Q1

Ans Random sample = $x_1, x_2, \ldots, x_n$ from a normal distribution with mean θ_1 ~~and~~ and variance θ_2.

→ Probability density function (PDF) is:-

$$f(x, \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{\left(-\frac{(x-\theta_1)^2}{2\theta_2}\right)}$$

→ Likelihood function $L(\theta_1, \theta_2)$ is joint PDF of sample:-

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i, \theta_1, \theta_2)$$

→ Taking log of Liklihood function, we get log liklihood function:-

$$\ln L(\theta_1, \theta_2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta_2) - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i - \theta_1)^2$$

→ differential now w.r.t to θ_1 and then setting it equal to zero:

$$\frac{\partial \ln L(\theta_1, \theta_2)}{\partial \theta_1} \Rightarrow \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i - \theta_1) = 0$$

$$\Rightarrow \sum_{i=1}^{n}(x_i - \theta_1) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - n\theta_1 = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i = n\theta_1$$

$$\Rightarrow \boxed{\theta_1 = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

→ So, MLE for θ_1 is Sample mean.
(Population)

Now differentiate w.r.t to θ_2 and then set it equal to zero.

→ $$\frac{\partial \ln L(\theta_1, \theta_2)}{\partial \theta_2} = \frac{-n}{2\theta_2} + \frac{1}{2\theta_2^2}\sum_{i=1}^{n}(x_i - \theta_1)^2 = 0$$

$$\Rightarrow -n + \theta_2^{-1}\sum_{i=1}^{n}(x_i - \theta_1)^2 = 0$$

$$n = \theta_2^{-1}\sum_{i=1}^{n}(x_i - \theta_1)^2$$

$$\boxed{\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \theta_1)^2}$$

→ So, MLE for θ_2 is Sample variance.
(Population)

Q2

Ans $m \rightarrow$ known positive integer

$\theta \rightarrow$ unknown parameter in interval $(0,1)$

$\rightarrow$ Probability Mass function (PMF) of binomial distribution:-

$$f(x, m, \theta) = {}^{m}C_{x}\, \theta^{x} (1-\theta)^{m-x}$$

where $x \rightarrow$ no. of success

${}^{n}C_{x} \rightarrow$ binomial coefficients.

$\rightarrow$ Liklihood function for sample is Joint PMF :-

$$L(\theta) = \prod_{i=1}^{n} f(x_i, m, \theta)$$

$\rightarrow$ Taking log of liklihood function.

$$\ln L(\theta) = \sum_{i=1}^{n} \left[\ln({}^{n}C_{x}) + x_i \ln(\theta) + (m - x_i) \ln(1-\theta) \right]$$

$\rightarrow$ differentiating with respect to θ and setting it equal to zero.

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \left[\frac{x_i}{\theta} - \frac{m - x_i}{1-\theta} \right] = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{x_i}{\theta} - \frac{n \cdot m - \sum_{i=1}^{n} x_i}{1-\theta} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{x_i}{\theta} = \frac{n \cdot m - \sum_{i=1}^{n} x_i}{1-\theta}$$

$$\Rightarrow (1-\theta) \sum_{i=1}^{n} x_i = \theta(n \cdot m) - \theta\left(\sum_{i=1}^{n} x_i\right)$$

$$\Rightarrow \theta(n \cdot m) = (1 - \theta + \theta) \sum_{i=1}^{n} x_i$$

$$\Rightarrow \theta = \frac{\sum_{i=1}^{n} x_i}{n \cdot m}$$

$$\Rightarrow \boxed{\theta = \left(\frac{\sum_{i=1}^{n} x_i}{n}\right) * \left(\frac{1}{m}\right)}$$

SO, MLE for θ is sample/population mean divided by m which's known number.