All sets will be connected unless said otherwise.

1 Schwarz Reflection

Schwarz reflection is a way to analytically continue a function.

Theorem 1.1 (Schwarz reflection)**:** thm:schwarz-reflection Suppose Ω is an open set symmetric about the real axis. Let $\Omega^+ = \Omega \cap \{z : \Im z > 0\}$ and similarly for Ω^0, Ω^-. Suppose f^+, f^- are holomorphic on Ω^+ and Ω^-, can be *continuously* extended to Ω^0, and are equal there.

Then the pasting f of f^+, f^- is holomorphic.

Proof. We use Morera's Theorem. By splitting, we just have to consider curves (or triangles) one of whose edges are on the real axis. Use continuity to move the path upwards/downwards slightly. □

The following special case is useful.

Corollary 1.2: cor:schwarz-reflection Let $\Omega, \Omega^\pm$ be as above. Suppose f is holomorphic on Ω^+ and real on Ω^0. Then f can be analytically continued to Ω.

Proof. Let $f^- = \overline{f(\overline{z})}$ in Schwarz reflection 1.1. (We have $\lim_{\varepsilon\to 0} \frac{\overline{f(\overline{z+\varepsilon})} - \overline{f(\overline{z})}}{\varepsilon} = \overline{f'(\overline{z})}$.) □

2 Schwarz's lemma

Let D be the open disc.

Lemma 2.1 (Schwarz)**:** lem:schwarz Let f be a map $D \to D$ with $f(0) = 0$.

If either $f'(0) = 1$ or $\sup_{z\in D} |f(z)| = 1$, then $f(z) = cz$ for some $|c| = 1$.

We can use this to find $\mathrm{Aut}(D)$.

Lemma 2.2: lem:autD The automorphism of D are in the form

$$f(z) = e^{i\theta} \underbrace{\frac{\alpha - z}{1 - \overline{\alpha} z}}_{:=\psi_\alpha}$$

for $\alpha \in D$. Here, $\frac{\alpha - z}{1-\overline{\alpha}z}$ is an involution switching 0, α, and $z \mapsto e^{i\theta} z$ is a rotation by θ.

The automorphisms of $\mathcal{H}$ are in the form $z \mapsto \frac{az+b}{cz+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. We have $\mathrm{Aut}(D) \cong \mathrm{Aut}(\mathcal{H}) \cong \mathrm{SL}_2(\mathbb{R})$.

Proof of Lemma 2.1. By the maximal modulus principle, $\left|\frac{f(z)}{z}\right| \le 1$ inside the disc. (Consider the circle C_r, and let $r \to 1^-$. This is necessary because f many not be defined on ∂D.)

If equality is attained anywhere, then $\frac{f(x)}{x}$ is constant. □

Proof of Lemma 2.2. Suppose $f \in \mathrm{Aut}(D)$ sends α to 0. Then f is the composition of $\frac{\alpha - z}{1-\overline{\alpha}z}$ and an automorphism fixing 0.

To get the automorphisms of $\mathcal{H}$, conjugate by the isomorphism between D and $\mathcal{H}$.

$$\begin{tikzcd} D \arrow[r, bend left, "i\frac{1-z}{1+z}"] & \mathcal{H} \arrow[l, bend left, "\frac{i-z}{i+z}"] \end{tikzcd}$$

The rest is calculation. □

Tip: How to remember/think about the maps?

1. On the boundary, the map $\mathcal{H} \to D$ is given by $\tan\theta \mapsto \cos 2\theta + i \sin 2\theta$. By trig identities, this is $\frac{1-z^2}{1+z^2} + \frac{2zi}{1+z^2} = \frac{i-z}{i+z}$.

3 On complex analytic maps

Theorem 3.1 (Open mapping theorem)**:** thm:omt If f is holomorpic on Ω and $\forall z \in \Omega, f'(z) \neq 0$, then f is open.

Moreover, $f'(z_0) \neq 0$ iff f is injective in some open set around z_0.

Proof. Use Rouché's Theorem on $f(z) \approx f'(z_0)(z - z_0) + f(z_0)$ to show there is exactly 1 solution to $f(z) = y$ for y close to $f(z_0)$. Thus the image of an open is open.

