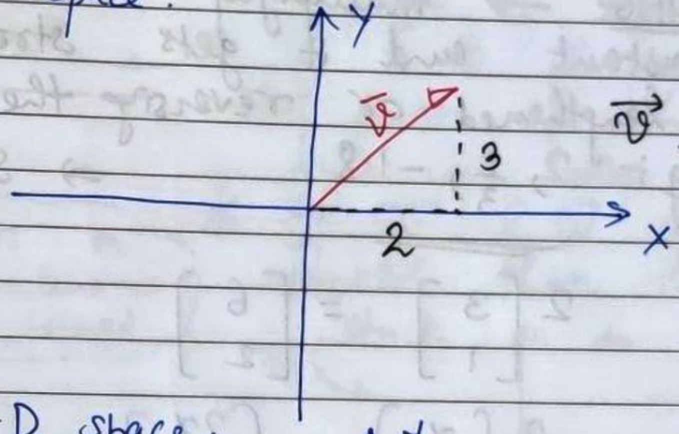
श्री महावीराय नमः

LINEAR ALGEBRA

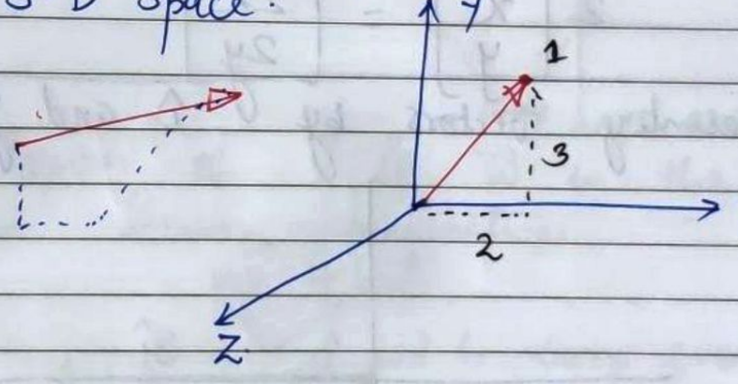
Representation of a vector.

* In 2-D space.



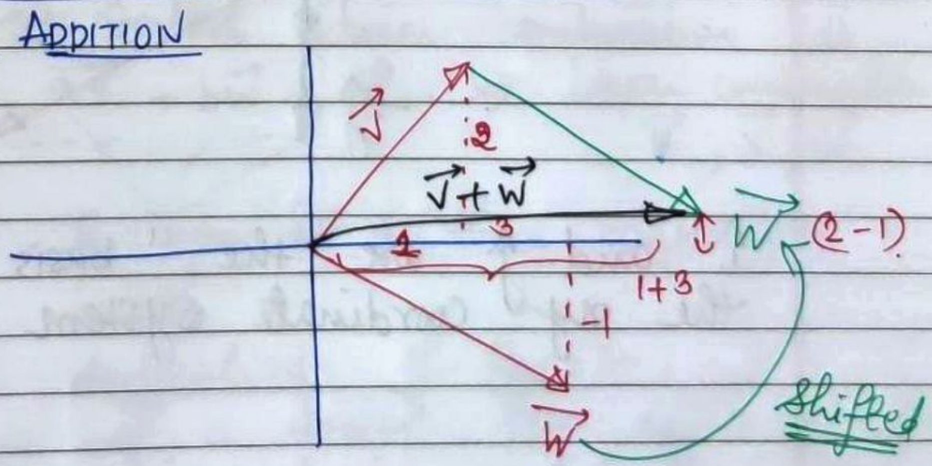
$$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

* In 3-D space.



$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

* VECTOR ADDITION



$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

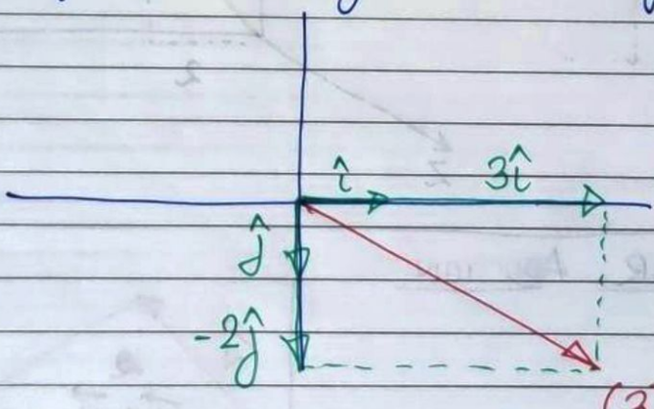
SCALING → multiplying a vector by a constant and it gets stretched or get lengthened or reversing the direction.

For eg :- $2, \frac{1}{3}, -1.8$ ⇒ SCALARS

$$2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Representing vectors by $\hat{i}$ and $\hat{j}$.



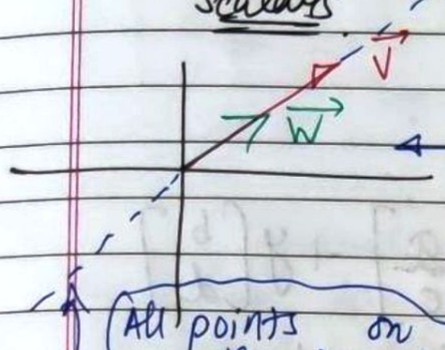
$(3)\hat{i} + (-2)\hat{j}$

$\hat{i}$ and $\hat{j}$ are the "basis vectors" of the xy coordinate system.

LINEAR COMBINATION OF VECTORS

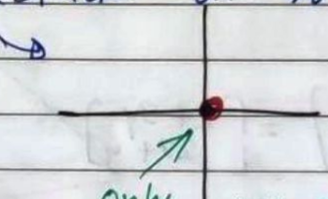
$a\bar{v} + b\bar{w}$

scalars



All points on this line is covered

* If $\bar{v}$ & $\bar{w}$ are two non-overlapping non-zero vectors, it can cover all the points of 2-D space
* If $\vec{v}$ & $\vec{w}$ overlap than the points of that specific line will only get covered.
* If $\vec{v}$ & $\vec{w}$ are zero vectors than the vector is stuck on to origin



only origin covered

The "span" of $\vec{v}$ & $\vec{w}$ is the set of all their linear combinations

$a\vec{v} + b\vec{w}$

a and b vary over all real nos.

for three vectors, linear combination is

$a\vec{v} + b\vec{w} + c\vec{u} \Rightarrow$ Linear combination of $\vec{u}, \vec{v}, \vec{w}$

for span let all these consts vary

Technical defination of Basis:-

The basis of a vector space is a set of linearly independent vectors that span the full space.

Linear Transformations

1) "2×2 Matrix" ↳ Multiplying a vector by a matrix

$$\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$$

where $\hat{i}$ lands. where $\hat{j}$ lands.

2) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}$$

$$= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+3y \\ 2x+y \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3) $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $L(\hat{j})$

$L(\hat{i})$ $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$x\begin{bmatrix} 2 \\ 2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$x\,L(\hat{i}) + y\,L(\hat{j})$$

Composition — eg
1. first rotate by 90°
2. Shear

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Shear}}\left(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}}\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Composition}}\begin{bmatrix} x \\ y \end{bmatrix}$$

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Product of 2 "2×2" matrix

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Shear}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{composition}}$$

$\longleftarrow$ Read right to left

geometrical meaning is first do one transf and than do another so its the composition of 2 transformations

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{M_2} \underbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}_{M_1} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+hd \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix} = e\begin{bmatrix} a \\ c \end{bmatrix} + g\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ae+bg \\ ce+dg \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix} = f\begin{bmatrix} a \\ c \end{bmatrix} + h\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} af+bh \\ cf+hd \end{bmatrix}$$

NOTE:-

1. $M_1 M_2 \neq M_2 M_1$
2. Associative $\rightarrow$ $A(BC) = (AB)C$

for 3×3 matrics

$$\underbrace{\begin{bmatrix} 0 & -2 & 2 \\ 5 & 1 & 5 \\ 1 & 4 & -1 \end{bmatrix}}_{\text{Second transformation}} \underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}}_{\text{first transformation}}$$

Transformation in 3-D

→ Rotation by 90°

$$\hat{i} \rightarrow \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \hat{j} \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \hat{k} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

X-axis goes to negative Z-axis — Y-axis stays as it is — Z-axis goes to +ve X-axis

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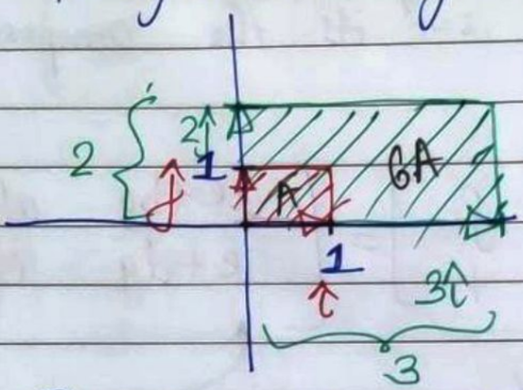
Transformation by a determinant.

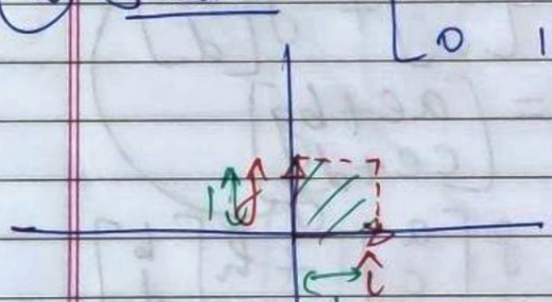
(1)

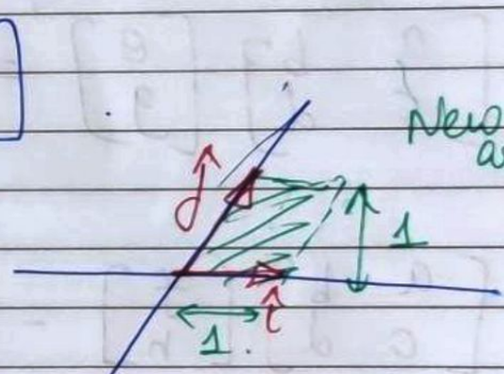


$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

New area $= 3 \times 2$
$= \underline{6}$

(2) Shear $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

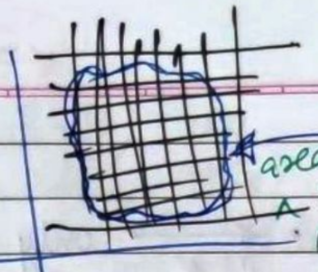




New area $= 1 \times 1$
$= 1$

Area remains same as ht & base of parallelogram & rect are same

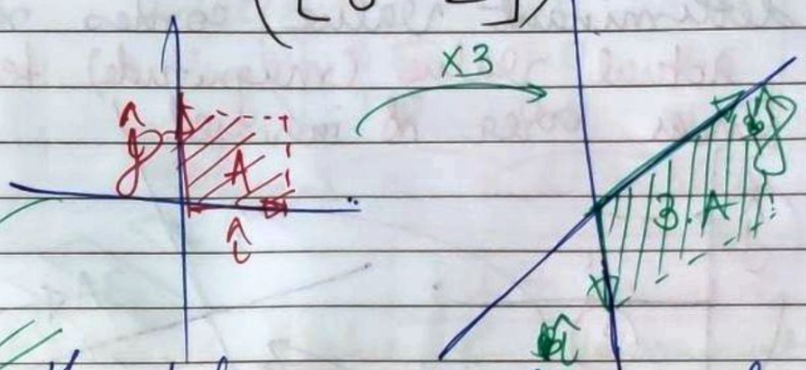
The grid lines are parallel and evenly spaced so whatever happens to one square it happens to any other square in the plane.



Any object (random shape) can be approximated by grid squares, smaller the grid squares, lesser the error

The factor (scaling factor) by which the grid changes area of one square is given by the determinant of that transformation

$$\det\left(\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\right) = 3 \times 2 = 6$$



$$\begin{bmatrix} 0.0 & 2.0 \\ -1.5 & 1.0 \end{bmatrix}$$

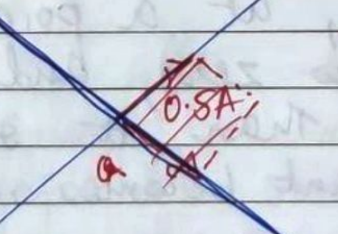
×½

The determinant of transformation is 3 if (it) increases the ~~for~~ area of the square by 3 times

transformation

$$\det\left(\begin{bmatrix} 0.0 & 2.0 \\ -1.5 & 1.0 \end{bmatrix}\right) = \underline{\underline{3.0}}$$

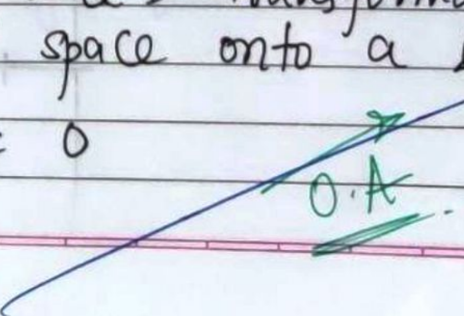
$$\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$$



$$\det\left(\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}\right) = \underline{\underline{0.5}}$$

→ The determinant of a 2-D transformation is 0 if it squzes all its space onto a line

