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Course: M.Tech (Aerospace Engineering)

Subject: AE 291 (Matrix Computations)

SAP No.: 6000007645

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Importing the necessary libraries

[1]:
```
import numpy as np
```

[2]:
```
from matplotlib import cm
```

[3]:
```
import matplotlib.pyplot as plt
```

Problem:

Solving 2D Poisson's problem using Successive Overrelaxation (SOR) iterative methods

Consider the 2D Poisson's equation in the domain Ω = [0,1] $\times$ [0, 1], the unit square:

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f \qquad in \qquad \Omega, \tag{1}$$

with the boundary condition

$$u = g \qquad on \qquad \partial\Omega \tag{2}$$

where f and g are given functions, and $\partial\Omega$ represents the boundary of Ω. Eq. 1 can be discretized using the centered Finite difference method (as explained in the class).

Consider the case where f = 0, and g is given as,

$$g(x,y) = \begin{cases} 0 & if\ x = 0 \\ y & if\ x = 1 \\ (x-1)sin(x) & if\ y = 0 \\ x(2-x) & if\ y = 1 \end{cases}$$

Solve the discretized Poisson's problem using Successive Overrelaxation (SOR) iterative method for various values of ω, the relaxation parameter (see lecture notes). Note that when $\omega = 1$, SOR method collapses to Gauss-Seidel method. Take the initial guess as $u^{(0)} = 0$. Consider three mesh intervals: h = 1/10, h = 1/20 and h = 1/40. The iterations should be continued until the relative change in the solution u from one iteration to another is less than 10^{-8} . More precisely, stop the iterations when

$$\frac{||u^{(k+1)} - u^{(k)}||_2}{||u^{(k+1)}||_2} < 10^{-8} \tag{3}$$

For each h,

1. Perform the iterations for $\omega = \{0.8, 1.0, 1.4, 1.6, 1.8, 1.9\}$. Plot the relative change in the solution (LHS of Eq. 3) versus the iteration index (k) for each ω on a single plot. In the plot, the relative change in the solution should be in base-10 logarithmic scale (For example, see the command "semilogy" in matlab). Report the optimal ω for which the iterations required to reach the convergence criteria (Eq. 3) is minimum.

Answer (1):

Set of values of the h

[4]:
```
H = [1/10,1/20,1/40]
```

Set of values of the ω

[5]:
```
W = [0.8,1.0,1.4,1.6,1.8,1.9]
```

Defining the domain: Ω = [0,1] $\times$ [0, 1]

[6]:
```
x_0 = 0
```

[7]:
```
x_1 = 1
```

[8]:
```
y_0 = 0
```

[9]:
```
y_1 = 1
```

Function to implement the boundary conditions

$$g(x,y) = \begin{cases} 0 & if\ x = 0 \\ y & if\ x = 1 \\ (x-1)sin(x) & if\ y = 0 \\ x(2-x) & if\ y = 1 \end{cases}$$

[10]:
```
def g(x,y):
    """
    g(x,y) sets the boundary conditions at based on the coordinates of the node,␣
↪x and y.
    If the given node does not lie on the boundary then 0 will be returned.
    x: x coordinate
    y: y coordinate
    """
    if x == 0:
        return 0
    if x == 1:
        return y
    if y == 0:
        return (x-1)*np.sin(x)
    if y == 1:
        return x*(2-x)
    else:
        return 0
```

[11]:
```
def f(x,y):
    """
    f(x,y) evaluated the function f of the question based on the coordinates of␣
↪the node, x and y.
    x: x coordinate
    y: y coordinate
    """
    return 0
```

Stopping criteria:

$$\frac{||u^{(k+1)} - u^{(k)}||_2}{||u^{(k+1)}||_2} < 10^{-8} \tag{3}$$

[12]:
```
def Error_Function(u_new,u):
    """
    Error_Function(u_new,u) evaluates the error.
    u_new: u(k+1)
    u: u(k)
    """

    # Calculating the numerator
    temp = (((u_new.flatten()-u.flatten())**2).sum())**0.5

    # Calculating the denominator
    temp_1 = (((u_new.flatten())**2).sum())**0.5

    # Calculating the error
```

```
    temp = temp/temp_1

    return temp
```

[13]:
```
Tolerance = 1e-8
```

A dictionary to store errors

[14]:
```
Error = {}
```

A dictionary to store u

[15]:
```
U = {}
```

A dictionary to store meshgrids

[16]:
```
X_Y_dict = {}
```

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f \qquad in \qquad \Omega$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f \qquad in \qquad \Omega$$

$$\Delta x = \Delta y = h$$

$$\frac{u_{(i+1,j)} - 2u_{(i,j)} + u_{(i-1,j)}}{h^2} + \frac{u_{(i,j+1)} - 2u_{(i,j)} + u_{(i,j-1)}}{h^2} = f_{(i,j)}$$

or,

$$\frac{u_{(i+1,j)} - 2u_{(i,j)} + u_{(i-1,j)} + u_{(i,j+1)} - 2u_{(i,j)} + u_{(i,j-1)}}{h^2} = f_{(i,j)}$$

or,

$$\frac{u_{(i+1,j)} + u_{(i,j+1)} - 4u_{(i,j)} + u_{(i-1,j)} + u_{(i,j-1)}}{h^2} = f_{(i,j)}$$

or,

$$u_{(i+1,j)} + u_{(i,j+1)} - 4u_{(i,j)} + u_{(i-1,j)} + u_{(i,j-1)} = h^2 f_{(i,j)}$$

or,

$$u_{(i+1,j)} + u_{(i,j+1)} + u_{(i-1,j)} + u_{(i,j-1)} - h^2 f_{(i,j)} = 4u_{(i,j)}$$

or,

$$\frac{u_{(i+1,j)} + u_{(i,j+1)} + u_{(i-1,j)} + u_{(i,j-1)} - h^2 f_{(i,j)}}{4} = u_{(i,j)}$$

or,

$$u_{(i,j)} = \frac{u_{(i+1,j)} + u_{(i,j+1)} + u_{(i-1,j)} + u_{(i,j-1)} - h^2 f_{(i,j)}}{4}$$

SOR Iteration:

$$u^{k+1}_{(i,j)} = u^{k}_{(i,j)} + w\delta$$

or,

$$u^{k+1}_{(i,j)} = u^{k}_{(i,j)} + w(\hat{u}^{k}_{(i,j)} - u^{k}_{(i,j)})$$

Let,

$$I = (n \times i) + j$$

So,

$$u^{k+1}_{I} = u^{k}_{I} + w(\hat{u}^{k}_{I} - u^{k}_{I})$$

As,

$$\hat{u}^{k}_{I} = \frac{b_I - \sum_{J=I+1}^{n} a_{IJ} u^{k}_{J} - \sum_{J=1}^{I-1} a_{IJ} u^{k+1}_{J}}{a_{II}}$$

