All sets will be connected unless said otherwise.

1 Schwarz Reflection

Schwarz reflection is a way to analytically continue a function.

Theorem 1.1 (Schwarz reflection)**:** thm:schwarz-reflection Suppose Ω is an open set symmetric about the real axis. Let $\Omega^+ = \Omega \cap \{z : \Im z > 0\}$ and similarly for Ω^0, Ω^-. Suppose f^+, f^- are holomorphic on Ω^+ and Ω^-, can be *continuously* extended to Ω^0, and are equal there.

Then the pasting f of f^+, f^- is holomorphic.

Proof. We use Morera's Theorem. By splitting, we just have to consider curves (or triangles) one of whose edges are on the real axis. Use continuity to move the path upwards/downwards slightly. □

The following special case is useful.

Corollary 1.2: Let $\Omega, \Omega^\pm$ be as above. Suppose f is holomorphic on Ω^+ and real on Ω^0. Then f can be analytically continued to Ω.

Proof. Let $f^- = \overline{f(\overline{z})}$ in Schwarz reflection 1.1. □

2 Schwarz's lemma

Let D be the open disc.

Lemma 2.1 (Schwarz)**:** lem:schwarz Let f be a map $D \to D$ with $f(0) = 0$.

If either $f'(0) = 1$ or $\sup_{z\in D} |f(z)| = 1$, then $f(z) = cz$ for some $|c| = 1$.

We can use this to find $\mathrm{Aut}(D)$.

Lemma 2.2: lem:autD The automorphism of D are in the form

$$f(z) = e^{i\theta} \underbrace{\frac{\alpha - z}{1 - \overline{\alpha}z}}_{:=\psi_\alpha}$$

for $\alpha \in D$. Here, $\frac{\alpha - z}{1-\overline{\alpha}z}$ is an involution switching 0, α, and $z \mapsto e^{i\theta}z$ is a rotation by θ.

The automorphisms of $\mathcal{H}$ are in the form $z \mapsto \frac{az+b}{cz+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. We have $\mathrm{Aut}(D) \cong \mathrm{Aut}(\mathcal{H}) \cong \mathrm{SL}_2(\mathbb{R})$.

Proof of Lemma 2.1. By the maximal modulus principle, $\left|\frac{f(z)}{z}\right| \le 1$ inside the disc. (Consider the circle C_r, and let $r \to 1^-$. This is necessary because f many not be defined on ∂D.)

If equality is attained anywhere, then $\frac{f(x)}{x}$ is constant. □

Proof of Lemma 2.2. Suppose $f \in \mathrm{Aut}(D)$ sends α to 0. Then f is the composition of $\frac{\alpha - z}{1 - \bar{\alpha} z}$ and an automorphism fixing 0.

To get the automorphisms of $\mathcal{H}$, conjugate by the isomorphism between D and $\mathcal{H}$.

$$\begin{tikzcd} D \arrow[r, bend left, "i\frac{1-z}{1+z}"] & \mathcal{H} \arrow[l, bend left, "\frac{i-z}{i+z}"] \end{tikzcd}$$

The rest is calculation. $\square$

Tip: How to remember/think about the maps?

1. On the boundary, the map $\mathcal{H} \to D$ is given by $\tan\theta \mapsto \cos 2\theta + i \sin 2\theta$. By trig identities, this is $\frac{1-z^2}{1+z^2} + \frac{2zi}{1+z^2} = \frac{i-z}{i+z}$.

3 On complex analytic maps

Theorem 3.1 (Open mapping theorem)**:** thm:omt If f is holomorpic on Ω and $\forall z \in \Omega, f'(z) \neq 0$, then f is open.

Moreover, $f'(z_0) \neq 0$ iff f is injective in some open set around z_0.

Proof. Use Rouché's Theorem on $f(z) \approx f'(z_0)(z - z_0) + f(z_0)$ to show there is exactly 1 solution to $f(z) = y$ for y close to $f(z_0)$. Thus the image of an open is open.

For the second part, use Rouché on $f \approx f(z_0) + a_k(z - z_0)^k$. $\square$

Lemma 3.2: If $f : U \to V$ satisfies $\forall z \in U, f'(z) \neq 0$ and is bijective, then the inverse map is holomorphic.

Proof. We can certainly define f^{-1}. Theorem 3.1 shows it's continuous. The standard calculation shows that $(f^{-1})'(f(z)) = \frac{1}{f'(z)}$, which is defined.

$\square$

Define a biholomorphic map to be $f : U \to V$ that is bijective holomorphic with $\forall z \in U, f'(z) \neq 0$.

4 Normal families

Say a property holds compact-locally if it holds for every compact subset.

Theorem 4.1 (Arzela-Ascoli)**:** thm:arzela-ascoli

1. Suppose $X \subseteq \mathbb{R}^n$ is compact. A closed and bounded equicontinuous family of functions in $C(X)$ is compact. In other words, if $\mathcal{F}$ is an infinite family of pointwise bounded equicontinuous functions in $C(X)$, then any sequence in $\mathcal{F}$ has a uniformly convergent subsequence.

