Finite Element for Steady-State Advection-Diffusion Problem

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Abstract

The problem of Steay state Advection-Diffusion Equation is simulated using the Galerkin Method, Full Upwind, Streamline Upwind (SU) and Streamline Upwind Petrov-Galerkin method for the $1-D$ and $2-D$ case. Galerkin methods are a class of methods for converting a strong form of differential equation to a discrete problem. In principle, it is the equivalent of applying the method of variation of parameters to a function space, by converting the equation to a weak formulation. But it has been observed that Galerkin Method has certain deficiencies in convection dominated problem. To overcome these deficiencies, Petrov-Galerkin Methods is designed to produce stable and accurate results which include Streamline Upwind and Streamline Upwind Petrov-Galerkin (SUPG) Method.

Keywords: Advection-Diffusion Equation, Galerkin Method, Petrov-Galerkin Method

1. Problem Statement

1.1. Strong Form

Let us consider the transport by convection and diffusion of a scalar quantity $u = u(x)$ in a domain $\Omega \subset \mathbb{R}^{n_{sd}}$, with $n_{sd} = 1, 2$ or 3 with smooth boundary Γ. The boundary is assumed to be the portion of Dritchlet boundary Γ_D or the Neuman Boundary Γ_N. The boundary value problem associated with advection-diffusion equation are as:

$$\mathbf{a}.\nabla u - \nabla(\nu\nabla u) = s \quad in \quad \Omega \tag{1}$$

$$u = u_D \quad on \quad \Gamma_D \tag{2}$$

$$n.\nu\nabla u = h \quad on \quad \Gamma_N \tag{3}$$

where u is the unknown scalar, $a(x)$ is the convection viscosity, $\nu > 0$ is the coefficient of diffusivity and $s(x)$ is a volumetric source term. The unit outward normal vector to Γ is denoted by n.

1.2. Weak Form

Let the trial solution space $\mathbf{S}$ consists of real-valued functions, u, defined on Ω such that all members of $\mathbf{S}$ satisfy the Dirichlet condition in Eqn (2).Similarly, the space $\mathbf{V}$ of the weighting functions, w, is chosen such that $w = 0$ on Γ_D.

$$S := \{u \in \mathbf{H}^1(\Omega)\,|u = u_D \quad on \quad \Gamma_D\} \tag{4}$$

$$V := \{w \in \mathbf{H}^1(\Omega)\,|w = 0 \quad on \quad \Gamma_D\} \tag{5}$$

The weak form of advection-diffusion problem of Eqn (1) takes the following form:

$$\int_\Omega w(\mathbf{a}.\nabla u)d\Omega - \int_\Omega w\nabla.(\nu\nabla u)d\Omega = \int_\Omega wsd\Omega \quad \text{for all} \quad w \in \mathbf{V} \tag{6}$$

Using the divergence theorem on Eqn. (6) and applying $w = 0$ on Γ_D

$$\int_\Omega w(\mathbf{a}.\nabla u)d\Omega + \int_\Omega \nabla w.(\nu\nabla u)d\Omega = \int_\Omega wsd\Omega + \int_{\Gamma_N} whd\Gamma \quad \text{for all} \quad w \in \mathbf{V} \tag{7}$$

Rewriting the Eqn. (7) in compact form:

$$a(w,u) + c(\mathbf{a}; w, u) = (w, s) + (w, h)_{\Gamma_N} \quad \text{for all} \quad w \in \mathbf{V} \tag{8}$$

where

$$a(w,u) = \int_\Omega \nabla w.(\nu\nabla u)d\Omega \tag{9}$$

$$c(\mathbf{a}; w, u) = \int_\Omega w(\mathbf{a}.\nabla u)d\Omega \tag{10}$$

$$(w,s) = \int_\Omega wsd\Omega \tag{11}$$

$$(w,h)_{\Gamma_N} = \int_{\Gamma_N} whd\Gamma \tag{12}$$

2. Galerkin Approximation

Let $\mathbf{S}^h$ and $\mathbf{V}^h$ be finite dimensional spaces subsets of $\mathbf{S}$ and $\mathbf{V}$ respectively.The weighting functions $w^h \in \mathbf{S}^h$ and vanish on Γ_D. The approximation u^h lies in $\mathbf{S}^h$ and satisfies, with the precision given by the characteristic mesh size "h", the boundary condition u_D on Γ_D.The Galerkin formulation is obtained by restricting the weak form (8) to the finite dimensional spaces, namely, find $u^h \in \mathbf{S}^h$ such that

$$a(w^h, u^h) + c(\mathbf{a}; w^h, u^h) = (w^h, s) + (w^h, h)_{\Gamma_N} \quad \text{for all} \quad w^h \in \mathbf{V}^h \tag{13}$$

