1 Hardy-Littlewood Maximal Function

Throughout we work in $\mathbb{R}^n$.

Definition 1.1: Given a function f, define the **maximal function** as the maximal average of absolute value over balls centered at f.

$$Uf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy.$$

Theorem 1.2 (Maximal inequality)**:** 1. (Weak L^1 bound) For all $\alpha > 0$,

$$m(\{x : Mf(x) > \alpha\}) \precsim \frac{1}{\alpha} \int |f|\, dx.$$

This essentially says that if $f \in L^1$, then $Mf \in L^{1,\infty}$.

2. If $f \in L^p$ and $1 < p \le \infty$, then $Mf \in L^p$, and

$$\|Mf\|_{L^p} \precsim \|f\|_{L^p}.$$

To prove this we need the covering lemma.

Lemma 1.3 (Vitali covering lemma)**:** Let E be a nonempty measurable set, covered by balls $B_1, \ldots, B_M$. There exists a subcollection $\{B_1, \ldots, B_N\}$ that is mutually disjoint and

$$\sum_{k=1}^{N} m(B_k) \ge 3^n m(E).$$

Suppose we have some weird set E. If we can cover it by a finite collection of balls, we can find a disjoint subcollection that still tells us something about the measure of E.

Proof. Use the greedy method ("analysis is being greedy"). Let $B_1 \in \{B_\alpha\}$ have the largest radius. Let B_2 be the ball disjoint from B_1 which has largest radius, and so forth.

They are mutually disjoint by construction. Suppose $B \in \{B_\alpha\}$ which is not one of $B_1, \ldots, B_N$. Then for some l, $B \cap B_\ell \ne \phi$. Suppose B_l is the largest (first) ball that it intersects. Then $r(B_l) > r(B)$ because otherwise B would have been chosen instead of B_l at that stage. Then the dilation of B_l by a factor of 3 covers B: $B \subseteq 3B_l$.

Then $3B_1, \ldots, 3B_N$ cover E, so

$$m(E) \le \sum_{k=1}^{N} m(3B_k) = 3^n \sum_{k=1}^{N} m(B_k).$$

□

Proof of maximal inequality 1.2. Define the **uncentered maximal function** by

$$\widetilde{M}f(x) = \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y)|\, dy.$$

It's clear that $Mf(x) \le \widetilde{M}f(x)$.

It suffices to prove the theorem for $\widetilde{M}f$.

Set

$$E_\alpha = \left\{x : \widetilde{M}f(x) > \alpha\right\}.$$

For all $x \in E_\alpha$, there exists $B_x \ni x$ such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|\, dy > \alpha,$$

i.e. $m(B_x) \le \frac{1}{\alpha} \int_{B_x} |f(y)|\, dy$. Let measure of E can be made arbitrarily close to that of E_α because of inner regularity.) Let $\{B_x\}_{x\in E_\alpha}$ be a collection of balls that cover E_α. By compactness we can choose a finite number $B_1, \dots, B_M$ that cover E. By the Covering Lemma 1.3 we can choose $B_1, \dots, B_N$ mutually disjoint and $\sum_{k=1}^N m(B_k) \ge 3^{-n} m(E)$. We have an upper bound on this already:

$$\begin{aligned}
m(E) &\le 3^n \sum_{k=1}^N m(B_k) \\
&\le \frac{3^n}{\alpha} \sum_{k=1}^N \int_{B_f} |f(y)|\, dy \\
&\le \frac{3^n}{\alpha} \int |f|\, dx \\
\implies m(E_\alpha) &\precsim \frac{1}{\alpha} \int |f|\, dx.
\end{aligned}$$

$\square$

The norm is independent of the dimension.

Definition 1.4: Let f be measurable. The **distribution function** of f is a function $\lambda_f(\alpha) : [0,\infty) \to [0,\infty)$ given by

$$\lambda_f(\alpha) = m(\{|f(x)| > \alpha\}).$$

This gives us useful information. For example,

$$\int |f|^p\, dx = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, dx.$$

"Cut the cake" horizontally.

