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Lecture 15

4.6 Change of Basis

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Change of Basis

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are two bases of V. Then the *transition matrix P from $\mathcal{C}$ to $\mathcal{B}$* is the matrix P such that

$$[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{C}} .$$

P is also called the *change of coordinates matrix.*

In the following, in the special case $V = \mathbb{R}^n$, we denote $\mathcal{S} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ the standard basis of $\mathbb{R}^n$.

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Thm: If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are two bases of V, expand for each i,

$$\mathbf{b}_i = a_{1i}\mathbf{c}_1 + a_{2i}\mathbf{c}_2 + \cdots + a_{ni}\mathbf{c}_n\,.$$

Then the transition matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$Q = [\,a_{ij}\,] = \begin{bmatrix} [\,\mathbf{b}_1\,]_{\mathcal{C}} & \cdots & [\,\mathbf{b}_n\,]_{\mathcal{C}} \end{bmatrix}.$$

Proof. If $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{b}_i$, then $\mathbf{x} = \sum_{i=1}^n \alpha_i (\sum_{j=1}^n a_{ji}\mathbf{c}_j) = \sum_{j=1}^n (\sum_{i=1}^n a_{ji}\alpha_i)\mathbf{c}_j$. So

$$Q[\,\mathbf{x}\,]_{\mathcal{B}} = [\,a_{ij}\,]\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j}\alpha_j \\ \vdots \\ \sum_j a_{nj}\alpha_j \end{bmatrix} = [\,\mathbf{x}\,]_{\mathcal{C}}.$$

□

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The transition matrix P from $\mathcal{C}$ to $\mathcal{B}$ is invertible. Moreover, P^{-1} is the transition matrix from $\mathcal{B}$ to $\mathcal{C}$.

Proof. We have $[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{C}} = PQ[\mathbf{x}]_{\mathcal{B}}$ for any $\mathbf{x} \in V$. This implies $\mathbf{v} = PQ\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^n$ as the coordinate map is onto. Since $PQ = I_n$, then $Q = P^{-1}$ (thm). $\square$

Find the transition matrix from the standard basis $\mathcal{S}$ of $\mathbb{R}^2$ to $\mathcal{C} = \{(1,1),(1,-1)\}$.

Thm: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be two bases of $\mathbb{R}^n$. The transition matrix Q from $\mathcal{B}$ to $\mathcal{C}$ arises in the row equivalence $\begin{bmatrix} \mathcal{C} & \mathcal{B} \end{bmatrix} \sim \begin{bmatrix} I_n & Q \end{bmatrix}$.

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Here $\mathbf{c}_j, \mathbf{b}_j$ are arranged as columns to form $\begin{bmatrix} \mathcal{C} & \mathcal{B} \end{bmatrix}$.

Proof. The matrix $\mathcal{C}$ is invertible since $\mathbf{c}_i$ form a basis, so $\mathcal{C} \sim I_n$ (thm), so there are elementary matrices E_j such that $I_n = E_k \cdots E_1 \mathcal{C}$. Let $M = E_k \cdots E_1$. Then $M\mathcal{C} = I_n$ implies $\begin{bmatrix} M\mathbf{c}_1 & \cdots & M\mathbf{c}_n \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$, i.e. $M\mathbf{c}_j = \mathbf{e}_j$.

Now if $\mathbf{b}_i = \sum_{j=1}^n a_{ji}\mathbf{c}_j$, then

$$M\mathbf{b}_i = \sum_{j=1}^n a_{ji} M\mathbf{c}_j = \sum_{j=1}^n a_{ji}\mathbf{e}_j = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix} = [\,\mathbf{b}_i\,]_{\mathcal{C}}.$$ Thus,

$$M\mathcal{B} = \begin{bmatrix} M\mathbf{b}_1 & \cdots & M\mathbf{b}_n \end{bmatrix} = \begin{bmatrix} [\,\mathbf{b}_1\,]_{\mathcal{C}} & \cdots & [\,\mathbf{b}_n\,]_{\mathcal{C}} \end{bmatrix} = Q.$$

Recalling that $M = E_k \cdots E_1$, this says that the row operations reducing $\mathcal{C}$ to I_n also reduce $\mathcal{B}$ to Q. $\square$

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Find the transition matrix from $\mathcal{B}$ to $\mathcal{C}$ if $\mathcal{B}$ is the standard basis $\mathcal{S}$ of $\mathbb{R}^3$ and $\mathcal{C} = \{(1,2,3),(4,5,6),(0,0,1)\}$.