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Discretizing domain Ω into finite elements Ω^e, $1 \le e \le n_{el}$, u^h can be rewritten as:

$$u^h(x) = \sum_{A\in\eta\backslash\eta_D} N_A(x)u_A + \sum_{A\in\eta_D} N_A(x)u_D(x_A) \qquad (14)$$

where N_A is the shape function associated with the node number A and u_A is the nodal unknown. The test function w^h is defined as:

$$w^h \in V^h := \underset{A\in\eta\backslash\eta_D}{span}\ \{N_A\} \qquad (15)$$

Combining Eqn (13), (14) and (15) we obtain the discrete weak form:

$$\sum_{B\in\eta\backslash\eta_D} [a(N_A, N_B) + c(\mathbf{a}; N_A, N_B)]u_B = (N_A, s)+ (N_A, h)_{\Gamma_N} - \sum_{B\in\eta_D} [a(N_A, N_B) + c(\mathbf{a}; N_A, N_B)]u_B \quad \text{for all} \quad A \in \eta\backslash\eta_D \qquad (16)$$

2.1. Piecewise Linear Interpolation in 1-D

A linear element in 1D is defined by two nodes, $n_{en} = 2$ locally denoted as 1 and 2. The shape function of the linear element is given by:

$$N_1(\xi) = \frac{1}{2}(1+\xi) \qquad (17)$$

$$N_2(\xi) = \frac{1}{2}(1-\xi) \qquad (18)$$

where ξ is in the normalized coordinate $-1 \le \xi \le 1$. Assuming constant $\mathbf{a}$ and ν and assembling the element contributions to this weak form, we obtain the algebraic system governing the nodal values of the discrete solution of the advection-diffusion problem, For any particular internal node j, the discretized equation can be written as:

$$\mathbf{a}\left(\frac{u_{j+1} - u_{j-1}}{2h}\right) - \nu\left(\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}\right) = \frac{1}{6}\left(s_{j-1} + 4s_j + s_{j+1}\right) \qquad (19)$$

2.2. Analysis of discretized equation

Let us introduce the mesh Peclet number defined by Eqn(20) which defines the ratio of convective to diffusive transport.

$$Pe = \frac{\mathbf{a}h}{2\nu} \qquad (20)$$

Rewriting the Eqn (16) in the form:

$$\frac{\mathbf{a}}{2h}\left(\frac{Pe-1}{Pe}u_{j+1} + \frac{2}{Pe}u_j - \frac{Pe+1}{Pe}u_{j-1}\right) = \frac{1}{6}\left(s_{j-1} + 4s_j + s_{j+1}\right) \qquad (21)$$

Let us consider the boundary value problem in 1D as:

$$\mathbf{a}u_x - \nu u_{xx} = 1 \quad \text{in} \quad]0, L[\qquad (22)$$

$$u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \qquad (23)$$

The exact solution of the above mentioned problem (22)is given as:

$$u(x) = \frac{1}{\mathbf{a}}\left(x - \frac{1 - exp(\gamma x)}{1 - exp(\gamma)}\right) \qquad (24)$$

where $\gamma = \mathbf{a}/\nu$. The numerical approximation to Eqn(24) has been computed with a mesh of 10 uniform element using the Galerkin scheme for Pe = 0.25, 0.9 and 5. The result are shown in Figure (1). The Galerkin solution is corrupted by non-physical oscillations when the $Pe \ge 1$. To obtain an exact scheme, we want to find the constant

