Finite Element Simulations for Unsteady Advection-Diffusion Problems

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Abstract

The problem of Unsteady state Advection-Diffusion is simulated using the semi-discrete formulation and space-time formulation for 1-D case. The semi-discrete formulation with Crank Nicolson scheme for time discretization and Galerkin scheme for spatial discretization is considered. The effect of adding stabilization technique in spatial direction is studied. Finally, the space-time formulation, a discontinuous Galerkin method is considered to simulate the problem which makes use of finite element interpolation in both time and space direction.

Keywords: Advection-Diffusion Equation, Galerkin Method, Discontinuous Galerkin Method, Space-time Formulation, Stabilization-techniques.

1. Introduction

Let us consider the general form of unsteady state convection - diffusion - reaction initial/boundary value problem in the strong form which can be stated as : for a given velocity field $\mathbf{a}(\mathbf{x}$,t), diffusion coefficient $\nu(\mathbf{x}$,t), the reaction $\sigma(\mathbf{x}$,t), the source term $s(\mathbf{x}$,t) and the necessary initial and boundary condition, find $u(\mathbf{x}$,t) such that:

$$
\begin{aligned}
u_t + \mathbf{a}.\nabla u - \nabla.(\nu\nabla u) + \sigma u &= s && \textit{in } \Omega\times]0,T[&& (1)\\
u(\mathbf{x},0) &= u_0(\mathbf{x}) && \textit{on } \Omega && (2)\\
u &= u_D && \textit{on } \Gamma_D && (3)\\
\nu(\mathbf{n}.\nabla)u &= h && \textit{on } \Gamma_N && (4)
\end{aligned}
$$

The bilinear form associated with the spatial operator (in the Glaerkin formulation) is continuous and coercive. Thus, the Lax-Milgram lemma can be applied to show the existence and uniqueness of the solution [1].

The unsteady state convection-diffusion-reaction equation (1) is parabolic. Because of the presence of physical diffusivity, the solutions here are continuous, unlike the case with pure convection problem [1] and hence the boundary condition must be imposed everywhere on the domain boundary. However on account of the Dirichlet Boundary condition, solutions may result in the steep gradients in both internal and boundary layers. Thus, in order to correctly simulate the problem, one has to take into consideration the stability issues of the Galerkin formulation and simultaneously make use of the time-stepping algorithm that are capable of simulating the role of the characteristics.

The numerical solution of the above equation (1) involves the discretization both in space and time. The two alternatives for numerical formulation of such type of problem are semi-discrete formulation and space-time formulation which are discussed in the corresponding sections.

2. Semi-Discrete Formulation

In semi-discrete formulation time and spatial derivatives are discretized separately. Spatial derivatives are discretized leading to the system of coupled first order ODEs, which can be solved by well known ODE solution techniques, followed by the time-integration. Shape functions are taken to be functions of space only. It is noteworthy that for linear spatial operator with constant coefficients and Galerkin formulation, the order of discretization is not a problem. However, for non-linear problem and stabilization techniques for transient analysis, it is preferred that the time discretization be followed by space discretization. [1]

2.1. Time Discretization

Generally finite differences are employed in time discretization. The most common methods for parabolic problems are θ family of discretization. The value of unknown u^{n+1} is determined by the value at u^n by taking the weighted average of u_t^{n+1} and u_t^n as in (5), where u^n represents the value of the u at the time step t^{n+1} and $\triangle t$ represents the time step.

$$
\frac{u^{n+1}-u^n}{\triangle t} = \theta u_t^{n+1} + (1-\theta)u_t^n + O((1/2-\theta)\triangle t, \triangle t^2) \quad (5)
$$

Different methods can be obtained for different values of θ. For $\theta > 1/2$, the schemes are unconditionally stable. Also, only for $\theta = 1/2$ (Crank - Nicolson Scheme), the scheme is second order accurate.

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Introducing incremental unknown $\triangle u = u^{n+1} - u^n$ and solving for $\triangle u$ rather than u^{n+1}, Eqn. (5) becomes

$$\frac{\triangle u}{\triangle t} - \theta \triangle u_t = u_t^n \tag{6}$$

Replacing u_t in Eqn (1) by Eqn (6), we get:

$$\begin{aligned}\frac{\triangle u}{\triangle t} + \theta[\mathbf{a}.\nabla - \nabla.(\nu\nabla) + \sigma]\triangle u = \\ \theta s^{n+1} + (1-\theta)s^n - [\mathbf{a}.\nabla - \nabla.(\nu\nabla) + \sigma]u^n\end{aligned} \tag{7}$$

The second order derivatives of u are present due to the diffusion operator. This method involves only first order time derivatives and therefore can be implemented in connection with standard $\mathbf{C}^0$ continuous finite elements.

2.2. Space Discretization

Let us discuss the Galerkin formulation associated with the strong form of unsteady state convection - diffusion - reaction problem (Eqn (1)). Galerkin formulation can be applied directly to equation (1) or on the time discretized equation (7).

