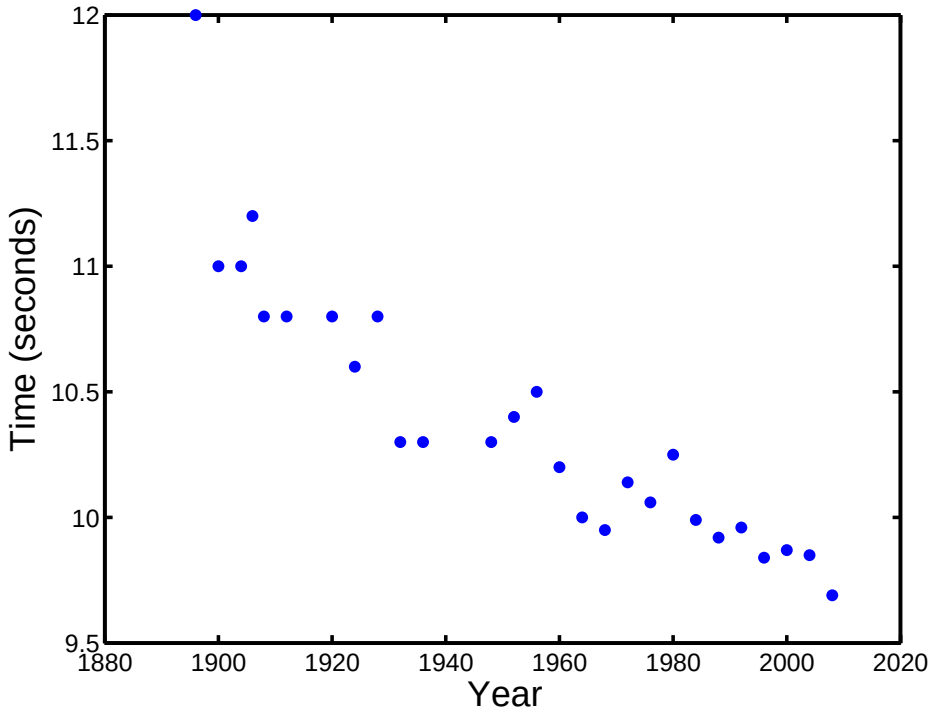
Some data and a problem

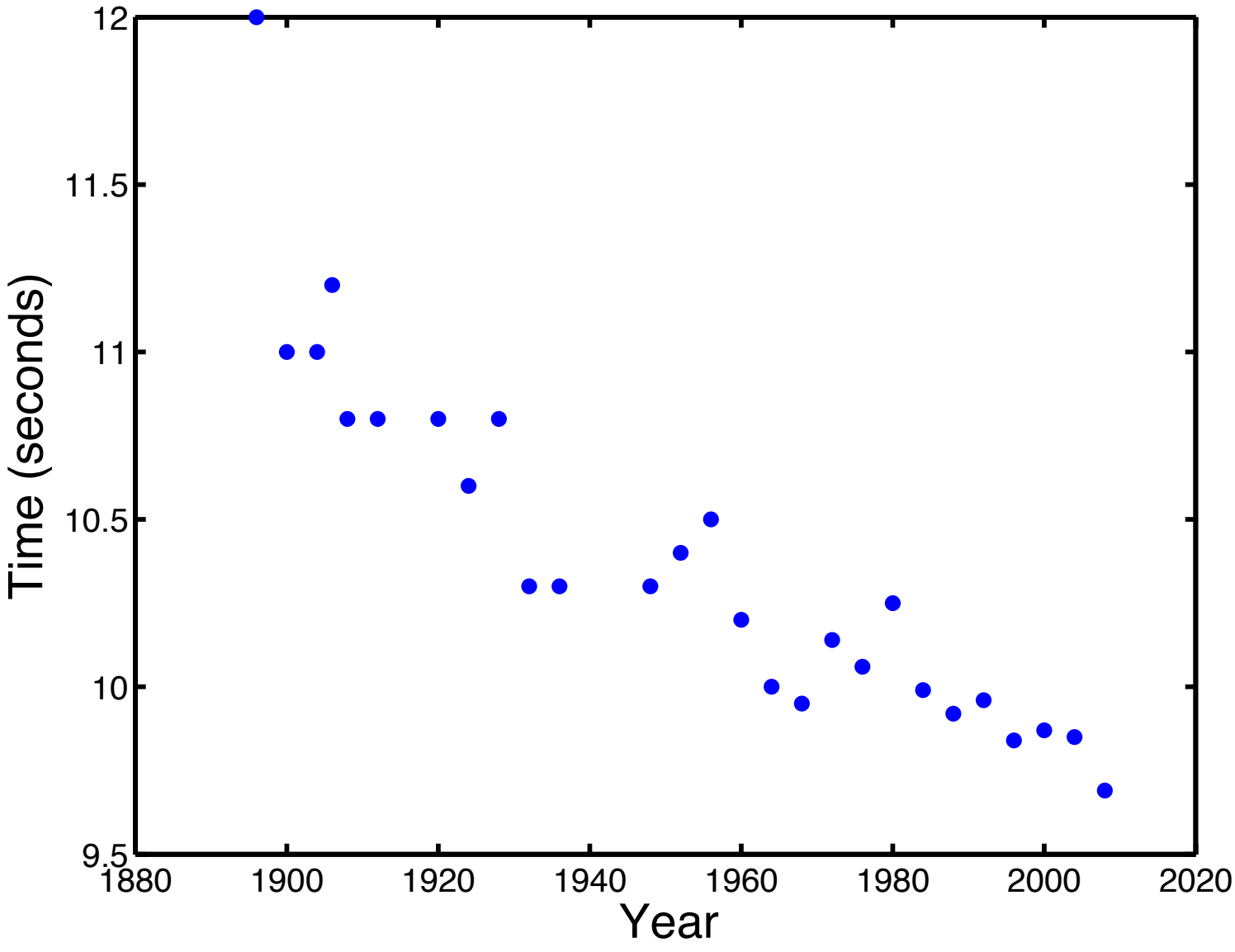


Winning times for the men's Olympic 100m sprint, 1896-2008.

In this lecture, we will use this data to predict the winning time in London 2012

Reading: Section 1.1 of FCML

Draw a line through it!



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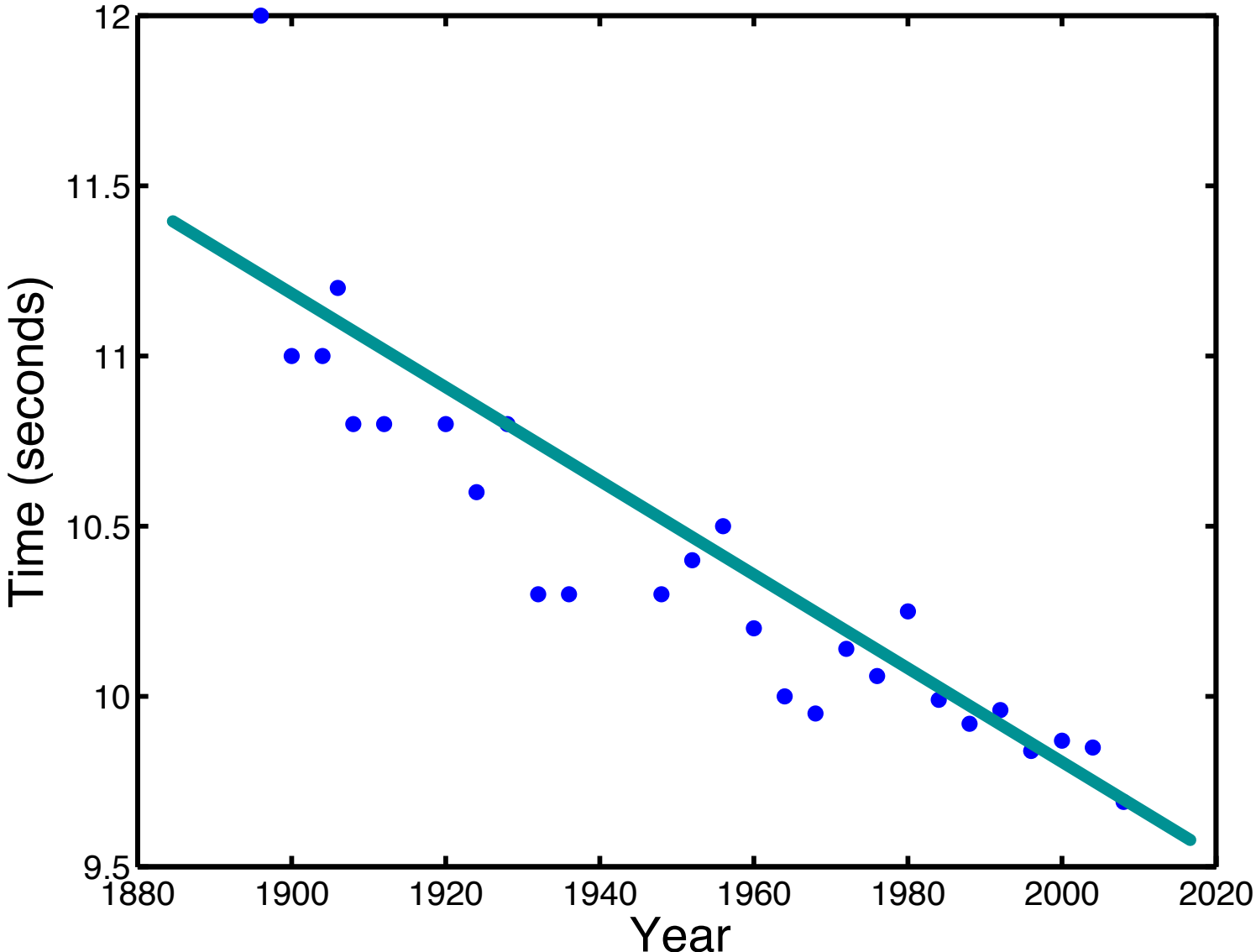

