18.997 Probabilistic Method Problem Set #1

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Problem 1 *(1.1)*

(A)
Consider a complete graph with n vertices. Call the first color red and the second blue. Color each edge in the graph red with probability p and blue with probability $1-p$. The probability that a given set of k vertices forms a red K_k is $p^{\binom{k}{2}}$ and the probability that a given set of t vertices forms a blue K_t is $(1-p)^{\binom{t}{2}}$. There are $\binom{n}{k}$ groups of k vertices and $\binom{n}{t}$ groups of t vertices. Hence by the union bound the probability that there is a red K_k or blue K_t is

$$P(\text{there is a red } K_k \text{ or } K_t) \le \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1.$$

Hence there exists a coloring of K_n such that there is no red K_k and no blue K_t, giving $r(k,t) > n$.

(B)
Suppose $t \ge 2$ and $n \ge \frac{ct^{3/2}}{(\ln t)^{3/2}}$, where c is a constant to be chosen. Let

$$p = \frac{\ln t}{t-1} = \frac{\frac{t}{2}\ln t}{\binom{t}{2}}.$$

Then

$$\begin{aligned}
\binom{n}{4}p^6 + \binom{n}{t}(1-p)^{\binom{t}{2}} &\le \binom{n}{4}p^6 + \binom{n}{t}e^{-p\binom{t}{2}} \\
&\le \frac{n^4}{24}\left(\frac{\ln t}{t-1}\right)^6 + \frac{n^t}{t!}t^{-t/2} \\
&\le \frac{c^4}{24}\frac{t^6}{(\ln t)^6}\left(\frac{\ln t}{t-1}\right)^6 + \frac{c^t t^{3t/2}}{(\ln t)^{3t/2}}\frac{1}{t!}t^{-t/2} \\
&\sim \frac{c^4}{24} + \frac{c^t t^{3t/2}}{(\ln t)^{3t/2}}\frac{e^t}{t^t\sqrt{2\pi t}}t^{-t/2} \\
&\sim \frac{c^4}{24} + \left(\frac{ce}{(\ln t)^{3/2}}\right)^t \frac{1}{\sqrt{2\pi t}} && \text{as } t \to \infty.
\end{aligned}$$

The latter term goes to 0 as $t \to \infty$, and the left term is constant. Thus choosing $c > 0$ so that $c^4 < 24$, we find that $\binom{n}{4}p^6 + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$ for large n, and hence that $r(4,n) \geq \frac{ct^{3/2}}{(\ln t)^{3/2}}$ for sufficiently large n, as needed.

Problem 2 *(1.2)*

Color each vertex of H with one of the four colors, independently with probability $\frac{1}{4}$. Given an edge in H, the probability that none of its n incident vertices are colored with color i is $\frac{3^n}{4^n}$ (for fixed $i = 1, 2, 3$, or 4). Hence the probability that its vertices are colored with at most three colors is less than $4 \cdot \frac{3^n}{4^n}$. (Strict inequality holds because we overcount the probability in the cases where they are colored in at most 2 colors.) The probability that some edge has its vertices colored with at most three colors is

$$P < 4 \cdot \frac{3^n}{4^n} \cdot |E| \leq 4 \cdot \frac{3^n}{4^n} \cdot \frac{4^{n-1}}{3^n} \leq 1.$$

Hence there exists a coloring such that in every edge all four colors are represented.

Problem 3 *(1.4)*

Fix $p \in [0,1]$. Pick randomly and independently each vertex with probability p. Let X be the set of picked vertices. Let Y be the set in vertices in $V - X$ with no neighbors in X, and let $X' = X \cup Y$. Let Z be the set of vertices in $V - X'$ all of whose neighbors are in X'. Let $A = X' \cup Z$.

We show that $A, B = V - A$ works—i.e. every vertex in B is adjacent to some vertex of A and some vertex of B.

A vertex in B cannot have neighbors in Z because the vertices in Z have only vertices in X' as neighbors. A vertex in B cannot only have neighbors in X' because then it would be in Z instead. Hence a vertex in B cannot only have neighbors in A.

A vertex with only neighbors in B *a fortiori* only has neighbors in $V - X$, so must be in Y, and hence is not in B. This proves our claim.

