18.997 Probabilistic Method Problem Set #3

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Problem 1 *(3.1, $R(k,k)$)*

Lemma 1.1:

$$\binom{n}{k} \le \frac{1}{e}\left(\frac{en}{k}\right)^k.$$

Proof. By integral estimation,

$$\begin{aligned}
\ln k! &= \sum_{m=1}^{k} \ln m \\
&\ge \int_1^k \ln x\,dx \\
&= k\ln k - k + 1.
\end{aligned}$$

Exponentiating gives $k! > e\left(\frac{k}{e}\right)^k$. Hence

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \le \frac{n^k}{k!} \le \frac{1}{e}\left(\frac{en}{k}\right)^k.$$

□

Let $a = \left\lfloor \frac{k}{e}2^{\frac{k}{2}} \right\rfloor \Big/ \frac{k}{e}2^{\frac{k}{2}}$, and put $n = \left\lfloor \frac{k}{e}2^{\frac{k}{2}} \right\rfloor = a\frac{k}{e}2^{\frac{k}{2}}$ in

$$R(k,k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$$

and use the above estimate to get

$$\begin{aligned}
R(k,k) &> a\frac{k}{e}2^{\frac{k}{2}} - \frac{1}{e}\left(\frac{en}{k}\right)^k 2^{1-\binom{k}{2}} \\
&= \left(a - \frac{1}{k}\left(\frac{en}{k}\right)^k 2^{1-\frac{k^2}{2}}\right)\frac{k}{e}2^{\frac{k}{2}}. \\
&= \left(a - \frac{2a^k}{k}\right)\frac{k}{e}2^{\frac{k}{2}}.
\end{aligned}$$

For $k \ge 3$, we have $1 - \frac{1}{2^{\frac{k}{2}}} < a \le 1$, and $\frac{1}{2^{\frac{k}{2}}} = o(1)$. Since $\frac{2a^k}{k} \le \frac{2}{k} = o(1)$ as well, $R(k,k) \ge (1-o(1))\frac{k}{e}2^{\frac{k}{2}}$.

Problem 2 *(3.2, $R(4,k)$)*

Consider a complete graph with n vertices. Call the first color red and the second blue. Color each edge in the graph red with probability p and blue with probability $1-p$. The probability that a given set of 4 vertices forms a red K_4 is $p^{\binom{4}{2}}$ and the probability that a given set of k vertices forms a blue K_k is $(1-p)^{\binom{k}{2}}$.

Let X be the total number of red K_4's and blue K_k's. By linearity of expectation, since there are $\binom{n}{4}$ groups of 4 vertices and $\binom{n}{k}$ groups of k vertices.

$$\mathbb{E}(X) = \binom{n}{4}p^6 + \binom{n}{k}(1-p)^{\binom{k}{2}}.$$

There exists a coloring with at most $\mathbb{E}(X)$ red K_4's and blue K_k's. Pick a vertex from each red K_4 and blue K_k and delete it. We obtain a graph with at least $n - \mathbb{E}(X)$ vertices and no red K_4 or blue K_k. This shows that for any $n \in \mathbb{N}$ and any $p \in [0,1]$,

$$R(4,k) > n - \binom{n}{4}p^6 - \binom{n}{k}(1-p)^{\binom{k}{2}}.$$

Assume $k \geq 3$. Now pick $n = a\left(\frac{k}{\ln k}\right)^2$ and $p = \frac{2\ln k}{k}$, where a is to be chosen (depending on k to make n an integer, but close to a constant). Using Lemma 1.1,

$$\begin{aligned}
R(4,k) &\geq n - \frac{n^4}{24}p^6 - \left(\frac{en}{k}\right)^k (1-p)^{\binom{k}{2}} \\
&\geq n - \frac{n^4}{24}p^6 - \left(\frac{en}{k}\right)^k e^{-p\binom{k}{2}} \\
&= a\left(\frac{k}{\ln k}\right)^2 - \frac{1}{24}a^4\left(\frac{k}{\ln k}\right)^8 2^6\left(\frac{\ln k}{k}\right)^6 - \left(\frac{eak}{(\ln k)^2}\right)^k e^{-\frac{2\ln k}{k}\binom{k}{2}} \\
&= a\left(\frac{k}{\ln k}\right)^2 - \frac{2^6a^4}{24}\left(\frac{k}{\ln k}\right)^2 - \left(\frac{eak}{(\ln k)^2}\right)^k k^{-(k-1)} \\
&= \left(a - \frac{8}{3}a^4\right)\left(\frac{k}{\ln k}\right)^2 - \frac{k(ea)^k}{(\ln k)^{2k}}.
\end{aligned}$$

