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Lecture 7

2.2 The inverse of a matrix

2.3 Characterizations of Invertible Matrices

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Definition of the Inverse of a Matrix

An $n \times n$ matrix A is *invertible* (or *nonsingular*) if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is the (multiplicative) inverse of A. A matrix that does not have an inverse is *noninvertible* (or *singular*).

If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1}.

<u>Proof of uniqueness.</u> If B and C are two inverses of A, then $AB = I \implies (CA)B = C \implies B = C$. $\square$

Examples

Thm: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible iff $ad - bc \neq 0$. Moreover, if A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. If $ad - bc \neq 0$ and B is the matrix on RHS, then $AB = BA = I_2$. The proof that A is not invertible if $ad - bc = 0$ appears later in the lecture. $\square$

2. Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Finding the inverse of a larger Matrix

For larger matrices we use Gauss-Jordan elimination.

Let A be a square matrix of order n. To find A^{-1},

1. Write the $n \times 2n$ matrix $\begin{bmatrix} A & I_n \end{bmatrix}$, where I_n is the identity matrix. This process is called adjoining matrix I_n to matrix A.
2. If possible, row reduce A to I_n using elementary row operations on the entire matrix $\begin{bmatrix} A & I_n \end{bmatrix}$. The result will be the matrix $\begin{bmatrix} I_n & A^{-1} \end{bmatrix}$.
   If this is not possible, then A is not invertible.

The proof of this scheme appears later in the lecture.

Examples

1. Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$, if it exists.

2. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$, if it exists.

3. Find the inverse of $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, if it exists.

Properties of inverses

Thm: If A is an invertible matrix, k is a positive integer, and $c \neq 0$ is a scalar, then A^{-1}, A^k, cA, and A^T are invertible. Moreover,

1. $(A^{-1})^{-1} = A$.
2. $(A^k)^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}$.
4. $(A^T)^{-1} = (A^{-1})^T$.

<u>Proof.</u> 1. The inverse C of A^{-1} should satisfy $CA^{-1} = I = A^{-1}C$. Since $C = A$ satisfies these properties, then $C = A$ by uniqueness.

The inverse of a product

2. $(A^{-1})^k A^k = A^{-1} \cdots A^{-1} A \cdots A = I = A^k (A^{-1})^k$.

3. $\frac{1}{c} A^{-1} (cA) = I = (cA) \frac{1}{c} A^{-1}$ as required.

4. $(A^{-1})^T A^T = (AA^{-1})^T = I$, $A^T (A^{-1})^T = (A^{-1}A)^T = I$. $\square$

Thm: If A and B are invertible matrices of order n, then AB is invertible and

$$(AB)^{-1} = B^{-1} A^{-1}$$

Proof. $B^{-1} A^{-1} AB = B^{-1} B = I$ and $AB(B^{-1} A^{-1}) = AA^{-1} = I$ as required. $\square$

The inverse of a product

Thm: Let C be an invertible matrix.

1. If $AC = BC$, then $A = B$. Right cancellation
2. If $CA = CB$, then $A = B$. Left cancellation

Proof. Multiply by C^{-1} to the right in 1, to the left in 2. $\square$

Thm: If A is invertible, then each column $\mathbf{b}_j$ of A^{-1} is the solution of $A\mathbf{x} = \mathbf{e}_j$.

Proof. Indeed, such $\mathbf{x}$ satisfies

$$\mathbf{x} = A^{-1}\mathbf{e}_j = 0\mathbf{b}_1 + 0\mathbf{b}_2 + \cdots + 1\mathbf{b}_j + 0\mathbf{b}_{j+1} + \cdots + 0\mathbf{b}_n = \mathbf{b}_j.$$

Systems of equations

Thm: If A is an invertible matrix, then for each $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof. $A^{-1}\mathbf{b}$ is a solution since $A(A^{-1}\mathbf{b}) = \mathbf{b}$. If $\mathbf{y}$ is another solution, then $A\mathbf{y} = \mathbf{b}$ implies, by taking A^{-1} on the left to both sides, that $\mathbf{y} = A^{-1}\mathbf{b}$, same sol. $\square$

Solve the following systems:

$$\begin{cases} x - y = 1 \\ x - z = 0 \\ -6x + 2y + 3z = -1 \end{cases} \qquad \begin{cases} x - y = -1 \\ x - z = 1 \\ -6x + 2y + 3z = 2 \end{cases}$$

Elementary matrices

An $n \times n$ matrix is an *elementary matrix* if it can be obtained from the identity matrix I_n by a single elementary row operation.

Recall that row operations are interchanging rows, multiplying a row by a nonzero scalar, and adding a multiple of one row to another.

Which of the following matrices are elementary ? Explain why.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Examples

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

e.g. $$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \qquad (R_1 \longleftrightarrow R_2)$$

Examples

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 & 3/2 \\ 4 & 5 & 6 \end{bmatrix} \qquad (R_1/2)$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix} \qquad (R_2 + 2R_1 \to R_2)$$

> If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

To find E^{-1}, simply reverse the operation that E does.