2. The same is true of any open set U, if we ask these conditions hold compact-locally. (Define **normal** to mean every sequence has a subsequence converging compact-locally.)

Proof. See Real Analysis notes. For (2) diagonalize over an exhaustion by compact subsets (e.g., in $\mathbb{R}^n$). $\square$

In the $\mathbb{C}$ world, much looser criteria force $\mathcal{F}$ to be an family of pointwise bounded equicontinuous functions.

Theorem 4.2 (Montel)**:** thm:montel If $\mathcal{F}$ is compact-locally uniformly bounded, then $\mathcal{F}$ is compact-locally equicontinuous and hence normal.

Counterexample for $\mathbb{R}$: $\sin(nx)$.

Proof. One way to show good continuity/limit/boundedness properties in complex analysis is to use the integral representation. Given K, for points $|x-y|<\varepsilon_1$, write $f(x) = \frac{1}{2\pi}\int_{C_{\varepsilon_2}} \frac{f(z)}{z-x}\,dz$ over a circle containing x, y. Now bound the difference $f(x)-f(y)$ uniformly $\to 0$ in terms of $\sup_{z\in K, f\in\mathcal{F}} f$ as $\varepsilon_1 \to 0$.

Now use Arzela-Ascoli 4.1. $\square$

We'll take $\mathcal{F}$ to be an injective family of functions below, so we'll want this to be preserved under limit.

Lemma 4.3: lem:inj-conv (Pr. 8.3.5) If Ω is a connected open subset, f_n are injective, and $f_n \to f$ compact-locally uniformly, then f is injective or constant.

Proof. Idea: we can count the number of solutions of $f_n(z)-w$ using a holomorphic function. By convergence, the number of solutions must stay constant.

Suppose $f(0)=0$ is a problem point, WLOG $f_n(0)=0$. Just use $\frac{1}{2\pi i}\int_\gamma \frac{f_n'}{f_n}\,dz$. $\square$

5 Riemann Mapping Theorem

Theorem 5.1: All proper simply connected open set in $\mathbb{C}$ are isomorphic (complex analytically).

It suffices to prove that if Ω is a simply connected open set then there is a biholomorphism $f:\Omega\to D$.

Proof.

1. Construct an injective map $f:\Omega\to D$ with $f'\neq 0$ and with 0 in the image.
   (a) Find $f_1:\Omega\to\Omega_1$ where Ω_1 avoids an open set around a point. Let $f_1 = \ln(x-a)$ where $a\notin\Omega$; this is well-defined because Ω is simply connected. Note $w\in f(\Omega)$ implies $w+2\pi i\notin\overline{f(\Omega)}$. (To see it's not in the closure, note otherwise (becuase f is open) there is a point close by with $w', w'+2\pi i\in f(\Omega)$, contradiction. Alternatively, replace 2 by $2+\varepsilon$ so this argument is unnecessary.)
   (b) Now Ω_1 avoids an open set around β. Let $f_2 = \frac{1}{z-\beta}$; $f_2(\Omega_1)$ is bounded in norm.

(c) Translate and dilate so it fits in the unit disc and includes 0.

2. We can now assume $0 \in \Omega \subseteq D$. Recast the existence problem as a maximization problem. Consider $\mathcal{F} := \{f : \Omega \to D : f \text{ injective}, f' \neq 0\}$, and $s = \sup_{f\in\mathcal{F}} |f'(0)|$. Claim 1: The maximum is attained. Claim 2: f attaining the maximum is an isomorphism $\Omega \to D$.

   Proof of Claim 1: $\mathcal{F}$ is uniformly bounded so normal by Theorem 4.2. Thus there exists a subsequence $f_n \to f$ with $f_n'(0) \to s$. In the real world, this is not enough to say $f'(0) = s$. In the complex world it is, because we can express f_n' again *in terms of* f_n using the integral representation, and just use uniform convergence of the f_n. We have $f \in \mathcal{F}$ by Lemma 4.3 (f can't be constant because $f'(0) \neq 0$.)

3. Proof of Claim 2: Otherwise (if f is not surjective) we exhibit g with larger $|g'(0)|$. Idea: If f is not surjective, write

$$f = \Phi \circ F$$

   where $|\Phi'(0)| < 1$ (proved using Schwarz's lemma 2.1) forcing $|F'(0)| > |f'(0)|$. (Use chain rule.)

   If Φ is a map $D \to D$ that is not injective, it can't be in the form $cz, |c| = 1$, so by Schwarz must have $|\Phi'(0)| < 1$. Thus we let F be a function whose inverse is multivalued on D but that is injective on Ω. Use the square root!

   BWOC, let $\alpha \notin f(\Omega)$. $F = \psi_{\sqrt{\alpha}} \circ \surd \circ \psi_\alpha \circ f$ and let Φ be the inverse $\psi_{\sqrt{\alpha}} \circ \surd \circ \psi_\alpha$. This gives $F \in \mathcal{F}$ with $|F'(0)|$ larger, contradiction.

$\square$

6 Schwarz-Christoffel formula