So,

$$u^{k+1}_{I} = u^{k}_{I} + w \left(\frac{b_I - \sum_{J=I+1}^{n} a_{IJ} u^{k}_{J} - \sum_{J=1}^{I-1} a_{IJ} u^{k+1}_{J}}{a_{II}} - u^{k}_{I} \right)$$

But the equation for grid point (i,j) only involves (i+1,j), (i-1,j), (i,j+1) and (i,j-1).

$$u_{(i,j)} = \frac{u_{(i+1,j)} + u_{(i,j+1)} + u_{(i-1,j)} + u_{(i,j-1)} - h^2 f_{(i,j)}}{4}$$

Thus, the equation for grid point I (I $= (n \times i) + j$) only involves:

$$I_1 = (n \times (i+1)) + j > I$$
$$I_2 = (n \times (i-1)) + j < I$$
$$I_3 = (n \times (i)) + j - 1 < I$$
$$I_4 = (n \times (i)) + j + 1 > I$$

So,

$$\hat{u}^{k}_{I} = \frac{b_I - \sum_{J=I+1}^{n} a_{IJ} u^{k}_{J} - \sum_{J=1}^{I-1} a_{IJ} u^{k+1}_{J}}{a_{II}}$$

or,

$$\hat{u}_I^k = \frac{b_I - \sum_{J>I}^{n} a_{IJ} u_J^k - \sum_{J<I} a_{IJ} u_J^{k+1}}{a_{II}}$$

or,

$$\hat{u}_I^k = \frac{b_I - \sum_{J=I_1,I_4} a_{IJ} u_J^k - \sum_{J=I_2,I_3} a_{IJ} u_J^{k+1}}{a_{II}}$$

As,

$$u_{(i,j)} = \frac{u_{(i+1,j)} + u_{(i,j+1)} + u_{(i-1,j)} + u_{(i,j-1)} - h^2 f_{(i,j)}}{4}$$

So,

$$\hat{u}_I = \frac{u_{I_1} + u_{I_4} + u_{I_2} + u_{I_3} - h^2 f_I}{4}$$

So,

$$\hat{u}_I^k = \frac{u_{I_1}^k + u_{I_4}^k + u_{I_2}^{k+1} + u_{I_3}^{k+1} - h^2 f_I}{4}$$

So,

$$\hat{u}_{(i,j)}^k = \frac{u_{(i+1,j)}^k + u_{(i,j+1)}^k + u_{(i-1,j)}^{k+1} + u_{(i,j-1)}^{k+1} - h^2 f_{(i,j)}}{4}$$

As,

$$u_I^{k+1} = u_I^k + w(\hat{u}_I^k - u_I^k)$$

So,

$$u_{(i,j)}^{k+1} = u_{(i,j)}^k + w(\hat{u}_{(i,j)}^k - u_{(i,j)}^k)$$

So,

$$u_{(i,j)}^{k+1} = u_{(i,j)}^k + w\left(\left(\frac{u_{(i+1,j)}^k + u_{(i,j+1)}^k + u_{(i-1,j)}^{k+1} + u_{(i,j-1)}^{k+1} - h^2 f_{(i,j)}}{4}\right) - u_{(i,j)}^k\right)$$

[17]:
```python
def SOR_Solver(u,U,Error,Tolerance,f,x,y,w,h):
    """
    SOR_Solver(u,U,Error,Tolerance,f,x,y,w): Solves the given poisson equation␣
 ↪using the SOR method
    u: Initial value of u in the computational domain
    U: Dictionary to store u for different values of h
    Error: Dictionary to store error at each iterations for different values of h
    Tolerance: Stopping Criteria
    f: RHS of the equation
    x: X Meshgrid
    y: Y Meshgrid
    w: the relaxation parameter
```

```
    h: mesh interval
    """
    # u at the next iteration
    u_old = u.copy()

    # Intializing the tolerance achieved
    temp = Tolerance + 1

    # Error over iterations for some particular h and w
    error = []

    # While loop until the stopping criteria is met
    while temp >= Tolerance:

        # Updating u
        u_old = u.copy()

        # NOTE: We are not calculating the values of u at the boundary nodes
        for i in range(1,u.shape[0]-1):

            # NOTE: We are not calculating the values of u at the boundary nodes
            for j in range(1,u.shape[1]-1):

                # SOR Iteration step
                u[i][j] = u[i][j] + (w*((0.25*(u[i+1][j] + u[i][j+1] + u[i-1][j]␣
↪+ u[i][j-1] - ((h**2)*f(x[i,j],y[i,j])))) - u[i][j]))

        # Calculating the relative error
        temp = Error_Function(u,u_old)

        # Storing the errors corresponding to each iteration
        error.append(temp)

    # Storing u in the dictionary for some particular h and w
    U[(h,w)] = u_old

    # Storing error in the dictionary for some particular h and w
    Error[(h,w)] = error
```

[18]:
```
# Iterating over different values of h (Mesh Interval)
for h in H:

    # Iterating over different values of w (Successive Overrelaxation Factor)
    for w in W:

        # Mesh intervals
        dx = h
```

```python
        dy = h

        # Number of the grid points
        n = (int((x_l-x_0)/dx)-1)*(int((y_l-y_0)/dy)-1)

        # Creating a meshgrid
        x = np.arange(x_0,x_l+dx,dx)
        y = np.arange(y_0,y_l+dy,dy)
        X,Y = np.meshgrid(x,y,indexing = "ij")

        # Storing meshgrids
        X_Y_dict[h] = (X,Y)

        # Intializing the u at the current iteraion with zeros
        u = np.zeros(((int((x_l-x_0)/dx))+1,(int((y_l-y_0)/dy))+1))

        # Applying the boundary conditions
        for i in range(u.shape[0]):
            for j in range(u.shape[1]):
                u[i,j] = g(x[i],y[j])

        # Solving the system of equation using SOR Method
        SOR_Solver(u,U,Error,Tolerance,f,X,Y,w,h)

        iteraions = len(Error[(h,w)])

        print(f"For h = {h} and w = {w} :")
        print(f"Number of iterations required: {iteraions}\n\n")
```

```
For h = 0.1 and w = 0.8 :
Number of iterations required: 231


For h = 0.1 and w = 1.0 :
Number of iterations required: 157


For h = 0.1 and w = 1.4 :
Number of iterations required: 66


For h = 0.1 and w = 1.6 :
Number of iterations required: 41


For h = 0.1 and w = 1.8 :
Number of iterations required: 84
```

