EXAMPLE 8.2.1

Let

$$A = \neg\exists y\forall z(P(z,y) \equiv \neg\exists x(P(z,x) \land P(x,z))).$$

First, we negate A and eliminate $\equiv$. We obtain the sentence

$$\begin{gathered}\exists y\forall z[(\neg P(z,y) \lor \neg\exists x(P(z,x) \land P(x,z)))\land\\ (\exists x(P(z,x) \land P(x,z)) \lor P(z,y))].\end{gathered}$$

Next, we put in this formula in NNF:

$$\begin{gathered}\exists y\forall z[(\neg P(z,y) \lor \forall x(\neg P(z,x) \lor \neg P(x,z)))\land\\ (\exists x(P(z,x) \land P(x,z)) \lor P(z,y))].\end{gathered}$$

Next, we eliminate existential quantifiers, by the introduction of Skolem symbols:

$$\begin{gathered}\forall z[(\neg P(z,a) \lor \forall x(\neg P(z,x) \lor \neg P(x,z)))\land\\ ((P(z,f(z)) \land P(f(z),z)) \lor P(z,a))].\end{gathered}$$

We now put in prenex form:

$$\begin{gathered}\forall z\forall x[(\neg P(z,a) \lor (\neg P(z,x) \lor \neg P(x,z)))\land\\ ((P(z,f(z)) \land P(f(z),z)) \lor P(z,a))].\end{gathered}$$

We put in CNF by distributing $\land$ over $\lor$:

$$\begin{gathered}\forall z\forall x[(\neg P(z,a) \lor \neg P(z,x) \lor \neg P(x,z))\land\\ (P(z,f(z)) \lor P(z,a)) \land (P(f(z),z)) \lor P(z,a))].\end{gathered}$$

Omitting universal quantifiers, we have the following three clauses:

$$\begin{aligned}&C_1 = (\neg P(z_1,a) \lor \neg P(z_1,x) \lor \neg P(x,z_1)),\\ &C_2 = (P(z_2,f(z_2)) \lor P(z_2,a)) \text{ and}\\ &C_3 = (P(f(z_3),z_3) \lor P(z_3,a)).\end{aligned}$$

We will now show that we can prove that $B = \neg A$ is unsatisfiable, by instantiating C_1, C_2, C_3 to ground clauses and use the resolution method of Chapter 4.

8.3 Ground Resolution

The *ground resolution method* is the resolution method applied to sets of ground clauses.

EXAMPLE 8.3.1

Consider the following ground clauses obtained by substitution from C_1, C_2 and C_3:

$$
\begin{aligned}
G_1 &= (\neg P(a,a)) \text{ (from } C_1, \text{ substituting } a \text{ for } x \text{ and } z_1)\\
G_2 &= (P(a,f(a)) \lor P(a,a)) \text{ (from } C_2, \text{ substituting } a \text{ for } z_2)\\
G_3 &= (P(f(a),a)) \lor P(a,a)) \text{ (from } C_3, \text{ substituting } a \text{ for } z_3).\\
G_4 &= (\neg P(f(a),a) \lor \neg P(a,f(a))) \text{ (from } C_1, \text{ substituting } f(a)\\
&\qquad \text{for } z_1 \text{ and } a \text{ for } x).
\end{aligned}
$$

The following is a refutation by (ground) resolution of the set of ground clauses G_1, G_2, G_3, G_4.

$$
\begin{tikzcd}
G_2 \arrow[dr, dash] & & G_1 \arrow[dl, dash] \arrow[dr, dash] & & G_3 \arrow[dl, dash] & & G_4 \arrow[ddl, dash] \\
& \{P(a,f(a))\} \arrow[ddrr, dash] & & \{P(f(a),a)\} \arrow[drr, dash] & & & \\
& & & & & \{\neg P(a,f(a))\} \arrow[dll, dash] & \\
& & & \square & & &
\end{tikzcd}
$$

We have the following useful result.

Lemma 8.3.1 (Completeness of ground resolution) The ground resolution method is complete for ground clauses.

Proof: Observe that the systems G' and $GCNF'$ are complete for quantifier-free formulae of a first-order language without equality. Hence, by theorem 4.3.1, the resolution method is also complete for sets of ground clauses. $\square$

However, note that this is not the case for quantifier-free formulae with equality, due to the need for equality axioms and for inessential cuts, in order to retain completeness.

Since we have shown that a conjunction of ground instances of the clauses C_1, C_2, C_3 of example 8.2.1 is unsatisfiable, by the Skolem-Herbrand-Gödel theorem, the sentence A of example 8.2.1 is valid.

Summarizing the above, we have a method for finding whether a sentence B is unsatisfiable known as *ground resolution*. This method consists in converting the sentence B into a set of clauses B', instantiating these clauses to ground clauses, and applying the ground resolution method.

By the completeness of resolution for propositional logic (theorem 4.3.1), and the Skolem-Herbrand-Gödel theorem (actually the corollary to theorem 7.6.1 suffices, since the clauses are in CNF, and so in NNF), this method is complete.