Let us consider the semi-discrete scheme. The weighting function space $\mathbf{V}$ satisfies the homogeneous boundary condition on Γ_D whereas the function w in $\mathbf{V}$ does not depend on time. The time dependency of approximate solution u can be translated into trial space S_t ([1, Section 3.4.2.1]), which varies as a function of time:

$$\begin{aligned}\mathbf{S}_t := \{u | u(.,t) \in \mathbf{H^1}(\Omega), t \in [0,T] \quad \text{and} \\ u(\mathbf{x},t) = u_D \quad \text{for} \quad \mathbf{x} \in \Gamma_D\}\end{aligned} \tag{8}$$

The weak form of the initial boundary value problem (1) is defined as follows : Given s, u_D, h, u_e and u_0, for any $t \in [0,T]$ we find $u(\mathbf{x},t) \in S_t$ such that for all $w \in \mathbf{V}$

$$\begin{aligned}(w, u_t) + c(\mathbf{a}; w, u) + a(w,u) + (w, \sigma u) = \\ (w,s) + (w,h)_{\Gamma_N}\end{aligned} \tag{9}$$

where:

$$a(w,u) = \int_\Omega \nabla w.(\nu\nabla u) d\Omega \tag{10}$$

$$c(\mathbf{a}; w, u) = \int_\Omega w(\mathbf{a}.\nabla u) d\Omega \tag{11}$$

$$(w,s) = \int_\Omega w s d\Omega \tag{12}$$

$$(w,h)_{\Gamma_N} = \int_{\Gamma_N} w h d\Gamma \tag{13}$$

The spatial discretization of by means of the Galerkin formulation consists of defining two finite dimensional spaces $\mathbf{S^h}$ and $\mathbf{V^h}$ as subsets of S and V, such that

$$\begin{aligned}\mathbf{V}_t^h &:= \{w \in \mathbf{H^1}(\Omega), w|_{\Omega_e} \in \mathbf{P}_p(\Omega^e) \quad \text{and } w = 0 \quad \text{on } \Gamma_D\} \\ \mathbf{S}_t^h &:= \{u | u(.,t) \in \mathbf{H^1}(\Omega), u(.,t)|_{\Omega_e} \in \mathbf{P}_p(\Omega^e) \ t \in [0,T] \forall e \\ &\qquad \text{and} \quad u(\mathbf{x},t) = u_D \text{ on } \Gamma_D\}\end{aligned}$$

where $\mathbf{P}_p$ is the finite element interpolating function space consisting of polynomials of order p. The semi-discrete Galerkin formulation is obtained by restricting the weak form of these finite dimensional spaces which leads to the system of Ordinary Differential Equation ([1, Section 3.4.2.2]). The shape functions $N_A(\mathbf{x})$ does not depend on time. Thus, the time dependent solution can be expressed as:

$$u^h(\mathbf{x},t) = \sum_{A \in \eta \setminus \eta_D} N_A(\mathbf{x}) u_A(t) + \sum_{A \in \eta_D} N_A(\mathbf{x}) u_D(\mathbf{x}_A, t) \tag{14}$$

where η refers to the global node numbers in the finite element mesh and $\eta_D \subset \eta$, the subset of the nodes belonging to the Dirichlet portion of the boundary Γ_D. The test functions are defined as $w^h \in \mathbf{V}^h = span_{B \in \eta \setminus \eta_D}\{N_B\}$.

The assembly procedure, assuming σ to be uniform and constant, leads to the semi-discrete system of ordinary differential equations:

$$\mathbf{M}\dot{\mathbf{u}} + (\mathbf{C} + \mathbf{K} + \sigma\mathbf{M}) = \mathbf{f} \tag{15}$$

where, the vectors $\mathbf{u}$ and $\dot{\mathbf{u}}$ contain the nodal values of the unknown u and it's time derivative respectively, while $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are the consistent mass matrix, the convection matrix and the diffusion matrix respectively. These matrices are obtained by the topological assembly of element contributions as follows:

$$\begin{aligned}\mathbf{M} = \mathbf{A}^e\,\mathbf{M^e} \quad & M_{ab}^e = \int_{\Omega^e} N_a\, N_b\, d\Omega \\ \mathbf{C} = \mathbf{A}^e\,\mathbf{C^e} \quad & C_{ab}^e = \int_{\Omega^e} N_a\, (\mathbf{a}.\nabla N_b)\, d\Omega \\ \mathbf{K} = \mathbf{A}^e\,\mathbf{K^e} \quad & K_{ab}^e = \int_{\Omega^e} \nabla N_a.\, (\nu \nabla N_b)\, d\Omega\end{aligned} \tag{16}$$

where $\mathbf{A}$ denotes the assembly operator, $1 \leq a, b \leq n_{en}$ where n_{en} is the number of element nodes. The r.h.s vector, $\mathbf{f}$, considers the contribution of the source term s, the prescribed flux h and the Dirichlet data u_D which results from the assembly of the nodal contributions of the form $\mathbf{f} = \mathbf{A^e}\,\mathbf{f^e}$ and

$$\begin{aligned}f_a^e = (N_a, s)_{\Omega^e} + (N_a, h)_{\partial\Omega^e \cap \Gamma_N} - \sum_{b=1}^{n_{en}} \Big[(N_a, N_b)_{\Omega^e} \frac{\partial u_{Db}^e(t)}{\partial t} \\ + \Big(c(\mathbf{a}; N_a, N_b)_{\Omega^e} + a(N_a, N_b)_{\Omega^e} + (N_a, \sigma N_b)_{\Omega^e} \Big) u_{Db}^e(t) \Big]\end{aligned}$$

where $u_{Db}^e(t) = u_D(x_b, t)$ if u_D is prescribed at node number b and zero otherwise.

