Engineering Mathematics-I (NMCI101)
IIT (ISM) Dhanbad
Tutorial Sheet 3(b)- Solutions

1. Question-Answer

Question1: Find the volume inside the unit sphere $x^2 + y^2 + z^2 = 1$.
To find the volume inside the unit sphere $x^2+y^2+z^2 = 1$, we convert to spherical coordinates. In spherical coordinates, we have the following relations:

$$x = \rho \sin\theta \cos\phi, \quad y = \rho \sin\theta \sin\phi, \quad z = \rho \cos\theta$$

where: $\rho \in [0, 1]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$.

The volume element in spherical coordinates is:

$$dV = \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$$

The volume of the unit sphere is given by:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$$

Now, we compute the integrals step by step.
1. **Integration with respect to** ρ:

$$\int_0^1 \rho^2 \, d\rho = \left[\frac{\rho^3}{3}\right]_0^1 = \frac{1}{3}$$

2. **Integration with respect to** θ:

$$\int_0^{\pi} \sin\theta \, d\theta = [-\cos\theta]_0^{\pi} = 2$$

3. **Integration with respect to** ϕ:

$$\int_0^{2\pi} d\phi = 2\pi$$

Finally, multiplying all the results:

$$V = \frac{1}{3} \times 2 \times 2\pi = \frac{4\pi}{3}$$

Thus, the volume of the unit sphere is:

$$V = \frac{4\pi}{3}$$

Question 2: Find the volume inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution:
We are tasked with finding the volume inside the ellipsoid given by the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

We can use a change of variables to transform the ellipsoid into a unit sphere. Let:

$$x = au, \quad y = bv, \quad z = cw,$$

which transforms the ellipsoid equation into:

$$u^2 + v^2 + w^2 = 1.$$

The Jacobian determinant of this transformation is abc, and the volume element becomes:

$$dx\,dy\,dz = abc\,du\,dv\,dw.$$

The volume integral then becomes:

$$V = \int\int\int_{\text{inside the unit sphere}} abc\,du\,dv\,dw.$$

Since the volume of the unit sphere is $\frac{4\pi}{3}$, the total volume inside the ellipsoid is:

$$V = abc \times \frac{4\pi}{3}.$$

Thus, the volume of the ellipsoid is:

$$V = \frac{4\pi}{3} \times abc.$$

Question 3: Find the volume of tetrahedron T bounded by the inequalities $x, y, z \geq 0$ and $2x + 3y + z \leq 6$.
A tetrahedron T is defined by the inequalities $x, y, z \geq 0$ and $2x + 3y + z \leq 6$.
The tetrahedron has three faces which are triangles in the coordinate planes. For example, the face of T in the xy-plane is given by $x, y \geq 0$ and $2x + 3y \leq 6$. The remaining face of T is the triangle with vertices $(3, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 6)$. If we want to describe T as a standard region for the order of integration $dx\,dy\,dz$, then T is defined by the inequalities:

$$0 \leq x \leq 3, \quad 0 \leq y \leq \frac{1}{3}(6 - 2x), \quad 0 \leq z \leq 6 - 2x - 3y.$$

If we use a triple integral to compute the volume of T, we have:

$$\text{volume}(T) = \iiint_T 1\,dV = \int_0^3 \int_0^{\frac{1}{3}(6-2x)} \int_0^{6-2x-3y} 1\,dz\,dy\,dx.$$

First, integrate with respect to z:

$$= \int_0^3 \int_0^{\frac{1}{3}(6-2x)} (6 - 2x - 3y)\,dy\,dx.$$

Next, integrate with respect to y:

$$= \int_0^3 \left[(6 - 2x)y - \frac{3}{2}y^2\right]_0^{\frac{1}{3}(6-2x)} dx = \int_0^3 \left[\frac{1}{3}(6 - 2x)^2 - \frac{3}{2} \cdot \frac{(6 - 2x)^2}{9}\right] dx.$$

Simplifying the integrand:

$$= \frac{1}{6} \int_0^3 (6 - 2x)^2\,dx.$$

Finally, integrate with respect to x:

$$= \frac{1}{6} \cdot \frac{-1}{6} \left[(6 - 2x)^3\right]_0^3 = \frac{1}{6} \cdot \frac{-1}{6} \left[0^3 - 6^3\right] = \frac{6^3}{6 \times 6} = 6.$$

Thus, the volume of the tetrahedron is volume$(T) = 6$.

