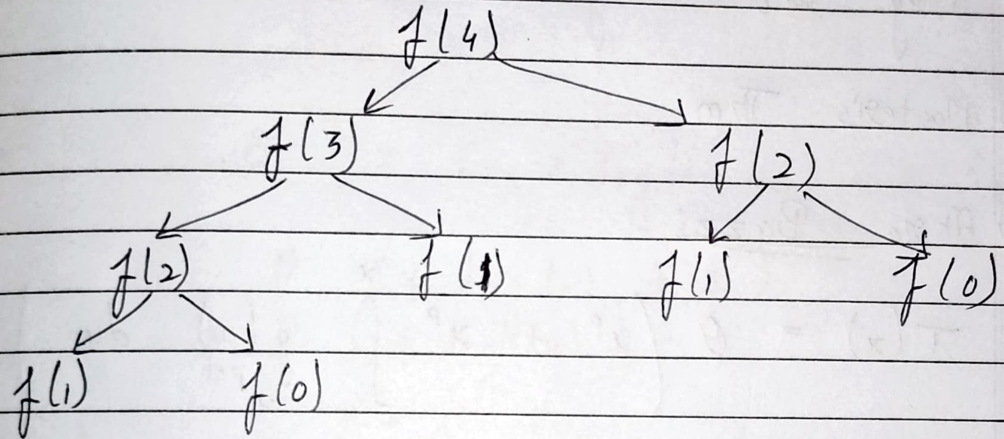
08/12/2021

Recursion Time and Space Complexity.

"At an particular point of time no two f^n call of same level of recursion will be in stack at same time."

Note: Only call that are interlinked will be in the stack at same time.



Space Complexity = Height of Tree
(Maximum f^n call that can be made is the space complexity)

2 types of recursion

1. linear
2. Divide and Conquer

Divide and Conquer Recurrence:

Form:

$$T(x) = a_1 T(b_1 x + \varepsilon_1(x)) + a_2 T(b_2 x + \varepsilon_2(x)) + \dots + a_k (b_k x + \varepsilon_k(x)) + g(x)$$

for $x \geq x_0$

$x_0 \Rightarrow$ constant

Binary

$$T(N) = T\left(\frac{N}{2}\right) + c$$

~~For bi~~ $a_1 = 1$ $\qquad \varepsilon_1(x) = 0$

$b_1 = 1/2$ $\qquad g(x) = c$

How to actually solve to get complexity:

1. Plug and chug.
2. Master's Thm
3. Akra Bazzi:

$$T(x) = \theta\left(x^p + x^p \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)$$

What is p?

$$a_1 b_1^p + a_2 b_2^p + \dots = 1$$

$$\boxed{\sum_{i=1}^{K} a_i b_i^p = 1}$$

Eg $T(N) = 2T\left(\frac{N}{2}\right) + (N-1)$

$a_1 = 2$ $\qquad g(n) = n-1$

$b_1 = 1/2$

$$a_1 b_1^p = 1$$

$$2 \times \left(\frac{1}{2}\right)^p = 1$$

$$\boxed{p = 1}$$

P. in formula

$$T(N) = \theta\left(x + x\int_1^x \frac{u-1}{u^2} \cdot du\right)$$

$$= \theta\left(x + x\int_1^x \frac{1}{u} - \frac{1}{u^2}\, du\right)$$

$$= \theta\left(x + x\left[\log u + \frac{1}{u}\right]_1^x\right)$$

$$= \theta\left(x + x\left[\log(x) + \frac{1}{x} - (0+1)\right]\right)$$

$$= \theta\left(x + x\log(x) + 1 - x\right)$$

$$= \theta\left(x\log(x) + 1\right)$$

$$= \theta\left(x\log(x)\right)$$

Q $T(N) = 2T\left(\frac{N}{2}\right) + \frac{8}{9}T\left(\frac{3N}{4}\right) + N^2$

