EECS 16B Designing Information Devices and Systems II
Fall 2019 Discussion Worksheet Discussion 1A

1 Transistor Introduction

Transistors (as presented in this course) are 3 terminal, voltage controlled switches. This means that, when a transistor is "on," it connects the Source (S) and Drain (D) terminals via a low resistance path (short circuit). When a transistor is "off," the Source and Drain terminals are disconnected (open circuit).

Two common types of transistors are NMOS and PMOS transistors. Their states (shorted or open) are determined by the voltage difference across the Gate (G) and Source (S) terminals, compared to a "threshold voltage." Transistors are extremely useful in digital logic design since we can implement Boolean logic operators using switches.

Recall that in this class, V_{tn} denotes how much **higher** the gate needs to be relative to the source for the NMOS to be on, and that $|V_{tp}|$ denotes how much **lower** the gate needs to be relative to the source for the PMOS to be on.

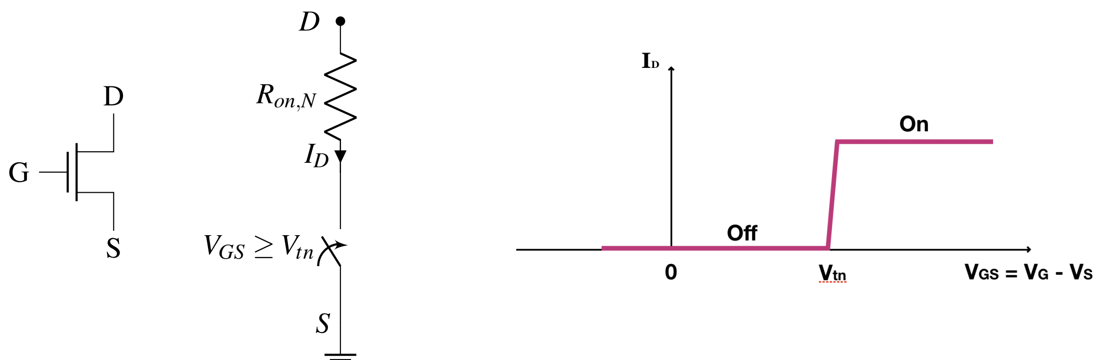

Figure 1: NMOS Transistor Resistor-switch model

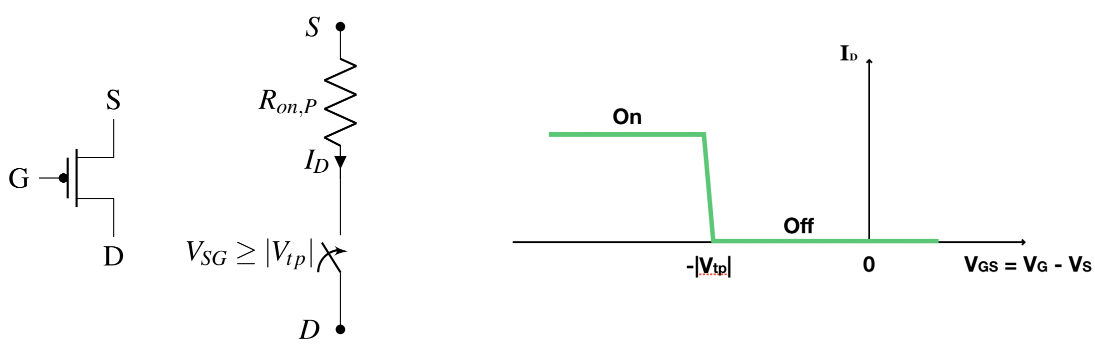

Figure 2: PMOS Transistor Resistor-switch model

Transistors can be connected together to perform boolean algebra. For example, the following circuit is called an "inverter" and represents a NOT gate.

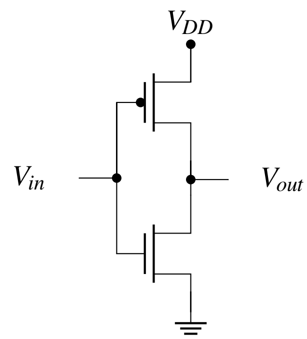

Figure 3: CMOS Inverter

When the input is high ($V_{in} \geq V_{tn}$, $V_{in} \geq V_{DD} - |V_{tp}|$), then the NMOS transistor is on, the PMOS transistor is off, and $V_{out} = 0$. When the input is low ($V_{in} \leq V_{tn}$, $V_{in} \leq V_{DD} - |V_{tp}|$), the NMOS transistor is off, the PMOS transistor is on, and $V_{out} = V_{DD}$. When working with digital circuits like the one above, we usually only consider the values of $V_{in} = 0, V_{DD}$. This yields the following truth table:

V_{in}	V_{out}	NMOS	PMOS
V_{DD}	0	on	off
0	V_{DD}	off	on

If you think of V_{DD} being a logical 1 and 0 V being a logical 0, we have just created the most elementary logical operation using transistors!

