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Lecture 19

6.1 Inner Product, Length and Orthogonality

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Inner Product in $\mathbb{R}^n$

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we define the *inner product* or *dot product* of $\mathbf{u}$ and $\mathbf{v}$ by

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i \, .$$

Find $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Dot product in $\mathbb{R}^n$

Thm: If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (symmetric)
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ (linear)
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$, (linear)
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$
   $\mathbf{u} \cdot \mathbf{u} = 0 \iff \mathbf{u} = \mathbf{0}$ (positive definite)

Proof. Follows easily from the definition. Try it.

The *length* or *norm* of $\mathbf{v}$ is the nonnegative scalar

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$$

Length of a vector

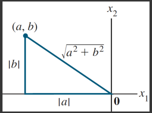


Figure: If $\mathbf{v} = (a, b)$ then $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$.

From the definition we get for $c \in \mathbb{R}$,

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

If $\|\mathbf{v}\| = 1$ we say that $\mathbf{v}$ is a *unit vector*.

Given $\mathbf{v} \in \mathbb{R}^n$, the vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector *in the same direction as* $\mathbf{v}$. This process to create $\mathbf{u}$ from $\mathbf{v}$ is called *normalizing* $\mathbf{v}$.

Distance

1. If $\mathbf{v} = (1, 0, -3, 5, -1)$, find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$.

2. Let $H = \text{Span}\{(2, 3)\}$. Find a unit vector $\mathbf{u}$ which is a basis for H.

The *distance between* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is defined by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| .$$

3. Find the distance between $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, 4)$.

4. Find the distance between $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

Orthogonal vectors

Thm: for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we have

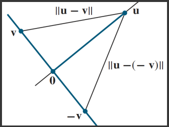
$$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \pm 2\mathbf{u} \cdot \mathbf{v}.$$

Proof. We have $\|\mathbf{u} \pm \mathbf{v}\|^2 = (\mathbf{u} \pm \mathbf{v}) \cdot (\mathbf{u} \pm \mathbf{v}) =$ $\mathbf{u} \cdot \mathbf{u} \pm \mathbf{u} \cdot \mathbf{v} \pm \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \pm 2\mathbf{u} \cdot \mathbf{v}$ $\square$

We say that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

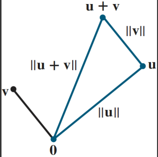
One intuition behind this definition is that in $\mathbb{R}^2$, two vectors $\mathbf{u}, \mathbf{v}$ are perpendicular iff $\mathrm{dist}(\mathbf{u}, \mathbf{v}) = \mathrm{dist}(\mathbf{u}, -\mathbf{v})$, see the next figure.

The theorem of Pythagoras



This, in turn, happens iff $\mathbf{u} \cdot \mathbf{v} = 0$.

> Pythagorean Theorem: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal iff $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.



Orthogonal Complements

Given a subset $W \subset \mathbb{R}^n$, we define *the orthogonal complement of W* by

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \quad \forall \mathbf{w} \in W\}$$

e.g. In $\mathbb{R}^3$, if W is the xy plane, then $W^\perp$ is the z axis.

Thm: Let $W \subset \mathbb{R}^n$ be a subset.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.
2. If $W = \text{Span}(S)$, then $\mathbf{v} \in W^\perp \iff \mathbf{v} \in S^\perp$.

Orthogonal Complements

Proof. 1. We have $\mathbf{0} \in W^{\perp}$ since $\mathbf{0} \cdot \mathbf{w} = 0$ for any $\mathbf{w} \in W$. If $\mathbf{u}, \mathbf{v} \in W^{\perp}$, then $\mathbf{u} + \mathbf{v} \in W^{\perp}$ since $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} = 0 + 0 = 0 \ \forall \mathbf{w} \in W$. Finally, if $\mathbf{u} \in W^{\perp}$ and $c \in \mathbb{R}$ then $c\mathbf{u} \in W^{\perp}$ since $(c\mathbf{u}) \cdot \mathbf{w} = c(\mathbf{u} \cdot \mathbf{w}) = c0 = 0$ for any $\mathbf{w} \in W$.

2. If $\mathbf{v} \in W^{\perp}$ then $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$, in particular, for all $\mathbf{w} \in S$.

Conversely, if $\mathbf{v} \in S^{\perp}$ and $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, then $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all i. Now let $\mathbf{w} \in W$. Then $\mathbf{w} = \sum_{i=1}^{m} \alpha_i \mathbf{w}_i$ for some $\alpha_i \in \mathbb{R}$. So $\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot (\sum_{i=1}^{m} \alpha_i \mathbf{w}_i) = \sum_{i=1}^{m} \alpha_i (\mathbf{v} \cdot \mathbf{w}_i) = \sum_{i=1}^{m} 0 = 0$. Hence, $\mathbf{v} \in W^{\perp}$. $\square$

Orthogonal Complements

Thm: Let A be an $m \times n$ matrix. Then

$$(\mathrm{Row}(A))^{\perp} = \mathrm{Nul}(A) \qquad \text{and} \qquad (\mathrm{Col}(A))^{\perp} = \mathrm{Nul}(A^T).$$

Proof. 1. We have

$$\mathbf{w} \in \mathrm{Nul}(A) \iff A\mathbf{w} = \mathbf{0} \iff \begin{bmatrix} \mathrm{row}_1(A) \cdot \mathbf{w} \\ \vdots \\ \mathrm{row}_m(A) \cdot \mathbf{w} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\iff \mathbf{w} \cdot \mathrm{row}_i(A) = 0 \ \forall i \iff \mathbf{w} \in S^{\perp} \iff w \in (\mathrm{Row}(A))^{\perp}$, where $S = \{\mathrm{row}_1(A), \ldots, \mathrm{row}_m(A)\}$ and we used the previous theorem.

2. Using 1., $\mathrm{Nul}(A^T) = (\mathrm{Row}(A^T))^{\perp} = (\mathrm{Col}(A))^{\perp}$. $\square$

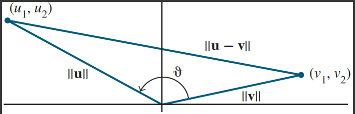
Angles

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we define the *angle θ between* $\mathbf{u}$ *and* $\mathbf{v}$ by

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta .$$

In other words, if $\mathbf{u}, \mathbf{v}$ are nonzero, then $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

This definition is consistent with our intuition in $\mathbb{R}^2$:



Angles

In fact, the law of cosines says that

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta\,.$$

On the other hand

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}\,.$$

Simplifying yields the formula we gave.

Thm: If W is a subspace then $W \cap W^{\perp} = \{\mathbf{0}\}$.

Proof. Let $\mathbf{v} \in W \cap W^{\perp}$. Then $\mathbf{v}\cdot\mathbf{v} = 0$ if we think of the first $\mathbf{v}$ as belonging to W and the second as belonging to $W^{\perp}$. It follows that $\mathbf{v} = \mathbf{0}$. This proves $W \cap W^{\perp} \subseteq \{\mathbf{0}\}$. But $\mathbf{0} \in W \cap W^{\perp}$ as it is a subspace. So $W \cap W^{\perp} = \{\mathbf{0}\}$. $\square$