

Adjoint and Duals

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RECALL

Some basis definition and blue print of Dual space.

1. V is a f.d.v.s. over \mathbf{F} .

the dual space $V' = \mathcal{L}(V, \mathbf{F})$ is defined as the set of all linear maps $\varphi : V \rightarrow \mathbf{F}$ (linear functionals). Thus

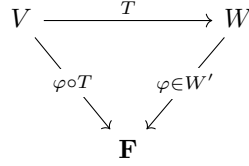
$$\dim V' = \dim V$$

2. Suppose V, W are f.d.v.s over \mathbf{F} .

Then there is an isomophic between $T \in \mathcal{L}(V, W)$ and $T' \in \mathcal{L}(W', V')$

For any $\varphi \in W'$, define $T'(\varphi)$ as

$$(T'(\varphi))(v) = \varphi(Tv)$$



3. Let V be a f.d.i.p.s over \mathbf{F}

Recall Riesz representation theorem, define $L^V : \overline{V} \rightarrow V'$ as

$$(L^V(v))(u) = \varphi_v^V(u) := \langle u, v \rangle$$

Under this definition, we claim that L' is a linear function.

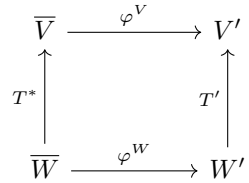
For any $v \in \overline{V}$, $u \in V$ and $\lambda \in \mathbf{F}$, we have

$$\begin{aligned}
 (L'(\lambda * v))u &= \langle u, \bar{\lambda}v \rangle \\
 &= \lambda \langle u, v \rangle \\
 &= \lambda \varphi_v(u) \\
 &= \lambda (L'(v))(u)
 \end{aligned}$$

4. Let V, W be f.d.i.p.s., and $T \in \mathcal{L}(V, W) = \mathcal{L}(\overline{V}, \overline{W})$ (\star)

The “=” in (\star) is true since

$$T(\lambda * v) = T(\bar{\lambda}v) = \bar{\lambda}Tv = \lambda * Tv$$



THEOREM

$$T' \circ \varphi^W = \varphi^V \circ T^*$$

PROOF

Let $w \in \overline{W}$, $v \in V$ be arbitrary

$$\begin{aligned}
 (T' \varphi^W(w))(v) &= T'(\varphi_w^W)(v) = (\varphi_w^W(Tv)) = \langle Tv, w \rangle \\
 ((\varphi^V T^*)(w))(v) &= \varphi^V(T^*(w))(v) = \langle v, T^*(w) \rangle = \langle Tv, w \rangle
 \end{aligned}$$

□