Proof of part 2 of Theorem 1.2. Let $f \in L^p$, $1 < p < \infty$. Then

$$\int |\widetilde{M}f(x)|^p\, dx = p \int_0^\infty \alpha^{p-1} m\left(\left\{x : \widetilde{M}f(x) > \alpha\right\}\right)\, dx \le p \int_0^\infty \alpha^{p-1} m\left(\left\{x : \widetilde{M}g(x) > \frac{\alpha}{2}\right\}\right)\, dx.$$

where

$$g(x) := \begin{cases} |f(x)|, & |f(x)| > \frac{\alpha}{2} \\ 0, & \text{otherwise.} \end{cases}$$

We show the last inequality. The new function g contains all the information we need. We know

$$\begin{aligned} |f(x)| &\le \max\{g(x), \frac{\alpha}{2}\} \\ &\le |g(x)| + \frac{\alpha}{2}. \end{aligned}$$

This translates to the maximal function.

$$\begin{aligned} \widetilde{M}f(x) &\le \widetilde{M}g(x) + \frac{\alpha}{2} \\ \widetilde{M}f(x) > \alpha &\implies \widetilde{M}g(x) > \frac{\alpha}{2} \\ \{\widetilde{M}f(x) > \alpha\} &\subseteq \{\widetilde{M}g(x) > \frac{\alpha}{2}\}. \end{aligned}$$

We have

$$\begin{aligned} \int |\widetilde{M}f(x)|^p\,dx &\precsim \int_0^\infty \alpha^{p-1}\frac{1}{\alpha}\left(\int_{\mathbb{R}^n} |g(x)|\,dx\right)\,d\alpha \\ &= \int_0^\infty \alpha^{p-2}\left(\int_{\left\{x:|f(x)|>\frac{\alpha}{2}\right\}} |f(x)|\,dx\right)\,d\alpha \\ &= \int_{\mathbb{R}^n} |f(x)|\left(\int_0^{2|f(x)|} \alpha^{p-1}\,d\alpha\right)\,dx \precsim \int |f|^p\,dx. \end{aligned}$$

□

Let's build up what we need for the Calderon-Zygmund decomposition. First we need another (more annoying) covering lemma.

We can cover the complement of a closed set with cubes, where the diameter of the cube is proportion to the distance from the cube to the set.

Lemma 1.5 (Whitney decomposition)**:** Let F be a nonempty closed set. There exists a sequence of almost disjoint cubes (intersecting only on a set of measure 0, i.e., their boundary) $\{Q_k\}$ such that

$$F^c = \bigcup Q_k,$$

and there exists $C > 0$ such that

$$\operatorname{diam}(Q_k) \le d(Q_k, F) \le C\operatorname{diam}(Q_k).$$

Proof. Let M_0 be cubes of unit side length with vertices in $\mathbb{Z}^n$.

Define M_k by bisecting each cube in M_{k-1}, so that M_k consists of cubes with side length 2^{-k}, vertices $2^{-k}\mathbb{Z}^n$. (This is the "dyadic decomposition.")

Let $\Omega = F^c$; set

$$\Omega_k = \left\{x \in \mathbb{R}^n : C2^{-k} \le d(x, F) \le 2C2^{-k}\right\},$$

where C is a constant to be chosen. Think of them as bands around the original set.

For $Q \in M_k$, include $Q \in \mathcal{F}$ if $Q \cap \Omega_k \neq \phi$. We need a lower bound of the distance of the cube to the original set F. Let $x \in Q$. Then

$$\begin{aligned} d(x, F) &\geq C2^{-k} - \underbrace{\mathrm{diam}(Q)}_{\sqrt{n}2^{-k}} \\ &= (C - \sqrt{n})2^{-k} \geq \sqrt{n}2^{-k} = \mathrm{diam}(Q). \end{aligned}$$

where we take $C = 2\sqrt{n}$. This gives one direction.

For the other,

$$\begin{aligned} d(Q, F) &\leq 2C2^{-k} && \text{since it intersects the band} \\ &= 2 \cdot 2\sqrt{n}2^{-k} \\ &= 4d(Q). \end{aligned}$$

For all $Q \in \mathcal{F}$, we have

$$\mathrm{diam}(Q) \leq d(Q, F) \leq 4\,\mathrm{diam}(Q).$$

We also have to remove redundant cubes. Proof omitted.

