Definition 1 (Piling)**.** *Let $\Gamma(G)$ be a RAAG with canonical generators*[1] $v_1, ..., v_n$. *Given a word w in the generators a piling $\Pi(w)$ is a tuple of n strings in $\{0, +, -\}^*$ defined inductively as follows:*

- $\Pi(e)$ *is a tuple of empty strings.*
- *Given* $\Pi(w)$, $\Pi(wv_i)$ *is defined as follows:*
    1. *If the ith string is empty, ends in '0', or ends in '+', then $\Pi(wv_i)$ is obtained from $\Pi(w)$ by appending a '+' to the ith string, and appending a 0 to the jth string for all j such that v_j and v_i don't commute*
    2. *If the ith string ends in a '$-$', then this character is removed, as well as the final '0' on each jth string where v_j and v_i do not commute.* [2]
- *The piling $\Pi(wv_i^{-1})$ is obtained likewise, by swapping the roles of '+' and '$-$'.*

Definition 2. *Given an element $g \in \Gamma(G)$, the terminal clique term(g) is the set of vertices v_i in the graph G such that the ith string in the piling $\Pi(g)$ ends in a '+' or '$-$'. The initial clique init(g) is the terminal clique of g^{-1}.*

Given two pilings $\Pi(g)$ and $\Pi(h)$, the concatenation $\Pi(g)\Pi(h)$ is defined componentwise by concatenating the ith strings of each tuple for all $1 \leq i \leq n$.

Lemma 1. *Let g, h be words in the RAAG $\Gamma(G)$ such that term$(g) \cap$ init$(h) = \varnothing$. Then $\Pi(gh) = \Pi(g)\Pi(h)$.*

Proof. We induct on the length of h. First we observe that h can be represented by some reduced word in the generators, in the sense that it contains no subword of the form $v_i x v_i^{-1}$ where v_i commutes with all letters of x. Indeed, if such a subword appears, we can apply our relations to replace it by x. As each substitution strictly decreases the length of the word, we can eliminate all such occurences by finitely many moves.

Now write h in such a form and suppose that the word begins with v_i^k or v_i^{-k} where k is maximal (i.e. the word does not have v_i^{k+1} or $v_i^{-(k+1)}$ as a prefix). Without loss of generality, suppose it starts with v_i^k (the v_i^{-k} case follows from the same argument).

Let us consider the piling of $\Pi(gv_i^k)$ – we claim that $\Pi(gv_i^k) = \Pi(g)\Pi(v_i^k)$. Since h can be represented by a reduced word beginning with v_i, then the ith string in the piling $\Pi(h^{-1})$ ends in a '+' or '-', hence v_i is in the initial clique of h. Therefore v_i is not in the terminal clique of g. Then the ith string in $\Pi(g)$ is either empty or ends in a '0'. Hence the piling $\Pi(gv_i^k)$ is obtained from $\Pi(g)$ by appending k '+'s and appending k '0''s to all jth strings where v_j does not commute with v_i. This is precisely the concatenation $\Pi(g)\Pi(v_i^k)$.

Now let $g' = gv_i^k$ and let $h' = v_i^{-k}h$. As we chose k to be maximal, then v_i is not in the initial clique of h'. Indeed, as the only other occurence of $v_i^{\pm}$ in the chosen word for h'

[1] here $v_1, ..., v_n$ are the vertices of the graph G

[2] Since the ith string ends in a '$-$', then the jth string ends in a 0 if v_j and v_i don't commute.

occurs after some $v_j^{\pm}$ not commuting with v_i, then the corresponding word for $(h')^{-1}$ has no occurences of $v_i^{\mp}$ after $v_j^{\mp}$. Hence the ith string in $\Pi((h')^{-1})$ ends in a '0'.

Also observe that the terminal clique of g', which consists of all v_i such that $\Pi(g')$ the ith string ends in a '+' or '−', is contained in $term(g) \cup \{v_i\}$. Hence $term(g') \cap init(h') = \varnothing$. As $|h'|$ is strictly less than $|h|$, we know by our induction hypothesis that $\Pi(g')\Pi(h') = \Pi(g'h')$.

Since $|h'| < |h|$ and v_i is not in the initial clique of h', we can apply our induction hypothesis again to get $\Pi(v_i^k)\Pi(h') = \Pi(v_i^k h') = \Pi(h)$. Putting this together, we get

$$\begin{aligned}
\Pi(gh) &= \Pi(g'h') \\
&= \Pi(g')\Pi(h') \\
&= \Pi(gv_i^k)\Pi(h') \\
&= \Pi(g)\Pi(v_i^k)\Pi(h') \\
&= \Pi(g)\Pi(h).
\end{aligned}$$

□

1 Main argument

Let $d > 14$ and consider a RAAG $\Gamma(G)$ with connection graph a cycle with d vertices, labelled $a_1, ..., a_d$. That is to say, $\Gamma(G)$ is presented as

$$\langle a_1, ..., a_d | [a_i, a_{i+1}] = e \rangle$$

where the indices are taken modulo d. Let μ_S be the uniform measure on $\{a_1^{\pm}, ..., a_d^{\pm}\}$. Further suppose that ν is a measure with $\nu(e) = 0$. Consider a random walk $\mu_S * \nu$. In other words, our random walk is given by $Z_n = s_1 w_1 ... s_n w_n$.

