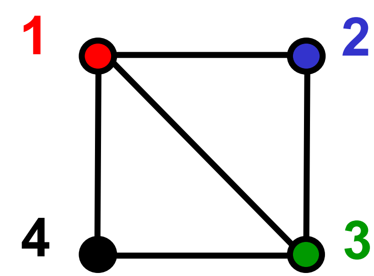
Inapproximability of NPO problems

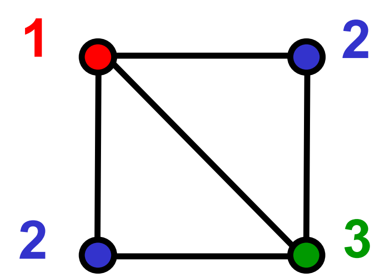
Let ∏ be a problem in NPO that is NP-hard

- **We seek results of the form: if P≠NP then ∏ cannot be approximated within a factor of c for some constant c**
  - **i.e. if P≠NP then ∏ has no approximation algorithm with performance guarantee d, for any $1 \le d \le c$**
  - **Gives an indication of the limit to how well ∏ can be approximated**



Example 1: Minimum Graph Colouring

- **Instance:** Graph G=(V,E)
- **Feasible solutions:** Any colouring of V
- **Measure:** Number of colours
- **Goal:** min



Min Graph Colouring is NP-hard

Inapproximability of Min Graph Colouring

The problem of deciding whether a graph can be coloured with 3 colours is known to be NP-complete

Suppose that there is an approximation algorithm A for Min Graph Colouring with performance guarantee $c < 4/3$

Let G be any graph

(i) If G is 3-colourable, then $m^*(G) \leq 3$ so A returns a colouring using $\leq 3c < 4$ colours, i.e. 3 colours

(ii) If G is not 3-colourable, then $m^*(G) \geq 4$ so A returns a colouring using ≥ 4 colours

So the existence of A would decide the 3-colourability of G in polynomial time

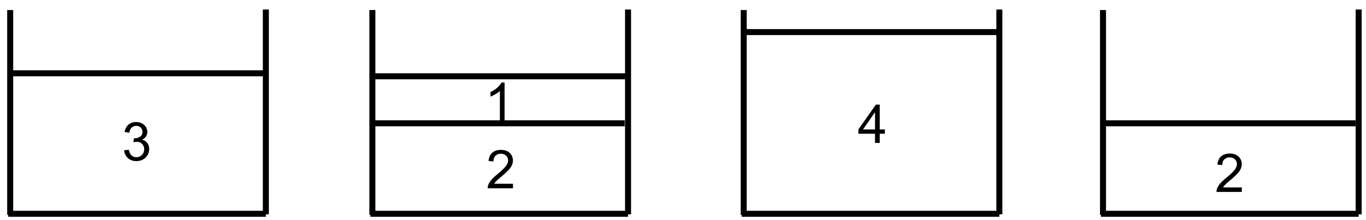
Theorem: If $P \neq NP$ then Minimum Graph Colouring has no c-approximation algorithm with $c < 4/3$

Example 2: Minimum Bin Packing

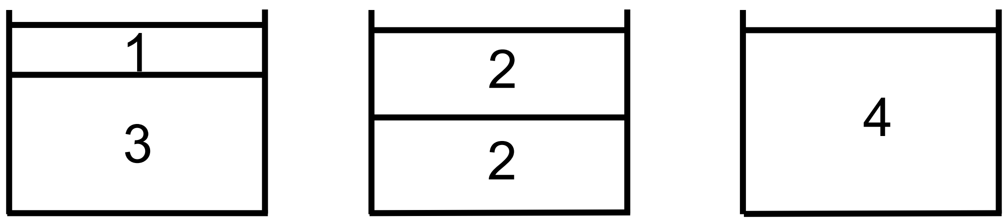
- **Instance:** Collection C of n objects, of sizes a_i ($1 \leq i \leq n$) and a bin capacity b
- **Feasible solutions:** Any partition of C into k bins such that the sum of sizes of the objects in each bin does not exceed b
- **Measure:** Number of bins k
- **Goal:** min

