Network Science

2019. október 8.

Network models


THE ERDŐS-RÉNYI MODEL

The Erdős-Rényi model (1959)

The Erdős-Rényi model (classical random graphs)

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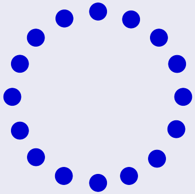
- Take N nodes.
- Uniformly link each pair independently of each other with probability p.



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- The $G(N,M)$ model is almost the same: distribute M links independently amongst the N nodes with uniform probability.

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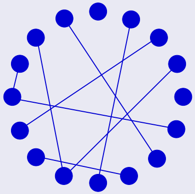

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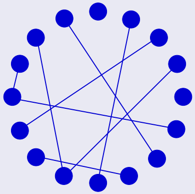

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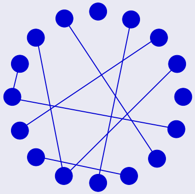

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- average degree: $\langle k \rangle = (N-1)p \simeq Np$,
- number of expected links: $M = pN(N-1)/2$.

Degree distribution of the E-R graph

Degree distribution of the E-R graph

We have already derived the $p(k)$ of the E-R graph earlier,

$$\begin{aligned} p(k) &= \binom{N-1}{k} p^k (1-p)^{N-1-k} \simeq \binom{N}{k} p^k (1-p)^{N-k} && \text{(binomial)} \\ &\simeq \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} && \text{(Poisson)} \end{aligned}$$

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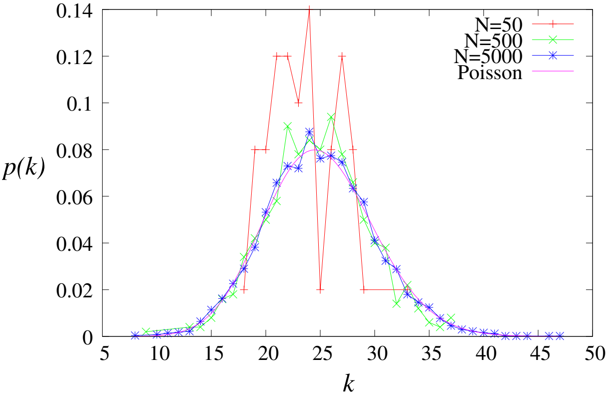

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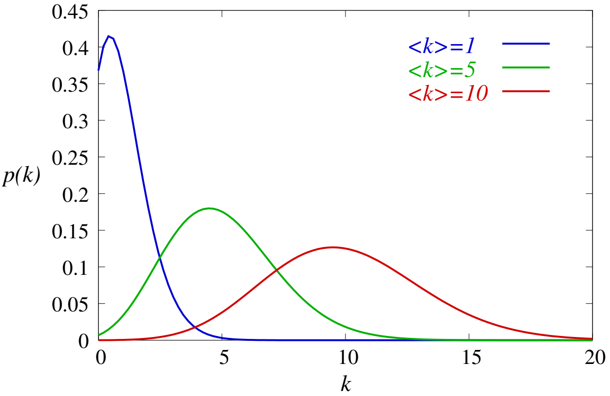

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Variance

- What is the variance of $p(k)$?

The degree distribution is binomial,

$$p(k) = \binom{N}{k} p^k (1-p)^{N-k},$$

→ the average and variance for a binomial distribution in general is

$$\begin{aligned}
\langle k \rangle &= Np, \\
\langle k^2 \rangle &= Np(1-p) + p^2 N^2, \\
\mathrm{Var}(k) &= \langle k^2 \rangle - \langle k \rangle^2 = Np(1-p), \\
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The variance is **constant**, thus, it becomes **negligible compared to the system size**!

- In the continuum formalism:

$$\mathcal{P}(k > k_0) = \int_{k_0}^{\infty} p(k)dk = \int_{k_0}^{\infty} \frac{\langle k\rangle^k}{k!} e^{-\langle k\rangle} dk.$$

- E.g, for $\langle k\rangle = 10$:
  - the prob. to find a node with $k \geq 20$ is 0.00158826,
  - the prob. to find a node with $k \leq 1$ is 0.00049,
  - the prob. to find a node with $k \geq 100$ is $1.79967152 \times 10^{-13}$.

- According to sociologists, for a typical individual $k \sim 1000$.

→ the prob. to find someone with $k \geq 2000$ is roughly 10^{-27}!

