Network Science

2019. október 14.

Network models

Percolation and generating functions

Quick overview on generating functions in general:

Percolation and generating functions

Quick overview on generating functions in general:

- For a discrete random variable X that can take non-negative integer values with a probability distribution given by $P(X = k) = p_X(k)$, the corresponding **generating function** is defined as

$$G_X(z) = \sum_{k=0}^{\infty} z^k p_X(k).$$

Percolation and generating functions

Quick overview on generating functions in general:

- For a discrete random variable X that can take non-negative integer values with a probability distribution given by $P(X=k)=p_X(k)$, the corresponding **generating function** is defined as

$$G_X(z)=\sum_{k=0}^{\infty} z^k p_X(k).$$

- The $p_X(k)$ probabilities defining the distributions can be obtained from the generating function as

$$p_X(k)=\frac{1}{k!}\left.\frac{d^k G_X(z)}{dz^k}\right|_{z=0}.$$

Percolation and generating functions

Quick overview on generating functions in general:

- For a discrete random variable X that can take non-negative integer values with a probability distribution given by $P(X=k)=p_X(k)$, the corresponding **generating function** is defined as

$$G_X(z)=\sum_{k=0}^{\infty} z^k p_X(k).$$

- The $p_X(k)$ probabilities defining the distributions can be obtained from the generating function as

$$p_X(k)=\frac{1}{k!}\left.\frac{d^k G_X(z)}{dz^k}\right|_{z=0}.$$

- The m-th moment of the distribution can be obtained from the generating function as

$$\langle X^m\rangle=\langle k^m\rangle=\left[z\frac{d}{dz}\right]^m G_X(z)\Big|_{z=1},$$

Percolation and generating functions

Quick overview on generating functions in general:

- For a discrete random variable X that can take non-negative integer values with a probability distribution given by $P(X=k)=p_X(k)$, the corresponding **generating function** is defined as

$$G_X(z)=\sum_{k=0}^{\infty} z^k p_X(k).$$

- The $p_X(k)$ probabilities defining the distributions can be obtained from the generating function as

$$p_X(k)=\frac{1}{k!}\left.\frac{d^k G_X(z)}{dz^k}\right|_{z=0}.$$

- The m-th moment of the distribution can be obtained from the generating function as

$$\langle X^m\rangle=\langle k^m\rangle=\left[z\frac{d}{dz}\right]^m G_X(z)\Big|_{z=1},$$

which for the case of the expected value is equivalent to

$$\langle X\rangle=\langle k\rangle=G'_X(z=1).$$

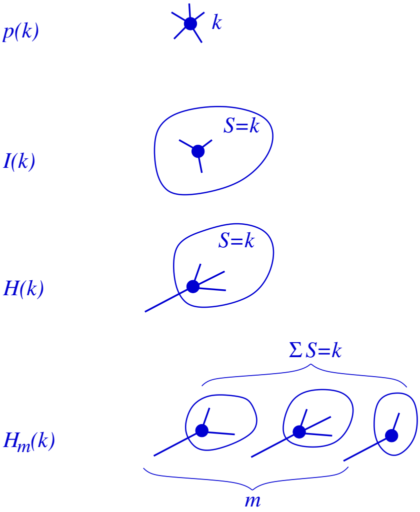
Percolation and generating functions

- If $X_1, X_2, \ldots X_n$ are independent variables and $Z = \sum_{i=1}^{n} X_i$, then

$$G_Z(x) = \prod_{i=1}^{n} G_X(x).$$

Percolation and generating functions

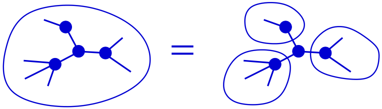
The discrete distributions in our game:



- $p(k)$ is the degree distribution.
- $I(k) = P($a rand. chosen node is in a component of size $k)$.
- $H(k) = P($a rand. chosen link has a component of size k at one end $)$.
- $H_m(k) = P($the sum of the comps. at one end of m rand. chosen links adds up to $k)$.

