Exercises, chapter 1, solutions

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1 Problem

Let $F(x) = \sum_{k=0\cdots n} a_k x^k$ be a polynomial equation, where $a_0 \cdots a_n$ are stored in an array $A[0 \cdots n]$. Given an input integer x, we can find the corresponding output value $F(x)$. For example, the coefficients of the polynomial $F(x) = 1 + x + 2x^2$ can be stored in an array $< 1, 1, 2 >$. When $x = 2$, $F(x)$ equals to 11. Assume that we do not have access to a function for computing x^k directly.

The basic algorithm (illustrated in Algorithm 1) is designed for solving this problem. The idea is that, for every k (line 2), compute $A[k] \cdot x^k$ (line 3-5) first and then sum up the results (line 7).

Algorithm 1: Basic Algorithm

```
  Input: coefficient array A[0···n] representing a_0···a_n
  Input: x
  Output: corresponding value F(x)
1 sum = 0
2 for k=0 to n do
3     temp = A[k]
4     for i = 1 to k do
5         temp = temp · x
6     sum = sum + temp
7 return sum
```

2 Time Complexity

Analyze the time complexity of Algorithm 1.

Solution:

Table 1 illustrates the time complexity of every line. Sum up the cost of every line gives $O(n^2)$.

Table 1: Time Complexity of Each Line

Line	**Time Complexity**
1	$O(1)$
2	$n + 1 = O(n)$
3	same as Line 2
4	$\sum_{k=0..n} k = \frac{n(n+1)}{2} = O(n^2)$
5	same as Line 4
6	same as Line 2
7	$O(1)$

3 Incremental Algorithm

- Design an incremental algorithm for solving the problem.
  hint: $F(x) = \sum_{k=0\cdots n} a_k x^k$ can be reorganized as $F(x) = a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + xa_n)\cdots))$
- Consider the polynomial equation $F(x) = 4 + 2x + 3x^2 + x^3$. Show the running steps of your algorithm when $x = 2$.
- Analyze the time complexity of your algorithm.

Solution:

- Refer to Algorithm 2.
- Refer to Table 2.
- Time complexity is $O(n)$.

Algorithm 2: Incremental Algorithm

Input: coefficient array $A[0\cdots n]$ representing $a_0 \cdots a_n$
Input: x
Output: corresponding value $F(x)$
$sum = 0$
for $k = n$ *downto 0* **do**
    $sum = x \cdot sum + A[k]$
return sum

Table 2: Running Steps

k	sum
3	$0 \cdot 2 + A[3] = 1$
2	$1 \cdot 2 + A[2] = 5$
1	$5 \cdot 2 + A[1] = 12$
0	$12 \cdot 2 + A[0] = 28$