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Clustering coefficient in the E-R graph

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  - The E-R graph is very democratic, and we expect all nodes to have more or less the same C, thus, $C_i \simeq \langle C \rangle$.
  - C_i can be also interpreted as the probability of the neighbors of i being connected. Since in the E-R model we link every pair independently with uniform probability p, the neighbors of any node shall be linked also with probability p.
  - Thus, in the E-R graph $\langle C \rangle = p$.

Giant component in the E-R graph

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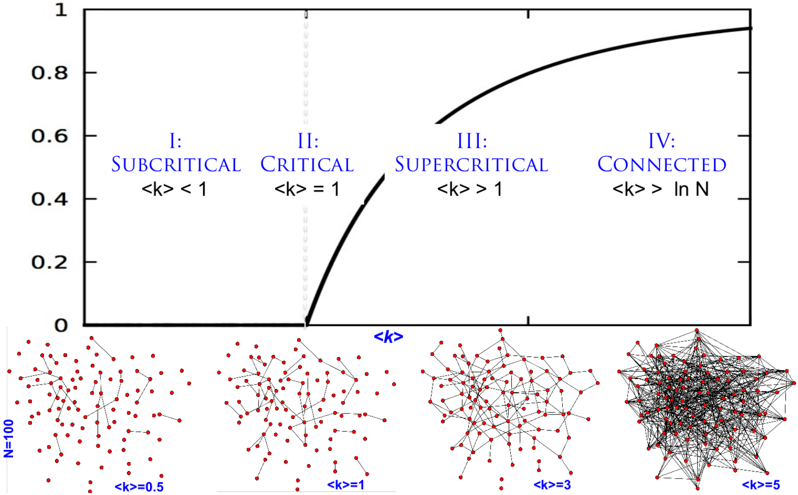
Relative size of the largest component:



(from the slides of A.-L. Barabási)

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- However, the probability that i is NOT in the giant component is u by definition, thus

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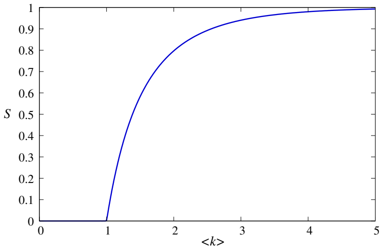
This equation can be solved numerically for any $\langle k \rangle$, giving the corresponding S:



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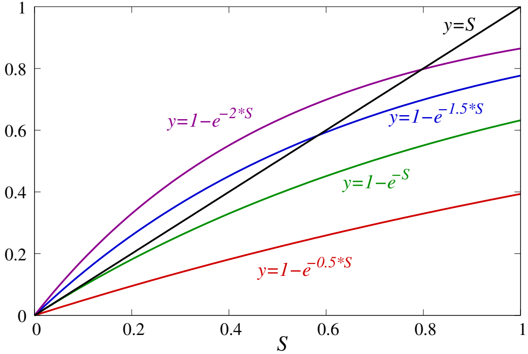

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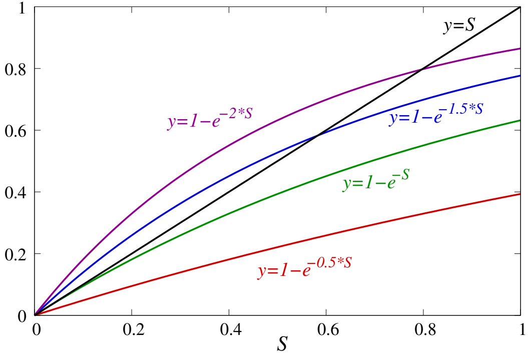

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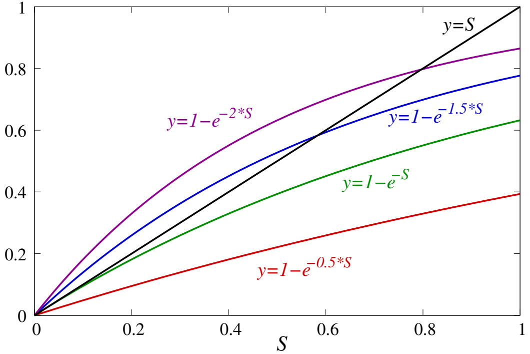

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$$\rightarrow \langle k\rangle \geq 1$$

- **Thus, the critical point is at $k=1$, and for $k\geq 1$ we have a giant component in the E-R graph.**

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  - we approach the critical point of the percolation transition from below, i.e., from the dispersed phase where the network is sparse and can be assumed to be **localy tree-like**.

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- The $p_X(k)$ probabilities defining the distributions can be obtained from the generating function as

$$p_X(k) = \frac{1}{k!} \left. \frac{dG_X(z)}{dz^k} \right|_{z=0} .$$