1 Base for nilpotent operators

Definition 1.1**(basic): We say N "basic" with $(e_1, ..., e_2)$ if the basis $(e_1, ..., e_n)$ satisfy**

$$Ne_1 = 0, Ne_2 = e_1, ..., Ne_n = e_{n-1}$$

Lemma 1.2: **Suppse $\dim \operatorname{Null}(N^i) = i$ for $i \in \{1, ..., n\}$, then**

N is basic with some basis

Proof. proof by two steps.

- Step 1: Claim: for $1 \leq i \leq n$, $N(\operatorname{Null}(N^i)) = \operatorname{Null}(N^{i-1})$
  Proof of step 1. If $v \in \operatorname{Null} N^i$, then $0 = N^i v = N^{i-1}(Nv)$. Thus, $Nv \in \operatorname{Null} N^{i-1}$.
  i.e, $N(\operatorname{Null}(N^i)) \subseteq \operatorname{Null}(N^{i-1})$
  Denote $N_0 \in \mathcal{L}(\operatorname{Null} N^i, \operatorname{Null} N^{i-1})$ as $N_0 v = Nv$

$$\begin{aligned} \dim \operatorname{Range} N_0 + \dim \operatorname{Null} N_0 &= \dim \operatorname{Null} N^i \\ \dim \operatorname{Range} N_0 + 1 &= \dim \operatorname{Null} N^i \end{aligned}$$

  Therefore $\dim \operatorname{Range} N_0 = i - 1$, it follows that N_0 is surjective. i.e, $\operatorname{Null}(N^{i-1}) \subseteq N(\operatorname{Null}(N^i))$ □
- Step 2: Let $e_1 \in \operatorname{Null}(N) \setminus \{0\}$.
  Pick $e_2 \in \operatorname{Null}(N^2)$ such that $Ne_2 = e_1$. Repeat picking, we get

$$E = (e_1, ..., e_n) \text{ such that } e_i \in \operatorname{Null}(N^i), Ne_i = e_{i-1} i \geq 2$$

  rewrite it, we see there is exists e such that $E = (e, Ne, ..., N^{n-1}e)$

- Step 3: Check it is linearly independent.
  Suppose it is not linearly independent, then there is a non-trivial solution $a_1, ..., a_n$ such that

$$\sum_{i=1}^{n} a_i e_i = 0$$

  Let m be the largest index such that $a_m \neq 0$.
  Then

$$0 = \sum a_i e_i \Rightarrow 0 = N^{m-1} \sum a_i e_i = a_m e_1$$

  Contradiction.

□

2 Jordan for Nilpotent

Theorem 2.1 *Suppose $V \in \mathbf{F}^n$ be f.d.v.s., $N \in \mathcal{L}(V)$ is a nilpotent, then we can decompose V into subspace $V_1, ..., V_m$ such that*

$$V = \bigoplus_{i=1}^{m} V_i$$

1. *N preserves each V_i*
2. *$N|V_i$ are basic nilpotent operator*

Example 2.1. $\dim V = 3, N^2 = 0, \dim \operatorname{Range}(N) = 1$
Suppose

$$\operatorname{Range}(N) = \bigoplus_{i=1}^{m} \operatorname{Range}(N|V_i)$$

Therefore, wlog $\operatorname{Range}(N|V_1) = \operatorname{Range}(N)$ and $N|V_i = 0|V_1$ for $i > 1$.
Denote e, f as $\operatorname{Range}(N) = \operatorname{span}(e)$, and $e = Nf$, then $V_1 := \operatorname{span}(e, f)$
We know that $\operatorname{Null}(N) = 2$ and $e \in \operatorname{Null}(N)$, then denote g as $\operatorname{Null}(N) = \operatorname{span}(e, g)$. Then $V_2 := \operatorname{span}(g)$

Proof of Theorem 2.1. Proceed by induction on $\dim V$.

- Step 1: $\dim \operatorname{Range}(N) < \dim V$, and N **preserves** $\operatorname{Range}(N)$.
  Thus, we apply induction to $(\operatorname{Range}(N), N|\operatorname{Range}(N))$
- Step 2: By induction hypthesis, we can find a basis for $\operatorname{Range}(N)$ as follows:

$$e_1, ..., e_k \text{ such that } \operatorname{Range}(N) = \bigoplus_{i=1}^{k} W_i \ ,$$

  where $W_i = \operatorname{span}(e_i, Ne_i, ...N^{m_i}e^i)$, where m_i satisfy $N^{m_i}e_i \neq 0$ but $N^{m_i+1}e_i = 0$
  Denote $\epsilon = \{N^j e_i | \forall 0 \leq j \leq m_i, 1 \leq i \leq k)\}$. Then we see that

$$\operatorname{Range}(N) = \operatorname{span}(r)$$

- Step 3: Let $\sigma = (f_1, ..., f_k), f_i \in V$, be such that $Nf_i = e_i$.
  Claim (ϵ, σ) are linearly independent.
  Proof of the claim. We already know that ϵ is linearly independent, so suppose f_1 is involved, i.e., has non-zero coefficient.
  Then, after apply N, we find a linearly dependent that does not involving fs, which comes a Contradiction. □
- Step 4: For $1 \leq i \leq k$, define

$$V_i := \operatorname{span}(f_i, e_i, Ne_i, ..., N^{m_i}e_i) = \operatorname{span}(f_i, Nf_i, ..., N^{m_i+1}f_i),$$

  where $N^{m_i+2}f_i = 0$
- Let $f_{k+1}, ..., f_m \in V$ be such that $(f_1, ..., f_m, e_1, ..., N^{m_1}e_1, ..., e_k, ..., N^{m_k}e_k)$ is the basis of V.