Neglecting the truncation error associated with time discretization, the equation (7) can be interpreted as spatial differential operator representing a strong form that must be solved at each time step. Considering this rationale, the variational form associated with the θ family methods can be determined from Eqn(7), as:

$$\left(w, \frac{\Delta u}{\Delta t}\right) + \theta[c(\mathbf{a}; w, \Delta u) + a(w, \Delta u) + (w, \sigma\Delta u)]$$
$$= -[c(\mathbf{a}; w, \Delta u^n) + a(w, \Delta u^n) + (w, \sigma\Delta u^n)]$$
$$+ \left(w, \theta s^{n+1} + (1-\theta)s^n\right) + \left(w, \theta h^{n+1} + (1-\theta)h^n\right) \tag{17}$$

Since, time is already discretized in the above equation, therefore, depending on the value of θ, the Eqn (17) will be either first order or second order in time.

2.3. Stabilization Techniques

Galerkin formulation lack stabilization in the convection dominated problem (high Peclet Number). Hence, there is a need to stabilize the convective term in a consistent manner ensuring that the solution of the differential equation is also a solution of the weak form. A stabilized formulation of the convection-diffusion-reaction problem (Eqn. 1) can be written as follows

$$\left(w, u_t + \mathbf{a}.\nabla u + \sigma u\right) + a(w,u) + \sum_{e=1}^{n_{el}} \left(\mathbf{P}(w), \tau R(u)\right)_{\Omega^e}$$
$$= (w,s) + (w,h)_{\Gamma_N} \tag{18}$$

where the perturbation operator P characterises the stabilization method. The stabilization term involves the residual $R(u) = u_t + \mathbf{a}.\nabla u + \sigma u - \nabla.(\nu\nabla u)$ of the governing equation, which in turn ensures a consistent formulation [1, Chapter 2]. Except for the θ family methods where u_t is replaced by $(u^{n+1} - u^n)/\Delta t$, or for space-time formulations (discussed later in Section (3)),the time derivative u_t results in a complex implementation.

The consistently stabilized weak form of the time discretized problem can be written as [1, Section 5.4.6]:

$$\left(w, \frac{\Delta u}{\Delta t}\right) - (w, W\Delta \mathbf{u}_t) + \underbrace{\sum_e \left(\tau \mathbf{P}(w), R(\Delta u)\right)_{\Omega^e}}_{\textbf{Stabilization} \quad \textbf{Term}} = \left(w, \mathbf{w} u_t^n\right) \tag{19}$$

For Crank-Nicolson method we can define $\mathbf{W} = 1/2$ and $\mathbf{w}$ = 1 [1, Section 5.3.3.5].

2.3.1. Streamline Upwind Petrov Galerkin (SUPG)

The perturbation operator $\mathbf{P}$ for SUPG stabilization technique is defined as

$$\mathbf{P}(w) := W(a.\nabla)w \tag{20}$$

The weak form for the SUPG method is obtained after substitution of the perturbation operator from Eqn (20) to Eqn (19). The non-symmetric nature of the stibilization leads to some technical difficulties induced in the stability analysis. This drawback is removed in the GLS stabilization technique.

2.3.2. Galerkin Least Square Method (GLS)

The GLS stabilization technique introduces a symmetric stabilization term in a consistent manner. The perturbation operator $\mathbf{P}$ for GLS is the spatial differential operator of the strong form, which in this case, is affected by the time discretization. For linear advection-diffusion-reaction equation with constant coefficients, $\mathbf{P}$ is defined as

$$\mathbf{P}(w) := \frac{w}{\Delta t} + WL(w) \tag{21}$$

where L is differential operator associated with the differential equation, $L(u) = \mathbf{a}.\nabla u - \nabla.(\nu\nabla u) + \sigma u$.

3. Space-Time Formulation

We start the discussion by briefly introducing the Discontinuous Galerkin Method.

3.1. Discontinuous Galerkin Method

Because of the compactness of the formulation, discontinuous Galerkin Method is commonly used for solving the unsteady state convection or convection - diffusion problem [1]. Let us briefly consider discontinuous Galerkin method for a convection problem, stated as follows:

$$u_t + \mathbf{a}.\nabla u = s(x,t) \quad in\ \Omega\times]0,T[\tag{22}$$
$$u(\mathbf{x},0) = u_0(\mathbf{x}) \quad on\ \Omega \tag{23}$$
$$u = u_D \quad on\ \Gamma_D^{in}\times]0,T[\tag{24}$$

The discontinuous approximation belong to *broken-space*. For a given regular partition, $\mathbf{T^h}$, of the computational domain Ω into sub-domains Ω^e, the test function belonging to the broken space $\mathbf{V}(\mathbf{T^h})$ are continuous and smooth in every element $\Omega^e \in \mathbf{T^h}(\Omega)$, but discontinuous across inter-element boundaries. Also the trial space varies with time as:

$$\mathbf{V}(\mathbf{T^h}) := \{w \in \mathbf{L_2}(\Omega), w|_{\Omega_e} \in \mathbf{H^2}(\Omega^e)\ \forall \Omega^e \in \mathbf{T^h}(\Omega)\}$$
$$\mathbf{S}_t(\mathbf{T^h}) := \{u | u(.,t) \in \mathbf{L_2}(\Omega), u(.,t)|_{\Omega_e} \in \mathbf{H}^2(\Omega^e)$$
$$t \in [0,T] \forall \Omega^e \in \mathbf{T^h}(\Omega)\}$$