Fix $a' \in \left(0, \sqrt[3]{\frac{3}{8}}\right)$, and choose a to be as close to a' as possible, so that n is an integer. Since $\left(\frac{k}{\ln k}\right)^2 \to \infty$, we have $a \to a'$ as $k \to \infty$.

Note the last term above goes to 0 as $k \to \infty$ because $\frac{k}{(\ln k)^k} \to 0$ and $\left(\frac{ea}{\ln k}\right)^k \to 0$. Since a converges to a' as $k \to \infty$, for large k, $a - \frac{8}{3}a^4$ is bounded below by some $c > 0$. Hence $R(4,k) = \Omega(\left(\frac{k}{\ln k}\right)^2)$.

Problem 3 *(3.3, Independent set in 3-uniform hypergraph)*

Let G be a 3-uniform hypergraph with n vertices and $m \geq \frac{n}{3}$ edges. Take a random subset A by placing each vertex of G in A independently with probability p. Let $X = |A|$; then

$\mathbb{E}(X) = np$. Let Y be the number of edges in the subgraph induced by A. The probability that a given edge is in A is p^3, since each of its vertices, independently, has probability p of being in A. Since there are m edges, by linearity of expectation, $\mathbb{E}(Y) = mp^3$.

Now $\mathbb{E}(X - Y) = np - mp^3$. There exists a subset A such that $X - Y \geq np - mp^3$. For each edge in the subgraph induced by A, choose one of its vertices. Upon removing these vertices, we get a set of at least $X - Y \geq np - mp^3$ vertices with no edges between them, i.e. an independent set of size at least $np - mp^3$.

Now take $p = \left(\frac{n}{3m}\right)^{\frac{1}{2}}$ (legal since $m \geq \frac{n}{3}$). Then we get an independent set of size at least

$$np - mp^3 = n\left(\frac{n}{3m}\right)^{\frac{1}{2}} - m\left(\frac{n}{3m}\right)^{\frac{3}{2}} = \frac{2n^{\frac{3}{2}}}{3\sqrt{3}\sqrt{m}}.$$

Problem 4 *(3.4, Even directed cycle)*

We show that in fact, the statement holds when each outdegree is at least $\log_2 n - \alpha \log_2 \log_2 n$ where $\alpha \in [0, \frac{1}{2})$.

Lemma 4.1: Let G be a directed graph, whose vertices are colored in two colors such that for every vertex v, there exists a vertex w such that there is an edge from v to w, and w is colored oppositely from v. Then G has a directed even cycle.

Proof. Choose any vertex v_1. Once v_k is chosen, choose v_{k+1} to be adjacent to v_k along an outgoing edge, of the opposite color as v_k. At some point, a vertex will be repeated. Say that the first repeated vertex is v_k, and $v_k = v_j$, $j < k$. Then $v_j, v_{j+1}, \ldots, v_k$ is a simple cycle, since v_k is the first repeated vertex. Since the colors of vertices in the cycle alternate, it must have even length. $\square$

The following is Corollary 3.5.2 in the text.

Theorem 4.2:

$$m(d) = \Omega\left(2^d \left(\frac{d}{\ln d}\right)^{\frac{1}{2}}\right).$$

In other words, there exists C such that for every $d \geq 2$, any d-uniform hypergraph with at most $C2^d \left(\frac{d}{\ln d}\right)^2$ edges can be colored with two colors, so that no edge is monochromatic.

