Problem Set 2: Suggested Answers

Chen Li   Introduction to Game Theory

June 23, 2025

1. Mixed Strategy Nash Equilibrium. Suppose player 1 is playing strategy σ_1 such that $\sigma_1(x_1) = p$ and $\sigma_1(y_1) = 1 - p$. Also suppose player 1's belief is π_1 such that $\pi_1(x_2) = q$ and $\pi_1(y_2) = 1 - q$.

The expected payoff of pure strategy x_1 is $v_1(x_1, \pi_1) = 2 \times q + 0 \times (1 - q) = 2q$, and the expected payoff of pure strategy y_1 is $v_1(y_1, \pi_1) = 1 \times q + 3 \times (1 - q) = 3 - 2q$. Thus the expected payoff of σ_1 is $v_1(\sigma_1, \pi_1) = p \times 2q + (1 - p) \times (3 - 2q) = (4q - 3)p + 3 - 2q$ and the value of p when player 1 is playing best response is

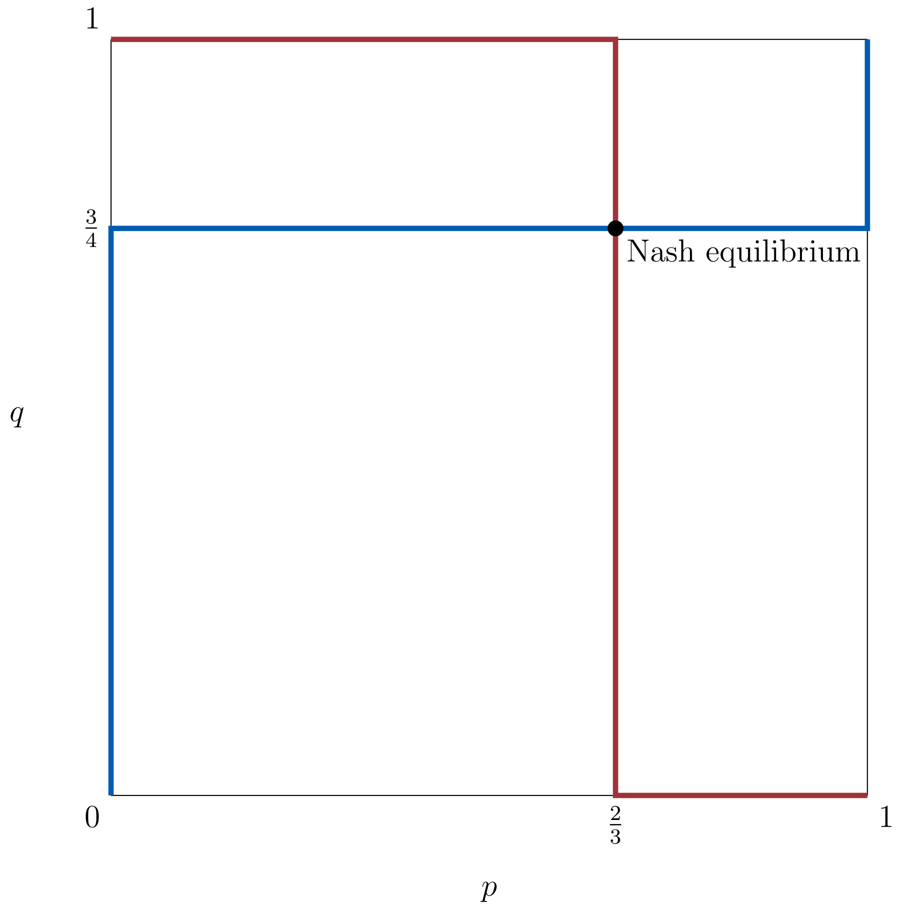
- $p = 0$ if $q < \frac{3}{4}$;
- any $0 \leq p \leq 1$ if $q = \frac{3}{4}$;
- $p = 1$ if $q > \frac{3}{4}$.

Similarly, suppose player 2 is playing strategy σ_2 such that $\sigma_2(x_2) = q$ and $\sigma_2(y_2) = 1 - q$. Also suppose player 2's belief is π_2 such that $\pi_2(x_1) = p$ and $\pi_2(y_1) = 1 - p$.

The expected payoff of pure strategy x_2 is $v_2(x_2, \pi_2) = 1 \times p + 2 \times (1 - p) = 2 - p$, and the expected payoff of pure strategy y_2 is $v_2(y_2, \pi_2) = 2 \times p + 0 \times (1 - p) = 2p$. Thus the expected payoff of σ_2 is $v_2(\sigma_2, \pi_2) = q \times (2 - p) + (1 - q) \times 2p = (2 - 3p)q + 2p$ and the value of q when player 2 is playing best response is

- $q = 1$ if $p < \frac{2}{3}$;
- any $0 \leq q \leq 1$ if $p = \frac{2}{3}$;
- $q = 0$ if $p > \frac{2}{3}$.