Question 4: Evaluate the triple integral

$$\iiint_G \sqrt{x^2 + z^2}\,dV,$$

where G is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.
Solution: We want to evaluate the triple integral

$$\iiint_G \sqrt{x^2 + z^2}\, dV,$$

where G is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. We switch to cylindrical coordinates, where the paraboloid becomes $y = r^2$ and the volume element is $r\, dr\, d\theta\, dy$. The integral becomes:

$$\iiint_G \sqrt{x^2 + z^2}\, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2\, dy\, dr\, d\theta.$$

First, integrate with respect to y:

$$\int_{r^2}^4 r^2\, dy = r^2(4 - r^2).$$

Next, integrate with respect to r:

$$\int_0^2 r^2(4 - r^2)\, dr = \int_0^2 (4r^2 - r^4)\, dr = \left[\frac{4r^3}{3}\right]_0^2 - \left[\frac{r^5}{5}\right]_0^2 = \frac{32}{3} - \frac{32}{5} = \frac{64}{15}.$$

Finally, integrate with respect to θ:

$$\int_0^{2\pi} d\theta = 2\pi.$$

Thus, the value of the integral is:

$$\frac{128\pi}{15}.$$

Question 5: Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

Solution : To find the limits of integration, we first analyze the geometry of the tetrahedron.

The vertices are given as:

$$(0, 0, 0),\ (2, 0, 0),\ (0, 2, 0),\ (0, 0, 2)$$

The base of the tetrahedron lies in the xy-plane and is the right triangle with vertices $(0, 0, 0)$, $(2, 0, 0)$, and $(0, 2, 0)$. The vertex $(0, 0, 2)$ is the peak of the tetrahedron in the z-direction.

We now determine the limits for x, y, and z.

1. The variable z varies between 0 and 2:

$$0 \le z \le 2$$

2. For each value of z, y varies from 0 to $2 - z$ (since the upper boundary of the triangle in the yz-plane is a line from $(0, 2)$ to $(0, 0)$):

$$0 \le y \le 2 - z$$

3. Finally, for each value of y and z, x varies from 0 to $2 - y - z$, forming the upper boundary of the tetrahedron:

$$0 \le x \le 2 - y - z$$

The triple integral of the function $F(x, y, z)$ over the tetrahedron D can be written as:

$$\iiint_D F(x, y, z)\, dV = \int_{z=0}^{2} \int_{y=0}^{2-z} \int_{x=0}^{2-y-z} F(x, y, z)\, dx\, dy\, dz$$

These are the limits of integration for the triple integral over the given tetrahedron.

Question 6: Evaluate the following integrals.

$$\textbf{(i)} \int_0^a \int_0^a \int_0^a (xy + yz + zx)\, dx\, dy\, dz \quad \textbf{(ii)} \int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z - x^2}} dy\, dx\, dz$$

$$\textbf{(iii)} \int_0^2 \int_0^2 \int_0^z (4 - x^2)(2x + y)\, dz\, dy\, dx$$

Solutions (i) Solution: We can break this integral into three separate integrals due to the linearity of integration:

$$\begin{aligned}\int_0^a \int_0^a \int_0^a (xy + yz + zx)\, dx\, dy\, dz = \int_0^a \int_0^a \int_0^a xy\, dx\, dy\, dz + \int_0^a \int_0^a \int_0^a yz\, dx\, dy\, dz \\ + \int_0^a \int_0^a \int_0^a zx\, dx\, dy\, dz\end{aligned}$$

Each of these integrals will yield the same value due to symmetry, so we will compute one of them. Let's calculate $\int_0^a \int_0^a \int_0^a xy\, dx\, dy\, dz$:

$$\int_0^a \int_0^a \int_0^a xy\, dx\, dy\, dz = \int_0^a \left(\int_0^a xy\, dx\, dy\right) dz$$

First, compute the integral with respect to x:

$$\int_0^a xy\, dx = y\left[\frac{x^2}{2}\right]_0^a = y \cdot \frac{a^2}{2}$$

Next, integrate with respect to y:

$$\int_0^a \frac{a^2}{2} y\, dy = \frac{a^2}{2}\left[\frac{y^2}{2}\right]_0^a = \frac{a^2}{2} \cdot \frac{a^2}{2} = \frac{a^4}{4}$$

