Distribution Theory and Applications

Part III Course

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Introduction

Andrew Ashton (acla2@damtp.cam.ac.uk) taught a course on Distribution Theory and Applications at the University of Cambridge in Lent (Winter) 2014. Example sheets are available at `http://damtp.cam.ac.uk/user/examples/indexP3.html`. These are my "live-TeXed" notes from the course.

Book recommendations (low-tech to high-tech):

1. Introduction to Fourier Analysis and Generalized Functions, Lighthill, [3]
2. Introduction to the Theory of Distributions, Friedlander, Joshi, [1]
3. The Analysis of Linear Partial Differential Operators, Lars Hörmander, [2]

Chapter 0

Motivation

1 Dirac delta function

What is the derivative of the Dirac delta?

You may have seen the mysterious "**Dirac delta function**" defined by

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0), \qquad \delta(x) = 0,\ x \neq 0. \tag{1}$$

eq:dist0-1

This emerged from Fourier's classical treatise on heat. It was there implicitly in his work. Cauchy and Dirac noticed it. It is used in math, applied math, physics, engineering.

But there is no function in any sense of the word that does this job! It makes no mathematical sense!

However, looking at (1) in an abstract sense, the "process" which takes $\delta(x - x_0)$ and the "nice" function $f(x)$, and spits out $f(x_0)$ is well-defined.

However, people don't just talk about the delta function, they also talk about its derivative! Trying to differentiate something that doesn't exist...?

How can we define the derivative $\delta'(x - x_0)$? A first attempt might be (assuming f is nice)

$$\begin{aligned}
\int \delta'(x - x_0) f(x)\, dx &= \lim_{h \to 0} \int \left(\frac{\delta(x - x_0 + h) - \delta(x - x_0)}{h} \right) f(x)\, dx \\
&= \lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{h} \\
&= -f'(x_0) \\
&= -\int \delta(x - x_0) f'(x)\, dx. \qquad (2)
\end{aligned}$$

eq:dist0-2

The equality (2) suggests that we could have simply integrated by parts

$$\int \delta'(x - x_0) f(x)\, dx = -\int \delta(x - x_0) f'(x)\, dx + \underbrace{\text{boundary term}}_{0}.$$

This function that doesn't exist seems to follow the usual rules of calculus! This suggests there is a way of interpreting all the integrals in a consistent way We can make all this rigorous using the theory of distributions.

2 Fourier transforms of polynomials

The **Fourier transform** is defined by (abbreviate $\int := \int_0^\infty$)

$$\mathcal{F} : f \mapsto \widehat{f}(x) = \int e^{-i\lambda x} f(x)\, dx.$$

This makes sense if f is absolutely integrable:

$$\left| \int e^{-i\lambda x} f(x) \right| \le \int |e^{-i\lambda x} f(x)|\, dx = \int |f(x)|\, dx < \infty.$$

What if $f \notin L^1$, in particular, what if $|f| \not\to 0$ as $|x| \to \infty$? You may have seen

$$\text{eq:dist0-3}\;\delta(\lambda) = \frac{1}{2\pi} \int e^{-i\lambda x}\, dx. \tag{3}$$

There is a way of interpreting this so that it makes sense. This seems to suggest that the Fourier transform of $\frac{1}{2\pi}$ is equal to $\delta(\lambda)$. What if $f(x) = x^n$? Can we take the Fourier Transform?

$$\begin{aligned} \int e^{-i\lambda x} x^n\, dx &= \int \left(i \frac{\partial}{\partial \lambda} \right)^n e^{-i\lambda x}\, dx \\ &= \left(i \frac{\partial}{\partial \lambda} \right)^n \int e^{-i\lambda x}\, dx \\ &= 2\pi e^{-i\pi n/2} \delta^{(n)}(\lambda). \end{aligned}$$

If we can make sense on the derivatives of δ, then we can define the Fourier transform of polynomials.

An alternative way of defining Fourier transform of $f(x) = x^n$ would be to use Parseval's Theorem, which states

$$\text{eq:dist0-4}\;\int f(x) \widehat{g}(x)\, dx = \int \widehat{f}(\lambda) g(\lambda)\, d\lambda \tag{4}$$

for all "nice" functions f and g. We could define $\hat{f}(\lambda)$ where $f(x) = x^n$, as the function for which

$$\text{eq:dist0-5}\;\int x^n \hat{g}(x)\, dx = \int \hat{f}(\lambda) g(\lambda)\, d\lambda \text{ for all nice functions } g \tag{5}$$

we could say that $\hat{f}(\lambda)$ is the Fourier transform of $f(x) = x^n$. Note $\hat{g}(x)$ decays quickly, so this makes sense. This can be done rigorously using the theory of distributions.

"Everything's easy when you know the answer." It's only perfectly natural when you've been shown its perfectly natural. To prove consistency is not quite so easy.

3 Discontinuities to Differential Equations

There are some important genuinely important physical scenarios in which we would like a solution to a PDE to have discontinuities. For example, in acoustics we want the pressure $p(x,t)$ to solve the wave equation

$$\text{eq:dist0-6}\; p_{xx} - p_{tt} = 0, \tag{6}$$

but for p to jump either side of a shock wave.

Is there any meaningful way to say that the function

$$u(x,y) = \alpha(x-y) + \beta(x+y)$$

is a solution to the wave equation $u_{xx} - u_{yy} = 0$ if $\alpha, \beta \notin C^2$? Assume $\alpha, \beta \in C^2$ and $u_{xx} - u_{yy} = 0$ so that for any nice function $f(x,y)$ (say $f = 0$ when $x^2 + y^2$ is sufficiently large),

$$\begin{aligned} 0 &= \int\int f(x,y)\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) dx\, dy \\ &= \int\int u(x,y)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) dx\, dy \text{integration by parts twice.} \end{aligned}$$

If we can find $u(x,y)$ such that

$$\text{eq:dist0-7}\; \int\int u(x,y)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) dx\, dy = 0 \text{ for all nice functions } f, \tag{7}$$

we say that $u = u(x,y)$ is a **weak solution** to the PDE $u_{xx} - u_{yy} = 0$. We can use distribution theory to study weak solutions to PDE's.

4 Summary

In each motivating example we introduced a family of "nice" functions that allowed us to extend classical definitions to a wider class of problems. This is the underlying idea in distribution theory. Given a vector space V of "nice" functions we define the distributions on V to be the topological dual V^*, which consists of all the continuous linear forms $V \to \mathbb{C}$.

For example, if $V = C(\mathbb{R})$ then we can define Dirac delta by the linear form[1]

$$\text{eq:dist0-8}\; \langle \delta_{x_0}, f \rangle = f(x_0). \tag{8}$$

In general, any $u \in V^*$ is linear, so $\langle u, \alpha f + \beta g \rangle = \alpha \langle u, f \rangle + \beta \langle u, g \rangle$ for arbitrary constants α, β and $f, g \in V$. We need functional analysis because we require continuity. (The algebraic dual is too big to be interesting. Hence we supplement V with a topology, i.e. a notion of convergence $f_n \to V$ in V. This is the motion of w^*-convergence, $\langle u, f_n \rangle \to \langle u, f \rangle$.

Lecture 2

[1] You may be more familiar with the notation $\langle x, y \rangle = x \cdot y$ for finite-dimensional vector spaces.

Chapter 1

Distributions

Recap:

- Delta function doesn't make sense.
- Way to define distributions is to first define a "nice" space of functions V (having all the properties we want it to have) and define distributions as continuous linear maps from V to $\mathbb{C}$.

We'll always work with continuous functions, so we can define continuity of functions $V \to \mathbb{C}$ very explicitly.

1 Notation and preliminaries

An element of $\mathbb{R}^n$ will be written $x, y, z, \ldots$ so that

$$x = (x_1, x_2, \ldots, x_n)$$

and we will use $dx = dx_1\, dx_2\, \cdots\, dx_n$ to denote the volume element in $\mathbb{R}^n$. Capitals X, Y, Z will always denote open subsets of $\mathbb{R}^n$ and K will always denote a compact (closed and bounded) subset of $\mathbb{R}^n$. Integrals over all $\mathbb{R}^n$ or over $X \subseteq \mathbb{R}^n$ will be denoted by $\int [\cdot]\, dx$ and $\int_X [\cdot]\, dx$, respectively. We will use multi-index notation α, β, γ (Greek letters) will denote multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n = \{0, 1, 2, 3, \ldots\}^{\times n}$. Multi-index notation reads as follows.

$$\begin{aligned} &\partial^\alpha \equiv \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, &\qquad &x^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n} \\ &\alpha! \equiv \alpha_1! \cdots \alpha_n! &\qquad &|\alpha| \equiv \alpha_1 + \cdots + \alpha_n. \end{aligned}$$

We will often write ∂_x^α to make it clear what we're differentiating with respect to. We will also use $D = -i\partial$ when we do Fourier analysis. Define the **support** of a function f by

$$\mathrm{Supp}(f) = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

We will often want to take limits inside integrals. To do this we refer to the dominated convergence theorem: (See for instance https://dl.dropboxusercontent.com/u/27883775/math%20notes/18.125.pdf, Theorem 15.1.)

Theorem 1.1.1 (Dominated convergence theorem)**:** thm:dct Given a sequence of absolutely integrable functions $\{f_m\}_{m\geq 1}$ such that $f_m(x) \to f(x)$ for each x and $|f_m(x)| \leq g(x)$, where g is absolutely integrable, then

$$\lim_{m\to\infty}\int_X f_m(x) = \int_X \left[\lim_{m\to\infty} f_m(x)\right]\, dx = \int_X f(x)\, dx.$$

If f is absolutely integrable on X, i.e.,

$$\int_X |f|\, dx < \infty,$$

we say that $f \in L^1(X)$.

2 Test functions and distributions

We need to define our first vector space of test functions.

Definition 1.2.1: The space $D(X)$ consists of all the smooth functions from X to $\mathbb{C}$ that have compact support. We say that a sequence $\{\varphi_m\}_{m\geq 1}$ tends to zero in $D(X)$ if there exists some compact set $K \subseteq X$ such that $\mathrm{Supp}(\varphi_m) \subseteq K$ and $\partial^\alpha \varphi_m \to 0$ uniformly for each multi-index α.

The space $D(X)$ is often written $C_c^\infty(X)$.

(For convergence, the function is not allowed to have its mass moving away to infinity.)

Since the functions in $D(X)$ vanish at the boundary of X, we have

$$\int_X \varphi \partial^\alpha \psi\, dx = (-1)^\alpha \int_X \psi \partial^\alpha \varphi\, dx, \qquad \varphi, \psi \in D(X)$$

by integration by parts $|\alpha|$ times. We have Taylor's Theorem to any order

$$\text{eq:taylor}\,\varphi(x+h) = \sum_{|\alpha|\subseteq N} \frac{h^\alpha}{\alpha!}\partial^\alpha \varphi(x) + R_N(x,h) \tag{1.1}$$

where the remainder is $o(|h|^N)$ uniformly in x.

Definition 1.2.2: df:distribution A linear map $u : D(X) \to \mathbb{C}$ is a **distribution** (on X) if for each compact $K \subseteq X$, there exist C, N such that

$$\text{eq:dist1-1}\,|\langle u, \varphi\rangle| \leq C \sum_{|\alpha|\leq N} \sup |\partial^\alpha \varphi| \tag{1.2}$$

for each $\varphi \in D(X)$ with $\mathrm{Supp}(\varphi)$ with $\mathrm{Supp}(\varphi) \subseteq K$. The space of all such maps is denoted $D'(X)$. If we can use the same N for all $\varphi \in D(X)$, we call the least such N the order of u, denoted $\mathrm{ord}(u)$.

Remark: Thinking of $D(X)$ as a locally convex space (in fact, Fréchet space) with seminorms defined by $\sup|\partial^\alpha \varphi|$, we have $D'(X) = D(X)^*$. See Example 2.2.4 in Functional Analysis[1]. The example there is actually a larger space ($\mathcal{E}(X)$ of the next chapter), but it contains our $D(X)$.

Example 1.2.3: 1. We check that the Dirac delta δ_x is a distribution. δ_x is defined by

$$\langle \delta_x, \varphi \rangle = \varphi(x) \qquad \left[\int \delta(x-y)\varphi(y)\, dy = \varphi(x)\right].$$

We want to check if (1.2) holds:

$$|\langle \delta_x, \varphi\rangle| = |\varphi(x)| \leq \sup|\varphi|$$

so (1.2) holds with $C = 1$, $N = 0$, no matter what φ we choose. So δ_x is a distribution of order 0.

2. Here is a more useful example. Consider the linear map T on $D(X)$ defined by

$$\langle T, \varphi \rangle := \sum_{|\alpha| \leq M} \int_X f_\alpha \partial^\alpha \varphi \, dx,$$

$f_\alpha \in C(X)$. Now for $\varphi \in D(X)$, with $\mathrm{Supp}(\varphi) \subseteq K$. By definition of T,

$$\begin{aligned}
|\langle T, \varphi\rangle| &= \left| \sum_{|\alpha|\leq M} \int_K f_\alpha \partial^\alpha \varphi \, dx \right| \\
&\leq \sum_{|\alpha| \leq M} \int_K |f_\alpha||\partial^\alpha \varphi| \, dx \\
&\leq \sum_{|\alpha|\leq M} \sup_{|\alpha|\leq M} |\partial^\alpha \varphi| \int_K |f_\alpha| \, dx \\
&\leq \left(\max_\alpha \int_K |f_\alpha| \, dx\right) \sum_{|\alpha|\leq M} \sup|\partial^\alpha\varphi|
\end{aligned}$$

Note that the test functions have compact support, so it doesn't matter if f_α blows up at the boundary. So we have estimate (1.2) with

$$C = \max_{|\alpha|\leq M} \int_K |f_\alpha| \, dx, \qquad N = M.$$

Note that $C = C_K$. We could have done this only assuming that $\{f_\alpha\}$ were locally integrable on X (integrable on every compact subset of X), written $f_\alpha \in L^1_{\mathrm{loc}}(X)$.

Note here the constant C depends on the support test function. N can also depend on it.

[1] `https://dl.dropboxusercontent.com/u/27883775/math%20notes/part_iii_functional.pdf`

Lemma 1.2.4: lem:dist1-1 A linear map $u : D(X) \to \mathbb{C}$ belongs to $D'(X)$ if $\langle u, \varphi_m \rangle \to 0$ for every sequence $\{\varphi_m\}_{m \geq 1}$ in $D(X)$ that tends to 0.

Lecture 3 (27 Jan)

Remark: That Definition 1.2.2 and Lemma 1.2.4 are equivalent conditions for continuity is a special case of Lemma 2.2.5 in the functional analysis notes.

Proof.

- $\implies$: If $u \in D(X)$ and $\varphi_m \to 0$ in $D(X)$ then

$$|\langle u, \varphi_m \rangle| \leq \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi_m| \to 0.$$

- $\Leftarrow$: Assume not, so there exists a compact set $K \subseteq X$ such that estimate (1.2) does not hold for any C, N. In particular, it doesn't hold for $C = N = m$. So there exist $\phi_m \in D(X)$ such that $|\langle u, \phi_m \rangle| > m \sum_{|\alpha| \leq m} \sup |\partial^\alpha \phi_m|$. WLOG, we can assume that $\langle u, \phi_m \rangle = 1$, by setting $\widetilde{\phi_m} = \frac{\phi_m}{\langle u, \phi_m \rangle}$. This implies

$$\begin{aligned} \implies \sum_{|\alpha| \leq m} \sup |\partial^\alpha \phi_m| &< \frac{1}{m} \\ \implies \sup |\partial^\alpha \phi_m| &< \frac{1}{m}, \qquad |\alpha| \leq m \\ \implies \phi_m &\to 0 \text{ in } D(X). \end{aligned}$$

But this is a contradiction.

□

3 Limits in $D'(X)$

Often we have sequence of distributions $\{u_m\}_{m \geq 1}$. If there exist some $u \in D'(X)$ such that $\langle u_m, \varphi \rangle \to \langle u, \varphi \rangle$ for all $\varphi \in D(X)$, then we say that $u_m \to u$ in $D'(X)$.

Limits in $D'(X)$ often look strange.

Example 1.3.1: For instance, define the distribution $u_m \in D'(\mathbb{R})$ by the locally integrable function

$$u_m(X) := \sin(mx).$$

Then $u_m \to 0$ in $D'(\mathbb{R})$.

Proof: We have using integration by parts

$$\begin{aligned} \langle u_m, \varphi \rangle &= \int \sin(mx) \varphi(x)\, dx \\ &= \frac{1}{m} \int \cos(mx) \varphi'(x)\, dx \end{aligned}$$

Hence $u_m \to 0$ in $D'(\mathbb{R})$.

Theorem 1.3.2 (Closure under pointwise convergence)**:** thm:dist1-1 If $u_m \in D'(X)$ is such that $\lim_{m\to\infty} \langle u_m, \varphi \rangle$ exists for every $\varphi \in D(X)$, then the linear map

$$\langle u, \varphi \rangle := \lim_{m\to\infty} \langle u_m, \varphi \rangle$$

is an element of $D'(X)$.

It's obvious that the LHS will satisfy the estimate or the definition. Use the principle of uniform boundedness (See the Banach-Steinhaus Theorem, 4.2.7 in FA notes).

4 Basic operations

4.1 Differentiation and multiplication by smooth functions

If $u \in C^\infty(X)$, then $\partial^\alpha u$ defines an element of $D'(X)$ for every multi-index α by

$$\begin{aligned}
\langle \partial^\alpha u, \varphi \rangle &= \int_X \partial^\alpha u \varphi \, dx \\
&= (-1)^{|\alpha|} \int_X u \partial^\alpha \varphi \, dx && \text{integration by parts} \\
&= \left\langle u, (-1)^{|\alpha|} \partial^\alpha \varphi \right\rangle .
\end{aligned}$$

The RHS is well-defined for any $u \in D'(X)$. We arrive at the following.

Definition 1.4.1: df:dist1-3 For $f \in C^\infty(X)$, $u \in D'(X)$ and any multi-index α we define $\partial^\alpha(fu)$ by

$$\langle \partial^\alpha (fu), \varphi \rangle := \left\langle u, (-1)^{|\alpha|} f \partial^\alpha \varphi \right\rangle .$$

We call $\partial^\alpha u$ the **distributional derivatives** of u.

Example 1.4.2: Take the Dirac delta δ_x. Then $\partial^\alpha \delta_x$ is defined by

$$\langle \partial^\alpha \delta_x, \varphi \rangle := \left\langle \delta_x, (-1)^{|\alpha|} \partial^\alpha \varphi \right\rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(x).$$

Consider the Heaviside function

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0. \end{cases}$$

Then this defines an element of $D'(\mathbb{R})$. We compute H':

$$\langle H', \varphi \rangle := \langle H, -\varphi' \rangle = \int_0^\infty -\varphi'(x) \, dx = -\varphi|_0^\infty = \varphi(0) = \langle \delta_0, \varphi \rangle .$$

Hence $H' = \delta_0$. In general if $\langle u, \varphi \rangle = \langle v, \varphi \rangle$ for all $\varphi \in D(X)$ then we say $u = v$ in $D'(X)$.

Now we ask: how does the calculus for normal functions carries over to calculus for distributions?

Lemma 1.4.3: If $u' = 0$ in $D'(\mathbb{R})$ then u is a constant.

Proof. Note that $u' = 0$ means

$$0 = \langle u', \psi \rangle = -\langle u, \psi' \rangle .$$

Idea: we'd like to say that given φ, we can write $\varphi = \frac{d}{dx} \int_{-\infty}^{x} \varphi(y)\, dy = \psi'$, and use the above to conclude $0 = \langle u, \varphi \rangle$, so u is constant. The problem is that when we integrate a test function, we don't necessarily get a test function. We need to adjust our function so that the integral is 0 for large x.

Fix $\theta \in D(\mathbb{R})$ with $\langle 1, \theta \rangle = \int \theta\, dx = 1$. For arbitrary $\varphi \in D(\mathbb{R})$ write

$$\begin{aligned} \varphi &= (\varphi - \langle 1, \varphi \rangle \theta) + \langle 1, \varphi \rangle \theta \\ &\equiv \varphi_A + \varphi_B \end{aligned}$$

Then

$$\langle 1, \varphi_A \rangle = \langle 1, \varphi \rangle - \langle 1, \varphi \rangle \langle 1, \theta \rangle = 0.$$

This is helpful because

$$\psi_A(x) = \int_{-\infty}^{x} \varphi_A(y)\, dy \in D(\mathbb{R}).$$

We have $\varphi_A = \psi_A'$. So

$$\begin{aligned} \langle u, \varphi \rangle &= \langle u, \varphi_A \rangle + \langle u, \varphi_B \rangle \\ &= \langle u, \psi_A' \rangle + \langle 1, \varphi \rangle \underbrace{\langle u, \theta \rangle}_{b} = 0 + c \langle 1, \varphi \rangle = \langle c, \varphi \rangle . \end{aligned}$$

This implies that u is a constant. □

4.2 Reflection and translation

Definition 1.4.4: For $\varphi \in D(\mathbb{R}^n)$ then we can define its **translation** by $h \in \mathbb{R}^n$ by

$$(\tau_h \varphi)(x) := \varphi(x - h)$$

and **reflection**

$$\check{\varphi}(x) = \varphi(-x).$$

(Motivation: $\langle u, \varphi \rangle = \int u\varphi\, dx$ gives $\langle \tau_h u, \varphi \rangle = \int u(x-h)\varphi(x)\, dx = \int u(x)\varphi(x+h)\, dx = \langle u, \tau_{-h}\varphi \rangle$.) By duality, the definitions of these operations on $u \in D'(\mathbb{R}^n)$,

$$\begin{aligned} \langle \tau_h u, \varphi \rangle &:= \langle u, \tau_{-h} \varphi \rangle \\ \langle \check{u}, \varphi \rangle &:= \langle u, \check{\varphi} \rangle . \end{aligned}$$

Lemma 1.4.5: lem:dist1-3 For $u \in D'(\mathbb{R}^n)$ define

$$v_h = \frac{\tau_{-h}u - u}{|h|}.$$

Then $v_h \to n \cdot \partial u$, where $\lim_{h\to 0} \frac{h}{|h|} = n \in S^{n-1}$.

Proof. We have

$$\begin{aligned} \langle v_h, \varphi \rangle &= \left\langle \frac{\tau_{-h}u - u}{|h|}, \varphi \right\rangle \\ &= \left\langle u, \frac{\tau_h \varphi - \varphi}{|h|} \right\rangle. \end{aligned}$$

By Taylor's Theorem,

$$\begin{aligned} \tau_h\varphi(x) - \varphi(x) &= -h\partial\varphi(x) + \underbrace{R(x,h)}_{o(|h|) \text{ in } D(\mathbb{R}^n)} \\ \left\langle u, \frac{\tau_h\varphi - \varphi}{|h|} \right\rangle &= \left\langle u, -\frac{h}{|h|}\partial\varphi \right\rangle + \underbrace{\left\langle u, \frac{(R(\cdot,h))}{h} \right\rangle}_{\to 0 \text{ as } |h| \to 0} \\ &= n \cdot \langle \partial u, \varphi \rangle. \end{aligned}$$

□

This shows the distributional derivative coincides with the normal notion of derivative as difference quotient.

Lecture 4 (29 Jan)

4.3 Convolution between $D(\mathbb{R}^n)$ and $D'(\mathbb{R}^n)$

If we combine the operations of reflection and translation, we get

$$(\tau_x\check{\varphi})(y) = \check{\varphi}(y-x) = \varphi(x-y).$$

If $u \in D(\mathbb{R}^n)$, we define the **convolution** of u and $\varphi \in D(\mathbb{R}^n)$ with

$$(u * \varphi)(x) := \int u(x-y)\varphi(y)\,dy = \int \varphi(x-y)u(y)\,dy = (\varphi * u)(x).$$

Using τ_h and $\check{\bullet}$, we can write

$$u * \varphi(x) = \langle u, \tau_x\check{\varphi} \rangle.$$

This is well defined for all $u \in D'(\mathbb{R}^n)$.

Definition 1.4.6: df:dist1-5 For $u \in D'(\mathbb{R}^n)$ and $\varphi \in D(\mathbb{R}^n)$, define their convolution by

$$u * \varphi(x) = \langle u, \tau_x\check{\varphi} \rangle.$$

It is clear that $u * \varphi(x)$ is just some function of $x \in \mathbb{R}^n$. It is actually smooth.