Elements of $\mathbf{S}_t(\mathbf{T}^h)$ and $\mathbf{V}(\mathbf{T}^h)$ are non-zero functions only on one element Ω^e and zero everywhere else. The discontinuous Galerkin formulation of problem (22) can be stated as follows : given $u_0(\mathbf{x})$, for any $t \in]0,T[$, find $u \in \mathbf{S}_t(\mathbf{T}^h)$, such that $u(\mathbf{x},0) = u_0(\mathbf{x})$ and

$$\left(w, u_t + \nabla.(\mathbf{a}u) - s\right)_{\Omega^e} - \left(w, (u^+ - u^-)(\mathbf{a}.\mathbf{n_e})\right)_{\partial\Omega^{e,in}\setminus\Gamma^{in}}$$
$$- \left(w, (u^+ - u_D)(\mathbf{a}.\mathbf{n_e})\right)_{\partial\Omega^{e,in}\cap\Gamma^{in}} = 0$$

for all $w \in \mathbf{V}(\mathbf{T}^h)$ and $\Omega^e \in \mathbf{T}^h(\Omega)$. Here, n_e is the outward normal to the element e. This weak form is local, defined only over one element. This is possible because the test functions are discontinuous along inter-element boundaries. This doesn't imply that each element can be solved independently. The boundary integral introduces a weakly enforced continuity across the boundaries, which couples the unknowns of the adjacent elements. To cope with the discontinuity of the field variables across inter-element boundaries, one defines

$$u^{\pm} = \lim_{\epsilon\to 0^+} u(\mathbf{x} \pm \epsilon\mathbf{a}) \quad \text{for} \quad \mathbf{x} \in \partial\Omega^e \tag{25}$$

whereas, the inlet portion of the element boundary, $\partial\Omega^e$, is defined by

$$\partial\Omega^e = \{\mathbf{x} \in \partial\Omega^e | \mathbf{a}.\mathbf{n_e}(\mathbf{x}) < 0\} \tag{26}$$

3.2. Space Time Formulation Basics

In the space-time formulation, the time and space are considered in a similar fashion. Shape functions are considered to be the function of both space and time. Assuming, a linear interpolation over a time slab $]t^n, t^{n+1}[$, the local space-time interpolation is written in product form as:

$$u^h(\mathbf{x}, t) = \sum_{A=1}^{n_{np}} N_A(\mathbf{x})\big((1-\theta)u_A^n + \theta u_A^{n+1}\big) \tag{27}$$

where $\theta = (t - t^n)/(t^{n+1} - t^n)$.

Let us now extend the space-time domain of the finite element methods for transient convection problems. Consider piecewise continuous approximations in space and discontinuous approximations in time which enables us to solve independently for each time slab instead of solving a global problem over the whole time domain. Such procedure is already discussed in space discretization when we discussed about discontinuous Galerkin method. The time domain is partitioned in n_{st} sub-intervals, where each subinterval is defined as $I^n =]t^n, t^{n+1}[$, n = 0,1,....,n$_{st}$-1. Space-time slabs are then obtained of the form as

$$Q^n = \Omega \times I^n$$

for the considered space-time slab Q^n, the spatial domain Ω is subdivided into n_{el} elements, Ω^e, e = 1,2,....,n_{el}, giving space-time elements domains as

$$Q_e^n = \Omega^e \times I^n, \quad e = 1, ..., n_{el}$$

3.3. Time Discontinuous Galerkin Method

Since the finite element interpolation is discontinuous at the space-time interfaces, it is useful to employ the notion

$$u^h(t_{\pm}^n) = \lim_{\epsilon\to 0^+} u^h(t^n \pm \epsilon)$$

The finite element spaces for the trial and weighting functions are defined as follows

$$\begin{aligned} \mathbf{S}^h &= \bigcup_{n=0}^{n_{st}-1} \mathbf{S_n^h},\ \mathbf{S_n^h} = \{u^h \mid u^h \in \mathbf{C^0}(Q^n), \\ &\qquad u^h|_{Q_e^n} \in \mathbf{P_k}(Q_n^e), u^h|_{\Gamma^{in}} = u_D\} \\ \mathbf{V}^h &= \bigcup_{n=0}^{n_{st}-1} \mathbf{V}_n^h,\ \mathbf{V}_n^h = \{w^h \mid w^h \in \mathbf{C}^0(Q^n), \\ &\qquad w^h|_{Q_e^n} \in \mathbf{P}_k(Q_n^e), w^h|_{\Gamma^{in}} = 0\} \end{aligned} \tag{28}$$

