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Lecture 18

5.4 Eigenvectors and Linear Transformations

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Eigenvectors of Linear Transformations

A scalar λ is an *eigenvalue* of a linear map $T : V \to V$ if there exists $\mathbf{x} \neq \mathbf{0}$ such that $T(\mathbf{x}) = \lambda \mathbf{x}$. We call $\mathbf{x}$ an *eigenvector* of T corresponding to λ.

e.g. In the vector space of bounded infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$, let $T(\mathbf{x}) = (x_2, x_3, x_4, \ldots)$. Then $T(\mathbf{x}) = \lambda \mathbf{x} \iff x_{j+1} = \lambda x_j \ \forall j \iff \mathbf{x} = x_1(1, \lambda, \lambda^2, \ldots)$. This $\mathbf{x}$ is a bounded sequence iff $|\lambda| \leq 1$. So this linear map has infinitely many eigenvalues.

e.g. For $V = C[0, 1]$, let $(Tf)(x) = xf(x)$. Then T has no eigenvalues: $Tf = \lambda f \iff (x - \lambda)f(x) = 0 \ \forall x \iff f(x) = 0 \ \forall x \neq \lambda \iff f = 0$ as f is continuous.

The matrix of T in different bases (Extra)

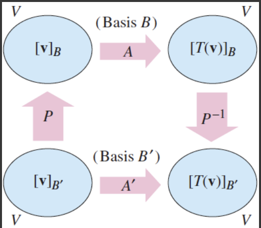
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $T(x, y) = (3x - y, x + 2y)$.
Find the matrix of T relative to the bases
$\mathcal{B} = \{(1, 1), (1, -1)\}$ and $\mathcal{S} = \{(1, 0), (0, 1)\}$.

Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases for V. Let $T : V \to V$ be linear. If

1. A is the matrix of T relative to $\mathcal{B}$, i.e. $[T(\mathbf{v})]_{\mathcal{B}} = A[\mathbf{v}]_{\mathcal{B}}$,
2. A' is the matrix of T relative to $\mathcal{B}'$, i.e. $[T(\mathbf{v})]_{\mathcal{B}'} = A'[\mathbf{v}]_{\mathcal{B}'}$
3. P is the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$

Then $A' = P^{-1}AP$.

The matrix of T in different bases (Extra)



1. Find the matrix of $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (2x + 3y, 4y - 5x)$ relative to $\mathcal{B}' = \{(1, 1), (1, -1)\}$.

2. Recall the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$ if $\mathcal{B} = \{(1, 2), (3, 4)\}$, $\mathcal{B}' = \{(1, 1), (2, 0)\}$ is $P^{-1} = \begin{bmatrix} 2 & 4 \\ \frac{-1}{2} & \frac{-1}{2} \end{bmatrix}$. Suppose T has matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ relative to $\mathcal{B}$. Find the matrix of T relative to $\mathcal{B}'$.

Diagonalization and Linear Transformations

Thm: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear map. If the standard matrix A of T is diagonalizable, then the eigenvectors of A form a basis $\mathcal{B}$ such that the matrix of T relative to $\mathcal{B}$ is diagonal.

<u>Proof.</u> We showed in Lec17 that if A is diagonalizable, then its eigenvectors form a basis $\mathcal{B} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$. Let M be the matrix of T relative to $\mathcal{B}$. Then

$$M = \left[[T(\mathbf{p}_1)]_{\mathcal{B}} \quad \cdots [T(\mathbf{p}_n]_{\mathcal{B}} \right] = \left[[A\mathbf{p}_1]_{\mathcal{B}} \quad \cdots [A\mathbf{p}_n]_{\mathcal{B}} \right] = \left[\lambda_1 [\mathbf{p}_1]_{\mathcal{B}} \quad \cdots \quad \lambda_n [\mathbf{p}_n]_{\mathcal{B}} \right] = \left[\lambda_1 \mathbf{e}_1 \quad \cdots \quad \lambda_n \mathbf{e}_n \right] = D.$$

Let $T(x, y, z) = (x + y + z, 2y - z, 3z)$. Is there a basis $\mathcal{B}$ of $\mathbb{R}^3$ such that the matrix of T relative to $\mathcal{B}$ is diagonal ?