of B_X.

The dual of weak star separable does not imply that the space is separable. We can actually prove this now.

Proof. Let $K = \overline{B}_X^{w^*}$. Then K is w^*-compact by the Banach-Alaoglu Theorem 4.4.3. Assume $K \neq B_{X^{**}}$. Fix $x_0^{**} \in B_{x^{**}} \backslash K$. We can finish in 2 ways.

1. By Theorem 4.3.4(2) (Hahn-Banach separation for convex sets) there exists x^* such that $\sup_K x^* < x_0^{**}(x^*)$. Since $K \supseteq B_X$, $\|x^*\| \le \sup_K x^* < x_0^{**}(x^*) \le \|x^*\|$.

2. K is w^*-closed, so there exists a w^*-neighborhood U of x_0^{**} such that $U \cap K = \phi$. WLOG $U = \{y^{**} \in X^{**} : |y^{**}(x_i^*) - x_0^{**}(x_i^*)| < \varepsilon \text{ for } 1 \le i \le n\}$ for some $n \in \mathbb{N}$, $x_1^*, \ldots x_n^*$, $\varepsilon > 0$. Let $F = \operatorname{span}\{x_1^*, \ldots, x_n^*\}$. By Lemma 4.4.7, there exists $x \in X$ such that $\|x\| \le 1 + \varepsilon$, $\hat{x}|_F = x_0^{**}|_F$. Then $\frac{x}{1+\varepsilon} \in B_X \subseteq K$,

$$\left| \frac{x}{1+\varepsilon}(x_i^*) - x_0^{**}(x_i^*) \right| = |x_0^{**}(x_i^*)| \left| \frac{1}{1+\varepsilon} - 1 \right| \le \varepsilon \text{ for all } i,$$

so $\frac{x}{1+\varepsilon} \in U$. (We don't need the Hahn-Banach Theorem in full generality, we just needed it for normed spaces.)

$\square$

[1]This is not a direct application as we don't have Banach spaces. We have linear $T : X \to \mathbb{R}^n$, $x \mapsto (x_i^*(x))_{i=1}^n$, T onto. But because $\mathbb{R}^n$ is finite-dimensional there exists $E \subseteq X$, $\dim E < \infty$ such that $T(E) = \mathbb{R}^n$. By the Open Mapping Theorem on $E \to \mathbb{R}^n$, $T(B_E)$, and hence $T(B_X)$ is a neighborhood of 0 in $\mathbb{R}^n$.

Theorem 4.4.9: thm:4-19 Let X be a Banach space. TFAE:

1. X is reflexive.
2. (B_X, w) is compact.
3. X^* is reflexive.

It follows that if Y is a closed subspace of X with X reflexive, then Y is reflexive.

Proof. (1) $\implies$ (2) If X is reflexive, $(B_X, w) = (B_{X^{**}}, w^*)$ and $(B_{X^{**}}, w^*)$ is compact by Banach-Alaoglu 4.4.3.

(2) $\implies$ (1) Since the restriction to X of the w^*-topology on X^{**} is the w-topology, we get B_X is w^*-compact on X^{**}, and so it's w^*-closed. By Goldstine's Theorem 4.4.8, $B_X = \overline{B}_X^{w^*} = B_{X^{**}}$. So $X = X^{**}$.

(1) $\implies$ (3) We have $\sigma(X^*, X) = \sigma(X^*, X^{**})$ since X is reflexive, so $(B_{X^*}, w) = (B_{X^*}, w^*)$ is compact by Banach-Alaoglu 4.4.3. By (2) $\implies$ (1), X^* is reflexive.

(3) $\implies$ (1) By (1) $\implies$ (3), X^{**} is reflexive, so by (1) $\implies$ (2), $(B_{X^{**}}, w)$ is compact. B_X is convex and $\|\cdot\|$-closed in X^{**}, and hence w-closed by Mazur's Theorem 4.4.1. So (B_X, w) is w-compact. So by (2) $\implies$ (1), X is reflexive. □

Remark: If X is separable and reflexive, then (B_X, w) is a compact metric space (Proposition 4.4.6).

Theorem 4.4.10: thm:f4-20 If X is a separable Banach space, then $X \hookrightarrow C[0,1]$ isometrically.

We've used Hahn-Banach to show we can embed into ℓ^∞ (Theorem 2.1.17) but that's not separable, and $C[0,1]$ is.

Lemma 4.4.11: lem:f4-21 If K is a compact metric space, then there exists a continuous surjection $\varphi : \Delta \to K$, where Δ is the Cantor set.

Proof. Δ is homeomorphic to $\{0,1\}^{\mathbb{N}}$ with the product topology, with the homeomorphism given by $(\varepsilon_i)_{i=1}^m \mapsto \sum_{i=1}^\infty \varepsilon_i 2 \cdot 3^i$.

We introduce some notation. For $m \le n$, $\varepsilon = (\varepsilon_i)_{i=1}^n \in \{0,1\}^n$, let $\varepsilon|_m = (\varepsilon_i)_{i=1}^m \in \{0,1\}^m$.

Our plan is to start with the compact set, divide it into closed sets K_0, K_1 that are smaller, and then divide K_{00} into $K_{00} \cup K_{01}$, and so forth. Any 0-1 sequence corresponds to a nested sequence of compact sets, whose intersection is a singleton x.

K is a compact metric space, so there exist finitely many closed subsets with diameter less than 1 that cover K; we can assume the the number is a power of 2: There exists $n_1 \in \mathbb{N}$ and nonempty closed subsets K_ε of K, $\varepsilon \in \{0,1\}^{n_1}$ such that $K = \bigcup_{\varepsilon \in \{0,1\}^{n_1}} K_\varepsilon$, $\operatorname{diam}(K_\varepsilon) < 1$.

There exists $n_2 > n_1$ such that for each $\varepsilon \in \{0,1\}^{n_1}$, there exist nonempty closed sets $K_\delta \subseteq K_\varepsilon$, $\delta \in \{0,1\}^{n_2}$, $\delta|_{n_1} = \varepsilon$ and $\bigcup_{\delta \in \{0,1\}^{n_2}, \delta|_{n_1} = \varepsilon} K_\delta = K_\varepsilon$, $\operatorname{diam} K_\delta \le \frac{1}{2}$. Continue inductively to get $n_1 < n_2 < n_3 < \cdots \neq \phi$, closed sets K_ε, $\varepsilon \in \{0,1\}^{n_t}$, $k \in \mathbb{N}$, $\bigcup_{\delta \in \{0,1\}^{n_{k+1}}, \delta|_{n_k} = \varepsilon} K_\delta = K_\varepsilon$, $\dim(K_\delta) < \frac{1}{k+1}$ $(k \in \mathbb{N}, \varepsilon \in \{0,1\}^{n_d})$.

Define $\varphi : \{0,1\}^{\mathbb{N}} \to K$ by $\varphi((\varepsilon_i)_{i=1}^{\infty})$, the unique point in $\bigcap_{k=1}^{\infty} K_{\varepsilon_1,\dots,\varepsilon_{n_k}}$. Given $\varepsilon, \delta \in \{0,1\}^{\mathbb{N}}$, if $\eta = \varepsilon|_{n_k} = \delta|_{n_k}$, then $\varphi(\varepsilon)$, $\varphi(\delta) \in K_\eta$ so $d(\varphi(\varepsilon), \varphi(\delta)) < \frac{1}{k}$. So φ is continuous. Given $x \in K$ such that $x \in K_{\varepsilon_1,\dots,\varepsilon_{n_1}}$, in turn there exist $\varepsilon_{n_1+1}, \dots, \varepsilon_{n_2}$ such that $x \in K_{\varepsilon_{n_1},\dots,\varepsilon_{n_2}}$, etc. Then $x = \varphi((\varepsilon_i)_{i=1}^{m})$, showing surjectivity. □

Proof of Theorem 4.4.10. X is separable, so $K = (B_{X^*}, w^*)$ is a compact metric space (Banach-Alaoglu 4.4.3, Proposition 4.4.4). We saw in the proof of Proposition 4.4.4 that $X \hookrightarrow C(K)$ isometrically ($x \mapsto \hat{x}|_K$). $C(K) \hookrightarrow C(\Delta)$ isometrically by the map $f \mapsto f \circ \varphi$ where $\varphi : \Delta \to K$ is continuous surjective (Lemma 4.4.11).

Finally define the map $C(\Delta) \hookrightarrow C[0,1]$ by $f \mapsto \widetilde{f}$, where $\widetilde{f}$ is obtained by linearly extending f to $[0,1]$; it is an isometry. Take the composition

$$X \hookrightarrow C(K) \hookrightarrow C(\Delta) \hookrightarrow C[0,1].$$

□

One reason why this is useful is that now we can think of the class of separable Banach spaces as the set of closed subspaces of $C[0,1]$. We can put a Borel structure on this set of subspaces and get a Polish space; there is a connection between Banach space properties and set theory.

Useful for us is the fact that our abstract spaces become concrete. (This also says $C[0,1]$ is a very hard space to understand.)

5 Additional material (non-examinable)

Theorem 4.5.1 (Eberlein-Šmulian)**:** Let X be a Banach space and $K \subseteq X$. Then K is w-compact iff K is w-sequentially compact.

Proof. $\implies$: WLOG X is separable. Then (K, w) is metrizable (Proposition 4.4.6).

$\Leftarrow$: If K is w-sequentially compact, then K is bounded (as it's weakly bounded). So $\overline{K}^{w^*}$ is w^* compact. We need $K \subseteq X$.

Let $x^{**} \in \overline{K}^{w^*}$. Pick $x_1^* \in S_{X^*}$. There exist $x_i \in K$ such that $|w^{**}(x_1^*) - x_1^*(x_1)| < 1$. Let $F_1 = \operatorname{span}\{x^{**}, x_1\}$. There exist $x_2^*, \dots, x_{n_2}^* \in S_{X^*}$ such that

$$\|y^{**}\| \, 2 \leq \sup_{i \leq n_2} |y^{**}(x_i)|.$$

There exist $x_2 \in K$ such that $|x^{**}(x_i^*) - x_i^*(x_2)| < \frac{1}{2}$ for all $i \leq n_2$. Let $F_2 = \operatorname{span}(x^{**}, x_1, x_2)$. There exist $x_{n_2+1}^*, \dots, x_{n_3}^* \in S_{X^*}$ such that $\frac{\|y^{**}\|}{2} \leq \sup_{i \leq n_3} |y^{**}(x_i^*)|$. There exists $x_3 \in K$ with $|x^{**}(x_i^*) - x_i^*(x_3)| < \frac{1}{3}$ for all $i \leq n_3$, etc. "The sequences chase each other, at ∞ they somehow meet." We obtain $(x_n) \subset K, (x_n^*) \subset S_{X^*}$. There exist $b_1 < b_2 < \cdots$ wuch that $x_{k_n} \xrightarrow{w} x \in K$. By Corollary 4.4.2, $x^{**} - x \in \overline{\operatorname{span}}\{x^{**}\} \cup \{x_n : n \in \mathbb{N}\}$. If $y^{**} \in \operatorname{span}\{x^{**}\} \cup \{x_n : n \in \mathbb{N}\}$, $\frac{\|y^{**}\|}{2} \leq \sup_{n \in \mathbb{N}} |y^{**}(x_n^*)|$. The same holds for $y^{**} \in \overline{\operatorname{span}}\{x^{**}\} \cup \{x_n : n \in \mathbb{N}\}$. We have

$$|(x^{**} - x)(x_m^*)| \leq \underbrace{|(x^{**} - x_{k_n})(x_m^*)|}_{< \frac{1}{n} \to 0 \text{ as } n \to \infty} + |(x - x_{k_n})(x_m^*)| \to 0 \text{ as } n \to \infty$$

$\square$

Lecture 13-11

Theorem 4.5.2 (Krein-Šmulian Theorem)**:** thm:krein-smulian If K is w-compact in a Banach space X, then $\overline{\text{conv}}(K)$ is w-compact.

(Note "K $\|\cdot\|$-compact $\implies$ $\overline{\text{conv}}(K)$ is $\|\cdot\|$-compact" is easy.)
We first need some background on vector-valued integration.

5.1 Vector-valued integration

Definition 4.5.3: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and X be a Banach space.

1. $f : \Omega \to X$ is a **simple function** if $f = \sum_{i=1}^n x_i 1_{A_i}$ where $x_i \in X$ and $A_i \in \mathcal{F}$.

2. $f : \Omega \to X$ is **μ-measurable** if there exist simple functions $f_n, n \in \mathbb{N}$ such that $f_n \to f$ μ-a.e. (i.e., $\mu(\{\omega \in \Omega : f_n(\omega) \not\to f(\omega)\} = 0)$).

Theorem 4.5.4 (Pettis measurability)**:** For X separable, let $f : \Omega \to X$. TFAE:

1. f is μ-measurable.

2. $x^* \circ f : \Omega \to \mathbb{R}$ is measurable foir all $x^* \in X^*$.

Definition 4.5.5: Let $f = \sum_{i=1}^n x_i 1_{A_i}$ be a simple function. Define $\int_\Omega f_1(\omega)\, d\mu(\omega) = \sum_{i=1}^n x_i \mu(A_i) \in X$. We say $f : \Omega \to X$ is **Bochner integrable** if there exist simple functions $f_n, n \in \mathbb{N}$ such that $\int_\Omega \|f(\omega) - f_n(\omega)\|\, d\mu(\omega) \to 0$ as $n \to \infty$. We set $\int_\Omega f\, d\mu = \lim_{n\to\infty} \int_\Omega f_n\, d\mu$. We say $f : \Omega \to X$ is integrable iff $\int_\Omega \|f(\omega)\|\, d\mu(\omega) < \infty$.

The idea is that for $x_1, \ldots, x_n \in K$, $t_1, \ldots, t_n \geq 0$, $\sum_{i=1}^n t_i = 1$, $\sum_{i=1}^n t_i x_i = \int_K x\, d\mu$, $\mu = \sum_{i=1}^n t_i \delta_{x_i}$.

Proof. WLOG X is separable (Eberlein-Šmulian). Then K is compact in the w-topology, $f : K \to X$, $f(x) = x$. For all $x^* \in X^*$, $x^* \circ f = x^*|_K$ is w-continuous, so measurable for the Borel σ-field of (K, w). So f is μ-measurable for all $\mu \in C(K)^* = M(K)$. Also K is w-compact implies K is $\|\cdot\|$-bounded, so f is Bochner integrable for all $\mu \in M(K)$. Consider $T : C(K)^* \to X$, $T(\mu) = \int_K f\, d\mu$. Then T is linear, w^*-w continuous ???:

$$x^* \circ T(\mu) = x^* \left(\int_K x\, d\mu \right) = \int_K x^*(x)\, d\mu = \langle x^*(x), \mu \rangle .$$

(It's clear for simple functions; use the staandard argument to get it for all functions.) So $x^* \circ T$ is w^*-continuous. $B_{C(K)^*}$ is w^*-compact, so $T(B_{C(K)^*})$ is w-compact. For all $x \in K$, $T(\delta_x) = x$, so $T(B_{C(K)^*}) \supseteq K$. Hence $T(B_{C(K)^*}) \supseteq \overline{\text{conv}}(K)$. $\square$

Chapter 5

Krein–Milman Theorem

1 Krein-Milman Theorem

Definition 5.1.1: Let X be a real (or complex) vector space. Let $K \subseteq X$ be convex. An **extreme point** of K is a point $x \in K$ such that if $x = (1-t)y + tz$ for $y, z \in K$, $t \in (0,1)$, then $y = z = x$.

Let $\mathrm{Ext}(K)$ be the set of extreme points of K.

Example 5.1.2: 1. $B_{\ell_1^2}$: $\mathrm{Ext}(B_{\ell_1^2}) = \{\pm e_1, \pm e_2\}$.

2. $B_{\ell_2^2}$: $\mathrm{Ext}(B_{\ell_2^2}) = S_{\ell_2^2}$.

3. B_{C_0}: $\mathrm{Ext}(B_{C_0}) = \phi$. (!) Given $x = (x_n) \in B_{C_0}$, there exists n such that $|x_n| < \frac{1}{2}$. Let

$$y_m = \begin{cases} x_m, & m \neq n \\ x_n + \frac{1}{2}, & m = n \end{cases}, \qquad z_n = \begin{cases} x_m, & m \neq n \\ x_n - \frac{1}{2}, & m = n \end{cases}.$$

We have $y = (y_m), z = (z_m) \in B_{C_0}$. Then $y \neq z$ and $x = \frac{y+z}{2}$; every point is in the middle of a line segment.

A natural question is, when do the extreme points determing the topology? They do in the first two examples but not the third.

Theorem 5.1.3 (Krein-Milman Theorem)**:** thm:krein-milman Let $(X, \mathcal{P})$ be a LCS. Let K be a compact, convex set in X. Then $K = \overline{\mathrm{conv}}(\mathrm{Ext}(K))$. In particular, if $K \neq \phi$, then $\mathrm{Ext}(K) \neq \phi$.

Corollary 5.1.4: For a normed space X, $B_{X^*} = \overline{\mathrm{conv}}^{w^*}(\mathrm{Ext}(B_{X^*}))$. In particular, if $K \neq \phi$, then $\mathrm{Ext}(K) \neq \phi$.

The set of extreme points makes sense without any topology, so in a sense this theorem connects algebra and topology.

How do we find any extreme points to start with? We might as well take a continuous functional (from Hahn-Banach). Slice this convex set, and shift the affine hyperplane to the edge. Take another functional and push it to the boundary. In 2-D we're done, but in general we need to take infinitely many hyperplanes and use Zorn's lemma.

Let X and K be as in Theorem 5.1.3, $K \neq \phi$. A **face** of K is a nonempty compact convex $F \subseteq K$ such that for all $y, z \in K$ for all $t \in (0,1)$, if $(1-t)y + tz \in F$, then $y, z \in F$. (For example, a face of a square is an edge.)

We start with the following preliminary observations.

1. For $x \in K$, $x \in \text{Ext}(K)$ iff $\{x\}$ is a face of K.
2. Given $f \in X^*$, let $\alpha = \sup_K f$. Then $F = \{x \in K : f(x) = \alpha\}$ is a face of K. We check
   (a) $F \neq \phi$: K is compact and f is continuous.
   (b) F is convex (f is linear).
   (c) F is compact (f is continuous so F is a closed subset of a compact set).
   (d) If $y, z \in K$, $t \in (0,1)$, $(1-t)y + tx \in F$, then
   $$\alpha = f((1-t)y + tz) = (1-t)f(y) + tf(z) \leq \alpha,$$
   so we have equality throughout, i.e., $f(y) = f(z) = \alpha$, so $y, z \in F$.
3. If F is a face of K, E is a face of F, then E is a face of K.

Proof of Theorem 5.1.3. First we show that every face contains a minimal (with respect to inclusion) face. Let F be a face of K. The set P of all faces E of K with $E \subseteq F$ is partially ordered by reverse inclusion: $E_1 \geq E_2 \iff E_1 \subset E_2$. Note $P \neq \phi$, since $F \in P$. Given $E_i, i \in I$ in P ($I \neq \phi$) a chain, let $E = \bigcap_{i \in I} E_i$, it's easy to check $E \in P$, and clearly $E \geq E_i$ for all i. So by Zorn's lemma P has a maximal element.

Our next step is that every minimal face is a singleton. Assume F is a face and $|F| > 1$. Let $x \neq y$ be in F. By Hahn-Banach for LCS (Theorem 2.2.7), there exists a separating functional $f \in X^*$ such that $f(x) \neq f(y)$, WLOG $f(x) < f(y)$. Let $\alpha = \sup_F f$. Then $E = \{z \in F : f(z) = \alpha\}$ is a face of F (and hence of K by observation 3) and $E \subsetneq F$ since $x \in F \backslash E$, from $f(x) < f(y) \leq \alpha$.

So $\text{Ext}(K) \neq \phi$. Let $L = \overline{\text{conv}}(\text{Ext}(K))$. Clearly $L \subseteq K$. Assume there exists $x_0 \in K \backslash L$. By Hahn-Banach Separation 4.3.4(2), (take $A = L, B = \{x_0\}$), there exists $f \in X^*$ such that $\sup_L f < f(x_0)$. Let $\alpha = \sup_K f$. Then $E = \{x \in K : f(x) = \alpha\}$ is a face of K so there exists $x \in E \cap \text{Ext}(K)$. However $E \cap L \neq \phi$, contradiction. □

Lemma 5.1.5 (Slices form a subbase)**:** lem:f5-3 Let $(X, \mathcal{P})$ be a LCS, K be a compact set, $x \in K$. Then for every neigborhood U of x in K, there exists $n \in \mathbb{N}$, $f_1, \ldots, f_n \in X^*$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that

$$x \in \{y \in K : f_i(y) < \alpha_i \text{ for all } i\} \subseteq U$$

Recall that the topology of a LCS is defined by seminorms. The lemma says we can find a neighborhood that is defined as intersection of half-spaces.

Note the statement is clearly true in the weak topology because seminorms are given by functionals.

Proof. WLOG $U = V \cap K$ where

$$V = \{y \in X : p_i(x-y) < \varepsilon, 1 \le i \le m\} \text{ for some } m \in \mathbb{N}, p_1, \dots, p_n \in \mathcal{P}, \varepsilon > 0.$$

V is convex open, so by Hahn-Banach Separation (Theorem 4.3.3), for every $z \in K \backslash U = K \backslash V$ there exists $f_z \in X^*$, $\lambda \in \mathbb{R}$ such that $f_z(y) < \lambda = f_z(z)$ for all $y \in V$. So $f_z(y) < \lambda \le f_z(z)$ for $\alpha_z \in \mathbb{R}$, $f_z(x) < \alpha_z \le f_z(z)$. Since $K \backslash U$ is compact, there exists $n \in \mathbb{N}$, $z_1, \dots, z_n$ such that $K \backslash U \subseteq \bigcup_{i=1}^n \{y \in K : f_{z_i}(y) > \alpha_{z_i}\}$.

Set $f_i = f_{z_i}$, $\alpha_i = \alpha_{z_i}$, we have

$$x \in \{y \in K : f_i(y) < \alpha_i \forall i\} \subseteq U.$$

□

A **slice** of a set K in a LCS $(X, \mathcal{P})$ is a nonempty subset of K of the form $\{y \in K : f(y) < \alpha\}$ for some $f \in X^*, \alpha \in \mathbb{R}$.

Lemma 5.1.6 (Slices form a neighborhood base)**:** lem:f5-4 Let $(X, \mathcal{P})$ be a LCS, K convex, compact, $x \in \mathrm{Ext}(K)$. Then the slices of K containing x form a neighborhood base at x in K.

Proof. Let U be a neighborhood of x in K. By Lemma 5.1.5, WLOG

$$U = \{y \in K : f_i(y) < \alpha_i, 1 \le i \le k\}$$

for some $n \in \mathbb{N}$, $f_1, \dots, f_n \in X^*$, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Set $K_i = \{y \in K : f_i(y) \ge \alpha_i\}$. So $\bigcup_{i=1}^n K_i = K \backslash U$. Each K_i is convex and compact.

$$\mathrm{conv}\left(\bigcup_{i=1}^n K_i\right) = \left\{\sum_{i=1}^n t_i x_i : t_i \ge 0 \text{ for all } i, \sum_{i=1}^n t_i = 1, x_i \in K_i \text{ for all } i\right\}.$$

This is compact since it is the continuous image of the compact set $K_1 \times K_2 \times \cdots \times K_n \times S$ where $S = \{(t_i)_{i=1}^n \in \mathbb{R}^n : t_i \ge 0 \text{ for all } i, \sum_{i=1}^n t_i = 1\}$. Since $x \in \mathrm{Ext}(K)$, $x \notin \mathrm{conv}\left(\bigcup_{i=1}^n K_i\right)$. By Theorem 4.3.4(2), there exists $f \in X^*$, $\alpha \in \mathbb{R}$ such that $f(x) < \alpha < \inf_{\mathrm{conv} \bigcup_{i=1}^n K_i}$. So $x \in \{y \in K : f(y) < \alpha\} \subseteq U$. □

Theorem 5.1.7 (Partial converse of Krein-Milman)**:** thm:f5-5 Let $(X, \mathcal{P})$ be a LCS, $K \neq \phi$, convex and compact. Assume $K = \overline{\mathrm{conv}}(S)$ for some $S \subseteq K$. Then $\overline{S} \supseteq \mathrm{Ext}(K)$.

Remark:

1. We need $\overline{S}$, for example consider a disc in $\mathbb{R}^2$ with one point on the boundary removed.
2. $\mathrm{Ext}(K)$ need not be closed. Consider a vertical line segment connected to a horizontal circle.

Proof. Suppose not. Suppose there exists $x \in \mathrm{Ext}\, K$, $x \notin S$. Then $K \backslash \overline{S}$ is a neighborhood of x. Hence, by lemma 5.1.6, there exists $f \in X^*$, $\alpha \in \mathbb{R}$ such that $x \in \{y \in K : f(y) < \alpha\} \subseteq K \backslash \overline{S}$. Then $K = \overline{\mathrm{conv}} S \subseteq \{y \in K : f(y) \ge \alpha\} \neq K$. ($K$ is convex and compact.) □

Note that in the complex case, a slice would be $\Re f(y) < \alpha$. Everything goes through in $\mathbb{C}$ except with using $\Re f$ when needed.

We use this to compute the set of extreme points in the dual of balls for certain Banach spaces.

Proposition 5.1.8: pr:f5-6 Let K be a compact Hausdorff space $X = C(K)$. Then $\mathrm{Ext}(B_{X^*}) = \{\pm\delta_k : k \in K\}$ where $\delta_k(f) = f(k)$, $f \in X = C(K)$. (In the complex case, $\mathrm{Ext}(B_{X^*}) = \{\lambda\delta_k : k \in K, |\lambda| = 1\}$.)

Proof. Let $S = \{\pm\delta_k : k \in K\}$. The map $k \mapsto \delta_k : K \to (B_{X^*}, w^*)$ is continuous and injective (In the complex case, take $(k, \lambda) \mapsto \delta_k : K \times S^1 \to (B_{X^*}, w^*)$.), so K is homeomorphic to $\{\delta_k : k \in K\}$. If follows that S is compact.

