3.Growth of Functions

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Outline

- **Asymptotic notation**
- Standard notations
- Common functions

$$O \approx \leq$$
$$\Omega \approx \geq$$
$$\Theta \approx =$$
$$o \approx <$$
$$\omega \approx >$$

一種描述 function
"被侷限在什麼範圍"的表示法。

利用"抽象化"觀念
"常數不管它"，"次方小的不管它"

可以看出 function
"長得多快"。

進而用"人工智慧"
預測程式執行時間。

3.1 Asymptotic notation 漸近表示法

$$\Theta(g(n)) = \{f(n) \mid \exists c_1, c_2, n_0 \text{ s.t. } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\}$$

可找到3個常數, 使得 在 n 長得夠大時, f(n)會被包在g(n)乘上兩個常數之間

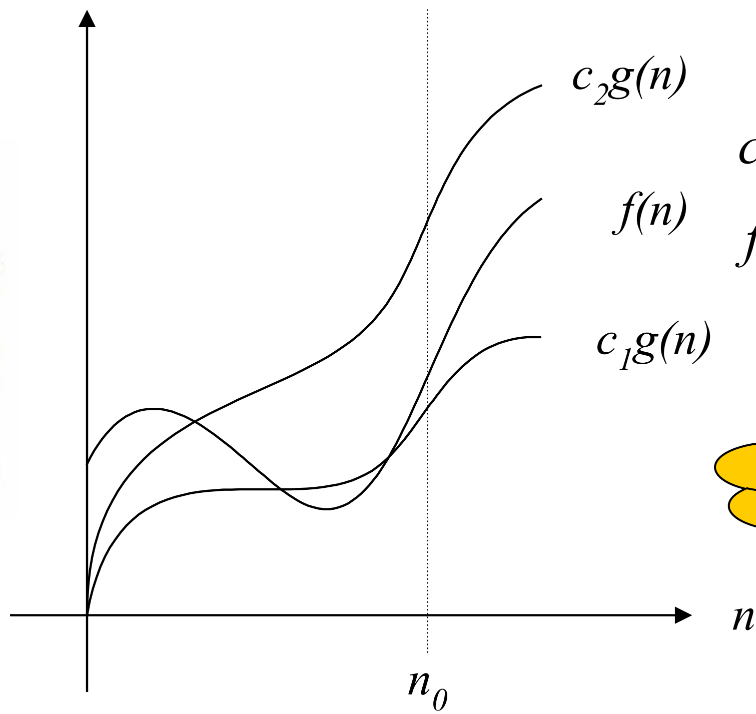
$$f(n) = \Theta(g(n))$$

哇麥咁妳攬條條: 太棒!

➔ $g(n)$ is an asymptotic **tight bound** for $f(n)$.

"=" abuse

- The definition of $\Theta g(n)$ required every member of $f(n) \in \Theta g(n)$ be asymptotically nonnegative.



$$c_1 g(n) \le f(n) \le c_2 g(n)$$

$$f(n) = \Theta(g(n))$$

f 在 c_1g 和 c_2g 之間

Example:

$$\frac{1}{2}n^2 - 3n = \Theta(n^2)$$

$$c_1 n^2 \le \frac{1}{2}n^2 - 3n \le c_2 n^2$$

$$c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2$$

$$\Longrightarrow \boxed{\frac{1}{2}n^2 - 3n = \Theta(n^2) \qquad n \ge 7}$$

$c_1 \le 1/14$

$n \ge 7$

$c_1 \ge 1/2$

$n \ge 1$

Example:

$$\frac{n^2}{14} \le \frac{n^2}{2} - 3n \le \frac{n^2}{2} \; if \; n>7.$$

$$6n^3 \ne \Theta(n^2)$$

$$f(n) = an^2 + bn + c, \quad a, b, c \text{ constants, } a > 0.$$

$$\Rightarrow f(n) = \Theta(n^2).$$

In general,

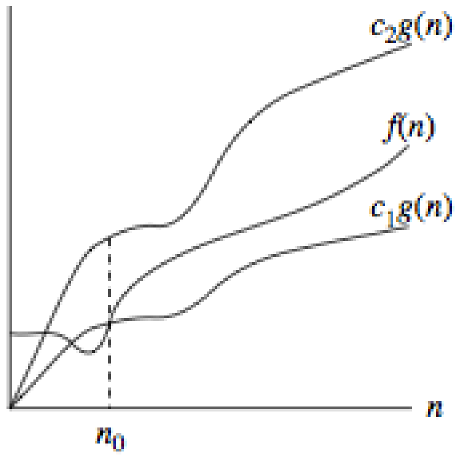
$$p(n) = \sum_{i=0}^{d} a_i n^i \text{ where } a_i \text{ are constant with } a_d > 0.$$

$$\text{Then } P(n) = \Theta(n^d).$$

常數不管它
次方小的不管它

Exercise

$$\frac{n^2}{2} - 2n = \Theta(n^2)$$



$$c_1 = 1/4$$

$$c_2 = 1/2$$

$$n_0 = 8$$

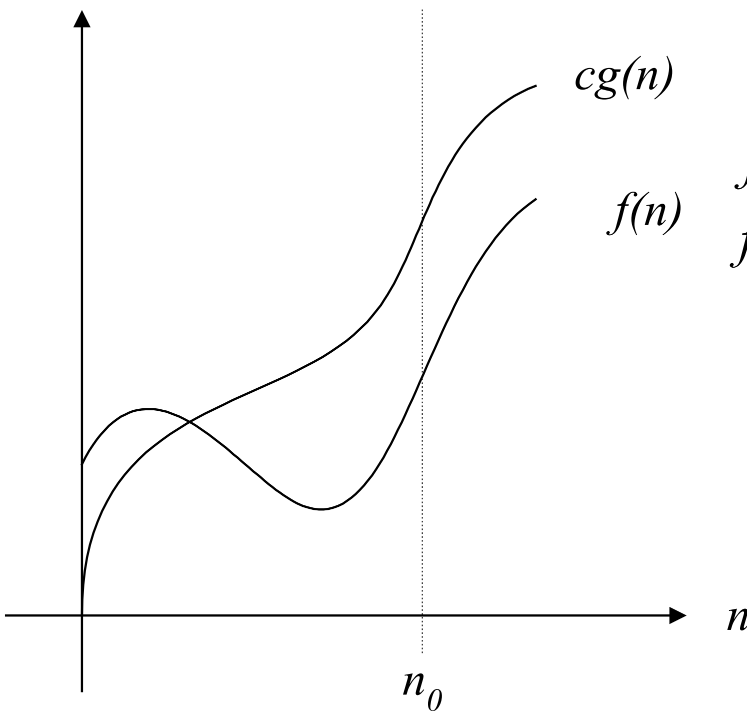
從定義來解決問題

$$f(n) = \Theta(g(n))$$

$$\Theta(g(n)) = \{f(n) \mid \exists c_1, c_2, n_0 \text{ s.t. } 0 \le c_1 g(n) \le f(n) \le c_2 g(n), \text{ for all } n \ge n_0\}$$

asymptotic upper bound

$$O(g(n)) = \{ f(n) \mid \exists c, n_0 \text{ s.t. } 0 \le f(n) \le cg(n)\ \forall n \ge n_0 \}$$