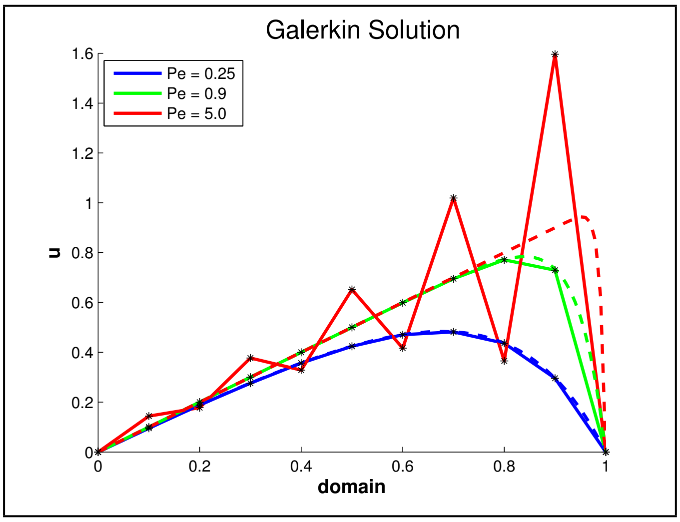


Figure 1: Galerkin solution (solid lines) of the advection- diffusion problem with $u_D = 0$ at $x = 0$ and $x = 1$ and $s = 1$ using a uniform mesh of 10 linear elements. The dashed line represent the exact solution given by Eqn(24) of the respective Pe.

α_1,α_2 and α_3 such that

$$\alpha_1 u_{j-1} + \alpha_2 u_j + \alpha_3 u_{j-1} = 1 \qquad (25)$$

for all nodal coordinates x_j, mesh size h and Peclet number Pe. From the exact solution Pe, we have:

$$\begin{cases} u_{j-1} = \frac{1}{\mathbf{a}}\left(x_j - h - \frac{1-exp(\gamma x_j)exp(-2Pe)}{1-exp(\gamma)}\right) \\ u_j = \frac{1}{\mathbf{a}}\left(x_j - \frac{1-exp(\gamma x_j)}{1-exp(\gamma)}\right) \\ u_{j+1} = \frac{1}{\mathbf{a}}\left(x_j + h - \frac{1-exp(\gamma x_j)exp(2Pe)}{1-exp(\gamma)}\right) \end{cases} \qquad (26)$$

By introducing Eqn (26) in Eqn (24), we get:

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_3 = {}^{\mathbf{a}}/_{h} \\ \alpha_1 exp(-2Pe) + \alpha_2 + \alpha_3 exp(2Pe) = 0 \end{cases} \qquad (27)$$

Solving for α_1, α_2 and α_3, we obtain

$$\begin{cases} \alpha_1 = -\frac{\mathbf{a}(1+coth(Pe))}{2h} \\ \alpha_2 = \frac{\mathbf{a}coth(Pe)}{h} \\ \alpha_3 = \frac{\mathbf{a}(1-coth(Pe))}{2h} \end{cases} \tag{28}$$

Introducing the Eqn. (28) in Eqn. (25) we get:

$$\frac{a}{2h}[(1-coth(Pe))u_{j+1} + (2coth(Pe))u_j - (1+coth(Pe))u_{j-1}] = 1 \tag{29}$$

The above equation Eqn. (29) can be rewritten in two forms:

$$\mathbf{a}\frac{u_{j+1}-u_{j-1}}{2h} - (\nu+\bar{\nu})\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = 1 \tag{30}$$

$$\frac{1-\beta}{2}\left(\mathbf{a}\frac{u_{j+1}-u_j}{h}\right) + \frac{1+\beta}{2}(\mathbf{a}\frac{u_j-u_{j-1}}{h}) - \nu\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = 1 \tag{31}$$

where

$$\bar{\nu} = \beta\frac{ah}{2} = \beta\nu Pe \tag{32}$$

$$\beta = coth(Pe) - \frac{1}{Pe} \tag{33}$$

In Eqn. (30) we added a numerical diffusion $\bar{\nu}$, also called "artificial diffusion" which only depends on the parameters of the governing differential equation and the element size h. In Eqn. (31) the discretization of the convective term appears as a weighted average of the fluxes (convection) of the solution to the left and to the right of node j. That is, the convection term is not discretized using a centered scheme.

The analysis forms the two techniques for improving the Galerkin method: adding an artificial diffusion and concerned with non-centered discretizations of the convection operator, also called upwind schemes.

3. Early Petrov Galerkin Methods

3.1. Upwind approximation to the convective term

First order upwinding is defined as:

$$u_x(x_j) = \begin{cases} \frac{u_j-u_{j-1}}{h} & a > 0 \\ \frac{u_{j+1}-u_j}{h} & a < 0 \end{cases} \tag{34}$$

The upwind derivative of the convective term introduces a numerical dissipation which comes in addition to the physical diffusion ν, as can be seen from the Taylor series development of the convective term around x_j:

$$\mathbf{a}\frac{u_j-u_{j-1}}{h} = \mathbf{a}u_x(x_j) - \frac{\mathbf{a}h}{2}u_{xx}(x_j) + O(h^2) \tag{35}$$

From the Eqn. (35), we get

$$\mathbf{a}u_x - \left(\nu + \frac{\mathbf{a}h}{2}\right)u_{xx} = s \tag{36}$$

which includes an added numerical diffusion of magnitude $\mathbf{a}h/2$. For large Peclet numbers the solution is stable and close to the exact one. However, for low values of the Peclet number, when the Galerkin approximation is accurate, the full upwind solution is over diffusive.

