The Dundas–Goodwillie–McCarthy theorem

$A = A_1 \cdots A_{dg}$

V A-bimodule N,

$A \mapsto$ split sq-zero extension $A \oplus N$ $\to k$

G-gives

$F: A\text{-bimod} \to Sp$.

not exact-like ($F(0) \neq 0$)

Goodwillie derivative stabilizes

& produce a functor

commutes w/ all colimits.

$\partial_1 K$: stable K-thy

$= \mathrm{ddt}$ of $THH(A, -)$

$A\text{-bimod} \to Sp$

stable TC:

& cyclotomic trace induces

$\partial_1 K \xrightarrow{\sim} \partial_1 TC$

structural properties.

K & TC commutative of certain colimits & is homotopical

bounded below.

Iterate: prove above the

filtration tr_n $K \to TC$

"stable"

K: Schur or calculus

expansion

soft argument that

$\partial_1 K = THH$

TC: almost entirely normal

an explicit formula for TC

in the sq-zero extension case

Ω-cat

$= IV$-functors

$\mathrm{Hom}_C(F, G)$. $F, G \in C \subset \mathrm{Cat}_{\infty}$

$\mathrm{Hom}_C(C, D)$. category

the then we indicate the functor category

Cat_{∞} is $(\infty,2)$-Cat instead of

nerve & groupoid

$F[1] = \Sigma F$.

$F[-1] = \Omega F$

kernel = fibers

SiCat_{cont}: category of

cocomplete stable cats

continuous exact functors

& commutes of filtered colimits)

Lurie 4.4.

Has a sym mon structure ⊗

property:

$C \otimes D \to E$, functor

$\Updownarrow$ equivalent to

functor $C \times D \to E$.

commutes w/ colimits in

each variable separately

Canonical (Non-exact) functor

$C \times D \to C \otimes D$

by: $(F, G) \mapsto F \boxtimes G \in C \otimes D$

$F \in C$, $G \in D$

$Sp \in \mathrm{SiCat}_{cont}$, unit in the

sym mon structure

$\otimes$: $Sp \to Sp$.

$\Sigma^{\infty}: Spl \rightleftarrows Sp : \Omega^{\infty}$

C: stable cat.

$F, G \in C$

$\underline{\mathrm{Hom}}_C(F, G)$: spectrum (negative) Ban Fin G.

$\mathrm{Hom}_C(F, G)$. Spl of maps

$\mathrm{Hom}_C(F, G) = \Omega^{\infty} \underline{\mathrm{Hom}}_C(F, G)$

$C \in \mathrm{SiCat}_{cont}$ is dualizable

if its dualizable resp. $\otimes$.

dual cat functor $C^{\vee}$

$C^{\vee} \otimes D \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{SiCat}_{cont}}(C, D)$

evaluation map:

tr: $\mathrm{End}_{\mathrm{SiCat}_{cont}}(C)$

$C^{\vee} \otimes C \xrightarrow{ev} Sp \in \mathrm{SiCat}_{cont}$

trace

For $F \in C$ a compact object

$\mathrm{Hom}_C(F, -): C \to Sp$

commutes of colimits,

$\Rightarrow$ defines an obj of $C^{\vee}$.

denote this by DF

Compact-

C compactly gen,

& dualizable

$\mathrm{Indcp}(C^{op}) = C^{\vee}$.

$C^{op} \to C^{\vee}$

$F \mapsto DF$.

$(\mathrm{Ind}\, C)^{\vee} = A^{op}\text{-mod}$.

$\mathrm{End}_{\mathrm{SiCat}_{cont}}(A\text{-mod})$

$= A\text{-bimod}$

Sifted colimit:

cat I is sifted if

I non empty &

$I \to I \times I$ cofinal.

A functor commutes w/ sifted

colimits $\longleftrightarrow$ continuous (comm w/ filtered)

and commutes of geometric realization

Commutes w/ all colimits

$\Leftrightarrow$ commutes w/ sifted colimits

& finite coproducts

Top:

C ∞-category. commutes w/

$C \to C^{\otimes n}$ sifted colimits

$f \mapsto f^{\otimes n}$ but not general colimits

Goodwillie derivative

Φ stable cats

& comm of sifted colimits

$\Psi: C \to D$.

between cocomplete stable

the Goodwillie derivative

$\partial \Psi: C \to D$ of Ψ is

the initial category continuous exact

functors under Ψ

(receiving a natural transf

$\Psi \to \Psi$)

By Lurie: prop 1.4.2.13

$\partial \Psi$?