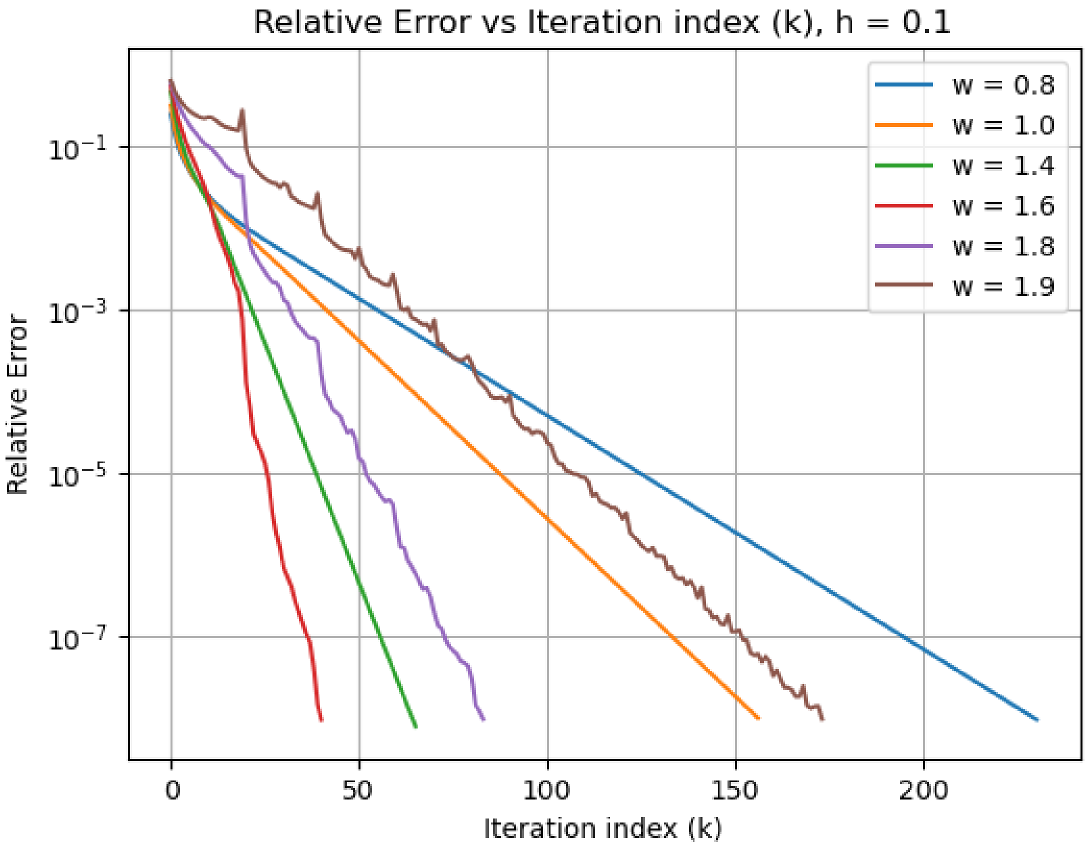
```
For h = 0.1 and w = 1.9 :
Number of iterations required: 174


For h = 0.05 and w = 0.8 :
Number of iterations required: 836


For h = 0.05 and w = 1.0 :
Number of iterations required: 574


For h = 0.05 and w = 1.4 :
Number of iterations required: 259


For h = 0.05 and w = 1.6 :
Number of iterations required: 150


For h = 0.05 and w = 1.8 :
Number of iterations required: 88


For h = 0.05 and w = 1.9 :
Number of iterations required: 176


For h = 0.025 and w = 0.8 :
Number of iterations required: 3000


For h = 0.025 and w = 1.0 :
Number of iterations required: 2069


For h = 0.025 and w = 1.4 :
Number of iterations required: 949


For h = 0.025 and w = 1.6 :
Number of iterations required: 574


For h = 0.025 and w = 1.8 :
Number of iterations required: 250
```

```
For h = 0.025 and w = 1.9 :
Number of iterations required: 179
```

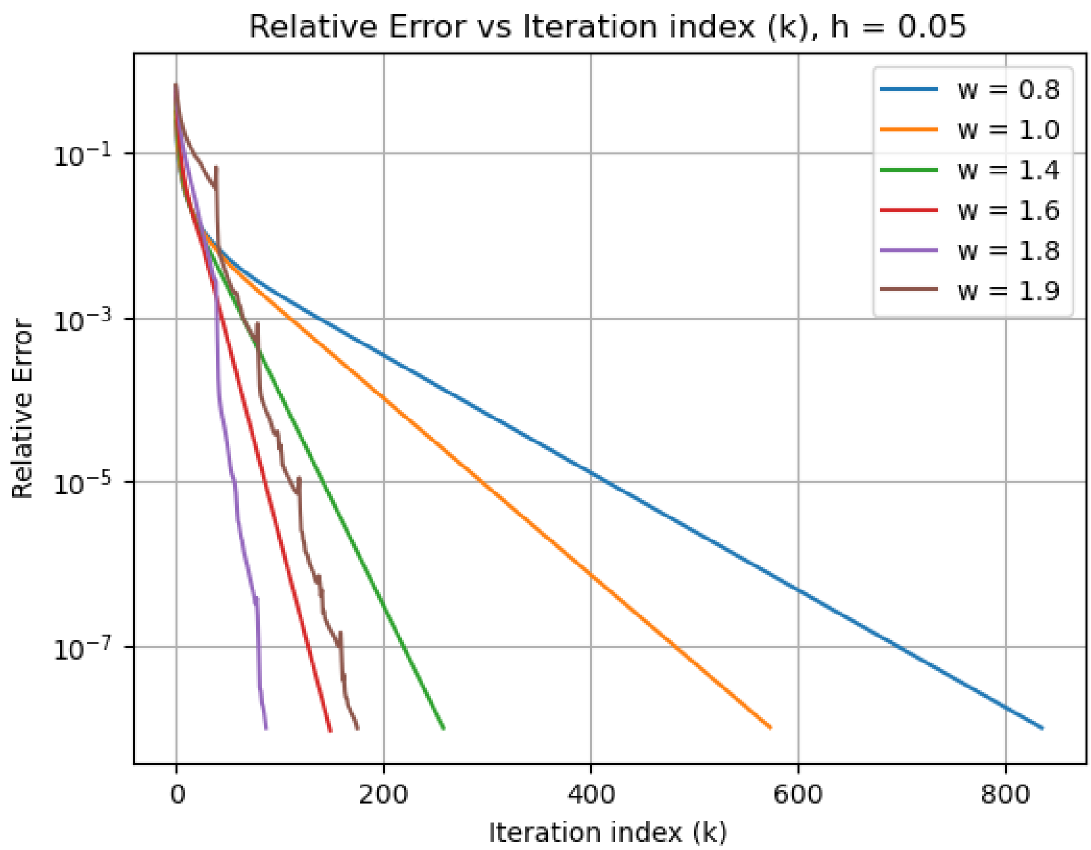
[19]:

```
for h_1 in H:
    temp = len(Error[list(Error.keys())[0]])
    for (h,w),error in Error.items():
        if h == h_1:
            temp_new = len(error)
            if temp_new <= temp:
                temp = temp_new
                w_optimum = w
            plt.semilogy(error,label = "w = "+str(w))
    plt.legend()
    # NOTE: Here, the iteration index k starts from 0
    plt.xlabel("Iteration index (k)")
    plt.ylabel("Relative Error")
    plt.title(f"Relative Error vs Iteration index (k), h = {h_1}")
    plt.grid()
    print(f"For h = {h_1}, optimal w is {w_optimum}")
    plt.show()
```

