CLASSICAL MECHANICS PROJECT

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Analysis of Foucault's pendulum using Lagrangian Mechanics

THEORY

To understand the theoretical background of Foucault's pendulum we must understand the following points:

- Historical background and the experimental Setup of Foucault's pendulum.
- Physics of Foucault's pendulum
- Derivation of EOM using Lagrangian Mechanics

Historical background:

From ages there was an ongoing debate, weather, the earth rotated or the universe rotated around the Earth. While there was ample of proof given by physicist that Earth rotated, the problem with their experimental evidence was that it required observation of celestial bodies, especially stars for years. This needed meticulous note making and was prone to a lot of errors as it depended on climatic conditions which cannot be controlled.

Even though by the time of Foucault the notion of earth rotating was widely accepted, there still lacked a simple experiment to demonstrate this effect. And Foucault with his pendulum was able to achieve that.

The idea was very simple and elegant, if we let a pendulum to oscillation on any place on the earth except for the equator, the plane of oscillation of the pendulum will change its direction with time and at poles the plan of oscillation will complete an entire rotation in 24 Hrs.(in a day)

Experimental setup of Foucault's pendulum:

Foucault's pendulum consists of a light and long string (ideally inextensible) a heavy blob of mass, a table or surface wherein degrees from 0° to 360° is marked. A motor which counter acts the effect of damping without causing any disturbance to the plane of oscillation. This motor is necessary for anyone to be able to make observations of the Foucault's pendulum.

Caution that need to be taken during constructing an experimental setup:

I. The string should be long and should go as less longitudinal deformation as possible.
II. The mass should be heavy and should not deform
III. At the start while starting the oscillation care must be taken so that it is not a jerk start , for this purpose it is always recommended to attach the pendulum to a string at the side and at the time of starting the experiment this side string should be burn rather than cut .Example image

Physics of Foucault's pendulum:

To understand why does a Foucault's pendulum change the direction of plane of oscillation we must understand the forces that act on a body in rotating co-ordinate system.

The equation of motion for a body in rotating co-ordinate system is as follows:

m$\ddot{r}$ = m$\ddot{r}'$ + 2m $(\omega \times v)$ + m $(\omega(\omega \times r))$ + m $(\dot{\omega} \times r)$

here,

m$\ddot{r}$ → force in fixed frame of reference

m$\ddot{r}'$ → force in fixed moving /rotating frame of reference

2m $(\omega \times v)$ → Coriolis force (it is a pseudo force arising out of the interaction between earth and object in motion)

m $(\omega(\omega \times r))$ → centrifugal force

m $(\dot{\omega} \times r)$ → Euler force

We know that Foucault pendulum is present in the earths gravitational field, hence apart from the force mentioned above, force of gravity also acts on the Foucault pendulum.

Because we consider the amplitude of oscillation to be small the effect of Centrifugal force is negligible.

We also know that earth rotates with an almost constant angular velocity, hence,

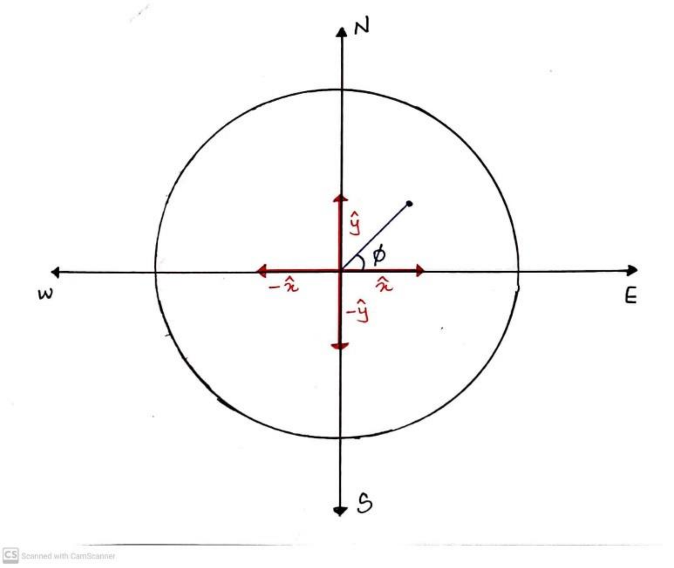
$\dot{\omega} = 0 \rightarrow \ \omega = constant$

So, Euler force (m $(\dot{\omega} \times r)$) is zero.

