EECS 16B Designing Information Devices and Systems II
Fall 2019 Discussion Worksheet

5A: Inputs

1. **A system governed by differential equations being controlled with piecewise constant inputs**

Working through this question will help you understand better differential equations with inputs and the sampling of a continuous-time system of differential equations into a discrete-time view. This is important for control, since it is often easier to think about doing what we want in discrete-time.

(a) Consider the scalar system

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t). \tag{1}$$

Suppose that our $u(t)$ of interest is *constructed* to be piecewise constant over durations of width Δ, which we assume to be 1 for this problem. In other words:

$$u(t) = u(i) \text{ if } t \in [i, i+1) \tag{2}$$

Given that we start at $x(i)$, where do we end up at $x(i+1)$?

Answer: Our differential equation takes the form,

$$\frac{d}{dt}x(t) = \lambda x(t) + u(i) \tag{3}$$

where $u(i)$ is a constant value of some input function $u(t)$ at time $t = i$.
First we solve the differential equation by guessing

$$x(t) = \alpha e^{\lambda(t-i)} + \beta$$

This gives,

$$\frac{d}{dt}x(t) = \lambda \alpha e^{\lambda(t-i)}$$

We know that this should equal to the right hand side of (3), so we get,

$$\lambda \alpha e^{\lambda(t-i)} = \lambda x(t) + u(i) = \lambda(\alpha e^{\lambda(t-i)} + \beta) + u(i)$$
$$\implies \lambda \alpha e^{\lambda(t-i)} = \lambda \alpha e^{\lambda(t-i)} + \lambda \beta + u(i)$$

Now using $u(i) = u(i)$, we get,

$$\beta = \frac{-u(i)}{\lambda}$$

Further, we get,

$$x(i) = \alpha e^{\lambda(i-i)} + \beta = \alpha + \beta$$

And using, $\beta = -\frac{u(i)}{\lambda}$ we get,

$$x(i) = \alpha + \frac{-u(i)}{\lambda}$$

$$\implies \alpha = x(i) + \frac{u(i)}{\lambda}$$

So, we get that,

$$x(t) = (x(i) + \frac{u(i)}{\lambda})e^{\lambda(t-i)} - \frac{u(i)}{\lambda}$$

$$\implies x(i) = x(i)e^{\lambda(t-i)} + (\frac{e^{\lambda(t-i)} - 1}{\lambda})u(i)$$

Thus,

$$x(i+1) = x((i+1)) = x(i)e^{\lambda} + (\frac{e^{\lambda} - 1}{\lambda})u(i)$$

(b) Suppose that $x(0) = x_0$. **Unroll the implicit recursion you derived in the previous part to write $x(i+1)$ as a sum that involves x_0 and the $u(j)$ for $j = 0, 1, \ldots, i$.**
For this part, feel free to just consider the discrete-time system in a simpler form

$$x(i+1) = ax(i) + bu(i) \tag{4}$$

and you don't need to worry about what a and b actually are in terms of λ and Δ.
Your derivation here is actually an example of a simple proof by induction.

Answer: Let's look at the pattern starting with $x(1)$, given that $x(i+1) = ax(i) + bu(i)$,

$$x(1) = ax(0) + bu(0)$$

$$x(2) = ax(1) + bu(1)$$

$$\implies x(2) = a(ax(0) + bu(0)) + bu(1) = a^2(x(0)) + b(u(0))a + bu(1)$$

$$x(3) = ax(2) + bu(2) = a(a^2(x(0)) + b(u(0))a + bu(1)) + bu(2)$$

$$\implies x(3) = a^3x(0) + b(u(0)a^2 + u(1)a + u(2))$$

So, given this pattern, if we guess,

$$x(i) = a^i x(0) + b(\sum_{j=0}^{i-1} u(j)a^{i-1-j}) \tag{5}$$

Then, let's see what we get for $x(i+1)$,

$$x(i+1) = ax(i) + bu(i) = a(a^i x(0) + b(\sum_{j=0}^{i-1} u(j)a^{i-1-j})) + bu(i)$$

$$\implies x(i+1) = a^{i+1}x(0) + b((\sum_{j=0}^{i-1} u(j)a^{i-j}) + u(i)) = a^{i+1}x(0) + b(\sum_{j=0}^{i} u(j)a^{i-j})$$

This satisfies (9), for i+1 and hence our guess was correct!