The previous theorem directly implies:

Cor: The transition matrix from $\mathcal{S} \to \mathcal{C}$ is $\mathcal{C}^{-1}$.

The transition matrix from $\mathcal{B} \to \mathcal{S}$ is $\mathcal{B}$.

Find the transition matrix from $\mathcal{B}$ to $\mathcal{C}$ if

1. $\mathcal{B} = \{(1,1),(1,-1)\}, \mathcal{C} = \mathcal{S}$.
2. $\mathcal{B} = \{(1,2),(3,4)\}, \mathcal{C} = \{(1,1),(2,0)\}$.

Change of bases in P_n and $M_{m,n}$

Thm: Let $\mathcal{B} = \{\mathbf{p}_0, \ldots, \mathbf{p}_n\}$ and $\mathcal{C} = \{\mathbf{q}_0, \ldots, \mathbf{q}_n\}$ be two bases of P_n. Let $\mathcal{S} = \{1, x, \ldots, x^n\}$ be the standard basis. Define $\tilde{\mathcal{B}} = \{[\mathbf{p}_0]_{\mathcal{S}}, \ldots, [\mathbf{p}_n]_{\mathcal{S}}\}$ and $\tilde{\mathcal{C}} = \{[\mathbf{q}_0]_{\mathcal{S}}, \ldots, [\mathbf{q}_n]_{\mathcal{S}}]\}$. Then the transition matrix from $\mathcal{B}$ to $\mathcal{C}$ is the transition matrix from $\tilde{\mathcal{B}}$ to $\tilde{\mathcal{C}}$.

This result says that to find the transition matrix between bases in P_n, it suffices to consider the corresponding coordinate vectors in $\mathbb{R}^{n+1}$ and find the transition matrix using the recipe $\begin{bmatrix} \tilde{\mathcal{C}} & \tilde{\mathcal{B}} \end{bmatrix}$.
The same method works for $M_{m,n}$.

Consider P_2. Find the transition matrix from the bases $\mathcal{B}$ to $\mathcal{C}$ if $\mathcal{B} = \{1 + x + x^2, 1, x\}$ and $\mathcal{C} = \{2 + x + 6x^2, 5 + x, x^2\}$

Change of bases in P_n and $M_{m,n}$

Proof. The transition matrix Q from $\mathcal{B}$ to $\mathcal{C}$ is

$Q = \begin{bmatrix} [\mathbf{p}_0]_{\mathcal{C}} & \cdots & [\mathbf{p}_n]_{\mathcal{C}} \end{bmatrix}$. Now $[\mathbf{p}]_{\mathcal{C}} = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{bmatrix} \iff \mathbf{p} =$

$\sum_{i=0}^{n} \alpha_i \mathbf{q}_i \iff [\mathbf{p}]_{\mathcal{S}} = \sum_{i=0}^{n} \alpha_i [\mathbf{q}_i]_{\mathcal{S}} \iff [[\mathbf{p}]_{\mathcal{S}}]_{\tilde{\mathcal{C}}} = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{bmatrix}$

Thus, $Q = \begin{bmatrix} [[\mathbf{p}_0]_{\mathcal{S}}]_{\tilde{\mathcal{C}}} & \cdots & [[\mathbf{p}_n]_{\mathcal{S}}]_{\tilde{\mathcal{C}}} \end{bmatrix}$, which is the transition matrix from $\tilde{\mathcal{B}}$ to $\tilde{\mathcal{C}}$. $\square$

Consider $M_{2,2}$. Find the transition matrix from $\mathcal{B} = \{E_i\}$ the standard basis of $M_{2,2}$ to

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} -2 & -5 \\ -5 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} -2 & -3 \\ -5 & 11 \end{bmatrix} \right\}.$$