ILAS Seminar in Probability: Exercise Sheet *3*

Exercises from Norris, Markov Chains

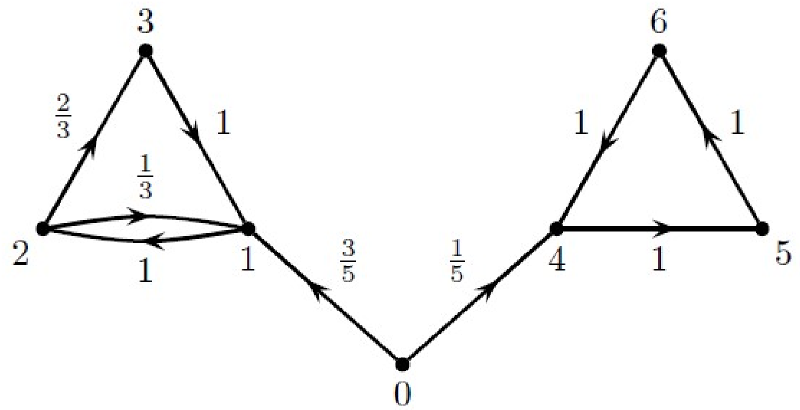
1.2.1 Identify the communicating classes of the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Which classes are closed?

1.3.1 Prove the claims (a), (b) and (c) made in example (v) of the Introduction.

(v) (*Discrete time*)



We use this example to anticipate some of the ideas discussed in detail in Chapter 1. The states may be partitioned into *communicating classes*, namely $\{0\}$, $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Two of these classes are *closed*, meaning that you cannot escape. The closed classes here are *recurrent*, meaning that you return again and again to every state. The class $\{0\}$ is *transient*. The class $\{4, 5, 6\}$ is *periodic*, but $\{1, 2, 3\}$ is not. We shall show how to establish the following facts by solving some simple linear equations. You might like to try from first principles.

(a) Starting from 0, the probability of hitting 6 is $1/4$.

(b) Starting from 1, the probability of hitting 3 is 1.

(c) Starting from 1, it takes on average three steps to hit 3.

1.3.2 A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if a player bets on the right side, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10.

Let $X_0 = 2$ and let X_n be his capital after n throws. Prove that the gambler will achieve his aim with probability $1/5$.

What is the expected number of tosses until the gambler either achieves his aim or loses his capital?