$0 \to E \to 0$

$\Rightarrow \Psi(E) = \Psi_{red}(E) \oplus \Psi(0)$

reduced

$0 \mapsto 0$

$A \to$

$\downarrow \quad \downarrow$

$0 \to \Sigma A$

$\Sigma \circ \Psi_{red} \Rightarrow \Psi_{red} \circ \Sigma$

$\Psi_{red}(0) \to 0$

$\downarrow \quad \downarrow$

$0 \to \Sigma \Psi_{red}(A) \to \Psi_{red}(\Sigma A)$

$\Rightarrow \Psi_{red} \Rightarrow \Omega \Psi_{red} \Sigma$.

$\partial \Psi = \mathrm{colim}(\Psi_{red} \to \Omega \Psi_{red} \Sigma \to \Omega^2 \Psi_{red} \Sigma^2 \to \cdots)$

Variant: Suppose C is equipped of a tensor.

compatible w/ filtered colimits and

$\Psi: C^{\leq 0} \to D$ comm of sifted colimits

$\cdots \partial \Psi: C \to D$ initial.

& $\Psi \mapsto \partial \Psi|_{C^{\leq 0}}$.

$\partial \Psi(F) = \mathrm{colim}_m \mathrm{colim}_{n \geq m} \Omega^n \Psi_{red}(\Sigma^m C^{\leq m} F)$

If t-str C right complete

& $\Psi: C \to D$ comm of sifted colimits,

then $\partial \Psi$ in this sense. coincides w/ $\partial(\Psi|_{C^{\leq 0}})$

Alg: cat of $\mathbb{E}_1$-alg.

Alg_{comm}: cat of commutative $\mathbb{E}_1$-alg

$A \in \mathrm{Alg}$. $M \in A\text{-bimod}$.

split sq-zero extension.

$\mathrm{Sqzero}(A, M) \in \mathrm{Alg}$.

underlying is $A \oplus M$

mult: $(a, m) \cdot (b, n)$

$= (ab, a \cdot n + m \cdot b)$

2.5.1

$\Psi: \mathrm{Alg}_{comm} \to Sp$.

$\forall A \in \mathrm{Alg}_{comm}$.

$\Psi_A: A\text{-bimod}^{\leq 0} \to Sp$

$M \mapsto \Psi(\mathrm{Sqzero}(A, M))$

comm of sifted colimits and

has vanishing Goodwillie derivative.

Suppose more about.

$\Psi_{A,red}(A\text{-bimod}^{\leq 0}) \subseteq Sp^{\leq 0}$

Then $\forall A \in \mathrm{Alg}_{comm}$, $M \in A\text{-bimod}^{\leq -1}$

$\Psi(\mathrm{Sqzero}(A, M)) \to \Psi(A)$ is iso

2.6.1 Categorical version (implying 2.5.1)

C, D: cocomplete stable cats

w/ t-str. compatible of filtered colimits

and suppose the t-str on D is left separated.

$\Psi: C^{\leq 0} \to D^{\leq 0}$ be a reduced

functor. comm of sifted colimits,

If $\partial(\Psi(F \oplus -)) = 0$ $\forall F \in C^{\leq -1}$

then $\Psi(F) = 0$ $\forall F \in C^{\leq -1}$

Proof of 2.6.1 $\Rightarrow$ 2.5.1: (in the input of 2.6.1)

Fix $A \in \mathrm{Alg}_{comm}$. define $\Psi: A\text{-bimod}^{\leq 0} \to Sp^{\leq 0}$

$M \mapsto \Psi_{A,red}$

To use 2.6.1, we need (in the input of 2.5.1)

to show that

$\partial(\Psi(M \oplus -)) = 0$ for $M \in A\text{-bimod}^{\leq -1}$ (currently ✓ for $M \in A\text{-bimod}^{\leq 0}$)

For $M = 0$ ✓ (by assumptions in 2.5.1) the input of

In general

$\Psi(M \oplus -) \oplus \Psi(-) = \Psi(\mathrm{Sqzero}(A, M \oplus -)) = \Psi(\mathrm{Sqzero}(\mathrm{Sqzero}(A, M), -))$

