Lectures 10

Sec A,B : 26/09, 06/10

Sec C,D : 09/10 , 11/10

Discussion continued from Lec. 9

→ Cylindrical Polar Coordinates

→ Some example problems on calculation of Electric field.

Written in Lecture - 9 notes.

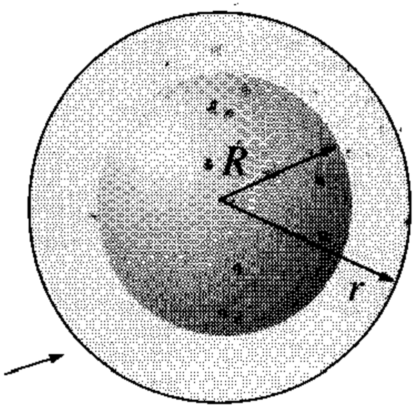
Dirac - delta function written in Lecture - 9 notes

Find the field outside a uniformly charged solid sphere of radius R and total charge q.

Griffiths Ex 2.2.

Symmetry tells us that $\vec{E}$ is directed radially outward $\rightarrow$ no $\hat{\theta}$ or $\hat{\phi}$ in $\vec{E}$.

$\vec{E} = E\,\hat{r}$ $\qquad d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$



Gauss's law

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

$$\int_S |\mathbf{E}|\, da = |\mathbf{E}| \int_S da = |\mathbf{E}|\, 4\pi r^2.$$

E is independent of θ & ϕ.

$$|\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} q,$$

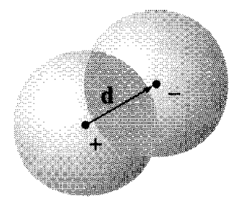
$$\int_S \mathbf{E} \cdot d\mathbf{a} = \int_S |\mathbf{E}|\, da,$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}.}$$

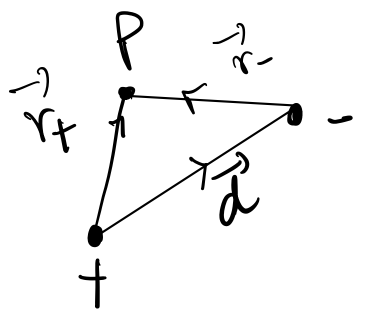
HW: Find electric field inside a solid sphere of charge 'q'.

Problem 2.18 in Griffiths

Problem 2.18 Two spheres, each of radius R and carrying uniform charge densities $+\rho$ and $-\rho$, respectively, are placed so that they partially overlap (Fig. 2.28). Call the vector from the positive center to the negative center $\mathbf{d}$. Show that the field in the region of overlap is constant, and find its value. [*Hint:* Use the answer to Prob. 2.12.]



From Problem 2.12, field inside a positively charged sphere : $\vec{E}_+ = \frac{\rho}{3\epsilon_0}\vec{r}_+$



'P' is a pt. inside the spheres in the overlapping region.

$\vec{E}_- = -\frac{\rho}{3\epsilon_0}\vec{r}_-$, So total field $\vec{E} = \frac{\rho}{3\epsilon_0}(\vec{r}_+ - \vec{r}_-)$

$$\vec{E} = \frac{\rho}{3\epsilon_0}\vec{d}$$

$$\underbrace{Q_1 = Q_2}_{\text{Charge of each sphere}} = \frac{4}{3}\pi R^3 \rho = Q \Rightarrow \rho = \frac{3Q}{4\pi R^3}$$

↳ Radius of the individual Spheres

Answer $\Rightarrow \vec{E} = \frac{Q\vec{d}}{4\pi\epsilon_0 R^3}$

Convention:

$\vec{d}$: from +ve to -ve is +ve direction

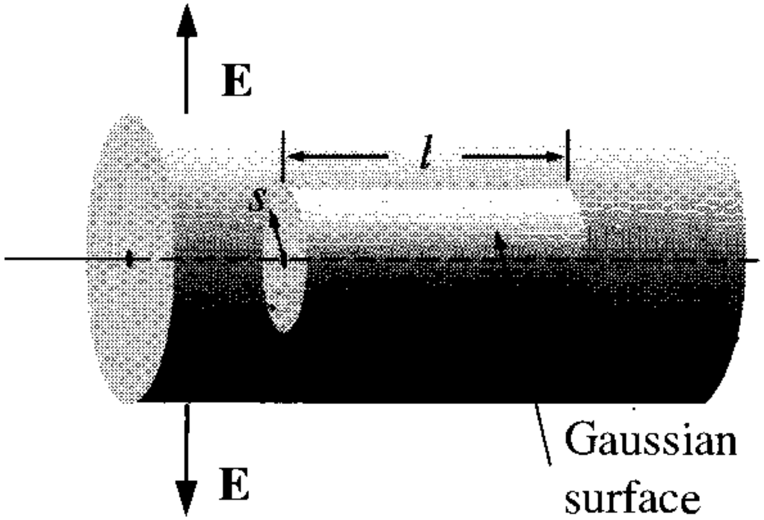
A long cylinder carries a charge distribution that is proportional to the distance from the axis. Find the electric field inside the cylinder.

