Exercises, chapter 16, solutions

Answer 1

a) See Chapter 16.1 in the textbook or the slides from Lecture 13.

b) After examining t elements for any $t \geq m$, where m is the size of the reservoir, each element has to be in the reservoir with probability m/t. When examining element $(t+1)$, the algorithm will therefore let it replace some element currently in the reservoir with probability $m/(t+1)$. The value of m is fixed, but t and hence $m/(t+1)$ will change as the algorithm proceeds.

Answer 2

A table analogous to Table 16.2 is shown on the right.

According to the table, voter B is critical in 3 coalitions, so the Banzhaf score $\eta(B)$ is 3. Then, since $n = 4$, the Banzhaf measure $\beta'(B)$ is $\frac{\eta(B)}{2^{n-1}} = \boxed{3/8}$.

Coalitions without B	Coalitions with B	Votes	Winning	Critical
$\emptyset$	$\{B\}$	2		
$\{A\}$	$\{A, B\}$	6	✓	✓
$\{C\}$	$\{B, C\}$	3		
$\{D\}$	$\{B, D\}$	5		
$\{A, C\}$	$\{A, B, C\}$	7	✓	✓
$\{A, D\}$	$\{A, B, D\}$	9	✓	
$\{C, D\}$	$\{B, C, D\}$	6	✓	✓
$\{A, C, D\}$	$\{A, B, C, D\}$	10	✓	

Answer 3

Define $\mathcal{X}'$ to be an algorithm that runs the algorithm $\mathcal{X}$ independently t times. If $\mathcal{X}$ returns FALSE at least once then let $\mathcal{X}'$ return FALSE; otherwise, let $\mathcal{X}'$ return TRUE.

To bound the error probability of $\mathcal{X}'$, observe that if q is not prime then the probability that $\mathcal{X}$ incorrectly returns TRUE t times is $\leq (\frac{1}{4})^t$. On the other hand, if q is prime then $\mathcal{X}$ always returns TRUE, and $\mathcal{X}'$ correctly returns TRUE. Hence, the error probability of $\mathcal{X}'$ is $\leq (\frac{1}{4})^t$ in every case.

The time complexity of $\mathcal{X}'$ is $O(t \cdot \lg^3 q)$ because it runs $\mathcal{X}$ t times.

Answer 4

Algorithm:

1. Repeat
2. $\quad X \leftarrow Random - Bias()$
3. $\quad Y \leftarrow Random - Bias()$
4. Until $X \neq Y$
5. Return X

Correctness proof:

Let A be the event "$X = 0$" and B the event "$X + Y = 1$". By definition, the conditional probability $Pr\{A \mid B\} = \frac{Pr\{A \cap B\}}{Pr\{B\}}$. This gives $Pr\{output = 0\} = Pr\{X = 0 \mid X + Y = 1\} = \frac{Pr\{(X=0) \cap (X+Y=1)\}}{Pr\{X+Y=1\}}$ $= \frac{Pr\{(X=0) \cap (Y=1)\}}{Pr\{(X=0) \cap (Y=1)\} + Pr\{(X=1) \cap (Y=0)\}} = \frac{(1-p)p}{(1-p)p + p(1-p)} = \frac{1}{2}$. In the same way, $Pr\{output = 1\} = \frac{1}{2}$.

Time complexity analysis:

- In the loop, each iteration takes $O(1)$ time.
- The probability of success at Line 4 is $2p(1–p)$
- By the geometric distribution formula, the expected number of iterations is $\frac{1}{(2p(1-p))}$

$\Rightarrow$ The algorithm's expected running time is $\Theta\big(\frac{1}{2p(1-p)}\big)$.