Problem Set 1: Suggested Answers

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1. Iterated Elimination of Strictly Dominated Strategies and Nash Equilibrium.

(a) Neither player 1 nor player 2 has a strictly dominant strategy. Player 1's strategy B is strictly dominated by strategy T.

(b) Strategy profiles that survive iterated elimination of strictly dominated strategies are (T, L), (M, L), (T, R), (M, R). First, we eliminate player 1's stragegy B, which is strictly dominated by strategy T. Then we eliminate player 2's strategy C, which is strictly dominated by strategy R.

(c) $BR_1(L) = \{M\}$, $BR_1(C) = \{T, M\}$, $BR_1(R) = \{T\}$.
$BR_2(T) = \{R\}$, $BR_2(M) = \{L\}$, $BR_2(B) = \{L\}$.

(d) Pure strategy Nash equilibria of this game: (T, R), (M, L).

2. Iterated Elimination of Weakly Dominated Strategies.

(a) Player 1's strategy M and strategy B are both weakly dominated by strategy T.

(b) If we eliminate player 1's strategy M and strategy B at the same time, there is no weakly dominated strategy in the "smaller" game. Thus the strategy profiles that survive iterated elimination of weakly dominated strategies are (T, L) and (T, R).

(c) We can first eliminate player 1's strategy M, or first eliminate player 1's strategy B. In the former case, player 2's strategy L is now weakly dominated by strategy R. In the smaller game, if we first eliminate player 1's strategy B, then we go back to (b). If we eliminate player 2's strategy L before eliminating player 1's strategy B, then the strategy profile that survives iterated elimination of weakly dominated strategies is (T, R).

In the latter case, player 2's strategy R is now weakly dominated by strategy L. In the smaller game, if we first eliminate player 1's strategy B, then we go back to (b) again. If we eliminate player 2's strategy R before eliminating player 1's strategy M, then the strategy profile that survives iterated elimination of weakly dominated strategies is (T, L).

3. Second Price Auction. Suppose that player i is bidding $b_i = v_i$. We only have to confirm that player i cannot strictly improve her/his payoff by submitting a different bid $b'_i \neq v_i$. There are several cases to consider.

Case 1: Player i wins by bidding $b_i = v_i$ and s/he is the only player who submits the highest bid. Suppose the second highest bid is $b_j (j \neq i)$. Thus in this case, player i's payment is b_j and her/his payoff is $v_i - b_j > 0$.

- If player i bids $b'_i > b_j$ with $b'_i \neq v_i$, b'_i will still be the only highest bid, and player i's payoff is still $v_i - b_j$.
- If player i bids $b'_i < b_j$ with $b'_i \neq v_i$, s/he will lose and get a payoff of 0, which is strictly lower than $v_i - b_j$.
- If player i bids $b'_i = b_j (\neq v_i)$, whether s/he wins or loses. If s/he wins, that is because s/he is the one with the lower (lowest) index. The payment will be b_j and her/his payoff will be $v_i - b_j$. If s/he loses, her/his payoff will be 0, which is strictly lower than $v_i - b_j$.

Case 2: Player i wins by bidding $b_i = v_i$ because s/he has the lowest index among those who submit the highest bid. In this case the payment is $b_i (= v_i)$ and player i's payoff is $v_i - b_i = 0$.

- If player i bids $b'_i > v_i$, s/he will win and the payment will still be $b_i (= v_i)$, so the payoff will still be 0.
- If player i bids $b'_i < v_i$, s/he will lose and get a payoff of 0.

Case 3: Player i loses by bidding $b_i = v_i$. In this case player i's payoff is 0.

- If player i bids $b'_i > v_i$, whether s/he wins or loses. If s/he wins, since the second highest bid is higher than v_i, her/his payoff will be negative. If s/he loses, her/his payoff is still 0.
- If player i bids $b'_i < v_i$, s/he loses and gets a payoff of 0.

To summarize, regardless of the bids of the other players, player i cannot make her/himself a strictly higher payoff by deviating from $b_i = v_i$, which means that v_i is a weakly dominant action.

Inefficiency: There are equilibria in which the winner is not player 1. Such equilibria are inefficient because the object is not given to the one who values it most, so social welfare is not maximized. Here is an example of an inefficient equilibrium: $b^* = (b_1^*, b_2^*, ..., b_n^*)$ where $b_1^* = v_n$ and $b_i^* = v_{i-1}$ for $i \geq 2$. b^* is a strategy profile where player 1 bids v_n (the lowest value among all the players), and each player $i (i \geq 2)$ bids the value of player $i-1$. In b^*, player 1's bid is the lowest, so s/he does not obtain the object. In fact, the winner is player 2 (who bids v_1). Let us confirm that b^* is indeed a Nash equilibrium.

