Special Topics in CFD

DAY 4

Solving Advection Equation

Kumaresh

Contents

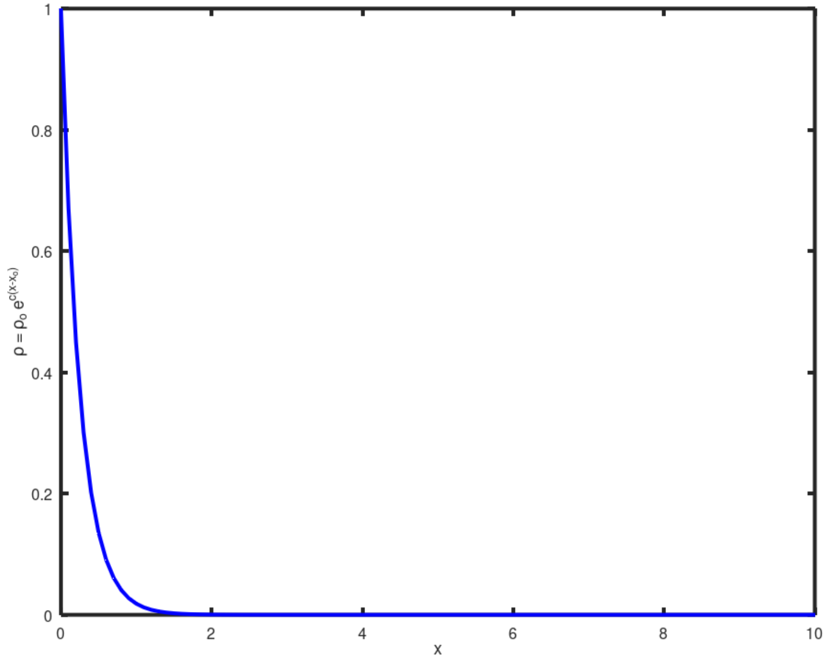
Numerical Stability

- Numerical approach should not magnify the error that appears in the solution.

$$\frac{d\rho}{dx} = -c\rho$$

$$\int_{\rho_o}^{\rho} \frac{d\rho}{\rho} = \int_{x_o}^{x} -c\,dx$$

Analytical $\boxed{\rho = \rho_o e^{-c(x - x_o)}}$



$c = 4, x_o = 0, \rho_o = 1, x \in [0,10]$

Numerical Stability

- Numerical discretization

$$\frac{d\rho}{dx} = -c\rho$$

$$\frac{\rho_{i+1} - \rho_i}{\Delta x} = -c\rho_i$$

Numerical $\boxed{\rho_{i+1} = \rho_i(1 - c\Delta x)}$

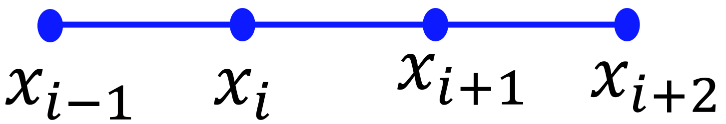
Numerical Stability

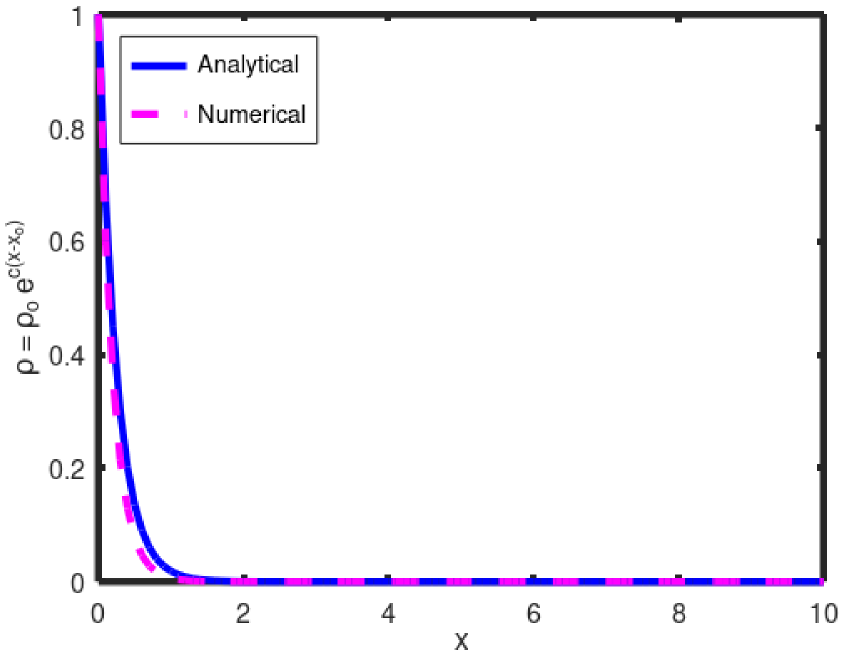
- Numerical discretization

$$\boxed{\frac{d\rho}{dx} = -c\rho}$$

$$\frac{\rho_{i+1} - \rho_i}{\Delta x} = -c\rho_i$$

$$\rho_{i+1} = \rho_i(1 - c\Delta x)$$



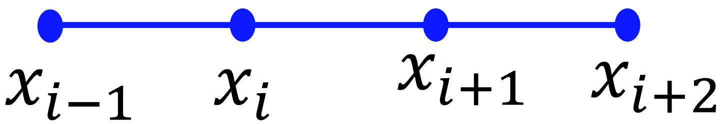


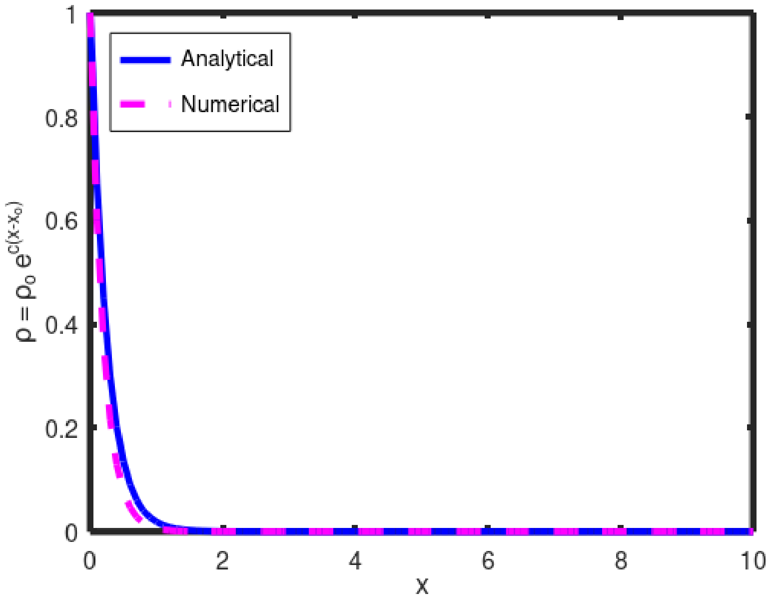
$c = 4, x_o = 0, \rho_o = 1, x \in [0,10], \Delta x = 0.1$