$\Psi_{A,red}(C) = \Psi_{A,red}(C)$

$\Psi_A(C) = \Psi_{A,red}(C) + \Psi_A(0)$

$\Psi_A(0) = \Psi(\mathrm{Sqzero}(A, 0)) = \Psi(A)$

$A \oplus (M \oplus N) = (A \oplus M) \oplus N$

(a, m, n)
(b, m', n')

$= (ab, a \cdot n' + b \cdot n + m \cdot m')$

$m \cdot m' \to 0$

$\Psi(\mathrm{Sqzero}(A, M))$

By hypothesis of 2.5.1 has vanishing derivative Goodwillie

2.6.1 is cheap:

it replace $C^{\leq 1}$ w/ $C^{\leq 0}$.

Ψ may not be 0:

take $\Psi: \mathrm{Sym}^{perf}(-): \mathrm{Sp}_{perf}^{\leq 0} \to \mathrm{Sp}_{perf}^{\leq 0}$

V^{perf}. colim V

fin.

V: Sp. perf

$\mathrm{Sym}^{perf}(-)$ vanishes in $\mathrm{Sp}_{perf}^{\leq -1}$

rec ∂ vanishes.

$\mathrm{Sym}^{perf}(V \oplus w)$

$= -V \oplus -W$.

for $V \in \mathrm{Sp}_{perf}^{\leq 0}$, $W \mapsto \mathrm{Sym}^{perf}(V \oplus w)$

also have ∂ vanishes

Bilinear obstruction co-linearity

Notations:

$\mathrm{Lin}, \mathrm{Cat}$.

note that Ψ commutes w/ all colimits iff it commutes w/ pairwise direct sums

Because Ψ is reduced and sifted

By: obstruction to Ψ commuting w/ all colimits

$F, G \in C^{\leq 0}$

$B_\Psi(F, G) = \mathrm{Cofiber}(\Psi(F) \oplus \Psi(G) \to \Psi(F \oplus G)) \in D$

$\Psi(F \oplus G) = \Psi(F) \oplus \Psi(G) \oplus B_\Psi(F, G)$

because

$\Psi(F) \oplus \Psi(G) \to \Psi(F \oplus G) \to \Psi(F \times G) \to \Psi(F) \oplus \Psi(G)$

is the identity

(this map splits)

Ψ commutes w/ colimits iff $B_\Psi = 0$

Simplicial version

prove 2.6.1 using some standard simplicial methods

$F_\bullet$: simplicial obj.

$|F_\bullet|$: geometric realization colim)

$|F_\bullet|_{\leq n}$ partial geom realization.

$\mathrm{colim}_{\Delta^{op}_{\leq n}} F_\bullet$ $\Delta_{\leq n} \subseteq \Delta$ supplies order sn.

c.f. the proof of Dold-Kan thm:

Lemma 2.8.1. for $n \geq 0$. $\mathrm{Cofiber}(|F_\bullet|_{\leq n} \to |F_\bullet|_{\leq n+1})$

is a direct summand of $F_{n+1}[n+1]$

proof:

$\Delta^{op}_{\leq n} \to \Delta^{op}_{\leq n+1}$

$F_{\leq n}$ $\downarrow$ $\nearrow$ Lan $F_{\leq n}$

C

by HA Rmk 1.2.4.3

$(\mathrm{Lan}\, F_{\leq n})(n+1) =$ $n+1$-th homo object of F

$(\mathrm{Lan}\, F_{\leq n})(i) \xrightarrow{\sim} F(i)$ $i \leq n$

main tech output

Lemma 2.9.1

Suppose $\Psi: C^{\leq 0} \to D$ comm of sifted colimits

For any $F \in C$. $\Psi(\Sigma F)$ admits an increasing

exhaustive filtration $\mathrm{fil}_\bullet \Psi(\Sigma F)$

s.t.

1. $\mathrm{fil}_i \Psi(\Sigma F) = 0$ for $i < 0$

2. for $i \geq 0$ $\mathrm{gr}_i \Psi(\Sigma F)$ is a direct summand of $\Psi(F^{\otimes i})[i]$

3. $\mathrm{gr}_0 \Psi(\Sigma F) = \Psi(0)$

$\mathrm{gr}_1 \Psi(\Sigma F) = \Psi_{red}(F)[1]$

$\mathrm{gr}_2 \Psi(\Sigma F) = B_{\Psi_{red}}(F, F)[2]$

proof: canonical simplicial diagram

$\cdots \rightrightarrows F \oplus F \rightrightarrows F \to 0$

with geometric realization ΣF.