Finally, integrate with respect to z:

$$\int_0^a \frac{a^4}{4}\, dz = \frac{a^4}{4} \cdot a = \frac{a^5}{4}$$

Thus, the value of the first integral is:

$$3 \cdot \frac{a^5}{4} = \frac{3a^5}{4}$$

(ii) Solution: Evaluate the integral:

$$\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$$

The innermost integral can be evaluated first:

$$\int_0^{\sqrt{4z-x^2}} dy = \sqrt{4z - x^2}$$

Thus, we have:

$$\int_0^4 \int_0^{2\sqrt{z}} \sqrt{4z - x^2}\, dx\, dz$$

Next, we evaluate the integral with respect to x:

$$\int_0^{2\sqrt{z}} \sqrt{4z - x^2}\, dx$$

This is a standard integral that represents the area of a quarter circle:

$$= \frac{1}{2}\left(4z \cdot \frac{\pi}{2}\right) = 2\pi z$$

Now integrating with respect to z:

$$\int_0^4 2\pi z\,dz = 2\pi\left[\frac{z^2}{2}\right]_0^4 = 2\pi\cdot\frac{16}{2} = 16\pi$$

(iii) Solution:

$$\int_0^2\int_0^2\int_0^z (4-x^2)(2x+y)\,dx\,dy\,dz.$$

Expanding $(4-x^2)(2x+y)$ gives:

$$= \int_0^2\int_0^2\int_0^z (8x+4y-2x^3-x^2y)\,dx\,dy\,dz.$$

Now, integrating each term with respect to x from 0 to z:

$$= \int_0^2\int_0^2\left[4x^2+4yx-\frac{x^4}{2}-\frac{x^3y}{3}\right]_0^z dy\,dz.$$

Substituting x = z:

$$= \int_0^2\int_0^2\left(4z^2+4yz-\frac{z^4}{2}-\frac{z^3y}{3}\right)dy\,dz.$$

Next, integrate with respect to y:

$$= \int_0^2\left[4z^2y+2y^2z-\frac{z^4y}{2}-\frac{z^3y^2}{6}\right]_0^2 dz.$$

Substitute y = 2:

$$= \int_0^2\left(8z^2+8z-z^4-\frac{2z^3}{3}\right)dz.$$

Now, integrate with respect to z:

$$= \left[\frac{8z^3}{3}+4z^2-\frac{z^5}{5}-\frac{z^4}{6}\right]_0^2.$$

Substitute z = 2:

$$= \frac{8\cdot 8}{3}+4\cdot 4-\frac{32}{5}-\frac{16}{6}.$$

Simplify each term:

$$= \frac{64}{3}+16-\frac{32}{5}-\frac{8}{3}.$$

Combine terms over a common denominator:

$$= \frac{160}{15}+\frac{240}{15}-\frac{96}{15}-\frac{40}{15} = \frac{264}{15} = \frac{88}{5}.$$

Thus, the final answer is:

$$\int_0^2\int_0^2\int_0^z (4-x^2)(2x+y)\,dx\,dy\,dz = \frac{88}{5}.$$

Question 7 A solid "trough" of constant density si bounded below by the surface $z = 4y$, above by the plane $z = 4$, and on the ends by the planes $x = 1$ and $x = -.1$ Find the center of mass and the moments of inertia with respect to the three axes.

Solution We are given a solid trough with constant density ρ that is bounded by the following surfaces: Below by the surface $z = 4y^2$, Above by the plane $z = 4$, On the ends by the planes $x = 1$ and $x = -1$.

We are tasked with finding the center of mass and the moments of inertia with respect to the three coordinate axes.

1. Volume of the Solid

The volume of the solid is defined by the limits: $x \in [-1, 1]$, $y \in [-1, 1]$, $z \in [4y^2, 4]$.