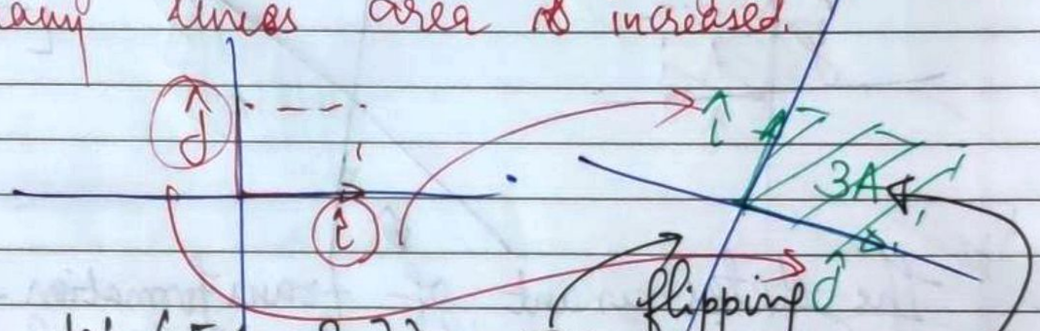
$$\det\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right) = 0$$



$\rightarrow \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$

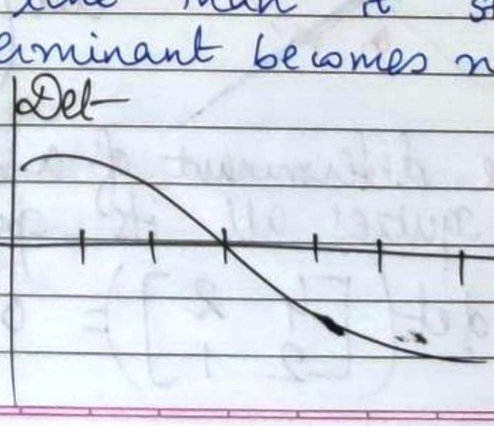
• $\xleftarrow{}$ O.A · only single point at origin

* NOTE :- Determinant also gives a negative value what does it mean?
It means it flips the axis
If first the $\hat{j}$ is on the left of $\hat{i}$ than it comes on the right of $\hat{i}$ than determinant value comes negative
Though actual value (magnitude) tells how many times area is increased.



$\det\left(\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\right) = -3$

* As $\hat{i}$ & $\hat{j}$ gets closer the determinant goes on decreasing, at a point of time the determinant is zero, and the grid is just one line than it starts flipping over and determinant becomes negative.



* In 3-D, we consider a cube, with $\hat{i}, \hat{j}, \hat{k}$ and after the transformation it results into a Parallelopiped. So the DETERMINANT gives the Volume of PARALLELOPIPED

* If the determinant is zero, everything is squised into a single plane or a ~~pla~~ line or in most extreme case a point. (It results in something which has no volume)

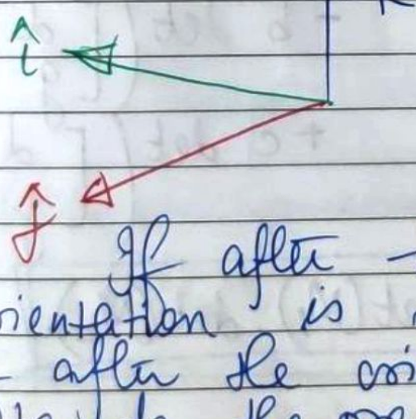
$$\det\left(\begin{array}{c} \begin{array}{ccc} \bar{u} & \bar{v} & \bar{w} \end{array} \\ \begin{bmatrix} 1.0 & 0.0 & 1.0 \\ 0.5 & 1.0 & 1.5 \\ 1.0 & 0.0 & 1.0 \end{bmatrix} \end{array}\right) = 0$$

columns of such matrix are linearly dependent

$$\boxed{1\bar{u} + 1\bar{v} - 1\bar{w} = 0}$$

* Negative Value of determinant in 3-D

RIGHT HAND RULE



fore finger $\rightarrow \hat{i}$
Middle finger $\rightarrow \hat{j}$
Thumb $\rightarrow \hat{k}$

If after the transformation, the orientation is not changed, determinant is +ve if after the orientation you only manage to do the process by left hand than orientation is changed and determinant is negative

Note

1] $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$

① When b & c = 0, than geometrically it forms ~~a~~ rectangle of length and breadth a & d, determinant gives the area of rect.

② When any one of a, b, c, d is zero it forms a parallelogram with a as base & d is ht and determinant gives area of parallelogram

③ If b or c term is not zero, it tells u how much the parallelogram is stretched or squized in diagonal direct.

2]

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

3] $\boxed{\det(M_1 M_2) = \det(M_1) \cdot \det(M_2)}$

LINEAR EQ^N'S

$2x + 5y + 3z = 0$
$4x + 0y + 8z = 0$
$1x + 3y + 0z = 2$

$$\Rightarrow \underbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}_{\substack{\text{Coefficients} \\ A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\substack{\text{Variables} \\ \vec{x}}} = \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}_{\substack{\text{Constants.} \\ \vec{v}}}$$

$$\boxed{A\vec{x} = \vec{v}}$$

When the determinant of coeff matrix A is non-zero, than $\vec{x}$ directly lies on to the $\vec{v}$. This can be verified by inverse transformation.

→ If the A is rotated by $90°$ anticlockwise, than inverse of A rotates clockwise by $90°$

→ If the A is rotated leftward by 1 unit inverse rotates A rightward by 1 unit

→ When A and A^{-1} is multiplied, we get back to our original position

$$\text{So } AA^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This transformation does nothing

→ $A\vec{x} = \vec{v}$

Once you find A^{-1} you can get value of $\vec{x}$ by multiplying both sides by A^{-1}

$$A^{-1}A\vec{x} = A^{-1}\vec{v} \Rightarrow \boxed{\vec{x} = A^{-1}\vec{v}}$$

→ When $|A| = 0$, then A^{-1} DNE

When $|A|$ becomes 0, you squizzed the grid into a single plane, line or point in 3-D now u cannot unsquize

If the $\det(A) = 0$, then also solution exists but under ~~unter~~ certain condition

(1) When output of transformation is a line then it means RANK 1.

(2) If output of transformation is a plane than transformation has a rank of 2.

RANK → No of dimensions in the output of the transformation

(3) Non-zero determinant → Rank 3.

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Span of columns

Known as → Column space

(4) The vectors that land on origin is called the "Null space" or "kernel".

DOT PRODUCT ~~product~~

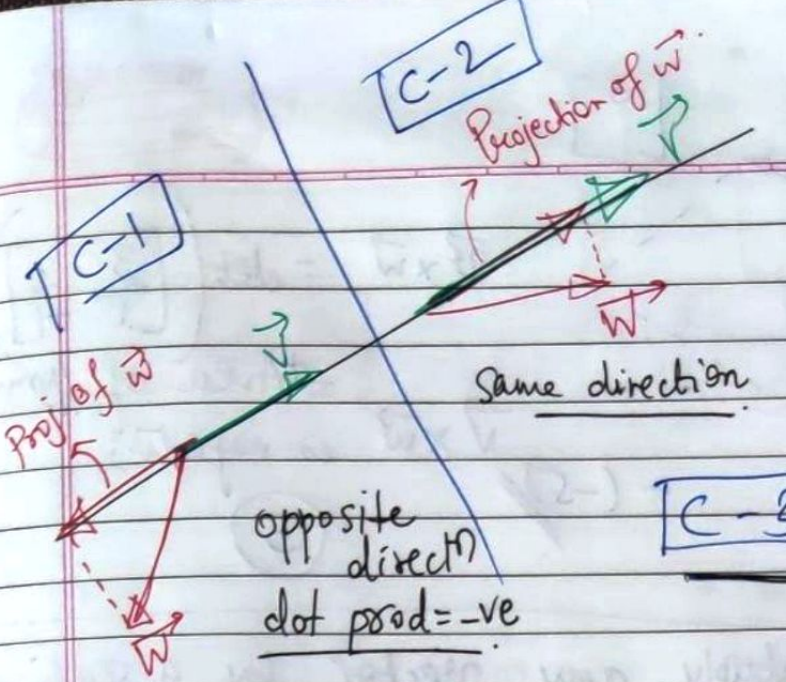
Dot product is the ~~projection~~ of (projection of one vector on another) × (projection of another vector).

