7.Quicksort

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112學年度第2學期

Outline

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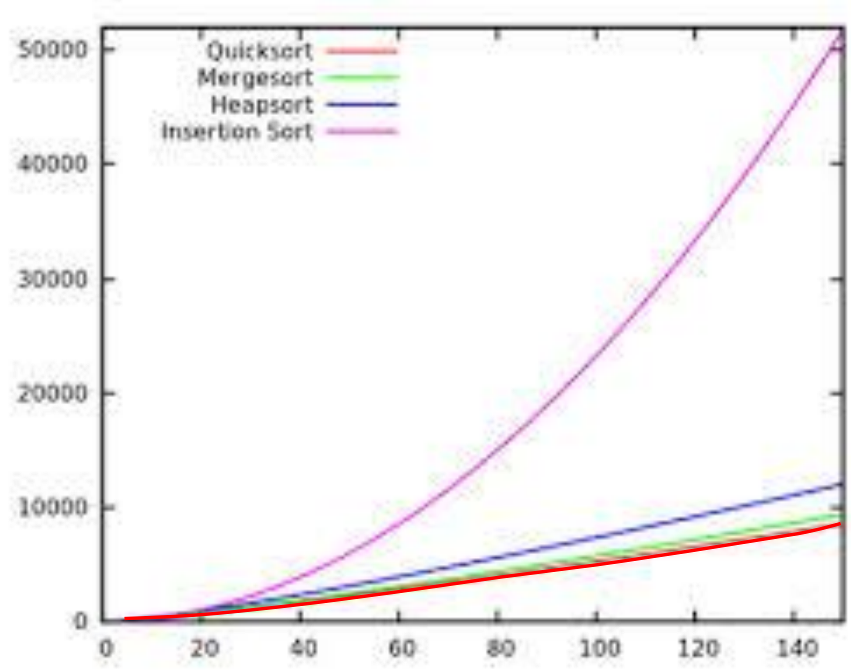
7.2 Analysis of Quick Sort 快速排序分析
7.3 Randomized Quick Sort 隨機快速排序
7.4 Analysis of Randomized Quick Sort 隨機快速排序分析



7.1 Quicksort

- Proposed by C.A.R. Hoare in 1962.
- *Divide-and-conquer* algorithm. 屬於各個擊破法
- Sorts "in place" (like insertion sort, but not like merge sort). 就地排序
- Very practical (with tuning). 實用性高

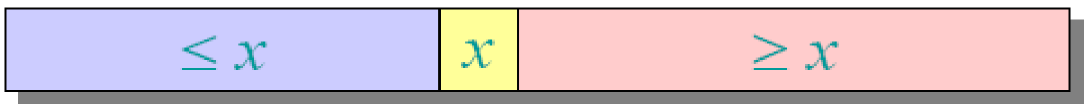
➢ Important results: 重要結果
  - ✓ Worst case $\Theta(n^2)$
  - ✓ Average case $\Theta(n \lg n)$
  - ✓ Hidden constants are small 隱藏的常數很小

Divide and conquer

Quicksort an *n*-element array: QUICKSORT(N) 演算法

1. *Divide*: Partition the array into two subarrays around a ***pivot x*** such that

Elements in lower subarray ≤ *x* ≤ elements in upper subarray.



2. *Conquer*: Recursively sort the two subarrays.

3. *Combine*: Trivial.

Key: ***Linear-time partitioning subroutine.***

1. **分割**: 選一個「基準值」*x*, 將陣列分割成左右兩邊, 使得 $left \le x \le right$
2. **各個擊破**: 左右兩邊再分別遞迴呼叫 QUICKSORT
3. **合併**: **不需要(已自動合併)**

關鍵步驟: 第1步的 ***分割子程序必須屬於線性O(N)***

Partitioning subroutine

分割子程序

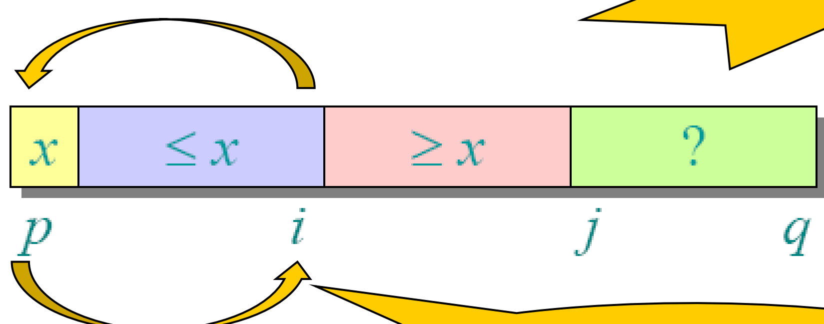
PARTITION(A, p, q) ▷ $A[p \,..\, q]$
  $x \leftarrow A[p]$ ▷ pivot = $A[p]$
  $i \leftarrow p$
  for $j \leftarrow p + 1$ **to** q
    do if $A[j] \le x$
      then $i \leftarrow i + 1$
        exchange $A[i] \leftrightarrow A[j]$
  exchange $A[p] \leftrightarrow A[i]$
  return i

1. 選第一個當 Pivot $O(1)$

2. 接著從第二個比到最後一個
如果比 Pivot 小就往前搬 $O(n)$

Running time = $O(n)$
for n elements

invariant:

