Asynchronous Finite Difference Scheme for PDEs

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FD Schemes

Finite Difference Schemes

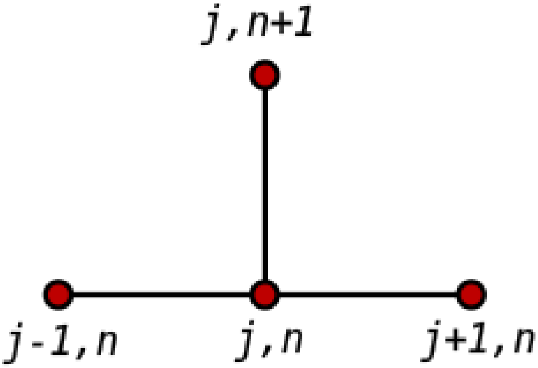
FD method can be used to discretize and solve PDEs.
Consider the heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

which can be discretized as:

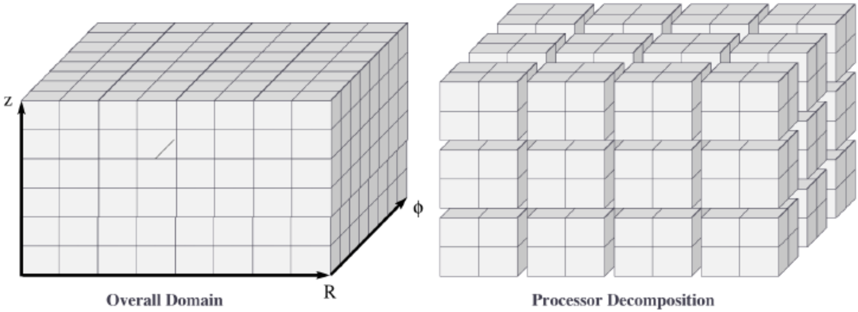
$$\frac{u_i^{n+1} - u_i^n}{\triangle t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\triangle x^2} + \mathscr{O}(\triangle x^2, \triangle t) \tag{2}$$

FD Stencil



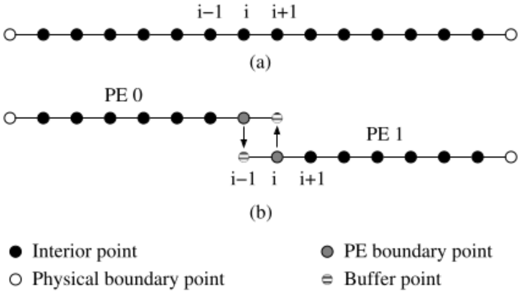
Parallel Computing

Divide Workload among different processor.



Synchronization

Stencil needs information not on the PE for boundary nodes.



Communication between PEs is required.

Algorithm

```
for(t = 0; t < final_T; t+=delta_t){
        for(i = 0; i < ne; i++){
        //Calculate according to the scheme
        //ne: number of elements on each
   processor
        .......
        }
        MPI_Send(boundary nodes);
        MPI_Recv(boundary nodes);
        MPI_Barrier(MPI_COMM_WORLD)

}
```

- Can we somehow avoid synchronization and reduce communication overhead?

Consider the explicit form of the heat equation:

$$u_i^{n+1} = u_i^n + \frac{\alpha \triangle t}{\triangle x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{3}$$

Problem: To compute u_i^{n+1} we need values of u that are possibly not on the PE

Solution:

- For the left boundary

$$u_i^{n+1} = u_i^n + \frac{\alpha \triangle t}{\triangle x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^{\tilde{n}}) \tag{4}$$

- For the right boundary

$$u_i^{n+1} = u_i^n + \frac{\alpha \triangle t}{\triangle x^2}(u_{i+1}^{\tilde{n}} - 2u_i^n + u_{i-1}^n) \tag{5}$$

where $\tilde{n}$ is the last available value for a particular node.

