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Lecture 4

1.8 Introduction to Linear Transformations

1.9 The Matrix of a Linear Transformation

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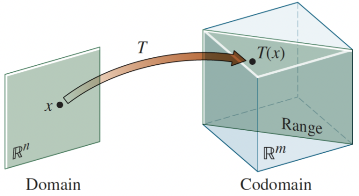
Linear Transformations

A transformation (or function or map) $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x} \in \mathbb{R}^n$ a vector $T(\mathbf{x}) \in \mathbb{R}^m$. Here, the *domain* of T is $\mathbb{R}^n$ and the *codomain* is $\mathbb{R}^m$. We call $T(\mathbf{x})$ the *image* of $\mathbf{x}$ under T. The set of all images $\{T(\mathbf{x})\}$ is called the *range* of T.



Matrix transformations

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Find $T(\mathbf{u})$ for $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

b. Find $\mathbf{x} \in \mathbb{R}^2$ whose image under T is $\mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.

c. Is there more than one $\mathbf{x}$ whose image is $\mathbf{b}$?

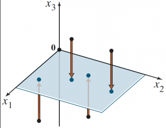
d. Is $\mathbf{c} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$ in the range of T ?

Matrix transformations

> Thm: The range of a matrix transformation is the span of its columns.

Proof. Indeed, $A\mathbf{x} = \sum_{i=1}^{n} x_i \boldsymbol{\alpha}_i$. As $\mathbf{x}$ varies, we get all linear combinations of $\boldsymbol{\alpha}_i$. $\square$

e.g. The projection of $\mathbb{R}^3$ onto the xy plane is given by the transformation $T(\mathbf{x}) = A\mathbf{x}$, for $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.



Linear transformations

The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ for $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a *shear* transformation.

A transformation (or mapping) T is *linear* if for any $\mathbf{u}, \mathbf{v}$ in the domain of T, and any scalar c,

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$.

Linear transformations

Hence, linear transformations preserve the operations of vector addition and scalar multiplication.

Any matrix transformation is linear since $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ and $A(c\mathbf{u}) = cA\mathbf{u}$.

Thm: If T is linear, then $T(\mathbf{0}) = \mathbf{0}$ and $T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$, for any $\mathbf{u}, \mathbf{v}$ in the domain of T and scalars c, d.

Proof. $T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}$. Next, $T(c\mathbf{u}+d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$. $\square$

Linear transformations

Repeating, we get $T\left(\sum_{i=1}^{p} c_i \mathbf{v}_i\right) = \sum_{i=1}^{p} c_i T(\mathbf{v}_i)$.

The map $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = c\mathbf{x}$ is a *dilation* if $c > 1$ and a *contraction* if $0 \leq c \leq 1$.

The map $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(\mathbf{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a *rotation counterclockwise by θ.*

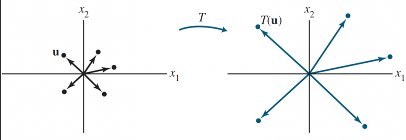


FIGURE 5 A dilation transformation.

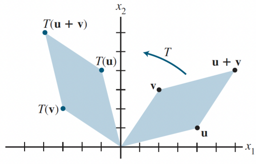


FIGURE 6 A rotation transformation.

The matrix of a linear transformation

Show the previous maps are linear.

In the following, $\mathbf{e}_j$ is the vector with all entries equal to 0 except the j-th entry, which is equal to 1.

Thm: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

In fact, A is the $m \times n$ matrix $A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$.

The matrix of a linear transformation

Proof. Write $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$. Then by linearity of T, $T(\mathbf{x}) = T\left(\sum_{i=1}^n x_i\mathbf{e}_i\right) = \sum_{i=1}^n x_i T(\mathbf{e}_i)$, and by definition,

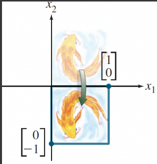
$$A\mathbf{x} = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^n x_i T(\mathbf{e}_i).$$

So $T(\mathbf{x}) = A\mathbf{x}$ as required. $\square$

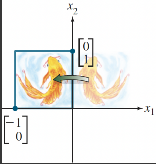
The matrix A is called the *standard matrix* for T.

Find the standard matrix for the dilation transformation $T(\mathbf{x}) = c\mathbf{x}$.