□

Theorem 1.6 (Calderon-Zygmund decomposition)**:** thm:c-z Let $f \in L^1$. Fix $\alpha > 0$. Then there exists a decomposition $f = g + \sum_k b_k$ and a sequence of almost disjoint cubes $\{Q_k\}$ such that

1. (the good part is bounded) $|g(x)| \leq \alpha$,
2. $\mathrm{Supp}(b_k) \subseteq Q_k$, $\int_{Q_k} b_k = 0$, and $\int |b_k(x)| \precsim \alpha \cdot m(Q_k)$,
3. $\sum m(Q_k) \precsim \frac{1}{\alpha} \int |f|\, dx$.

Bounds ($\precsim$) depend only on the dimension.

This is very useful. To construct g, b_k, it's tempting to just cut off f at α. But the correct thing to do is to cut off the maximal function at α.

Idea: The bad set will be covered by a bunch of cubes; if you dilate the cubes by a fixed factor it intersects the good set, so you have a bound on the maximal function.

Proof. We cutoff where $\widetilde{M}f(x) > \alpha$. Let $E_\alpha = \left\{x : \widetilde{M}f(x) > \alpha\right\}$. E_α^c is closed (WLOG nonempty and not all of $\mathbb{R}^n$). Apply Whitney decomposition 1.5 to cover E_α by $\{Q_k\}$. Set

$$g(x) = \begin{cases} |f(x)|, & x \in E_\alpha^c \\ \frac{1}{m(Q_k)} \int_{Q_k} f(y)\, dy, & x \in Q_k \\ 0, & \text{elsewhere.} \end{cases}$$

On E_α^c we have $|f(x)| \precsim |\widetilde{M}f(x)| \leq \alpha$.

Claim: We have $|g(x)| \precsim \alpha$ almost everywhere.

If $x \in Q_k$ (This is a set completely contained in the bad region, but we can blow it up so that it overlaps the good region.),

$$\begin{aligned}|g(x)| &\le \frac{1}{m(Q_k)}\int_{Q_k}|f(y)|\,dy\\ &\le \frac{4^n}{m(Q_k^*)}\int_{Q_k^*}|f(y)|\,dy && Q_k^* := 4Q_k\\ &\lesssim \alpha.\end{aligned}$$

Let

$$b_k(x) = \chi_{Q_k}(x)\left(f(x) - \frac{1}{m(Q_k)}\int_{Q_k} f(y)\,dy\right);$$

note that $\int b_k(x) = 0$; we have $f = g + \sum b_k$.

Now we need to estimate the L^1 norm of each. We find (the "$\le \alpha$" comes from taking $x \in Q_k^* \cap E_\alpha$ and noting $\frac{1}{m(Q_k^*)}\int_{Q_k^*}|f(y)|\,dy \le \widetilde{M}f(x)$ by definition of $\widetilde{M}f$)

$$\begin{aligned}\int |b_k(x)|\,dx &\le 2\int_{Q_k}|f(y)|\,dy\\ &\le \int_{Q_k^*}|f(y)|\,dy\\ &= m(Q_k^*)\cdot\underbrace{\frac{1}{m(Q_k^*)}\int_{Q_k^*}|f(y)|\,dy}_{\le\alpha}\\ &\lesssim \alpha m(Q_k^*) \lesssim \alpha m(Q_k)\\ \sum m(Q_k) &= m(\{x : \widetilde{M}f(x) > \alpha\})\\ &\lesssim \frac{1}{\alpha}\int |f|\,dx.\end{aligned}$$

□

What can you do with this decomposition? You can estimate in singular integrals. This comes up in L^p elliptic regularity results for Laplace's equation. There's an integral kernel that can be estimated using Calderon-Zygmund.

2-17-15

2 Singular integrals

2.1 Approximation technique

We first discuss the Lebesgue differentiation theorem. A closely related topic is approximation to the identity. They use an approximation theorem that is very important and will occur many times in proofs about singular integrals.

Theorem 2.1 (Lebesgue differentiation theorem)**:** Assume $f \in L^1_{\text{loc}}$ (locally integrable function, i.e., integrable when restricted to any ball).