Ψ comm w/ geom real. we have

$\Psi(\Sigma F) = |\Psi(F^{\otimes \bullet})|$

set $\mathrm{fil}_i \Psi(\Sigma F) := |\Psi(F^{\otimes \bullet})|_{\leq i}$

1. tautologically ✓

2. by lemma 2.8.1

3. almost by def

In the terminology of Goodwillie calculus

we generally have

$\mathrm{gr}_i \Psi(\Sigma F) = \mathrm{cr}_i \Psi(F, \ldots, F)[i]$

no exploit need for higher B_Ψ is $\mathrm{cr}_2 \Psi$ cross products

$\mathrm{cr}_i \Psi$ is the i-th cross product of Ψ

As a first consequence, we obtain:

Corollary 2.9.2.

suppose $\Psi: C^{\leq 0} \to D^{\leq 0}$ is reduced and comm of sifted colimits,

then

$\forall i \geq 0$. $\Psi(C^{\leq -n}) \subseteq D^{\leq -n}$

by induction. suffices to show

$\Psi(C^{\leq -1}) \subseteq D^{\leq -1}$.

Suppose $F \in C^{\leq 0}$. goal: $\Psi(\Sigma F) \in D^{\leq -1}$

we use the 2c suffices to show

filtration

$\mathrm{gr}_0 \Psi(\Sigma F) = 0$

since Ψ is reduced

and for $i > 0$. $\mathrm{gr}_i \Psi(\Sigma F)$

is a summand of $\Psi(F^{\otimes i})[i]$

clearly in degrees $\leq -i < 0$

We now prove theorem 2.6.1

proof of 2.6.1.

We will show by induction on n that these hypotheses in Ψ force

$\Psi(C^{\leq -1}) \subseteq D^{\leq -n}$ The ind hypo is [hypotheses in 2.6.1 $\Rightarrow \Psi(C^{\leq -1}) \subseteq D^{\leq -n}$]

$n = 1$ ✓ by 2.9.3

assume inductive hypothesis for n and deduce it for $n+1$

Step 1. First. for $F \in C^{\leq -1}$

$B_\Psi(F, -)[-1]: C^{\leq 0} \to D$

satisfy the hypotheses of 2.6.1

Clearly it reduced and comm w/ sifted colimits

(equal at 0) (since Ψ is)

Let us show this functor maps $C^{\leq 0}$ to $D^{\leq 0}$

fix $G \in C^{\leq 0}$ $B_\Psi(-, G)$ reduced comm of sifted colimits and maps $C^{\leq 0}$ to $D^{\leq 0}$.

Note that for any $F' \in C^{\leq -1}$.

$\Omega B_\Psi(F, F' \oplus -): C^{\leq 0} \to D$

has vanishing Goodwillie derivative

(i.e. is a summand of $\Psi(F \oplus F' \oplus -)[-1]$)

hypothesis of 2.6.1

suppose that $\partial(\Psi(F \oplus -)) = 0$

for any $F \in C^{\leq -1}$

Ψ here $= B_\Psi(F, -)[-1]$

By 2.9.3. $B_\Psi(F, G) \in D^{\leq -1}$.

$\overset{?}{C^{\leq -1}}$

$B_\Psi(F, G)[-1] \in D^{\leq 0}$

In particular $\Rightarrow B_\Psi(F, -)[-1] \in D^{\leq -n}$

$B_\Psi(F, F) \in D^{\leq -n-1} \Rightarrow B_\Psi(F, -) \in D^{\leq -n-1}$

Step 2: we claim that

$\mathrm{Cofiber}(\Sigma \Psi(F) \to \Psi(\Sigma F)) \in D^{\leq -n-3}$

by construction of 2.9.1. (fil with gr cross product)

$\Sigma \Psi(F) = \mathrm{gr}_1 \Psi(\Sigma F) = \mathrm{fil}_1 \Psi(\Sigma F) \to \Psi(\Sigma F)$