This turns out to be a proof by induction, with base case $x(1) = ax(0) + bu(0)$. Going from i to $(i+1)$ is the inductive step. This is how we transform a recursively found pattern into a rigorous proof!

(c) Suppose we have a system of differential equations with an input that we express in vector form:

$$\frac{d}{dt}\vec{x}_c(t) = A\vec{x}_c(t) + \vec{b}u(t) \tag{6}$$

where $\vec{x}_c(t)$ is n-dimensional.
Suppose further that the matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Collect the eigenvectors together into a matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$.
If we apply a piecewise constant control input $u(t)$ as in (2), and sample the system $\vec{x}(i) = \vec{x}_c(i)$, **what are the corresponding A_d and $\vec{b}_d$ in:**

$$\vec{x}(i+1) = A_d\vec{x}(i) + \vec{b}_d u(i). \tag{7}$$

Answer: First, we change coordinates so that $\vec{x}_c(t) = V\vec{\widetilde{x}}(t)$ and $\vec{\widetilde{x}}(t) = V^{-1}\vec{x}_c(t)$.
We have,

$$(\widetilde{x}(i+1))[j] = (e^{\lambda_j})(\widetilde{x}(i))[j] + (\frac{(e^{\lambda_j})-1}{\lambda_j})(V^{-1}\vec{b})[j](u(i))$$

$$\widetilde{x}(i+1) = (\begin{bmatrix} e^{\lambda_1} & 0 & \ldots \\ \vdots & \ddots & 0 \\ 0 & \ldots & e^{\lambda_n} \end{bmatrix})\widetilde{x}(i) + (\begin{bmatrix} \frac{e^{\lambda_1}-1}{\lambda_1} & 0 & \ldots \\ \vdots & \ddots & 0 \\ 0 & \ldots & \frac{e^{\lambda_n}-1}{\lambda_n} \end{bmatrix})V^{-1}\vec{b}u(i)$$

Now we define the following notations,

$$E_\Lambda = (\begin{bmatrix} e^{\lambda_1} & 0 & \ldots \\ \vdots & \ddots & 0 \\ 0 & \ldots & e^{\lambda_n} \end{bmatrix})$$

$$\Lambda^{-1} = (\begin{bmatrix} \frac{1}{\lambda_1} & 0 & \ldots \\ \vdots & \ddots & 0 \\ 0 & \ldots & \frac{1}{\lambda_n} \end{bmatrix})$$

So,

$$x(i+1) = V\widetilde{x}(i+1) = (VE_\Lambda V^{-1})x(i) + (V\Lambda^{-1}(E_\Lambda - I)V^{-1}\vec{b})u(i)$$

Hence,

$$A_d = (VE_\Lambda V^{-1})$$

and

$$\vec{b}_d = (V\Lambda^{-1}(E_\Lambda - I)V^{-1}\vec{b})$$

(d) Suppose that $\vec{x}(0) = \vec{x}_0$. **Unroll the implicit recursion you derived in the previous part to write $\vec{x}(i+1)$ as a sum that involves $\vec{x}_0$ and the $u(j)$ for $j = 0, 1, \ldots, i$.**
For this part, feel free to just consider the discrete-time system in a simpler form

$$\vec{x}(i+1) = A\vec{x}(i) + \vec{b}u(i) \tag{8}$$

and you don't need to worry about what A and $\vec{b}$ actually are in terms of the original parameters.