Percolation and generating functions

Main idea:



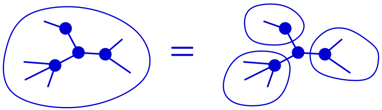
$$
\begin{aligned}
\rightarrow I(k) &= p(0)H_0(k-1) + p(1)H_1(k-1) + p(2)H_2(k-1) + \ldots \\
&\quad + p(m)H_m(k-1) + \ldots = \sum_{m=0}^{\infty} p(m)H_m(k-1)
\end{aligned}
$$

By taking the generating function of both sides we obtain

$$
\begin{aligned}
G_I(x) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m)H_m(k-1)x^k = \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m)\frac{1}{(k-1)!}\left.\frac{d^{k-1}}{dx^{k-1}}G_{H,m}(x)\right|_{x=0} x^k
\end{aligned}
$$

Percolation and generating functions

Main idea:



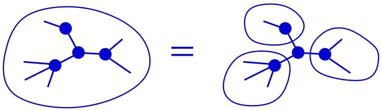
$$\rightarrow I(k) = p(0)H_0(k-1) + p(1)H_1(k-1) + p(2)H_2(k-1) + \ldots$$
$$+p(m)H_m(k-1) + \ldots = \sum_{m=0}^{\infty} p(m)H_m(k-1)$$

By taking the generating function of both sides we obtain

$$G_I(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m)H_m(k-1)x^k =$$
$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m)\frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}G_{H,m}(x)\bigg|_{x=0} x^k$$

Percolation and generating functions

Main idea:



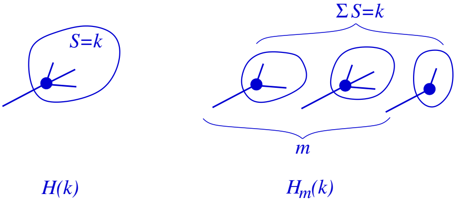
$$\begin{aligned}\rightarrow I(k) &= p(0)H_0(k-1) + p(1)H_1(k-1) + p(2)H_2(k-1) + \ldots \\ &+ p(m)H_m(k-1) + \ldots = \sum_{m=0}^{\infty} p(m)H_m(k-1)\end{aligned}$$

By taking the generating function of both sides we obtain

$$\begin{aligned}G_I(x) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m)H_m(k-1)x^k = \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m)\frac{1}{(k-1)!}\left.\frac{d^{k-1}}{dx^{k-1}}G_{H,m}(x)\right|_{x=0} x^k\end{aligned}$$

Percolation and generating functions

- The $G_{H,m}(x)$ can be expressed based on $G_H(x)$ as



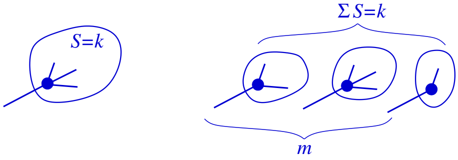
$$\Rightarrow \; G_{H,m}(x) = [G_H(x)]^m$$

$$\begin{aligned}
\rightarrow G_I(x) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m) \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} G_{H,m}(x)\Big|_{x=0} x^k \\
& \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m) \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} [G_H(x)]^m\Big|_{x=0} x^k \\
&= \sum_{m=0}^{\infty} p(m)\, [G_H(x)]^m\, x = xG(G_H(x)),
\end{aligned}$$

where $G(x)$ is the generating function of the degree distribution $p(k)$.

Percolation and generating functions

- The $G_{H,m}(x)$ can be expressed based on $G_H(x)$ as



$$\Rightarrow G_{H,m}(x) = [G_H(x)]^m$$

$H(k)$ $\quad$ $H_m(k)$

$$\begin{aligned}
\rightarrow G_I(x) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m)\frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}G_{H,m}(x)\bigg|_{x=0} x^k \\
& \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m)\frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}[G_H(x)]^m\bigg|_{x=0} x^k \\
&= \sum_{m=0}^{\infty} p(m)\,[G_H(x)]^m\, x = xG(G_H(x)),
\end{aligned}$$

where $G(x)$ is the generating function of the degree distribution $p(k)$.

