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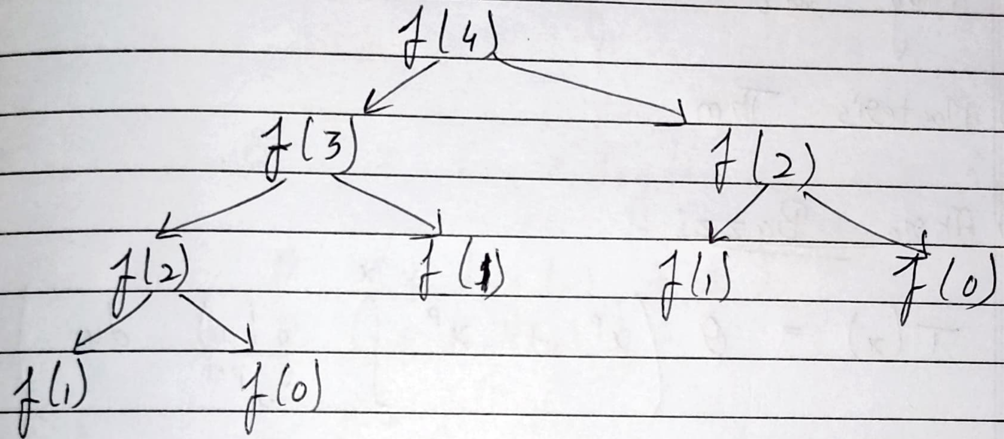
Recursion Time and Space Complexity.

"At an particular point of time no two f^n call of same level of recursion will be in stack at same time."

Note: Only call that are interlinked will be in the stack at same time.



Space Complexity = Height of Tree
(Maximum f^n call that can be made is the space complexity)

2 types of recursion

1. linear
2. Divide and Conquer

Divide and Conquer Recurrence:

Form:

$$T(x) = a_1 T(b_1 x + \varepsilon_1(x)) + a_2 T(b_2 x + \varepsilon_2(x)) + \dots + a_k (b_k x + \varepsilon_k(x)) + g(x)$$

for $x \geq x_0$

$x_0 \Rightarrow$ constant

Binary

$T(N) = T\left(\frac{N}{2}\right) + c$

~~For bi~~ $a_1 = 1$ $\quad \varepsilon_1(x) = 0$

$b_1 = 1/2$ $\quad g(x) = c$

How to actually solve to get complexity:

1. Plug and chug.
2. Master's Thm
3. Akra Bazzi:

$$T(x) = \theta\left(x^p + x^p \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)$$

What is p?

$a_1 b_1^p + a_2 b_2^p + \dots = 1$

$$\boxed{\sum_{i=1}^{k} a_i b_i^p = 1}$$

Eg $T(N) = 2T\left(\frac{N}{2}\right) + (N-1)$

$a_1 = 2$ $\quad g(n) = n-1$

$b_1 = 1/2$

$a_1 b_1^p = 1$

$2 \times \left(\frac{1}{2}\right)^p = 1$

$$\boxed{p = 1}$$

P. in formula

$$T(N) = \theta\left(x + x \int_1^x \frac{u-1}{u^2} \cdot du \right)$$

$$= \theta\left(x + x \int_1^x \frac{1}{u} - \frac{1}{u^2}\, du \right)$$

$$= \theta\left(x + x \left[\log u + \frac{1}{u} \right]_1^x \right)$$

$$= \theta\left(x + x \left[\log(x) + \frac{1}{x} - (0+1) \right] \right)$$

$$= \theta\left(x + x\log(x) + 1 - x \right)$$

$$= \theta\left(x\log(x) + 1 \right)$$

$$= \theta\left(x\log(x) \right)$$

Q $T(N) = 2\,T\left(\frac{N}{2}\right) + \frac{8}{9}\,T\left(\frac{3N}{4}\right) + N^2$

Sol^n $a_1 = 2$ $\qquad b_1 = 1/2$

$a_2 = 8/9$ $\qquad b_2 = 3/4$

$g(N) = N^2$

$$a_1 b_1^p + a_2 b_2^p = 1$$

$$2\left(\frac{1}{2}\right)^p + \frac{8}{9}\left(\frac{3}{4}\right)^p = 1$$

Let $p = 2$

$$\cancel{2}^1 * \frac{1}{\cancel{4}_2} + \frac{8}{9} * \frac{9}{16} = 1$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

$$1 = 1$$

$$\boxed{\therefore\ p = 2}$$

p in formula

$$T(N) = \theta\left(x^2 + x^2 \int_1^x \frac{u^2}{u^3} \, du \right)$$

$$= \theta\left(x^2 + x^2 \int_1^x \frac{1}{u} \, du \right)$$

$$= \theta \quad x^2 + x^2 \left[\log u \right]_1^x$$

$$= \theta\left(x^2 + x^2 \log x \right)$$

$$= \theta\left(x^2 \log x \right)$$

$\because$ Ignoring less dominant

☆ **If you can't find value of p :-**

$$T(x) = 3T\left(\frac{x}{3}\right) + 4T\left(\frac{x}{4}\right) + x^2$$

Let's $p = 1$

$$3 \times \frac{1}{3} + 4 \times \frac{1}{4} = 1$$

$$2 \neq 1$$

We have to ↑ power to ↑ denominator.

$\therefore p > 1$

Let $p = 2$

$$3 \times \frac{1}{9} + 4 \times \frac{1}{16} = 1$$

$$\frac{1}{3} + \frac{1}{4} = 1$$

$$\frac{7}{12} < 1$$

$\therefore p < 2$

Note : When $p <$ power of $[g(x)]$
then ans $= g(x)$.

Here $g(x) = x^2$
$p < 2$

Hence ans $= o(g(x))$

Linear Recurrences

Form:-

$$f(x) = a_1 f(x-1) + a_2 f(x-2) + a_3 f(x-3) + \ldots + a_n f(x-n)$$

$$f(x) = \sum_{i=1}^{n} a_i f(x-i)$$

$n \Rightarrow$ n is fixed $\Rightarrow$ order of recurrence.

Solution for fibonacii no.:

$$f(n) = f(n-1) + f(n-2)$$

Steps

① Put $f(n) = \alpha^n$ for some constant α

$$\Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$$

$$\alpha^n - \alpha^{n-1} - \alpha^{n-2} = 0$$

divide by α^{n-2}

$$\alpha^2 - \alpha - 1 = 0$$

$\Rightarrow$ roots of quad eqn

$$\alpha = \frac{1 \pm \sqrt{5}}{2}$$

$\alpha_1 = \frac{1+\sqrt{5}}{2}$ $\alpha_2 = \frac{1-\sqrt{5}}{2}$

② $f(n) = c_1 \alpha_1^n + c_2 \alpha_2^n$ is a sol^n for fibo

$$f(n) = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \quad — ②$$

③ <u>Fact</u> : no. of roots = no. of ans you have already.

∵ 2 roots
∴ we should have 2 ans already.

$F(0) = 0$ & $F(1) = 1$

$f(0) = 0 = c_1 + c_2 \Rightarrow c_1 = -c_2$ — ③

$f(1) = 1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right)$

from ③

$1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right) - c_1 \left(\frac{1-\sqrt{5}}{2}\right)$

$c_1 = \frac{1}{\sqrt{5}}$ $c_2 = \frac{-1}{\sqrt{5}}$

In Eq^n ②

$$f(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n \cancel{+} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$f(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

less dominating

$$f(n) = O\left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$= O\ (1.618)^n$$

Golden Ratio