Numerical Stability

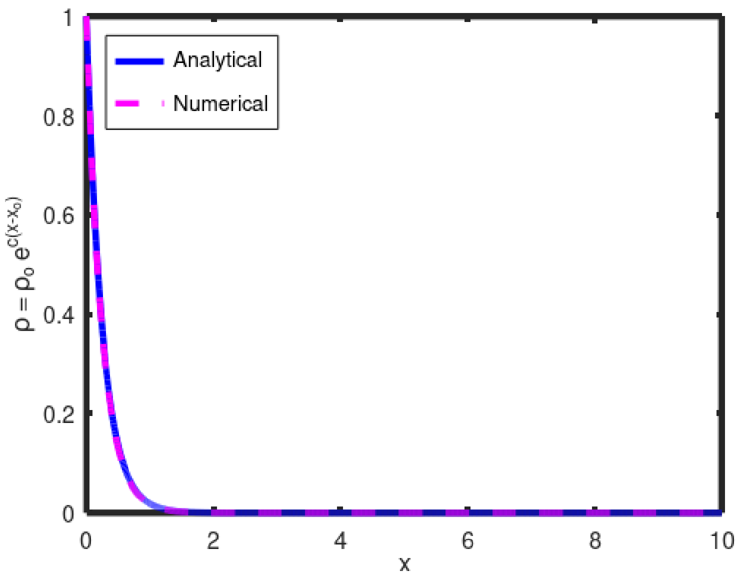
$$\boxed{\frac{d\rho}{dx} = -c\rho}$$

$$\boxed{\rho_{i+1} = \rho_i(1 - c\Delta x)}$$





$c = 4, x_o = 0, \rho_o = 1, x \in [0,10]$,
$\Delta x = 0.1$



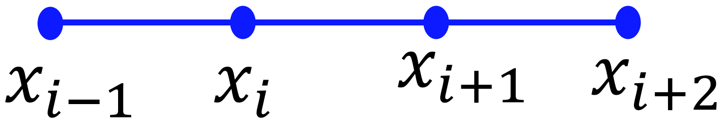
$c = 4, x_o = 0, \rho_o = 1, x \in [0,10]$,
$\Delta x = 0.01$