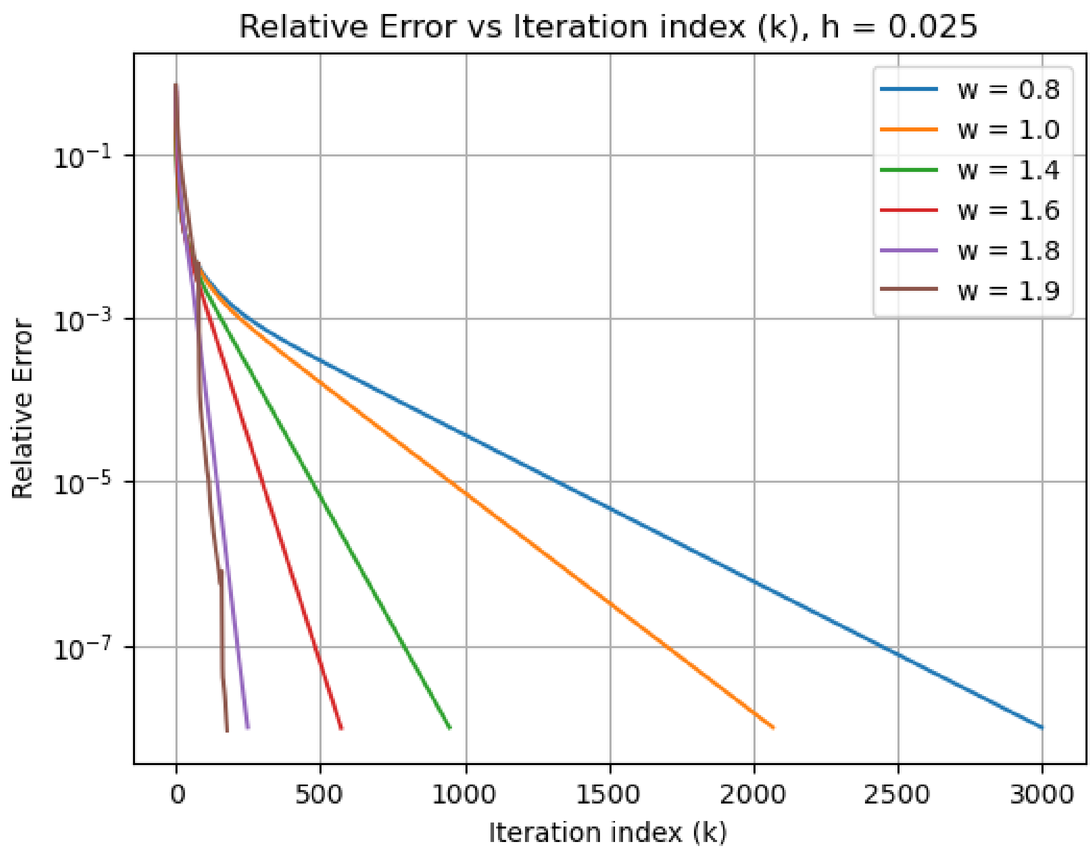
```
For h = 0.1, optimal w is 1.6
```




```
For h = 0.05, optimal w is 1.8
```




```
For h = 0.025, optimal w is 1.9
```

