Homework 5

1 Eigenvalues + Eigenvectors.

a) $M = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

$\therefore Mx = \lambda x.$

$(Mx - \lambda x) = 0.$

$(M - \lambda I)x = 0.$

$\therefore \det(M - \lambda I) = 0.$

$\therefore \det \begin{bmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = 0.$

$\therefore (5-\lambda)(2-\lambda) = 0$

$10 - 5\lambda - 2\lambda + \lambda^2 = 0$

$\lambda^2 - 7\lambda + 10 = 0$

$\therefore (\lambda-2)(\lambda-5) = 0$

$\lambda = \boxed{5}, \lambda = \boxed{2}$

$\therefore$ Subst. $\lambda = 5$

$\therefore (M - 5I)x = 0.$

$\therefore \left[\begin{array}{cc|c} 5-5 & 0 & 0 \\ 0 & 2-5 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -3 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -3 & 0 \end{array}\right]$

$\left.\begin{array}{l} x = \epsilon \\ y = 0 \end{array}\right] \rightarrow \therefore V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Let $\lambda = 2$

$\therefore \left[\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$

$\sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$

$\therefore x = 0$

$y = \epsilon.$

$\therefore V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Eigenspace $= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

b) $M = \begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

$\therefore Mx = \lambda x.$

$(M - \lambda I)x = 0.$

$\therefore \begin{bmatrix} 22-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix} x = 0.$

$\therefore \det \begin{pmatrix} 22-\lambda & 6 \\ 6 & 13-\lambda \end{pmatrix} = 0.$

$\therefore (22-\lambda)(13-\lambda) - 36 = 0.$

$286 - 22\lambda - 13\lambda + \lambda^2 - 36 = 0.$

$\lambda^2 - 35\lambda + 250 = 0.$

$\lambda^2 - 25\lambda - 10\lambda + 250 = 0$

$(\lambda - 25)(\lambda - 10) = 0$

$\lambda = 25, \lambda = 10.$

Let $\lambda = 25$

$\therefore (M - 25I)x = 0$

$$\left[\begin{array}{cc|c} -3 & 6 & 0 \\ 6 & -12 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x = 2t$
$y = t$

$\therefore V_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Let $\lambda = 10$

$(M - 10I)x = 0$

$$\therefore \left[\begin{array}{cc|c} 12 & 6 & 0 \\ 6 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 12 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x = -\frac{1}{2}t$
$y = t$

$\therefore V_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

$\therefore$ Eigenspace $= \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$

c) $M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\therefore Mx = \lambda x$
$(M - \lambda I)x = 0$
$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\right) = 0$

$\therefore (4-\lambda)(1-\lambda) - 4 = 0$
$4 - 4\lambda - \lambda + \lambda^2 - 4 = 0$
$\lambda^2 - 5\lambda + 0 = 0$
$\lambda(\lambda - 5) = 0$
$\therefore \lambda = 5, \quad \lambda = 0$

$\therefore$ Let $\lambda = 5$

$$\therefore \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore x = \frac{1}{2}t$
$y = t$

$\therefore V_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Let $\lambda = 0$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

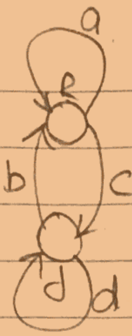
$\therefore x = -2t$
$y = t$

$V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Eigenspace $= \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

2. $R[n+1] = aR[n] + bJ[n]$

$J[n+1] = cR[n] + dJ[n]$



a) Case: $a+b = c+d$

Aim: $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an Eigenvector of A.

$\therefore \quad Ax = \lambda x$

$\therefore \quad A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ By substitution.

$\begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 \end{bmatrix} \Rightarrow \begin{bmatrix} a+b \\ a+b \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 \end{bmatrix}$

$\therefore \boxed{a+b = \lambda_1}$

$(a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \therefore \lambda_1 = a+b$.