Stability Condition

$$\left|\frac{\rho_{i+1}}{\rho_i}\right| < 1$$

Numerical Stability

$$\frac{d\rho}{dx} = -c\rho$$



Stability Condition

$$\boxed{\left|\frac{\rho_{i+1}}{\rho_i}\right| < 1}$$

$$\rho_{i+1} = \rho_i(1 - c\Delta x)$$

$$\left|\frac{\rho_{i+1}}{\rho_i}\right| = |(1 - c\Delta x)| < 1$$

$$0 < \Delta x < 2/c$$

$$\boxed{\Delta x < 2/c}$$

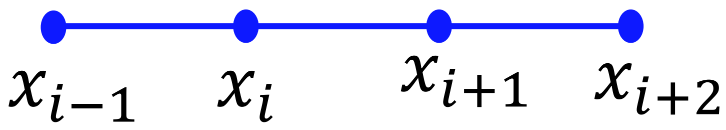
$$\Delta x < \frac{2}{c}$$

$$\frac{2}{c} = \frac{2}{4} = 0.5$$

Numerical Stability

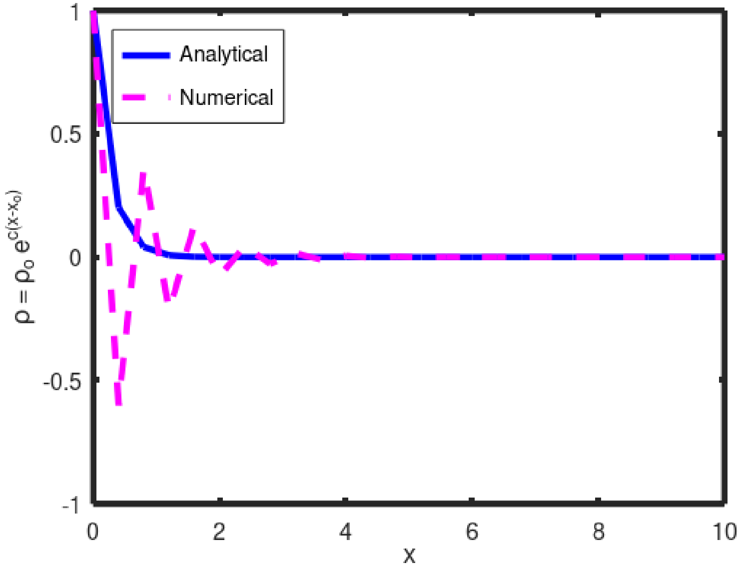
$$\boxed{\frac{d\rho}{dx} = -c\rho}$$

$$\boxed{\rho_{i+1} = \rho_i(1 - c\Delta x)}$$

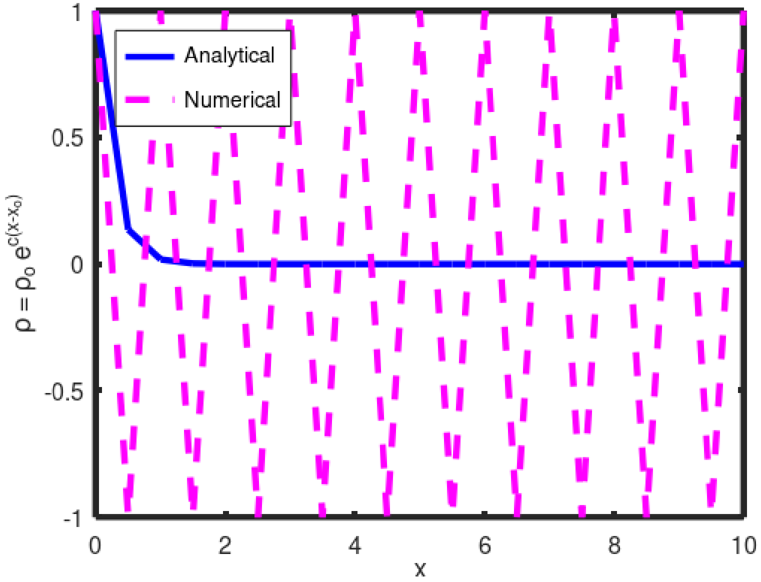


$$\Delta x < \frac{2}{c}$$

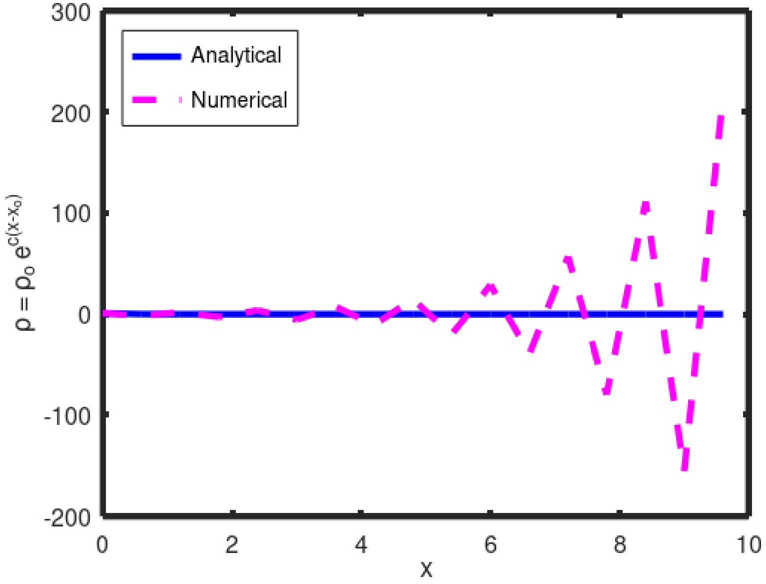
$$\frac{2}{c} = \frac{2}{4} = 0.5$$



$c = 4, x_o = 0, \rho_o = 1, x \in [0,10],$
$\Delta x = 0.4$

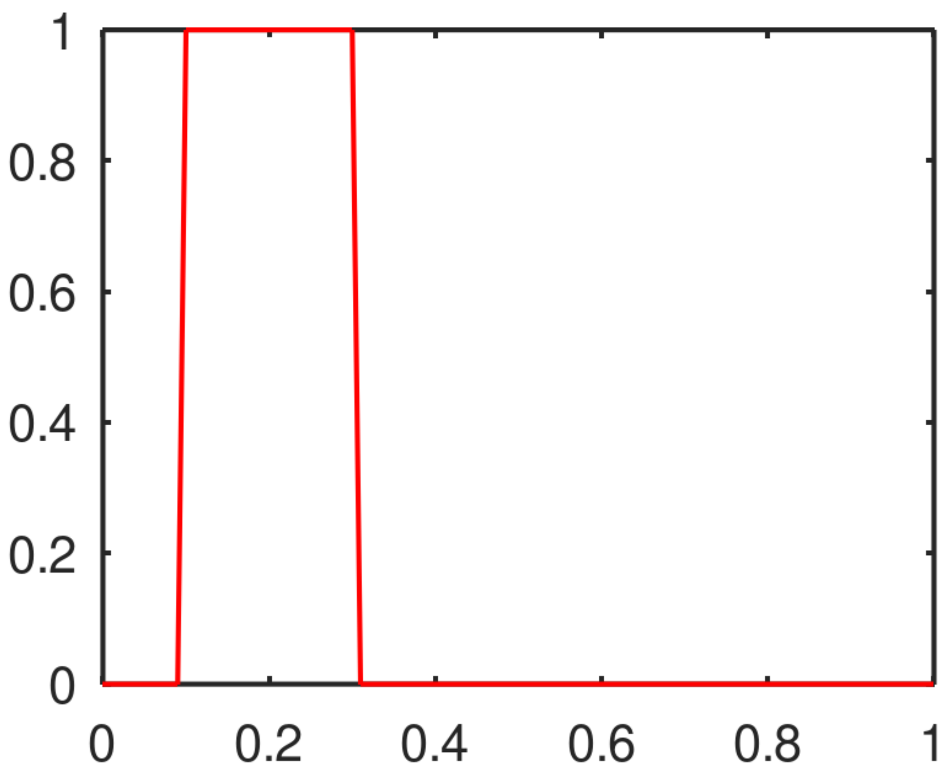


$c = 4, x_o = 0, \rho_o = 1, x \in [0,10],$
$\Delta x = 0.5$



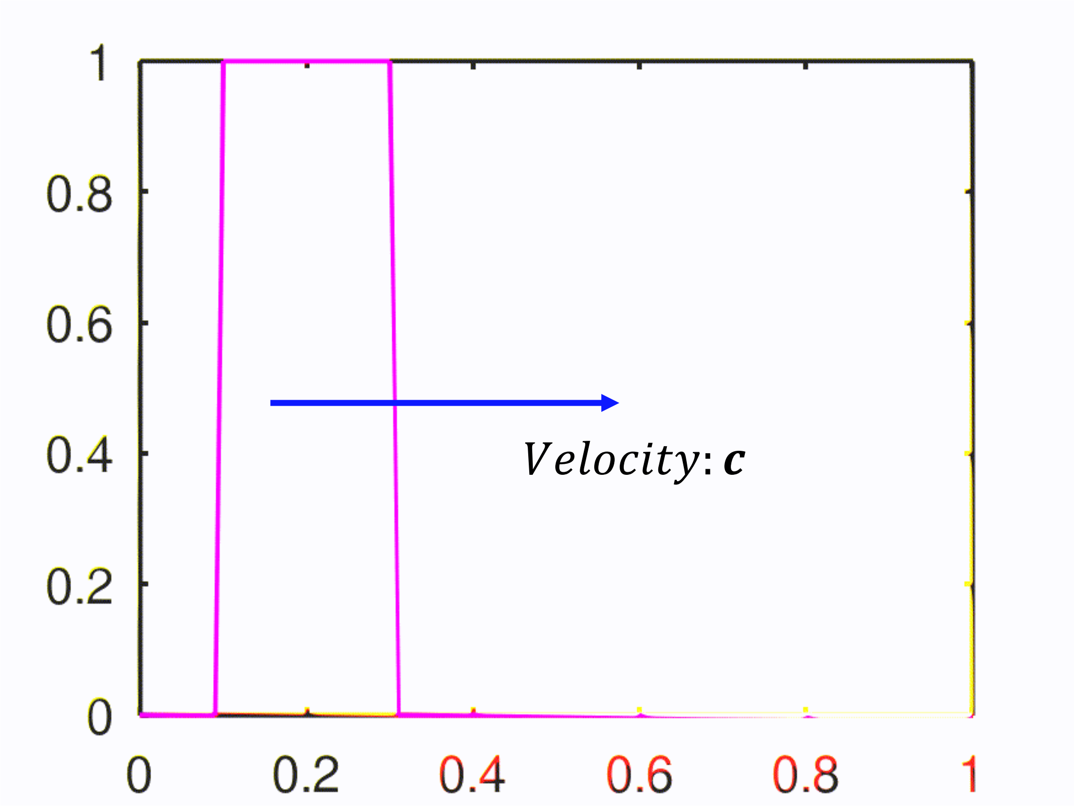
$c = 4, x_o = 0, \rho_o = 1, x \in [0,10],$
$\Delta x = 0.6$