We show

$$\text{eq:f5-1}\quad \overline{\mathrm{conv}}^{w^*}(S) = B_{X^*}. \tag{5.1}$$

If not then $\overline{\mathrm{conv}}^{w^*}(S) \subsetneq B_{X^*}$, so fix $\varphi \in B_{X^*} \backslash \overline{\mathrm{conv}}^{w^*} S$. By Hahn-Banach Separation (Theorem 4.3.4(2)), there exists $f \in X$ such that

$$\sup_{\overline{\mathrm{conv}}^{w^*} S} f < \varphi(f).$$

Why can we get $f \in X$, rather than something in X^{**}? Is $C(K)$ reflexive? But then

$$\|f\|_\infty \le \sup_{k\in K}(\pm\delta_k(f)) \overset{\pm\delta_k \in S}{\le} \sup_{\overline{\mathrm{conv}}^{w^*} S} f < \varphi(f) \overset{\varphi\in B_{X^*}}{\le} \|f\|_\infty,$$

contradiction. This shows 5.1.

By the converse to Krein-Milman (Theorem 5.1.7), $\overline{S} = S \supseteq \mathrm{Ext}(B_{X^*})$. Fix $k \in K$. By Urysohn's Lemma, for every open neighborhood U of k, there exists $f_n : K \to [0,1]$ continuous, $f_U(k) = 1$,$f_U \equiv 0$ on $K\backslash U$.

The map $\varphi \mapsto \varphi(f_U)$ is a w^*-continuous functional on X^*, whose supremum on B_{X^*} is 1 (Take $\varphi = \delta_k$). So $\{\varphi \in B_{X^*} : \varphi(f_U) = 1\}$ is a face of B_{X^*}. So

$$F = \bigcap_{U \text{ open neighborhood of } k} \{\varphi \in B_{X^*} : \varphi(f_U) = 1\}$$

is also a face of B_{X^*}. So there exists an extreme point in F, i.e., there exists $\ell \in K$ such that δ_ℓ or $\delta_{-\ell}$ is in F. If $\ell \ne k$, there exists an open neighborhood U of k such that $\ell \notin U$, so $\delta_\ell(f_U) = 0$. So δ_k is this extreme point. $\square$

Theorem 5.1.9 (Banach, Stone)**:** thm:f5-7 Let K, L be compact Hausdorff spaces. Then $C(K) \cong C(L)$ iff L is homeomorphic to K.

Proof.

1. $\Leftarrow$: If $\varphi : L \to K$ is a homeomorphism, then $\varphi^* : C(K) \to C(L)$, $f \mapsto f \circ \varphi$ is an isometric isomorphism.

2. $\implies$: The idea here is that T^* will send δ_ℓ to $\pm\delta_{\varphi(\ell)}$ for some φ; then φ will be the desired homeomorphism.

Let $T : C(K) \to C(L)$ be an isometric isomorphism. Then $T^* : C(L)^* \to C(K)^*$ is also an isometric isomorphism. So

$$T^*(\mathrm{Ext}(B_{C(L)^*})) = \mathrm{Ext}(B_{C(K)^*}).$$

Hence, for every $\ell \in L$, $T^*(\delta_\ell) = \varepsilon(\ell)\delta_{\varphi(\ell)}$ for some $\varepsilon(\ell) \in \{-1, 1\}$, $\varphi(\ell) \in K$. We need to "get rid" of ε; we do this by showing ε is continuous. We have

$$\varepsilon(\ell) = \varepsilon(\ell)\delta_{\varphi(\ell)}(1_K) = (T^*\delta_\ell)(1_K) = T(1_K)(\ell),$$

i.e., $\varepsilon = T(1_K)$ is continuous. We get $\ell \mapsto \varepsilon(\ell)T^*(\delta_\ell) = \delta_{\varphi(\ell)}$ is also continuous (T^* is w^*-w^* continuous, which can be checked by the universal property). Since $k \mapsto \delta_k$ is a homeomorphism between K and $\{\delta_k : k \in K\}$, it follows that $\ell \mapsto \varphi(\ell)$ is continuous. Since T^* is injective, and $T^*(\mathrm{Ext}(B_{C(L)^*})) = \mathrm{Ext}(B_{C(K)^*})$, φ is a bijection. Finally, a continuous bijection from a compact to a Hausdorff space is a homeomorphism

□

Theorem 5.1.9 tells us $C(K)$ has all the topological information about K. So to study compact Hausdorff space, we can study this class of Banach spaces, or actually Banach algebras, or actually commutative C^*-algebras. Generalizing, we can study noncommutative C^*-algebras; we call this field **noncommutative geometry**.

Chapter 6

Banach algebras

1 Banach algebras

Definition 6.1.1: Let A be a real or complex algebra. (An algebra is a vector field with multiplication, satisfying associativity and distributivity with scalars.) An **algebra norm** on A is a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\| \|b\|$. The pair $(A, \|\cdot\|)$ is a **normed algebra**. A complete algebra is called a **Banach algebra**.

Note that multiplication in a normed space is continuous: if $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.

Definition 6.1.2: A normed algebra A is **unital** (normed algebra) if it has an element 1 such that $1a = a1 = a$ for all $a \in A$ and $\|1\| = 1$.

Note that if A has an identity $1 \neq 0$ then $a \mapsto \sup_{b \in A, \|b\| \leq 1} \|ab\|$ is an equivalent norm on A on which A is a unital normed algebra.

Remark:

1. A homomorphism between algebras is a linear map $\varphi : A \to B$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$. If A, B are unital, then φ is a unital homomorphism if $\varphi(1) = 1$.
2. From now on, every algebra is complex.

Example 6.1.3: ex:f6-1

1. $C(K)$ is compact Hausdorff, with pointwise multiplication and $\|\cdot\|_\infty$ is a commutative, unital Banach algebra.
2. A **uniform algebra** is a closed subalgebra of $C(K)$ (K compact Hausdorff) such that $1 \in A$ and A separates the points of K. For example, $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ with the **disc algebra** $A(\Delta) := \{f \in C(\Delta) : \text{ analytic on } \operatorname{int}(\Delta)\}$.

   More generally, for a compact $K \subseteq \mathbb{C}$, we have

$$P(K) \subseteq R(K) \subseteq O(K) \subseteq A(K) \subseteq C(K)$$

where $P(K), R(K), O(K)$ are the closures in $C(K)$ of respectively, polynomials, rational functions (without poles in K), functional analytic on some open neighborhood of K, and

$$A(K) = \{f \in C(K) : f \text{ analytic on } \mathrm{int}(K)\}.$$

Later we will see

$$\begin{aligned} &P(K) = R(K) \iff \mathbb{C}\backslash K \text{ is connected.} \\ &R(K) = O(K) \text{ always} \\ &R(K) \neq A(K) \text{ in general} \\ &A(K) = C(K) \iff \mathrm{int}(K) \neq \phi. \end{aligned}$$

3. $L_1(\mathbb{R})$ with the product $(f * g)(x) = \int_{\mathbb{R}} f(g)g(x-y)\,dy$ (convolution) is a commutative Banach algebra without an identity. This is studied in harmonic analysis.

4. If X is a Banach space, then $B(X)$ is a unital Banach algebra, and noncommutative (unless $\dim(X) = 1$). The special case $B(H)$, H a Hilbert space, is important. For example, $M_n(\mathbb{C}) \cong B(\ell_2^n)$.

We give some standard constructions.

1. A subalgebra of a normed algebra is a normed algebra. The closure of a subalgebra of a normed algebra is a subalgebra. A closed subalgebra of a Banach algebra is a Banach algebra. If A is a unital algebra, then a **unital subalgebra** is a subalgebra B of A that contains the identity of A.

2. (**unitication**) Let A be an algebra. We let $A_+ = A \oplus \mathbb{C}$ with multiplication

$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu), \qquad a, b \in A, \qquad \lambda, \mu \in \mathbb{C}.$$

   Identifying A with $\{(a, 0) : a \in A\}$, we have $A \triangleleft A_+$ (A is an ideal in A_+). Setting $1 = (0, 1)$, we have $1x = x1 = x$ for all $x \in A_+$. When A is a normed algebra, we define $\|(a, \lambda)\| = \|a\| + |\lambda|$. Then A_+ is a unital normed algebra, and A is a closed ideal of A_+. If A is a Banach algebra, then so is A_+.

3. If A is a normed algebra, and $J \triangleleft A$ is a 2-sided ideal, then $\overline{J} \triangleleft A$. If J is a closed ideal in A, then A/S is a normed algebra in the quotient norm. When A is unital and $J \neq A$, A/J is unital. ($\|1 + J\| = 1$ will follow from Lemma 6.1.4.) When A is a Banach algebra, so is A/J.

4. The completion $\overline{A}$ of a normed algebra A is a Banach algebra. Given $a, b \in \overline{A}$, let $(a_n), (b_n) \in A$ such that $a_n \to a$, $b_n \to b$, and define $ab = \lim_{n\to\infty} a_n b_n$.

5. If A is a unital normed algebra, define for $a \in A$ $L_a : A \to A$ by $L_a(b) = ab$. The map $a \mapsto L_a : A \to B(A)$ is an isometric isomorphism. So *every Banach algebra is the closed subalgebra of $B(X)$ for some Banach space X.*

1.1 Units and inverses

Lemma 6.1.4: lem:f6-1 Let A be a unital Banach algebra, $x \in A$. If $\|1-x\| < 1$, then x is invertible, and $\|x^{-1}\| \le \frac{1}{1-\|x\|}$.

We construct the inverse by writing down a geometric series, and showing it converges.

Proof. We have
$$\sum_{n=0}^{\infty} \|1-x\|^n \le \frac{1}{1-\|1-x\|}$$
since $\|1-x\| < 1$, so $\sum_{n=0}^{\infty}(1-x)^n$ converges absolutely. If it t converges to z, say, then
$$\begin{aligned} xz &= \lim_{n\to\infty} \sum_{k=0}^{n} x(1-x)^n = \lim_{n\to\infty} (1-(1-x))(1-x)^n \\ &= \lim_{n\to\infty} (1-(1-x)^{n+1}) = 1. \end{aligned}$$
since $\|(1-x)^n\| \le \|1-x\|^n \to 0$ as $n \to \infty$. Similarly, $zx = 1$. □

Definition 6.1.5: For a unital algebra A, we let $G(A)$ be the set of units, i.e., invertible elements of A.

Corollary 6.1.6: cor:f6-2 Let A be a unital Banach algebra. Then

1. $G(A)$ is open in A.
2. $x \mapsto x^{-1} : G(A) \to G(A)$ is continuous (so $G(A)$ is a topological group).
3. If $x_n \in G(A), n \in \mathbb{N}$, $x_n \to x \in A\backslash G(A)$, then $\|x_n^{-1}\| \to \infty$ as $n \to \infty$.
4. If $x \in \partial G(A)$, then there exist $z_n \in A, n \in \mathbb{N}, \|z_n\| = 1, z_n x \to 0, xz_n \to 0$ as $n \to \infty$. In particular, x has no left or right inverse, even in a normed algebra B that contains A as a closed subalgebra.[1]

There are some errors I here I haven't gotten around to fixing. For now, please see `https://www.dpmms.cam.ac.uk/~az10000/resume-on-hilbert.pdf`.

Proof.

1. Let $x \in G(A)$. Assume $\|y-x\| \le \frac{1}{\|x^{-1}\|}$. We have $\|1-x^{-1}y\| = \|x^{-1}(x-y)\| \le \|x^{-1}\| \|x-y\| < 1$, so by Lemma 6.1.4, $x^{-1}y \in G(A)$ and so $y = x(x^{-1}y) \in G(A)$, so $B\left(x, \frac{1}{\|x^{-1}\|}\right) \subseteq G(A)$.

2. Let $x, y \in G(A)$. Consider $y^{-1} - x^{-1} = x^{-1}(x-y)y^{-1}$. Assume $\|y-x\| < \frac{1}{\|x^{-1}\|}$. Then $\|1-x^{-1}y\|$. Then $\|1-x^{-1}y\| < 1$, so $x^{-1}y$ is invertible, so $\|y^{-1}x\| < \frac{1}{1-\|1-y^{-1}x\|}$ by Lemma 6.1.4. Now
$$\left\|y^{-1}-x^{-1}\right\| \le \left\|x^{-1}\right\| \|x-y\| \left\|y^{-1}x^{-1}\right\| \le \left\|x^{-1}\right\|^2 \|x-y\| \frac{1}{1-\|x^{-1}\|\|x-y\|} \to 0 \text{ as } y \to \infty.$$

[1]Here ∂A denotes the boundary of a set A in a topological space X: $\partial A = \overline{A}\backslash A^\circ$.

3. For all n, $x \notin B(x_n, \frac{1}{\|x_n^{-1}\|})$ by proof of (1), so $\underbrace{\|x - x_n\|}_{\to 0 \text{ as } n \to \infty} \geq \frac{1}{\|x_n^{-1}\|}$, so $\|x_n^{-1}\| \to \infty$ as $n \to \infty$.

4. There exists $x_n \in G(A)$ such that $x_n \to x$. Set $z_n = \frac{x_n^{-1}}{\|x_n^{-1}\|}$, $n \in \mathbb{N}$. Then $\|z_n\| = 1$ for all n. We get

$$z_n x = \frac{x_n^{-1}}{\|x_n^{-1}\|}(x_n - (x_n - x)) = \frac{1}{\|x_n^{-1}\|} - \frac{\|xx_n\|}{\|x_n^{-1}\|} \to 0.$$

Similarly, $xz_n \to 0$ as $n \to \infty$. If say, there exists $w \in B$ suchthat $x_w = 1_B$ then $z_n = z_n x w \to 0$.

□

20 Nov.

2 Spectra

Definition 6.2.1: Let A be a unital Banach algebra, $x \in A$. The **spectrum** of x in A is the set $\sigma(x) = \sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \text{ is not invertible}\}$. If A is non-unital, then we define $\sigma_A(x)$ to be $\sigma_{A_+}(x)$, where A_+ is the unitization of A. Note that in this case, for any $x \in A$, $0 \in \sigma_A(x)$.

Example 6.2.2:

1. $A = M_n(\mathbb{C})$: $\sigma_A(x)$ is the set of eigenvalues of x.
2. $A = C(K)$: $\sigma_A(f) = f(K)$.

Theorem 6.2.3: thm:f6-3 Let A be a Banach algebra, $x \in A$. Then $\sigma(x)$ is a non-empty, compact subset of $\{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$.

Proof. WLOG A is unital. If $|\lambda| > \|x\|$ then $\left\|\frac{x}{\lambda}\right\| < 1$, so $1 - \frac{x}{\lambda} \in G(A)$ by Lemma 6.1.4, so $\lambda\left(1 - \frac{x}{\lambda}\right) = \lambda 1 - x$ is invertible, so $\lambda \notin \sigma_A(x)$. The function $\mathbb{C} \to A$, $\lambda \mapsto \lambda 1 - x$ is continuous. $G(A)$ is open, so $\sigma_A(x)$ is closed, and so compact.

Let $\rho_A(x) = \mathbb{C} \backslash \sigma_A(x)$ (the **resolvent set**). Consider

$$\begin{aligned}
R : \rho_A(x) &\to A \\
R(\lambda) &= (\lambda 1 - x)^{-1} \\
R(\lambda) - R(\mu) &= (\mu 1 - x)^{-1}((\mu 1 - x) - (\lambda 1 - x))(\lambda 1 - x)^{-1} \\
&= (\mu - \lambda) R(\lambda) R(\mu) \\
\implies \frac{R(\lambda) - R(\mu)}{\lambda - \mu} &= -R(\lambda)R(\mu) \to -R(\mu)^2 \text{ as } \lambda \to \mu.
\end{aligned}$$

Thus R is analytic. For $|\lambda| > \|x\|$,

$$\|R(\lambda)\| = \frac{1}{|\lambda|}\left\|\left(1 - \frac{x}{\lambda}\right)^{-1}\right\| \leq \frac{1}{|\lambda|}\frac{1}{1 - \left\|\frac{x}{\lambda}\right\|} = \frac{1}{|\lambda| - \|x\|} \to 0 \text{ as } |\lambda| \to \infty.$$

If $\sigma_A(x) = \phi$, then R is an entire function and bounded, so by vector-valued Liouville's Theorem 2.1.18, R is constant, so $R = 0$, contradiction. □

Example 6.2.4: Let A be an algebra of complex-valued functions on some set K. Assume there exists a Banach algebra norm $\|\cdot\|$ on A. Then all functions in A are bounded.

Indeed, for $f \in A$, $x \in K$, if $f(x) \neq 0$, then $f(x) \in \sigma_A(f)$. So $|f(x)| < \|f\|$.

Proposition 6.2.5: pr:xy-inverse If A is an algebra with $1 \neq 0$, $x, y \in A$, $xy = yx$, and xy is invertible, then x, y are invertible.

Proof. Suppose xy has inverse $w \in A$,

$$w(xy) = (xy)w = 1.$$

Then

$$x(yw) = 1, \quad (yw)x = ywx(yxw) = ywxyxw = yxw = xyw = 1.$$

□

Corollary 6.2.6 (Gelfand-Mazur Theorem)**:** cor:f6-4 If A is a complex unital normed division algebra, then $A \cong \mathbb{C}$.

Definition 6.2.7: A **division algebra** is an algebra A such that if $x \in A$ and $x \neq 0$, then x is invertible.

Proof. Let $x \in A$. Let B be the completion of A. We can pick $\lambda \in \sigma_B(x)$ as this is nonempty by Theorem 6.2.3. So $\lambda 1 - x$ is not invertible in B, and hence *a fortiori* is not invertible in A. Since A is a division algebra, $\lambda 1 - x = 0$, i.e., $x = \lambda 1$. □

Theorem 6.2.8 (Spectral mapping theorem for polynomials)**:** lem:f6-5 Let A be a unital Banach algebra, $x \in A$. Then for a polynomial p,

$$\sigma(p(x)) = \{p(\lambda) : \lambda \in \sigma(x)\}.$$

Proof. The result is clear for constant polynomials. Suppose $n := \deg(p) > 1$. For $\lambda \in \mathbb{C}$, we have by the Fundamental Theorem of Algebra that

$$\lambda - p(t) = c \prod_{k=1}^{n} (\mu_k - t)$$

for some $c, \mu_1, \ldots, \mu_n \in \mathbb{C}$ and $c \neq 0$, so $\lambda 1 - p(x) = c \prod_{k=1}^{n} (\mu_k 1 - x)$. We show:

1. $\lambda \notin \sigma(p(x)) \implies \lambda \notin p(\sigma(x))$: If $\lambda \notin \sigma(p(x))$, then $\lambda 1 - p(x)$ is invertible. Because the elements $\mu_k 1 - x$ pairwise commute, by Proposition 6.2.5, $\mu_k 1 - x$ is invertible for all k, i.e. $\mu_k \notin \sigma(x)$ for all k. So there does not exist $\mu \in \sigma(x)$ such that $\lambda = p(\mu)$.

2. $\lambda \notin p(\sigma(x)) \implies \lambda \notin \sigma(p(x))$: Conversely, if $\lambda \neq p(\mu)$ for any $\mu \in \sigma(x)$, then $\mu_k 1 - x$ is invertible for all k, so $\lambda 1 - p(x)$ is also invertible. So $\lambda \notin \sigma(p(x))$.

□

One can prove, using this lemma plus some convergence estimates, the following: for any analytic function f defined (as a power series) on a disc containing $\sigma(x)$, we have $\sigma(f(x)) = f(\sigma(x))$. To get a result on the domain of f and not just a disc, we need the holomorphic functional calculus; see Theorem **??**.

Definition 6.2.9: Let A be a Banach algebra, $x \in A$. The **spectral radius** of x in A is

$$r(x) = r_A(x) := \sup\{|\lambda| : \lambda \in \sigma_A(x)\}.$$

By Theorem 6.2.3, this is well-defined, and $r(x) \le \|x\|$.

Theorem 6.2.10 (Spectral radius formula)**:** thm:f6-6 Let A be a Banach algebra, $x \in A$. Then $r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_n \|x^n\|^{\frac{1}{n}}$.

The idea is basically uses the "root test": the radius of convergence of $\sum a_n z^n$ is $\lim_{n\to\infty} |a_n|^{-\frac{1}{n}}$. Some care is needed: (1) we need to compose with $\Lambda \in A^*$, (2) we need to pass from the result for $\Lambda \circ R$, (6.1), to the result R; one way is the result on weak boundedness implying norm boundedness.

Proof. WLOG A is unital. For $n \in \mathbb{N}$, $\lambda \in \sigma(x)$, $\lambda^n \in \sigma(x^n)$ by Lemma 6.2.8, so $|\lambda^n| \le \|x^n\|$. If follows that $r(x) \le \|x^n\|^{\frac{1}{n}}$. Hence $r(x) \le \inf_n \|x^n\|^{\frac{1}{n}}$.

Consider

$$R : \rho_A(x) = \mathbb{C}\backslash\sigma_A(x) \to A$$
$$R(\lambda) = (\lambda 1 - x)^{-1} = \frac{1}{\lambda}\left(1 - \frac{x}{\lambda}\right)^{-1} \text{ for } \lambda \ne 0.$$

Fix $\Lambda \in A^*$. Then $\Lambda \circ R$ is analytic on $\rho_A(x)$:

$$\rho_A(x) \supseteq \{\lambda \in \mathbb{C} : |\lambda| > r(x)\} \supseteq \{\lambda \in \mathbb{C} : |\lambda| > \|x\|\}$$

so $\Lambda \circ R$ has Laurent expansion on $\{\lambda \in \mathbb{C} : |\lambda| > r(x)\}$. For $|\lambda| > \|x\|$,

$$\Lambda \circ R(\lambda) = \Lambda\left(\sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}\right) = \sum_{n=0}^{\infty} \frac{\Lambda(x^n)}{\lambda^{n+1}}.$$

This is also an expansion of $\Lambda \circ R$ on $\{\lambda \in \mathbb{C} : |\lambda| > r(x)\}$.[2] Fix $|\lambda| > r(x)$. We have $\frac{\Lambda(x^n)}{\lambda^{n+1}} \to 0$ as $n \to \infty$, so in particular there exists $c > 0$ such that $|\Lambda(x^n)|\,\lambda^{n+1} \le C$ for all n. It follows that $\limsup_{n\to\infty} |\Lambda(x^n)|^{\frac{1}{n}} \le |\lambda|$. Hence

$$\text{eq:srf1} \limsup_{n\to\infty} |\Lambda(x^n)|^{\frac{1}{n}} \le r(x). \tag{6.1}$$

We want the LHS to not have Λ, to transfer this "weak" bound to a bound involving $\|x\|$. To do anything we want to look at something that converges; fortunately, to prove a lower

[2]We used the following fact about analytic functions: the radius of convergence of its power series around a point is exactly the radius of the largest disc inside which the function is defined.

bound on $r(x)$ is the same as proving a lower bound on all $K > r(x)$. Fix $K > r(x)$, so $\limsup_{n\to\infty} \left|\frac{\Lambda(x^n)}{K^n}\right|^{\frac{1}{n}} < 1$. Hence $\left(\Lambda\left(\frac{x^n}{K^n}\right)\right)_{n=1}^{\infty}$ is bounded. So

$$\left\{\frac{x^n}{K^n} : n \in \mathbb{N}\right\}$$

is weakly bounded and by Proposition 4.2.6 it is norm bounded. There exists $M \geq 0$ with $\left\|\frac{x^n}{K^n}\right\| \leq M$ for all $n \in \mathbb{N}$, so $\limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq K$. Hence

$$\limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq r(x) \leq \inf_n \|x^n\|^{\frac{1}{n}} \leq \liminf_{n\to\infty} \|x^n\|^{\frac{1}{n}}.$$

□

Theorem 6.2.11: thm:f6-7 Let A be a unital Banach algebra and B be a closed unital subalgebra of A. Then for $x \in B$, we have

$$\sigma_B(x) \supseteq \sigma_A(x)$$
$$\partial\sigma_B(x) \subseteq \partial\sigma_A(x).$$

It follows that $\sigma_B(x)$ is the union of $\sigma_A(x)$ together with some of the bounded components of $\mathbb{C}\backslash\sigma_A(x)$.

Proof. It is clear that $\sigma_B(x) \supseteq \sigma_A(x)$. Let $x \in \partial\sigma_B(x)$. There exists $\lambda_n \to \lambda$ such that $\lambda_n \notin \sigma_B(x)$ for all n. We have

$$\underbrace{\lambda_n 1 - x}_{\in G(B)} \to \underbrace{\lambda 1 - x}_{\notin G(B)}$$

so $\lambda 1 - x \in \partial G(B)$. By Corollary 6.1.6(4), $\lambda 1 - x \notin G(A)$, so $\lambda \in \sigma_A(x)$. Since $\lambda_n \notin \sigma_A(x)$ for all n, $\lambda \in \partial\sigma_A(x)$. □

Let K be a non-empty, compact subset of $\mathbb{C}$. We use the temporary notation

$$\widetilde{K} := K \cup (\text{all bounded components of } \mathbb{C}\backslash K).$$

Definition 6.2.12: The **polynomial hull** of K is

$$\widehat{K} = \{z \in \mathbb{C} : |p(z)| \leq \|p\|_K \text{ for all polynomials } p\}$$

where $\|p\|_K = \sup_{\lambda \in K} |p(\lambda)|$.

It is easy to check $K \subseteq \widehat{K}$ and $\widehat{K}$ is compact.[3]

Definition 6.2.13: We say K is **polynomially convex** if $K = \widehat{K}$. In general $\widehat{K}$ is the smallest polynomially convex compact set containing K.

[3] It is bounded by taking $p = 1$. It is closed as it can be written $\bigcap_{\text{polynomial } p} \{z \in \mathbb{C} : |p(z)| \leq \|p\|_K\}$. Note the convex hull is $\{z \in \mathbb{C} : |p(z)| \leq \|p\|_K \text{ for all linear } p\}$.

For a unital Banach algebra A, $x \in A$, let $A(x)$ be the closed unital subalgebra of A generated by x,

$$A(x) = \overline{\{p(x) : p \text{ polynomial}\}}.$$

Theorem 6.2.14: thm:f6-8 If A is a unital Banach algebra and $x \in A$, then

1. $\sigma_{A(x)}(x) = \widehat{\sigma_A(x)} = \widetilde{\sigma_A(x)}$.