Hence the forces that majorly act on the Foucault pendulum are:

I. Force of gravity
II. Coriolis force (2m $(\omega \times v)$)

Coriolis force can be represented in a more suggestive manner in a following way:

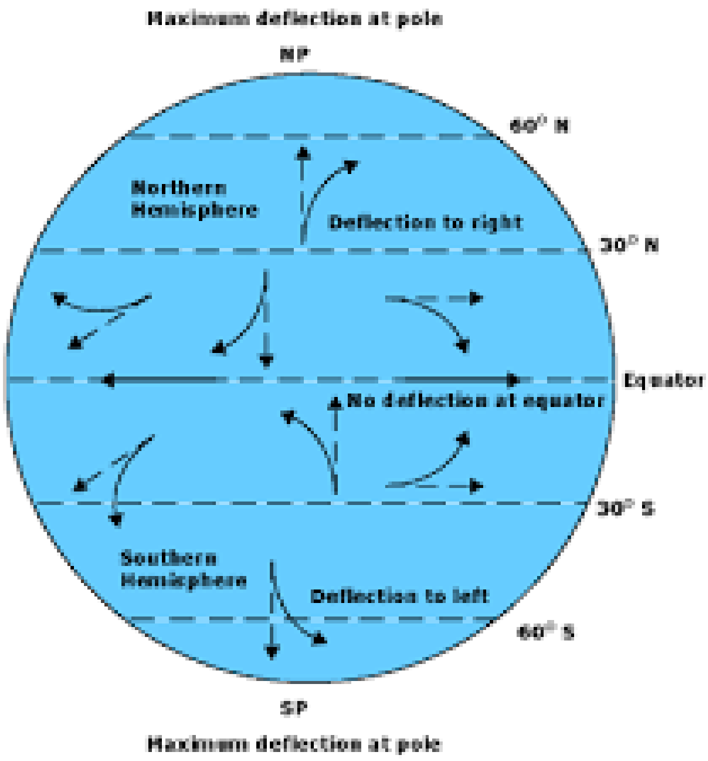


$$F_c = 2m[\omega \sin \emptyset \,(v_y\hat{x} - v_x\hat{y})]$$

Here we write v in terms of its x and y components.

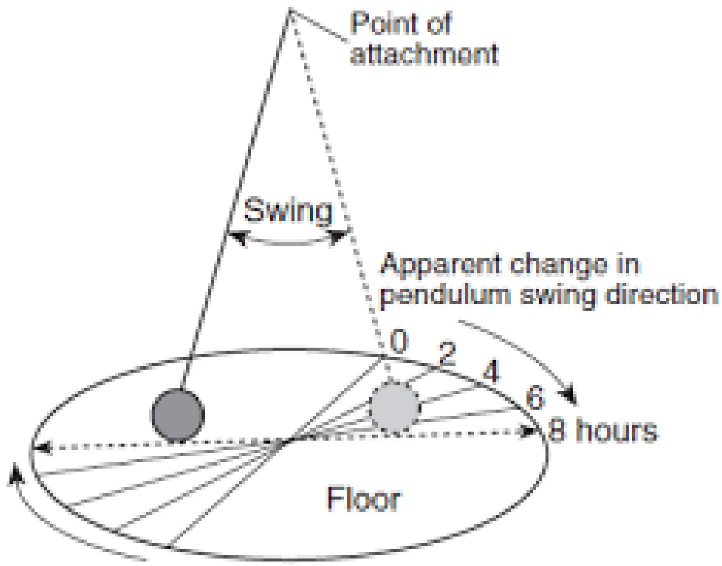
$\emptyset$ is the latitude. In the northern hemisphere it is taken as positive and in the southern hemisphere it is taken as negative.

This way of representing Coriolis force helps us understand its effect. Now imagine a cannon ball being shot, towards the north direction, in northern hemisphere. If Coriolis force was not present the ball would have travelled straight. But because of Coriolis force the ball will curve towards east. This is because the Coriolis force will act in East direction. This is shown clearly in the depiction below.



The Coriolis effect is the main reason why the direction of plane of oscillation changes for a Foucault pendulum.

Now let's finally understand how Foucault pendulum works.



Let the pendulum be in the northern hemisphere for which the latitude $\emptyset$ is positive.

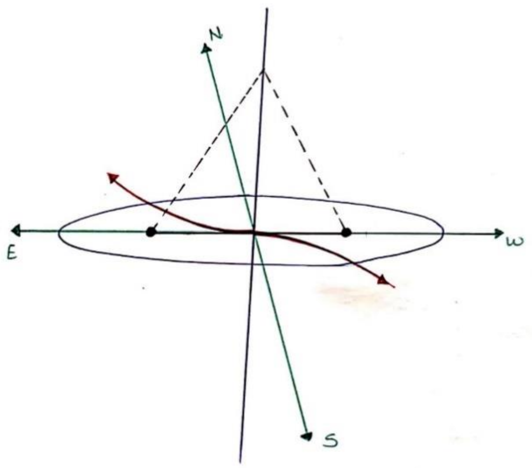
The velocity is v.

Let's consider the motion for one oscillation. When the blob of mass goes from west to east.