2 RC Circuit Theory

The RC circuit is a fundamental component of any real world circuit. Many electronic systems' specifications, like clock speed and bandwidth, are direct results of RC circuits. We will use differential equation methods to find the time domain behavior of RC systems. We first set up our problem by defining two functions of time: $I_C(t)$ is the current into the capacitor at time t, and $V_C(t)$ is the voltage across the capacitor at time t.

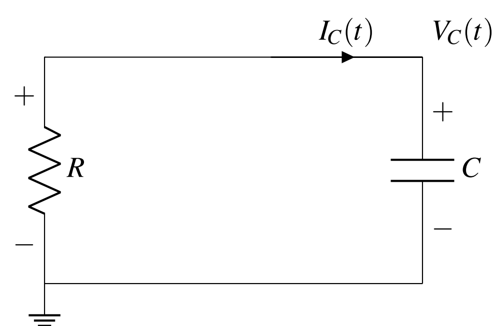

Figure 4: RC Circuit

Let's consider the RC circuit above in Figure 4. Assume that the capacitor is initially fully charged to V_{DD} at time $t = 0$. Current will flow out of the capacitor through the resistor: as the current flows out, the charge stored in the capacitor decreases. This causes the voltage across the capacitor to decrease. How can we describe this behavior mathematically?

The current across the resistor, I_R, is

$$I_R(t) = \frac{V_C(t)}{R}.$$

From KCL, $I_R(t) = I_C(t)$, and we also know that

$$I_C(t) = C \cdot \frac{d}{dt} V_C(t)$$

Equating our expressions for $I_R(t)$ and $I_C(t)$, we get

$$C \cdot \frac{d}{dt} V_C(t) = -\frac{V_C(t)}{R}$$

We end up with $-\frac{V_C(t)}{R}$ because the current through the resistor is flowing against the direction we defined for $I_C(t)$. Rearranging terms, we get

$$\frac{d}{dt} V_C(t) = -\frac{1}{RC} V_C(t)$$

How can we solve this differential equation to get $V_C(t)$? One way to solve this is to notice that $\frac{d}{dt} k \cdot e^{-at} = -k \cdot a \cdot e^{-t}$, and hence, a reasonable guess to the solution is

$$V_C(t) = k e^{-\frac{1}{RC} t}$$

where k is a constant; how do we get k explicitly? This requires initial conditions of our problem: we know at $t = 0$, $V_C(t) = V_{DD}$. Hence, we have $k = V_{DD}$. This gives us

$$V_C(t) = V_{DD} e^{-\frac{t}{RC}}$$

Now, we can start to ask some interesting questions. One of them is: How long does it take so that the voltage in the capacitor is halved? The answer is

$$t_{\text{half life}} = \ln(2) RC \approx 0.693 RC,$$

which is derived by setting $V_C(t) = \frac{1}{2} V_{DD}$ and solving for t. We see that the bigger the values of RC, the longer it takes for the voltage to drop. Because of this reason, RC is also called the time constant τ.

1. NAND Circuit

Let us consider a NAND logic gate. This circuit implements the boolean function $\overline{(A \cdot B)}$.

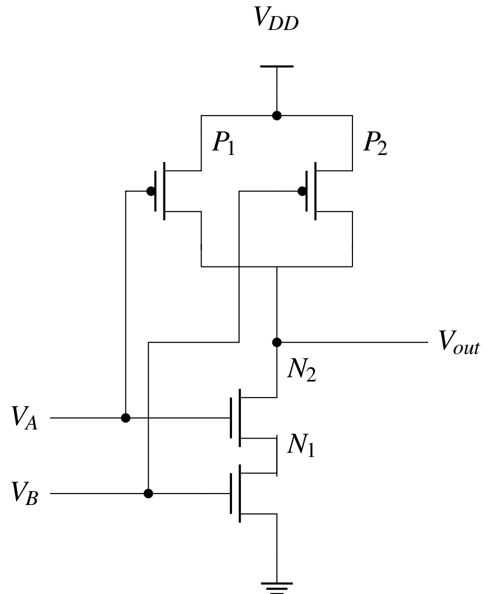

Figure 5: NAND

V_{tn} and V_{tp} are the threshold voltages for the NMOS and PMOS transistors, respectively. Assume that $V_{DD} > V_{tn}$ and $|V_{tp}| > 0$.