2. If K is a nonempty compact subset of $\mathbb{C}$, then $\widetilde{K} = \widehat{K}$.

Proof. 1. By the maximum modulus principle[4],

$$K \subseteq \widehat{K} \subseteq \widetilde{K}.$$

(Indeed, if $z \in \widehat{K}$ then $p(z) < \|p\|_{\partial \widehat{K}} \leq \|p\|_K$.) From Theorem 6.2.11,

$$\sigma_A(x) \subseteq \sigma_{A(x)}(x) \subseteq \widehat{\sigma_A(x)} \subseteq \widetilde{\sigma_A(x)}.$$

Assume $\lambda \notin \sigma_{A(x)}(x)$; we show that $\lambda \notin \widetilde{\sigma_A(x)}$. We have that $\lambda 1 - x$ is invertible in $A(x)$. So there exists a polynomial p such that

$$\|p(x)(\lambda 1 - x) - 1\| < \frac{1}{2},$$

say. Consider $q(t) = p(t)(\lambda - t) - 1$. If $\mu \in \sigma_A(x)$ then $q(\mu) \in \sigma_A(q(x))$ by Lemma 6.2.8 (spectral mapping theorem), so $|q(\mu)| \leq \|q(x)\|$ (Theorem 6.2.3) and $\|q(x)\| < \frac{1}{2}$ so

$$\|q\|_{\sigma_A(x)} \leq \|q(x)\| < \frac{1}{2}.$$

But by plugging in, $|q(\lambda)| = 1$, so $\lambda \notin \widetilde{\sigma_A(x)}$, so $\widehat{\sigma_A(x)} \subseteq \sigma_{A(x)}(x)$, and hence

$$\sigma_{A(x)}(x) = \widehat{\sigma_A(x)} = \widetilde{\sigma_A(x)}.$$

2. Let $A = C(K)$, $x(z) = z, z \in K$. Then $\sigma_A(x) = K$. By part (1), $\widehat{K} = \widetilde{K}$.

□

Proposition 6.2.15: pr:f6-9 Let A be a unital Banach algebra and C be a maximal commutative subalgebra. Then $\sigma_C(x) = \sigma_A(x)$ for all $x \in C$.

Note that C is closed and unital by its maximality.

Proof. Proof? missing □

[4] Let K be a compact subset of $\mathbb{C}$. Let f be an analytic function. Then the $\sup_K |f|$ is obtained on the boundary of K.

3 Commutative Banach algebras

3.1 Characters and maximal ideals

Let A be a Banach algebra.

Definition 6.3.1: A **character** on A is a nonzero algebra homomorphism $\varphi : A \to \mathbb{C}$.

If A has $1 \neq 0$, then $\varphi(1) = 1$. We let Φ_A be the set of all characters on A.

Lemma 6.3.2: lem:f6-10 Let A be a Banach algebra, $\varphi \in \Phi_A$. Then $\|\varphi\| \leq 1$ (so φ is continuous). Moreover, if A is unital, then $\|\varphi\| = 1 = \varphi(1)$.

Proof. First consider the unital case. If $x \in A$ and $\|x\| \leq 1 < |\varphi(x)|$, then $\left\|\frac{x}{\varphi(x)}\right\| < 1$, so $1 - \frac{x}{\varphi(x)}$ is invertible (Lemma 6.1.4): there exists $z \in A$ such that $z\left(1 - \frac{x}{\varphi(x)}\right) = 1$. Apply φ to both sides:

$$\varphi(z)\varphi\left(1 - \frac{x}{\varphi(x)}\right) = \varphi(1) = 1$$

so $\varphi(1 - \frac{x}{\varphi(x)}) \neq 0$. This is a contradiction.

Hence $\|\varphi\| \leq 1$. Also $\varphi(1) = 1$, $\|\varphi\| \leq 1$, so $\|\varphi\| = 1$.

If A is nonunital, define $\varphi_+ : A_+ \to \mathbb{C}$ by

$$\varphi_+(x + \lambda 1) = \varphi(x) + \lambda, \quad x \in A, \lambda \in \mathbb{C}.$$

Then $\varphi_+ \in \Phi_{A_+}$, so $\|\varphi\| \leq \|\varphi_+\| = 1$. □

Example 6.3.3: Consider the disc algebra $A(\Delta)$. Let $A_0 = \{f \in A(\Delta) : f(0) = 0\}$ and the character $\varphi(f) = f(z)$. Then $\varphi \in \Phi_{A_0}$. We get $\|\varphi\| = |z|$ by Schwarz's Lemma. (Schwarz's Lemma: If $f(z)$ is analytic for $|z| < 1$ and satisfies $|f(z)| \leq 1$, $f(0) = 0$, then $|f(z)| \leq |z|$.)

Lemma 6.3.4: lem:f6-11 Let A be a unital Banach algebra and let J be a proper ideal ($J \neq A$). Then $\overline{J}$ is also a proper ideal. In particular, it follows that maximal ideals are closed.

Proof. Since J is proper ($J \neq A$), $J \cap G(A) = \phi$. By by Corollary 6.1.6, $G(A)$ is open, so $\overline{J} \cap G(A) = \phi$.

If M is a maximal ideal, then M is proper, so $\overline{M}$ is also proper and $M \subseteq \overline{M}$, and by maximality $M = \overline{M}$. □

Let $\mathcal{M}_A$ be the set of all maximal ideals of A.

Theorem 6.3.5: thm:f6-12 Let A be a commutative unital Banach algebra. Then

$$\begin{aligned} \Phi_A &\to \mathcal{M}_A \\ \varphi &\mapsto \ker\varphi \end{aligned}$$

is a bijection.

Proof. We check that it is...

1. <u>well-defined:</u> For $\varphi \in \Phi_A$, $\ker \varphi$ is a proper ideal, since $\varphi \neq 0$ and φ is a homomorphism. Also φ is a linear functional, so $\ker \varphi$ is 1-codimensional in A, so $\ker \varphi \in \mathcal{M}_A$.

2. <u>injective:</u> If $\ker \varphi = \ker \psi$, then for $x \in A$, $x - \varphi(x)1 \in \ker \varphi = \ker \psi$ so $\psi(x - \varphi(x)1) = 0$, giving $\psi(x) = \varphi(x)$, so $\varphi = \psi$.

3. <u>onto:</u> If $M \in \mathcal{M}_A$, then A/M is a unital Banach Algebra (using the fact that M is closed and proper) and A/M is also a field. In particular it is a division algebra. By Gelfand-Mazur (Corollary/Theorem 6.2.6), $A/M \cong \mathbb{C}$ so the quotient map $q : A \to A/M$ "is" a character. Of course $\ker q = M$.

□

Corollary 6.3.6: cor:f6-13 Suppose A is a commutative unital Banach algebra. Then

1. for $x \in A$, $x \in G(A)$ iff $\varphi(x) \neq 0$ for all $\varphi \in \Phi_A$.

2. for $x \in A$, $\sigma_A(x) = \{\varphi(x) : \varphi \in \Phi_A\}$

3. for $x \in A$, $r(x) = \sup\{|\varphi(x)| : \varphi \in \Phi_A\}$.

Proof. For (1) $\Longrightarrow$, if there exists $z \in A$, $zx = 1$ then $\varphi(z)\varphi(x) = \varphi(1) = 1$ for all $\varphi \in \Phi_A$ so $\varphi(x) \neq 0$ for all $\varphi \in \Phi_A$.

For (1)$\Leftarrow$, if $x \in G(A)$, then $J = \{ax : a \in A\}$ is a proper ideal, with $x \in J$. By Zorn, there exists $M \in \mathcal{M}_A$ such that $J \subseteq \mathcal{M}$. By Theorem 6.3.5, there exists $\varphi \in \Phi_A$, $\ker \varphi = M$, so $\varphi(x) = 0$.

(2) and (3) follow easily. □

Corollary 6.3.7: cor:f6-14 Let A be a Banach algebra (not necessarily commutative). Let $x, y \in A$ which commute $xy = yx$. Then $r(x + y) \leq r(x) + r(y)$, and $r(xy) \leq r(x)r(y)$.

Proof. WLOG, A is unital. Replacing A with a maximal commutative subalgebra containing x, y, WLOG we can assume A is commutative. By Corollary 6.3.6, $r(x + y) = \sup\{|\varphi(x + y)| : \varphi \in \Phi_A(x)\}$ and clearly

$$\begin{aligned} r(x+y) &\leq \sup\{|\varphi(x)| : \varphi \in \Phi_A\} + \sup\{|\varphi(y)| : \varphi \in \Phi_A\} \\ r(x+y) &\leq r(x) + r(y) \end{aligned}$$

and similarly we obtain $r(xy) \leq r(x)r(y)$. □

Example 6.3.8: ex:f6-2

1. $A = C(K)$, K compact Hausdorff. For $k \in K$, let $\delta_k(f) = f(k)$. Clearly $\delta_k \in \Phi_A$ for all $k \in K$. Let M be a maximum ideal of $C(K)$. If for all $k \in K$, there exists $f \in M$ such that $f(k) \neq 0$, then an easy compactness argument shows that there exists $f \in M$, $f(k) \neq 0$ for all $k \in K$. So $f \in G(A)$, contradiction.

   So there exists $k \in K$ with $M \subseteq \ker(\delta_k)$. But M is maximal, so $M = \ker(\delta_k)$. Hence $\Phi_{C(K)} = \{\delta_k : k \in K\}$.

2. Disc algebra $A(\Delta)$. For any $\alpha \in \Delta$, $\delta_\alpha \in \Phi_A$. Let $\varphi \in \Phi_A$. Let $u(z) = z$, $z \in \Delta$. Put $\alpha = \varphi(u)$. Then $|\alpha| = |\varphi(u)| \leq \|u\| = 1$. So $\alpha \in \Delta$, $\delta_\alpha(u) = \varphi(u)$. So $\delta_\alpha(p) = \varphi(p)$ for all polynomials p. Polynomials are dense in $A(\Delta)$, so δ_α, φ agree on a dense subspace and hence $\delta_\alpha = \varphi$. In conclusion,

$$\Phi_{A(\Delta)} = \left\{\delta_\alpha : \alpha \in \Delta\right\}.$$

3. For $K \subseteq \mathbb{C}$, $K \neq \phi$ compact, it's easy to show (exercise)

$$\Phi_{R(K)} = \left\{\delta_k : k \in K\right\}.$$

4. Define the **Wiener algebra** to be

$$W = A(\mathbb{T}) = \left\{ f \in C(\mathbb{T}) : \sum_{n \in \mathbb{Z}} |\widehat{f}_n| < \infty \right\}.$$

Let $\|f\|_1 = \sum_{n\in\mathbb{Z}} |\widehat{f}_n|$ for $f \in A(\mathbb{T})$. Then $(A(\mathbb{T}), \|\cdot\|_1)$ is a (commutative, unital) Banach algebra with pointwise multiplication. It is isometrically isomorphic to $(\ell_1(\mathbb{Z}), \|\cdot\|_1)$ with convolution, $a = (a_n)_{n\in\mathbb{Z}}$, $b = (b_n)_{n\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ where $(a * b)_n = \sum_{r+s=n} a_r b_s$ via

$$\begin{aligned} A(\mathbb{T}) &\to \ell_1(\mathbb{Z}) \\ f &\mapsto (\hat{f}_n)_{n\in\mathbb{Z}} \\ \left(\sum_{n\in\mathbb{Z}} a_n e^{int}\right) &\leftarrow\!\shortmid (a_n)_{n\in\mathbb{Z}}. \end{aligned}$$

25th Nov We compute the characters of $A(\mathbb{T})$. For all $a \in \mathbb{T}$, $\delta_\alpha \in \Phi_W$. Conversely, let $\varphi \in \Phi_W$, $u(z) = z$. For $z \in \mathbb{T}$, we have $u \in W$, and in fact $u \in G(w)$, $u^{-1}(z) = \frac{1}{z}$. So $\alpha = \varphi(u) \neq 0$ and $|\alpha| \leq \|u\| = 1$. We have $\left\|\frac{1}{\alpha}\right\| = |\varphi(u^{-1})| \leq \|u^{-1}\| = 1$, so $\alpha \in \mathbb{T}$. If $p(z) = \sum_{n=-N}^N \lambda_n z^n$ is a trigonometric polynomial, then $p = \sum_{n=-N}^N \lambda_n u^n$, so $\varphi(p) = \sum_{n=-N}^N \lambda_n \alpha^n = p(\alpha) = \delta_\alpha(p)$. The trigonometric polynomials are dense in W, so $\varphi = \delta_\alpha$. Hence

$$\Phi_W = \left\{\delta_\alpha : \alpha \in \mathbb{T}\right\}.$$

Applying Corollary 6.3.6(1) we get the following.

Theorem 6.3.9 (Wiener's Theorem)**:** thm:wiener If $f \in C(\mathbb{T})$ has absolutely summable Fourier coefficients, and $f(z) \neq 0$ for all $z \in \mathbb{T}$, then $\frac{1}{f}$ also has absolutely summable Fourier coefficients.

This theorem was proved in the 1930's, and generated a lot of interest in algebraic methods in analysis.

3.2 Gelfand representation

Let A be a commutative unital Banach algebra. Then

$$\begin{aligned}\Phi_A &= \{\varphi \in A^* : \varphi(ab) = \varphi(a)\varphi(b)\,\forall a, b \in A, \varphi \neq 0\} \\ &= \{\varphi \in B_{A^*} : \varphi(ab) = \varphi(a)\varphi(b)\,\forall a, b \in A, \varphi(1) = 1\} \qquad \text{Lemma 6.3.2.}\end{aligned}$$

This is a w^*-closed subset of B_{A^*}, since $\varphi \mapsto \varphi(ab) - \varphi(a)\varphi(b)$, $a, b \in A$ and $\varphi \mapsto \varphi(1)$ are w^*-continuous on A^*. So Φ_A with the relative w^*-topology is a compact Hausdorff space. It is called the **spectrum** of A (or **character space** or **maximal ideal space** of A). The relative w^*-topology on Φ_A is the **Gelfand topology**.

For $x \in A$, define $\widehat{x} : \Phi_A \to \mathbb{C}$ by $\widehat{x}(\varphi) = \varphi(x)$ for $\varphi \in \Phi_A$. (This is the restriction to Φ_A of the canonical embedding of x into A^{**}.) Note that $\widehat{x} \in C(\Phi_A)$.

Theorem 6.3.10 (Gelfand's Representation Theorem)**:** thm:f6-15 Let A be a commutative unital Banach algebra. Then the map $A \to C(\Phi_A)$, $x \mapsto \widehat{x}$, is a continuous algebra homomorphism called the **Gelfand representation**. For $x \in A$,

$$\begin{aligned}\|\widehat{x}\|_\infty &= \sup\{|\varphi(x)| : \varphi \in \Phi_A\} = r(x) \leq \|x\| \\ \sigma_{C(\Phi_A)}(\widehat{x}) &= \{\varphi(x) : \varphi \in \Phi_A\} = \sigma_A(x).\end{aligned}$$

so $\widehat{x} \in G(C(\Phi_A))$ iff $x \in G(A)$.

In general, the Gelfand representation need not be injective or surjective. The kernel is given by

$$\begin{aligned}\{x \in A : \sigma_A(x) = \{0\}\} &= \left\{x \in A : r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = 0\right\} \\ &= \bigcap_{\varphi \in \Phi_A} \ker\varphi = \bigcap_{\mathfrak{m} \in M_A} M = J(A),\end{aligned}$$

the **Jacobson radical** of A. A is called **semisimple** if $J(A) = \{0\}$, i.e., if the Gelfand representation is injective.

Lemma 6.3.11: lem:f6-16 Let K be a compact Hausdorff space, A be a subalgebra of $C(K)$ that separates the points of K and $1 \in A$. Assume that A is given some Banach algebra norm. Then $k \mapsto \delta_k$ is a homeomorphism of K onto a closed subset of Φ_A.

Note that letting $\|\cdot\|$ be the Banach algebra norm on A, $\|\cdot\|_\infty \leq \|\cdot\|$. Indeed, $f(k) \in \sigma_A(f)$ for all $f \in A$ and all $k \in K$.

Proof. We show $k \mapsto \delta_k$ is continuous: given $f \in A$, the composite of $k \mapsto \delta_k$ with evaluation at f is $k \mapsto f(k)$ i.e., it is f which is continuous as $f \in A \subseteq C(K)$. (This is enough as we're using the w^*-topology on Φ_A.)

$k \mapsto \delta_k$ is injective since A separates the points of K.

The image is a closed subset of Φ_A, as it is the continuous image of a compact set in Hausdorff space. $k \mapsto \delta_k$ is a continuous bijection from a compact to a Hausdorff space, hence a homeomorphism. □

Example 6.3.12: We have seen that the above map is onto Φ_A for the following algebras.

- $A = C(K)$, K compact Hausdorff
- $A = A(\Delta)$ disc algebra
- $A = W$ Weiner algebra
- $A = R(K)$, $K \subseteq \mathbb{C}$ nonempty and compact (the closure in $C(K)$ of rational functions without poles in K)

Identifying Φ_A with the underlying compact set ($K, \Delta, \mathbb{T}$ as appropriate), the Gelfand representation is just inclusion. We get

$$C(K) \to C(K), \quad A(\Delta) \to C(\Delta), \quad W \to C(\mathbb{T}), \quad R(K) \to C(K).$$

So all these algebras are semisimple. These give examples when the Gelfand representation is not surjective (the last three), or when the image is not even closed (the Weiner algebra W is not closed; it's dense in $C(\mathbb{T})$).

Example 6.3.13: Let $V = L_1[0,1]$ with $\|\cdot\|_{L^1}$ and the multiplication being "chopped off" convolution:

$$(f * g)(x) = \int_0^x f(t)g(x-t)\,dt, x \in [0,1].$$

V is a commutative Banach algebra with no identity. Let $A = V_+$ be the unitization of V. If $f \in V$, $f = 0$ on $[0,\varepsilon]$ for some $\varepsilon > 0$, then $f * f = 0$ on $[0, 2\varepsilon]$, etc., so there exists n such that $f^n = 0$, and $f \in J(A)$. However, the set of these f is dense in V, so $J(A) = V$.

So there is only 1 maximal ideal, and hence only 1 character, $\varphi(x + \lambda 1) = \lambda$, $x \in V, \lambda \in \mathbb{C}$.

The Gelfand representation will be useful for studying C^*-algebras. As we will see in the Gelfand-Naimark Theorem 8.1.9, for a commutative unital C^*-algebra, the Gelfand transform is an isometric isomorphism.

4 Spectral theory for linear operators

We now specialize to the case $A = \mathcal{B}(X)$ where X is a (non-zero) complex Banach space. We have that the spectrum of $T \in \mathcal{B}(X)$ is

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ not invertible}\}.$$

Recall that $\sigma(T)$ is a non-empty compact subset of $\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$ (Theorem 6.2.3), and the spectral radius $r(T)$ satisfies

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}$$

(Theorem 6.2.10).

For $A = \mathcal{B}(X)$, we can draw upon our knowledge of eigenvalues for finite-dimensional operators to give intuition for spectra. In particular, we can define eigenvalues. When X is finite-dimensional, the spectrum of a linear operator is exactly its eigenvalues. For infinite-dimensional operators, this is not quite true anymore.

Definition 6.4.1: Let X ba a complex Banach space and $T \in \mathcal{B}(X)$.

1. We say that λ is an **eigenvalue** for T if there exists $x \in X$ (an **eigenvector**) such that
$$(\lambda I - T)x = 0.$$
The **point spectrum** of T is the set of eigenvalues of T, denoted $\sigma_p(T)$.

2. We say λ is an **approximate eigenvalue** of T if there exists a sequence (x_n) in X with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ such that
$$(\lambda I - T)x_n \to 0 \text{ as } n \to \infty.$$
The sequence (x_n) is an **approximate eigenvector** for λ. The **approximate point spectrum** of T is the set of all approximate eigenvalues of T, and is denoted by $\sigma_{\text{ap}}(T)$.

One clearly has
$$\sigma_p(T) \subseteq \sigma_{\text{ap}}(T) \subseteq \sigma(T).$$
In general, these inclusions can be strict and the point spectrum can be empty (unlike the spectrum). However, we have the following result.

Theorem 6.4.2: We have $\partial\sigma(T) \subseteq \sigma_{\text{ap}}(T)$. In particular, $\sigma_{\text{ap}}(T) \neq \phi$.

Proof. Add me. □

Definition 6.4.3: Let $T \in \mathcal{B}(X)$. We say T is **compact** if whenever U is bounded, $T(U)$ is relatively compact (has compact closure).

Example of a compact operator that's not finite-rank?

Theorem 6.4.4: Let $T \in \mathcal{B}(X)$ be a compact operator. Let $\lambda \in \sigma_{\text{ap}}(T)$ and $\lambda \neq 0$. Then λ is an eigenvalue of T.

Chapter 7

Holomorphic functional calculus

ch:hfc

1 Holomorphic functional calculus

Let U be a nonempty open subset of $\mathbb{C}$. Recall from Chapter 2 that $\mathcal{O}(U)$ is the space of holomorphic (i.e., analytic functions)

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} : f \text{ analytic}\}.$$

This is a LCS with seminorms

$$\|f\|_K = \sup_{z \in K} |f(z)|,$$

where $K \subseteq U$ is compact. It is an algebra with pointwise multiplication, which is continuous with respect to the topology. The topology on $\mathcal{O}(U)$ is the topology of local uniform convergence. It's metrizable but it is not a Banach algebra[1]; it's a Fréchet algebra.

Theorem 7.1.1 (Holomorphic Functional Calculus)**:** thm:f7-1 Let A be a commutative, unital Banach algebra $x \in A$, U an open subset of $\mathbb{C}$ with $\sigma(x) \subseteq U$. Let $u(z) = z, z \in U$ be the identity map[2].

Then there's a unique, continuous, unital homomorphism $\theta_x : \mathcal{O}(U) \to A$ such that $\theta_x(u) = x$. Moreover, $\varphi(\theta_x(f)) = f(\varphi(x))$ for all $\varphi \in \Phi_A$ and all $f \in \mathcal{O}(U)$, so $\sigma(\theta_x(f)) = \{f(\lambda) : \lambda \in \sigma(x)\}$.

This is a generalization of the spectral mapping theorem for analytic functions, not just polynomials (Theorem 6.2.8).

(We will write $f(x) = \theta_x(f)$, $\varphi(f(x)) = f(\varphi(x))$, and $\sigma(f(x)) = f(\sigma(x))$.)

For example, if $U = \{z : |z| < R\}$, $f \in \mathcal{O}(U)$ has Taylor expansion $f = \sum_{n=0}^{\infty} a_n z^n$, then $f(x) = \sum_{n=0}^{\infty} a_n x^n$, since $\sum_{n=0}^{\infty} a_n z^n$ converges locally uniformly to f.

[1] See remark after Theorem 6.2.3: any algebra of complex analytic functions that is a Banach algebra has to be bounded. However there are analytic functions on an open set that are not bounded.

[2] as opposed to the identity $e(z) = 1$

Theorem 7.1.2 (Runge's Theorem)**:** thm:f7-2 If $K \neq \phi$ is a compact subset of $\mathbb{C}$, then $R(K) = O(K)$, i.e., every analytic function on an open set containing K can be uniformly approximated on K by a rational function with no poles in K.

More precisely, suppose Λ contains exactly one point from each bounded component of $\mathbb{C}\backslash K$, then if f is analytic on some open neighborhood of K, then for every $\varepsilon > 0$ there exists a rational function r with poles in Λ such that $\|f - r\|_K < \varepsilon$.

Remark: If $\mathbb{C}\backslash K$ is connected, then $\Lambda = \phi$, so we have a polynomial approximation theorem, $O(K) = P(K)$.

The idea of the proof of Theorem 7.1.1 is the following analogue for Cauchy's integral theorem for Banach algebras:

$$f(x) = \frac{1}{2\pi i} \int_\Gamma f(z)(z1 - x)^{-1}\, dz.$$

There are two ingredients to the proof, given in the next two subsections.

1.1 Cycles and winding numbers

First, a definition.

Definition 7.1.3: A **cycle** consists of paths $\gamma_1, \ldots, \gamma_n$ (where $\gamma_k : [a_k, b_k] \to \mathbb{C}$ is continuously differentiable) such that there exists a permutation π such that

$$\gamma_k(b_k) = \gamma_{\pi(k)}(a_{\pi(k)}) \text{ for all } k.$$

We define the **winding number** of a cycle to be ths sum of the winding numbers of the paths:

$$n(\Gamma, w) = \sum_k n(\gamma_k, w) = \sum_k \frac{1}{2\pi i} \int_{\gamma_k} \frac{dz}{z - w} \in \mathbb{Z},$$

and denote

$$[\Gamma] = \bigcup \{\gamma_k(t) : a_k \leq t \leq b_k\} \subseteq U\backslash K.$$

Let $K \subset U \subset \mathbb{C}$ with K compact and U open. Then there exists a cycle Γ in $U\backslash K$ such that

$$n(\Gamma, w) = \begin{cases} 1, & \text{for all } w \in K, \\ 0, & \text{for all } w \notin K. \end{cases}$$

1.2 Integration on Banach spaces

We need to generalize integration to Banach spaces.

Definition 7.1.4:

1. (Integration on $[a, b]$) For $[a, b] \subseteq \mathbb{R}$, X a Banach space, $f : [a, b] \to X$ continuous, define $\int_a^b f(t)\, dt$ as follows (basically a Riemann sum). For $n \in \mathbb{N}$, take a dissection D_n of $[a, b]$,

$$a = t_0^n < t_1^n < \cdots < t_{k_n}^n = b$$

such that

$$\max\left\{t_k^n - t_{k-1}^n : 1 \le k \le k_n\right\} \to 0 \text{ as } n \to \infty.$$

Define

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{k=1}^{k_n} (t_k^n - t_{k-1}^n) f(t_k^n).$$

This exists and is independent of D_n (use the uniform continuity of f). It is immediate from the definition that for all $\varphi \in X^*$,

$$\varphi\left(\int_a^b f(t)\,dt\right) = \int_a^b \varphi(f(t))\,dt.$$

2. (Integration on γ) If $\gamma : [a,b] \to \mathbb{C}$ is continuously differentiable, and

$$f : [\gamma] = \{\gamma(t) : t \in [a,b]\} \to X$$

is continuous, then define

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

The usual theorems in complex analysis generalize.