When the blob of mass goes from west to east (in the terms of figure from left to right) with the velocity v it is acted upon by the Coriolis force upon it and tries to curve/bent it in southern direction. This force acts infinitesimally hence the effect is not observable to naked eye.

Now again when the blob is on its return journey, from east to west (in the terms of figure from right to left) with velocity v, it is acted upon by Coriolis force again but this time it tries to curve/bent it in northern direction.

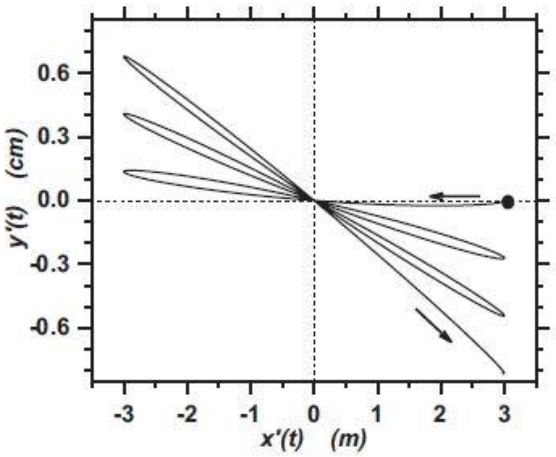
Here is a figure depicting the above process for one oscillation.



This infinitesimal deviation accumulates over time and visible changes in the plane of oscillation can be spotted typically after few hours.

Now because the Coriolis force changes sign when we do the same experiment in southern hemisphere, hence we can see why the Foucault pendulum will move in a clockwise sense in the northern hemisphere, whereas move in an anticlockwise sense in southern.

Thus, the Foucault's pendulum traces a path shown by the figure below:



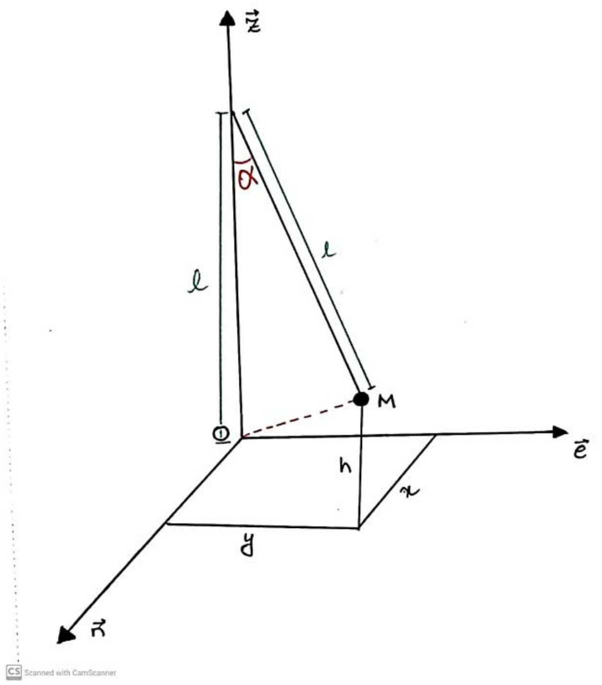
Derivation using Lagrangian Mechanics:

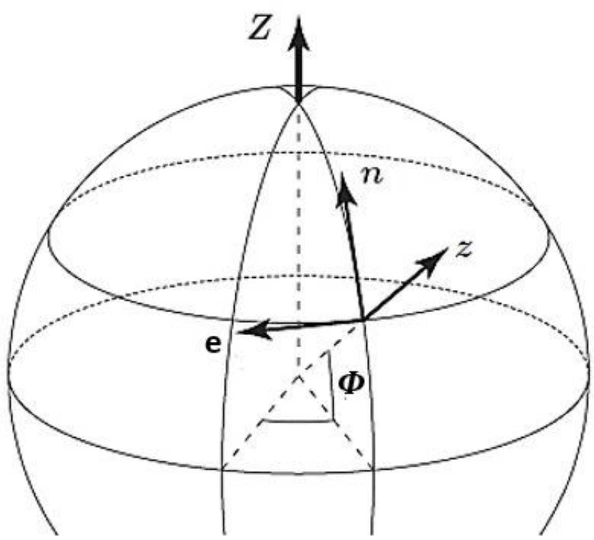
Let us take a simple pendulum having mass m and length l , which is located at some latitude ϕ on the Earth's surface – ϕ is the complementary angle between the vertical at that point and the rotation axis of the Earth.

We wish to study the effect of Earth's rotation on the motion of the pendulum. We neglect the revolution of Earth around the sun.

We take x, y as out generalized coordinates.

