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Lecture 1

1.1 Systems of Linear Equations

1.2 Row Reduction and Echelon Forms

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Systems of Linear Equations and Their Solutions

A linear equation in n variables takes the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b\,.$$

The a_i are known coefficients, the x_i are the variables.

A system of linear equations is a set of equations in the same variables, in which each equation is linear.

A solution of a system is an assignment of values $x_i = s_i$ that makes each equation true. The set of all possible solutions is the solution set.

Here is a system of linear equations in two variables:

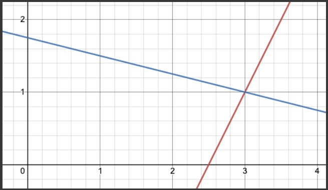
$$\begin{cases} 2x - y = 5 \\ x + 4y = 7 \end{cases}$$

Systems of Linear Equations and Their Solutions

We can check that $x = 3$ and $y = 1$ is a solution of this system.

The solution can be written as the ordered pair $(3, 1)$.

The graphs of each equation is a line. Since $(3, 1)$ satisfies each equation, the point $(3, 1)$ lies on each line. So it is the point of intersection of the two lines.



Substitution Method

1. Solve for one variable in the first equation.
2. Substitute in the second equation.
3. Back-substitute.

Find all solutions of the system

$$\begin{cases} 2x + y = 1 \\ 3x + 4y = 14 \end{cases}$$

Elimination Method

1. Adjust the coefficients.
2. Add or subtract the equations.
3. Back-substitute.

Find all solutions of the system

$$\begin{cases} 3x + 2y = 14 \\ x - 2y = 2 \end{cases}$$

Graphical Method

1. Graph each equation.
2. Find the intersection point(s).

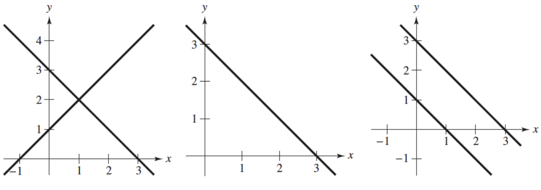
Find all solutions of the system

$$\begin{cases} 6x - 2y = -10 \\ 8x + y = -6 \end{cases}$$

The Number of Solutions of a Linear System

A system of linear equations may have

1. Exactly one solution.
2. Infinitely many solutions.
3. No solution (Inconsistent system).



a. Two intersecting lines:
$$x + y = 3$$
$$x - y = -1$$

b. Two coincident lines:
$$x + y = 3$$
$$2x + 2y = 6$$

c. Two parallel lines:
$$x + y = 3$$
$$x + y = 1$$

Examples

Solve each system.

$$\begin{cases} 3x - y = 0 \\ 5x + 2y = 22 \end{cases}$$

$$\begin{cases} 8x - 2y = 5 \\ -12x + 3y = 7 \end{cases}$$

$$\begin{cases} 3x - 6y = 12 \\ 4x - 8y = 16 \end{cases}$$

When there are infinitely many solutions, we label the free variable by t, s, etc. This is a *parametric description of the solution.*

Matrices

An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

The numbers a_{ij} are called the entries of the matrix. We denote this matrix by $[a_{ij}]$ for short.

$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix}$ is a 2×3 matrix. $\begin{bmatrix} 6 & -5 & 0 & 1 \end{bmatrix}$ is a 1×4 matrix.

The Augmented Matrix of a Linear System

We write a system of linear equations as an *augmented matrix* by copying only the coefficients and constants:

$$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases} \qquad \begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$$

A missing variable in an equation corresponds to a 0 entry in the augmented matrix. If we remove the last column, we obtain the *coefficient matrix* instead.

Write the augmented matrix of $\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$

Elementary Row Operations

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

Performing any of these operations on the augmented matrix does not change its solution. Notation:

$R_i + kR_j \to R_i$	Change the ith row by adding k times row j to it, and then put the result back in row i.
kR_i	Multiply the ith row by k.
$R_i \longleftrightarrow R_j$	Interchange the ith and jth rows.

(Reduced) Row-Echelon Form

Row operations can simplify the augmented matrix.

A matrix is in row-echelon form if :

1. All entries below the first nonzero number, called the *leading entry*, are zero.
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

A matrix is in *reduced row-echelon form* if moreover,

4. Each leading entry is 1 and every number above and below each leading entry is 0.

(Reduced) Row-Echelon Form

Not in row-echelon form

$$\begin{bmatrix} 0 & 1 & \frac{-1}{2} & 0 \\ 1 & 0 & 3 & -3 \\ 0 & 0 & 1 & 0.4 \end{bmatrix}$$

Row-echelon form

$$\begin{bmatrix} 1 & 3 & -6 & 10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Two systems of equations sharing the same solution set are called *equivalent*. By performing row operations, we arrive at a *row equivalent* matrix with the same solution set.

(Reduced) Row-Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

A *pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A *pivot column* is a column of A that contains a pivot position.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Figure: The $\square$ identify the pivot positions.

Steps to put a matrix in row-echelon form

1. First obtain 1 in the top left corner. Then obtain zeros below that 1 by adding multiples of the first row to the rows below it.
2. Next, obtain a leading 1 in the next row, and then obtain zeros below that 1.
3. At each stage make sure that every leading entry is to the right of the leading entry in the row above it–rearrange the rows if necessary.
4. Continue until reaching a row-echelon form.
5. Solve the system using back substitution

In the process, get rid of any common factor in a row by dividing or multiplying. This procedure is called *Gaussian elimination*, in honor of Gauss.

Example

Solve the following systems using Gaussian elimination:

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases} \qquad \begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

The Gauss-Jordan elimination is a step further to get a reduced row-echelon form. After the row-echelon, we obtain zeros above each leading entry by adding multiples of the row containing that entry to the rows above it. Begin with the last leading entry and work up.

Solve the last system using Gauss-Jordan elimination.

Remarks

After performing row operations on the augmented matrix we may arrive at the following situations:

1. All entries in a row except the last one are zeroes (the last column is a pivot). In this case, the system has no solution (we get $0 = c$ with c nonzero).
2. A row consists entirely of zeroes. In this case, if there are more variables than nonzero rows, and if the system is consistent, then the system has infinitely many solutions, described with parameters t,s,...etc.

No solution

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

One solution

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

Infinitely many sols

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find complete sols of each consistent system

$$\begin{cases} x-3y+2z=12 \\ 2x-5y+5z=14 \\ x-2y+3z=20 \end{cases}$$

$$\begin{cases} -3x-5y+36z=10 \\ -x+7z=5 \\ x+y-10z=-4 \end{cases}$$

$$\begin{cases} x+2y-3z-4w=10 \\ x+3y-3z-4w=15 \\ 2x+2y-6z-8w=10 \end{cases}$$

A system of linear equations is *homogeneous* if each of the constant terms is zero. This always has the trivial solution where all $x_i = 0$, but it may also have infinitely many solutions. This occurs if there are more variables than equations.