Theorem 7.1.5 (Cauchy's Theorem for Banach spaces)**:** thm:cauchy-Banach Let $U \subseteq \mathbb{C}$ with U open and $f : U \to X$ **analytic**, i.e.,

$$\lim_{z\to w} \frac{f(z) - f(w)}{z - w}$$

exists for all $w \in \mathbb{C}$. Let Γ be a cycle in U. Then, provided $n(\Gamma, w) = 0$ for all $w \notin U$,

$$\int_\Gamma f(z)\,dz = 0.$$

Proof. We have by scalar Cauchy that

$$\varphi\left(\int_\Gamma f(z)\,dz\right) = \int_\Gamma \varphi(f(z))\,dz = 0.$$

This is true for all $\varphi \in X^*$, so by Hahn-Banach, $\int_\Gamma f(z)\,dz = 0$. □

2 Proof

Lemma 7.2.1: lem:f7-3 Let A, x, U be as in Theorem 7.1.1. Fix a cycle Γ in $U \backslash \sigma(x)$ such that

$$n(\Gamma, w) = \begin{cases} 1, & \text{for all } w \in \sigma(x), \\ 0, & \text{for all } w \notin U. \end{cases}$$

It winds around everything in $\sigma(x)$ exactly once. Define $\theta_x : \mathcal{O}(U) \to A$ by

$$\theta_x(f) = \frac{1}{2\pi i} \int_\Gamma f(z)(z1 - x)^{-1}\,dz.$$

Then

1. θ_x is linear and continuous.
2. If r is a rational function with no poles in U, then $\theta_\alpha(r) = r(\alpha)$ in the usual sense.
3. $\varphi(\theta_x(f)) = f(\varphi(x))$ for all $f \in \mathcal{O}(U)$, for all $\varphi \in \Phi_A$. So $\sigma(\theta_x(f)) = f(\sigma(x))$.

Proof. First we show θ_x is well-defined: $[\Gamma] \subseteq U \backslash \sigma(x)$ so $z1 - x$ is invertible for all $z \in [\Gamma]$. Moreover, $z \mapsto f(z)(z1-x)^{-1}$ is continuous by Corollary 6.1.6(2).

1. θ_x is clearly linear. Since $[\Gamma]$ is compact, there exists $M \geq 0$ such that $\|(z1-x)^{-1}\| \leq M$ for all $z \in [\Gamma]$. ($[\Gamma]$ is compact as a finite union of images of compact sets). So
$$\|\theta_x(f)\| \leq \frac{1}{2\pi} \text{length}(\Gamma) \cdot \|f\|_{[\Gamma]} \cdot M,$$
and hence θ_x is continuous by Lemma 2.2.5.

2. Let $e(z) = 1$ for all $z \in U$, i.e., e is the identity of $\mathcal{O}(U)$. We'll show that $\theta_x(e) = 1 \in A$. Fix $R > 0$ large enough so that $\sigma(x) \cup [\Gamma] \subseteq \{z \in \mathbb{C} : |z| < R\}$ and $R > \|x\|$. By Cauchy's Theorem,
$$\theta_x(e) = \frac{1}{2\pi i} \int_{|z|=R} (z1-x)^{-1}\, dz.$$
For $|z| = R \geq \|x\|$,
$$(z1-x)^{-1} = \sum_{n=1}^{\infty} \frac{x^n}{z^{n+1}}$$
converges uniformly on $|z| = R$, where the equality was shown in the proof of Lemma 6.1.4.
So we can integrate it term-by-term to get
$$\theta_x(e) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int \frac{dz}{z^{n+1}} \right) x^n = 1 \in A.$$
Let r be a rational function with no poles in U, i.e., $r \in \mathcal{O}(U)$. So $r = \frac{p}{q}$, where p, q are polynomials, and q has no zeros in U. By Lemma 6.2.8,
$$\sigma(q(x)) = \{q(\lambda) : \lambda \in \sigma(x)\}.$$
So $0 \notin \sigma(q(x))$, so $q(x)$ is invertible.
So we can define $r(x) = p(x)q(x)^{-1}$, and we can write (by putting $r(z), r(x)$ over a common denominator)
$$r(z)1 - r(x) = (z1-x) \sum_{k=1}^{m} s_k(z) t_k(x)$$
where s_k, t_k are rational functions with no poles in U. So
$$\begin{aligned}
\theta_x(r) &= \frac{1}{2\pi i} \int_\Gamma r(z)(z1-x)^{-1}\, dz \\
&= \frac{1}{2\pi i} \int_\Gamma (z1-x)^{-1} r(x)\, dz + \frac{1}{2\pi i} \int_\Gamma \sum_{k=1}^{m} s_k(z) t_k(x)\, dz \\
&= \theta_x(e) r(x) + 0 \\
&= r(x)
\end{aligned}$$

using Cauchy's Theorem and the above.

3. Since for all $\varphi \in \Phi_A$, $\varphi((z1-x)^{-1}) = \frac{1}{\varphi(z1-x)} = \frac{1}{z-\varphi(x)}$,

$$\varphi(\theta_x(f)) = \frac{1}{2\pi i}\int_\Gamma f(z)(z-\varphi(x))^{-1}\,dz = f(\varphi(x))$$

by Cauchy's integral formula and $n(\Gamma, \varphi(x)) = 1$. (Note f has no poles inside of U, and we assumed Γ winds 0 times around anything in U.)

Finally, use the characterization $\sigma(x) = \{\varphi(x)\}$ (Corollary 6.3.6).

□

Remark: If A is semisimple, then θ_x is an algebra homomorphism. Indeed, for every $\varphi \in \Phi_A$,

$$\varphi(\theta_x(fg) - \theta_x(f)\theta_y(f)) = (fg)(\varphi(x)) - f(\varphi(x))g(\varphi(x)) = 0.$$

Now we use the fact that for semisimple algebras, $\varphi(a) = 0$ for all $\varphi \in \Phi_A$ implies $a = 0$.

Proof of Runge's Theorem 7.1.2. Suppose $K \neq \phi$ is compact, and $K \subseteq \mathbb{C}$. Let $A = R(K)$. Assume $f : U \to \mathbb{C}$ is analytic for some open $U \supseteq K$. Let, as usual, $u \in \mathcal{O}(U)$ be $u(z) = z$ for all $z \in U$. Let $x = u|_K \in A$. Note $\sigma(x) = K$ and $\Phi_A = \{\delta_k : k \in K\}$. Apply Lemma 7.2.1 to get $\theta_x : \mathcal{O}(U) \to A$. What is $\theta_x(f)$? We have

$$\theta_x(f)(k) = \delta_k(\theta_x(f)) = f(\delta_k(x)) = f(k)$$

for all $k \in K$. So

$$\theta_x(f) = f|_K \in R(K).$$

This shows $R(K) = O(K)$; i.e., for all $\varepsilon > 0$, there exists r a rational function without poles in K, such that $\|f - r\|_K < \varepsilon$.

29 Nov. Now we prove the "more precisely part," where we only allow rational functions with poles in Λ, where Λ a set consisting of exactly one point from each bounded component of $\mathbb{C}\backslash K$. Set B to be the closed subalgebra of A generated by 1, x, $(\lambda 1 - x)^{-1}, \lambda \in \Lambda$. Then $\sigma_B(x) = \sigma_A(x) \cup S$ where S is the union of some bounded components of $\mathbb{C}\backslash\sigma_A(x) = \mathbb{C}\backslash K$ (Theorem 6.2.11). If V is a bounded component of $\mathbb{C}\backslash K$, then there exists $\lambda \in \Lambda \cap V$, so $(\lambda 1 - x)^{-1} \in B$, i.e., $\lambda \notin \sigma_B(x)$, so $V \cap \sigma_B(x) = \phi$. Hence $S = \phi$. This shows

$$\sigma_B(x) = \sigma_A(x) = K.$$

Lemma 7.2.1 gives the same $\theta_x : \mathcal{O}(U) \to B$. So $\theta_x(f) = f|_K \in B$. But B is the closure in $C(K)$ of rational functions all of whose poles lie in Λ. □

Corollary 7.2.2: cor:f7-4 If $\mathbb{C}\backslash K$ is connected, then for every analytic function f on some neighborhood of K for every $\varepsilon > 0$, there exists a polynomial p such that $\|f - p\|_K < \varepsilon$.

Proof. Here $\Lambda = \phi$. □

Corollary 7.2.3: cor:f7-5 Let U be a nonempty open subset of $\mathbb{C}$. Then the set of rational functions without poles in U is dense in $\mathcal{O}(U)$ (in the topology of local uniform convergence).

The difference between this and Runge's Theorem 7.1.2 is that here we're looking at functions defined on U, not on a compact K.

Proof. Given $f \in \mathcal{O}(U)$, compact $K \subset U$, $\varepsilon > 0$, we need a rational function $r \in \mathcal{O}(U)$ such that $\|r - f\|_K < \varepsilon$. Let

$$\widetilde{K}_U = K \cup \left\{ \begin{matrix} \text{all bounded components of } \mathbb{C}\backslash K \\ \text{that are contained in } U \end{matrix} \right\}.$$

Clearly, $K \subseteq \widetilde{K}_U \subseteq U$, so $\widetilde{K}_U$ is compact. The bounded components of $\mathbb{C}\backslash\widetilde{K}_U$ are those bounded components V of $\mathbb{C}\backslash K$ such that $V\backslash U \neq \phi$. We can pick $\lambda_v \in V\backslash U$. Let Λ be the set of all λ_v's. By Theorem 7.1.2, there exist rational functions r all of whose poles lie in Λ, such that $\|r - f\|_{\widetilde{K}_u} < \varepsilon$. □

Proof of Theorem 7.1.1. Existence: Lemma 7.2.1 gives $\theta_x : \mathcal{O}(U) \to A$ satisfying (1), (2), and (3). All that is left to prove is that $\theta_x(fg) = \theta_x(f)\theta_x(g)$ for all $f, g \in \mathcal{O}(U)$.[3] This holds for rational functions $f, g \in \mathcal{O}(U)$. Since the rational functions are dense in $\mathcal{O}(U)$ (Corollary 7.2.3) and θ_x is continuous (Lemma 7.2.1(1)), we're done.

Uniqueness: Suppose $\theta : \mathcal{O}(U) \to A$ is a continuous, unital algebra homomorphism such that $\theta(u) = x$. Then $\theta(p) = p(x)$ for all polynomials p and hence $\theta(r) = r(x)$ for all rational functions $r \in \mathcal{O}(U)$. So $\theta = \theta_x$ on a dense set, and they must be equal. □

See the examples sheet for some applications (existence of idempotents, etc.).

[3] If A is semisimple, this already follows from (3), as remarked.

Chapter 8

C^*-algebras

1 $*$-algebras

Definition 8.1.1: A **$*$-algebra** is a complex algebra A with an involution, i.e., a map $* : A \to A$ such that for every $x, y \in A$ for all $\lambda, \mu \in \mathbb{C}$.

1. $(\lambda x + \mu y)^* = \overline{\lambda} x^* + \overline{\mu} y^*$
2. $(xy)^* = y^* x^*$.
3. $x^{**} = x$.

A **$*$-subalgebra** B is a subalgebra with $x \in B \implies x^* \in B$.

Note that if A has an identity 1 then $1^* = 1$.

Definition 8.1.2: A **C^*-algebra** is a Banach algebra with an involution (Banach $*$-algebra) such that

$$\|x^* x\| = \|x\|^2 \text{ for all } x \in A.$$

This innocent-looking equation is extremely powerful. Note that if A has ean identity $1 \neq 0$, then $\|1\| = 1$ so A is then a unital C^* algebra.

Example 8.1.3:

1. $C(K)$ with K compact and Hausdorff is a commutative unital C^* algebra with involution $f^*(z) = \overline{f(z)}, z \in K, f \in C(K)$.
2. $\mathcal{B}(H)$ with H a Hilbert space with involution T^* the adjoint of T.
3. If A is a closed $*$-subalgebra of a C^*-algebra B, then A is a C^*-algebra. (A is called a **C^*-subalgebra** of $\mathcal{B}(H)$.)

The following describes all C^*-algebras.

Theorem 8.1.4 (Gelfand-Naimark)**:** thm:g-n

1. If A is a commutative unital C^*-subalgebra, then A is isometrically $*$-isomorphic to $C(K)$ for some compact Hausdorff K.

2. If A is a (not necessarily commutative) C^*-algebra, then A is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.

We will prove the first part of the theorem.

Definition 8.1.5: df:hun Let A be a C^*-algebra and $x \in A$. Then

- x is **hermitian** if $x^* = x$,
- x is **unitary** if $x^*x = xx^* = 1$,
- x is **normal** if $x^*x = xx^*$.

For example, projections in a Hilbert space are hermitian. Unitary operators in a Hilbert space are those that are isometric and bijective.

There is a process of unitization; it works the same as in Banach algebra except we have to check the C^* relation holds.

We have the following simple algebraic properties.

Proposition 8.1.6: pr:c* Let A be a C^*-algebra, $x \in A$. Then

1. We can write $x = h + ik$ for unique hermitian h, k ($h = \frac{x+x^*}{2}, k = \frac{x-x^*}{2}$.) Then x is normal iff $hk = kh$.

2. $\|x^*\| = \|x\|$. Therefore, involution is continuous with respect to the norm.

3. $x \in G(A)$ iff $x^* \in G(A)$, and then $(x^*)^{-1} = (x^{-1})^*$. Hence
   (a) $\lambda \in \sigma(x)$ iff $\overline{\lambda} \in \sigma(x^*)$.
   (b) $\sigma(x^*) = \left\{\overline{\lambda} : \lambda \in \sigma(x)\right\}$, and $r(x^*) = r(x)$.

Proof. For 2, note

$$\|x\|^2 = \|x^*x\| \le \|x^*\| \, \|x\|$$

so $\|x\| \le \|x^*\|$. Hence

$$\|x^*\| \le \|x^{**}\| = \|x\| \, .$$

□

These propeties will allow us to emulate many constructions in linear algebra, for example positive square roots of postiive operators, and the polar decomposition as a unitary times a positive.

Lemma 8.1.7: lem:f8-1 If $x \in A$ is normal, then $r(x) = \|x\|$.

Proof. First assume $h \in A$ is hermitian. Then $\|h^2\| = \|h^*h\| = \|h\|^2$. By induction,

$$\left\|h^{2^n}\right\| = \|h\|^{2^n} \text{ for all } n \in \mathbb{N}.$$

So by Theorem 6.2.10,

$$\lim_{n\to\infty} \left\|h^{2^n}\right\|^{2^{-n}} = \|h\|.$$

Let $x \in A$ be normal. Then $(x^*x)^* = x^*x^{**} = x^*x$, so x^*x is hermitian. Then

$$\|x\|^2 = \|x^*x\| = r(x^*x) \overset{\text{Cor. 6.3.7}}{\leq} r(x^*)r(x) = r(x)^2 \leq \|x\|^2.$$

So we have equality all the way through, and $\|x\| = r(x)$. $\square$

We saw that positive linear functionals on compact space take real values on real-valued functions. The C^*-algebra equivalent is that positive linear functionals take hermitian elements to real elements.

Lemma 8.1.8: lem:f8-2 Let A be a commutative C^*-algebra. Then $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in A$ and all $\varphi \in \Phi_A$.

We see that characters are not just algebra homomorphisms, but also C^*-algebra homomorphisms.

Proof. Since $x = h + ik$ for hermitian h, k we have

$$\varphi(x^*) = \varphi(h - ik) = \varphi(h) - i\varphi(k).$$

So it suffices to show $\varphi(h) \in \mathbb{R}$ for hermitian h. Let $\varphi(h) = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$. Consider $h + it1$. We have

$$\|h + it1\|^2 = \|(h + it1)^*(h + it1)\| = \left\|h^2 + t^2 1\right\| \leq \|h\|^2 + t^2$$

for $t \in \mathbb{R}$. So using Lemma 6.3.2,

$$|\alpha + i(\beta + t)|^2 = |\varphi(h + it1)|^2 \leq \|h + it1\|^2 \leq \|h\|^2 + t^2$$

for all $t \in \mathbb{R}$. We deduce $\beta = 0$. $\square$

The statement of Gelfand-Naimark we will prove is given below.

Theorem 8.1.9 (Gelfand-Naimark)**:** thm:f8-4 If A is a commutative unital C^*-algebra, then $A \cong C(K)$ as $*$-algebras for some compact Hausdorff K. More precisely, the Gelfand transform $A \to C(\Phi_A)$, $x \mapsto \widehat{x}$ is an isometric $*$-isomorphism onto $C(\Phi_A)$.

Recall that all our C^*-algebras are unital.

Corollary 8.1.10: cor:f8-3 Let A be a C^*-algebra.

1. If $h \in A$ is hermitian, then $\sigma_A(h) \subseteq \mathbb{R}$.

2. If $u \in A$ is unitary, then $\sigma_A(u) \subseteq \mathbb{T}$.

Proof. 1. Let B be the closed subalgebra of A generated by 1, h. Then B is a commutative C^*-subalgebra of A. So

$$\sigma_B(h) \overset{\text{Cor. 6.3.6}}{=} \{\varphi(h) : \varphi \in \Phi_B\} \overset{\text{Lem. 8.1.8}}{\subseteq} \mathbb{R}.$$

So $\sigma_A(h) \subseteq \sigma_B(h) \subseteq \mathbb{R}$ by Theorem 6.2.11. (Passing to a subalgebra, it's harder to be invertible.)

2. Let B be the closed subalgebra of A generated by $1, u, u^*$. This is a commutative C^*-subalgebra of A. So $\sigma_A(u) \subseteq \sigma_B(u) = \{\varphi(u) : \varphi \in \Phi_B\} \subseteq \mathbb{T}$, where the last inequality holds because

$$\begin{aligned} & 1 = \varphi(1) = \varphi(u^*u) = \varphi(u^*)\varphi(u) \\ \implies & 1 = \overline{\varphi(u)}\varphi(u) = |\varphi(u)|^2 . \end{aligned}$$

$\square$

Remark: 1. Both for $x = h$, $x = u$ above, we have

$$\sigma_B(x) = \partial\sigma_B(x) \overset{\text{Thm. 6.2.11}}{\subseteq} \partial\sigma_A(x) \subseteq \sigma_A(x)$$

so we actually have $\sigma_A(x) = \sigma_B(x)$.

2. More generally, if C is any C^*-subalgebra containing x ($x = h$ or $x = u$ above), then $B \subseteq C \subseteq A$, so $\sigma_C(x) = \sigma_B(x) = \sigma_A(x)$, so $\partial\sigma_C(x) = \sigma_C(x)$. This also holds for normal x.

   For $\lambda \in \mathbb{C}$, $\lambda 1 - x$ is invertible in C iff $(\lambda 1 - x)(\overline{\lambda} 1 - x^*)$ is invertible in C (as $\lambda 1 - x$, $\overline{\lambda} 1 - x^*$ commute). This holds for any C, and $(\lambda 1 - x)(\overline{\lambda} 1 - x^*)$ is hermitian, so its invertibilty does not depend on C.

3. This applies to elements of $\mathcal{B}(H)$, where H is a Hilbert space. If $T \in \mathcal{B}(H)$ is hermitian or unitary, then $\sigma(T) = \partial\sigma(T) = \sigma_{\text{ap}}(T)$.[1] If fact, T is normal, then $\sigma(T) = \sigma_{\text{ap}}(T)$ as well.

Proof of Gelfand-Naimark Theorem 8.1.9. We know the Gelfand transform $x \mapsto \hat{x}$ is an algebra homomorphism $A \to C(\Phi_A)$. We check it is...

1. $*$-homomorphism: for $x \in A$, by Lemma 8.1.8,

$$\widehat{x^*}(\varphi) = \varphi(x^*) = \overline{\varphi(x)} = \overline{\hat{x}}(\varphi).$$

[1] The approximate spectrum is defined as follows. λ is an approximate eigenvalue of T if there exists a sequence (x_n) in X with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ such that $(\lambda I - T)x_n \to 0$ as $n \to \infty$. The **approximate point spectrum** of T is the set of all approximate eigenvalues of T. We use the result that $\sigma_{\text{ap}}(T) \subseteq \sigma(T)$.

2. isometric:
$$\|\hat{x}\|_\infty = \sup\{|\varphi(x)| : \varphi \in \Phi_A\} \overset{\text{Cor. 6.3.6}}{=} r(x) = \|x\|$$
by Lemma 8.1.7 (x is normal as A is commutative).

3. onto: Note
   (a) $\hat{A} = \{\hat{x} : x \in A\}$ is a closed subalgebra of $C(\Phi_A)$ that separates points of Φ_A (if $\varphi \neq \psi$ then there exists $x \in A$, $\varphi(x) \neq \psi(x)$, i.e., $\hat{x}(\varphi) \neq \hat{x}(\psi)$).
   (b) $\hat{A}$ contains $1_{\Phi_A} = \hat{1}$, and
   (c) $\hat{A}$ is closed under complex conjugation (Lemma 8.1.8), so by Stone-Weierstrass, $\hat{A} = C(\Phi_A)$.

□

2 Applications

We give two applications.

2.1 Positive elements and square roots

Definition 8.2.1: An element a of a C^*-algebra A is **positive** if $a^* = a$, $\sigma_A(a) \subseteq \mathbb{R}^+ = [0, \infty)$, e.g., $T \in \mathcal{B}(H)$ is positive iff $T^* = T$ and $\langle Tx, x \rangle \geq 0$ for all $x \in H$.

Theorem 8.2.2: Let A be a (not necessarily commutative) C^*-algebra and a be positive.

There exists a unique positive $b \in A$ such that $b^2 = a$ (the unique positive square root of a, denoted by $a^{\frac{1}{2}}$.)

Proof. Let B the the closed subalgebra generated by $1, a$. This is a commutative C^*-subalgebra. Now $\hat{a} \in C(\Phi_B)$ has $\hat{a} \geq 0$ so by Theorem 8.1.9, there exists $b \in B$, $\hat{b} = (\hat{a})^{\frac{1}{2}}$. So b is positive, $b^2 = a$. (By Remark 1 after Corollary 8.1.10, $\sigma_B(a) = \sigma_A(a)$.)

For uniqueness, suppose $c \in A$ is positive and $c^2 = a$. Then c commutes with a, so $cp(a) = p(a)c$ for all polynomials p. So $cb = bc$. (By construction above, $b \in \overline{\{p(a) : p \text{ polynomial}\}}$.) Let C be the closed subalgebra of A generated by $1, a, c$, we have that C is commutative. By Theorem 8.1.9, $\hat{b} = \hat{c} = \hat{a}^{\frac{1}{2}}$, so $b = c$. □

2.2 Polar decomposition

Theorem 8.2.3 (Polar decomposition for invertible operators)**:** If $T \in \mathcal{B}(H)$ is invertible, then there exists a unique positive $R \in \mathcal{B}(H)$, unitary $U \in \mathcal{B}(H)$ such that $T = RU$.

Compare with complex numbers: a nonzero complex number can be written as a product of a complex number of absolute value 1 and a positive real number.

Proof. Note that TT^* is positive: indeed, it is hermitian, and for $\lambda < 0$,

$$\|(\lambda I - TT^*)x\| \geq |\langle(\lambda I - TT^*)x, x\rangle| = \left|\lambda - \|T^*x\|^2\right| \geq |\lambda|.$$

This is always bounded away from 0, so $\lambda \notin \sigma_{\mathrm{ap}}(TT^*) = \sigma(TT^*)$. There is no negative number in the spectrum, so the spectrum is positive.

Let $R = (TT^*)^{\frac{1}{2}}$. Since T is invertible, so is R, and we can set $U = R^{-1}T$. Since T is invertible, so is R, and we can set $U = R^{-1}T$. Then U is invertible,

$$UU^* = R^{-1}TT^*R^{-1} = R^{-1}R^2R^{-1} = I.$$

Uniqueness: For $T = RU$, $TT^* = RUU^*R = R^2$, so $R = (TT^*)^{\frac{1}{2}}$, $U = R^{-1}T$. $\square$

! For a, b hermitian, we write $a \leq b$ to mean $b - a$ is positive. If a, b are positive, $a \leq b$ implies $a^{\frac{1}{2}} \leq b^{\frac{1}{2}}$, but $a \leq b$ does not imply $a^2 \leq b^2$. So not everything is true that seems like it should by analogy with $\mathbb{C}$.

Chapter 9

Spectral Theory

1 Spectral Theory

Throughout, H is a Hilbert space, K is a compact Hausdorff space, $\mathcal{B}$ is a Borel σ-field on K.

Definition 9.1.1: df:res-id A **resolution of the identity** of $\mathcal{B}(H)$ over K is a function $P : \mathcal{B} \to \mathcal{B}(H)$ such that

1. $P(\phi) = 0$, $P(K) = I$.
2. $P(E)$ is an orthogonal projection for all $E \in \mathcal{B}$.
3. If $E \cap F = \phi$, then $P(E \cup F) = P(E) + P(F)$.
4. $P(E \cap F) = P(E)P(F)$ for all $E, F \in \mathcal{B}$
5. For all $x, y \in H$, $P_{x,y} : \mathcal{B} \to \mathbb{C}$, $P_{x,y}(E) = \langle P(E)x, y\rangle$ is a regular complex measure.

Example 9.1.2: Let $H = L_2[0, 1]$, $K = [0, 1]$, $P(E)(f) = 1_E f$.

Here is a list of simple properties.

Proposition 9.1.3:

1. $P(E)$, $P(F)$ commute for all E, F.
2. $E \cap F = \phi$ implies $P(E)P(F) = 0$.
3. For $x \in A$, $P_{x,x}(E) = \langle P(E)x, x\rangle = \|P(E)x\|^2$, so $P_{x,x}$ is a positive measure and $P_{x,x}(K) = \|x\|^2$.
4. P is finitely additive (Definition 9.1.1(3)), but in general, P is not countably additive (if $P(E) \neq 0$ then $\|P(E)\| = 1$).
5. For $x \in H$, $E \mapsto P(E)(x)$ is countably additive. (Use Definition 9.1.1(5).) Informally, P is a projection-valued mesure which is weakly countably additive.

From (5), if $P(E_n) = 0$ for all $n \in \mathbb{N}$, then $P\left(\bigcup_{n=1}^{\infty} E_n\right) = 0$.

Definition 9.1.4: For a Borel function $f : K \to \mathbb{C}$, say f is P**-essentially bounded** if there exists $E \in \mathcal{B}$, $P(E) = 0$,

$$\|f\|_\infty = \inf\left\{\|f\|_{K\setminus E} : P(E) = 0\right\}.$$

Let $L_\infty(P)$ be the space of all P-essentially bounded functions.

It is easy to see that $L_\infty(P)$ is a commutative unital C^*-algebra.

The first theorem is that we can integrate with respect to this.

Theorem 9.1.5: Let H, K, P be as above. Then there exists an isometric $*$-isomorphism $\Phi : L_\infty(P) \to \mathcal{B}(H)$ such that

1. $\langle \Phi(f)x, y\rangle = \int_K f\, dP_{x,y}$.

2. $\|\Phi(f)x\|^2 = \int_K |f|^2\, dP_{x,x}$.

3. $S \in \mathcal{B}(H)$ commutes with all $\Phi(f)$, $f \in L_\infty(P)$ and S is commutes with all $P(E)$, $E \in \mathcal{B}$.