Regarding $\tilde{n}$

- Synchronous when $\tilde{n} = n$
- $\tilde{n}$ can be $n, n-1, n-2, \ldots$
- Concrete value of $\tilde{n}$ depends on hardware, network, traffic, (possible) unpredictable factors,...
- $\tilde{n}$ is in fact a principle random variable.

Results from Theoretical Analysis

- Stability is independent of statistics of delay.
- Stability is independent of the number of PEs.
- However, accuracy depends on both delay and number of PEs.

Results from Theoretical Analysis

- Second order convergence in space without delays
- First order convergence in space with delay

Numerical Results

Problem

Advection - Diffusion Equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2} \tag{6}$$

Initial Condition:

$$u(x,0) = \sum_{\kappa} A(\kappa) sin(\kappa x) \tag{7}$$

which has the analytical result:

$$u_a(x,t) = \sum_{\kappa} \exp^{(-\alpha\kappa^2 t)} A(\kappa) sin(\kappa(x - ct)) \tag{8}$$

Discretization

Interior nodes:

$$\frac{1}{\triangle t}(u_i^{n+1} - u_i^n) + \frac{c}{2\triangle x}(u_{i+1}^n - u_{i-1}^n) = \frac{\alpha}{\triangle x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{9}$$

Left boundary nodes:

$$\frac{1}{\triangle t}(u_i^{n+1} - u_i^n) + \frac{c}{2\triangle x}(u_{i+1}^n - u_{i-1}^{\tilde{n}}) = \frac{\alpha}{\triangle x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^{\tilde{n}}) \tag{10}$$

Right boundary nodes:

$$\frac{1}{\triangle t}(u_i^{n+1} - u_i^n) + \frac{c}{2\triangle x}(u_{i+1}^{\tilde{n}} - u_{i-1}^n) = \frac{\alpha}{\triangle x^2}(u_{i+1}^{\tilde{n}} - 2u_i^n + u_{i-1}^n) \tag{11}$$

Parameters for Simulation

```
len = 2*pi; % length of domain
A = 1;
k = 2;
alpha = 1;
r_alpha = 0.1; % CFL
c = 1;
final_t = 0.08*len/c
```