So generally.

→ Vectors pointing in same direction, dot product → +ve

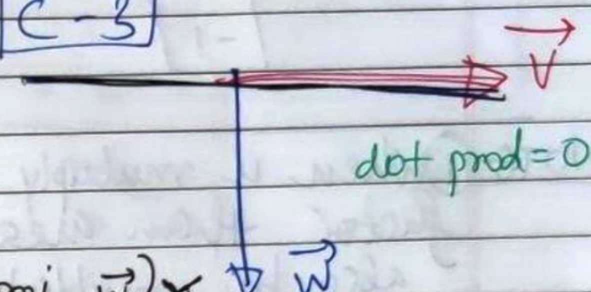
→ Vectors $\perp^r$ → Dot product = 0

→ Vectors pointing in opposite direction, dot prod = -ve



$$\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\vec v}\underbrace{\begin{bmatrix}2\\-1\end{bmatrix}}_{\vec w}$$

$2\cdot1 + (-1)1 = \underline{\underline{1}}$ (+ve)



Dot prod = (Length of proj $\vec w$) x (Length of $\vec v$)

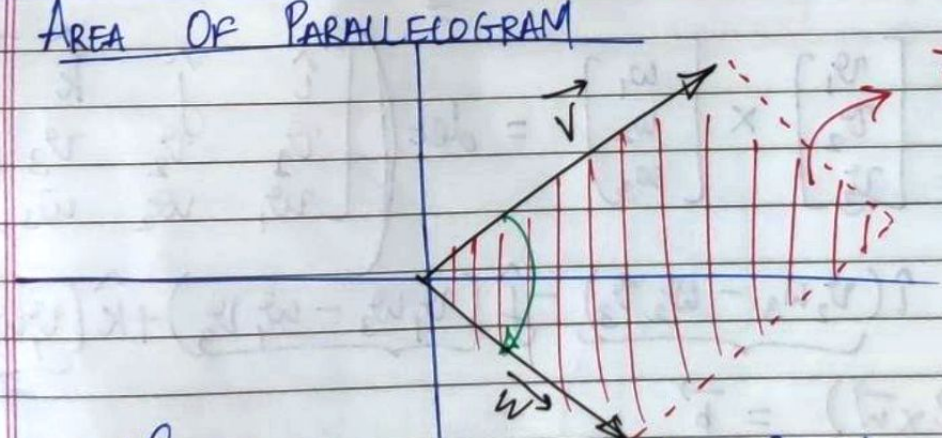
OR

Order don't matter

Dot prod = (Length of proj $\vec v$) x (Length of $\vec w$)

this will also give same result

AREA OF PARALLELOGRAM



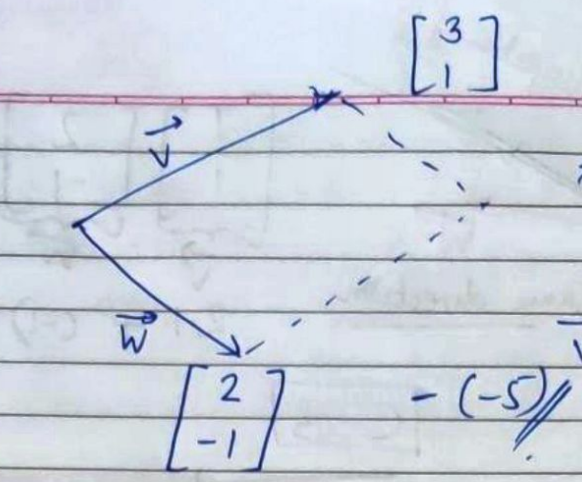
$\vec v$ is on left of $\vec w$

$\vec v \times \vec w$ = (−) Area of Parallelogram

$\vec v$ is on left of $\vec w$ so negative!

done by thumb rule

If $\vec v$ on the right of $\vec w$ cross product is positive.



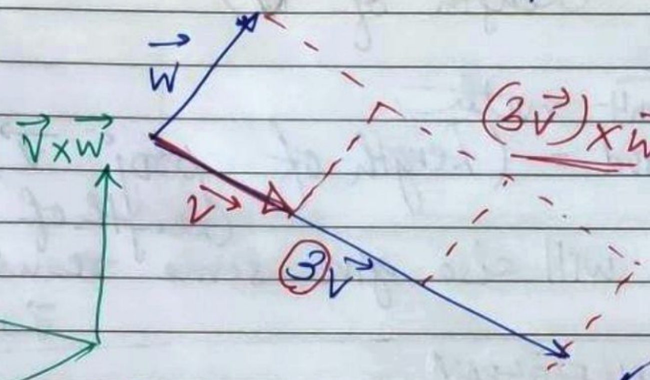
$\vec{v}\times\vec{w} = \det\left(\begin{bmatrix}3 & 2\\1 & -1\end{bmatrix}\right)$

$= \pm$ Area of ||gm

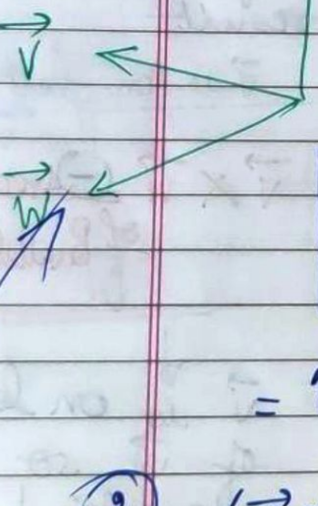
$\vec{v}\times\vec{w}$ is negative

$-(-5)$ 5

NOTE ① When u multiply any vector by a scaling factor then area of parallelogram will also be multiplied by 3.



$(3\vec{v})\times\vec{w} \rightarrow$ Area of ||gm $3(\vec{v}\times\vec{w})$



$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\times\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \det\left(\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{bmatrix}\right)$$

$$= \hat{i}\underbrace{(v_2w_3 - w_2v_3)} - \hat{j}\underbrace{(v_1w_3 - w_1v_3)} + \hat{k}\underbrace{(v_1w_2 - v_2w_1)}$$

② $(\vec{v}\times\vec{w}) = \vec{p}$ vector.

→ Length of $\vec{p}$ = (parallelogram's area)

→ Points in direction ⊥r to $\vec{v}$ & $\vec{w}$ both.

