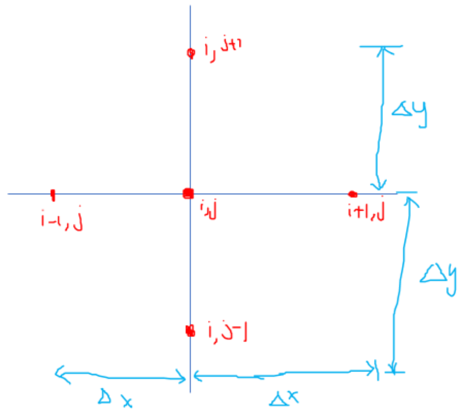
Discretization of Stream Function and vorticity Equation:



Stream function equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega$$

Apply central difference approximation

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-i,j}}{(\Delta x)^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{(\Delta y)^2} = -\omega_{i,j}$$

Assume $\beta = \frac{\Delta x}{\Delta y}$

$$\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-i,j} + \beta^2\left(\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}\right) = -\omega_{i,j}(\Delta x)^2$$

$$\beta^2\psi_{i,j-1} + \psi_{i-i,j} - 2\,(1+\beta^2)\psi_{i,j} + \psi_{i+1,j} + \beta^2\psi_{i,j+1} = -\omega_{i,j}(\Delta x)^2$$

$$\psi_{i,j} = \frac{[\,\psi_{i+1,j} + \psi_{i-i,j} + \beta^2(\psi_{i,j+1} + \psi_{i,j-1}) + \omega_{i,j}(\Delta x)^2]}{2\,(1+\beta^2)}$$

Vorticity Equation:

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \gamma\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

$$u = \frac{\partial \psi}{\partial y}\;;\;\; u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\,\Delta y}$$

$$v = -\frac{\partial \psi}{\partial x}\;;\;\; v_{i,j} = \frac{-\psi_{i+1,j} - \psi_{i-1,j}}{2\,\Delta x}$$

$$u_{i,j}\ \frac{\omega_{i+1,j}-\omega_{i-1,j}}{2\Delta x}+\ v_{i,j}\ \frac{\omega_{i,j+1}-\omega_{i,j-1}}{2\Delta y}=\ \gamma\ \left(\frac{\omega_{i+1,j}-2\omega_{i,j}+\omega_{i-1,j}}{(\Delta x)^2}+\frac{\omega_{i,j+1}-2\omega_{i,j}+\omega_{i,j-1}}{(\Delta y)^2}\right)$$

Multiply both side by $(\Delta x)^2$ *and divide both* side by γ

$$-\left(1-\frac{u_{i,j}\Delta x}{2\gamma}\right)w_{i+1,j}-\ \left(1+\frac{u_{i,j}\Delta x}{2\gamma}\right)w_{i-1,j}-\left(1-\frac{v_{i,j}\Delta y}{2\gamma}\right)\beta^2 w_{i,j+1}-\left(1+\frac{v_{i,j}\Delta y}{2\gamma}\right)\beta^2 w_{i,j+1}\ \ = \text{-}2(1+\beta^2)\omega_{i,j}$$

$$\omega_{i,j}=\frac{\left(1-\frac{u_{i,j}\Delta x}{2\gamma}\right)\omega_{i+1,j}+\left(1+\frac{u_{i,j}\Delta x}{2\gamma}\right)\omega_{i-1,j}\ +\left(1-\frac{v_{i,j}\Delta y}{2\gamma}\right)\beta^2\omega_{i,j+1}+\left(1+\frac{v_{i,j}\Delta y}{2\gamma}\right)\beta^2\omega_{i,j-1}}{2(1+\beta^2)}$$

$$u_{i,j}=\frac{\psi_{i,j+1}-\psi_{i,j-1}}{2\Delta y}$$

$$v_{i,j}=-\frac{\psi_{i+1,j}-\psi_{i-1,j}}{2\Delta x}$$

$$\omega_{i,j}=\frac{1}{2(1+\beta^2)}\left[\left(1-\frac{\psi_{i,j+1}-\psi_{i,j-1}}{2\Delta y}\frac{\Delta x}{2\gamma}\right)\right]\omega_{i+1,j}\ +\left[\left(1+\frac{\psi_{i,j+1}-\psi_{i,j-1}}{2\Delta y}\frac{\Delta x}{2\gamma}\right)\right]\omega_{i-1,j}\ +\left[\left(1+\frac{\psi_{i+1,j}-\psi_{i-1,j}}{2\Delta x}\frac{\Delta y}{2\gamma}\right)\right]\beta^2\omega_{i,j+1}\ \ +\left[\left(1-\frac{\psi_{i+1,j}-\psi_{i-1,j}}{2\Delta x}\frac{\Delta y}{2\gamma}\right)\right]\beta^2\omega_{i,j-1}$$

$$\omega_{i,j} = \frac{1}{2(1+\beta^2)}\left[\left[1-\left(\psi_{i,j+1}-\psi_{i,j-1}\right)\frac{\beta}{4\gamma}\right]\omega_{i+1,j}\right.$$
$$+\left[1+\left(\psi_{i,j+1}-\psi_{i,j-1}\right)\frac{\beta}{4\gamma}\right]\omega_{i-1,j}$$
$$+\left[1+\left(\psi_{i+1,j}-\psi_{i-1,j}\right)\frac{1}{\boldsymbol{4\beta\gamma}}\right]\beta^2\omega_{i,j+1}$$
$$\left.+\left[1-\left(\psi_{i+1,j}-\psi_{i-1,j}\right)\frac{1}{\boldsymbol{4\beta\gamma}}\right]\beta^2\omega_{i,j-1}\right]$$

$$\psi_{i,j} = \frac{1}{2(1+\beta^2)}\left[\psi_{i+1,j}+\psi_{i-1,j}+\beta^2\left(\psi_{i,j+1}+\Psi_{i,j-1}\right)\right.$$
$$\left.+(\Delta x)^2\omega_{i,j}\right]$$