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Course: M.Tech. (Course)
Subject: Numerical Solution of Differential Equations (DS 289)

Assignment No.: 3
SAP No.: 6000007645
S.R. No.: 05-01-00-10-42-22-1-21061

Problem ①

Consider the viscous Burger's equation $u_t + uu_x = \alpha u_{xx}$, where $u(x,t)$ is the velocity component along the x-direction and α is the kinematic viscosity. Solve for this 1D equation in a periodic domain of size 1.0 for the following cases.

Use $\Delta t = 0.0004$, $t_{end} = 0.075$ and

$u(x,0) = \sin(4\pi x) + \sin(6\pi x) + \sin(10\pi x)$

ⓐ Use Euler and first order upwind schemes to solve the equation with $\alpha = 0$ for the grid resolution of 64 and 1024. Compare the two results by plotting the solution for a few timesteps, and comment on the errors.

ⓑ Use Euler and second order central difference schemes to solve the equation with $\alpha = 0.001$ for a grid resolution of 1024. Compare with part ⓐ solution and comment on the nature of solution.

Answer ① ⓐ:

Burger's Equation:

$$u_t + uu_x = \alpha u_{xx} \quad — ①$$

$$\alpha = 0 \text{ (Given)} \quad — ②$$

So,

$$u_t + uu_x = 0 \quad — ③$$

Euler and first order upwind scheme:

$$\left(\frac{u_j^{n+1} - u_j^n}{\Delta t}\right) + u_j^n\left(\frac{u_{j+1}^n - u_j^n}{\Delta x}\right) = 0 \qquad (\text{if, } u_j^n < 0)$$

$$— ④$$

Also,

$$\left(\frac{u_j^{n+1} - u_j^n}{\Delta t}\right) + u_j^n\left(\frac{u_j^n - u_{j-1}^n}{\Delta x}\right) = 0 \quad (\text{if}, u_j^n \geq 0) \quad — ⑤$$

So, using ④,

$$\left(\frac{u_j^{n+1} - u_j^n}{\Delta t}\right) + u_j^n\left(\frac{u_{j+1}^n - u_j^n}{\Delta x}\right) = 0 \quad (\text{if}, u_j^n < 0)$$

So,

$$u_j^{n+1} - u_j^n + \left(\frac{\Delta t}{\Delta x}\right) u_j^n \left(u_{j+1}^n - u_j^n\right) = 0$$

Let $m = \left(\frac{\Delta t}{\Delta x}\right)$ — ⑥

So, $u_j^{n+1} - u_j^n + m u_j^n (u_{j+1}^n - u_j^n) = 0$

So, $u_j^{n+1} = u_j^n - m u_j^n (u_{j+1}^n - u_j^n)$

So, $u_j^{n+1} = u_j^n (1 - m(u_{j+1}^n - u_j^n))$ $(\text{if}, u_j^n < 0)$ — ⑦

Similarly, using ⑤,

$$\left(\frac{u_j^{n+1} - u_j^n}{\Delta t}\right) + u_j^n\left(\frac{u_j^n - u_{j-1}^n}{\Delta x}\right) = 0 \quad (\text{if}, u_j^n \geq 0)$$

So,

$$u_j^{n+1} - u_j^n + \left(\frac{\Delta t}{\Delta x}\right) u_j^n \left(u_j^n - u_{j-1}^n\right) = 0$$

So, $u_j^{n+1} - u_j^n + m\, u_j^n (u_j^n - u_{j-1}^n) = 0$

So, $u_j^{n+1} = u_j^n - m\, u_j^n (u_j^n - u_{j-1}^n)$

So, $u_j^{n+1} = u_j^n (1 - m(u_j^n - u_{j-1}^n))$ (if, $u_j^n \geq 0$) — (8)

Using (7) and (8), we have

$$u_j^{n+1} = u_j^n (1 - m(u_{j+1}^n - u_j^n)) \quad (\text{if, } u_j^n < 0)$$

and

$$u_j^{n+1} = u_j^n (1 - m(u_j^n - u_{j-1}^n)) \quad (\text{if, } u_j^n \geq 0)$$

[NOTE:- u_j^n = Value of u on jth grid point at nth time step.]

Modified Equation:

If $u_j^n < 0$

So, $u_j^{n+1} = u_j^n(1 - m(u_{j+1}^n - u_j^n))$

As $u_j^{n+1} = u_j^n + (\Delta t) \frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u}{\partial t^2}\Big|_j^n + \frac{(\Delta t)^3}{3!} \frac{\partial^3 u}{\partial t^3}\Big|_j^n + \cdots$

As, $u_{j+1}^n = u_j^n + (\Delta x) \frac{\partial u}{\partial x}\Big|_j^n + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2}\Big|_j^n + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3}\Big|_j^n + \cdots$

So, $u_j^{n+1} = u_j^n(1 - m(u_{j+1}^n - u_j^n))$

$$\Rightarrow u_j^n + (\Delta t) \frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u}{\partial t^2}\Big|_j^n + \frac{(\Delta t)^3}{3!} \frac{\partial^3 u}{\partial t^3}\Big|_j^n + \cdots$$

$$= u_j^n\left(1 - m\left(\Delta x \frac{\partial u}{\partial x}\Big|_j^n + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2}\Big|_j^n + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3}\Big|_j^n + \cdots\right)\right)$$

$$\Rightarrow (\Delta t) \frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u}{\partial t^2}\Big|_j^n + \frac{(\Delta t)^3}{3!} \frac{\partial^3 u}{\partial t^3}\Big|_j^n + \cdots$$

$$= -u_j^n(\Delta t)\left(\frac{\partial u}{\partial x}\Big|_j^n + \frac{(\Delta x)}{2} \frac{\partial^2 u}{\partial x^2}\Big|_j^n + \frac{(\Delta x)^2}{3!} \frac{\partial^3 u}{\partial x^3}\Big|_j^n + \cdots\right)$$

As all terms are about nth time step and jth grid point
So, we can drop them.

So, $(\Delta t) u_t + \frac{(\Delta t)^2}{2!} u_{tt} + \frac{(\Delta t)^3}{3!} u_{ttt} + \cdots$

$$= -u(\Delta t)\left(u_x + \frac{(\Delta x)}{2} u_{xx} + \frac{(\Delta x)^2}{3!} u_{xxx} + \cdots\right)$$

So, $u_t + \frac{(\Delta t)}{2} u_{tt} + \frac{(\Delta t)^3}{3!} u_{ttt} + \cdots$

$$= -u u_x + \left(\frac{\Delta x}{2}\right) u u_{xx} + \frac{(\Delta x)^2}{3!} u u_{xxx} + \cdots$$

So, $u_t + u u_x = \left(\frac{\Delta x}{2}\right) u u_{xx} + \left(\frac{\Delta t}{2}\right)\left(-u_{tt}\right) + H.O.T.$

As $u_t + u u_x = 0$

So, $u_t = -u u_x$

$$u_{tt} = -\frac{\partial}{\partial t}(u u_x)$$

$$= -u_t u_x - u\frac{\partial}{\partial x}(u_t)$$

$$= -(-u u_x) u_x - u\frac{\partial}{\partial x}(-u u_x)$$

$$= u(u_x)^2 + u(u_x)^2 + u^2 u_{xx}$$

So, $\underline{\underline{u_{tt} = 2u(u_x)^2 + u^2 u_{xx}}}$

So, $u_t + u u_x = \left(\frac{\Delta x}{2}\right) u u_{xx} - \left(\frac{\Delta t}{2}\right)\left(2u(u_x)^2 + u^2 u_{xx}\right)$
$+ H.O.T.$

So, $u_t + (u + u\,\Delta t\, u_x)\, u_x = \left(\frac{\Delta x}{2}\right) u u_{xx} - \frac{(\Delta t)^2 u^2}{2} u_{xx}$
$+ H.O.T.$

This is the <u>modified equation</u>.

So, the wave speed becomes $\left(u + (u)(\Delta t)(u_x)\right)$.

This is the modified wave speed.

As the truncation error has leading order term with even derivative in space. So, the errors will be primarily <u>Dissipative</u>.

Modified Equation :

If $u_j^n \geq 0$

So, $u_j^{n+1} = u_j^n\left(1 - m\left(u_j^n - u_{j-1}^n\right)\right)$

As, $u_j^{n+1} = u_j^n + (\Delta t)\frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}\Big|_j^n + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3}\Big|_j^n + \cdots$

As, $u_{j-1}^n = u_j^n - (\Delta x)\frac{\partial u}{\partial x}\Big|_j^n + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_j^n - \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_j^n + \cdots$

So, $u_j^{n+1} = u_j^n\left(1 - m\left(u_j^n - u_{j-1}^n\right)\right)$

So, $u_j^n + (\Delta t)\frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}\Big|_j^n + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3}\Big|_j^n + \cdots$

$= u_j^n\left(1 - m\left((\Delta x)\frac{\partial u}{\partial x}\Big|_j^n - \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_j^n + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_j^n - \cdots\right)\right)$

So, $(\Delta t)\frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}\Big|_j^n + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3}\Big|_j^n + \cdots$

$= -(\Delta t)u_j^n\left(\frac{\partial u}{\partial x}\Big|_j^n - \frac{(\Delta x)}{2}\frac{\partial^2 u}{\partial x^2}\Big|_j^n + \frac{(\Delta x)^2}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_j^n - \cdots\right)$

As all terms are about nth time step and jth gridpoint

So, we can drop them.