→ obeys right hand rule.

This is real cross prod

$p_1x+p_2y+p_3z = x(v_2w_3-v_3w_2) + y(v_3w_1-v_1w_3) + z(v_1w_2-v_2w_1)$

$$\boxed{\overrightarrow{p}\;\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \det\left(\begin{bmatrix}x & v_1 & w_1\\ y & v_2 & w_2\\ z & v_3 & w_3\end{bmatrix}\right)}$$

$\uparrow \vec{v}$ $\uparrow \vec{w}$

$$\boxed{\begin{aligned}p_1 &= v_2w_3 - v_3w_2\\ p_2 &= v_3w_1 - v_1w_3\\ p_3 &= v_1w_2 - v_2w_1\end{aligned}}$$

(Area of ||gm) × (Component of $\begin{bmatrix}x\\y\\z\end{bmatrix}$ ⊥r to $\vec{v}$ and $\vec{w}$)

[This is same as taking dot prod b/t $\begin{bmatrix}x\\y\\z\end{bmatrix}$ and vector ⊥r to $\vec{v}$ & $\vec{w}$ with a length equal to Area of ||gm]

<u>for a 3-D vector.</u>

$\vec{u} = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$ $\vec{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ $\vec{W} = \begin{bmatrix}W_1\\W_2\\W_3\end{bmatrix}$

<u>then</u>

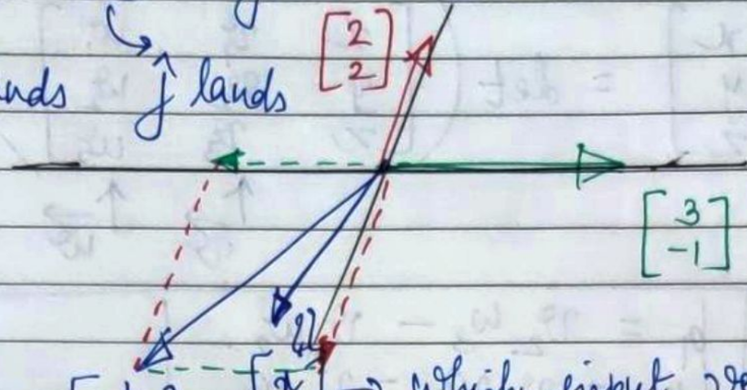
$$\vec{u}\times\vec{v}\times\vec{w} = \det\left(\begin{bmatrix}u_1 & v_1 & W_1\\ u_2 & v_2 & W_2\\ u_3 & v_3 & W_3\end{bmatrix}\right)$$

Not the real cross prod

This gives the volume of <u>parallelopiped</u>

$$\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$\hat{i}$ lands $\quad$ $\hat{j}$ lands



$\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow$ which input vector $\begin{bmatrix} x \\ y \end{bmatrix}$ will land on output vector $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$.

$$x \begin{bmatrix} 3 \\ -1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Consider case where $A\vec{x} = \vec{v}$
$\det(A) \neq 0$

every input has one & only one output and every output has one & only one input

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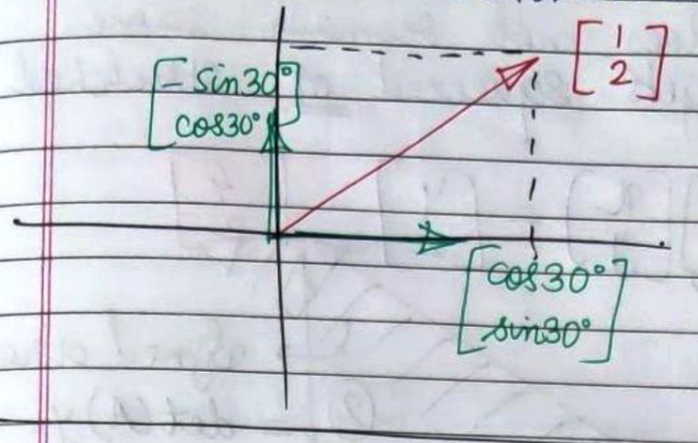
If $T(\vec{v}) . T(\vec{w}) = \vec{v} . \vec{w}$ for all $\vec{v}$ & $\vec{w}$ then T is "Orthonormal"

Generally vectors before taking dot product and after taking dot product differ, so they don't remain same, but for ORTHONORMAL vectors, before & after transformatn there is no change as they are unit and have angle 90°

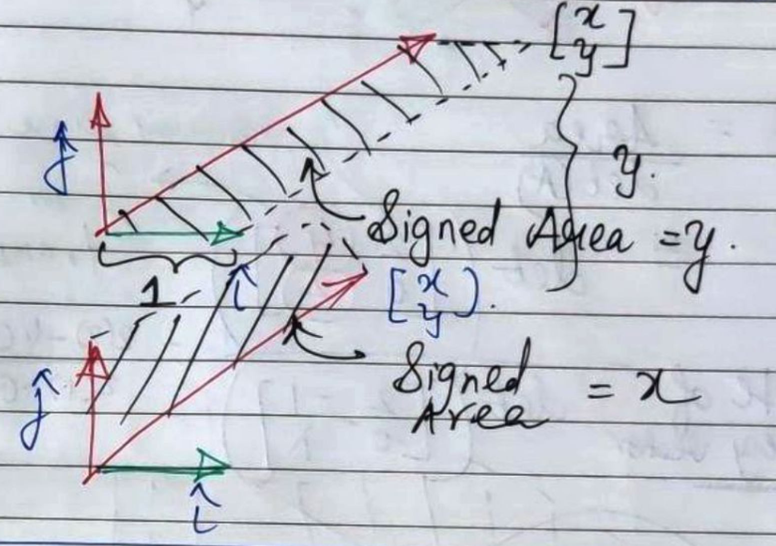
ONLY FOR ORTHONORMAL VECTORS



$$\underbrace{\begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix}}_{\text{ORTHONORMAL}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} \cos 30^\circ \\ \sin 30^\circ \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} -\sin 30^\circ \\ \cos 30^\circ \end{bmatrix}$$



FOR 3-D

$$z = \det\left(\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{bmatrix}\right)$$

$$y = \det\left(\begin{bmatrix} 1 & x & 0 \\ 0 & y & 0 \\ 0 & z & 1 \end{bmatrix}\right)$$

$$x = \det\left(\begin{bmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{bmatrix}\right)$$

Volume of // piped.

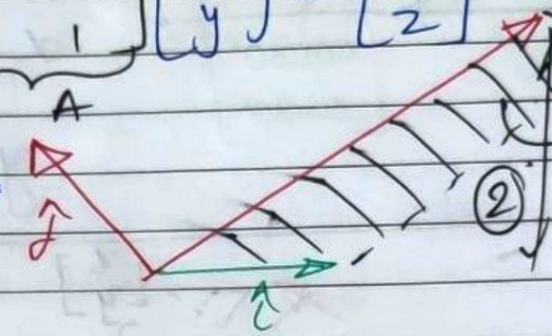
CRAMER'S RULE

When we apply transformation the area of ||gm does not remain same
They either gets squeezed or stretched

Apply $\underbrace{\begin{bmatrix}2 & -1\\0 & 1\end{bmatrix}}_{A}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$

$\begin{bmatrix}4\\2\end{bmatrix}$

After transformation



② Signed area $= \det(A)y$.

