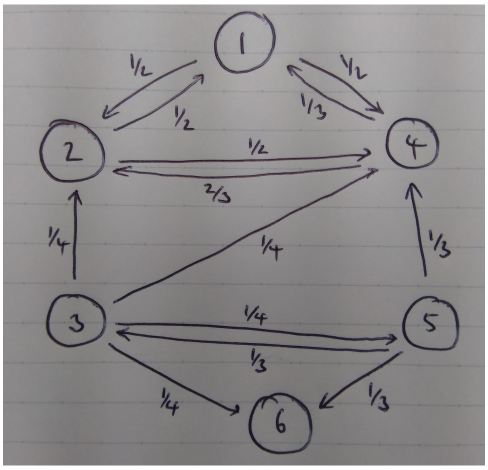
Introduction to Probability (E2): Sample exam solutions

1. (a) The chain is sketched as follows.



[2 marks]

(b) $\{1,2,4\}$ closed; $\{3,5\}$ open; $\{6\}$ closed. Not irreducible, since the state space is not one closed communicating class. [3 marks]

(c) Boundary conditions: $h_1^{\{1\}} = 1$, $h_6^{\{1\}} = 0$. Equations from first step decomposition:

$$h_2^{\{1\}} = \tfrac{1}{2} + \tfrac{1}{2}h_4^{\{1\}}, \qquad h_3^{\{1\}} = \frac{1}{4}h_2^{\{1\}} + \frac{1}{4}h_4^{\{1\}} + \frac{1}{4}h_5^{\{1\}},$$
$$h_4^{\{1\}} = \tfrac{1}{3} + \tfrac{2}{3}h_2^{\{1\}}, \qquad h_5^{\{1\}} = \frac{1}{3}h_3^{\{1\}} + \frac{1}{3}h_4^{\{1\}}.$$

Solving these gives $h_3^{\{1\}} = \frac{7}{11}$. [4 marks]

(d) Boundary condition: $k_1^{\{1,6\}} = k_6^{\{1,6\}} = 0$. Equations from first step decomposition:

$$\begin{aligned} k_2^{\{1,6\}} &= 1 + \frac{1}{2}k_4^{\{1,6\}}, \\ k_3^{\{1,6\}} &= 1 + \frac{1}{4}k_2^{\{1,6\}} + \frac{1}{4}k_4^{\{1,6\}} + \frac{1}{4}k_5^{\{1,6\}}, \\ k_4^{\{1,6\}} &= 1 + \frac{2}{3}k_2^{\{1,6\}}, \\ k_5^{\{1,6\}} &= 1 + \frac{1}{3}k_3^{\{1,6\}} + \frac{1}{3}k_4^{\{1,6\}}. \end{aligned}$$

We first solve for: $k_2^{\{1,6\}} = \frac{9}{4}$, $k_4^{\{1,6\}} = \frac{5}{2}$. Solving the remaining equations gives $k_3^{\{1,6\}} = \frac{127}{44}$. [4 marks]

(e) i. The transition matrix of Y is given by

$$\begin{pmatrix} \frac{1}{6} & \frac{5}{6} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

[2 marks]

ii. Using that any such limit is given by the relevant component of an invariant distribution, we compute that it must be equal to $\frac{4}{9}$. [2 marks]

2. (a) We use a first step decomposition and the strong Markov property to deduce that

$$\phi(s) = \frac{\alpha}{\alpha+1} \times s + \frac{1}{\alpha+1} \times s \times \mathbf{E}\left(s^{H_0} \middle| X_0 = 3\right) = \frac{\alpha}{\alpha+1} \times s + \frac{1}{\alpha+1} \times s \times \phi(s)^3.$$

Rearranging gives the result. [3 marks]

(b) The probability $\mathbf{P}_0(H_0 < \infty)$ is given by $\ell = \lim_{s\uparrow 1} \phi(s)$. We note that taking limits in the equation of part (a) yields that ℓ satisfies

$$0 = \ell^3 - (\alpha+1)\ell + \alpha.$$

This has solutions

$$\ell = 1, \frac{-1 \pm \sqrt{1+4\alpha}}{2}.$$

Since $\frac{-1-\sqrt{1+4\alpha}}{2} < 0$ and $\frac{-1+\sqrt{1+4\alpha}}{2} \geq 1$ for $\alpha \geq 2$, the only solution in $[0,1]$ is 1. So $\ell = 1$. [3 marks]