So, $(\Delta t)u_t + \frac{(\Delta t)^2}{2}u_{tt} + \frac{(\Delta t)^3}{3!}u_{ttt} + \cdots$

$= -(\Delta t)u\left(u_x - \frac{\Delta x}{2}u_{xx} + \frac{(\Delta x)^2}{3!}u_{xxx} - \cdots\right)$

So, $u_t + u u_x = -\frac{(\Delta t)}{2}u_{tt} + \left(\frac{\Delta x}{2}\right)u_{xx} + \text{H.O.T.}$

$= -\frac{\Delta t}{2}\left(2u(u_x)^2\right) - \frac{(\Delta t)}{2}u^2 u_{xx} + \left(\frac{\Delta x}{2}\right)u_{xx}$
$+ \text{H.O.T.}$

$= -u(\Delta t)(u_x)^2 - \left(\frac{\Delta t}{2}\right)u^2 u_{xx} + \left(\frac{\Delta x}{2}\right)u_{xx} + \text{H.O.T.}$

So, $u_t + (u + u\Delta t\, u_x)u_x = \left(-\frac{(\Delta t)}{2}u^2 + \left(\frac{\Delta x}{2}\right)\right)u_{xx} + \text{H.O.T.}$

So, the modified wave speed is $u + u(\Delta t)\,u_x$

As the ~~trun~~ truncation error has leading order term with even derivative in space. So the errors will be predominantly Dissipative

Post_processing

April 30, 2023

1 Question (1)

1.1 Consider the viscous Burgers' equation $u_t + uu_x = \alpha u_{xx}$, where u(x, t) is the velocity component along the x−direction, and is the kinematic viscosity. Solve this 1D equation in a periodic domain of size 1.0 for the following cases. Use $\Delta t = 0.0004$, $t_{end} = 0.075$, and $u(x, 0) = sin(4\pi x) + sin(6\pi x) + sin(10\pi x)$.

1.2 (a) Use Euler and first order upwind schemes to solve the equation with $\alpha = 0$ for a grid resolution of 64 and 1024. Compare the two results by plotting the solution for a few timesteps, and comment on the errors.

2 Answer (1)(a):

3 Import the necessary libraries

[1]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

4 Reading the name of the files generated

[2]:
```
Output_files = pd.read_csv("Output_file_names.csv",delimiter=",",header=None).
 ↪to_numpy()
Output_files = np.squeeze(Output_files)
Output_files = Output_files.tolist()
```

4.1 Output files generated

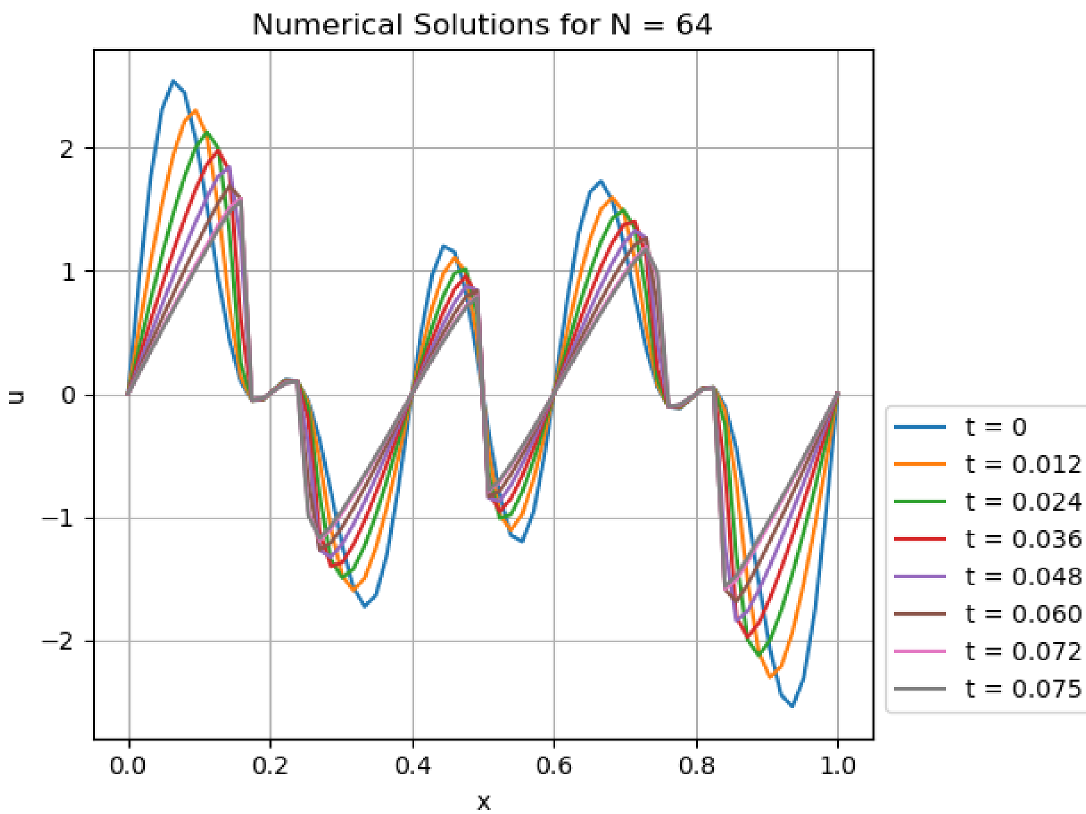
[3]:
```
Output_files
```

[3]:
```
['Question_1_a_u_n_t_0.000000_N_64_.csv',
 'Question_1_a_u_n_t_0.012000_N_64_.csv',
 'Question_1_a_u_n_t_0.024000_N_64_.csv',
 'Question_1_a_u_n_t_0.036000_N_64_.csv',
 'Question_1_a_u_n_t_0.048000_N_64_.csv',
```

```
 'Question_1_a_u_n_t_0.060000_N_64_.csv',
 'Question_1_a_u_n_t_0.072000_N_64_.csv',
 'Question_1_a_u_n_t_0.075000_N_64_.csv',
 'Question_1_a_u_n_t_0.000000_N_1024_.csv',
 'Question_1_a_u_n_t_0.012000_N_1024_.csv',
 'Question_1_a_u_n_t_0.024000_N_1024_.csv',
 'Question_1_a_u_n_t_0.036000_N_1024_.csv',
 'Question_1_a_u_n_t_0.048000_N_1024_.csv',
 'Question_1_a_u_n_t_0.060000_N_1024_.csv',
 'Question_1_a_u_n_t_0.072000_N_1024_.csv',
 'Question_1_a_u_n_t_0.075000_N_1024_.csv']
```

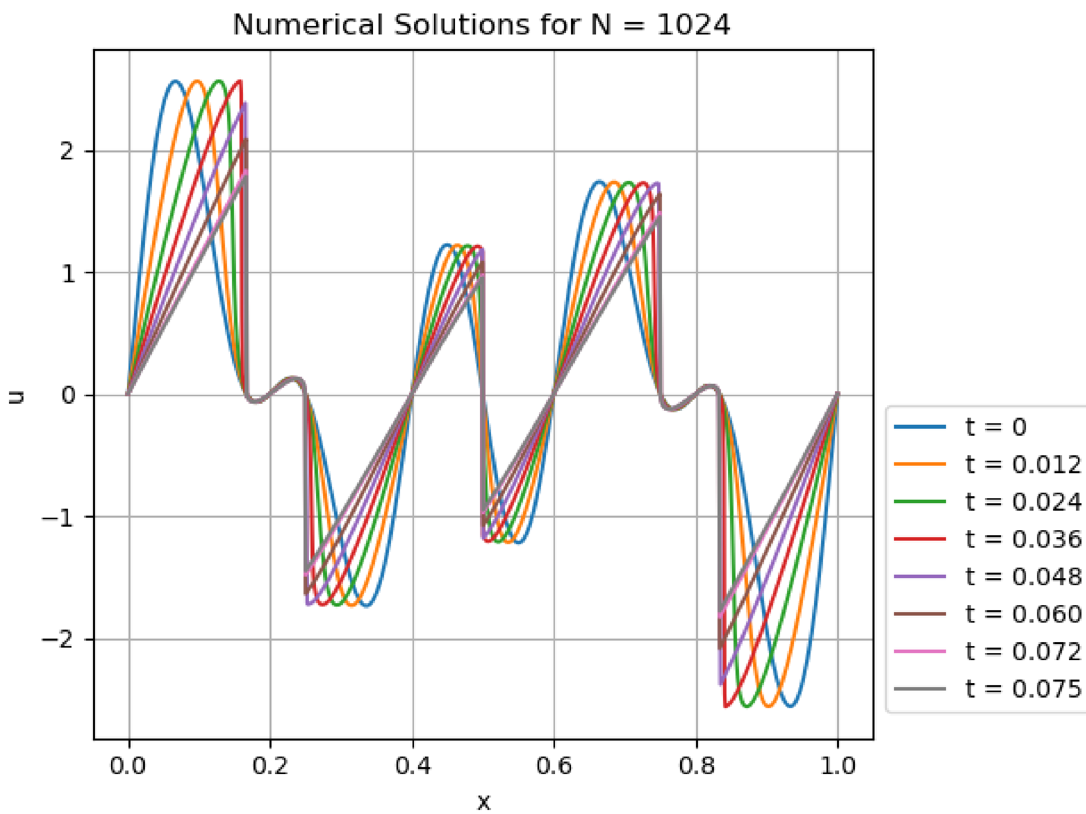
4.2 Numerical solution for grid resolution of 64