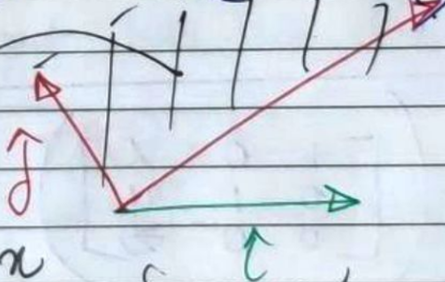
we can find y-coordinate of mystery vector

$$y = \frac{\text{Area}}{\det(A)} = \frac{\det\left(\begin{bmatrix}2 & 4\\0 & 2\end{bmatrix}\right)}{\det\left(\begin{bmatrix}2 & -1\\0 & 1\end{bmatrix}\right)} = \frac{2(2)-4(0)}{2(1)-(-1)0} = \frac{4}{2} = 2$$

$\hat{i}$ remains same

$\hat{j}$ is transformed

$\begin{bmatrix}4\\2\end{bmatrix}$



Signed Area $= \det(A)\,x$

③

$\hat{i}$ is transformed.

we can find x-coor of mystery vector

$$x = \frac{\text{Area}}{\det(A)} = \frac{\det\left(\begin{bmatrix}4 & -1\\2 & 1\end{bmatrix}\right)}{\det\left(\begin{bmatrix}2 & -1\\6 & -1\end{bmatrix}\right)}$$

$$\frac{4(1)-(-1)2}{(2)(1)-(-1)0} = \frac{6}{2} = 3$$

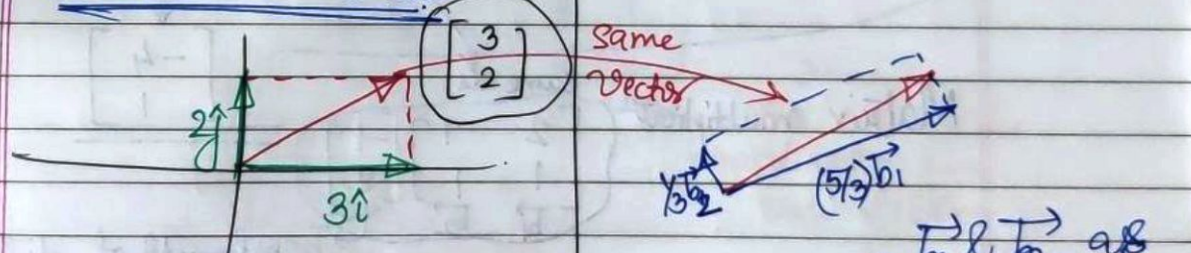
FOR 3-D

$$\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 2 \\ -4 & 6 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix}$$

$$x = \frac{\det\left(\begin{bmatrix} 7 & 2 & 3 \\ -8 & 0 & 2 \\ 3 & 6 & -9 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 2 \\ -4 & 6 & -9 \end{bmatrix}\right)}$$

$$y = \frac{\det\left(\begin{bmatrix} -4 & 7 & 3 \\ -1 & -8 & 2 \\ -4 & 3 & -9 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 2 \\ -4 & 6 & -9 \end{bmatrix}\right)} \qquad z = \frac{\det\left(\begin{bmatrix} -4 & 2 & 7 \\ -1 & 0 & -8 \\ -4 & 6 & 3 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 2 \\ -4 & 6 & -9 \end{bmatrix}\right)}$$

CHANGE OF BASIS

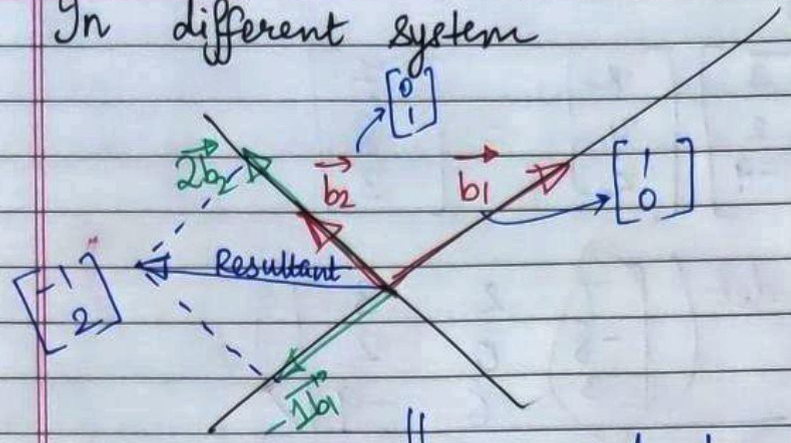


General system

Different system

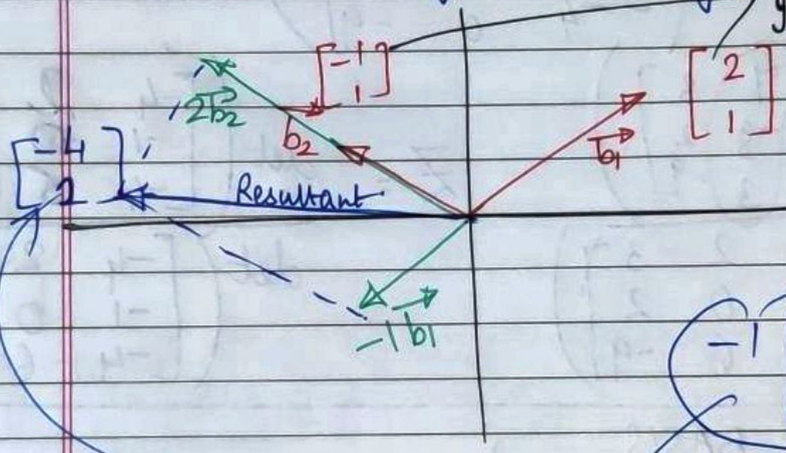
$\vec{b_1}$ & $\vec{b_2}$ as basis vectors

In different system



converted to our system

In our system coordinates of $\vec{b_1}$ & $\vec{b_2}$



$$-1b_1 + 2b_2.$$

$$-1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Matrix multiplicatn same as

$$\begin{array}{cc} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ \vec{b_1} \quad \vec{b_2} & \end{array} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Transformatn of different system to be done.

Our grid $\longrightarrow$ Other system's grid.

Our language $\longleftarrow$ Different System's language.

Taking Inverse.

Other System's grid ⟶ Our grid.

Other system's language ⟵ Our language

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

So to view any vector eg $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in other system we multiply the inverse change of base matrix to original matrix

$$\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$$

Conclusion

$\cancel{A = \begin{bmatrix} 2 & -1 \\ 1 & \end{bmatrix}}$

Matrix Multiplication

① $A\begin{bmatrix} x_d \\ y_d \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \end{bmatrix}$

same vectors with coordinates in our system

Vector with coordinates in different system.