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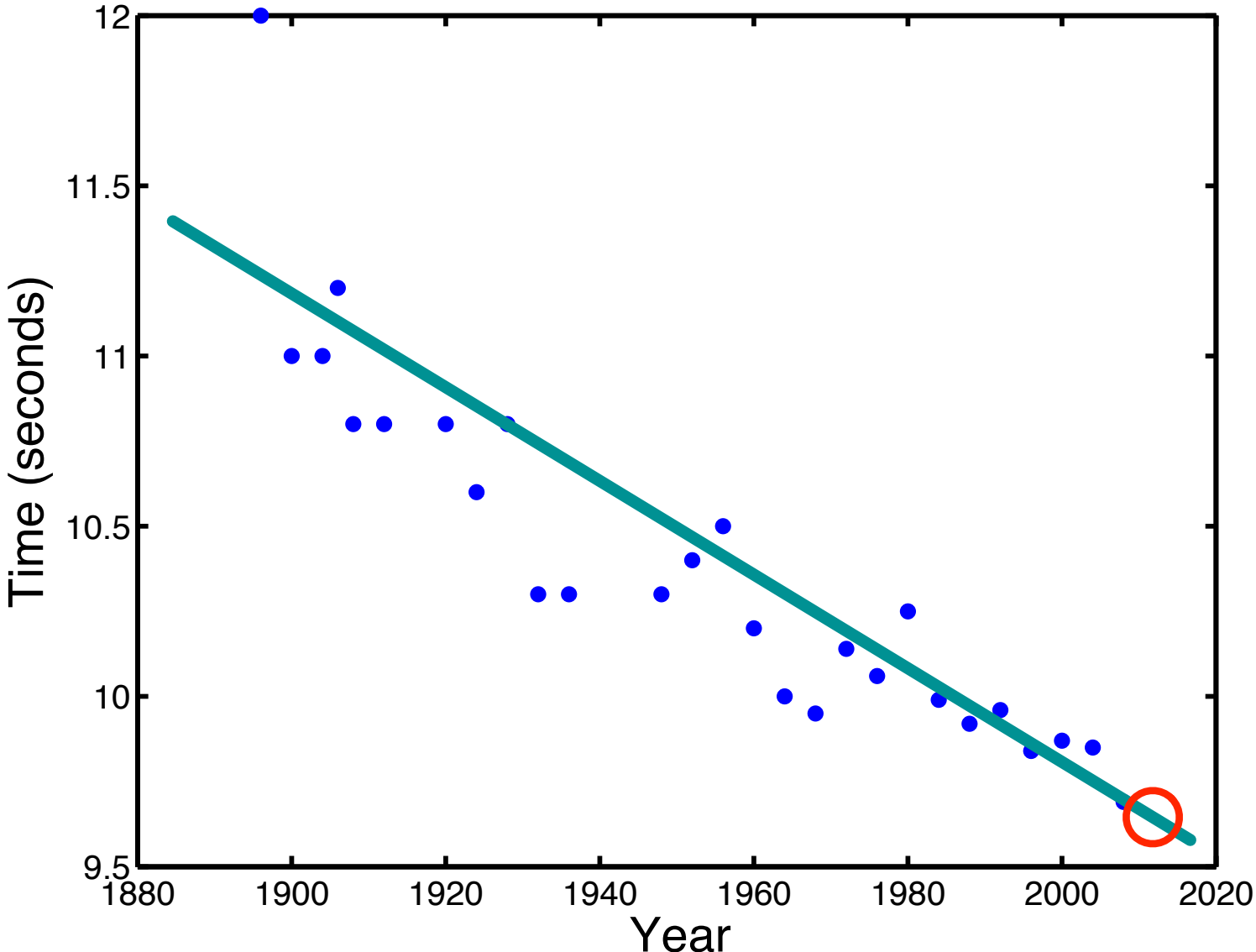

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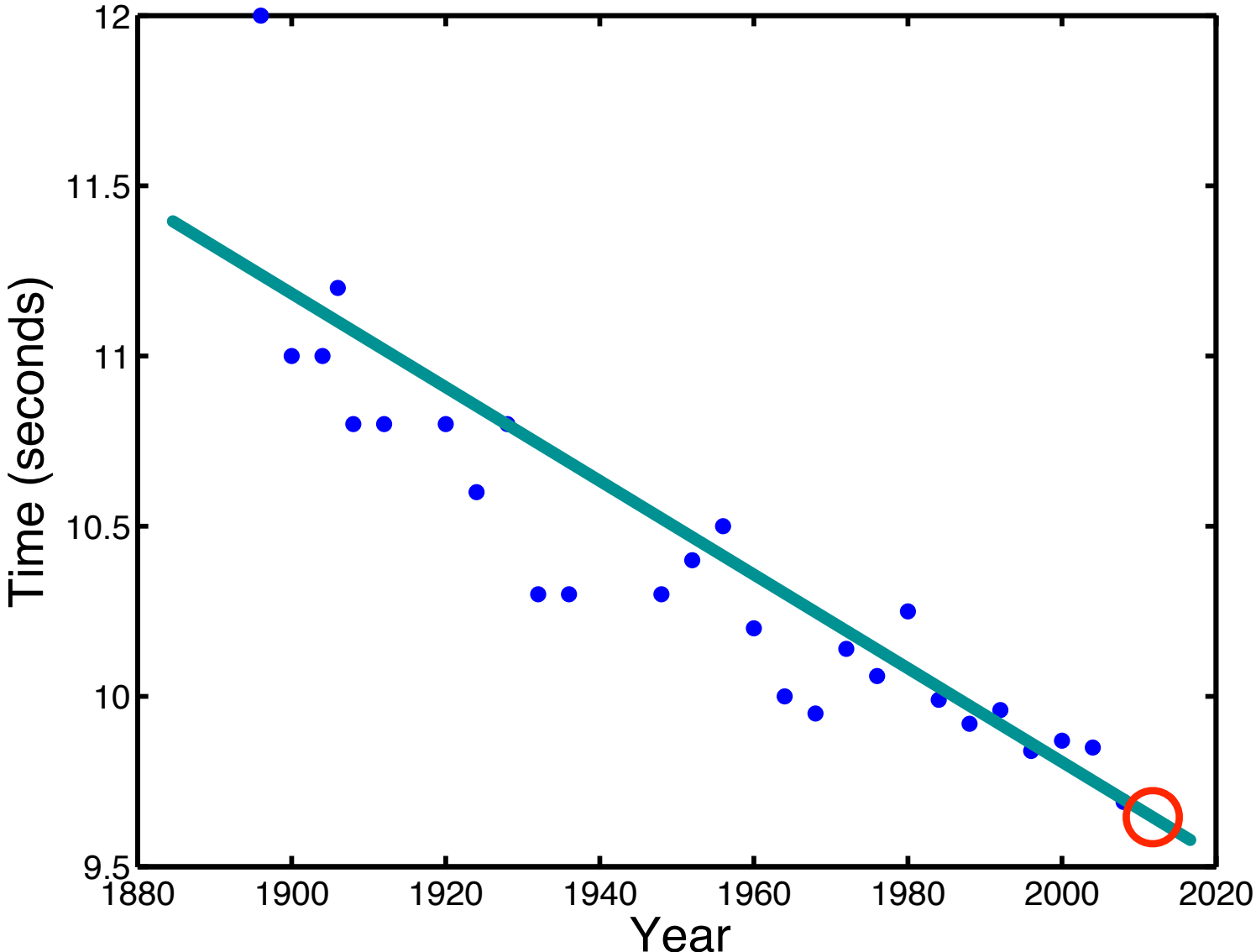

Overview

Linear models

- Introduce the idea of **building models**.
- Talk about **assumptions**.
- Use a **linear** model.
- What constitutes a **good** model?
- Find the **best** linear model.
- Use it to **predict** the winning time in 2012.

What did we do?

Basically:

- **Decided to draw a line through our data.**
- Chose a straight line.
- Drew a good straight line.
- Extended the line to 2012.
- Read off the winning time for 2012.

Technically

- Decided we needed a model.
- Chose a linear model.
- Fitted a linear model.
- Evaluated the model at 2012.
- Used this as our prediction.

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So far..

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The rest of the lecture...

- Mathematically define the model.
- Fit the model – there are many possible lines!
- Make our prediction.
- Some discussion about the assumptions.

Definitions

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We're going to create the model from data:

- N attribute-response pairs, (x_n, t_n)
- e.g. $(1896, 12s), (1900, 11s), \ldots, (2008, 9.69s)$
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Often called **training** data

A linear model

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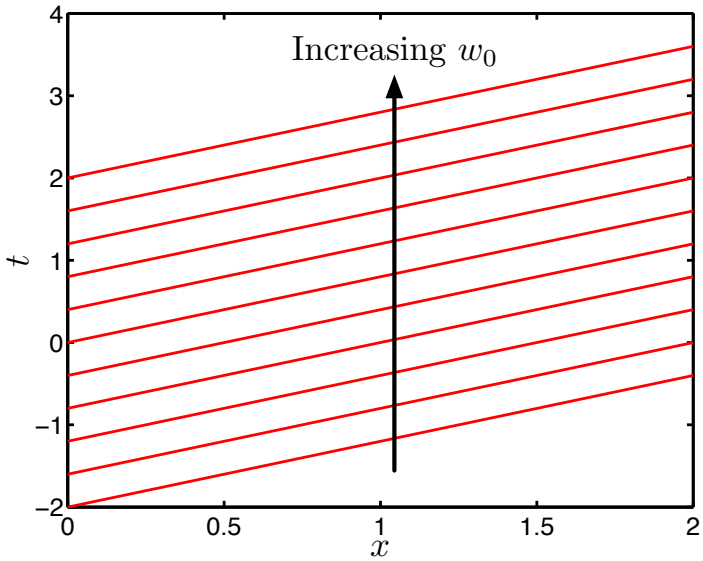
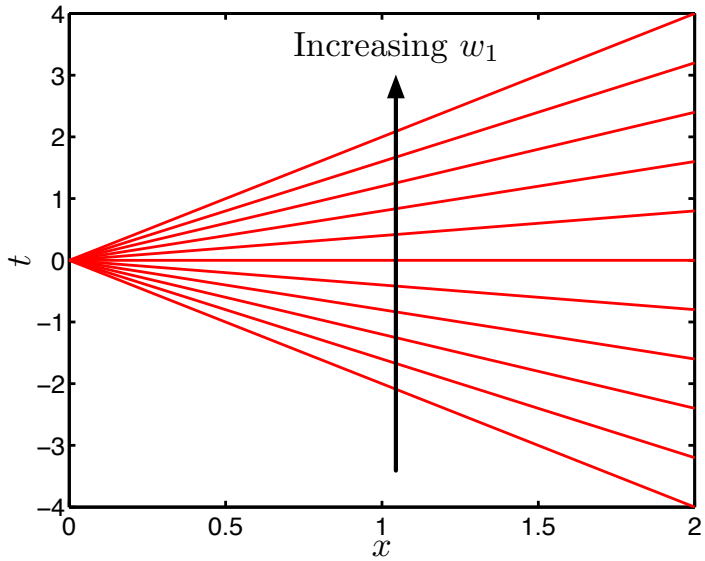
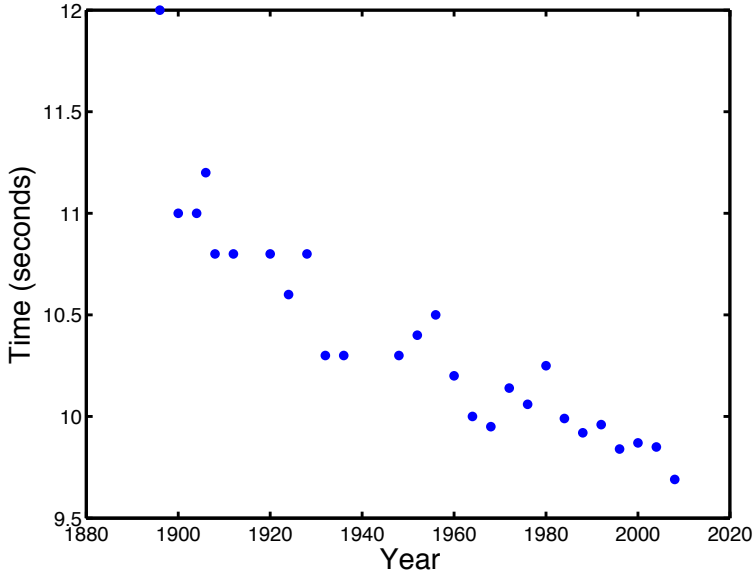