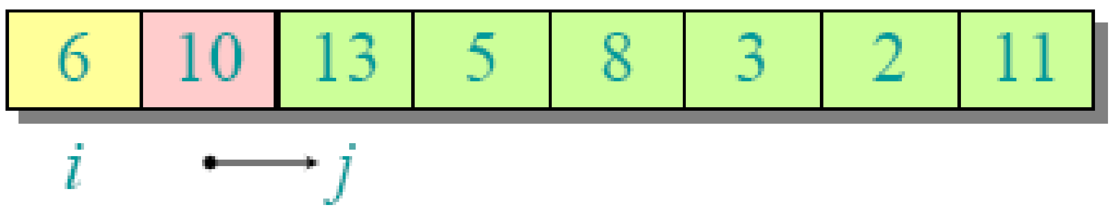


3. 最後第i個再跟第1個交換 $O(1)$

Example of partitioning

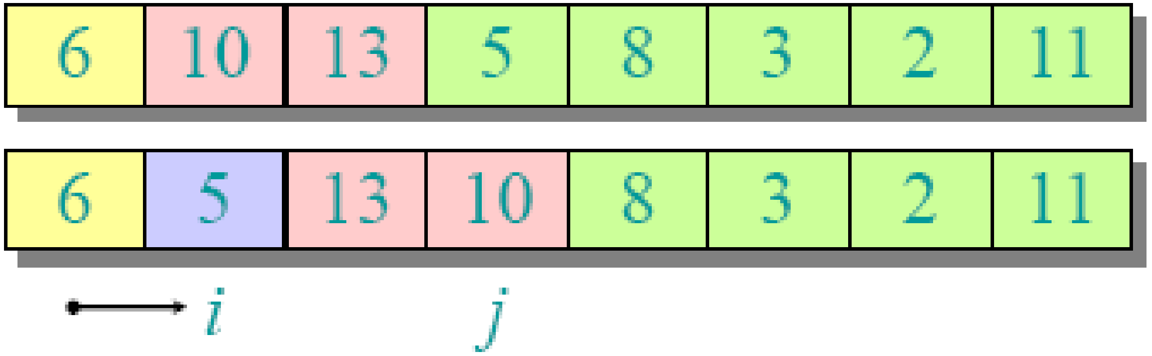
6	10	13	5	8	3	2	11
i	j						

Example of partitioning

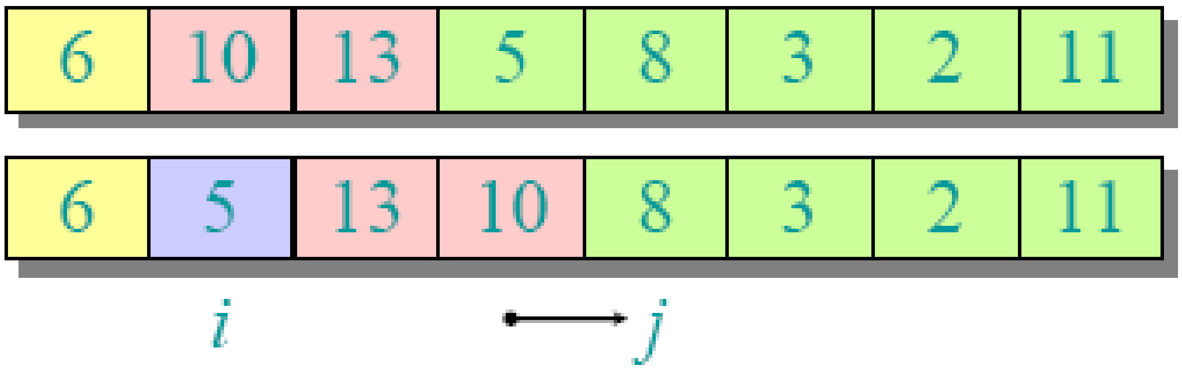


Example of partitioning

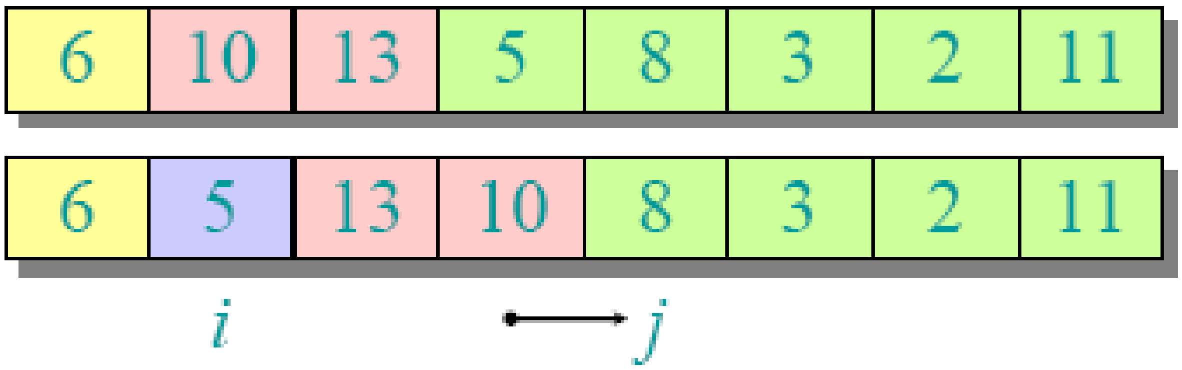
Example of partitioning



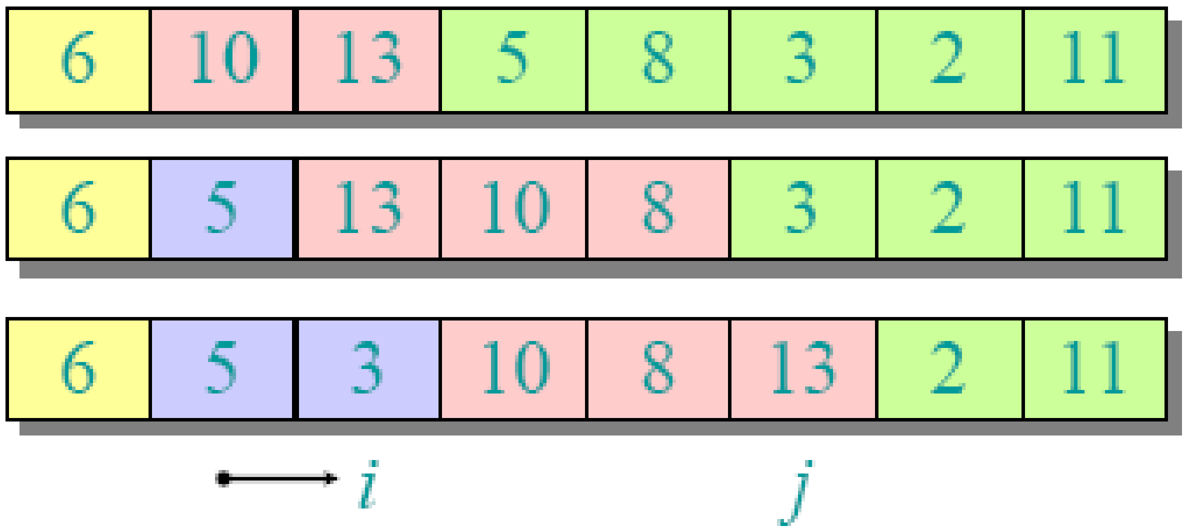
Example of partitioning



Example of partitioning



Example of partitioning



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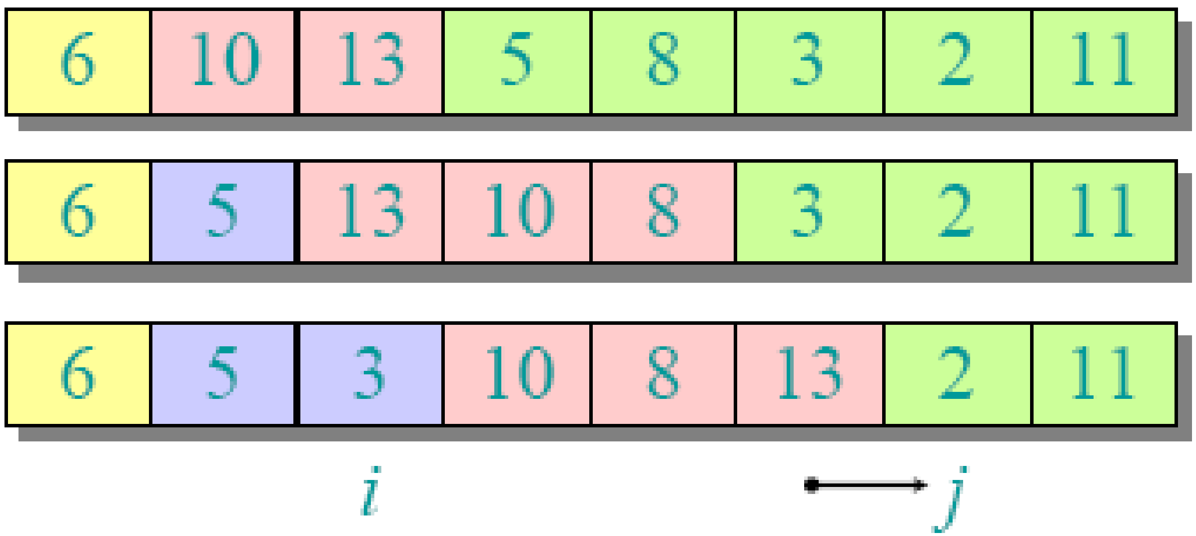
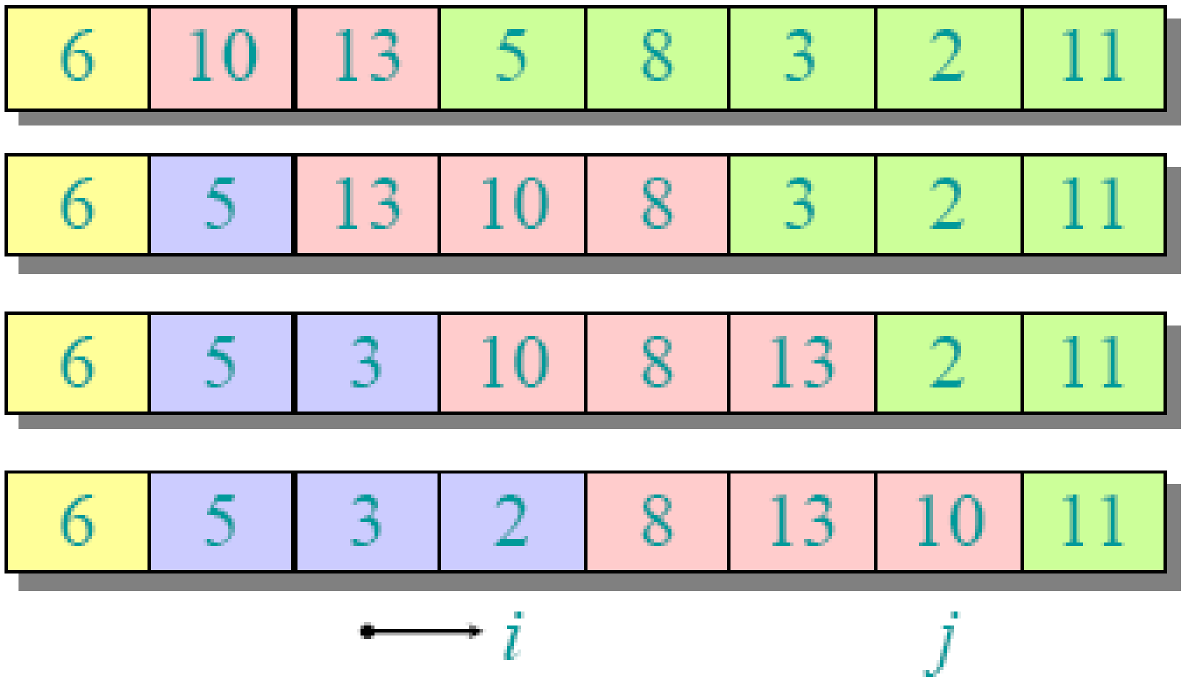
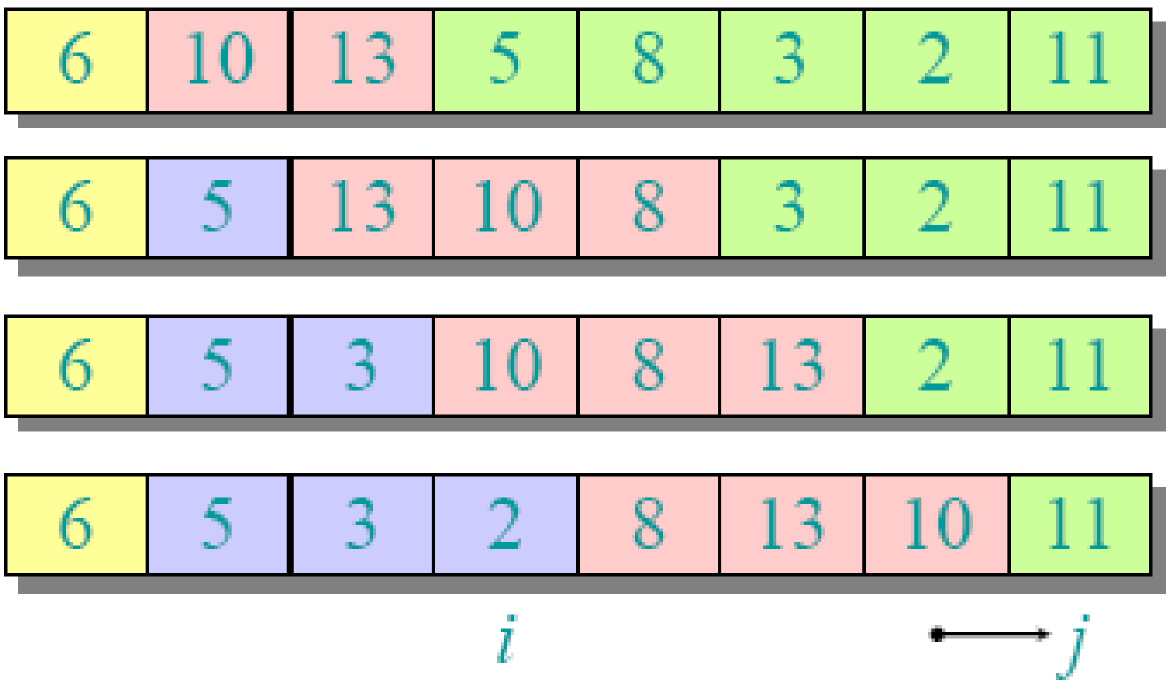


Example of partitioning



Example of partitioning



Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11

i $\qquad\qquad\qquad \longrightarrow j$

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
2	5	3	6	8	13	10	11

$$left \le x \le right$$

PARTITION C++ implementation

```cpp
unsigned partition(vector<int> & A, unsigned p, unsigned q)
{
    int x = A[p]; // pivot