Percolation and generating functions

- $I(k) = P($ rand. chosen node is in a component of size $k)$.

→ The mean of the component size for a randomly chosen node is

$$\langle S \rangle = G_I'(1) = [xG(G_H(x)]'\big|_{x=1} = G(G_I(1)) + G'(1)G_H'(1) = \\ = 1 + \langle k \rangle G_H'(1),$$

where $G'(1) = \langle k \rangle$ is the average degree.

- How could we calculate $G_H'(1)$?

Percolation and generating functions

- $I(k) = P($ rand. chosen node is in a component of size $k)$.

→ The mean of the component size for a randomly chosen node is

$$\langle S\rangle = G_I'(1) = [xG(G_H(x)]'\big|_{x=1} = G(G_I(1)) + G'(1)G_H'(1) =$$
$$= 1 + \langle k\rangle\, G_H'(1),$$

where $G'(1) = \langle k\rangle$ is the average degree.

- How could we calculate $G_H'(1)$?

Percolation and generating functions

- $I(k) = P($ rand. chosen node is in a component of size $k)$.

$\rightarrow$ The mean of the component size for a randomly chosen node is

$$\begin{aligned}\langle S\rangle &= G_I'(1) = [xG(G_H(x)]'\big|_{x=1} = G(G_I(1)) + G'(1)G_H'(1) = \\ &= 1 + \langle k\rangle\, G_H'(1),\end{aligned}$$

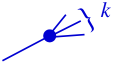
where $G'(1) = \langle k\rangle$ is the average degree.

- How could we calculate $G_H'(1)$?

Percolation and generating functions

- To obtain $G'_H(1)$ we have to introduce a further distribution:

$q(k)$



$q(k) = P($ we can proceed to k further links at one end of a rand. chosen link $)$.

- Idea:

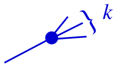
$$\begin{aligned}\rightarrow H(k) &= q(0)H_0(k-1) + q(1)H_1(k-1) + q(2)H_2(k-1) + \ldots \\ &\quad + q(m)H_m(k-1) + \ldots = \sum_{m=0}^{\infty} q(m)H_m(k-1)\end{aligned}$$

Again, we take the generating function of both sides.

Percolation and generating functions

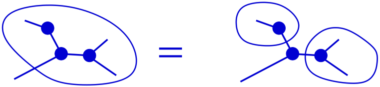
- To obtain $G'_H(1)$ we have to introduce a further distribution:

$q(k)$



$q(k) = P($ we can proceed to k further links at one end of a rand. chosen link $)$.

- Idea:



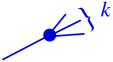
$$\rightarrow H(k) = q(0)H_0(k-1) + q(1)H_1(k-1) + q(2)H_2(k-1) + \dots$$
$$+ q(m)H_m(k-1) + \dots = \sum_{m=0}^{\infty} q(m)H_m(k-1)$$

Again, we take the generating function of both sides.

Percolation and generating functions

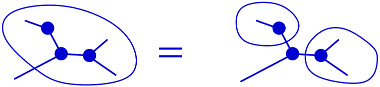
- To obtain $G'_H(1)$ we have to introduce a further distribution:

$q(k)$



$q(k) = P($ we can proceed to k further links at one end of a rand. chosen link $)$.

- Idea:



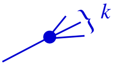
$$\rightarrow H(k) = q(0)H_0(k-1) + q(1)H_1(k-1) + q(2)H_2(k-1) + \ldots$$
$$+q(m)H_m(k-1) + \ldots = \sum_{m=0}^{\infty} q(m)H_m(k-1)$$

Again, we take the generating function of both sides.

Percolation and generating functions

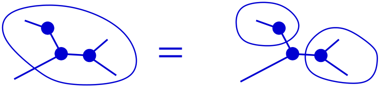
- To obtain $G'_H(1)$ we have to introduce a further distribution:

$q(k)$



$q(k) = P($ we can proceed to k further links at one end of a rand. chosen link $)$.