What next?

We have data and a family of models:



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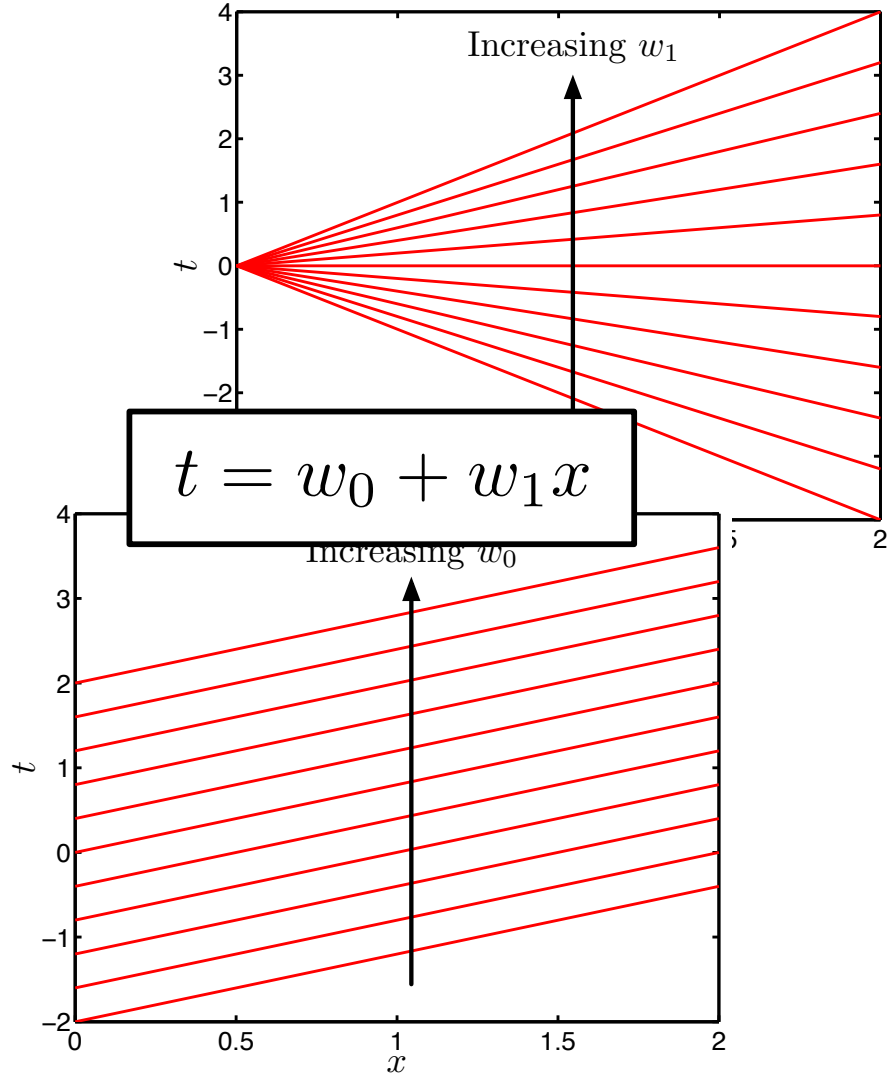

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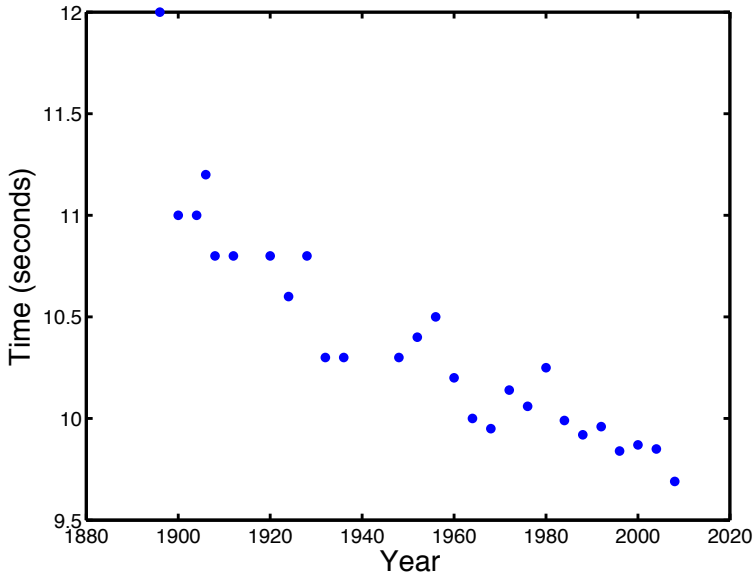

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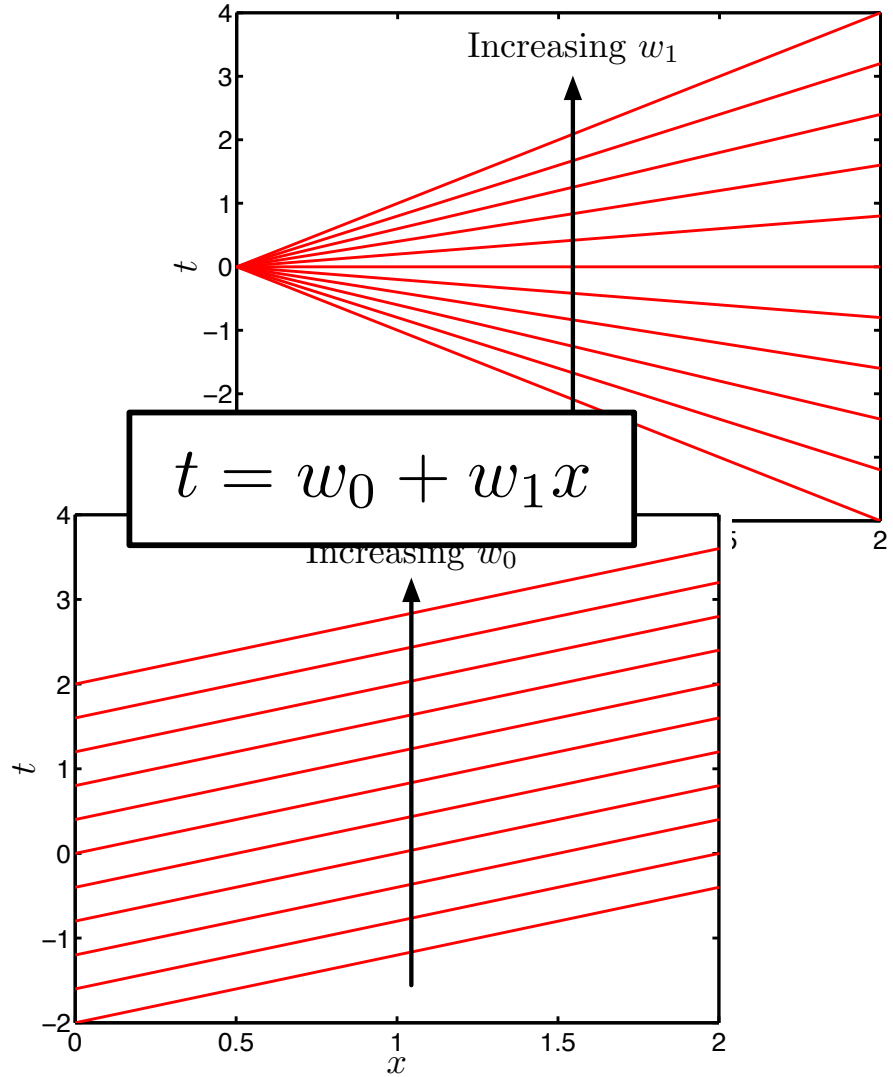

