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Lecture 25

7.2 Quadratic Forms

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Quadratic Forms and Equations

A *quadratic form* on $\mathbb{R}^n$ is a function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for some symmetric $n \times n$ matrix A.

e.g. If $A = I_n$ then $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2$.

Compute $Q(\mathbf{x})$ for $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

A general *quadratic equation* in $\mathbb{R}^2$ has the form

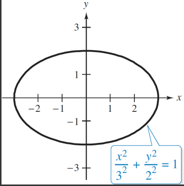
$$ax^2 + bxy + cy^2 + dx + ey + f = 0\,.$$

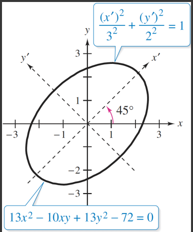
Define $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$. Then

Quadratic Forms and Equations

$$\mathbf{x}^T A\mathbf{x} + \begin{bmatrix} d & e \end{bmatrix}\mathbf{x} + f = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} ax + b/2y \\ b/2x + cy \end{bmatrix} + dx + ey + f$$
$$= ax^2 + bxy + cy^2 + dx + ey + f\,.$$

We thus see that the quadratic form in $\mathbb{R}^2$ corresponds to $ax^2 + bxy + cy^2$. We call A the *matrix of the quadratic form*. It is diagonal iff there is no xy term.





Quadratic Forms and Equations

Find the matrix of the quadratic form associated with:

1. $4x^2 + 9y^2 - 36 = 0$.
2. $13x^2 - 10xy + 13y^2 - 72 = 0$.

It is easier to deal with quadratic forms that have no xy terms. We can always reduce it to this case after performing a rotation, as in the previous figure. In formulas, this amounts to diagonalizing A. More precisely, we know $P^T AP = D$ is diagonal. Define $\mathbf{x}' = P^T \mathbf{x}$. Then
$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T (PDP^T)\mathbf{x} = (P^T \mathbf{x})^T D(P^T \mathbf{x}) = (\mathbf{x}')^T D\mathbf{x}'$.

We need to choose P carefully. As an orthogonal matrix, $\det(P) = \pm 1$. We construct P such that $\det(P) = 1$, which amounts to switching the columns if necessary.

Quadratic Forms and Equations

Then P is indeed a rotation, $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for some $0 \le \theta < 2\pi$. Prove this.

Principal Axes Thm: For a conic with equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the rotation $\mathbf{x} = P\mathbf{x}'$ eliminates the xy term, where P is the orthogonal matrix diagonalizing $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ such that $|P| = 1$. If $P^TAP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ are the eigenvalues of A, then the equation for the rotated conic is $\lambda_1(x')^2 + \lambda_2(y')^2 + \begin{bmatrix} d & e \end{bmatrix} P\mathbf{x}' + f = 0$.

Quadratic Forms and Equations

The proof follows by summarizing our discussion, since $(\mathbf{x}')^T D\mathbf{x}' = \lambda_1 (x')^2 + \lambda_2 (y')^2$.

If the quadratic equation has no xy term, we say its graph is in *standard position*. We showed that we can always do that with a *change of variables*.

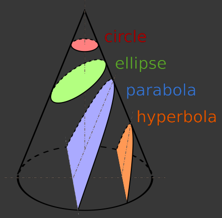
1. Perform a rotation of axes to eliminate the xy term in $13x^2 - 10xy + 13y^2 - 72 = 0$, then write the equation for the graph in standard position.

2. Same question for $3x^2 - 10xy + 3y^2 + 16\sqrt{2}x - 32 = 0$.

Classification of Conic Sections

Thm: The graph of a quadratic equation in $\mathbb{R}^2$ is either a parabola, an ellipse, a hyperbola or a degenerate conic.

By a degenerate conic, we mean an empty set, a point or one or two lines. The proof consists of putting the equation in standard form, then completing the square. The cases depend on the coefficients being positive, negative or zero. We omit the details.