[4]:
```
for output_file in Output_files:
    if "_N_64_" in output_file:
        Numerical_Solution = pd.read_csv(output_file,delimiter=",",header=None).
 ↪to_numpy()
        x = np.linspace(0, 1,Numerical_Solution.shape[0])
        plt.plot(x,Numerical_Solution)
plt.xlabel("x")
plt.ylabel("u")
plt.legend(["t = 0","t = 0.012","t = 0.024","t = 0.036","t = 0.048","t = 0.
 ↪060","t = 0.072","t = 0.075"],bbox_to_anchor=(1,0.5))
plt.title("Numerical Solutions for N = 64")
plt.grid()
plt.tight_layout()
plt.savefig("Question_1_a_N_64.png",dpi = 500)
plt.show()
```



4.3 Numerical solution for grid resolution of 1024

[5]:
```
for output_file in Output_files:
    if "_N_1024_" in output_file:
        Numerical_Solution = pd.read_csv(output_file,delimiter=",",header=None).
 ↪to_numpy()
        x = np.linspace(0, 1,Numerical_Solution.shape[0])
        plt.plot(x,Numerical_Solution)
plt.xlabel("x")
plt.ylabel("u")
plt.legend(["t = 0","t = 0.012","t = 0.024","t = 0.036","t = 0.048","t = 0.
 ↪060","t = 0.072","t = 0.075"],bbox_to_anchor=(1,0.5))
plt.title("Numerical Solutions for N = 1024")
plt.grid()
plt.tight_layout()
plt.savefig("Question_1_a_N_1024.png",dpi = 500)
plt.show()
```



4.4 Comment on the errors:

4.4.1 1. From the above graph it is apparent that the error is dissipative in nature.

4.4.2 2. In case of grid resolution of 64 the dissipation is more than that of grid resolution of 1024.

4.4.3 3. We are also getting the same result about the nature of the error using the modified equation.

Answer 1 b:

Burger's Equation:

$$u_t + u u_x = \alpha u_{xx}$$

$\alpha = 0.001$ (Given)

Euler and Second Order Central Difference Scheme;

$$\left(\frac{u_j^{n+1} - u_j^n}{\Delta t}\right) + u_j^n\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right)$$

$$= \alpha\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}\right)$$

So,

$$(u_j^{n+1} - u_j^n) + \left(\frac{\Delta t}{\Delta x}\right) u_j^n (u_{j+1}^n - u_{j-1}^n)\left(\frac{1}{2}\right)$$

$$= \alpha\left(\frac{\Delta t}{(\Delta x)^2}\right)(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

Let $m_1 = \left(\frac{\Delta t}{\Delta x}\right)\left(\frac{1}{2}\right) = \left(\frac{\Delta t}{2\Delta x}\right)$

$P = \left(\frac{\Delta t}{(\Delta x)^2}\right)(\alpha)$

So,

$$u_j^{n+1} - u_j^n + m_1 u_j^n (u_{j+1}^n - u_{j-1}^n) = P(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

So,

$$u_j^{n+1} = u_j^n - m_1 u_j^n (u_{j+1}^n - u_{j-1}^n) + P(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

<u>Modified Equation:</u>

$$u_j^{n+1} = u_j^n + (\Delta t) u_t\big|_j^n + \frac{(\Delta t)^2}{2!} u_{tt}\big|_j^n + \frac{(\Delta t)^3}{3!} u_{ttt}\big|_j^n + \cdots$$

$$u_{j+1}^n = u_j^n + (\Delta x) u_x\big|_j^n + \frac{(\Delta x)^2}{2!} u_{xx}\big|_j^n + \frac{(\Delta x)^3}{3!} u_{xxx}\big|_j^n + \cdots$$

$$u_{j-1}^n = u_j^n - \Delta x\, u_x\big|_j^n + \frac{(\Delta x)^2}{2!} u_{xx}\big|_j^n - \frac{(\Delta x)^3}{3!} u_{xxx}\big|_j^n + \cdots$$

So,

$$u_j^{n+1} = u_j^n - m_1 u_j^n (u_{j+1}^n - u_{j-1}^n) + P(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

So,

$$u_j^n + (\Delta t) u_t\big|_j^n + \frac{(\Delta t)^2}{2!} u_{tt}\big|_j^n + \frac{(\Delta t)^3}{3!} u_{ttt}\big|_j^n + \cdots$$
$$= u_j^n - m_1 u_j^n \left(2\Delta x\, u_x\big|_j^n + \frac{2(\Delta x)^3}{3!} u_{xxx}\big|_j^n + \cdots\right) + P\left((\Delta x)^2 u_{xx}\big|_j^n + \frac{(\Delta x)^4}{4!} 2\, u_{xxxx}\big|_j^n + \cdots\right)$$

As all terms are about time step $= n$ and grid point $= j$
So, we can drop them.

So,

$$(\Delta t) u_t + \frac{(\Delta t)^2}{2!} u_{tt} + \frac{(\Delta t)^3}{3!} u_{ttt} = -m_1 u \left(2\Delta x\, u_x + \frac{(\Delta x)^3 u_{xxx}}{3}\right)$$

~~$-m_1 u_j^n (2\Delta$~~

$$+ \frac{P(\Delta x)^2}{(\Delta x)^2}\left((\Delta x)^2 u_{xx} + \frac{(\Delta x)^4 u_{xxxx}}{12}\right)$$
$$+ H.O.T.$$

So,

$$u_t + \frac{(\Delta t)}{2} u_{tt} + \frac{(\Delta t)^2}{6} u_{ttt} = -u u_x - \frac{(\Delta x)^2}{6} u\, u_{xxx}$$
$$+ \frac{\alpha \Delta t}{\Delta t}\left((\Delta x)^2 \frac{u_{xx}}{(\Delta x)^2} + \frac{(\Delta x)^4}{12}\frac{u_{xxxx}}{(\Delta x)^2}\right)$$
$$+ H.O.T.$$

So, $u_t + uu_x = \alpha u_{xx} - \frac{(\Delta x)^2}{6} u u_{xxx} + \frac{\alpha(\Delta x)^2}{12} u_{xxxx}$

$$- \frac{(\Delta t)}{2} u_{tt} - \frac{(\Delta t)^2}{6} u_{ttt} + H.O.T.$$

As $u_t + uu_x = \alpha u_{xx}$

So, $u_t = \alpha u_{xx} - uu_x$

So, $u_{tt} = \alpha \frac{\partial^2 u_t}{\partial x^2} - \frac{\partial (u u_x)}{\partial t}$

$$= \alpha \frac{\partial^2}{\partial x^2}(\alpha u_{xx} - uu_x) - u_t u_x - u \frac{\partial u_t}{\partial x}$$

$$= \alpha^2 u_{xxxx} - \alpha u_{xx} u_x - \alpha u u_{xxx}$$
$$- (\alpha u_{xx} - uu_x) u_x - u(\alpha u_{xxx} - u u_{xx} - (u_x)^2)$$

$$= \alpha^2 u_{xxxx} - \alpha u_{xx} u_x - \alpha u u_{xxx} - \alpha u_x u_{xx}$$
$$+ u(u_x)^2 - \alpha u u_{xxx} + u^2 u_{xx} + u(u_x)^2$$

So, $u_{tt} = \alpha^2 u_{xxxx} - 2\alpha u_x u_{xx} - 2\alpha u u_{xxx}$
$$+ 2u(u_x)^2 + u^2 u_{xx}$$

So, $u_t + uu_x = \alpha u_{xx} - \frac{(\Delta x)^2}{6} u u_{xxx} + \frac{\alpha (\Delta x)^{12}}{12} u_{xxxx}$

$$- \frac{(\Delta t)}{2}(\alpha^2 u_{xxxx} - 2\alpha u_x u_{xx} - 2\alpha u u_{xxx}$$
$$+ 2u u_x^2 + u^2 u_{xx})$$
$$\underline{+ H.O.T.}$$

As we have both even and odd order derivative in space in the truncation error. So, the error will be <u>both dispersive and dissipative</u>.

