Lecture-14

1. Maxwell's Correction to Ampere's law
2. Wave Equation in Electrodynamics
3. Energy density and Poynting vector.
   ↳ Included in notes but not in syllabus

Maxwell's Equations in free space: [So far]

(i) $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ Gauss's law

(ii) $\vec{\nabla}\cdot\vec{B} = 0$ (no name)

(iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law)

(iv) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ (Ampere's law)

We know that $\vec{\nabla}\cdot(\vec{\nabla}\times \text{Vector}) = 0$
(Mathematics)

Let's take divergence of Eqs. (iii) & (iv).

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\cdot\left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B})$$
$$= 0 \quad \checkmark$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot(\mu_0\vec{J}) = \mu_0(\vec{\nabla}\cdot\vec{J})$$
$$\overset{?}{=} 0 \quad \text{Always?}$$

Inconsistency with Mathematics. $\times$

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True only for steady currents.

I.e. if we go beyond Magnetostatics Ampere's law fails.

So Maxwell introduced a correction. But before we introduce the correction let us try to understand the problem in a standard setting.

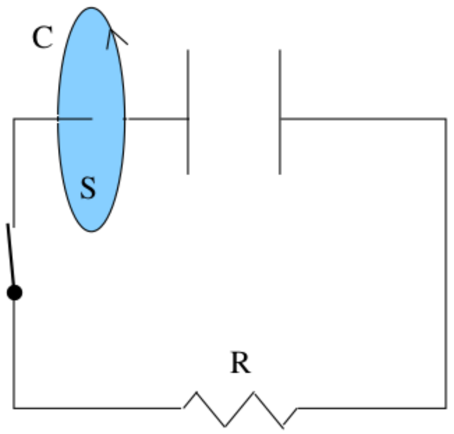


Figure 42: This choice of surface suggests there is a magnetic field

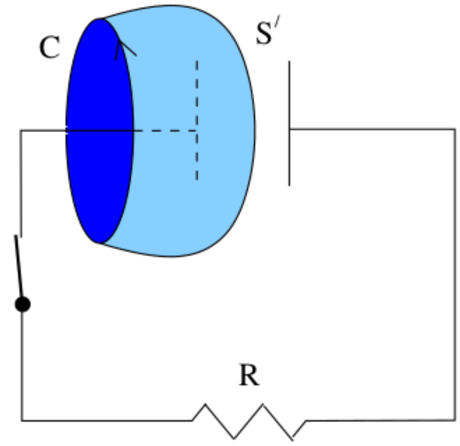


Figure 43: This choice of surface suggests there is none.

The circuit includes a switch such that when the switch is closed, the current will flow out of the capacitor and through the circuit, ultimately heating the resistor.

(1) If we choose 'C' & 'S' surface:

$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I$$

↳ Current through the wire that is changing with time. (Non-steady current).

(2) Suppose we choose to bind the curve 'C' using another surface that looks like S′. The surface now lies inside the gap b/w the capacitor plates.

Then there is no current passing through S'. So, Ampere's law: $\int_C \vec{B} \cdot d\vec{r} = 0$?

WE CANNOT USE AMPERE'S LAW HERE BECAUSE OF NON-STEADY CURRENT.

SOLUTION: NEED TO ADD A TERM CALLED THE DISPLACEMENT CURRENT TO FIX THE AMPERE'S LAW FOR NON-STEADY CURRENT.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \boxed{\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Take Divergence on both sides:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \left(\vec{\nabla} \cdot \vec{J} + \epsilon_0 \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} \right)$$

$$= \mu_0 \left(\vec{\nabla} \cdot \vec{J} + \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) \right)$$

$$= \mu_0 \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right)$$

Recall Continuity equation $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

This gives, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$

Consistent with Mathematics.

$$\epsilon_0 \frac{\partial \vec{E}}{\partial t} \equiv \text{Displacement Current}$$

Connecting back to the capacitor problem:

The build up of charge on the capacitor plates leads to a time dependent electric field b/w the plates. This changing electric field is related to the current $I(t)$ flowing through the circuit.

The changing electric field further leads to a magnetic field.

- The electric displacement current introduced by Maxwell as a correction to Ampere's law is of great importance to understand 'light' as an electromagnetic wave.

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Potential formulation:

Electrostatics + Magnetostatics

$$\nabla^2 V = -\rho/\epsilon_0 \qquad \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Electromagnetic Wave

Let's now look at the Maxwell's equations in vacuum, i.e., $\rho = 0$; $J = 0$

Absence of external charge and current.

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Change in electric field causes a change in magnetic field and vice-versa.

Let's calculate the time derivative :

$$\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) = \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = \underline{\underline{-\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right)}}$$

Use the identity:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E} \checkmark$$

But in vacuum, $\vec{\nabla} \cdot \vec{E} = 0$

So,

$$\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla^2 \vec{E} \quad ; \quad c = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$$

$$\Rightarrow \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \nabla^2 \vec{E} = 0$$

Wave equation and speed of the waves is 'c'. Put in $\mu_0 \epsilon_0$ values & you will get the speed of light.