The volume element in Cartesian coordinates is $dV = dz\,dy\,dx$. Hence, the total mass M is:

$$M = \rho \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=4y^2}^{4} dz\,dy\,dx.$$

First, we compute the innermost integral:

$$\int_{z=4y^2}^{4} dz = 4 - 4y^2.$$

Now, substituting this result into the next integral:

$$M = \rho \int_{x=-1}^{1} \int_{y=-1}^{1} (4 - 4y^2) dy\,dx.$$

We compute the y-integral:

$$\int_{y=-1}^{1} (4 - 4y^2) dy = 4 \int_{y=-1}^{1} dy - 4 \int_{y=-1}^{1} y^2 dy = 4(2) - 4\left(\frac{2}{3}\right) = 8 - \frac{8}{3} = \frac{16}{3}.$$

Thus, the total mass becomes:

$$M = \rho \int_{x=-1}^{1} \frac{16}{3} dx = \rho \cdot \frac{16}{3} \cdot 2 = \frac{32\rho}{3}.$$

2. Center of Mass

The coordinates of the center of mass are given by:

$$\bar{x} = \frac{1}{M} \int\int\int x \rho\,dV, \quad \bar{y} = \frac{1}{M} \int\int\int y \rho\,dV, \quad \bar{z} = \frac{1}{M} \int\int\int z \rho\,dV.$$

By symmetry, the solid is symmetric about the yz-plane, so:

$$\bar{x} = 0.$$

The solid is also symmetric about the xz-plane, so:

$$\bar{y} = 0.$$

We now calculate $\bar{z}$:

$$\bar{z} = \frac{1}{M} \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=4y^2}^{4} z \rho\,dz\,dy\,dx.$$

First, we compute the z-integral:

$$\int_{z=4y^2}^{4} z\,dz = \frac{1}{2} \left[z^2\right]_{4y^2}^{4} = \frac{1}{2} \left(16 - 16y^4\right) = 8(1 - y^4).$$

Now, substitute this into the y-integral:

$$\bar{z} = \frac{\rho}{M} \int_{x=-1}^{1} \int_{y=-1}^{1} 8(1 - y^4) dy\,dx.$$

First, compute the y-integral:

$$\int_{y=-1}^{1} 8(1-y^4)dy = 8\left(2-\frac{2}{5}\right) = 8\times\frac{8}{5} = \frac{64}{5}.$$

Thus, the z-coordinate of the center of mass is:

$$\bar{z} = \frac{\rho}{M}\cdot\frac{64}{5}\cdot 2 = \frac{64\rho}{5M}.$$

Since $M = \frac{32\rho}{3}$, we substitute this value into the equation:

$$\bar{z} = \frac{64\rho}{5\times\frac{32\rho}{3}} = \frac{64\times 3}{5\times 32} = \frac{192}{160} = 1.2.$$

Thus, the center of mass is:

$$(\bar{x},\bar{y},\bar{z}) = (0,0,1.2).$$

3. Moments of Inertia

The moments of inertia are given by:

$$I_x = \int\int\int \rho(y^2+z^2)\,dV, \quad I_y = \int\int\int \rho(x^2+z^2)\,dV, \quad I_z = \int\int\int \rho(x^2+y^2)\,dV.$$

These can be computed using similar methods of integration as outlined above for the volume and center of mass. The exact calculations would depend on evaluating these integrals over the specified bounds.

Question 8 Find the moment of inertia of a solid sphere W of uniform density and radius a about the z-axis.

Solution: Here, density is given to be a constant, say k, i.e., $\delta(x,y,z) = k$ for all (x,y,z). According to the definition, the moment of inertia about the z-axis is

$$\begin{aligned} I_z &= \iiint_W (x^2+y^2)\,\delta(x,y,z)\,dx\,dy\,dz \\ &= \iiint_W k(x^2+y^2)\,dx\,dy\,dz \end{aligned}$$

The region of integration W in this case can be easily described by spherical coordinates. In these coordinates, the region W can be described as

$$W = \{(r,\theta,\phi) \mid 0\le r\le a,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi\}$$

Further, $x^2+y^2 = r^2\sin^2\phi$, and hence,

$$\begin{aligned} I_z &= \int_0^a \int_0^{2\pi} \int_0^{\pi} k\,r^2\sin^2\phi\, r^2\sin\phi\,d\theta\,d\phi\,dr \\ &= k\int_0^a \int_0^{2\pi} \int_0^{\pi} r^4\sin^3\phi\,d\theta\,d\phi\,dr \\ &= 2\pi\int_0^a \int_0^{\pi} r^4\sin^3\phi\left[\int_0^{2\pi} d\theta\right]d\phi\,dr \end{aligned}$$

But

$$\begin{aligned} \int_0^{\pi}\sin^3\phi\,d\phi &= \int_0^{\pi}\sin\phi\,(1-\cos^2\phi)\,d\phi \\ &= -\left[\cos\phi - \frac{\cos^3\phi}{3}\right]_0^{\pi} \end{aligned}$$

$$= \frac{4}{3}$$

Thus,

$$I_z = \int_0^a 2\pi k \cdot \frac{4}{3} \cdot r^4 \, dr$$

$$= \frac{8\pi k}{3} \left[\frac{r^5}{5} \right]_0^a$$

$$= \frac{8\pi k a^4}{15}$$

Question 9 Find the center of gravity (centroid) of a solid object bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ using triple integration.

