18.785 Analytic Number Theory Problem Set #5

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Problem 1

(A)
Since $\Gamma_0(N)$ acts on $S_k(\Gamma_1(N))$ by the slash operation and the subgroup $\Gamma_1(N)$ fixes $S_k(\Gamma_1(N))$, $S_k(\Gamma_1(N))$ is a representation of $\Gamma_0(N)/\Gamma_1(N)$. The kernel of the map $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto d \pmod{n}$$

is the set of elements $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$ where $d \equiv 1 \pmod{n}$. However, since $c \equiv 0 \pmod{n}$ and the determinant is 1, this forces $a \equiv 1 \pmod{n}$, and $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_1(N)$. Hence the kernel is $\Gamma_1(N)$ and $\Gamma_0(N)/\Gamma_1(N) = (\mathbb{Z}/N\mathbb{Z})^\times$ by the isomorphism sending the class of $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to $d \pmod{n}$.

Since $S_k(\Gamma_1(N))$ is a finite-dimensional representation of $\Gamma_0(N)/\Gamma_1(N) = (\mathbb{Z}/N\mathbb{Z})^\times$, it decomposes into irreducible representations of $(\mathbb{Z}/N\mathbb{Z})^\times$. Thus we can write

$$S_k(\Gamma_1(N)) = \bigoplus_{\rho_N} V_{\rho_N}$$

where the sum is over all irreducible representations of $(\mathbb{Z}/N\mathbb{Z})^\times$, and V_{ρ_N} is the subspace where $\Gamma_0(N)/\Gamma_1(N) = (\mathbb{Z}/N\mathbb{Z})^\times$ acts by ρ_N. (Specifically, $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)/\Gamma_1(N)$, corresponding to $d \pmod{p} \in (\mathbb{Z}/N\mathbb{Z})^\times$, acts by $\rho_N(d)$.) These are the same as the characters χ_N since all irreducible representations of $(\mathbb{Z}/N\mathbb{Z})^\times$ are one-dimensional. If χ_N corresponds to ρ_N, then by definition, $V_{\rho_N} = S_k(\Gamma_0(N), \chi_N)$, giving the desired result.

(B)
Let f be a modular form of weight k on a congruence subgroup containing $\Gamma(n)$.

Let $N = n^3$, and let $\alpha = \left[\begin{smallmatrix} n & 0 \\ n & \frac{1}{n} \end{smallmatrix}\right]$. Then for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1(N)$, we have

$$\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} \alpha^{-1} = \begin{bmatrix} a - n^2 b & n^2 b \\ a + \frac{c}{n^2} - n^2 b - d & n^2 b + d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{n}$$

since $a \equiv d \equiv 1 \pmod{n^3}$, and $c \equiv 0 \pmod{n^3}$. This shows that for any $\beta \in \Gamma_1(N)$, $\alpha\beta\alpha^{-1} \in \Gamma(n)$ so

$$f|[\alpha\beta\alpha^{-1}]_k = f \text{ for all } \beta \in \Gamma_1(N).$$

Then

$$f|[\alpha]_k|[\beta]_k = f|[\alpha]_k \text{ for all } \beta \in \Gamma_1(N)$$

so $f|[\alpha]_k \in M_k(\Gamma_1(N))$.

Problem 2

Consider $f(z) = e^{e^{-iz}}$ on the strip $\Re z \in [-\pi, \pi]$. We verify:

1. $f(z) = O(1)$ when $\Re(z) = \pm\pi$: If $z = \pm\pi + iy$ with $y \in \mathbb{R}$, then

$$f(z) = e^{e^{-y \pm \pi i}} = e^{-e^{-y}} < e^0 = 1.$$

2. $f(z) \neq O(1)$ when $\Re(z) = 0$: If $z = iy$, $y \in \mathbb{R}$, then

$$f(z) = e^{e^{-i(iy)}} = e^{e^y}$$

   which is clearly not bounded.

Problem 3

Lemma 3.1 (Schreier's subgroup lemma)**:** Let G be a group, H a subgroup, and T a right transversal of H in G containing 1. For every $g \in G$, let $\overline{g}$ be the unique element $t \in T$ such that $Hg = Ht$.

Suppose G is generated by the set S. Then

$$\{ts(\overline{ts})^{-1} : s \in S, t \in T\}$$

generates H.