3.2. Finite Element of Upwind Type

The first upwind finite element formulations were based on modified weighting functions such that the element upstream of a node is weighted more heavily than the element downstream of a node. An example of this is shown in figure (2).

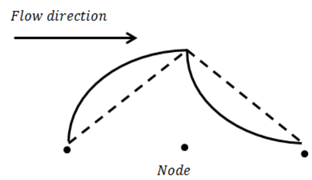


Figure 2: The solid line represent upwind-type weighting function while dashed line represent standard Galerkin weighting function. Upwind weighting function has more weight when the velocity is towards the node and less weight when the velocity is away from the node.

3.3. Stream-line Upwind (SU)

Consider the modified advection-diffusion equation:

$$\mathbf{a}u_x - (\nu+\bar{\nu})u_{xx} = 0 \tag{37}$$

where $\bar{\nu}$ is given according to Eqn. (33). Considering the weak form of Eqn. (37), we get:

$$\int_0^L (w\mathbf{a}u_x + w_x(\nu+\bar{\nu})u_x)dx = 0 \tag{38}$$

$$\int_0^L \left(\left[w + \frac{\beta h}{2}w_x\right]\mathbf{a}u_x + w_x\nu u_x\right)dx = 0 \tag{39}$$

which shows that the balancing diffusion method uses a modified weighting junction, given by $\tilde{w} = w + \beta(h/2)w_x$ for the convective term only. This does not correspond to a consistent Petrov-Galerkin formulation because the modified weighting function is only applied to the convective term.

3.3.1. Multidimensional Extension

The convective transport takes place along the streamlines and adding diffusion transversely to the flow leads to overly diffusive results, due to an excess of crosswind diffusion. So the balancing diffusion should be added in the flow direction only, and not transversely.This leads to the concept of SU schemes, which account for the directional character of the convective term. The idea of adding diffusion along the streamlines are described as anisotropic balancing dissipation. This is achieved by replacing the scalar artificial diffusion coefficient $\bar{\nu}$ in Eqn (39) by the tensor diffusivity:

$$\bar{\nu} = \bar{\nu}\frac{a_i a_j}{\|a\|^2} \tag{40}$$

The SU test function, which only affects the convective term, is defined as:

$$\tilde{w} = w + \frac{\bar{\nu}}{\|a\|^2}(\mathbf{a}.\nabla w) \tag{41}$$

The weak form for the generalized advection-diffusion problem associated with the SU stabilization becomes:

$$a(w,u) + c(\mathbf{a}; w, u) + \underbrace{\sum_e \int_{\Omega_e} \frac{\bar{\nu}}{\|a\|^2}(\mathbf{a}\nabla w)(\mathbf{a}\nabla u)d\Omega}_{\text{SU stabilization term}} = (w,s) + (w,h)_{\Gamma_N} \quad \text{for all} \quad w \in \mathbf{V} \tag{42}$$

3.4. Stabilization Techniques

In order to stabilize the convective term in a consistent manner (consistent stabilization), ensuring that the solution of the differential equation is also a solution of the weak form, in a structure similar to Equation (42). That is, an extra term over the element interiors is added to the Galerkin weak form. This term is a function of the residual of the differential equation to ensure consistency. These methods are especially designed for the steady state convection -diffusion-reaction equation, namely

$$\mathbf{a}.\nabla u - \nabla(\nu\nabla u) + \sigma u = s \quad \text{in} \quad \Omega \tag{43}$$

with the usual Dirichlet and Neumann boundary conditions. The result of the differential equation (42) is

$$\mathcal{R}(u) = \underbrace{\mathbf{a}.\nabla u - \nabla(\nu\nabla u) + \sigma u}_{\mathcal{L}(u)} - s = \mathcal{L}(u) - s \tag{44}$$

where $\mathcal{R}$ is the differential operator associated with the differential equation. The general form of these (consistent) *stabilization techniques* is

$$a(w,u) + c(\mathbf{a}; w, u) + (w, \sigma u) + \underbrace{\sum_e \int_{\Omega_e} \mathcal{P}(w)\mathcal{R}(u)d\Omega}_{\text{Stabilization term}} = (w,s) + (w,h)_{\Gamma_N} \tag{45}$$

where $\mathcal{P}(w)$ is a certain operator applied to the test function, τ is the stabilization parameter (also called intrinsic time), and $\mathcal{R}(u)$ is the residual of the differential Eqn (44). The stabilization techniques are characterized by the definition of $\mathcal{P}(w)$.

