MTH 204 Quiz 4

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Total Points: 30
Total Time: 20 mins

Question 1.

[10 pts] Solve the following IVP

$$y''' - 5y'' - 22y' + 56y = 0, \quad y(0) = 1, y'(0) = -2, y''(0) = -4.$$

Hint. One root of the corresponding characteristic equation is 2.

Question 2.

[10 pts] Find the general solution of the following ODE

$$y''' - 12y'' + 48y' - 64y = 12 - 32e^{-8t} + 2e^{4t}.$$

Hint. One root of the corresponding characteristic equation is 4.

Question 3.

[10 pts] Find the general solution of the following system of ODEs:

$$\begin{aligned} y_1' &= 3y_1 \\ y_2' &= y_1 + y_2 \\ y_3' &= y_1 + y_3 \end{aligned}$$

<u>Q.1.</u> Characteristic eq. is

$$\lambda^3 - 5\lambda^2 - 22\lambda + 56 = 0$$

Since 2 is a root, to find other roots, we will do long division

$$\begin{array}{r|l} & \lambda^2 - 3\lambda - 28 \\ \hline \lambda - 2 & \lambda^3 - 5\lambda^2 - 22\lambda + 56 \\ & \underline{\lambda^3 - 2\lambda^2} \quad (-\,+) \\ & -3\lambda^2 - 22\lambda \\ & \underline{-3\lambda^2 + 6\lambda} \quad (+\,-) \\ & -28\lambda + 56 \\ & \underline{-28\lambda + 56} \\ & 0 \end{array}$$

$$\lambda^2 - 3\lambda - 28 = 0$$

$$\lambda^2 - 7\lambda + 4\lambda - 28 = 0$$

$$\lambda(\lambda - 7) + 4(\lambda - 7) = 0$$

$$\lambda = 7, -4$$

So, general solution is

$$y = c_1 e^{-4t} + c_2 e^{2t} + c_3 e^{7t}$$

$$y' = -4c_1 e^{-4t} + 2c_2 e^{2t} + 7c_3 e^{7t}$$

$$y'' = 16c_1 e^{-4t} + 4c_2 e^{2t} + 49c_3 e^{7t}$$

168

$y(0) = 1 \Rightarrow c_1 + c_2 + c_3 = 1 \Rightarrow c_3 = 1 - c_1 - c_2$

$y'(0) = -2 \Rightarrow -4c_1 + 2c_2 + 7c_3 = -2 \Rightarrow -11c_1 - 5c_2 = -9$

$y''(0) = -4 \Rightarrow 16c_1 + 4c_2 + 49c_3 = -4 \Rightarrow -33c_1 - 45c_2 = -53$

$+\cancel{33}c_1 + 15c_2 = 27$

$-30c_2 = -26$

$c_2 = \frac{26}{30} = \frac{13}{10}$

$-11c_1 - \frac{13}{2} = -9$

$\Rightarrow -11c_1 = \frac{13}{2} - 9 = \frac{-5}{2}$

$\Rightarrow c_1 = \frac{5}{22}$

$c_3 = 1 - \frac{5}{22} - \frac{13}{10}$

$= \frac{100 - 25 - 143}{110} = \frac{-68}{110} = \frac{-34}{55}$

So, solution is $\boxed{y = \frac{5}{22} e^{-4t} + \frac{13}{10} e^{2t} - \frac{34}{55} e^{7t}}$

Q.2. Charectrishic eq. of H. eq. is

$$\lambda^3 - 12\lambda^2 + 48\lambda - 64 = 0$$

$$\begin{array}{r|l} & \lambda^2 - 8\lambda + 16 \\ \lambda - 4 & \lambda^3 - 12\lambda^2 + 48\lambda - 64 \\ & \underline{\cancel{\lambda^3} - 4\lambda^2} \\ & -\cancel{8\lambda^2} + 48\lambda \\ & \underline{-8\lambda^2 + 32\lambda} \\ & 16\lambda - \cancel{64} \\ & \underline{16\lambda - 64} \\ & 0 \end{array}$$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0$