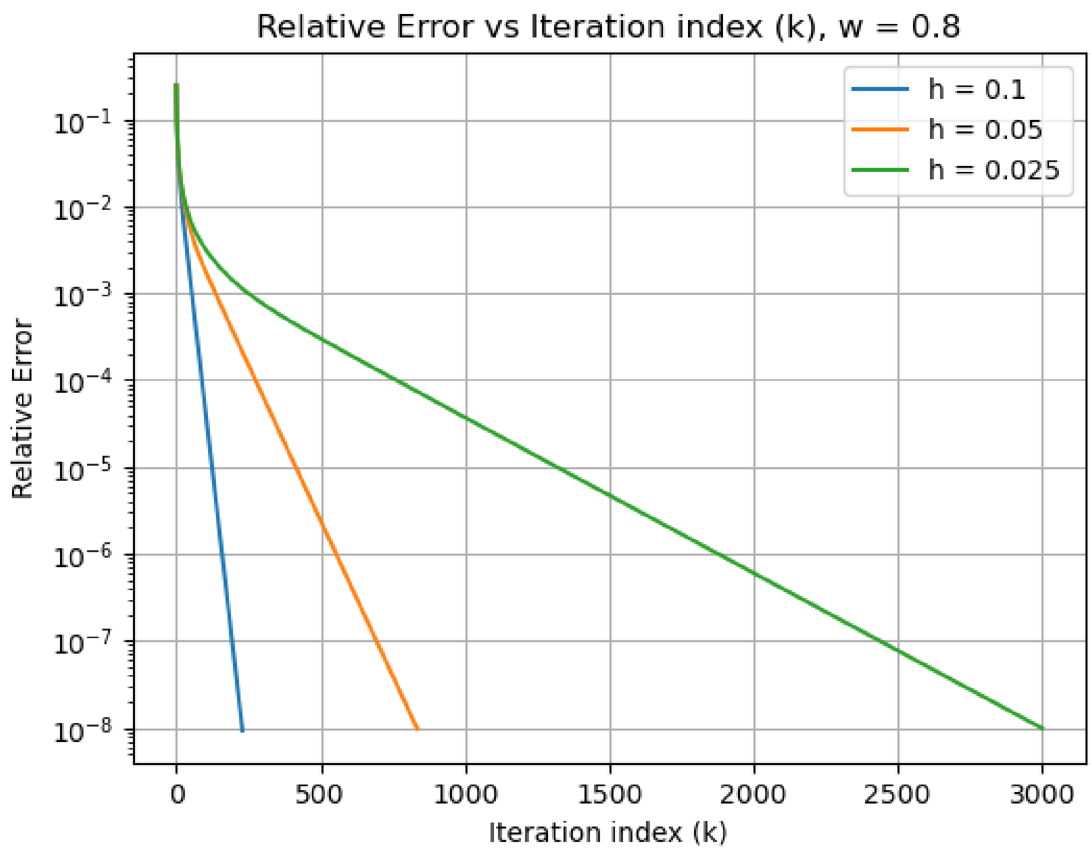


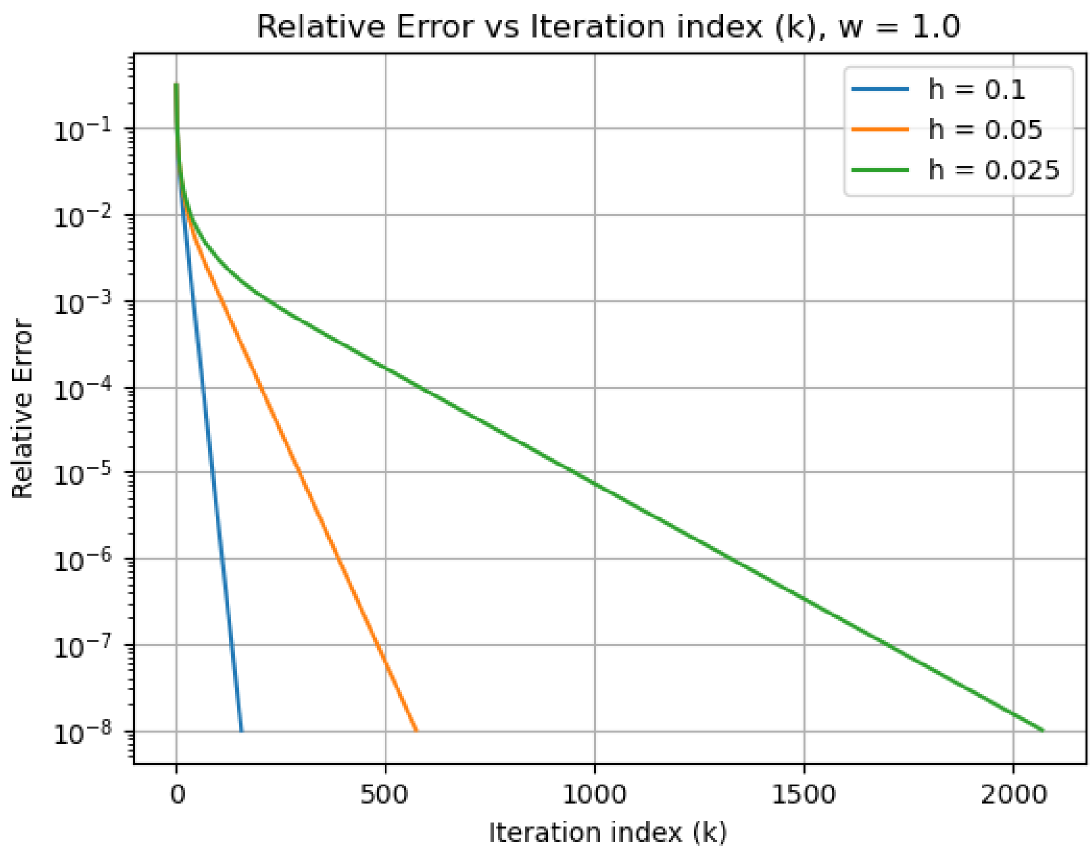
For h = 0.1, optimal ω is 1.6

For h = 0.05, optimal ω is 1.8

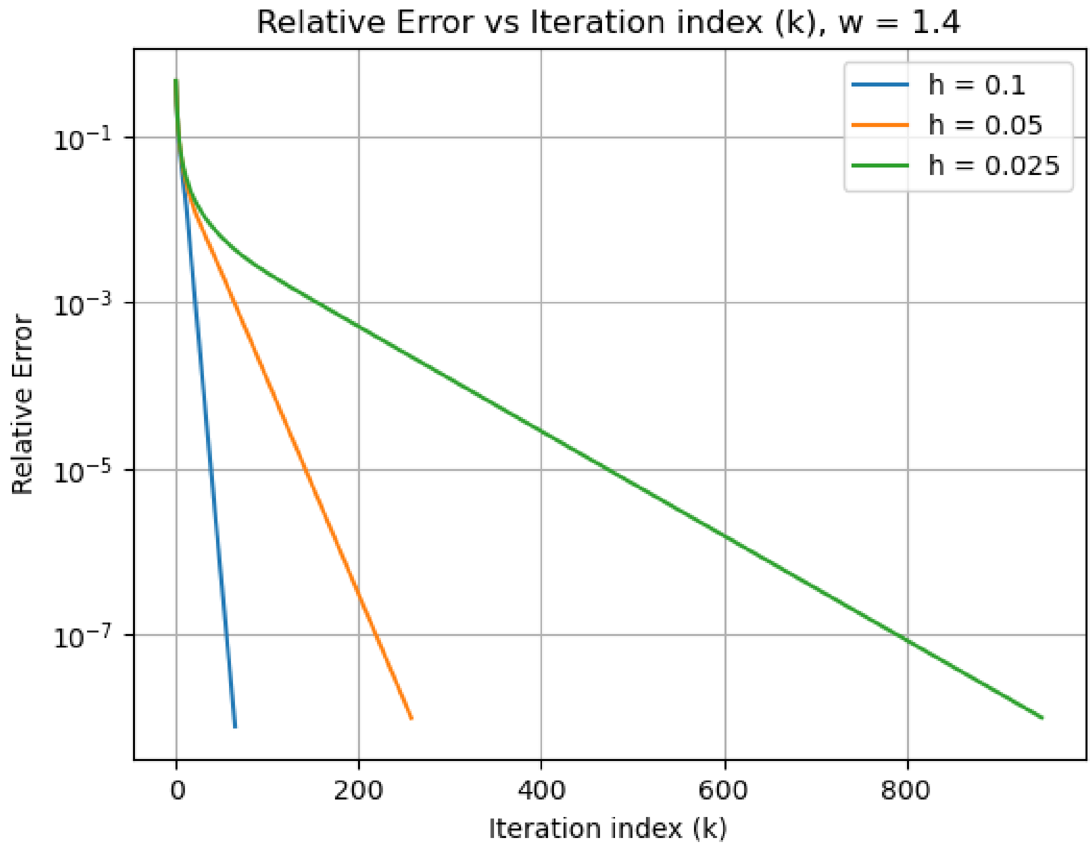
For h = 0.025, optimal ω is 1.9

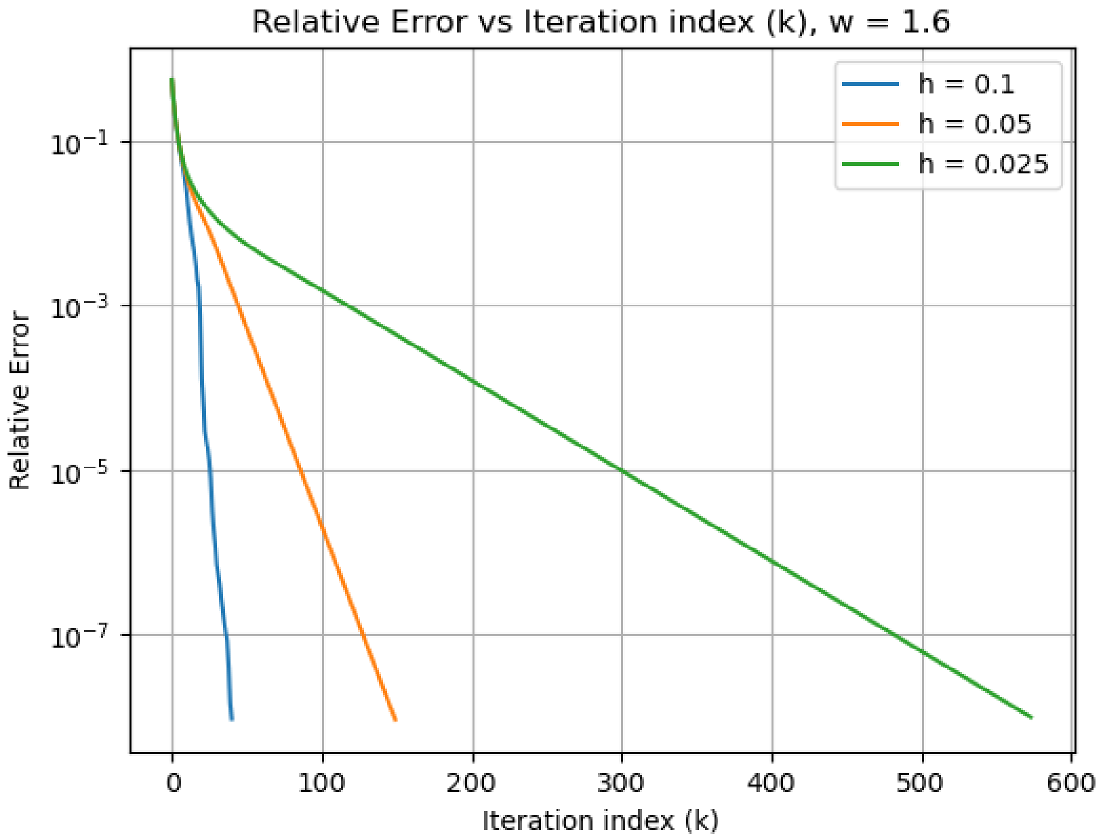
[20]:
```
for w_1 in W:
    for (h,w),error in Error.items():
        if w == w_1:
            plt.semilogy(error,label = "h = "+str(h))
    plt.legend()