We want to show that there exists some $\kappa > 0$ such that

$$\mathbb{P}(|Z_n| \leq \kappa n) \leq e^{-\kappa n},$$

where $|Z_n|$ denotes the word length of Z_n with respect to the symmetric generating set given above.

To prove this, we condition on the w_n's and keep the randomness coming from the s_n's. We find a working κ that is independent of our conditioning, so this will prove the claim.

Definition 3. *Say that a time k is pivotal with respect to n if $|Z_k|, |Z_k s_k|, |Z_k s_k w_k|$ is strictly increasing, and if the piling for Z_k is a prefix of the piling for Z_j for all $k \leq j \leq n$.*

Denote by P_n the set of pivotal times w.r.t n and let $A_n = |P_n|$. We have $|Z_n| \geq A_n$. We want to show that there exists some $\kappa > 0$ such that $\mathbb{P}(A_n \leq \kappa n) \leq e^{-\kappa n}$. To do this, we show that A_{n+1} stochastically dominates[3] $A_n + U_n$ where U_n's are i.i.d with positive expectation and exponential tail.

To discuss how the choices of s_i affect the number of pivotal times, we use the notion of a 'piling' for an element of a RAAG (taken from "'The conjugacy problem in Right-Angled Artin groups and their subgroups" by Crisp, Godelle, and West). We don't discuss them here, but we note that the authors prove that to each word in the RAAG there is a well-defined clique in the connection graph such that $|Z_{n-1} s_n| \leq |Z_{n-1}|$ only if s_n corresponds to a vertex in the clique. Likewise, for each word w_n, by considering s_n^{-1} and w_n^{-1}. we have that a unique well-defined clique such that $|w_n| \leq |s_n w_n|$ only if s_n corresponds to a vertex in this clique. We call these cliques the *terminal clique* of Z_n and the *initial clique* of w_n, respectively.

First consider the probability of adding another pivotal time. There are at most three ways for this to avoid adding another pivotal time:

1. If s_n corresponds to a vertex in the initial clique of w_n. For example, $s_n = a_1^{-1}$ and $w_n = a_1$.

2. If s_n corresponds to a vertex in the terminal clique of Z_{n-1}. For example, if $Z_{n-1} = a_1 a_2$ and $s_n = a_1^{-1}$.

[3] in the sense that for any i we have $\mathbb{P}(A_{n+1} \geq i) \geq \mathbb{P}(A_n + U \geq i)$

3. If s_n corresponds to a vertex which is adjacent to both the initial clique of Z_{n-1} and the terminal clique of w_n. For example, if $Z_{n-1} = a_2^2$, $s_n = a_1$, and $w_n = a_2^{-2}$.

As our graph is a cycle of length greater than 6, then each clique has size at most 2, and for any two cliques K_1, K_2 there is at most one vertex in $C_d \setminus (K_1, K_2)$ adjacent to both. Therefore there are at most 5 possible vertices in the connection graph for which we don't add another pivotal time.

As our measure is uniform on our symmetric generating set, this means that $\mathbb{P}(A_{n+1} = A_n + 1) \geq \frac{2d-2\cdot 5}{2d} = \frac{d-5}{d}$.

Now consider the probability of backtracking (i.e. the probability that $A_{n+1} = A_n - q$ for some $q > 0$).

First consider the probability that $A_{n+1} \leq A_n - 1$. This first requires that our choice of s_n does not cause the number of pivotal times to increase, with happens with probability at most $\frac{5}{d}$. Now let $k = \max\{P_{n-1}\}$ be the last pivotal time. For this time to no longer be pivotal, it requires that the vertex corresponding to s_k is now adjacent to the initial clique of $w_k ... s_n w_n$. As $|s_n| = 1$ and w_k, w_n are nontrivial, then this can occur with probability at most $\frac{4}{d-5}$. The 4 comes from requiring adjacency to the initial clique, and the $d-5$ comes from requiring that s_k was already pivotal to begin with. Therefore we lose 1 pivotal time with probability at most $\frac{5}{d} \cdot \frac{4}{d-5}$.

By the same argument, we have probability at most $\frac{5}{d}\left(\frac{4}{d-5}\right)^j$ of losing j pivotal times, for $j > 0$.

Therefore A_{n+1} stochastically dominates $A_n + U_n$ where $\mathbb{P}(U = 1) = \frac{d-5}{d}$ and $\mathbb{P}(U \leq -j) = \frac{5}{d}\left(\frac{4}{d-5}\right)^j$ for $j \geq 0$. This has expectation $\frac{(d-5)(d-14)}{d(d-9)}$, which is positive when $d > 14$. As the random variable U has exponential tail, we know by the large deviations bound that there exists some $\kappa > 0$ such that $\mathbb{P}(|Z_n| \leq \kappa n) \leq e^{-\kappa n}$.

As the random variables $U = U(d)$ are increasing with d, then we can choose $\kappa(14)$ as a universal bound.