$f(n) \le cg(n)$

$f(n) = O(g(n))$

g 乘上一個常數
會 永遠在 f 上面

Exercise

- 以下何者不屬於 $O(n^2)$

$$2n^3$$

$$n^2$$

$$n^2 + n$$

$$1000n^2 + 1000n$$

$$n$$

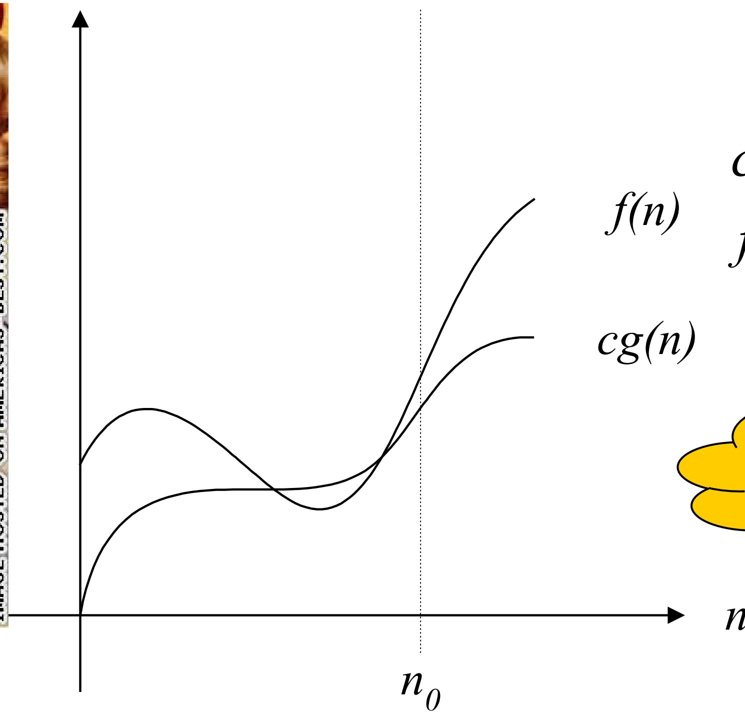
$$n/1000$$

$$n^{1.99999}$$

$$n^2/\lg\lg\lg n$$

asymptotic lower bound

$$\Omega(g(n)) = \{f(n) \mid \exists c, n_0 \text{ s.t. } 0 \le cg(n) \le f(n)\ \forall n \ge n_0\}$$



$cg(n) \le f(n)$

$f(n) = \Omega(g(n))$

g 乘上一個常數
會 永遠在 f 下面

Exercise

- 以下何者不屬於 $\Omega(n^2)$

$$\sqrt{n}$$

$$n^2$$

$$n^2 + n$$

$$n^2 - n$$

$$1000n^2 + 1000n$$

$$0.00001n^2 - 100000n$$

$$n^3$$

$$n^{2.00001}$$

$$n^2 \lg \lg \lg n$$

$$2^{2^n}$$

Theorem 3.1.

今天最重要的一頁！

- For any two functions *f*(*n*) and *g*(*n*), 任意兩函數f, g,

$$f(n) = \Theta(g(n))$$ f=Θ(g) 若且唯若

$$\iff f(n) = \mathrm{O}(g(n)) \wedge f(n) = \Omega(g(n))$$

f=O(g) 且 f=Ω(g)

- Example

$$f(n) = an^2 + bn + c \qquad g(n) = n^2$$

$$f(n) = \Theta(n^2)$$

$$\iff f(n) = O(n^2) \wedge f(n) = \Omega(n^2)$$

舉一反三，融會貫通

Exercise

$$
\begin{aligned}
2n^2 + 3n + 1 &= 2n^2 + \Theta(n) \\
&= \Theta(n^2)
\end{aligned}
$$

$$\forall f(n) \in \Theta(n), \exists g(n) \in \Theta(n^2) \implies 2n^2 + f(n) = g(n)$$

o-notation & ω-notation

$$o(g(n)) = \{f(n) \mid \forall c, \exists n_0 \forall n > n_0, 0 \le f(n) < cg(n)\}$$

$$f(n) = o(g(n)) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

g長得很快

$$\omega(g(n)) = \{f(n) \mid \forall c, \exists n_0 \forall n > n_0, 0 \le cg(n) < f(n)\}$$