    # NOTE: Here, the iteration index k starts from 0
    plt.xlabel("Iteration index (k)")
    plt.ylabel("Relative Error")
    plt.title(f"Relative Error vs Iteration index (k), w = {w_1}")
    plt.grid()
    plt.show()
```

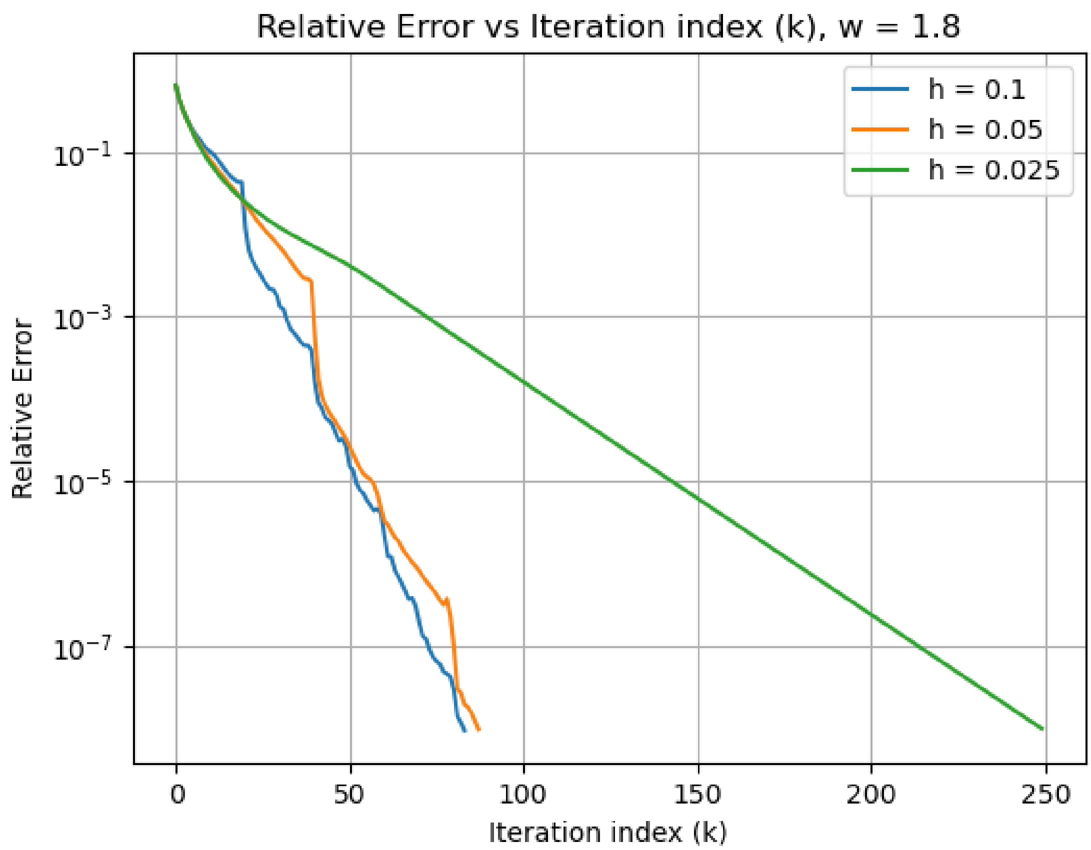

Relative Error vs Iteration index (k), w = 0.8
h = 0.1
h = 0.05
h = 0.025
Relative Error
Iteration index (k)
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
0
500
1000
1500
2000
2500
3000