Numerical Stability - Advection equation




```
for i = 1 : length(x)
  if (x(i, 1) >= 0.1) && (x(i, 1) <= 0.3)
    u(i, 1) = 1;
  endif
end
```

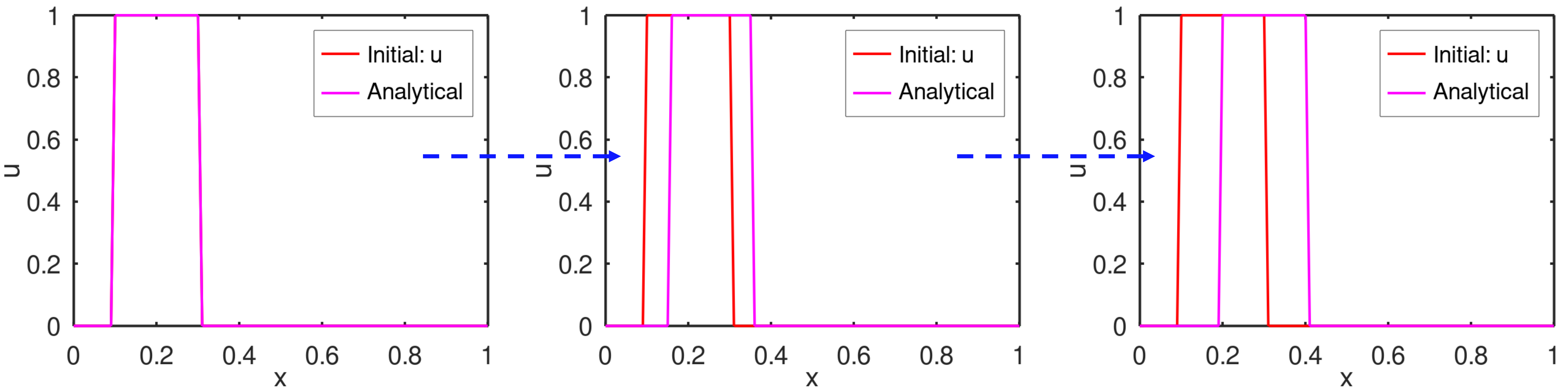
Numerical Stability - Advection equation



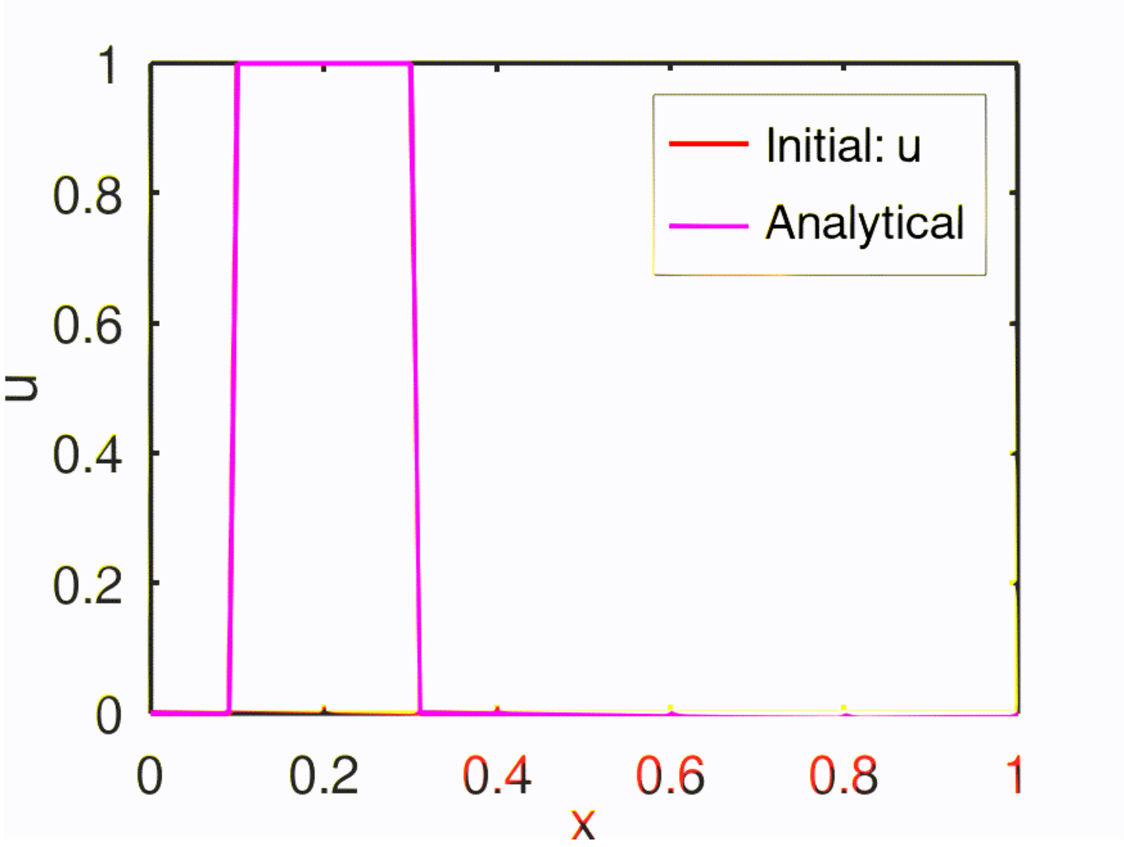
Numerical Stability - Advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \longleftarrow$$ Advection equation

```
for i = 1 : length(x)
  if (x(i, 1) >= 0.1+c*t) && (x(i, 1) <= 0.3+c*t)
    u_analytical(i, 1) = 1;
  endif
end
```