Lemma 1.4.7: lem:dist1-4 Define $\Phi_x(y) = \phi(x,y)$ where $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $\phi(\cdot, y) = 0$ for y outside some compact $K \subseteq \mathbb{R}^n$. Then for $u \in D'(\mathbb{R}^n)$,

$$\partial_x^\alpha \langle u, \Phi_x \rangle = \langle u, \partial_x^\alpha \Phi_x \rangle .$$

We can take the derivatives inside the bracket.

Proof. By Taylor's theorem,

$$\Phi_{x+h}(y) - \Phi_x(y) = \sum_i h_i \frac{\partial \phi}{\partial x_i}(x,y) + R_x(y,h)$$

It is not difficult to show that $R_x(y,h) = o(|h|)$ in $D(\mathbb{R}^n)$ for each $x \in \mathbb{R}^n$. Hence

$$\langle u, \Phi_{x+h} \rangle - \langle u, \Phi_x \rangle = \sum_i h_i \langle u, \partial \Phi_x x_i \rangle + \langle u, R_x(\cdot, h) \rangle .$$

Since $R_x(\cdot, h) = o(|h|)$ in $D(\mathbb{R}^n)$, dividing by $|h|$ and taking $h \to 0$ gives ($u = \frac{h}{|h|}$)

$$n \cdot \langle u, \Phi_x \rangle = n \cdot \langle u, \partial_x \Phi \rangle .$$

So the result follows. □

Corollary 1.4.8: cor:dist1-1 If $u \in D'(\mathbb{R}^n)$ and $\varphi \in D(\mathbb{R}^n)$ then $u * \varphi$ is smooth and $\partial^\alpha(u * \varphi) = u * \partial^\alpha \varphi$.

Proof. By Lemma 1.4.7, $\partial^\alpha(u * \varphi) = \partial_x^\alpha \langle u, \tau_x \check{\varphi} \rangle = u * \partial^\alpha \varphi$. □

4.4 Density of $D(\mathbb{R}^n)$ in $D'(\mathbb{R}^n)$

We have just seen the following.

If $u \in D'(\mathbb{R}^n)$ and $\varphi \in D(\mathbb{R}^n)$ then $u * \varphi$ is smooth.

This is extremely useful. No matter how wild u is, $u * \varphi$ is nice. For this reason, $u * \varphi$ is often called a **regularisation** of the distribution u. We will use this fact to prove $D(\mathbb{R}^n)$ is dense in $D'(\mathbb{R}^n)$, i.e., for each $u \in D^1(\mathbb{R}^n)$ there exists a sequence of test functions $\{\varphi_m\}_{m \geq 1}$ in $D(\mathbb{R}^n)$ such that $\varphi_m \to u$ in $D'(\mathbb{R}^n)$, i.e., $\langle \varphi_m, \theta \rangle \to \langle u, \theta \rangle$ for all $\theta \in D(\mathbb{R}^n)$.

How to apply this: Suppose we have a problem about a distribution we don't know anything about. We know for some sequence of ϕ_m, $\varphi_m = u * \phi_m \to u$. We replace u with φ_m, do manipulations there, run the argument and take a limit.

We need a technical lemma.

Lemma 1.4.9: lem:dist1-5 For $\varphi, \psi \in D(\mathbb{R}^n)$ and $u \in D'(\mathbb{R}^n)$ we have

$$(u * \varphi) * \psi = u * (\varphi * \psi).$$

Proof. The LHS is

$$\begin{aligned}
(u * \varphi) * \psi(x) &= \int (u * \varphi)(x-y)\psi(y)\,dy \\
&= \int \langle u, \tau_{x-y}\widehat{\varphi}\rangle \,\psi(y)\,dy \\
&= \int \langle u(z), \varphi(x-y-z)\psi(y)\rangle \,dy && z \text{ is “dummy” variable} \\
&= \lim_{h\to 0} \sum_{m\in\mathbb{Z}^n} \langle u(z), \varphi(x-z-hm)\psi(hm)h^n\rangle && \text{Riemann sum} \\
&= \lim_{h\to 0} \left\langle u(z), \sum_{m\in\mathbb{Z}^n} \varphi(x-z-hm)\varphi(hm)h^n \right\rangle
\end{aligned}$$

(We want to take the $\int$ inside the $\langle\cdot\rangle$, and we do this by turning it into a Riemann sum. Note the sum only has finitely many terms for each m, so this is legal.) It is not hard to show that

$$\sum_{m\in\mathbb{Z}^n} \varphi(x-hm)\psi(m)h^n \to \varphi * \psi(x) \text{ in } D(\mathbb{R}^n) \text{ as } h \to 0.$$

Continuing the above calculation,

$$\begin{aligned}
&= \langle u(z), \varphi * \psi(x-z)\rangle \\
&= \langle u, \tau_x(\varphi * \psi)\check{}\,\rangle \\
&= u * (\varphi * \psi)(x).
\end{aligned}$$

□

Theorem 1.4.10: thm:dist1-2 $D(\mathbb{R}^n)$ is dense in $D'(\mathbb{R}^n)$.

We would like to say

$$u(x) = \int \delta(x-y)u(y)\,dy \stackrel{?}{=} \lim_{m\to\infty} \int \delta_m(x-y)u(y)\,dy$$

The $\delta_m(x)$ are such that $\int \delta_m(x)\,dx = 1$ and get squashed closer and closer to δ.

Proof. Fix $\psi \in D(\mathbb{R}^n)$ with $\int \psi\,dx = 1$ and set

$$\phi_m(x) = m^n\psi(mx) \qquad \left(\int \phi_m\,dx = \int \psi\,dx = 1\right).$$

Also introduce the bump function $\chi \in D(\mathbb{R}^n)$ with $\chi = 1$ on $|x| < 1$ and $\chi = 0$ on $|x| > 1$. Now set $\chi_m\left(\frac{x}{m}\right)$ and

$$\varphi_m = \chi_m(x)(u * \phi_m)(x).$$

(The purpose of $\chi_m(x)$ is to make φ_m have compact support.) Choose $\langle \varphi_m, \theta \rangle$ for $\theta \in D(\mathbb{R}^n)$ arbitrary, giving

$$\begin{aligned}
\langle \varphi_m, \theta \rangle &= \langle u * \phi_m, \chi_m \theta \rangle \\
&= (u * \phi_m) * (\chi_m \theta)\check{}(0) \\
&= u * (\phi_m * (\chi_m \theta)\check{})(0) \\
\phi_m * (\chi_m \theta)\check{}(x) &= \int \phi_m(x-y) \chi_n(-y) \theta(-y)\, dy \\
&= \int m^n \phi(m(x-y)) \chi \left(\frac{-y}{m} \right) \theta(-y)\, dy && y' = m(x-y) \implies y = x - \frac{y'}{m} \\
&= \int \phi(y) \chi \left(\frac{y}{m^2} - \frac{x}{m} \right) \theta \left(\frac{y}{m} - x \right)\, dy \\
&= \theta(-x) + \underbrace{\int \phi(y) \chi \left(\frac{y}{m^2} - \frac{x}{m} \right) \left[\theta \left(\frac{y}{m} - x \right) - \theta(-x) \right]\, dy}_{=: R_m(-x)} \\
&= (\check{\theta} + \check{R}_m)(x).
\end{aligned}$$

We can show $R_m \to 0$ in $D(\mathbb{R}^n)$. So

$$\langle \varphi_m, \theta \rangle = \langle u, \theta \rangle + \underbrace{\langle u, R_m \rangle}_{\to 0 \text{ as } m \to \infty}.$$

Hence $\langle \varphi_m, \theta \rangle \to \langle u, \theta \rangle, \theta \in D(\mathbb{R}^n)$, giving $\varphi_m \to u$ in $D'(\mathbb{R}^n)$ i.e. $D(\mathbb{R}^n)$ dense in $D'(\mathbb{R}^n)$. $\square$

Chapter 2

Distributions of compact support

Lecture 6 (3 Feb)

To talk about distributions of compact support, we need a different space of test functions, and a new notion of convergence.

For $u \in D'(X)$, we say that u vanishes on $Y \subseteq X$ if $\langle u, \varphi \rangle = 0$ for all $\varphi \in D(Y)$; it kills test functions supported on Y.

Definition 2.0.11: df:dist2-1 For $u \in D'(X)$, we define the **support** of u, written $\mathrm{Supp}(u)$, as the complement of the largest open set on which u vanishes.

The definition looks a little cumbersome, but in practice it is very easy to establish $\mathrm{Supp}(u)$. Usually we have a guess, and we just verify it. For example, $\mathrm{Supp}(\delta_x) = \{x\}$. Note that as in the case of normal functions, $\mathrm{Supp}(u)$ is closed.

1 Test functions and distributions

We can now define a new space of test functions.

Definition 2.1.1: df:dist2-2 The space $\mathcal{E}(X)$ consists of smooth functions from X to $\mathbb{C}$. We say a sequence $\{\varphi_m\}_{m \geq 1}$ tends to 0 in $\mathcal{E}(X)$ if $\partial^\alpha \varphi_m \to 0$ uniformly on compact subsets of X, for each multi-index α.

Note *we haven't mentioned the support!* The functions don't have to vanish outside the compact set. We have little control on what it does in the limit. Thus it makes sense that for continuous linear maps on $\mathcal{E}(X)$, we have to say something about the support of those maps.

Definition 2.1.2: df:dist2-3 A linear map $u : \mathcal{E}(X) \to \mathbb{C}$ belongs to $\mathcal{E}'(X)$ if there exist constants C, N and a compact set $K \subseteq X$ such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi|$$

for all $\varphi \in \mathcal{E}(X)$. They are called **distributions of compact support**.[1]

Since $\mathcal{E}(X)$ is bigger than $D(X)$, we expect $\mathcal{E}'(X)$ to be smaller than $D'(X)$.

Lemma 2.1.3: lem:dist2-1 A linear map $u : \mathcal{E}(X) \to \mathbb{C}$ belongs to $\mathcal{E}'(X)$ iff $\langle u, \varphi_m \rangle \to 0$ for every sequence $\{\varphi_m\}_{m \geq 1}$ that tends to 0 in $\mathcal{E}(X)$.

Proof. Same as the $D'(X)$ case. □

Lemma 2.1.4: lem:dist2-2 For each $u \in \mathcal{E}'(X)$, u restricted to $D(X)$ defines a distribution of compact support and finite order. Conversely, if $u \in D'(X)$ has compact support, then there exists a unique $\tilde{u} \in \mathcal{E}'(X)$ such that $u = \tilde{u}$ in $D'(X)$.

In other words, we can think of $\mathcal{E}'(X) \subset D'(X)$; it is the subspace of distributions with compact support.

Proof. Since $D(X)$ is contained in $\mathcal{E}(X)$ the restriction is well-defined. There exist constants C, N and a compact set $K \subseteq X$ such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi| \text{ for all } \varphi \in D(X).$$

So u certainly has finite order, and $\operatorname{ord}(u) \leq N$. And $\operatorname{Supp}(u)$ is contained inside K, since $\langle u, \varphi \rangle = 0$ for φ supported outside K.

If $u \in D'(X)$ has compact support, define $\tilde{u}$ by

$$\langle \tilde{u}, \varphi \rangle := \langle u, \rho\varphi \rangle \quad \forall \varphi \in \mathcal{E}(X).$$

where $\varphi \in D(X)$ is such that $\rho = 1$ on $\operatorname{Supp}(u)$. Since $u \in D'(X)$ we have

$$|\langle \tilde{u}, \varphi \rangle| = |\langle u, \rho\varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha(\rho\varphi)| \leq C' \sum_{|\alpha| \leq N} \sup_{\operatorname{Supp}(\rho)} |\partial^\alpha \varphi|, \varphi \in \mathcal{E}(X).$$

(We expanded the derivatives.) So $\tilde{u} \in \mathcal{E}'(X)$. For uniqueness, suppose $\tilde{v}$ is another extension of u such that $\tilde{u} = \tilde{v}$ on $D(X)$. Write

$$\varphi = \rho\varphi + (1-\rho)\varphi =: \varphi_1 + \varphi_2,$$

where $\rho \in \mathcal{E}(X)$ and $\varphi \in D(X)$ with $\rho = 1$ on $\operatorname{Supp}(u)$. We get

$$\begin{aligned}
\langle \tilde{v}, \varphi \rangle &= \langle \tilde{v}, \varphi_1 \rangle + \langle \tilde{v}, \varphi_2 \rangle && (2.1)\\
\text{eq:dist2-1} &= \langle \tilde{u}, \varphi_1 \rangle + 0 && (2.2)\\
&= \langle \tilde{u}, \varphi_1 + \varphi_2 \rangle && (2.3)\\
&= \langle \tilde{u}, \varphi \rangle. && (2.4)
\end{aligned}$$

In (2.2) we use the fact that if $\tilde{u}$ vanishes on $Y \subseteq X$, then so does $\tilde{v}$ since for arbitrary $\varphi \in D(Y)$, so we have $\langle \tilde{v}, \varphi_2 \rangle = \langle \tilde{u}, \varphi_2 \rangle = 0$. So $\tilde{u} = \tilde{v}$ in $\mathcal{E}'(X)$. □

[1] Warning: we can't take $K = \operatorname{Supp}(u)$. See Example 4.5.1.

We can define differentiation and multiplication by smooth functions just as we did for $D'(X)$.

Remark: We know $D(X) \subset \mathcal{E}(X)$; in fact $D(X) \hookrightarrow \mathcal{E}(X)$, meaning $D(X)$ is continuously embedded in $\mathcal{E}(X)$, i.e., if $\{\varphi_m\}_{m\geq 1}$ in $D(X)$ tends to 0 in $D(X)$, then $\varphi_m \to 0$ in $\mathcal{E}(X)$ also. From this one can show $\mathcal{E}'(X) \hookrightarrow D'(X)$.

2 Convolution between $D'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$

Just as in definition 1.4.6, we can define convolution between $\mathcal{E}'(\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R}^n)$ via

$$u * \varphi(x) := \langle u, \tau_x \check{\varphi} \rangle, \qquad \varphi \in \mathcal{E}(\mathbb{R}^n).$$

Again this is smooth and $\partial^\alpha(u*\varphi) = u*\partial^\alpha\varphi$. If $\varphi \in D(\mathbb{R}^n)$ then we also have $u*\varphi \in D(\mathbb{R}^n)$, where $u \in \mathcal{E}'(\mathbb{R}^n)$. This is true because $u * \varphi(x)$ vanishes unless $x - y \in \mathrm{Supp}(\varphi)$ for some $y \in \mathrm{Supp}(\varphi)$, i.e.,

$$\mathrm{Supp}(u * \varphi) \subseteq \mathrm{Supp}(u) + \mathrm{Supp}(\varphi).$$

So $\mathrm{Supp}(u * \varphi)$ is compact.

Definition 2.2.1: If u_1 and u_2 are distributions on $\mathbb{R}^n$, one of which has compact support, then we can define $u_1 * u_2$ to be the unique $u \in D'(\mathbb{R}^n)$ such that

$$u_1 * (u_2 * \varphi) = u * \varphi \quad \forall \varphi \in D(\mathbb{R}^n).$$

This definition makes sense, since $u_2 * \varphi \in D(\mathbb{R}^n)$ if $\mathrm{Supp}(u_2)$ is compact and $u_2 * \varphi \in \mathcal{E}(\mathbb{R}^n)$ otherwise.

Lemma 2.2.2: For u_1, u_2 distributions on $\mathbb{R}^n$, one of which has compact support, we have

$$u_1 * u_2 = u_2 * u_1.$$

Proof. For $\varphi, \psi \in D(\mathbb{R}^n)$ we have

$$\begin{aligned}(u_1 * u_2) * (\varphi * \psi) &= u_1 * (u_2 * (\varphi * \psi)) \\ &= u_1 * ((u_2 * \varphi) * \psi) \\ &= u_1 * (\psi * (u_2 * \varphi)) \\ &= (u_1 * \psi) * (u_2 * \varphi).\end{aligned}$$

Computing $(u_2 * u_1) * (\varphi * \psi) = (u_2 * u_1) * (\psi * \varphi)$ we find

$$(u_1 * u_2) * (\varphi * \psi) - (u_2 * u_1) * (\varphi * \psi) = 0.$$

Set $u_1 * u_2 - u_2 * u_1 := u$. Then

$$\begin{aligned}
0 &= u * (\varphi * \psi) \\
&= (u * \varphi) * \psi && \text{by Lemma 1.4.9} \\
\langle u * \varphi, \psi \rangle &= (u * \varphi) * \check{\psi}(0) = 0 \\
\implies u * \varphi &= 0 \quad \forall \varphi \in D(\mathbb{R}^n) \\
\implies u &= 0 \text{ in } D'(\mathbb{R}^n), \text{ i.e, } u_1 * u_2 = u_2 * u_1.
\end{aligned}$$

□

Lecture 7 (5 Feb)

As an example take δ_0 and arbitrary $u \in D'(\mathbb{R}^n)$. Because δ_0 has compact support we can take their convolution. So $u * \delta_0 = \delta_0 * u$ is defined by $(u * \delta_0) * \varphi := u * (\delta_0 * \varphi)$ for all $\varphi \in D(\mathbb{R}^n)$. We have that

$$\begin{aligned}
\delta_0 * \varphi(x) &:= \langle \delta_0, \tau_x \check{\varphi} \rangle \\
&= (\tau_x \check{\varphi})(0) \\
&= \check{\varphi}(0 - x) \\
&= \varphi(x),
\end{aligned}$$

i.e., $\delta_0 * \varphi = \varphi$. (Going back to old, nonrigorous language, $u(x) = \int \delta(x-y) u(y)\, dy$.) So we find $(u * \delta_0) * \varphi = u * \varphi$, giving $u * \delta_0 = u$ for all $u \in D'(\mathbb{R}^n)$. This is exactly how we hope convolution with δ_0 would work.

Chapter 3

Fourier analysis and tempered distributions

Whenever we do anything significant about distributions, we'll end up doing Fourier analysis.

1 Test functions and distributions

The relevant space of test functions to work on when using the Fourier transform is the space of functions with rapid decay.

Definition 3.1.1: df:dist3-1 The **Schwartz space** $S(\mathbb{R}^n)$ consists of smooth functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|\varphi\|_{\alpha,\beta} := \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| < \infty$$

for all multi-indices α, β. A sequence $\{\varphi_m\}_{m\geq 1}$ in $S(\mathbb{R}^n)$ tends to 0 if $\|\varphi\|_{\alpha,\beta} \to 0$ for all multi-indices α, β.

We can differentiate φ as many times as we want and multiply by any power of x, and this will be bounded; hence the name rapid decay.

So the elements of $S(\mathbb{R}^n)$ decay faster than the reciprocal of any polynomial, as do their derivatives. The archetypal element of $S(\mathbb{R}^n)$ is the Gaussian

$$\varphi(x) = e^{-|x|^2} = e^{-x_1^2 - \cdots - x_n^2}.$$

Definition 3.1.2: df:dist3-2 A linear map $u : S(\mathbb{R}^n) \to \mathbb{C}$ is called a **tempered distribution** if there exists a constant C, N such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha|,|\beta|\leq N} \|\varphi\|_{\alpha,\beta}$$

for all $\varphi \in S(\mathbb{R}^n)$. Denote the space of all such linear maps by $S'(\mathbb{R}^n)$.

We can show an equivalent definition in terms of sequential continuity, i.e., $u \in S'(\mathbb{R}^n)$ iff $\langle u, \varphi_m \rangle \to 0$ for all sequences $\{\varphi_m\}_{m\geq 1}$ in $S(\mathbb{R}^n)$ that tend to 0.

We can define all the usual operations on tempered distributions, i.e., differentiation, etc. In fact, we can view $S'(\mathbb{R}^n)$ as a subspace of $D'(\mathbb{R}^n)$:

$$D(\mathbb{R}^n) \hookrightarrow S(\mathbb{R}^n) \hookrightarrow \mathcal{E}(\mathbb{R}^n)$$

where the inclusions are continuous. The distributions (dual spaces) satisfy

$$\mathcal{E}'(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow D'(\mathbb{R}^n).$$

(The smaller the spce of distributions, the larger the space of distributions.)

2 The Fourier transform on $S(\mathbb{R}^n)$

We can now define the Fourier transform.

Definition 3.2.1: df:dist3-3 For $f \in L^1(\mathbb{R}^n)$, we define the **Fourier transform** by

$$\hat{f}(\lambda) = \int_{\lambda \in \mathbb{R}^n} e^{-i\lambda \cdot x} f(x)\, dx.$$

So the Fourier transform is well-defined on $S(\mathbb{R}^n)$ since if $\varphi \in S(\mathbb{R}^n)$,

$$\begin{aligned}\int |\varphi(x)|\, dx &= \int (1+|x|)^N |\varphi(x)| (1+|x|)^{-N}\, dx \\ &\leq C \sum_{|\alpha|\leq N} \|\varphi\|_{\alpha,0} \int (1+|x|)^{-N}\, dx \\ &< \infty\end{aligned}$$

for N sufficiently large.

Lemma 3.2.2: lem:dist3-1 If $f \in L^1(\mathbb{R}^n)$ then $\hat{f} \in C(\mathbb{R}^n)$.

Proof. If $\{\lambda_n\}_{n\geq 0}$ is a sequence in $\mathbb{R}^n$ such that $\lambda_n \to \lambda$, then

$$\begin{aligned}\lim_{n\to\infty} \hat{f}(\lambda_n) &= \lim_{n\to\infty} \int e^{-i\lambda_n \cdot x} f(x)\, dx &(3.1)\\ \text{eq:dist3-1} &= \int \lim_{n\to\infty} e^{-i\lambda_n \cdot x} f(x)\, dx &(3.2)\\ &= \int e^{-i\lambda \cdot x} f(x)\, dx &(3.3)\\ &= \hat{f}(\lambda). &(3.4)\end{aligned}$$

In (3.2) we used the dominated convergence theorem 1.1.1, justified since $|e^{-i\lambda_n \cdot x} f(x)| \leq |f(x)|$ and $f \in L^1(\mathbb{R}^n)$. ☐

This is a general property of the Fourier transform, and one reason it's useful in analysis. It tends to interchange the properties of smoothness/differentiability and decay. easy to check decay. (It's usually more work to check the properties of smoothness than decay.) For example, if you know the Fourier transform $\hat{f}$ is very smooth, we can deduce f decays rapidly. This is exemplified by the following lemma.

Lemma 3.2.3: lem:df3-2 For $\varphi \in S(\mathbb{R}^n)$ we have (remember $D := -i\partial$, so $D^\alpha = \left(-i\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(-i\frac{\partial}{\partial x_n}\right)^{\alpha_n}$)

$$(D^\alpha \varphi)\hat{}(\lambda) = \lambda^\alpha \hat{\varphi}(\lambda), \qquad (x^\beta \varphi)\hat{}(\lambda) = (-D)^\beta \hat{\varphi}(\lambda)$$

for all multi-indices α, β.

Proof. We have using integration by parts

$$\begin{aligned}(D^\alpha \varphi)\hat{} &= \int e^{-i\lambda\cdot x}(D_x^\alpha \varphi)(x)\,dx \\ &= \int \varphi(x)(-D_x)^\alpha e^{-i\lambda\cdot x}\,dx \\ &= \int \varphi(x)\lambda^\alpha e^{-i\lambda\cdot x}\,dx \\ &= \lambda^\alpha \hat{\varphi}(\lambda).\end{aligned}$$

For the second equation, differentiate under the integral sign (justified by the dominated convergence theorem 1.1.1)

$$(-D_\lambda)^\beta \hat{\varphi}(\lambda) = \int (-D_\lambda^\beta e^{-i\lambda\cdot x}\varphi(x)\,dx = \int e^{-i\lambda\cdot x} x^\beta \varphi(x)\,dx = (x^\beta\varphi)^n(\lambda).$$

□

The following theorem demonstrates why $S(\mathbb{R}^n)$ is the correct space to work on if you want to do Fourier analysis.

Theorem 3.2.4: The Fourier transform defines a continuous isomorphism on $S(\mathbb{R}^n)$.

Proof. If $\varphi \in S(\mathbb{R}^n)$ then we have

$$\begin{aligned}|D^\beta \lambda^\alpha \hat{\varphi}| &= \left|\int e^{-i\lambda x} x^\beta D^\alpha \varphi\,dx\right| \\ &< \infty \qquad \text{since } x^\beta D^\alpha\varphi \in S(\mathbb{R}^n) \text{ for all } \alpha, \beta.\end{aligned}$$

Also, since $x^\alpha \in L^1(\mathbb{R}^n)$ for all α, this implies $D_\lambda^\alpha \hat{\varphi}$ is continuous for all α (Lemma 3.2.2), so $\hat{\varphi}$ is smooth and $\|\hat{\varphi}\|_{\alpha,\beta} < \infty$ for all multi-indices α, β, so $\hat{\varphi} \in S(\mathbb{R}^n)$.

