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Lecture 22

6.7 Inner Product Spaces

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Inner Products

An inner product on a vector space is a generalization of the dot product of $\mathbb{R}^n$.

An *inner product* $\langle \cdot, \cdot \rangle$ on a vector space V is a map $(\mathbf{u}, \mathbf{v}) \mapsto \langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{R}$ such that $\forall \mathbf{u}, \mathbf{v} \in V$, $c \in \mathbb{R}$,

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$,
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$,
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$,
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ iff $\mathbf{u} = \mathbf{0}$.

A vector space endowed with an inner product is called an *inner product space.*

Inner Products

Items 1-3 say that the map is bilinear and symmetric. Item 4 says it is positive-definite.

1. On $\mathbb{R}^2$, define $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2$. Is this an inner product ?

2. On $\mathbb{R}^3$, define $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 - 2u_2 v_2 + u_3 v_3$. Is this an inner product ?

3. On $C[a, b]$, define $\langle f, g \rangle = \int_a^b f(t) g(t)\, dt$. Is this an inner product ?

If V is an inner product space, we define the *norm* or *length* of a vector $\mathbf{v}$ by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}\,.$$

Length, Distances, Orthogonality

Let $t_0, \dots, t_n$ be distinct real numbers. For $p, q \in P_n$, define $\langle p, q \rangle = \sum_{i=0}^{n} p(t_i)q(t_i)$. Show that this is an inner product on P_n. In case $n = 2$, and $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, compute $\|p\|$ for $p(t) = 1 + t + t^2$.

A *unit vector* in V is a vector of length 1.

The *distance between* $\mathbf{u}$ *and* $\mathbf{v}$ is $\|\mathbf{u} - \mathbf{v}\|$.

We say that $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Lem: Given $\mathbf{u}, \mathbf{v} \in V$, let $W_{\mathbf{u}} = \mathrm{Span}\{\mathbf{u}\}$ and define

$$\mathrm{proj}_{W_{\mathbf{u}}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}\,.$$

Then $\mathbf{z} = \mathbf{v} - \mathrm{proj}_{W_{\mathbf{u}}}(\mathbf{v})$ is orthogonal to $\mathbf{u}$.

Cauchy-Schwarz Inequality

Indeed, $\langle \mathbf{z}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle = 0$.

Lem: $\|\mathrm{proj}_{W_{\mathbf{u}}}(v)\| \leq \|\mathbf{v}\|$.

Indeed, $\mathrm{proj}_{W_{\mathbf{u}}}(\mathbf{v}) = \alpha \mathbf{u}$, so by the previous lemma and Pythagoras,

$$\|\mathbf{v}\|^2 = \|\mathrm{proj}_{W_{\mathbf{u}}}(v)\|^2 + \|\mathbf{v} - \mathrm{proj}_{W_{\mathbf{u}}}(\mathbf{v})\|^2 \geq \|\mathrm{proj}_{W_{\mathbf{u}}}(v)\|^2 .$$

Thm (Cauchy-Schwarz): For any $\mathbf{u}, \mathbf{v} \in V$,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\| .$$

Proof. $\|\mathrm{proj}_{W_{\mathbf{u}}}(v)\| = \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|}{\|\mathbf{u}\|^2} \|\mathbf{u}\|$ so this follows from the lemma. $\square$

Triangle inequality, Gram-Schmidt

($\triangle$ inequality): For any $\mathbf{u}, \mathbf{v} \in V$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Proof. As in Lec19, $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle$. By Cauchy-Schwarz, $2\langle \mathbf{u}, \mathbf{v}\rangle \leq 2\|\mathbf{u}\|\|\mathbf{v}\|$. So
$\|\mathbf{u} + \mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$. $\square$

Thm: (Gram-Schmidt) In an inner product space, if $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ is a basis for W, $\mathbf{v}_1 = \mathbf{x}_1$ and for $p > 1$,

$$\mathbf{v}_p = \mathbf{x}_p - \frac{\langle \mathbf{x}_p, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 - \cdots - \frac{\langle \mathbf{x}_p, \mathbf{v}_{p-1}\rangle}{\langle \mathbf{v}_{p-1}, \mathbf{v}_{p-1}\rangle}\mathbf{v}_{p-1}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W.

Orthogonal decomposition

The proof is the same as before. Consequently, any subspace has an orthonormal basis.

Thm: Given a subspace $W \subseteq V$, any $\mathbf{v} \in V$ has a unique orthogonal decomposition

$$\mathbf{v} = \mathrm{proj}_W(\mathbf{v}) + \mathbf{z}$$

where $\mathrm{proj}_W(\mathbf{v}) \in W$ and $\mathbf{z} \in W^\perp$.

Same old proof with $\mathrm{proj}_W(\mathbf{v}) = \sum_{i=1}^{p} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$, for an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ of W.

Let U, W be subspaces of V. Then V is said to be the *direct sum* of U and W, and we write $V = U \oplus W$, if $V = U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}$ and $U \cap W = \{\mathbf{0}\}$.

Direct sums, best approximations

Thm: If $W \subseteq V$ is a subspace, then $V = W \oplus W^{\perp}$.

Proof. By the orthogonal decomposition, any $\mathbf{v} \in V$ has the form $\mathbf{v} = \widehat{\mathbf{v}} + \mathbf{z}$ for $\widehat{\mathbf{v}} \in W$ and $\mathbf{z} \in W^{\perp}$. This implies that $V = W + W^{\perp}$. Next, $W \cap W^{\perp} = \{\mathbf{0}\}$ as we saw before. So the claim follows. $\square$

Thm: $\text{proj}_W(\mathbf{v})$ is the closest element in W to $\mathbf{v}$.

i.e. $\|\mathbf{v} - \text{proj}_W(\mathbf{v})\| < \|\mathbf{v} - \mathbf{w}\|\ \forall \mathbf{w} \in W,\ \mathbf{w} \neq \text{proj}_W(\mathbf{v})$.

Same proof as before.

Examples

1. Apply the Gram-Schmidt process to the vectors $F = \{1, t, t^2\}$ if

   (i) $F \subset P_3$, where P_3 is endowed with the inner product

   $$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) + p(2)q(2),$$

   (ii) $F \subset C[0,1]$, where $C[0,1]$ is endowed with the inner product $\langle p, q\rangle = \int_0^1 p(t)q(t)\,dt$.

2. Endow P_3 with the inner product of point (i). Let $W = \mathrm{Span}\{1, t, t^2\} \subset P_3$. Find the best approximation of $p(t) = 4t^3 - 2t + 1$ by elements in W.