Solution

The center of gravity (centroid) $(\bar{x}, \bar{y}, \bar{z})$ of a solid object in three-dimensional space is given by:

$$\bar{x} = \frac{1}{V} \iiint_D x \, dV, \quad \bar{y} = \frac{1}{V} \iiint_D y \, dV, \quad \bar{z} = \frac{1}{V} \iiint_D z \, dV,$$

where V is the volume of the region D.

Step 1: Set up the Region

The region D is bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Step 2: Convert to Cylindrical Coordinates

In cylindrical coordinates, where $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$:

- The paraboloid becomes $z = r^2$,
- The plane $z = 4$ remains $z = 4$,
- The region D is described by $0 \le r \le 2$, $0 \le \theta \le 2\pi$, and $r^2 \le z \le 4$.

Step 3: Find the Volume V

$$V = \iiint_D dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dz \, dr \, d\theta.$$

Integrate with respect to z:

$$V = \int_0^{2\pi} \int_0^2 [rz]_{r^2}^4 \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r(4 - r^2) \, dr \, d\theta.$$

Integrate with respect to r:

$$V = \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta = \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4} \right]_0^2 d\theta.$$

$$V = \int_0^{2\pi} (8 - 4) \, d\theta = \int_0^{2\pi} 4 \, d\theta = 8\pi.$$

Step 4: Calculate $\bar{x}$ and $\bar{y}$

By symmetry, $\bar{x} = 0$ and $\bar{y} = 0$.

Step 5: Calculate $\bar{z}$

$$\bar{z} = \frac{1}{V}\iiint_D z\,dV = \frac{1}{8\pi}\int_0^{2\pi}\int_0^2\int_{r^2}^4 z\cdot r\,dz\,dr\,d\theta.$$

Integrate with respect to z:

$$\bar{z} = \frac{1}{8\pi}\int_0^{2\pi}\int_0^2\left[\frac{z^2}{2}\cdot r\right]_{r^2}^4 dr\,d\theta = \frac{1}{8\pi}\int_0^{2\pi}\int_0^2 \frac{r(16-r^4)}{2}\,dr\,d\theta.$$

$$\bar{z} = \frac{1}{8\pi}\int_0^{2\pi}\int_0^2 (8r - \frac{r^5}{2})\,dr\,d\theta.$$

Integrate with respect to r:

$$\bar{z} = \frac{1}{8\pi}\int_0^{2\pi}\left[4r^2 - \frac{r^6}{12}\right]_0^2 d\theta = \frac{1}{8\pi}\int_0^{2\pi}\left(16 - \frac{64}{12}\right)d\theta.$$

$$\bar{z} = \frac{1}{8\pi}\int_0^{2\pi}\left(\frac{128}{12}\right)d\theta = \frac{16}{3}.$$

Final Answer

The center of gravity (centroid) is:

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{16}{3}\right).$$

Question 10 Find the mass of a solid hemisphere of radius R with a density function $\rho(x, y, z) = kz$, where k is a constant. The hemisphere is located above the xy-plane (i.e., $z \geq 0$).

Solution

In three-dimensional space, the mass M of a solid object with a density function $\rho(x, y, z)$ is given by:

$$M = \iiint_D \rho(x, y, z)\,dV,$$

where D is the region occupied by the object.

Step 1: Set up the Region D

The region D is the upper hemisphere of radius R, described by the inequality $x^2 + y^2 + z^2 \leq R^2$ and $z \geq 0$.

Step 2: Convert to Spherical Coordinates

In spherical coordinates, where $x = \rho\sin\theta\cos\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\theta$:

- The density function $\rho(x, y, z) = kz$ becomes $\rho(\rho, \theta, \phi) = k\rho\cos\theta$,
- The volume element dV becomes $\rho^2\sin\theta\,d\rho\,d\theta\,d\phi$.