We will only need the weaker bound

$$m(d) = \Omega\left(2^d d^{\alpha'}\right), \text{ for any } \alpha' \in \left[0, \frac{1}{2}\right).$$

Given a directed graph G all of whose vertices have outdegree at least $\delta = \log_2 n - \alpha \log_2 \log_2 n$, for each vertex v let S_v be a set of vertices consisting of v and $\lceil \delta \rceil - 1$ vertices adjacent along an outgoing edge. Choose α' so that $\alpha < \alpha' < \frac{1}{2}$. Take C such that

$m(d) > C2^d d^{\alpha'}$ for $d \geq 2$. Let D be a positive constant less than 1. For large enough n,

$$\begin{aligned} m(\lceil\delta\rceil) &\geq C2^{\lceil\delta\rceil} \lceil\delta\rceil^{\alpha'} \\ &\geq C2^{\delta}\delta^{\alpha'} \\ &= Cn(\log_2 n)^{-\alpha}(\log_2 n - \alpha \log_2 \log_2 n)^{\alpha'} \\ &\geq Cn(\log_2 n)^{-\alpha} D(\log_2 n)^{\alpha'} \\ &= CDn(\log_2 n)^{\alpha'-\alpha} \geq n. \end{aligned}$$

Consider the $\lceil\delta\rceil$-uniform hypergraph whose vertices are the vertices of G and whose edges are the n sets S_v. By the above calculations, (for large enough n) there exists a coloring so that none of the S_v are monochoromatic, i.e. so that each vertex leads to a vertex of a different color. By the lemma, G has an even cycle.

Problem 5 *(4.1, $P(X = 0)$)*

Let $p_k = P(X = k)$. By the Cauchy-Schwarz inequality,

$$\left(\sum_{k\geq 0} kp_k\right)^2 \leq \left(\sum_{k>0} p_k\right)\left(\sum_{k\geq 0} k^2 p_k\right).$$

(Note the $k = 0$ terms for kp_k and k^2p_k are 0.) We rewrite this as

$$\begin{aligned} &\mathbb{E}(X)^2 \leq (1 - P(X = 0))\mathbb{E}(X^2) \\ \implies &P(X = 0)\mathbb{E}(X^2) \leq \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ \implies &P(X = 0)\mathbb{E}(X^2) \leq \mathrm{Var}(X) \\ \implies &P(X = 0) \leq \frac{\mathrm{Var}(X)}{\mathbb{E}(X^2)}. \end{aligned}$$

Problem 6 *(4.2)*

We show the inequality with $c = \frac{\sqrt{5}-2}{2}$.

Lemma 6.1: Let $a_1, \ldots, a_k$ be any real numbers, and $\varepsilon_1, \ldots, \varepsilon_k$ independent random variables taking the values -1 and 1 each with probability $\frac{1}{2}$. Let $X = \varepsilon_1 a_1 + \cdots + \varepsilon_k a_k$. Then

$$\mathrm{Var}(X) = a_1^2 + \cdots + a_k^2$$

and

$$P(|X| \leq 1) \geq 1 - (a_1^2 + \cdots + a_k^2).$$

Proof. Let $X_i = \varepsilon_i a_i$. Note $\mathrm{Var}(X_i) = \mathbb{E}((\varepsilon_i a_i)^2) = a_i^2$. Since the X_i are independent, we have

$$\mathrm{Var}(X) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_k) = a_1^2 + \cdots + a_k^2.$$

Clearly, $\mathbb{E}(X_i) = 0$ so $\mathbb{E}(X) = 0$. By Chebyshev's inequality with $\lambda = \frac{1}{\sqrt{a_1^2+\cdots+a_k^2}}$ and $\sigma = \sqrt{a_1^2 + \cdots + a_k^2}$, we get

$$P(|X| \geq 1) = P(|X - \mathbb{E}(X)| \geq \lambda\sigma) \leq \frac{1}{\lambda^2} = a_1^2 + \cdots + a_k^2.$$

Since $P(|X| \leq 1) \geq 1 - P(|X| \geq 1)$, this proves the second part. □

Let $\lambda = \sqrt{5} - 2$. Note $\frac{1-\lambda^2}{8} = \frac{\lambda}{2}$ and $c = \frac{\lambda}{2}$. Consider two cases.