Need to find w_0, w_1 from $(x_1, t_1), (x_2, t_2), \ldots, (x_N, t_N)$

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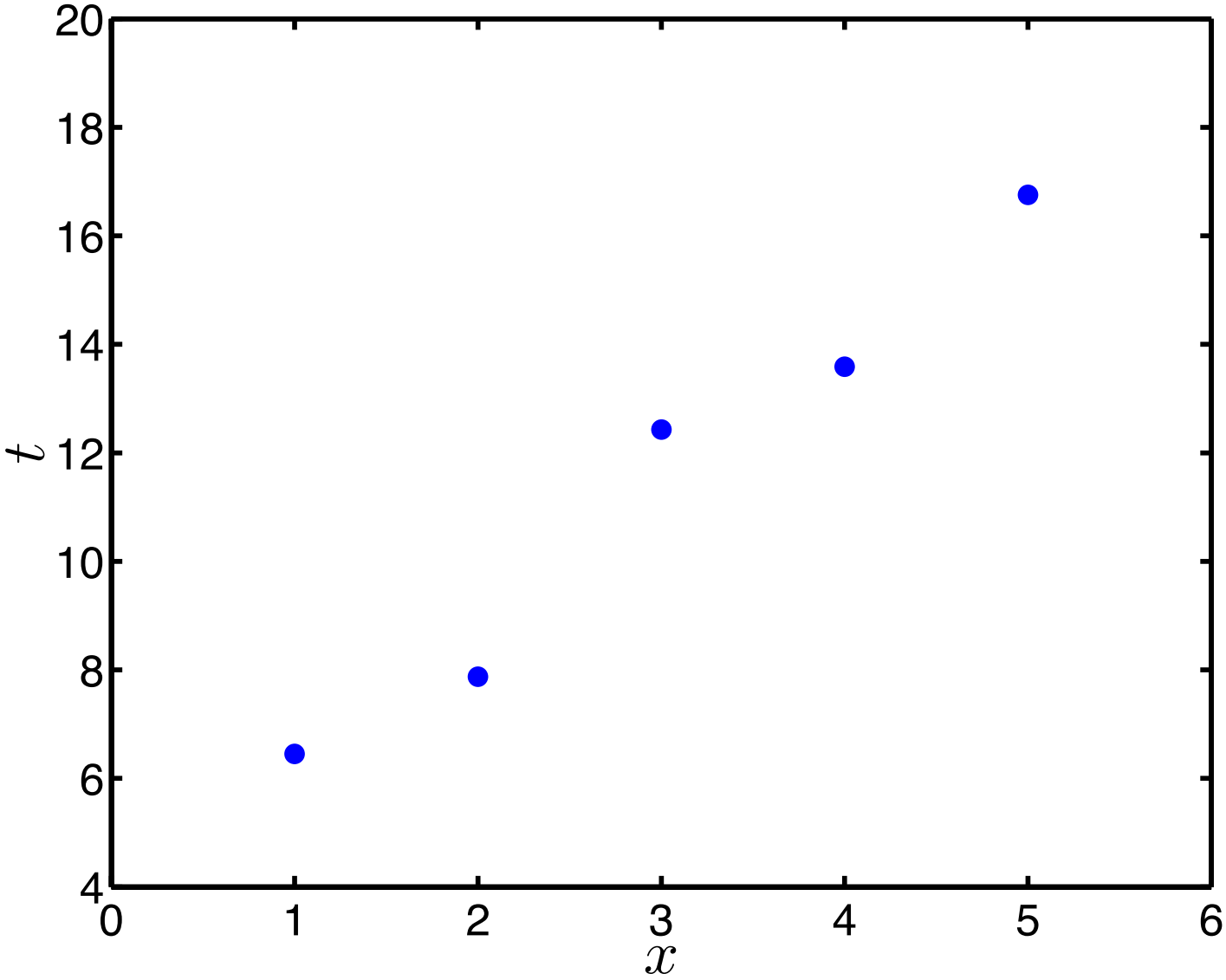
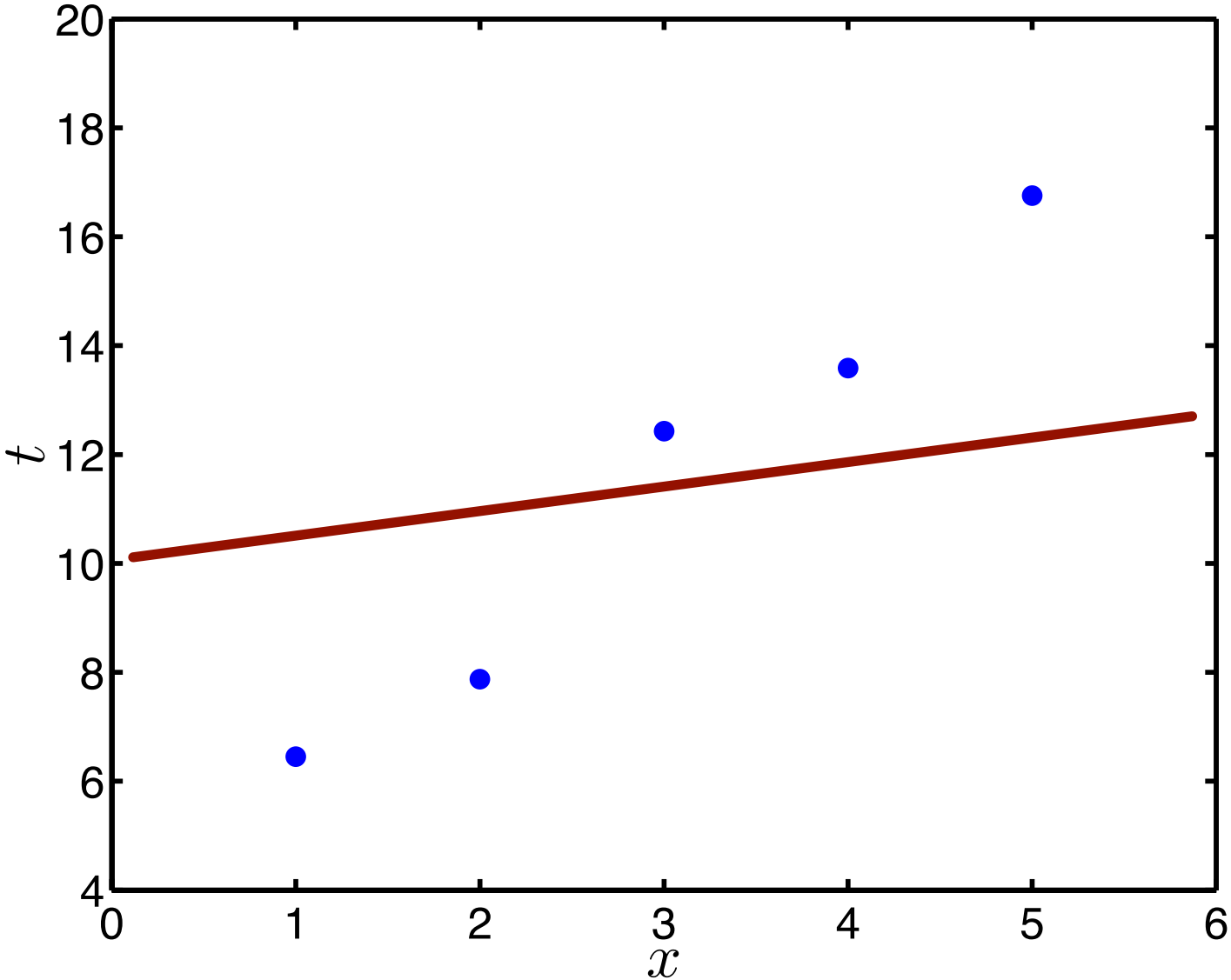
- Once we can answer these questions, we can search for the best w_0, w_1 pair.