The continuity over Q^n implies continuity in space but doesn't require a continuous interpolation between time slabs. The degree $\mathbf{P_k}$ can be chosen independently in space and time. The weighted residual formulation of the homogeneous linear convection equation with Dirichlet inlet conditions is: for n = 0,1,...n$_{st}$-1, find $u^h \in S_n^h$, such that for all $w^h \in \mathbf{V}_n^h$,

$$\begin{aligned} &\iint_{Q^n} w^h(u_t^h + \mathbf{a}.\nabla u^h)d\Omega dt + \\ &\int_{\Omega} w^h(t_+^n)(u^h(t_+^n) - u^h(t_-^n))d\Omega = 0 \end{aligned} \tag{29}$$

with initial condition $u^h(\mathbf{x}, t_-^0) = u_0(\mathbf{x})$. The last integral in equation (29) implies a jump condition which imposes a weakly enforced continuity condition across the slab interfaces by which information is propagated from one slab to another. In view of the above mentioned developments, the partitioned matrix system for the nodal unknowns $\mathbf{u}^{n+1}$ and $\mathbf{u}^{n^+}$ is written as follows

$$\begin{aligned} \Big(\mathbf{M} + \frac{2}{3}\triangle t\mathbf{C}\Big)\mathbf{u}^{n+1} - \Big(\mathbf{M} - \frac{1}{3}\triangle t\mathbf{C}\Big)\mathbf{u}^{n^+} &= 0 \\ \Big(\mathbf{M} + \frac{1}{3}\triangle t\mathbf{C}\Big)\mathbf{u}^{n+1} + \Big(\mathbf{M} + \frac{2}{3}\triangle t\mathbf{C}\Big)\mathbf{u}^{n^+} &= 2\,\mathbf{M}\,\mathbf{u}^{n^-} \end{aligned}$$

where $\mathbf{M}$ and $\mathbf{C}$ represents the mass matrix and convection matrix respectively. The above mentioned scheme is third order accurate in time and is unconditionally stable. [1]

Extending the formulations for the pure convection problem to the convection-diffusion problem, results for the nodal unknowns $\mathbf{u}^{n+1}$ and $\mathbf{u}^n$ in the following partitioned matrix system

$$\begin{aligned} &\Big(\mathbf{M} + \frac{2}{3}\triangle t\mathbf{C} + \frac{2}{3}\nu\triangle t\mathbf{K}\Big)\mathbf{u}^{n+1} \\ &\quad - \Big(\mathbf{M} - \frac{1}{3}\triangle t\mathbf{C} - \frac{1}{3}\nu\triangle t\mathbf{K}\Big)\mathbf{u}^{n^+} = 0 \\ &\Big(\mathbf{M} + \frac{1}{3}\triangle t\mathbf{C} + \frac{1}{3}\nu\triangle t\mathbf{K}\Big)\mathbf{u}^{n+1} \\ &\quad + \Big(\mathbf{M} + \frac{2}{3}\triangle t\mathbf{C} + \frac{2}{3}\nu\triangle t\mathbf{K}\Big)\mathbf{u}^{n^+} = 2\,\mathbf{M}\,\mathbf{u}^{n^-} \end{aligned}$$

4. Results and Discussion

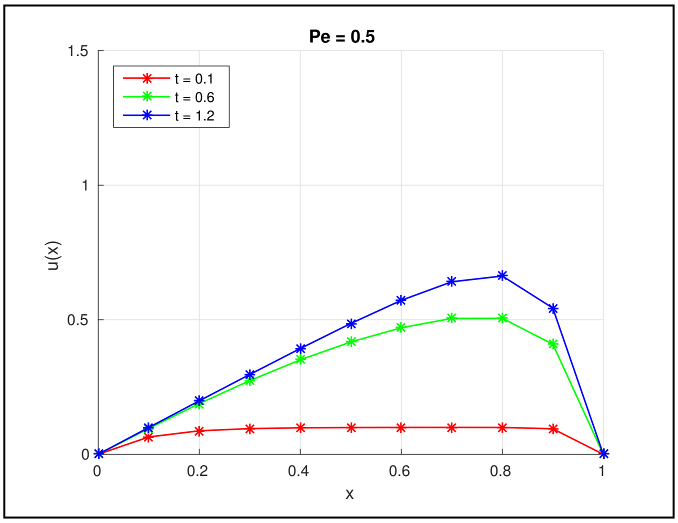
To carry out the numerical simulations, XNS code was used. [2]

4.1. Advection-Diffusion Model Problem

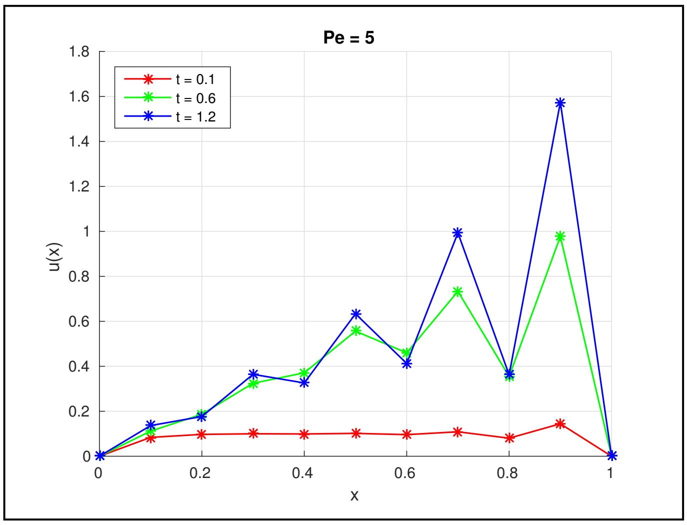
Let us consider the following advection-diffusion model problem:

$$\begin{aligned} u_t + au_x - \nu u_{xx} = 1 \quad &\text{for} \quad (x,t) \in]0,1[\times\mathbf{R}^+ \\ u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0 \quad &\text{for} \quad t \in \mathbf{R}^+ \\ u(x,0) = 0 \quad &\text{for} \quad x \in]0,1[\end{aligned} \tag{30}$$

