1 Basic Definition of Complexification

Definition 1.1: **Let V be a real vector space. We define $V_{\mathbb{C}}$ as follows:**

1. **Set: The underlying set is $V \times V$. (any vector in it can be writen as (v_1, v_2) for any $v_1, v_2 \in V$)**
2. **Addtion:** $(v_1, v_2) + (v_1', v_2') := (v_1 + v_1', v_2 + v_2')$
3. **Scalar:** $(a + bi)(u_1, u_2) := (au_1 - bu_2, bu_1 + au_2)$

Claim *This makes $V_{\mathbb{C}}$ into a complex vector space. (i.e., This definition of $V_{\mathbb{C}}$ satisfy the axiom of complex vector space)*

Proof. The "HARD" part is : Let $\lambda_1, \lambda_2 \in \mathbb{C}, (v_1, v_2) \in V_{\mathbb{C}}$
Then, $\lambda_1(\lambda_2(v_1, v_2)) = (\lambda_1\lambda_2)(v_1, v_2)$
□

Now we have a natural injection $V \hookrightarrow V_{\mathbb{C}}$ such that $v \mapsto (V, 0)$
In this way, we can think of V as a subset of $V_{\mathbb{C}}$

Remark. $(0, v) = i(v, 0) = iv$
i.e., $(v_1, v_2) = (v_1, 0) + (0, v_2) = v_1 + iv_2$
Tensor Product $V_{\mathbb{C}} = V \bigotimes_{\mathbb{R}} \mathbb{C}$

Lemma 1.2: **if $(e_1, ..., e_n)$ is a basis for V, then it is also a basis for $V_{\mathbb{C}}$**

Proof. Suppose we have $v_1 + iv_2 \in V_{\mathbb{C}}$, write $v_j = \sum a_j e_j, v_2 = \sum b_j e_j i$
where $a_j, b_j \in \mathbb{R}$
Then we have $v_1 + iv_2 = \sum(a_j + ib_j)e_j$.
Since $(a_j, +ib_j) \in \mathbb{C}$, $\text{span}(e_1, ..., e_n) = V_{\mathbb{C}}$

Suppose $0 = \sum \lambda_j e_j = \sum a_j e_j + ib_j e_j$
Since $e_1, ..., e_n$ are linearly independent over $\mathbb{R}$, it follows that $a_j = b_j = 0$
Therefore, $\lambda_j = 0$. i.e., $(e_1, ..., e_n)$ linearly independent over $V_{\mathbb{C}}$ □

2 Linear Maps Over $V_{\mathbb{C}}$

Suppose V, W are real vector space and $T \in \mathcal{L}(V, W)$,
we define $T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}}, W_{\mathbb{C}})$ as follows:

$$T_{\mathbb{C}}(v_1 + iv_2) = Tv_1 + iTv_2$$

Claim *Under the same basis in real vector space, the matrix of T and $T_{\mathbb{C}}$ are same.*
(Notice: we can not choose a basis in complex vecotr space for T)

Theorem 2.1 *Let V be f.d.v.s. over $\mathbb{R}$, $T \in \mathcal{L}(V)$. Then there exists an invariant subspace U that has dimension 1 or 2.*
i.e.,

$$\exists U \subseteq V.(T(U) \subset U \wedge \dim U = 1 \text{ or } 2)$$

Proof. $T_{\mathbb{C}}$ has an eigenvector $u + iv$ with eigenvalue $\lambda = a + bi \in \mathbb{C}$
$T_{\mathbb{C}}(u + iv) = Tu + iTv = (a + bi)(u + iv)$ implies $Tu = au - bv \wedge Tv = av + bu$
Therefore, $U = \operatorname{span}(u, v)$ $\square$

Claim *There is a natural isomophism between $V_{\mathbb{C}}$ and $\overline{V_{\mathbb{C}}}$. This is used below.*

$$\overline{(a + ib)(v_1 + iv_2)} = (a - ib)(v_1 - iv_2)$$

Conside it as rotation, this isomophism will become intuitively.

Theorem 2.2 *Let V be 2-D i.p.v.s over $\mathbb{R}$, $T \in \mathcal{L}(V)$ is normal. Then in any orthonormal basis $\vec{e}$.*

1. *if T self-adjoint ($T = T^*$), then $M(T, \vec{e}) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$*
2. *if T is not self-adjoint ($T \neq T^*$), then $M(T, \vec{e}) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$*

Proof. (1) is easy.
WTS (2), Assume $T \neq T^*$
Using complex spectral theorem, there exists an ONB such that $\vec{f} = (f_1, f_2)$ of eigenvecotrs for $T_{\mathbb{C}}$ with eigenvalue λ_1, λ_2.
Since $T_{\mathbb{C}}$ is not self-adjoint, λ_1, λ_2 are not both in $\mathbb{R}$
WLOG (without loss of generality) $\lambda_1 \notin \mathbb{R}$,
we claim $\lambda_2 = \overline{\lambda_1}$ by change eigenvectors of λ_1 into conjugate vector space
Then we conclude that

$$M(T_{\mathbb{C}}, \vec{f}) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{bmatrix}$$
$$M(T_{\mathbb{C}} + T_{\mathbb{C}}^*, \vec{f}) = \begin{bmatrix} 2\operatorname{Re}\lambda_1 & 0 \\ 0 & 2\operatorname{Re}\{\lambda_1\} \end{bmatrix}$$

