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Lecture 12

4.3 Linearly Independent Sets; Bases

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Linear independence

When does a linear combination vanish ? Let V be a vector space, $\mathbf{v}_i \in V$ and $c_i \in \mathbb{R}$. Consider the equation

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}\,. \tag{1}$$

This is always possible by taking $c_i = 0$ for all i.

If this is the *only way*, we say the $\mathbf{v}_i$ are *linearly independent*. If we can choose the c_i such that not all are zero and (1) holds, we say the $\mathbf{v}_i$ are *linearly dependent.*

To test for independence, we solve equation (1) for the variables c_i.

Linear independence

In P_n, two polynomials $\sum_{k=0}^{n} c_k x^k$ and $\sum_{k=0}^{n} d_k x^k$ are equal iff all coefficients are equal, i.e. $c_k = d_k$ for all k. For example, $p(x) = a + bx + cx^2$ and $q(x) = d + ex + fx^2$ iff $a = d$ and $b = e$ and $c = f$.

This implies that the vectors $\{1, x, x^2, \dots, x^n\}$ are linearly independent: if $c_0 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n = \mathbf{0}$, then $\sum_{k=0}^{n} c_k x^k = \sum_{k=0}^{n} 0 x^k$ and thus $c_k = 0$ for all k.

Are the vectors $\{1 + x + 3x^2, -2 + x, 1 + x^2\}$ in P_2 linearly independent ?

Are the vectors $\left\{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ in $M_{2,2}$ linearly independent ?

Linear independence

Thm: A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent iff at least one of the vectors $\mathbf{v}_i$ is a linear combination of the $\mathbf{v}_j$ preceding it, $j < i$.

In particular, two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent iff one of them is a scalar multiple of the other one.

Are the vectors $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ linearly independent in $M_{2,2}$?

Basis for a vector space

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is a *basis* for V if S is linearly independent and spans V.

1. Is $S = \{(1,1), (1,-1)\}$ a basis for $\mathbb{R}^2$?
2. Is $S = \{1, x, \ldots, x^n\}$ a basis for P_n ?
3. Suggest a standard basis for $M_{2,2}$.

Spanning Set Thm. If $\mathbf{u} \in \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ then $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{u}\} = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

If $\{\mathbf{0}\} \neq H = \mathrm{Span}(S)$, then a subset of S is a basis for H.

Prove this theorem.

Bases for Col(A) and Row(A)

A basis is the smallest set which spans the vector space. Adding vectors to a basis makes it dependent. Removing vectors from a basis makes it lose the spanning property. e.g. among $\{\mathbf{e}_1\}$, $\{\mathbf{e}_1, \mathbf{e}_2\}$, $\{\mathbf{e}_1, \mathbf{e}_2, (1,1)\}$ in $\mathbb{R}^2$, only $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis.

> Reminder: To find a basis for Col(A), reduce $A \sim B$ to row-echelon form, find the pivot columns, the corresponding columns in A form a basis of Col(A).

> Thm: If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space of A.

Row Spaces

Proof. We know $\text{Row}(A) = \text{Row}(B)$. Moreover, the nonzero vectors of B are linearly independent because their pivots shift in each row, so a linear combination $\sum_{i=1}^{k} c_i \mathbf{r}_i(B)$ of the nonzero rows takes the form $(0, c_1 p_1, c_1 *, c_2 p_2 + c_1 *, \ldots, c_k p_k + c_{k-1} * + \cdots + c_1 *)$, where p_i are the pivots, and equals $(0, \ldots, 0)$ iff $c_1 = \cdots = c_k = 0$. Thus, $S = \{\mathbf{r}_1, \ldots \mathbf{r}_k\}$ is linearly independent and spans $\text{Row}(B)$ by the previous thm. $\square$

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Methods to obtain a basis

1. Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix}.$$

To find a basis for the subspace H of $\mathbb{R}^n$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_p$, consider the matrix A

1. Method 1. with rows $\mathbf{v}_i$ and find a basis $\mathcal{B}$ for $\mathrm{Row}(A)$. Then $\mathcal{B}$ is a basis for H.
2. Method 2. with columns $\mathbf{v}_i$ and find a basis $\mathcal{B}$ for $\mathrm{Col}(A)$. Then $\mathcal{B}$ is a basis for H.

2. Find a basis for $\mathrm{Span}\{(1, 1, 3), (-2, 1, 0), (1, 0, 1)\}$.