Finiteness and Ambidextrosity in Korea Local Stable Homotopy Theory

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Def (Atiyah)

An n-dimensional topological quantum field theory is a tensor functor

$Z : \left\{ \begin{matrix} (n+1)\text{-dim cobordisms} \\ \text{bordisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{G-vector spaces} \\ \text{linear maps} \end{matrix} \right\}$

no boundary, say, oriented

$Z(M^{n-1}) \leftarrow$ G-vect space

$Z(B) : Z(M) \to Z(N)$

Great given a section

def

not allowed to $\frac{1}{|G|}$

$Z(M \sqcup N) \simeq Z(M) \otimes_{\mathbb{C}} Z(N)$

$Z(\emptyset) \simeq \mathbb{C}$

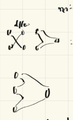
Ex. $n=1$

$Z(S^1) \overset{\text{def}}{=} A$

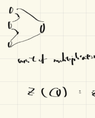
$Z\left(\begin{matrix} \circ \\ \circ \end{matrix}\right) : Z\left(\begin{matrix} \circ \\ \circ \end{matrix}\right) \to Z(\circ)$

$\cong$ $\cong$

$\mathbb{C}$ A



: m is commutative



unit of multiplication

$Z(\bigcirc) : Z(\bullet) \to Z(S^1)$

$\mathbb{C} \to A$

$Z(\bigcirc) : Z(S^1) \to Z(\bullet)$

$A \to \mathbb{C}$

$A \otimes A \xrightarrow{m} A \xrightarrow{\varepsilon} \mathbb{C}$

is an homomorphism.

2D TQFT - determines commutative Frobenius algebra

(Todd Trimble and others)

$Z\left(\begin{matrix} \circ \\ \circ \end{matrix}\right)$

$n^* : A^* \to A^* \otimes A^*$

$Z(n^*) : \mathbb{C} \to \mathbb{C}$

$\frac{1}{\det}$

$Z(n^*) \in \mathbb{C}$

manifold invariant

$Z(\bigcirc) = Z(\bullet) \circ Z(\bigcirc)$

$Z(\bigcirc) = \varepsilon_A(1)$

$Z(\bigcirc) = Z(\bigcirc) \circ (\bigcirc)$

$\mathbb{C} \to A \otimes A \to \mathbb{C}$

$\mathbb{C} \to A \otimes A^* \to \mathbb{C}$

$1 \mapsto \varepsilon_A(1)$ (contraction)

$= \dim_{\mathbb{C}}(A)$

diagrammatic written dig.

G - finite group

G bundles on

$X = \tilde{X} \to X$

if acyclic

free

G-bundles on $G \curvearrowright \tilde{X}$

$X / \cong$ $\quad \tilde{X}/G \cong X$

$\cong$ $X \to BG$ / hpy

$\pi_i(BG) \cong \begin{cases} G & i=1 \\ 0 & i \neq 1 \end{cases}$

$Z(M^n) = \frac{\#\{\text{homomorphisms } \pi_1(M) \to G\}}{|G|}$

M connected.

M compact

$Z(M^n) = \#$ of G bundles on M counted with mass

G bundles of X / iso $= \frac{\text{homomorphisms } \pi_1 M \to G}{\text{conjugacies}}$

$\#$ G bundles of X / iso, counted with mass $= \sum_{\text{G bundles } \tilde{X} \to X} \frac{1}{|\mathrm{Aut}(\tilde{X})|}$

$= \sum_{\text{conjugacy classes } \alpha : \pi_1 M \to G} \frac{1}{|\text{centralizer of } \alpha|}$

G-vect space of dim $= \#$ G bundles of M / iso

$Z(M) =$ [only constant functions on $\mathrm{Map}(M, BG)$]

space of G bundles on M

$\sum_{\alpha : \pi_1 M \to G} \frac{1}{|\text{centralizer of } \alpha|} \frac{1}{\#\text{ conjugacies in } \alpha}$