Numerical Stability - Advection equation




```
for i = 1 : length(x)
  if (x(i, 1) >= 0.1+c*t) && (x(i, 1) <= 0.3+c*t)
    u_analytical(i, 1) = 1;
  endif
end
```

Exercise – 4 (i)

1. Solve the following advection equation analytically in octave

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

2. Upload in GitHub

Numerical Stability - Advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\left(\frac{\partial u}{\partial x}\right)_i^n = 0$$

$$\left(\frac{d\rho}{dx}\right)_i \approx \frac{\rho(x_{i+1}) - \rho(x_i)}{\Delta x_i}$$

$$\left(\frac{d\rho}{dx}\right)_i \approx \frac{\rho(x_{i+1}) - \rho(x_{i-1})}{2\Delta x_i}$$

$$u_i^{n+1} = u_i^n - c\Delta t\boxed{\left(\frac{\partial u}{\partial x}\right)_i^n}$$

$$\left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_{i+1}^n - u_i^n}{\Delta x_i}$$

Simple forward difference scheme

$$\left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x_i}$$

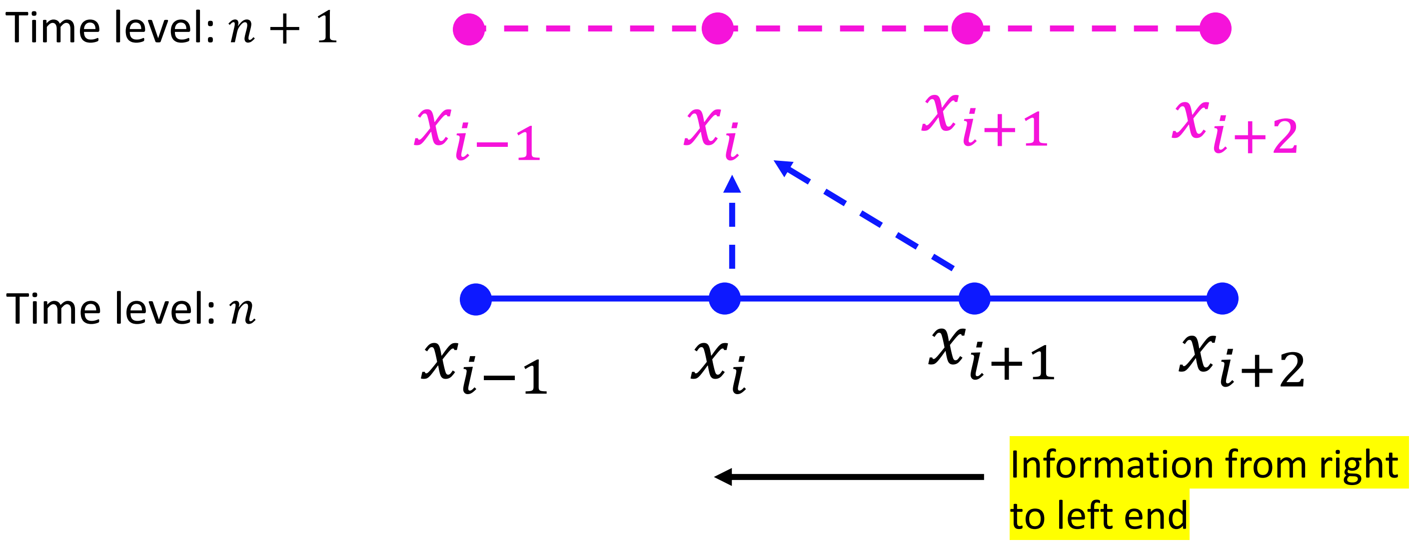
Central difference

(Explicit) First order - Forward Euler

(only one unknown (n+1) with other knowns at n^{th} node)

→ conditionally stable

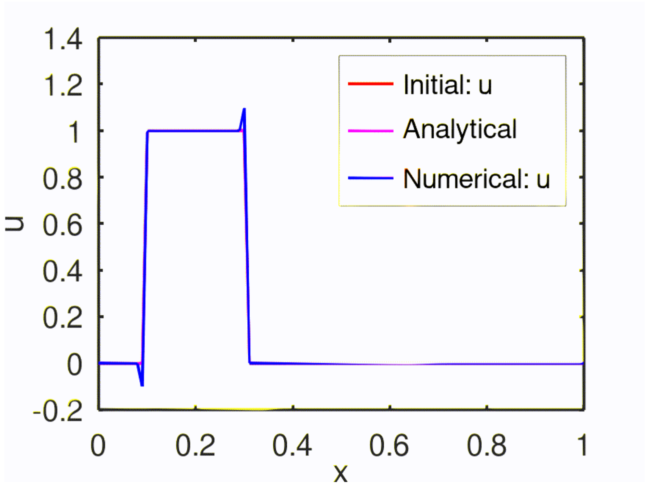
Numerical Stability - Advection equation

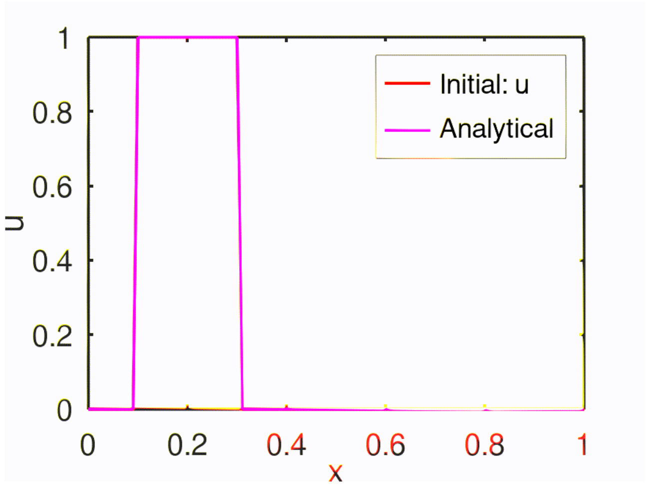


$$u_i^{n+1} = u_i^n - c\Delta t \boxed{\left(\frac{\partial u}{\partial x}\right)_i^n}$$

$$\left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_{i+1}^n - u_i^n}{\Delta x_i}$$ Simple forward difference scheme

