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Lecture 13

4.4 Coordinates Systems

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Unique Representation Theorem

Thm: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis of a vector space V, then any element $\mathbf{w} \in V$ can be written uniquely as $\mathbf{w} = \sum_{i=1}^{p} c_i \mathbf{v}_i$, for some $c_i \in \mathbb{R}$.

The proof is the same as we saw in Lec5: existence of the expansion follows from the spanning property, while uniqueness follows from linear independence.

Find the unique expansion of $\mathbf{u} = (u_1, u_2)$ in the basis $\mathcal{C} = \{(1,1), (1,-1)\}$.

Coordinate systems

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a vector space V. For each $\mathbf{x} \in V$, the *coordinates of* $\mathbf{x}$ *relative to the basis* $\mathcal{B}$ are the weights $c_1, \ldots, c_p$ such that $\mathbf{x} = \sum_{i=1}^{p} c_i \mathbf{b}_i$. The vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \in \mathbb{R}^p$ is called the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

If $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$.

If $\mathcal{C} = \{(1,1), (1,-1)\}$, $\mathbf{u} = (u_1, u_2)$, $[\mathbf{u}]_{\mathcal{C}} = \begin{bmatrix} \frac{u_1+u_2}{2} \\ \frac{u_1-u_2}{2} \end{bmatrix}$.

Coordinate systems

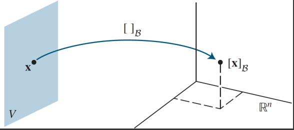

Figure: The coordinate mapping from V to $\mathbb{R}^n$.

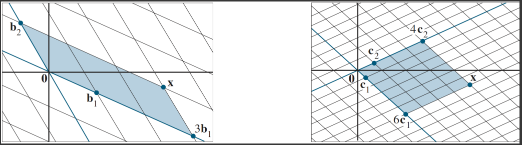

Figure: Even in $V = \mathbb{R}^n$, it can be useful to consider different coordinate systems $\mathcal{B}$ and $\mathcal{C}$. Here $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

In crystallography, we try to choose a coordinate system adapted to the crystal lattice

Coordinate systems

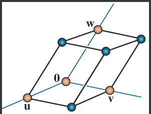

Figure: It is natural to consider a slanted coordinate system to analyze this crystal.

1. If $\mathcal{B} = \{(1,2),(3,4)\}$ and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find the coordinate vector of $\mathbf{x}$ relative to the standard basis.

2. Let $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 3 \\ -4 & -4 \end{bmatrix}$. Is B in $H = \text{Span}\{A_1, A_2\}$? If yes, find $[B]_{\mathcal{B}}$, for $\mathcal{B} = \{A_1, A_2\}$.

In the last example, $B \in M_{2,2}$, but only needs 2 coordinates to be fully described in the plane H.

Isomorphism

Let V, W be vector spaces. We say that $T : V \to W$ is an *isomorphism* if it linear, injective and surjective. If such a T exists, we say V and W are *isomorphic*.

Prove that T^{-1} exists and is an isomorphism.

Thm: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an isomorphism of V onto $\mathbb{R}^n$.

Proof. If $\mathbf{x}, \mathbf{y} \in V$, then $\mathbf{x} = \sum_{i=1}^n c_i \mathbf{b}_i$ and $\mathbf{y} = \sum_{i=1}^n d_i \mathbf{b}_i$ for some $c_i, d_i \in \mathbb{R}$. So $\mathbf{x} + \mathbf{y} = \sum_{i=1}^n (c_i + d_i)\mathbf{b}_i$. Also, if $\alpha \in \mathbb{R}$, $\alpha\mathbf{x} = \alpha \sum_{i=1}^n c_i \mathbf{b}_i = \sum_{i=1}^n \alpha c_i \mathbf{b}_i$. Thus,

Isomorphism

$$[\mathbf{x}+\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}} + [\mathbf{y}]_{\mathcal{B}}$$

$$[\alpha\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha c_1 \\ \vdots \\ \alpha c_n \end{bmatrix} = \alpha \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \alpha[\mathbf{x}]_{\mathcal{B}}$$

so the map is linear. Next, if $[\mathbf{x}]_{\mathcal{B}} = \mathbf{0}$, then $\mathbf{x} = 0\mathbf{b}_1 + 0\mathbf{b}_2 + \cdots + 0\mathbf{b}_n = \mathbf{0}$. Thus, the map is one-to-one. Finally, given $\mathbf{u} \in \mathbb{R}^n$, let $\mathbf{x} = u_1\mathbf{b}_1 + u_2\mathbf{b}_2 + \cdots + u_n\mathbf{b}_n$. Then $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{u}$. So the map is surjective. □

Examples

If $\mathcal{B} = \{1, x, x^2, \dots, x^n\}$ is the standard basis of P_n and
$p \in P_n$, $p(x) = \sum_{k=0}^{n} c_k x^k$, then $[p]_{\mathcal{B}} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$.