It is also clear from this that the map $\varphi \mapsto \hat{\varphi}$ is continuous.

To prove the isomorphism, we will use the inverse Fourier transform. We want to prove

$$\varphi(x) = \frac{1}{(2\pi)^n}\int e^{i\lambda\cdot x}\hat{\varphi}(x)\,d\lambda \tag{3.5}$$

$$\text{eq:dist3-2} = \frac{1}{(2\pi)^n}\int e^{i\lambda\cdot x}\Big[\int e^{-i\lambda\cdot y}\varphi(y)\,dy\Big]\,d\lambda. \tag{3.6}$$

We cannot interchange the order of integration[1] because $\iint dy\, d\lambda$ is not absolutely convergent, since there is no decay in the λ-direction. We need to insert regularisation.

By the dominated convergence theorem,

$$\int e^{i\lambda\cdot x}\hat{\varphi}\, d\lambda = \lim_{\varepsilon\to 0}\int e^{-i\lambda\cdot x-\varepsilon|\lambda|^2}\hat{\varphi}(\lambda)\, d\lambda$$

since $|e^{i\lambda x-\varepsilon|\lambda|^2}\hat{\varphi}(\lambda)| \le |\hat{\varphi}(\lambda)| \in L^1$. We can now interchange the order of integration since the double integral is absolutely convergent for every $\varepsilon > 0$.

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Equation (3.6) now becomes

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^n}\int e^{i\lambda\cdot x}e^{-\varepsilon|\lambda|^2}\int e^{-i\lambda\cdot y}\varphi(y)\, dy = \lim_{\varepsilon\to 0}\frac{1}{(2\pi)^n}\int\varphi(y)\underbrace{\left[\int e^{-i\lambda\cdot(x-y)-c|\lambda|^2}\right]}_{A} dy. \tag{3.7}$$

eq:dist3-3

We have

$$\begin{aligned}
A &= \prod_{i=1}^{n}\int e^{-i\lambda_i(x_i-y_i)-\varepsilon\lambda_i^2}\, d\lambda_i \\
&= \prod_{i=1}^{n} e^{-\varepsilon\left[\lambda_i-\left(\frac{i(x_i-y_i)}{2\varepsilon}\right)\right]^2-\frac{(x_i-y_i)^2}{4\varepsilon}}\, d\lambda_i && \text{completing square} \\
&= \prod_{i=1}^{n} e^{-\frac{(x_i-y_i)^2}{4\varepsilon}}\int e^{-\varepsilon(\lambda_i-i(\frac{x_i-y_i}{2\varepsilon}))^2}\, d\lambda_i \\
&= \prod_{i=1}^{n} e^{-\frac{(x_i-y_i)^2}{4\varepsilon}}\int e^{-\varepsilon\lambda_i^2}\, d\lambda_i \\
&= \prod_{i=1}^{n} e^{-\frac{(x_i-y_i)^2}{4\varepsilon}}\sqrt{\frac{\pi}{\varepsilon}} \\
&= e^{-\frac{|x-y|^2}{4\varepsilon}}\frac{\pi^{n/2}}{\varepsilon^{n/2}}
\end{aligned}$$

where we used that (integrating along the rectangle from $-R \to R \to R + \left(\frac{x_i-y_i}{2\varepsilon}\right)i \to -R + \left(\frac{x_i-y_i}{2\varepsilon}\right)i \to -R$),

$$\begin{aligned}
&\oint_\gamma e^{-\varepsilon\lambda_i^2}\, d\lambda_i = 0 \\
\implies &\int e^{-\varepsilon(\lambda_i-(\frac{x_i-y_i}{2\varepsilon})i)^2}\, d\lambda_i = \int e^{-\varepsilon\lambda_i^2}\, d\lambda_i.
\end{aligned}$$

Then (3.7) becomes (letting $y' = \frac{x-y}{2\sqrt{\varepsilon}}$)

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^n}\int\varphi(y)e^{-\frac{|x-y|^2}{4\varepsilon}}\frac{\pi^{n/2}}{\varepsilon^{n/2}}\, dy \\
=&\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^n}\int\varphi(x-2\sqrt{\varepsilon}y')e^{-|y'|^2}\frac{\pi^{n/2}}{\varepsilon^{n/2}}\varepsilon^{n/2}\, dy'.
\end{aligned}$$

[1] Fubini's Theorem says that if $\iint |f(x,y)|\, dx\, dy < \infty$ then $\int dx \int dy\, f(x,y) = \int dy \int dx\, f(x,y)$.

Taking the limit, justified by dominated convergence 1.1.1,

$$\frac{1}{(2\pi)^n}\int e^{i\lambda\cdot x}\hat{\varphi}(\lambda)\,d\lambda = \varphi(x).$$

So the Fourier transform is injective on $S(\mathbb{R}^n)$, i.e., $\hat{\varphi}=0$ implies $\varphi=0$. By changing sign of x, find

$$\hat{\varphi}(\lambda) = \int e^{-i\lambda\cdot x}\left[\frac{1}{(2\pi)^n}\int e^{i\mu\cdot x}\hat{\varphi}(\mu)\,d\mu\right]\,dx.$$

So the Fourier transform is also surjective. Hence it is a bijection on $S(\mathbb{R}^n)$. □

A nicer proof is due to Fokas and Gelfand, and uses "spectral analysis" of PDE, $\frac{\partial\mu}{\partial x} - ik\mu = \varphi(x)$. Their proof applies to nonlinear Fourier isomorphisms and PDE's as well.

3 Fourier transform on $S'(\mathbb{R}^n)$

To define the Fourier transform on $S'(\mathbb{R}^n)$, we need the following famous lemma.

Lemma 3.3.1 (Parseval's Theorem)**:** lem:dist3-3 For $\varphi,\psi \in S(\mathbb{R}^n)$, we have

$$\int \hat{\varphi}(x)\psi(x)\,dx = \int \varphi(x)\hat{\psi}(x)\,dx.$$

Proof. The LHS is

$$\begin{aligned}\int \hat{\varphi}(x)\psi(x)\,dx &= \int\left[\int e^{-i\lambda\cdot x}\varphi(\lambda)\,d\lambda\right]\psi(x)\,dx\\ &= \int \varphi(\lambda)\left[\int e^{-i\lambda\cdot x}\psi\,dx\right]\,d\lambda\\ &= \int \varphi(\lambda)\hat{\psi}(\lambda)\,d\lambda\end{aligned}$$

by Fubini. We can interchange the order of integration because the Schwartz functions φ,ψ decay fast in the λ, x directions. □

If $u \in S(\mathbb{R}^n)$, then we can write Parseval as

$$\langle \hat{u},\varphi\rangle = \langle u,\hat{\varphi}\rangle\,, \qquad \varphi\in S(\mathbb{R}^n).$$

($\hat{\bullet}$ is self-adjoint.) But the RHS makes sense for any $u \in S'(\mathbb{R}^n)$, since $\varphi \in S(\mathbb{R}^n)$ implies $\hat{\varphi}\in S(\mathbb{R}^n)$. This gives rise to the following definition.

Definition 3.3.2: df:dist3-4 For $u \in S'(\mathbb{R}^n)$ define the Fourier transform $\hat{u}$ by

$$\langle \hat{u},\varphi\rangle := \langle u,\hat{\varphi}\rangle$$

for all $\varphi \in S(\mathbb{R}^n)$.

For instance, we can compute the Fourier transform of δ_0.

$$\left\langle \widehat{\delta_0}, \varphi \right\rangle := \langle \delta_0, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int \varphi(x)\, dx = \langle 1, \varphi \rangle .$$

So $\widehat{\delta_0} = 1$. We can take the Fourier transform of a constant.

$$\left\langle \hat{1}, \varphi \right\rangle = \langle 1, \hat{\varphi} \rangle = \int 1 \cdot \hat{\varphi}(\lambda)\, d\lambda = (2\pi)^n \varphi(0) = \langle (2\pi)^n \delta_0, \varphi \rangle$$

by the Fourier inversion theorem. So $\delta_0 = \frac{1}{(2\pi)^n} \hat{1}$. In old-fashioned language, $\delta(x) = \frac{1}{(2\pi)^n} \int e^{-i\lambda \cdot x}\, d\lambda$.

It is straightforward to show (using duality arguments) that

$$(D^\alpha)\hat{} = \lambda^\alpha \hat{u}, \qquad (x^\beta)\hat{}=(-D)^\beta \hat{u}$$

for $u \in S'(\mathbb{R}^n)$.

Theorem 3.3.3: The Fourier transform defines a continuous isomorphism on $S'(\mathbb{R}^n)$.

Proof. We check that $\check{u} = (2\pi)^{-n} (\hat{u})\hat{}$:

$$\langle \check{u}, \varphi \rangle = \langle u, \check{\varphi} \rangle = \underbrace{\left\langle u, (2\pi)^{-n} (\hat{\varphi})\hat{} \right\rangle}_{(*)} = \left\langle (2\pi)^{-n} (\hat{u})\hat{}, \varphi \right\rangle$$

where in $(*)$ we used Fourier inversion

$$\varphi(-x) = \frac{1}{(2\pi)^n} \int e^{-i\lambda \cdot x} \left[e^{-i\lambda \cdot y} \varphi(y)\, dy \right] d\lambda.$$

So we have $u = (2\pi)^{-n} [(\hat{u})\hat{}]\check{}$. Continuity is proved as follows: if $u_m \to u$ in $S'(\mathbb{R}^n)$, ($\langle u_m, \varphi \rangle \to \langle u, \varphi \rangle$ for all $\varphi \in S(\mathbb{R}^n)$), then $\widehat{u_m} \to \hat{u}$, since $\langle \widehat{u_m}, \varphi \rangle = \langle u_m, \hat{\varphi} \rangle \to \langle u, \hat{\varphi} \rangle = \langle \hat{u}, \varphi \rangle$. So the FT is continuous. $\square$

4 Sobolev spaces

One of the main reasons to use FT is that they give a nice way to define Sobolev spaces.

Definition 3.4.1: df:dist3-5 If $u \in S'(\mathbb{R}^n)$ such that $\hat{u}(\lambda)$ is a function and

$$\|u\|_{H^s}^2 := \int |\hat{u}(\lambda)|^2 (1 + |\lambda|^2)^s\, d\lambda < \infty$$

then we say $u \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$.

It is handy to use the notation $\langle \lambda \rangle = (1 + |\lambda|^2)^{\frac{1}{2}}$. Then $u \in H^s(\mathbb{R}^n)$ if $\hat{u}(\lambda) \langle \lambda \rangle^s \in L^2(\mathbb{R}^n)$.

We can also define Sobolev spaces on $X \subseteq \mathbb{R}^n$ by localisation: for $u \in D'(X)$, say that $u \in H^s_{\text{loc}}(X)$ if $u\varphi \in H^s(\mathbb{R}^n)$ for every $\varphi \in D(X)$.

Lemma 3.4.2: lem:dist3-4 If $u \in H^s(\mathbb{R}^n)$ and $s > \frac{n}{2}$ then $u \in C(\mathbb{R}^n)$.

Proof. We will show $\hat{u} \in L^1(\mathbb{R}^n)$:

$$\begin{aligned}\int |\hat{u}(\lambda)|\, d\lambda &= \int \langle\lambda\rangle^{-s} \langle\lambda\rangle^{s} |\hat{u}(\lambda)|\, d\lambda \\ &\leq \underbrace{\left(\int \langle\lambda\rangle^{-2s}\, d\lambda\right)^{\frac{1}{2}}}_{(*)} \underbrace{\left(\int |\hat{u}(\lambda)|^2 \langle\lambda\rangle^{2s}\, d\lambda\right)^{\frac{1}{2}}}_{\|u\|_{H^s}} \qquad \text{C-S inequality.}\end{aligned}$$

Since $s > \frac{n}{2}$, write $s = \frac{n}{2} + \varepsilon$, $\varepsilon > 0$. Then

$$(*) = \int_{S^{n-1}} d\sigma_n \int r^{n-1}\, dr (1+r^2)^{-\frac{n}{2}-\varepsilon} = \int O(r^{n-1-n-2\varepsilon}) = \int O(r^{-1-2\pi}) < \infty.$$

So we have $\hat{u} \in L^1(\mathbb{R}^n)$, so

$$\begin{aligned}\langle\hat{u}, \varphi\rangle &= \int \hat{u}(\lambda)\varphi(\lambda)\, d\lambda \\ &= \int \hat{u}(\lambda) \left[\frac{1}{(2\pi)^n} \int e^{i\lambda\cdot x} \hat{\varphi}(x)\, dx\right] d\lambda \\ &= \int \hat{\varphi}(x) \left[\frac{1}{(2\pi)^n} \int e^{i\lambda\cdot x} \hat{u}(\lambda)\, d\lambda\right] dx \\ &= \langle u, \hat{\varphi}\rangle\end{aligned}$$

where $u(x) = \frac{1}{(2\pi)^n} \int e^{i\lambda\cdot x}\hat{u}(\lambda)\, d\lambda$. Since $\hat{u} \in L^1(\mathbb{R}^n)$, we know that $u \in C(\mathbb{R}^n)$ by Lemma 3.2.2. □

Given information how quickly FT transform decays, you can say something about the smoothness of the function.

Corollary 3.4.3: If $u \in H^s(\mathbb{R}^n)$ for each $s > \frac{n}{2}$ then $u \in C^\infty(\mathbb{R}^n)$.

Chapter 4

Applications of the Fourier transform

1 Elliptic regularity

We will be interested in the solution to PDE's of the form

$$P(D)u = f \qquad (D \equiv -i\partial)$$

where u, f are distributions and P is a polynomial, e.g., if $P(\lambda) = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$ then $P(D) = -\Delta$.

We're not interested in writing down explicit solutions; we're interested in *regularity* properties of the solution.

Problem: Given we know how smooth f is, can we deduce how smooth u must be without solving the PDE?

Given f is k times differentiable, we might expect that u is $n+k$ times differentiable if P has degree n. This isn't true, but we will answer the question for a large class of elliptic differential operators known as elliptic operators.

Context: One of the millennium problems is to establish regularity properties of solutions. the Navier-Stokes equations. Knowing regularity is important because you know how hard you have to work to approximate a solution. If a solution is wildly bad, then you need a cleverer numerical routine. Thus these are problems of practical interest.

Definition 4.1.1: df:dist4-1 A Nth order partial differential operator (PDO) is

$$P(D) = \sum_{|\alpha| \le N} c_\alpha D^\alpha,$$

where c_α are constants. It has **principal symbol** $\sigma_P(\lambda)$, defined by

$$\sigma_P(\lambda) = \sum_{|\alpha| = N} c_\alpha \lambda^\alpha.$$

We say that $P(D)$ is **elliptic** if $\sigma_P(\lambda) \neq 0$ for any $\lambda \neq 0$.

We know regularity is related to how quickly Fourier transforms decay. Pretending for a moment that we can take Fourier transforms, we get $\hat{u} = f/P$, so we care about how big $P(\lambda)$ gets.

Here are some examples of PDO's.

Example 4.1.2: 1. The **Laplacian**

$$\begin{aligned} P(\lambda) &= \lambda_1^2 + \cdots + \lambda_n^2 \\ \sigma_P(\lambda) &= \lambda_1^2 + \cdots + \lambda_n^2. \end{aligned}$$

2. The **Cauchy-Riemann operator**

$$\begin{aligned} P(\lambda) &= i\lambda_1 - \lambda_2 \\ \sigma_P(\lambda) &= i\lambda_1 - \lambda_2. \end{aligned}$$

An example of a non-elliptic PDO is the heat operator $\frac{\partial}{\partial t} - \Delta$.

Lemma 4.1.3: lem:dist4-1 If $P(D)$ is an elliptic, Nth order PDO then there exist constants $C, R > 0$ such that $|P(\lambda)| \geq C \langle \lambda \rangle^N$ for all $|\lambda| > R$.

From now on, if there exist $C > 0$ such that $f \leq Cg$ then we write $f \lesssim g$. The implied constants from one time to another are subject to change.

For N sufficiently large $|P(\lambda)| \gtrsim \langle \lambda \rangle^N$.

Proof. Since $|\lambda_p(\lambda)| > 0$ on $S^{n-1} = \{\lambda \in \mathbb{R}^n : |\lambda| = 1\}$, and $S^{n-1} \subseteq \mathbb{R}^n$ is compact, it follows that $|\sigma_P(\lambda)|$ attains minimum on S^{n-1}, so

$$\text{eq:dist4-1} |\sigma_P(\lambda)| \gtrsim 1 \text{ on } S^{n-1}. \tag{4.1}$$

Now for general $\lambda \in \mathbb{R}^n$ we can write

$$\begin{aligned} |\sigma_P(\lambda)| = \left| \sum_{|\alpha|=N} c_\alpha \lambda^\alpha \right| &= |\lambda|^N \left| \sum_{|\alpha|=N} c_\alpha \frac{\lambda^\alpha}{|\lambda|^N} \right| \\ &= |\lambda|^N \underbrace{\left| \sum_{|\alpha|=N} c_\alpha \left(\frac{\lambda}{|\lambda|} \right)^\alpha \right|}_{\gtrsim 1 \text{ by 4.1, since } \frac{\lambda}{|\lambda|} \in S^{n-1}} . \end{aligned}$$

So we have

$$|\sigma_P(\lambda)| \gtrsim |\lambda|^N, \qquad \lambda \in \mathbb{R}^n.$$

By the triangle inequality,

$$|\sigma_P(\lambda)| \leq |\sigma_P(\lambda) - P(\lambda)| + |P(\lambda)|$$

so

$$\begin{aligned}|P(\lambda)| &\geq |\sigma_P(\lambda)| - |P(\lambda) - \sigma_P(\lambda)| \\ &\gtrsim \Big(1 - \underbrace{\frac{P(\lambda) - \sigma_P(\lambda)}{|\lambda|^N}}_{(**)}\Big)|\lambda|^N.\end{aligned}$$

In $(**)$ note that $P(\lambda) - \sigma_P(\lambda)$ is a polynomial of order $< N$. Consequently, one can make $(**)$ arbitrarily small by choosing $|\lambda|$ sufficiently large. Hence

$$|P(\lambda)| \gtrsim |\lambda|^N \gtrsim \langle\lambda\rangle^N$$

for $|\lambda|$ sufficiently large. □

To prove regularity properties of solutions to PDE's, the natural space to work on is the Sobolev spaces. If $P(D)u = f$ in X, we want to know about u away from the boundary of X, so we work with local Sobolev spaces $H^s_{\text{loc}}(X)$. ($u \in D'(X)$ belongs to $H^s_{\text{loc}}(X)$ if $\varphi u \in H^s(\mathbb{R}^n)$ for all $\varphi \in D(X)$.)

Theorem 4.1.4 (Elliptic regularity)**:** thm:dist4-1 Let $P(D)$ be a Nth order partial differential operator that is elliptic. If $u \in D'(X)$ satisfies $P(D)u = f$ and $f \in H^s_{\text{loc}}(X)$, then $u \in H^{s+N}_{\text{loc}}(X)$.

Corollary 4.1.5: cor:dist4-1 If $P(D)u \in C^\infty(X)$ then $u \in C^\infty(X)$.

We'll first prove a much easier version when $u \in D'(\mathbb{R}^n)$ and has compact support. We do this by constructing a parametrix for $P(D)$.

Definition 4.1.6: A distribution $E \in D'(\mathbb{R}^n)$ is called a **parametrix** for $P(D)$ if

$$P(D)E = \delta_0 + \omega$$

where $\omega \in \mathcal{E}(\mathbb{R}^n)$.

It's like a Green's function, but with an extra term $\omega \in \mathcal{E}(\mathbb{R}^n)$ allowed.

Lemma 4.1.7: lem:dist4-2 Every elliptic PDO has a parametrix belonging to $S'(\mathbb{R}^n)$.

Proof. Since $P(D)$ is Nth order elliptic, we know $|P(\lambda)| \gtrsim \langle\lambda\rangle^N$ for $|\lambda|$ sufficiently large ($|\lambda| > R$, say). Introduce $\chi_R \in D(\mathbb{R}^n)$ with $\chi_R = 1$ in $|\lambda| \leq R$ and $\chi_R = 0$ in $|\lambda| > R + 1$. Consider the function

$$\hat{E}(\lambda) = \frac{1 - \chi_R(\lambda)}{P(\lambda)}.$$

Take the inverse FT to get

$$P(D)E = \delta_0 + \omega$$

where $\hat{\omega} = -\chi_R$. Then $\omega \in S(\mathbb{R}^n) \subseteq E(\mathbb{R}^n)$, so E is a parametrix.

To show that E is smooth away from the origin, consider

$$\begin{aligned}
|(D^\alpha(x^\beta E))\hat{}(\lambda)| &= |\lambda^\alpha D^\beta \hat{E}(\lambda)| \\
&= \left|\lambda^\alpha D^\beta \left(\frac{1}{P(\lambda)}\right)\right| && \text{for } |\lambda| > R+1 \\
&\lesssim |\lambda|^{|\alpha|-|\beta|-N} && \text{(exercise: induction).}
\end{aligned}$$

So for each α, the FT of $D^\alpha(x^\beta E)$ is in $L^1(\mathbb{R}^n)$ if we choose each $|\beta|$ sufficiently big. So $D^\alpha(x^\beta E)$ is continuous for each α, with β chosen sufficiently large. Hence E is smooth away from the origin. $\square$

We get an easy version of Theorem 4.1.4: if $P(D)u = f$, $u \in D'(\mathbb{R}^n)$ has compact support and $f \in H^s(\mathbb{R}^n)$, then

$$\begin{aligned}
u = \delta_0 * u &= (P(D)E - \omega) * u \\
&= P(D)E * u - \omega * u \\
&= E * P(D)u - \omega * u \\
&= E * \underbrace{f}_{\in H^s(\mathbb{R}^n)} - \underbrace{\omega * u}_{\in S(\mathbb{R}^n)}.
\end{aligned}$$

Since $u \in D'(\mathbb{R}^n)$ has compact support, $\omega * u \in S(\mathbb{R}^n)$. To get regularity of $E * f$, take the FT

$$|(E * f)\hat{}(\lambda)| = |\hat{E}(\lambda)\hat{f}(\lambda)| = \left|\frac{\hat{f}(\lambda)}{P(\lambda)}\right| \lesssim \langle\lambda\rangle^{-N} \hat{f}(\lambda).$$

using $|\lambda| > R+1$ in the second inequality. So $E * f \in H^{s+N}(\mathbb{R}^n)$.

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$$\begin{aligned}
\|E * f\|^2_{H^{s+N}(\mathbb{R}^n)} &= \int \langle\lambda\rangle^{2s+2n} \|(E * f)\hat{}(\lambda)\|^2 \, d\lambda \\
&\lesssim \int \langle\lambda\rangle^{2s+\cancel{2N}} \cancel{\lambda}^{-2N} |\hat{f}(\lambda)|^2 \, d\lambda \\
&= \|f\|^2_{H^s(\mathbb{R}^n)} < \infty.
\end{aligned}$$

Proof of Theorem 4.1.4. Use the following elementary facts about Sobolev spaces.

1. If $u \in \mathcal{E}(\mathbb{R}^n)$, then $\exists t \in \mathbb{R}$ such that $u \in H^t(\mathbb{R}^n)$.
2. If $u \in H^s(\mathbb{R}^n)$, then $D^\alpha u \in H^{s-|\alpha|}(\mathbb{R}^n)$.
3. If $s > t$ then $H^s(\mathbb{R}^n) \subseteq H^t(\mathbb{R}^n)$.

(See example sheet 2.) We'll use a "bootstrap" argument. (Have a go at proving this theorem. If you set out trying to avoid the bootstrap argument, you'll find yourself having to fudge things until you are basically doing a bootstrap argument.)

We want to show that $\varphi u \in H^s(\mathbb{R}^n)$ for arbitrary $\varphi \in D(X)$. Fix $\varphi \in D(X)$ and introduce test functions $\psi_0, \psi_1, \ldots, \psi_M \in D(X)$ such that

$$\operatorname{Supp}(\varphi) \subseteq \operatorname{Supp}(\psi_M) \subseteq \cdots \subseteq \operatorname{Supp}(\psi_0)$$

and also that $\psi_{i-1} = 1$ on $\operatorname{Supp}\psi_i$. Here's the picture.

Figure (see math post-it).