3.4.1. The SUPG Method

This stabilization technique is defined by taking by taking

$$\mathcal{P}(w) = \mathbf{a}.\nabla w \tag{46}$$

which in fact corresponds to the perturbation of the test function introduced in the SU method, see Eqn (42). Since the space of the test function does not coincide with the space of the interpolation function, this in fact is the Petrov-Galerkin formulation. With the perturbation definition in Eqn (46), the usual finite dimensional subspaces leads to the discrete problem that must be solved: find $u^h \in S^h$ such that

$$a(w^h, u^h) + c(\mathbf{a}; w^h, u^h) + (w^h, \sigma u^h) + \sum_e \int_{\Omega_e} (\mathbf{a}.\nabla w^h)\tau[\mathbf{a}.\nabla u^h - \nabla(\nu\nabla u^h) + \sigma u^h - s]d\Omega = (w^h, s^h) + (w^h, h)_{\Gamma_N} \quad for \quad all \quad w^h \in \vee^h \tag{47}$$

where the stabilization parameter τ can be defined as

$$\tau = \bar{\nu}/||\mathbf{a}||^2 \tag{48}$$

with $\bar{\nu}$ given as $\bar{\nu} = \beta a h/2$ in 1-D.

4. Results and Numerical Simulation

For all the simulations presented, XNS code was used.

4.1. 1-D Simulations

For the numerical simulations in 1-D, first consider the Eqn. (49) with $Pe = 0.25, 0.9$ and 5.0.

$$\begin{aligned} &\mathbf{a}u_x + \nu u_{xx} = 1 \\ &u_D = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1 \end{aligned} \tag{49}$$

The Galerkin Solution of the above equation is shown in Figure (1). The Galerkin solution was observed to be corrupted for $Pe \geq 1$. Further to obtain the stable solution, the method of full upwinding and Streamline Upwind (SU) was used. The results are shown in the figure (3) and (4) respectively. For these simulations, the viscosity was changed as $\nu_{new} = \nu + \bar{\nu}$. For full upwind, $\bar{\nu} = {}^{\mathbf{a}h}/_2$ and for streamline upwind case, $\bar{\nu} = \beta^{\mathbf{a}h}/_2$, where β is given by Eqn (33).

A full upwind treatment of the convective term leads to stable, but overly diffusive, as can be seen from Figure (3). For large Peclet numbers the solution is stable and close to the exact one. However, for low values of the Peclet number, when the Galerkin approximation is accurate, the full upwind solution is clearly over diffusive. The Streamline upwind solution tends to approximate the analytic solution in a much better way for both high and low values of Peclet number.

Let us consider the Eqn 50) with different source term.

$$\begin{aligned} &\mathbf{a}u_x + \nu u_{xx} = 5e^{-100\left(x-\frac{1}{8}\right)^2} - 5e^{-100\left(x-\frac{1}{4}\right)^2} \\ &u_D = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1 \end{aligned} \tag{50}$$

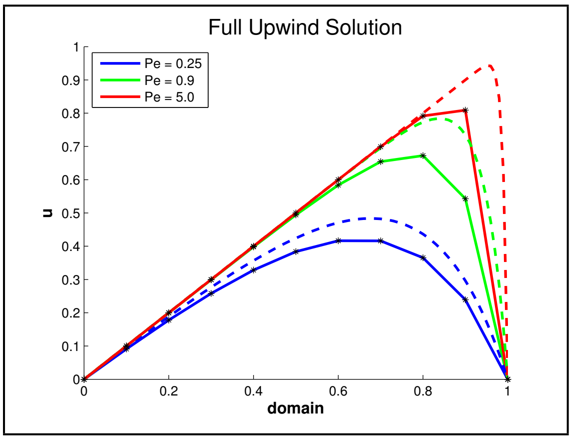


Figure 3: Full Upwind solution (solid lines) of the advection- diffusion problem for Eqn. (49) using a uniform mesh of 10 linear elements. The dashed line represent the exact solution given by Eqn(24) of the respective Pe.