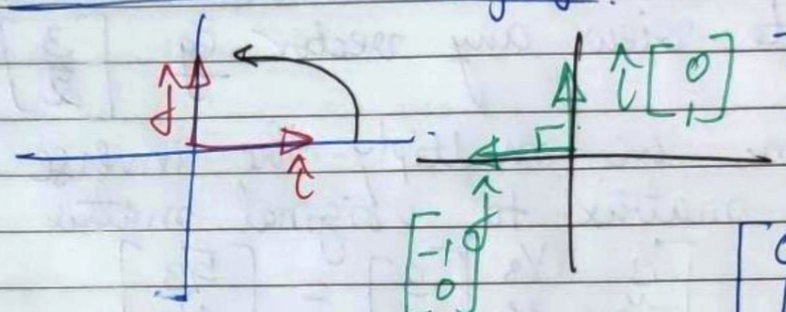
② $\begin{bmatrix} x_d \\ y_d \end{bmatrix} = A^{-1}\begin{bmatrix} x_o \\ y_o \end{bmatrix}$

Inverse matrix does the opposite.

If we want to transform the vector in other ~~language~~ coordinate system to over coordinate system.

2 system < our system / Jennifer's system

90° rotation in our language ⇒



Transformation in our language

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

To get the same Transformation of 90° in another system.

$$\underbrace{\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}^{-1}}_{\text{Inverse change of base matrix}} \overbrace{\underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}_{\text{Transformat}^n\text{ matrix in our language}} \underbrace{\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}}_{\text{Change of basis matrix}} \underbrace{\begin{bmatrix}-1\\ 2\end{bmatrix}}_{\text{Vector in Other system's language}}}^{\text{TRANSFORMED VECTOR IN OUR LANGUAGE}}$$

Same Vector in our language

TRANSformed vector in another language.

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→ EIGEN VECTORS AND EIGEN VALUES

Eigen Vectors are the span of vectors of the transformation and eigen value is factor by which it is stretched or squashed during transformation

→ Can Eigen Values be Negative?

YES! If eigen value is $-\frac{1}{2}$, than its meaning is vector gets flipped and squished by $\frac{1}{2}$.

Matrix Vector Multiplication

$$A\vec{v} = \lambda\vec{v}$$

Transformation matrix   Eigen Vector   EigenValue   SCALAR Multiplication

Solve for λ and $\vec{v}$

Scaling by λ.

$$\lambda \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

Matrix multiplication by

$$A\vec{v} = (\lambda I)\vec{v}$$

$$A\vec{v} - (\lambda I)\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

This matrix looks something like

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 1 & 5-\lambda & 9 \\ 2 & 6 & 5-\lambda \end{bmatrix}$$

This $(A-\lambda I)\vec{v}=\vec{0}$ is only possible if the system squishes to a lower dimension into a line for the Squishification $\Rightarrow \boxed{\det(A-\lambda I)=0}$

As ~~the~~ the value of λ changes, the value of determinant $(A-\lambda I)$ changes. If the determinant becomes negative, the space squishes down. There is a value of λ, where the determinant becomes 0, it squishes space into a line, at that time $(A-\lambda I)\vec{v} = \vec{0}$ is satisfied. $\rightarrow$ The value of λ when det $(A-\lambda I)$ becomes zero (0) is known as eigen value (λ).

eg :- $$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Only possibles $(3-\lambda)(2-\lambda) = 0$

Eigen Values are $\rightarrow$ $\boxed{\lambda = 2}$ or $\boxed{\lambda = 3}$

Putting value of λ.

$$\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When $\lambda = 3$ we get solution of all vectors at the line (diagonal) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

this corresponds to the fact that $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ unaltered matrix has the capability of stretching the system by a factor of 2

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eg $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow A - \lambda I \Rightarrow \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

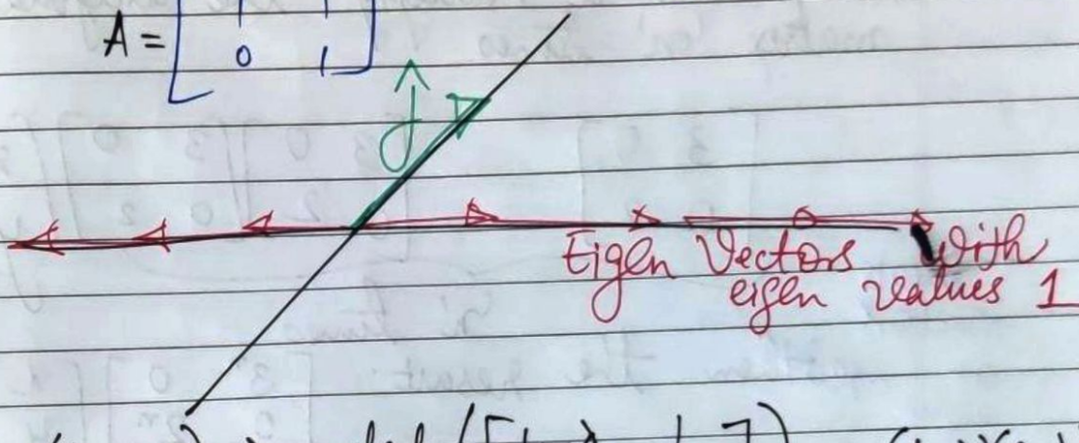
$$\det(A - \lambda I) = \lambda^2 + 1 = 0$$

$$\boxed{\lambda = \pm i}$$

Imaginary Roots $\Rightarrow$ There are No Eigen Vectors.

eg:- Shear $\hat{i}$ remains same, $\hat{j}$ changes by 1

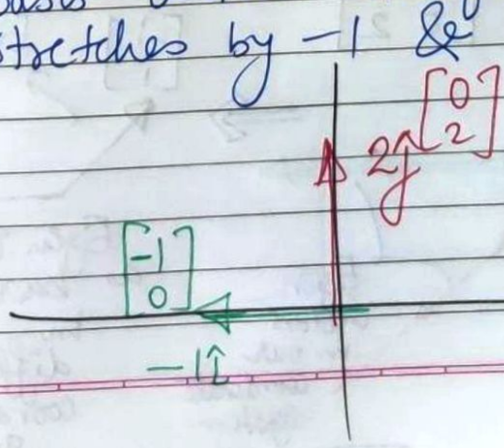
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



$\det(A - \lambda I) \Rightarrow \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = \underbrace{(1-\lambda)(1-\lambda) = 0}_{\boxed{\lambda = 1}}$

A single eigen value can have more that a line full of eigen vectors.

Eigen Basis:- If the eigen vectors are the basis vectors itself and if eg $\hat{i}$ gets stretches by -1 & $\hat{j}$ by 2 then



$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

both the eigen values of $\hat{i}$ & $\hat{j}$ placed on diagonal values.

