Mostafa Sabri

Lecture 3

1.5 Solution sets of Linear Systems

1.7 Linear Independence

جامعة نيويورك أبوظبي
NYU ABU DHABI

Homogeneous Linear Systems

A system of equations is *homogeneous* if all constant terms are zero, i.e. it can be written as $A\mathbf{x} = \mathbf{0}$. This always has the trivial solution $\mathbf{x} = \mathbf{0}$. If it has more solutions $\mathbf{x} \neq \mathbf{0}$, we call them nontrivial solutions.

> Thm: The equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution iff the equation has at least one free variable.

Proof. As the system is consistent, it has a solution other than $\mathbf{x} = \mathbf{0}$ iff it has infinitely many solutions, which occurs iff it has some free variable(s). $\square$

Parametric Vector Form

Do the following systems have nontrivial solutions?

$$\begin{cases} x+y+z=0 \\ 3x-2y+5z=0 \\ 6x-y+13z=0 \end{cases} \qquad \begin{cases} x+y+z=0 \end{cases}$$

When there are free variables as in the last example, and we write the solution as $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, we say it is in *parametric vector form*. Note that $\mathbf{x}$ is decomposed as a linear combination of some vectors $\mathbf{u}, \mathbf{v}$, using the free variables as parameters.

Find all solutions of $\begin{cases} x+y+z=3 \\ 3x-2y+5z=6 \\ 6x-y+13z=18 \end{cases}$

Solutions of Nonhomogeneous Systems

Thm: Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

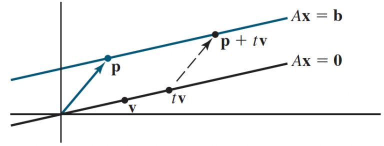

Figure: If $A\mathbf{x} = \mathbf{0}$ has as solutions the line $\{t\mathbf{v}\}$, then $A\mathbf{x} = \mathbf{b}$ has solutions the parallel line $\{\mathbf{p} + t\mathbf{v}\}$.

Solutions of Nonhomogeneous Systems

Proof. If $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{p}$ and $\mathbf{v}_h$ satisfy the assumptions, then $A\mathbf{w} = A\mathbf{p} + A\mathbf{v}_h = \mathbf{b}$. So $\mathbf{w}$ is a solution.

We should now show all solutions have the form $\mathbf{w}$. So suppose $\mathbf{q}$ is another solution of $A\mathbf{x} = \mathbf{b}$. Then $A(\mathbf{q} - \mathbf{p}) = A\mathbf{q} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$. So $\mathbf{v} = \mathbf{q} - \mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{0}$ and $\mathbf{q} = \mathbf{p} + \mathbf{v}$ has the form of $\mathbf{w}$. $\square$

Solve the systems of linear equations

$$\begin{cases} x - y + 3z = 0 \\ 2x + y + 3z = 0 \end{cases} \qquad \begin{cases} x - y + 3z = 3 \\ 2x + y + 3z = 6 \end{cases}$$

Linear independence

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is *linearly independent* if the following implication holds:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0} \implies x_i = 0 \quad \forall i$$

i.e. the equation has only the trivial solution. The set is *linearly dependent* otherwise. Hence, the vectors $\{\mathbf{v}_i\}$ are linearly dependent if there exist c_i, not all zero, such that $\sum_{i=1}^{p} c_i\mathbf{v}_i = \mathbf{0}$. We call $\sum_{i=1}^{p} c_i\mathbf{v}_i = \mathbf{0}$ a *linear dependence relation* among $\mathbf{v}_i$.

Linear independence

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. If not, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

> The columns of a matrix A are linearly independent iff the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

This is clear by writing $A\mathbf{x} = \mathbf{0}$ as $x_1\boldsymbol{\alpha}_1 + \cdots + x_n\boldsymbol{\alpha}_n = \mathbf{0}$.

This is the main test for linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_p$. Form the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_p \end{bmatrix}$, consider $A\mathbf{x} = \mathbf{0}$. If the only solution is $\mathbf{x} = \mathbf{0}$, then $\{\mathbf{v}_i\}$ are linearly independent. If there is a solution $\mathbf{x} \neq \mathbf{0}$ they are dependent.

