Adjoint and Duals

22.3.12 *Kunlong Wu*

RECALL Some basis definition and blue print of Dual space.

1. V is a f.d.v.s. over $\mathbf{F}$.
   the dual space $V' = \mathcal{L}(V, \mathbf{F})$ is defined as the set of all linear maps $\varphi : V \to \mathbf{F}$ (linear functionals). Thus
   $$\dim V' = \dim V$$

2. Suppose V, W are f.d.v.s over $\mathbf{F}$.
   Then there is an isomophic between $T \in \mathcal{L}(V, W)$ and $T' \in \mathcal{L}(W', V')$
   For any $\varphi \in W'$, define $T'(\varphi)$ as
   $$(T'(\varphi))(v) = \varphi(Tv)$$
   $$\begin{tikzcd} V \arrow[rr, "T"] \arrow[dr, "\varphi\circ T"'] & & W \arrow[dl, "\varphi\in W'"] \\ & \mathbf{F} & \end{tikzcd}$$

3. Let V be a f.d.i.p.s over $\mathbf{F}$
   Recall Riesz representation theorem, define $L^V : \overline{V} \to V'$ as
   $$(L^V(v))(u) = \varphi_v^V(u) := \langle u, v\rangle$$
   Under this definition, we claim that L' is a linear function.
   For any $v \in \overline{V}$, $u \in V$ and $\lambda \in \mathbf{F}$, we have
   $$\begin{aligned}(L'(\lambda * v))u &= \langle u, \overline{\lambda} v\rangle \\ &= \lambda\langle u, v\rangle \\ &= \lambda\varphi_v(u) \\ &= \lambda(L'(v))(u)\end{aligned}$$

4. Let V, W be f.d.i.p.s., and $T \in \mathcal{L}(V, W) = \mathcal{L}(\overline{V}, \overline{W})$ $(\star)$
   The " $=$ " in $(\star)$ is true since
   $$T(\lambda * v) = T(\overline{\lambda} v) = \overline{\lambda} Tv = \lambda * Tv$$
   $$\begin{tikzcd} \overline{V} \arrow[r, "\varphi^V"] & V' \\ \overline{W} \arrow[u, "T^*"] \arrow[r, "\varphi^W"] & W' \arrow[u, "T'"'] \end{tikzcd}$$

THEOREM
$$T' \circ \varphi^W = \varphi^V \circ T^*$$

PROOF Let $w \in \overline{W}$, $v \in V$ be arbitrary
$$(T'\varphi^W(w))(v) = T'(\varphi_w^W)(v) = (\varphi_w^W(Tv)) = \langle Tv, w\rangle$$
$$((\varphi^V T^*)(w))(v) = (\varphi^V(T^*(w))(v) = \langle v, T^*(w)\rangle = \langle Tv, w\rangle$$

□