ILAS Seminar in Probability: Exercise sheet 2

1. Consider the Markov chain of Example 1.1.4, with transition matrix:

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

   Does P^n have a limit as $n \to \infty$? If so, what is it?
   Hint. You might want to consider the cases $\alpha + \beta = 0$, $\alpha + \beta \in (0, 2)$ and $\alpha + \beta = 2$ separately.

2. Consider a Markov chain on three vertices, with transition matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

   Give a graphical representation of the chain. Moreover, compute P^n for $n \geq 1$.
   Hint. The form of P^n is different for odd and even n. In particular, try showing that $p_{11}^{(n+2)} = Cp_{11}^{(n)}$ for some constant C, and similarly for $p_{12}^{(n)}$, $p_{21}^{(n)}$, $p_{22}^{(n)}$. Then use that

$$P^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   and $P^1 = P$ to deduce the answer.

3. Suppose that $Z_1, Z_2, \ldots$ are independent, identically-distributed random variables such that

$$\mathbf{P}(Z_i = 1) = p, \qquad \mathbf{P}(Z_i = 0) = 1 - p$$

   for some $p \in (0, 1)$. Moreover, define $S_0 = 0$, and $S_n = Z_1 + \cdots + Z_n$ for $n \geq 1$. In each of the following cases, determine whether $(X_n)_{n \geq 0}$ is a Markov chain:

   (a) $X_n = Z_n$;
   (b) $X_n = S_n$;
   (c) $X_n = S_0 + S_1 + \cdots + S_n$.

   In the cases where $(X_n)_{n \geq 0}$ is a Markov chain, find its state space and transition matrix. In the cases where it is not a Markov chain, give an example where the conditional probability $\mathbf{P}(X_{n+1} = i \mid X_n = j,\ X_{n-1} = k)$ depends on k.