$\frac{1}{|G|}$

$\frac{1}{|G|}$ $\mathrm{Aut}(\tilde{X})$

isom class of G-bundle $\tilde{X} \to M$

$\mathrm{Map}(M, BG)$

$\mathrm{Map}(M, BG)$ $\quad$ $\mathrm{Map}(M, BG)$

$Z(M)$

$\alpha : \mathrm{Map}(M, BG) \to \mathbb{C}$

$Z(B)(\alpha) \in Z(N)$

$Z(B)(\alpha)(x) = \sum_{\substack{\text{connected} \\ \text{components} \\ \alpha(x)}} \frac{\alpha(\varphi(C))}{|\pi_1 C|}$

Ex. $M = N = \varphi$

$Z(M) = \mathbb{C}$ $\quad B = B^n$

$Z(N) = \mathbb{C}$

$\mathrm{Map}(B, BG)$

$Z(S^1) = \mathrm{Map}(S^1, BG) \to G$

space of class functions on G

$= \mathrm{Rep}(G) \otimes \mathbb{C}$

In fields char S^1 / iso $\cong$ classes of / conjugation.

Axiomatic

like organization principle.

information equate exactly in 2 different ways.

$Z\left(\begin{matrix} \bigcirc \\ \bigcirc \end{matrix}\right) = \frac{\#\text{ homomorphisms } \pi_1 T \to G}{|G|}$

$\#$ pairs of commuting elements $\in G$ / $|G|$ $= Z(\bigcirc) = \#$ of conjugacy classes in G

Formulas to play with. diagram with $\frac{1}{\#}$

along:

finite group G

(replace BG by other space?)

G $\Rightarrow$ some other ring

"finite" $\Rightarrow$ finite group

$\mathrm{Vol} \to$ inv for fin group

$\mathrm{Trace} \to$ inv for mfd

Great given a section

def

not allowed to $\frac{1}{|G|}$

D-W strategy

Trace group G

$\eta \in H^n(BG, \mathbb{C}^\times)$

$Z(M) = \sum_{d : M \to BG} \frac{1}{|\mathrm{Aut}(d)|} (d^*\eta)[M]$

up to homotopy

twisted case

unoriented: indep of orientation

count # of G bundles

$\#(\mathrm{Map}(M, BG))$

combinatorial

X introduced

Def 1:

$\# X^n = |\pi_n X|$

Def 2:

$\# X^n = \sum_{c \in \pi_0 X} \frac{1}{|\pi_1 c|}$

$\tilde{X} \to X$

covering space

(they have cardinality n.

Hope $\# \tilde{X}^n = \# X^n \cdot n$.

Ex. $X = BG$ $\quad \tilde{X} = EG$

$\# \tilde{X} = 1$

$\# X = \frac{1}{|G|}$

homotopy invariant.

$\boxed{\text{Part 2}}$

Comm Frob algebra

$\mathrm{tr} \in A^\vee$

$A \xrightarrow{\sim} A^\vee$

A trivially

Grothendieck ring of cat o

$G[x_1, x_2, \dots, x_n] / f_1, f_2, \dots, f_n$

if finite rep G-vector spaces

when Frobenius only

$\mathrm{tr}(\rho) = \mathrm{Res}\left(\frac{\det(x_i - A\rho)}{f_1 f_2 \cdots f_n} \right)$

Symmetry cube dos

X : top space.

C : category

Def: A local system on X w/ values

in C is

1. $\forall$ point $x \in X$ an obj $\mathcal{L}_x \in C$

2. $\forall$ path $p : [0,1] \to X$ an iso $\mathcal{L}_p : \mathcal{L}_{p(0)} \xrightarrow{\sim} \mathcal{L}_{p(1)}$

| If X is mfd. vector bundle with connection

3. $\forall$ 2-simplex σ:

$\mathcal{L}_r = \mathcal{L}_q \circ \mathcal{L}_p$

| If X is mfd. vector bundle with fiber connection

Equivalently A local sys on X is functor

$\mathcal{L} : \pi_{\leq 1} X \to C$

Concretely. If X is connected w/ base point $x \in X$

Local systems on X

$\Updownarrow$

objects of C w/ action of $\pi_1(X, x)$

C := cat of f-dim G-vector spaces

1. Assign $\mathbb{C}$ to every point
2. Assign $\mathbb{C}^\times$ to every path
3. $\mathcal{L}_p$ ..

} 1-complex in X w/ edges in $\mathbb{C}$

local sys w/ triintersection $\leftarrow$ 1-complex

If M is a sub group acted by finite group H

$M_H \xrightarrow{\mathrm{Nm}} M^H$

is not always iso

often has no inverse given by

$x \mapsto \frac{x}{|H|}$

if $|H| : M \to M$ is an iso

Question. When can we say that the homology (deg 0) of a local system is iso to its cohomology (deg 0)

C is any category w/ limits & colimits

For any space X, the mod orientation collection of local sys on X forms a cat C^X

$f : X \to Y$

determines a functor

$f^* : C^Y \to C^X$

$(f^*\mathcal{L})_x = \mathcal{L}_{f(x)}$

f^* has a left adjoint

$f_! : C^X \to C^Y$

& right adjoint

$f_* : C^X \to C^Y$

If f is a fibration

$(f_!\mathcal{L})_y = H_0(f^{-1}(y), \mathcal{L}_{f^{-1}(y)})$

$(f_*\mathcal{L})_y = H^0(f^{-1}(y), \mathcal{L}_{f^{-1}(y)})$

Ex: If $Y = *$

$C^Y \simeq C$

$f : X \to Y$.

$f_*\mathcal{L} = H^0(Y, \mathcal{L})$

$f_!\mathcal{L} = H_0(Y, \mathcal{L})$

When can we say that

$f_! \simeq f_*$

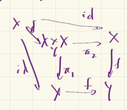
for a map $f : X \to Y$

Attempt to define an iso

$f_! \xrightarrow{\sim} f_*$

W/Otto homotopy arrange.

f is a fibration.



Assume $f_! \simeq f_*$

Analogous to the norm, not necessarily iso

correspondence to invariance homology to cohomology

We will construct a natural transformation

$\mathrm{Nm} : f_! \to f_*$

$\mathrm{Hom}(f_!\mathcal{L}, f_*\mathcal{L}) \simeq \mathrm{Hom}(\mathcal{L}, f^*f_*\mathcal{L})$ norm map: $f_! \xrightarrow{\exists!} f_*$

$\mathrm{id}_{C^X} = \mathrm{id}_* \mathrm{id}^* = \pi_{2*}\delta_! \delta^* \pi_1^* \simeq \pi_{2*}\delta_!\delta^*\pi_1^*$

$\downarrow$

$\pi_{2*} \circ \mathrm{id} \circ \pi_1^*$

$\wr\wr$

f^*f_*

Def: f is antidextrous if

1. $\delta : X \to X \times_Y X$ is antidextrous
2. $\mathrm{Nm} : f_! \to f_*$ is an isomorphism

coinductive def.

smallest collection of maps

contains all homotopy equivalences

and have the closure property

opposite

what condition on C do you need to make f antidextrous

Part 3

Let $f : X \to Y$ be a fibration

suppose given $d : f_* \to f_!$

construct norm map:

$\mathrm{Nm} : f_! \simeq f_! \mathrm{id}_* \mathrm{id}^* \simeq f_! \pi_{2*}\delta_!\delta^*\pi_1^* \xrightarrow{d} f_!\pi_{2*}\delta_!\delta^*\pi_1^*$

$\downarrow$ commutativity

$f_!\pi_{2*}\pi_1^*$

$\wr\wr$

$f_! f^* f_*$

$\downarrow$ commutativity

f_*

Def: we define a class of maps

$f : X \to Y$ called antidextrous maps

& $\forall$ antidextrous map f

a norm map $\mathrm{Nm} : f_! \xrightarrow{\sim} f_*$

1. homotopy equiv are antidextrous Nm is the identity map.
2. If $f : X \to Y$ is a fibration and δ is antidextrous

define $\mathrm{Nm} : f_! \to f_*$ using $d = \mathrm{Nm}^{-1}$

f is antidextrous if Nm_f is an iso

Per almost all purpose suffices an entire Yoga.

