Introduction to Probability (E2): Sample exam questions

1. Consider a Markov chain $X = (X_n)_{n\geq 0}$ on $\{1,2,3,4,5,6\}$ with transition matrix:

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) Draw a graphical representation of the state space and transition probabilities of the Markov chain.

   (b) Determine the communicating classes of the Markov chain, and state which classes are open, and which are closed. Is the chain irreducible?

   (c) Let $h_i^{\{1\}}$ be the probability the Markov chain ever hits state 1 when started from state i. Compute $h_3^{\{1\}}$.

   (d) Let $k_i^{\{1,6\}}$ be the expected time that the Markov chain takes to hit the set $\{1,6\}$ when started from state i. Compute $k_3^{\{1,6\}}$.

   (e) Suppose now that $X_0 \in \{1,2\}$ and $Y = (Y_n)_{n\geq 0}$ is the Markov chain X observed on $\{1,2\}$, i.e. $Y_n = X_{\tau(n)}$, where $\tau(0) = 0$ and

$$\tau(n) := \inf\{m > \tau(n-1) : \ X_m \in \{1,2\}\}, \qquad n \geq 1.$$

      i. What is the transition matrix of Y?

      ii. Assuming that

$$\lim_{n\to\infty} \mathbf{P}(Y_n = 1 \,\big|\, Y_0 = i)$$

      exists and is independent of $i \in \{1,2\}$, what must it be equal to?

[17 marks]

2. Let $X = (X_n)_{n\geq 0}$ be the Markov chain on $\mathbb{Z}$ with transition probabilities determined by:

$$p_{i,i+2} = \frac{1}{\alpha+1}, \qquad p_{i,i-1} = \frac{\alpha}{\alpha+1},$$

   where $\alpha \geq 2$. Moreover, let $H_0 := \inf\{n \geq 0 : \ X_n = 0\}$ be the hitting time of 0 by X.

   (a) Show that the generating function

$$\phi(s) = \mathbf{E}\left(s^{H_0} \,\big|\, X_0 = 1\right), \qquad s \in [0,1),$$

   satisfies

$$0 = s\phi(s)^3 - (\alpha+1)\phi(s) + \alpha s.$$

   (b) What is the probability that $H_0 < \infty$ given that $X_0 = 1$?

   (c) What is the expected value of H_0 given that $X_0 = 1$?

[9 marks]

3. (a) Show that if the quadratic equation $ax^2 + bx + c = 0$ (with $a, c \neq 0$) has two distinct roots, λ_1 and λ_2 say (which are necessarily non-zero), then $x_n = A\lambda_1^n + B\lambda_2^n$ solves the difference equation

$$ax_{n+2} + bx_{n+1} + cx_n = 0, \qquad n \geq 0,$$

for any constants A, B. Hence give the unique solution to the above difference equation with $x_0 = \alpha$, $x_1 = \beta$.

(b) Let $X = (X_n)_{n \geq 0}$ be a Markov chain on $\{0, 1, 2, \dots\}$ with transition probabilities determined by:

$$p_{0,1} = p, \qquad p_{0,0} = 1 - p,$$

$$p_{1,2} = p, \qquad p_{1,0} = 1 - p,$$

$$p_{i,i+1} = p, \qquad p_{i,i-1} = q, \qquad p_{i,0} = r, \qquad \forall i \geq 2,$$

where $p, q, r > 0$ are such that $p+q+r = 1$. Compute the unique invariant probability distribution $(\pi_j)_{j \geq 0}$ of X. In particular, confirm that

$$\pi_0 = 1 - \frac{p}{1 - q\lambda},$$

where λ is a root of a certain quadratic equation that you should identify.

[14 marks]