Griffiths Ex 2.3

charge density $\rho = k s$.

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

$\int dz' = l$, $\int d\phi' = 2\pi$



$$Q_{\text{enc}} = \int \rho \, d\tau = \int (ks')(s' \, ds' \, d\phi \, dz) = 2\pi k l \int_0^s s'^2 \, ds' = \frac{2}{3}\pi k l s^3.$$

$d\tau' = s' ds' d\phi' dz'$

da on curved surface $= s \, d\phi' dz'$

$$\int \mathbf{E} \cdot d\mathbf{a} = \int |\mathbf{E}| \, da = |\mathbf{E}| \int da = |\mathbf{E}| \, 2\pi s l, \qquad |\mathbf{E}| \, 2\pi s l = \frac{1}{\epsilon_0} \frac{2}{3} \pi k l s^3.$$

from symmetry $\vec{E}$ must point radially outward! & $\vec{E}$ is independent of ϕ, z.

$$\boxed{\mathbf{E} = \frac{1}{3\epsilon_0} k s^2 \hat{\mathbf{s}}.}$$

Potential difference :

Evidently, the potential *difference* between two points **a** and **b** is

$$\begin{aligned} V(\mathbf{b}) - V(\mathbf{a}) &= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l} \\ &= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}. \end{aligned} \tag{2.22}$$

Now, the fundamental theorem for gradients states that

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l},$$

so

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

Since, finally, this is true for *any* points **a** and **b**, the integrands must be equal:

$$\boxed{\mathbf{E} = -\nabla V.} \tag{2.23}$$

READ SECTION 2.3.2

Poisson's Equation and Laplace's Equation :

$$\vec{E} = -\vec{\nabla} V$$

We have,

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla} \times \vec{E} = 0$$

So,

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla} V) = -\nabla^2 V$$

Thus, $\nabla^2 V = -\rho/\epsilon_0$ $\Leftarrow$ Poisson's Equation

$$\nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \Rightarrow \nabla^2 = \text{LAPLACIAN}$$

If $\rho = 0$, $\nabla^2 V = 0 \Leftarrow$ Laplace's Eqn

For the 'curl' law:

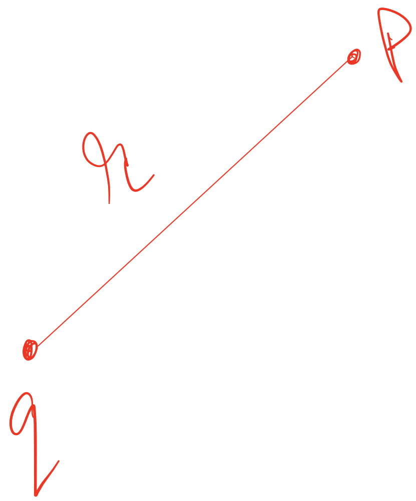
$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times (-\vec{\nabla} V)$$

But this is not a new info.

In other words,

. But that's no condition on V—curl of gradient is *always* zero. Of course, we *used* the curl law to show that $\mathbf{E}$ could be expressed as the gradient of a scalar, so it's not really surprising that this works out: $\nabla \times \mathbf{E} = 0$ *permits* $\mathbf{E} = -\nabla V$; in return, $\mathbf{E} = -\nabla V$ *guarantees* $\nabla \times \mathbf{E} = 0$. It takes only *one* differential equation (Poisson's) to determine V, because V is a scalar; for $\mathbf{E}$ we needed *two,* the divergence and the curl.