Loss



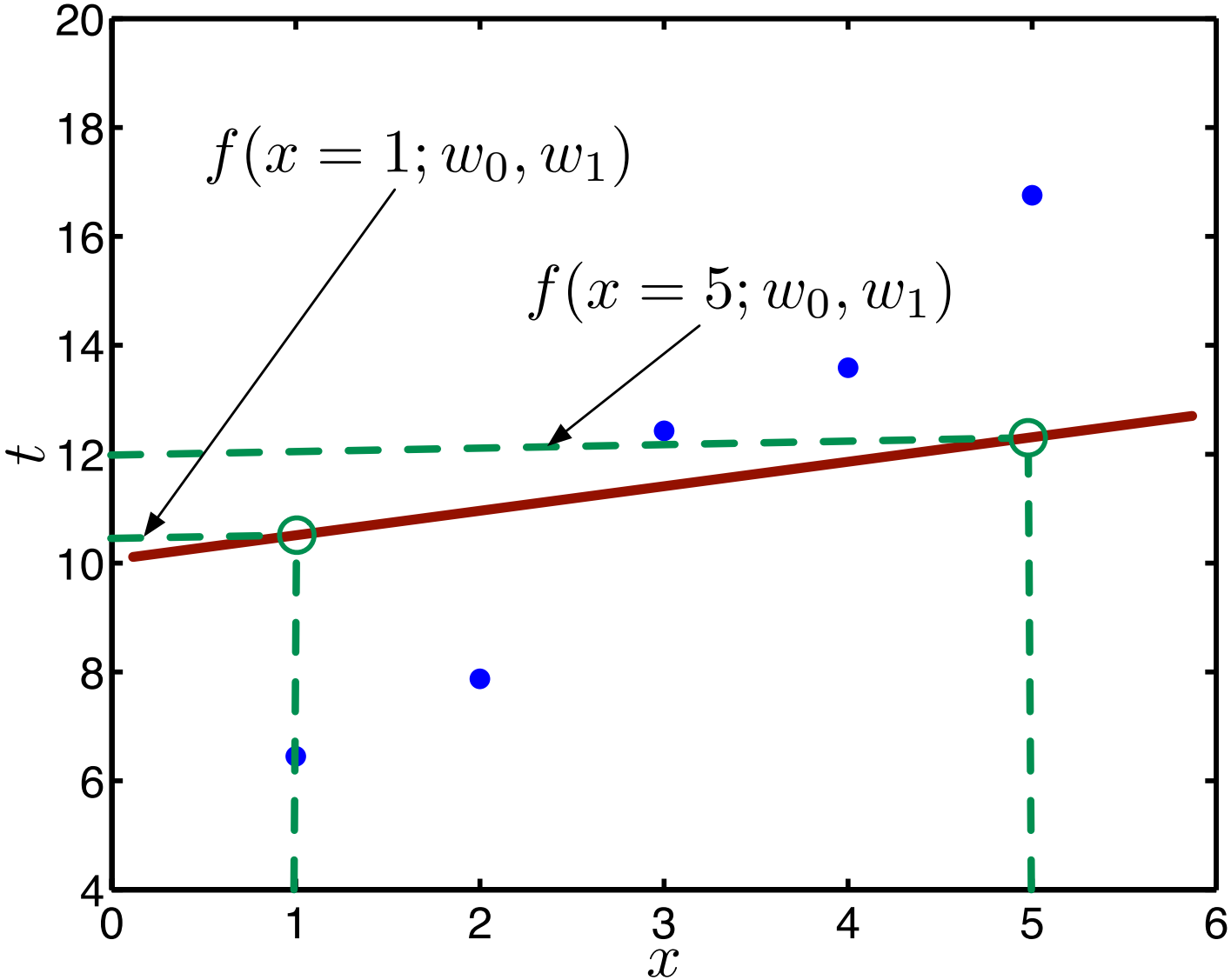
Some different data

Loss



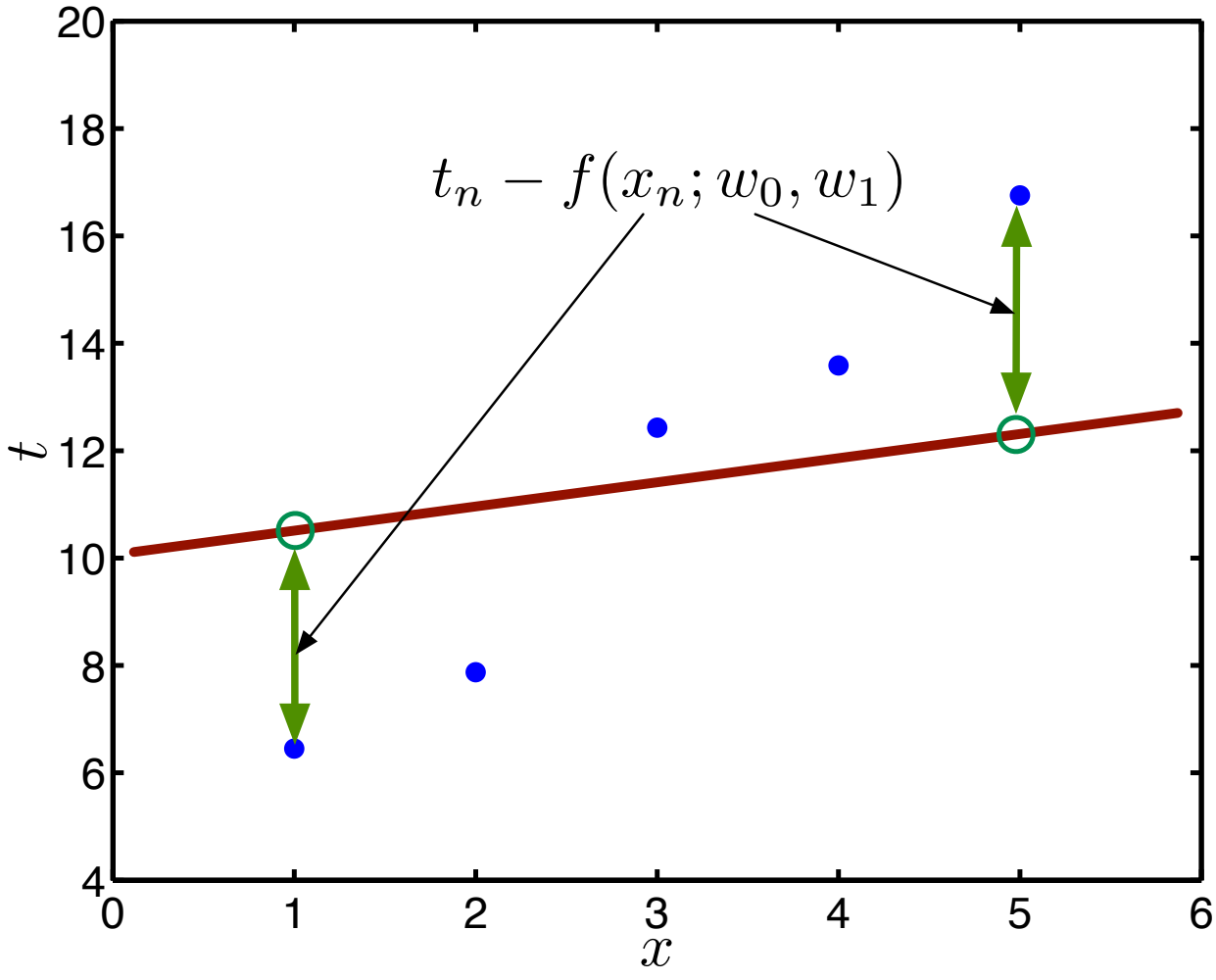
Given w_0 and w_1 you can draw a line

Loss



This means that we can compute $f(x_n; w_0, w_1)$ for each x_n

Loss



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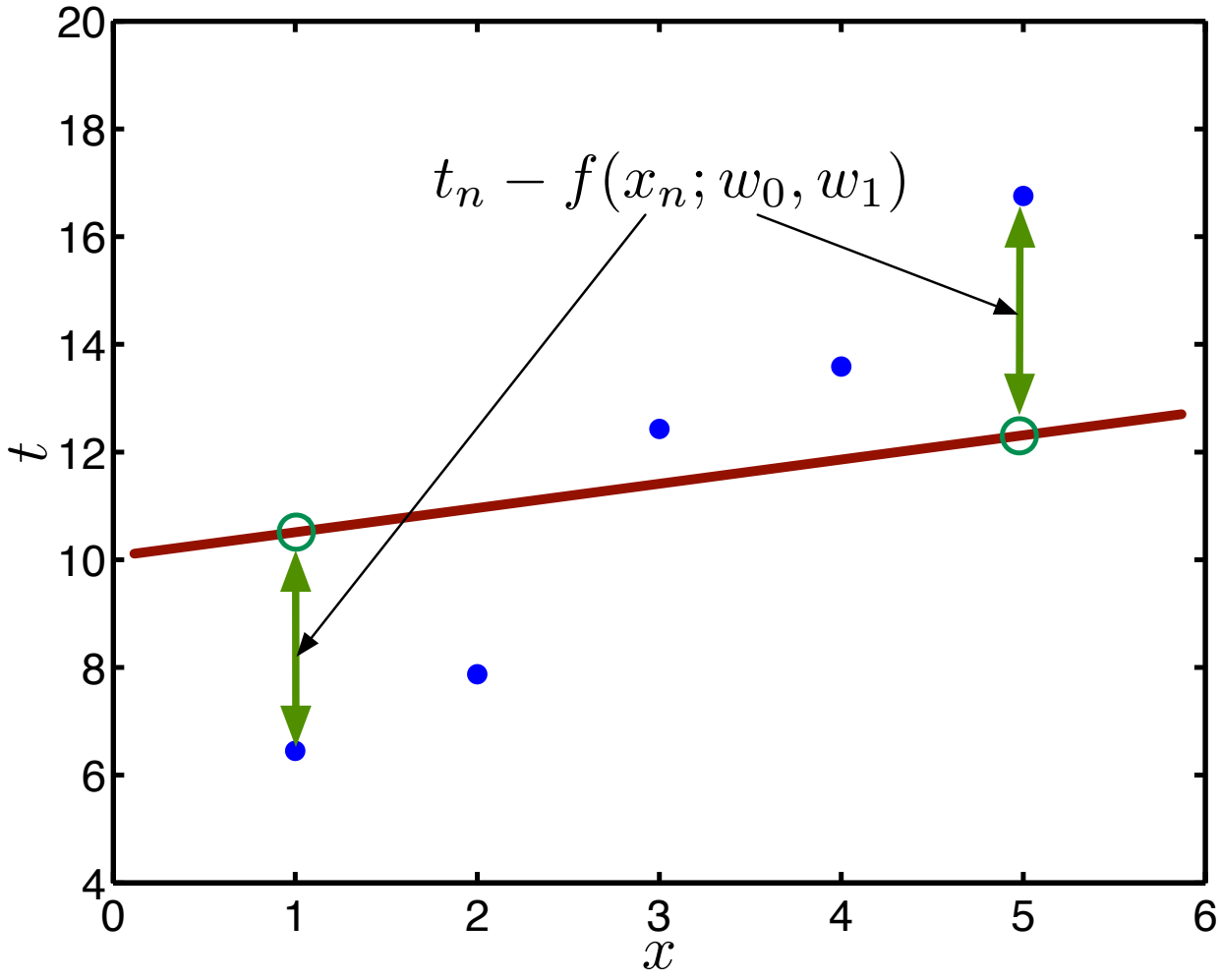

$f(x_n; w_0, w_1)$ can be compared with the truth, t_n

$|t_n - f(x_n; w_0, w_1)|$ tells us how *badly* we model (x_n, t_n)