We take the deviations w.r.t the vertical along the south-north direction **n** and along east-west direction **e.**





In the lab frame of reference, i.e. the rotating frame, the point of suspension of the pendulum lies at the latitude ϕ, at equilibrium let the pendulum mass M be located at the origin. For small deviation α, w.r.t the vertical, the altitude is: $h = l(1 - \cos\alpha) \approx \frac{l\alpha^2}{2}$ (small angle approximation).

Thus, potential energy: $V = mgh \approx \frac{(mgl\alpha^2)}{2}$

$$\alpha^2 \approx \sin^2\alpha = \frac{0M^2}{l^2} = \frac{x^2+y^2}{l^2}$$

Thus, we get, $\boldsymbol{V(x,y) = mg\frac{x^2+y^2}{2l}}$

Since the coordinates in the frame attached to the Earth are defines as (x, y, z), thus the velocity in this frame is given by $(\dot{x}, \dot{y}, \dot{z})$, the order of deviation from the equilibrium for z coordinate is la^2, as compared to la for x and y coordinate. Thus, we assume that the motion only takes place along the horizontal plane and the relative velocity is given by: $V_r = \dot{x}\boldsymbol{n} + \dot{y}\boldsymbol{e}$

Since **n** lies in the plane defined by the pole axis (**Z**) and the true vertical **z**, thus, unit vector along the pole axis **Z**

Is formed by (**n**, **z**), such that: $\boldsymbol{Z} = \cos\phi\,\boldsymbol{n} + \sin\phi\,\boldsymbol{z}$

The velocity of a point M, which is part of a solid and which rotates with angular velocity Ω around the axis u is

$\omega\boldsymbol{u} \times \boldsymbol{OM}$ where **O** is any point on the rotation axis. The instantaneous rotation vector is directed along the pole axis:
ω = ω**Z**.
Let M0 represent the coincident pendulum point at a given time. The driving velocity is simply expressed as:
$\boldsymbol{v_e = \omega \times OM_0}$**......(i)**

$$OM_0 = x\boldsymbol{n} + y\boldsymbol{e} + (R_E + z)\boldsymbol{z}$$

From (i):

$$\boldsymbol{v_e} = -\omega y\,sin(\phi)\,\boldsymbol{n} + \omega(x\,sin(\phi) - (R_E + z)\,cos(\phi))\boldsymbol{\omega} + \omega y\,cos(\phi)\,\boldsymbol{z}$$

The absolute velocity of the pendulum is given by: $\boldsymbol{v_a} = \boldsymbol{v_e} + \boldsymbol{v_r}$ and the kinetic energy is given by: $T = \frac{1}{2}m\boldsymbol{v_a^2}$

Since the value of omega for Earth is of the order of 10^{-5} and the order of $R_E\ is\ 10^6 m$. The order of x and y is unity and the value of z is very small.

Thus, for the expression of T we neglect the terms containing ω^2x^2, ω^2y^2, ωz, $\omega^2 xz$

⇨ $\boldsymbol{v_a} = \dot{x}^2 + \dot{y}^2 - 2\omega\dot{x}y\sin\phi + \boldsymbol{2\omega\dot{y}(x\,sin(\phi)} - R_E\,cos(\phi)) - R_E\omega^2 x\,sin(2\phi)$

Thus, L = T – V

⇨ $L = \frac{1}{2}m\,[\dot{x}^2 + \dot{y}^2 - 2\omega\dot{x}y\sin\phi + \boldsymbol{2\omega\dot{y}(x\,sin(\phi)} - R_E\,cos(\phi)) - R_E\omega^2 x\,sin(2\phi)] - m\tilde{g}\frac{(x^2+y^2)}{2l}$

$$\tilde{g}\ is\ the\ effective\ gravitational\ field, \tilde{g} = g - \omega^2 R_E\cos^2\phi\ , modified\ due\ to\ centrifugal\ force.$$

For motion in the horizontal plane we solve for:

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ and $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{y}}\right) - \frac{\partial L}{\partial y} = 0$

We get:

$\ddot{x} - 2\omega\dot{y}\sin\emptyset + \frac{\tilde{g}}{l}x + \frac{1}{2}R_E\omega^2\sin(2\phi) = 0 \quad and \quad \ddot{y} + 2\omega\dot{x}\sin\phi + \frac{\tilde{g}}{l}y = 0$ ……(ii)

Let, $\tilde{x} = x - x_e$, $where\ x_e = -\frac{R_E l\omega^2\sin(2\phi)}{2g}$; $where\ \tilde{g} \cong g$

x_e, is chosen in such a way that it cancels the constant term in (ii)

Thus, we get EOM as a coupled differential equation:

$$\ddot{\tilde{x}} - 2\omega\dot{y}\sin\phi + \frac{\tilde{g}\tilde{x}}{l} = 0 \quad and \quad \ddot{y} + 2\omega\dot{\tilde{x}}\sin\phi + \frac{\tilde{g}y}{l} = 0$$

STATEMENT OF PROBLEM

Analyzing the Foucault's pendulum using Lagrangian Mechanics and studying the effects of variation of parameters of the system.