Let $G = \mathrm{SL}_2(\mathbb{Z})$ and $H = \Gamma_0(5)$. By the algorithm in PSet 2, problem 5, we find coset representatives of H in G to be

$$I, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}.$$

Note G is generated by $S = \{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\}$. Thus the lemma gives the following generators for $\Gamma_0(5)$:

$s \backslash r$	I_2	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	I_2	I_2	$\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$	I_2	I_2
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	I_2	$-I_2$	$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$	$-\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

To show f is a modular form for $\Gamma_0(N)$, i.e. invariant under all elements of $\Gamma_0(N)$, it suffices to show f is invariant under all the matrices in the above table. It is clear that f

is invariant under $\pm I_2$. f is invariant under the translation $\left[\begin{smallmatrix}1 & \pm 1\\ 0 & 1\end{smallmatrix}\right]$ because it has a Fourier series expansion. It suffices to show f is invariant under $\left[\begin{smallmatrix}1 & 0\\ -5 & 1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}-2 & -1\\ 5 & 2\end{smallmatrix}\right]$.

First we show that

$$f|\left[\begin{smallmatrix}0 & -1\\ N & 0\end{smallmatrix}\right]_k = f, \tag{1}$$

i.e.

$$f(z) = \sqrt{N}^{-k/2} z^{-k} f\left(\frac{-1}{Nz}\right).$$

By the given functional equation,

$$\begin{aligned}
L(s) &= \frac{L^*(s)(2\pi)^s}{\sqrt{N}^s\Gamma(s)}\\
&= \frac{i^{-k}L^*(k-s)(2\pi)^s}{\sqrt{N}^s\Gamma(s)}\\
&= \frac{i^{-k}\cdot[(2\pi)^{-(k-s)}\sqrt{N}^{k-s}\Gamma(k-s)L(k-s)]\cdot(2\pi)^s}{\sqrt{N}^s\Gamma(s)}\\
&= \frac{i^{-k}\Gamma(k-s)L(k-s)(2\pi)^{2s-k}}{\sqrt{N}^{2s-k}} \qquad (2)
\end{aligned}$$

By Proposition 5.1.2, for σ large enough, for $y\in\mathbb{R}$,

$$\begin{aligned}
f(iy)-a_0 &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)L(s)(2\pi y)^{-s}\,ds\\
&= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{i^{-k}\Gamma(k-s)L(k-s)(2\pi)^{2s-k}}{\sqrt{N}^{2s-k}}\Gamma(s)(2\pi y)^{-s}\,ds \qquad \text{by (2)}\\
&= \sqrt{N}^{-k}(iy)^{-k}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(k-s)L(k-s)\left(\frac{2\pi}{Ny}\right)^{-(k-s)}\,ds\\
&= \sqrt{N}^{-k}(iy)^{-k}\left(\frac{1}{2\pi i}\int_{(k-\sigma)-i\infty}^{(k-\sigma)+i\infty}\Gamma(k-s)L(k-s)\left(\frac{2\pi}{Ny}\right)^{-(k-s)}\,ds-\sqrt{N}^k(iy)^k a_0\right) \qquad (3)\\
&= \sqrt{N}^{-k}(iy)^{-k}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)L(s)\left(\frac{2\pi}{Ny}\right)^{-s}\,ds-a_0\\
&= \sqrt{N}^{-k}(iy)^{-k}f\left(\frac{i}{Ny}\right)-a_0
\end{aligned}$$

We have $|\Gamma(\sigma+it)|\sim\sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\pi|t|/2}\to 0$ as $|t|\to\infty$, $L(s)$ bounded in vertical strips by assumption, and that the absolute value of $\left(\frac{2\pi}{Ny}\right)^{-s}$, $s=\sigma+it$ is determined by σ. Hence $\Gamma(s)L(s)\left(\frac{2\pi}{Ny}\right)^{-s}\to 0$ as $|t|\to\infty$, so $\Gamma(k-s)L(k-s)\left(\frac{2\pi}{Ny}\right)^{-(k-s)}\to 0$ as $|t|\to\infty$. Thus by Phragmén-Lindelöf, $\Gamma(k-s)L(k-s)\left(\frac{2\pi}{Ny}\right)^{-(k-s)}$ is bounded on vertical strips, and our use of Cauchy's Theorem to move the path of integration in (3) is justified. This shows (1).

Now note that
$$\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}^{-1}.$$
Since f is invariant under slashing by both $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$, it is invariant under $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$.