1. There exists a_i with $a_i^2 \geq \lambda$. Without loss of generality, $a_1^2 \geq \lambda$. Then by the lemma,

$$P(|\varepsilon_2 a_2 + \cdots + \varepsilon_k a_k| \leq 1) \geq 1 - (a_2^2 + \cdots + a_n^2) = a_1^2 \geq \lambda.$$

Since $|a_1| \leq 1$, given that $|\varepsilon_2 a_2 + \cdots + \varepsilon_k a_k| \leq 1$, if ε_1 is such that $\varepsilon_1 a_1$ has opposite sign from $\varepsilon_2 a_2 + \cdots + \varepsilon_k a_k$, then we also have $|X| \leq 1$. Thus

$$P(|X| \leq 1) \geq \frac{1}{2} P(|\varepsilon_2 a_2 + \cdots + \varepsilon_k a_k| \leq 1) \geq \frac{\lambda}{2}.$$

2. There does not exist a_i with $a_i^2 \geq \lambda$. Let k be the greatest index so that

$$a_1^2 + a_2^2 + \cdots + a_k^2 \leq \frac{1+\lambda}{2}.$$

(Note $k \geq 1$ since $a_1^2 < \lambda < \frac{1+\lambda}{2}$.) Let $A = a_1^2 + \cdots + a_k^2$. By the maximality assumption, $a_1^2 + \cdots + a_k^2 + a_{k+1}^2 > \frac{1+\lambda}{2}$. Since $a_{k+1}^2 < \lambda$, we conclude $A > \frac{1-\lambda}{2}$. Thus by the lemma,

$$P(|\varepsilon_1 a_1 + \cdots + \varepsilon_k a_k| \leq 1) \geq 1 - (a_1^2 + a_2^2 + \cdots + a_k^2) = 1 - A.$$
$$P(|\varepsilon_{k+1} a_{k+1} + \cdots + \varepsilon_n a_n| \leq 1) \geq 1 - (a_{k+1}^2 + \cdots + a_n^2) = A.$$

By symmetry,

$$P(\varepsilon_{k+1} a_{k+1} + \cdots + \varepsilon_n a_n \in [0,1]) = P(\varepsilon_{k+1} a_{k+1} + \cdots + \varepsilon_n a_n \in [-1,0]) \geq \frac{A}{2}.$$

Now, noting $A \in [\frac{1-\lambda}{2}, \frac{1+\lambda}{2}]$ implies $A(1-A) \geq \left(\frac{1-\lambda}{2}\right)\left(\frac{1+\lambda}{2}\right) = \frac{1-\lambda^2}{4}$,

$$\begin{aligned}
P(|X| \leq 1) &\geq P(\varepsilon_1 a_1 + \cdots + \varepsilon_k a_k \in [0,1]) P(\varepsilon_{k+1} a_{k+1} + \cdots + \varepsilon_n a_n \in [-1,0]) \\
&\quad + P(\varepsilon_1 a_1 + \cdots + \varepsilon_k a_k \in [-1,0)) P(\varepsilon_{k+1} a_{k+1} + \cdots + \varepsilon_n a_n \in [0,1]) \\
&\geq P(\varepsilon_1 a_1 + \cdots + \varepsilon_k a_k \in [0,1]) \left(\frac{A}{2}\right) + P(\varepsilon_1 a_1 + \cdots + \varepsilon_k a_k \in [-1,0)) \left(\frac{A}{2}\right) \\
&= P(|\varepsilon_1 a_1 + \cdots + \varepsilon_k a_k| \leq 1) \left(\frac{A}{2}\right) \\
&\geq (1-A)\left(\frac{A}{2}\right) \geq \frac{1-\lambda^2}{8} = \frac{\lambda}{2}
\end{aligned}$$

as needed.

Problem 7 *(4.3)*

We need the following estimate. The proof is similar to Chebyshev's inequality.

Lemma 7.1: Let $a_1, \ldots, a_n$ be vectors in $\mathbb{R}^2$, and $\varepsilon_1, \ldots, \varepsilon_n$ be independently chosen to be ± 1 with probability $\frac{1}{2}$. Then

$$P\left(\left\|\sum_{k=1}^{n} \varepsilon_k a_k\right\| \geq R\right) \leq \frac{\sum_{k=1}^{n} \|a_k\|^2}{R^2}.$$

Proof. First we calculate $\mathbb{E}\left(\|\sum_{k=1}^{n} \varepsilon_k a_k\|^2\right)$. Let $a_k = (x_k, y_k)$. Now

$$\mathbb{E}\left(\left\|\sum_{k=1}^{n} \varepsilon_k a_k\right\|^2\right) = \mathbb{E}((\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n)^2 + (\varepsilon_1 y_1 + \cdots + \varepsilon_n y_n)^2)$$