Post_processing

April 30, 2023

1 Question (1)

1.1 Consider the viscous Burgers' equation $u_t + uu_x = \alpha u_{xx}$, where u(x, t) is the velocity component along the x−direction, and is the kinematic viscosity. Solve this 1D equation in a periodic domain of size 1.0 for the following cases. Use $\Delta t = 0.0004$, $t_{end} = 0.075$, and $u(x, 0) = sin(4\pi x) + sin(6\pi x) + sin(10\pi x)$.

1.2 (b) Use Euler and second order central difference schemes to solve the equation with $\alpha = 0.001$ for a grid resolution of 1024. Compare with part (a) solution and comment on the nature of solution.

2 Answer (1)(b):

3 Import the necessary libraries

[1]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

4 Reading the name of the files generated

[2]:
```
Output_files = pd.read_csv("Output_file_names.csv",delimiter=",",header=None).
 ↪to_numpy()
Output_files = np.squeeze(Output_files)
Output_files = Output_files.tolist()
```

4.1 Output files generated

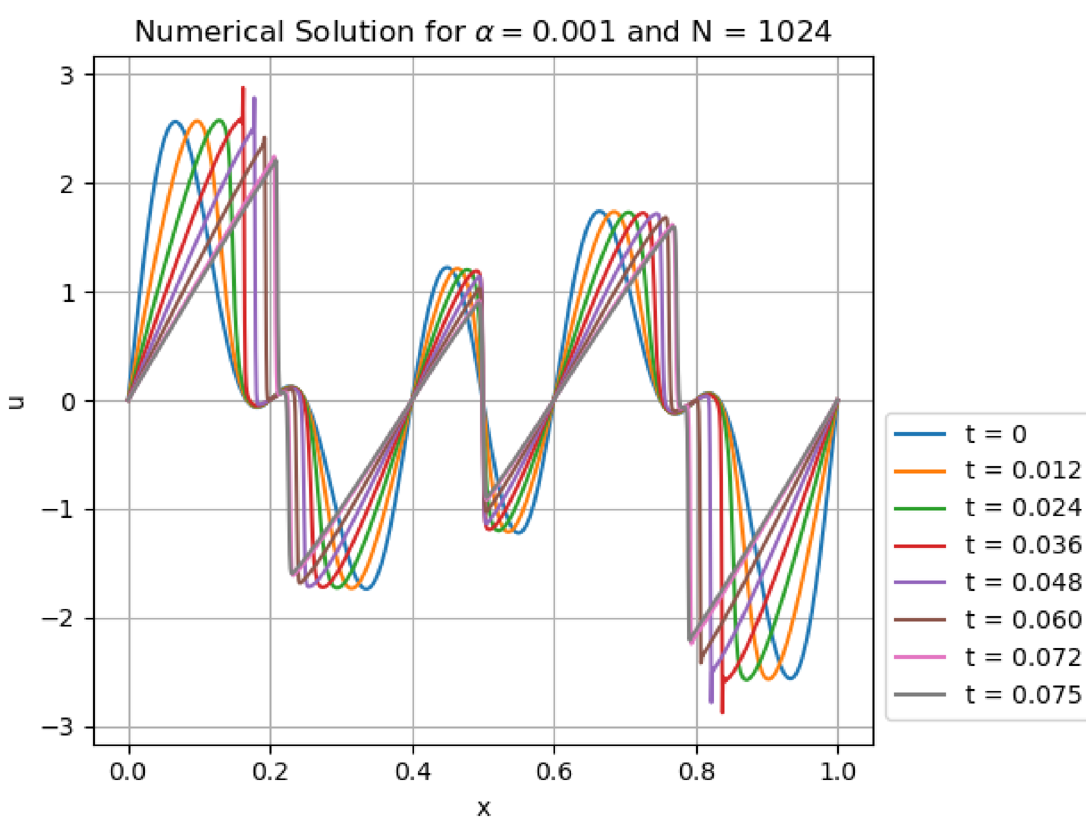
[3]:
```
Output_files
```

[3]:
```
['Question_1_b_u_n_t_0.000000_N_1024_.csv',
 'Question_1_b_u_n_t_0.012000_N_1024_.csv',
 'Question_1_b_u_n_t_0.024000_N_1024_.csv',
 'Question_1_b_u_n_t_0.036000_N_1024_.csv',
 'Question_1_b_u_n_t_0.048000_N_1024_.csv',
```

```
'Question_1_b_u_n_t_0.060000_N_1024_.csv',
'Question_1_b_u_n_t_0.072000_N_1024_.csv',
'Question_1_b_u_n_t_0.075000_N_1024_.csv']
```

4.2 Numerical solution for $\alpha = 0.001$ and grid resolution of 1024

[4]:
```
for output_file in Output_files:
    Numerical_Solution = pd.read_csv(output_file,delimiter=",",header=None).
 ↪to_numpy()
    x = np.linspace(0, 1,Numerical_Solution.shape[0])
    plt.plot(x,Numerical_Solution)
plt.xlabel("x")
plt.ylabel("u")
plt.legend(["t = 0","t = 0.012","t = 0.024","t = 0.036","t = 0.048","t = 0.
 ↪060","t = 0.072","t = 0.075"],bbox_to_anchor=(1,0.5))
plt.title(r"Numerical Solution for $\alpha = 0.001$ and N = 1024")
plt.grid()
plt.tight_layout()
plt.savefig("Question_1_b.png",dpi = 500)
plt.show()
```



4.3 Comment on the errors:

4.3.1 1. From the above graph it is apparent that the error in this case is both dissipative and dispersive in nature. In part (a) solution the nature of the error was dissipative and NOT dispersive.

4.3.2 2. We are also getting the same result about the nature of the error using the modified equation.

4.3.3 3. We can observe small kinks in the solution (associated with t = 0.036, 0.048 and 0.060), which may be because the absolute value of negative viscosity term becoming higher, resulting in negative damping. Interestingly, the solution does not blow up, indicating that the negative viscosity is not very high and the overall viscosity remains mostly positive.

Problem ②:

Consider the linear wave equation $u_t + c u_x = 0$. Using leap frog method for time derivative and second order central difference for space derivative, obtain the stability condition. Derive the modified equation and comment on the dispersive and dissipative errors.

Answer ②:

Linear Wave Equation:

$$u_t + c u_x = 0 \quad — ①$$

Leap Frog for time derivative and second order central difference for space derivative.

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + c\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) = 0 \quad — ②$$

So, $u_j^{n+1} - u_j^{n-1} + c\left(\frac{\Delta t}{\Delta x}\right)\left(u_{j+1}^n - u_{j-1}^n\right) = 0$

Let $r = \left(\frac{c\Delta t}{\Delta x}\right) \quad — ③$

So, $u_j^{n+1} - u_j^{n-1} + r\left(u_{j+1}^n - u_{j-1}^n\right) = 0$

So, $u_j^{n+1} = u_j^{n-1} - r\left(u_{j+1}^n - u_{j-1}^n\right) \quad — ④$

von Neumann Stability analysis:

Let $u_j^n \sim g^n e^{ij\theta}$, $i = \sqrt{-1}$

$\theta = K\Delta x$

$x_j = j\Delta x$

So, $u_j^n \sim g^n e^{ij\theta} = g^n e^{ijK\Delta x}$

So, $u_j^n \sim g^n e^{ij\theta} = g^n e^{iKx_j}$

Similarly, $u_j^{n+1} \sim g^{n+1} e^{ij\theta}$

$u_j^{n-1} \sim g^{n-1} e^{ij\theta}$

$u_{j+1}^n \sim g^n e^{i(j+1)\theta}$

$u_{j-1}^n \sim g^n e^{i(j-1)\theta}$

So, $u_j^{n+1} = u_j^{n-1} - r(u_{j+1}^n - u_{j-1}^n)$

So, $g^{n+1} e^{ij\theta} = g^{n-1} e^{ij\theta} - r(g^n e^{i(j+1)\theta} - g^n e^{i(j-1)\theta})$

So, $g = g^{-1} - r(e^{i\theta} - e^{-i\theta})$

(Dividing both sides by $g^n e^{ij\theta}$)

So, $g = g^{-1} - r(\cos\theta + i\sin\theta - (\cos(-\theta) + i\sin(-\theta)))$