- Idea:



$$\rightarrow H(k) = q(0)H_0(k-1) + q(1)H_1(k-1) + q(2)H_2(k-1) + \ldots$$
$$+q(m)H_m(k-1) + \ldots = \sum_{m=0}^{\infty} q(m)H_m(k-1)$$

Again, we take the generating function of both sides.

Percolation and generating functions

$$
\begin{aligned}
G_H(x) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q(m) H_m(k-1) x^k = \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q(m) \frac{1}{(k-1)!} \left.\frac{d^{k-1}}{dx^{k-1}} G_{H,m}(x)\right|_{x=0} x^k \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q(m) \frac{1}{(k-1)!} \left.\frac{d^{k-1}}{dx^{k-1}} [G_H(x)]^m\right|_{x=0} x^k \\
&= \sum_{m=0}^{\infty} q(m) [G_H(x)]^m x = x G_q(G_H(x)) \\
G'_H(1) &= G_q(G_H(1)) + G'_q(1) G'_H(1) = 1 + G'_q(1) G'_H(1) \\
\rightarrow G'_H(1) &= \frac{1}{1 - G'_q(1)}
\end{aligned}
$$

Percolation and generating functions

- By substituting back we obtain

$$\langle S\rangle = 1 + \langle k\rangle G_H'(1) = 1 + \frac{\langle k\rangle}{1 - G_q'(1)}$$

→ The critical point is where

$$G_q'(1) = 1$$

- $G_q(x)$ can be expressed based on the degree distribution: $q(k) = P($ we have a node with degree $k+1$ at one end of a rand. chosen link $)$,

$$P(\text{node with degree } k \text{ at end of the link}) = \frac{kN_k}{\sum_{k'} k'N_{k'}} = \frac{kp(k)}{\sum_{k'} k'p(k')} = \frac{kp(k)}{\langle k\rangle}$$

$$\rightarrow \quad q(k) = \frac{k+1}{\langle k\rangle} p(k+1)$$

$$G_q(x) = \frac{1}{\langle k\rangle}\sum_{k=0}^{\infty}(k+1)p(k+1)x^k = \frac{1}{\langle k\rangle}\sum_{l=1}^{\infty} lp(l)x^{l-1} = \frac{G'(x)}{\langle k\rangle}$$

Percolation and generating functions

- By substituting back we obtain

$$\langle S\rangle = 1 + \langle k\rangle\, G_H'(1) = 1 + \frac{\langle k\rangle}{1 - G_q'(1)}$$

→ The critical point is where

$$G_q'(1) = 1$$

- $G_q(x)$ can be expressed based on the degree distribution: $q(k) = P($ we have a node with degree $k+1$ at one end of a rand. chosen link $)$,

$$P(\text{node with degree } k \text{ at end of the link}) = \frac{kN_k}{\sum_{k'} k'N_{k'}} = \frac{kp(k)}{\sum_{k'} k'p(k')} = \frac{kp(k)}{\langle k\rangle}$$

$$\rightarrow \quad q(k) = \frac{k+1}{\langle k\rangle} p(k+1)$$

$$G_q(x) = \frac{1}{\langle k\rangle}\sum_{k=0}^{\infty}(k+1)p(k+1)x^k = \frac{1}{\langle k\rangle}\sum_{l=1}^{\infty} l p(l) x^{l-1} = \frac{G'(x)}{\langle k\rangle}$$

Percolation and generating functions

- By substituting back we obtain

$$\langle S\rangle = 1 + \langle k\rangle\, G_H'(1) = 1 + \frac{\langle k\rangle}{1 - G_q'(1)}$$

→ The critical point is where

$$G_q'(1) = 1$$

- $G_q(x)$ can be expressed based on the degree distribution:
  $q(k) = P($ we have a node with degree $k+1$ at one end of a rand. chosen link $)$,

$$P(\text{node with degree } k \text{ at end of the link}) = \frac{kN_k}{\sum_{k'} k'N_{k'}} = \frac{kp(k)}{\sum_{k'} k'p(k')} = \frac{kp(k)}{\langle k\rangle}$$

$$\rightarrow \quad q(k) = \frac{k+1}{\langle k\rangle} p(k+1)$$

$$G_q(x) = \frac{1}{\langle k\rangle}\sum_{k=0}^{\infty}(k+1)p(k+1)x^k = \frac{1}{\langle k\rangle}\sum_{l=1}^{\infty} lp(l)x^{l-1} = \frac{G'(x)}{\langle k\rangle}$$