$\therefore$ Aim: $V_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$ is an Eigenvector.

$a+b = c+d$

$a-c = d-b$

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} b \\ -c \end{bmatrix} = \lambda_2 \begin{bmatrix} b \\ -c \end{bmatrix}$

$\begin{bmatrix} ab - bc \\ bc - dc \end{bmatrix} = \begin{bmatrix} b\lambda_2 \\ -c\lambda_2 \end{bmatrix}$

$\begin{bmatrix} b(a-c) \\ -c(d-b) \end{bmatrix} = \begin{bmatrix} b\lambda_2 \\ -c\lambda_2 \end{bmatrix}$

$\therefore \boxed{\lambda_2 = a-c}$

$\Rightarrow \begin{bmatrix} b(a-c) \\ -c(a-c) \end{bmatrix} = \begin{bmatrix} b\lambda_2 \\ -c\lambda_2 \end{bmatrix}$

2. b) $A = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$

$\therefore \lambda_1 = a+b = 1.$ $\therefore \lambda_2 = a-c = 0.5$

$\therefore V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\therefore V_2 = \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}$

c) Steady states $\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}$

d) $Ax = \lambda x$

$\therefore AV_2 = \lambda_2 V_2$

$\therefore A \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix} = 0.5 \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}$

$\therefore \underset{(n \to \infty)}{S[n]} = (0.5)^n \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ This means that as $n \to \infty$, $S[n]$ tends to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Therefore, as $n \to \infty$, Romeo and Juliet have Neutral feelings.

e) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore \lambda_1 = a+b = 2$ $\therefore \lambda_2 = a-c = 0$

$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2.f) $s[0] = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\therefore \lambda = 0$.

Using equation $Av = \lambda v$,

As $n \to \infty$, $0^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ Romeo and Juliet have neutral feelings as $n \to \infty$.

g) $s[0] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda = 2$

As $n \to \infty$, $\lambda^n s[0] = 2^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} \infty \\ \infty \end{bmatrix}$

$\therefore$ As $n \to \infty$, Romeo likes Juliet.

h) $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

$Ax = \lambda x$.

$(A - \lambda I)x = 0$

$\det(A - \lambda I) = 0$.

$\therefore \det\left(\begin{bmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}\right) = 0$

$\therefore (1-\lambda)(1-\lambda) - 4 = 0$

$1 - \lambda - \lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$\therefore (\lambda - 3)(\lambda + 1) = 0$.

$\therefore \lambda = 3$ or $\lambda = -1$.

Continued. $\Longrightarrow$

Let $\lambda = 3$

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore x = -t$

$y = t$

$\therefore v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Let $\lambda = -1$

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x = t$

$y = t$

$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

i) $\lambda_1 = 3$

$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow S[0] = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

As $n \to \infty$, $3^n \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -\infty \\ \infty \end{bmatrix}$

$\therefore$ if $R[0] < 0$ and $J[0] > 0$, then they continued in that same feeling state but with greater feelings. In other words, Juliet likes Romeo.

if $R[0] > 0$ and $J[0] < 0$, then Romeo likes Juliet as $n \to \infty$ but with greater feelings.

2.j) $\lambda = -1$

$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\therefore s[0] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

As $n \to \infty$, $(-1)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\therefore$ They oscillate between liking and disliking each other as $n \to \infty$.

3.

a) $S = Hi + \omega$.

Aim: Express i in terms of H, S, ω.

$\therefore \quad Hi = S - \omega$

$H^{-1}Hi = H^{-1}(S-\omega)$

$i = H^{-1}(S-\omega)$

b) Noise is scaled $\Rightarrow$ $\hat{\omega} = \boxed{H^{-1}\omega}$

Let $\underbrace{\omega = \alpha_1 b_1 + \dots + \alpha_n b_n}$

Change of base.

ω is represented as the Eigenvectors of H^{-1}.