2 Reading guide

Please see `https://www.dpmms.cam.ac.uk/~az10000/spectral-theory.pdf` for the rest of the notes. Here I'll instead put notes I took while reading.

2.1 Preliminaries

sec:spectral-prelim Before reading this chapter, it's good to recall facts about finite-dimensional linear operators. This is because the spectral theorem will be a generalization of the spectral theorem from linear algebra (although at first glance there seems to be so much measure-theoretic notation that this is not clear). In general, in functional analysis we draw intuition from finite-dimensional linear algebra and prove structure theorems about function spaces.

Problem 9.2.1: Do the following. (If you prefer, think of T as a matrix.) Everything is over $\mathbb{C}$.

1. (The spectral theorem) State the spectral theorem for normal operators on a finite-dimensional vector space.
   (a) Be sure to state the entire thing—i.e., with the "resolution of the identity."
   (b) What are some important classes of normal operators, and what happens in those cases?
2. (Operators as functions on their spectra)
   (a) Let T be a finite-dimensional diagonalizable linear operator. Consider the algebra $\mathbb{C}[T, T^*]$. Describe $\hat{T}$.
   (b) Let A be the algebra of linear operators on a finite-dimensional V that are diagonalizable with respect to a fixed basis. Describe the Gelfand transform $\hat{\bullet}$.
3. (Commuting linear operators) When do two diagonalizable linear operators commute? Prove this.
4. (Defining functions on linear operators) Let T be normal, and f be a function.
   (a) When does it make sense to define $f(T)$? How do we define it?
   (b) Why do we restrict to T normal?

1. The most compact version of the spectral theorem is probably the following.

   Theorem 9.2.2 (Spectral theorem, I)**:** Let T be a normal operator on a finite-dimensional Hilbert space H (vector space with inner product). Then T is unitarily diagonalizable, i.e., there exist U unitary and D diagonal so that

   $$T = UDU^{-1}.$$

   If T is unitary, then T and hence D has eigenvalues on the unit circle $\mathbb{T}$. If T is hermitian, then T and hence D has real eigenvalues.

   The "expanded" version of the spectral theorem involves several more statements—all of which are relatively easy to derive from the above, so that we often forget about them. But they are important because they will suggest the right way to state the spectral theorem in the infinite-dimensional case.

   Theorem 9.2.3 (Spectral theorem, II)**:** Let T be a normal operator on a finite-dimensional Hilbert space H. Let $\sigma(T)$ denote the eigenvalues of T. Let E_λ be the eigenspace of T with eigenvalue λ, and P_λ be the orthogonal projection onto E_λ (so that $P_{\lambda_1}P_{\lambda_2} = 0$ for $\lambda_1 \neq \lambda_2$). Then the following hold.

(a) (Diagonalizability) $H = \bigoplus_{\lambda\in\sigma(T)} E_\lambda$, where the eigenspaces E_λ are mutually orthogonal.

(b) (Resolution of the identity) $I = \sum_{\lambda\in\sigma(T)} P_\lambda$

(c) (Spectral decomposition of T) $T = \sum_{\lambda\in\sigma(T)} \lambda P_\lambda$

It is not hard to see that in this finite-dimensional case, (a), (b), and (c) are just different ways of saying the same thing.

2. The characters correspond to eigenspaces E of T: $\varphi_E(T')$ is the eigenvalue of T' on E. Thus $\hat{T} : \Phi_A \to \mathbb{C}$ is simply the function sending the character corresponding to E_λ, to λ:
$$\hat{T}(\varphi_{E_\lambda}) = \lambda.$$
For (b), the φ correspond to basis elements v; $\hat{T}$ is the function sending φ_v to the eigenvalue of v.

3. A, B commute when they are simultaneously diagonalizable. One way is to find a simultaneous eigenbasis step by step. Another is to write the spectral decomposition as in (1) and see that B has to commute with the projections to the eigenspaces of A. (This is actually the criteria we'll use in the infinite-dimensional case.)

4. f needs to be defined at the eigenvalues of T. Writing $T = UDU^{-1}$, we define $f(T) = Uf(D)U^{-1}$. In other words, if the spectral decomposition of T is $\sum_{\lambda\in\sigma(T)} \lambda P_\lambda$, we define
$$f(T) := \sum_{\lambda\in\sigma(T)} f(\lambda)P_\lambda.$$
cf. (b) and (c) in (1), where $f(x) = 1, x$ respectively. We restrict to T normal so that we have a spectral decomposition of T. (Nothing really goes wrong if T is diagonalizable but not normal.)

Note that $f(T)g(T) = (fg)(T)$: this is since we multiply termwise
$$\sum_{\lambda\in\sigma(T)} f(\lambda)P_\lambda \sum_{\lambda\in\sigma(T)} g(\lambda)P_\lambda = \sum_{\lambda\in\sigma(T)} (fg)(\lambda)P_\lambda,$$
using $P_{\lambda_1}P_{\lambda_2} = 0$ for $\lambda_1 \neq \lambda_2$.

Combining (1) and (2), we can write the spectral decomposition in a fancy way:
$$T = \sum_{\lambda\in\sigma(T)} \hat{T}(\varphi_{E_\lambda})P_\lambda.$$
Or identifying φ_{E_λ} with λ,
$$\text{eq:f9b-1}\, T = \sum_{\lambda\in\sigma(T)} \hat{T}(\lambda)P_\lambda. \tag{9.1}$$
(We wouldn't actually want to do this, but stretching your brain for what's coming up...)

2.1.1 Preparing for infinite dimensions

Things we have to consider:

1. There may be an infinite number of eigenspaces. Everything seems to fall apart: how do we write our space as a direct sum of eigenspaces? How do we write I as a sum of projections? How do we write T as a sum of functions that are nonzero only on eigenspaces?

2. In the finite-dimensional case, an eigenspace is λ such that $T - \lambda I$ is not invertible, or equivalently, it's not injective. In the infinite-dimensional case, $T - \lambda I$ may be not invertible, and still injective. difference something to do with discrete/continuous spectrum?

To solve (1), we have to take an integral rather than a sum!

3. How do we take an integral of *operators*? What is the measure space?

We'll have to address these questions!

2.2 Reading notes

See Section 9 for a possibly more coherent summary.

As a quick example, consider $H = C[0, 1]$, continuous functions on $[0, 1]$. Given f, consider the linear operator on H that is simply multiplication by f. Can you "see" that f is like a diagonal matrix? Think of functions on $[0, 1]$ as functions with an infinite number of components, one for each $x \in [0, 1]$; multiplication by f multiplies the x-component by $f(x)$.[1] Then $\sigma(f)$ is simply the image of f.

When we have a sum it makes sense to isolate points, but when we have an integral it doesn't—so we don't really want to just define $P_\lambda = P_{E_\lambda}$ for $\lambda \in \sigma(T)$. We want to define $P(E)$ where E is a *subset* of $\sigma(T)$.

Thus we define resolution of the identity (9.1) where (iii)-(v) describe how the different $P(E)$ relate to one another. Think of K as being $\sigma(T)$; the formulation here is more general. Now we want to be able to say that P is a measure in some sense; however P is a function with image $\mathcal{B}(H)$; thus instead we say $P_{x,y}$ is a regular complex Borel measure for all K.

Returning to our example, see (9.2).

(9.3) Note (i) is by 9.1(iii), (ii) is by 9.1(ii) and (iv), (iii) is by taking $x = y$ in 9.1(v) and noting $P(K) = I$. Still haven't resolved 9.3(iv)-(v). (iii) is important: we don't just have a complex measure here, we have a positive measure.

(9.4) Compare this with the notation $L_\infty(\mu)$ for a measure μ. The only difference is we're dealing with P which is an "operator-valued measure."

(9.5) Given f, $\int_K f \, dP_{x,y}$ looks like it should be bilinear in x, y, so we should be able to write it as $\langle \Phi(f)x, y \rangle$. This gives us a different way to express the integral. (For the proof, we will actually show how to "calculate" what $\Phi(f)$ is, as opposed to just saying it exists.)

[1] What if $H = L^2[0, 1]$? We aren't allowed to evaluate at a point anymore.

Proof: Try to define Φ for simple functions first. It is easy to show that Φ is a *unital $*$-homomorphism* on simple functions using the properties of P, since we are only dealing with finite sums.

In order to pass from the finite to the infinite case, we need to deal with convergence, i.e., we need a bound on $\|\Phi(s)\|$. We find we actually have an equality $\|\Phi(s)\| = \|s\|_\infty$, so Φ is isometric on simple functions. Now knowing how Φ respects norms for simple functions, we find $\Phi(s_n)$ is Cauchy for s_n Cauchy, and extend the definition in a straightforward way to all f. (iii) is a again a "true for simple functions, hence true for all functions" argument.

Important remark (9.6): given P we can determing Φ. Φ is such that (i) holds. We have $\langle \Phi(1_E)x, y\rangle = \int_K 1_E \, dP_{x,y} = P_{x,y}(E)$, so $\Phi(1_E) = P(E)$. Φ as an extension of P to non-simple functions, in the way that an integral generalizes a measure in a unique way.

(9.8) Stare at (9.1) for a few seconds, and this will make sense.

Think of $\hat{T}$ as crystallizing what T does on various eigenspaces E_λ, as separating T along its various components; we just have to combine them with an integral to get T.

Proof:

1. Our main tool for dealing with the integral is 9.5(i). 9.5(i) tells us that given a resolution of the identity there is Ψ such that

   $$\Psi(\hat{T}) = \int_K \hat{T} \, dP;$$

   actually, 9.5(i) defines T for *all* $f \in L_\infty(P)$, not just $C(\Psi_A) \cong A$:

   $$\Psi(f) = \int_K f \, dP.$$

   Summarizing, $\hat{\bullet} : A \xrightarrow{\cong} C(\Phi_A)$, and we'd like $\Psi : L^\infty \to A$. We want $\Psi(\hat{T}) = T$, Ψ is like an inverse Gelfand transform.

   Note here, however, we don't know what P is yet. (The whole point of the spectral theorem is to get a resolution of the identity!) So we'd like to define Ψ such that $\Psi(\hat{T}) = T$, and then get P out of it. We'd like to reverse 9.5. The key to getting a measure from a functional is the Riesz Representation Theorem; we get a family $P_{x,y}$, and then check these come from a single P.

   That's the motivation. Let's do everything in order now: in order to define Ψ on $L^\infty(K)$ extending $\hat{T} \mapsto T$, i.e. define a nice map when we know the map for continuous functions,

   (a) we use RRT to express $\hat{T} \mapsto \langle Tx, y\rangle$ as an integral wrt $\mu_{x,y}$,

   (b) then define $\Psi(f)$ using the integral (using sesquilinearity).

   We make sure Ψ is well-behaved by looking at norms. (Review chapter 3 if you need help on the total variation norm.) (Some comments here: the norm of $(x, y) \mapsto \int_K f \, d\mu_{x,y}$ is by definition $\max\left(\frac{\int_K f \, d\mu_{x,y}}{\|x\|\|y\|}\right)$.

   We have defined Ψ now.

2. We show that $\Psi(fg) = \Psi(f)\Psi(g)$. We think about what this is saying in terms of measures:

$$\langle \Psi(fg)x, y\rangle = \int fg\, d\mu_{x,y}$$
$$\langle \Psi(f)\Psi(g)x, y\rangle = \langle \Psi(g)x, \Psi(f)^* y\rangle = \int g\, d\mu_{x,\Psi(f)^* y}$$

(I'm trusting that it's easier to move the $\Psi(f)$ first. Or maybe it doesn't matter.) So we need $f\, d\mu_{x,y} = d\mu_{x,\Psi(f)^* y}$. To test equality of measures, by uniqueness in RRT it suffices to test equality of measures by looking at integrals of continuous functions; $\hat{T}$ are all the continuous functions on Φ_A so we test using those. This is the 2 lines of calculations on the bottom of page 3. (We're basically trying to take $\widehat{ST} = \widehat{S}\widehat{T}$ and turn it into $\Phi(fg) = \Phi(f)\Phi(g)$.)

3. Define P from Ψ. As remarked in 9.6 above, we should have $P(E) = \Psi(1_E)$. We do need to verify P is a resolution of the identity (because we're going the opposite way around from 9.5); this is straightforward. Now 9.5 applies immediately.

4. Uniqueness: simple RRT.

5. Moreover: Use Urysohn. Note $\hat{T}$ is real (i.e., T is Hermitian). (This is an example of when it's easier to look at the Gelfand transform—we can treat T like a function!)

6. (ii) Two options: use 9.5(iii), and uniqueness in RRT, or do it directly by calculating $\langle STx, y\rangle$ and using uniqueness in RRT.

(9.10) Note $\lambda x^* - \overline{\lambda} x$ is has $\sigma() \subseteq i\mathbb{R}$ because $a\overline{b} - \overline{a}b \in i\mathbb{R}$. Why is f bounded analytic? Idea: "complete" into 1-parameter group.

(9.11) Proof: We want to apply 9.8. The difference is that the integral is over $\sigma(T)$ not K. The thing to note is that Φ_A is homeomorphic to $\sigma(T)$: we have

$$\{\varphi(T) : \varphi \in \Phi_A\} = \sigma(T)$$

so consider the map $\varphi \mapsto \varphi(T)$. Since Φ_A separates points of $\overline{\mathbb{C}[T, T^*]}$, this is a bijection; since it is a continuous map between compact Hausdorffs, it is a homeomorphism. (Note that for normal operators, $\sigma(T)$ doesn't get bigger when we restrict the algebra.)

Uniqueness: use uniqueness in RRT and Stone-Weierstrass.

Commutating: use the lemma.

9.12 cf. Problem 4. Note that for normal operators, we can define $f(T)$ for f not necessarily analytic! Before, we could only define it for f analytic.

(iv) follows from Lemma 1(iii). Why do we have equality in (ii)?

9.13 This is ingenious! From the fact for $\mathbb{C}$, and noting we have multiplicativity in the integral just like in Problem 4, the result follows immediately.

9.14 Unitary is e^{iH}: Apply the functional calculus with the logarithm, remembering to show that $e^{\ln}$ converges.

9.15 Connectedness of $G(\mathcal{B}(H))$: find a path to the identity.

Appendix A

Problems

I took these down very quickly during the examples class, so these are probably unreadable right now. Need to fix!

1. Show that for all $x \in K$ there exists $f_x \in A$ such that $f_x(x) \neq 0$. There exists a neighborhood U_x of x, $f_x \neq 0$ on U_n. There exists $x_1, \dots, x_n$, $\bigcup_{i=1}^n U_{x_i} = K$, $f = \sum_{i=1}^n f_{x_i}^2 \in A$, $f > 0$ on K. Write $B = A \oplus \mathbb{C}1$, $\overline{B} = C(K)$. Given $\varepsilon > 0$ there exists $a \in A$, $\lambda \in \mathbb{C}$, $\left\|(a+\lambda) - \frac{1}{f}\right\| < \varepsilon$. (Replaced f by $\frac{f}{\|f\|}$. $\|(af + \lambda f) - 1\| < \varepsilon$. So $\overline{A} = C(K)$.

2. $q : X \to X/Y$ $q(B_X^\circ)$, so q is an open map. Assume $q^{-1}(U)$ is open in X, and $U = q(q^{-1}(U))$ is open in the norm topology. q is continuous, so $\|\cdot\|$-topology is weaker than quotient topology.

   Take a Cauchy sequence. $x_n + Y$ is Cauchy in X/Y, so $x_n + Y \to x + Y$. For all n, there exists $y_n \in Y$ such that $\|x_n - x + y_n\| < \|x_n - x + Y\| + 2^{-n}$. The difference tends to 0, so y_n is Cauchy, so $y_n \to y$, and $x_n \to x + y$.

3. Factor $T : X \to Y$ with $X \to X/\ker T \xrightarrow{\widetilde{T}} Y$. $E \subseteq X/\ker T$ is closed and bounded. Say $\|x\| < C$ for all $x \in E$. Then $q(q^{-1}(E) \cap CB_X) = E$ by definition of the quotient norm. (lift by arbitrary small amount) Then $\widetilde{T}(E) = T(q^{-1}(E) \cap CB_X)$. we've kept our property. So WLOG T is injective.

   Suppose $Tx_n \to y$. Then $y \in \overline{T(E)}$, $E = \overline{\{x_n : n \in \mathbb{N}\}}$. If bounded set, image is closed, so $y \in T(E)$, and we're done. Assume E is unbounded. Then $F := \overline{\left\{\frac{x_n}{\|x_n\|} : n \in \mathbb{N}\right\}} \subseteq S_X$ is closed. Then $0 \in \overline{T(F)} = T(F)$.

   (The injective step isn't essential, but instead of normalize you perturb by members of the kernel, it's slightly messier.)

4. (Grothendieck) We'll use this theorem when we study compact operators.

   Pick a finite number of points around which balls of radius 1 cover K. (There exist finite $F_1 \subseteq K$ such that $K \subseteq F_1 + B_X$.) Then compact $K_1 = \bigcup_{x \in F_1} (K - x) \cap B_X \subseteq F_z + \frac{1}{4} B_X$. (There exist $F_2 \subseteq K_1$.) Let $K_n = \bigcup_{x \in F_n} [(K_{n-1} - x) \cap \frac{1}{4^{n-1}} B_X] \subseteq F_n + \frac{1}{4^{n-1}}$.

For $x \in K$, can keep approximating, $x - x_1 - x_2 - x_3 - \cdots$, so $x = \sum_{n=1}^{\infty} x_n$ where $x_n \in F_n$. $x = \sum_{n=1}^{\infty} \frac{1}{2^n}(2^n x_n)$. Infinite convex combination of these points.

Then $y \in F_n$ implies $\|y\| \leq \frac{1}{4^{n-2}}$ for $n \geq 2$. Then $2x, x \in F_1$, $2^n x, x \in F_n$, is a null sequence; its closed convex hull covers K.

5. Start with $\{f_n : n \in \mathbb{N}\}$ dense in Y. For all n there exists $(x_{n_i})_i \subseteq S_X$ with $f_n(x_{n_i}) \to \|f_n\|$ as $n \to \infty$. Let $Z = \overline{\operatorname{span}}\{x_{n_i} : n, i \in \mathbb{N}\}$ separable. $Y \to Z^*$, $f \mapsto f|_Z$. Take $y^* \in Y$, $\varepsilon > 0$, there exists n such that $\|y^* - f_n\| < \frac{\varepsilon}{3}$. There exists $k \in \mathbb{N}$ such that $f_n(x_{nk}) > \|f_n\| - \frac{\varepsilon}{3}$. Use the triangle inequality to show $|y^*(x_{nk})| > \|y^*\| - \varepsilon$.

6. $T : X \to Y$, $T(X) = Y$. By the open mapping theorem there exists $C > 0$ such that $\overline{T(CB_X) \supseteq B_Y}$. Choose dense $\{y_n\} \subseteq B_Y$. Pick x_n such that $Tx_n = y_n$. Let $Z = \overline{\operatorname{span}}\{x_n : n \in \mathbb{N}\}$. $T|_Z : Z \to Y$, $\overline{T|_Z(CB_Z)} \supseteq B_Y$. By Open mapping lemma, $T|_Z$ is onto.

7. $f : X \to \mathbb{R}$ factors as $X \to X/\ker f \xrightarrow{\widetilde{f}} \mathbb{R}$. We have $\left\|\widetilde{f}\right\| = \|f\| = 1$. We have $|f(x)| = |\widetilde{f}(x + \ker f)| = \|x + \ker f\| = d(x, \ker f)$.

   Use fact $\ker f$ is 1-D space, so copy of reals, just multiplication by scalar. So for some λ, $\widetilde{f}(x + \ker f) = t\lambda$. There exist $\widetilde{f} \in S_{(X/Y)^*}$ such that $\widetilde{f}(x_0 + Y) = \|x_0 + Y\|$ by Hahn-Banach. Let $f = \widetilde{f} \circ q$.

   (You can do without quotient maps just by writing inequalities, but this is neater.)

8. Unit sphere is compact in finite-dimensional space. Consider $Y \cap Z$. Define... $d(S_F, Y \cap Z) = d(x, Y \cap Z) > 0$ for some $x \in S_F$. Projection onto $Y \cap Z$ is continuous. In finite dimension everything is continuous. By triangle inequality we have the bound on the RHS.

   Or let $f = 0$ on $Y \cap Z$ and $f(y_i) = 1$, $f(y_j) = 0$ for all $j \neq 1$, $f(z_i), f(w_i) = 0$. Check continuous functional by checking kernel is closed, you do the same work.

   For example, to prove if $C[0,1] \sim \{f \in C[0,1] : f(1) = 0\}$.) Sufficiently nasty Banch spaces are not isomorphic to their 1-codimensional subspaces.

9. Hahn-Banach allows us to extend the functionals. Consider Y. Consider a, b with $a|||y||| \leq \|y\| \leq b|||y|||$. Check with the operator norms induced by this. We get

$$aB_Y^{\|\cdot\|} \subseteq B_Y^{|||\cdot|||} \subseteq bB_Y^{\|\cdot\|}.$$

   so

$$a\left\|y^*\right\| \leq |\left\|y^*\right\|| \leq b\left\|y^*\right\|.$$

   and we get

$$bB_Y^{|||\cdot|||} \subseteq B_{Y^*}^{\|\cdot\|} \subseteq aB_Y^{|||\cdot|||}.$$

   For all $f \in B_{Y^*}^{|||\cdot|||}$ let $\widetilde{f}$ be a Hahn-Banach extension to X such that $\left\|\widetilde{f}\right\| = \|f\|$. Define something that we'll show is a norm on the whole space.

$$|\left\|x\right\||' = \sup\left(\left\{|\widetilde{f}(x)| : f \in B_{Y^*}^{|||\cdot|||}\right\} \cup \frac{1}{b}B_{X^*}\right) = \sup\left\{|f(x)| : f \in S\right\}.$$

$y \in Y, |\,\|y\|\,|' = |\,\|y\|\,|$. $\frac{1}{b}B_{X^*} \subseteq S \subseteq \frac{1}{a}B_{X^*}$. $\{f|_Y : f \in S\} = B_{Y^*}^{|||\cdot|||}$.

OR Define a convex symmetric set, using the Minkowski functional. Let $C = \left\{conv : B_Y^{|||\cdot|||} \cup aB_X^{\|\cdot\|}\right\}$. We will define μ_C. $C \cap Y = B_Y^{|||\cdot|||}$.

OR Zorn—a more "helpful" version of Hahn-Banach, 1 dimension at a time.

10. (a) $\overline{\text{span}} A = (A^\perp)_\perp$.

    i. $\subseteq$: From definition, $\subseteq$ is clear.

    ii. $\supseteq$: The RHS is a closed linear subspace because it is a intersection of kernels. ($B_\perp = \{x \in X : b(x) = 0 \forall b \in B\} = \bigcap_{b\in B} \ker b$.)
    What might go wrong? There might not be enough functionals, so maybe if we take all f such that $f(A) = 0$, there might be stuff outside the $\overline{\text{span}}(A)$ such that $f(x) = 0$ as well. But there *are* enough functionals, by Hahn-Banach. (Whenever we worry about lack of functionals, use Hahn-Banach!)
    Pick $y \notin \overline{\text{span}}(A)$. By Hahn-Banach, there exists $f \in X^*$ such that $f \equiv 0$ on $\text{span}(A)$[1] and $f(y) = 1$. Now $f \in A^\perp$ so $y \notin (A^\perp)_\perp$.

    (b) $\overline{\text{span}} B \stackrel{?}{=} (B_\perp)^\perp$

    i. $\subseteq$: We have $\overline{\text{span}}(B) \subseteq (B_\perp)^\perp$.

    ii. $\supseteq$: NOT NECESSARILY. We try to emulate the above proof. What goes wrong? Take $f \notin \overline{\text{span}}(B)$. By Hahn-Banach, there exists $F \in X^{**}$ such that $F(f) = 1$, $F(B) = \{0\}$. Uh-oh: if $F = \widehat{x}$ we get $f(x) = 1, \{f(y) : y \in B\} = \{0\}$ and $f \notin (B_\perp)^\perp$.
    So what goes wrong is that we're trying to do the above with X^* in place of X, but maybe $X \neq X^{**}$!
    (?) So to disprove this, take Y not reflexive, for instance, $Y = \ell_\infty$, $B = c_0$. Then $B_\perp = \ell_1$ but $(B_\perp)^\perp = \ell_\infty$.

    Question: How can we modify this statement to be true?

    Take weak star closure: $\overline{\text{span}}^{w^*} B = (B_\perp)^\perp$. See Problem 7 in Example sheet 2.

11. $X^*/Y^\perp \to Y^*$ by mapping $f + Y^\perp \mapsto f|_Y$. Use Hahn-Banach.

    Start with $R : X^* \to Y^*$, $R(f) = f|_Y$. $R(B_{X^*}) = B_{Y^*}$. Exact quotient map. $\ker R = Y^\perp$. So this factors through $X^* \to X^*/Y^\perp \xrightarrow{\widetilde{R}} Y^*$.

    $(X/Y)^* \to Y^\perp$ Take composite of f and quotient map $f \mapsto f \circ q$ where $q : X \to X/Y$ is the quotient map. This is well defined because q dies on Y. This is the fact that $q(B_X^\circ) = B_{X/Y}^\circ$. Map open ball to same thing so same norm. Onto because everything dies on Y factors through quotient map. $g : X \to \mathbb{R}$ and $\ker g \supseteq Y$, then $X \xrightarrow{q} X/Y \xrightarrow{\exists g} \mathbb{R}$.

12. Suppose x_n is continuous for all n. Form $y = \frac{x_n}{2^n\|x_n\|}$. It's a finite sum $\sum_{y\in F} \lambda_y y$ where F is finite. We get $\varepsilon_{x_n}(y) = \frac{1}{2^n\|x_n\|}$ for all n, this is 0 for large n, get a contradiction.

[1] Since $\text{span}(A)$ is a subspace, $f(\text{span}(A))$ is a subspace of $\mathbb{R}$ and hence 0

Using Zorn can construct continuous functional.

Kernels are closed, intersection of kernels $\bigcap_n \ker \varepsilon_{x_n}$. The quotient is also complete. X/Y has countable basis. $x_n + Y$ is a basis. There is no Banach space with countable basis.

13. Maximal element must have $C \cup D$ to be the whole space. Can take discontinuous functional, kernel is dense.

    $\{f > 0\}$, $\{f \leq 0\}$. $\ker f$ is dense. 1-codimensional. Kernel dense, translate of kernel dense.

    Need Zorn to get discontinuous functional.