$$f(n) = \omega(g(n)) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

f長得很快

Properties

- Transitivity (遞移律)

$$f(n)=\Theta(g(n))\wedge g(n)=\Theta(h(n))\Rightarrow f(n)=\Theta(h(n))$$
$$f(n)=O(g(n))\wedge g(n)=O(h(n))\Rightarrow f(n)=O(h(n))$$
$$f(n)=\Omega(g(n))\wedge g(n)=\Omega(h(n))\Rightarrow f(n)=\Omega(h(n))$$
$$f(n)=o(g(n))\wedge g(n)=o(h(n))\Rightarrow f(n)=o(h(n))$$
$$f(n)=\omega(g(n))\wedge g(n)=\omega(h(n))\Rightarrow f(n)=\omega(h(n))$$

- Reflexivity (反射律)

$$f(n)=\Theta(f(n))$$
$$f(n)=O(f(n))$$
$$f(n)=\Omega(f(n))$$

- Symmetry (對稱律)

$$f(n)=\Theta(g(n))\Leftrightarrow g(n)=\Theta(f(n))$$

- Transpose symmetry (轉置對稱律)

$$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$$
$$f(n) = o(g(n)) \Leftrightarrow g(n) = \omega(f(n))$$

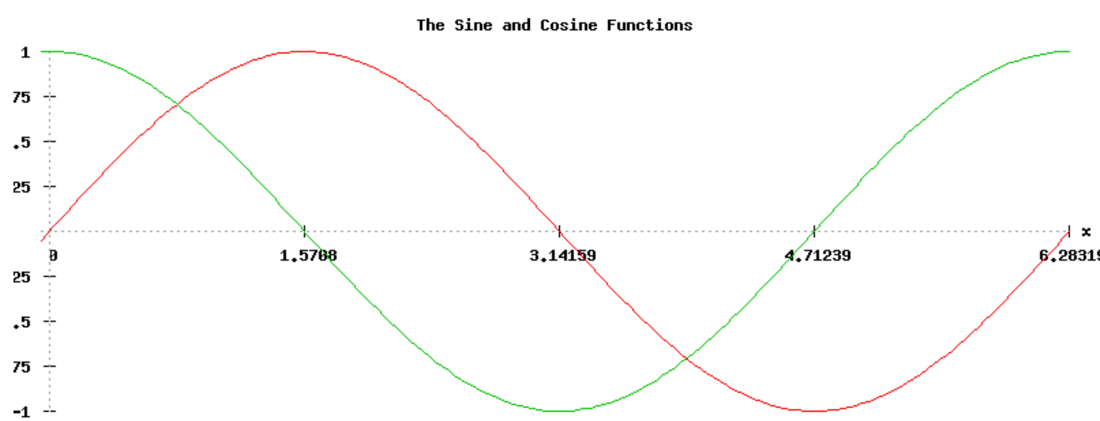
$$\left.\begin{aligned} f(n) &= O(g(n)) \approx a \le b \\ f(n) &= \Omega(g(n)) \approx a \ge b \\ f(n) &= \Theta(g(n)) \approx a = b \\ f(n) &= o(g(n)) \approx a < b \\ f(n) &= \omega(g(n)) \approx a > b \end{aligned}\right\} \text{用 } >, =, < \text{ 的關係來想}$$

Trichotomy 三分法

- 任意兩實數之間比大小，以下只有一種狀況成立:
  - $a < b$, $a = b$, or $a > b$.
- 任意兩個函數不一定能比大小
  - e.g.,