Expanding, noting that $\mathbb{E}(\varepsilon_i \varepsilon_j x_i x_j) = \mathbb{E}(\varepsilon_i \varepsilon_j y_i y_j) = 0$ for $i \neq j$ (since $\varepsilon_i \varepsilon_j$ has equal probability of being ± 1), the expected value equals

$$\mathbb{E}(\varepsilon_1^2 x_1^2) + \cdots + \mathbb{E}(\varepsilon_n^2 x_n^2) + \mathbb{E}(\varepsilon_1^2 y_1^2) + \cdots + \mathbb{E}(\varepsilon_n^2 y_n^2) = x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2 = \sum_{k=1}^{n} \|a_k\|^2.$$

By Markov's inequality,

$$\begin{aligned} P\left(\left\|\sum_{k=1}^{n} \varepsilon_k a_k\right\| \geq R\right) &= P\left(\left\|\sum_{k=1}^{n} \varepsilon_k a_k\right\|^2 \geq R^2\right) \\ &\leq \frac{\mathbb{E}\left(\|\sum_{k=1}^{n} \varepsilon_k a_k\|^2\right)}{R^2} \\ &= \frac{\sum_{k=1}^{n} \|a_k\|^2}{R^2}. \end{aligned}$$

□

Lemma 7.2: Let $a_1, \ldots, a_n$ be vectors in $\mathbb{R}^2$, all of length at most r. Then there exist $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ so that

$$\|\varepsilon_1 a_1 + \cdots + \varepsilon_n a_n\| \leq \sqrt{2} r.$$

Proof. For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ and $i \neq j$, let $\mathcal{P}_{i,j}(\varepsilon)$ be the (possibly degenerate) parallelogram bounded by the 4 vertices $v \pm a_i \pm a_j$, where $v = \sum_{k \neq i,j; 1 \leq k \leq n} \varepsilon_i a_i$. Let $\mathcal{P} = \bigcup_{1 \leq i < j \leq n, \varepsilon \in \{-1,1\}^n} \mathcal{P}_{i,j}(\varepsilon)$. Note that if $Q_1 Q_2 Q_3 Q_4$ is one of these parallelograms, then we have $Q_2 = Q_1 \pm 2a_i$ for some i, and similarly for the other adjacent pairs of vertices.

We claim that $\mathcal{P}$ contains the origin. First we show that $\mathcal{P}$ is convex. Let Q be a vertex on the boundary of $\mathcal{P}$, and QQ_1 and QQ_2 be edges of $\mathcal{P}$, with $Q_1 \neq Q_2$ (i.e. QQ_1, QQ_2 are edges of some $\mathcal{P}_{i,j}(\varepsilon)$.) As mentioned, $Q_1 = Q \pm 2a_i$ for some i and $Q_2 = Q \pm 2a_j$ for some j, for $i \neq j$. Suppose the directed angle $\angle Q_1 Q Q_2$ is in the range $[0°, 180°]$. Let

$Q' = Q \pm 2a_i \pm 2a_j$, where the two signs match the signs in Q_1 and Q_2, respectively. Then Q_1QQ_2Q' is one of the parallelograms, in particular, P contains the angle $\angle Q_1QQ_2$. This shows that $\mathcal{P}$ has no reflex angle on the boundary. $\mathcal{P}$ has a well-defined outer boundary that traces a convex polygon, and has no "holes" (because holes would cause reflex angles as well). Hence $\mathcal{P}$ is convex. Since $\mathcal{P}$ is clearly symmetric around the origin, it must contain the origin.

Hence we can take a parallelogram $\mathcal{P}_{i,j}(\varepsilon)$ containing the origin. Suppose by way of contradiction that all its vertices P_1, P_2, P_3, P_4 are at a distance greater than $\sqrt{2}r$ from the origin O. One of the angles $\angle P_1OP_2, \angle P_2OP_3, \angle P_3OP_4, \angle P_4OP_1$ is at least $90°$, say WLOG P_1OP_2. Then by Pythagorean's inequality $|P_1P_2|^2 \geq |OP_1|^2 + |OP_2|^2 > 2(\sqrt{2}r)^2$ so $|P_1P_2| > 2r$. But $|P_1P_2| = 2a_i$ for some i, and $a_i > r$, contradiction. Thus one of P_1, P_2, P_3, P_4 is at most a distance of $\sqrt{2}r$ from O, proving the lemma. $\square$