1. Check that P_n is isomorphic to $\mathbb{R}^{n+1}$ and $M_{2,2}$ is isomorphic to $\mathbb{R}^4$.

2. Let $\mathcal{B}$ be the standard basis of $M_{2,2}$ and
$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Find $[A]_{\mathcal{B}}$ and $[A]_{\mathcal{C}}$.

3. Prove that if $T_1 : V \to W$ and $T_2 : W \to H$ are isomorphisms, then $T_2 \circ T_1 : V \to H$ is an isomorphism.

Isomorphism properties

Thm: If $T : V \to W$ is an isomorphism, then T carries lin. independent sets to lin. independent sets, spanning sets to spanning sets, bases to bases.

Thus, a set S is independent iff its image $T(S)$ is. Why ?

<u>Proof.</u> Let $S = \{\mathbf{v}_i\}_{i=1}^n$, its image $T(S) = \{T(\mathbf{v}_i)\}_{i=1}^n$. If $\sum_{i=1}^n \alpha_i T(\mathbf{v}_i) = \mathbf{0}$ then $T\left(\sum_{i=1}^n \alpha_i \mathbf{v}_i\right) = \mathbf{0}$ by linearity. As T is injective, we must have $\sum_{i=1}^n \alpha_i \mathbf{v}_i = \mathbf{0}$. But S is independent, so $\alpha_i = 0\ \forall i$. Thus, $T(S)$ is independent.

Next, if $V = \mathrm{Span}(S)$ and $\mathbf{w} \in W$, then $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V$, so $\mathbf{v} = \sum_{i=1}^n c_i \mathbf{v}_i$ for some $c_i \in \mathbb{R}$, so $\mathbf{w} = T(\sum c_i \mathbf{v}_i) = \sum_{i=1}^n c_i T(\mathbf{v}_i) \in \mathrm{Span}(T(S))$. As $\mathbf{w}$ is arbitrary, then $\mathrm{Span}(T(S)) = W$. $\square$

Exercises

Use the coordinate map to study the linear independence of $S = \{t + t^2, 2 + 3t^2, 1 + 2t\}$ in P_2.

Use the coordinate map to find a basis for the space $H = \text{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right\}$.

Let $\mathcal{B} = \{1 + t, 1 + t^2, t + t^2\}$. Show that $\mathcal{B}$ is a basis of P_2 and find $[p]_{\mathcal{B}}$ if $p(t) = 6 + 3t - t^2$.

If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of $\mathbb{R}^n$ and $\mathcal{S}$ is the standard basis, then the matrix $P = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$ satisfies that $\mathbf{x} = P[\mathbf{x}]_{\mathcal{B}}$, i.e. P is the *transition matrix* from $\mathcal{B}$ to $\mathcal{S}$. We explain this in detail in Lec15.

The matrix of a linear transformation

Thm: Let V and W be vector spaces with bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. Say $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.

If $T : V \to W$ is linear then the matrix

$$A = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{B}'} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}'} \end{bmatrix}$$

satisfies that $[T(\mathbf{v})]_{\mathcal{B}'} = A[\mathbf{v}]_{\mathcal{B}}$ for all $\mathbf{v} \in V$.

Proof. Let $\mathbf{v} \in V$, so $\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{b}_i$. Then by linearity,
$T(\mathbf{v}) = \sum_{i=1}^{n} c_i T(\mathbf{b}_i)$, so
$[T(\mathbf{v})]_{\mathcal{B}'} = \sum_{i=1}^{n} c_i [T(\mathbf{b}_i)]_{\mathcal{B}'} = A[\mathbf{v}]_{\mathcal{B}}$. $\square$

Let $D_x : P_n \to P_{n-1}$ be the differentiation operator. Find the matrix of D_x relative to the bases $\mathcal{B} = \{1, x, \ldots, x^n\}$ and $\mathcal{B}' = \{1, x, \ldots, x^{n-1}\}$ of P_n and P_{n-1}.