Percolation and generating functions

- By substituting back we obtain

$$\langle S\rangle = 1 + \langle k\rangle G_H'(1) = 1 + \frac{\langle k\rangle}{1 - G_q'(1)}$$

→ The critical point is where

$$G_q'(1) = 1$$

- $G_q(x)$ can be expressed based on the degree distribution: $q(k) = P($ we have a node with degree $k+1$ at one end of a rand. chosen link $)$,

$$P(\text{node with degree } k \text{ at end of the link}) = \frac{kN_k}{\sum_{k'} k'N_{k'}} = \frac{kp(k)}{\sum_{k'} k'p(k')} = \frac{kp(k)}{\langle k\rangle}$$

$$\rightarrow \quad q(k) = \frac{k+1}{\langle k\rangle}p(k+1)$$

$$G_q(x) = \frac{1}{\langle k\rangle}\sum_{k=0}^{\infty}(k+1)p(k+1)x^k = \frac{1}{\langle k\rangle}\sum_{l=1}^{\infty} lp(l)x^{l-1} = \frac{G'(x)}{\langle k\rangle}$$

Percolation and generating functions

- The $G_q'(1)$ is closely related to the expected value of the number of second neighbours.
- We introduce

$q_m(k)$

$q_m(k) = P($we can proceed to k new links at the end of m links chosen at random$)$.

$n_2(k) = P($the number of 2nd neighbours of a rand. chosen node is $k)$.

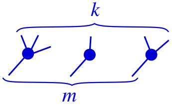
$$\begin{aligned} \rightarrow n_2(k) &= p(0)q_0(k) + p(1)q_1(k) + p(2)q_2(k) + \ldots \\ &\quad + p(m)q_m(k) + \ldots = \sum_{m=0}^{\infty} p(m)q_m(k) \end{aligned}$$

(We take the generating function of both sides as usual).

Percolation and generating functions

- The $G_q'(1)$ is closely related to the expected value of the number of second neighbours.
- We introduce

$q_m(k)$



$q_m(k) = P($we can proceed to k new links at the end of m links chosen at random$)$.

$n_2(k) = P($the number of 2$^{\text{nd}}$ neighbours of a rand. chosen node is $k)$.

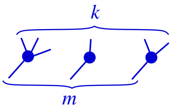
$$\begin{aligned}\to n_2(k) &= p(0)q_0(k) + p(1)q_1(k) + p(2)q_2(k) + \ldots \\ &\quad + p(m)q_m(k) + \ldots = \sum_{m=0}^{\infty} p(m)q_m(k)\end{aligned}$$

(We take the generating function of both sides as usual).

Percolation and generating functions

- The $G_q'(1)$ is closely related to the expected value of the number of second neighbours.
- We introduce

$q_m(k)$



$q_m(k) = P($we can proceed to k new links at the end of m links chosen at random$)$.

$n_2(k) = P($the number of 2$^{\text{nd}}$ neighbours of a rand. chosen node is $k)$.

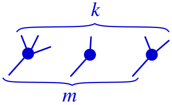
$$\rightarrow n_2(k) = p(0)q_0(k) + p(1)q_1(k) + p(2)q_2(k) + \dots$$
$$+p(m)q_m(k) + \dots = \sum_{m=0}^{\infty} p(m)q_m(k)$$

(We take the generating function of both sides as usual).

Percolation and generating functions

- The $G_q'(1)$ is closely related to the expected value of the number of second neighbours.
- We introduce

$q_m(k)$



$q_m(k) = P($we can proceed to k new links at the end of m links chosen at random$)$.