Lemma 3.2:
$$f|\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}_k + f|\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}_k + f|\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}_k = 2^{1-\frac{k}{2}}a(2)a(n).$$

Proof. We calculate

$$\begin{aligned} f|\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}_k + f|\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}_k + f|\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}_k &= \sum_{n=0}^{\infty} a(n)\left[2^{\frac{k}{2}}e(2nz) + 2^{-\frac{k}{2}}e\left(\frac{nz}{2}\right) + 2^{-\frac{k}{2}}e\left(\frac{n(z+1)}{2}\right)\right] \\ &= \sum_{n=0}^{\infty} a(n)\left[2^{\frac{k}{2}}e(2nz) + 2^{-\frac{k}{2}}e\left(\frac{nz}{2}\right)\left(1+e\left(\frac{n}{2}\right)\right)\right] \\ &= \sum_{n=0}^{\infty} a(n)2^{\frac{k}{2}}e(2nz) + \sum_{n\geq 0,\, n \text{ even}} 2^{1-\frac{k}{2}}e\left(\frac{nz}{2}\right)\left(1+e\left(\frac{n}{2}\right)\right) \\ &= \sum_{n=0}^{\infty}\left(a\left(\frac{n}{2}\right)2^{\frac{k}{2}} + a(2n)2^{1-\frac{k}{2}}\right)e(nz) \end{aligned}$$

where, for convenience, we set $a(n) = 0$ for $n \notin \mathbb{N}$. Let $b(n) = a\left(\frac{n}{2}\right)2^{\frac{k}{2}} + a(2n)2^{1-\frac{k}{2}}$.

Now consider the $p = 2$ term in the Euler product:

$$\frac{1}{1 - a(2)2^{-s} + 2^{k-1}4^{-s}} = \sum_{n=0}^{\infty} c_n 2^{-ns}$$

Rewriting this with $2^{-s} = x$,

$$\frac{1}{1 - a(2)x + 2^{k-1}x^2} = \sum_{n=0}^{\infty} c_n x^n$$
$$\implies 1 = (1 - a(2)x + 2^{k-1}x^2)\sum_{n=0}^{\infty} c_n x^n$$

Matching the coefficients of x^{j+1} on both sides gives

$$c_{j+1} - a(2)c_j + 2^{k-1}c_{j-1} = 0, \quad j \geq 0$$

(where $c_{-1} = 0$). Since $c_j = a(2^j)$, this rewrites to

$$2^{k-1}a(2^{j-1}) + a(2^{j+1}) = a(2)a(2^j).$$

Since f has an Euler product expansion, $a(m)$ is multiplicative. Given m, suppose $2^{j-1}||m$. Then multiplying the above by $a\left(\frac{m}{2^{j-1}}\right)$ gives

$$2^{k-1}a(m)+a(4m)=a(2)a(2m).$$

Thus $b(n)=a\left(\frac{n}{2}\right)2^{\frac{k}{2}}+a(2n)2^{1-\frac{k}{2}}=2^{1-\frac{k}{2}}a(2)a(n)$. (If n is odd, then $b(n)=a(2n)2^{1-\frac{k}{2}}=2^{1-\frac{k}{2}}a(2)a(n)$ as well.) $\square$

Let $w=\left[\begin{smallmatrix}0&1\\-5&0\end{smallmatrix}\right]$. Now $f|[w]_k=f$ from (1), so

$$f\left|\left[w\begin{bmatrix}2&0\\-5&1\end{bmatrix}w^{-1}\right]_k\right.=f.$$

Hence

$$\begin{aligned}f|\begin{bmatrix}2&0\\0&1\end{bmatrix}_k+f|\begin{bmatrix}1&0\\0&2\end{bmatrix}_k+f|\begin{bmatrix}1&1\\0&2\end{bmatrix}_k&=f\left|\left[w\begin{bmatrix}2&0\\0&1\end{bmatrix}w^{-1}\right]_k\right.+f\left|\left[w\begin{bmatrix}1&0\\0&2\end{bmatrix}w^{-1}\right]_k\right.+f\left|\left[w\begin{bmatrix}1&1\\0&2\end{bmatrix}w^{-1}\right]_k\right.\\&=f|\begin{bmatrix}1&0\\0&2\end{bmatrix}_k+f|\begin{bmatrix}2&0\\0&1\end{bmatrix}_k+f|\begin{bmatrix}2&0\\-5&1\end{bmatrix}_k.\end{aligned}$$

This shows $f|\left[\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right]=f|\left[\begin{smallmatrix}2&0\\-5&1\end{smallmatrix}\right]$. Slashing by $\frac{1}{2}\left[\begin{smallmatrix}2&1\\0&1\end{smallmatrix}\right]$ gives

$$f|\begin{bmatrix}2&1\\-5&-2\end{bmatrix}=f|\begin{bmatrix}1&1\\0&1\end{bmatrix}=f.$$

Similar results would probably hold for other N, with similar proof.