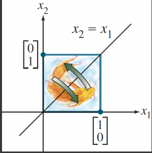
Illustrations



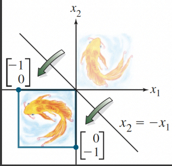
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



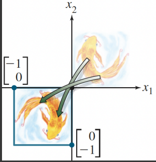
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



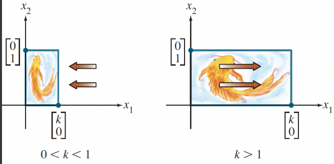
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



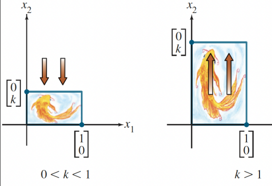
$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

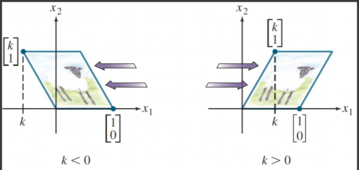


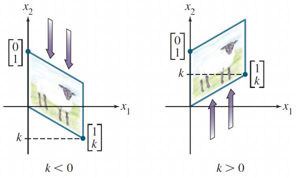
$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

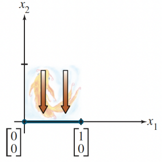
Illustrations

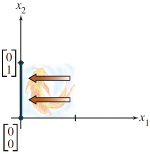




$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$



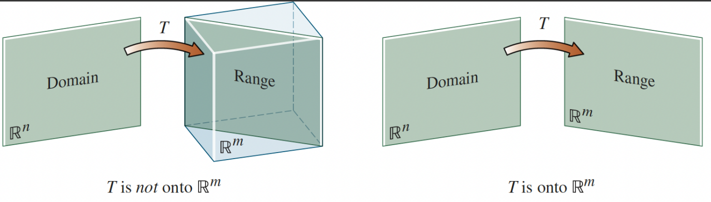


$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

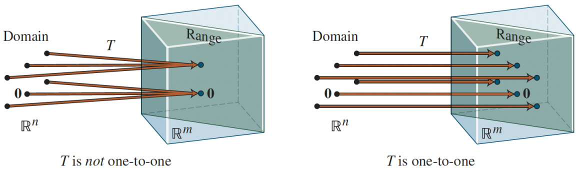
Surjective maps

A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is surjective, or onto $\mathbb{R}^m$, if the range of T is all of $\mathbb{R}^m$. In other words, for each $\mathbf{b} \in \mathbb{R}^m$, there exists some $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.



Injective maps

A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is injective, or one-to-one, if distinct elements in $\mathbb{R}^n$ have distinct images in $\mathbb{R}^m$. In other words, $\mathbf{u} \neq \mathbf{v} \implies T(\mathbf{u}) \neq T(\mathbf{v})$. Equivalently, $T(\mathbf{u}) = T(\mathbf{v}) \implies \mathbf{u} = \mathbf{v}$.



Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map defined by the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$. Is T onto $\mathbb{R}^2$? Is it one-to-one ?

Injective maps

Thm: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one iff the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Proof. ($\Longrightarrow$) As T is linear, $T(\mathbf{0}) = \mathbf{0}$. If T is one-to-one, then any $\mathbf{x} \neq \mathbf{0}$ has a different image, i.e. $T(\mathbf{x}) \neq \mathbf{0}$. Hence, the equation has only the trivial solution.

($\Longleftarrow$) If $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution, let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and suppose that $T(\mathbf{u}) = T(\mathbf{v})$. Then $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$ by linearity. So by hypothesis, we get $\mathbf{u} - \mathbf{v} = \mathbf{0}$, i.e. $\mathbf{u} = \mathbf{v}$. $\square$

Injective and surjective maps

Thm: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$.
2. T is one-to-one iff the columns of A are linearly independent.

Proof. 1. T is onto $\mathbb{R}^m$ iff $\mathrm{Range}(T) = \mathbb{R}^m$. The range is the span of the columns of A (earlier slide), so we get 1.

2. T is one-to-one iff $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution (previous slide), which occurs iff the columns of A are linearly independent (previous lecture), proving 2.

Injective and surjective maps

Cor: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear.

1. If $n > m$ then T is never one-to-one.
2. If $n < m$ then T is never onto $\mathbb{R}^m$.

Proof. Here A is $m \times n$, so we have n column vectors in $\mathbb{R}^m$. If $n > m$, they must be linearly dependent (theorem), so T is not 1-1. If $n < m$ then we have fewer vectors than the dimension m of the space, hence they cannot span $\mathbb{R}^m$, so T is not surjective. $\square$

Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x, y) = (x + 2y, 2x + 4y, 3x - 5y)$. Is T one-to-one ? Is it onto $\mathbb{R}^3$?