The probability that a vertex is in Y is (since there is probability $1 - p$ that a given vertex is in $V - X$; we care about the vertex and its neighbors)

$$P(v \in Y) = (1-p)^{\deg(v)+1} \leq (1-p)^{\delta+1}.$$

Now we estimate the probability that a vertex is in Z. Now $v \in Z$ means that $v \in B - X$ and v is only adjacent to vertices in $X' = X \cup Y$. Take any vertex w adjacent to v. Now $w \in X$ with probability p and $w \in Y$ with probability $(1-p)^{\deg(w)+1} \leq (1-p)^{\delta+1}$ as above. Hence

$$P(v \in Z) \leq p + (1-p)^{\delta+1}.$$

Hence by linearity of expectation, for any vertex v,

$$\begin{aligned}\mathbb{E}(|A|) &= \mathbb{E}(|X|) + \mathbb{E}(|Y|) + \mathbb{E}(|Z|)\\ &= n(P(v \in X) + P(v \in Y) + P(v \in Z))\\ &\le n(p + (1-p)^{\delta+1} + (p + (1-p)^{\delta+1}))\\ &= 2n(p + (1-p)^{\delta+1})\\ &\le 2n(p + e^{-p(\delta+1)}).\end{aligned}$$

Putting in $p = \frac{\ln(\delta+1)}{\delta+1}$, we get

$$E(|A|) \le 2n\frac{\ln(\delta+1)+1}{\delta+1} = O\left(\frac{\ln\delta}{\delta}\right).$$

Problem 4 *(1.6)*

Lemma 4.1: Let G be a tournament with n vertices. Let $S_v = \{v\} \cup \{w | v \text{ dominates } w\}$. There exists a vertex v such that $|S_v| > \frac{n}{2}$.

Proof. If we sum the outdegree of each vertex, we count all the edges once:

$$\sum_{v \text{ vertex}} \text{out}(v) = \frac{n(n-1)}{2}.$$

Hence there exists v such that $\text{out}(v) \ge \frac{n-1}{2}$, and $|S_v| = \text{out}(v) + 1 \ge \frac{n+1}{2}$. □

Suppose G has less than $n = \frac{1}{2} \cdot k2^k$ vertices. We show that G has a dominating set of k vertices. Suppose by way of contradiction that it does not.

Let V be the set of vertices. We pick v_1 so that $|S_{v_1}| > \frac{n}{2}$. Now given $v_1, \ldots, v_i$, $(i < k-1)$, let $W_i = V - \bigcup_{j=1}^{i} S_{v_j}$. Given that

$$m := |W_i| < \frac{n}{2^i},$$

we choose v_{i+1} inductively as follows. Consider the induced subgraph with vertex set W_i. It has m vertices, so by the lemma we can choose v_{i+1} among these vertices so that $|S_{v_{i+1}} \cap W_i| > \frac{m}{2}$. Then

$$|W_{i+1}| = |W_i - S_{v_{i+1}}| < \frac{m}{2} < \frac{n}{2^{i+1}}.$$

Hence we can choose $v_1, \ldots, v_k$ so that

$$|W_{k-1}| < \frac{n}{2^{k-1}} = k.$$

Since W_{k-1} has less than k vertices, by assumption there exists a vertex v_k such that v_k dominates W_{k-1}. Then $\{v_1, \ldots, v_k\}$ is a dominating set of k vertices (since $\bigcup_{j=1}^{k} S_{v_j} = V$), a contradiction.

Hence a tournament with no dominating k-set contains at least $\frac{1}{2}k2^k$ vertices.

Problem 5 *(1.8)*

Let W be a random infinite binary string, where each digit is equal to 0 or 1 with (independent) probability $\frac{1}{2}$. For a binary string S, let $l(S)$ denote the length of S. Then the probability that W starts with $l(S)$ is $\frac{1}{2^{l(S)}}$. Now, the fact that no two member of F is a prefix of another one means that the events "W starts with S" and "W starts with T," for distinct $S, T \in F$, are disjoint. Hence the probability that W starts with some string in F is

$$\begin{aligned} P(W \text{ starts with some } S \in F) &= \sum_{S \in F} P(W \text{ starts with } S \in F) \\ &= \sum_{S \in F} \frac{1}{2^l(S)} \\ &= \sum_{i=1}^{\infty} \sum_{S \in F, l(S)=i} \frac{1}{2^i} \\ &= \sum_{i=1}^{\infty} \frac{N_i}{2^i}. \end{aligned}$$

Since this is a probability, it is at most 1, as needed.