Consider player 1: Now s/he loses and her/his payoff is 0. S/he can deviate from b^* by bidding some $b_1' \geq b_2^*$ so that s/he wins. S/he can also deviate by bidding some $b_1' < b_2^*$ so that s/he still loses.

- If s/he wins, the payment will be the "second" highest bid $b_2^* (= v_1)$, so her/his payoff will be $v_1 - v_1 = 0$.
- If s/he loses, her/his payoff is still 0.

In both cases, deviation cannot yield a strictly higher payoff, which means that b_1^* is a best response to b_{-1}^*.

Consider player 2: Now s/he wins and the payment is the second highest bid $b_3^* = v_2$. So her/his payoff is 0. S/he can deviate from b^* by bidding some $b_2' \geq b_3^*$ so that s/he still wins. S/he can also deviate by bidding some $b_2' < b_3^*$ so that s/he loses.

- If s/he wins, the payment is still $b_3^* (= v_2)$, so her/his payoff is still 0.
- If s/he loses, her/his payoff will be 0.

In both cases, deviation cannot yield a strictly higher payoff, which means that b_2^* is a best response to b_{-2}^*.

Consider player i, $i \geq 3$: Now s/he loses and her/his payoff is 0. Similar to player 1, s/he can deviate from b^* by bidding some $b_i' > b_2^*$ so that s/he wins. S/he can also deviate by bidding some $b_i' \leq b_2^*$ so that s/he still loses.

- If s/he wins, the payment will be $b_2^* (= v_1)$, so her/his payoff will be $v_i - v_1 < 0$.
- If s/he loses, her/his payoff is still 0.

In both cases, deviation cannot yield a strictly higher payoff, which means that b_i^* is a best response to b_{-i}^*, $i \geq 3$.

Since every player is playing a best response, we conclude that b^* is indeed a Nash equilibrium. (You can construct other inefficient Nash equilibria. Can you figure out the condition(s) that an inefficient Nash equilibrium has to satisfy?)

4. Splitting the Dollar. Pure strategy Nash equilibria of this game: All (s_1, s_2) such that $s_1 + s_2 = 1$ and $(s_1, s_2) = (1, 1)$.

Consider player i, $i = 1, 2$. The best response to a belief $s_{-i} < 1$ is $1 - s_{-i}$: If $s_{-i} < 1$, then by naming a share of $1 - s_{-i}$, player i could receive a higher payoff than naming any share s_i such that $s_i + s_{-i} < 1$ or $s_i + s_{-i} > 1$. The best responses to a belief $s_{-i} = 1$ are all s_i satisfying $0 \leq s_i \leq 1$, because player i would receive a payoff of 0 anyway.

Since Nash equilibria are the strategy profiles in which players are best responding to each other, it is straightforward to conclude that all (s_1, s_2) such that $0 < s_1 < 1, 0 < s_2 < 1, s_1 + s_2 = 1$ are Nash equilibria of this game. (Confirm that there is no incentive for either player to deviate from such strategy profiles by yourself.)

If $s_i = 1$ for some i, in a Nash equilibrium it cannot be the case that $0 < s_{-i} < 1$: Although such strategies are indeed best responses for player $-i$ to belief $s_i = 1$, if player i believes that $0 < s_{-i} < 1$ then $s_i = 1$ is not a best response. Hence in the Nash equilibrium we must have $s_{-i} = 0$ or $s_{-i} = 1$. (Confirm that there is no incentive for either player to deviate from such strategy profiles by yourself.)

5. Hotelling's Location Game.

(a)
- Set of players: {Group A, Group B}
- Strategy set of each player: $[0, 1]$. Let s_A stand for Group A's strategy and s_B stand for Group B's strategy.
- Payoff function of Group A:

$$u_A(s_A, s_B) = \begin{cases} \frac{s_A + s_B}{2} & , s_A < s_B \\ \frac{1}{2} & , s_A = s_B \\ 1 - \frac{s_A + s_B}{2} & , s_A > s_B \end{cases}$$

Payoff function of Group B:

$$u_B(s_A, s_B) = \begin{cases} 1 - \frac{s_A + s_B}{2} & , s_A < s_B \\ \frac{1}{2} & , s_A = s_B \\ \frac{s_A + s_B}{2} & , s_A > s_B \end{cases}$$

(b) The only Nash equilibrium is $(\frac{1}{2}, \frac{1}{2})$. We first show that any other strategy profile cannot be a Nash equilibrium, then we confirm that $(\frac{1}{2}, \frac{1}{2})$ is indeed a Nash equilibrium.

