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Lecture 11

4.2 Null Spaces, Column Spaces, Row Spaces and Linear Transformations

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Reminders

If A is an $m \times n$ matrix, the null space

$$\mathrm{Nul}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

is a subspace of $\mathbb{R}^n$. The column space

$$\mathrm{Col}(A) = \mathrm{Span}\{\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_n\}$$

is a subspace of $\mathbb{R}^m$, equal to the range of A when considered as a matrix transformation $\mathbb{R}^n \to \mathbb{R}^m$.

The student should revise the relations between these spaces and invertibility (Lec7). Also recall $\mathbf{b} \in \mathrm{Col}(A)$ iff $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is consistent.

Row Space

Let A be an $m \times n$ matrix. The *row space* $\mathrm{Row}(A)$ is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

If B is obtained from A by row operations, then the rows of B are a linear combination of those of A, and vice versa. Hence,

If an $m \times n$ matrix A is row-equivalent to B, then $\mathrm{Row}(A) = \mathrm{Row}(B)$.

Linear Transformations

Let V, W be vector spaces. A function (map) $T : V \to W$ is linear if for any $\mathbf{u}, \mathbf{v} \in V$ and any $c \in \mathbb{R}$,

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$,
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

As before, it follows that for linear T,

1. $T(\mathbf{0}) = \mathbf{0}$,
2. $T\left(\sum_{i=1}^{p} c_i \mathbf{v}_i\right) = \sum_{i=1}^{p} c_i T(\mathbf{v}_i)$.

The *kernel* of a linear $T : V \to W$ is the set

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}.$$

Linear Transformations

The *range* of T is the set

$$\text{Range}(T) = \{T(\mathbf{v}) \in W : \mathbf{v} \in V\}$$

Thm: For linear $T : V \to W$, $\ker(T)$ is a subspace of V and $\text{Range}(T)$ is a subspace of W.

Proof. The proof that the kernel is a subspace is the same as the proof that $\text{Nul}(A)$ is a subspace. To see that $\text{Range}(T)$ is a subspace, let $\mathbf{y}, \mathbf{w} \in \text{Range}(T)$ and $c \in \mathbb{R}$. Then $\mathbf{y} = T(\mathbf{x})$ and $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{x}, \mathbf{v} \in V$. So $\mathbf{y} + \mathbf{v} = T(\mathbf{x}) + T(\mathbf{v}) = T(\mathbf{x} + \mathbf{v}) \in \text{Range}(T)$ since $\mathbf{x} + \mathbf{v} \in V$, as V is a vector space. Next, $c\mathbf{y} = cT(\mathbf{x}) = T(c\mathbf{x}) \in \text{Range}(T)$ since $c\mathbf{x} \in V$. $\square$

Examples

1. Let $T : M_{m,n} \to M_{n,m}$ be the map $T(A) = A^T$. Is T linear ?
2. Let $T : C^1[a,b] \to C[a,b]$ be the map $T(f) = f'$, the derivative. Is T linear ?
3. Let $T : C[a,b] \to \mathbb{R}$ be the map $T(f) = \int_a^b f(x)dx$. Is T linear ?
4. Find $\ker(T)$ if $T : V \to W$ is the zero transformation, and if $T : V \to V$ is the identity transformation.
5. Find $\ker(T)$ if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the projection $T(x,y,z) = (x,y,0)$.
6. Find $\ker(T)$ if $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (x-y, y+z)$.
7. Express the set of solutions of $y'' + \omega^2 y = 0$ as the kernel of some linear T.