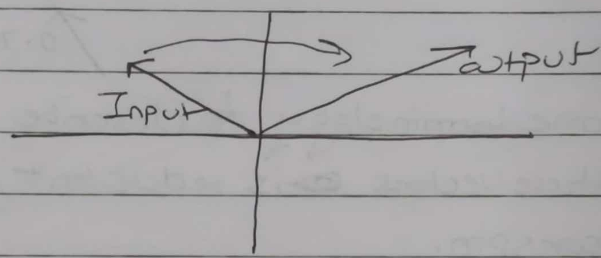
Linear transformation and matrices

The word 'transformation' suggest that you think using movement.

If a transformation takes some input vector to some output vector, we imagine that input vector moving over to output vector



A transformation is linear if it has 2 properties:

- All lines must remain lines, without getting curved
- Origin must remain fixed in place.

You should think of linear transformations as keeping gridlines parallel and evenly spaced

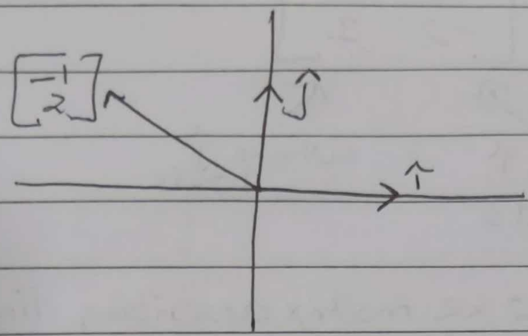
Simple linear transformation → Rotation about origin

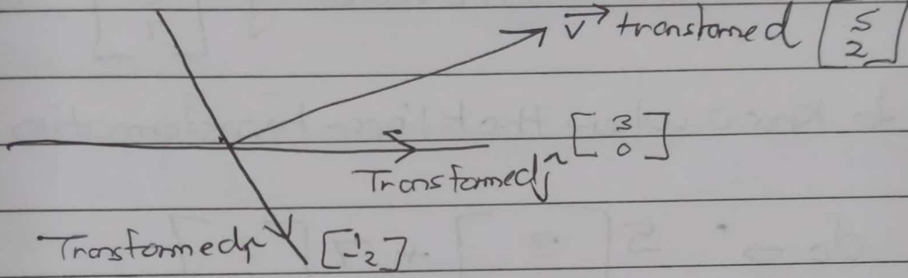
→ What formula would you give to computer so that if you give it coordinates of vector, it can give you coordinates of where that vector lands?

- For this, you only need to record where the two basis vectors, $\hat{i}$ and $\hat{j}$ lands. and everything else will follow from that.

For example, consider,

$\vec{v} = -1\hat{i} + 2\hat{j}$





The place where $\vec{v}$ lands will be -1 times the vector where $\hat{i}$ landed $+ 2$ times the vector where $\hat{j}$ landed

$$\text{Transformed } \vec{v} = -1(\text{Transformed } \hat{i}) + 2(\text{Transformed } \hat{j})$$

$$= -1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Generalized

$\hat{i} \rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\quad \hat{j} \rightarrow \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$

In matrix form,

2x2 matrix, $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$

where $\hat{i}$ lands   where $\hat{j}$ lands

If you are given 2x2 matrix describing linear transformation, and some specific vector eg. $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$, and you want to know where that linear transformation takes the vector,

do → $5\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

These corresponds with idea of adding the scaled version of our new basis vectors.

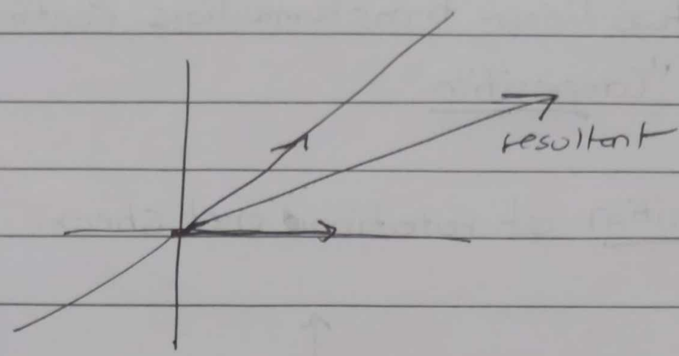
General case

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix}$

where first basis vector lands   where 2nd basis vector lands

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}}_{\text{where all the intuition is}} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

Special case ⇒ shear



In this $\hat{i}$ remains fixed. $(1,0)$ but $\hat{j}$ moves over to coordinates $(1,1)$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

shear

Summary:

- Linear transformations are way to move around space such that grid lines remain parallel and evenly spaced and origin remains fixed.