    "When you use Zorn in hidden way usually right."

14. Take $Z = C[0,1]$. A finite-dimensional space imbeds isometrically into ℓ_∞. $T : F \hookrightarrow \ell_\infty$. Embed almost isometrically? $P_N \circ T : H \hookrightarrow c_\infty$, $1 + \varepsilon$ for all $\varepsilon > 0$. There exists an ε-net in S_F with $x_1, \ldots, x_n$ there exists N such that $\|x_i - P_N x_i\| < \varepsilon$ for all i. $(P_N(a_i) = (a_1, \ldots, a_N, \ldots))$ c_{00}

    There are countably many norms, so we can approximate every norm. Look at $M_n = \{(\mathbb{R}^n, \|\cdot\|)\}$, d Banach. M_n is compact, so separable. Take a countable dense set, do it for every n. Thus there exists a sequence $F_1, F_2, \ldots,$ of fintie dimensional spaces so that every finite-dimensional space is close to one of these. For all ε there exists n such that $d(F, F_n) < 1 + \varepsilon$. Take $\left(\bigoplus f_n\right)_{\ell_2}$. Each is reflexive.

Appendix B

Problems 2

1 Problems

2 Solutions

I will write up motivations and ideas for the solutions before the solutions themselves. (If you just want to see the solution, jump to Solution; everything before is just motivation.)

Complete: 1, 2, 3, 4, 5, 6, 7, 10, 13, 14, 16, 17

Incomplete: 8, 9, 11, 12, 15, 18, 19, 20, 21, 22.

Need solutions to:

1. By definition,

   - f_n is weakly null if for all $F \in C(K)^*$, $F(f_n) \to 0$.
   - f_n is pointwise null if for all $x \in K$, $f_n(x) \to 0$.

   Idea: The first is a statement about $C(K)^*$. Recall that we described $C(K)^*$ in terms of measures by Riesz Representation. We use this to transfer (a) to a statement about measures, and then apply our knowledge of measure theory.

   Riesz representation characterizes $C(K)^*$ completely in terms of measures.

   Proof:

   (a) If (f_n) is weakly null, then it is pointwise null by applying the evaluation-at-a-point functionals $\delta_k(f) = f(k), k \in K$.

   (b) Suppose (f_n) is pointwise null.

   We want to show $F(f_n) \to 0$ for all $F \in C(K)^*$. By the Riesz Representation Theorem, functionals in $C(K)^*$ correspond to regular complex Borel measures μ. Thus is suffices to show

   $$\text{eq:fex2-1} \quad \lim_{n\to\infty} \int f_n \, d\mu = 0 \text{ for all regular complex Borel } \mu. \tag{B.1}$$

But (f_n) is bounded (say $|f_n| \leq C$), so $|f_n|$ is bounded by a μ-integrable function (namely $C \in L_1(\mu)$) for any regular complex Borel measure μ. By Lebesgue's dominated convergence theorem, we obtain (B.1).

2. <u>Solution 1:</u> Idea: Use the definition of a weakly open/closed set and Urysohn's Lemma.

   Lemma (Urysohn's Lemma)**:** A normal space X is (completely) regular, i.e., for given a closed subset $A \subseteq X$ and a point $x \notin A$ and $a, b \in \mathbb{R}$, there exists a continuous function such that $f(A) = \{a\}$ and $f(x) = b$.

   What does it mean for C to be weakly closed and $p \notin C$? It means there is a (basic) open neighborhood around p not intersecting C, i.e., there are $f_1, \ldots, f_k$ and open $U \subseteq C$ such that letting $f = (f_1, \ldots, f_k) : A \to \mathbb{R}^k$, $f(p) \in U$ but $f(C)$ doesn't intersect U. (Make sure you see this.) In other words, letting $Z = \mathbb{R}^k \backslash U$,

   $$f(C) \subseteq Z, \; Z \text{ closed}, \qquad f(p) \notin Z.$$

   Now we can apply Urysohn's lemma, not on our original space X, but on $\mathbb{R}^k$ (which is in fact a metric space). We get a function g on $\mathbb{R}^k$ separating $f(C)$ and $f(p)$. Now take $g \circ f$.

   <u>Solution 2:</u> We define the function explicitly. As before, we have a basic open $U(x_1^*, \ldots, x_n^*, \varepsilon, p) = \{y \in X : |x_i^*(y) - x_i^*(p)| \leq \varepsilon\}$ disjoint from C. Let

   $$f(x) = \min\left(\varepsilon, \max_i |x_i^*(x) - x_i^*(p)|\right).$$

   At p this is 0, and on C is ε. Now compose with linear map to get it to equal 1 and 0 on $\{p\}$ and C, respectively.

3. Why is this not obvious? $q : X \to X/Y$ is an open map with respect to the usual topology, and it would remain an open map still if we weaken the topology of X/Y, but *not necessarily if we weaken the topology of X!*

   Let's write it out. Let's first check q of a sub-basic open set is open.

   $$q(\{x : f(x) \in U\}) = \{q(x) : f(x) \in U\};$$

   we need to show the RHS can be replaced by $g(q(x)) \in U$ for some $g \in (X/Y)^*$. Assume $U \neq \phi$. The difficulty is that f may not be constant on Y.

   But we ask, *how could it act on* Y? There are *only* two cases, since f is linear!

   (a) $f(Y) = 0$. Then we are done; f factors through q.

   (b) $f(Y) \neq \{0\}$: f is linear so the image of $f(Y)$ is a subspace of $\mathbb{R}$, i.e., $f(Y) = \mathbb{R}$. Then $\{q(x) : f(x) \in U\} = X/Y$: indeed, given $\overline{x} \in X/Y$, choose any lift $x \in X$, then $f(x + Y) = \mathbb{R}$ so there exists $y \in Y$ with $f(x + y) \in U$.

We're not done, because those were subbasic open sets, and in general $q(A \cap B) \neq q(A) \cap q(B)$.

Those were subbasic opens; we need all basic opens!

<u>Solution:</u> We follow the same idea; the proof is just slightly more complicated. Let $f = (f_1, \dots, f_k)$, where $f : X \to \mathbb{R}^k$ and $U \subseteq \mathbb{R}^k$ is open. We know $f(Y)$ is a subspace; by the same idea as (b) above (given $\overline{x} \in X/Y$, choose any lift $x \in X$, then $f(x+Y) = f(x) + f(Y)$), we have

$$q(\{x : f(x) \in U\}) = \{q(x) : f(x) \in U\} = \{q(x) : f(x) \in U\} + Y.$$

Consider the map $X \xrightarrow{f} \mathbb{R}^k \xrightarrow{p} \mathbb{R}^k/f(Y)$. Then because $Y = \ker q \subseteq \ker(p \circ f)$, this map factors

$$X \xrightarrow{q} X/Y \xrightarrow{g} \mathbb{R}^k/f(Y).$$

But $p : \mathbb{R}^k \to \mathbb{R}^k/f(Y)$ is open (because it is a quotient of topological groups, or you can check this directly), so $p(U)$ is open; the $q(U) = g^{-1}(p(U))$ is open.

4. <u>Solution:</u> Let
$$K := \bigcap \ker g_i.$$
Consider
$$\bullet|_K : X^* \to K^*.$$
If f then $f|_K$ of norm at most ε. By Hahn-Banach there exists $h \in X^*$ such that $h|_K = f|_K$ and $\|h\| \leq \varepsilon$.
$$\begin{tikzcd} f \arrow[r, mapsto] & f|_K \\ h \arrow[ur, mapsto] & \end{tikzcd}$$
Now $h - f|_K = 0$ so by Lemma, $f - h \in \operatorname{span}\{g_1, \dots, g_n\}$, so $d(f, \operatorname{span}(g_1, \dots, g_n)) \leq \varepsilon$.

5. Two ideas on why X should not be metrizable.

   (a) In an infinite-dimensional normed space, the weak neighborhoods are unbounded in the usual metric. We want to use this in some way.

   (b) A metric space has a countable neighborhood base. We don't know if X has a countable neighborhood base, but somehow if it were a metric space, maybe balls with radius going to 0 fail to capture this topology.

   <u>Solution:</u> Assume that the w-topology is metrizable. Let $x_n \in B(0, \frac{1}{n})$ such that $\|x_n\| > n$, where the ball B is with respect to the metric, and $\|\bullet\|$ is the usual norm. But $x_n \xrightarrow{w} 0$, contradicting the fact that weakly convergent sequences are norm-bounded (Proposition 4.2.8).

   Similarly, if the w^*-topology is metrizable, let $x_n^* \in B(0, \frac{1}{n})$, $\|x_n^*\| > n$. We get a contradiction from Proposition 4.2.8 again, *noting that this time X is required to be complete.*

   <u>Solution 2:</u> We start with the definition of what it means for topologies to be equivalent.

(a) In one direction we have

$$B\left(0,\frac{1}{n}\right) \supseteq U(F_n,\varepsilon,0) \supseteq \cap_{y^*\in F_n} \ker y^*,$$

for F_n some finite subset of X^*. (Notation: $U(F,\varepsilon,x) := \{y : |f(y-x)| < \varepsilon \text{ for all } f \in F\}$.)

(b) In the other direction, given x^*, there is some n such that

$$\boxed{U(x^*,1,0) \supseteq B\left(0,\frac{1}{n}\right)} \supseteq U(F_n,\varepsilon_n,0) \supseteq \bigcap_{y^*\in F_n} \ker y^*.$$

So for every x^*, we have for some n that $\ker x^* \supseteq \bigcap_{y^*\in F_n} \ker y^*$, so $x^* \in \operatorname{span} F_n$. Thus

$$X^* = \operatorname{span} \bigcup F_n.$$

This is a contradiction. (X^* as an infinite-dimensional complete space, can't have a countable spanning set.)

The proof for X^* is analogous. This time, we need X to be complete; before we automatically had X^* complete.

6. Idea:

> To show $a \notin A$ where A is convex compact, use Hahn-Banach separation to find f such that $\sup f(x) < f(a)$. Then take norms to obtain a contradiction.

Suppose $(1-\varepsilon)B_X \not\subseteq \overline{\operatorname{conv}} A$. Take $x \in (1-\varepsilon)B_X \backslash \overline{\operatorname{conv}} A$. Because $(1-\varepsilon)B_X$ is closed, we can require $\|x\| < 1-\varepsilon$, say $\|x\| = 1-\varepsilon-\varepsilon_1$.

By HB separation we get f such that $f(x) > \sup_{\overline{\operatorname{conv}} A} f$. By scaling assume $\|f\| = 1$. Choose $y \in B_X$ such that $f(y) > \frac{\varepsilon}{\varepsilon+\varepsilon_1}$, i.e. $(\varepsilon+\varepsilon_1)f(y) > \varepsilon\|y\|$. [1] Why is this a contradiction? We show that this means some point of B_X is far away from a point of A, so A is not maximally ε-separated.

Because $x \in (1-\varepsilon)B_X$, we have (by the triangle inequality) $x + (\varepsilon+\varepsilon_1)y \in B_X$. We get

$$f(x+(\varepsilon+\varepsilon_1)y) = f(x) + \varepsilon > \sup_{\overline{\operatorname{conv}} A} f + \varepsilon.$$

Thus for any $a \in A$,

$$\|x + \hat{x}\varepsilon - a\| \ge f(x+\hat{x}\varepsilon) - f(a) > \varepsilon$$

Thus $x + \hat{x}\varepsilon$ is too far away from A, contradiction.

Simplification of my argument (?) $1-\varepsilon \ge x^*((1-\varepsilon)x) > \sup_{y\in\overline{\operatorname{conv}} A} \langle y, x^*\rangle \ge \sup_{y\in A} |\langle y, x^*\rangle| \ge 1-\varepsilon$.

Nina: cf. Riesz's lemma. There exists $a_0 \in A$, $\|x-a_0\| < \varepsilon$, $(1-\varepsilon)x = \sum_{n=0}^{\infty} \varepsilon^n a_n$, $x = \sum_n^{\infty} \frac{\varepsilon^n}{1-\varepsilon} a_n \in \overline{\operatorname{conv}} A$.

[1](Can we make it $= \|y\|$ always? Probably infinite-dimensionality ruins this.)

7. [N.B. Typo: should be $\overline{\text{span}}$ not conv.]

   (0) $\overline{\text{span}}^{w^*}(B) = (B_\perp)^\perp$:

      i. $\subseteq$: Clearly $B \subseteq (B_\perp)^\perp$. Note $(B_\perp)^\perp$ is a vector subspace, and it is w^* closed because it is the intersection of kernels of evaluation maps, and such kernels are w^*-closed by definition.

      ii. $\bullet^c \subseteq \bullet^c$. Now we proceed as in Question 1.10. Take $x \notin \overline{\text{span}}^{w^*}(B)$. We seek to find $\varphi \in B_\perp$ such that $\varphi(x) \neq 0$. Then $x \notin (B_\perp)^\perp$.
      We can define a linear map $\varphi : \text{span}(\{x^*\} \cup \overline{\text{span}}^{w^*} B) \to \mathbb{R}$ with $\varphi|_{\overline{\text{span}}^{w^*} B} = 0$, $\varphi(x^*) = 1$. This is a w^*-continuous map because $\ker \varphi = \overline{\text{span}}^{w^*}(B)$ is w^*-closed. Now extend φ to B by Hahn-Banach. Then $\varphi \in B_\perp$ is the desired map.

   (i) We have

$$\begin{aligned} \ker T &= \{x : T(x) = 0\} \\ T^*(Y^*)_\perp &= \{f \circ T : f \in Y^*\}_\perp = \{x : f \circ T(x) = 0 \forall f \in Y^*\} \end{aligned}$$

   The RHS are equal because of Hahn-Banach 2.1.8 (existence of norming functionals) For the second part, note

$$\ker T^* = \{f : f \circ T = 0\} = \{f : f \circ T(X) = \{0\}\} = T(X)^\perp.$$

   (ii) We use Question 1.10 and the 0th part.

$$\begin{aligned} \ker(T^*)_\perp &= (T(X)^\perp)_\perp \overset{1.10}{=} \overline{T(X)} \\ \ker(T)^\perp &= (T^*(Y^*)_\perp)^\perp \overset{(0)}{=} \overline{T^*(Y^*)}^{w^*} \end{aligned}$$

8. incomplete

   (a) $\Leftarrow$ is HB. $\implies$: given $f \in X^*$, it is the image of $f \circ T^{-1}$, okay since imbedding.
   $\Leftarrow$: If T is onto then $T(B_X) \supseteq \delta B_Y$ by Open Mapping Theorem.

$$\|T^* y^*\| = \sup_{x \in B_X} |\langle x, T^* y^* \rangle| = \sup_{x \in B_X} |\langle Tx, y^* \rangle| \geq \sup_{y \in \delta B_Y} |\langle y, y^* \rangle| = \delta \|y^*\|.$$

   Or use Q7, it doesn't give a quantitative version—it gives injective but don't get isomorphism.
   $\implies$: Assume $\|T^* y^*\| \geq \delta \|y^*\|$ for all $y^* \in Y^*$.
   Claim: $TB_X \supseteq \delta B_Y$. Enough to show $\overline{TB_X} \supseteq \delta B_Y$. (Open mapping theorem) If not then by Hahn-Banach there exists $y \in B_Y$, $y^* \in B_{Y^*}$, $y^*(\delta y) > \sup_{y \in \overline{TB_X}} |\langle y, y^* \rangle|$. We get

$$\delta \geq y^*(\delta y) > \sup_{x \in B_X} |\langle Tx, y^* \rangle| = \|T^* y^*\| \geq \delta.$$

(b) $\Leftarrow$ if clear. $\implies$: by first part, T^{**} is onto. The preimage of $\hat{y}$ must be $\hat{x}$ for some x since why? $\Leftarrow$: T^* onto gives T^{**} into isomorphism, $T = T^{**}|_X$.

$\implies$: Assume T is into isomorphism. Say $\|Tx\| \geq \delta \|x\|$ for all x. Given $x^* \in X^*$, tell how to define y^* on the image of T. Define $y^* : TX \to F$ (F scalars) by $y^*(Tx) = x^*(x)$. Then $y^* \in (TX)^*$, $\|y^*\| \leq \frac{1}{\delta}\|x^*\|$. Done by Hahn-Banach.

(c) For $\implies$, just use the same argument as in 3, but with $\ker T$ instead of Y. For $\Leftarrow$, look at the inverse image of a ball and use Mazur's Theorem, that the closure of a convex set is the same in the usual and weak topologies. So a closed ball contained in the preimage is also a weak closed ball.

Factor through

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, "\widetilde{T}"'] & Y \\ & TX. \arrow[u, hook, "J"'] \end{tikzcd} \quad \xrightarrow{\bullet^*} \quad \begin{tikzcd} X^* & Y^* \arrow[l, "T^*"'] \arrow[d, two heads, "J^*"] \\ & \overline{TX}^* \arrow[ul, "\widetilde{T}^*"] \end{tikzcd}$$

TX is closed gives $\widetilde{T}$ onto, so $\widetilde{T}^*$ is an into isomorphism. J^* is onto as J is an into isomorphism. $T^*(Y^*) = \widetilde{T}^*(\overline{TX}^*)$ is closed.

Suppose $T^*(Y^*)$ is closed. Use Q7. So $\widetilde{T}^*(\overline{TX}^*)$ is closed. $\ker \widetilde{T}^* = (\widetilde{T}X)^\perp = \{0\}$. Now use the fact this is injective. By the inversion/isomorphism thereom, $\widetilde{T}^*$ is an into isomorphism.

9. incomplete TFAE. ??

(a) $T : X \to Y$ is continuous.

(b) $T : Y^* \to X^*$ is continuous w.r.t. to the usual

$T : X \to Y$. w-w continuous iff for all $y^* \in Y^*$, $x \mapsto \langle Tx, y^* \rangle = \langle x, T^*y^* \rangle$ is in X^*. If T is continuous, then $T^*y^* \in X^*$. If $\{\langle Tx, y^* \rangle : x \in B_X\}$ bounded, so $\{Tx : x \in B_X\}$ is weakly bounded, so norm-bounded.

Alternate solution: $T : Y^* \to X^*$ is w^*-w^* continuous. $y^* \to \langle x, T^*y^* \rangle$ is w^*-continuous so there exists $Sx \in Y$ such that $\langle x, T^*y^* \rangle = \langle Sx, y^* \rangle$ for all y^*. $\|S\| = \sup_{x \in B_X, y^* \in B_{Y^*}} \langle Sx, y^* \rangle = \|T^*\| < \infty$ and $S^* = T$.

10. (a) Not norm compact: this is an isolated set of points (take a ball of radius $\frac{1}{2}$ around each).

(b) Weakly compact: $\ell_\infty^* = \ell_1$. A functional f is dot product by $(a_1, a_2, \ldots)$. $f^{-1}(U)$ where $0 \in U$ (this is why we throw in 0) is $\{e_n : a_n \in U\}$; note $a_n \to 0$ so this contains e_n for n large enough. So one open set contains e_n for all n large enough.

! $\ell_\infty^* \neq \ell_1$!

$f|_{c_0} \in \ell_1$, so $f(e_n) = a_n \to 0$.

ℓ_∞ commutative unital C^*-algebra, so is $C(K)$ where $K = \beta\mathbb{N}$. One way to define Stone-Čech compactifiction. So ℓ^∞ is measures on K.

cf. $\{\frac{1}{n}\} \cup \{0\}$.

11. incomplete It suffices to show ℓ_∞^* is w^* separable. Then it is norm separable by Pr. 16.

    ℓ_∞^* is w^*-separable by Goldstine's Theorem ($\overline{\ell_1}^{w^*} = \ell_\infty^* = \ell_1^{**}$). (? ℓ_1 isn't separable though) So (K, w), weakly compact, is metrizable. So (K, w) is w-metrizable. There exists $(x_n) \subseteq K$ such that $\overline{\{x_n : n \in \mathbb{N}\}} = K$. We have $\overline{\{x_n : n \in \mathbb{N}\}}^w = K$. $\overline{\text{conv}}\{x_n : x \in \mathbb{N}\} = \overline{\text{conv}}^w \{x_n : x \in \mathbb{N}\} \supseteq K$. So $\overline{\text{conv}}\{x_n : x \in \mathbb{N}\}$ is norm-separable.

12. incomplete What might make a weakly convergent sequence not norm-convergent? If it *isn't* norm-convergent, such as $(1, 0, \ldots), (0, 1, \ldots), (0, 0, 1, \ldots), \ldots$. But a norm-Cauchy sequence should somehow *be* convergent. And we shouldn't be able to have x_n converge to different things under the norm vs. the weak topology! We just have to unravel this argument so we aren't going in circles.

    Solution 1: Note the weak topology is Hausdorff (given 2 points, there is a functional separating them) so x_n can only converge to 1 point in the weak topology. If it converges in the norm topology it must converge to the same point.

    Let $\overline{X}$ be the completion of X. Given $x \in \overline{X}$ the limit of a Cauchy sequence, we have $x_n \to y$ in the weak topology for some $y \in X$. Hence by the above, $x = y \in X$.

    What is this mess? (x_n) is Cauchy. WLOG $\|x_n - x_{n+1}\| < 2^{-n-1}$ so $\|x_n - x_m\| < 2^{-n}$ for all $m > n$. We have $x_n \xrightarrow{w} x$, say given $\varepsilon > 0, N$

    $$\left|\sum_{i=1}^{n} \lambda_i x_i - x_i\right| < \varepsilon.$$

    Note $\sum_{i=1}^{n} \lambda_i x_i \approx x_N$. Get convergent subsequence.

    > “convex combinations trick”—what is this?

    For all $n \geq m$, $x_n \in x_m + \varepsilon B_X$, convex, so w-closed, so $x = \text{w} \lim x_n \in x_m + \varepsilon B_X$.

13. Idea: “for all B_{X^*}” and the inequality condition suggest that we'll cover B_{X^*} by open sets where the condition holds for various n and then choose a finite subcover by Banach-Alaoglu.

    Solution: For a given i let

    $$U_i = \{x^* \in B_{X^*} : |x^*(x_i)| < \varepsilon\}.$$

    By definition of weak convergence, for all $x^* \in B_{X^*}$ there exists $i > m$ such that $|x^*(x_i)| < \varepsilon$. Thus

    $$\bigcup_{i > m} U_i = B_{X^*}.$$

    Note each U_i is w^*-open (by definition). By Banach-Alaoglu, B_{X^*} is w^*-compact, so there exists $n > m$, $\bigcup_{m<i<n} U_i = B_{X^*}$.

14. Suppose BWOC that (x_n) is norm null but not weakly null.

By passing to a subsequence, we may assume $|x_n| > \varepsilon$ for all n. We seek to define $x^* \in \ell_1^* = \ell_\infty$ such that $x^*(x_n) \not\to 0$. Idea: if the x_n concentrate near beginning terms, then we are okay, since we reduce to finite dimensions. Otherwise, they get more and more spread out, and we can define x^* term by term (thinking of it in ℓ_∞) to make sure $x^*(x_n)$ is large for infinitely many n.

For $x = (a_1, a_2, \ldots)$, define $x|_n = (a_1, \ldots, a_n)$ and

$$\|x|_n\| := |a_1| + \cdots + |a_n|.$$

Define $i_k : \mathbb{R}^k \hookrightarrow \ell_\infty$ by $i_k(b_1, \ldots, b_n) = (b_1, \ldots, b_n, 0, \ldots)$.

Choose any n_1, and consider x_{n_1}. Choose k_1 so that $\|x_{n_1}|_{k_1}\| > \frac{4}{5}\|x_{n_1}\|$. Define $y_1 \in \mathbb{R}^{k_1} \subseteq \ell_\infty$ so that

$$y_1(x_{n_1}) = \|x_{n_1}|_{k_1}\|.$$

We inductively show the following:

Claim: For each i, one of the following holds.

(a) There is y^* so that $y^*(x_n) \not\to 0$, or

(b) There exists k_i and $y_i \in \mathbb{R}^{k_i} \subset \ell_\infty$ and the following hold.

   i. (k_i is large enough to capture most of x_{n_i})

$$\text{eq:fex2-14} \quad \|x_{n_i}|_{k_i}\| \geq \frac{4}{5}\|x_{n_i}\|. \tag{B.2}$$

   ii. ($y_i^*(x_{n_i})$ is large)

$$\text{eq:fex2-14-2} \quad \|y_i^*(x_{k_i})\| \geq \frac{3}{5}\|x_{k_i}\| \geq \frac{3}{5}\varepsilon. \tag{B.3}$$

   iii. (Compatibility) $y_i|_{k_j} = y_j$ for $j < i$.

Proof. Suppose this holds for i. Consider 2 cases.

(a) (Concentration near beginning) For all $m > n_i$, $\|x_m|_{k_i}\| \geq \frac{1}{5}\|x_{n_i}\| \geq \frac{1}{5}\varepsilon$. Then for each $m > n_i$ there exists $z_m^* \in [-1,1]^{k_i}$ such that

$$z_m^*(x_m) \geq \frac{1}{5}\|x_m\|.$$

Now by sequential compactness of $[-1,1]^{k_i}$, some subsequence $z_{i_j}^*$ has a limit point z^*. Then for any $0 < \varepsilon'$ there exists m so that

$$z^*(x_m) \geq \left(\frac{1}{5} - \varepsilon'\right)\|x_m\| > \left(\frac{1}{5} - \varepsilon'\right)\varepsilon.$$

(b) (Spreading out) For some $m > n_i$, $\|x_m|_{k_i}\| < \frac{1}{5}\|x_{n_i}\|$. Let n_{i+1} be this value. Choose k_{i+1} such that (B.2) holds for $i+1$. Now choose y_{i+1} so that $y_{i+1}|_{k_i} = y_i$, and (B.3) holds. This completes the induction.

□

If case (b) holds in all cases, define y so that $y|_{k_i} = y_i$. Then (B.2) and (B.3) imply

$$|y^*(x_{n_i})| \geq \frac{3}{5}\|x_{k_i}\| - (y^*(x_{n_i} - x_{n_i}|_{k_i})) \geq \frac{2}{5}\varepsilon.$$

This is called the "gliding hump" argument.

Remarks:

(a) I wanted to do the sequential compactness argument in case (a) above directly, but compactness does not imply sequential compactness in general. (We have w^*-compactness by Banach-Alaoglu, but not sequential compactness.) Indeed, such abstract arguments can't work because the statement does not hold for general X, so we have to use stuff about ℓ_1.