Sol^n $a_1 = 2$ $\qquad b_1 = 1/2$

$a_2 = 8/9$ $\qquad b_2 = 3/4$

$g(N) = N^2$

$$a_1 b_1^p + a_2 b_2^p = 1$$

$$2\left(\frac{1}{2}\right)^p + \frac{8}{9}\left(\frac{3}{4}\right)^p = 1$$

Let $p = 2$

$$\cancel{2}^1 * \frac{1}{\cancel{4}_2} + \frac{8}{9} * \frac{9}{16} = 1$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

$$1 = 1$$

$$\boxed{\therefore\ p = 2}$$

p in formula

$$T(N) = \theta \left(x^2 + x^2 \int_1^x \frac{u^2}{u^3} \cdot du \right)$$

$$= \theta \left(x^2 + x^2 \int_1^x \frac{1}{u} \cdot du \right)$$

$$= \theta \quad x^2 + x^2 \left[\log u \right]_1^x$$

$$= \theta \left(x^2 + x^2 \log x \right)$$

$$= \theta \left(x^2 \log x \right)$$

$\because$ Ignoring less dominant

☆ **If you can't find value of p :-**

$$T(x) = 3T\left(\frac{x}{3}\right) + 4T\left(\frac{x}{4}\right) + x^2$$

Let's $p = 1$

$$3 \times \frac{1}{3} + 4 \times \frac{1}{4} = 1$$

$$2 \neq 1$$

We have to $\uparrow$ power to $\uparrow$ denominator.

$\therefore p > 1$

Let $p = 2$

$$3 \times \frac{1}{9} + 4 \times \frac{1}{16} = 1$$

$$\frac{1}{3} + \frac{1}{4} = 1$$

$$\frac{7}{12} < 1$$

$\therefore p < 2$

Note : When $p <$ power of $[g(x)]$
then ans $= g(x)$.

Here $g(x) = x^2$
$p < 2$

Hence ans $= O(g(x))$

Linear Recurrences

Form:-

$$f(x) = a_1 f(x-1) + a_2 f(x-2) + a_3 f(x-3) + \dots + a_n f(x-n)$$

$$f(x) = \sum_{i=1}^{n} a_i f(x-i)$$

$n \Rightarrow n$ is fixed $\Rightarrow$ order of recurrence.

Solution for fibonacii no:

$$f(n) = f(n-1) + f(n-2)$$

Steps

① Put $f(n) = \alpha^n$ for some constant α

$\Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$

$\alpha^n - \alpha^{n-1} - \alpha^{n-2} = 0$

$\cancel{\frac{\alpha^n}{\alpha^{n-2}}}$ divide by α^{n-2}

$\alpha^2 - \alpha - 1 = 0$

$\Rightarrow$ roots of quad eq^n

$$\alpha = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \qquad \alpha_2 = \frac{1-\sqrt{5}}{2}$$

(2) $f(n) = c_1 \alpha_1^n + c_2 \alpha_2^n$ is a sol^n for fib.

$$f(n) = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \quad — (2)$$

(3) <u>Fact</u> : no. of roots = no. of ans you have already.

2 roots

$\therefore$ we should have 2 ans already.

$$F(0) = 0 \quad \& \quad F(1) = 1$$

$$f(0) = 0 = c_1 + c_2 \Rightarrow c_1 = -c_2 \quad — (3)$$

$$f(1) = 1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

from (3)

$$1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right) - c_1 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$c_1 = \frac{1}{\sqrt{5}} \qquad c_2 = -\frac{1}{\sqrt{5}}$$

In Eq^n (2)

$$f(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$f(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

less dominating

$$f(n) = O\left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$= O\ (1.618)^n$$

Golden Ratio

Equal roots

$$f(n) = 2f(n-1) + f(n-2)$$

① $f(n) = \alpha^n$

$$\alpha^n = 2\alpha^{n-1} + \alpha^{n-2}$$

$$\alpha^n - 2\alpha^{n-1} - \alpha^{n-2} = 0$$

Divide by α^{n-2}

$$\alpha^2 - 2\alpha + 1 = 0 \Rightarrow \alpha = 1$$

General case: If α is repeated r times, then, α^n, $n\alpha^n$, $n^2\alpha^n$ ... $n^{r-1}\alpha^n$ are all sol^n

