Mostafa Sabri

Lecture 8

3.1 Introduction to Determinants

3.2 Properties of Determinants I

جامعة نيويورك أبوظبي
NYU ABU DHABI

The Determinant of a Matrix

The *determinant* of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = |A| = ad - bc$.

Find the determinant of $\begin{bmatrix} 1 & 0 \\ 5 & 10 \end{bmatrix}$.

For order higher than 2, we will need the following.

If A is a square matrix, then the *minor* M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The *cofactor* C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Minors and cofactors

Minor of a_{21}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Minor of a_{22}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

The cofactor may only differ from its minor by a sign, according to the sign pattern:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

e.g. $C_{11} = M_{11}$, $C_{12} = -M_{12}$ and so on.

Determinant of a Square Matrix

Find all minors and cofactors of $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & -4 & 5 \\ -1 & 0 & -2 \end{bmatrix}$.

If A is a square matrix of order $n \geq 2$, then the determinant of A is the sum of the entries in the first row of A multiplied by their respective cofactors:

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Laplace expansion theorem

Find the determinant of $\begin{bmatrix} 0 & 1 & 2 \\ 3 & -4 & 5 \\ -1 & 0 & -2 \end{bmatrix}$.

Thm: If A has size $n \times n$, then

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$
$$= \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

This allows us to expand from any ith row, or any jth column.

Expansion of the determinant

The proof is admitted (it uses permutations and is not particularly enlightening); the statement is very useful.

The best choice for expansion is the row or column that contains the most zeroes, as we get for them $a_{ij}C_{ij} = 0C_{ij} = 0$.

Find the determinant of $\begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -2 \end{bmatrix}$.

Rule of Sarrus for 3×3 matrices

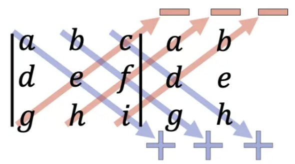

Figure: $\det(A) = aei + bfg + cdh - ceg - bdi - afh$

Find the determinant of $\begin{bmatrix} 0 & 1 & 2 \\ 3 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

Only works for 3×3 matrices !

Determinant of triangular matrices

If A is a triangular $n \times n$ matrix, then its determinant is the product of the entries on the main diagonal:

$$\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn}$$

Find the determinant of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ -7 & 1000 & 2 & 0 \\ -\sqrt{2} & 10 & 3 & 7 \end{bmatrix}$.

Row operations and determinants

Thm: Let A and B be square matrices.

1. If B is obtained from A by interchanging two rows of A, then $\det(B) = -\det(A)$.
2. If B is obtained from A by adding a multiple of a row of A to another row of A, then $\det(B) = \det(A)$.
3. If B is obtained from A by multiplying a row of A by a nonzero constant c, then $\det(B) = c\det(A)$.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}.$$

Row and column operations and determinants

Find $\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$ using row operations.

Thm: The same rules apply if we perform analogous operations on the columns.

Find $\begin{vmatrix} -1 & 2 & 3 \\ 2 & -4 & 5 \\ -7 & 14 & 99 \end{vmatrix}$.

We prove both theorems later in the lecture.

Zero determinants

Thm: If A is a square matrix and any one of the conditions below is true, then $\det(A) = 0$.

1. An entire row (or column) consists of zeros.
2. Two rows (or columns) are equal.
3. One row (or column) is a multiple of another row (or column).

Proof. 1. Expand in this row or column.

2. If B is the matrix A with the two rows/columns interchanged, then $B = A$ but $\det B = -\det A$ by the previous theorems. So $\det A = -\det A$ and $\det A = 0$.

3. Reduces to 2 using the property $\det B = c \det A$. $\square$

Determinant of large matrices

Find $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 12 & 24 \end{vmatrix}$.

For square matrices A of size ≥ 5, it is always faster to find $\det A$ using row/column operations, than expanding in cofactors. This is how computers calculate.

Find $\begin{vmatrix} 4 & 2 & 0 & -2 & 1 \\ 1 & 1 & 2 & 4 & 0 \\ 2 & -1 & 0 & 4 & 1 \\ 5 & 2 & 1 & -2 & 4 \\ 1 & 5 & 0 & 2 & 3 \end{vmatrix}$.

Proof of the row operation theorem

1. (Interchange) This is clear for 2×2 matrices. If A is a 3×3 matrix and B is obtained by interchanging two of its rows, expand $|A|$ and $|B|$ in a row different than these exchanged rows. Then the cofactors of B will be the negatives of the cofactors of A, as they have two rows interchanged, and we know the property for 2×2 matrices. So we proved the property for matrices of order 3. Repeating, we can go to any higher order (this is formalized by *mathematical induction*).

3. If B is obtained from A by multiplying R_i by c, then expanding in this row we get $|B| = \sum_j c a_{ij} C_{ij} = c \sum_j a_{ij} C_{ij} = c|A|$.

Proof of the column operation theorem

2. If B is obtained from A via $R_i + cR_j \to R_i$, expand $|B|$ in row i. Then $|B| = \sum_k (a_{ik} + ca_{jk})C_{ik} = |A| + c\sum_k a_{jk}C_{ik}$. Now the second sum is the determinant of a matrix having two rows equal (i and j). This determinant is zero by a previous theorem (we didn't use the present property 2 to prove it). Hence, $|B| = |A|$. $\square$

The proof of the column operation theorem is exactly the same by expanding through columns instead of rows. $\square$