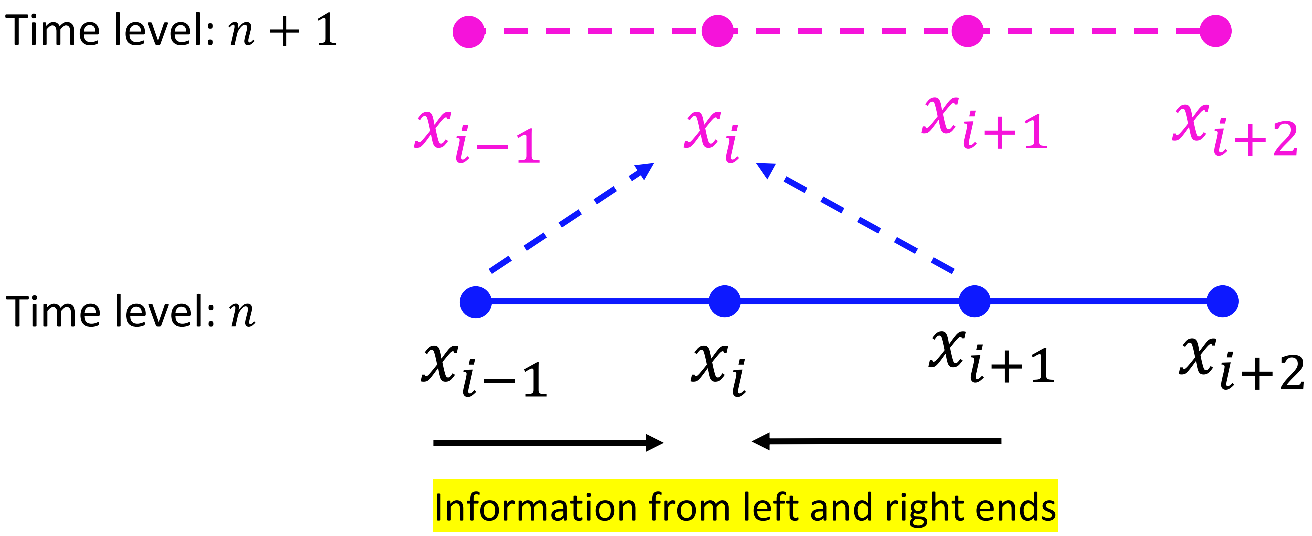
$$\left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x_i}$$ Central difference

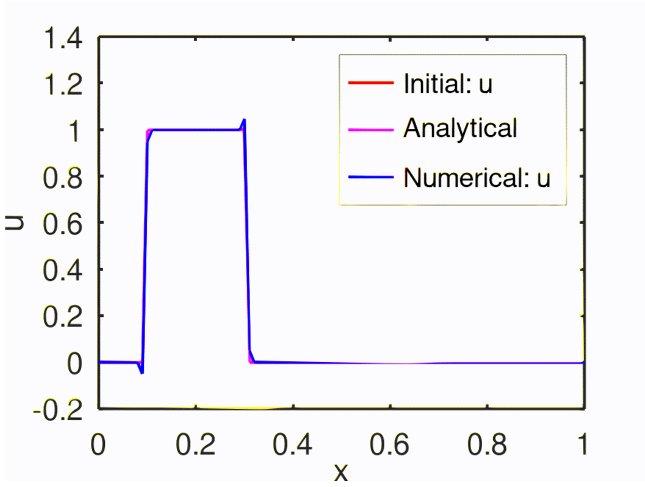


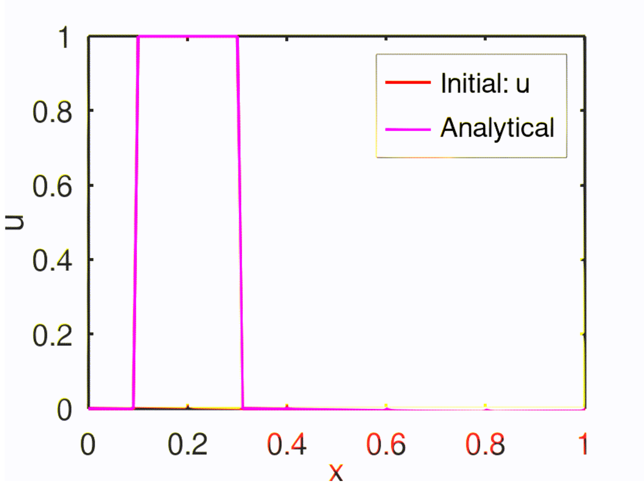


Application: Differential equation used in physics, weather forecasting, stoke markets behaviors (based on probability, past and present information and predicting future information)

Numerical Stability - Advection equation





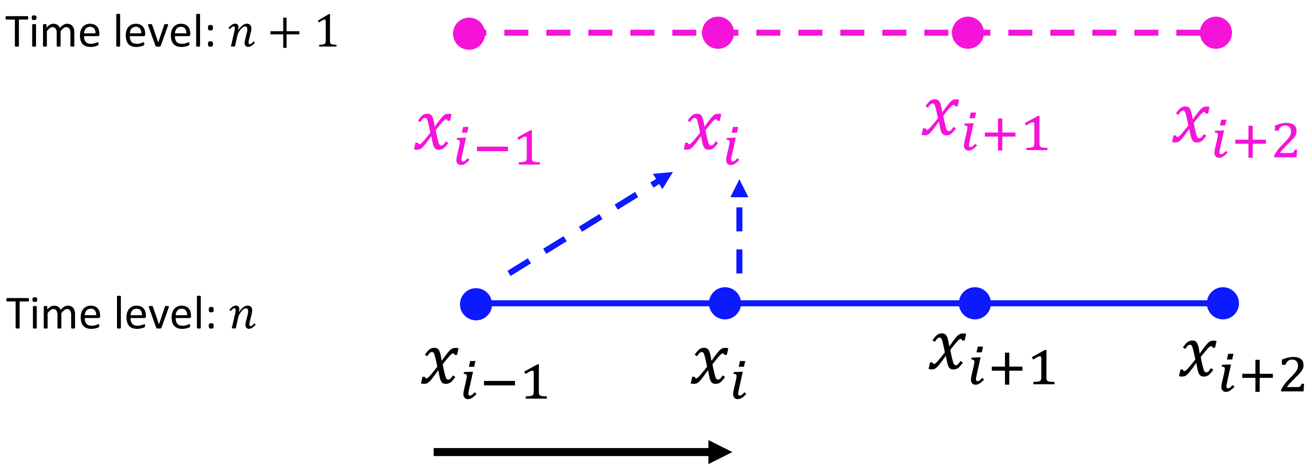


$$u_i^{n+1} = u_i^n - c\Delta t \boxed{\left(\frac{\partial u}{\partial x}\right)_i^n}$$

$$\left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_{i+1}^n - u_i^n}{\Delta x_i}$$ Simple forward difference scheme

$$\left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x_i}$$ Central difference