Hence, I can take 2 roots as

$1,\ n\alpha^n \Rightarrow 1, n$ $\quad$ ($\because$ As $\alpha = 1$)

$$f(n) = c_1\alpha^n + c_2\, n\alpha^n$$

$$= c_1 + c_2 n$$

$f(0) = 0$ & $f(1) = 1$

$f(0) = 0$ & $\quad$ | $\quad f(1) = 1 = c_1 + c_2$

$c_1 = 0$ $\quad$ $c_2 = 1$

Ans $f(N) = N$ $\quad$ T.C $\Rightarrow O(N)$

Non Homogenous Linear Recurrence

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + a_3 f(n-3) + \dots + a_d f(n-d) + \underbrace{g(n)}$$

When this extra f^n is there, it is known as Non Homogeneous.

How to Solve:

① Replace $g(n)$ by 0 & solve usually

Q. $f(n) = 4f(n-1) + 3^n$, $\quad f(1) = 1$

$$f(n) = 4 f(n-1)$$

$$\alpha^n = 4 \alpha^{n-1}$$

$$\alpha^n - 4\alpha^{n-1} = 0$$

$$\alpha - 4 = 0$$

$$\boxed{\alpha = 4}$$

Homogeneous sol^n =

$$f(n) = c_1 4^n$$

② Take $g(n)$ on one side and find particular soln :

$$f(n) - 4f(n-1) = 3^n$$

Guess something that is similar to $g(n)$

If $g(n) = n^2$, guess a polynomial of degree 2.

My guess :- $f(n) = c3^n$

$$c3^n - 4c3^{n-1} = 3^n \qquad c = -3$$

~~$c3 - 4 = 3^n$~~

Particular soln $\Rightarrow f(n) = -3^{n+1}$

③ Add both soln together

$$f(n) = c_1 4^n + (-3^{n+1})$$

$f(1) = 1 \Rightarrow c_1 4 - 3^2$

$$c_1 = \frac{5}{2}$$

$$f(n) = \frac{5}{2} 4^n - 3^{n+1}$$

How do we guess Particular soln?

* If $g(n)$ is exponential, guess of same type

Eg: $g(n) = 2^n + 3^n$

guess $= f(n) = a2^n + b3^n$

* If it is polynomial $(g(n))$, guess of same degree.

Eg: $g(n) = n^2 - 1 \Rightarrow$

guess $\Rightarrow an^2 + bn + c = f(n)$

* $g(n) = 2^n + n$

guess:

$f(n) = a2^n + (bn + c)$

Let say you guessed $f(n) = a2^n$ and if fails, then try $(an + b)2^n$, if this also fails increase the degree, $(a^2n + bn + c)2^n$.

Eg $f(n) = 2f(n-1) + 2^n$, $f(0) = 1$

Put $2^n = 0$

$f(n) = 2f(n-1)$

$\alpha^n - 2\alpha^{n-1} = 0$

$\alpha - 2 = 0$

$\boxed{\alpha = 2}$

② Guess Particular soln

$$g(n) = 2^n$$

Guess $\Rightarrow$ $f(n) = a2^n$

$$a2^n = 2a2^{n-1} + 2^n$$

$$a = a + 1 \quad \times \text{ wrong}$$

Hence guess another one from our rules.

$$f(n) = (an+b)2^n$$

$$(an+b)2^n = 2(an+b)2^{n-1} + 2^n$$

$$an+b = a(n-1)+b + 1$$

$$\cancel{an} + \cancel{b} = \cancel{an} - a + \cancel{b} + 1$$

$$\boxed{a = 1} \quad \text{// discard } b$$

$$f(n) = n2^n$$

③ General ans:-

$$f(n) = c_1 2^n + n2^n$$

$$f(0) = 1 \Rightarrow 1 = c_1 + 0$$

$$\boxed{c_1 = 1}$$

$$\Rightarrow f(n) = 2^n + n2^n$$

Complexity :- $O(n2^n)$