Squared loss

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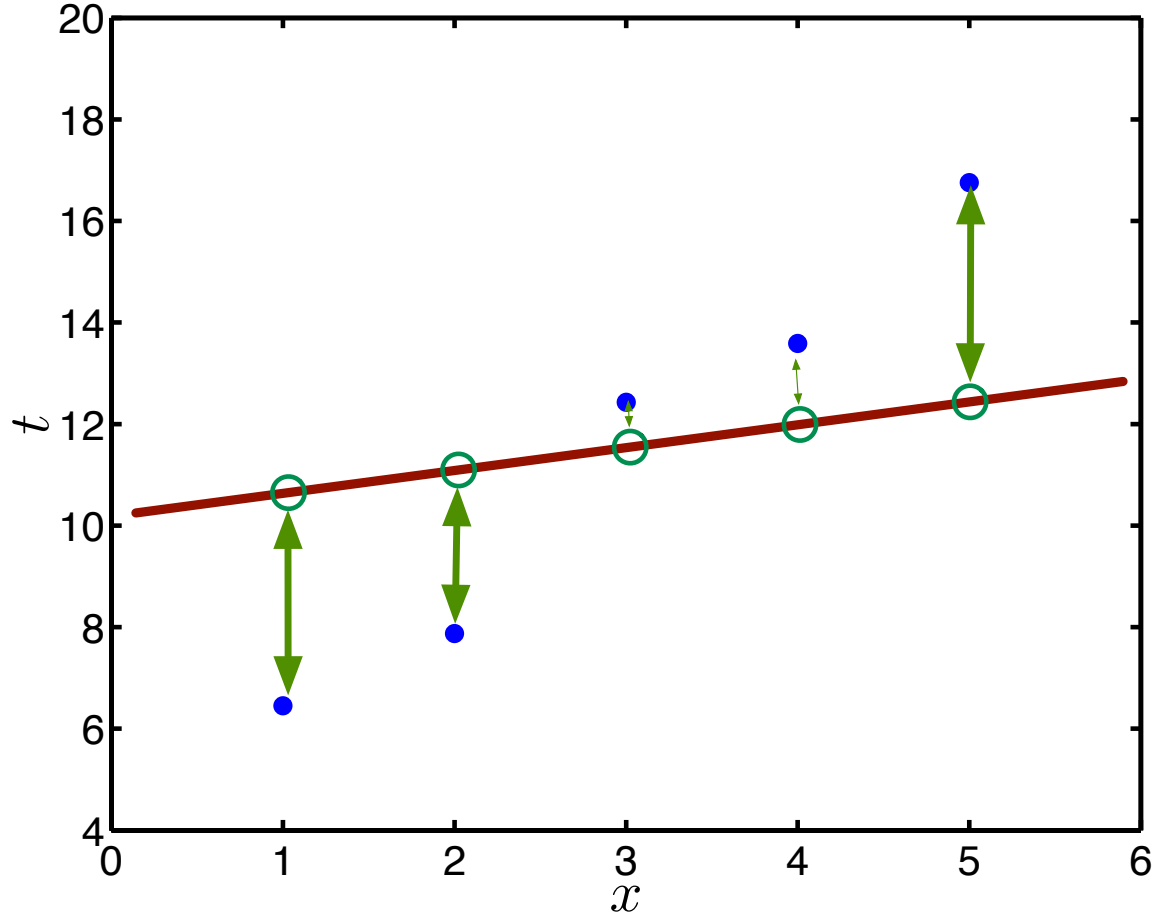
- The lower $\mathcal{L}_n$, the closer the line at x_n passes to t_n

Mean squared loss



Average the loss at each training point to give single figure for all data:

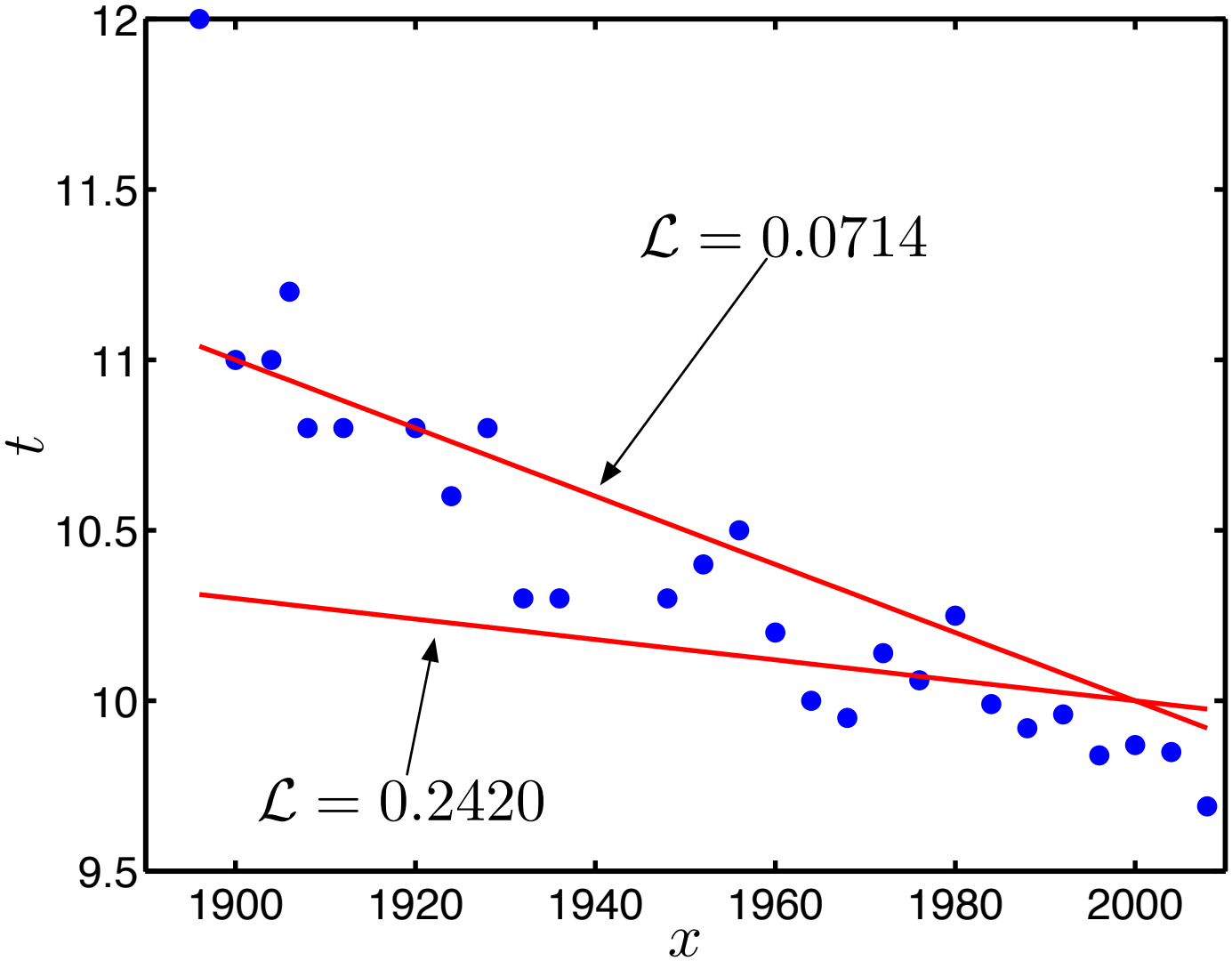
$$\mathcal{L} = \frac{1}{N}\sum_{n=1}^{N}(t_n - f(x_n; w_0, w_1))^2$$

- The average loss:

$$\mathcal{L} = \frac{1}{N} \sum_{n=1}^{N} (t_n - f(x_n; w_0, w_1))^2$$

- $\mathcal{L}$ tells us how good the model is as a function of w_0 and w_1.
  - Remember that lower is better!

Example


12
11.5
11
10.5
10
9.5
1900
1920
1940
1960
1980
2000
t
x
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- We've derived an expression for how good the model is for any w_0 and w_1.

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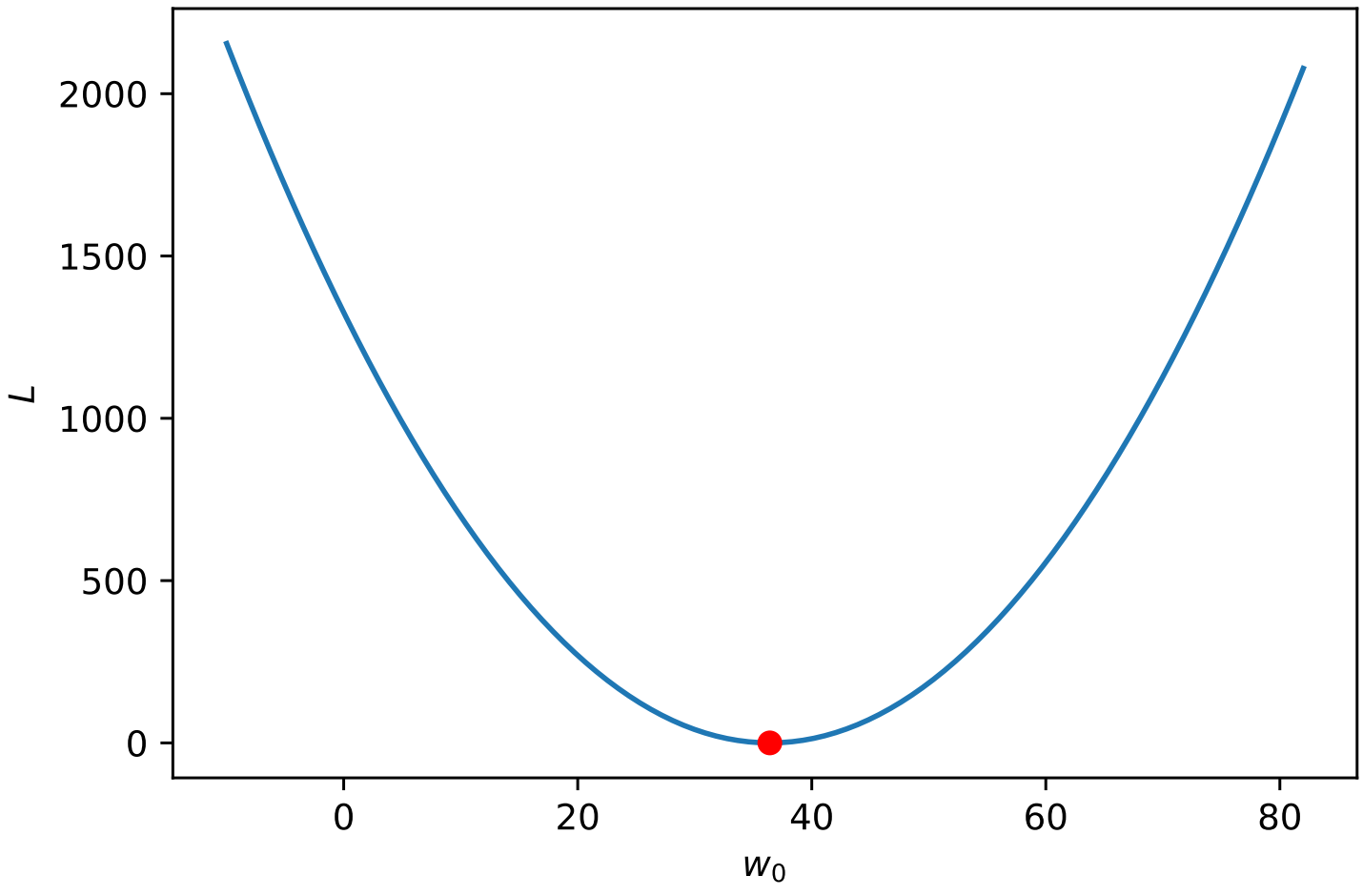
- Could use trial and error to find a good w_0, w_1 combination.
- Can we be **more efficient**?