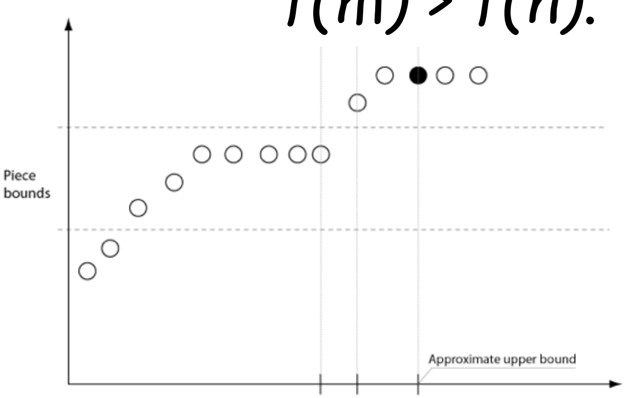
$$n \qquad n^{1+\sin n}$$

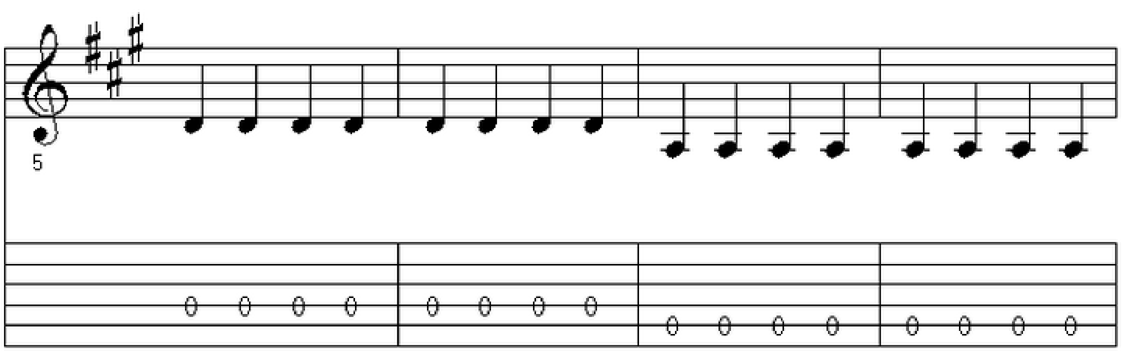
$$n^0 \quad ? \quad n^2$$



3.2 Standard notations and common functions

- **Monotonicity**: (單調性)
  - A function f is *monotonically increasing* if $m \le n$ implies $f(m) \le f(n)$. 單調遞增
  - A function f is *monotonically decreasing* if $m \le n$ implies $f(m) \ge f(n)$.
  - A function f is *strictly increasing* if $m < n$ implies $f(m) < f(n)$. 嚴格遞增
  - A function f is *strictly decreasing* if $m < n$ implies $f(m) > f(n)$.





Floor and ceiling 地板 & 天花板

$$x-1<\lfloor x\rfloor \le x \le \lceil x\rceil < x+1$$

$\lfloor x\rfloor$ 地板: 不大於x的最大整數

$\lceil x\rceil$ 天花板: 不小於x的最小整數

$$\lceil n/2\rceil + \lfloor n/2\rfloor = n$$

小學程度的題目：
每件貨品售價6元，某人有40元，最多可買多少件貨品？

$$\lceil \lceil n/a\rceil /b\rceil = \lceil n/ab\rceil$$

$$\lfloor \lfloor n/a\rfloor /b\rfloor = \lfloor n/ab\rfloor$$

$$\lceil a/b\rceil \le (a+(b-1))/b$$

$$\lfloor a/b\rfloor \ge (a-(b-1))/b$$

Modular arithmetic 除餘運算

- For any integer *a* and any positive integer *n*, the value *a* mod *n* is the **remainder** (or **residue**) of the quotient *a/n*：a 除以 n 之後的餘數

$$a \mod n = a - \lfloor a/n \rfloor n$$

- If(*a* mod *n*) = (*b* mod *n*). We write $a \equiv b \pmod{n}$ and say that a is **equivalent** to *b*, modulo *n*.
  a, b 除餘 n 相等
- We write $a \not\equiv b \pmod{n}$ if *a* is not equivalent to *b* modulo *n*.