So, $g = g^{-1} - r(\cos\theta + i\sin\theta - \cos\theta + i\sin\theta)$

So, $g = g^{-1} - r(2i\sin\theta)$

So, $g = \frac{1}{g} - r(2i\sin\theta)$

So, $g^2 = 1 + g(-2ri\sin\theta)$

So, $g^2 + g(2r\sin\theta)i - 1 = 0$

So,

$$g = \frac{-(2r\sin\theta)i \pm \sqrt{(i2r\sin\theta)^2 - 4(1)(-1)}}{2(1)}$$

So,

$$g = \frac{(-2r\sin\theta)i \pm \sqrt{-4r^2\sin^2\theta + 4}}{2}$$

So,

$$g = (-r\sin\theta)i \pm \sqrt{1 - r^2\sin^2\theta}$$

If the scheme is to be stable, then $|g| \leq 1$

Case ①:

If $1 - r^2\sin^2\theta > 0$

So, $|g| = \sqrt{(-r\sin\theta)^2 + (1 - r^2\sin^2\theta)^{\frac{1}{2}\times 2}}$

$$\cancel{= \sqrt{+r^2\sin^2\theta + 1}}$$

So, $|g| = \sqrt{r^2\sin^2\theta + 1 - r^2\sin^2\theta}$

So, $\underline{\underline{|g| = 1}}$

<u>In this case the given scheme is Stable</u>

Case ②:

If $1 - r^2\sin^2\theta < 0$

So, $g = \left(-r\sin\theta \pm \sqrt{r^2\sin^2\theta - 1}\right)i$

Consider, $g = i\left(-r\sin\theta - \sqrt{r^2\sin^2\theta - 1}\right)$

So, $|g| = r\sin\theta + \sqrt{r^2\sin^2\theta - 1}$

As $r = \frac{c\Delta t}{\Delta x} > 0$

We need $|g| \leq 1$

So, $r\sin\theta + \sqrt{r^2\sin^2\theta - 1} \leq 1$

So, $\sqrt{r^2\sin^2\theta - 1} \leq 1 - r\sin\theta$

So, $r^2\sin^2\theta - 1 \leq (1 - r\sin\theta)^2$

So, $r^2\sin^2\theta - 1 \leq 1 + r^2\sin^2\theta - 2r\sin\theta$

So, $-1 \leq 1 - 2r\sin\theta$

So, $2r\sin\theta \leq 2$

So, $r\sin\theta \leq 1$

This is true for any θ.

We are taking the maximum value of $\sin\theta$

$(\sin\theta = 1)$

So, $r \leq 1$

So, $r = \frac{c\Delta t}{\Delta x} \leq 1$

So, $\underline{\underline{\Delta t \leq \left(\frac{\Delta x}{c}\right)}}$

Consider, $g = \left(-r\sin\theta + \sqrt{r^2\sin^2\theta - 1}\right) i$

So, $|g|^2 = \left(-r\sin\theta + \sqrt{r^2\sin^2\theta - 1}\right)^2$

We need $|g| \leq 1$

So, $|g|^2 \leq 1$

So, $\left(-r\sin\theta + \sqrt{r^2\sin^2\theta - 1}\right)^2 \leq 1$

So, $r^2\sin^2\theta + r^2\sin^2\theta - 1 - 2r\sin\theta\sqrt{r^2\sin^2\theta - 1} \leq 1$

So, $2r^2\sin^2\theta - 2r\sin\theta\sqrt{r^2\sin^2\theta - 1} \leq 2$

So, $r^2\sin^2\theta - r\sin\theta\sqrt{r^2\sin^2\theta - 1} \leq 1$

So, $-r\sin\theta\sqrt{r^2\sin^2\theta - 1} \leq 1 - r^2\sin^2\theta$

So, $(r^2\sin^2\theta)(r^2\sin^2\theta - 1) \leq (1 - r^2\sin^2\theta)^2$

So, $r^4\sin^4\theta - r^2\sin^2\theta \leq 1 + r^4\sin^4\theta - 2r^2\sin^2\theta$

So, $-r^2\sin^2\theta \leq 1 - 2r^2\sin^2\theta$

So, $r^2\sin^2\theta \leq 1$

This is true for any value of θ.

So, we are taking maximum value of $\sin\theta$.

So, $r^2 \leq 1$

As $r > 0$

So, $r \leq 1$

So, $\frac{c\Delta t}{\Delta x} \leq 1$

So, $\Delta t \leq \left(\frac{\Delta x}{c}\right)$

This is the same condition as before.

So, the stability condition is $(\Delta t) \leq \left(\frac{\Delta x}{c}\right)$

Modified Equation :-

$$u_j^{n+1} = u_j^{n-1} - r\left(u_{j+1}^n - u_{j-1}^n\right)$$

So, $u_j^{n+1} = u_j^n + (\Delta t)\left.\frac{\partial u}{\partial t}\right|_j^n + \frac{(\Delta t)^2}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_j^n + \cdots$

So,

$$u_j^{n-1} = u_j^n - (\Delta t)\left.\frac{\partial u}{\partial t}\right|_j^n + \frac{(\Delta t)^2}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_j^n - \cdots$$

So,

$$u_{j+1}^n = u_j^n + (\Delta x)\left.\frac{\partial u}{\partial x}\right|_j^n + \frac{(\Delta x)^2}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_j^n + \cdots$$

So,

$$u_{j-1}^n = u_j^n - (\Delta x)\left.\frac{\partial u}{\partial x}\right|_j^n + \frac{(\Delta x)^2}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_j^n - \cdots$$

So, using

$$u_j^{n+1} = u_j^{n-1} - r\left(u_{j+1}^n - u_{j-1}^n\right)$$

So, $$\left(u_j^n + (\Delta t)\frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}\Big|_j^n + \cdots\right)$$

$$= \left(u_j^n - (\Delta t)\frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}\Big|_j^n - \cdots\right)$$

$$- r\Bigg(\left(u_j^n + (\Delta x)\frac{\partial u}{\partial x}\Big|_j^n + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_j^n + \cdots\right)$$

$$- \left(u_j^n + (-\Delta x)\frac{\partial u}{\partial x}\Big|_j^n + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_j^n - \cdots\right)\Bigg)$$

So, $$2\left((\Delta t)\frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3}\Big|_j^n + \cdots\right)$$

$$= -r(2)\left(\Delta x\frac{\partial u}{\partial x}\Big|_j^n + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_j^n + \cdots\right)$$

So, $$(\Delta t)\left(\frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^2}{3!}\frac{\partial^3 u}{\partial t^3}\Big|_j^n + \cdots\right)$$

$$= -\left(\frac{c\Delta t}{\Delta x}\right)(\Delta x)\left(\frac{\partial u}{\partial x}\Big|_j^n + \frac{(\Delta x)^2}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_j^n + \cdots\right)$$

So, $$\left(\frac{\partial u}{\partial t}\Big|_j^n + \frac{(\Delta t)^2}{3!}\frac{\partial^3 u}{\partial t^3}\Big|_j^n + \cdots\right)$$

$$= -c\left(\frac{\partial u}{\partial x}\Big|_j^n + \frac{(\Delta x)^2}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_j^n + \cdots\right)$$

As all the terms are about j and n.
So, we are dropping it without loss of generallity.

So,

$$\frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{3!}\frac{\partial^3 u}{\partial t^3} + \cdots$$

$$= -c\left(\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots\right)$$

So, $$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\left(\frac{(\Delta t)^2}{3!}\frac{\partial^3 u}{\partial t^3} + \cdots\right) - c\left(\frac{(\Delta x)^2}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots\right)$$

So, $$u_t + c u_x = -\frac{(\Delta t)^2}{3!} u_{ttt} - \frac{c(\Delta x)^2}{3!} u_{xxx} + \text{H.O.T.}$$

As $u_t + c u_x = 0$ (Given Equation)

So, $u_t = -c u_x$

So, $$u_{tt} = -c u_{xt} = -c\frac{\partial}{\partial x}(u_t) = c^2 \frac{\partial u_x}{\partial x}$$

So, $u_{tt} = c^2 u_{xx}$

So, $u_{ttt} = c^2 \frac{\partial^2}{\partial x^2}(u_t)$

So, $u_{ttt} = -c^3 u_{xxx}$

So, $U_t + cU_x = -\frac{(\Delta t)^2}{3!}\left(-c^3 U_{xxx}\right) - c\frac{(\Delta x)^2}{3!} U_{xxx}$
$+ H.O.T.$

So, $U_t + cU_x = \left(\frac{c^3(\Delta t)^2}{3!} - \frac{c(\Delta x)^2}{3!}\right) U_{xxx} + H.O.T.$

This is the <u>modified equation</u>.

As the leading order term has odd order derivative in space (in truncation Error)
So, the <u>nature of the error</u> is <u>Dispersive</u>.