$n_2(k) = P($the number of 2$^{\text{nd}}$ neighbours of a rand. chosen node is $k)$.

$$\begin{aligned}\rightarrow n_2(k) \quad &= \quad p(0)q_0(k) + p(1)q_1(k) + p(2)q_2(k) + \ldots \\ &+p(m)q_m(k) + \ldots = \sum_{m=0}^{\infty} p(m)q_m(k)\end{aligned}$$

(We take the generating function of both sides as usual).

Percolation and generating functions

$$
\begin{aligned}
G_{n,2}(x) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m) q_m(k) x^k = \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m) \frac{1}{k!} \left. \frac{d^k}{dx^k} G_{m,q}(x) \right|_{x=0} x^k = \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(m) \frac{1}{k!} \left. \frac{d^k}{dx^k} \left[G_q(x)\right]^m \right|_{x=0} x^k = \\
&= \sum_{m=0}^{\infty} p(m) \left[G_q(x)\right]^m = G(G_q(x)) \\
z_2 = \langle n_2 \rangle &= G'_{n,2}(1) = G'(1) G'_q(1) = \langle k \rangle \, G'_q(1) \\
\rightarrow G'_q(1) &= \frac{z_2}{\langle k \rangle} = \frac{z_2}{z_1}
\end{aligned}
$$

Percolation and generating functions

The critical point of percolation

- Based on the above, the expected value of the component size is

$$\begin{aligned}\langle S\rangle &= 1+\frac{\langle k\rangle}{1-G_q'(1)} = 1+\frac{\langle k\rangle}{1-z_2/\langle k\rangle} = \\ &= 1+\frac{\langle k\rangle^2}{\langle k\rangle - z_2} = 1+\frac{z_1^2}{z_1-z_2}\end{aligned}$$

- Consequences:

$z_1 > z_2 \quad \rightarrow \quad$ small, isolated clusters

$z_1 = z_2 \quad \rightarrow \quad$ CRITICAL POINT!

$z_1 < z_2 \quad \rightarrow \quad$ giant component

Percolation and generating functions

The critical point of percolation

- Based on the above, the expected value of the component size is

$$\begin{aligned}\langle S\rangle &= 1+\frac{\langle k\rangle}{1-G_q'(1)} = 1+\frac{\langle k\rangle}{1-z_2/\langle k\rangle} = \\ &= 1+\frac{\langle k\rangle^2}{\langle k\rangle - z_2} = 1+\frac{z_1^2}{z_1-z_2}\end{aligned}$$

- Consequences:

$$\begin{aligned} z_1 > z_2 &\quad\rightarrow\quad \text{small, isolated clusters} \\ z_1 = z_2 &\quad\rightarrow\quad \text{CRITICAL POINT!} \\ z_1 < z_2 &\quad\rightarrow\quad \text{giant component}\end{aligned}$$

Percolation and generating functions

Percolation in the E-R-model

- What does this give for the E-R graph?
- The degree distribution can be approximated by a Poisson distribution,

$$
\begin{aligned}
p(k) &= \binom{N}{k} p^k (1-p)^{N-k} \simeq \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}, \\
\rightarrow G(x) &= e^{\langle k \rangle (x-1)} \\
G_q(x) &= \frac{G'(x)}{\langle k \rangle} = e^{\langle k \rangle (x-1)} = G(x),
\end{aligned}
$$

thus, for the critical point we receive

$$G'_q(1) = G'(1) = \langle k \rangle = 1.$$

Percolation and generating functions

Percolation in the E-R-model

- What does this give for the E-R graph?
- The degree distribution can be approximated by a Poisson distribution,

$$
\begin{aligned}
p(k) &= \binom{N}{k} p^k (1-p)^{N-k} \simeq \frac{\langle k\rangle^k}{k!} e^{-\langle k\rangle}, \\
\rightarrow G(x) &= e^{\langle k\rangle (x-1)} \\
G_q(x) &= \frac{G'(x)}{\langle k\rangle} = e^{\langle k\rangle (x-1)} = G(x),
\end{aligned}
$$