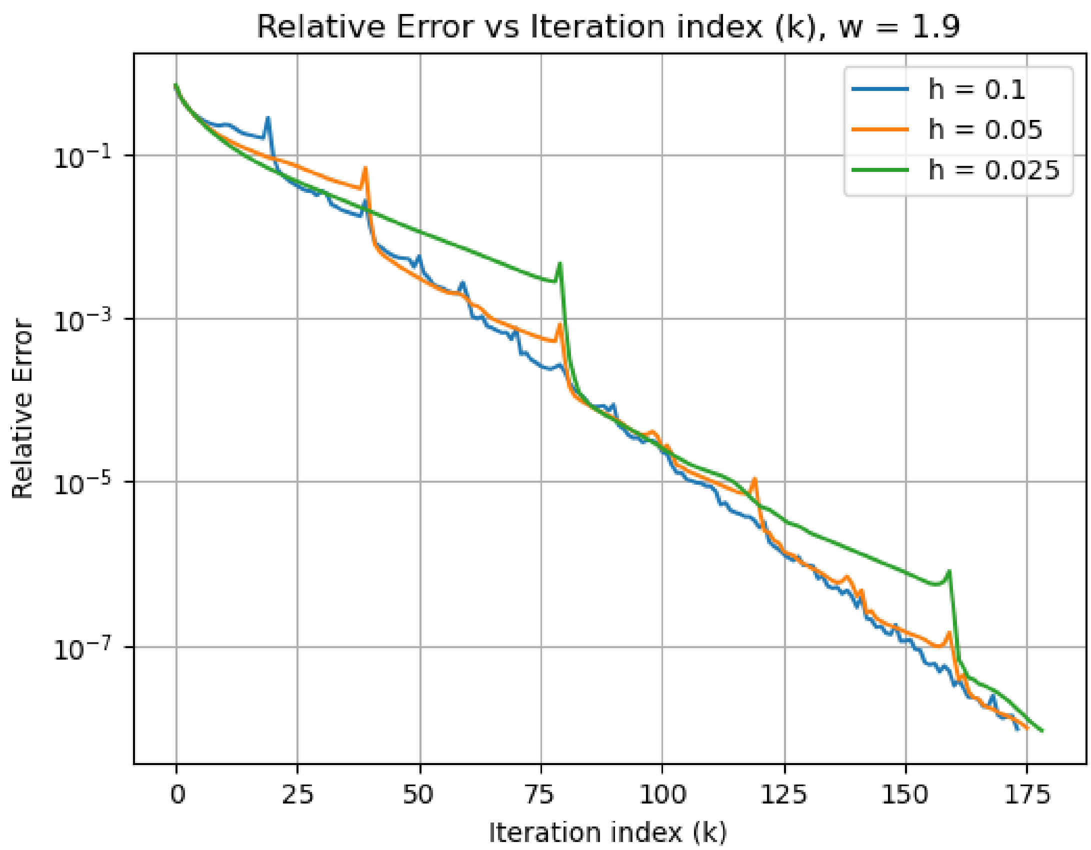


Relative Error vs Iteration index (k), w = 1.0
h = 0.1
h = 0.05
h = 0.025
Relative Error
Iteration index (k)


Relative Error vs Iteration index (k), w = 1.4
h = 0.1
h = 0.05
h = 0.025
Relative Error
Iteration index (k)
10^-1
10^-3
10^-5
10^-7
0
200
400
600
800


Relative Error vs Iteration index (k), w = 1.6
h = 0.1
h = 0.05
h = 0.025
10^{-1}
10^{-3}
10^{-5}
10^{-7}
Relative Error
0
100
200
300
400
500
600
Iteration index (k)

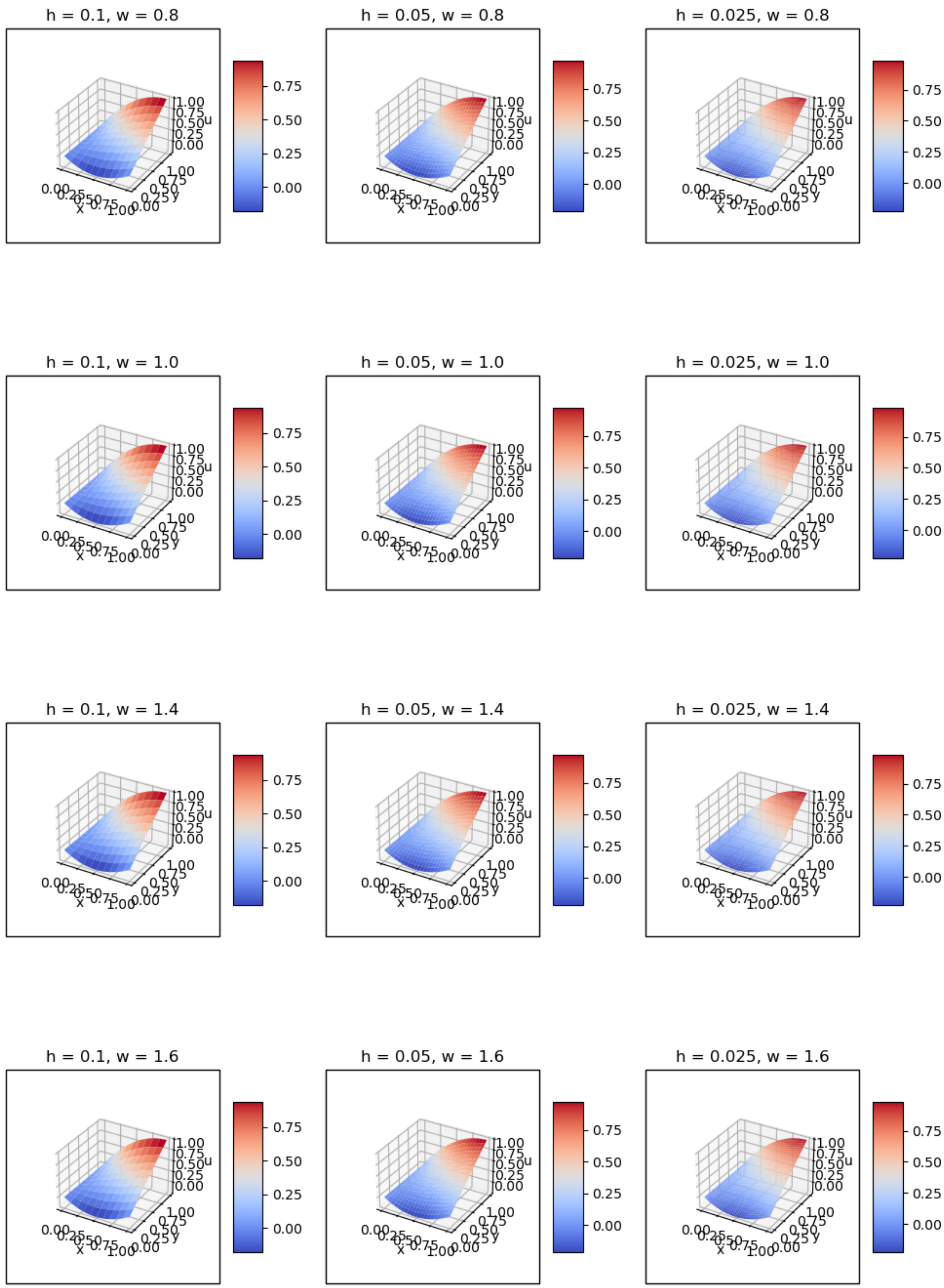

Relative Error vs Iteration index (k), w = 1.8
h = 0.1
h = 0.05
h = 0.025
Relative Error
10^{-1}
10^{-3}
10^{-5}
10^{-7}
0
50
100
150
200
250
Iteration index (k)