As all the terms on the Truncation Error has only odd order derivative in space. So, the nature of the error is <u>Dispersive</u> and <u>NOT Dissipative</u>.

[NOTE: All odd order derivative in time can be converted to odd order derivative in space]

Problem ③:

Construct the weak form for the following equations

ⓐ Beam on elastic foundation:

$$\frac{d^2}{dx^2}\left(b\,\frac{d^2w}{dx^2}\right) + Kw = f \quad \text{for } 0 < x < L$$

$$w = b\,\frac{d^2w}{dx^2} = 0 \text{ at } x = 0, L$$

where $b = EI$ and f are functions of x and K is a constant.

ⓑ A non-linear equation:

$$-\frac{d}{dx}\left(u\,\frac{du}{dx}\right) + f = 0 \quad \text{for } 0 < x < 1$$

$$\left.\frac{du}{dx}\right|_{x=0} = 0 \qquad u(1) = \sqrt{2}$$

Answer ③ ⓐ:

$$\frac{d^2}{dx^2}\left(b\,\frac{d^2w}{dx^2}\right) + Kw = f \quad , \quad 0 < x < L$$

Let $I = [0, L]$

$w = 0$ at $x = 0$, $x = L$

$b\,\frac{d^2w}{dx^2} = 0$ at $x = 0$, $x = L$

$\mathcal{Q}$ = Set of the test functions

$$\mathcal{Q} = \{q : q \in H^2(I) \mid q(0) = q(L) = q''(0) = q''(L) = 0\}$$

Here $H^2(I)$ is Sobolov Space

q is twice weakly differentiable.

So, $$\int_0^L q\left[\frac{d^2}{dx^2}\left(b\,\frac{d^2 w}{dx^2}\right) + Kw\right] dx = \int_0^L q f\, dx$$

So, $$\int_0^L q\,\frac{d^2}{dx^2}\left(b\,\frac{d^2 w}{dx^2}\right) dx + \int_0^L q K w\, dx = \int_0^L q f\, dx$$

So, $$q\,\frac{d}{dx}\left(b\,\frac{d^2 w}{dx^2}\right)\Bigg|_{x=0}^{L} - \int_0^L \frac{dq}{dx}\,\frac{d}{dx}\left(b\,\frac{d^2 w}{dx^2}\right) dx$$
$$+ \int_0^L q K w\, dx = \int_0^L q f\, dx$$

As $q(0) = q(L) = 0$

So,
$$-\int_0^L \frac{dq}{dx}\,\frac{d}{dx}\left(b\,\frac{d^2 w}{dx^2}\right) dx + \int_0^L q K w\, dx = \int_0^L q f\, dx$$

So, $-\frac{dq}{dx}\left(b\frac{d^2w}{dx^2}\right)\Big|_{x=0}^{L} + \int_0^L \frac{d^2q}{dx^2} b \frac{d^2w}{dx^2}\,dx + \int_0^L qKw\,dx$

$= \int_0^L qf\,dx$

As $b\frac{d^2w}{dx^2} = 0$ at $x=0$ and $x=L$

So, $\int_0^L \frac{d^2q}{dx^2} b \frac{d^2w}{dx^2}\,dx + \int_0^L qKw\,dx = \int_0^L qf\,dx$

So, $\int_0^L b\frac{d^2q}{dx^2}\frac{d^2w}{dx^2}\,dx + \int_0^L qKw\,dx = \int_0^L qf\,dx$

So, $\int_0^L bq''w''\,dx + \int_0^L Kqw\,dx = \int_0^L qf\,dx$

So, $\int_0^L (bq''w'' + Kqw)\,dx = \int_0^L qf\,dx$

Weak Form:-

Find $w \in \mathcal{Q}$ such that

$\int_0^L (bq''w'' + Kqw)\,dx = \int_0^L qf\,dx$ for all $q \in \mathcal{Q}$

Answer ③ ⓑ :

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right)+f=0 \qquad \text{for } 0<x<1$$

$$\left.\frac{du}{dx}\right|_{x=0}=0$$

$$u(1)=\sqrt{2}$$

Let $q \in \mathcal{Q}$ where

$$\mathcal{Q}=\{q : q\in H^1(I) \mid q'|_0=0\}$$

So, $\displaystyle\int_0^1 q\left(-\frac{d}{dx}\left(u\frac{du}{dx}\right)+f\right)dx=0$ for all $q\in\mathcal{Q}$

So, $\displaystyle -\int_0^1 q\frac{d}{dx}\left(u\frac{du}{dx}\right)dx+\int_0^1 qf\,dx=0$

So, $\displaystyle -qu\frac{du}{dx}\Big|_{x=0}^{1}+\int_0^1\frac{dq}{dx}u\frac{du}{dx}dx+\int_0^1 qf\,dx=0$

So, $\displaystyle -q(1)u(1)\left.\frac{du}{dx}\right|_{x=1}+q(0)u(0)\left.\frac{du}{dx}\right|_{x=0}$

$$+\int_0^1 u\frac{dq}{dx}\frac{du}{dx}dx+\int_0^1 qf\,dx=0$$

So, $\left(-q(1)\right)(\sqrt{2})\left.\frac{du}{dx}\right|_{x=1} + 0 + \int_0^1 \frac{dq}{dx}\, u \frac{du}{dx}\, dx + \int_0^1 q f\, dx = 0$

So, $\int_0^1 u \frac{dq}{dx}\frac{du}{dx}\, dx + \int_0^1 q f\, dx = (\sqrt{2})(q(1))\left.\frac{du}{dx}\right|_{x=1}$

for all $q \subset Q$

~~Weak~~

<u>Weak Form:-</u>

Find $u \in Q$

such that

$\int_0^1 u\, q' u'\, dx + \int_0^1 q f\, dx = (\sqrt{2})(q(1))\left.\frac{du}{dx}\right|_{x=1}$

for all $q \in Q$

Answer 3 b:

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right)+f=0 \quad \text{— (1)} \qquad , \quad 0<x<1, \quad I=(0,1)$$

$$\left.\frac{du}{dx}\right|_{x=0}=0$$

$$u(1)=\sqrt{2}$$

Let $w=u-\sqrt{2}$

So, $\left.\frac{dw}{dx}\right|_{x=0}=\left.\frac{du}{dx}\right|_{x=0}=0$

So, $w(1)=u(1)-\sqrt{2}=\sqrt{2}-\sqrt{2}=0$

So, $w(1)=0$

Substituting $u=w+\sqrt{2}$ in (1)

$$-\frac{d}{dx}\left((w+\sqrt{2})\frac{d}{dx}(w+\sqrt{2})\right)+f=0$$

So, $-\frac{d}{dx}\left((w+\sqrt{2})\frac{dw}{dx}\right)+f=0$

So, $-\frac{d}{dx}\left(w\frac{dw}{dx}\right)-\sqrt{2}\,\frac{d^2w}{dx^2}+f=0$

So, $\left.\frac{dw}{dx}\right|_{x=0}=0$ and $w(1)=0$

So, now we have homogeneous Boundary Condition

Let $v \in V = \{v \in H'(I) \mid v(1)=0,\ v'(0)=0\}$

So,

$$-\int_0^1 v \frac{d}{dx}\left(w \frac{dw}{dx}\right) dx - \sqrt{2}\int_0^1 v \frac{d^2 w}{dx^2} dx + \int_0^1 v f\, dx = 0$$

So,

$$-vw\frac{dw}{dx}\Big|_{x=0}^{1} + \int_0^1 \frac{dv}{dx} w \frac{dw}{dx} dx - v\sqrt{2}\frac{dw}{dx}\Big|_{x=0}^{1}$$

$$+\sqrt{2}\int_0^1 \frac{dv}{dx}\frac{dw}{dx} dx + \int_0^1 v f\, dx = 0$$

So, $$\int_0^1 w \frac{dv}{dx}\frac{dw}{dx} dx + \sqrt{2}\int_0^1 \frac{dv}{dx}\frac{dw}{dx} dx + \int_0^1 v f\, dx = 0$$

So, $$\int_0^1 (w+\sqrt{2})\frac{dv}{dx}\frac{dw}{dx} dx + \int_0^1 v f\, dx = 0$$

<u>Weak Form</u> :-

Find $w \in V$ such that

$$\int_0^1 (w+\sqrt{2})\, v' w'\, dx + \int_0^1 v f\, dx = 0$$

for all $v \in V$

Problem ④:

A version of the Poisson equation that occurs in mechanics is the following model for the vertical deflection of a bar with a distributed load $P(x)$:

$$A_c E \frac{d^2u}{dx^2} = P(x)$$

where A_c = Cross-sectional area, E = Young's modulus, u = Deflection and x = distance measured along the bar's length. If the bar is rigidly fixed $(u=0)$ at both ends use FEM to model ~~the~~ its deflection for $A_c = 0.1\,m^2$, $E = 200 \times 10^9\,N/m^2$, $L = 10\,m$ and $P(x) = 100\,N/m$. Employ a value of $\Delta x = 0.5\,m$.