Example instance: bin capacity 4; object sizes 3,2,1,4,2

(i)
4 bins



(ii)
3 bins



Minimum Bin Packing is NP-hard

Inapproximability of Minimum Bin Packing

Recall the Partition problem (known to be NP-complete):

- **Instance:** a collection S of positive integers
- **Question:** can S be partitioned into two sub-collections of equal sum?

Example: $a_1=7$, $a_2=4$, $a_3=11$, $a_4=4$, $a_5=9$, $a_6=7$ - answer is no

Change a_6 to 5 - answer is yes since

$$a_3 + a_5 = 11 + 9 = 7 + 4 + 4 + 5 = a_1 + a_2 + a_4 + a_6$$

Suppose that there is a polynomial-time approximation algorithm A for Minimum Bin Packing with performance guarantee $c < 3/2$

Let S be a collection of n integers in an instance of Partition. Define an instance x of Minimum Bin Packing with n objects with sizes given by S and define the capacity of each bin to be $b = ½\sum\{a_i : a_i \text{ is in } S\}$

Inapproximability of Minimum Bin Packing (continued)

(i) If S has a partition into two sets of equal sum, then $m^*(x) = 2$ so A returns a packing into $2c < 3$ bins, i.e. 2 bins

(ii) If S does not have a partition into two sets of equal sum, then A returns a packing into $\geq m^*(x) \geq 3$ bins

Therefore the existence of A would decide the whether S has a partition into two sets of equal sum in polynomial time

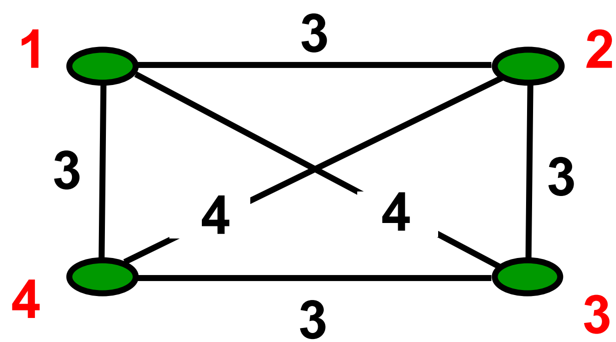
Theorem: If $P \neq NP$ then Minimum Bin Packing has no c-approximation algorithm with $c < 3/2$

(NB Minimum Bin Packing *is* approximable within 3/2)

Example 3: TSP

- **Instance:** Set S of n cities, distance $d(c_i,c_j)$ between each pair of cities $c_i, c_j \in S$
- **Feasible solutions:** Any travelling salesman tour of S, i.e. a permutation of $1, ..., n$
- **Measure:** Total length of the tour
- **Goal:** min

Example instance:

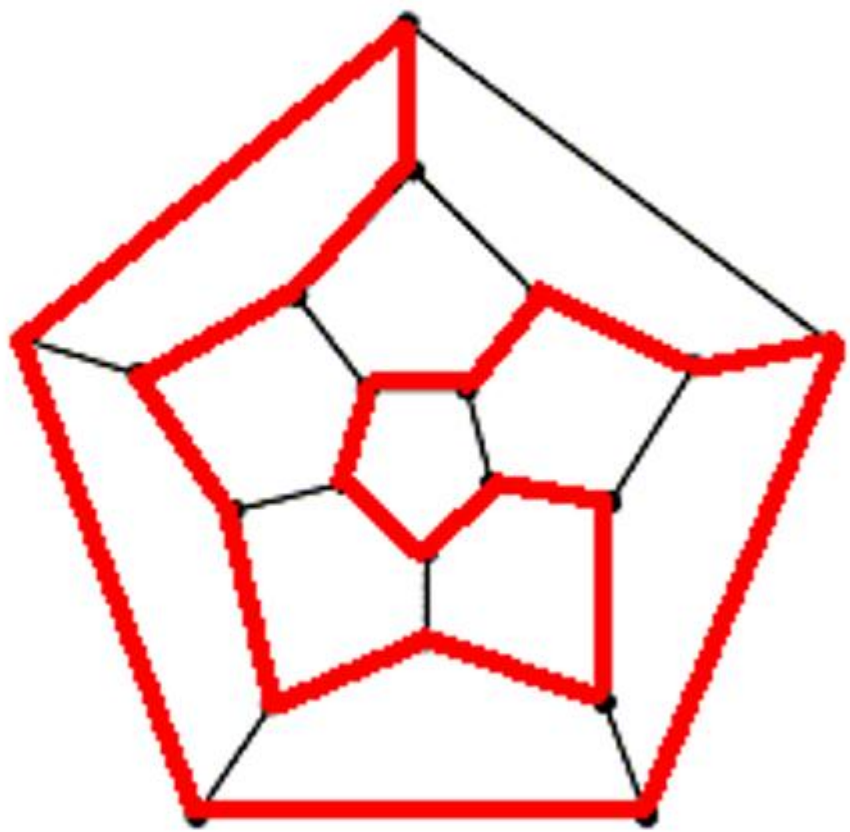


- Tour 1-2-4-3 is a feasible solution of measure 14
- Tour 1-2-3-4 is an optimal solution - measure is 12

TSP is NP-hard

Inapproximability of TSP

Problem of deciding whether a graph G has a Hamiltonian Cycle is NP-complete (HC)



Inapproximability of TSP

Problem of deciding whether a graph G has a Hamiltonian Cycle is NP-complete (HC)

Suppose that there is an approximation algorithm A for TSP with performance guarantee $c \geq 1$

Let $G=(V,E)$ be a graph in an instance of HC, where $n=|V|$

Define an instance x of TSP by letting V be the set of cities in x with distance function $d : V \times V \to N$ as follows:

$$
\begin{array}{ll}
d(u,v)=1, & \text{if } \{u,v\} \in E \\
d(u,v)=1+nc, & \text{if } \{u,v\} \notin E
\end{array}
$$

(i) If G has a Hamiltonian Cycle then $m^*(x)=n$ so A returns a tour of length $\leq nc$

(ii) If G does not have a Hamiltonian Cycle, then A returns a tour of length $\geq m^*(x) \geq (n-1) + (1+nc) = (c+1)n > nc$

Inapproximability of TSP (continued)

Therefore the existence of A would decide the whether G has a Hamiltonian Cycle in polynomial time

Theorem: If P≠NP then TSP has no c-approximation algorithm with $c \geq 1$

Approximation algorithms have been the subject of much recent research

- Recall PO $\subseteq$ FPTAS $\subseteq$ PTAS $\subseteq$ APX $\subseteq$ NPO
- Each of these inclusions is strict (if P≠NP)
- A major research aim has been to classify problems within this hierarchy

Some NPO problems – what is known

- Not in APX
  - Minimum Graph Colouring, Maximum Clique, Travelling Salesman, Longest Common Subsequence
- In APX but not in PTAS (with bounds on best guarantee)
  - Max SAT ($c \leq 1.2551$)
  - Vertex cover ($1.1666 \leq c \leq 2$)
  - Bin Packing ($c = 1.5$)
  - Shortest common superstring ($c \leq 2.4783$)
  - Maximum 3D-matching ($c \leq 1.5 + \varepsilon$)
  - Max Cut ($1.0624 \leq c \leq 1.1383$)
- In PTAS but not in FPTAS
  - Geometric TSP
- In FPTAS but not in PO
  - Subset Sum Optimization
  - Knapsack
- (All of the above under the assumption that $P \neq NP$)