For the second part, use Rouché on $f \approx f(z_0) + a_k(z - z_0)^k$. □

Lemma 3.2: If $f : U \to V$ satisfies $\forall z \in U, f'(z) \neq 0$ and is bijective, then the inverse map is holomorphic.

Proof. We can certainly define f^{-1}. Theorem 3.1 shows it's continuous. The standard calculation shows that $(f^{-1})'(f(z)) = \frac{1}{f'(z)}$, which is defined.

□

Define a biholomorphic map to be $f : U \to V$ that is bijective holomorphic with $\forall z \in U, f'(z) \neq 0$.

4 Normal families

Say a property holds compact-locally if it holds for every compact subset.

Theorem 4.1 (Arzela-Ascoli)**:** thm:arzela-ascoli

1. Suppose $X \subseteq \mathbb{R}^n$ is compact. A closed and bounded equicontinuous family of functions in $C(X)$ is compact. In other words, if $\mathcal{F}$ is an infinite family of pointwise bounded equicontinuous functions in $C(X)$, then any sequence in $\mathcal{F}$ has a uniformly convergent subsequence.

2. The same is true of any open set U, if we ask these conditions hold compact-locally. (Define **normal** to mean every sequence has a subsequence converging compact-locally.)

Proof. See Real Analysis notes. For (2) diagonalize over an exhaustion by compact subsets (e.g., in $\mathbb{R}^n$). $\square$

In the $\mathbb{C}$ world, much looser criteria force $\mathcal{F}$ to be an family of pointwise bounded equicontinuous functions.

Theorem 4.2 (Montel)**:** thm:montel If $\mathcal{F}$ is compact-locally uniformly bounded, then $\mathcal{F}$ is compact-locally equicontinuous and hence normal.

Counterexample for $\mathbb{R}$: $\sin(nx)$.

Proof. One way to show good continuity/limit/boundedness properties in complex analysis is to use the integral representation. Given K, for points $|x-y|<\varepsilon_1$, write $f(x)=\frac{1}{2\pi}\int_{C_{\varepsilon_2}}\frac{f(z)}{z-x}\,dz$ over a circle containing x, y. Now bound the difference $f(x)-f(y)$ uniformly $\to 0$ in terms of $\sup_{z\in K, f\in\mathcal{F}} f$ as $\varepsilon_1 \to 0$.

Now use Arzela-Ascoli 4.1. $\square$

We'll take $\mathcal{F}$ to be an injective family of functions below, so we'll want this to be preserved under limit.

Lemma 4.3: lem:inj-conv (Pr. 8.3.5) If Ω is a connected open subset, f_n are injective, and $f_n \to f$ compact-locally uniformly, then f is injective or constant.

Proof. Idea: we can count the number of solutions of $f_n(z)-w$ using a holomorphic function. By convergence, the number of solutions must stay constant.

Suppose $f(0)=0$ is a problem point, WLOG $f_n(0)=0$. Just use $\frac{1}{2\pi i}\int_\gamma \frac{f_n'}{f_n}\,dz$. $\square$

5 Riemann Mapping Theorem

Theorem 5.1 (Riemann mapping theorem)**:** thm:rmt All proper simply connected open set in $\mathbb{C}$ are isomorphic (complex analytically).

It suffices to prove that if Ω is a simply connected open set then there is a biholomorphism $f:\Omega\to D$.

Proof.

1. Construct an injective map $f:\Omega\to D$ with $f'\neq 0$ and with 0 in the image.
   (a) Find $f_1:\Omega\to\Omega_1$ where Ω_1 avoids an open set around a point. Let $f_1=\ln(x-a)$ where $a\notin\Omega$; this is well-defined because Ω is simply connected. Note $w\in f(\Omega)$ implies $w+2\pi i\notin\overline{f(\Omega)}$. (To see it's not in the closure, note otherwise (becuase f is open) there is a point close by with $w', w'+2\pi i\in f(\Omega)$, contradiction. Alternatively, replace 2 by $2+\varepsilon$ so this argument is unnecessary.)
   (b) Now Ω_1 avoids an open set around β. Let $f_2=\frac{1}{z-\beta}$; $f_2(\Omega_1)$ is bounded in norm.