(fiberwise constructions)

Def We say X is antidextrous if $X \to *$ is antidextrous.

(For any local sys $\mathcal{L}$ on X.

get an iso $H_0(X; \mathcal{L}) \xrightarrow{\sim} H^0(X; \mathcal{L})$

relating

$X \xrightarrow{f} Y$ fibration

as each fiber will be an antidextrous space.

converse $\rightsquigarrow$ almost true

if $\checkmark$ and some upper bound on homotopy groups then the map is...

If the antidextrous

$\Downarrow$

C is pointed

(C has zero object)

Integration rule.

$f : \mathcal{G} \to \mathrm{Hom}(C, D)$

get $\int_{\mathcal{G}} f \, d\mu = 0 \in \mathrm{Hom}(C, D)$

$C \xrightarrow{\exists!} 0 \xrightarrow{\exists!} D$

$f_*\mathcal{L} = \prod_{y \in Y} \mathcal{L}_y$

$f_!\mathcal{L} = \coprod_{x \in X} \mathcal{L}_x$ $\quad \mathrm{Nm}$

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ & & 1 & 1 \\ & & 0 & 1 \end{bmatrix}$

X is discrete $\Rightarrow$ f is antidextrous.

$C^X = \prod_{x \in X} C$

$f_! f_* : C^X \to C$

X is antidextrous $\Leftrightarrow$ the identity $X \times X$ matrix is an iso

if C is Imp.

$\#\{*\}$ is not antidextrous

if $|X| = \infty$. All.

$\bigoplus \mathcal{L}_x \xrightarrow{\mathrm{Nm}} \prod \mathcal{L}_x$

not iso usually

Assume C has the property that finite sets are antidextrous

then C is "semiadditive"

$\mathrm{Hom}_C(C, D)$

$\Updownarrow$

commutative monoid

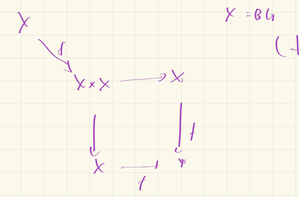
$C \to \prod_{x \in X} C \to \prod_{y \in X} D \to D$

$C \to \begin{matrix} C \\ \oplus \\ C \end{matrix} \to \begin{matrix} D \\ \times \\ D \end{matrix} \to D$

addition on Hom sets

For example

f is antidextrous if $X = BG$



(fiber is finite covering map)

$E(G \times G)/G$

$\downarrow$

$E(G \times G)/G \times G$

C^{BG} cat of obj / C w/ an action of G

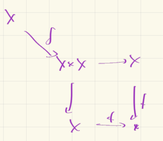
Assume BG is antidextrous (G of C)

Say BG is antidextrous G abelian

$\Downarrow$

$X = K(G, 2)$

f is antidextrous $\pi_i X = \begin{cases} G & i = 2 \\ 0 & i \neq 2 \end{cases}$



fibers of f looks like copies of BG

$\mathcal{L}$ a local sys on $X = K(G, 2)$

$\mathrm{Nm}_{\mathcal{L}} : H_0(\mathcal{L}) \to H^0(\mathcal{L})$

not interesting

multiplication by $\frac{1}{|G|}$

slightly interesting

C : $(\infty, 1)$-category

$(\infty, 1)$ functors $X \to C$

Let k be a field. C = chain complexes of k-vector spaces.

f is antidextrous

fin sets are antidextrous

G fin group

is BG antidextrous

X space.

$f : X \to *$

$f_!\mathcal{L} = C_*(X; k)$

$f_*\mathcal{L} = C^*(X; k)$

$\mathcal{L}$: const local sys w/ value k

failure is measured by Tate cohomology

char $k = p$. all p-groups are bad.

Sometimes ($k = \mathbb{C}$)

unconditionally typical for higher $K(G, n)$

Interesting local systems, antidextrosity ✓

mysterious duality phenomena not because of chain

$\Downarrow$

$P_K(n)$.

"poincare duality" like mfd of dim

X is antidextrous has some