1. Then
$$\lim_{r\to 0} \frac{1}{m(B(r))} \int_{B(X,r)} |f(y) - f(x)|\, dy = 0$$
for almost every x.

2. $$\lim_{r\to 0} \frac{1}{m(B(r))} \int_{B(X,r)} f(y)\, dy \to f(x)$$
for almost every x.

Proof. 1. Assume $f \in C_c$ (continuous and compactly supported). Then f is uniformly continuous:
$$\forall \varepsilon > 0 \qquad \exists \delta, \qquad \forall |x-y| < \delta, \qquad |f(x) - f(y)| < \varepsilon.$$
Then for $r < \delta$, the integral is $< \varepsilon$.

2. Assume $f \in L^1$. We approximate it with a continuous function
$$\forall \varepsilon > 0, \qquad \exists g \in C_c \text{ such that } \|f - g\|_1 < \varepsilon.$$
By the triangle inequality,
$$|f(y) - f(x)| \le |g(y) - g(x)| + |f(y) - g(y)| + |f(x) - g(x)|.$$
Averaging over a ball, taking the limsup, and using (1),
$$\begin{aligned}
\limsup \frac{1}{m(B(r))} \int_{B(x,r)} |f(y) - f(x)| &\le \sup_{r\to 0} \frac{1}{m(B(r))} \int |f(y) - g(y)|\, dy + |f(x) - g(x)| \\
&\le M(f-g)(x) + |f-g|(x) \\
m\left(\limsup_{r\to 0} \frac{1}{m(B(r))} \int |f(y) - f(x)|\, dy \ge \alpha\right) &\le m(M(f-g)(x) > \frac{\alpha}{2}) + m(|f-g|(x) > \frac{\alpha}{2}) \\
&\lesssim \frac{1}{\alpha} \|f-g\|_1 < \frac{\varepsilon}{\alpha}.
\end{aligned}$$
where we used the weak L^1 bound for the maximal function, Theorem 1.2(1). Now take $\varepsilon \to 0$, so we get the LHS is 0. Now take $\alpha = \frac{1}{n}, n \to \infty$.

□

Theorem 2.2 (Approximation to identity)**:** Let $|K(x)| < (1+|x|)^{-n-\varepsilon}$. Let
$$K_t(x) = \frac{1}{t^n} K\left(\frac{x}{t}\right)$$
(note this is normalized so $\int K_t = \int K$). Then the following hold.

1. $\sup_{t>0} |f * K_t(x)| \precsim Mf(x)$
2. As $t \to 0$, $f * K_t(x) \to f(x) \int K$ for almost all x.

We use the same approximation argument.

Proof. WLOG assume $f > 0, K > 0$.

1. First we do the easier case where $K = \sum_{j=1}^N c_j 1_{B_j(0,r_j)}$. (For example, something that looks like stacked cylinders.) By linearity you can consider K to the characteristic function of a single ball, $K = 1_{B(1)}, K_t = \frac{1}{t^n} 1_{B(t)}$. Then $f * K_t$ is the maximal function, so this follows from the maximal inequality, Theorem 1.2. Then

$$\begin{aligned} f * K_t(x) &= \int f(x-y) K_t(y)\,dy \\ &= \frac{1}{t^n} \int_{B(x,t)} f(y)\,dy \\ &\precsim c_n Mf(x) \|K\|_\infty . \end{aligned}$$

   Now take $\sup_{t>0}$.

2. For general k, choose $K_j > 0$, simple $K^{(j)} \nearrow K$. By the Monotone Convergence Theorem, as $j \to \infty$,

$$c * K_t^{(j)}(x) \to f * K_t(x)$$

   (We check that $c * K_t^{(j)}(x) \le c_n Mf(x) \left\| K^{(j)} \right\|_\infty \le c_n Mf(x) \|K\|_\infty$, $c * K_t^{(j)}(x) \le c_n Mf(x) \|K\|_\infty$.)

3. For continuous functions we have that as $t \to 0, f * K_t(x) \to f(x) \int K$ for almost all x. Now approximate L^1 functions by continuous functions.