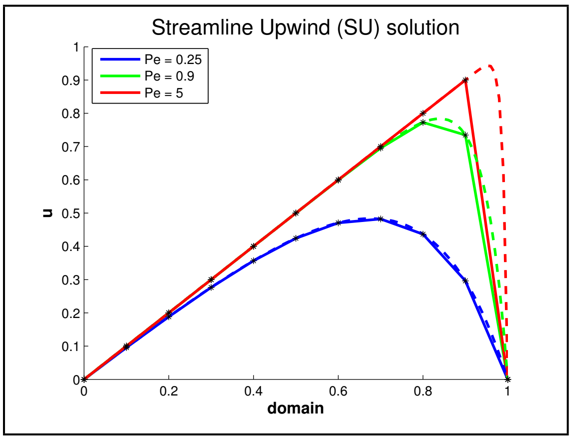


Figure 4: Streamline Upwind solution (solid lines) of the advection-diffusion problem for Eqn. (49) using a uniform mesh of 10 linear elements. The dashed line represent the exact solution given by Eqn(24) of the respective Pe.

The result of the simulation are shown in Figure (5). The difference in the result of SU and SUPG is due to the the consistent formulation associated with SUPG. For the simulation of SUPG case `tau_momentum_factor` is set to 1.0.

4.2. 2-D Simulation

The problem statement is depicted in Figure (6) where the unit square is taken as the computational domain, $\bar{\Omega} = [0, 1] \times [0.1]$. A mesh of 10 by 10 equal-sized bilinear elements is considered. The flow is unidirectional and constant, $\|a\| = 1$, but the convective velocity is skew to the mesh with an angle of 30°. The diffusivity coefficient is taken to be 10^{-4} , corresponding to a mesh Peclet number of 10^4 . The inlet boundary data are discontinuous and two types of boundary conditions are considered at the outlet, namely Downwind homogeneous natural boundary conditions and Downwind homogeneous essential boundary conditions.

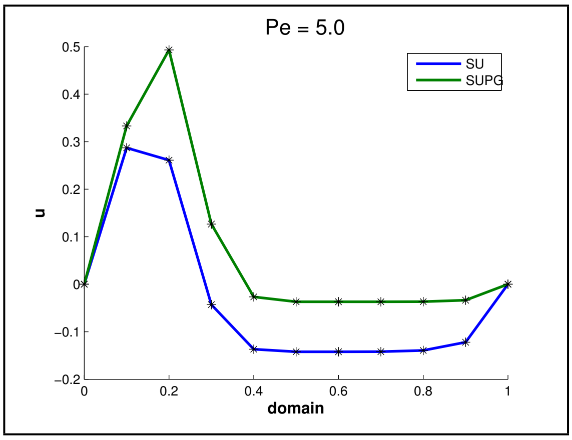


Figure 5: SUPG and SU solutions of the advection diffusion problem Eqn (50) at Pe = 5 using a uniform mesh of 10 linear elements

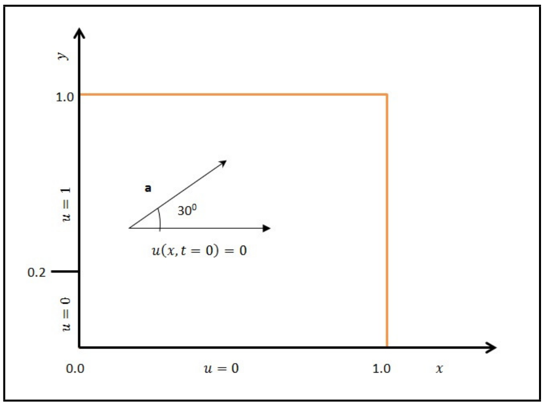


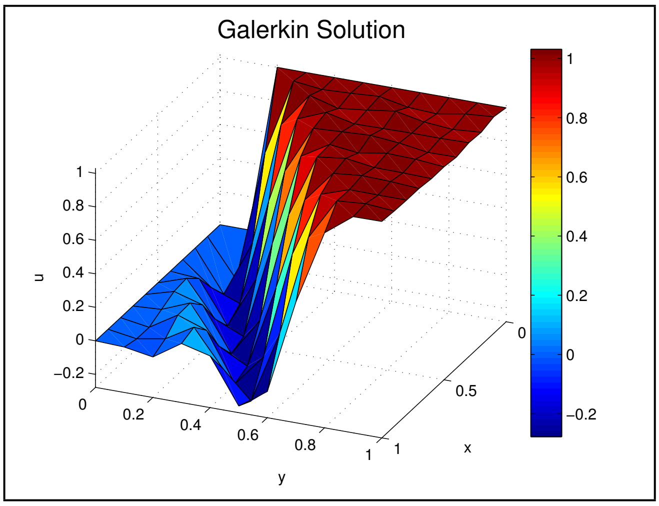
Figure 6: Domain for 2D numerical simulation with discontinuous inlet data