(b) This shows that w-convergent sequences are exactly norm-convergent sequences in ℓ_1 (there is nothing special about the point 0), but the topologies are clearly not the same! So sequences very much fail to determine the topology. Does this mean ℓ_1 is non-sequential? See `http://math.stackexchange.com/questions/44907/whats-going-on-with-compact-implies-sequentially-co` `http://en.wikipedia.org/wiki/Sequential_space`, `http://en.wikipedia.org/wiki/Sequentially_compact_space`. cf. how the set of ultrafilters is very weird.

(c) Here is another fact that uses the "gliding hump" argument.

In ℓ_p, if $x_n \xrightarrow{w} 0$, but $x_n \not\to 0$, then there exist $k_1 < k_2 < \cdots$ such that (x_{k_n}) is like (e_n), i.e., $\|\sum a_k x_{k_n}\| \approx (\sum_n |a_n|^p)^{\frac{1}{p}}$.

For any subspace of ℓ^p, you can find a subspace isomorphic to ℓ^p and complemented in ℓ^p; there exists a norming functional in ℓ^q such that $x = \sum_n a_n e_n \to \sum_n \langle x, v_n \rangle u_n$.

15. incomplete Use Proposition 4.2.6. (Recall that this was because $x_n \in X \subseteq X^{**} = \mathcal{B}(X^*, \mathbb{R})$, $x_n \to 0$ pointwise.

(a) The dual of ℓ_2 is ℓ_2. Take a subsequence so that the dot product between 2 elements in the subsequence is quite small, say $|\langle x_m, x_n \rangle| < 2^{-\min(m,n)}$ for all $m \neq n$.

$((x_n)$ is bounded, WLOG by 1.) Then $\frac{1}{N^2}\left\|\sum_{n=1}^N x_n\right\|^2 \leq \sum_{m,n=1}^N |\langle x_m, x_n \rangle| \leq \frac{1}{N}(1 + o(1))$.

$\|u_n - x_{k_n}\| < \varepsilon_n$, $\|u_n\| \leq 2$ for all n.

$\left\|\frac{1}{n}\sum_{i=1}^n u_i\right\|^2 = \frac{1}{n^2}\sum_{i=1}^n \|u_i\|^2 \to 0$. so $\frac{1}{n}\sum u_i - x_{k_i} \to 0$.

(b) fixme The sequence e_n is such that $\frac{1}{n}\sum_{i=1}^n e_i \not\to 0$. We claim it is not weakly null. (This is tricky!)

(a) Define the **Schreier family**

$$S = \{A \subseteq \mathbb{N} : |A| \le \min A\} \cup \{\phi\}.$$

Not generalize Ramsey's Theorem to coloring all finite subsets n??? Then $\|(x_n)\| = \sup_{A\in S}\sum_{n\in A}|x_n|$.

Now $S \subseteq \mathcal{P}\mathbb{N} = \{0,1\}^{\mathbb{N}}$ is a closed subset. We can embed $X \to C(S)$, $x \mapsto \hat{x}$. $\hat{x} = \sum_{n\in A} x_n$, $\frac{1}{2}\|x\| \le \|\hat{x}\|_\infty \le \|x\|$; so this is an isomorphic embedding.

Now $\widehat{e_n} \to 0$ pointwise. $\|\widehat{e_n}\| \le 1$ for all n. By Riesz Representation Theorem, by LDCT, $\widehat{e_n} \xrightarrow{w} 0$ in $C(S)$, so $e_n \xrightarrow{w} 0$.

(b) X is c_0**-saturated**: every infinite-dimensional closed subspace contains a subspace isomorphic to c_0.

Now if $e_n \not\to 0$ weakly, there exists $f \in X^*$, $\varepsilon > 0$ and an infinite $M \subseteq \mathbb{N}$, $f(e_n) > \varepsilon$ for all $m \in M$.

Choose $F_n \in S$, $F_n \subseteq M$, $F_1 < F_2 < \cdots$, where $A < B$ means $\max A < \min B$. The size grows rapidly (ε's): $|F_1| \ll |F_2| \ll \cdots$. Take $x_n = \frac{1}{|F_n|}\sum_{i\in F_n} e_i$. Schreier set, norm and get 1. $\|x_n\| = 1$. $f(x_n) > \varepsilon$. Adding them together, $\left\|\sum_{i=1}^N x_i\right\| \approx 1$. $f\left(\sum_{i=1}^N x_i\right) > \frac{1}{\varepsilon}$. Where does the Schreier set start? Number of elements at most max of x_3. Can make later ones flatter.

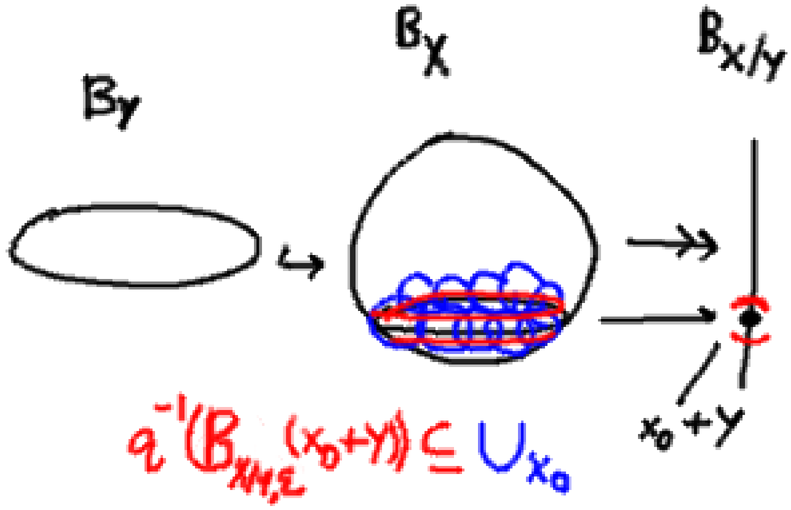
16. We use the characterization given by Theorem 4.4.9: X is reflexive iff (B_X, w) is compact.

(a) The theorem gives that B_X is weak compact. A closed subset of a compact set is compact, so B_Y is weak compact. The image of a compact set is compact, so $B_{X/Y}$ is weak compact. (Recall that $B_X \twoheadrightarrow B_{X/Y}$ by a Hahn-Banach argument.) Now use the theorem in the other direction.

(b) We're given that $B_Y, B_{X/Y}$ are compact, and we need to show B_X is compact.

$$B_Y \hookrightarrow B_X \twoheadrightarrow B_{X/Y}.$$

Let q be the second map. We might think to write $B_X = B_Y \times B_{X/Y}$, and then use the fact that products of compact sets are compact; unfortunately, this sequence doesn't necessarily split (does it?). Instead we imitate the proof that products of compact sets are compact. Recall that one way to prove this is to use the "tube lemma." (See Munkres, Topology.) Here we'll want our "tube" to be around $x_0 + Y$.



Given x_0, we have that

$$A_{x_0} := \{x_0 + y : y \in Y, x_0 + y \in B_X\} \hookrightarrow (1 + |x_0|)B_Y$$
$$x \mapsto x - x_0$$

injectively, by the triangle inequality. The latter is compact by assumption; A_{x_0} is a closed subset of a compact set so compact.

Consider an open cover of B_X. By compactness of B_X, there is a finite subcover for A_{x_0}, say $\{U_{x_0,i}\}$. Let $U_{x_0} = \bigcup_i U_{x_0,i}$.

Lemma 2.2.1: Let K be a compact set in a topological set and U_i be open sets such that $U_1 \supseteq U_2 \supseteq \cdots \supseteq K$ and $\bigcap_i \overline{U_i} = K$. Let U be an open set containing K. Then there exists n such that $U_n \subseteq U$.

Proof. Consider $\overline{U_n} - U$. If $U_n \not\subseteq U$ for all n, then these sets are nonempty. These sets are closed and have the finite intersection property. Suppose $a \in \bigcap \overline{U_n} - U$. Since $\bigcap_i U_i = K$, we have $a \in K$. But $K = \bigcap_i \overline{U_i}$, so K does not intersect $\overline{U_n} - U$, contradiction.

Hence $U_n \subseteq U$ for some n. □

Apply the lemma to the open sets

$$f^{-1}(B_{X/Y,\varepsilon}(x_0 + Y)), \qquad \varepsilon = \frac{1}{n}$$

to get that there exists $\varepsilon(x_0) > 0$ such that $f^{-1}(B_{Y,\varepsilon(x_0)}(x_0 + Y)) \subseteq U_{x_0}$. Now the sets $B_{Y,\varepsilon(x_0)}(x_0 + Y)$ form an open cover for $B_{X/Y}$, so by compactness of $B_{X/Y}$, there is a finite subcover.

Alternate solution: incomplete $Y = Y^{**} \cong (X^*/Y^\perp)^* \cong Y^{\perp\perp}$. $X/Y = (X/Y)^{**} \cong (Y^\perp)^* \cong X^* * /Y^\perp$. You need to show these are the right isomorphisms though.

We have a short exact sequences with canonical embeddings.

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, two heads] & X \arrow[r, "q"] \arrow[d] & X/Y \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & Y^{*}* \arrow[r, "i^{**}"] & X^{**} \arrow[r, "q^{**}"] & (X/Y)^{**} \arrow[r] & 0.
\end{tikzcd}$$

Use the 5-lemma to get the middle map is onto.

Does taking the dual preserve short exact sequences? Yes, by question 8.

$$0 \to X \xrightarrow{S} Y \xrightarrow{S} Z \to 0$$
$$0 \to Z^* \xrightarrow{T^*} Y^* \xrightarrow{S^*} X^* \to 0.$$

$\ker T = \mathrm{im} S$ closed, S injective, so S is an into isomorphism. $\ker S^* = (\mathrm{im} S)^\perp = (\ker T)^\perp = \overline{\mathrm{im} T^*}^{w^*}$ by question 7. But because T^* is into, image is w^* closed. Why? Note $T^*(B_{Z^*})$ is w^*-compact. T^* into, $T^*(B_{Z^*}) \supseteq \delta B_{T^*(Z^*)}$. Use 19, part 3.

17. We have $px + (1-p)y \in X \implies pT(x) + (1-p)T(y)$, using bijectivity of T.

    For the second part, suppose $y \in \mathrm{Ext}(T(C))$. Now $T^{-1}(y) \cap C$ is a closed subspace of C, so compact. Hence it has an extreme point x. We claim x is also an extreme point of C. Suppose $x = px_1 + (1-p)x_2$. Then $y = pT(x_1) + (1-p)T(x_2)$, so by $y \in \mathrm{Ext}(T(C))$, $T(x_1) = T(x_2) = y$; since $x \in \mathrm{Ext}(T^{-1}(y) \cap C)$; $x_1 = x_2 = x$. Hence $y = T(x) \in T(\mathrm{Ext}(C))$.

18. incomplete

    First we claim the extreme points are $(\pm a, \pm a, \ldots)$.

    We can approximate within $\frac{a}{2^{k-1}}$ by

    $$\sum_{j=1}^{2^k} \frac{1}{2^k}((a_i \geq j)a + (a_i < j)(-a)).$$

    where we use the notation $(P) = \begin{cases} 1, & P \text{ true} \\ 0, & P \text{ false}. \end{cases}$.

    Extremal points are $\{\lambda \delta_k\}$. We claim that Lebesgue measure λ is not in the convex hull.

    $$\lambda \notin \overline{\mathrm{conv}} \left\{\pm\delta_k : k \in [0,1]\right\}.$$

    Take a function that spikes.

19. incomplete Take U an open neighborhood of 0. There exists a finite F_0 such that

    $$\{x^* \in B_{X^*} : x^*(x) \leq 1 \text{ for all } F_0\} \subseteq U.$$

Can we do better? We claim that there exists finite $F_1 \subseteq B_X$ such that

$$\{x^* \in 2B_{X^*} : x^*(x) \leq 1 \text{ for all } F_0 \cup F_1\} \subseteq U.$$

(With the condition on F_1 it's not obvious.) This is a compactness argument. If not, for all finite $F \subseteq B_X$, $\{x^* \in 2B_{X^*} : |x^*(x)| \leq 1 \forall x \in F_0 \cup F\} \setminus U \neq \phi$. Note $K_F \cap K_{F'} = K_{F \cup F'}$. The K_F's are w^*-compact with the FIP (finite intersection property), so there exists $x^* \in \bigcap K_F$, $\|x^{**}\| \leq 1$. So $x^* \in U$ by choice of F_0. Continue inductively. For all n, there exists $F_n \subseteq \frac{1}{2^{n-1}} B_X$ such that

$$\{x^* \in 2^n B_{X^*} : |x^*(x)| \leq 1 \forall x \in F_0 \cup F_1 \cup \cdots \cup F_n\} \subseteq U.$$

Similar to show that if you have a linear function that is w^* continuous, must be evaluation, factor through...

Difference here is factoring through infinite sequence space.

(a) Assume $\varphi : X^* \to \mathbb{R}$ is σ-continuous. There exists $x_n \to 0$ such that

$$\{x^* \in X^* : |x^*(x_n)| < 1 \forall n\} \implies |\varphi(x^*)| < 1.$$

finite set, factor through $\mathbb{R}^n$. This time the obvious map to look at is through c_0!

$$\begin{tikzcd} X^* \arrow[rr, "\varphi"] \arrow[dr, "T"'] & & \mathbb{R} \\ & c_0 & \end{tikzcd}$$

We get $T^*(x^*) = 0$, so $T^*(\lambda x^*) = 0$ for all λ. We get $|\varphi(\lambda x^*)| < 1$ for all λ, so $\varphi(x^*) = 0$. We can factor through

$$\begin{tikzcd} X^* \arrow[rr, "\varphi"] \arrow[dr, "T"'] & & \mathbb{R} \\ & \operatorname{im} T \arrow[ur, "\widetilde{\varphi}"'] \arrow[d, hook] & \\ & c_0 & \end{tikzcd}$$

We have $|\widetilde{\varphi}(T(x^*))| < 1$ if $\|(Tx^*)\|_\infty < 1$, so $\|\widetilde{\varphi}\| \leq 1$. $\widetilde{\varphi}$ extends to c_0 by Hahn-Banach, so there exists $(a_n) \in \ell_1 \cong c_0^*$. We get $\widetilde{\varphi}(y_n) = \sum_n a_n y_n$. $\varphi(x^*) = \sum a_n x^*(x_n) = x^*(\sum a_n x_n)$, so φ is w^*-continuous.

(b) incomplete B_Y is w^*-compact. C convex implies (C is w^*-closed iff C is σ-closed).

(c) incomplete

To show Y is w^*-closed, it suffices to show it's σ-closed. So if S is bounded, then $S \cap Y = S \cap NB_Y$ for some N. (w^*-closed in S.)

20. incomplete Counterexample: ℓ_1 $e_1, -e_1, e_2, -e_2, \ldots$.

21. incomplete Consider $X \xrightarrow{T} Y$ where T is 1-to-1. w^* dense, so separating set, we want $T^*(Y^*)$ not 1-norming. Any such thing will work. $c_0 \to \ell_1$, with $1/2, -1/2$, next row $0, 1/4, -1/4$, and so forth.

22. incomplete

    (a) We have $T^{**} : X^{**} \to Y^{**}$ with $T^{**}(B_{X^{**}}) = T^{**}(\overline{B_X}^{w^*}) \subseteq \overline{T^{**}(B_X)}^{w^*} = \overline{T(B_X)}^{w^*} = \overline{T(B_X)} \subseteq Y$ by Goldstine and w^* continuity of T. w^* topology to X get w topology.

    Look at $y^* \mapsto \langle x^{**}, T^* y^* \rangle$. $w^* - w$ continuous. Is w^* continuous on y^*? $\langle T^{**}x^{**}, y^* \rangle$ w^*-continuous.

    (ii) $\implies$ (iii): $T^*(B_{Y^*})$ is w-compact.

    (iii) $\implies$ (i): $\overline{T(B_X)} \subseteq \overline{T^{**}(B_{X^{**}})}$.

    Weakly compact means weakly closed.

Appendix C

Problems 3

Progress:

Done: 2, 3, 5, 7

1 Problems

1. (*The Swiss Cheese: a compact set $K \subseteq \mathbb{C}$ such that $R(K) \neq A(K)$.*) Let Δ be the closed unit disc in $\mathbb{C}$. Show that one can choose a sequence (D_n) of non-overlapping open discs in the interior of Δ such that $\sum_n \operatorname{diam}(D_n) < \infty$ and $\bigcup_n D_n$ is dense in Δ. Set $K = \Delta \setminus \bigcup_n D_n$. ($K$ is the Swiss cheese.) Show that the formula

$$\theta(f) = \int_{\partial\Delta} f(z)\,dz - \sum_{n=1}^{\infty} \int_{\partial D_n} f(z)\,dz$$

defines a nonzero, bounded linear functional on $C(K)$. Deduce that $R(K) \neq A(K)$.

2. Let p be an idempotent element of a Banach algebra A (which means that $p^2 = p$). Show that if p is in the closure of an ideal J of A, then p does in fact belong to J.

3. Let K be an non-empty compact subset of $\mathbb{C}$. Verify that the only characters of $R(K)$ are the point evaluations δ_z with $z \in K$.

4.

5. Give an example of 2×2 matrices x, y with $r(xy) > r(x)r(y)$ and $r(x+y) > r(x)+r(y)$.

6.

7. Let A be a Banach algebra such that every element of A is nilpotent: for each $x \in A$ there exists $n(x) \in \mathbb{N}$ such that $x^{n(x)} = 0$. Prove that there exists $N \in \mathbb{N}$ such that $x^N = 0$ for all $x \in A$.

2 Solutions

1. (a) *Choosing the discs.* Enumerate the rational points Q_j in the disc. For each Q_j, if Q_j is not already covered by (or on the boundary of) a disc, put a disc of radius $r_j < \frac{1}{2^j}$ around Q_j not intersecting any previous disc. For convenience, relabel so that the discs are D_n with radius r_n.

   We have $\sum_{n=1}^{\infty} r_n < 1$. The discs are dense in Δ because $\Delta \cap \mathbb{Q}^2$ is dense in Δ.

   (b) *Non-zero, bounded linear functional.* The functional is clearly linear. To see it's nonzero, let $f(z) = \overline{z}$. Then for a disc D with center a and radius r,

   $$\int_{\partial D} f(z)\,dz \overset{z=a+e^{2\pi it}}{=} \int_0^1 (\overline{a} + re^{-2\pi it}) 2\pi i e^{2\pi it}\,dt = 2\pi i r.$$

   Thus

   $$\theta(f) = \int_{\partial\Delta} f(z)\,dz - \sum_{n=1}^{\infty} \int_{\partial D_n} f(z)\,dz = 2\pi i - 2\pi i \sum_{n=1}^{\infty} r_n > 0.$$

   θ is bounded because $\theta(f) \le \|f\| \left(2\pi + 2\pi \sum_{n=1}^{\infty} r_n\right)$.

   (c) $R(K) \neq A(K)$. Because K has empty interior, $A(K) = C(K)$. Hence $\theta \in C(K) \backslash R(K) = A(K) \backslash R(K)$.

   Pick rational points, one-by-one, if covered, done. If not, put in another disc. r^2 not rational. Choose sufficiently small.

   Estimate of integrals. $z \mapsto \sum \varphi(z_i)$.

   Checking nonzero: $z \mapsto \overline{z}$. Assume $\sum \ell(\partial D_n) < 1$. Otherwise multiply by something.

   $R(K) \neq A(K) = C(K)$.

2. Idea: the set of units is open, and p acts "like" a unit.

   <u>Solution:</u> Consider the ideal pAp. This is an algebra, and $p = ppp$ is the identity element of pAp, since $p(pap) = pap$. Note $G(pAp)$ is open in pAp.

   If $p \in \overline{J}$, then there is a sequence

   $$a_n \to p \text{ with } a_n \in J,$$

   and hence

   $$pa_np \to p \text{ with } pa_np \in pJp.$$

   Since $p \in G(pAp)$ is open in pAp, we get that pJp intersects $G(pAp)$, and hence $pJp = pAp$. Thus $p \in J$.

3. Suppose φ is a character. Let $u(x) = x$. Suppose $\varphi(u) = a$. If $a \in K$, then $\varphi(u-a) = 0$, contradicting the fact that $\frac{1}{u-a} \in R(K)$. Hence $a \notin K$. For any $f(x) = \prod(x-a_i) \prod \frac{1}{x-b_i}$ we have

   $$\varphi(f) = \prod(a-a_i) \prod \frac{1}{a-b_i} = f(a).$$

   Since $\varphi(f) = f(a)$ for any rational function f and θ_x is continuous, we get $\varphi(f) = f(a)$ for any $f \in R(K)$, i.e., $\varphi = \delta_a$.

4. Let $f_n \in A$ and $f_n \to f \in C(\Delta)$. Let $g_n \in A(\Delta)$ with $g_n|_{\mathbb{T}} = f_n|_{\mathbb{T}}$.

   By the *maximum modulus principle*, $\|f_n - f_m\|_{\mathbb{T}} = \|g_n - g_m\|_{\mathbb{T}} = \|g_m - g_n\|_{\Delta}$ so g_n is Cauchy in $A(\Delta)$, and $g_n \to g$. So $g|_{\mathbb{T}} = f|_{\mathbb{T}}$.

   Define $\pi : A \to A(\Delta) \subseteq A$ by letting $\pi(f)$ be the unique $g \in A(\Delta)$ with $g|_{\mathbb{T}} = f|_{\mathbb{T}}$. (It is unique by the maximum modulus principle.) If $f \in A(\Delta)$, then $\pi(f) = f$. π is a projection and algebra homomorphism.

   Now
$$A = A(\Delta) \oplus B, \qquad B := \{f \in C(\Delta) : f|_{\pi} = 0\}.$$
   For all $z \in \Delta$, $\delta_z \circ \pi \in \Phi_A$, then $\delta_z \circ \pi \neq \delta_z$ if $z \notin \mathbb{T}$. For all $z \in \Delta$, $\delta_z \in \Phi_A$. We check there are no others. For $\varphi \in \Phi_A$,
$$B \oplus \mathbb{C}1 = \{f \in C(\Delta) : f|_{\pi} \text{ is constant}\} \cong C(\mathbb{S}^2).$$
   Thus $\varphi|_{B\oplus\mathbb{C}1} = \delta_z$, and $\varphi|_{A(\Delta)} = \delta_w$. If $z \in \mathbb{T}$, then $\varphi = \delta_w \circ \pi$. What if $z \neq w, z \notin \mathbb{T}$. This can't happen, because letting $f \in B$ with $f(z) = 1$,
$$\begin{aligned} \varphi((u-f)^2) &= \varphi(u^2 - 2uf + f^2) = w^2 - 2zf(z) + f^2(z) = w^2 - 2z + 1 \\ \varphi(u-f)^2 &= (w-1)^2 = w^2 - 2w + 1. \end{aligned}$$
   Hence $w = z$.

   We get $\Delta \sqcup \Delta\Phi_A$. Onto, continuous (weak): valuation at f. Agree on the boundary so it factors through
$$\begin{tikzcd} \Delta \sqcup \Delta \arrow[rr, "\theta"] \arrow[dr, "q"'] & & \Phi_A . \\ & \Delta \sqcup \Delta/\pi \arrow[ur, "\overline{\theta}"'] & \end{tikzcd}$$
   Continuous bijection from compact to Hausdorff.

   Φ_A consists of 2 discs glued together at the edge, i.e., a sphere.

5. Let $x = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$. We have $r(x) = r(y) = 2$, $r(x)r(y) = r(x) + r(y) = 4$. But $xy = \left(\begin{smallmatrix} 2 & 4 \\ 4 & 0 \end{smallmatrix}\right)$ and $r(xy) = 1 + \sqrt{17} > 4$.

   Easier: Let $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then $r(x) = r(y) = 0$. But the sum is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and the product is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, with $r(x+y) = r(xy) = 1$.

6. Suppose $\lambda \neq 0$ and $\lambda \notin \sigma(xy)$. Then $\lambda - xy$ is invertible, we have to show $\lambda - yx$ is likewise invertible. Let $z = (\lambda - xy)^{-1}$. $(\lambda - yx)y = y(\lambda - xy)$. Take y out on other side. $(\lambda - yx)yz = y$. $(\lambda - yx)yzx = yx - \lambda + \lambda$. We get
$$(\lambda - yx)(yzx + 1) = \lambda \neq 0 \implies (\lambda - yx)^{-1} = \frac{1}{\lambda}(yzx + 1).$$
   so we're done.

   If $xy - yx = \lambda 1$ for $\lambda \neq 0$, then $\sigma(xy) = \sigma(yx + \lambda 1) = \sigma(yx) + \lambda$. This contradicts the first part.

7. Let $C_n := \{x : x^n = 0\}$. Because A is nilpotent,

$$\bigcup_{n\in\mathbb{N}} C_n = A.$$

However, the Baire category theorem says that a complete metric space is not a countable union of nowhere dense sets. Hence C_n (since it is closed) has nonempty interior for some n. There exists $x_0 \in A$ such that $x_0 + rB_A \subseteq C_n$. In the commutative case we are done, because if $x \in C_n$ then $\lambda x \in C_n$, and if $xy = yx$ then $x + y \in C_{2n}$ (expand by the binomial theorem). Hence $C_{2n} = \operatorname{span} C_n$, and hence $C_{2n} = A$.

Take the continuous path $p(t) = (x_0 + t(x - x_0))^N$, there exists δ such that for all t, $|t| < \delta$, $p(t) = 0$. Then $p \equiv 0$, $p(1) = x^N = 0$. To apply this complex result to Banach algebras, apply a functional and use Hahn-Banach.

8. (a) This definition makes sense because $\left\| \frac{x^n}{n!} \right\| \le \frac{\|x\|^n}{n!}$, so the sum converges absolutely.

   (b) When x, y commute, multiply $e^x e^y$ out, rearrange terms (legal since the series converge absolutely), commute, and collect terms using the binomial theorem to see that it equals e^{x+y}.

   (c) $\sigma(e^x) = e^{\sigma(x)}$ follows from holomorphic functional calculus applied to e^x. (Recall that if we take C to be the maximual commutative subalgebra containing x, then the spectrum over C is the same; we proved this using characters.)

   (d) First note $e^x \in G_0$ as $t \mapsto e^{tx}, 0 \le t \le 1$ is a path from 1 to e^x; G_0 is closed under multiplication since if $g, h \in G_0$, then there is a path $\gamma(t)$ from 1 to g, and $\gamma(t)h$ is a path from h to gh. Hence $\langle e^x : x \in A\rangle \subseteq G_0$.