- Any strategy profile (s_A, s_B) with $s_A < s_B$ cannot be a Nash equilibrium. It suffices to show that at least one player has an incentive to deviate (is not playing the best response). Consider Group A: Given that Group B's location is s_B, Group A's payoff is $\frac{s_A+s_B}{2}$ when it chooses s_A. However, by moving its location a little bit closer to that of Group B, Group A can ensure a higher payoff. For example, if Group A chooses a location s'_A satisfying $s_A < s'_A < s_B$ instead of s_A, then Group A's payoff becomes $\frac{s'_A+s_B}{2} > \frac{s_A+s_B}{2}$. Hence we know that, given Group B's strategy, Group A is not choosing the best response. We can show that Group B also has an incentive to deviate from such a strategy profile. (How?) You can draw a picture to confirm the argument above.
- Similarly we can show that any strategy profile (s_A, s_B) with $s_A > s_B$ cannot be a Nash equilibrium.
- Any strategy profile (s_A, s_B) with $s_A = s_B < \frac{1}{2}$ cannot be a Nash equilibrium. It suffices to show that at least one player has an incentive to deviate (is not playing the best response). Consider Group A: Given that Group B's location is s_B, Group A's payoff is $\frac{1}{2}$ when it chooses $s_A = s_B$. However, by moving its location a little bit further to the right of Group B, Group A can ensure a higher payoff. For example, if Group A chooses a location s'_A satisfying both $s'_A > s_B$ and $s'_A + s_B < 1$ instead of s_A, then Group A's payoff becomes $1 - \frac{s'_A+s_B}{2} > \frac{1}{2}$. Hence we know that, given Group B's strategy, Group A is not choosing the best response. We can show that Group B also has an incentive to deviate from such a strategy profile. (How?) You can draw a picture to confirm the argument above.
- Similarly we can show that any strategy profile (s_A, s_B) with $s_A = s_B > \frac{1}{2}$ cannot be a Nash equilibrium.
- $(s_A, s_B) = (\frac{1}{2}, \frac{1}{2})$ is a Nash equilibrium. It suffices to show that players are best responding to each other, that is, there is no incentive for deviation. Consider Group A: Given that Group B's location is $s_B = \frac{1}{2}$, Group A's payoff is $\frac{1}{2}$ when it chooses $s_A = \frac{1}{2}$. Group A cannot achieve a strictly higher payoff by deviating.

Indeed, if Group A chooses a strategy s'_A satisfying $s'_A < \frac{1}{2}$ instead of $s_A = \frac{1}{2}$, its payoff is $\frac{s'_A + \frac{1}{2}}{2} < \frac{1}{2}$. If it chooses a strategy s'_A satisfying $s'_A > \frac{1}{2}$ instead of $s_A = \frac{1}{2}$, its payoff is $1 - \frac{s'_A + \frac{1}{2}}{2} < \frac{1}{2}$. To conclude, $s_A = \frac{1}{2}$ is a best response to $s_B = \frac{1}{2}$.

Similarly, we can show that $s_B = \frac{1}{2}$ is a best response to $s_A = \frac{1}{2}$. Since players are mutually best responding to each other, $(s_A, s_B) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is a Nash equilibrium.

(c) There is no Nash equilibrium. Writing down the payoff functions for all the players can be hard work. However, we can confirm the nonexistence of Nash equilibrium by checking every possible location pattern. In fact, the location patterns can be divided into two cases. In each case, we can find at least one group that is not playing the best response.

Case 1: Not all groups choose the same location. In this case, there must exist one group whose location is different from the other two, and who is closer to 0 or 1 than the other two. This group can achieve a strictly higher payoff by getting closer to the other groups.

- If this group's location is closer to 0, it can achieve a strictly higher payoff by deviating to the right, that is, getting closer to the group that is on its right.
- If this group's location is closer to 1, it can achieve a strictly higher payoff by deviating to the left, that is, getting closer to the group that is on its left.

Case 2: All groups choose the same location so the payoff of each group is $\frac{1}{3}$. In this case, each group can achieve a strictly higher payoff by getting away from the other groups.

- If the common location is not $\frac{1}{2}$, each group can achieve a strictly higher payoff by deviating to a location that is closer to $\frac{1}{2}$.
- If the common location is $\frac{1}{2}$, each group can achieve a strictly higher payoff by deviating to a location slightly different from $\frac{1}{2}$. (My calculation says that this works as long as the new location falls into the interval $(\frac{1}{6}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{5}{6})$. Is it right?)

Draw some pictures to confirm the arguments above.