Consider $\psi_0 u$. By (1) $\exists t \in \mathbb{R}$ such that $\psi_0 u \in H^t(\mathbb{R}^n)$. Now consider $\psi_1 u$. We have

$$P(D)(\psi_1 u) = \psi_1 P(D)u + P(D)(u\psi_1) - \psi_1 P(D)u \tag{4.2}$$

$$\text{eq:dist4-2} = \psi_1 f + [P(D), \psi_1](\psi_0 u). \tag{4.3}$$

The final step is allowed because $\psi_0 = 1$ on $\operatorname{Supp}\psi_1$. The final step is allowed because $\psi_0 = 1$ on $\operatorname{Supp}(\psi_1)$. From (4.3) we have

$$P(D)(\psi_1 u) = \underbrace{\psi_1 f}_{\in H^s(\mathbb{R}^n)} + \underbrace{[P(D), \psi_1](\psi_0 u)}_{\in H^{t-N+1}(\mathbb{R}^n)}$$

by property (2). So

$$P(D)(\psi_1 u) \in H^{\widetilde{A_1}}(\mathbb{R}^n), \qquad \widetilde{A_1} = \min\{S, t - N + 1\}.$$

By the "easy" version of elliptic regularity for H^m,

$$\psi_1 u \in H^{A_1}(\mathbb{R}^n), \qquad A_1 = \widetilde{A}_1 + N = \min\{s + N, t + 1\}.$$

Now apply the same idea to $\psi_2 u$. We have

$$\begin{aligned} P(D)(\psi_2 u) &= \psi_2 P(D)u + [P(D), \psi_2](u) \\ &= \psi_2 P(D)u + [P(D), \psi_2](\psi_1 u), \end{aligned}$$

the second equality follows from the fact $\psi_1 = 1$ on $\operatorname{Supp}(\psi_2)$. So we find

$$\psi_2 u \in H^{A_2}(\mathbb{R}^n), \qquad A_2 = \min\{s + N, A_1 + 1\}$$

since we replaced condition $\psi_0 u \in H^t(\mathbb{R}^n)$ with $\psi_1 u \in H^{A_1}(\mathbb{R}^n)$.

By associativity of the min-function,

$$\begin{aligned} A_2 &= \min\{s + N, A_1 + 1\} \\ &= \min\{s + N, \min\{s + N, t + 1\} + 1\} \\ &= \min\{s + N, s + N + 1, t + 2\} \\ &= \min\{s + N, t + 2\}. \end{aligned}$$

Carrying on like this we find $\psi_M u \in H^{A_M}(\mathbb{R}^n)$ where

$$A_M \equiv \min\{s+N, t+M\}.$$

Choose $M > s-t+N$. We have $A_M = s+N$, i.e., $\psi_M u \in H^{s+N}(\mathbb{R}^n)$. But since $\psi_M = 1$ on $\operatorname{Supp}\varphi$, it follows that $\varphi u \in H^{s+N}(\mathbb{R}^n)$. Since $\varphi \in D(X)$ was arbitrary, we conclude that $u \in H^{s+N}_{\text{loc}}(X)$. $\square$

Example 4.1.8 (Laplace equation)**:** If $\Delta u = 0$ in X, i.e.,

$$\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_n}^2} = 0$$

then u is smooth in X by elliptic regularity.

On the other hand the wave equation

$$\frac{\partial^2 u}{\partial {x_1}^2} - \frac{\partial^2 u}{\partial {x_2}^2} = 0$$

is not elliptic. We can show any locally integrable function $F(x_1,x_2) = f(x_1-x_2)$ will solve the wave equation.

2 Fundamental solutions

In this section we again look at PDE's

$$P(D)u = f$$

where $f, u \in D'(\mathbb{R}^n)$. We will solve it by constructing a fundamental solution for $P(D)$, otherwise known as **Green's functions**.

Definition 4.2.1: A distribution $E \in D'(\mathbb{R}^n)$ is called a **fundamental solution** for $P(D)$ if

$$P(D)E = \delta_0.$$

Fundamental solutions can be used to solve $P(D)u = f$ if $f \in \mathcal{E}^1(\mathbb{R}^n)$. Since

$$\begin{aligned} P(D)(E * f) &= (P(D)E) * f \\ &= \delta_0 * f = f, \end{aligned}$$

i.e., $u = E * f$ satisfies $P(D)u = f$.

Lemma 4.2.2: Let $P(D)$ be given by the Cauchy-Riemann operator

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$

(where $z = x_1 + ix_2$ using $\mathbb{C} \cong \mathbb{R}^2$). Then $E = \frac{1}{\pi z}$ is a fundamental solution.

Proof. By definition, for $\varphi \in D(\mathbb{R}^2)$,

$$\begin{aligned}\left\langle \frac{\partial}{\partial \bar{z}} E, \varphi \right\rangle &= -\left\langle E, \frac{\partial \varphi}{\partial \bar{z}} \right\rangle \\ &= -\int E \frac{\partial \varphi}{\partial \bar{z}}\, dx & dx := dx_1\, dx_2 \\ &= -\lim_{\varepsilon \to 0} \int_{|z|>\varepsilon} E \frac{\partial \varphi}{\partial \bar{z}}\, dx. & \left(E \sim \frac{1}{z} = O\left(\frac{1}{R}\right), dx \sim R\, dR\, d\theta\right).\end{aligned}$$

Since $\frac{\partial E}{\partial \bar{z}} = 0$ on $|z| > \varepsilon$ (as E is analytic there), we have this equals

$$\begin{aligned}&= -\lim_{\varepsilon \to 0} \int_{|z|>\varepsilon} \frac{\partial}{\partial \bar{z}} (\varphi E)\, dx \\ &= \frac{1}{2i} \oint_{|z|=\varepsilon} \varphi E\, dz, & dz := dx_1 + i dx_2\end{aligned}$$

where we used Green's Theorem

$$\iint\limits_A \left(\frac{\partial P}{\partial x_1} - \frac{\partial Q}{\partial x_2} \right) dx_1\, dx_2 = \int_{\partial A} P\, dx_2 + Q\, dx_1.$$

Continuing,

$$\begin{aligned}&= \lim_{\varepsilon \to 0} \frac{1}{2i} \int_0^{2\pi} \varphi(\varepsilon e^{i\theta}) \frac{1}{\pi \varepsilon e^{-i\theta}} i \varepsilon e^{i\theta}\, d\theta \\ &\quad \big(\text{used } z = \varepsilon e^{i\theta}, 0 < \theta \le 2\pi \text{ to parameterize integral}\big) \\ &= \frac{1}{2\pi} 2\pi \varphi(0).\end{aligned}$$

So $\left\langle \frac{\partial E}{\partial \bar{z}}, \varphi \right\rangle = \varphi(0) = \langle \delta_0, \varphi \rangle$, so $\frac{\partial E}{\partial \bar{z}} = \delta$. □

Lemma 4.2.3: lem:dist4-4 Let $P(D) = \frac{\partial}{\partial t} - \Delta_x$ be the heat operator. Define the function

$$E(x,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} \exp\left(\frac{-|x|^2}{4t}\right), & t > 0 \\ 0, & t \le 0. \end{cases}$$

Then $P(D)E = \delta_0$.

Proof. For $\varphi \in D(\mathbb{R}^{n+1})$ we have

$$\begin{aligned}\langle(\partial_t - \Delta_x)E, \varphi\rangle &= -\langle E, \partial_t\varphi + \Delta_x\varphi\rangle \\ &= -\int_0^\infty dt \int dx\, E(x,t)(\varphi_t + \Delta_x\varphi) \\ &= -\lim_{\varepsilon\to 0}\int_\varepsilon^\infty dt \int dx\, E(\varphi_t + \Delta_x\varphi) \\ &= -\lim_{\varepsilon\to 0}\int_\varepsilon^\infty dt \int dx \left[(E\varphi)_t - \underbrace{(E_t - \Delta_x E)\varphi}_{=0 \text{ for } t>0}\right] \\ &= \lim_{\varepsilon\to 0}\int E(x,\varepsilon)\varphi(x,\varepsilon) \\ &= \varphi(0)\end{aligned}$$

by substituting $x' = 2\sqrt{\varepsilon}x$ (calculations omitted). ☐

13 Feb Recap

- We defined fundamental solutions for $P(D)$: $E \in D'(\mathbb{R}^n)$ such that $P(D)E = \delta_0$.
- Let $E = \frac{1}{\pi z}$, $z = x_1 + ix_2$ is the fundamental solution for the Cauchy-Riemann operator $\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$.
- $E(x,t) = \begin{cases}(4\pi t)^{-\frac{n}{2}}\exp\left(\frac{-|x|^2}{4t}\right), & t > 0 \\ 0, & t \le 0.\end{cases}$ is the fundamental solution for $P = \frac{\partial}{\partial t} - \Delta_x$.

We would like to construct the fundamental solution for arbitrary $P(D)$. A good first guess would be to define

$$\varphi \mapsto \frac{1}{(2\pi)^n}\int \frac{\hat\varphi(-\lambda)}{P(\lambda)}\,d\lambda \overset{?}{=} \langle E, \varphi\rangle, \qquad \varphi \in D(\mathbb{R}^n).$$

In that case,

$$\begin{aligned}\langle P(D)E, \varphi\rangle &:= \frac{1}{(2\pi)^n}\langle E, P(-D)\varphi\rangle = \frac{1}{(2\pi)^n}\int \frac{[P(-D)\varphi]^n(-\lambda)}{P(\lambda)}\,d\lambda \\ &= \frac{1}{(2\pi)^n}\int \frac{P(\lambda)\hat\varphi(-\lambda)}{P(\lambda)}\,d\lambda = \frac{1}{(2\pi)^n}\int \hat\varphi(-\lambda) = \varphi(0) = \langle \delta_0, \varphi\rangle.\end{aligned}$$

The problem is that the function $\lambda \mapsto \frac{1}{P(\lambda)}$ may not be locally integrable! We will avoid the zeros of $P(\lambda)$ by instead integrating over "Hörmander's staircase," rather than $\mathbb{R}^n$. We need the following lemma.

Lemma 4.2.4: lem:dist4-5 For $x \in \mathbb{R}^n$ write $x = (x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$. Then for $\varphi \in D(\mathbb{R}^n)$ the function $\hat\varphi(\lambda', z)$ is complex analytic in $z \in \mathbb{C}$ for each $\lambda' \in \mathbb{R}^{n-1}$ and

$$|\hat\varphi(\lambda', z)| \lesssim_m (1 + |z|)^{-m} e^{\delta|\Im z|}, \qquad m \in \mathbb{N}_0.$$

for some $\delta > 0$, where the implied constant depends on m but not on $\lambda' \in \mathbb{R}^{n-1}$.

Proof. By definition of the Fourier Transform,

$$\hat{\varphi}(\lambda', z) = \int \left[\int e^{-izx_n}\varphi(x', x_n)\, dx_n\right] e^{-i\lambda'\cdot x'}\, dx'.$$

We can see that $z \mapsto \hat{\varphi}(\lambda', z)$ is complex analytic (e.g., differentiate with respect to z under the integral, or use Morera, etc.). We also have for $m \in \mathbb{N}_0$ that

$$\begin{aligned}
|z^m \hat{\varphi}(\lambda', z)| &= \left|\int \left\{\int \left[\left(-i\frac{\partial}{\partial x_n}\right)^m e^{-izx_n}\right] \varphi(x', x_n)\, dx_n\right\} e^{-i\lambda'\cdot x'}\, dx'\right| \\
&= \left|\int \left[\int e^{-izx_n}\frac{\partial^m \varphi}{\partial x_n^m}(x', x_n)\, dx_n\right] e^{-i\lambda'\cdot x'}\, dx'\right| && \text{IbP} \\
&\leq \int e^{\delta|\Im z|}\frac{\partial^m \varphi}{\partial x_n^m}\, dx \\
&\qquad \text{where } \delta > 0 \text{ is such that } \varphi \equiv 0 \text{ on } |x_n| > \delta \\
&\gtrsim_m e^{\delta|\Im z|}
\end{aligned}$$

as desired. $\square$

We can now prove the celebrated Malgrange-Ehrenpries theorem.

Theorem 4.2.5: thm:dist4-2 Every non-zero, constant coefficient partial differential operator has a fundamental solution.

Proof. By a rotation of coordinates, we can assume that our PDO takes the form

$$P(\lambda', \lambda_n) = \lambda_n^M + \sum_{i=0}^{M-1} a_i(\lambda')\lambda_n^i$$

where the $a_i(\lambda')$ are polynomials in λ'. We want to construct distributions of the form

$$\langle E, \varphi\rangle = \int_\Gamma \frac{\hat{\varphi}(-\lambda)}{P(\lambda)}\, d\lambda$$

where Γ avoids the zero set of $P(\lambda)$. Fix $\mu' \in \mathbb{R}^{n-1}$. Then we can write

$$P(\mu', \lambda_n) = \prod_{i=1}^{M}(\lambda_n - \tau_i(\mu'))$$

so the $\tau_i(\mu')$ are the roots of the polynomial $\lambda_n \mapsto P(\mu', \lambda_n)$. We claim that there is a horizontal line $\Im\lambda_n = c$ in the complex λ_n-plane, with $|\Im\lambda_n| \leq M + 1$ such that

$$|\Im(\lambda_n - \tau_1(\mu'))| > 1$$

for $i = 1, \ldots, M$. Indeed, inside $|\Im\lambda_n| \leq M + 1$, there are $2M + 2$ strips of width 1, and since each of the M roots can lie in at most 1 strip, there must be two adjacent strips containing no roots. By choosing the horizontal line that separates these strips, we have the desired estimate $|\Im(\lambda_i - \tau_i(\mu'))| > 1$ for all $i = 1, \dots, M$ on $|\Im\lambda_n| = c$. By a simple continuity argument, the same result holds for λ' in a small neighborhood $N(\mu')$ of μ'. Since this argument works for any $\mu' \in \mathbb{R}^{n-1}$, we can cover $\mathbb{R}^{n-1}$ with open sets $N(\mu')$ such that on $\Im\lambda_n = c(\mu')$, we have

$$|P(\lambda', \lambda_n)| = \left|\prod_i^M (\lambda_n - \tau_i(\lambda'))\right| \geq \prod_{i=1}^M |\Im(\lambda_n - \tau_i(\lambda'))| > 1.$$

Because any open cover of a compact set contains a finite subcover and $\mathbb{R}^{n-1}$ is locally compact, we can extract a countable, locally finite subcover[1] $N_1 = N(\mu_1), N_2 = N(\mu_2), \dots$ of $\mathbb{R}^{n-1}$ and we know for $\lambda' \in N_p$ we can choose $c_i \equiv c(\mu_i)$ such that $|P(\lambda', \lambda_n)| > 1$ when $\lambda' \in N_i$ and $\Im\lambda_n = c_i$. To make sets disjoint, write $\Delta_i = N_i \backslash \bigcup_{j=1}^{i-1} \overline{N_j}$. Then the $\{D_i\}_{i\geq 1}$ are disjoint, and their closures form a locally finite cover of $\mathbb{R}^{n-1}$, $\bigcup \overline{\Delta_i} = \mathbb{R}^{n-1}$. In particular,

$$\int_{\mathbb{R}^n} d\lambda' = \sum_{i=1}^\infty \int_{\Delta_i} d\lambda'.$$

Now define $E \in D'(\mathbb{R}^n)$ via

$$\langle E, \varphi \rangle = \frac{1}{(2\pi)^n} \sum_{i=1}^\infty \int_{\Delta_i} d\lambda' \int_{\Im\lambda_n = c_i} d\lambda_n \, \frac{\hat\varphi(-\lambda_1' - \lambda_n)}{P(\lambda)}.$$

(Exercise: $E \in D'(\mathbb{R}^n)$.) Then

$$\begin{aligned}
\langle P(D)E, \varphi \rangle &= \frac{1}{(2\pi)^n} \sum_{i=1}^\infty \int_{\Delta_i} d\lambda' \int_{\Im\lambda_n = \varphi} d\lambda_n \hat\varphi(-\lambda_1' - \lambda_n) \frac{\cancel{P(\lambda)}}{\cancel{P(\lambda)}} \\
&= \frac{1}{(2\pi)^n} \sum_{i=1}^\infty \int_{\Delta_i} d\lambda' \int \hat\varphi(-\lambda_1' - \lambda_n) \, d\lambda_n && \text{using Cauchy's Theorem and Lemma 4.2.4} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat\varphi(-\lambda) \, d\lambda \\
&= \varphi(0) = \langle \delta_0, \varphi \rangle .
\end{aligned}$$

So E is a fundamental solution of $P(D)$. □

24 Feb
Recap

- We constructed Hörmander's staircase.

  We defined $E \in D'(\mathbb{R}^n)$ with

  $$\langle E, \varphi \rangle := \sum_{i=1}^\infty \int_\Delta \int_{\Im\lambda_n = c} \frac{\varphi(-\lambda)}{P(\lambda)} \, d\lambda.$$

  This stays away from all the zeros of $P(\lambda)$.

[1] this means that only finitely many cover any compact set

Let's focus on the case where $P = P(D)$ is an ordinary differential operator. In this case, Hörmander's construction shows that there is always a fundamental solution for $P(D)$ of the form Hu where u is a classical solution to $P(D)u = 0$ and H is the Heaviside function.

We follow our nose. As in the previous theorem, define

$$\langle E, \varphi \rangle := \int_{\Im\lambda = c} \frac{\varphi(-\lambda)}{P(\lambda)} \, d\lambda$$

where $c \in \mathbb{R}$ is chosen so that $\Im\lambda = c$ lies below all the roots of $P(\lambda) = 0$. (In 1 dimension, we can avoid all the bad bits by swooping underneath them all.) It is easy to see that $P(D)E = \delta_0$. Consider the cases $x > 0$ and $x < 0$ separately. We have

$$\begin{aligned} \langle E, \varphi \rangle &= \int_{\Im\lambda = c} \frac{\widehat{\varphi}}{P(\lambda)} \, d\lambda && (4.4) \\ &= \lim_{R\to\infty} \int_{-R-ic}^{R+ic} \frac{\widehat{\varphi}(-\lambda)}{P(\lambda)} \, d\lambda_{\text{eq:dist4-3}} && (4.5) \\ &= \lim_{R\to\infty} \int_{-R-ic}^{R+ic} \frac{1}{P(\lambda)} \left[\int e^{i\lambda x} \varphi(x) \, dx \right] d\lambda && (4.6) \\ &= \lim_{R\to\infty} \int \varphi(x) \left[\int_{-R-ic}^{R+ic} \frac{e^{i\lambda x}}{P(\lambda)} \, d\lambda \right] dx_{\text{eq:dist4-4}} && (4.7) \end{aligned}$$

where (4.5) follows by Dominated Convergence 1.1.1 and (4.7) follows from Fubini.

- If $x < 0$, close contours in the lower half of the λ-plane: figure 3

  $$\int_{-R+ic}^{R+ic} = \iint_{\gamma_R} + \int_{\cup}.$$

  By Cauchy's Theorem, $\iint_{\gamma_R} = 0$. Also since $x < 0$,

  $$|e^{i\lambda x}| = e^{-x\Im\lambda} \to 0$$

  along the lower semicircle of radius R as $R \to \infty$. So

  $$\lim_{R\to\infty} \int_{-R-ic}^{R+ic} \frac{e^{i\lambda x}}{P(\lambda)} \, d\lambda = 0$$

  when $x < 0$.

- If $x > 0$, instead close in upper half plane figure 4

  $$\int_{-R+ic}^{R+ic} = \iint_{\gamma_R} + \int_{\cap}.$$

  By the residue theorem, for R sufficiently large,

  $$\iint_{\gamma_R} = 2\pi i \sum \Re s \left(\frac{e^{i\lambda x}}{P(\lambda)} \right).$$

We have $\int_\cap \to 0$ as $R \to \infty$ since $|e^{-i\lambda x}| = e^{-x\Im\lambda} \to 0$ as $R \to \infty$. Hence

$$E = \begin{cases} 0, & x < 0 \\ 2\pi i \sum \operatorname{Res}\left[\frac{e^{i\lambda x}}{P(\lambda)}\right], & x > 0. \end{cases}$$

Since each term in the sum satisfies $P(D)u = 0$, we conclude $E = Hu$, as desired.

This shows that Hörmander's proof can be used to get explicit fundamental solutions (Green's functions).

3 Structure theorem for $\mathcal{E}'(X)$

We already know that if $f \in C(X)$, then $\partial^\alpha f$ defines elements of $D'(X)$ via

$$\langle \partial^\alpha f, \varphi \rangle := \left\langle f, (-1)^{|\alpha|} \partial^\alpha \varphi \right\rangle, \qquad \varphi \in D(X).$$

A natural question to ask is

Problem: Do all distributions (in $D'(X)$) arise as a linear combination of derivatives of continuous functions? I.e., if $u \in D'(X)$ does there exist $\{f_\alpha\}$ of continuous functions such that $u = \sum_\alpha \partial^\alpha f_\alpha$ in $D'(X)$?

We will answer for distributions of compact support.

We will use properties of the Fourier Transform of elements of $\mathcal{E}'(\mathbb{R}^n)$. We can define Fourier Transform of $u \in \mathcal{E}'(\mathbb{R}^n) \subseteq S'(\mathbb{R}^n)$ by

$$\hat{u}(\lambda) = \left\langle u(x), e^{-i\lambda \cdot x} \right\rangle, \qquad \lambda \in \mathbb{R}^n.$$

This coincides with the usual definition of the Fourier Transform on $S'(\mathbb{R}^n)$:

$$\begin{aligned} \langle \hat{u}, \varphi \rangle &= \langle u(x), \hat{\varphi}(x) \rangle \\ &= \left\langle u(x), \int e^{-i\lambda x} \varphi(\lambda)\, d\lambda \right\rangle \\ &\overset{?}{=} \int \left\langle u(x), e^{-i\lambda \cdot x} \varphi(\lambda) \right\rangle d\lambda \qquad \forall \varphi \in S(\mathbb{R}^n), \end{aligned}$$

where $\overset{?}{=}$ is left as an exercise (example sheet 2). By differentiating under $\langle \cdot, \cdot \rangle$ brackets, we see that $u(\lambda)$ is a smooth function of $\lambda \in \mathbb{R}^n$. Also by definition 2.1.2, there exist constants C, N and compact $K \subseteq \mathbb{R}^n$ such that

$$\begin{aligned} |\hat{u}(\lambda)| &= \left| \left\langle u(x), e^{-i\lambda \cdot x} \right\rangle \right| \\ &\le C \sum_{|\alpha| \le N} \sup_K |\partial_x^\alpha (e^{-i\lambda \cdot x})| \\ &\lesssim \langle \lambda \rangle^N. \end{aligned}$$

So $\hat{u}(\lambda)$ is a smooth function that has polynomial growth.

Theorem 4.3.1 (Structure theorem for $\mathcal{E}'(X)$)**:** thm:dist-structure If $u \in \mathcal{E}'(X)$ then there exists a collection $\{f_\alpha\}$ in $C(X)$ with $\operatorname{Supp} f_\alpha \subseteq X$ such that $u = \sum_\alpha \partial^\alpha f_\alpha$ in $\mathcal{E}'(X)$.

There is also a structure theorem for $D'(X)$. The structure theorem is handy in a practical sense, because it's easier to deal with continuous functions than distributions.