(a) Label the gate, source, and drain nodes for the NMOS and PMOS transistors above.

Answer: In an NMOS, the terminal at the higher potential is always the drain, and the terminal at the lower potential is always the source. Therefore, the drain is at the top of N_2 (connected to V_{out}) and the top of N_1 (connected to N_2). The source is at the bottom of N_2 (connected to N_1) and the bottom of N_1 (connected to ground). The gate terminal of N_2 is connected to V_A; the gate of N_1 is connected to V_B. In an PMOS, the terminal at the higher potential is always the source, and the terminal at the lower potential is always the drain. Therefore, the source is at the top of P_1 and P_2 (connected to V_{DD}). The drain is at the bottom of P_1 and P_2 (connected to V_{out}). The gate terminal of P_1 is connected to V_A; the gate of P_2 is connected to V_B.

(b) If $V_A = V_{DD}$ and $V_B = V_{DD}$, which transistors act like open circuits? Which transistors act like closed circuits? What is V_{out}?

Answer: P_1 and P_2 are off, creating an open circuit. N_1 and N_2 are on, creating a closed circuit. $V_{out} = 0V$ because it is connected by closed circuit to ground.

(c) If $V_A = 0V$ and $V_B = V_{DD}$, what is V_{out}?

Answer: P_2 and N_2 are off, creating an open circuit. P_1 and N_1 are on, creating a closed circuit. $V_{out} = V_{DD}$ because it is connected by closed circuit to V_{DD}.

(d) If $V_A = V_{DD}$ and $V_B = 0V$, what is V_{out}?

Answer: P_1 and N_1 are off, creating an open circuit. P_2 and N_2 are on, creating a closed circuit. $V_{out} = V_{DD}$ because it is connected by closed circuit to V_{DD}.

(e) If $V_A = 0V$ and $V_B = 0V$, what is V_{out}?

Answer: N_1 and N_2 are off, creating an open circuit. P_1 and P_2 are on, creating a closed circuit. $V_{out} = V_{DD}$ because it is connected by closed circuit to V_{DD}.

2. RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor over time in an RC circuit. We set up our problem by first defining three functions over time: $I(t)$ is the current at time t, $V(t)$ is the voltage across the circuit at time t, and $V_o(t)$ is the voltage across the capacitor at time t.

Recall from 16A that the voltage across a resistor is defined as $V_R = RI_R$ where I_R is the current across the resistor. Also, recall that the voltage across a capacitor is defined as $V_o = \frac{Q}{C}$ where Q is the charge across the capacitor.

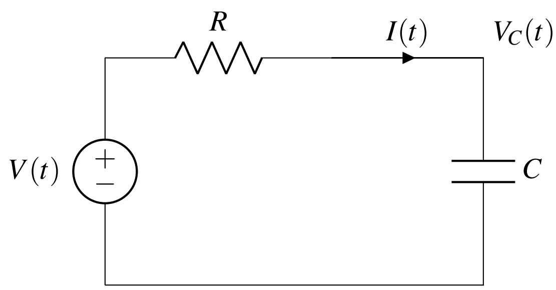

Figure 6: Example Circuit

(a) First, find an equation that relates the current across the capacitor $I(t)$ with the voltage across the capacitor $V_C(t)$.

Answer:

Differentiating $V_C(t) = \frac{Q(t)}{C}$ in terms of t, we get

$$\frac{dV_C(t)}{dt} = \frac{dQ(t)}{dt}\frac{1}{C}$$

By definition, the change in charge is the current across the capacitor, so

$$\frac{d}{dt}V_C(t) = I(t)\frac{1}{C}$$

(b) Write a system of equations that relates the functions $I(t)$, $V_C(t)$, and $V(t)$.

Answer:

Kirchhoff's law states that the voltage across a closed loop is 0.

$$RI(t) + V_C(t) - V(t) = 0$$

$$RI(t) + V_C(t) = V(t) \tag{1}$$

(c) So far, we have three unknown functions and only one equation, but we can remove $I(t)$ from the equation using what we found in part (a). Rewrite the previous equation in part (b) in the form of a differential equation.

