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Lecture 9

3.2 Properties of Determinants II

3.3 Cramer's Rule, Volume, and Linear Transformations

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Determinants and inverses

Thm: A square matrix A is invertible iff $\det A \neq 0$.

Proof. Let B be the reduced row-echelon form of A. By the row operation theorem, $\det(B)$ is a nonzero multiple of $\det(A)$, say $\det(B) = q\det(A)$. If A is invertible, then $B = I_n$, so $\det(A) = \frac{1}{q} \cdot 1 \neq 0$. If A is not invertible, it has less than n pivots, and the last row of B is identically zero, so $\det(A) = \frac{1}{q} \cdot 0 = 0$. $\square$

Are the following matrices invertible ?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -6 & -9 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Matrix products and scalar multiples

Thm: If A is invertible then $\det(A^{-1}) = \frac{1}{\det(A)}$.

This follows from the next result on multiplications.

Find $|A^{-1}|$ if $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 6 \\ 0 & 1 & -1 \end{bmatrix}$.

Thm: Let A, B be $n \times n$ matrices and $c \in \mathbb{R}$. Then $\det(cA) = c^n \det(A)$ and $\det(AB) = \det(A)\det(B)$.

Proof. If C is obtained from A by multiplying one row by c, then $|C| = c|A|$. Since cA is obtained by multiplying each row by c, repeating n times gives $|cA| = c^n|A|$.

Matrix products and scalar multiples

If A is not invertible then $\mathbf{x} \mapsto A\mathbf{x}$ is not onto, so $\mathbf{x} \mapsto AB\mathbf{x}$ is not onto (as $\mathrm{Range}(AB) \subseteq \mathrm{Range}(A)$), so AB is not invertible. In particular, $|A||B| = 0 = |AB|$.

So assume henceforth that A is invertible. Then A is row equivalent to I_n, i.e. $I_n = E_k E_{k-1} \cdots E_1 A$ for some elementary matrices E_j.

If E is an elementary matrix, then $|E| = -1$ if E is a row interchange, $|E| = k$ if E multiplies a row by k, and $|E| = 1$ if E operates as $R_i + kR_j \to R_i$. This all follows from the row operation theorem, since E is obtained from I_n by performing the operation, and $|I_n| = 1$.

So the row operation theorem can be restated as: if

Determinant of an invertible Matrix

$C = EA$ for an elementary matrix E, then $|C| = |E||A|$.

Finally, $A = E_1^{-1} \cdots E_k^{-1}$, where E_j^{-1} are elementary matrices. So $AB = E_1^{-1} \cdots E_k^{-1} B$. Applying the previous paragraph k times, we get
$|AB| = |E_1^{-1}||E_2^{-1}| \cdots |E_k^{-1}||B| = |E_1^{-1} \cdots E_k^{-1}||B| = |A||B|$. $\square$

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Find $\det(A)$, $\det(B)$, $\det(AB)$ and $\det(-6A)$.

Unfortunately there is no similar rule for $\det(A + B)$.

Find $\det(A + B)$ for the above matrices.

Determinant of the transpose

Thm: If A is a square matrix, then $\det(A) = \det(A^T)$.

<u>Proof.</u> This is clear for size 2. If A has size 3, then expanding on first row, $|A| = \sum_{i=1}^{n} a_{1j} C_{1j}$. Expanding on first column, $|A^T| = \sum_{i=1}^{n} A^T_{j1} C_{j1}(A^T) = \sum_{i=1}^{n} a_{1j} C_{1j}$, where we used the property for 2×2 matrices in the last equality. We thus see the property holds for size 3. Repeating (induction) gives the result for all n.

Thm: The determinant is linear in each column. That is, if $T(\mathbf{x}) = \det \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{j-1} & \mathbf{x} & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{bmatrix}$, then T is linear.

The adjugate of a matrix

Proof. $T(c\mathbf{x}) = cT(\mathbf{x})$ by the column operation theorem.
Next, expanding on the jth column,
$T(\mathbf{x}+\mathbf{y}) = \sum_i (x_i + y_i)C_{ij} = \sum_i x_i C_{ij} + \sum_i y_i C_{ij} = T(\mathbf{x}) + T(\mathbf{y})$.

