Mid-semester Exam 1 Solution

Q.1. Let $y(t)$ be the number of bacteria present at time t where t is in hours then

$$y(t) = y(0)e^{kt}$$

$\underline{t=3}$ $\quad 10000 = y(0)e^{3k} \Rightarrow (10^4)^5 = y(0)^5 e^{15k}$

$\underline{t=5}$ $\quad 40\,000 = y(0)\, e^{5k} \Rightarrow (4\times 10^4)^3 = y(0)^3 e^{15k}$

$$\Rightarrow \frac{(10^4)^5}{4^3 \times (10^4)^3} = y(0)^2$$

$$\Rightarrow \frac{10^8}{2^6} = y(0)^2 \Rightarrow \boxed{y(0) = \frac{10^4}{2^3} = 1250}$$

Q.2. $2I' + 6I = 100 \Rightarrow I' + 3I = 50$

For steady state $I' = 0 \Rightarrow \boxed{I = \frac{50}{3}}$

Q.3. $I' + 3I = 50 \Rightarrow \displaystyle\int \frac{dI}{\frac{50}{3} - I} = \int 3\,dt$

$$\Rightarrow I(t) = \frac{50}{3} - Ce^{-3t} \qquad \text{since } I(0) = 0 \Rightarrow C = \frac{50}{3}$$

$$\Rightarrow I(t) = \frac{50}{3}\left(1 - e^{-3t}\right)$$

$$I(t) = 0.99 \times \frac{\cancel{50}}{\cancel{3}} = \frac{\cancel{50}}{\cancel{3}}\left(1 - e^{-3t}\right)$$

$$\Rightarrow e^{-3t} = 0.01 \Rightarrow -3t = \ln(0.01)$$

$$\Rightarrow -3t = \ln(10^{-2}) = -2\ln(10) = -4.6$$

$$\Rightarrow \boxed{t = 1.5}$$

Q.4. $(4x^3y^3-2xy)\,dx + (3x^4y^2-x^2+2y)\,dy = 0$

$$M = 4x^3y^3-2xy, \quad N = 3x^4y^2-x^2+2y$$

$$\frac{\partial M}{\partial y} = 12x^3y^2-2x, \quad \frac{\partial N}{\partial x} = 12x^3y^2-2x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y} \Rightarrow \boxed{\text{Yes}}$, eq. is exact.

Q.5. $\frac{\partial u}{\partial x} = M = 4x^3y^3-2xy$

$$\Rightarrow u(x,y) = x^4y^3 - x^2y + f(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \cancel{3x^4y^2} - \cancel{x^2} + f'(y) = N = \cancel{3x^4y^2} - \cancel{x^2} + 2y$$

$$\Rightarrow f(y) = y^2$$

$$\Rightarrow \boxed{u(x,y) = x^4y^3 - x^2y + y^2}$$

Q.6. If substitution is $v = y^n$

$$\Rightarrow \frac{dv}{dx} = ny^{n-1}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{ny^{n-1}}\frac{dv}{dx} + 2xy = -xy^4$$

$$\Rightarrow \frac{dv}{dx} + 2nx\underbrace{y^n}_{\text{this is } v} = -nx\underbrace{y^{n+3}}_{\text{this should be 1}}$$

so, $n+3=0$

$$\Rightarrow \boxed{n=-3}$$