Problem Set 3: Suggested Answers

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1. Subgame Perfect Equilibrium.

(a) There are four possible outcomes for Stage 1, each followed by a subgame. We first find the Nash equilibrium of each subgame.

- When the outcome of Stage 1 is (x_1, x_2), players face the following subgame:

		Player 2	
		z_2	w_2
Player 1	z_1	(2, 0)	(1, 1)
	w_1	(1, 1)	(3, 3)

  The Nash equilibrium of this subgame is (w_1, w_2).

- When the outcome of Stage 1 is (x_1, y_2), players face the following subgame:

		Player 2	
		z_2	w_2
Player 1	z_1	(1, 0)	(0, 1)
	w_1	(0, 1)	(2, 3)

  The Nash equilibrium of this subgame is (w_1, w_2).

- When the outcome of Stage 1 is (y_1, x_2), players face the following subgame:

		Player 2	
		z_2	w_2
Player 1	z_1	(2, -1)	(1, 0)
	w_1	(1, 0)	(3, 2)

The Nash equilibrium of this subgame is (w_1, w_2).

- When the outcome of Stage 1 is (y_1, y_2), players face the following subgame:

		Player 2	
		z_2	w_2
Player 1	z_1	(1, -1)	(0, 0)
	w_1	(0, 0)	(2, 2)

The Nash equilibrium of this subgame is (w_1, w_2).

Thus at Stage 1, we can replace the entire game with the following game:

		Player 2	
		x_2	y_2
Player 1	x_1	(3, 3)	(2, 3)
	y_1	(3, 2)	(2, 2)

There are four Nash equilibria: (x_1, x_2), (x_1, y_2), (y_1, x_2), (y_1, y_2).

To conclude, there are four subgame perfect equilibria of the entire game:

- $((x_1, (w_1, w_1, w_1, w_1)), (x_2, (w_2, w_2, w_2, w_2)))$
- $((x_1, (w_1, w_1, w_1, w_1)), (y_2, (w_2, w_2, w_2, w_2)))$
- $((y_1, (w_1, w_1, w_1, w_1)), (x_2, (w_2, w_2, w_2, w_2)))$
- $((y_1, (w_1, w_1, w_1, w_1)), (y_2, (w_2, w_2, w_2, w_2)))$

where the strategy profile should be read as (player 1's strategy, player 2's strategy). For each player, the actions are written in the following order: (action at Stage 1, (action at Stage 2 when the outcome of Stage 1 is (x_1, x_2), action at Stage 2 when the outcome of Stage 1 is (x_1, y_2), action at Stage 2 when the outcome of Stage 1 is (y_1, x_2), action at Stage 2 when the outcome of Stage 1 is (y_1, y_2))).

(b) (There are 40 subgame perfect equilibria.)

2. Finitely Repeated Game. First we can confirm that the stage game has three Nash equilibria: (P_1, P_2), (R_1, R_2), (S_1, S_2). Thus the given strategy (played by both players) does require players to play a Nash equilibrium in the second stage. It remains to check when can (Q_1, Q_2) be a Nash equilibrium in the first stage, taking what will happen in stage two as given.

Suppose players follow the given strategy. In the first stage, adding the payoffs players can achieve in the second stage yields the following payoff matrix:

		Player 2			
		P_2	Q_2	R_2	S_2
Player 1	P_1	(4, 4)	(x, 2)	(1, 2)	(2, 2)
	Q_1	(2, x)	(6, 6)	(1, 0)	(2, 0)
	R_1	(2, 2)	(0, 2)	(2, 4)	(2, 2)
	S_1	(2, 1)	(0, 1)	(1, 1)	(4, 2)

It can be confirmed that the stage game Nash equilibria, (P_1, P_2), (R_1, R_2), (S_1, S_2), are still Nash equilibria here. For (Q_1, Q_2) to be a Nash equilibrium too, $x < 6$ must hold (otherwise player i, $i = 1, 2$ would like to deviate to P_i). So the given strategy is a subgame perfect equilibrium if $4 < x < 6$.

3. Repeated Game.

(a) The subgame perfect equilibrium is $((D, (D, D, D, D)), (D, (D, D, D, D)))$. That is, player 1 chooses D at stage 1 and D at stage 2, independent of the outcome of stage 1. Player 2 chooses D at stage 1 and D at stage 2, independent of the outcome of stage 1.

(b) Tit-for-Tat strategy profile can form a subgame perfect equilibrium only when $\delta = \frac{2}{5}$. First, consider an arbitrary subgame following a history of cooperation at every stage. (This means that players are "on the equilibrium path".) To continue cooperating forever yields a payoff of $0 + 0\delta + 0\delta^2 + \cdots = 0$. If player 1 (player 2) deviates to D for the first period, the outcomes of the subgame would be $(D, C), (C, D), (D, C), (C, D), \cdots$ $((C, D), (D, C), (C, D), (D, C), \cdots)$. Thus for each player, one-shot deviation yields a payoff of $2 - 5\delta + 2\delta^2 - 5\delta^3 + 2\delta^4 \cdots = \frac{2}{1-\delta^2} - \frac{5\delta}{1-\delta^2}$. Therefore, one-shot deviation cannot yield a strictly higher payoff if and only if $\frac{2}{1-\delta^2} - \frac{5\delta}{1-\delta^2} \leq 0$, that is, $\delta \geq \frac{2}{5}$.

However, a subgame perfect equilibrium must yield a Nash equilibrium in every subgame, even if it is not on the equilibrium path. Consider a subgame following a history where the previous stage play was (C, D). For player 1, the Tit-for-Tat strategy suggests her/him to play D at the current stage, which will yield a total payoff of $2 - 5\delta + 2\delta^2 - 5\delta^3 +$

$2\delta^4 \cdots = \frac{2}{1-\delta^2} - \frac{5\delta}{1-\delta^2}$. But if player 1 deviates to C for the first period, the outcomes of the subgame would be (C, C) forever, and the total payoff will be 0. Thus for player 1, one-shot deviation cannot yield a strictly higher payoff if and only if $\frac{2}{1-\delta^2} - \frac{5\delta}{1-\delta^2} \geq 0$, that is, $\delta \leq \frac{2}{5}$.

Thus we conclude that Tit-for-Tat strategy profile can only be a subgame perfect equilibrium when $\delta = \frac{2}{5}$. (In fact, we have to check the conditions to prevent one-shot deviations in other subgames, where the previous stage play was (D, C) or (D, D). Check the conditions by yourself.)

(c) (Defect Forever, Grim Trigger) cannot form a subgame perfect equilibrium. From stage 2, there is no incentive for one-shot deviation for both players in any subgame: The outcomes of the subgame would be (D, D) for every stage forward with or without a deviation. Therefore whether one-shot deviation can raise total payoff depends on the payoff for the first stage. Since deviating from D to C will give a player a strictly lower payoff for the first stage, no player would like to deviate after stage 2.

However, at stage 1, one-shot deviation can raise player 2's total payoff for every possible δ. If player 2 deviates by playing D instead of C, the outcomes of the game would be (D, D) for every stage. This will raise player 2's total payoff since her/his first stage payoff increases from -5 to -3, while her/his payoff from stage 2 onward is unchanged.