Student Information

Name : Yaşar Cahit Yıldırım

ID : 2310647

Answer 1

a)

Expectation is a weighted sum of probabilities. So, with the help of law of total probability, the expectation function of random variable X can be expressed as:

$$\mathbf{E}(X) = \sum_x x \sum_y P(x,y) = \sum_x x\big(P(x,0) + P(x,2)\big)$$

$$\mathbf{E}(X) = 0 * \big(P(0,0) + P(0,2)\big) + 1 * \big(P(1,0) + P(1,2)\big) + 2 * \big(P(2,0) + P(2,2)\big) = 1$$

Variance is the variability or the distance of the values of random variables from their expected values.

$$\mathrm{Var}(X) = \sum_x (\mu - x)^2 \sum_y P(x,y) = \sum_x (\mu - x)^2 \big(P(x,0) + P(x,2)\big)$$

$$\mathrm{Var}(X) = (0-1)^2 * \big(P(0,0) + P(0,2)\big) + (1-1)^2 * \big(P(1,0) + P(1,2)\big) + (2-1)^2 * \big(P(2,0) + P(2,2)\big) = 1/2$$

b)

Let Z denote $X + Y$. Probability mass function of Z is sum of probabilities $P(x,y)$ where $z = x + y$.

$P(z)$	z				
	0	1	2	3	4
	1/12	4/12	3/12	2/12	2/12
x, y pair	0, 0	1, 0	0, 2 2, 0	1, 2	2, 2

c)

Covariance is the relative relation between two random variables.

$$\text{Cov}(X,Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) = \sum_x \sum_y xyP(x,y) - \mathbf{E}(X)\mathbf{E}(Y)$$

We know **E**(X) from **(a)**. Calculating **E**(Y) with the same method gives us:

$$\mathbf{E}(Y) = 0 * \big(P(0,0) + P(1,0) + P(2,0)\big) + 2 * \big(P(0,2) + P(1,2) + P(2,2)\big) = 1$$

Since the rows and columns where either x or y is 0 will be 0, those multiplications are omitted from the equation for the sake of simplicity.

$$\text{Cov}(X,Y) = 1 * 2 * P(1,2) + 2 * 2 * P(2,2) - \mathbf{E}(X) * \mathbf{E}(Y)$$

$$\text{Cov}(X,Y) = 1 * 2 * (2/12) + 2 * 2 * (2/12) - 1 * 1 = 0$$

d)

From the **Properties of expectations, pg.49**, we know that:

If X and Y are independent then $\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y)$.

So, substituting that to the equation of covariance as:

$$\text{Cov}(X,Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) = \mathbf{E}(X)\mathbf{E}(Y) - \mathbf{E}(X)\mathbf{E}(Y) = 0$$

e)

Let us give a counterexample to show non-independence. Such as:

$P(x,y) \neq P(x)P(y)$ for some x, y.

$$P(X=0,y) = P(0,0) + P(0,2) = 1/12 + 2/12 = 3/12$$
$$P(x,Y=0) = P(0,0) + P(1,0) + P(2,0) = 1/12 + 4/12 + 1/12 = 6/12$$

$$P(0,0) = 1/12 \neq P(X=0) * P(Y=0) = 3/12 * 6/12 = 1/8$$

So, this shows that X and Y are non-independent.

Answer 2

Let us consider being broken as a successful outcome for this question.

a)

We will use Binomial Distribution with 12 trials, at least 3 successes, probability of success as 0.2 and probability of failure as $1 - 0.2 = 0.8$.

$$P(X \geq 3) = 1 - P(X < 3)$$

$$P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$P(X = 0) = \binom{12}{0} * (0.2)^0 * (0.8)^{12-0} = 0.0687$$

$$P(X = 1) = \binom{12}{1} * (0.2)^1 * (0.8)^{12-1} = 0.2062$$

$$P(X = 2) = \binom{12}{2} * (0.2)^2 * (0.8)^{12-2} = 0.2835$$

$$P(X < 3) = 0.0687 + 0.2062 + 0.2835 = 0.5584$$

$$P(X \geq 3) = 1 - 0.5584 = 0.4416$$

b)