(c) By the result of part (b), we have that $\mathbf{E}_0(H_0) = \lim_{s\uparrow 1} \phi'(s)$. Differentiating the equation of part (a) yields

$$0 = \phi(s)^3 + 3s\phi(s)^2\phi'(s) - (\alpha+1)\phi'(s) + \alpha.$$

Rearranging gives

$$\phi'(s) = \frac{\phi(s)^3 + \alpha}{\alpha + 1 - 3s\phi(s)^2}.$$

And taking limits $s \uparrow 1$ gives

$$\mathbf{E}_0(H_0) = \frac{\alpha+1}{\alpha-2},$$

which is interpreted as $+\infty$ when $\alpha = 2$. [3 marks]

3. (a) It holds that

$$\begin{aligned} & ax_{n+2} + bx_{n+1} + cx_n \\ &\quad = aA\lambda_1^{n+2} + aB\lambda_2^{n+2} + bA\lambda_1^{n+1} + bB\lambda_2^{n+1} + cA\lambda_1^n + cB\lambda_2^n \\ &\quad = A\lambda_1^n\left(a\lambda_1^2 + b\lambda_1 + c\right) + B\lambda_2^n\left(a\lambda_2^2 + b\lambda_2 + c\right) \\ &\quad = 0. \end{aligned}$$

[2 marks]

Moreover, note that the equations

$$A + B = \alpha, \qquad A\lambda_1 + B\lambda_2 = \beta$$

have unique solution given by

$$A = \frac{\beta - \alpha\lambda_2}{\lambda_1 - \lambda_2}, \qquad B = \alpha - A.$$

In particular, if A and B are defined in this way and $x_n := A\lambda_1^n + B\lambda_2^n$, then $(x_n)_{n\geq 0}$ solves the difference equation and satisfies $x_0 = \alpha$, $x_1 = \beta$. By induction, any sequence $(y_n)_{n\geq 0}$ satisfying these properties must be equal to $(x_n)_{n\geq 0}$, and so the solution in question is unique.

[4 marks]

(b) From the equation $\pi P = \pi$, we have that

$$\pi_0 = \pi_0(1-p) + \pi_1(1-p) + \sum_{i\geq 2} \pi_i r = \pi_0(1-p) + \pi_1(1-p) + (1-\pi_0-\pi_1)r,$$

which implies

$$\pi_0(p+r) = q\pi_1 + r.$$

Moreover, for $i \geq 1$, we have that

$$\pi_i = q\pi_{i+1} + p\pi_{i-1}.$$

This is a difference equation with corresponding quadratic:

$$q\lambda^2 - \lambda + p = 0,$$

which has solutions

$$\lambda_\pm = \frac{1 \pm \sqrt{1-4pq}}{2q}.$$

The roots are distinct and one can check that

$$\begin{aligned} \lambda_+ &= \frac{1+\sqrt{1-4p(1-p-r)}}{2q} \\ &> \frac{1+\sqrt{1-4p(1-p)}}{2q} \\ &= \frac{1+|1-2p|}{2q} \\ &\geq \frac{1+(1-2p)}{2q} \\ &= 1. \end{aligned}$$

Hence, any solution of the difference equation taking values in $[0,1]$ must be of the form

$$\pi_i = A\lambda^i, \qquad i \geq 1,$$

where $\lambda = \lambda_- \in (0,1)$. Note that we also must have

$$1 = \sum_{i\geq 0} \pi_i = \pi_0 + \frac{A\lambda}{1-\lambda}.$$

And, from the equation for π_0 above,

$$\pi_0(p+r) = qA\lambda + r.$$

Solving the latter two equations for π_0 and A, we find that

$$\pi_0 = 1 - \frac{p}{1-q\lambda}, \qquad A = \frac{p(1-\lambda)}{\lambda(1-q\lambda)},$$

which gives the result. [8 marks]