4.2.1. Downwind homogeneous natural boundary conditions

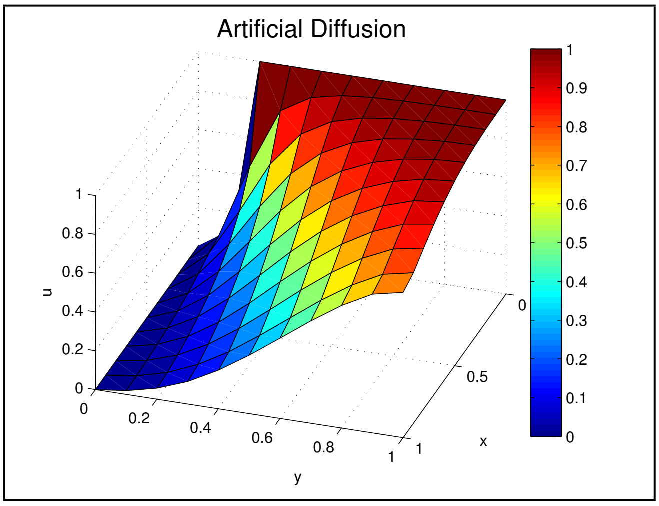
The results for this case are displayed in Figure (7). Given the elevated value of the Peclet number, the solution is practically one of pure convection. The Galerkin method is not able to satisfactorily resolve the discontinuity and produces spurious oscillations. The artificial diffusion method and the SUPG method yield better results, but SUPG introduces less crosswind diffusion as compared to artificial diffusion. During the simulation of artificial diffusion, additional diffusivity is introduced as $\bar{\nu} = \tau \|\mathbf{a}\|^2$, where $\tau = {}^{h}/{}_{2\mathbf{a}}(coth(Pe) - {}^{1}/{}_{Pe})$.

4.2.2. Downwind homogeneous essential boundary conditions

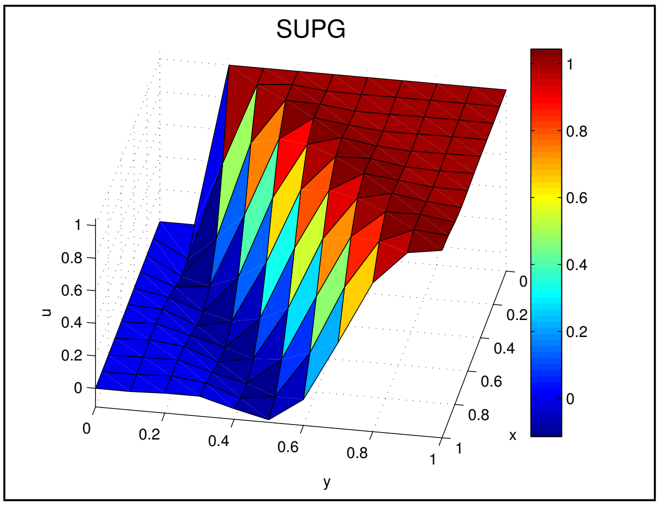
Here, we impose $u = 0$ on the outlet portion of the boundary. The solution now involves a thin boundary layer at the outlet. As shown in Figure (8), the crude 10 by 10 mesh has difficulty capturing the details of the solution. The Galerkin results are wildly oscillatory and bear no resemblance to the exact results. Better results are obtained with the stabilized formulations. The artificial diffusion technique introduces excessive numerical diffusion.