Proof. Given $u \in \mathcal{E}'(X)$, fix $\rho \in D(X)$ with $\rho = 1$ on $\operatorname{Supp}(u)$. Then for arbitrary $\varphi \in \mathcal{E}(X)$,

$$\langle u, \varphi \rangle = \langle u, \rho\varphi \rangle .$$

By extension by zero, we can treat $\varphi\rho$ as an element of $D(\mathbb{R}^n)$. Since $D(\mathbb{R}^n) \subseteq S(\mathbb{R}^n)$, we can write $\rho\varphi = (\hat{\psi})\hat{}$ for some $\psi \in S(\mathbb{R}^n)$. Hence

$$\begin{aligned}\langle u, \varphi \rangle &= \left\langle u, (\hat{\psi})\hat{} \right\rangle \\ &= \left\langle \hat{u}, \hat{\psi} \right\rangle .\end{aligned}$$

Using the Laplacian $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$ we can always write

$$\begin{aligned}\hat{\psi}(\lambda) &= \langle \lambda \rangle^{-2m} \langle \lambda \rangle^{2m} \hat{\psi}(\lambda) \\ &= \langle \lambda \rangle^{-2m} ((1-\Delta)^m \psi)\hat{}(\lambda) \\ \langle u, \varphi \rangle &= \left\langle \hat{u}(\lambda), \langle \lambda \rangle^{-2\lambda} ((1-\Delta)^m \psi)\hat{}(\lambda) \right\rangle \\ &\stackrel{1}{=} \left\langle \langle \lambda \rangle^{-2m} \hat{u}(\lambda), ((1-\Delta)^m \psi)\hat{}(\lambda) \right\rangle .\end{aligned}$$

By choosing m sufficiently large, we can guarantee that $\langle \lambda \rangle^{-2m} \hat{u}(\lambda) \in L^1(\mathbb{R}^n)$. So we can define a continuous function $f = f(x)$ via

$$\begin{aligned}\check{f}(x) &= (2\pi)^n (\langle \lambda \rangle^{-2m} \hat{u})\hat{}(x) \\ &= (2\pi)^n \int e^{-i\lambda\cdot x} \langle \lambda \rangle^{-2m} \hat{u}(\lambda)\, d\lambda.\end{aligned}$$

Continuing the calculation,

$$\begin{aligned}&\stackrel{1}{=} \left\langle (\langle \lambda \rangle^{-2m} \hat{u})\hat{}(x), (1-\Delta)^m \psi(x) \right\rangle \\ &\stackrel{2}{=} \left\langle (2\pi)^n \check{f}, (1-\Delta)^m \psi \right\rangle .\end{aligned}$$

Note that because $\rho\varphi = (\check{\psi})\check{}$, we have $\psi = (2\pi)^{-n}(\rho\varphi)\check{}$ by inverse Fourier transform. So

$$\stackrel{2}{=} \left\langle \cancel{(2\pi)^n} \check{f}, (1-\Delta)^m (\rho\varphi)\check{}\, \cancel{(2\pi)^{-n}} \right\rangle \qquad \stackrel{3}{=} \langle f, (1-\Delta)^m (\rho\varphi) \rangle .$$

By expanding out the derivative, $(1-\Delta)^m(\rho\varphi) = \sum_\alpha (-1)^{|\alpha|} \rho_\alpha \partial^\alpha \varphi$ where $\operatorname{Supp}(\rho_\alpha) \subseteq X$.

$$\begin{aligned}&\stackrel{3}{=} \left\langle f, \sum_\alpha (-1)^{|\alpha|} \rho_\alpha \partial^\alpha \varphi \right\rangle \\ &= \left\langle \sum_\alpha \partial^\alpha (\rho_\alpha f), \varphi \right\rangle \\ &= \left\langle \sum_\alpha \partial^\alpha f_\alpha, \varphi \right\rangle .\end{aligned}$$

where each f_α is continuous and $\operatorname{Supp} f_\alpha \subseteq X$. □

Note this is a constructive proof. If we're willing to just have an existence proof, Hahn-Banach offers a much shorter proof.

26 Feb Recap

- Proved that if $u \in \mathcal{E}'(X)$ then there exists $\{f_\alpha\}$ continuous with $\mathrm{Supp}(\rho_\alpha) \subseteq X$ such that $u = \sum \partial^\alpha f_\alpha$ in $\mathcal{E}'(X)$.
- Missed factor of $(2\pi)^{-n}$ in definition of E in Malgrange-Ehrenpries Theorem.

4 The Paley-Wiener-Schwartz Theorem

The theorem was originally proven by Paley and Wiener. Schwartz extended it to distributions. This is useful in applied math: sometimes we have more unknowns than equations, but we can show that 1 equation tells you more than 1 equation, almost 2. For example, in acoustics, was ask what happens when you fire waves at a plank? Some bounce off, some keep going, and some are lost in the shadow. We get one equation in two unknowns. We use an idea like the PWS theorem (Weiner-Hoftakster (?)) to solve the equation.

We saw in the previous section that if $u \in \mathcal{E}'(\mathbb{R}^n)$ then the Fourier Transform $\hat{u}(\lambda) = \left\langle u(x), e^{-i\lambda\cdot x}\right\rangle$ was a smooth function of $\lambda \in \mathbb{R}^n$. We can extend this definition for complex $z \in \mathbb{C}^n$. The complex function $z \mapsto \hat{u}(z)$ is called the Fourier-Laplace transform of u. This function is an entire function of $z \in \mathbb{C}^n$ (complex analytic everywhere). See this from the Cauchy-Riemann equations

$$\frac{\partial}{\partial \overline{z_i}} \left\langle u(x), e^{-iz\cdot x}\right\rangle = 0$$

for $i = 1, \ldots, n$ and Hartog's Theorem: a complex analytic function in each variable z_i is complex analytic in $z = (z_1, \ldots, z_n)$. Thus, we can think of $\hat{u}$ not just as a function of $\lambda \in \mathbb{R}^n$, but also $z \in \mathbb{C}^n$. We can also estimate the size of $\hat{u}(z)$ for large $|z|$.

Lemma 4.4.1: lem:dist4-6 If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\mathrm{Supp}(u) \subseteq \overline{B_\delta} = \{x \in \mathbb{R}^n : |x| \le \delta\}$ then there exists $N \ge 0$ such that

$$|\hat{u}(z)| \lesssim (1 + |z|)^N e^{\delta|\Im z|}$$

for each $z \in \mathbb{C}^n$.

Proof. Fix $\psi \in \mathcal{E}(\mathbb{R})$ such that

$$\psi(x) = \begin{cases} 0, & \tau \le -1 \\ 1, & \tau \ge -\frac{1}{2}. \end{cases}.$$

Now for each $\varepsilon > 0$ define φ_ε via $\varphi_\varepsilon(x) = \psi(\varepsilon(\delta - |x|))$, so that

$$\varphi_\varepsilon(x) = \begin{cases} \psi(x), & |x| \le \delta + \frac{1}{2\varepsilon} \\ 0, & |x| \ge \delta + \frac{1}{\varepsilon}. \end{cases}$$

Since $\varphi_\varepsilon = 1$ on the Supp(u), we can write

$$\hat{u}(z) = \left\langle u(x), \varphi_\varepsilon(x) e^{-iz\cdot x} \right\rangle .$$

Now since $u \in \mathcal{E}'(\mathbb{R}^n)$, there exists $N \geq 0$ and compact $K \subseteq \mathbb{R}^n$ such that

$$\begin{aligned} |\hat{u}(z)| &= |\left\langle u(x), \varphi_\varepsilon(x) e^{-iz\cdot x} \right\rangle| \\ &\leq \sum_{|\alpha|\leq N} \sup_K |\partial_x^\alpha (\varphi_\varepsilon(x) e^{-iz\cdot x})|, \end{aligned}$$

We will expand out the bracket using the Leibniz rule. By the chain rule,

$$|\partial_x^\beta \varphi_\varepsilon(x)| \precsim |\varepsilon|^{|\beta|}$$

and we also have (noting the expression is 0 off the support of φ_ε)

$$|\partial_x^\gamma e^{-iz\cdot x}| \precsim |z|^{|\gamma|} |e^{-iz\cdot x}| = |z|^{|\gamma|} e^{\Im z \cdot x} \leq |z|^{|\gamma|} e^{(\delta + \frac{1}{\varepsilon})|\Im z|}$$

on Supp(φ_ε). So by Leibniz product rule,

$$|\hat{u}(z)| \precsim \sum_{|\beta|+|\gamma|\leq N} |z|^{|\gamma|} \varepsilon^{|\beta|} e^{(\delta+\frac{1}{\varepsilon})|\Im z|}.$$

Set $\varepsilon = |z|$, in which case What happened to the $\frac{1}{2\varepsilon}$ in the exponent?

$$|\hat{u}(z)| \precsim e^{\delta|\Im z|} \sum_{|\beta|+|\gamma|\leq N} |z|^{|\gamma|+|\beta|} \precsim (1+|z|)^N e^{\delta|\Im z|}.$$

□

So we know that if $u \in \mathcal{E}'(\mathbb{R}^n)$, $\operatorname{Supp} u \subseteq \overline{B_\delta}$, then $\hat{u}(z)$ is entire and

$$\text{eq:dist4-5} \quad |\hat{u}(z)| \precsim (1+|z|)^N e^{\delta|\Im z|}, \qquad z \in \mathbb{C}^n, \text{ some } N \geq 0. \tag{4.8}$$

The Paley-Wiener-Schwartz theorem is about the converse: if $U = U(z)$ is an entire function of $z \in \mathbb{C}^n$ and $|U(z)| \precsim (1+|z|)^N e^{\delta|\Im z|}$ for some $N \geq 0$ and $\delta > 0$, then is $U = \hat{u}$ for some $u \in \mathcal{E}'(\mathbb{R}^n)$ with Supp$(u) \subseteq \overline{B_\delta}$? Yes. We can generalize to supports in convex sets; see Hörmander's book.

Theorem 4.4.2 (Paley-Wiener-Schwartz Theorem)**:** thm:dist4-4

1. If $u \in D(\mathbb{R}^n)$ and Supp$(u) \subseteq \overline{B_\delta}$ then $\hat{u}(z)$ is entire and

$$\text{eq:dist4-6} \quad |\hat{u}(z)| \precsim_N (1+|z|)^{-N} e^{\delta|\Im z|}, \qquad N \in \mathbb{N}_0. \tag{4.9}$$

   Conversely, if $U = U(z)$ is an entire function and satisfies an estimate of the form 4.9, then $U = \hat{u}$ where $u \in D(\mathbb{R}^n)$ with Supp$(u) \subseteq \overline{B_\delta}$.

2. If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\mathrm{Supp}(u) \subseteq \overline{B_\delta}$ then $\hat{u}(z)$ is entire and there exists $N > 0$ such that
$$\text{eq:dist4-7}|\hat{u}(z)| \precsim (1+|z|)^N e^{\delta|\Im z|}. \tag{4.10}$$
Conversely, if $U = U(z)$ is entire and satisfies an estimate of the form (4.10) then $U = \hat{u}$ where $u \in \mathcal{E}'(\mathbb{R}^n)$ with $\mathrm{Supp}(u) \subseteq \overline{B_\delta}$.

Proof. 1. If $u \in D(\mathbb{R}^n)$ then
$$\begin{aligned} |z^\alpha \hat{u}(z)| &= \left| \int (D^\alpha u)(x) e^{-iz\cdot x}\, dx \right| && \text{IbP} \\ &\le \int |D^\alpha u(x)| e^{x\cdot \Im z}\, dx \\ &\precsim_{|\alpha|} e^{\delta|\Im z|} && \text{since } \mathrm{Supp}(u) \subseteq \overline{B_\delta}. \end{aligned}$$
Thus
$$|\hat{u}(z)| \precsim_N (1+|z|)^{-N} e^{\delta|\Im z|}, \qquad N \in \mathbb{N}_0.$$
If $U = U(z)$ is entire and satisfies (4.9) then $U(\lambda)$ decays rapidly as $|\lambda| \to \infty$ in $\mathbb{R}^n$, so the function
$$u(x) = \frac{1}{(2\pi)^n} \int e^{i\lambda\cdot x} U(\lambda)\, d\lambda$$
is smooth. It remains to prove that $\mathrm{Supp}(u) \subseteq \overline{B_\delta}$. Since $U(\lambda)$ is entire and $|U(\lambda + i\eta)| \to 0$ rapidly if $|\lambda| \to \infty$ and $\eta \in \mathbb{R}^n$ is fixed. By using Cauchy's Theorem in each of the λ_i-integrals, we can shift the contour of integration onto the line $\lambda_j + i\eta_j$, η_n =constant, $-\infty < \lambda_j < \infty$. So
$$u(x) = \frac{1}{(2\pi)^n} \int e^{i(\lambda + i\eta)\cdot x} U(\lambda + i\eta)\, d\lambda.$$
2-26 figure 1 So we have
$$\begin{aligned} |u(x)| &\precsim \int e^{-\eta\cdot x} |e^{i\lambda\cdot x}| |U(\lambda + i\eta)|\, d\lambda \\ &\precsim_N e^{-\eta\cdot x} e^{\delta|\eta|} \int (1 + |\lambda + i\eta|)^{-N}\, d\lambda \\ &\precsim e^{\delta|\eta| - \eta\cdot x}. \end{aligned}$$
Set $\eta = t\frac{x}{|x|}$, where $t > 0$ then
$$|u(x)| \precsim e^{t(\delta - |x|)}.$$
If $|x| > \delta$, we can take $t \to \infty$ to conclude $u(x) = 0$ if $|x| > \delta$, i.e., $\mathrm{Supp}(u) \subseteq \overline{B_\delta}$.

2. By Lemma 4.6, the first part is already proved. For the converse, if $U = U(z)$ is entire and satisfies (4.10), the $U(\lambda)$ is polynomially bounded on $\mathbb{R}^n$, in particular it defines an element of $S'(\mathbb{R}^n)$. Since the Fourier tansform is an isomorphism on $S'(\mathbb{R}^n)$ we can write $U = \hat{u}$ for some $u \in S'(\mathbb{R}^n)$. To show $\mathrm{Supp}(u) \subseteq \overline{B_\delta}$, fix $\varphi \in D(\mathbb{R}^n)$ such that

$\int \varphi \, dx = 1$ and $\mathrm{Supp}(\varphi) \subseteq B_1$. Set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi\left(\frac{x}{\varepsilon}\right)$, so $\mathrm{Supp}(\varphi_\varepsilon) \subseteq \overline{B_\varepsilon}$. Consider regularization $u_\varepsilon = \varphi_\varepsilon * u$. Then we can show (example sheet 2) that

$$\widehat{u_\varepsilon}(\lambda) = \widehat{\varphi_\varepsilon}(\lambda)\hat{u}(\lambda) = \hat{\varphi}_\varepsilon U(\lambda).$$

By (4.9) and (4.10),

$$|\widehat{u_\varepsilon}(z)| \precsim_N (1+|z|)^{-N} e^{\varepsilon|\Im z|} e^{\delta|\Im z|}.$$

So by the first part, $\mathrm{Supp}(u_\varepsilon) \subseteq \overline{B_{\delta+\varepsilon}}$. But $\varphi_\varepsilon \to \delta_0$ in $S'(\mathbb{R}^n)$, so $u_\varepsilon \to u$ in $S'(\mathbb{R}^n)$, and we have $\mathrm{Supp}(u) \subseteq \overline{B_\delta}$.

□

Ex. class Monday 2-3, MR11

Lecture 3 Mar

Why is the Paley-Wiener-Schwartz Theorem useful? Suppose an applied maths problem came down to

$$G(\lambda) + H(\lambda) + F(\lambda) = 0, \qquad \lambda \in \mathbb{C}.$$

One equation in 3 unknowns is bad news. Obey some growth estimate than Fourier transform of some function in ball of radius δ. A similar proof shows that if $|e^{i\lambda a}U(\lambda)| \precsim (1+|\lambda|^N)e^{\delta|\Im\lambda|}$, then it is the Fourier transform of some function in a ball of radius δ around a. Suppose G, H, F obey some estimate with some a_1, a_2, a_3. If the balls are disjoint, their supports are disjoint and they must all be 0.

5 Aside: Can we take $K = \mathrm{Supp}(u)$ in the definition of $\mathcal{E}'$?

Recall (Definition 2.1.2) that a linear map $u : \mathcal{E}(X) \to \mathbb{C}$ was a distribution of compact support if there exist constants C, N and a compact set $K \subseteq X$ such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi|$$

for all $\varphi \in \mathcal{E}(X)$.

We wonder whether it is okay to take $K = \mathrm{Supp}(u)$ in this definition. Certainly $\mathrm{Supp}(u) \subseteq K = \phi$, then $\langle u, \varphi \rangle = 0$. If this was the case, then Lemma 4.4.1 would have been trivial. In general, the answer is no.

We give a famous counterexample due to L. Schwartz.

Example 4.5.1: ex:supp-counterex Define the linear map $u : \mathcal{E}(\mathbb{R}) \to \mathbb{C}$ by

$$\langle u, \varphi \rangle := \lim_{n\to\infty} \langle u_n, \varphi \rangle$$

where $u_n \in \mathcal{E}'(\mathbb{R})$ is defined by

$$\langle u_n, \varphi \rangle := \sum_{m=1}^{n} \varphi\left(\frac{1}{m}\right) - \varphi(0)n - \varphi'(0)\ln n.$$

To check $u \in \mathcal{E}'(X)$, it suffices to check that $\langle u_n, \varphi \rangle$ exists for all $\varphi \in \mathcal{E}(\mathbb{R})$ (Theorem 1.3.2 says that pointwise convergence is enough). For $\varphi \in \mathcal{E}(\mathbb{R})$ we can write by Taylor's Theorem

$$\varphi(x) = \varphi(0) + x\varphi'(0) + x^2\psi(x)$$

where ψ is an appropriate smooth function. Then

$$\begin{aligned}\langle u_n, \varphi \rangle &= \left\langle u_n, \varphi(0) + x\varphi'(0) + x^2\psi(x) \right\rangle \\ &= \sum_{n=1}^{m} \left[\varphi(0) + \frac{1}{m}\varphi'(0) + \frac{1}{m^2}\psi\left(\frac{1}{m}\right)\right] - \varphi(0)n - \varphi'(0)\ln n \\ &= \left[\sum_{m=1}^{n} \frac{1}{m} - \ln n\right]\varphi'(0) + \sum_{m=1}^{n} \frac{1}{m^2}\psi\left(\frac{1}{m}\right).\end{aligned}$$

Then it is clear that the limit $n \to \infty$ exists, since $\psi\left(\frac{1}{m}\right)$ will be bounded and $\sum \frac{1}{m^2}$ exists. So u is a distribution and clearly

$$\mathrm{Supp}(u) = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\},$$

which is compact. Now suppose that there exist C, N such that

$$\text{eq:dist4-ex}|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_{\mathrm{Supp}(u)} |\partial^\alpha \varphi|. \tag{4.11}$$

Consider sequences of smooth test functions $\{\varphi_k\}$ in $\mathcal{E}(\mathbb{R})$ with

$$\varphi_k(x) = \begin{cases} \frac{1}{\sqrt{k}}, & x \geq \frac{1}{k} \\ 0, & x \leq \frac{1}{k+1}. \end{cases}$$

Note that φ_k is constant on $x \geq \frac{1}{k}$ and $x \leq \frac{1}{k+1}$ so by continuity, all derivatives of φ_k vanish at $x = \frac{1}{k}, \frac{1}{k+1}$.

In particular, derivatives all vanish on $\mathrm{Supp}(u)$, so

$$\sum_{|\alpha| \leq N} \sup_{\mathrm{Supp}(u)} |\partial^\alpha \varphi_k| = \sup_{\mathrm{Supp}(u)} |\varphi_k| = \frac{1}{\sqrt{k}} \to 0$$

as $k \to \infty$.

However, we can evaluate $\langle u, \varphi_k \rangle$ explicitly:

$$\begin{aligned}\langle u, \varphi_k \rangle &= \lim_{n \to \infty} \sum_{m=1}^{n} \varphi_k\left(\frac{1}{m}\right) - n\varphi_k(0) - \ln n\varphi_k'(0) \\ &= \lim_{n \to \infty} \sum_{m=1}^{n} \varphi_k\left(\frac{1}{m}\right) \\ &= \sum_{m=1}^{k} \varphi_k\left(\frac{1}{m}\right) \\ &= \sum_{m=1}^{k} \frac{1}{\sqrt{k}} = \sqrt{k} \to \infty.\end{aligned}$$

So (4.11) cannot be true.

Whitney (of the embedding theorem) studied Taylor expansions of nice functions. If support is nice enough—that is, the boundary is nice in a definable way—then can get away can get away with setting $K = \mathrm{Supp}(u)$.

Chapter 5

Oscillatory integrals

1 Introduction

A the beginning of the course we played cavalier with some seemingly ill-defined integrals like

$$\int e^{-i\lambda x}\, d\lambda.$$

This is an example of an **oscillatory integral**. We will make sense of these creatures. We study very general objects of the form

$$\int e^{i\Phi(x,\theta)} a(x,\theta)\, d\theta$$

where Φ is called a **phase function** and a belongs to a class of functions called **symbols**. These integrals are not defined in a classical sense (Riemann/Lebesgue) because we will allow all the functions $a(x,\theta)$ to become large as $|\theta| \to \infty$. (For example, a could be polynomial in θ.) The key will be to make sure $a(x,\theta)$ does not oscillate quickly enough to cancel the effects of the $e^{i\Phi(x,\theta)}$ term.

We first develop some intuition. Why do oscillations help us? Recall the Riemann-Lebesgue Lemma (Example sheet 2).

Lemma 5.1.1 (Riemann-Lebesgue lemma)**:** lem:rl If $f \in L^1(\mathbb{R}^n)$ then

$$\int e^{-i\lambda\cdot x} f(x)\, dx = o(1) \text{ as } |\lambda| \to \infty.$$

If f is nice, the real/imaginary parts of $e^{-\lambda x} f(x)$ will look like

figure 1

Making the frequency large, the area of a positive blob will be almost exactly the same as the area of an adjacent negative blob.

figure 2

As λ increases, the areas above and below graph cancel, so the integral gets smaller. Mathematically, *oscillations mean we want to integrate by parts.*

(IbP for oscillatory) To estimate an oscillatory integral, integrate by parts.

To show the integral $\int e^{-i\lambda x} f(x)\,dx$ is small for large λ, we should integrate by parts: assuming f is nice,

$$\int e^{-i\lambda x} f(x)\,dx = \frac{1}{i\lambda}\int e^{-i\lambda x} f'(x)\,dx. \tag{5.1}$$

eq:dist5-1

So when dealing with oscillatory integrals,

$$\int e^{i\Phi(x,\theta)} a(x,\theta)\,d\theta,$$

we expect to gain control of this object by integration by parts. In (5.1) we wrote

$$\int e^{-i\lambda x} f(x)\,dx = \int \frac{1}{\frac{d}{dx}(-i\lambda x)}\frac{d}{dx}(e^{-i\lambda x})\varphi(x)\,dx$$

So for oscillatory integrals, we expect to do something similar. In particular, we will get away with this trick unless $\partial_\theta \Phi = 0$. So we expect to get major contributions from points satisfying $\partial_\theta\Phi = 0$. These are called points of **stationary phase**, an idea which goes back to Lord Kelvin. We will prove the stationary phase lemma.

Lecture 5 Mar Recap

- Noted that if $\varphi \in D(\mathbb{R})$ then integral $\int e^{-i\lambda x}\varphi(x)\,dx$ decays rapidly using integration by parts $\int \frac{1}{\frac{d}{dx}(-i\lambda x)}\frac{d}{dx}(e^{-i\lambda x})\varphi(x)\,dx$.
- Want to make sense of $\int e^{-i\Phi(x,\theta)} a(x,\theta)\,d\theta$.

These integrals will be defined as distributions.

Lemma 5.1.2 (Stationary phase)**:** lem:dist5-1 Let $\Phi \in C^\infty(\mathbb{R})$ such that $\Phi'(\theta) \neq 0$ on $\mathbb{R}\backslash\{0\}$ and $\Phi(0) = \Phi'(0) = 0$ and $\Phi''(\theta) \neq 0$ for all θ. Then for $\chi \in D(\mathbb{R})$ we have

$$\left|\int e^{i\lambda\Phi(\theta)}\chi(\theta)\,d\theta\right| \lesssim |\lambda|^{-\frac{1}{2}}, \quad |\lambda| \to \infty.$$

Proof. The statement of the lemma suggests that we split the integral

$$\begin{aligned}\int e^{i\lambda\Phi(\theta)}\chi(\theta)\,d\theta &= \int e^{i\lambda\Phi(\theta)}\chi(\theta)\rho\left(\frac{\theta}{\delta}\right)\,d\theta + \int e^{i\lambda\Phi(\theta)}\chi(\theta)\left(1-\rho\left(\frac{\theta}{\delta}\right)\right)\,d\theta \\ &=: I_1(\lambda) + I_2(\lambda)\end{aligned}$$

where $\rho \in D(\mathbb{R})$ is such that $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$. Note that $\rho\left(\frac{\theta}{\delta}\right) = 0$ on $|\theta| > 2\delta$. It is easy to estimate $I_1(\lambda)$:

$$|I_1(\lambda)| \le \int |\chi(\theta)\rho\left(\frac{\theta}{\delta}\right)|\,d\theta \lesssim \delta.$$

Why have we tried to estimate our integral by splitting into I_1, I_2? Recall that to show an oscillatory integral is small, we integrate by parts. On the integral I_2 we can integrate by parts because the derivative doesn't vanish.