Let's look at the average squared loss in 1D



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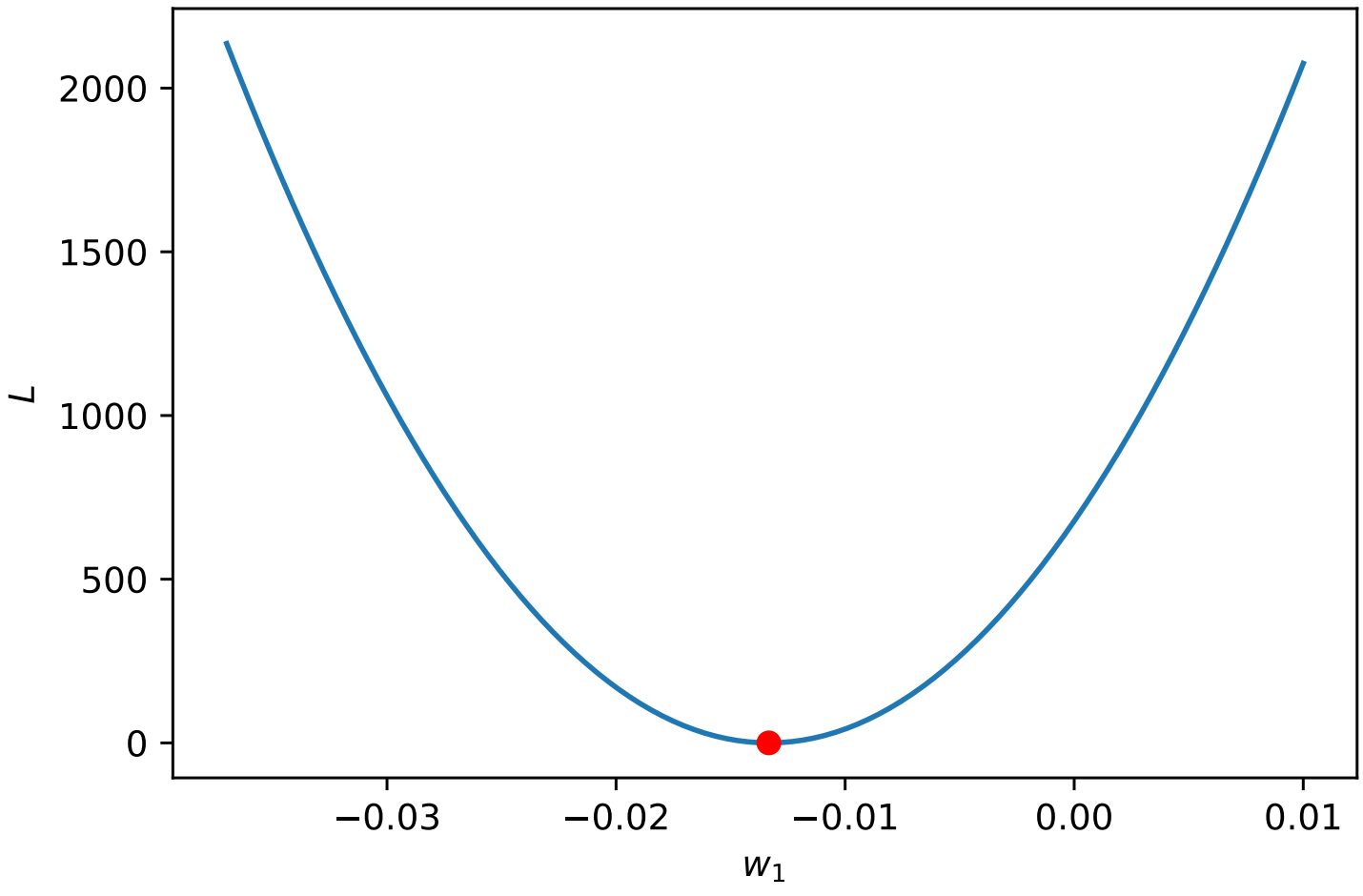
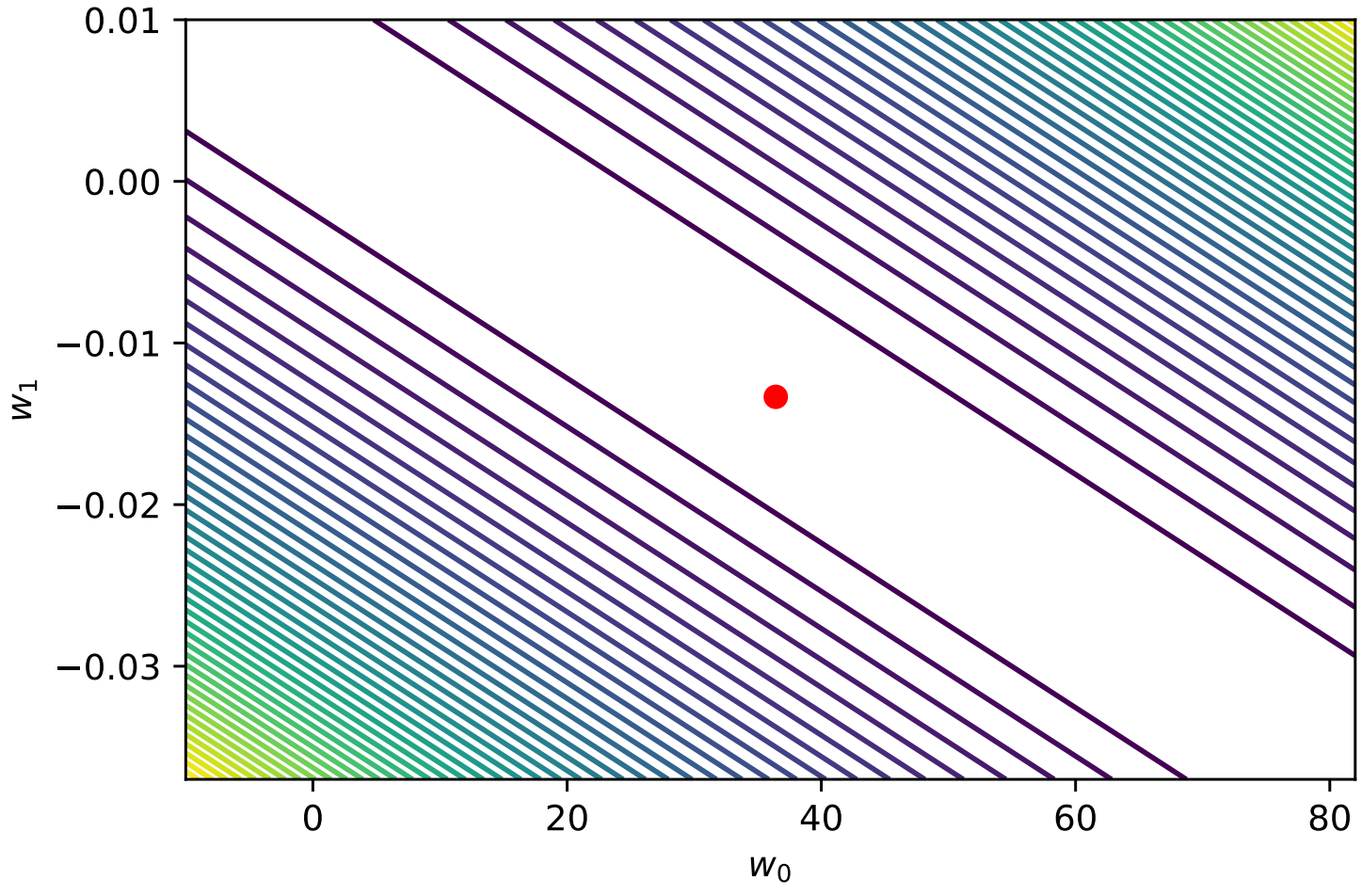


Let's look at the average squared loss in 2D



Maths Revision: Gradients

- if $f(x_1, x_2, \ldots)$ is a scalar-valued function of $(x_1, x_2, \ldots)$
- ...then gradient $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \end{pmatrix}$

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- always points in direction of **steepest ascent**

Gradient Descent

Initial guess w_0 w_1;
Set α;
while *Not converged* **do**
  for *i=0,1* **do**
    $g_i = \frac{\partial L(w_0, w_1)}{\partial w_i}$
  end
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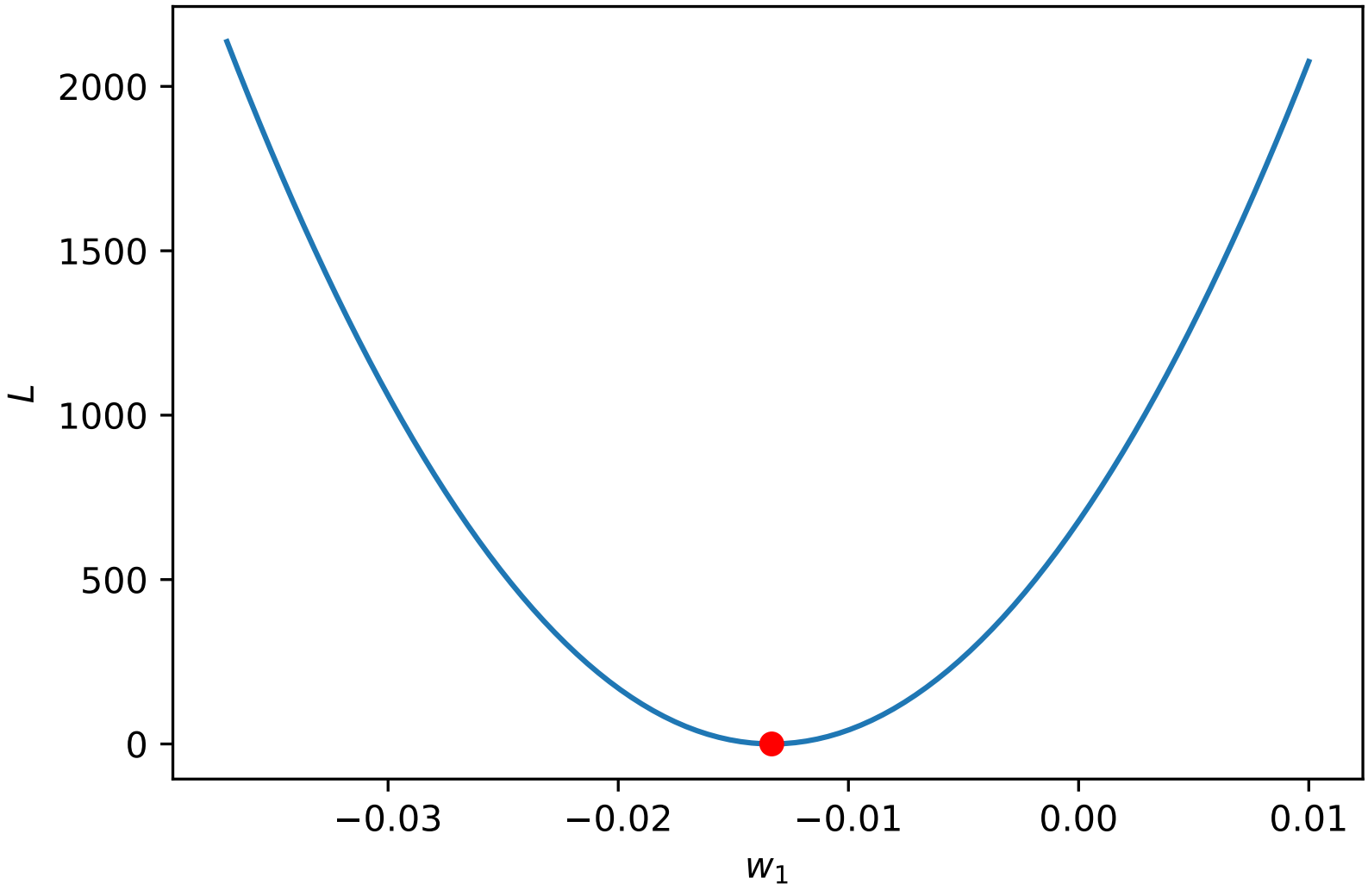

