Mostafa Sabri

Lecture 23

5.5 Complex Eigenvalues (Extra)

جامعة نيويورك أبوظبي
NYU ABU DHABI

Complex Eigenvalues

In this section we study $n \times n$ matrices A acting on $\mathbb{C}^n$ instead of $\mathbb{R}^n$, and allow the eigenvalues of A to be non-real numbers. So a nonzero $\mathbf{x} \in \mathbb{C}^n$ is an eigenvector if there exists $\lambda \in \mathbb{C}$ such that $A\mathbf{x} = \lambda\mathbf{x}$, in which case we call λ the corresponding eigenvalue.

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1-i & 0 \\ 1+i & 0 & 1-i \\ 0 & 1+i & 0 \end{bmatrix}$$

Complex Eigenvalues

> Thm: Any $n \times n$ matrix A over $\mathbb{C}^n$ has n (possibly complex) eigenvalues.

Proof. The characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n. So it has n roots, which are possibly not in $\mathbb{R}$. This is called the *fundamental theorem of algebra*. Math majors will learn its proof in Complex Analysis. Since the roots of $p(\lambda)$ are precisely the eigenvalues of A, we are done. $\square$

> Thm: If A has real entries, then each non-real eigenvalue (if any) comes in conjugate pairs.

Complex Eigenvalues

The theorem means that if $\lambda = a + ib$ is an eigenvalue, then $\overline{\lambda} = a - ib$ must also be an eigenvalue.

<u>Proof.</u> Suppose $\lambda = a + ib$ is an eigenvalue with corresponding eigenvector $\mathbf{x}$. Let $\overline{\mathbf{x}}$ be the vector $\mathbf{x}$ with complex conjugate entries, i.e. $\overline{\mathbf{x}} = (\overline{x}_1, \ldots, \overline{x}_n)^T$. Then $A\overline{\mathbf{x}} = \overline{A}\overline{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$. This shows that $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$. $\square$

Find the eigenvalues of $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

This matrix can be written as a composition of first rotation, then scaling: if $a = r\cos\theta$ and $b = r\sin\theta$, we have $A = DR$ for $D = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ and $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Complex Eigenvalues

Thm: Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - ib$, $b \neq 0$, and associated eigenvector $\mathbf{v}$. Then

$$A = PCP^{-1}, \quad \text{where } P = \begin{bmatrix} \operatorname{Re}\mathbf{v} & \operatorname{Im}\mathbf{v} \end{bmatrix}, \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Sketch of proof. 1. By calculation, $AP = PC$.
2. $\mathbf{v}$ cannot be real otherwise $A\mathbf{v}$ is real and cannot equal $\lambda\mathbf{v}$.
3. It follows that $\operatorname{Im}\mathbf{v} \neq \mathbf{0}$. It also follows that $\operatorname{Re}\mathbf{v} \neq \mathbf{0}$ by considering $\frac{1}{i}\mathbf{v}$.
4. If $\operatorname{Im}\mathbf{v} = z\operatorname{Re}\mathbf{v}$ for some $z \neq 0$ then $\mathbf{v} = (1 + iz)\operatorname{Re}\mathbf{v}$ with $1 + iz \neq 0$ since $\mathbf{v} \neq \mathbf{0}$. So $\mathbf{w} = \frac{1}{1+iz}\mathbf{v}$ is a real eigenvector for λ, a contradiction,
5. By 2-4 we see that P is invertible and we are done. $\square$

Complex Eigenvalues

Let $A = \begin{bmatrix} -7 & 1 \\ -5 & -3 \end{bmatrix}$. Factorize $A = PCP^{-1}$, where C has the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for some $a, b \in \mathbb{R}$.

The result holds in some sense in 3 dimensions as well. That is, if A is a 3×3 matrix with a non-real eigenvalue (and hence 2 of them), there is a plane in $\mathbb{R}^3$ on which A acts as a rotation followed by a scaling.