(a) Galerkin Solution for convection - diffusion problem with Downwind homogeneous natural boundary conditions

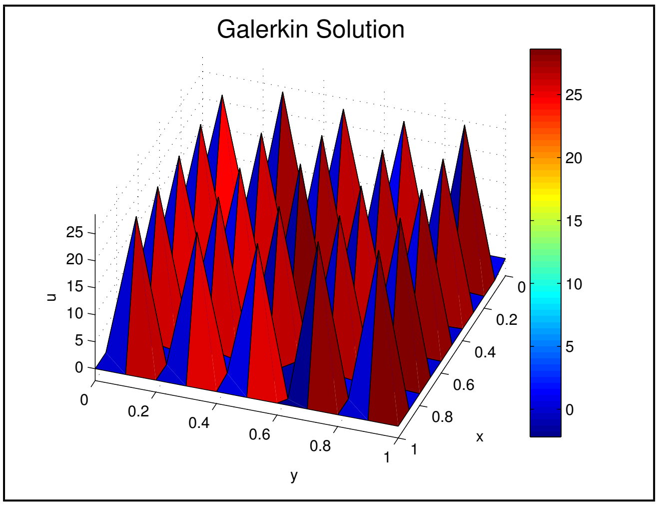


(b) Artificial Diffusion Solution for convection - diffusion problem with Downwind homogeneous natural boundary conditions

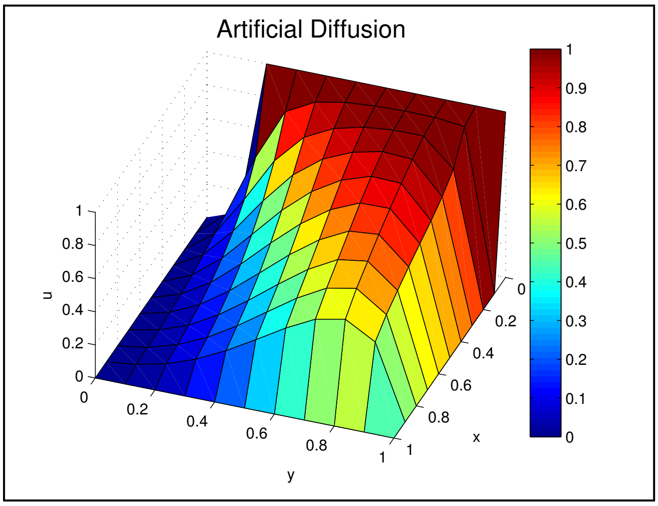


(c) SUPG Solution for convection - diffusion problem with Downwind homogeneous natural boundary conditions

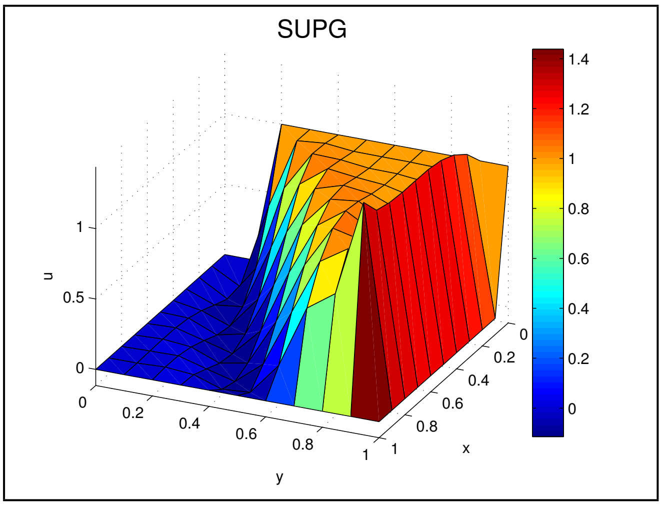
Figure 7: Solutions for 2D convection - diffusion problem with Downwind homogeneous natural boundary conditions.



(a) Galerkin Solution for convection - diffusion problem with Downwind homogeneous essential boundary conditions



(b) Artificial Diffusion Solution for convection - diffusion problem with Downwind homogeneous essential boundary conditions



(c) SUPG Solution for convection - diffusion problem with Downwind homogeneous essential boundary conditions

Figure 8: Solutions for the 2D convectiondiffusion problem with downwind essential boundary conditions.

5. Conclusion

The standard Galerkin method solution is corrupted by non-physical oscillations for advection dominated phenomenon (high Peclet number). To stabilize the solution of Galerkin method, we can use the method of artificial diffusion. But it adds the diffusion in both crosswind and streamline direction which leads to the excessive numerical diffusion. Though the addition of artificial diffusion stabilizes the solution for the high Peclet number, but for the low Peclet number solution is over-diffusive, To overcome this, we can use Streamline-upwind which adds diffusion only in the direction of the streamlines (anisotropic diffusion), but it faces the problem of consistency as the SU formulation, makes use of upwind test functions only for the convective term. The concept of adding diffusion along the streamlines in a consistent manner has been successfully exploited in the SUPG method.

6. References

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