ILAS Seminar in Probability: Assignment 8

1. Consider a Markov chain on $\{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 1-p & p & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

   for some $p \in (0, 1)$. Find the unique invariant distribution for this chain.

2. Consider a Markov chain on $\{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

   Find the unique invariant distribution for this chain.

3. Complete Norris Exercise 1.7.2: Gas molecules move about randomly in a box which is divided into two halves symmetrically by a partition. A hole is made in the partition. Suppose there are N molecules in the box. Show that the number of molecules on one side of the partition just after a molecule has passed through the hole evolves as a Markov chain. What are the transition probabilities? What is the invariant distribution of this chain?

   Hint: First determine expressions for $p_i := p_{i,i+1}$ and $q_i := p_{i,i-1}$. Then, show that the invariant measure satisfies the recursion:

$$\lambda_i = \lambda_{i+1} q_{i+1} + \lambda_{i-1} p_{i-1}.$$

   Use this to show that $\lambda_i / \lambda_{i-1} = p_{i-1}/q_i$, and thus deduce the result. For this, you might also recall $\sum_{i=0}^{N} \binom{N}{i} = 2^N$.