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Lecture 6

2.1 Matrix Operations

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Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal when they have the same size ($m \times n$) and $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

e.g. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix}$ iff $x = 1$. Also, $\begin{bmatrix} 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Addition, subtraction, scalar multiplication

If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size $m \times n$, we define their sum $A + B$ to be the $m \times n$ matrix $[a_{ij} + b_{ij}]$.

The sum of matrices of different sizes is undefined.

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} -2 & e & \pi \\ 1 & 1 & -5 \end{bmatrix} = \begin{bmatrix} -2 & 1+e & 2+\pi \\ 4 & 5 & 0 \end{bmatrix}$$

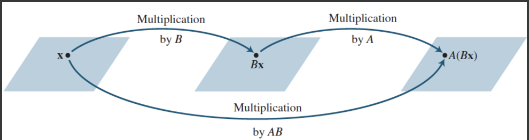
Subtraction is defined the same way, replacing + by − under the same assumptions.

Given a scalar $c \in \mathbb{R}$ and a matrix $A = [a_{ij}]$, we define the scalar multiplication $cA = [ca_{ij}]$.

Matrix multiplication

e.g. if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 5 \\ 3 & 9 \end{bmatrix}$ then
$A - 2B = \begin{bmatrix} 5 & -8 \\ -3 & -14 \end{bmatrix}$.

Matrix multiplication is more complicated. Viewing matrices as linear maps, when considering $AB\mathbf{x}$, we actually have a composition:



Matrix multiplication

If A has size $m \times n$, B has size $n \times p$ and $\mathbf{x} \in \mathbb{R}^p$, then denoting the columns of B by $\{\mathbf{b}_j\}$, we have $A(B\mathbf{x}) = A\left(\sum_{i=1}^p x_i \mathbf{b}_i\right) = \sum_{i=1}^p x_i \mathbf{A}\mathbf{b}_i = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix} \mathbf{x}$. Hence,

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}.$$

Note that each column $A\mathbf{b}_j$ of AB is a linear combination $\sum_{k=1}^n b_{kj} \boldsymbol{\alpha}_k$ by definition.

In particular, the ij-entry, $(AB)_{ij}$ is the ith entry of $A\mathbf{b}_j$, so $(AB)_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$.

Matrix multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, we define the product AB to be the $m \times p$ matrix $[c_{ij}]$ given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = c_{ij}$$

Matrix multiplication

Just like $AB = \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_p \end{bmatrix}$, also $AB = \begin{bmatrix} \mathrm{row}_1(A)B \\ \mathrm{row}_2(A)B \\ \vdots \\ \mathrm{row}_m(A)B \end{bmatrix}$.

Find the product AB if

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 5 & -2 \\ 1 & -4 & 6 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Matrix multiplication

$$A = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$$

Properties of Addition and Scalar Multiplication

If A, B, and C are $m \times n$ matrices, and c and d are scalars, then

1. $A + B = B + A$ — Commutative addition
2. $A + (B + C) = (A + B) + C$ — Associative addition
3. $(cd)A = c(dA)$ — Associative multiplication
4. $1A = A$ — Multiplicative identity
5. $c(A + B) = cA + cB$ — Distributive property
6. $(c + d)A = cA + dA$ — Distributive property

The proof is done by reducing these statements to the analogous properties of real numbers.

Some definitions

The *diagonal entries* of a matrix $A = [a_{ij}]$ are the entries $a_{11}, a_{22}, \ldots$ on the diagonal.

A *square matrix* is a matrix of size $n \times n$, i.e. $m = n$.

A square matrix A is a *diagonal matrix* if all nondiagonal entries are zero.

We denote by 0 the $m \times n$ matrix will all entries equal to zero.

Note that if A is an $m \times n$ matrix, then $A + 0 = A$, $A - A = 0$, and $cA = 0$ implies either $c = 0$ or $A = 0$.

Identity matrix

The $n \times n$ identity matrix is the diagonal matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \dots & & 1 \end{bmatrix}$$

For example $I_1 = [\,1\,]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

> If A has size $m \times n$ then $AI_n = A$ and $I_m A = A$.

Properties of Matrix Multiplication

<u>Proof.</u> Indeed, $AI_n = \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_1 & \cdots & \boldsymbol{\alpha}_n \end{bmatrix} = A$ and $I_m A = \begin{bmatrix} I_m\boldsymbol{\alpha}_1 & \cdots & I_m\boldsymbol{\alpha}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_1 & \cdots & \boldsymbol{\alpha}_n \end{bmatrix} = A$. $\square$

If A, B, and C are matrices (with sizes such that the matrix products are defined), and c is a scalar, then

1. $A(BC) = (AB)C$ Associative multiplication
2. $A(B+C) = AB + AC$ Distributive property
3. $(A+B)C = AC + BC$ Distributive property
4. $c(AB) = (cA)B = A(cB)$

Properties of Matrix Multiplication

Proof. 1. Matrix multiplication corresponds to composition of functions, which is associative.

2. The ij-th entry of $A(B+C)$ is $\sum_k a_{ik}(b_{kj}+c_{kj}) = \sum_k a_{ik}b_{kj} + \sum_k a_{ik}c_{kj}$, which is the ij-th entry of $AB+AC$.

3. and 4. are similar to 2. $\square$

Find the matrix product ABC if

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 3 & -4 \end{bmatrix}.$$

Powers of a matrix

Compute AB and BA if $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$.

Compute AC and BC if $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$.

If A is a square matrix, the matrix

$$A^k = A \cdots A$$

is well-defined.

Compute A^3 if $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$.

Transpose of a matrix

The transpose of a matrix is formed by writing its rows as columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Size: $m \times n$

then the transpose, denoted by A^T, is the $n \times m$ matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

Size: $n \times m$

Properties of transposes

Find the transpose of $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

If A and B are matrices (of sizes such that the matrix operations are defined) and c is a scalar, then

1. $(A^T)^T = A$ — Transpose of a transpose
2. $(A+B)^T = A^T + B^T$ — Transpose of a sum
3. $(cA)^T = c(A^T)$ — Transpose of a scalar multiple
4. $(AB)^T = B^T A^T$ — Transpose of a product

Transposes

Proof. 1,2,3 are immediate from the definition. For 4, note the ij-th entry of $(AB)^T$ is the ji-th entry of AB, which is $\sum_k a_{jk} b_{ki}$. Next, the ij-th entry of $B^T A^T$ is $\sum_k (B^T)_{ik}(A^T)_{kj} = \sum_k b_{ki} a_{jk}$, the same as $(AB)^T_{ij}$. $\square$

Compute $(AB)^T$ and $B^T A^T$ if $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

We say that a square matrix is *symmetric* if $A = A^T$.

2. Show that the matrices AA^T and $A^T A$ are symmetric.

3. Compute AA^T if $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$.