Potential of a Localized Charge distribution :



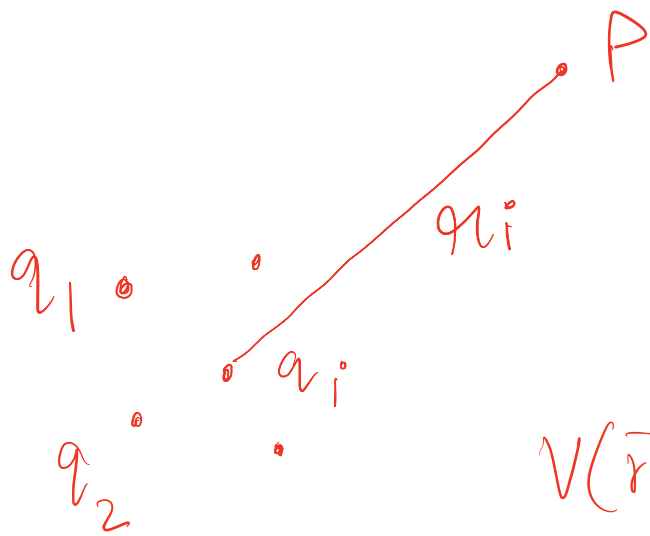
Set the reference point at infinity, potential of a point charge 'q' at the origin is,

$$V(r) = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr'$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{q}{r}$$

↳ distance from the charge to the point where potential is calculated.

If you read section 2.3.2, you will be convinced that potential also follows the superposition principle. Thus, the potential due to a collection of charges



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}}\, dq$$

For a volume charge

$$= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\, d\tau'$$

Similarly, for line charge:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r}\,')}{\mathscr{r}}\, dl'$$

For surface charge:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r}\,')\, da'}{\mathscr{r}}$$

Find the potential inside and outside a spherical shell of radius R (Fig. 2.31), which carries a uniform surface charge. Set the reference point at infinity.

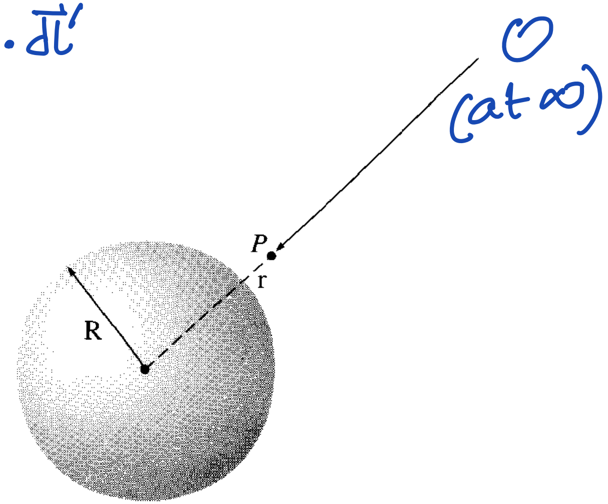
Griffiths Ex 2.6

using $V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E}(\vec{r}\,') \cdot d\vec{l}\,'$

for $d\vec{l}\,' = dr'\,\hat{r}$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}},$$

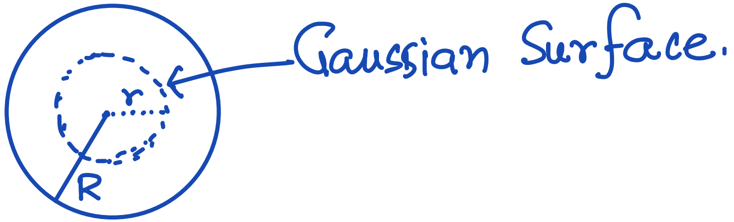
(outside)



$$V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$ (outside)

$$V(r) = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{R}\frac{q}{r'^2}\,dr' - \int_{R}^{r}(0)\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{R} + 0 = \frac{1}{4\pi\epsilon_0}\frac{q}{R}.$$ (inside)