(where Ψ_i is Ψ_{red}) is the canonical map used

in the definition of Goodwillie derivative

Note: the fil is increasing:

$\mathrm{fil}_0 = 0 \to \mathrm{fil}_1 \Psi(\Sigma F) \to \mathrm{fil}_2 \Psi(\Sigma F) \to \cdots \to \Psi(\Sigma F)$

gr_1 gr_2 gr_3

$\mathrm{gr}_2 \to \Psi \to \mathrm{gr}_3$

Σ^{n-3} Σ^{n-3} Σ^{n-3}

It suffices to show

that $\mathrm{gr}_i \Psi(\Sigma F) \in D^{\leq -n-3}$ for $i \geq 2$

$i \geq 3$: $\Psi(F^i) \in D^{\leq -n}$ by induction hyp

$\mathrm{gr}_i \Psi(\Sigma F) \in D^{\leq -n-i}$

$i = 2$:

$\mathrm{gr}_2 \Psi(\Sigma F)$

$= B_\Psi(F, F)[2]$ by step 1.

$\in D^{\leq -n-1}$

Step 3: step 2 implies

that for $F \in C^{\leq -1}$.

$\Psi(F) \to \Omega \Psi(\Sigma F)$

is iso on H^{-n}

Finally, $\forall m \geq 0$.

$\Omega^m \Psi \Sigma^m: C^{\leq 0} \to D^{\leq 0}$

satisfy by hypo

so $\Omega^m \Psi(\Sigma^m F) \to \Omega^{m+1} \Psi(\Sigma^{m+1} F)$

is iso on H^{-n}.

Finally we obtain $H^{-n}(\Psi(F)) \xrightarrow{\sim} H^{-n}(\partial \Psi(F))$ is an iso

$\partial \Psi = 0 \Rightarrow H^{-n} \Psi(F) = 0$

$\Rightarrow \Psi(F) \in D^{\leq -n-1}$ proving the inductive step.

Lemma 2.11.4:

extensible $\Rightarrow$ pseudo extensible

for $\Psi: C^{\leq 0} \to D^{\leq 0}$

proof: any $\varphi \in S_\Psi$ is extensive

(since Ψ is and S_Ψ is groupoid)

reduced to show that

for $F \in C^{\leq 0}$. $B_\Psi(F, -)[-1]$

maps $C^{\leq 0}$ to $D^{\leq 0}$

Let $\tilde{\Psi}$ be is the def of extensibility.

$B_{\tilde{\Psi}}$ maps $C^{\leq 0} \times C^{\leq 0}$ to $D^{\leq 1}$.

apply 2.9.3 on each variable.

$B_{\tilde{\Psi}}$ maps $C^{\leq 0} \times C^{\leq 0}$ to $D^{\leq -1}$.

coincides w/ B_Ψ, so $B_\Psi(F, -)[-1] \in D^{\leq 0}$

Flexibility, why we consider pseudo extensible functors

Lemma 2.11.5 in the setting of 2.11.4. suppose

the t-str on D is left complete.

Let J be a filtered cat. and given a diagram

$J^{op} \xrightarrow{i \mapsto \Psi_i} \mathrm{Hom}(C^{\leq 0}, D^{\leq 0})$ of pseudo extensive functors

suppose $\forall n$, $\exists i \in J$ s.t.

$\tau^{\geq -n} \Psi_j \xrightarrow{\sim} \tau^{\geq -n} \Psi_i$

$\forall j \to i \in J$

Then $\Psi = \lim_{i \in J^{op}} \Psi_i$ is pseudo extensive.

the value-wise limit

proof: For $F, G \in C^{\leq 0}$ we have

$B_\Psi(F, G) = \lim_i B_{\Psi_i}(F, G)$ (finite colim comm of filtered lim)

Reduces to show that $B_\Psi(F, G) \in D^{\leq -1}$

Left complete & truncation

Similarly, we have $\tau^{\geq -n} \lim_i B_{\Psi_i}(F, G)$

$= \tau^{\geq -n} B_{\Psi_i}(F, G)$

for i sufficiently large. $\Rightarrow$ ✓ for Ψ

so clearly $B_\Psi(F, G) \in D^{\leq -1}$

$B_{\Psi_i}(F, -)[-1] \in D^{\leq 0}$

$\Rightarrow \tau^{\geq -n} \Psi$ reduced

& comm of sifted colimits $\forall n$

$\Rightarrow \tau^{\geq -n} \Psi = \tau^{\leq -n} \Psi_i$

for sufficiently large $i \in J$ (depends on n)