We conclude:
Let $\vec{e}$ be any ONB for V and

$$M(T, \vec{e}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then, we see that

$$M(T + T^*, \vec{e}) = \begin{bmatrix} 2a & b + c \\ b + c & 2d \end{bmatrix}$$

Therefore, $b = -c$ and $a = d$. $\square$

Corollary 2.3 *If further T is an isomertry, then $M(T, \vec{e}) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where $\theta \in [0, 2\pi]$.*

Theorem 2.4 *Let T be an isometry on a real f.d.i.p.s. V, then there exists ONB $\vec{e}$ such that*

$$M(T,\vec{e}) = \begin{bmatrix} r_1 & & & & & & \\ & \ddots & & & \mathit{0} & & \\ & & r_n & & & & \\ & & & \begin{bmatrix} \cos\theta_{n+1} & \sin\theta_{n+1} \\ -\sin\theta_{n+1} & \cos\theta_{n+1} \end{bmatrix} & & & \\ & \mathit{0} & & & \ddots & & \\ & & & & & \begin{bmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{bmatrix} \end{bmatrix}$$

where $r_i = \pm 1, \theta \in [0, 2\pi]$ To understand this matrix, we see that $r_i = 1$ means do nothing, $r_i = -1$ means flipped and θ means rotation.

Proof. Prove by 6 steps

- STEP 1: By the theorem, there e.xists a subspace $W \subseteq V$ such that $\dim W = 1$ *or* 2 , $T(W) \subseteq W$.
- STEP 2: T is normal, so $W, W^\perp$ are invariant under T, T^*
- STEP 3: $(T|_W)^* = T^*|_W, (T|_W^\perp)^* = T^*|_W^\perp$
- STEP 4: Theorem true for $W^\perp$ by induction hypothesis there exists an ONB $B_{W^\perp}$ satisfy the $W^\perp$.
- STEP 5: If $\dim W = 1$ then true for $(W, T|W)$
  If $\dim W = 2$
  1. $T|_W$ is self-adjoint, then there exists ONB $B_W$for operator that let the matrix diagonal.
     Since $T|_W$ is isometry, it follows that all eigenvalue are ± 1.
  2. $T|_W$ is not self-adjoint, then we have already shown in above.
- STEP 6: Set $\vec{B} = \vec{B}_{W^\perp} \cup \vec{B}_W$

□

Theorem 2.5 *Let V be a f.d.v.s over $\mathbb{R}$, and $T \in \mathcal{L}(V)$. If $\dim V$ is odd, then T has an eigenvalue.*

LEMMA 2.6: **Let V be a f.d.v.s over $\mathbb{R}$, $T \in \mathcal{L}(V)$, then**

$$T \textbf{ is invertiable } \iff T_{\mathbb{C}} \textbf{ is invertiable}$$

LEMMA 2.7: **Let (V, T) as above, $r \in \mathbb{R}$, then**

LEMMA 2.8: **r is an eigenvalue of $T \iff r$ is an eigenvalue of $T_{\mathbb{C}}$**

LEMMA 2.9: **Let (V, T) as above, $\lambda \in \mathbb{C}$, then**

$$\dim E(\lambda, T_{\mathbb{C}}) = \dim E(\overline{\lambda}, T_{\mathbb{C}})$$

Proof. Let $v = v_1 + v_i 2i \in V \times V$, where $v_1, v_2 \in V$, then $\overline{v} = v_1 - v_2 i$.
$T_{\mathcal{C}}\overline{v} = TTv_1 - iTv_2 = \overline{Tv_1 + iTv_2} = \overline{T_{\mathcal{C}}v} = \overline{\lambda}\overline{v}$

□

Remark.

$$v_1, ..., v_m \ lin.ind. \iff \overline{v_1}, ..., \overline{v_m} \ lin.ind.$$

The conjugate of a basis for $E(\lambda, T_{\mathcal{C}})$ is a basis for $E(\overline{\lambda}, T_{\mathcal{C}})$

Lemma 2.10: $T \in \mathcal{L}(V), a \neq b$**, then**

$$\mathbf{null}((T - aI)(T - bI)) = \mathbf{null}(T - aI) \oplus \mathbf{null}(T - bI)$$

Lemma 2.11: $S, T \in \mathcal{L}(V)$**. Suppose** $ST = TS$**, then**

$$\mathbf{null}(S), \mathbf{Range}(S) \textbf{ are preserved by } T$$

Other things see in complexification2

3 Restriction of Scalars

Definition 3.1: **Suppose that** V **is a complex vector space. We denote the Restriction of Scalars , written** $\mathrm{Res}V$**, to be the real vector space.**
such that is same with complex vector space excepts we only scalar with real numbers.