Answer:

From part (a), we have

$$I(t) = \frac{dV_C(t)}{dt}C$$

Substituting this into Equation 1 gives us

$$RC\frac{dV_C(t)}{dt} + V_C(t) = V(t)$$

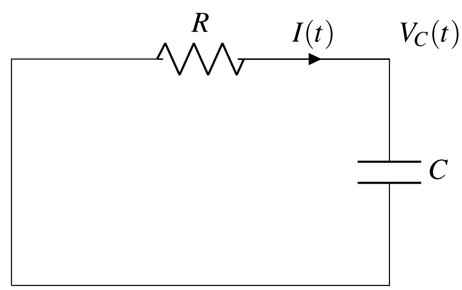

Figure 7: Circuit for part (d)

(d) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0$ for all $t \geq 0$. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Answer:

Because $V(t) = 0$, our differential equation simplifies to

$$RC\frac{dV_C(t)}{dt} + V_C(t) = 0$$

Doing some algebraic manipulations gives us

$$\frac{dV_C(t)}{dt} = -\frac{1}{RC}V_C(t)$$

This equation tells us that we are looking for some function $V_C(t)$ such that when we take its derivative, we get the same function $V_C(t)$ multiplied by a scalar $-\frac{1}{RC}$. Because the derivative is equal to a scalar times itself, we think that the solution $V_C(t)$ will probably be of the form Ae^{bt}, where A and b are both constants. In this case we see that $b = -\frac{1}{RC}$, and we find that

$$V_C(t) = Ae^{-\frac{1}{RC}t}$$

We still need to solve for the constant A in front of the exponential, and we use $V_C(0) = K$ to help us find A. Setting $t = 0$ in the equation gives us

$$\begin{aligned} V_C(0) &= Ae^{-\frac{1}{RC}0} \\ &= Ae^0 \\ &= A \\ &= V_{DD} \end{aligned}$$

Thus, we see that $A = V_{DD}$, and our solution is

$$V_C(t) = V_{DD} e^{-\frac{1}{RC}t}$$

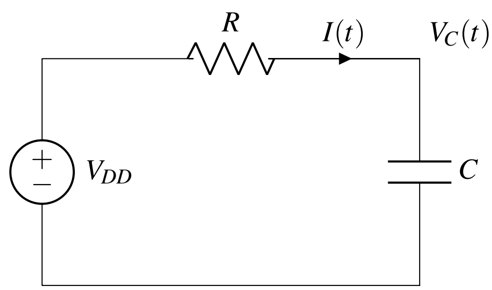

Figure 8: Circuit for part (e)

(e) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Answer:

Substituting $V(t) = V_{DD}$ into our solution from part (c):

$$RC\frac{dV_C(t)}{dt} + V_o(t) = V_{DD}$$

We want to arrange this equation to be in a form that we know how to solve:

$$\frac{d}{dt}V_C = \frac{V_{DD} - V_C(t)}{RC}$$

This is not quite the form we have seen before, as the term on the right is not equal to the term being differentiated. Let's instead define a new variable $\tilde{V}_C(t) = V_C(t) - V_{DD}$. Note that $\frac{d\tilde{V}_C(T)}{dt} = \frac{dV_C(t)}{dt}$. We can substitute these into our differential equation and obtain

$$RC\frac{dV_C(t)}{dt} + V_C(t) - V_{DD} = 0$$

$$RC\frac{d\tilde{V}_C(T)}{dt} + \tilde{V}_C(t) = 0$$

In this equation, we have now removed V_{DD} from the left hand because of how we defined $\tilde{V}_C(t)$. We can now solve the differential equation using the same method as in the previous part to get

$$\tilde{V}_C(t) = Ae^{-\frac{t}{RC}}$$

Substituting $V_C(t) = V_{DD} + \tilde{V}_C(t)$ back into this equation gives us

$$V_C(t) = V_{DD} + Ae^{-\frac{t}{RC}}$$

Using in the initial condition $V_C(0) = 0$, we get:

$$0 = V_{DD} + Ae^{-\frac{0}{RC}} = V_{DD} + A \implies A = -V_{DD}$$

Therefore,

$$\begin{aligned} V_C(t) &= V_{DD} - V_{DD}e^{-\frac{t}{RC}} \\ &= V_{DD}(1 - e^{-\frac{t}{RC}}) \end{aligned}$$

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