    unsigned i = p; // index to be swapped
    for (unsigned j=p+1; j<=q; ++j) {
        if (A[j] < x) {
            ++i;
            swap(A[i], A[j]);
        }
    }
    swap(A[p], A[i]);
    return i;
} // end of partition
```

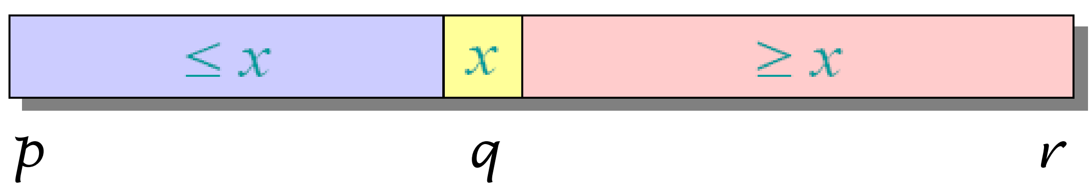
PARTITION(A, p, r)

$x = A[r]$
$i = p - 1$
for $j = p$ **to** $r - 1$
  if $A[j] \leq x$
    $i = i + 1$
    exchange $A[i]$ with $A[j]$
exchange $A[i + 1]$ with $A[r]$
return $i + 1$

Pseudocode for quicksort

```
QUICKSORT(A, p, r)
    if p < r
        then q ← PARTITION(A, p, r)
            QUICKSORT(A, p, q–1)
            QUICKSORT(A, q+1, r)
```

Initial call: QUICKSORT$(A, 1, n)$



QUICK_SORT C++ implementation

CLRS implementation:

```
void quick_sort(vector<int> & A, unsigned p, unsigned r)
{
    if (p < r) {
        unsigned q = partition(A, p, r);
        quick_sort(A, p, (q>0)?(q-1):0);
        quick_sort(A, q+1, r);
    }
} // end of quick_sort
```

QUICKSORT(A, p, r)

if $p < r$

    $q = $ PARTITION(A, p, r)