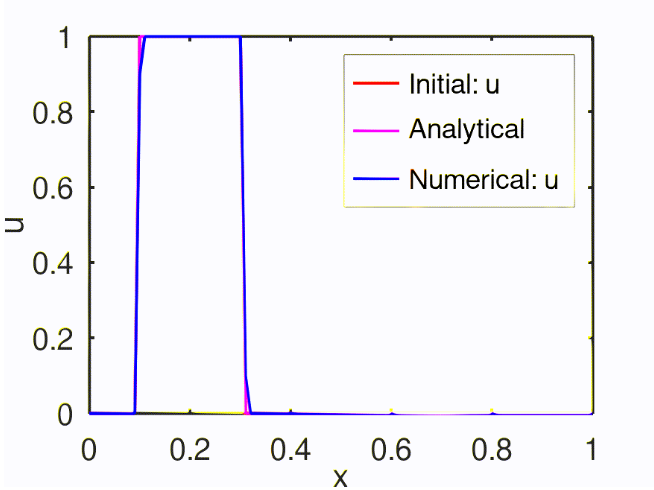
Numerical Stability - Advection equation

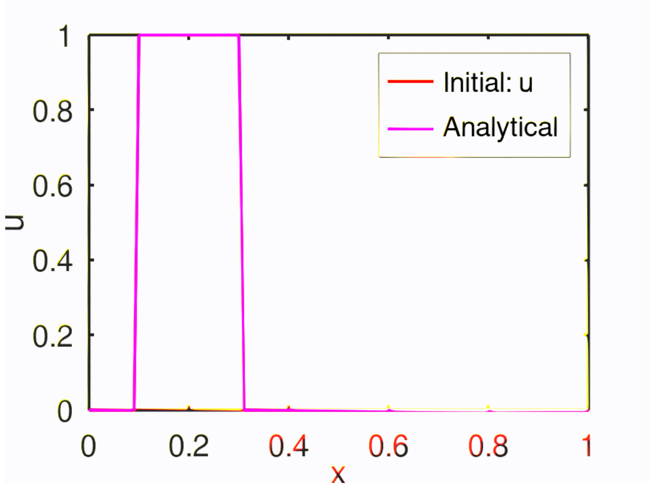


Information from left to right end
Wind is flowing from left end (bird moves from left to right)

$$u_i^{n+1} = u_i^n - c\Delta t \boxed{\left(\frac{\partial u}{\partial x}\right)_i^n} \longrightarrow \left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_i^n - u_{i-1}^n}{\Delta x_i}$$

Simple backward difference scheme



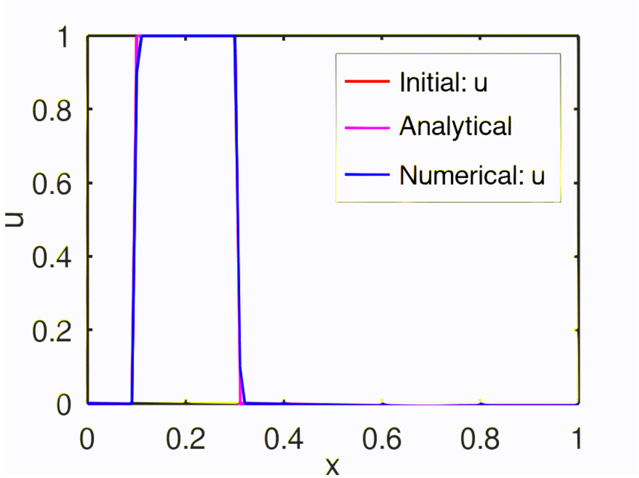


Numerical Stability - Advection equation

$$\boxed{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0}$$

$$u_i^{n+1} = u_i^n - c\Delta t\left(\frac{\partial u}{\partial x}\right)_i^n \rightarrow \boxed{\left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_i^n - u_{i-1}^n}{\Delta x_i}}$$

Simple backward difference scheme



CFL = 0.1

$CFL: \frac{c\Delta t}{\Delta x}$

Courant - Friedrichs - Lewy Number

Upwind scheme → Information from left to right end

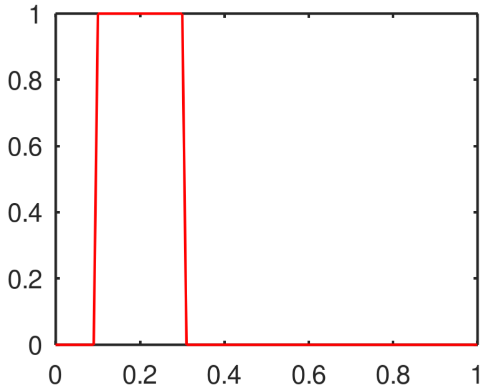
Wind is flowing from left end (Flowing towards the source of the wind)

$$u_i^{n+1} = u_i^n - c\Delta t\left(\frac{\partial u}{\partial x}\right)_i^n \rightarrow \left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_{i+1}^n - u_i^n}{\Delta x_i}$$