$\lambda = 4, 4$

General sol. of H. eq. is

$$y = (c_1 + c_2 t + c_3 t^2) e^{4t}$$

For particular sol. of NH eq., we use method of undetermined coefficients and guess a sol of the form

$$y_p = A + Be^{-8t} + (Ct^3 + Dt^2 + Et + F)e^{4t}$$

$$y_p' = -8Be^{-8t} + (3Ct^2 + 2Dt + E + 4Ct^3 + 4Dt^2 + 4Et + 4F)e^{4t}$$

$$= -8Be^{-8t} + (4Ct^3 + (3C+4D)t^2 + (2D+4E)t + E + 4F)e^{4t}$$

$$y_p'' = 64Be^{-8t} + (12Ct^2 + (6C+8D)t + 2D + 4E + 16Ct^3 + (12C+16D)t^2 + (8D+16E)t + 4E + 16F)e^{4t}$$

$$= 64Be^{-8t} + (16Ct^3 + (24C+16D)t^2 + (6C+16D+16E)t + 2D + 8E + 16F)e^{4t}$$

$$y_p''' = -512Be^{-8t} + (48Ct^2 + (48C+32D)t + 6C + 16D + 16E + 64Ct^3 + (96C+64D)t^2 + (24C+64D+64E)t + 8D + 32E + 64F)e^{4t}$$

$$= -512Be^{-8t} + (64Ct^3 + (144C+64D)t^2 + (72C+96D+64E)t + 6C + 24D + 48E + 64F)e^{4t}$$

-288
-66
354
144
210

8

Plugging this in eq. & collecting terms, we get

192
88
280

Const. coeff. $\quad -64A = 12 \Rightarrow \boxed{A = -\frac{12}{64} = -\frac{3}{16}}$

e^{-8t} $\quad -512B - 12\cdot 64B - 8\cdot 48B - 64B = -32$

120
88
32

$$\boxed{B = \frac{1}{16+24+12+2} = \frac{1}{54}}$$

280
120
-160

t^3e^{4t} $\quad \cancel{64C} - 12\cdot 16C + \cancel{4\cdot 48C} - \cancel{64C} = 0$

t^2e^{4t} $\quad 144C + 64D - 12\cdot 24C - 12\cdot 16D - 22\cdot 3C - 22\cdot 4D + 56D = 0$

$-160D = 210C$

te^{4t} $\cancel{72C}+\cancel{96D}+\cancel{64E}-\cancel{72C}-12\cdot16D-12\cdot\cancel{16E}$
$+\cancel{2\cdot48D}+\cancel{48\cdot4E}-\cancel{64E}=0$

e^{4t} $\cancel{24D}+\cancel{48E}+\cancel{64F}-\cancel{24D}-\cancel{96E}-12\cdot\cancel{16F}+\cancel{48E}+\cancel{4\cdot48F}$
$6C+\quad -\cancel{64F}=2$

$$\Rightarrow \boxed{C=\frac{1}{3}}$$

$$-16D=\cancel{21}^{7}\cdot\frac{1}{\cancel{3}} \Rightarrow \boxed{D=-\frac{7}{16}}$$

Take $E=F=0$

So, $y_p=-\frac{3}{16}+\frac{1}{54}e^{-8t}+\left(\frac{1}{3}t^3-\frac{7}{16}t^2\right)e^{4t}$

General solⁿ of given problem is

$$\boxed{y=(C_1+C_2t+C_3t^2)e^{4t}-\frac{3}{16}+\frac{1}{54}e^{-8t}+\frac{t^3}{3}e^{4t}}$$

Q.3. $\begin{pmatrix} y \\ y_2 \\ y_3 \end{pmatrix}'=\begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=:A\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Eigenvalues of A are $3,1,1$, Let's find eigenvectors.

$\lambda=3$ $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=3\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{array}{l} 3a=3a \\ a+b=3b \Rightarrow a=2b \\ a+c=3c \Rightarrow a=2c \end{array}$ $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\lambda=1$ $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{array}{l} 3a=a \Rightarrow a=0 \\ \left.\begin{array}{l} a+b=b \\ a+c=c \end{array}\right\} \to a=0 \end{array}$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

General solⁿ is $\boxed{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=C_1\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}e^{3t}+C_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^{t}+C_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{t}}$