Case of $\mathbb{R}^3$

In standard position, the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a parabola is $y^2 = 4ax$ (or $x^2 = 4ay$, which is a rotation of this), a hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

We now consider $\mathbb{R}^3$. A quadratic equation has the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0\,.$$

The quadratic part is captured by the symmetric matrix

$$A = \begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix}$$

Quadratic Forms in Higher Dimension

in the sense that the quadratic equation becomes

$$\mathbf{x}^T A\mathbf{x} + \begin{bmatrix} g & h & i \end{bmatrix}\mathbf{x} + j = 0\,.$$

For $\mathbf{x} = (x, y, z)$, let
$Q(\mathbf{x}) = x^2 + 4y^2 + 6z^2 + 4xy + 6xz + 10yz$. Find a symmetric matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

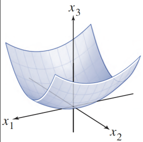
A quadratic form Q in $\mathbb{R}^n$ is

a. *positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$,

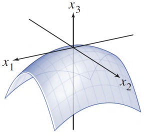
b. *negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$,

c. *indefinite* if $Q(\mathbf{x})$ takes both positive and negative values.

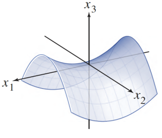
Quadratic Forms in Higher Dimension



Positive definite



Negative definite



Indefinite

Thm: Let A be an $n \times n$ symmetric matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Then the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ is

a. positive definite iff $\lambda_i > 0 \ \forall i$,

b. negative definite iff $\lambda_i < 0 \ \forall i$,

c. indefinite iff A has both positive and negative eigenvalues.

Quadratic Forms in Higher Dimension

Proof. The principal axes theorem generalizes to say that if $D = P^T A P$ is a diagonalization of A, then the change of variables $\mathbf{x} = P\mathbf{x}'$ satisfies that

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = (\mathbf{x}')^T D \mathbf{x}' = \lambda_1 (x_1')^2 + \cdots + \lambda_n (x_n')^2 \quad (1)$$

Since P is invertible, we can pass from $\mathbf{x}$ to $\mathbf{x}'$ and vice versa. So the possible values of $Q(\mathbf{x})$ are the possible values of the RHS of (1). The theorem clearly follows. $\square$

Is $Q(\mathbf{x}) = 2xy + 2xz + 2yz$ positive-definite ?

We sometimes say that A is a *positive definite matrix* if it is symmetric and its quadratic form is positive definite. Similar terminology for negative...etc are used.

Quadric Surfaces

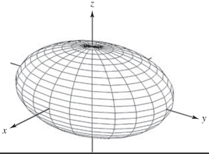


Figure: Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

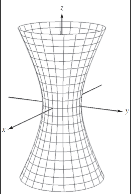


Figure: Hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

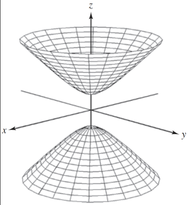


Figure: Hyperboloid of two sheets $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

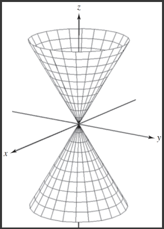


Figure: Elliptic cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

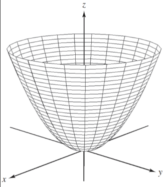


Figure: Elliptic paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

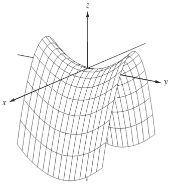


Figure: Hyperbolic paraboloid $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$

Final Remarks

- A quadratic form is *positive semidefinite* if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$. Using the usual change of variable, we see this occurs iff all eigenvalues λ_i of the associated matrix A satisfy $\lambda_i \geq 0 \ \forall i$. The form is *negative semidefinite* if $Q(\mathbf{x}) \leq 0$ for all $\mathbf{x}$, which occurs iff $\lambda_i \leq 0 \ \forall i$.
- When we speak of *classification of quadratic forms* in this course, we mean classifying if it is positive definite, positive semidefinite, indefinite,...etc. We do not mean the geometric classification like ellipsoid, hyperboloid and so forth.

Final Remarks

- The change of variables $\mathbf{x}' = P^T\mathbf{x}$ (equivalently $\mathbf{x} = P\mathbf{x}'$) works in any dimension to eliminate the cross terms $x_i x_j$. Here P is the orthogonal matrix diagonalizing A. We *do not* need $|P| = 1$ for this. We only required $|P| = 1$ in dimension two to have the geometric interpretation that this gives a rotation.