DETAILS OF METHOD OF STUDY/ALGORITHM

SOFTWARE USED: PYTHON

The EOM of Foucault pendulum obtained from Lagrangian are:

$$\ddot{\tilde{x}} - 2\omega\dot{y}\sin\phi + \frac{\tilde{g}\tilde{x}}{l} = 0 \quad and \quad \ddot{y} + 2\omega\dot{\tilde{x}}\sin\phi + \frac{\tilde{g}y}{l} = 0$$

To find the solution of this we multiply the equation for y with the complex no. I and then add it with the equation in x, we get the auxiliary variable $p = \tilde{x} + iy$, and EOM is defined as:

$$\ddot{p} + 2\omega\sin\emptyset\, i\dot{p} + \frac{\tilde{g}}{l}p = 0$$

- ⇨ $\ddot{p} = -\,2\omega\sin\emptyset\, i\dot{p} \,-\, \frac{\tilde{g}}{l}p$
- ⇨ $\ddot{p} \;=\; -A\dot{p} \;-\; Bp$

To solve this equation by numerical method (RK4), In python, we convert the 2^{nd} order ode into two 1^{st} order odes:

Let, $u = p \quad and \quad v \;=\; \dot{p}$

- ⇨ $\dot{u} = \frac{du}{dt} = \dot{p} = v$
- ⇨ $\dot{v} = \frac{dv}{dt} = \ddot{p} = -A\dot{p} - Bp = -Av - Bu$

We solve these equations using the RK-4 method:

1. Start

2. Define function f1(u, v) and f2(u, v)

3. Read values of initial condition (u0 and v0), number of steps and calculation point.

4. Calculate step size (h)

5. Set i=0

6. Loop

```
  k11 = h*f1(u0, v0)
  k21 = h*f2(u0, v0)
  k12 = h*f1(u0+(0.5*k11), v0+(0.5*k21))
  k22 = h*f2(u0+(0.5*k11), v0+(0.5*k21))
  k13 = h*f1(u0+(0.5*k12), v0+(0.5*k22))
  k23 = h*f2(u0+(0.5*k12), v0+(0.5*k22))
  k14 = h*f1(u0+(k13), v0+(k23))
  k24 = h*f2(u0+(k13), v0+(k23))
  u0 += (k11+(2*k12) +(2*k13) +k14)/6
  v0 += (k21+(2*k22) +(2*k23) +k24)/6
 While i < n
```

7. Display u and v as result

8. Stop

Solving these equations, we get the value for position co-ordinates by separating the real and imaginary parts of u

And the value for velocity coordinates is obtained by separating the real and imaginary parts of v.

RESULTS AND DISCUSSION

For Foucault's pendulum, we plot the following plots for various cases:

- Position of the bob in the horizontal plane – y(t) v/s x(t)

- And v_yv/s v_x

The values for the different parameters are taken as follow:

- g = 9.8 m/s^2
- l, length of pendulum = 98 m
- ω , angular velocity of Earth= 7.27$\times$ 10^{-5} rad/sec
- ϕ $=3$ rad

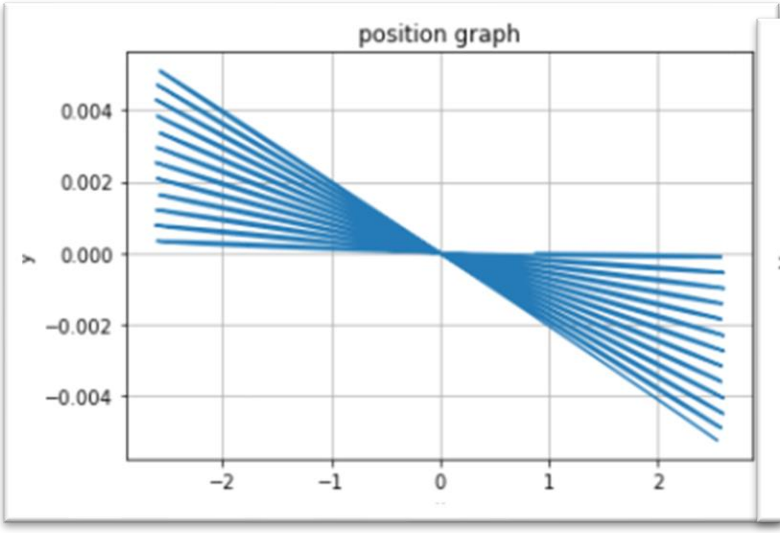
We took the initial condition of the bob as follows:

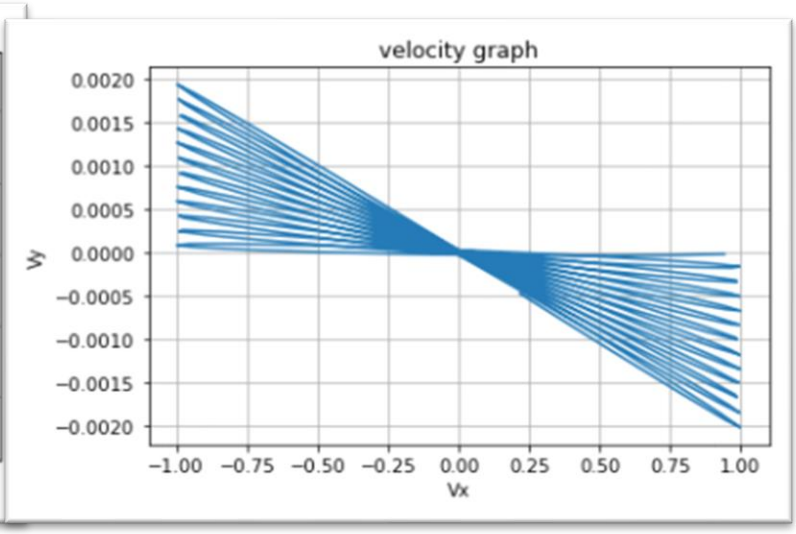
- $u_0 = 0 \Rightarrow x_0 = 0, y_0 = 0$
- $v_0 = 1 \Rightarrow v_{x_o} = 1, v_{y_0} = 0$

Plot for the above parameters is as follow:

1) Initial plots without changing any parameters.

The plot shows the path traced by Foucault pendulum for 200 sec.





This plot shows the change in the direction of plane of oscillation.

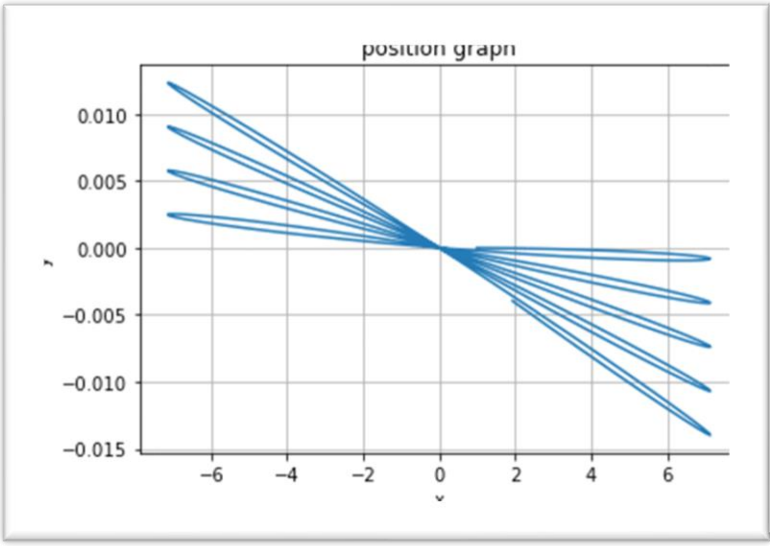
Here the spacing is small because the Coriolis force ($F_c = 2\mathrm{m}(\omega \times v)$) depends on ω and the ω for earth is very small.

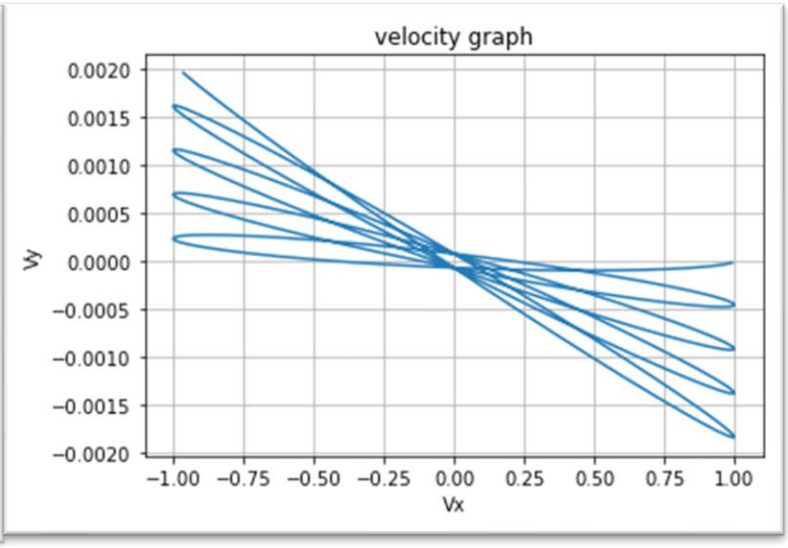
We now, vary different parameters for the pendulum, such as, length of the pendulum - l , latitude - ϕ and ω and observe how the graph changes.