$\vec{E}_{inside} = 0$



$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = 0$$

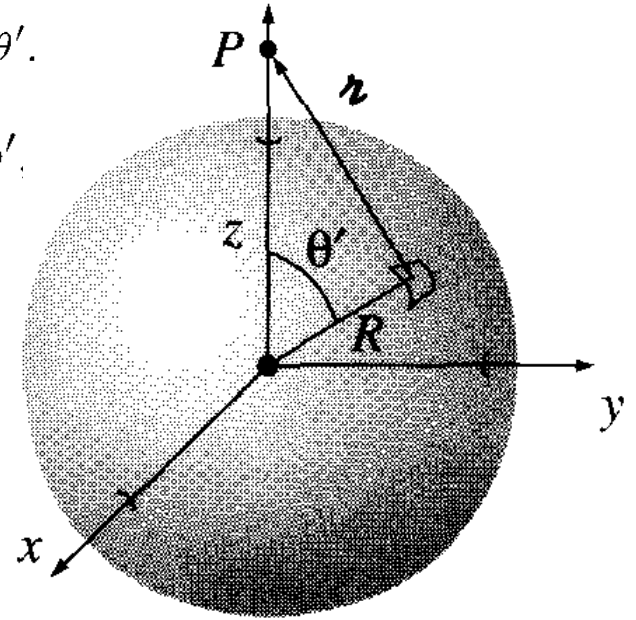
$$\Rightarrow E \cdot 4\pi r^2 = 0 \Rightarrow \boxed{E = 0}$$

Find the potential inside and outside a spherical shell of radius R (Fig. 2.31), which carries a uniform surface charge. Set the reference point at infinity. Griffiths Ex-2.7.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}}\,da'.$$

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'.$$

$$da' = R^2 \sin\theta'\,d\theta'\,d\phi'.$$



$$\begin{aligned}
4\pi\epsilon_0 V(z) &= \sigma\int \frac{R^2\sin\theta'\,d\theta'\,d\phi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}\\
&= 2\pi R^2\sigma\int_0^\pi \frac{\sin\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}\,d\theta'\\
&= 2\pi R^2\sigma\left(\frac{1}{Rz}\sqrt{R^2+z^2-2Rz\cos\theta'}\right)\Big|_0^\pi\\
&= \frac{2\pi R\sigma}{z}\left(\sqrt{R^2+z^2+2Rz}-\sqrt{R^2+z^2-2Rz}\right)\\
&= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2}-\sqrt{(R-z)^2}\right].
\end{aligned}$$

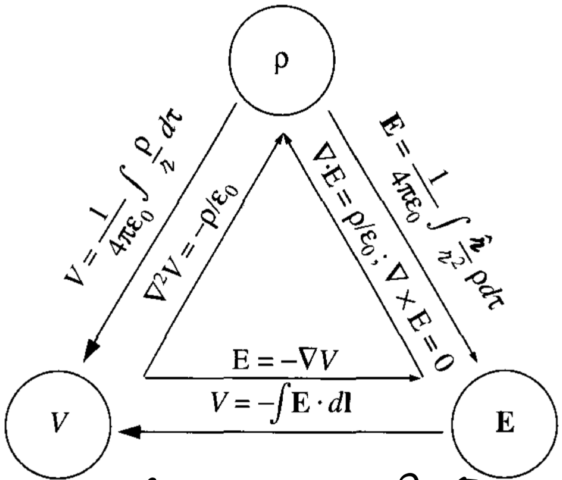
$$\boxed{q = 4\pi R^2\sigma,\ V(z) = (1/4\pi\epsilon_0)(q/z)}$$

$$\boxed{V(z) = (1/4\pi\epsilon_0)(q/R)}$$

$$\boxed{\begin{aligned}
V(z) &= \frac{R\sigma}{2\epsilon_0 z}[(R+z)-(z-R)] = \frac{R^2\sigma}{\epsilon_0 z}, &&\text{outside;}\\
V(z) &= \frac{R\sigma}{2\epsilon_0 z}[(R+z)-(R-z)] = \frac{R\sigma}{\epsilon_0}, &&\text{inside.}
\end{aligned}}$$

SUMMARY :

In the typical electrostatic problem you are given a source charge distribution ρ, and you want to find the electric field $\mathbf{E}$ it produces. Unless the symmetry of the problem admits a solution by Gauss's law, it is generally to your advantage to calculate the potential first, as an intermediate step. These, then, are the three fundamental quantities of electrostatics: ρ, $\mathbf{E}$, and V. We have, in the course of our discussion, derived all six formulas interrelating them. These equations are neatly summarized in Fig. 2.35. We began with just two experimental observations: (1) the principle of superposition—a broad general rule applying to *all* electromagnetic forces, and (2) Coulomb's law—the fundamental law of electrostatics. From these, all else followed.



HOMEWORK : READ SECTION 2.5 of Griffiths

(FACTS ABOUT CONDUCTORS)

EXAMPLE :
1) What is a conductor ?
2) What is $\vec{E}$ inside a conductor ?
3) What are the charges inside called ?
4) What is ρ inside a conductor ?

END OF LECTURE 10

Continue discussion on Dirac-delta function