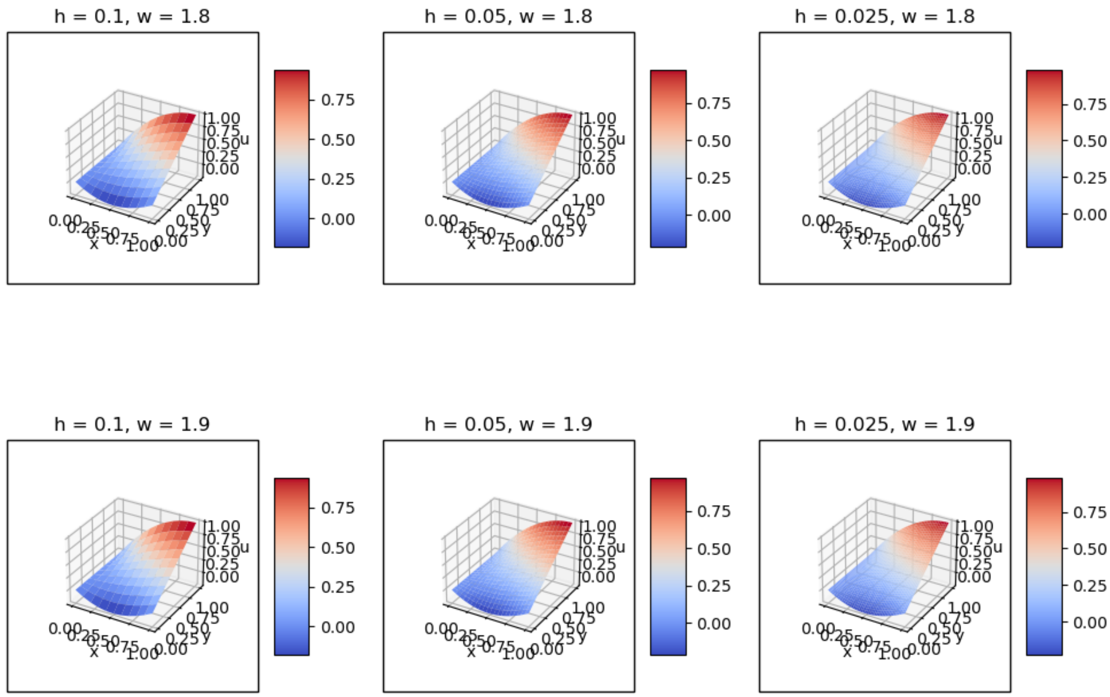


Observation: The total number of iterations required is approximately the same on increasing ω to a very large value such as 1.9, provided it is still less than 2.

[21]:
```
for w_1 in W:
    fig = plt.figure()
    fig.set_size_inches(12,4)
    ctr = 1
    for (h,w),u in U.items():
        if w == w_1:
            ax = fig.add_subplot(1,3,ctr,projection = "3d")
            ctr = ctr+1
            surf = ax.plot_surface(X_Y_dict[h][0], X_Y_dict[h][1], u, cmap=cm.
↪coolwarm,linewidth=0)
            fig.colorbar(surf, shrink=0.5, aspect=5)
            ax.set_xlabel("x")
            ax.set_ylabel("y")
            ax.set_zlabel("u")
            ax.set_title(f"h = {h}, w = {w}")
            ax.set_box_aspect(aspect=None, zoom=0.625)
```

```
            ax.patch.set_edgecolor('black')
            ax.patch.set_linewidth(1)
    plt.show()
```



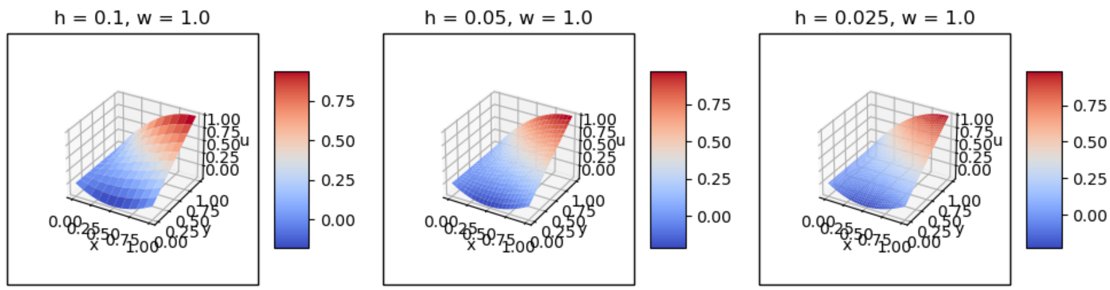


2. For $\omega = 1$ (Gauss-Seidel), show a 3D surface plot of the final solution u (as a function of x and y). Also compare the iterations required for the Gauss Seidel and the Jacobi method (from Assignment 2) to reach the convergence criteria.

Answer (2):

[22]:
```
fig = plt.figure()
fig.set_size_inches(12,4)
ctr = 1
for (h,w),u in U.items():
    if w == 1:
        ax = fig.add_subplot(1,3,ctr,projection = "3d")
        ctr = ctr+1
        surf = ax.plot_surface(X_Y_dict[h][0], X_Y_dict[h][1], u, cmap=cm.
↪coolwarm,linewidth=0)
        fig.colorbar(surf, shrink=0.5, aspect=5)
        ax.set_xlabel("x")
        ax.set_ylabel("y")
        ax.set_zlabel("u")
        ax.set_title(f"h = {h}, w = {w}")
        ax.set_box_aspect(aspect=None, zoom=0.625)
```

```
        ax.patch.set_edgecolor('black')
        ax.patch.set_linewidth(1)
plt.show()
```



Iteration required for Gauss-Siedel method:

If h = 0.1, then iterations = 157

If h = 0.05, then iterations = 574

If h = 0.025, then iterations = 2069

Iteration required for Jacobi method:

If h = 0.1, then iterations = 293

If h = 0.05, then iterations = 1077

If h = 0.025, then iterations = 3882

In conclusion, the number of iterations required for Gauss-Seidel method is approximately half as compared to Jacobi method.