where

$$r_\alpha = \frac{\alpha \triangle t}{\triangle x^2} \tag{12}$$

Results from Matlab Implementation

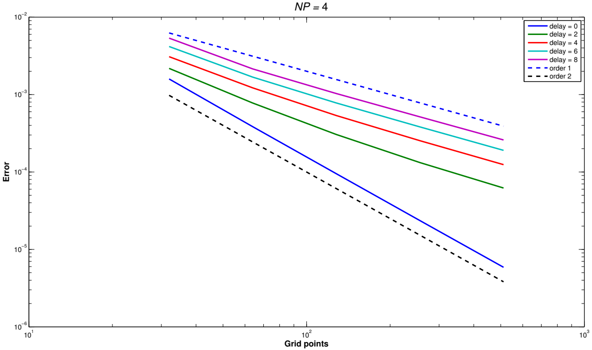


Figure : Number of PEs = 4

Results from Matlab Implementation

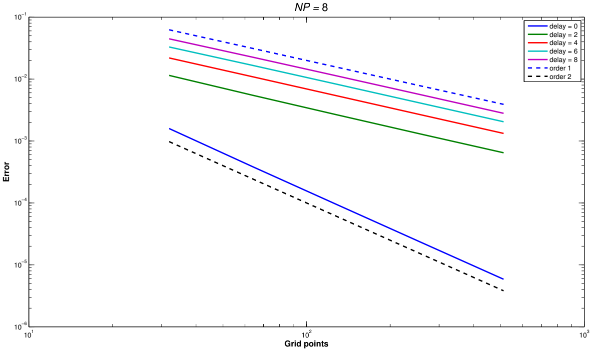
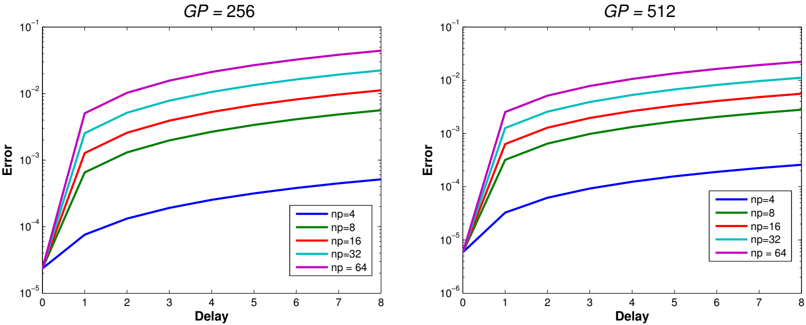
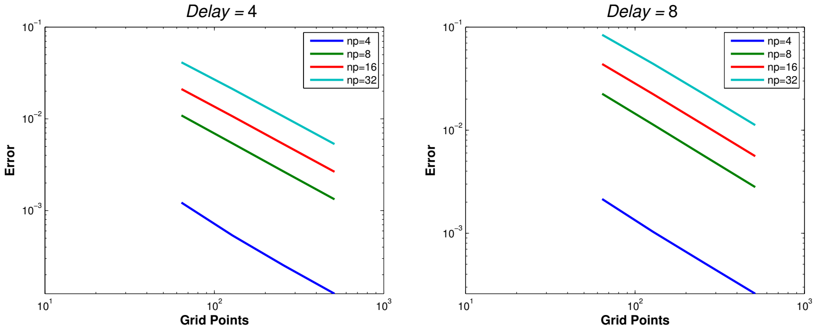


Figure : Number of PEs = 8



(a) Number of Grid Point = 256 (b) Number of Grid Point = 512

Figure : Influence of Delay on Grid Point



(a) Delay = 4 (b) Delay = 8

Figure : Influence of number of Grid Point for different number of PEs

C++ implementation

- Master Work Paradigm using MPI.
- OpenMP shared Memory
- MPI approach

Master Work Paradigm using MPI

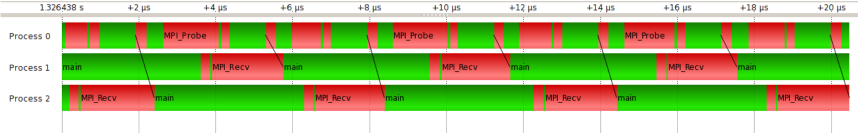


Figure : Vampir Trace with Grid Point 512.

MPI approach

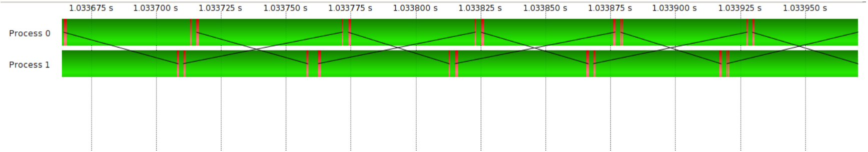
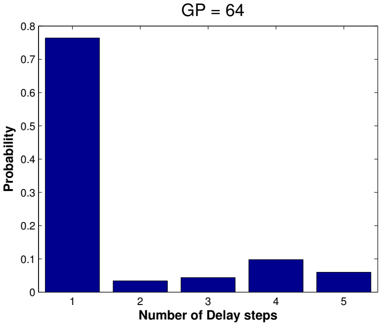
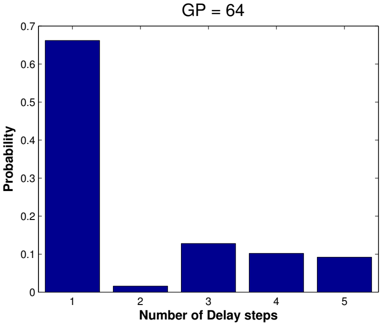


Figure : Vampir Trace with Grid Point 512.