First, we see for the variation of l *and* ϕ

2) Varying length of the pendulum. l = 500 m (keeping all the other parameters constant)

The plot shows the path traced by Foucault pendulum for 200 sec.





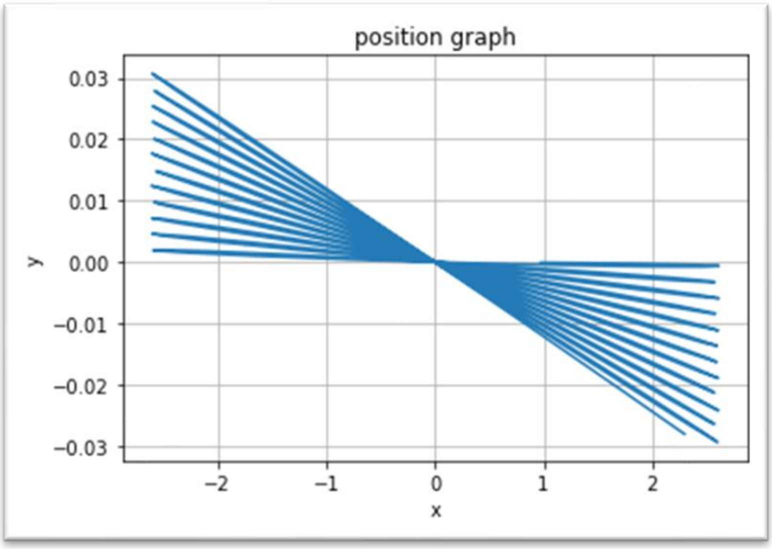
We can see the plot changes with the change in length of the pendulum.

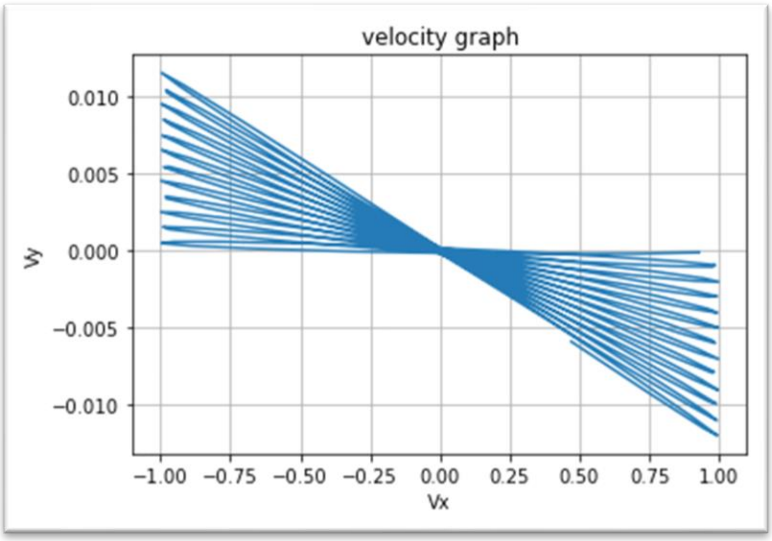
This due to the dependence of time period on length. The relation between time period and length is as follows: $T = 2\pi\sqrt{\frac{L}{g}}$. Thus, as the length increases, time period of the pendulum increases.

3) Varying the latitude, ϕ .(keeping all the other parameters constant)

The plot shows the path traced by Foucault pendulum for 200 sec.

a) $\phi = 1\,rad$





We know that, the Coriolis force, in the pendulum's plane of motion depends upon the latitude. In the frame of reference of an earth-bound observer, this Coriolis force, causes the pendulum to precess through a complete circle over a period of time.

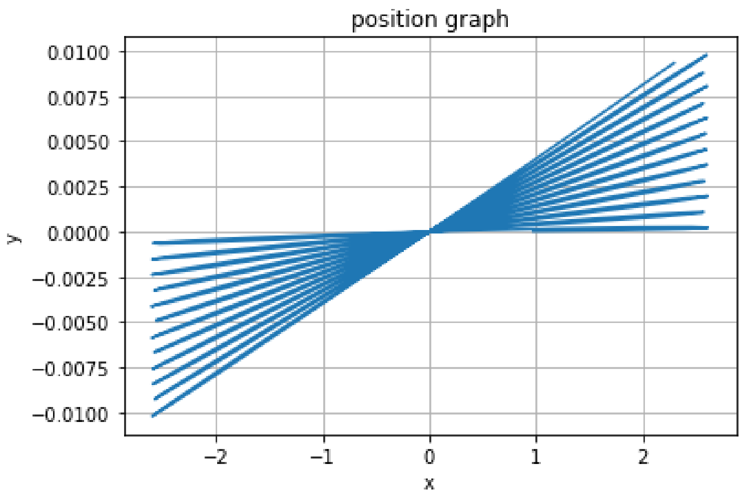
The Coriolis force is given by, $F_c = 2m\vec{v} \times \vec{\omega}_\theta$, where $\vec{v}$, is the velocity of the bob in the x-y plane.

