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Lecture 10

4.1 Vector Spaces and Subspaces

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Vector Spaces

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and every scalar $c, d \in \mathbb{R}$, then V is a *vector space*.

1. $\mathbf{u} + \mathbf{v} \in V$. (Closed under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutative)
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associative)
4. V has a zero vector $\mathbf{0}$ such that for any $\mathbf{u} \in V$, $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ (Additive identity)
5. For every $\mathbf{u} \in V$ there exists $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ (Additive inverse)

Vector Spaces

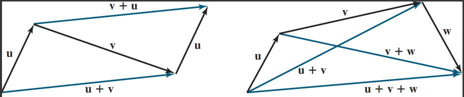
6. $c\mathbf{u} \in V$. (Closed under scalar multiplication)
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (Distributive)
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (Distributive)
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ (Associative)
10. $1\mathbf{u} = \mathbf{u}$ (Scalar identity)

e.g. $V = \mathbb{R}^n$ is a vector space. $V = \{\mathbf{0}\}$ as well.

e.g. The space $M_{m,n}$ of all $m \times n$ matrices is a vector space. This follows immediately from the properties of matrix addition and scalar multiplication (no need for matrix multiplication).

Examples

e.g. The set of vectors in the plane (arrows) with scalar multiplication and addition according to the parallelogram rule form a vector space.



The set $\mathbb{S}$ of all doubly infinite sequences of numbers

$$(y_k) = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$$

with coordinate-wise addition and scalar multiplication is a vector space.

Exercises

1. Let P_2 be the set of all polynomials of the form $p(x) = a_0 + a_1 x + a_2 x^2$, where $a_0, a_1, a_2 \in \mathbb{R}$. The sum of two polynomials $p(x) = a_0 + a_1 x + a_2 x^2$ and $q(x) = b_0 + b_1 x + b_2 x^2$ is defined in the usual way,

   $$(p+q)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

   and scalar multiplication by $c \in \mathbb{R}$ is defined by

   $$(cp)(x) = ca_0 + ca_1 x + ca_2 x^2 .$$

   Show that P_2 is a vector space.

2. Let $C(\mathbb{R})$ be the set of all real-valued continuous functions on $\mathbb{R}$. Define addition and scalar multiplication as usual by $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$. Show $C(\mathbb{R})$ is a vector space.

Further properties

More generally, the following are vectors spaces:

1. The space P_n of all polynomials of degree $\leq n$.
2. The space P of all polynomials.
3. The space $C[a,b]$ of continuous functions on $[a,b]$.

If V is a vector space, $\mathbf{v} \in V$ and $c \in \mathbb{R}$, we also have

1. The additive identity (zero) is unique.
2. Every $\mathbf{v} \in V$ has a unique additive inverse.
3. $0\mathbf{v} = \mathbf{0}$
4. $c\mathbf{0} = \mathbf{0}$
5. $c\mathbf{v} = \mathbf{0} \implies c = 0$ or $\mathbf{v} = \mathbf{0}$
6. $(-1)\mathbf{v} = -\mathbf{v}$

Proof

1. If $\mathbf{0}_1, \mathbf{0}_2$ are two additive identities then $\mathbf{0}_1 = \mathbf{0}_1 + \mathbf{0}_2$ (as $\mathbf{0}_2$ is a zero) and $\mathbf{0}_2 = \mathbf{0}_2 + \mathbf{0}_1$ (as $\mathbf{0}_1$ is a zero), hence $\mathbf{0}_1 = \mathbf{0}_2$.
2. If $\mathbf{v}$ has two additive inverses $\mathbf{u}$ and $\mathbf{w}$, then $\mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{0} + \mathbf{w} = \mathbf{w}$.
3. $0\mathbf{v} = (0+0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$ $\implies 0\mathbf{v} - 0\mathbf{v} = 0\mathbf{v} + 0\mathbf{v} - 0\mathbf{v} \implies \mathbf{0} = 0\mathbf{v} + \mathbf{0} = 0\mathbf{v}$.
4. $c\mathbf{0} = c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$ $\implies c\mathbf{0} - c\mathbf{0} = c\mathbf{0} + c\mathbf{0} - c\mathbf{0} \implies \mathbf{0} = c\mathbf{0} + \mathbf{0} = c\mathbf{0}$
5. If $c = 0$, we are done. If $c \neq 0$ then $\mathbf{v} = 1\mathbf{v} = (c^{-1}c)\mathbf{v} = c^{-1}(c\mathbf{v}) = c^{-1}\mathbf{0} = \mathbf{0}$ by 4.
6. $\mathbf{v} + (-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = (1-1)\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ by 3. Similarly $(-1)\mathbf{v} + \mathbf{v} = \mathbf{0}$. By 2, we get $(-1)\mathbf{v} = -\mathbf{v}$.

Exercises

Are the following sets vector spaces ?

1. The set of integers with standard operations.
2. The set Q of polynomials of degree 2.
3. $H = \{(x, y) : y = 3x + 2\}$.
4. The set $\mathbb{R}^3$ with standard addition and nonstandard scalar multiplication given by $c(x, y, z) = (cx, 0, 0)$.

A nonempty subset W of a vector space V is a *subspace* of V when W is a vector space under the operations of addition and scalar multiplication defined in V.

Subspaces

Test for a subspace. If W is a subset of a vector space V, then W is a subspace iff

1. $\mathbf{0} \in W$.
2. If $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$.
3. If $\mathbf{u} \in W$ and $c \in \mathbb{R}$, then $c\mathbf{u} \in W$.

The reason is that all the algebraic properties are automatically inherited from V.

Are the following sets subspaces ?

1. The subset of symmetric matrices in $M_{2,2}$.
2. The subset of invertible matrices in $M_{2,2}$.

Linear combinations

3. The subset of singular matrices in $M_{2,2}$.
4. $[0, +\infty) \subset \mathbb{R}$
5. The subset of differentiable functions in $C[a, b]$
6. $H = \{(x, y) : y = x^2\}$

We define linear combinations in vector spaces like in $\mathbb{R}^n$. Namely, $\mathbf{v} \in V$ is a linear combination of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_p \in V$ if there exist $c_i \in \mathbb{R}$ such that

$$\mathbf{v} = \sum_{i=1}^{p} c_i \mathbf{u}_i = c_1 \mathbf{u}_1 + \cdots + c_p \mathbf{u}_p \,.$$

Span

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$. We say $\mathbf{u} \in \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ if

$$\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

for some $c_i \in \mathbb{R}$, i.e. the span is the set of all linear combinations of $\mathbf{v}_i$.

We say that $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a *spanning set* of V if $V = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. We also say that S *spans* V, or that S is a *generating set* for V.

Span

> Thm: If V is a vector space, $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$, then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Proof is the same as before where V was $\mathbb{R}^n$.

Is $\left\{ \begin{bmatrix} a-b & a+b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ a subspace of $M_{2,2}$?