   Conversely, note e^x is invertible on a neighborhood of identity, since its inverse is

$$\ln(x-1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$$

   For $y \in H$ such that $\|y - z\| < \frac{1}{\|y^{-1}\|}$, $\|zy^{-1} - 1\| < 1$, spectrum contained in ball of radius 1, but $\sigma(zy^{-1}) \subseteq B(1,1) = \{z \in \mathbb{C} : |z-1| < 1\}$. Use 10(ii). (Analytic branch of log.) $zy^{-1} = e^x$,

$$z = e^x y.$$

   Then $y \in H \iff z \in H$. $G \backslash H$ open is the same argument. Then $H = \{e^x : x \in A\}$ is connected and both open and closed, and so must be G_0.

9. We have

$$\begin{aligned}
\Theta_x(f)\Theta_x(g) &= \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma f(z)(z1 - x)^{-1}\,dz \int_{\Gamma'} g(u)(u1 - z)^{-1}\,du \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \int_{\Gamma'} \frac{f(z)g(w)}{z - w}(w1 - x)^{-1}\,dw\,dz - \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \int_{\Gamma'} \frac{f(z)g(w)}{z-w}(z1 - x)^{-1}\,dw\,dz \\
&= \int_{\Gamma'} \left(\int_\Gamma \frac{f(z)}{z-w}\,dz \right) g(w)(w1 - x)^{-1} + \frac{1}{2\pi i} \int_\Gamma f(z)g(z)(z1 - x)^{-1}\,dz.
\end{aligned}$$

using (in the second line)

$$\frac{(z1-w)^{-1}(z-w)(w1-x)^{-1}}{z-w} = \frac{(w1-x)^{-1}-(z1-x)^{-1}}{z-w}.$$

10. (a) Suppose $\sigma(x) = U \cup V$. Consider $\Theta_x(1_U)$, we have $p^2 = p$. Then $\sigma(p) = \{0, 1\}$.
    nontrivial invariant subspaces.
    Invariant subspace problem.
    (b) Take a branch of the logarithm $e^{\ln z} = z$, $z \in \mathbb{C}\backslash(-\infty, 0]$. $u = \Theta_x(\ln) \sum_{n=0}^{\mu} \frac{L(z)^n}{n!} \to z$. $\sum_{n=0}^{\mu} \frac{y^n}{n!} \to x$.
    Power series converge locally uniformly.

11. No. On the unit disc, take the line at $\frac{1}{n}$, take the function that is $0 < \frac{1}{n}$, 1 on $> \frac{1}{2n}$, P_n approximates within $\frac{1}{n}$ on these compact sets. P_n converges to the function $0 \leq 0$ and $1 > 0$. (Runge's Theorem)

12. We have
$$\|f(\lambda)\| \leq Kr(e^{-\lambda x} y e^{-\lambda x}) = K.$$
(amplification!) Liouville: Bounded, so constant. Putting $\lambda = 0$, constant must be y. $ye^{\lambda x} = e^{\lambda x} y$ for all x, so $yx = xy$.

13. Take a closed ball $\overline{B}(z, r) \subseteq U$, $\|f\| = \sup_{z \in B(z,r)} |f(z)|$.

14. (a) $C(\mathbb{R})$, $\Phi = \{\delta_x : x \in \mathbb{R}\}$, $\varphi \in \Phi$. $a = \varphi(u)$.
    For any $f \in C(\mathbb{R})$, $\varphi(f) \in \mathrm{im} f$. Else $g := \varphi(f)1 - f$ is invertible, contradiction because $\varphi(g) = 0$.
    cf. restriction of analytic functions to boundary. $g = (f - \varphi(f))^2 + (u - a)^2$. $\varphi(g) = 0$, so g is not invertible. There exists x such that $g(x) = 0$. In other words, $x = a$. $f(a) = \varphi(f)$, so we're done.
    (b) $A = C(\mathbb{R})$. $(A, \|\cdot\|)$, B the completion. $\Phi_B = \Phi_C$. We can map $\Phi_B = \Phi_C \to \mathbb{R}$ by $\varphi :\mapsto \varphi(u)$. This is a homeomorphism onto its image. Compact space, so compact image. $\Phi_c = \{\delta_x : x \in K\}$.
    (c) Take a function g that is 1 on K, and 0 away from K. Then g is invertible in B. Now take any function f which doesn't vanish but vanishs on $\mathrm{Supp}(g)$. Then $fg = 0$, so $f = 0$, contradiction.

15. Like multiplying out.
    (a) Use $D^n(ab) = \sum \binom{n}{k} D^k(a) D^{n-k}(b)$.
$$\sum_n \frac{1}{n!} D^n(ab) = \sum_n \frac{1}{n!} \sum \binom{n}{k} D^k(a) D^{n-k}(b) = \sum_k \frac{1}{k!} D^k(a) \sum_l \frac{1}{l!} D^l(b).$$
    (Careful: $e^D(a) \neq e^{D(a)}$. $e^D(ab) = e^D(a) e^D(b)$. Isomorphism in the linear sense.

(b) multiple also derivation.

(c) Fix x, φ, $\lambda \mapsto \varphi(e^{\lambda D}(x))$ bounded by $\|x\|$ as $\varphi(e^{\lambda D})$ is a character. By Liouville it's constant.

(d) Assume $\|\cdot\|$ is a Banach algebra norm on $C^\infty[0,1]$. Look at $D(f) = f'$. If this was continuous, using the previous, maps into Jacobson radical, not possible because the derivative of x is 1.

Why is it continuous? eval at pt charIf $f_n \to 0$, $f_n \to g$ in $\|\cdot\|$, in $\|\cdot\|_\infty$ by the Closed Graph Theorem. (useful!)
Use $\|\cdot\|_\infty \le \|\cdot\|$. $\delta x \in \Phi$ for all $x \in [0,1]$.

16. Apply 20(b). spec radius norm form for herm op Plug in 1
$\|x\|^2 = \|x^*x\| = r(x^*x) = \|x^*x\|' = \|x\|'^2$.

17.

18. (a) SWAP: (L_1L_2, R_2R_1). We have

$$\begin{aligned} \|L(a)\|^2 = \|L(a)^*L(a)\| = \|R(L(a)^*)a\| &\qquad \le \|R\| \, \|L(a)\| \, \|a\| \\ \|L(a)\| \le \|R\| \, \|a\| & \\ \implies \|L\| \le \|R\| \, . & \end{aligned}$$

We get

$$\|L\| = \sup_{\|b\|=1} \|L(b)\| = \sup_{\|a\|=\|b\|=1} \|aL(b)\| \, .$$

19. $A^+ \to A \oplus \mathbb{C}$ in the sense of question 17. $x + \lambda \mapsto (x+\lambda, \lambda)$. LHS new unit, RHS old unit. If A unital, $M(A) = A$, $(\mathrm{id}, \mathrm{id})$. If unital $= (L_1, R_1)$. Unit belongs to A thus $M(A)$. When unital doesn't help unitize. Sitting outside, add that.

A nonunital: $A^+ \to M(A)$, $x + \lambda \mapsto x + \lambda$ (the respective units) $*$-homomorphism, pull back norm.

20. This is a remarkable theorem becuase algebraic conditions imply continuity. bollobas linear analysis, uniqueness of norm result. All operators on Banach space, unique. Map between certain banach with alg condition imposed. Automatic continuity theory.

(a) The characters of A are evaluations, so $\Phi_K \cong K$. $(A, \|\cdot\|_1)$. complete B.
$L = \{y \in K : \delta_\eta \text{ is continuous w.r.t. } \|\cdot\|_1\} = \Phi_B$. Claim: $\overline{L} = K$. $\|f\|_1 \ge r_B(f) = \sup_{y \in L} |f(y)| = \|f\|$.
If not, take $x \notin \overline{L}$. Compact Hausdorff space normal, put a neighborhood around it: there exists V of x, $\overline{V} \cap \overline{L} = \phi$. Let $g = 1$ on $\overline{L}$ and 0 on $\overline{V}$. g is invertible in B because applying the character, we get something nonzero. Take $f = 1$ at x, 0 on $K \backslash V$, $gf = 0$, $f = 0$, contradiction.

(b) By unitization we can assume A, B, θ are unital. (In 18 and 19 we checked that C^* algebras have unitization. This is not trivial.) We have $\theta_B(\theta(x)) \subseteq \sigma_A(x)$. We have

$$\|\theta(x)\|^2 = \|\theta(x)^*\theta(x)\| = \|\theta(x^*x)\| = r_B(\theta(x^*x)) \subseteq r_A(x^*x) = \|x^*x\| = \|x\|^2 .$$

Given x, let C be the subalgebra generated by $1, x^*x$, a C^*-subalgebra. $B|_C : C \to B$: if $\theta|_C$ isometric then $\|\theta(x)\|^2 = \|\theta(x^*x)\| = \|x^*x\| = \|x\|^2$. WLOG A is commutative.

$\theta : C(K) \to B$ is injective, unital $*$-homomorphism, we can define $\|f\|_1 = \|\theta(f)\|_1 \geq \|f\|$ by Kaplansky.

Appendix D

Summaries and additional notes

The first "index card" summaries are here.

1. https://dl.dropboxusercontent.com/u/27883775/math%20notes/index%20cards/an1.jpg
2. https://dl.dropboxusercontent.com/u/27883775/math%20notes/index%20cards/an2.jpg
3. https://dl.dropboxusercontent.com/u/27883775/math%20notes/index%20cards/an3.jpg
4. https://dl.dropboxusercontent.com/u/27883775/math%20notes/index%20cards/an4.jpg
5. https://dl.dropboxusercontent.com/u/27883775/math%20notes/index%20cards/an5.jpg

1 Introduction

2 Hahn-Banach Theorem

3 Riesz Representation Theorem

(Note: basics of real measure theory are omitted from this summary.)

Let K be compact Hausdorff. Define (3.1.1) spaces of functions $C(K), C^{\mathbb{R}}(K), C(K)^+$; spaces of functionals $M(K) = C(K)^*$, $M^{\mathbb{R}}(K) = C^{\mathbb{R}}(K)^*$, $M^+(K)$.

1. Basic lemma 3.1.2 on decomposing functionals: we can decompose *complex* functionals as $\varphi_1 + i\varphi_2$, $\varphi_i \in C^{\mathbb{R}}(K)$ *real*; and *real* functionals as $\varphi = \varphi^+ - \varphi^-$, $\varphi^+, \varphi^- \in M^+(K)$ *positive* with $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\|$.

(a) For real functionals, it suffices to consider norms over real functions (2). Positive functionals are those who attain their norm at the identity (3). [Proofs are somewhat annoying; omitted.]

2. Define **complex measures**, 3.2.7.

   (a) A Borel measure is **regular** if you can approach it on the inside by compact sets and the outside by open sets.

   (b) The **total variation** $|\mu|(X)$ is how big you can make the integral with simple functions on X. It is finite: $|\mu|$ is a measure by a "refining partitions" argument and use $\max_{A\subseteq S}\sum_{z\in A}|z| \geq \frac{1}{\pi}\sum_{z\in S}|z|$; if $|\mu|$ diverges, keep peeling off subsets where $|\mu|$ is large and show μ diverges too.

   (c) Hahn-Jordan decomposition: A real measure $\mu = \mu^+ - \mu^-$ with $|\mu| = \mu^+ + \mu^-$.

3. Riesz Representation 3.3.2: (Functionals correspond to measures.) For $\varphi \in M(K)$ there exists a unique regular complex Borel measure μ on K such that $\varphi(f) = \int_K f\,d\mu$, and $\|\varphi\| = \|\mu\|_1$.

   (a) Pf. Consider $\varphi \in M^+(K)$ first. We'd like to define $\mu(E) = \varphi(1_E)$ but 1_E is not continuous; approximate it using Urysohn: $\mu^*(U) = \sup\{\varphi(f) : f \ll U\}$ and $\mu^*(A) = \inf\{\mu^*(U) : U \supseteq A\}$.

   (b) Countable subadditivity for open sets: use partitions of unity. For arbitrary sets: approximate with opens (the $\sum_n \varepsilon_n = \varepsilon$ trick).

   (c) We show superadditivity: $\mu^*(A) \geq \mu^*(A\cap U) + \mu^*(A\backslash U)$ for all A and open U, to get a measure on all Borel sets.

   (d) Show $\varphi(f) = \int_K f\,d\mu$ by taking a partition of unity h_i on sets U_i where f is approximately constant, and taking φ of $f = \sum f h_i$.

   (e) For arbitrary φ: Break up $\varphi = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4$. To show $\|\varphi\| = \|\mu\|_1$, given $\bigsqcup E_i$, use PoU to get functions on U_i close to E_i, and take a linear combination of them.

4 Weak Topologies

5 Krein-Milman Theorem

6 Banach algebras

[Index card 6]

1. A **Banach algebra** is a complete normed algebra (with the norm submultiplicative). Some basic constructions are unitication, quotienting out by a closed ideal, and completion. We mostly consider unital algebras.

(a) Our first task was to understand when we can take inverses. Inverses exist when $\|1-x\| < 1$ (Lemma 6.1.4). $\bullet^{-1}$ is a homeomorphism (Corollary 6.1.6). $G(A) = A^\times$ is open, and when we approach its complement, inverses blow up; in fact, $x \in \partial G(A)$ has no left/right inverse in any superalgebra.

i. The key fact we use in the proof is that we can write down the inverse explicitly $\frac{1}{1-a} = \sum_{n=0}^{\infty} a^{-n}$ when the series converges. (Many facts about power series transfer over to the Banach algebra setting.)

2. The **spectrum** of x in A is $\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \text{ not invertible}\}$.

(a) $\sigma_A(x)$ is nonempty and compact (Theorem 6.2.3).

i. Compact: It is the inverse image of $A \backslash A^\times$ which is closed, under the map $\lambda \mapsto \lambda 1_A - x$.

ii. Nonempty: If $\sigma(x) = \phi$ then we can construct an entire function that's bounded (check that it is analytic):

$$\text{eq:spectrum-eq}\lambda \mapsto \frac{1}{\lambda 1_A - x} \in A^\times \tag{D.1}$$

is well-defined. By Liouville, it is constant.

(b) As a corollary, the only complex unital normed division algebra is $\mathbb{C}$.

(c) (Spectral mapping theorem for polynomials 6.2.8): $\sigma(p(x)) = p(\sigma(x))$. (Proof: Factor the polynomial and use xy invertible $\implies x, y$ invertible.)

(d) The spectral radius (Theorem 6.2.10) is

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_n \|x^n\|^{\frac{1}{n}}$$

i. Expand $\frac{1}{\lambda 1 - x} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{x}{\lambda}\right)^n$ and use the root test to see when it converges.

(e) When we pass to a subalgebra, the spectrum can only get bigger ($\sigma_B(x) \supseteq \sigma_A(x)$) but the boundary can only get smaller ($\partial\sigma_B(x) \subseteq \partial\sigma_A(x)$) (Theorem 6.2.11). (Proof: it's harder to be invertible; use the characterization of ∂G.)

i. In the special case $B = A(x)$, we have $\sigma_{A(x)}(x) = \widehat{\sigma_A(x)}$ (Theorem 6.2.14). (Proof: using maximum modulus, $\sigma_A(x) \subseteq \sigma_{A(x)}(x) \subseteq \widehat{\sigma_A(x)} \subseteq \widetilde{\sigma_A(x)}$; use an approximate inverse $p(x)$ of $\lambda 1 - x$ and construct q large at λ.)

ii. It suffices to consider a maximal commutative subalgebra C: $\sigma_C(x) = \sigma_A(x)$ (Proposition 6.2.15). ("Noncommutativity cannot furnish more inverses.") Proof?

3. Commutative Banach algebras are especially nice because we can understand the characters completely. (cf. We can understand characters of commutative algbras easily because the only irreducible representations are 1-dimensional.)

(a) Characters are in bijection with maximal ideals, $\Phi_A \cong \mathcal{M}_A$, $\varphi \mapsto \ker\varphi$ (Theorem 6.3.5). (Proof: Some basic lemmas, plus use Gelfand-Mazur to get $A/M \cong \mathbb{C}$).

(b) We can obtain $\sigma_A(x)$ by evaluating x at all characters: $\sigma_A(x) = \{\varphi(x) : \varphi \in \Phi_A\}$. (Corollary 6.3.6) (Proof: Show $x \in G(A) \iff \varphi(x) \neq 0$ for all φ.) *This characterization of $\sigma_A(x)$ is much easier to work with.* For example, we get facts about $r(x)$ (Corollary 6.3.7).

(c) The Gelfand topology on Φ_A is induced from the w^*-topology (so is compact Hausdorff). Gelfand representation: $A \to C(\Phi_A), x \to \hat{x}$ is a continuous algebra homomorphism (Theorem 6.3.10). (It is an isomorphism for c.u. C^* algebras.)

Examples (6.1.3 and 6.3.8): Canonical examples are spaces of functions and spaces of bounded linear operators.

1. For $C(K)$, K compact Hausdorff, and the disc algebra $A(\Delta)$, analytic functions on the unit disc, the characters are evaluation maps. (Proof: Use compactness. Look at $u(z) = z$ and use the fact polynomials are dense.) For $A \subseteq C(K)$, $K \to \Phi_A$, $k \mapsto \delta_k$ is an embedding (a homeomorphism if $A = C(K)$) (Lemma 6.3.11).

2. The Wiener algebra is functions with L^1 Fourier coefficients; characters are evaluations (same proof: trig polys are dense). Wiener's Theorem 6.3.9 is a non-obvious theorem that shows how algebra is useful in analysis.

7 Holomorphic functional calculus

[Index card 7]

(Abbreviate commutative unital Banach algebra as cuBa.)

Holomorphic functional calculus (Theorem 7.1.1) tells us that we can evaluate holomorphic functions at elements of a cuBa, i.e., there is a unique way to define "$f(x)$," $f \in \mathcal{O}(U)$, $x \in A$ such that $u(x) = x$, and such that $\theta_x(f) := f(x)$ is a continuous unital homomorphism $\mathcal{O}(U) \to A$. We have

1. $\varphi(f(x)) = f(\varphi(x))$ for all characters $\varphi \in \Phi_A$, and such that $u(x) = x$.

2. σ commutes with f: $\sigma(f(x)) = f(\sigma(x))$.

To prove this, we use **Runge's Theorem** (Theorem 7.1.2): for compact $K \subset\subset \mathbb{C}$, $R(K) = O(K)$ (analytic functions on an open set can be uniformly approximated by rational functions).

The idea is to define $\theta_x(f) = f(x)$ as an integral. (In complex analysis, by Cauchy's Theorem, every holomorphic function can be expressed as an integral. This is how a lot of theorems about analytic functions were proved in the first place: we can manipulate the integral to show that the function is nice.)

1. Define integration as a Riemann sum, and generalize to integration parametrized by γ in the usual way. We get Cauchy's Theorem for Banach spaces 7.1.5. (Prove using regular Cauchy after applying $\varphi \in X^*$.)

2. Define $\theta_x(f) = \frac{1}{2\pi i}\int_\Gamma f(z)(z1-x)^{-1}$, where Γ winds around everything in $\sigma(x)$ once and nothing outside of U. We check (Lemma 7.2.1)

   (a) $\theta_x(f)$ is linear and continuous in f: just remember that continuity means with respect to the seminorms on compact K

   (b) $\theta_x(1) = 1$: By Cauchy's Theorem, since 1 is defined not just on U but the whole space, we can change the contour to circles which wind around each point of $\sigma(x)$ once; now just expand in power series.

   (c) $\theta_x(r) = r(x)$ for rational functions: by writing

   $$\theta_x(r) = \theta_x(1)r(x) + \underbrace{\frac{1}{2\pi i}\int_\Gamma (r(z)-r(x))(z1-x)^{-1}}_{=0 \text{ by expanding}}.$$

   (d) $\varphi(f(x)) = f(\varphi(x))$: because we can commute φ with the integral.

   (e) $\sigma(f(x)) = f(\sigma(x))$: use $\sigma(x) = \{\varphi(x)\}$.

3. This proves Runge's Theorem: For any $f \in O(K)$, $f = \theta_x(f) \in R(K)$. $\theta_x(f) \in R(K)$, because it is defined as an integral in our algebra $R(K)$. (For a more precise statement, we can restrict to a subalgebra.)

4. Proof of holomorphic functional calculus: Show $R(U)$ is dense in $\mathcal{O}(U)$ (Corollary 7.2.3) (this follows from what we showed for K, and the fact the topology is defined in terms of $\|\cdot\|_K$). θ_x is an algebra homomorphism for rational functions, which are dense.

8 C^*-algebras

1. A **C^*-algebra** is a Banach algebra with involution such that $\|x^*x\| = \|x\|^2$. Define **hermitian** h**, unitary** u**, normal** (Definition 8.1.5). Note this meshes very nicely when x is hermitian or normal. Basic properties 8.1.6:

   (a) $x = h + ik$, h, k hermitian.

   (b) $\|x^*\| = \|x\|$.

   (c) (Commutes with $\bullet^{-1}$, σ, and φ (for commutative A)): $(x^*)^{-1} = (x^{-1})^*$, $\sigma(x^*) = \overline{\sigma(x)}$, $\varphi(x^*) = \overline{\varphi(x)}$ (Lemma 8.1.8).

      i. Proof: Suffices to show $\varphi(h) \in \mathbb{R}$. h acts like a real element, so consider $\|h+it\|^2$. But on the other hand look at $\varphi(h)$; h acting real forces $\varphi(h)$ to act real.

2. Results on spectrum (Lemma 8.1.7, 8.1.10): for x normal, $r(x) = \|x\|$. (Pf: $\|h\|^{2^n} = \left\|h^{2^n}\right\|$; x^*x is hermitian.) $\sigma_A(h) \subseteq \mathbb{R}$, $\sigma_A(u) \subseteq \mathbb{T}$ (Proof: pass to subalgebra, and use $\sigma(x) = \{\varphi(x)\}$.)

   (a) Note for hermitian and unitary, $\sigma = \partial\sigma$, so the subalgebra doesn't matter. For normal $T \in \mathcal{B}(H)$, $\sigma(T) = \sigma_{\text{ap}}(T)$.

3. Gelfand-Naimark Theorem 8.1.9: For cu C^*-algebras, the Gelfand transform is an isometric $*$-isomorphism $A \cong C(\Phi_A)$. (Proof: $*$ behaves nicely wrt $\|\cdot\|$, φ; Stone-Weierstrass for $\twoheadrightarrow$.)

4. Applications

> 🔑 To prove decompositions for $T \in A = \mathcal{B}(H)$, it helps instead to transfer the problem over to a problem about $C(\Phi_A)$.

   (a) Positive elements have square roots: because positive functions have square roots.

   (b) Polar decomposition for *invertible* elements: $T = RU$, $R \geq 0$. (Pf. Use approximate eigenvalues to see $\sigma(TT^*) > 0$, $T = (TT^*)^{\frac{1}{2}}[(TT^*)^{-\frac{1}{2}}T]$.)

9 Spectral theory

review9 The key example to have in mind is that of finite-dimensional matrices (See 2.1).

A **resolution of the identity** sends measurable sets to functions in $\mathcal{B}(H)$, satisfying compatibility conditions and with $P_{x,y}(E) = \langle P(E)x, y\rangle$ a regular complex Borel measure. Define $L_\infty(P)$.

1. 9.5: Boost $P_{x,y}(E) = \langle P(E)x, y\rangle$ to something defined for *functions* rather than sets:

   eq:r9-1
   $$\int_K dP_{x,y} = \langle \Phi(f)x, y\rangle . \tag{D.2}$$

   Moreover, $S \in \mathcal{B}(H)$ commutes with every $\Phi(f)$ iff it commutes with every $P(E)$.

   Proof: "Simple functions $\implies$ all functions" argument: Define Φ first for simple functions. Use $\|\Phi(s)\| = \|s\|_\infty$ to show we can define Φ for arbitrary f by taking simple $s_n \to f$.

2. Spectral Theorem for commutative C^*-algebras: Let $A \subseteq \mathcal{B}(H)$ be a commutative C^*-algebra. There is a unique resolution of the identity P of H over $K = \Phi_A$ such that

   $$T = \int_K \widehat{T}\, dP \text{ for all } T \in A.$$

   $P(U) \neq 0$ for nonempty open U, and S commutes with all $T \in A$ iff S commutes with all $P(E)$.

   (a) We have $\hat{\bullet} : A \xrightarrow{\cong} C(\Phi_A)$, and we'd like $\Psi : L^\infty \to A$. We would like $\Phi(\widehat{T}) = T$ in (D.2); by Riesz Representation we get a measure $\mu_{x,y}$ for each x, y.

   (b) Check for f, $\int_K f\, d\mu_{x,y}$ is a bounded sesquilinear form of norm $\leq \|f\|_K$, and obtain $\Psi(f)$. Check it is hermitian and respects $*$.

   (c) Check Ψ respects multiplication by converting to a problem about measures and using RRT.

(d) Define P from Ψ and verify it is a resolution of the identity. P is unique by RRT.

(e) For $P(U) \neq 0$ use Urysohn to get f for U and write $f = \widehat{T}^2$ using Gelfand-Naimark.

(f) For commutativity, calculate $\langle ST/TSx, y\rangle$; transfer to a problem on measures using RRT.

3. Spectral theorem for normal operators: $T = \int_{\sigma(T)} \lambda \, dP$. S commutes with every $P(E)$ iff it commutes with T.

   (a) Proof: Use $\Phi_A \cong \sigma(T)$, and pass to subalgebra $\overline{\mathbb{C}[T, T^*]}$.

   (b) Uniqueness: RRT and Stone-Weierstrass.

   (c) Commuting: Lemma (Fuglede-Putnam-Rosenblum) In a C^*-algebra, if x, y are normal and $xz = zy$, then $x^*z = zy^*$. Proof: $f(\lambda) = e^{\lambda x^* - \overline{\lambda} x} z e^{\overline{\lambda} y - \lambda y^*}$; use Liouville.

4. Borel functional calculus: For normal T, define for arbitrary $L_\infty(\sigma(T))$, $f(T) := \int_K f \, dP$. It is a unital $*$-homomorphism.

5. Applications

   (a) Polar decomposition: For normal T, $T = RU$. Pf. It's true for $\mathbb{C}$; we have multiplicativity in the integral.

   (b) Unitary operators can be written as e^{iH}. Pf. Apply the functional calculus with the logarithm.

   (c) $G(\mathcal{B}(H))$ is connected. Pf. Find a path to the identity using Polar decomposition and exponentials.

sectionMiscellaneous questions

1. How does holomorphic functional calculus fail for noncommutative Banach algebras?

Bibliography

[1] W. Rudin. *Real and Complex Analysis*.

List of Notations

Index