The best response curves (represented by p and q) are as follows. (Blue curve for player 1 and red curve for player 2.)



Hence, the mixed strategy Nash equilibrium of this game is (σ_1^*, σ_2^*) where $\sigma_1^* = (\sigma_1^*(x_1), \sigma_1^*(y_1)) = (\frac{2}{3}, \frac{1}{3})$, $\sigma_2^* = (\sigma_2^*(x_2), \sigma_2^*(y_2)) = (\frac{3}{4}, \frac{1}{4})$.

2. Backward Induction Solutions.

(a) Backward Induction Solution: Player 1 chooses x_1 at information set H_{11}, chooses a_1 at information set H_{12}, and player 2 chooses x_2 at information set H_{21}.

If we write a strategy profile as:

$((\text{player 1's action at } H_{11}, \text{player 1's action at } H_{12}), \text{player 2's action at } H_{21})$, then the backward induction solution can be written compactly as $((x_1, a_1), x_2)$.

(b) Backward Induction Solution: Player 1 chooses x_1 at information set H_{11}, chooses b_1 at information set H_{12}, chooses b_1 at information set H_{13}, and player 2 chooses x_2 at information set H_{21}, chooses z_2 at information set H_{22}.

Similar to (a), this solution can be written compactly as $((x_1, b_1, b_1), (x_2, z_2))$ (How to read this?).

3. Normal-Form Representation of Extensive Form Games.

(a) The normal-form representation:

Player 1 \ Player 2	x_2	y_2
(x_1, a_1)	(2, 0)	(2, 0)
(x_1, b_1)	(2, 0)	(2, 0)
(y_1, a_1)	(1, 1)	(3, 0)
(y_1, b_1)	(1, 1)	(0, 2)

Nash equilibria: $((x_1, a_1), x_2)$, $((x_1, b_1), x_2)$.

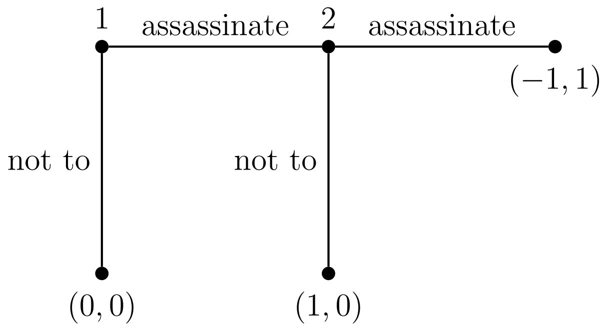
(b) The normal-form representation:

Player 1 \ Player 2	(x_2, x_2)	(x_2, y_2)	(x_2, z_2)	(y_2, x_2)	(y_2, y_2)	(y_2, z_2)
(x_1, a_1, a_1)	(1, 9)	(1, 9)	(1, 9)	(3, 1)	(3, 1)	(3, 1)
(x_1, a_1, b_1)	(1, 9)	(1, 9)	(1, 9)	(3, 1)	(3, 1)	(3, 1)
(x_1, b_1, a_1)	(2, 2)	(2, 2)	(2, 2)	(3, 1)	(3, 1)	(3, 1)
(x_1, b_1, b_1)	(2, 2)	(2, 2)	(2, 2)	(3, 1)	(3, 1)	(3, 1)
(y_1, a_1, a_1)	(2, 0)	(3, 8)	(0, 2)	(2, 0)	(3, 8)	(0, 2)
(y_1, a_1, b_1)	(2, 0)	(4, 1)	(0, 2)	(2, 0)	(4, 1)	(0, 2)
(y_1, b_1, a_1)	(2, 0)	(3, 8)	(0, 2)	(2, 0)	(3, 8)	(0, 2)
(y_1, b_1, b_1)	(2, 0)	(4, 1)	(0, 2)	(2, 0)	(4, 1)	(0, 2)
(z_1, a_1, a_1)	(1, 3)	(1, 3)	(1, 3)	(1, 3)	(1, 3)	(1, 3)
(z_1, a_1, b_1)	(1, 3)	(1, 3)	(1, 3)	(1, 3)	(1, 3)	(1, 3)
(z_1, b_1, a_1)	(1, 3)	(1, 3)	(1, 3)	(1, 3)	(1, 3)	(1, 3)
(z_1, b_1, b_1)	(1, 3)	(1, 3)	(1, 3)	(1, 3)	(1, 3)	(1, 3)

Nash equilibria: $((x_1, b_1, a_1), (x_2, x_2))$, $((x_1, b_1, a_1), (x_2, z_2))$, $((x_1, b_1, b_1), (x_2, x_2))$, $((x_1, b_1, b_1), (x_2, z_2))$.