Answer: Let's look at the pattern starting with $\vec{x}(1)$, given that $\vec{x}(i+1) = A\vec{x}(i) + \vec{b}u(i)$,

$$\vec{x}(1) = A\vec{x}(0) + \vec{b}u(0)$$

$$\vec{x}(2) = A\vec{x}(1) + \vec{b}u(1)$$
$$\implies \vec{x}(2) = A(A\vec{x}(0) + \vec{b}u(0)) + \vec{b}u(1) = A^2(\vec{x}(0)) + (u(0))A\vec{b} + \vec{b}u(1)$$

$$\vec{x}(3) = A\vec{x}(2) + \vec{b}u(2) = A(A^2(\vec{x}(0)) + (u(0))A\vec{b} + \vec{b}u(1)) + \vec{b}u(2)$$
$$\implies \vec{x}(3) = A^3\vec{x}(0) + (u(0)A^2 + u(1)A + u(2))\vec{b}$$

So, given this pattern, if we guess,

$$\vec{x}(i) = A^i\vec{x}(0) + (\sum_{j=0}^{i-1} u(j)A^{i-1-j})\vec{b} \tag{9}$$

Then, let's see what we get for $\vec{x}(i+1)$,

$$\vec{x}(i+1) = A\vec{x}(i) + \vec{b}u(i) = A(A^i\vec{x}(0) + (\sum_{j=0}^{i-1} u(j)A^{i-1-j})\vec{b}) + \vec{b}u(i)$$
$$\implies \vec{x}(i+1) = A^{i+1}\vec{x}(0) + ((\sum_{j=0}^{i-1} u(j)A^{i-j}) + u(i))\vec{b} = A^{i+1}\vec{x}(0) + (\sum_{j=0}^{i} u(j)A^{i-j})\vec{b}$$

This satisfies (9), for i+1 and hence our guess was correct!

This turns out to be a proof by induction, with base case $\vec{x}(1) = A\vec{x}(0) + \vec{b}u(0)$. Going from i to $(i+1)$ is the inductive step. This is how we transform a recursively found pattern into a rigorous proof!

2. Controlling states by designing sequences of inputs

This is something that you saw in 16A in the Segway problem. In that problem, you were given a semi-realistic model for a segway. Here, we are just going to consider a the following peculiar matrices chosen for intuitive ease of understanding what is going on:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let's assume we have a *discrete-time* system that follows the following "difference equation."

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{b}u(t).$$

(a) We are given the initial condition $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let's say we want to achieve $\vec{x}(m) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ for some specific $m \geq 0$. We don't need to stay there, we just want to be in this state at that time. **What is the smallest m such that this is possible? What is our choice of sequence of inputs $u(i)$?**

Answer: To ease notation, let

$$x(n) = \begin{bmatrix} x(n)[1] \\ x(n)[2] \\ x(n)[3] \\ x(n)[4] \end{bmatrix}.$$

Note that

$$\vec{x}(1) = A\vec{x}(0) + \vec{b}u(0) = \begin{bmatrix} x(0)[2] \\ x(0)[3] \\ x(0)[4] \\ u(0) \end{bmatrix}.$$

and so we see that if $n \geq 4$,

$$\vec{x}(n) = \begin{bmatrix} u(n-4) \\ u(n-3) \\ u(n-2) \\ u(n-1) \end{bmatrix}.$$

Hence, the smallest m is equal to 4, with $u(i) = (1, 2, 3, 4, \ldots)$ where the remaining terms are not relevant.

(b) **What if we started from $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest m and what is our choice of $u(i)$?**

Answer: We see that

$$\vec{x}(1) = A\vec{x}(0) + \vec{b}u(0) = \begin{bmatrix} x(0)[2] \\ x(0)[3] \\ x(0)[4] \\ u(0)[4] \end{bmatrix}.$$

so we only need $m = 1$ and input $u(i) = (4, \ldots)$.

(c) **What if we started from $\vec{x}(0) = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$? What is the smallest m and what is our choice of $u(i)$?**

Answer: We would still need $m \geq 4$ to achieve this. Input $u(i)$ should be equal to $(1, 2, 3, 4 \ldots)$.

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