The above problem is simulated with $Pe = 0.5$ and $Pe = 5$ using the semi-discrete formulation and the results of numerical result is presented in Figure (1). At low Peclet



(a) Crank Nicolson/Galerkin solution of Eqn (30) with Pe = 0.5



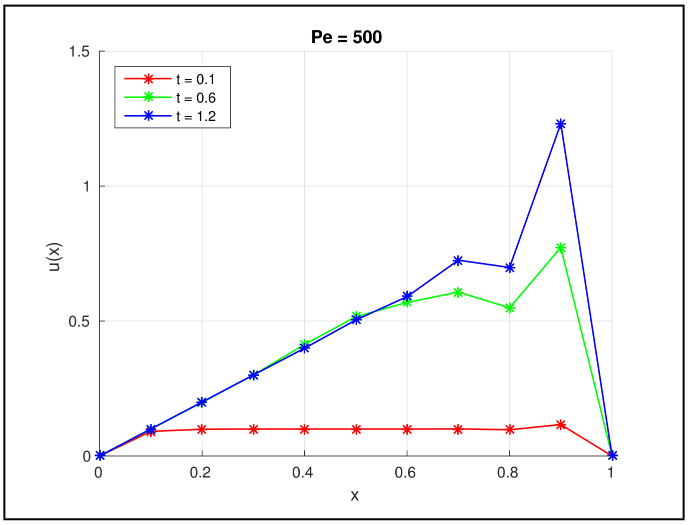
(b) Crank Nicolson/Galerkin solution of Eqn (30) with Pe = 5

Figure 1: Results of Semi-Discrete Formulation of Eqn (30) without stabilization.

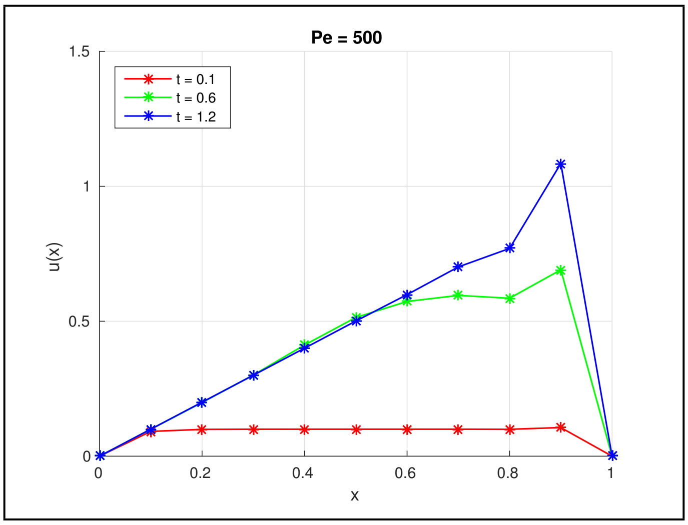
number (Figure (1a)), Crank Nicolson scheme, being a second order accurate scheme in time, is quite accurate. But as the Peclet number increases, the solution gets corrupted with non-physical oscillations (Figure (1b)) and Crank Nicolson scheme fails to dampen these oscillation. Since, the solution here is confronted with Dirichlet Boundary Condition, spatial instabilities will pollute the solution and hence further stabilization is required in spatial direction.

To study the effect of SUPG and GLS stabilization, Eqn (30) is simulated at $Pe = 500$ with different stablization techniques, the result of which are mentioned in Figure (2).

The SUPG and GLS spatial stabilization dampens the oscillations and hence produce better result as compared



(a) Crank Nicolson/Galerkin solution of Eqn (30) with GLS stabilization



(b) Crank Nicolson/Galerkin solution of Eqn (30) with SUPG stabilization

Figure 2: Results of Semi-Discrete Formulation of Eqn (30) with SUPG and GLS stabilization.

to the standard Galerkin method.

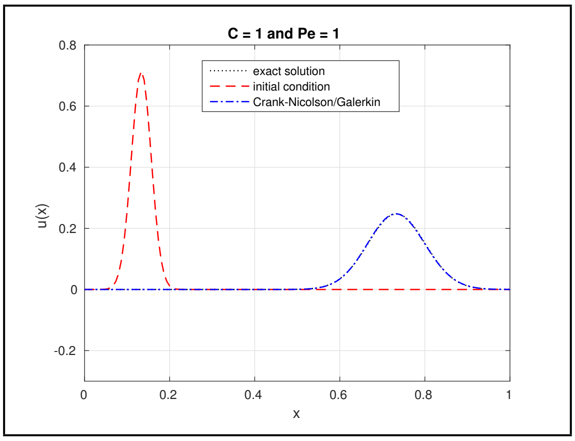
4.2. Advection-Diffusion of a Gaussian Hill

Let us consider the same problem (Eqn (30))but with different initial condition:

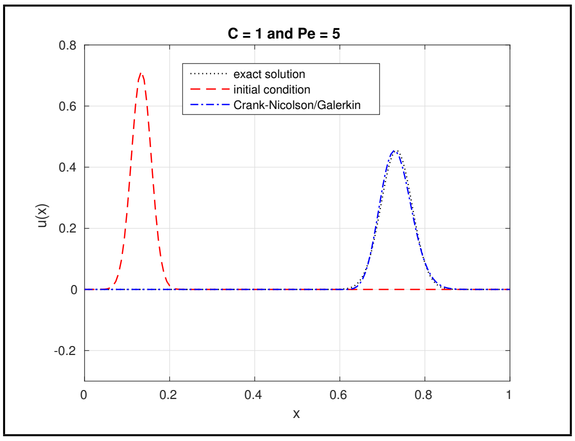
$$u(x,0) = \frac{5}{7}\, exp\left[-\left(\frac{x-x_0}{l}\right)^2\right] \tag{31}$$

where $x_0 = 2/15$ and $l = 7\sqrt{2}/300$. The exact solution for this problem with changed initial condition is given by

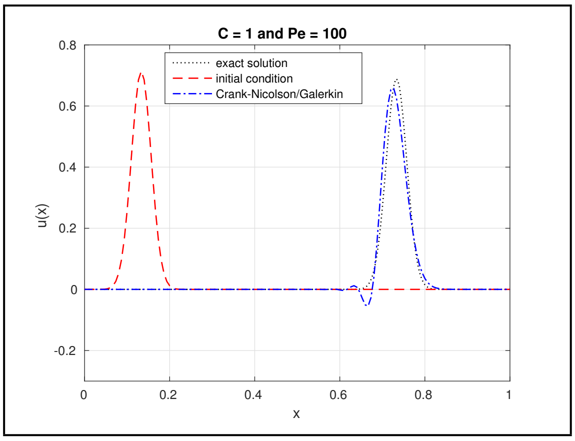
$$u(x,t) = \frac{5}{7\sigma(t)} exp\left[-\left(\frac{x-x_0-at}{l\sigma(t)}\right)^2\right] \quad \text{where} \quad \sigma(t) = \sqrt{1+4\nu t/(l^2)} \tag{32}$$



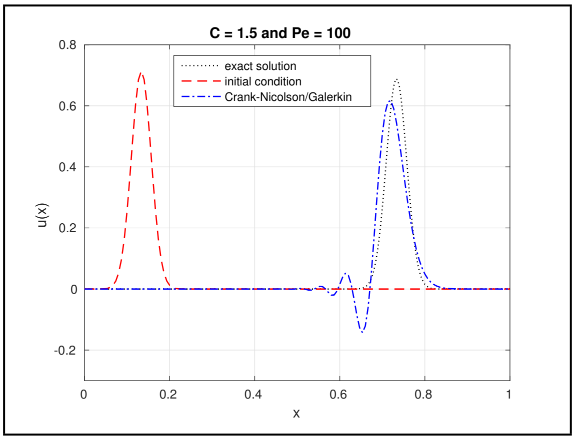
(a) No oscillation observed for low Pe and C



(b) No oscillations are observed for moderate Pe and low C



(c) Oscillations are observed for high Pe and low C



(d) Oscillations are observed for high Pe and high C

Figure 3: Results of Crank Nicolson/Galerkin Formulation of Gaussian Hill.

Figure (3) shows the comparison of exact solution of Gaussian Hill problem (Eqn (32)) with the standard Galerkin and Crank Nicolson formulation for different values of Peclet Number Pe and Courant number C.

For $C = 1$, the second order Crank Nicolson and Galerkin scheme is accurate enough to mimic the exact solution at low and moderate values of Pe (Figure (3a) and (3b)). But for high values of Pe, the oscillations in the solutions are observed (Figure (3c)). As we increase the value of time step, indicated by value of C, the solutions get even more corrupted by the non-physical oscillations when $C > 1$ (Figure(3d)).

Figure (5) shows the comparison of exact solution of Gaussian Hill problem (Eqn (32)) with the space-time formulation for different values of Peclet Number Pe. To test the unconditionally stable nature of this scheme, large time steps corresponding to $C = 3$ is chosen.

As seen from the Figure (5c), the scheme is able to curb the oscillation to a great extent even at such a high Courant number and mimic the exact solution in a much better way as compared with Crank-Nicolson/Galerkin formulation.

Finally, the comparison of the results of the simulation using Crank-Nicolson/Galerkin formulation and space-time/Galerkin formulation is presented in Figure (4). Clearly, the oscillations are dampened in the space-time formulation as compared with Crank-Nicolson formulation.