Back to the problem, let $i_0 = 0$, let i_1 be the largest integer so that $\|a_1\|^2 + \cdots + \|a_{i_1}\|^2 \leq \frac{1}{20}$, let i_2 be the largest integer so that $\|a_{i_1+1}\|^2 + \cdots + \|a_{i_2}\|^2 \leq \frac{1}{20}$, and so on. (Note $i_{j+1} > i_j$ because $\|a_i\|^2 \leq \frac{1}{100}$ for all i.) Suppose this divides the a_i into t groups. For $0 \leq j < t-1$, by the maximality assumption on i_{j+1}, $\|a_{i_j+1}\|^2 + \cdots + \|a_{i_{j+1}}\|^2 + \|a_{i_{j+1}+1}\|^2 > \frac{1}{20}$; since $\|a_{i_{j+1}+1}\|^2 \leq \frac{1}{100}$, we conclude $\|a_{i_j+1}\|^2 + \cdots + \|a_{i_{j+1}}\|^2 > \frac{1}{25}$. Thus we've divided the a_i into t groups, and in all of them except the last, the sum of the squares of the absolute values is in the interval $(\frac{1}{25}, \frac{1}{20}]$. This shows $t \leq 25$.

By Lemma 7.1,

$$P\left(\left\|\sum_{k=i_j+1}^{i_{j+1}} \varepsilon_k a_k\right\| \geq \frac{1}{\sqrt{18}}\right) \leq 18 \sum_{k=i_j+1}^{i_{j+1}} \|a_k\|^2 \leq \frac{18}{20}$$

so

$$P\left(\left\|\sum_{k=i_j+1}^{i_{j+1}} \varepsilon_k a_k\right\| \leq \frac{1}{\sqrt{18}}\right) \geq \frac{1}{10}.$$

Thus the probability that $\left\|\sum_{k=i_j+1}^{i_{j+1}} \varepsilon_k a_k\right\| \leq \frac{1}{\sqrt{18}}$ for each $0 \leq j < t$ is at least $\frac{1}{10^t} \geq \frac{1}{10^{25}}$. Let $v_j = \sum_{k=i_j+1}^{i_{j+1}} \varepsilon_k a_k$. Let S be the set of $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ such that $\|v_j\| \leq \frac{1}{\sqrt{18}}$ for all j; call $v = \sum_{k=1}^n \varepsilon_k a_k$ the vector associated to ε. We say two vectors $\varepsilon, \varepsilon'$ are equivalent if

$$(\varepsilon_{i_j+1}, \ldots, \varepsilon_{i_{j+1}}) = \pm(\varepsilon'_{i_j+1}, \ldots, \varepsilon'_{i_{j+1}})$$

for each j. This divides S into equivalence classes, each containing 2^t elements. Note that the vectors associated to the 2^t elements in the equivalence class of ε are in the form $\omega_0 v_0 + \cdots + \omega_{t-1} v_{t-1}$ where $\omega_i = \pm 1$. Since $\|v_i\| \leq \frac{1}{\sqrt{18}}$, by Lemma 7.2 there exists a choice of $\omega_0, \ldots, \omega_{t-1}$ so that $\|\omega_0 v_0 + \cdots + \omega_{t-1} v_{t-1}\| \leq \frac{\sqrt{2}}{\sqrt{18}} = \frac{1}{3}$ (in fact, there exist two choices, by symmetry). Hence in each equivalence class C of S, at least 2 of the 2^t elements of C have associated vectors with absolute value at most $\frac{1}{3}$. Thus

$$\begin{aligned} P\left(\left\|\sum_{k=1}^{n} \varepsilon_k a_k\right\| \leq \frac{1}{3}\right) &\geq P\left(\bigwedge_{j=0}^{t-1}\left(\left\|\sum_{k=i_j+1}^{i_{j+1}} \varepsilon_k a_k\right\| \leq \frac{1}{\sqrt{18}}\right)\right) \cdot \frac{2}{2^t} \\ &\geq \frac{1}{10^{25}} \cdot \frac{1}{2^{24}}. \end{aligned}$$