   (c) Translate and dilate so it fits in the unit disc and includes 0.

2. We can now assume $0 \in \Omega \subseteq D$. Recast the existence problem as a maximization problem. Consider $\mathcal{F} := \{f : \Omega \to D : f \text{ injective}, f' \neq 0\}$, and $s = \sup_{f\in\mathcal{F}} |f'(0)|$. Claim 1: The maximum is attained. Claim 2: f attaining the maximum is an isomorphism $\Omega \to D$.

   Proof of Claim 1: $\mathcal{F}$ is uniformly bounded so normal by Theorem 4.2. Thus there exists a subsequence $f_n \to f$ with $f_n'(0) \to s$. In the real world, this is not enough to say $f'(0) = s$. In the complex world it is, because we can express f_n' again *in terms of* f_n using the integral representation, and just use uniform convergence of the f_n. We have $f \in \mathcal{F}$ by Lemma 4.3 (f can't be constant because $f'(0) \neq 0$.)

3. Proof of Claim 2: Otherwise (if f is not surjective) we exhibit g with larger $|g'(0)|$. Idea: If f is not surjective, write
$$f = \Phi \circ F$$
   where $|\Phi'(0)| < 1$ (proved using Schwarz's lemma 2.1) forcing $|F'(0)| > |f'(0)|$. (Use chain rule.)

   If Φ is a map $D \to D$ that is not injective, it can't be in the form $cz, |c| = 1$, so by Schwarz must have $|\Phi'(0)| < 1$. Thus we let F be a function whose inverse is multivalued on D but that is injective on Ω. Use the square root!

   BWOC, let $\alpha \notin f(\Omega)$. $F = \psi_{\sqrt{\alpha}} \circ \sqrt{} \circ \psi_\alpha \circ f$ and let Φ be the inverse $\psi_{\sqrt{\alpha}} \circ \sqrt{} \circ \psi_\alpha$. This gives $F \in \mathcal{F}$ with $|F'(0)|$ larger, contradiction.

$\square$

The Riemann mapping theorem does not tell us what happens on the boundary of the open sets (if the map can be extended, etc.). We show the map behaves nicely in the case of a polygon.

Theorem 5.2 (Extending to the boundary)**:** thm:extend-boundary Let D, P be open sets with "regular" boundary, i.e., for every point $p \in \partial S$, there is a small open set $U \ni p$ with $U \cap \partial S$ topologically equivalent to a line segment. (Ex. D is a disc, P is a polygon.) Suppose $f : D \to P$ is an isomorphism. Then f can be extended to a continuous bijection function $\overline{D} \to \overline{P}$.

The idea is that if there is > 1 possible value for $\lim_{n\to\infty} f(z_n)$ given $z_n \to z \in \partial D$, then area would not be preserved. There would be z_n, z_n' such that $|z_n - z_n'| \to 0$ but $f(z_n) - f(z_n')$ is large.

Lemma 5.3: lem:area Let $f : U \to V$ be an isomorphism. Then $\text{Area}\, V = \int |f'(z)|^2 \, dx \, dy$.

Proof. Think of f as a map $\mathbb{R}^2 \to \mathbb{R}^2$. The factor is the Jacobian. Calculate it using the Cauchy-Riemann equations. $\square$

Proof of Theorem 5.2. 1. (4.5) First, to prove continuity at $z \in \partial D$, using the triangle inequality, it suffices to consider nearby points $z \in \partial D$ and $z' \in D$.

2. If $z_n \to z$ then by (sequential) compactness of $\overline{P}$, $f(z_n)$ has a limit point. If for each such sequence $f(z_n)$ has only 1 limit point it must be the actual limit. Thus if f does not extend continuously to z, then there are sequences z_n, z_n' such that $f(z_n), f(z_n')$ converge to different values w, w'. This suggests $f(x) - f(y)$ can stay away from 0 even though $x, y \to z$.

   To formalize this, suppose $f(z_n), f(z_n')$ converge to different values. We find continuous curves $\gamma(r), \gamma'(r)$ containing z_n, z_n' (r is distance to z) such that $|\gamma(r) - \gamma'(r)| > \delta$ for r, δ small enough. To do this, choose disjoint open balls around w, w' and take inverse images of curves containing $f(z_n), f(z_n')$ there.