□

2.2 Singular integrals

Definition 2.3: Singular integrals are integrals of the form

$$Tf(x) = \int K(x,y) f(y)\,dy$$

satisfying the following.

1. T is bounded on L^q for some $q > 1$.
2. (Regularity assumption K) For some $c > 1$, $\int_{|x-y|>c|y-y'|} |K(x,y) - K(x,y')|\,dx \le c$.[1]

[1] Commonly $K(x,y) \sim \frac{1}{|x-y|^n}$; then $DK(x) \sim \frac{1}{|x|^{n+1}}$. Then this condition is $\int_{C|y-y'|}^\infty \frac{|y-y'|}{r^{n+1}} r^{n-1}\,dr \le C$. It's like an L^1 bound on the gradient of K. "Away from the diagonal, you have good bounds." Ex. This shows up as the Newtonian potential when solving Laplace's equation.

3. If $f \in L^q$ is compactly supported, then $Tf(x) = \int K(x,y)f(y)\,dy$ is absolutely convergent.

Proposition 2.4: The following are true.

1. T is weak $(1,1)$:
$$\forall f \in L^1, \alpha > 0, \qquad \inf(Tf > \alpha) \lesssim \frac{1}{\alpha} \|f\|_1 .$$
2. For all $1 < p < q$, T is strong (p,p) (i.e., $\|Tf\|_p \lesssim_{p,n} \|f\|_p$).

We prove the first claim using Calderon-Zygmund, and the second claim using the first and Marcikiewicz interpolation.

Proof. Use the Calderon-Zygmund decomposition 1.6 to get

$$f = g + \underbrace{\sum_k b_k}_{b}$$
$$g \lesssim \alpha$$
$$\operatorname{Supp} b_k \subseteq Q_k, \qquad \int_{Q_k} |b_k| \lesssim \alpha m(Q_k)$$
$$\int_{Q_k} b_k = 0$$
$$\sum_k m(Q_k) \lesssim \frac{1}{\alpha} \|f\|_1 .$$

Let $F = \mathbb{R}^n \setminus \bigcup_k Q_k$. We bound

$$m(Tf > \alpha) \le m\left(Tg > \frac{\alpha}{2}\right) + m\left(Tb > \frac{\alpha}{2}\right).$$

1. For $m(Tg > \frac{\alpha}{2})$ we use the L^q bound for T. First we obtain a L^q bound for g. To do this we use our L^1 bound for g.
$$\begin{aligned}\int |g| &= \int_F |g| + \sum_k \int_{Q_k} |g| \\ &\le \int_F |f| + \alpha \sum_k m(Q_k) \\ &\lesssim \|f\|_1\end{aligned}$$
$$\int |g|^q \lesssim \alpha^{q-1} \|f\|_1$$
$$\implies \int |Tg|^q \lesssim \alpha^{q-1} \|f\|_1$$
$$\implies m\left(Tg > \frac{\alpha}{2}\right) \lesssim \frac{\alpha^{q-1}\|f\|_1}{\alpha^q}.$$
the last step by Chebyshev.

2. For the bad part, we have to get our hands dirty. Let the cube cQ_k have the same center as Q_k but be dilated by c. Let

$$F' = \mathbb{R}^n \backslash \bigcup_k cQ_k.$$

First look at the part bounded away from the cubes.

(a) Claim:

$$m(\{Tb > \frac{\alpha}{2}\} \cap F') \lesssim \frac{1}{\alpha} \|f\|_1 .$$

Proof:

$$\begin{aligned}
|Tb(x)| &= |\sum_k Tb_k(x)| \\
&= \left|\sum_k \int_{Q_k} K(x,y) b_k(y)\, dy\right| \\
(\text{fix } y_k \in Q_k) \quad &\le \sum_k \int_{Q_k} |K(x,y) - K(x,y_k)||b_k(y)|\, dy. \\
&\quad (\text{interchange integrals}) \\
\int_{F'} |Tb(x)| &\le \sum_k \int_{Q_k} |b_k(y)|\, dy \int_{F'} |K(x,y) - K(x,y_k)|\, dx \\
&\quad (\text{used zero-mean condition, take absolute value last}) \\
&\quad x \notin c_n(1+c)Q_k, \quad y \in Q_k \\
\implies |x-y| &\gtrsim \operatorname{diam}(Q_k) \gtrsim |y - y_k| \\
|x-y| &\lesssim \sum_k \int_{Q_k} |b_k(y)|\, dy \\
&\lesssim \alpha \sum_k m(Q_k) \lesssim \|f\|_1 .
\end{aligned}$$