Answer ④:

Exact Solution:

$$A_c E \frac{d^2u}{dx^2} = P$$

So, $$\frac{d^2u}{dx^2} = \frac{P}{A_c E}$$

So, $$\frac{du}{dx} = \frac{Px}{A_c E} + C_1$$

So, $$u(x) = \frac{Px^2}{2A_c E} + C_1 x + C_2$$

As $u(0) = 0$

So, $u(0) = \frac{P(0)^2}{2A_cE} + C_1(0) + C_2 = 0$

So, $\underline{\underline{C_2 = 0}}$

As $u(L) = 0$

So, $u(L) = \frac{PL^2}{2A_cE} + C_1 L = 0$

So, $C_1 = \frac{-PL}{2A_cE}$

So, $u(x) = \frac{Px^2}{2A_cE} - \frac{PLx}{2A_cE}$

So, $u(x) = \frac{Px(x-L)}{2A_cE}$

So, $\underline{\underline{u(x) = \frac{P(x)(x-L)}{2A_cE}}}$

This is the exact Solution.

As $A_cE \frac{d^2u}{dx^2} = P$

So, $A_cE\, u'' = P$

Let $v \in V = \{v : v \in H_0^1(I)\}$

$I = [0, 1]$

$H_0^1(I)$ = Vector space of functions which are weakly differentiable once and function values at the boundary of domain is zero.

So, $A_c E u'' v = P v$

So, $\int_0^L A_c E u'' v \, dx = \int_0^L P v \, dx$

So, $\int_0^L A_c E (u'v)' \, dx - \int_0^L A_c E u'v' \, dx = \int_0^L P v \, dx$

So, $A_c E u' v \Big|_{x=0}^{L} - \int_0^L A_c E u'v' \, dx = \int_0^L P v \, dx$

As, $v(0) = v(L) = 0$

So, $-\int_0^L A_c E u'v' \, dx = \int_0^L P v \, dx$

So, the weak form is

Find $u \in V$, such that

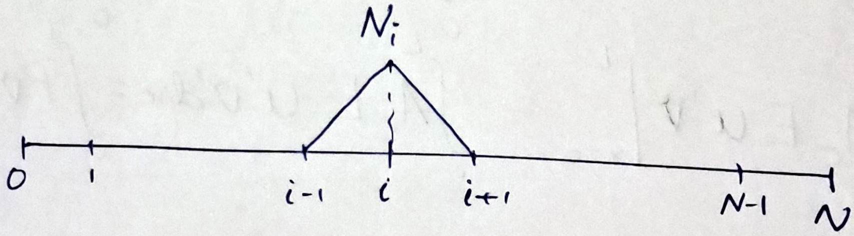
$$\int_0^L u'v' \, dx = -\frac{1}{A_c E} \int_0^L P v \, dx$$

for all $v \in V$

Basis function for V :

* We are taking hat function as our basis functions.

$$N_i = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & \text{if } x_{i-1} \le x \le x_i \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & \text{if } x_i \le x \le x_{i+1} \\[2ex] 0, & \text{otherwise.} \end{cases}$$



If we are using uniform grid then,

$\Delta x = x_i - x_{i-1} = h$

So, $$N_i = \begin{cases} \dfrac{x - x_{i-1}}{h}, & \text{if } x_{i-1} \le x \le x_i \\[2ex] \dfrac{x_{i+1} - x}{h}, & \text{if } x_i \le x \le x_{i+1} \\[2ex] 0, & \text{otherwise} \end{cases}$$

So, $\dfrac{dN_i}{dx} = N_i' = \begin{cases} \dfrac{1}{h}, & \text{if } x_{i-1} \leq x \leq x_i \\ \dfrac{-1}{h}, & \text{if } x_i \leq x \leq x_{i+1} \\ 0, & \text{Otherwise} \end{cases}$

Let $v = \alpha_i N_i$

$u = u_j N_j$

u_j = Value of u at node j.

<u>Weak Form:</u>

$$\int_0^L u' v' \, dx = \frac{-1}{A_c E} \int_0^L P v \, dx$$

Let $a(u, v) = \int_0^L u' v' \, dx$ (<u>Bilinear Form</u>)

Let $f(v) = \dfrac{-1}{A_c E} \displaystyle\int_0^L P v \, dx$

So, $a(u_j N_j, \alpha_i N_i) = f(\alpha_i N_i)$

So, $a(\alpha_i N_i, u_j N_j) = f(\alpha_i N_i)$

(As $a(u, v) = a(v, u)$ <u>is symmetric</u>)

So, $\alpha_i \, a(N_i, N_j) u_j = \alpha_i f(N_i)$

Let $K_{ij} = a(N_i, N_j)$

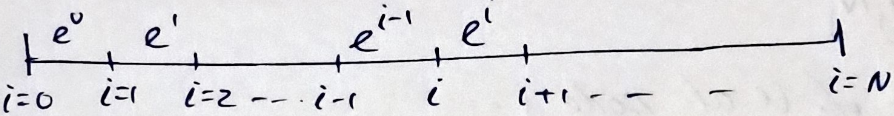
$f_i = f(N_i)$

So, $\alpha_i K_{ij} U_j = \alpha_i f_i$

So, $\alpha^T K U = \alpha^T F$

So, $\alpha^T (KU - F) = 0$

So, $KU = F$ (K = <u>Global Stiffness Matrix</u>)



As $u(0) = u_0 = 0$

$u(L) = u_N = 0$

<u>Element Stiffness Matrix for the element, e^i:</u>

$$k_{11}^i = \int_{x_i}^{x_{i+1}} N_1' N_1' \, dx$$

$$= \int_{x_i}^{x_{i+1}} \left(\frac{1}{h}\right)\left(\frac{1}{h}\right) dx$$

$$= \frac{x_{i+1} - x_i}{h^2} = \frac{1}{h}$$

So, $k_{11}^i = \frac{1}{h}$

Similarly, $k_{12}^i = \int_{x_i}^{x_{i+1}} N_1' N_2' \, dx = \int_{x_i}^{x_{i+1}} \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx$

$$= \frac{-1(x_{i+1} - x_i)}{h^2} = \frac{-1}{h}$$

So, $k_{12}^i = \frac{-1}{h}$

Similarly, $k_{21}^i = \int_{x_i}^{x_{i+1}} N_2' N_1' \, dx = \int_{x_i}^{x_{i+1}} \left(\frac{-1}{h}\right)\left(\frac{1}{h}\right) dx$

$$= \frac{-(x_{i+1} - x_i)}{h^2}$$

So, $\underline{k_{21}^i = \frac{-1}{h}}$

Similarly, $k_{22}^i = \int_{x_i}^{x_{i+1}} N_2' N_2' \, dx = \int_{x_i}^{x_{i+1}} \left(\frac{-1}{h}\right)\left(\frac{-1}{h}\right) dx$

$$= \frac{(x_{i+1} - x_i)}{h^2}$$

So, $\underline{\underline{k_{22}^i = \frac{1}{h}}}$

So, the element stiffness matrix for the element, e^i is

$$k^i = \begin{bmatrix} k_{11}^i & k_{12}^i \\ k_{21}^i & k_{22}^i \end{bmatrix} = \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} \\ \frac{-1}{h} & \frac{1}{h} \end{bmatrix}$$

So, $k^i = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

<u>Local force vector corresponding to element, e^i</u>

$$f_1^i = \frac{-P}{A_c E} \int_{x_i}^{x_{i+1}} N_1 \, dx = \frac{-P}{A_c E} \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{h} \, dx$$

$$= \frac{-P}{A_c E h} \left(x_{i+1} x - \frac{x^2}{2}\right)\Bigg|_{x = x_i}^{x_{i+1}}$$

So, $f_1^i = \dfrac{-P}{A_c E h}\left(x_{i+1}(x_{i+1}-x_i) - \frac{1}{2}(x_{i+1}^2 - x_i^2)\right)$

$$= \frac{-P}{A_c E h}(x_{i+1}-x_i)\left(x_{i+1} - \frac{1}{2}(x_{i+1}+x_i)\right)$$

$$= \frac{-P}{A_c E h}(h)\left(\frac{x_{i+1}-x_i}{2}\right)$$

So, $f_1^i = \dfrac{-Ph}{2A_c E}$

Similarly, $f_2^i = -\dfrac{P}{A_c E}\displaystyle\int_{x_i}^{x_{i+1}} N_2\, dx$

$$= \frac{-P}{A_c E}\int_{x_i}^{x_{i+1}} \frac{x-x_i}{h}\, dx$$

$$= \frac{-P}{A_c E h}\left(\frac{x^2}{2} - x x_i\right)\Bigg|_{x=x_i}^{x_{i+1}}$$

$$= \frac{-P}{A_c E h}\left(\left(\frac{x_{i+1}^2 - x_i^2}{2}\right) - (x_{i+1}-x_i)x_i\right)$$

$$= \frac{-P(x_{i+1}-x_i)}{A_c E h}\left(\frac{x_{i+1}+x_i-2x_i}{2}\right)$$

$$f_2^i = \frac{-Ph}{2A_c E}$$

So, $f^i = \begin{bmatrix} f_1^i \\ f_2^i \end{bmatrix} = \begin{bmatrix} \dfrac{-Ph}{2A_c E} \\ \dfrac{-Ph}{2A_c E} \end{bmatrix}$

