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Lecture 17

5.3 Diagonalization

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The Diagonalization Problem

An $n \times n$ matrix A is *diagonalizable* if A is similar to a diagonal matrix, i.e. there exists an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix.

Thm: An $n \times n$ matrix A is diagonalizable iff it has n linearly independent eigenvectors.

So A is diagonalizable iff its eigenvectors form a basis.

Proof. First note that a diagonal matrix D takes the form

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \lambda_2 \mathbf{e}_2 & \cdots & \lambda_n \mathbf{e}_n \end{bmatrix}.$$

Diagonalization

so, if P has columns $\mathbf{p}_i$, then

$$PD = \begin{bmatrix} \lambda_1 P\mathbf{e}_1 & \cdots & \lambda_n P\mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{p}_1 & \cdots & \lambda_n \mathbf{p}_n \end{bmatrix} \tag{1}$$

$$AP = \begin{bmatrix} A\mathbf{p}_1 & \cdots & A\mathbf{p}_n \end{bmatrix} \tag{2}$$

($\implies$) If A is diagonalizable, then $AP = PD$, so $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$ $\forall i$. Since P is invertible, then $\mathbf{p}_i \neq 0$, and $\mathbf{p}_i$ are linearly independent. Thus, A has n independent eigenvectors.

($\impliedby$) If A has n eigenvectors $\mathbf{p}_i$, let $P = \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_n \end{bmatrix}$. (1)-(2) then imply $PD = AP$. Independence of $\mathbf{p}_i$ implies P is invertible, so $D = P^{-1}AP$. $\square$

Steps for diagonalizing a square matrix

Cor: If all eigenvalues are distinct, then A is diagonalizable.

Diagonalization steps. Let A be an $n \times n$ matrix

1. Find n linearly independent eigenvectors $\mathbf{p}_1, \ldots, \mathbf{p}_n$ for A, if possible, with eigenvalues $\lambda_1, \ldots, \lambda_n$. If impossible, A is not diagonalizable.
2. Let $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$ be the matrix with columns $\mathbf{p}_i$. Then $D = P^{-1}AP$ will be a diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on its diagonal.

Diagonalizable matrices

1. Show that $A = \begin{bmatrix} 1 & 5 \\ 3 & -1 \end{bmatrix}$ is diagonalizable. Find the matrix P such that $P^{-1}AP$ is diagonal. Then compute A^k.

> Thm: A is diagonalizable iff $p(\lambda) = |\lambda I - A|$ has n real roots λ_i, and $\mu(\lambda_i) = \gamma(\lambda_i)$ for each eigenvalue λ_i.

Proof. Indeed, A is diagonalizable iff it has n linearly independent eigenvectors, which happens iff each eigenspace has the correct dimension $\mu(\lambda_i)$. $\square$

2. Diagonalize the following matrices if possible.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 2 & -2 \\ 0 & -2 & 3 \\ 0 & 0 & -2 \end{bmatrix}.$$

Examples

3. Is $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ diagonalizable ? And $D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$?

4. Check out the solved examples in the book.