Examples

If E interchanges $R_i \longleftrightarrow R_j$ then $E^{-1} = E$.

If E operates by multiplying k to R_j, then E^{-1} operates by multiplying $\frac{1}{k}$ to R_j.

If E operates by $R_i + kR_j$, then E^{-1} operates by $R_i - kR_j$.

e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{7} \end{bmatrix},$

$\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}.$

Proof of the criterion $\begin{bmatrix} A & I \end{bmatrix} \sim \begin{bmatrix} I & A^{-1} \end{bmatrix}$

Thm: An $n \times n$ matrix A is invertible iff A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

<u>Proof.</u> ($\Longrightarrow$) If A is invertible, then each equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, so A has a pivot in every row (lec. 2). As A is $n \times n$, the pivots must be on the diagonal, hence the reduced row echelon form is I_n.

($\Longleftarrow$) If $A \sim I_n$, then $I_n = E_k E_{k-1} \cdots E_1 A$, where E_j are the elementary matrices performing the row operations.

Proof of the criterion $\begin{bmatrix} A & I \end{bmatrix} \sim \begin{bmatrix} I & A^{-1} \end{bmatrix}$

But each E_j is invertible, hence so is the product, and $A = (E_k \cdots E_1)^{-1}$. It follows that A is invertible with $A^{-1} = E_k \cdots E_1$.

Finally, $A^{-1} = E_k \cdots E_1 I_n$ says that the same sequence of row operations $E_1, \ldots, E_k$ used to reduce A to I_n also transforms I_n to A^{-1}. $\square$

Characterizations of Invertible Matrices

Thm: Let A be $n \times n$. The following statements are equivalent.

1. A is invertible.
2. A is row equivalent to I_n.
3. A has n pivot positions.
4. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A are linearly independent.
6. The map $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The map $\mathbf{x} \mapsto A\mathbf{x}$ is onto $\mathbb{R}^n$.

Characterizations of Invertible Matrices

10. There is an $n \times n$ matrix C such that $CA = I_n$.
11. There is an $n \times n$ matrix D such that $AD = I_n$.
12. A^T is invertible.
13. The columns of A form a basis of $\mathbb{R}^n$.
14. $\mathrm{Col}(A) = \mathbb{R}^n$.
15. $\mathrm{rank}(A) = n$.
16. $\dim \mathrm{Nul}(A) = 0$.
17. $\mathrm{Nul}(A) = \{\mathbf{0}\}$.

Proof. This is a summary of the results we proved in several lectures, except for $10, 11 \implies 1$.

Characterizations of Invertible Matrices

More precisely, 1,2,3 are equivalent by the previous theorem. $1 \iff 12$, $1 \implies 4 \iff 5 \iff 6$ and $7 \iff 8 \iff 9$ were proved before. Also, $13 \implies 14 \implies 15 \iff 16 \iff 17$ by the rank theorem and $5 \iff 13$ and $8 \iff 13$ by the basis theorem. Since $15 \iff 3 \iff 1 \implies 5 \implies 13$, we conclude that 13 to 17 are all equivalent, and equivalent to 1,2,3,4,5,6,7,8,9,12.

Clearly $1 \implies 10, 11.$ So it remains to prove the converse. If 10 holds then $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = CA\mathbf{x} = C\mathbf{0} = \mathbf{0}$, hence 4 holds. Similarly, if 11 holds then $AD\mathbf{b} = \mathbf{b}$, so $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} = D\mathbf{b}$, so 7 holds. Hence, all statements are equivalent. $\square$

Invertible Linear Transformations

Is the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ invertible ?

Proof of the 2×2 case (continued). If $ad - bc = 0$, then $A\begin{bmatrix} d \\ -c \end{bmatrix} = \mathbf{0}$ is a nontrivial solution if d and c are not both zero, so A is not invertible by 4. If $d = c = 0$ then A has at most one pivot position, so not invertible by 3. $\square$

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is *invertible* if there exists a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S(T(\mathbf{x})) = T(S(\mathbf{x})) = \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$.

Invertible Linear Transformations

Thm: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear with standard matrix A. Then T is invertible iff A is invertible, and in this case, the inverse map S is given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$.

Proof. ($\Longrightarrow$) If T is invertible, then T is onto $\mathbb{R}^n$, since any $\mathbf{b} \in \mathbb{R}^n$ is the image of $\mathbf{x} = S(\mathbf{b})$ under T, i.e. $T(\mathbf{x}) = \mathbf{b}$. This implies $\mathbf{x} \mapsto A\mathbf{x} = T(\mathbf{x})$ is onto, hence A is invertible (thm).

($\Longleftarrow$) If A is invertible, define $S(\mathbf{x}) = A^{-1}\mathbf{x}$. Then, S is a linear transformation satisfying $T(S(\mathbf{x})) = S(T(\mathbf{x})) = \mathbf{x}$. e.g., $T(S(\mathbf{x})) = T(A^{-1}\mathbf{x}) = AA^{-1}\mathbf{x} = \mathbf{x}$. So T is invertible.

Show that a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one iff it is invertible.