   (4.3) Let $\gamma(r) = z + re^{i\theta_1(r)}$ and similarly for γ'. Let U be the region between γ, γ' from radius 0 to s. We have

$$\begin{aligned}
\int_{\theta_1(r)}^{\theta_2(r)} r|f'(z)|\,dz &= \int_{\gamma(r)}^{\gamma'(r)} |f'(z)|\,dz \ge \delta \\
\infty > \text{Area}(f(U)) &= \iint_U |f'(z)|^2\,dz \\
&= \int_0^s \int_{\theta_1(r)}^{\theta_2(r)} r|f'(z)|^2\,d\theta\,dr && (z = x + re^{i\theta})) \\
&\ge \int_0^s \frac{\left(\int_{\theta_1}^{\theta_2} r|f'(z)|\,d\theta\right)^2}{\int_{\theta_1}^{\theta_2} r\,d\theta}\,dr && \text{C-S} \\
&\ge \int_0^s \frac{c}{r}\,dr = \infty.
\end{aligned}$$

3. Now apply to $P \to D$ as well to get $g = f^{-1}$.

□

6 Schwarz-Christoffel formula

The Riemann Mapping Theorem does not give an explicit formula. We give an explicit formula for $\mathcal{H} \to P$, P a polygon.

How might we create one? The key is knowing how to the straight line $\mathbb{R}$ to a single sharp corner. Then multiplying functions together we can create a series of corners.

To get one corner, consider the function $f_\beta(z) = z^\beta$ which maps $\mathbb{R}$ to the rays of an angle $\beta\pi$. Now we would like a function on $\mathbb{R}$ such that $\arg F$ is a piecewise step function, so that if we integrate it we will move along the sides of a polygon. Just multiply translates of functions f_β and integrate. This motivates the formula below.

Theorem 6.1 (Schwarz-Christoffel): thm:schwarz-christoffel

1. Suppose $\beta_1, \ldots, \beta_k > 0$ with $1 < \sum_k \beta_k \le 2$ and $A_1 < \cdots < A_n$ be real. The following is an analytic map $\mathcal{H} \to P$ for some polygon P.

$$S(z) = \int_{-\infty}^{z} \prod_k (\zeta - A_k)^{-\beta_k}\,d\zeta.$$

P starts at $(0,0)$ in the $+\Re$ directions and its edges going around have exterior angles $\beta_k\pi$ and $\beta_0\pi := (2-\sum_k \beta_k)\pi$. A_kA_{k+1} gets mapped to the edge forming exterior angle $\beta_k\pi$ (absolute angle $\sum_k \beta_k\pi$), where $A_{n+1} = A_0 := \infty$.

2. For any such polygon, there is a isomorphism $\mathcal{H} \to P$ that is given by $c_1S(z) + c_2$ for some c_1, c_2.

Let a_k be the image of A_k. Note that if $\sum_k \beta_k = 2$, then the last exterior angle is 0, i.e. a_∞ is on the segment a_na_1.

Proof. We prove the case where $\sum_k \beta_k = 2$ first.

1. To see how it maps $\mathbb{R}$, note that

$$\arg \frac{d}{dz} = \arg \prod_k (z - A_k)^{-\beta_k} = \sum_{A_k > z} -\beta_k\pi.$$

   (The $-$ is so that the angle increases by $\beta_k\pi$ after passing A_k, and to make things convergent.)

   Note S is well-defined because $\beta_k < 1$, so the integral converges around A_k; $\sum b_k > 1$, so $\lim_{|z|\to\infty} S(z)$ exists (because integrating around a large arc gives something on the order of $r^{-\sum b_k + 1}$).

2. Let f be the isomorphism given by Riemann mapping 5.1, extend it to the boundary by 5.2, and let A_k be the preimages of the vertices (maybe one of them is ∞, in which we will be trying to match f up with S in the case $\sum \beta_k < 2$).

   We will use a Liouville-type argument to show f is given by the (a rotation+translation of the) Schwarz function. We can't hope to apply such an argument to f because f can't extend. We instead apply it to $\frac{f'}{f}$.