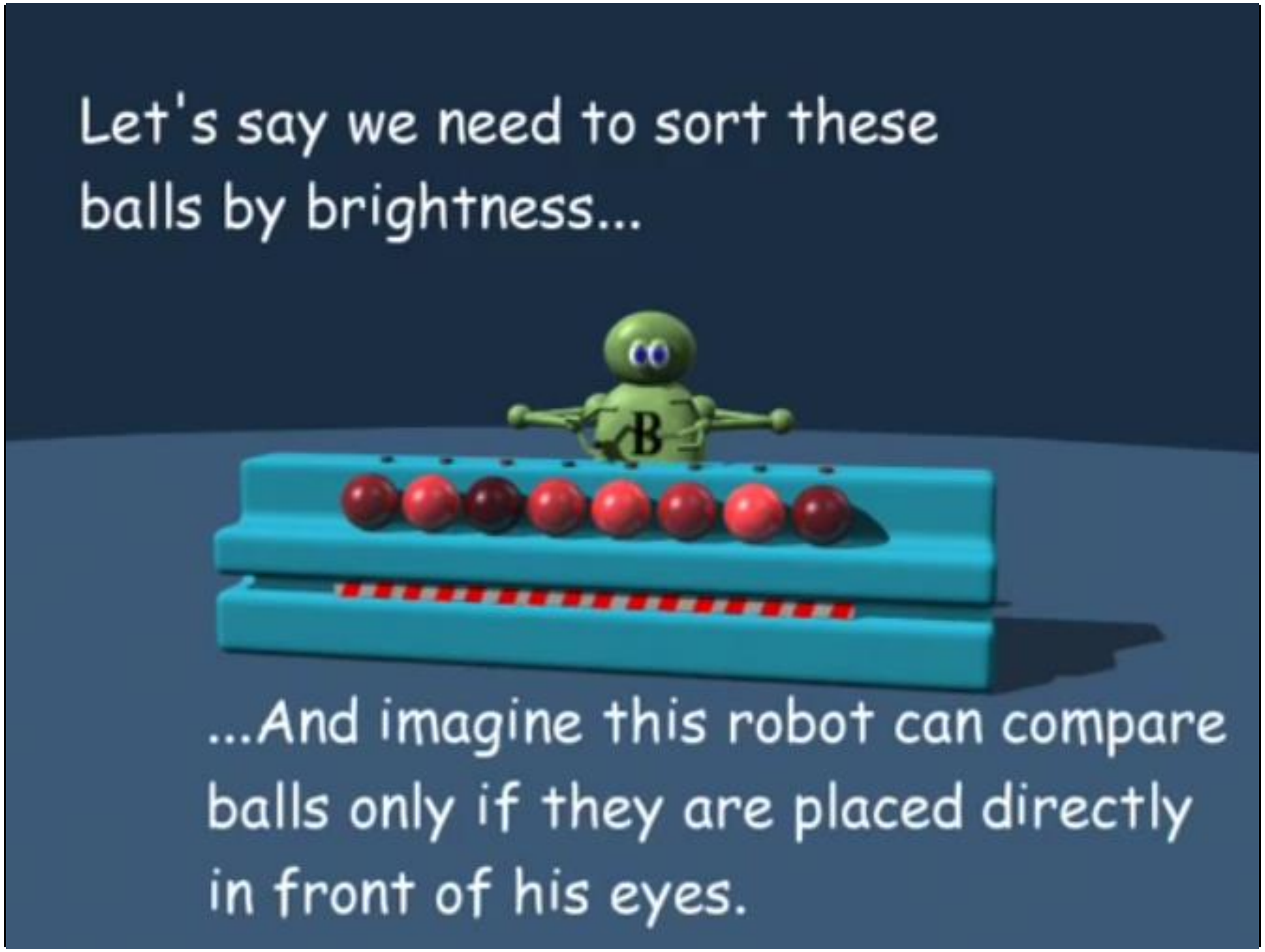
    QUICKSORT$(A, p, q-1)$

    QUICKSORT$(A, q+1, r)$

C++ STL implementation:

```
qsort(A.data(), A.size(), sizeof(int), [](const void * a, const void * b) { return ( *(int*)a - *(int*)b ); } );
```

QuickSort vs. BubbleSort



Exercise 7.1-1

- 請問以下陣列經過 PARTITION 運算後的結果為何?

  $A = \{13, 19, 9, 5, 12, 8, 7, 4, 21, 2, 6, 11\}$

[ANS]:

$A = \{ 11, 9, 5, 12, 8, 7, 4, 2, 6, \underline{13}, 21, 19\}$

7.2 Analysis of quicksort

- Assume all input elements are distinct.
  假設每個元素皆不同
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
  實務上若有相同元素可用更好的分割法
- Let $T(n)$ = worst-case running time on an array of n elements.

令$T(n)$ = worst-case 執行時間

Worst-case of quicksort 最壞情況

- Input sorted or reverse sorted.
  輸入元素已排序或已反向排序
- Partition around min or max element.
  以最大或最小元素為基準值做分割
- One side of partition always has no elements.
  一邊永遠沒有元素

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \quad \textbf{\textit{(arithmetic series)}}
\end{aligned}
$$

Worst-case recursion tree最壞情況

$$T(n) = T(0) + T(n-1) + cn$$

Worst-case recursion tree 最壞情況

$$T(n) = T(0) + T(n-1) + cn$$

$$T(n)$$

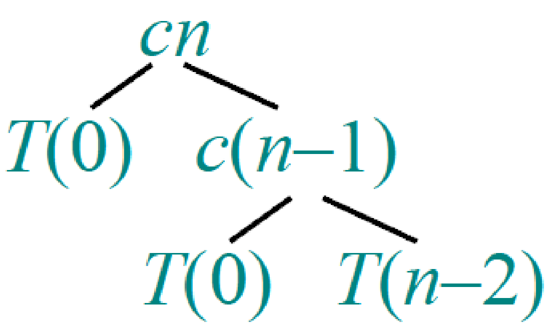
Worst-case recursion tree最壞情況

$$T(n) = T(0) + T(n-1) + cn$$

$$\begin{array}{ccc} & cn & \\ \swarrow & & \searrow \\ T(0) & & T(n-1) \end{array}$$