The bounds for the spherical coordinates are:

$$0 \leq \rho \leq R, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq 2\pi.$$

Step 3: Set up the Integral for Mass

$$M = \iiint_D \rho(x, y, z)\, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^R k\rho \cos\theta \cdot \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi.$$

Simplifying inside the integral:

$$M = k \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^R \rho^3 \cos\theta \sin\theta\, d\rho\, d\theta\, d\phi.$$

Step 4: Evaluate the Integral

Integrate with respect to ρ:

$$M = k \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\right]_0^R \cos\theta \sin\theta\, d\theta\, d\phi = k \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{R^4}{4} \cos\theta \sin\theta\, d\theta\, d\phi.$$

Separate constants and simplify:

$$M = \frac{kR^4}{4} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta\, d\theta\, d\phi.$$

Integrate with respect to θ, using the identity $\sin\theta \cos\theta = \frac{1}{2}\sin 2\theta$:

$$M = \frac{kR^4}{4} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2\theta\, d\theta\, d\phi = \frac{kR^4}{8} \int_0^{2\pi} \left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} d\phi.$$

$$M = \frac{kR^4}{8} \int_0^{2\pi} \left(\frac{1}{2}\right) d\phi.$$

Integrate with respect to ϕ:

$$M = \frac{kR^4}{8} \cdot \frac{1}{2} \cdot 2\pi = \frac{\pi k R^4}{8}.$$

Final Answer

The mass of the solid hemisphere is:

$$M = \frac{\pi k R^4}{8}.$$

Question 11 Evaluate the triple integral

$$\iiint_T x\, y\, z\, dx\, dy\, dz$$

where T is the region in the xyz-space bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$, and $z = x + y$. Use the transformation $u = x + y$, $v = x - y$, and $w = z$.

Solution

The given transformations are:

$$u = x + y, \quad v = x - y, \quad w = z.$$

To find the boundaries of T in the uvw-coordinate system, let's examine the planes in the xyz-space:

1. $x = 0$ becomes $u = y$ (since $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$). 2. $y = 0$ becomes $u = x$. 3. $z = 0$ becomes $w = 0$. 4. $x + y = 1$ becomes $u = 1$. 5. $z = x + y$ becomes $w = u$.

So the region T' in uvw-space is bounded by:

$$u = 0 \text{ to } 1, \quad v = -u \text{ to } u, \quad w = 0 \text{ to } u.$$

Solving for x, y, and z: Using the transformations, we find x, y, and z as follows:

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}, \quad z = w.$$

Determining the Jacobian The Jacobian determinant J is calculated as:

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{1}{2}.$$

Thus, $|J| = \frac{1}{2}$.

Transforming the Integral In the new variables, the integral becomes:

$$\iiint_T x\,y\,z\,dx\,dy\,dz = \iiint_{T'} \frac{u+v}{2} \cdot \frac{u-v}{2} \cdot w \cdot |J|\,du\,dv\,dw = \frac{1}{2} \iiint_{T'} \frac{(u^2-v^2)w}{4}\,du\,dv\,dw.$$

Simplifying, we get:

$$= \frac{1}{8} \iiint_{T'} u^2 w - v^2 w\,du\,dv\,dw.$$

Evaluating the Integral We split the integral:

$$= \frac{1}{8}\left(\int_0^1 \int_{-u}^u \int_0^u u^2 w\,dw\,dv\,du - \int_0^1 \int_{-u}^u \int_0^u v^2 w\,dw\,dv\,du \right).$$

1. For the first term:

$$= \frac{1}{8} \int_0^1 \int_{-u}^u \left[\frac{u^2 w^2}{2} \right]_0^u dv\,du = \frac{1}{8} \int_0^1 \int_{-u}^u \frac{u^4}{2}\,dv\,du.$$

2. Integrating with respect to v:

$$= \frac{1}{8} \int_0^1 \frac{u^4}{2}(2u)\,du = \frac{1}{8} \int_0^1 u^5\,du = \frac{1}{8} \cdot \frac{u^6}{6}\Big|_0^1 = \frac{1}{48}.$$

The second term can be shown to result in zero due to symmetry, so the final answer is:

$$\iiint_T x\,y\,z\,dx\,dy\,dz = \frac{1}{48}.$$