Problem 6 *(1.10)*

Fix l with $1 \leq l \leq n$. Take a permutation of the rows at random, with each permutation having $\frac{1}{n!}$ probability of being chosen. Fix a column C; take a subset S of numbers from that column with l elements. The probability that those numbers are in order down the column in the permuted matrix is $\frac{1}{l!}$ since all orderings of those numbers are equally likely. There are $\binom{n}{l}$ subsets of l numbers, so

$$P(C \text{ has a increasing sequence of length } l) \leq \binom{n}{l} \frac{1}{l!}.$$

There are n columns, so

$$P(\text{Some column has a increasing sequence of length } l) \leq \binom{n}{l} \frac{n}{l!}.$$

If this is less than 1, then there exists a permutation with no column containing an increasing sequence of length l.

Let $l = \lceil c\sqrt{n}\rceil$ where $c > e$. Note, using Stirling's formula for $n \to \infty$,

$$
\begin{aligned}
\binom{n}{l}\frac{n}{l!} &= \frac{n!n}{l!^2(n-l)!} \\
&= \Theta\left(\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n n^2}{2\pi c\sqrt{n}\left(\frac{c^2 n}{e^2}\right)^{c\sqrt{n}} \cdot \sqrt{2\pi(n-c\sqrt{n})}\left(\frac{n-c\sqrt{n}}{e}\right)^{n-c\sqrt{n}}}\right) \\
&= \Theta\left(\frac{n^{n-c\sqrt{n}+2}}{(n-c\sqrt{n})^{n-c\sqrt{n}+\frac{1}{2}}}\frac{e^{c\sqrt{n}}}{c^{1+2c\sqrt{n}}}\right) \\
&= \Theta\left(\left[\left(\frac{n}{n-c\sqrt{n}}\right)^{\sqrt{n}-c} \cdot \frac{e^c}{c^{2c}}\right]^{\sqrt{n}} n^{3/2}\right) \\
&= \Theta\left(\left[\frac{e^c n^{\frac{3/2}{\sqrt{n}}}}{c^{2c}\left(1-\frac{c}{\sqrt{n}}\right)^{\sqrt{n}-c}}\right]^{\sqrt{n}}\right).
\end{aligned}
$$

Note in the second line we used $(n-l)! = (n-l+1)!/(n-l+1)$, which is asymptotically at least Stirling's formula for $n - c\sqrt{n}$, divided by n.

Now note

$$\lim_{n\to\infty}\left(1-\frac{c}{\sqrt{n}}\right)^{\sqrt{n}-c} = \lim_{x\to\infty}\left(1-\frac{c}{x}\right)^x\left(1-\frac{c}{x}\right)^{-c} = e^{-c}$$

and

$$\lim_{n\to\infty} n^{\frac{3/2}{\sqrt{n}}} = \lim_{x\to\infty} x^{\frac{3}{x}} = e^{\lim_{x\to\infty}\frac{3\ln x}{x}} = e^0 = 1.$$

Hence

$$\lim_{n\to\infty}\frac{e^c n^{\frac{3/2}{\sqrt{n}}}}{c^{2c}\left(1-\frac{c}{\sqrt{n}}\right)^{\sqrt{n}-c}} = \left(\frac{e}{c}\right)^{2c} < 1.$$

Thus for $c > e$, $\binom{n}{l}\frac{n}{l!} \to 0$ as $n \to \infty$. We can find L so that $\binom{n}{l}\frac{n}{l!} < 1$ whenever $n > L$, so

$$P(\text{Some column has a increasing sequence of length } l = \lceil c\sqrt{n}\rceil) < 1 \tag{1}$$

for $n > L$. Now choose a larger value c' instead of c as necessary so that this holds for all n, for example, take $c' = \min(c, \sqrt{L}+1)$ (so that for $n \le L$, the probability above is trivially 0). Then there must exist a permutation so that no column has an increasing sequence of length $l \ge c'\sqrt{n}$.