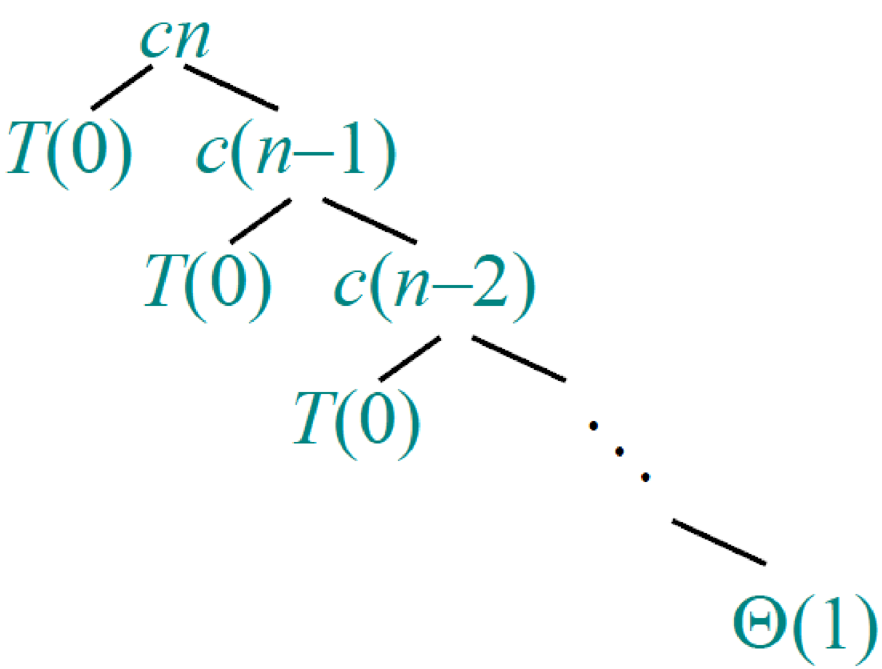
Worst-case recursion tree最壞情況

$$T(n) = T(0) + T(n-1) + cn$$



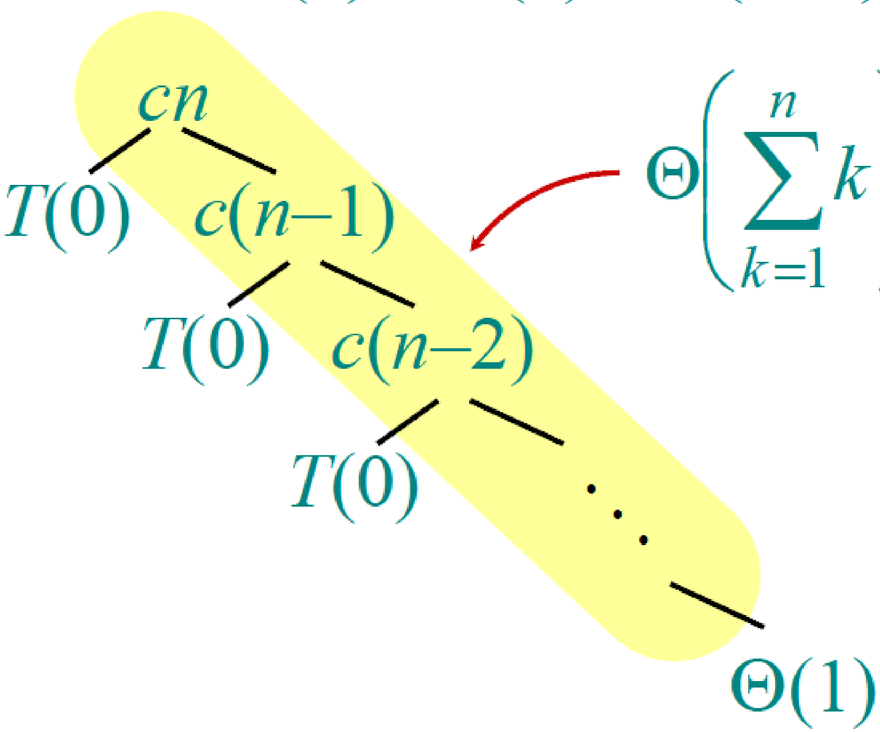
Worst-case recursion tree最壞情況

$$T(n) = T(0) + T(n-1) + cn$$



Worst-case recursion tree最壞情況

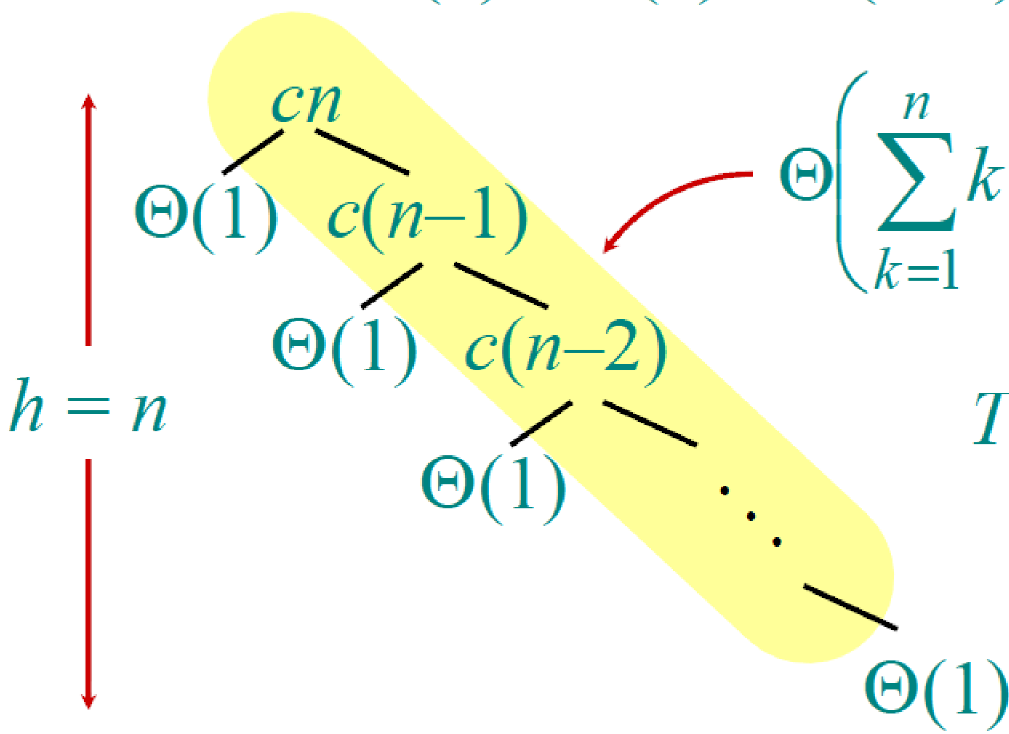
$$T(n) = T(0) + T(n-1) + cn$$



$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

Worst-case recursion tree最壞情況

$$T(n) = T(0) + T(n-1) + cn$$



$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$

Best-case analysis (For intuition only!)

最佳情況

If we're lucky, PARTITION splits the array evenly:

假設PARTITION很幸運地將array恰好切一半

$T(n) = 2T(n/2) + \Theta(n)$

$= \Theta(n \lg n)$ (same as merge sort)

What if the split is always $1/10:9/10$?

若每次分割都分成左:右=1/10:9/10

$T(n) = T(1/10n) + T(9/10n) + \Theta(n)$

What is the solution to this recurrence?