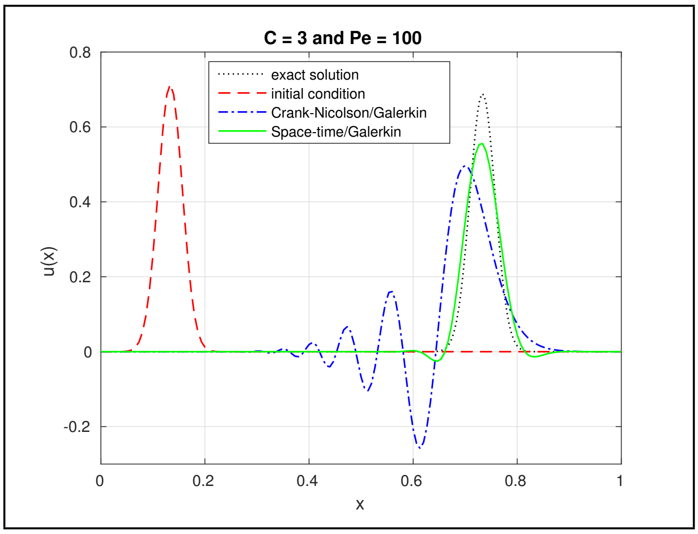
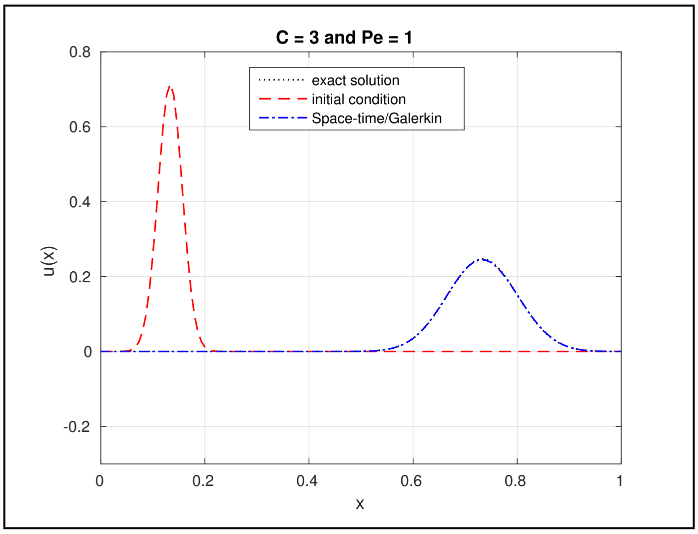


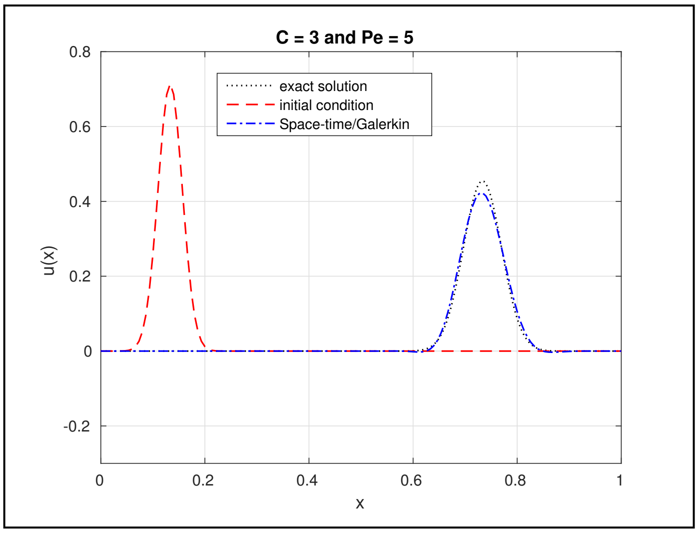
Figure 4: Comparison of Space-time/Galerkin Formulation with Crank Nicolson/Galerkin Formulation

5. Conclusion

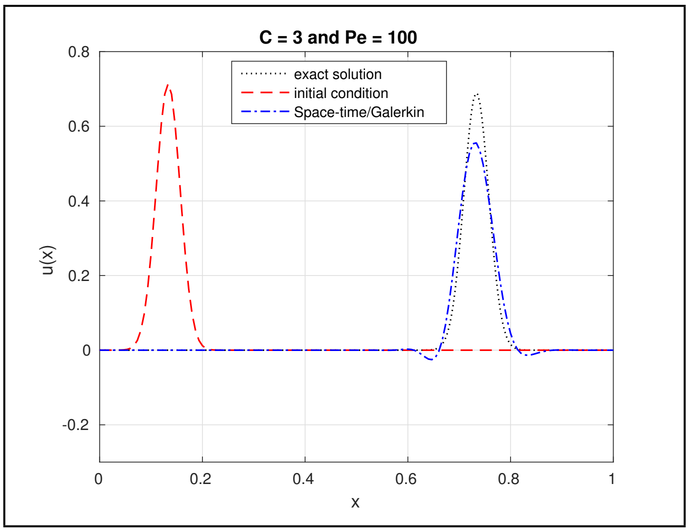
The standard Galerkin method solution is corrupted by non-physical oscillations for advection-dominated problem (high Peclet number). SUPG and GLS methods perform the task of adding the spatial stabilization in a consistent manner along the streamline. For the time discretization, among the θ family of methods, the Crank Nicolson, being the second order accurate in time was chosen. Though,



(a) No oscillations are observed for low Pe and high C



(b) No oscillations are observed for moderate Pe and high C



(c) Very slight oscillations are observed for high Pe and high C. The result is better from what is observed with semi-discrete formulations.

Figure 5: Results of Crank Nicolson/Galerkin Formulation of Gaussian Hill.

able to mimic the exact solution at low values of Pe and C, for the higher values it fails to dampen the oscillations. Therefore, the higher order accurate method such as space-time formulation were considered. Space-time formulation, being a third order accurate unconditionally stable scheme results in a more accurate simulations. Results obtained from the numerical simulation of Gaussian Hill problem suggests the space-time formulation to be more accurate than the semi-discrete formulation.

6. References

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