Sets of one or two vectors

Are the columns of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ lin. independent?

A singleton set $\{\mathbf{v}\}$ is linearly independent iff $\mathbf{v} \neq \mathbf{0}$.

A set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent iff one of the vectors is a multiple of the other.

Proof. If $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ with $c_1 \neq 0$ then $\mathbf{v}_1 = \alpha\mathbf{v}_2$ for $\alpha = \frac{-c_2}{c_1}$. Similarly, if $c_2 \neq 0$ then $\mathbf{v}_2 = \beta\mathbf{v}_1$ with $\beta = \frac{-c_1}{c_2}$. This proves ($\implies$). On the other hand, if $\mathbf{v}_1 = \alpha\mathbf{v}_2$ then $\mathbf{v}_1 - \alpha\mathbf{v}_2 = \mathbf{0}$ shows $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent. Similarly, $\mathbf{v}_2 = \beta\mathbf{v}_1$ implies $\mathbf{v}_1, \mathbf{v}_2$ are dependent.

Sets of two or more vectors

Next if $\mathbf{v} = \mathbf{0}$, then $1\mathbf{v} = \mathbf{0}$ shows $\{\mathbf{v}\}$ is dependent. If $\mathbf{v} \neq \mathbf{0}$, then $c\mathbf{v} = \mathbf{0} \implies c = 0$, so $\{\mathbf{v}\}$ is independent. $\square$

Are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ linearly independent?

Thm: A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent iff at least one of the vectors in S is a linear combination of the others.

If S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ with $j > 1$ is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Sets of two or more vectors

Proof. ($\Longrightarrow$) If S is linearly dependent, there is a relation $\sum_{j=1}^{p} c_j\mathbf{v}_j = \mathbf{0}$ with at least one $c_k \neq 0$. So $c_k\mathbf{v}_k + \sum_{j\neq k} c_j\mathbf{v}_j = \mathbf{0}$. So $\mathbf{v}_k = \sum_{j\neq k} d_j\mathbf{v}_j$ with $d_j = \frac{-c_j}{c_k}$.

($\Longleftarrow$) Conversely, if $\mathbf{v}_k = \sum_{j\neq k} c_j\mathbf{v}_j$ then $\sum_{j=1}^{p} c_j\mathbf{v}_j = \mathbf{0}$ with $c_k = -1$ nonzero, so S is linearly dependent.

For the second part, suppose $\sum_{i=1}^{p} c_i\mathbf{v}_i = \mathbf{0}$ and let j be the largest subscript with $c_j \neq 0$. Then $\sum_{i=1}^{j} c_i\mathbf{v}_i = \mathbf{0}$, so $c_j\mathbf{v}_j + \sum_{i=1}^{j-1} c_i\mathbf{v}_i = \mathbf{0}$. So, $\mathbf{v}_j = \sum_{i=1}^{j-1} d_i\mathbf{v}_i$ with $d_i = \frac{-c_i}{c_j}$. $\square$

Show that if $\mathbf{w} \in \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}\}$ is linearly dependent.

Linear dependence

Thm. A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ with $p > n$ is linearly dependent.

Proof. Let $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_p \end{bmatrix}$, of size $n \times p$. Consider $A\mathbf{x} = \mathbf{0}$, for $\mathbf{x} \in \mathbb{R}^p$. This gives a system of n equations in p variables $x_1, \ldots, x_p$. As $p > n$, there are more variables than equations, so there must be a free variable. Hence, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, so $\{\mathbf{v}_i\}$ are dependent. $\square$

Are the vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ linearly dependent ?

Linear dependence

If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then S is linearly dependent.

Proof. By renumbering the vectors, we may suppose $\mathbf{v}_1 = \mathbf{0}$. Then the equation $1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p = \mathbf{0}$ shows that S is linearly dependent. $\square$

Determine if the following sets are linearly dependent.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}, T = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \right\}.$$