求解以上遞迴式

Analysis of “almost-best” case

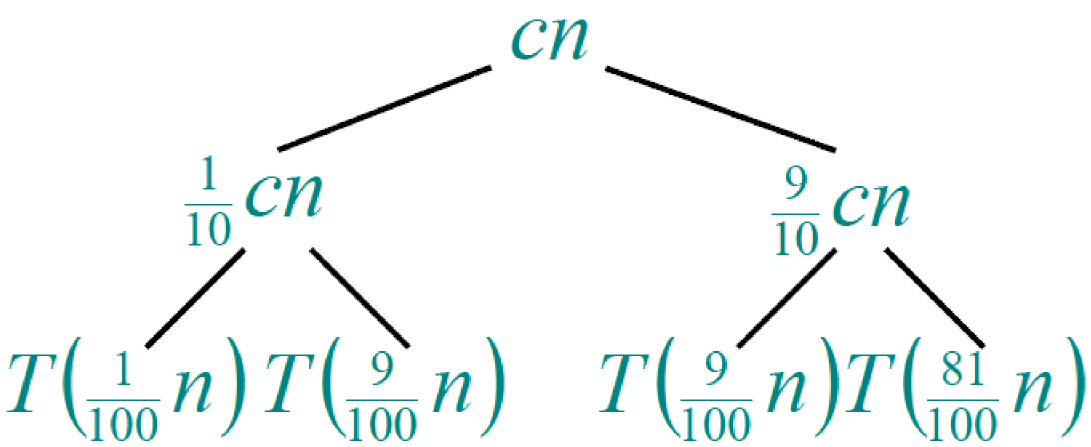
$T(n)$

Analysis of “almost-best” case

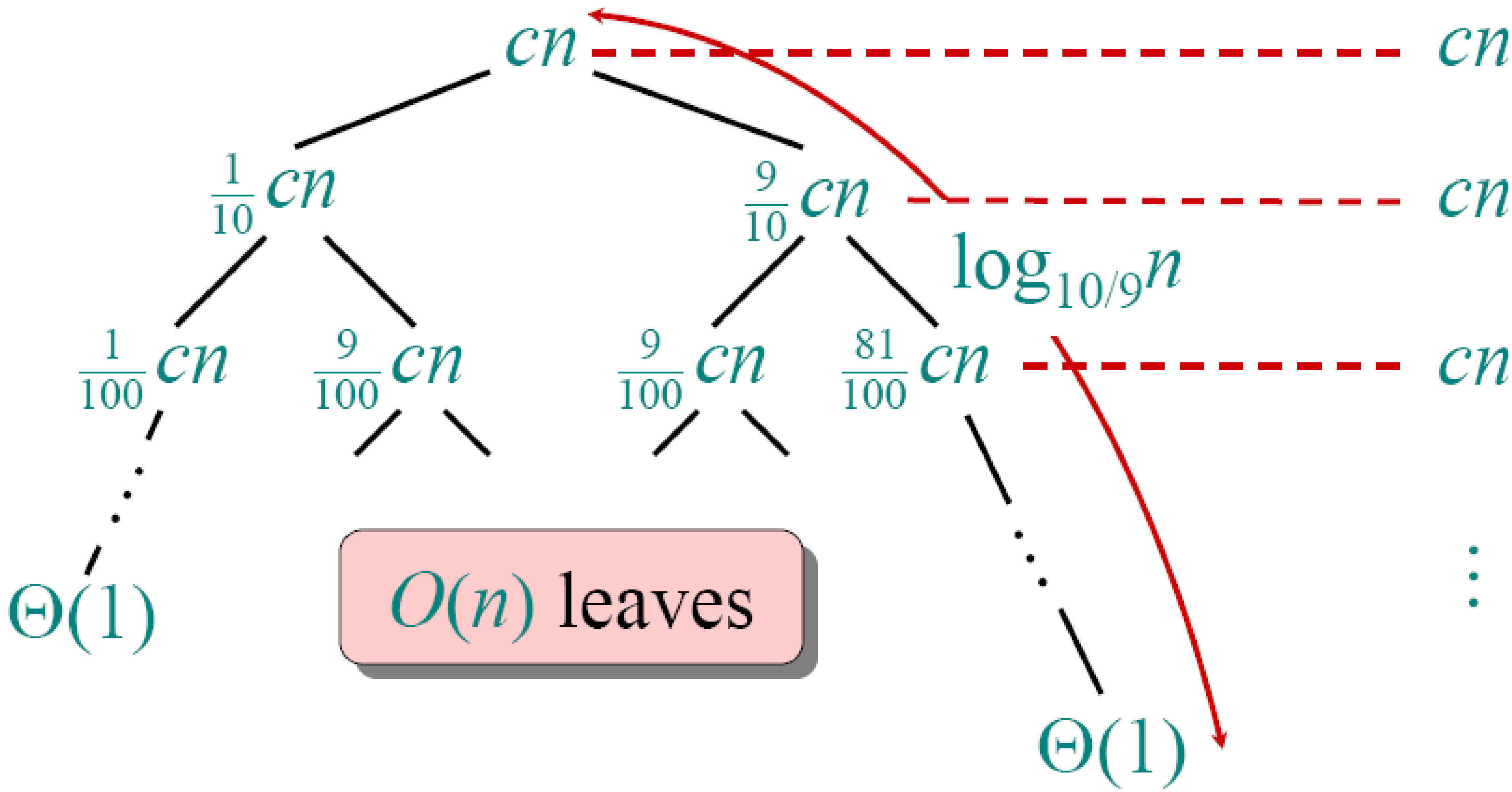
$$cn$$

$$T\left(\frac{1}{10}n\right) \qquad T\left(\frac{9}{10}n\right)$$

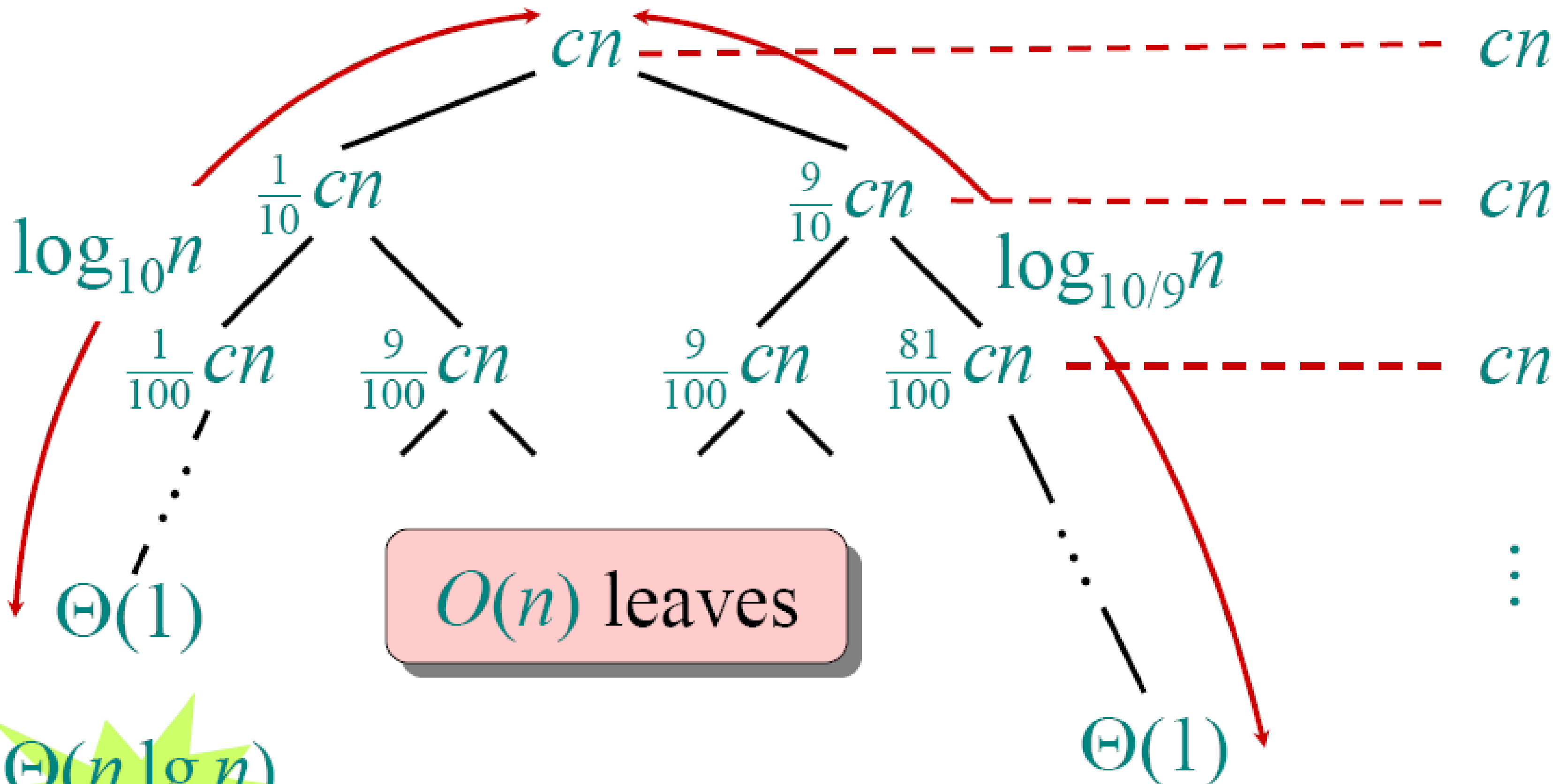
Analysis of "almost-best" case



Analysis of "almost-best" case



Analysis of “almost-best” case



$\Theta(n \lg n)$

Lucky!