We will use Negative Binomial Distribution with 2nd success at 5th trial, probability of success as 0.2 and probability of failure as $1 - 0.2 = 0.8$.

$$P(2) = \binom{5-1}{2-1}(0.8)^{5-2}(0.2)^2 = \binom{4}{1}(0.8)^3(0.2)^2 = 0.08192$$

c)

Since we know the number of successes and try to find the average number of trials, we will use $\mathbf{E}(X)$ of Negative Binomial Distribution. Number of successes is 4 and probability of success is 0.2.

$$\mathbf{E}(X) = \frac{k}{p} = \frac{4}{0.2} = 20$$

Answer 3

We will use Poisson Distribution with unit period 4 hours and λ as 1 call. Number of events occuring is phone calls.

a)

Since unit period for this part is 2, λ is now 0.5 from the equality $\frac{2}{4} = \frac{\lambda}{1}$ and the number of calls is 0.

$$P(X = 0) = \frac{(0.5)^0 * e^{-0.5}}{0!} = 0.6065$$

b)

Unit period for this part is 10 hours, so λ is now 2.5 from the equality $\frac{10}{4} = \frac{\lambda}{1}$ and the number of calls is ≤ 3.

$$\begin{aligned}
P(X \leq 3) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\
&= \frac{(2.5)^0 * e^{-2.5}}{0!} + \frac{(2.5)^1 * e^{-2.5}}{1!} + \frac{(2.5)^2 * e^{-2.5}}{2!} + \frac{(2.5)^3 * e^{-2.5}}{3!} \\
&= 0.0821 + 0.2052 + 0.2565 + 0.2138 \\
&= 0.7576
\end{aligned}$$

c)

There will be two solutions proposed, both with the same answer. Just for the sake of stronger argument and FUN.

Bayes' theorem

For the 10 hour unit period, the λ is 2.5; and for the 16 hour unit period, the λ is 4.

Let A denote: Bob did not get more than 3 phone calls for the first 16 hours.
Let B denote: Bob did not get more than 3 phone calls for the first 10 hours.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

It is straightforward that if Bob got at most 3 calls in the first 16 hours, he must have got at most 3 calls in the first 10 hours too, thus $P(B|A) = 1$. $P(B)$ is equal to the result of **(b)**.

$$\begin{aligned} P(A \leq 3) &= P(A = 0) + P(A = 1) + P(A = 2) + P(A = 3) \\ &= \frac{(4)^0 * e^{-4}}{0!} + \frac{(4)^1 * e^{-4}}{1!} + \frac{(4)^2 * e^{-4}}{2!} + \frac{(4)^3 * e^{-4}}{3!} \\ &= 0.0183 + 0.0733 + 0.1465 + 0.1954 \\ &= 0.4335 \end{aligned}$$

$$P(A|B) = \frac{1 * (0.4335)}{0.7576} = 0.5722$$

A python way

For every case of first 10 hours, there exists some cases for the 6 hours after that. Sum of conditional probabilities of these is the answer.

Let $P(X)$ denote the question itself. $P(B)$ is the answer from **(b)** since it is given.
Let A_i denote: Bob did not get more than i phone calls for the first 10 hours.
Let B_j denote: Bob did not get more than j phone calls for the last 6 hours.

$$P(X) = \sum_i \sum_j \frac{P(A_i \cap B_j)}{P(B)} \text{ where } i + j \leq 3.$$

```python
result = 0
for i in range(4): # 0 to 3 inclusive
    prob_first_10 = poisson(10/4, i) # Case of i calls from first 10 hours

    for j in range(4-i): # Remaining calls for last 6 hours
        prob_last_6 = poisson(6/4, j) # Case of j calls from remaining 6 hours
        result += prob_first_10 * prob_last_6
        # Sum of probabilities of i calls in 10 hours, j calls in 6 hours

result /= b_answer # Given 10 hours condition is true
# Result: 0.5722
```

Note: Variable *b_answer* is the answer from **(b)**. *Poisson* function is defined as $poisson(lambda, x)$.