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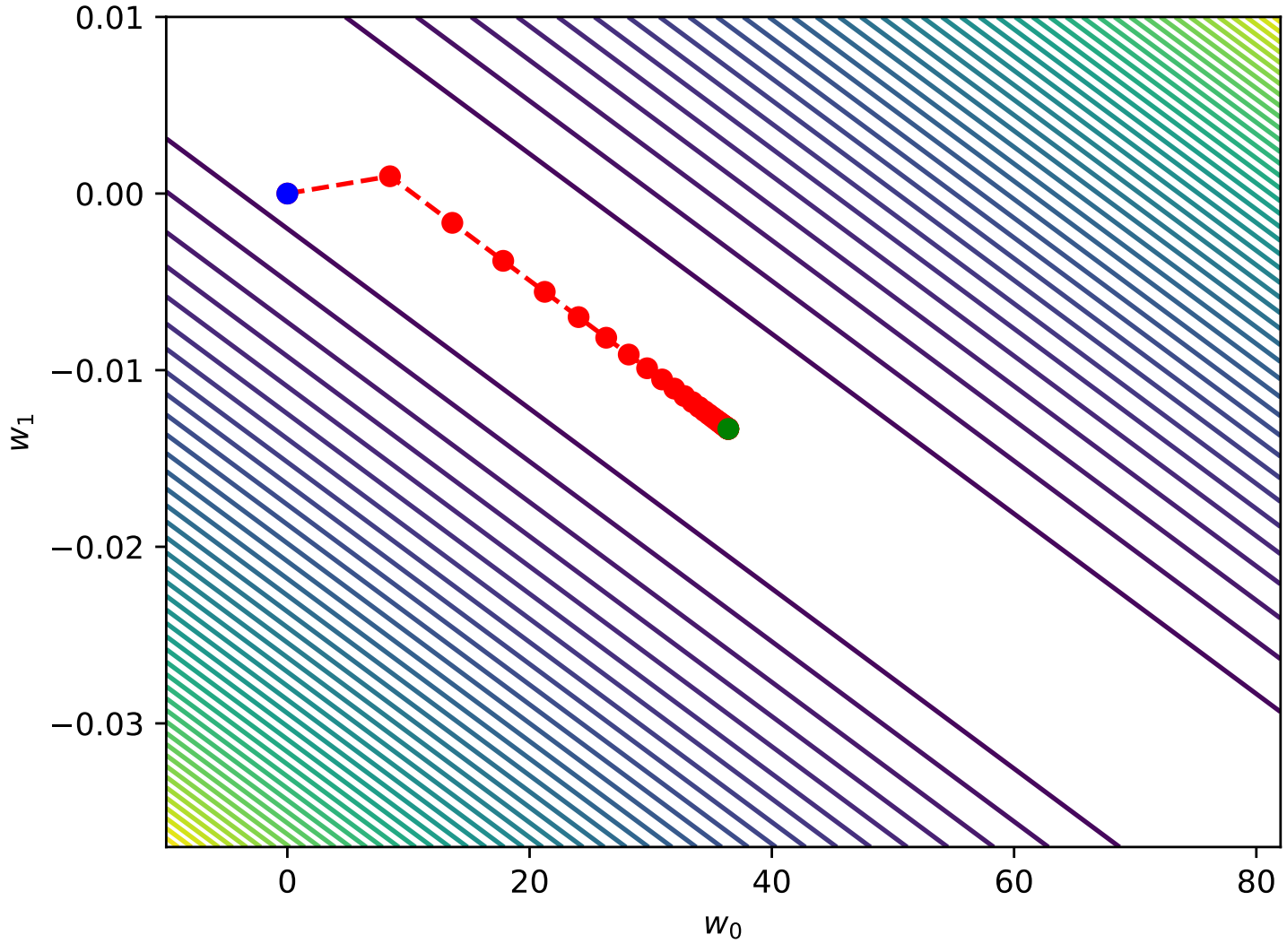
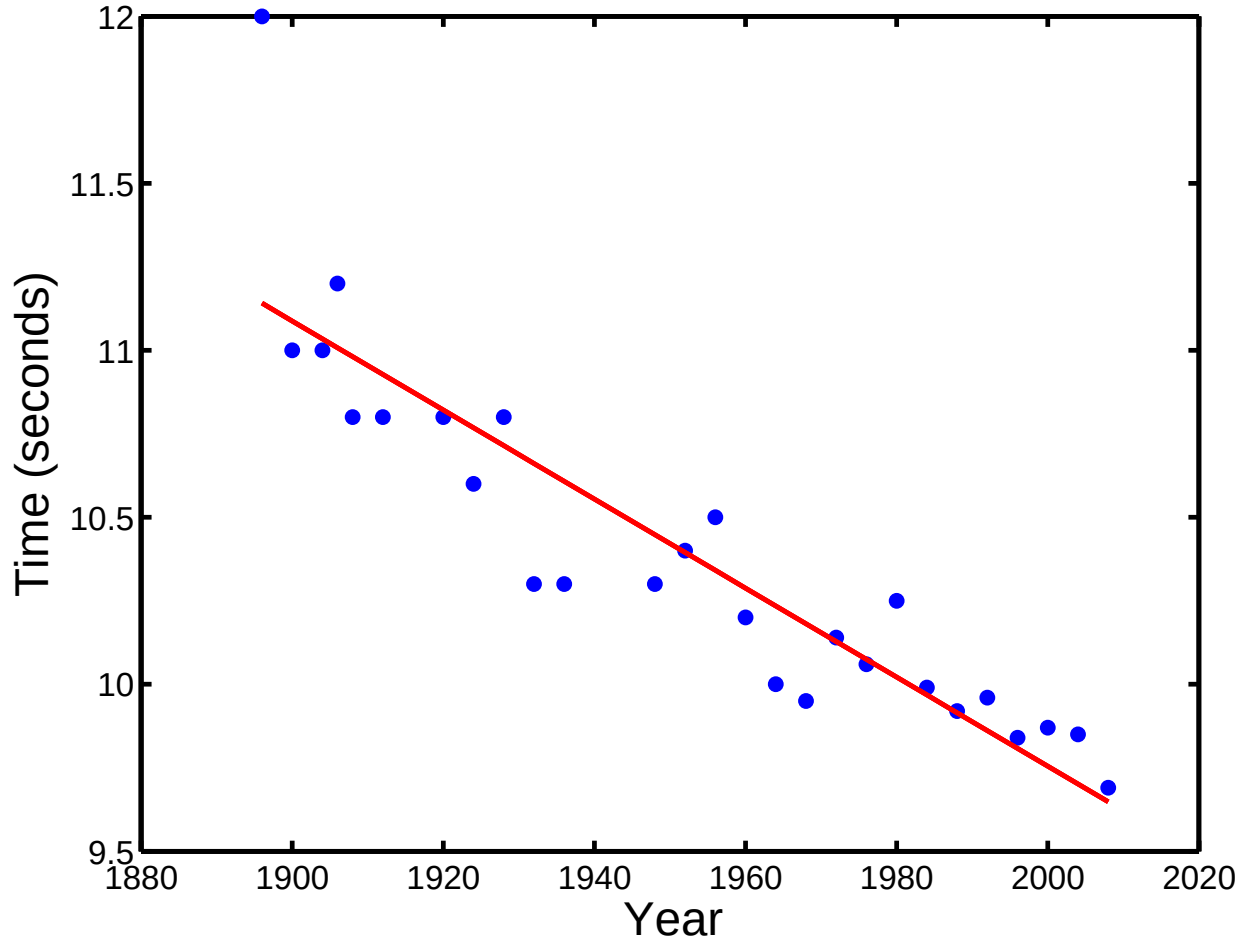


The model



$$t = 36.416 - 0.0133x$$

Our prediction

- We want to predict the winning time at London 2012.
- Substitute $x = 2012$ into our model.

$$\begin{aligned} t &= 36.416 - 0.0133x \\ t_{2012} &= 36.416 - 0.0133 \times 2012 \\ t_{2012} &= 9.5947\ s \end{aligned}$$

- Based on our modelling assumptions and the previous data, we predict a winning time of 9.5947 seconds.

Summary

- Introduced some ideas about modelling.
- Found some data.
- Derived a way of saying how good a model is.
- Found an expression for the best model.
- Used this to fit a model to the Olympic data.
- Made a prediction for the winning time in 2012.

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The model is 'wrong' but it might still be useful! How useful depends on the questions we wish to answer.