Note: for each j such that $k+1 \leq j \leq m$, we have

$$Nf_j \in \text{Range}(N) = \text{span}(N^j e_i)$$

Therefore, there exists $g_j \in \text{Range}(N)$ such that $Ng_j = Nf_j$. Define $f_j' := f_j - g_j$, then

$$(f_1, ..., f_k, f_{k+1}', ..., f_m', e_1, ..., N^{m_1}e_1, ..., e_k, ..., N^{m_k}e_k) \text{ is still a basis of } V$$

- Step 5: For $k+1 \leq j \leq m$, define $V_j := \text{span}(f_j')$, then

$$V = \bigoplus_{i=1}^{m} V_j$$

$$N|V_j \text{ is a basis for each } j$$

$\square$

3 Extend Jordan form to all linear operators

Let $T \in \mathcal{L}(V)$ be any arbitrary linear operator over V.

$$V = \bigoplus_{j=1}^{k} G(\lambda_j, T)$$

Then, we see that

$$(T - \lambda_j I)|G(\lambda_j, T) \text{ is nilpotent}$$

thus they can put it in Jordan form and preserves in there generalized eigenspace.
Therefore, T can generate a Jordan form matrix.

4 Invariant of Jordan Form

Theorem 4.1 *Let V be f.d.v.s., $N \in \mathcal{L}(V)$ be nilpotent operator*

$$\exists V = \bigoplus_{i=1}^{m} V_i. \text{ such that } N \text{ preserves } V_i.$$

Let a_j be the number of terms in the decomposition of dimention j.
The sequence $(a_1, ..., a_j)$ is determined by N.

Proof. Suppose $V = \text{span}(e_1, ..., e_k)$, such that $Ne_1 = 0, Ne_i = e_{i-1}$. Then we see that

$$\dim \text{Null}(N) = 1, \dim \text{Null}(N^2) = 2, \cdots, \dim \text{Null}(N^k) = k, \dim \text{Null}(N^{k+1}) = k.$$

Therefore, if (W, N) is such that $\dim W = k$ and N is a basic nilpotent, then

$$\dim \text{Null}(N^j) = \min(j, k).$$

$$\begin{aligned}\dim \mathrm{Null} N^j &= \sum \dim \mathrm{Null} N^j|V_i \\ &= a_1 \min(j,1) + a_2 \min(j,2) + \cdots\end{aligned}$$

Define $n_j = \dim \mathrm{Null} N^j$, then

$$\begin{aligned}n_1 &= a_1 + a_2 + \cdots \\ n_2 &= a_1 + 2a_2 + 2a_3 + \cdots \\ n_3 &= a_1 + 2a_2 + 3a_3 + 3a_4 + \cdots\end{aligned}$$

Then, subtract consecutive equations

$$\begin{aligned}n_2 - n_1 &= a_2 + a_3 + \cdots \\ n_3 - n_2 &= a_3 + a_4 + \cdots \\ n_4 - n_3 &= a_4 + a_5 + \cdots\end{aligned}$$

Then we got

$$\begin{aligned}2n_1 - n_2 &= a_1 \\ -n_1 + 2n_2 - n_3 &= a_2 \\ &\vdots \\ -n_k + 2n_{k+1} - n_{k+2} &= a_{k+1}\end{aligned}$$

□

Corollary 4.2

$$\dim \mathrm{Null}(N^{j+2}) - \dim \mathrm{Null}(N^{j+1}) \le \dim \mathrm{Null} N^{j+1} - \dim \mathrm{Null} N^j$$

Example 4.1. If $\dim \mathrm{Null} N = 3, \dim \mathrm{Null} N^2 = 5$, then $\dim \mathrm{Null} N^4 \le 9$

Alternative proof of 4.2. Idea: Apply rank-Nullity thm to subspaces of V.
It is easy to see that $\mathrm{Null}(N^j|\mathrm{Null}(N^{j+1})) = \mathrm{Null} N^j$
Thus

$$\dim \mathrm{Null} N^{j+1} - \dim \mathrm{Null} N^j = \dim \mathrm{Range}(N^j|\mathrm{Null}(N^{j+1}))$$

$$\dim \mathrm{Null} N^{j+2} - \dim \mathrm{Null} N^{j+1} = \dim \mathrm{Range}(N^{j+1}|\mathrm{Null}(N^{j+2}))$$

Claim: $\mathrm{Range}(N^{j+1}|\mathrm{Null}(N^{j+2})) \subset \mathrm{Range}(N^j|\mathrm{Null}(N^{j+1}))$

proof of claim. Let $v \in \mathrm{Null}(N^{j+2}) \Rightarrow Nv \in \mathrm{Null}(N^{j+1})$.Then,

$$N^{j+1}v = N^j(Nv) \in \mathrm{Range}(N^j|\mathrm{Null}(N^{j+1}))$$

□

Therefore,

$$\dim \mathrm{Range}(N^{j+1}|\mathrm{Null}(N^{j+2})) \leq \dim \mathrm{Range}(N^j|\mathrm{Null}(N^{j+1}))$$

$\square$