Polynomials v.s. Exponentials

(多項式) (指數)

- Polynomials: $P(n) = \sum_{i=0}^{d} a_i n^i$
  - ✓ A function is *polynomial bounded* if $f(n) = n^{O(1)}$
- Exponentials: $n^b = o(a^n) \qquad (a > 1)$
  - ✓ **Any positive exponential function grows faster than any polynomial.** 任何指數函數長得比多項式快

詳見CLRS p.55

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$1 + x \le e^x \le 1 + x + x^2 \qquad \text{if } |x| < 1$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Logarithms (對數)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots \quad \text{if } |x| < 1$$

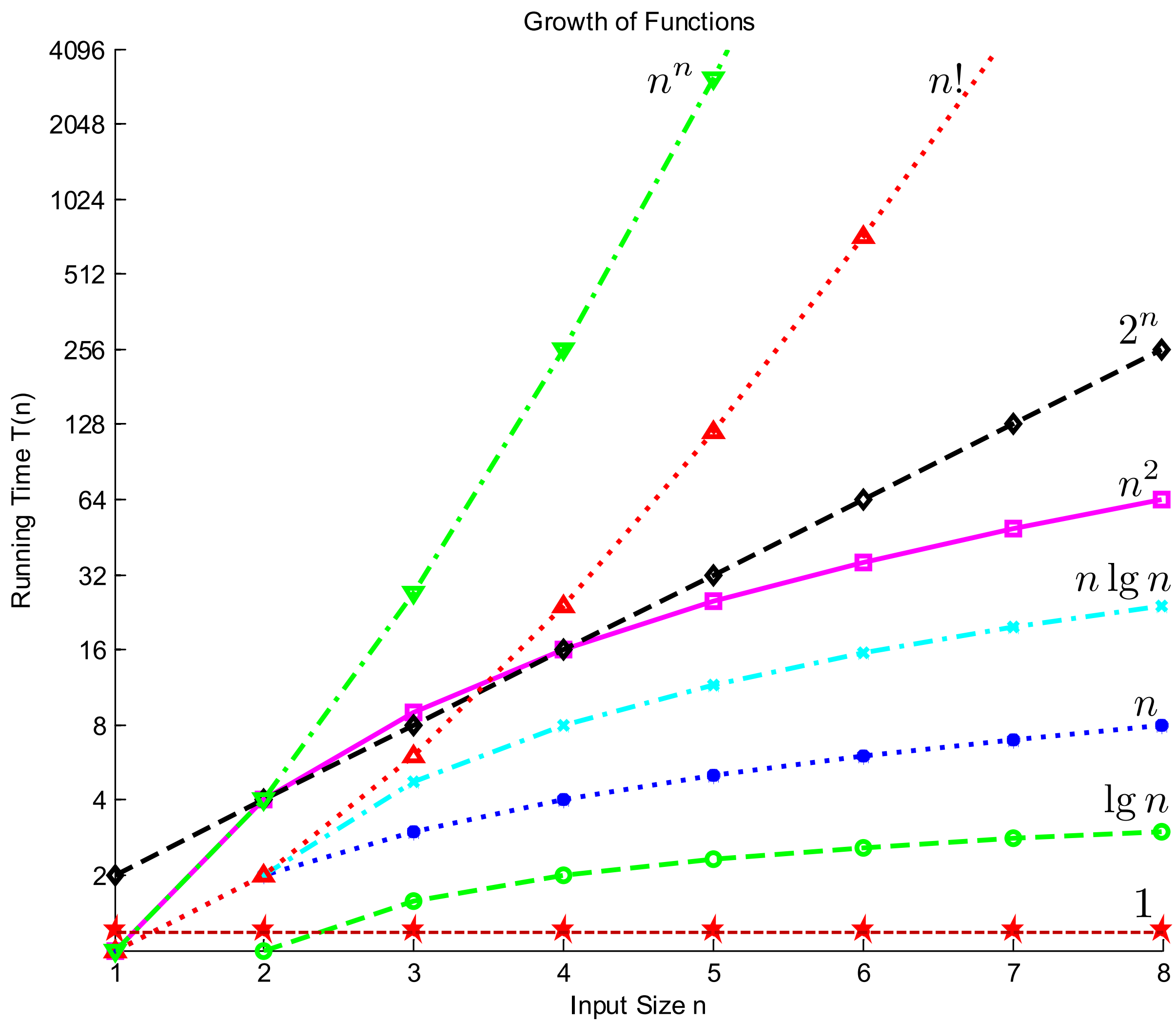
$$\frac{x}{1+x} \le \ln(1+x) \le x \quad \text{for } x > -1$$

