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Lecture 2

1.3 Vector Equations

1.4 The matrix equation $A\mathbf{x} = \mathbf{b}$

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Vectors in $\mathbb{R}^2$

A vector in $\mathbb{R}^2$ is denoted by a column vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ or an ordered pair (x_1, x_2), where $x_1, x_2 \in \mathbb{R}$.

Two vectors are equal iff their ordered entries are equal. Note that $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Addition and subtraction is defined component-wise:
$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$.

For $c \in \mathbb{R}$, we define scalar multiplication $c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$.

Geometric description of $\mathbb{R}^2$

Given $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, find $3\mathbf{u} - 2\mathbf{v}$.

Vectors are pictured by arrows. We can identify vectors in $\mathbb{R}^2$ geometrically with the point (a, b).

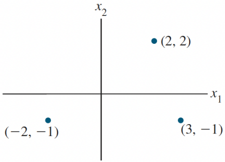


FIGURE 1 Vectors as points.

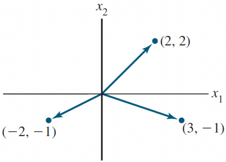
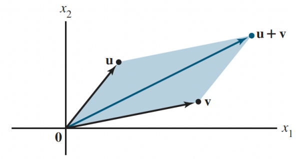


FIGURE 2 Vectors with arrows.

Parallelogram Rule for Addition

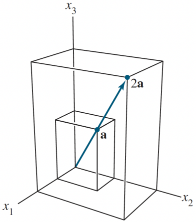
If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $\mathbf{v}$.



Let $\mathbf{u}=\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}-1\\1\end{bmatrix}$. Display $\mathbf{u}+\mathbf{v}$, $2\mathbf{u}$, $\frac{1}{2}\mathbf{u}$ and $-\mathbf{u}$ on the plane.

Vectors in $\mathbb{R}^n$

A vector in $\mathbb{R}^n$ is a column vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$.



The zero vector $\mathbf{0}$ is the vector with all entries zero.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars c, d,

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

Linear Combinations

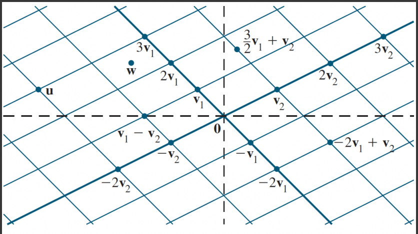
We say that $\mathbf{y} \in \mathbb{R}^n$ is a *linear combination* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$ if there exist scalars $c_1, \ldots, c_p \in \mathbb{R}$ such that

$$\mathbf{y} = \sum_{i=1}^{p} c_i \mathbf{v}_i = c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p$$

The c_i can be regarded as *weights*.

Linear Combinations

1. Based on the figure, write **u** and **w** as linear combinations of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.



2. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$. Is **u** a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$?

Span

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. We say that $\mathbf{u} \in \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ if

$$\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

for some $c_i \in \mathbb{R}$, i.e. if $\mathbf{u}$ is a linear combination of the $\mathbf{v}_i$.

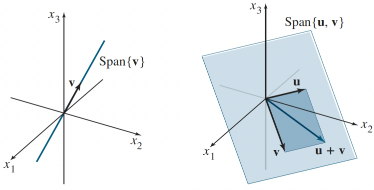


FIGURE 10 Span $\{\mathbf{v}\}$ as a line through the origin.

FIGURE 11 Span $\{\mathbf{u}, \mathbf{v}\}$ as a plane through the origin.

Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (0, 1, 2)$ and $\mathbf{u} = (1, -2, 2)$. Is $\mathbf{u}$ in the plane defined by $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

The matrix equation $A\mathbf{x} = \mathbf{b}$

If A is a matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{x} \in \mathbb{R}^n$, we define the product

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Compute $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Systems of equations

We note that given a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{cases} \tag{1}$$

if we define

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The matrix equation $A\mathbf{x} = \mathbf{b}$

then the system can be written as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}, \tag{2}$$

where $\mathbf{a}_j$ are the columns of A,

$$\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \dots \quad \mathbf{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

In other words,

$$A\mathbf{x} = \mathbf{b} \tag{3}$$

The matrix equation $A\mathbf{x} = \mathbf{b}$

Thm: The following statements are equivalent:

(a) $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) The vector $\mathbf{b}$ is a linear combination of the columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ of A.

(c) The augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ gives a consistent system.

<u>Proof.</u> Indeed, $(a) \iff (b)$ by equations (2) and (3).
Also, $(c) \iff (a)$ by equations (3) and (1).

The matrix equation $A\mathbf{x} = \mathbf{b}$

Thm. Let A be an $m \times n$ matrix. The following statements are equivalent.

(a) For each $\mathbf{b} \in \mathbb{R}^n$, equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b} \in \mathbb{R}^n$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^n$.

(d) A has a pivot position in every row.

Proof. $(a) \iff (b) \iff (c)$ by the previous theorem and the span definition. Let us prove $(d) \iff (a)$.

The matrix equation $A\mathbf{x} = \mathbf{b}$

Let U be an echelon form of A. So $\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \sim \begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ for some $\mathbf{d}$.

If (d) holds then each row of U has a pivot position, so there can be no pivot in the last augmented column, and the system has a solution for any $\mathbf{b}$.

Conversely, if (d) is false, then the last row of U is all zero. Choose $\mathbf{b}$ such that $\mathbf{d}$ has 1 in its last entry. Then the last row of $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ reads as $0 = 1$, hence the system is inconsistent.

This shows that $(d) \iff (a)$ and we are done. $\square$

Properties of $A\mathbf{x}$

Thm : If A is an $m \times n$ matrix, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$, then

1. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$,
2. $A(c\mathbf{u}) = c(A\mathbf{u})$.

Proof. 1. By definition, $A(\mathbf{u}+\mathbf{v}) = \sum_{i=1}^{n}(u_i + v_i)\boldsymbol{\alpha}_i = \sum_{i=1}^{n} u_i\boldsymbol{\alpha}_i + \sum_{i=1}^{n} v_i\boldsymbol{\alpha}_i = A\mathbf{u} + A\mathbf{v}$.

2. $A(c\mathbf{u}) = \sum_{i=1}^{n}(cu_i)\boldsymbol{\alpha}_i = c\sum_{i=1}^{n} u_i\boldsymbol{\alpha}_i = c(A\mathbf{u})$. $\square$

Computing $A\mathbf{x} = \mathbf{b}$

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all b_1, b_2 ? If not, describe the set of (b_1, b_2) for which it is compatible.

Note that by definition

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$$

Computing $A\mathbf{x} = \mathbf{b}$

This works for any size. Hence,

> When defined, the i-th entry of the column vector $A\mathbf{x}$ is
> $$\sum_{j=1}^{n} a_{ij}x_j = a_{i1}x_1 + \cdots + a_{in}x_n\,.$$

Compute $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.