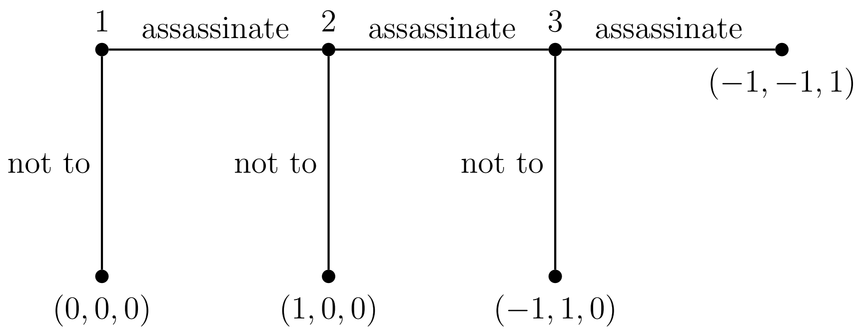
4. Assassination Game. Since the original king has no choice to make, we will omit his payoffs. For each minister, let the payoff be 1 if he becomes the king himself and is not assassinated, be 0 if he remains a minister, and be -1 if he becomes the king himself and is assassinated.

(a) The game tree when $n = 2$ (payoff should be read as (Minister 1's payoff, Minister 2's payoff)):



Backward induction solution: Minister 1 chooses "not to assassinate", Minister 2 chooses to "assassinate".

(b) The game tree when $n = 3$ (payoff should be read as (Minister 1's payoff, Minister 2's payoff, Minister 3's payoff)):



Backward induction solution: Minister 1 chooses to "assassinate", Minister 2 chooses "not to assassinate", Minister 3 chooses to "assassinate".

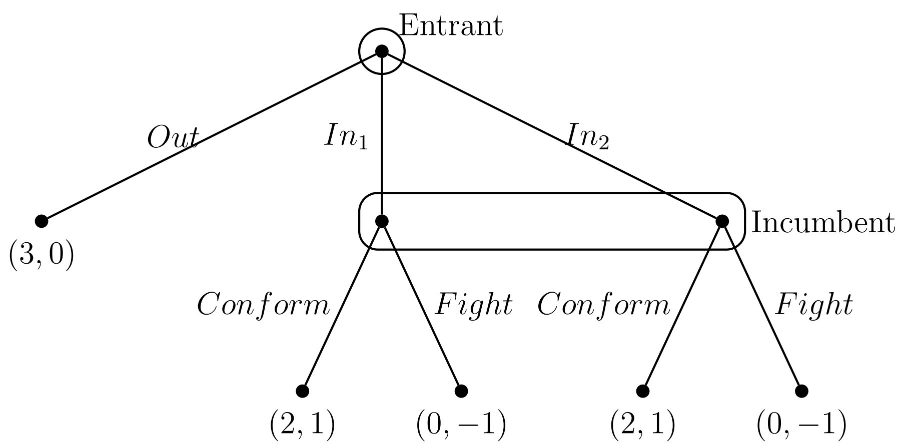
(c) The last minister, Minister n will choose to "assassinate" if he is rational. Taking this into account, Minister $n-1$ will choose "not to assassinate". In general, if Minister k chooses to "assassinate", then Minister $k-1$ will choose "not to assassinate". If Minister k chooses "not to assassinate", then Minister $k-1$ will choose to "assassinate" for $k = 2, ..., n$.

Thus we can conclude that the backward induction solution will be:

- If n is odd, Minister k with k odd will choose to "assassinate", and Minister k with k even will choose "not to assassinate". In this case, the original king will be assassinated.

- If n is even, Minister k with k odd will choose "not to assassinate", and Minister k with k even will choose to "assassinate". In this case, the original king will be alive.

5. Modified Entry Game. (a) The game tree is as follows. The payoff left is the Incumbent's, and the payoff right is the Entrant's.



(b) The only subgame of this game is the entire game. Thus, every Nash equilibrium of the normal-form representation of this game is a subgame perfect equilibrium. The normal-form representation of this game is as follows:

		Entrant		
		In_1	In_2	Out
Incumbent	$Conform$	(2, 1)	(2, 1)	(3, 0)
	$Fight$	(0, -1)	(0, -1)	(3, 0)

It can be readily confirmed that there are three subgame perfect equilibria (check this by yourself): ($Fight$, Out), ($Conform$, In_1), ($Conform$, In_2). Here, a strategy profile should be read as (Incumbent's strategy, Entrant's strategy).