Appendix A
Summations 加總公式

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A.1 Summation formulas and properties

- Σ-notation (加總符號)

$$\sum_{i=m}^{n} x_i = x_m + x_{m+1} + x_{m+2} + \cdots + x_{n-1} + x_n.$$

- Arithmetic series (算術/等差級數)

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1) = \Theta(n^2).$$

- Sum of squares and cubes (平方和與立方和)

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

- Geometric series (幾何/等比級數)

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}. \qquad \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1.$$

- Harmonic series (調和級數)

$$\begin{aligned} H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\ &= \sum_{k=1}^{n} \frac{1}{k} \\ &= \ln n + O(1). \end{aligned}$$

- Integrating & differentiating series (級數微分與積分): 可導出新的級數 (假設|x|<1)

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$ (幾何級數微分)

$$\sum_{k=1}^{\infty} kx^{k} = \frac{x}{(1-x)^2}$$

- Telescoping series (伸縮級數)

$$\sum_{k=1}^{n}(a_k - a_{k-1}) = a_n - a_0.$$

- Products (乘積)

$$\lg\left(\prod_{k=1}^{n} a_k\right) = \sum_{k=1}^{n} \lg a_k$$

Exercise

- 簡化以下公式

$$\sum_{k=1}^{n}(2k-1)$$

Exercise

- 簡化以下公式級數 (假設|x|<1)

$$\sum_{k=0}^{\infty}(2k+1)x^{2k}$$

A.2 Bounding summations 求解加總邊界值

1. Mathematical induction (數學歸納法)

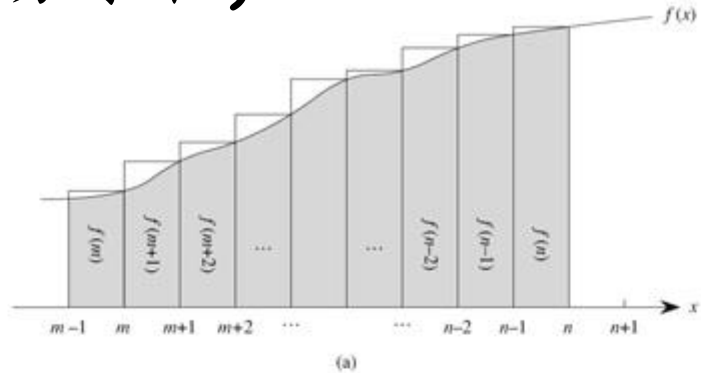
$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1)$$
$$= \frac{1}{2} n(n+2+(n+1)$$
$$= \frac{1}{2}(n+1)(n+2)$$

2. Bounding the terms (邊界條件)

$$\sum_{k=1}^{n} k \leq \sum_{k=1}^{n} n$$
$$= n^2.$$

3. Splitting summations (拆分求和)

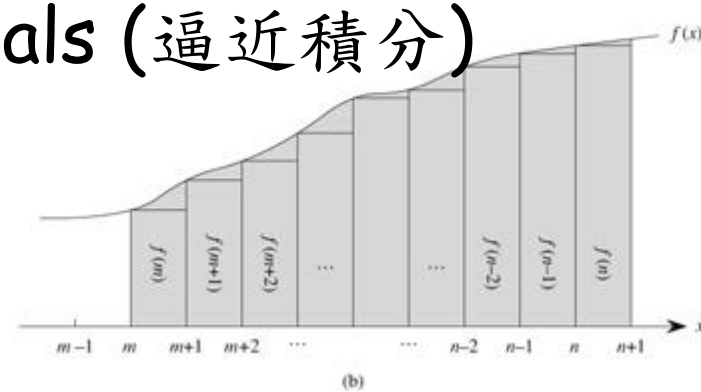
$$\sum_{k=1}^{n} k = \sum_{k=1}^{n/2} k + \sum_{k=n/2+1}^{n} k$$
$$\geq \sum_{k=1}^{n/2} 0 + \sum_{k=n/2+1}^{n} (n/2)$$
$$= (n/2)^2$$
$$= \Omega(n^2).$$



4. Approximation by integrals (逼近積分)

$$\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx.$$

$$\int_{m}^{n+1} f(x)\,dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)\,dx$$



$$\sum_{k=2}^{n} \frac{1}{k} \leq \int_{1}^{n} \frac{dx}{x}$$
$$= \ln n.$$

$$\sum_{k=1}^{n} \frac{1}{k} \leq \ln n + 1.$$

Exercise

- 求解以下公式

$$\sum_{k=1}^{n} 1/k$$

$$\sum_{k=1}^{n} 1/k^2$$