詳見CLRS p.56

- A function *f(n)* is *polylogarithmically bounded* if $f(n) = \log^{O(1)} n$ (綜合對數)
- $\log^b n = o(n^a)$ for any constant *a* > 0.
- Any positive polynomial function grows faster than any polylogarithmic function.

任何多項式函數長得比綜合對數快

Growth of Functions



Factorials (階乘)

- Stirling's approximation 史特林近似法

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

Tight upper & lower bound

$$n! = o(n^n)$$

Weak upper bound

$$n! = \omega(2^n)$$

Weak lower bound

$$\log(n!) = \Theta(n \log n)$$

可用 Stirling's approx 證明

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n}$$

where $\frac{1}{12n+1} < \alpha_n < \frac{1}{12n}$

Function iteration 函數疊代

將 funcion 中的 n 值用 f(n) 代入 i 次.

$$f^{(i)}(n) = \begin{cases} n & \text{if } i = 0, \\ f(f^{(i-1)}(n)) & \text{if } i > 0. \end{cases}$$

For example, if $f(n) = 2n$, then $f^{(i)}(n) = 2^i n$

The iterative logarithm function

$$\lg^{(i)}(n) = \begin{cases} n & \text{if } i = 0 \\ \lg(\lg^{(i-1)} n) & \text{if } i > 0 \text{ and } \lg^{(i-1)} n > 0 \\ \text{undefined} & \text{if } i > 0 \text{ and } \lg^{(i-1)} n \le 0 \\ & \text{or } \lg^{(i-1)} n \text{ is undefined.} \end{cases}$$

$$\lg^*(n) = \min\{i \ge 0 \mid \lg^{(i)} \le 1\}$$

$$\left.\begin{aligned} &\lg^* 2 = 1 \\ &\lg^* 4 = 2 \\ &\lg^* 16 = 3 \\ &\lg^* 65536 = 4 \\ &\lg^* 2^{65536} = 5 \end{aligned}\right\}$$

增加得非常非常慢

- Since the number of atoms in the observable universe is estimated to be about 10^{80}, which is much less than 2^{65536}, we rarely encounter a value of *n* such that $\lg^* n > 5$.

由於可觀測的宇宙中的原子數估計大約為10^{80}，遠小於2^{65536}，因此我們很少遇到 $\lg^* n > 5$ 的值。

Fibonacci numbers 費波南茲級數

$$F_0 = 0$$

$$F_1 = 1$$

$$F_i = F_{i-1} + F_{i-2}$$

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$$

$$\phi = \frac{1+\sqrt{5}}{2} = 1.61803...$$

$$\hat{\phi} = \frac{1-\sqrt{5}}{2} = -0.61803...$$

Summary

- Asymptotic notation
  - ✓ Θ-notation, O-notation, Ω-notation, o-notation, ω-notation
- Properties of asymptotic comparison
- Common functions
  - ✓ Monotone functions
  - ✓ Floor & ceiling
  - ✓ a mod b
  - ✓ Polynomials
  - ✓ Exponentials
  - ✓ Logarithms
  - ✓ Factorials
  - ✓ Function iteration
  - ✓ Iterative logarithm function
  - ✓ Fibonacci numbers