χ^2. ERROR PROPOGATION

let $S\sum x^2 - (\sum x)^2 = d$

$$a_1 = \frac{1}{d}\left(\sum y \sum x^2 - \sum x \sum xy\right)$$

$$\frac{\partial a_1}{\partial y_i} = \frac{1}{d}\left[\sum x^2 \cdot \frac{\partial(\sum y)}{\partial y_i} - \sum x \cdot \frac{\partial \sum xy}{\partial y_i}\right]$$

Now. $\sum y = \sum \frac{y_i}{\sigma_i^2} \Rightarrow \frac{\partial \sum y}{\partial y_i} = \frac{1}{\sigma_i^2}$

$\sum xy = \sum \frac{x_i y_i}{\sigma_i^2} \Rightarrow \frac{\partial \sum xy}{\partial y_i} = \frac{x_i}{\sigma_i^2}$

$$\Rightarrow \frac{\partial a_1}{\partial y_i} = \frac{1}{d}\left[(\sum x^2)\cdot\frac{1}{\sigma_i^2} - (\sum x)\frac{x_i}{\sigma_i^2}\right]$$

$$\frac{\partial a_1}{\partial y_i} = \frac{1}{d\sigma_i^2}\left[\sum x^2 - x_i \sum x\right]$$

$$\Rightarrow \sigma_{a_1}^2 = \sum_{i=1}^{m}\left(\frac{\partial a_1}{\partial y_i}\right)^2 \cdot \sigma_i^2$$

$$= \sum_{i=1}^{m} \frac{1}{d^2 \sigma_i^4}\left[\sum x^2 - x_i \sum x\right]^2 \sigma_i^2$$

$$\boxed{\sigma_{a_1}^2 = \frac{1}{d^2}\sum_{i=1}^{m}\frac{\left[\sum x^2 - x_i \sum x\right]}{\sigma_i^2}}$$

$S = \sum_{i=1}^{m}\frac{1}{\sigma_i^2}$

$\sum x = \sum_{i=1}^{m}\frac{x_i}{\sigma_i^2}$

$\sum x^2 = \sum_{i=1}^{m}\frac{x_i^2}{\sigma_i^2}$

$\sum y = \sum_{i=1}^{m}\frac{y_i}{\sigma_i^2}$

$\sum xy = \sum_{i=1}^{m}\frac{x_i y_i}{\sigma_i^2}$

$$a_2 = \frac{S\sum xy - \sum x \sum y}{S\sum x^2 - (\sum x)^2} = \frac{1}{d}\left[S\sum xy - \sum x \sum y\right]$$

$$\frac{\partial a_2}{\partial y_i} = \frac{1}{d}\left[S\frac{\partial \sum xy}{\partial y_i} - \sum x \frac{\partial \sum y}{\partial y}\right]$$

Now $\rightarrow$ $$\frac{\partial \sum xy}{\partial y_i} = \frac{x_i}{\sigma_i^2}$$

$$\frac{\partial \sum y}{\partial y} = \frac{1}{\sigma_i^2}$$

$$\Rightarrow \frac{\partial a_2}{\partial y_i} = \frac{1}{d}\left[S \cdot \frac{x_i}{\sigma_i^2} - (\sum x)\cdot\frac{1}{\sigma_i^2}\right]$$

$$\frac{\partial a_2}{\partial y_i} = \frac{1}{d\sigma_i^2}\left[Sx_i - \sum x\right]$$

$$\Rightarrow \sigma_{a_2}^2 = \cancel{\frac{\partial a_2}{\partial y_i}} \sum_{i=1}^{m}\left(\frac{\partial a_2}{\partial y_i}\right)^2 \sigma_i^2$$

$$= \sum_{i=1}^{m} \frac{1}{d^2\sigma_i^4}\left[Sx_i - \sum x\right]^2 \sigma_i^2$$

$$\boxed{\sigma_{a_2}^2 = \frac{1}{d^2}\sum_{i=1}^{m}\frac{(Sx_i - \sum x)^2}{\sigma_i^2}}$$