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Lecture 5

2.8 Subspaces of $\mathbb{R}^n$

2.9 Dimension and Rank

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Subspaces of $\mathbb{R}^n$

A *subspace* of $\mathbb{R}^n$ is any set $H \subseteq \mathbb{R}^n$ such that :

a. The zero vector is in H.

b. For each $\mathbf{u}, \mathbf{v} \in H$, the sum $\mathbf{u} + \mathbf{v} \in H$.

c. For each $\mathbf{u} \in H$ and each scalar c, $c\mathbf{u} \in H$.

$H = \{\mathbf{0}\}$ is a subspace, called the zero subspace.

1. Check that if $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$, then $\mathrm{Span}\{\mathbf{v}_1, , \ldots, \mathbf{v}_p\}$ is a subspace.

2. Are the following subsets of $\mathbb{R}^2$ subspaces ?
$H = \{(x, y) : y = 3x\}$, $V = \{(x, y) : y = 3x + 1\}$.

Column Space and Null Space of a matrix

The *column space* of an $m \times n$ matrix A is the set of all linear combinations of the columns $\{\mathbf{a}_i\}$ of A, i.e.

$$\mathrm{Col}(A) = \mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_p\} \subseteq \mathbb{R}^m .$$

The *null space* of A is the set of solutions of $A\mathbf{x} = \mathbf{0}$,

$$\mathrm{Nul}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} .$$

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & -1 \\ 2 & -5 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 9 \\ -6 \\ 17 \end{bmatrix}$. Is $\mathbf{b}$ in $\mathrm{Col}(A)$?

Clearly, $\mathrm{Col}(A)$ is a subspace, <u>equal to the range of A</u>.

Basis for a subspace

Thm: $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Proof. Since $A\mathbf{0} = \mathbf{0}$, then $\mathbf{0} \in \mathrm{Nul}(A)$. Next, if $\mathbf{u}, \mathbf{v} \in \mathrm{Nul}(A)$, then $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\mathbf{u} + \mathbf{v} \in \mathrm{Nul}(A)$. Finally, if $\mathbf{u} \in \mathrm{Nul}(A)$ and $c \in \mathbb{R}$, then $A(c\mathbf{u}) = cA\mathbf{u} = \mathbf{0}$, so $c\mathbf{u} \in \mathrm{Nul}(A)$. $\square$

A *basis* for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

e.g. The vectors $\mathbf{e}_j \in \mathbb{R}^n$ with all entries zero except the jth entry equal to 1, satisfy that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$. It is called the *standard basis of* $\mathbb{R}^n$. Check.

Bases for null spaces and column spaces

Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix}.$$

Thm: The pivot columns of a matrix A form a basis for $\mathrm{Col}(A)$.

What this theorem means is that if we want to find a basis for $\mathrm{Col}(A)$, we should first put A in row-echelon form, say $A \sim B$, identify the pivot columns of the reduced matrix B, then the corresponding columns <u>of A</u> (not B) will form a basis for $\mathrm{Col}(A)$.

Basis for the column space

Proof: Let $A \sim B \sim C$, with B in echelon form and C in reduced echelon form. If C has r pivots, its pivot columns must be $\mathbf{e}_1, \ldots, \mathbf{e}_r$.[1] Clearly these columns are linearly independent. Moreover, they span $\mathrm{Col}(C)$. In fact, if $\mathbf{c}_k$ is a non-pivot column, and if there are j pivot columns on its left, $\mathbf{e}_1, \ldots \mathbf{e}_j$, then $\mathbf{c}_k$ can only have nonzero entries in the first j components, otherwise it would be a pivot column. Thus, $\mathbf{c}_k$ is a linear combination of them $\mathbf{e}_1, \ldots \mathbf{e}_j$.

We have shown that the pivot columns of C are linearly independent and span $\mathrm{Col}(C)$, so they are a basis for $\mathrm{Col}(C)$.

[1] The only way to skip some $\mathbf{e}_i$ would be to have an intermediate zero row, which is impossible.

Basis for the column space

Next we note that the equations $A\mathbf{x} = \mathbf{0}$, $B\mathbf{x} = \mathbf{0}$ and $C\mathbf{x} = \mathbf{0}$ all have the same solutions. But nontrivial solutions give precisely the dependence relations among the columns. It follows that the dependence relations among the columns of A, B and C are the same. For example, if $\mathbf{c}_4 = 5\mathbf{c}_2 + 2\mathbf{c}_1$, then also $\boldsymbol{\alpha}_4 = 5\boldsymbol{\alpha}_2 + 2\boldsymbol{\alpha}_1$ and vice versa.

This shows that columns in A corresponding to the pivot columns of C are also linearly independent. Moreover, they span $\mathrm{Col}(A)$, as the pivot columns of C span $\mathrm{Col}(C)$.

Basis for the column space

This shows that the designated columns of A are a basis for $\mathrm{Col}(A)$. Finally, since the pivot columns of B and C are in the same positions, we are done. $\square$

$$\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Illustration of the proof. $A \sim B \sim C$, $\mathbf{c}_3 = -2\mathbf{c}_1 + \mathbf{c}_2$ implies $\boldsymbol{\alpha}_3 = -2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$. Independence of $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4$ implies that of $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4$, which are thus a basis of $\mathrm{Col}(A)$.

Find a basis for the column space of $A = \begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 0 & 7 & 8 \end{bmatrix}$.

