

**EXAMPLE 8.2.1**

Let

$$A = \neg \exists y \forall z (P(z, y) \equiv \neg \exists x (P(z, x) \wedge P(x, z))).$$

First, we negate  $A$  and eliminate  $\equiv$ . We obtain the sentence

$$\begin{aligned} \exists y \forall z [(\neg P(z, y) \vee \neg \exists x (P(z, x) \wedge P(x, z))) \wedge \\ (\exists x (P(z, x) \wedge P(x, z)) \vee P(z, y))]. \end{aligned}$$

Next, we put in this formula in NNF:

$$\begin{aligned} \exists y \forall z [(\neg P(z, y) \vee \forall x (\neg P(z, x) \vee \neg P(x, z))) \wedge \\ (\exists x (P(z, x) \wedge P(x, z)) \vee P(z, y))]. \end{aligned}$$

Next, we eliminate existential quantifiers, by the introduction of Skolem symbols:

$$\begin{aligned} \forall z [(\neg P(z, a) \vee \forall x (\neg P(z, x) \vee \neg P(x, z))) \wedge \\ ((P(z, f(z)) \wedge P(f(z), z)) \vee P(z, a))]. \end{aligned}$$

We now put in prenex form:

$$\begin{aligned} \forall z \forall x [(\neg P(z, a) \vee (\neg P(z, x) \vee \neg P(x, z))) \wedge \\ ((P(z, f(z)) \wedge P(f(z), z)) \vee P(z, a))]. \end{aligned}$$

We put in CNF by distributing  $\wedge$  over  $\vee$ :

$$\begin{aligned} \forall z \forall x [(\neg P(z, a) \vee \neg P(z, x) \vee \neg P(x, z)) \wedge \\ (P(z, f(z)) \vee P(z, a)) \wedge (P(f(z), z) \vee P(z, a))]. \end{aligned}$$

Omitting universal quantifiers, we have the following three clauses:

$$\begin{aligned} C_1 &= (\neg P(z_1, a) \vee \neg P(z_1, x) \vee \neg P(x, z_1)), \\ C_2 &= (P(z_2, f(z_2)) \vee P(z_2, a)) \text{ and} \\ C_3 &= (P(f(z_3), z_3) \vee P(z_3, a)). \end{aligned}$$

We will now show that we can prove that  $B = \neg A$  is unsatisfiable, by instantiating  $C_1, C_2, C_3$  to ground clauses and use the resolution method of Chapter 4.

**8.3 Ground Resolution**

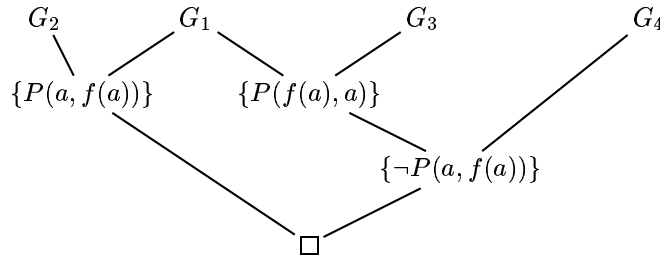
The *ground resolution method* is the resolution method applied to sets of ground clauses.

**EXAMPLE 8.3.1**

Consider the following ground clauses obtained by substitution from  $C_1$ ,  $C_2$  and  $C_3$ :

- $$\begin{aligned} G_1 &= (\neg P(a, a)) \text{ (from } C_1, \text{ substituting } a \text{ for } x \text{ and } z_1) \\ G_2 &= (P(a, f(a)) \vee P(a, a)) \text{ (from } C_2, \text{ substituting } a \text{ for } z_2) \\ G_3 &= (P(f(a), a) \vee P(a, a)) \text{ (from } C_3, \text{ substituting } a \text{ for } z_3). \\ G_4 &= (\neg P(f(a), a) \vee \neg P(a, f(a))) \text{ (from } C_1, \text{ substituting } f(a) \\ &\quad \text{for } z_1 \text{ and } a \text{ for } x). \end{aligned}$$

The following is a refutation by (ground) resolution of the set of ground clauses  $G_1, G_2, G_3, G_4$ .



We have the following useful result.

**Lemma 8.3.1** (Completeness of ground resolution) The ground resolution method is complete for ground clauses.

*Proof:* Observe that the systems  $G'$  and  $GCNF'$  are complete for quantifier-free formulae of a first-order language without equality. Hence, by theorem 4.3.1, the resolution method is also complete for sets of ground clauses.  $\square$

However, note that this is not the case for quantifier-free formulae with equality, due to the need for equality axioms and for inessential cuts, in order to retain completeness.

Since we have shown that a conjunction of ground instances of the clauses  $C_1, C_2, C_3$  of example 8.2.1 is unsatisfiable, by the Skolem-Herbrand-Gödel theorem, the sentence  $A$  of example 8.2.1 is valid.

Summarizing the above, we have a method for finding whether a sentence  $B$  is unsatisfiable known as *ground resolution*. This method consists in converting the sentence  $B$  into a set of clauses  $B'$ , instantiating these clauses to ground clauses, and applying the ground resolution method.

By the completeness of resolution for propositional logic (theorem 4.3.1), and the Skolem-Herbrand-Gödel theorem (actually the corollary to theorem 7.6.1 suffices, since the clauses are in CNF, and so in NNF), this method is complete.