$\hat{\omega} = H^{-1}(\alpha_1 b_1 + \dots \alpha_n b_n)$

$= H^{-1}\alpha_1 b_1 + \dots + H^{-1}\alpha_n b_n$

$= \alpha_1 H^{-1} b_1 + \dots + \alpha_n H^{-1} b_n$

$= \alpha_1 \lambda_1 b_1 + \dots + \alpha_n \lambda_n b_n$

* we multiply the eigenvector of a matrix with the matrix to get λb_1

For eigenvectors with larger Eigenvalues $\Rightarrow$ Noise Signal Amplified

For eigenvectors with smaller Eigenvalues $\Rightarrow$ Noise Signal Attenuated

c) Matrix H_1 performs best when recreating the image

so that the noise in the reconstructed image is Attenuated.

we want the eigenvalues of H^{-1} to be small. This means that we want the eigenvalues of H to be large.

↳ in ipynb, we are given $H_1 \ldots H_3$.

↳ H_1 has the largest smallest Eval.

↳ H_3 has the smallest of small Evals.

∴ Inverse ⇒ H_1 ⇒ smaller.

⇒ H_3 ⇒ larger.

d) Prove: Show that if λ is an Eigenval of H, then $1/\lambda$ is an eigenval. of H^{-1}.

Proof: we know that

$$Hv = \lambda v$$

$$H^{-1}Hv = H^{-1}\lambda v$$

$$v = \lambda H^{-1} v$$

$$\frac{1}{\lambda} v = H^{-1} v$$

$$\therefore H^{-1} v = 1/\lambda \, v$$

∴ Eigenvalue of H^{-1} is $\frac{1}{\lambda}$.

4 Cubic Polynomials

a) $p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

Dimension $\Rightarrow 3+1 = \boxed{4}$

b) $P(t) = c^T \varphi(t)$

$$\begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

$c^T = [c_0 \; c_1 \; c_2 \; c_3]$

$\therefore c^T \cdot \varphi(t) = [c_0 \; c_1 \; c_2 \; c_3] \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$

$= c_0 + c_1 t + c_2 t^2 + c_3 t^3$

$= p_0 + p_1 t + p_2 t^2 + p_3 t^3$

$\therefore c^T = [p_0 \; p_1 \; p_2 \; p_3]$

c) To show that $\varphi_k(t) = t^k$ for $k = 1, 2, 3, 0$ is a basis, show that they are linearly independent:

$\therefore$ For any t^i, where $i \in \{0, 1, 2, 3\}$, if we try to represent one base t^i in terms for the other bases, we get $\alpha_i t^i = \alpha_j t^j + \alpha_k t^k + \alpha_\ell t^\ell$. But we have coefficient α_i on one side and zero on the other

Therefore cannot represent one in terms of the other. $\therefore$ Linearly Independent

d) Basis Polynomials:

$\phi_0(t) = 1 \Rightarrow$ Deriv. $\Rightarrow 0$

$\phi_1(t) = t \Rightarrow$ Deriv. $\Rightarrow 1 \Rightarrow \phi_0(t)$

$\phi_2(t) = t^2 \Rightarrow$ Deriv. $\Rightarrow 2t \Rightarrow 2\,\phi_1(t)$

$\phi_3(t) = t^3 \Rightarrow$ Deriv. $\Rightarrow 3t^2 \Rightarrow 3\,\phi_2(t)$

e)

$$p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

$$\frac{\partial p}{\partial t} = c_1 + 2c_2 t + 3c_3 t^2 + 0 \cdot t^2$$

$$\Rightarrow \underbrace{\begin{bmatrix} c_1 & 2c_2 & 3c_3 & 0 \end{bmatrix}}_{(DC)^T} \cdot \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

$$\therefore D \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ 0 \end{bmatrix}$$

Continued.

$$\therefore D \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ 0 \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

7. I worked on question 1 and 2 on Monday and question 3 and 4 on wednesday. I worked alone but went to office hour on wednesday and friday to clarify some doubts.