To estimate $I_2(\lambda)$ we will need a very generic trick: by defining the differential operator

$$L = \frac{1}{i\lambda\Phi'(\theta)}\frac{d}{d\theta}$$

we see that $Le^{i\lambda\Phi} = e^{i\lambda\Phi}$, and more generally $L^N e^{i\lambda\Phi} = e^{i\lambda\Phi}$ for $N \geq 1$. The (formal) adjoint of L is

$$L^* = -\frac{d}{d\theta}\frac{1}{i\lambda\Phi'(\theta)} = -\frac{1}{i\lambda\Phi'(\theta)}\frac{d}{d\theta} + \frac{\Phi''(\theta)}{i\lambda\Phi'(\theta)^2}.$$

In particular, since $|\Phi'(\theta)| \gtrsim |\theta|$ on $\mathrm{Supp}(\chi)$ (since $\Phi''(\theta) \neq 0$), we have

$$L^* = -\frac{1}{i\lambda\Phi'(\theta)}\frac{d}{d\theta} + O(\lambda^{-1}|\theta|^{-2}).[1]$$

where we used $\left|\frac{1}{\Phi'(\theta)}\right| = \frac{1}{|\Phi'(\theta)|} \lesssim \frac{1}{|\theta|}$. By integration by parts,

$$\begin{aligned} I_2(\lambda) &= \int (L^N e^{i\Phi})\chi(\theta)\left(1 - \rho\left(\frac{\theta}{\delta}\right)\right) d\theta \\ &= \int e^{i\Phi}(L^*)^N\left(\chi(\theta)\left(1 - \rho\left(\frac{\theta}{\delta}\right)\right)\right) d\theta. \end{aligned}$$

We estimate the integrand

$$\left|(L^*)^N\left(1 - \rho\left(\frac{\theta}{\delta}\right)\right)\chi(\theta)\right| \lesssim \max\{\lambda^{-N}|\theta|^{-2N}, \delta^{-N}\lambda^{-N}|\theta|^{-N}\}.$$

To see this, note that when apply derivatives, by the product rule we will be applying derivatives to $\chi(\theta)$ or $1 - \rho\left(\frac{\theta}{\delta}\right)$.

1. For $\chi(\theta)$, θ can be small so the main contribution is from the $O(\lambda^{-1}|\theta|^{-2})$.
2. For $\rho\left(\frac{\theta}{\delta}\right)$, every time we differentiate we get a factor of $\frac{1}{\delta}$.

Thus

$$\int_{|\theta|>\delta}\left|(L^*)^N\left(1 - \rho\left(\frac{\theta}{\delta}\right)\right)\chi(\theta)\right| d\theta \lesssim \max\{\lambda^{-N}\delta^{-2N+1}, \delta^{-N}\lambda^{-N}\delta^{-N+1}\} \lesssim \lambda^{-N}\delta^{-2N+1}.$$

So we have

$$\left|\int e^{i\Phi(\theta)\lambda}\chi(\theta)\, d\theta\right| \lesssim \delta + \lambda^{-N}\delta^{-2N+1}.$$

Take $\delta = |\lambda|^{-\frac{1}{2}}$. Then $\delta^{-2N+1}\lambda^{-N} = \lambda^{-\frac{1}{2}}$, so $\left|\int e^{i\lambda\Phi(\theta)}\chi(\theta)\, d\theta\right| \lesssim |\lambda|^{-\frac{1}{2}}$. $\square$

[1]The O here not only means that the function is $O(\cdots)$, but all its derivatives are $O(\partial\cdot)$, i.e., we can differentiate inside the O. This is an important technicality.

This result is optimal: there exist test functions and Φ that achieves $\sim |\lambda|^{-\frac{1}{2}}$. Compare with $\int e^{-i\lambda x}$; here $\Phi(x) = x, \Phi'(x) = 1 \neq 0$; and our estimate on rapid decay was not optimal because we can continue to integrate by parts. But the second you let the derivative vanish, you only get decay of order $|\lambda|^{-\frac{1}{2}}$. Moral: we get large contributions when the phase has a derivative that vanishes.

Lord Kelvin studied these integrals in the context of ocean waves.

This highlights an important point: we expect oscillatory integrals

$$\int e^{i\Phi(x,\theta)} a(x,\theta)\, d\theta$$

to be "bad" at points x for which there exist θ such that $\nabla_\theta \Phi(x,\theta) := \left(\frac{\partial \Phi}{\partial \theta_1}, \dots, \frac{\partial \Phi}{\partial \theta_k}\right)$ vanishes. We can now define symbols and phase functions.

Definition 5.1.3: df:dist5-1 Let $X \subseteq \mathbb{R}^n$ be open. A smooth function $a(x,\theta) : X \times \mathbb{R}^k \to \mathbb{C}$ is called a **symbol** of order N if for each compact $K \subseteq X$ and each pair of multi-indices α, β,

$$\left| D_x^\alpha D_\theta^\beta a(x,\theta) \right| \precsim_{K,\alpha,\beta} \langle \theta \rangle^{N-|\beta|}$$

for $(x,\theta) \in K \times \mathbb{R}^n$. We call the space of all such symbols $\mathrm{Sym}(X, \mathbb{R}^n; N)$.

We just care how how large the functions get for large θ. For example, a could be a polynomial in θ whose coefficients are smooth functions in x. In particular, $a(x,\theta) = P(\theta)$, P a polynomial, defines a symbol. In general we will only care about the large $|\theta|$ behavior of $a(x,\theta)$, as we can always isolate what's going on in a compact set by introducing a bump function.

$$\int e^{i\Phi(x,\theta)} a(x,\theta)\, d\theta = \int e^{i\Phi(x,\theta)} \rho(\theta) a(x,\theta)\, d\theta + \int e^{i\Phi(x,\theta)} (1-\rho(\theta)) a(x,\theta)\, d\theta.$$

Now we define phase functions.

Definition 5.1.4: A real-valued function $\Phi : X \times \mathbb{R}^k \to \mathbb{R}$ is called a **phase function** if

1. $\Phi(x,\theta)$ is continuous on $X \times \mathbb{R}^k$ and homogeneous of degree 1 in θ. (This means $\Phi(x, t\theta) = t\Phi(x,\theta)$, $t > 0$.)

2. $\Phi(x,\theta)$ is smooth on $X \times (\mathbb{R}^k \backslash \{0\})$.

3. $$d\Phi := \nabla_x \Phi \cdot dx + \nabla_\theta \Phi \cdot d\theta = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}\, dx_i + \sum_{j=1}^k \frac{\partial \Phi}{\partial \theta_j} d\theta_j$$

   is nonvanishing on $X \times (\mathbb{R}^k \backslash \{0\})$.

Whenever you're given a very general definition or statement, it's good to find a very specific example to build intuition. A simple example of a phase function is $\Phi(x,\theta) = x \cdot \theta$. We see that $\nabla_x \Phi = 0$ so $d\Phi$ is nonvanishing if $\theta \in \mathbb{R}^k \backslash \{0\}$. This means that

$$D^\alpha \delta_0(x) \overset{?}{=} \frac{1}{(2\pi)^n} \int \theta^\alpha e^{ix\cdot\theta}\, d\theta$$

is precisely of the form

$$\int e^{i\Phi(x,\theta)} a(x,\theta)\, d\theta$$

with $a(x,\theta) = \theta^{|\alpha|}$ a symbol of order $|\alpha|$ and phase function $\Phi(x,\theta) = x\cdot\theta$.

Lemma 5.1.5: lem:dist5-2 We have:

1. If $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then $D_x^\alpha D_\theta^\beta a(x,\theta) \in \mathrm{Sym}(X,\mathbb{R}^k; N - |\beta|)$. If a_1, a_2 belong to $\mathrm{Sym}(X,\mathbb{R}^k; N_1)$ and $\mathrm{Sym}(X,\mathbb{R}^k; N_2)$ respectively then $a_1 a_2 \in \mathrm{Sym}(X,\mathbb{R}^k; N_1 + N_2)$.

Differentiating with respect to θ reduces the order of the symbol.

Thus symbols work just like polynomials in θ, except we allow smooth x dependence. Next time we will define oscillatory integrals as distributions, and we rely heavily on Lemma 5.1.5 for doing so.

Lecture 10 Mar We want to make sense of

$$\int e^{i\Phi(x,\theta)} a(x,\theta)\, d\theta.$$

by introducing the linear functional defined by

$$\varphi \mapsto \langle I_\Phi(a), \varphi\rangle = \iint e^{i\Phi(x,\theta)} a(x,\theta)\varphi(x)\, dx\, d\theta.$$

This definition is cumbersome because the double integral fails to be absolutely convergent (there's no decay in the θ direction), so we cannot apply Fubini. It is better to consider the limit $I_\Phi(a) = \lim_{\varepsilon\searrow 0} I_{\Phi,\varepsilon}(a)$ where

$$I_{\Phi,\varepsilon}(a) = \int e^{i\Phi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta)\, d\theta$$

where $\chi \in D(\mathbb{R}^n)$ is fixed with $\chi = 1$ on $|\theta| < 1$. For each $\varepsilon > 0$, $\chi(\varepsilon\theta)$ has compact support, so $I_{\Phi,\varepsilon}(a)$ makes sense classically.

Theorem 5.1.6: thm:dist5-1 If Φ is a phase function and $a \in \mathrm{Sym}(X,\mathbb{R}^k; N)$ then $I_\Phi(a) := \lim_{\varepsilon\searrow 0} I_{\Phi,\varepsilon}(a)$ belongs to $D'(X)$ and has order no greater than $N + k + 1$.

To prove this theorem we will need to find an integration by parts trick similar to that used in the proof of Lemma 5.1.2.

Lemma 5.1.7: lem:dist5-3 There exists a differential operator of the form

$$L = \sum_{j=1}^{k} a_j(x,\theta)\frac{\partial}{\partial\theta_j} + \sum_{j=1}^{n} b_j(x,\theta)\frac{\partial}{\partial x_j} + c(x,\theta)$$

where $a_j \in \mathrm{Sym}(X, \mathbb{R}^k; 0)$ and $b_j, c \in \mathrm{Sym}(X, \mathbb{R}^k; -1)$ such that $L^* e^{i\Phi} = e^{i\Phi}$. Here

$$L^* = \sum_{j=1}^{k} -\frac{\partial}{\partial\theta_j}(a_j(x,\theta)\bullet) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(b_j(x,\theta)\bullet) + c(x,\theta)$$

Proof. Note that

$$\frac{\partial}{\partial\theta_j}e^{i\Phi} = i\frac{\partial\Phi}{\partial\theta_j}e^{i\Phi}, \qquad \frac{\partial}{\partial x_j}e^{i\Phi} = i\frac{\partial\Phi}{\partial x_j}e^{i\Phi}.$$

Then

$$\begin{aligned}\left(-i\sum_{j=1}^{k}|\theta|^2\frac{\partial\Phi}{\partial\theta_j}\frac{\partial}{\partial\theta_j} - i\sum_{j=1}^{n}\frac{\partial\Phi}{\partial x_j}\frac{\partial}{\partial x_j}\right)e^{i\Phi} &= \left(\sum_{j=1}^{k}|\theta|^2\left|\frac{\partial\Phi}{\partial\theta_j}\right|^2 + \sum_{j=1}^{n}\left|\frac{\partial\Phi}{\partial x_j}\right|^2\right)e^{i\Phi} \\ &= (|\theta|^2|\nabla_\theta\Phi|^2 + |\nabla_x\Phi|^2)e^{i\Phi} \\ &= \frac{1}{\pi(x,\theta)}e^{i\Phi}\end{aligned}$$

where

$$\frac{1}{\pi(x,\theta)} = |\theta|^2|\nabla_\theta\Phi|^2 + |\nabla_x\Phi|^2.$$

Note that since Φ is a phase function,

$$\underbrace{\frac{\partial}{\partial x_j}\Phi(x,t\theta)}_{t\frac{\partial}{\partial x_j}\Phi(x,\theta)} = \frac{\partial\Phi}{\partial x_j}(x,t\theta), \qquad \text{i.e., } \frac{\partial\Phi}{\partial x_j} \text{ is homogeneous of degree 1}$$

so $|\nabla_x\Phi|^2$ is homogeneous of degree 2. Also

$$t\frac{\partial}{\partial\theta_j}\Phi(x,\theta) = \frac{\partial}{\partial\theta_j}\Phi(x,t\theta) = t\frac{\partial\Phi}{\partial\theta_j}(x,t\theta),$$

i.e., $\frac{\partial\Phi}{\partial\theta_j}$ is homogeneous of degree 0, so $|\theta|^2|\nabla_\theta\Phi|^2$ is homogeneous of degree 2. So $\pi(x,\theta)$ is $C^\infty(X\times(\mathbb{R}^k\backslash\{0\}))$ and is homogeneous of degree -2. Write

$$\widetilde{L}^* = \pi(x,\theta)\left(-i\sum_{j=1}^{k}|\theta|^2\frac{\partial\Phi}{\partial\theta_j}\frac{\partial}{\partial\theta_j} - i\sum_{j=1}^{n}\frac{\partial\Phi}{\partial x_j}\frac{\partial}{\partial x_j}\right),$$

so clearly $\widetilde{L}^*e^{i\Phi} = e^{i\Phi}$. However the coefficients of $\widetilde{L}^*$ can blow up near $\theta = 0$ so for fixed $\rho \in D(\mathbb{R}^k)$ write $L^* = (1-\rho(\theta))\widetilde{L}^* + \rho(\theta)$ with $\rho = 1$ in a neighborhood of $\theta = 0$. Then L^* has the desired properties. $\square$

We see that the operator L lowers the order of a symbol by 1, i.e., if $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then $L(a) \in \mathrm{Sym}(X, \mathbb{R}^k; N-1)$. More generally, for $\varphi \in D(X)$,

$$L^M(a(x,\theta)\varphi(x)) = \sum_{|\alpha| \le M} a_\alpha(x,\theta)\partial^\alpha \varphi(x)$$

where each a_α belongs to $\mathrm{Sym}(X, \mathbb{R}^k; N-M)$ (by a quick inductive proof). This is important.

Proof of Theorem 5.1.6. For each $\varepsilon > 0$,

$$\begin{aligned} \langle I_{\Phi,\varepsilon}(\theta), \varphi \rangle &= \iint e^{i\Phi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta)\varphi(x)\, dx\, d\theta \\ &= \iint (L^*)^M e^{i\Phi} a(x,t)\chi(\varepsilon\theta)\varphi(x)\, dx\, d\theta \\ &= \iint e^{i\Phi(x,\theta)} L^M(a(x,\theta)\chi(\varepsilon\theta)\varphi(x))\, dx\, d\theta. \end{aligned}$$

Note the simple estimate (given $0 < \varepsilon \le 1$)

$$\left| \frac{\partial}{\partial \theta^\alpha}[\chi(\varepsilon\theta)] \right| = \varepsilon^{|\alpha|} |(\partial^\alpha \chi)(\varepsilon\theta)| \lesssim_\alpha \varepsilon^{|\alpha|} \langle \varepsilon\theta \rangle^{-|\alpha|} \lesssim_\alpha \langle \theta \rangle^{-|\alpha|}$$

This is uniform in ε. So $\chi(\varepsilon\theta)$ can be treated as a symbol of order 0 uniformly in ε (we mean the estimates in the definition are independent of ε). Consequently,

$$L^M(a(x,\theta)\chi(\varepsilon\theta)\varphi(x)) = \sum_{|\alpha| \le M} a_\alpha(x,\theta;\varepsilon)\partial^\alpha \varphi(x)$$

where the a_α are symbols of order $N - M$. Choose M sufficiently large so that the θ-integral is absolutely convergent, i.e., $N - M < -k$ iff $M > N + k$, then the dominated convergence theorem 1.1.1 implies

$$\langle I_\Phi(a), \varphi \rangle = \iint e^{i\Phi(x,\theta)} L^M(a(x,\theta)\varphi(x))\, dx\, d\theta.$$

This defines a distribution of order $\le N + k + 1$. Now it is straightforward to see $I_\Phi(a) \in D'(X)$ and has order no larger than $N + k + 1$. $\square$

Lecture 12 Mar Now we know that $I_\Phi(a)$ defines an element of $D'(X)$, it is natural to ask when this distribution can be identified with a smooth function. When are they nice smooth functions and when are they bona fide distributions?

We will use the concept of singular support, written sing supp.

Definition 5.1.8: df:sing-supp The **singular support** of a distribution is defined to be the complement of the union of all the open sets on which the distribution is smooth:

$$\operatorname{sing\,supp}(u) = \left(\bigcup \{U \text{ open} : u \text{ smooth on } U\} \right)^c.$$

For example,

$$\operatorname{sing\,supp}(\delta_x) = \{x\}.$$

We expect from previous discussion that the distribution defined by $I_\Phi(a)$ will be worst at $x = x_0$ for which $\nabla_\theta \Phi(x_0, \theta) = 0$ for some $\theta \in \mathbb{R}^k$. We can make this more precise, but we need a simple lemma.

Lemma 5.1.9: lem:dist5-4 If Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then the function

$$x \mapsto \int e^{i\Phi(x,\theta)} a(x,\theta)\rho(\theta)\, d\theta$$

is smooth for any $\rho \in \mathcal{D}(\mathbb{R}^k)$.

Proof. Exercise. □

This makes sense as it is the large θ behavior that matters for the oscillatory integral. If we cut off outside some compact set, then we get a nice

Theorem 5.1.10: thm:dist5-2 If Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then $\operatorname{sing\,supp} I_\Phi(a) \subseteq M(\Phi)$ where

$$M(\Phi) = \left\{x : \nabla_\theta \Phi(x,\theta) = 0 \text{ for some } \theta \in (\mathbb{R}^k \backslash \{0\}) \cap \mathrm{Supp}[a(x,\theta)]\right\}$$

We only have θ's in the support of a, because if a is switched off can do whatever. We also remove $\theta = 0$, because recall a is homogeneous of degree 1 in θ, so bad things can happen to the derivative at the origin. There's a nice link to differential geometry, Lagrangian submanifolds in phase space. See the example sheet.

Proof. We may assume without loss of generality that $a(x,\theta)$ vanishes inside $|\theta| < 1$, since

$$I_\Phi(a) = \int e^{i\Phi(x,\theta)} a(x,\theta)\rho(\theta)\, d\theta + \int e^{i\Phi(x,\theta)} a(x,\theta)(1-\rho(\theta))\, d\theta$$

for $\rho \in D(\mathbb{R}^k)$ with $\rho = 1$ on $|\theta| \le 1$.

Because we're interested in sing supp, the only term we're interested is the second part. (The first term is smooth by Lemma 5.1.9.) Fix $x_0 \in X$ such that $|\nabla_\theta \Phi(x_0, \theta)| \neq 0$ for any $\theta \in \mathbb{R}^k \backslash \{0\}$. Since $\nabla_\theta \Phi$ is homogeneous of degree 0 and continuous in $|\theta| > 1$, the same must be true in a small neighborhood $N(x_0)$ about x_0., and on this neighborhood we have

$$|\nabla_\theta \Phi(x,\theta)| \gtrsim 1, \qquad |\theta| > 1.$$

Fix $\psi \in D(X)$ with $\mathrm{Supp}(\psi) \subseteq N(x_1)$. Consider

$$\begin{aligned} \psi(x) I_\Phi(\tilde{a}) &= \int e^{i\Phi(x,\theta)} a(x,\theta)\psi(x)\, d\theta \\ &= \int e^{i\Phi(x,\theta)} \tilde{a}(x,\theta)\, d\theta \end{aligned}$$

where $\widetilde{a}(x,\theta)$ is a symbol of same order as a, and $|\nabla_\theta \Phi| \gtrsim 1$ on $\operatorname{sing\,supp} \widetilde{a}$. So the differential operator

$$L = -i\frac{1}{|\nabla_\theta \Phi|^2}\sum_{j=1}^{k}\frac{\partial \Phi}{\partial \theta_j}\frac{\partial}{\partial \theta_j}$$

is well-defined on $\operatorname{Supp}\widetilde{a}$ and $Le^{i\Phi} = e^{i\Phi}$. (We can get away with doing less work than in Theorem 5.1.6, because know $|\nabla_\theta \Phi|$ doesn't vanish.) We have

$$\begin{aligned}
\langle \psi I_\Phi(a), \varphi\rangle &= \lim_{\varepsilon\to 0}\iint e^{i\Phi(x,\theta)}\widetilde{a}(x,\theta)\chi(\varepsilon\theta)\varphi(x)\,dx\,d\theta \\
&= \lim_{\varepsilon\to 0}\iint [L^M e^{i\Phi(x,\theta)}]\widetilde{a}(x,\theta)\chi(\varepsilon\theta)\varphi(x)\,dx\,d\theta \\
&= \lim_{\varepsilon\to 0}\iint e^{i\Phi(x,\theta)}(L^*)^M(\widetilde{a}(x,\theta)\chi(\varepsilon\theta)\varphi(x))\,dx\,d\theta \\
&= \lim_{\varepsilon\to 0}\iint e^{i\Phi(x,\theta)}\varphi(x)(L^*)^M(\widetilde{a}(x,\theta)\chi(\varepsilon\theta))\,dx\,d\theta
\end{aligned}$$

since L only acts on θ-coordinates. Again $(L^*)^M$ just lowers the order of the symbol $\widetilde{a}(x,\theta)\chi(\varepsilon\theta)$ by M. Choose M large enough so that the θ-integral becomes absolutely convergent. Then by dominated convergence 1.1.1,

$$\langle \psi I_\Phi(a), \varphi\rangle = \int \varphi(x)\Big(\int e^{i\Phi(x,\theta)}(L^*)^M(\widetilde{a}(x,\theta))\,d\theta\Big)\,dx$$

so $\psi I_\Phi(a)$ can be identified with

$$x \mapsto \int e^{i\Phi(x,\theta)}(L^*)^M(\widetilde{a}(x,\theta))\,d\theta$$

and by choosing M as large as we like, we can differentiate under the integral sign to show the function is smooth. □

Summary: if we go to a point x_0 tht isn't inside $M(\Phi)$, then looking at the distribution in a small neighborhood, it can be identified with a nice smooth funciton. If x is not in this set, it is not in sing supp.

This is useful in quantum field theory, in the theory of finite propagators. But unfortunately none of the integrals converge. Use distribution theory to make sense of them![2]

Example 5.1.11: If we consider $X = \mathbb{R}^n$ and $k = n$, then we have a simple oscillatory integral

$$u = \frac{1}{(2\pi)^n}\int e^{ix\cdot\theta}\theta^\alpha\,d\theta$$

[2]Mathematical physicists are some of the bravest mathmos; they don't care about ∞, singularities. It's sad that at Cambridge students are "separated at birth," pure or applied. It's a false dichotomy; if you get to do more, you get to see the uses of analysis. Some of the hardest problems in analysis come from mathematical physics. We still can't put QFT on a rigorous level—and QFT was initiated by Feynman, a long time ago.

We know

$$\begin{aligned}\operatorname{sing\,supp}(u) &\subseteq \left\{x : \nabla_\theta(x\cdot\theta) = 0,\ \theta \in \mathbb{R}^k\backslash\{0\}\right\}\\ &= \{0\},\end{aligned}$$

i.e., $\operatorname{sing\,supp}(u) \subseteq \{x = 0\}$. This is not surprising since $u = D^\alpha \delta_0$.

Here is a more involved example.

Example 5.1.12: We solve the wave equation

$$\begin{aligned}&\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \Delta_x E = 0 && (x,t) \in \mathbb{R}^n \times \mathbb{R}_+\\ &E = 0, \qquad \frac{\partial E}{\partial t} = \delta_0(x) && \text{when } t = 0.\end{aligned}$$

By Fourier transform in the x-variable,

$$\begin{aligned}&\frac{1}{c^2}\frac{\partial^2 \widehat{E}}{\partial t^2} + |\lambda|^2 \widehat{E} = 0, && (\lambda,t) \in \mathbb{R}^n \times \mathbb{R}_+\\ &\widehat{E} = 0, \qquad \frac{\partial \widehat{E}}{\partial t} = 1 && \text{when } t = 0.\end{aligned}$$

We quickly find

$$\begin{aligned}\widehat{E}(\lambda,t) &= \frac{\sin(c|\lambda|t)}{c|\lambda|} = \frac{1}{2ic|\lambda|}(e^{ic|\lambda|t} - e^{-ic|\lambda|t}),\\ \implies E(x,t) &= \frac{1}{(2\pi)^n}\int \frac{1}{2ic|\lambda|}(e^{ic|\lambda|t} - e^{-ic|\lambda|t})e^{i\lambda\cdot x}\,d\lambda\\ &= \frac{1}{(2\pi)^n}\left(\int \rho(\theta)(e^{\cdots} - e^{\cdots})\,d\theta + \int e^{i(x\cdot\lambda + c|\lambda|t)}a_+(\theta)\,d\theta + \int e^{i(x\cdot\lambda - c|\lambda|t)}a_-(\theta)\,d\theta\right)\end{aligned}$$

where

$$\begin{aligned}a_+(\theta) &= \frac{1}{2ic|\theta|}(1 - \rho(\theta))\\ a_-(\theta) &= -\frac{1}{2ic|\theta|}(1 - \rho(\theta)).\end{aligned}$$

where $\rho \in D(\mathbb{R}^n)$ with $\rho = 1$ on $|\theta| < 1$. We see that

$$\begin{aligned}\operatorname{sing\,supp} E &\subseteq \{(x,t) : \nabla_\theta(\theta\cdot x \pm c|\theta|t) = 0,\ \theta \in \mathbb{R}^n\backslash\{0\}\}\\ &= \left\{(x,t) : x \pm c\frac{\theta}{|\theta|}t = 0, \theta \in \mathbb{R}^n\backslash\{0\}\right\}\\ &= \{(x,t) : |x| = c|t|\}.\end{aligned}$$

This is the forward and backward light cone.