$$m(\{Tb > \frac{\alpha}{2}\} \cap \bigcup_k cQ_k) \lesssim_k m(Q_k) \lesssim \frac{1}{\alpha} \|f\|_1 .$$

Put all these together.

(b) We have to interpolate between a weak $(1,1)$ and strong (p,p) inequality. We use the identity (distribution formula for L_p norm)

$$\text{eq:cutcake} \quad \int |f|^p = p \int_0^\infty \alpha^{p-1} m(|Tf| > \alpha)\, d\alpha. \tag{1}$$

We decompose f into 1 pieces,

$$f = f_\alpha + f^\alpha, \qquad f_\alpha = 1_{|f| \le \alpha} f, \qquad f^\alpha = 1_{|f| > \alpha} f$$

Use the L^p bound to the first and the weak L^1 bound to the second. We have the inequalities

$$m(|g| > \alpha) \lesssim \frac{\|g\|_1}{\alpha}$$
$$m(|g| > \alpha) \lesssim \frac{\|g\|_1^p p}{\alpha^p}$$

useful when $\alpha \lesssim |g|$, $\alpha \gtrsim |g|$, respectively. Applying these 2 inequalities to the 2 parts,

$$\begin{aligned} m(|Tf| > \alpha) &\le m(|Tf_\alpha| > \frac{\alpha}{2}) + m(|Tf^\alpha| > \frac{\alpha}{2}) \\ &\lesssim \frac{1}{\alpha^q} \|Tf_\alpha\|_q^q + \frac{1}{\alpha} \|f^\alpha\|_1 \\ &\lesssim \frac{1}{\alpha^q} \int_{|f|\le\alpha} |f|^q + \frac{1}{\alpha} \int_{|f|>\alpha} |f|. \end{aligned}$$

Finally, plugging into (1)

$$\begin{aligned} \int |Tf|^p &\lesssim_p \int_0^\infty \left(\frac{\alpha^{p-1}}{\alpha^q} \int_{|f|\le\alpha} |f|^q + \frac{\alpha^{p-1}}{\alpha} \int_{|f|>\alpha} |f| \right) d\alpha \\ &= \int \int_{|f|}^\infty \alpha^{p-q-1} |f|^q \, d\alpha \, dx + \int \int_0^{|f|} \alpha^{p-2} |f| \, d\alpha \\ &\lesssim \int |f|^{p-q} |f|^q + \int |f|^{p-1} |f| \, dx \\ &\lesssim \|f\|_p^p. \end{aligned}$$

Bootstrap by getting L^q norm. As long as you have 1 L^q norm you can get all these results. How to get the L^q norm of T in the first case? Use Fourier analytic methods.

Assume $K(x, y) = K(x - y)$. We have $\widehat{f * K} = \widehat{f} * \widehat{K}$ so

$$\|f * K\|_2 = \left\|\widehat{f}\widehat{K}\right\|_2 \le \|f\|_2 \left\|\widehat{K}\right\|_\infty.$$

If $\widehat{K}$ is bounded then $Tf = f * K$ is bounded on L^2. Given the inequality for $1 < p < 2$, we get the inequality for $2 < p < \infty$ by a duality argument. Let $2 < p < \infty$ and p^* be such that $\frac{1}{p} + \frac{1}{p^*} = 1$. Then

$$\begin{aligned} \|Tf\|_p &= \sup_{\|g\|_{p^*}=1} \int Tf(x) g(x) \, dx \\ &= \sup \iint f(y) K(x, y) g(x) \, dy \, dx \\ &= \sup \int f(y) T^* g(y) \\ &\le \sup_{\|g\|_{p^*}=1} \|f\|_p \|g\|_{p^*}. \end{aligned}$$

□