Statistics of Delay. Maximum Allowable = 5 time steps



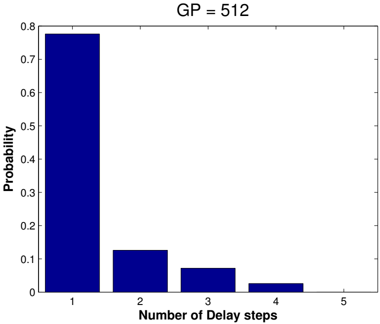
(a) Processor 1



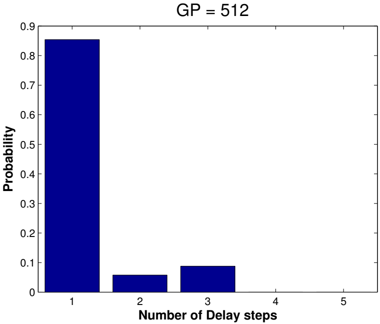
(b) Processor 2

Figure : Grid Points = 64

Statistics of Delay. Maximum Allowable = 5 time steps



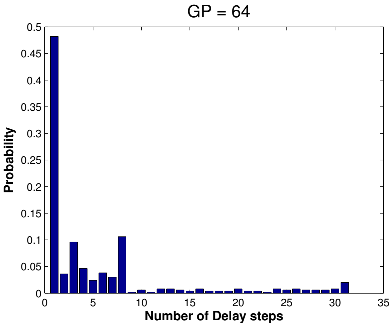
(a) Processor 1



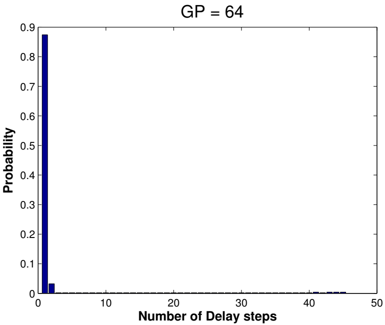
(b) Processor 2

Figure : Grid Points = 512

Statistics of Delay. No Waiting Time



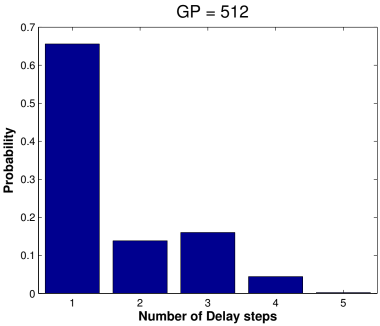
(a) Processor 1



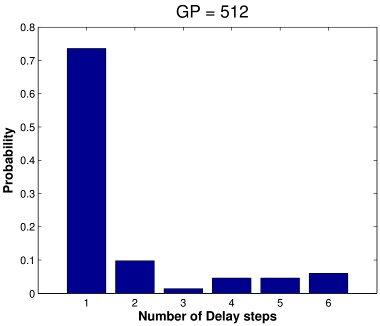
(b) Processor 2

Figure : Grid Points = 64

Statistics of Delay. No Waiting Time



(a) Processor 1



(b) Processor 2

Figure : Grid Points = 512

Remarks

- The statistics of delay is highly stochastic.
- The above results are obtained by 500 time step runs.
- The buffer size must be sufficiently large to hold values at all time steps.

Summary

- Avoid synchronization by asynchronous schemes.
- The statistics of delay is highly stochastic.

Future Work

- Extend the approach to multiple processor.
- Formally verify the difference in all the approaches.
- Order Recovery.
- Performance Analysis

Refrences

- Diego A. Donzis and Konduri Aditya. Asynchronous Finite Difference Scheme for Partial Difference Equations *Journal of Computational Physics*. 274(0):370-392,2014
- Thomas Camminady. CES Seminar Paper on Asynchronous Finite Difference Scheme for Partial Difference Equation. January 9,2015
- MPICH , `http://www.mpich.org/`, 4 12 2015.