Appendix A

Examples 1

1. Let
$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} e^{-\frac{1}{(1-x)^2}}, & 0 < x < 1 \\ 0, & \text{else.} \end{cases}$$
   Let $g(x) = f\left(\frac{|x|}{\varepsilon}\right)$.

2. (a) Choose h so that the neighborhood $N_{2h}(\mathrm{Supp}(\varphi)) \subset X$, possible since K is compact and X open. Then if $x \notin N_h(\mathrm{Supp}(\varphi))$, we have $\varphi(x + th) = 0$ for all $0 \le t \le 1$, so $R_N = 0$.

   (b) Choosing h as above, on $B = \overline{N_h(\mathrm{Supp}(\varphi))}$ we have $\sup_B |\partial^\gamma \varphi| = c_\gamma$ for some c_γ. We have
$$\begin{aligned} \lim_{|h|\to 0} \left(\frac{\sup_x |\partial^\alpha R_N|}{|h|^N} \right) &= \lim_{|h|\to 0} \sum_{|\beta|=N+1} \frac{h^\alpha}{\alpha! |h|^N} (N+1) \int_0^1 (1-t)^N (\partial^{\alpha+\beta}\varphi)(x+th)\, dt \\ &= \lim_{|h|\to 0} \sum_{|\beta|=N+1} \frac{h^\alpha}{\alpha! |h|^N} (N+1) \int_0^1 (1-t)^N c_{\alpha+\beta}\, dt \\ &= \lim_{|h|\to 0} \sum_{|\beta|=N+1} \frac{|h|}{\alpha!} (N+1) c_{\alpha+\beta}\, dt \\ &= 0. \end{aligned}$$

3. 0. First consider the 1-dimensional case. A function can be represented by power series iff it is analytic. If it is analytic and has compact support, it is 0 on $X \setminus \mathrm{Supp}(f)$, so by uniqueness of analytic continuation it is 0.

   Induct on the number of dimensions. Each $(n-1)$ dimensional slice is 0 by the induction hypothesis, so the function is 0.

4. Let $\chi = 1$ on $N_\varepsilon(K)$. Now take $\chi * \delta_\varepsilon$ for δ_ε supported on $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ with unit weight and maximum at 0.

(Remark:This is useful in differential geometry to construct metrics on manifolds. We use it to paste together charts with a partition of unity.)

5. We have for some C, N,

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \le C \sum_{|\gamma| \le N} \sup \partial^\gamma \partial^\alpha \varphi \le C \sum_{|\beta| \le N + |\alpha|} \sup \partial^\beta \varphi.$$

Thus $\operatorname{ord}(\partial^\alpha T) = m + |\alpha|$. (Need to check that it's $=$ not just $\le$, but it's not hard.)

6. We first check $|\alpha| = 1$.

$$\begin{aligned} \langle \partial_x (Tf), \varphi \rangle &\overset{?}{=} \langle (\partial_x T) f, \varphi \rangle + \langle (\partial_x f) T, \varphi \rangle \\ \Leftarrow - \langle Tf, \partial_x \varphi \rangle &= \langle T, \partial_x (f\varphi) \rangle - \langle T, (\partial_x f) \varphi \rangle \end{aligned}$$

by the normal product rule. Now induct in the flavor of the multinomial theorem.

7. We have $u_\alpha = \partial^\alpha \sum_{k=1}^\infty (-1)^{|\alpha|} \delta_{x_k}$. Only a finite number of x_k are inside a given compact K, so on K this reduces to a finite sum. The order is $|\alpha|$.

8. (a) <u>Solution 1:</u> Let θ be so that $\int \theta = 1$. Let $\Phi \in D(\mathbb{R})$ be so that $\Phi' = \varphi - \theta \langle 1, \varphi \rangle = \varphi - \theta \int \varphi$, i.e., $\Phi = \int_{-\infty}^x \varphi - \theta \langle 1, \varphi \rangle \, dx$. Let $c = \langle u, \varphi \rangle$. Then

$$\begin{aligned} \langle u, \varphi \rangle &= \langle u, \varphi - \theta \langle 1, \varphi \rangle \rangle + \langle 1, \varphi \rangle \langle u, \varphi \rangle \\ &= - \langle u', \Phi \rangle + \langle 1, \varphi \rangle \langle u, \varphi \rangle \\ &= - \int \Phi + \langle c, \varphi \rangle \\ &= - \int_{-\infty}^\infty \int_0^x (\varphi - \theta \int \varphi) \, dz \, dx + \langle c, \varphi \rangle \\ &= - \int_{-\infty}^\infty -x(\varphi - \theta \int \varphi) \, dx + \langle c, \varphi \rangle \\ &= \int x\varphi - \int x\theta \int \varphi + c \int \varphi \\ &= \int (x + c_2) \varphi. \end{aligned}$$

So $u = x + c_2$ for some c_2.

<u>Solution 2:</u> The space of solutions is $x + \ker(') = \{x + c\}$ since x is a particular solution.

(b) δ_0 is a solution:

$$\langle x\delta_0', \varphi \rangle = \langle \delta_0', x\varphi \rangle = \langle \delta_0, \varphi + x\varphi' \rangle = \varphi(0) = \langle \delta_0, \varphi \rangle .$$

Let $\check{H} = (x \le 0)$. Now $A = \ker(\check{H} \oplus 1)$ is 2-codimensional, as we can find θ_1, θ_2 so that the image of θ_1, θ_2 under $\check{H} \oplus 1$ is $(1,0), (0,1)$. If $\varphi \in A$ then $\langle x, \varphi \rangle = \left\langle xu', \Phi = \frac{\int_{-\infty}^x \varphi(y)\, dy}{x} \right\rangle$ so u agrees with δ_0 on A.

There's no requirement on $\operatorname{span}(\theta_1, \theta_2)$. Hence

$$u = \delta_0 + c_1 H + c_2.$$

(c) Let's state something general (of which (b) is a special case). Suppose L is a linear operator such that $\langle u, L\varphi\rangle = 0$ and $D(\mathbb{R}) \xrightarrow{L} D(\mathbb{R}) \xrightarrow{T} V \to 0$ is exact. Then the general solution of $\langle u, L\varphi\rangle = c$ is $u_{\text{part}} + (0 \oplus f)$ where we think of $D(\mathbb{R}) = \text{im}L \oplus (\text{im}L)^{\perp}$.

Here $T = ((\frac{\int_n^{n+1}\varphi}{e^{2\pi i x}-1},)_{n\in\mathbb{Z}}, \int\varphi)$ so the general solution is

$$\sum_{n\in\mathbb{Z}} b_n \chi_{[n,n+1]} + c$$

9. Calculate

$$\begin{aligned}
\left\langle \partial_x^2 u - \partial_y^2 u, \varphi\right\rangle &= \left\langle u, \partial_x^2\varphi\right\rangle - \left\langle u, \partial_y^2\varphi\right\rangle \\
&= \int_{x\geq y} \partial_x^2\varphi - \int_{x\geq y}\partial_y^2\varphi \\
&= \int_{-\infty}^{\infty} -\partial_x\varphi(y,y)\,dy - \int_{-\infty}^{\infty} -\partial_y\varphi(x,x)\,dx \\
&= -\int_{-\infty}^{\infty} \text{grad}\cdot\varphi(t,t)\,dt \\
&= -\frac{d}{dt}\int_{-\infty}^{\infty}\varphi(t,t)\,dt = 0.
\end{aligned}$$

10. Note $|x|^{2-n}$ is locally integrable (surface area scales as $|x|^{n-1}$, their product $|x|$ is integrable) and hence a distribution.

11. This is a good kernel as $\int \frac{1}{\pi}\frac{k}{(kx)^2+1}\,dx = 1$ (so it's L^1 and has constant mass), and for any c, $\lim_{k\to\infty}\int_{|x|>c}\cdots \to 0$.

(a) Integration by parts gives

$$\left\langle k^2 x e^{-k^2x^2}, f\right\rangle = \left\langle -\frac{1}{2}e^{-k^2x^2}, f\right\rangle \to 0 \text{ as } k\to 0.$$

(b) Integration by parts 3 times gives

$$\left\langle k^3 e^{ikx}, f\right\rangle = -\left\langle e^{-ikx}, f'''\right\rangle.$$

Now $\lim_{k\to\infty} e^{-ikx} = 0$ as a distribution. Indeed this is the Fourier analysis fact that $\lim_{\omega\to 0}\widehat{f}(\omega) \to 0$ when f is smooth (once differentiable is enough; this is proved by a single integration by parts).

(c) Write $f = f(0)\chi_\varepsilon + g(x)$ where χ_ε is a symmetric bump function around 0 with support in $[-\varepsilon,\varepsilon]$. (Take $\chi_\varepsilon = \chi\left(\frac{x}{\varepsilon}\right)$.) Now $\frac{g(x)}{x}$ is a C_0^∞ function so $\left\langle\frac{\sin kx}{\pi x}, g\right\rangle \to 0$ (again since $\widehat{h}\to 0$ for h once differentiable, or by integration by parts). But

$$\int_{-\varepsilon}^{\varepsilon}\frac{\sin kx}{\pi x} f(0)\,dx = \int_{-k\varepsilon}^{k\varepsilon}\frac{\sin y}{\pi y} f(0)\chi_\varepsilon\left(\frac{y}{k}\right)\,dx.$$

We can bound the error term from $(1-\chi_\varepsilon)$ (it's not going to contribute more than $\int_{r_1 x}^{r_2 x}\frac{1}{x}\,dx = \ln r_2/r_1$), so we get $\int_{-\infty}^{\infty}\frac{\sin x}{\pi x} = 1$, so the answer is δ_0.

12. Expand f into Fourier series as $\frac{1}{2} + \sum_{m>0} a_m \cos(\pi m x) + b_m \sin(\pi m x)$ (it converges since f is C^∞). Then
$$\lim_{k\to\infty} \left\langle \frac{1}{2} + \sum_{m=1}^{k} \cos(\pi m x), f \right\rangle = \frac{1}{2}c + \sum_{m=1}^{\infty} a_m = f(0).$$
So it's δ_0. (Note pointwise convergence is enough.)

13. (a) We have
$$\lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx = \underbrace{\int_{|x|>1} \frac{\varphi(x)}{x}\, dx}_{\leq \mu(K)\sup|\varphi|} + \int_{1>|x|>\varepsilon} \underbrace{\frac{\varphi(0)}{x}}_{0} + \underbrace{\frac{\varphi(x)-\varphi(0)}{x}}_{=\varphi'(y) \text{ for some } y}\, dx$$
using the mean value theorem.

(b)
$$\begin{aligned}
\left|\int_{|x|>\varepsilon'} \left(\frac{1}{x-i\varepsilon} - \frac{1}{x}\right)\varphi(x)\, dx\right| &= \left|\int_{|x|>\varepsilon'} \frac{i\varepsilon}{x(x-i\varepsilon)}\varphi(x)\right| \\
&\leq \left|\int_{|x|>\varepsilon'} \frac{i\varepsilon}{x^2}\varphi(x)\right| \\
&\leq \varepsilon \int_{|x|>\varepsilon'} \frac{1}{x^2} \sup|\varphi| \\
&\leq \frac{\varepsilon}{\varepsilon'} \sup|\varphi| \to 0 \text{ as } r := \frac{\varepsilon}{\varepsilon'} \to 0. \\
\int_{|x|\leq\varepsilon'} \frac{1}{x-i\varepsilon} &= \ln(x-i\varepsilon)|_{-\varepsilon'}^{\varepsilon'} = \ln\frac{1-ir}{-1-ir} \to i\pi \\
\int \frac{1}{\sqrt{|x|^2+\varepsilon^2}} &\leq \sinh^{-1}\left(\frac{x}{\varepsilon}\right)|_{-\infty}^{\infty} \leq 2
\end{aligned}$$
so for fixed r, $\frac{1}{x-i\varepsilon}$ is (almost—up to some error) a good kernel. Therefore
$$\lim_{\varepsilon\to 0}\left(\frac{1}{x-i\varepsilon}\right) = p.v.\left(\frac{1}{x}\right) + \lim_{r\to 0} \ln\frac{\varepsilon(1/r-i)}{\varepsilon(1/r+i)} = p.v.\left(\frac{1}{x}\right) + i\pi\delta_0.$$

14. We have
$$\int \ln|x| f\, dx = \int (x\ln|x| - x) f'$$
with $x\ln|x| - x$ locally bounded, so $\ln|x|$ is a distribution. (Or: it is locally integrable, so it is a distribution.) Now
$$\int \ln|x| f\, dx = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \ln|x| f\, dx$$
and we can differentiate the latter to get $\int_{|x|>\varepsilon} pv\left(\frac{1}{x}\right)\, dx$.

15.

16.

Appendix B

Review

chapter.0 section.0.1 section.0.2 section.0.3 section.0.4 chapter.1 section.1.1 section.1.2 section.1.3 section.1.4 subsection.1.4.1 subsection.1.4.2 subsection.1.4.3 subsection.1.4.4 chapter.2 section.2.1 section.2.2 chapter.3 section.3.1 section.3.2 section.3.3 section.3.4 chapter.4 section.4.1 section.4.2 section.4.3 section.4.4 section.4.5 chapter.5 section.5.1 appendix.A appendix.B

1 Distributions

1.1 The spaces

Problem: Define the 3 spaces of functions $\mathcal{D}, \mathcal{S}, \mathcal{E}$ and of the corresponding distributions. What are the inclusions? Do they have the subspace topology?

1. The function spaces are

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n).$$

In short,

- $\mathcal{D}(\mathbb{R}^n)$: bounded derivatives+compact support
- $\mathcal{S}(\mathbb{R}^n)$: bounded derivatives+decay
- $\mathcal{E}(\mathbb{R}^n)$: bounded derivatives.

Formally, $\mathcal{S}, \mathcal{E}$ are locally convex spaces with the following seminorms (each is defined as the space of functions where all the norms exist and are finite, so $\mathcal{D} = C_c^\infty$, the functions of compact support, while $\mathcal{S}, \mathcal{D}$ are subsets of C^∞). $\mathcal{D}$ is a little different; see below.

Space	Seminorms
$\mathcal{D}(\mathbb{R}^n)$	$(\sup \lvert\partial^\alpha \varphi\rvert)_\alpha$
$\mathcal{S}(\mathbb{R}^n)$	$(\sup \lvert x^\alpha \partial^\beta \varphi\rvert)_{\alpha,\beta}$
$\mathcal{E}(\mathbb{R}^n)$	$(\sup_K \lvert\partial^\alpha \varphi\rvert)_{\alpha,K}$

! NOTE: The topology we use on $\mathcal{D}(\mathbb{R}^n)$ is actually stronger than the one given by the seminorms: for $f_n \to 0$ we require $\bigcup \mathrm{Supp}(f_n)$ to be contained in a compact set. A function that keeps spreading out does NOT converge to 0. Hence $\mathcal{D} \to \mathcal{S}$ is a continuous map but not homeomorphic onto its image. The inclusion $\mathcal{S} \to \mathcal{E}$ is also continuous but not homeomorphic. (So in both cases, the topology is strictly stronger than the subspace topology.)

2. The spaces of distributions are the dual spaces,

$$\mathcal{D}'(\mathbb{R}^n) \supset \mathcal{S}'(\mathbb{R}^n) \supset \mathcal{E}'(\mathbb{R}^n).$$

They have the w^*-topology.

We use the following from functional analysis (Lemma 2.2.5).

Lemma 2.1.1: If X is a locally convex space, then X^* is the set of linear functionals u for which there exists C and a finite subset S of seminorms such that

$$|ux| \le C \max_{\|\cdot\| \in S} \|x\| .$$

Equivalently, it is the set of linear functionals such that $x_n \to 0$ implies $ux_n \to 0$ (sequential continuity).

Space	$\forall u \ldots$
$\mathcal{D}'(\mathbb{R}^n)$	$\forall K \exists C, N : \langle u, \varphi\rangle \le C \sum_{\lvert\alpha\rvert \le N} \sup \lvert\partial^\alpha \varphi\rvert$
$\mathcal{S}'(\mathbb{R}^n)$	$\exists C, N : \langle u, \varphi\rangle \le C \sum_{\lvert\alpha\rvert, \lvert\beta\rvert \le N} \sup \lvert x^\alpha \partial^\beta \varphi\rvert$
$\mathcal{E}'(\mathbb{R}^n)$	$\exists C, N, K : \langle u, \varphi\rangle \le C \sum_{\lvert\alpha\rvert \le N} \sup_K \lvert\partial^\alpha \varphi\rvert$

We also have the inclusions $\mathcal{D} \subset \mathcal{E}'$ (because functions in $\mathcal{D}$ have compact support), $\mathcal{S} \subset \mathcal{S}'$, $\mathcal{E} \subset \mathcal{D}'$ (just use Cauchy-Schwarz to show sequential continuity).

1.2 Derivatives and convolution

Problem: Define the translation, reflection, and derivative of a distribution. Relate the derivative to the typical notion of a derivative. Show that if $u' = 0$ then $u = 0$.

1. Define $\tau_h f(x) = f(x-h)$, $\check{f}(x) = f(-x)$. Define the derivative by $\langle \partial u, f \rangle = -\langle u, \partial f \rangle$. This is consistent with the definition for functions by integration by parts.
2. The derivative is continuous. (Easy.)
3. Lemma: $\langle \partial u, f \rangle = \lim_{h \to 0} \left\langle \frac{\tau_{-h} u - u}{h}, f \right\rangle$. Proof: Bounce the difference quotient to the other side, and then use the fact that the difference quotient converges uniformly (the remainder in the Taylor expansion is uniformly bounded).
4. See Lemma 1.4.3.
5. Computing: use ample use of FToC. Use intuition to guess, ex. $H' = \delta_0$.

Problem:
1. Define convolution, and extend its definition as far as you can. What are all the pairs of spaces that you can take convolution between? What are the continuity properties of $*$? How can we write $\langle\rangle$ as $*$?
2. How does differentiation act with $*$?
3. Prove that $*$ is associative and commutative. (What do you need to show?)
4. Prove that $\mathcal{D} \subset \mathcal{D}'$ is dense.

1. We have

$$\text{eq:dist-conv1}\,\mathcal{D} * \mathcal{E}' \subseteq \mathcal{D} \qquad \text{b/c both of compact support} \tag{B.1}$$
$$\text{eq:dist-conv2}\,\mathcal{D} * \mathcal{D}' \subseteq \mathcal{E} \tag{B.2}$$
$$\text{eq:dist-conv3}\,\mathcal{E} * \mathcal{E}' \subseteq \mathcal{E} \tag{B.3}$$
$$\text{eq:dist-conv4}\,\mathcal{D}' * \mathcal{E}' \subseteq \mathcal{D}'. \tag{B.4}$$

   (a) Original definition: $f * g = \int f(x-y) g(y)\, dy$.
   (b) Extend: $u * \varphi(x) = \langle u, \tau_x \check{\varphi} \rangle$.
   (c) Extend by $(u_1 * u_2) * \varphi = u_1 * (u_2 * \varphi)$ to (B.4).

   We have
$$\langle u, \varphi \rangle = u * \check{\varphi}(0)$$
   (this is useful because now we can use associativity).

2. 1.4.8. $\partial^\alpha(u * \varphi) = u * \partial^\alpha \varphi$ by 1.4.7; in particular, convolution is smooth.

3. 1.4.9. Schematic of argument:

$$(u * \varphi) * \psi = \int \langle u, ...\varphi\rangle \psi = \int \langle u, \varphi \ldots \psi\rangle = \lim \sum \langle u, \varphi \ldots \psi\rangle = \left\langle u, \lim \sum \varphi \ldots \psi \right\rangle.$$

   We want the integral to go inside the brackets, so we have to turn it into a Riemann sum.

   For commutativity, we need to show $u_1 * u_2 = u_2 * u_1$ if one is in $\mathcal{E}'$, the other in $\mathcal{D}'$. Note $(u_1 * u_2) * (\varphi * \psi) = (u_1 * \psi) * (u_2 * \varphi)$. Key: we can commute φ, ψ.

4. Convolute u with a "good kernel" and then cutoff: $\varphi_m = \chi_m(u * \phi_m) \to u$ where $\chi_m = \chi\left(\frac{x}{m}\right)$ is a bump function and $\phi_m(x) = m^n \psi(mx)$. Now use $\langle u, \varphi\rangle = u * \check{\varphi}(0)$ and associativity to get $u * (\phi_m * (\chi_m \theta)\check{})(x)$. Now $\phi_m * (\chi_m\theta)\check{} = \langle \chi_m \tau_{-x}\phi_m, \theta\rangle$, and $\chi_m \tau_{-x}\phi_m$ is like a good kernel at x, so $\langle \varphi_m, \theta\rangle \to \langle u, \theta\rangle$.

1.3 The Fourier transform

Problem:
1. How do you extend the definition of the Fourier transform? Give an alternate definition that resembles the original definition; when is it valid? What is the connection between the Fourier and Laplace transform?
2. Prove the Fourier inversion formula.
3. Fill in the table below. (Be careful with the normalizations.) What are the Fourier transforms of $e^{-\varepsilon x^2}, 1, \delta_0$?

1. Define the Fourier transform by $\widehat{f} = \int_{\lambda \in \mathbb{R}^n} e^{-i\lambda\cdot x} f(x)\, dx$. (See p. 136, SS.)

   We have

$$\widehat{u} = \left\langle u, e^{-i\lambda\cdot x}\right\rangle$$

   by exchanging double integrals.

2.

3.

f	$\widehat{f} = \int_{\lambda \in \mathbb{R}^n} e^{-i\lambda \cdot x} f(x)\, dx$	$\widehat{f} = \int_{\lambda \in \mathbb{R}^n} e^{-2\pi i\lambda \cdot x} f(x)\, dx$
Fourier inversion	$f = \frac{1}{2\pi} \int \widehat{f}(\lambda) e^{i\lambda \cdot x}\, dx = \frac{1}{2\pi} \widehat{\widehat{f}\,\tilde{}}$	$f = \int \widehat{f}(\lambda) e^{2\pi i\lambda \cdot x}\, dx = \widehat{\widehat{f}\,\tilde{}}$
Parseval	$\langle \widehat{f}, g \rangle = \langle f, \widehat{g} \rangle$, $\langle f, g \rangle = \frac{1}{2\pi} \langle (\widehat{f})\check{}, \widehat{g} \rangle$	$\langle \widehat{f}, g \rangle = \langle f, \widehat{g} \rangle$, $\langle f, g \rangle = \langle (\widehat{f})\check{}, \widehat{g} \rangle$
$f(x+h)$	$\widehat{f}(\lambda) e^{ih\lambda}$	$\widehat{f}(\lambda) e^{2\pi ih\lambda}$
$f(x)e^{-ihx}$	$\widehat{f}(\xi + h)$	$\widehat{f}\left(\lambda + \frac{h}{2\pi}\right)$
$f(\delta x)$	$\delta^{-1}\widehat{f}(\delta^{-1}\lambda)$	$\delta^{-1}\widehat{f}(\delta^{-1}\lambda)$
f'	$i\lambda\widehat{f}$	$2\pi i\lambda\widehat{f}$
$Df = -if'$	$\lambda\widehat{f}$	$2\pi\lambda\widehat{f}$
xf	$-D\widehat{f}' = i\widehat{f}'$	$-\frac{1}{2\pi i}\widehat{f}'$
δ_0	1	1
1	$2\pi\delta_0$	δ_0
$e^{-\varepsilon x^2}$	$\sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{\lambda^2}{4\varepsilon}}$	$\sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{\pi^2}{\varepsilon}\lambda^2}$

Problem:
1. Define Sobolev space and local Sobolev space.
2. What are the inclusions?
3. "Fourier transform converts smoothness into decay." Make this statement precise and quantitative.