thus, for the critical point we receive

$$G_q'(1) = G'(1) = \langle k\rangle = 1.$$

Percolation and generating functions

Percolation in the E-R-model

- What does this give for the E-R graph?
- The degree distribution can be approximated by a Poisson distribution,

$$\begin{aligned} p(k) &= \binom{N}{k} p^k (1-p)^{N-k} \simeq \frac{\langle k\rangle^k}{k!} e^{-\langle k\rangle}, \\ \rightarrow G(x) &= e^{\langle k\rangle(x-1)} \\ G_q(x) &= \frac{G'(x)}{\langle k\rangle} = e^{\langle k\rangle(x-1)} = G(x), \end{aligned}$$

thus, for the critical point we receive

$$G_q'(1) = G'(1) = \langle k\rangle = 1.$$

Percolation and generating functions

- What about **scale-free** networks?

Percolation and generating functions

- What about **scale-free** networks?

$\rightarrow$ For scale-free networks let us examine the general result that

$$
\begin{array}{lcl}
z_1 > z_2 & \rightarrow & \text{small, isolated clusters} \\
z_1 = z_2 & \rightarrow & \text{CRITICAL POINT!} \\
z_1 < z_2 & \rightarrow & \text{giant component}
\end{array}
$$

Percolation and generating functions

- What about **scale-free** networks?

→ For scale-free networks let us examine the general result that

$$z_1 > z_2 \quad \rightarrow \quad \text{small, isolated clusters}$$
$$z_1 = z_2 \quad \rightarrow \quad \text{CRITICAL POINT!}$$
$$z_1 < z_2 \quad \rightarrow \quad \text{giant component}$$

How can we calculate z_2?

Percolation and generating functions

- What about **scale-free** networks?

→ For scale-free networks let us examine the general result that

$$z_1 > z_2 \quad \rightarrow \quad \text{small, isolated clusters}$$
$$z_1 = z_2 \quad \rightarrow \quad \text{CRITICAL POINT!}$$
$$z_1 < z_2 \quad \rightarrow \quad \text{giant component}$$

How can we calculate z_2?

→ Let's use that $z_2 = z_1 G_q'(1) = \langle k \rangle \langle q(k) \rangle$. By using our former results for $q(k)$ we can write that

Percolation and generating functions

- What about **scale-free** networks?

→ For scale-free networks let us examine the general result that

$$\begin{array}{lcl} z_1 > z_2 & \rightarrow & \text{small, isolated clusters} \\ z_1 = z_2 & \rightarrow & \text{CRITICAL POINT!} \\ z_1 < z_2 & \rightarrow & \text{giant component} \end{array}$$

How can we calculate z_2?

→ Let's use that $z_2 = z_1 G_q'(1) = \langle k \rangle \langle q(k) \rangle$. By using our former results for $q(k)$ we can write that

$$\begin{aligned} z_2 \quad = \quad & \langle k \rangle \langle q(k) \rangle = \langle k \rangle \sum_{k=0}^{\infty} k q(k) = \langle k \rangle \sum_{k=0}^{\infty} k \frac{k+1}{\langle k \rangle} p(k+1) = \\ & \sum_{k=0}^{\infty} (k+1-1)(k+1) p(k+1) = \\ & \underbrace{\sum_{k=0}^{\infty} (k+1)^2 p(k+1)}_{\langle k^2 \rangle} - \underbrace{\sum_{k=0}^{\infty} (k+1) p(k+1)}_{\langle k \rangle} = \left\langle k^2 \right\rangle - \langle k \rangle . \end{aligned}$$

Percolation and generating functions

- What about **scale-free** networks?

→ For scale-free networks let us examine the general result that

$$
\begin{aligned}
z_1 > z_2 &\quad\rightarrow\quad \text{small, isolated clusters} \\
z_1 = z_2 &\quad\rightarrow\quad \text{CRITICAL POINT!} \\
z_1 < z_2 &\quad\rightarrow\quad \text{giant component}
\end{aligned}
$$

- Since $\langle k^2 \rangle$ is diverging in a scale-free network, according to this calculation they always contain a giant component!