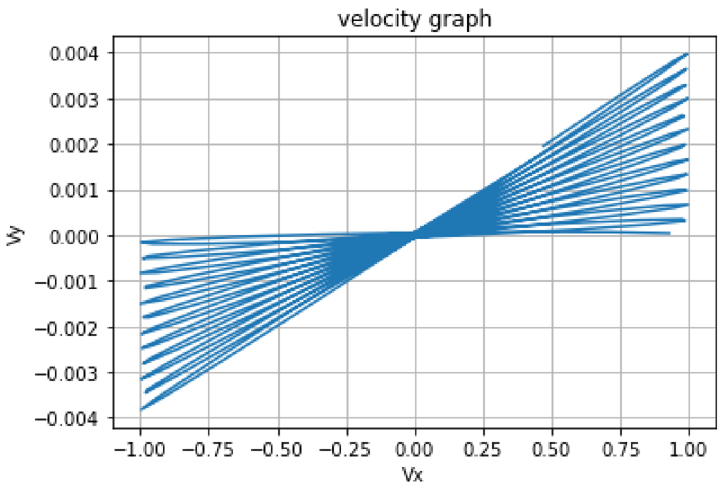
$\vec{\omega}_\theta$, is the effective angular velocity, that depends on the latitude, it is the component of Earth's axial rotation that's directed along the z-axis (i.e. perpendicular to bob's plane of motion.)

- ⇨ In the previous case, where $\phi = 3\,rad = 171.887°$, the Foucault's pendulum should take about, $\frac{1}{\sin\phi} = (\frac{1}{\sin 171.887^0})$ sidereal days (1 sidereal day = 23.9345hrs) to turn through a complete rotation. That is about 169.6hrs to complete one rotation. (since it lies very close to the equator the rotation is very slow)

- ⇨ In this case, where $\phi = 1\,rad = 57.2958°$, the Foucault's pendulum should take about, $\frac{1}{\sin\phi} = (\frac{1}{\sin 57.2958^0})$ sidereal days to turn through a complete rotation. That is about 28.48hrs to complete one rotation.

b) $\phi = 6\,rad$





For ϕ = 6 rad= 343.775°, it lies in the southern hemisphere, hence the pendulum moves in an anti-clockwise direction and this can be seen by compare these plots to the original plots.

4) Variation of angular velocity (ω) of the planet (keeping all the other parameters constant)

The variation is shown with the help of animated plots.

a) $\omega = 0.3$ rad/sec

The animation has been shown for the motion of 20sec.

For position y v/s x plot: Click here

For velocity v_yv/s v_x plot: Click here

b) $\omega = -0.3$ rad/sec

The animation has been shown for the motion of 20sec.

For position y v/s x plot: Click here

For velocity v_yv/s v_x plot: Click here

c) $\omega = 0.1$ rad/sec

The animation has been shown for the motion of 60 sec.

For position y v/s x plot: Click here

For velocity v_yv/s v_x plot: Click here

we can see from the animation that when the direction of angular velocity of the planet is flipped,the pendulum also rotates in the opposite direction and when the angular velocity of the planet is decreased to 0.1 the pendulum also rotates more slowly. This is again because of dependence of Coriolis force on ω.

Limitations:

1) It is experimentally difficult to perform. Even while performing the experiment there is always some error induced due to the following reasons:

⇨ Since the motion is damped there should be a motor installed to keep the pendulum going, this induces error in the system.

⇨ Some inherent errors as elasticity of string, error in calculating the degrees tranced accurately can't be removed

2) While performing the numerical calculation to solve the EOM using RK-4, it has a error of the order of – O(h^4) , h is the step size taken for the calculation.

Relevance of Study:

- Lagrangian formulation – Goldstein Chapter 1
- Oscillations – Goldstein Chapter 6

Bibliography:

1) Classical mechanics - Goldstien, Poole, Safko
2) numerical method for engineers- Steven C. Chapra, Raymond P. Canale
3) Marion, Jerry B.; Thornton, Stephen T. (1995). *Classical dynamics of particles and systems* (4th ed.)

4) https://www.youtube.com/playlist?list=PLq-Gm0yRYwTjpY9BlDxFGNXIaQJIOQRdo

All the codes and animations have been uploaded to the drive link given below:

https://drive.google.com/drive/folders/16sN8QStUXYkMlZnTChzSQWkLFgOqB3Gh?usp=sharing