So, $f^i = \dfrac{-Ph}{2A_c E}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Assembling the element stiffness matrix to the Global stiffness matrix.

$$K = \begin{bmatrix} k_{11}^0 & k_{12}^0 & 0 & \cdots & & & 0 \\ k_{21}^0 & k_{22}^0 + k_{11}^1 & k_{12}^1 & & & & \vdots \\ 0 & k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 & & & \vdots \\ \vdots & 0 & k_{21}^2 & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \\ \vdots & & & & k_{21}^{N-2} & (k_{22}^{N-2} + k_{11}^{N-1}) & k_{12}^{N-1} \\ 0 & \cdots & & & 0 & k_{21}^{N-1} & k_{22}^{N-1} \end{bmatrix}$$

So,

$$K = \frac{1}{h} \begin{bmatrix} 1 & -1 & 0 & \cdots & & & 0 \\ -1 & 2 & -1 & & & & \vdots \\ & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & & -1 & 2 & -1 \\ & & & & 0 & -1 & 1 \end{bmatrix}$$

Assembling the local force vector to global force vector

$$F = \begin{bmatrix} f_1^0 \\ f_2^0 + f_1^1 \\ \vdots \\ f_2^{N-2} + f_1^{N-1} \\ f_2^{N-1} \end{bmatrix} = \frac{-Ph}{2A_cE} \begin{bmatrix} 1 \\ 2 \\ 2 \\ \vdots \\ \vdots \\ 2 \\ 1 \end{bmatrix}$$

Also, $$u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_N \end{bmatrix}$$

So, $KU = F$

$$\frac{1}{h}\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 2 \\ 1 \end{bmatrix}\left(\frac{-Ph}{2A_cE}\right)$$

So,
$$\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \frac{-Ph^2}{2A_cE}\begin{bmatrix} 1 \\ 2 \\ 2 \\ \vdots \\ 2 \\ 1 \end{bmatrix}$$

As $u(0) = u_0 = 0$
$u(L) = u_N = 0$

So, our system of equation ~~eq~~ reduces to

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} = \frac{-Ph^2}{2A_cE}\begin{bmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{bmatrix}$$

So,
$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} = \frac{-Ph^2}{A_cE}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

So, we need to solve this equation to evaluate $u_1, u_2, \ldots, u_{N-1}$

We will do this using Jacobi Algorithm.

Jacobi Method :

Equations :

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1K}x_K = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2K}x_K = b_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_{K1}x_1 + a_{K2}x_2 + \cdots + a_{KK}x_K = b_K$$

Step ① :

Assume some seed value of $x_1, x_2, \ldots, x_K$.

Step ② :

$$x_i^{new} = \left(\frac{b_i - \left(\sum_{\substack{j=1 \\ j \neq i}}^{K} a_{ij} x_j \right)}{a_{ii}} \right)$$

$i = 1, 2, \ldots, K$

Step ③ :

Check for the tolerance

$$\text{Tolerance} = \sum_{i=1}^{K} \left| x_i - x_i^{new} \right|$$

Step ④ :

Replace all x_i with x_i^{new} $\quad (i = 1, 2, \ldots, K)$

Step ⑤ :

Stop if tolerance is less than the required tolerance.

Otherwise go to step ② with new values of x_i.

$(i = 1, 2, \ldots, K)$

Post_processing

April 30, 2023

1 Question (4)

1.1 A version of the Poisson equation that occurs in mechanics is the following model for the vertical deflection of a bar with a distributed load P (x):

$$A_c E \frac{d^2u}{dx^2} = P(x)$$

where A_c = cross-sectional area, E = Young's modulus, u = deflection, and x = distance measured along the bar's length. If the bar is rigidly fixed (u = 0) at both ends, use the finite-element method to model its deflections for Ac = 0.1m^2 , E = 200 × 109 N/m^2 , L = 10m, and P (x) = 100N/m. Employ a value of Δx = 0.5m.

2 Answer (4):

3 NOTE: Using the tolerance of 10^{-20} for jacobi solver

4 Import the necessary libraries

[1]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

5 Reading the name of the files generated

[2]:
```
Output_files = pd.read_csv("Output_file_names.csv",delimiter=",",header=None).
    ↪to_numpy()
Output_files = np.squeeze(Output_files)
Output_files = Output_files.tolist()
```

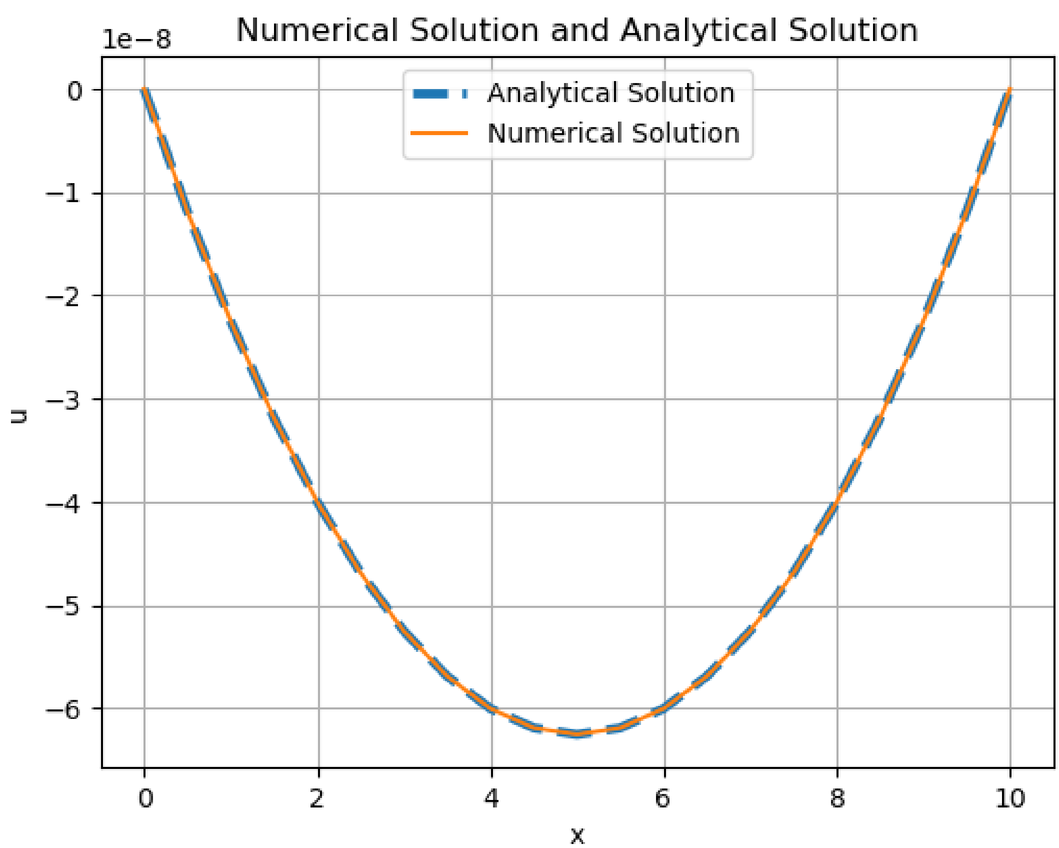
5.1 Output files generated

[3]:
```
Output_files
```

[3]:
```
['Question_4_u_Numerical_Solution.csv', 'Question_4_u_Analytical_Solution.csv']
```

5.2 Numerical solution for $\Delta x = 0.5m$

[4]:
```
Numerical_Solution = pd.read_csv(Output_files[0],delimiter=",",header=None).
  ↪to_numpy()
Analytical_Solution = pd.read_csv(Output_files[1],delimiter=",",header=None).
  ↪to_numpy()
x = np.linspace(0, 10,Numerical_Solution.shape[0])
plt.plot(x,Analytical_Solution,linestyle='dashed',linewidth=3.5)
plt.plot(x,Numerical_Solution)
plt.xlabel("x")
plt.ylabel("u")
plt.legend(["Analytical Solution","Numerical Solution"])
plt.title("Numerical Solution and Analytical Solution")
plt.grid()
plt.savefig("Question_4.png",dpi = 500)
plt.show()
```



5.3 It is evident from the above graph that the numerical solution using FEM is in good agreement with the analytical solution.