The *adjugate* $\mathrm{adj}(A)$ of a matrix A is the transpose of its matrix of cofactors:

$$\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & \vdots & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

The adjugate of a matrix

Find the adjugate of $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

Thm: If A is a square invertible matrix, then $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$.

<u>Proof.</u> The ij-entry $(A \operatorname{adj}(A))_{ij} = \sum_k a_{ik} C_{jk}$. If $j = i$, this gives $|A|$ by Laplace's expansion. If $j \neq i$, $\sum_k a_{ik} C_{jk} = 0$. To see this, note that C_{jk} is unchanged by modifying row j. Replace row j in A by a copy of row i. Expanding the determinant of this new matrix B in row j gives $\sum_k a_{jk} C_{jk} = \sum_k a_{ik} C_{jk}$. This determinant is zero because B has two equal rows. This shows $A \operatorname{adj}(A) = |A| I_n$. $\square$

Cramer's rule

Find A^{-1} for the previous matrix A.

Cramer: If A is an $n \times n$ invertible matrix, the unique solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det(A)}, \ldots, x_n = \frac{\det A_n(\mathbf{b})}{\det(A)},$$

where $A_i(\mathbf{b})$ is obtained from A by replacing its ith column with $\mathbf{b}$.

Proof. $\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{|A|}\text{adj}(A)\mathbf{b}$, so $x_i = \frac{1}{|A|}(\text{adj}(A)\text{b})_i = \frac{1}{|A|}\sum_j C_{ji}b_j = \frac{|A_i(\mathbf{b})|}{|A|}$, where in the last step we expanded $A_i(\mathbf{b})$ in the ith column. $\square$

Determinants as Areas and Volumes

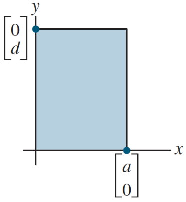
Use Cramer's rule to solve $\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases}$

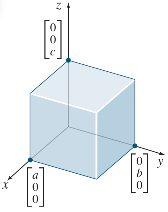
> If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.
> If A is 3×3, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

<u>Proof.</u> We may assume $|A| \neq 0$. In fact, if $|A| = 0$, the columns are linearly dependent, meaning the points are collinear (coplanar) and the area (volume) is 0 as well.

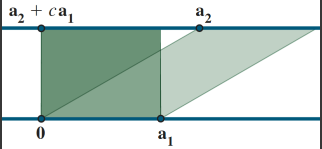
Determinants as Areas and Volumes

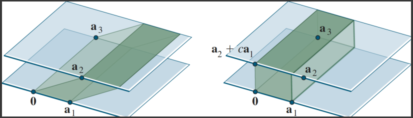
Assume $|A| \neq 0$. If A is diagonal, the statement is clear:





If A is not diagonal, we diagonalize A through column operations to reduce it to the previous case.





Determinants as Areas and Volumes

Find the area of the parallelogram determined by the points $(1,1)$, $(4,3)$, $(2,7)$, $(5,9)$.

Thm: If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear with standard matrix A and if $R \subset \mathbb{R}^2$ has a finite area, then

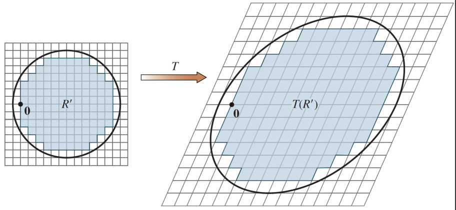
$$\mathrm{Area}(T(R)) = |\det A| \cdot \mathrm{Area}(R).$$

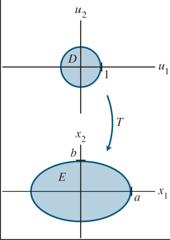
If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear with standard matrix A and if $R \subset \mathbb{R}^3$ has a finite volume, then

$$\mathrm{Volume}(T(R)) = |\det A| \cdot \mathrm{Volume}(R).$$

Determinants as Areas and Volumes

The proof goes by first checking this for parallelepipeds (parallelograms) and then approximating R using parallelepipeds (parallelograms), as in the figure.





We may use this to find the area of an ellipse $\mathcal{E}$ with semi-axes a, b. The matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ maps the unit disk $\mathcal{D}$ onto $\mathcal{E}$, so $\mathrm{Area}(\mathcal{E}) = |A| \mathrm{Area}(\mathcal{D}) = ab\pi$.