We now have

Theorem 2.11.6 in setting of 2.6.1 Suppose

Ψ is pseudo extensible and $\partial(\Psi(F \oplus -)) = 0$

$\forall F \in C^{\leq 0}$.

then Ψ maps $C^{\leq 0}$ to $\bigcap D^{\leq -n}$

proof. First, we claim $\Psi(F) \in D^{\leq -1}$ $\forall F \in C^{\leq 0}$

Note. suffices to show $\mathrm{gr}_i \Psi(\Sigma F) \in D^{\leq -3}$

for Ψ is Sym pseudo-ext ✓ for $i \geq 3$.

by induction on ∂ of $\Psi(F \oplus -)$

for $i = 2$. $\mathrm{gr}_2 \Psi(\Sigma F) = B_\Psi(F, F)[2]$.

and $B_\Psi(F, F) \in D^{\leq -1}$

vanish for any $F \in C^{\leq 0}$. by the above argument. $\varphi \in S_\Psi$ maps $C^{\leq 0}$ to $D^{\leq -1}$

Finally. $\Psi(-)$ is pseudo-extensible

by induction. suffice to 2.6.1.

Corollary 2.11.7 (in the setting of 2.5.1)

suppose that $\Psi_{A,red}: A\text{-bimod}^{\leq 0} \to Sp$

the functors are pseudo-extensible.

then $\Psi(A \oplus M) \xrightarrow{\sim} \Psi(A)$ for any $M \in A\text{-bimod}^{\leq 0}$

Ψ_A: $M \mapsto \Psi(\mathrm{Sqzero}(A, M))$

$\Psi_{A,red}$ vanish $\Rightarrow$ $\Psi_A(M) \simeq \Psi_{A,red}(M) \oplus \Psi_A(0)$

$\simeq \Psi_A(0)$

the split sq-zero case of main thm:

sqzero case for K: 1. $A \in \mathrm{Alg}_{comm}$ $M \mapsto K(\mathrm{Sqzero}(A, M))$

$A\text{-bimod}^{\leq 0} \to Sp^{\leq 0}$

2.12.1 comm of sifted colimits, the underlying reduced functor is cofinitely

2. for $A \in \mathrm{Alg}_{comm}$, the Goodwillie derivative:

$A\text{-bimod}^{\leq 0} \to Sp$

$M \mapsto K(A \oplus M)$

In canonically isomorphic to

$M \mapsto THH(A, M)[1]$

sqzero case for TC: 1. for $A \in \mathrm{Alg}_{comm}$.

$A\text{-bimod}^{\leq 0} \to Sp$

2.12.2 $M \mapsto TC(\mathrm{Sqzero}(A, M))$

$M \in A\text{-bimod}^{\leq 0}$ is pseudo-extensible

2. For $TC_{red}(\mathrm{Sqzero}(A, M)) := \mathrm{Ker}(TC(\mathrm{Sqzero}(A, M)) \to TC(A)) \in Sp^{\leq -1}$

3. the Goodwillie derivative of $TC_{red}(\mathrm{Sqzero}(A, -))$ canonically isomorphic

to $THH(A, -)[1]$ & compatible w/ cyclotomic trace & the isom

$\partial(K(A \oplus -)) \xrightarrow{\sim} THH(A, -)[1]$

$\Rightarrow$ K denotes connective K-thy

Define the functor

$\Psi^{Dun}: \mathrm{Alg}_{comm} \to Sp$

$A \mapsto \mathrm{Cofiber}(K(A) \xrightarrow{tr} TC(A))$

Assume 2.12.1 & 2.12.2 in sufficient show Ψ^{Dun} satisfies

the hypothesis of 2.5.1 & 2.4.7 (part automatic)

$\Psi^{Dun}_{red}(\mathrm{Sqzero}(A, -)): A\text{-bimod}^{\leq 0} \to Sp$ is pseudo extensible

by 2.12.1 & 2.12.2 (some pseudo extensibility is preserved under cofib)

maps into $Sp^{\leq 0}$ by 2.12.2 2.

and ∂ vanishes by 2.12.1 & 2.12.2 3.