Simple forward difference scheme

Downwind scheme

Numerical Stability - Advection equation

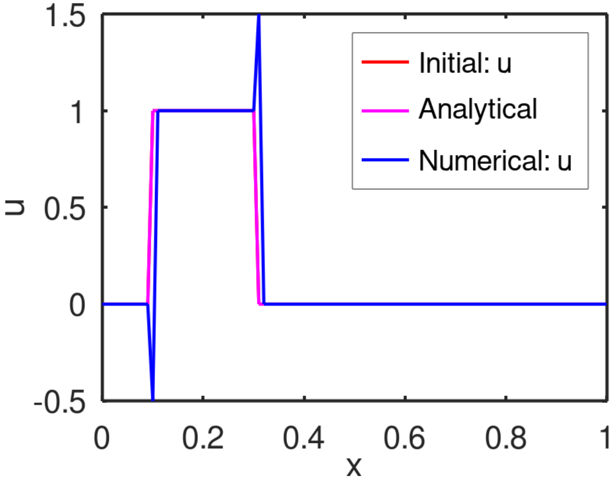


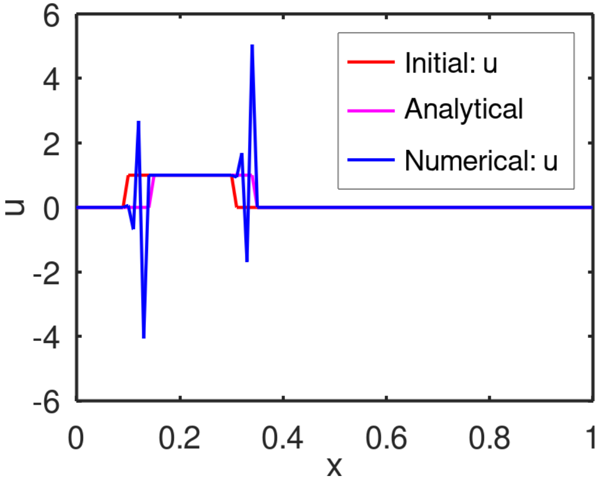
$$u_i^{n+1} = u_i^n - c\Delta t\left(\frac{\partial u}{\partial x}\right)_i^n \longrightarrow \left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_i^n - u_{i-1}^n}{\Delta x_i}$$

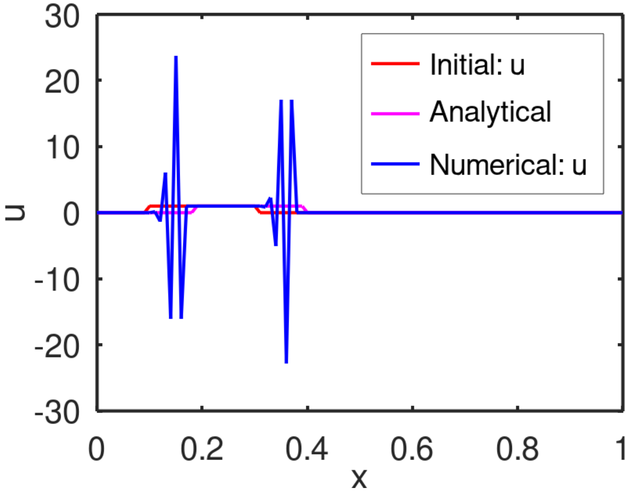
Simple backward difference scheme

Upwind scheme

$$CFL: \frac{c\Delta t}{\Delta x}$$



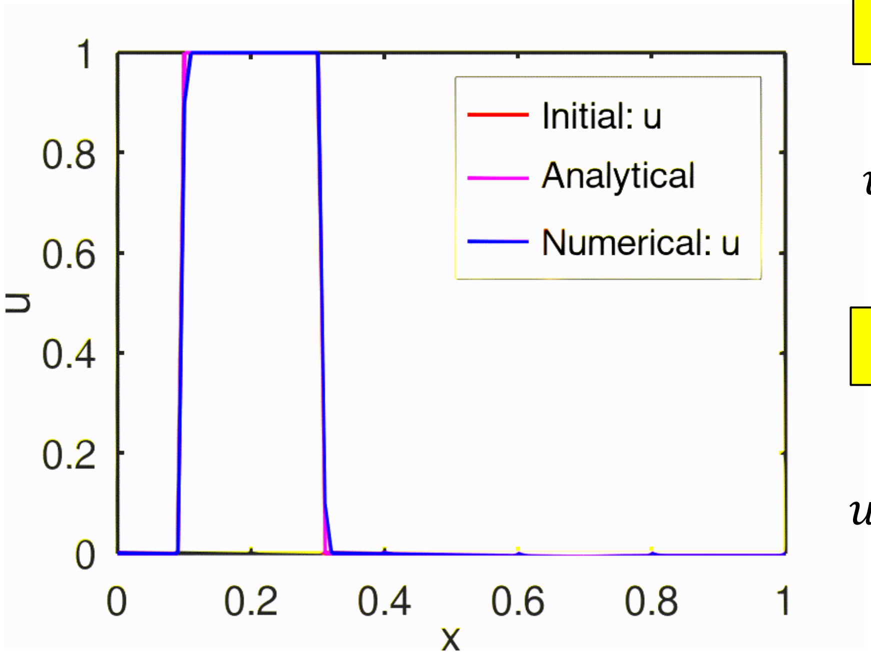




CFL = 1.5 $\longrightarrow$ Δt **increase**

What did we discuss ?

- Proper discrete approximations need to be chosen based on the velocity field.
- CFL number is critical to ensure numerical stability.



Upwind scheme

$$u_i^{n+1} = u_i^n - c\Delta t \boxed{\left(\frac{\partial u}{\partial x}\right)_i^n} \rightarrow \left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_i^n - u_{i-1}^n}{\Delta x_i}$$

Simple backward difference scheme

Downwind scheme

$$u_i^{n+1} = u_i^n - c\Delta t \boxed{\left(\frac{\partial u}{\partial x}\right)_i^n} \rightarrow \left(\frac{\partial u}{\partial x}\right)_i^n \approx \frac{u_{i+1}^n - u_i^n}{\Delta x_i}$$

Simple forward difference scheme

Exercise – 4 (ii)

1. Solve the following advection equation numerically in octave

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

   a) Central difference with CFL = 0.1 (dx = 0.01, c = 0.01, dt = 0.1, t_final = 5)

2. Upwind scheme (backward difference) with CFL = 0.1 (dx = 0.01, dt = 0.1, t_final = 5). Change the ”c” value between 0.01 and -0.01 and analyze the stability. Hint: Upwind scheme with c = -0.01 becomes unstable and act as downwind.

3. Downwind scheme (forward difference) with CFL = 0.1 (dx = 0.01, dt = 0.1, t_final = 5). Change the ”c” value between 0.01 and -0.01 and analyze the stability. Hint: Downwind scheme with c = -0.01 becomes stable and act as upwind.

4. Examine CFL numbers. Analyse the upwind scheme with CFL = 0.1, 1.0, and 10. Analyze the stability. (dx = 0.01, c = 0.01, dt = 0.1, t_final = 5)

5. Upload in GitHub.