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Lecture 16

5.1 Eigenvectors and Eigenvalues

5.2 The Characteristic Equation

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The Eigenvalue Problem

Let A be an $n \times n$ matrix. We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of A if there exists $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. In this case we call $\mathbf{x}$ an *eigenvector* of A corresponding to λ.

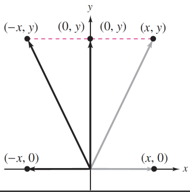

Figure: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflects vectors w.r.t. y axis. $A\mathbf{x} = \begin{bmatrix} -x \\ y \end{bmatrix}$.

For very specific vectors $(x, 0)$ and $(0, y)$, A acts as a scalar multiple. These are eigenvectors of A.

Eigenspaces

Check $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has eigenvectors $\mathbf{x}_1 = (1, 1)$ and $\mathbf{x}_2 = (-1, 1)$. Find the corresponding eigenvalues.

If an $n \times n$ matrix A has an eigenvalue λ, we call

$$E_\lambda = \left\{ \mathbf{x} \in \mathbb{R}^n : (A - \lambda I)\mathbf{x} = \mathbf{0} \right\}$$

the *eigenspace* of λ. It is a subspace of $\mathbb{R}^n$.

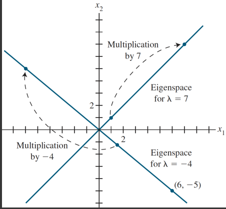

Figure: Eigenspaces of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. $A\mathbf{x} = \begin{bmatrix} x + 6y \\ 5x + 2y \end{bmatrix}$.

Characteristic polynomial

The action of A in the eigenspaces is very simple. Ideally, one would like to find a basis of $\mathbb{R}^n$ consisting of eigenvectors, in order to understand the global action of A from the simpler action in the eigenspaces. This is not always possible (it is possible if A is symmetric).

Note that $A\mathbf{x} = \lambda\mathbf{x}$ has nontrivial solutions iff $\det(\lambda I - A) = 0$. If A has size $n \times n$, then

$$\det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

is a polynomial of degree n in λ called the *characteristic polynomial*. The eigenvalues of A are the roots of this polynomial.

Finding eigenvalues and eigenvectors

Let A be an $n \times n$ matrix.

1. Find the real roots of the characteristic equation $\det(A - \lambda I) = 0$. These are the eigenvalues of A.
2. For each eigenvalue λ_i, find the eigenvectors corresponding to λ_i by solving the system $(A - \lambda_i I)\mathbf{x} = \mathbf{0}$. This can require row reducing an $n \times n$ matrix. The reduced row-echelon form must have at least one row of zeros.

Let $A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$. Is 5 an eigenvalue of A ?

Multiplicity

An eigenvalue λ_i has *algebraic multiplicity* k if it occurs as a multiple root k times in the characteristic polynomial. Equivalently, $(\lambda - \lambda_i)^k$ is a factor of the polynomial, but $(\lambda - \lambda_i)^{k+1}$ is not.

The *geometric multiplicity* of λ_i is the dimension of the corresponding eigenspace, $\dim E_{\lambda_i}$.

If $\mu(\lambda_i)$ is the algebraic multiplicity, $\gamma(\lambda_i)$ is the geometric multiplicity and A has size n, then

$$1 \leq \gamma(\lambda_i) \leq \mu(\lambda_i) \leq n\,.$$

We say that λ_i is a *simple eigenvalue* if $\mu(\lambda_i) = 1$. In this case we must have $\gamma(\lambda_i) = 1$, i.e. a basis for the eigenspace will be a single eigenvector.

Exercises

1. Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Compute $A\mathbf{x}$ if $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then find the eigenvalues of A and bases for the eigenspaces.

2. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = I_2$.

3. Find the algebraic and geometric multiplicities of eigenvalues in 1. and 2.

Check. The sum of eigenvalues must be equal to the sum of diagonal entries a_{ii} of A. This sum is called the *trace* of A. (We may prove this later).

Triangular matrices

Thm: If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

Proof. $\lambda I - A$ is also triangular, so $|\lambda I - A| = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$. The roots are $\lambda = a_{ii}$. $\square$

Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Linear independence of eigenvectors

Thm: Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Proof. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors of A for distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. If they are dependent, then some $\mathbf{v}_k = \sum_{i \neq k} c_i \mathbf{v}_i$, i.e. $\mathbf{v}_k \in H = \mathrm{Span}\{\mathbf{v}_i\}_{i \neq k}$. By the spanning set thm, a subset of these vectors forms a basis of H, say $\{\mathbf{v}_i\}_{i \in I}$. Then $\mathbf{v}_k \in H = \mathrm{Span}\{\mathbf{v}_i\}_{i \in I}$ implies $\mathbf{v}_k = \sum_{i \in I} d_i \mathbf{v}_i$ for some d_i. So

$$\lambda_k \mathbf{v}_k = A\mathbf{v}_k = \sum_{i \in I} d_i A \mathbf{v}_i = \sum_{i \in I} d_i \lambda_i \mathbf{v}_i \,.$$

But we also have

$$\lambda_k \mathbf{v}_k = \lambda_k \sum_{i \in I} d_i \mathbf{v}_i = \sum_{i \in I} d_i \lambda_k \mathbf{v}_i \,.$$

Similar matrices

Subtracting, we get $\mathbf{0} = \sum_{i \in I} d_i(\lambda_i - \lambda_k)\mathbf{v}_i$. But $\{\mathbf{v}_i\}_{i \in I}$ is a basis, so linearly independent. Since $\lambda_i - \lambda_k \neq 0\ \forall i$, we must have $d_i = 0\ \forall i$. But then $\mathbf{v}_k = \sum_{i \in I} d_i \mathbf{v}_i = \mathbf{0}$, a contradiction since $\mathbf{v}_k$ is an eigenvector.
Conclusion: the eigenvectors must be independent. $\square$

Thm: A is invertible iff 0 is not an eigenvalue of A.

This is clear since A is invertible iff $|A| \neq 0$, meaning $|0I - A| \neq 0$ i.e. 0 is not an eigenvalue.

Let A, B be $n \times n$ matrices. A is *similar to B* if there exists an invertible matrix P such that $B = P^{-1}AP$.

Similar matrices

Let A, B, C be square matrices of order n. Then

1. A is similar to A.
2. If A is similar to B then B is similar to A.
3. If A is similar to B and B is similar to C then A is similar to C.

Prove this.

Thm: If A and B are similar, they have the same characteristic polynomial, so the same eigenvalues.

Proof. $|\lambda I - B| = |\lambda I - P^{-1}AP| = |P^{-1}(\lambda I - A)P| = |P^{-1}|\,|\lambda I - A|\,|P| = |P^{-1}P|\,|\lambda I - A| = |I|\,|\lambda I - A| = |\lambda I - A|$. $\square$