   We want a function we can extend by Schwarz reflection. Thus we get rid of the corner. Let $h_k = (f(z) - a_k)^{\frac{1}{\alpha_k}}$. Then h_k sends $A_{k-1}A_{k+1}$ to a straight line. By Schwarz reflection 1.2 we can extend h_k to the strip between A_{k-1} and A_{k+1}. (We are in the case of 1.2 once we map the straight line into the real axis.) (Check $h_k' \neq 0$ by checking it is injective on small balls around real points (This suffices by Theorem 3.1.). Check the latter by examining the Schwarz construction.)

   Computing the derivatives gives an expression for $\frac{f''}{f'}$ in terms of h_k (without all those exponents). (We expect $\frac{f''}{f'} = (\ln f)'$ to be nice because if $f = S(z)$ this would be nice.) Its only pole in the strip is A_k. Since the strips cover $\mathbb{C}$, this analytically continues $\frac{f''}{f'}$.

   In the case of $[A_n, \infty), (-\infty, A_1]$, we can in fact analytically continue f across these, and find its continuation there to be bounded (because P is bounded, and the construction is just reflection), so f is holomorphic at ∞. Expanding in power series gives: if f is holomorphic at ∞ then $\lim_{|z|\to\infty} \frac{f''}{f'} \to 0$.

   Using Liouville, $\frac{f''}{f'} = -\sum \frac{\beta_k}{z - A_k}$. Integrating gives $F' = cS'$.

3. In the case where one of the vertices is mapped from ∞, use a Möbius map to reduce to the above case.

□

7 Examples, Elliptic integrals

Theorem 7.1: The function

$$\int_0^z \frac{1}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}\,d\zeta$$

is a function from $\mathcal{H}$ to a rectangle of side lengths

$$A = 2\int_0^1 \frac{1}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}\,d\zeta$$
$$B = \int_1^{\frac{1}{k}} \frac{1}{\sqrt{-(1-\zeta^2)(1-k^2\zeta^2)}}\,d\zeta.$$

It maps

$$\begin{aligned} \infty &\mapsto -Bi \\ -\frac{1}{k} &\mapsto \frac{A}{2} - Bi \\ -1 &\mapsto \frac{A}{2} \\ 0 &\mapsto 0 \\ 1 &\mapsto -\frac{A}{2} \\ \frac{1}{k} &\mapsto -\frac{A}{2} + Bi. \end{aligned}$$

The inverse function can be meromorphically extended to the whole plane and is elliptic with lattice $\langle 2A, 2Bi\rangle$. In fact it equals $\frac{c}{\wp(z)-\wp(Bi)}$ for some c. Find it!

Note we can also get a version for $\zeta(\zeta-1)(\zeta-\lambda)$. (These 2 forms of an elliptic curve have always puzzled me, but now it's clear: the corresponding functions on $\mathcal{H}$ are related by a $\frac{az+b}{cz+d}$ transformation that takes $-\frac{1}{k}, -1, 1, \frac{1}{k}$ to $1, \lambda, \infty, 0$.)

Proof. This function does what we claim by Schwarz-Christoffel 6.1. To extend the inverse function, use Schwarz reflection. These rectangles tile the plane. After reflecting 2 times we get back to the original function, so it is periodic with respect to the lattice $\langle 2A, 2Bi\rangle$. Let f be the inverse function.

Consider the function $(\wp, \wp') : \mathbb{C} \to E := \{y^2 = (x^2-1)(k^2x^2-1)\}$ (include the point at infinity). We have maps

$$\begin{tikzcd} E \arrow[d, "(x,y)\mapsto x"'] & \\ \mathcal{H} & \mathbb{C} \arrow[l, "f"'] \arrow[ul, "(\wp,\wp')"'] \end{tikzcd}$$

We have $(\wp, \wp')^* \frac{dx}{y} = dz$; restricting to the original domain of f, we find this means $\wp^* \frac{d\zeta}{\sqrt{\cdots}} = dz$. By FTC, $(\int_0^z \frac{1}{\sqrt{\cdots}} d\zeta = \cdots (z))^* dz = \frac{d\zeta}{\sqrt{\cdots}}$. The differentials agree on an open set so must agree everywhere. This tells us that $\wp = f$ up to translation by some constant. But what about the fact the pole changed? $\square$