$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

More intuition

進一步觀察

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ….

假設分割時運氣時好時壞

$L(n) = 2U(n/2) + \Theta(n)$ **lucky**

$U(n) = L(n-1) + \Theta(n)$ **unlucky**

Solving:

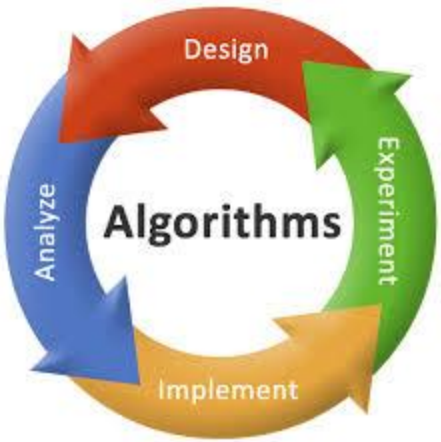
$$\begin{aligned} L(n) &= 2U(n/2) + \Theta(n) \\ &= 2(L(n/2 - 1) + \Theta(n/2)) + \Theta(n) \\ &= 2L(n/2 - 1) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

How can we make sure we are usually lucky?

如何保證分割時通常都很好運?

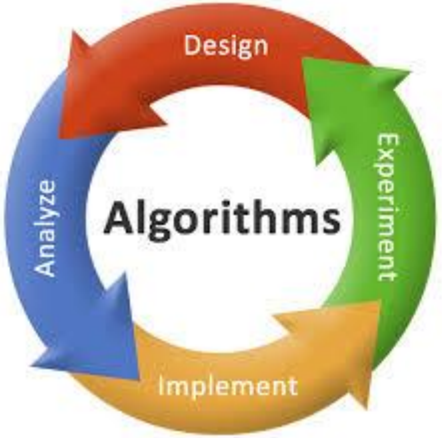
Exercise 7.2-2

- 當陣列*A*中元素的值都相同時, QUICKSORT 演算法執行時間複雜度以 Θ 表示是多少?



Exercise 7.2-3

- 證明當陣列*A*中元素已排序時, QUICKSORT 演算法執行時間為 $\Theta(n^2)$



7.3 Randomized quicksort

IDEA: Partition around a ***random*** element.
想法：隨機取一個基準元素做PARTITION

- Running time is *independent of the input* order. 執行時間與輸入無關
- No assumptions need to be made about the input distribution. 無需事先得知輸入分佈
- No specific input elicits the worst-case behavior. 賭最壞情況幾乎不會發生
- The worst case is determined only by the output of a random-number generator.
  最壞情況僅跟亂數產生步驟有關

RANDOMIZED_PARTITION(A,p,r)

$i \leftarrow RANDOM(p,r)$

exchange $A[p] \leftrightarrow A[i]$

return PARTITION(A,p,r)

隨機分割: RANSOMIZED_PARTITION

1. 呼叫RANDOM 隨機選一個 *Pivot*
2. 把*Pivot*和第1個元素交換
3. 呼叫 PARTITION 子程序

RANDOMIZED_QUICKSORT(A,p,r)

if $p < r$

then

$q \leftarrow RANDOMIZED\_PARTITION(A, p, r)$

RANDOMIZED_QUICKSORT(A,p,q)

RANDOMIZED_QUICKSORT(A,$q+1$,r)

隨機快速排序:

RANDOMIZED_QUICKSORT(N)

1. 呼叫RANSOMIZED_PARTITION
2. RANDOMIZED_QUICKSORT(左)
3. RANDOMIZED_QUICKSORT(右)

C++ RANDOMIZED_QUICKSORT implementation

```cpp
unsigned randomized_partition(vector<int> & A, unsigned p, unsigned r)
{
    unsigned i = random(p, r);
    swap(A[p], A[i]);
    return partition(A, p, r);
} // end of randomized_partition
```

RANDOMIZED-PARTITION(A, p, r)

```
i = RANDOM(p, r)
exchange A[r] with A[i]
return PARTITION(A, p, r)
```

```cpp
void randomized_quicksort(vector<int> & A, unsigned p, unsigned r)
{
    if (p < r) {
        unsigned q = randomized_partition(A, p, r);
        randomized_quicksort(A, p, (q>0)?(q-1):0);
        randomized_quicksort(A, q+1, r);
    }
} // end of randomized_quicksort
```

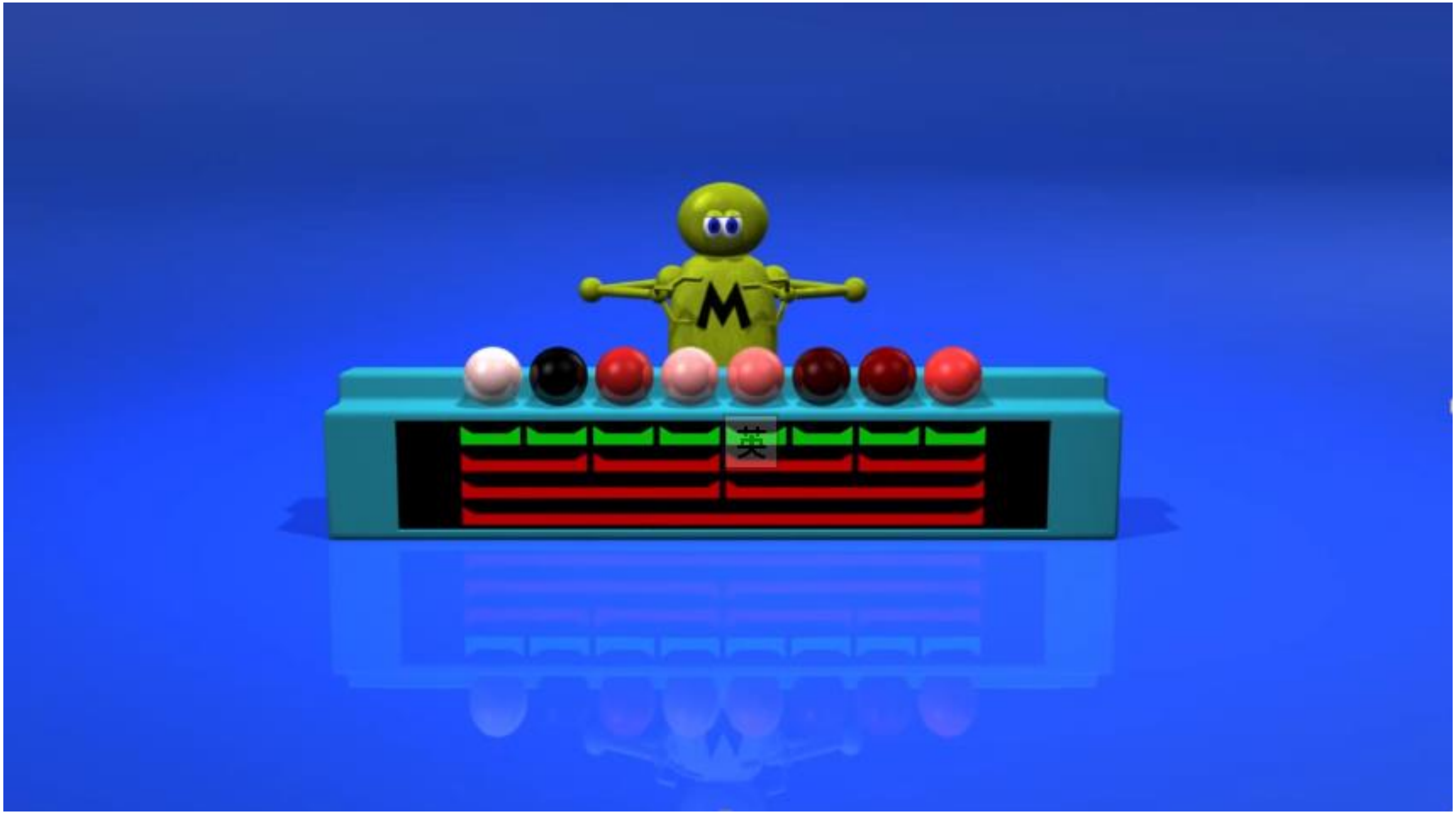
RANDOMIZED-QUICKSORT(A, p, r)

```
if p < r
    q = RANDOMIZED-PARTITION(A, p, r)
    RANDOMIZED-QUICKSORT(A, p, q - 1)
    RANDOMIZED-QUICKSORT(A, q + 1, r)
```

```cpp
unsigned random(unsigned p, unsigned r)
{
    std::default_random_engine generator = my_engine();
    std::uniform_int_distribution<unsigned> distribution(p, r); // define the range
    return distribution(generator); // generate a number
} // end of random
```

