Mostafa Sabri

Lecture 14

4.5 The Dimension of a Vector Space

جامعة نيويورك أبوظبي
NYU ABU DHABI

Dimension

Thm: If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors is linearly dependent.

Proof. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\} \subset V$ with $p > n$. Then $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}\} \subset \mathbb{R}^n$. The latter set is dependent by Lec3. As the coordinate map is an isomorphism, we deduce that $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is dependent as well (Lec 13).

Thm: If a vector space V has a basis of n vectors, then every basis of V must consist of n vectors.

The proof is identical to Lec5.

Finite and infinite dimension

If V is a vector space that has a basis with n vectors, we say that $\dim V = n$. If no finite set spans V, we say it is infinite dimensional.

e.g. $\dim P_n = n + 1$, $\dim M_{m,n} = mn$, while $C[a, b]$ is infinite dimensional.

Find the dimension of $H = \left\{ \begin{bmatrix} a - c \\ b + d \\ c - d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$.

Dimension and Subspaces

Thm: Let V be a finite dimensional vector space and $H \subseteq V$ a subspace. Then

1. Any linearly independent set in H can be expanded, if necessary, to a basis for H.
2. $\dim H \leq \dim V$.
3. If $\dim H \neq \dim V$, then $H \neq V$.
4. If $\dim H = \dim V$, then $H = V$.

<u>Proof.</u> Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset H$ be linearly independent. If S spans H, it is a basis for H. If not, then some $\mathbf{v} \in H$ is not in $\mathrm{Span}(S)$. It follows that $S' = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}\}$ is linearly independent. Indeed if S' was dependent, some vector in S' would be a combination of those preceding

Dimension and Subspaces

it, which is not possible here. Now, if S' is a basis for H we stop. Otherwise, we continue adding independent vectors until reaching $\dim H$, then the expanded set S_+ must be a basis for H, since any more vectors $\mathbf{v}$ would make S_+ dependent, implying $\mathbf{v} \in \mathrm{Span}(S_+)$.

2. Let $F \subseteq H$ be a basis for H. Then F is linearly independent in V. If it spans V, then $\dim V = \dim H$. If it does not span V, then we expand F until it becomes a basis for V, using 1. Then $\dim H < \dim V$.

3. This is trivial as the dimension is well-defined.

4. We showed in 2. that if $F \subseteq H$ is a basis for H, and if it does not span V, then $\dim H < \dim V$. So $\dim H = \dim V$ implies F must span V. Hence, $V = \mathrm{Span}(F) = H$.

Basis Theorem

Basis Thm. Let $\dim V = p$. If p vectors in V are linearly independent, they must be a basis for V. If p vectors in V span V, they must be a basis for V.

Proof. Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$. If S is linearly independent, let $H = \mathrm{Span}(S)$. Then S is a basis for H, so $\dim H = p = \dim V$, hence $H = V$. On the other hand, if S spans V, then a subset S' of S is a basis for V (Lec12), so $p = |S'| = |S|$, hence $S' = S$. $\square$

Let us turn back to the column and row spaces and discuss their dimension. Recall that the rank of a matrix A is the dimension of the column spaces, also the number of pivot columns.

Rank and Row Spaces

Thm: The row and column spaces of an $m \times n$ matrix A have the same dimension.

Proof. Let B be the reduced row echelon form of A. If $r = \dim \operatorname{Col}(A)$, then B has r pivot columns. Let s be the number of nonzero rows in B. Since the pivots move to the right when going down the rows, we must have $s \geq r$. But each nonzero row in B has a pivot position, so the corresponding columns are pivot columns. Thus, $s = r$. $\square$

Corollary: $\operatorname{rank}(A) = \dim \operatorname{Col}(A) = \dim \operatorname{Row}(A)$.

Rank Theorem

Rank Thm. If A has n columns, then
$\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = n$.

Proof. Let B be a row echelon of A and let r be the number of nonzero rows in B. We have $\mathrm{Nul}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} : B\mathbf{x} = \mathbf{0}\}$. But back-substitution using B gives r equations in n variables. So we have $n - r$ free variables. This implies we can write $\mathrm{Nul}(A)$ as the span of $n - r$ linearly independent vectors using the usual procedure. Hence, $\dim \mathrm{Nul}(A) = n - r$. $\square$

We sometimes call $\dim \mathrm{Nul}(A)$ the *nullity* of A.

Let A have size 4×10. Can A have rank 7 ? nullity 10 ?

Rank and Linear Transformations

In the following, V and W are vector spaces

If $T : V \to W$ is linear, we define
$\text{rank}(T) = \dim \text{Range}(T), \qquad \text{nullity}(T) = \dim \ker(T).$

Thm: If $T : V \to W$ is linear and $\dim V = n$, then

$$\text{rank}(T) + \text{nullity}(T) = n\,.$$

Proof. This follows from the rank theorem since T can be represented by a matrix A with n columns, so $\dim \text{Range}(T) = \dim \text{Range}(A) = \dim \text{Col}(A)$, and $\dim \ker(T) = \dim \text{Nul}(A)$. $\square$

Rank and Linear Transformations

Thm: Let $T : V \to W$ be linear, where $\dim W = m$.

1. T is surjective (onto W) iff $\mathrm{rank}(T) = m$.
2. If $\dim V = \dim W = m$, then T is surjective iff it is injective.

Proof. 1. If T is surjective then $\mathrm{Range}(T) = W$, so $\mathrm{rank}(T) = \dim W = m$. Conversely, if $\mathrm{rank}(T) = m$, then $\mathrm{Range}(T) \subset W$ is a subspace of W of the same dimension as W, so $\mathrm{Range}(T) = W$ (earlier thm).

2. $\mathrm{rank}(T) + \dim\ker(T) = m$. If T is surjective, then $\mathrm{rank}(T) = m$, so $\dim\ker(T) = 0$, so $\ker(T) = \{\mathbf{0}\}$ and T is injective. On the other hand, if T is injective, then $\dim\ker(T) = 0$, so $\mathrm{rank}(T) = m$ and T is surjective. $\square$

Isomorphism

Show that if V and W are both isomorphic to a vector space H, then V is isomorphic to W.

> Thm: Two finite dimensional vectors spaces V, W are isomorphic iff they have the same dimension.

<u>Proof.</u> If V, W are isomorphic, then the isomorphism $T : V \to W$ carries a basis of V to a basis of W, so $\dim V = \dim W$.[1] Conversely, if $\dim V = \dim W = m$, then V and W are both isomorphic to $\mathbb{R}^m$ via the coordinate map, so they are isomorphic to each other. $\square$

e.g. $\mathbb{R}^4$, $M_{4,1}$, $M_{2,2}$ and P_3 are all isomorphic.

[1] Find an alternative proof using the rank theorem.