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→ If all the other elements of a matrix is zero except the diagonal elements than the matrix is called diagonal matrix ~~with~~ and all the diagonal elements are the eigen values of basis vectors.

→ Also when u multiply the diagonal matrix 'n' times

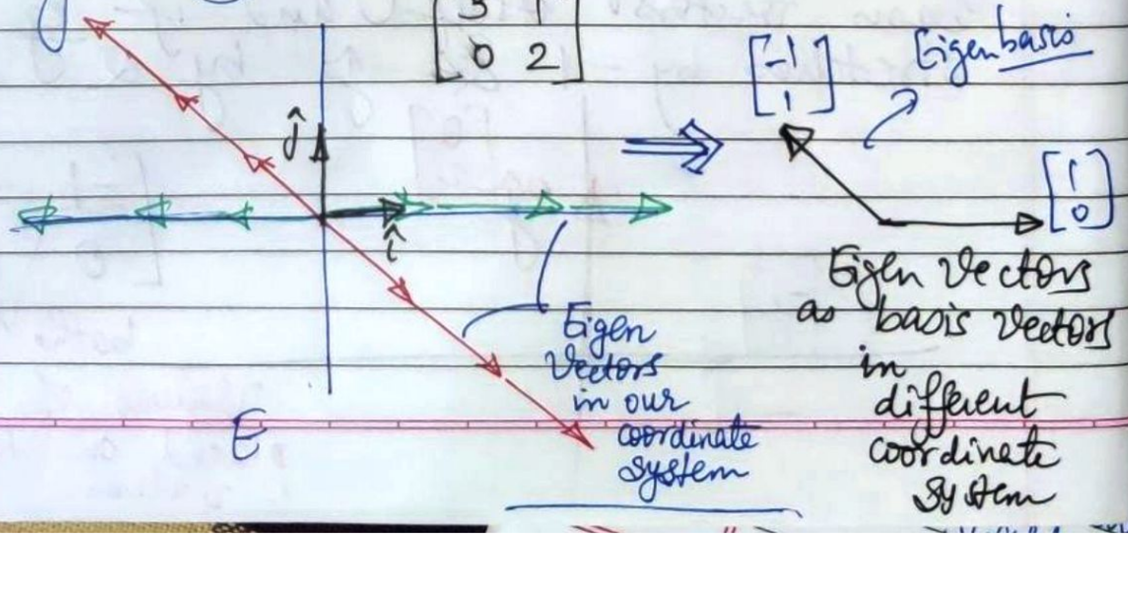
$$\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdots\cdots \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}_{\text{'n' times}}\begin{bmatrix} x \\ y \end{bmatrix}$$

basis vectors as eigen vectors

then the result. $\begin{bmatrix} 3^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 3^n x \\ 2^n y \end{bmatrix}$

If there are lot of eigen vectors, then we can change the coordinate system and make eigen vectors as your Basis Vectors.

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$



WORLD STAR™ Date: Page:

$$\left[\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{\text{change of basis matrix}}\right] = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

This matrix is guaranteed to be diagonal!.

It is easy to calculate $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^{100}$ in other coordinate system

If u are asked $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^{100}$ then first convert it to eigen basis compute 100 th power in that system than $\boxed{\text{convert}}$ back to our system. ??

$A = \underline{P D^n P^{-1}}$.

$$\boxed{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3^{100} & 0 \\ 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}$$

To find Eigen values of $\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$

Traditional method.

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}\right) = (3-\lambda)(1-\lambda) - (1)(4)$$

$$= \lambda^2 - 4\lambda - 1 = 0$$

$$\lambda_1, \lambda_2 = \frac{4 \pm \sqrt{4^2 - 4(1)(-1)}}{2} = \frac{4 \pm \sqrt{20}}{2}$$

$$= 2 \pm \sqrt{5}$$

1 Direct Method Steps

(1) $\frac{1}{2} tr\left(\begin{bmatrix} \boxed{a} & b \\ c & \boxed{d} \end{bmatrix}\right) = \frac{a+d}{2} = \frac{\lambda_1+\lambda_2}{2} = m$ (mean)

(2) $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc = \lambda_1\lambda_2 = p$ (product)

(3) $\lambda_1, \lambda_2 = \boxed{m \pm \sqrt{m^2-p}}$

Ex (1) $\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$

$m = \frac{3+1}{2} = 2$

$p = 3-4 = -1$

$\lambda_1, \lambda_2 = 2 \pm \sqrt{4+1}$

$= 2 \pm \sqrt{5}$

Ex (2) $\begin{bmatrix} 2 & 7 \\ 1 & 8 \end{bmatrix}$

$m = \frac{2+8}{2} = 5$

$p = 16-7 = 9$

$\lambda_1, \lambda_2 = 5 \pm \sqrt{25-9}$

$\lambda_1, \lambda_2 = 5 \pm 4$

$\boxed{\lambda_1, \lambda_2 = 9, 1}$

Formal Defination Of Linearity

Additivity :- $L(\vec{v}+\vec{w}) = L(\vec{v}) + L(\vec{w})$

Scaling :- $L(c\vec{v}) = cL(\vec{v})$

Linear transformations preserve addition and scalar multiplication.

Basis Functions

$b_0(x) = 1$
$b_1(x) = x$
$b_2(x) = x^2$
$b_3(x) = x^3$
$\vdots$

Polynomials

$$1x^2 + 3x + 5.1 \rightarrow \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$5\,b_0(x) + 3\,b_1(x) + 1b_2(x)$

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1 x + a_0 = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \\ 0 \\ \vdots \end{bmatrix}$$

$$\frac{d}{dx}(1x^3+5x^2+4x+5) = 3x^2+10x+4$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1\cdot 4 \\ 2\cdot 5 \\ 3\cdot 1 \\ 0 \\ \vdots \end{bmatrix}$$

How to form this matrix is on next page

$$\frac{d}{dx} b_0(x) = \frac{d}{dx}(1) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{d}{dx} = \begin{bmatrix} 1 & & & \\ & 2 & 3 & 4 \\ & & & \\ & & & \end{bmatrix}$$

$$\frac{d}{dx}(b_1 x) = \frac{d}{dx}(x) = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{d}{dx}(b_2 x) = \frac{d}{dx}(x^2) = 2x = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{d}{dx}(b_3 x) = \frac{d}{dx}(x^3) = 3x^2 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

	Linear Algebra Concepts	Alternate names when applied to functions
1	Linear Transformations	Linear operators
2	Dot products	Inner products
3	Eigen Vectors	Eigenfunctions

8 Axioms of Linear Algebra

1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
2. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. There is a vector 0 such that $0 + \vec{v} = \vec{v}$ for all $\vec{v}$
4. For every vector $\vec{v}$ there is a vector $-\vec{v}$ so that $\vec{v} + (-\vec{v}) = 0$
5. $a(b\vec{v}) = (ab)\vec{v}$

6 $1\vec{v} = \vec{v}$

7 $a(\vec{v}+\vec{w}) = a\vec{v} + a\vec{w}$

8 $(a+b)\vec{v} = a\vec{v} + b\vec{v}$