1. Definition 3.4.1 If $u \in S'(\mathbb{R}^n)$ such that $\hat{u}(\lambda)$ is a function and
$$\|u\|_{H^s}^2 := \int |\hat{u}(\lambda)|^2 (1 + |\lambda|^2)^s \, d\lambda < \infty$$
then we say $u \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. We say $u \in H^s_{\text{loc}}$ if $u\varphi \in H^s$ for every $\varphi \in \mathcal{D}$.
2. $H^{s > \frac{n}{2}}(\mathbb{R}^n) \subseteq C(\mathbb{R}^n)$ by Cauchy-Schwarz. (Careful: we need to show $\widehat{u}$ as a function agrees with $\widehat{u}$ as a distribution.)

   $\cap H^2(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n)$. (More generally, $H^{s > \frac{n}{2}+k}(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$. (I think so.) This shows $\widehat{u}\,\langle\lambda\rangle^{-k} \in L^1$.)

   $D^\alpha : H^s \to H^{s-|\alpha|}$.

   $\mathcal{E}(\mathbb{R}^n) \subseteq \bigcup_s H^s(\mathbb{R}^n)$.
3. See above.

2 Applications of distributions

Problem: Define an elliptic operator.
Show the existence of a parametrix for an elliptic PDO. What space is it in?
State the elliptic regularity theorem.

1. (Definition 4.1.1) $P(D), D = -i\partial$ corresponds to the polynomial $\sigma_P(\lambda)$. $P(D)$ is elliptic if for all $\sigma \neq 0$, $\sigma_P(\lambda) \neq 0$.

2. Define a parametrix to be $E \in \mathcal{D}'$ with $P(D)E = \delta_0 + \omega$, "fundamental solution up to $\mathcal{E}$." (Definition **??**.

   Lemma 4.1.7:

   Lemma 2.2.1: Every elliptic PDO has a parametrix$\in \mathcal{S}'(\mathbb{R}^n)$.

   Proof: $\mathcal{F}^{-1}\left(\frac{1-\chi_R(\lambda)}{P(\lambda)}\right)$. Check $P(\lambda) \gtrsim \langle\lambda\rangle^N$ (Lemma 4.1.3).

3. Note $E * f$ is more regular than f (compute the Sobolev norm). We get by writing $u = \delta_0 * u$ the following.

   Lemma 2.2.2: If $P(D)$ is elliptic of degree N; $P(D)u = f$ where $f \in H^s$; $u \in \mathcal{D}'$ with compact support (i.e., in $\mathcal{E}'$), then $u \in H^{N+s}$.

   Why is $\omega * u \in \mathcal{S}$? We don't have $\mathcal{E} * \mathcal{E}' \subseteq \mathcal{S}$!

4. Idea: Use the lemma to bootstrap.

   By definition of H^m_{loc}: it suffices to show $u\varphi \in H^{s+N}$ for every $u \in \mathcal{D}$.

   Use lemma to get

$$P(D)(u\varphi_M) = \underbrace{(P(D)u)}_{f\in H^s}\varphi + \underbrace{[P(D),\varphi]}_{-(N-1)}\underbrace{(\varphi_{M-1}u)}_{t}.$$

$$t_M \geq \min\{s, t_{M-1} - N + 1\} + N.$$

   where t_k is the regularity of $u\varphi_k$. (Warning: $P(D)(uv) \neq vP(D)u + uP(D)v$. For example, for $P(D) = D^2$ it is $(u''v) + (2u'v' + uv'')$.) We inserted φ_M because then we can iterate the process. At each stage,

$$t_k \geq \min\{s + N, t_{k-1} + 1\}$$

   so pick $M = s + N - t$.

Problem: Define a fundamental solution. Find the fundamental solution of the Cauchy-Riemann and heat operator. Prove the Malgrange-Ehrenpries Theorem.

1. A fundamental solution for a PDO $P(D)$ is $E \in \mathcal{D}'$ such that $P(D)E = \delta_0$. Then for any f, $E * f$ solves $P(D)u = f$. (Warning: uniqueness is a different issue.)

2. The fundamental solution of $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)^1$ is $\frac{1}{\pi z}$.

   (Proof: Use Green's Theorem $\oint F \cdot dv = \iint_A \nabla \times F \, dA$ on $\frac{\partial}{\partial \bar{z}}(\varphi E) = E\frac{\partial}{\partial \bar{z}}\varphi$. Note we get a $-$ since the region $|z| > \varepsilon$ is *outside*.)

   The fundamental solution of $P(D) = \frac{\partial}{\partial t} - \Delta_x$ is $(t > 0) \cdot (4\pi t)^{-\frac{n}{2}} \exp\left(\frac{-|x|^2}{4t}\right)$.

   (Pf. E basically solves the heat equation except at $t = 0$. To get over this, bounce to φ, write $\int = \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty$, and then bounce back to E. Note we get a boundary term in the t direction, $(E\varphi)_t$) in the last IbP.)

3.

Theorem 2.2.3: Every constant coefficient PDO has a fundamental solution.

Proof.

- Guess the solution. $\widehat{E} = \frac{1}{P(\lambda)}$, so

$$\langle E, \varphi \rangle = \frac{1}{(2\pi)^n} \left\langle \widehat{E}, \widehat{\tilde{\varphi}} \right\rangle = \frac{1}{(2\pi)^n} \int \frac{\widehat{\varphi}(-\lambda)}{P(\lambda)} \, d\lambda.$$

- Establish an estimaate for $\widehat{\varphi}$, when the last variable is thought of as complex. If $P(D)$ is elliptic and $\varphi \in \mathcal{D}$, then

$$|\hat{\varphi}(\lambda', z)| \lesssim_m (1 + |z|)^{-m} e^{\delta|\Im z|}, \qquad m \in \mathbb{N}_0.$$

  Key: We don't care about the exponential term outside the (compact) support of φ, so it is bounded by $e^{\delta|\Im z|}$. Bound $|z^m \widehat{\varphi}|$, and use IbP to get the derivative onto φ.

- By Cauchy's Theorem,

$$\langle E, \varphi \rangle = \frac{1}{(2\pi)^n} \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \int_{\Im \lambda_n = c_i} d\lambda_n \, \frac{\hat{\varphi}(-\lambda_1' - \lambda_n)}{P(\lambda)}$$

  is a solution provided we can bound $P(\lambda)$ away from 0 and have an estimate on $\widehat{\varphi}$. Where do we use the last condition? Also check this is a valid distribution.

  Figure of Hörmander's staircase.

  Given a point μ', we can bound $P(\mu', \lambda_n)$ away from 0 on some line $\Im \lambda_n = c$ because P is not identically 0 by ellipticity so P has finitely many roots. We can extend to neigborhoods by continuity. Now choose get Δ_i by a compactness argument.

[1]We have $(z, \bar{z}) = (x + iy, x - iy)$ so $\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial \bar{z}}{\partial x} \\ \frac{\partial z}{\partial y} & \frac{\partial \bar{z}}{\partial y} \end{pmatrix}$. The Jacobian is $\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ with inverse $\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} \frac{1}{-2i} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, so $\begin{pmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$.

4. Example of use: see example at the end of §4.2.

Problem: Prove the structure theorem for $\mathcal{E}'(X)$.

Theorem 2.2.4 (Structure Theorem for $\mathcal{E}'(X)$, 4.3.1)**:** Every $u \in \mathcal{E}$ can be written as a linear combination of derivatives of continuous functions (supported in X). ("$\mathcal{E} \subseteq \operatorname{span}_{f\in C(X),\alpha} \partial^\alpha f$.")

Proof. Idea: The FT takes differential operators to polynomials. First show $u \in H^s$, it does not grow too fast. Transfer to a problem in the Fourier domain. Considering $\langle u, \varphi\rangle$, take a polynomial term away from $\widehat{u}$; this results in derivatives on φ back in the original domain.

1. We know $\widehat{u} \in H^s$ for some s. WHY? Thus $\widehat{u}\,\langle\lambda\rangle^{-2m} \in L^1 \cap C$ for $m > \frac{s-n}{4}$ by C-S, and it has a Fourier transform/inverse. (The Fourier inversion holds on continuous L^1 functions whose transform is also L^1 (the FT is always continuous).)

2. Note $\varphi \in \mathcal{E}(X)$ may not be the FT of a function. But all functions in $\mathcal{D} \subseteq \mathcal{S}$ are FT's of functions. First step: because u has compact support, for a bump funtion ρ, $\langle u, \varphi\rangle = \langle u, \rho\varphi\rangle$.

3. Now

$$\begin{aligned}
\langle u, \rho\varphi\rangle &= \frac{1}{(2\pi)^n}\,\langle (\widehat{u})\check{}, \widehat{\rho\varphi}\rangle \\
&= \frac{1}{(2\pi)^n}\left\langle \underbrace{\langle\lambda\rangle^{-2m}\,\widetilde{u}}_{\in L^1}, \widehat{\rho\varphi}\,\langle\lambda\rangle^{2m}\right\rangle \\
&= \langle \cdots, P(D)(\rho\varphi)\rangle = \cdots.
\end{aligned}$$

□

Problem: State and prove the Paley-Wiener-Schwartz Theorem.

Theorem 2.2.5 (4.4.2)**:** Write $\mathcal{D}_X(\mathbb{R}^n)$ for the functions in $\mathcal{D}(\mathbb{R}^n)$ supported on X[2]. The Fourier-Laplace transform is a bijection

$$\begin{aligned}
\mathcal{D}_{\overline{B_\delta}} &\xrightarrow{\widehat{\bullet}} \left\{\text{entire } u : \forall N, \frac{\widehat{u}(z)}{e^{\delta|\Im z|}} \prec \langle\lambda\rangle^N\right\} \\
\mathcal{E}'_{\overline{B_\delta}}(\mathbb{R}^n) &\xrightarrow{\widehat{\bullet}} \left\{\text{entire } u : \exists N, \frac{\widehat{u}(z)}{e^{\delta|\Im z|}} \precsim \langle\lambda\rangle^N\right\}.
\end{aligned}$$

[2] we may as well write $\mathcal{D}(\mathbb{R}^n)$, right?

Proof. 1. Bound the growth of $\hat{u}$ as a complex analytic function (Lemma 4.4.1): If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\operatorname{Supp}(u) \subseteq \overline{B_\delta} = \{x \in \mathbb{R}^n : |x| \leq \delta\}$ then there exists $N \geq 0$ such that

$$|\hat{u}(z)| \precsim (1+|z|)^N e^{\delta|\Im z|}$$

for each $z \in \mathbb{C}^n$.

Proof: If u were a function, we would write out the integral and note that it vanishes outside the support of u. Here we have to treat u as a distribution, and use

$$\left\langle u, e^{-ix\cdot z}\right\rangle \leq C \sum_{|\alpha|\leq N} \sup_K |\partial^\alpha e^{-ix\cdot z}|.$$

Can we take $K = \overline{B_\delta}$? In general, we can't always take $K = \operatorname{Supp}(u)$, but in this case we almost can. To do this, replace u by $u\rho_\varepsilon$ where $\rho_\varepsilon = \psi(\frac{1}{\varepsilon}(|x|-\delta))$ where $\psi = \begin{cases} 1, & x \leq 0 \\ 0, & x \geq 1. \end{cases}$. We get

$$\left\langle u, e^{-ix\cdot z}\right\rangle \leq \sum_{|\alpha_1+\alpha_2|\leq N} C_{\alpha_1,\alpha_2} \sup_K \partial^{\alpha_1}\rho_\varepsilon \partial^{\alpha_2}\varphi \precsim \left(\frac{1}{\varepsilon}\right)^{|\alpha_1|} e^{(\delta+\varepsilon)|\Im z|}(|\sup_K \partial^{\alpha_2}\varphi|).$$

Take $\varepsilon = \frac{1}{|z|}$.

2. (1) To show rapid decay, consider $z^\alpha \hat{u}$ for all α.

$$|z^\alpha \hat{u}(z)| \precsim \int (D^\alpha u) e^{-i\lambda\cdot z}\, dx = \int u(-ix)^\alpha \underbrace{e^{-i\lambda\cdot z}}_{\text{compact support}} \leq e^{\delta|\Im z|}$$

Conversely, define u by the inverse Fourier transform

$$u = \frac{1}{(2\pi)^n}\int_{\mathbb{R}} \underbrace{\hat{u}(z)}_{|\cdot|\leq(1+|z|^{-N})e^{\delta 0}} e^{ix\cdot z}\, dz.$$

convergent by rapid decay. To use the bound, $e^{\delta\Im z}$ we have to shift the contour of integration to $\mathbb{R}+i\eta$, so that we get the integrand ($\Im z = \eta$)

$$|\cdot| \leq e^{-x\cdot\eta+\delta\eta}(1+|z|)^{-N}.$$

Given $|x| > \delta$, choose η in the direction of x, $\to \infty$. (N.B. η is a vector.)

(2) We showed the forward direction. For the reverse, regularize by $\varphi_\varepsilon = \varphi\left(\frac{x}{\varepsilon}\right)$.

$$|\widehat{u * \varphi_\varepsilon}| \precsim |U(\lambda)\widehat{(\varphi_\varepsilon)}| \leq U(\lambda)\varepsilon e^{\varepsilon\lambda}|$$

so $\operatorname{Supp}(u_\varepsilon) \subseteq \overline{B_{\delta+\varepsilon}}$. Now continuity of $*$ gives $\varphi_\varepsilon \xrightarrow{\mathcal{S}'} \delta_0 \implies u * \varphi_\varepsilon \xrightarrow{\mathcal{E}'} u$.

□

Problem: Can we take $K = \operatorname{Supp}(u)$ in the definition of $\mathcal{E}'$?

3 Oscillatory integrals

Problem: 1. Define an oscillatory integral. (Define a phase function and space of symbols first.)

2. State and prove the lemma of stationary phase.

3. Write $I_\Phi(a)$ as the limit of a sequence of functions in $\mathcal{D}$.

4. Define singular support and find a set that it's included in.

5. Find the singular support for the solution E to the wave equation $\frac{1}{c^2}E_{tt} - \Delta_x E = 0$, $E(0,x) = 0$, $E_t(0,x) = \delta_0(x)$.

"Prerequisite": Recall the IbP argument for: if all the derivatives of f up to degree k are L^1, then $\widehat{f} = O\left(\frac{1}{|f|^k}\right)$.

1. An **oscillatory integral** is denoted

$$I_{\Phi,a} = \int e^{i\Phi(x,\theta)} a(x,\theta)\, d\theta.$$

and is the distribution $\in \mathcal{D}'$ defined by

$$\langle I_{\Phi,a}, \varphi\rangle = \iint e^{i\Phi(x,\theta)} a(x,\theta)\varphi(x)\, dx\, d\theta$$

where

- $\Phi : X \times \mathbb{R}^k \to \mathbb{R}$ is a **phase function** (i.e., homogeneous of degree 1 in θ, smooth on $X \times \mathbb{R}^k\backslash 0$, continuous on $X \times \mathbb{R}^k$, $d\Phi \neq 0$). Example: $x \cdot \theta$.
- $a \in \mathrm{Sym}(X, \mathbb{R}^n; N)$ is a **symbol** (i.e., a function $X \times \mathbb{R}^n \to \mathbb{R}$ with the growth condition
$$|\partial_x^\alpha \partial_\theta^\beta a| \le \langle\theta\rangle^{N-\beta})$$
Example: a polynomial in θ of degree N. (It is easy to see $\mathrm{Sym}(X, \mathbb{R}^n)$ is a graded algebra, D_x^α doesn't change the order, D_θ^β reduces it by $|\beta|$.)

Notes for remembering:

(a) The domain for x does not have to be $\mathbb{R}^m$, it can be some $X \subseteq \mathbb{R}^m$; the domain for θ has to be $\mathbb{R}^n$.

(b) Note that the integral HAS to be taken over x first because compact support of φ makes the integral converge. You cannot use Fubini to interchange limits because the integrand does not converge absolutely—it oscillates in θ.

2. Lemma of stationary phase (Lemma 5.1.2): Let $\Phi \in C^\infty(\mathbb{R})$ such that

   - $\Phi(0) = \Phi'(0) = 0$,
   - $\Phi'(\theta) \neq 0$ on $\mathbb{R}\backslash\{0\}$, and
   - $\Phi''(\theta) \neq 0$ for all θ.

   Then for $\chi \in D(\mathbb{R})$ we have

$$\left|\int e^{i\lambda\Phi(\theta)}\chi(\theta)\,d\theta\right| \lesssim |\lambda|^{-\frac{1}{2}}, \quad |\lambda| \to \infty.$$

   Relation to oscillatory integrals: This is NOT an OI because Φ is not homogeneous of degree 1 in θ (ex. $\Phi = \theta^2$). This shows what goes wrong: if we then integrate over λ now the integral won't converge. "We expect major contributions from points satisfying $\partial_\theta \Phi = 0$," so?

   Proof: (1) Use a bump function to kill the bad point 0. Let ρ be bump at 0; shrink it to 0. Write

$$\chi(\theta) = \rho\left(\frac{\theta}{\delta}\right)\chi + \underbrace{\left(\rho\left(\frac{\theta}{\delta}\right)\right)\chi}_{\varphi}.$$

   The integral of the first part is bounded by $\boxed{\delta}$. (2) Invent L that is constant on the first term, and then IbP N times. Let

$$L = \frac{1}{i\lambda\Phi'(\theta)}\frac{d}{d\theta} \implies L* = \frac{\Phi''}{i\lambda\Phi'^2} - \frac{1}{i\lambda\Phi'}\frac{d}{d\theta} = -\frac{1}{i\lambda\Phi'}\frac{d}{d\theta} + O(\lambda^{-1}|\theta|^{-2}).$$

   Iterate N times to get terms of the form

$$\sum_{\alpha+\beta=N} \lambda^{-N}\Phi'^{-N-\alpha}\frac{d^\beta}{d\theta^\beta}P_{\alpha,\beta}(\Phi'', \Phi''', \ldots),$$

   $P_{\alpha,\beta}$ polynomials (induct). Now use

   (a) $|\Phi'| \gtrsim |\theta|$, by $\Phi'' \neq 0$ (hence it doesn't change sign and has a minimum $|\cdot|$).

   (b) ∂_θ throws off $\frac{1}{\delta}$.

   (c) ψ is only nonzero when $|\theta| > \delta$.

   Taking $(\alpha, \beta) = (N, 0), (0, N)$, get the integrand

$$\lesssim \max(\lambda^{-N}|\theta|^{-2N}, \delta^{-N}\lambda^{-N}|\theta|^{-N}) \lesssim \boxed{\lambda^{-N}\delta^{-2N+1}}.$$

   Balancing with δ, let $\delta = |\lambda|^{-\frac{1}{2}}$.

3. (Theorem 5.1.6) If Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then $I_\Phi(a) := \lim_{\varepsilon \searrow 0} I_{\Phi,\varepsilon}(a)$ belongs to $D'(X)$ and has order no greater than $N + k + 1$. Here

$$I_{\Phi,\varepsilon}(a) = \int e^{i\Phi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta)\, d\theta$$

where $\chi \in D(\mathbb{R}^n)$ is fixed with $\chi = 1$ on $|\theta| < 1$.

(a) Find a differential operator with $L^* e^{i\Phi} = e^{i\Phi}$ and iterate it. Baby example:

$$\iint e^{i\theta x} f\, dx\, d\theta = -\iint e^{i\theta x}\frac{1}{i\theta} f'\, dx\, d\theta = -\int \frac{1}{i\theta}\widehat{f'}(\theta)\, d\theta = \int \widehat{f}(\theta)\, d\theta = 2\pi f(0)$$

so $I_{\theta x}(1) = 2\pi\delta_0$.

(Lemma 5.1.7) Find a differential operator L of the form

$$L = \sum_{j=1}^{k} a_j(x,\theta)\frac{\partial}{\partial\theta_j} + \sum_{j=1}^{n} b_j(x,\theta)\frac{\partial}{\partial x_j} + c(x,\theta)$$

such that $L^* e^{i\Phi} = e^{i\Phi}$. We want it to be a "differential operator on $\mathrm{Sym}(X, \mathbb{R}^k)$ of some order." (We can think of differential operators on $\mathrm{Sym}(X, \mathbb{R}^k)$ as $\mathrm{Sym}(X, \mathbb{R}^k)[\partial_{x_j}, \partial_{\theta_j}]$ where ∂_x are order 0 and ∂_θ are order -1.)

We are given the condition $d\Phi \neq 0$ which we use. Note that $\Phi_{\theta_j} \in \mathrm{Sym}(X, \mathbb{R}^k; 0)$, $\Phi_{x_j} \in \mathrm{Sym}(X, \mathbb{R}^k; 1)$. In the below, the purple is what we get from $\partial_{\theta_j}, \partial_{x_j}$. The rest is the a_j and b_j. The underbrace is the order of the differential operator; the ones underneath the purple represent the order of $\partial_{\theta_j}, \partial_{x_j}$.

$$\widetilde{L}\frac{1}{\underbrace{|\theta|^2}_{2}\underbrace{|\nabla_\theta\Phi|^2}_{0} + \underbrace{|\nabla_x\Phi|^2}_{2}}\left(\underbrace{|\theta|^2}_{2}\underbrace{\sum(-i\Phi_{\theta_j})}_{-1}\underbrace{(i\Phi_{\theta_j})e^{i\Phi}}_{\text{(from diff.) }-1} + \underbrace{\sum(-i\Phi_{x_j})}_{0}\underbrace{(i\Phi_{x_j})}_{\text{(from diff.) }0}\right)$$

To fix blowup of the denominator near 0, use instead $L = (1-\rho)\widetilde{L} + \rho$ for a bump function ρ. (Behavior near 0 doesn't affect the order.)

(b) We have

$$I_{\Phi,\varepsilon}(a) = \iint e^{i\Phi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta)\varphi(x)\, dx\, d\theta$$

. Put in L^M, integrate by parts, noting that since L decrease order by 1, taking $M = N + k + 1$ suffices (as a has order N, $\chi(\varepsilon\theta)\varphi(x)$ has order 0 uniformly in ε (easy), and $\int_{\mathbb{R}^k} \langle\theta\rangle^{-k-1}\, d\theta$ converges).

4. (Definition 5.1.8) The **singular support** of a distribution is defined to be the complement of the union of all the open sets on which the distribution is smooth:

$$\operatorname{sing\,supp}(u) = \left(\bigcup\{U \text{ open} : u \text{ smooth on } U\}\right)^c.$$

(Theorem 5.1.10) If Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then $\operatorname{sing\,supp} I_\Phi(a) \subseteq M(\Phi)$ where

$$M(\Phi) = \left\{x : \nabla_\theta \Phi(x,\theta) = 0 \text{ for some } \theta \in (\mathbb{R}^k \backslash \{0\}) \cap \mathrm{Supp}[a(x,\theta)]\right\}$$

(a) Lemma 5.1.9: If Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then the function

$$x \mapsto \int e^{i\Phi(x,\theta)} a(x,\theta)\rho(\theta)\, d\theta$$

is smooth for any $\rho \in \mathcal{D}(\mathbb{R}^k)$.

(b) Write $a = a\rho + a(1-\rho)$, so we may assume $a = 0$ in a neighborhood of $x = 0$.

(c) Take x_0 such that $|\nabla_\theta \Phi(x_0,\theta)| \neq 0$, and a neighborhood N where it's bounded below. We need to show that for $\mathrm{Supp}(\psi) \in N$ that $\psi I_\Phi(a)$ is smooth.

(d) Find $Le^{i\Phi} = e^{i\Phi}$. We can take $L = -i\frac{1}{|\nabla_\theta \Phi|^2}\Phi_\theta \cdot \nabla_\theta$ (We're using that $|\nabla_\theta \Phi(x_0,\theta)| \gtrsim 1$ here, so L can be chosen to not involve x.) Write $I = \lim_{\varepsilon \to 0^+} I_{\Phi,\varepsilon}(a)$ and IbP M times. Choose M large to make the integral absolutely convergent.

5. Take a Fourier transform in x to obtain an ODE in t. Take the Fourier inverse and separate out behavior near 0 to get

$$E(x,t) = \frac{1}{(2\pi)^n}\left(\int \rho(\theta)(e^{\cdots} - e^{\cdots})\, d\theta + \int e^{i(x\cdot\lambda + c|\lambda| t)} a_+(\theta)\, d\theta + \int e^{i(x\cdot\lambda - c|\lambda| t)} a_-(\theta)\, d\theta\right).$$

Answer: $\operatorname{sing\,supp} E \subseteq \{(x,t) : |x| = c|t|\}$. How to show equality?

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