Coordinate systems

Thm: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis of a subspace H, then any element $\mathbf{w}$ of H can be written uniquely as $\mathbf{w} = \sum_{i=1}^{p} c_i \mathbf{v}_i$, for some $c_i \in \mathbb{R}$.

Proof. We know that $\mathbf{w}$ has at least one expansion $\mathbf{w} = \sum_{i=1}^{p} c_i \mathbf{v}_i$ since the $\{\mathbf{v}_i\}$ span H. If $\mathbf{w} = \sum_{i=1}^{p} d_i \mathbf{v}_i$ is another one, then $\mathbf{0} = \mathbf{w} - \mathbf{w} = \sum_{i=1}^{p} c_i \mathbf{v}_i - \sum_{i=1}^{p} d_i \mathbf{v}_i = \sum_{i=1}^{p} (c_i - d_i) \mathbf{v}_i$. But the $\{\mathbf{v}_i\}$ are linearly independent. So we must have $c_i - d_i = 0$ for all i. So $c_i = d_i \ \forall i$ and the expansion is unique. $\square$

Coordinate systems

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a subspace H. For each $\mathbf{x} \in H$, the *coordinates of* $\mathbf{x}$ *relative to the basis* $\mathcal{B}$ are the weights $c_1, \ldots, c_p$ such that $\mathbf{x} = \sum_{i=1}^{p} c_i \mathbf{b}_i$. The vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \in \mathbb{R}^p$ is called the coordinate vector of $\mathbf{x}$ (relative to $\mathcal{B}$) or the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix}$. Is $\mathbf{x}$ in $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$? If yes, find $[\mathbf{x}]_{\mathcal{B}}$, for $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$.

Coordinate systems

The idea in the previous example is that although vectors in H have 3 coordinates, the space H is actually a plane. The map $H \to \mathbb{R}^2$ given by $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an *isomorphism* (i.e. an identification).[2]

Thm: If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for H, then every set containing more than n vectors in H is linearly dependent.

Proof. Let $\{\mathbf{u}^1, \ldots, \mathbf{u}^m\}$ be m vectors in H, with $m > n$. Let us show that

$$\sum_{i=1}^{m} k_i \mathbf{u}^i = \mathbf{0} \iff \sum_{i=1}^{m} k_i [\mathbf{u}^i]_{\mathcal{B}} = \mathbf{0}. \tag{1}$$

[2] This will be proved later in the course.

The dimension of a subspace

Indeed,
$\mathbf{u}^i = \sum_{j=1}^n c_{ji}\mathbf{v}_j \implies \sum_{i=1}^m k_i\mathbf{u}^i = \sum_{j=1}^n \left(\sum_{i=1}^m k_i c_{ji}\right)\mathbf{v}_j$. But $\{\mathbf{v}_j\}$ are linearly independent, hence
$\sum_{i=1}^m k_i\mathbf{u}^i = \mathbf{0} \iff \sum_{i=1}^m k_i c_{ji} = 0 \ \forall j$. On the other hand,

$$\sum_{i=1}^m k_i[\mathbf{u}^i]_{\mathcal{B}} = \mathbf{0} \iff \sum_{i=1}^m k_i \begin{bmatrix} c_{1i} \\ \vdots \\ c_{ni} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \iff$$

$\sum_{i=1}^m k_i c_{ji} = 0 \ \forall j$. This proves (1).

But $[\mathbf{u}^1]_{\mathcal{B}}, \dots, [\mathbf{u}^m]_{\mathcal{B}}$ are m vectors in $\mathbb{R}^n$, $m > n$. So they are dependent (lecture 3). By the previous paragraph, we deduce $\mathbf{u}^1, \dots, \mathbf{u}^m$ are dependent. $\square$

Dimension Theorem

Thm: If H has one basis with n vectors, then every basis of H has n vectors.

Proof. Let $\mathcal{B}_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for H and let $\mathcal{B}_2 = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be another basis. Then by the previous theorem, $m \leq n$, because $\mathcal{B}_1$ is a basis and $\mathcal{B}_2$ is linearly independent. Similarly, $n \leq m$ because $\mathcal{B}_2$ is a basis and $\mathcal{B}_1$ is linearly independent. Thus, $n = m$. $\square$

The dimension of a subspace

The *dimension* of a nonzero subspace H, denoted by $\dim H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.

e.g. $\dim(\mathbb{R}^n) = n$ since $\mathcal{B} = \{\mathbf{e}^1, \ldots, \mathbf{e}^n\}$ is a basis.

The *rank* of a matrix A is defined by

$$\mathrm{rank}(A) = \dim \mathrm{Col}(A).$$

This is also the number of pivot columns of A, and also the dimension of the range of A.

Rank and Basis Theorems

Find the rank of $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix}$.

Rank Theorem: If a matrix A has n columns, then

$$\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = n\,.$$

Basis Theorem: Let $H \subset \mathbb{R}^n$ have dimension p.

If p vectors in H are linearly independent, they are a basis for H.

If p vectors in H span H, they are a basis for H.

Rank and Basis Theorems

These theorems allow us to save time. If we know the rank, we can immediately deduce $\dim \mathrm{Nul}(A)$, and vice versa.

Similarly, to test if a set of p vectors is a basis for a p-dimensional subspace, it suffices to test either linear independence or spanning, no need to test both.

These theorems will be proved later in the course in more generality.