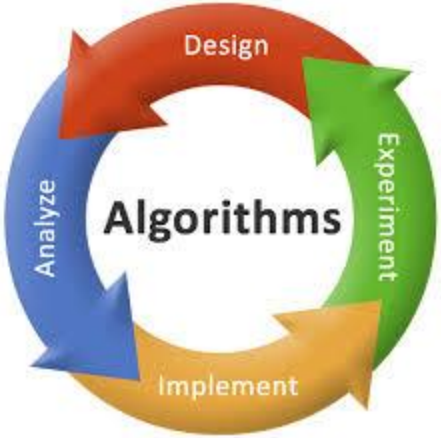
```cpp
std::default_random_engine & my_engine()
{
    static std::default_random_engine e{std::random_device{}()};
    return e;
} // end of my_engine
```

QuickSort vs. MergeSort



Exercise 7.3-1

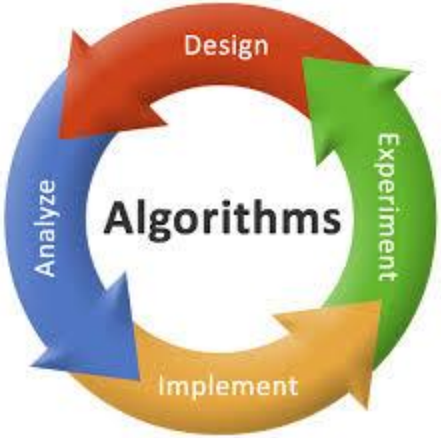
- 請將RANDOMIZED_QUICKSORT 演算法worst case執行時間複雜度以Θ表示。



Exercise 7.3-2

- 當RANDOMIZED_QUICKSORT執行時,

1. 在最佳狀況時 RANDOM函式會被呼叫幾次?
2. 在最差狀況時 RANDOM函式會被呼叫幾次?



7.4 Randomized quicksort analysis 分析隨機快速排序法

Let $T(n)$ = the **random variable** for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

令T(n)為隨機快速排序n 個元素的執行時間之隨機變數

For $k = 0, 1, \ldots, n-1$, 定義指標隨機變數 X_k

define the ***indicator random variable***

$X_k = \begin{cases} 1 & \text{若 PARTITION 成 } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$ (在第k個元素分割) (不在第k個元素分割)

$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.

因為分割在第幾個元素的機會都一樣,

所以X_k 的期望值= $1/n$

Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right).$$

Calculating expectation

$$T(n) = \sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n))$$

等號兩邊取期望值

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

期望值線性特性:
E[X+Y] = E[X] + E[Y]

$$= \sum_{k=0}^{n-1} E\big[X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

期望值為線性;
E[XY] = E[X] E[Y]

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

每個 X_k 隨機選項為獨立
$E[X_k] = 1/n$.

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

前兩項加總式其實一樣.

$$= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n)$$

Hairy recurrence

$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(k = 0, 1 項可被 $\Theta(n)$ 吸收.)

Prove: $E[T(n)] \le an\lg n$ for constant $a > 0$.

- 選一個夠大的 a 值, 使得 $an\lg n$ 大於 $E[T(n)]$ for sufficiently small $n \ge 2$.

Use fact: $\displaystyle\sum_{k=2}^{n-1} k\lg k \le \tfrac{1}{2}n^2\lg n - \tfrac{1}{8}n^2$ (exercise).

Substitution method

$$E[T(n)] \leq \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

取代歸納假設. *(假設* $E[T(n)] \leq an \lg n$ *)*

Substitution method

$$E[T(n)] \leq \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\leq \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.

Use fact: $\sum_{k=2}^{n-1} k\lg k \leq \frac{1}{2}n^2\lg n - \frac{1}{8}n^2$ (exercise).

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

移項將上式轉換成以 ***desired* – *residual*** 表示

(把欲證明項在一起, 再減掉其餘項)

Substitution method

$$
\begin{aligned}
E[T(n)] &\le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n) \\
&= \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n) \\
&= an\lg n - \left(\frac{an}{4} - \Theta(n)\right) \\
&\le an\lg n,
\end{aligned}
$$

當 *a* *夠大時*, *an*/4 支配 Θ(*n*) 項 (使得 *an*/4 - Θ(*n*) > 0).

Exercise 7.4-1

- 證明 $\sum_{k=2}^{n-1} k\lg k \le \frac{1}{2}n^2\lg n - \frac{1}{8}n^2$

[ANS]:

$$\sum_{k=1}^{n-1} k\lg k = \sum_{k=1}^{\lceil n/2\rceil-1} k\lg k + \sum_{k=\lceil n/2\rceil}^{n-1} k\lg k$$

拆分求和(詳附錄A)

$$\le \sum_{k=1}^{\lceil n/2\rceil-1} k\lg n + \sum_{k=\lceil n/2\rceil}^{n-1} k\lg(n-1)$$

每項分別取最大

$$\le \sum_{k=1}^{\lceil n/2\rceil-1} k(\lg n - 1) + \sum_{k=\lceil n/2\rceil}^{n-1} k\lg n$$

前後項最大對調

$$= (\lg n - 1)\sum_{k=1}^{\lceil n/2\rceil-1} k + \lg n \sum_{k=\lceil n/2\rceil}^{n-1} k$$

最大項與K值無關

$$= \lg n \sum_{k=1}^{n-1} k - \sum_{k=1}^{\lceil n/2\rceil-1} k \le \frac{n(n-1)\lg n}{2} - \frac{1}{2}\left(\frac{n}{2}-1\right)\frac{n}{2}$$

$$\le \frac{n^2\lg n}{2} - \frac{n^2}{8}\text{, if } n \ge 2.$$

Quicksort in practice *實務上應用*

- Quicksort is a great general-purpose sorting algorithm. 絶佳通用排序演算法
- Quicksort is typically over twice as fast as merge sort. 通常比 MergeSort 快一倍
- Quicksort can benefit substantially from ***code tuning***. 藉由程式碼最佳化可大幅提昇效率
- Quicksort behaves well even with caching and virtual memory. 在快取和虛擬記憶體環境中效率表現尤佳

Summary

- **QUICKSORT**
  - Worst-case $\Theta(n^2)$
  - Expected $\Theta(n \lg n)$
  - Divide & Conquer
    - **PARTITION**
- **RANDOMIZED_QUICKSORT**
  - **RANDOMIZED PARTITION**
- Analysis of Quicksort Algorithm
  - Worst-case $\Theta(n^2)$
  - Average-case $\Theta(n \lg n)$