Identical Manipulations for the magnetic field yields :

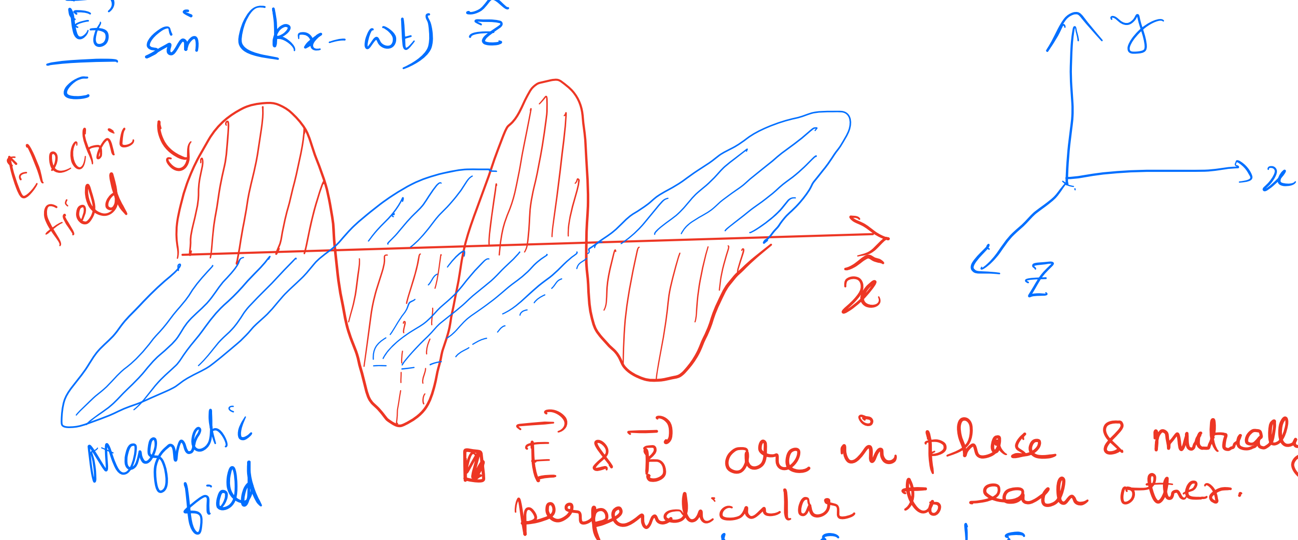
$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{B}$$

Electromagnetic waves are light.

On solving the wave equations we can see that, $\vec{E}$ & $\vec{B}$ are orthogonal to each other but oscillate in phase.

$$\vec{E} = E_0 \sin\left(\omega\left(\frac{x}{c} - t\right)\right)\hat{y} = \vec{E}_0 \sin(kx - \omega t)\,\hat{y}$$

$$\vec{B} = \frac{\vec{E}_0}{c} \sin(kx - \omega t)\,\hat{z}$$



- $\vec{E}$ & $\vec{B}$ are in phase & mutually perpendicular to each other.
- $B_0 = k/\omega \, E_0 = \frac{1}{c} E_0$.

Finally, Electromagnetic waves carry energy.

Starting pt. Energy stored in electric and magnetic fields:

$$U = \int_{vol.} d\tau \underbrace{\left(\frac{\epsilon_0}{2} \vec{E}\cdot\vec{E} + \frac{1}{2\mu_0} \vec{B}\cdot\vec{B}\right)}_{\text{Energy density.}}$$

Transport of energy:

$$\frac{dU}{dt} = \int_{vol.} d\tau \left(\epsilon_0 \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu_0} \vec{B}\cdot\frac{\partial \vec{B}}{\partial t}\right)$$

$$= \int_{vol.} d\tau \left(\frac{1}{\mu_0} \vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{E}\cdot\vec{J} - \frac{1}{\mu_0} \vec{B}\cdot(\vec{\nabla}\times\vec{E})\right)$$

$\vec{\nabla}\cdot(\vec{A}_1 \times \vec{A}_2) = \vec{A}_2\cdot(\vec{\nabla}\times\vec{A}_1) - \vec{A}_1\cdot(\vec{\nabla}\times\vec{A}_2)$; Put $\vec{A}_1 = \vec{E}$, $\vec{A}_2 = \vec{B}$

Now we use the vector identity:

$$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Fill in this step using divergence theorem: [To get second term]

$$-\frac{du}{dt} = +\int_{\text{Vol. `V'}} d\tau\ \vec{J}\cdot\vec{E} + \frac{1}{\mu_0}\oint_{\text{Surface}} (\vec{E}\times\vec{B})\cdot d\vec{A}$$

Surface (bounding the volume 'V')

$$\Rightarrow -\frac{d}{dt}\int_V \frac{1}{2}(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B})\,d\tau$$

$\Rightarrow$ EQ. [Poynting]

Poynting theorem

Define:

$$\frac{1}{\mu_0}\vec{E}\times\vec{B} = \vec{S} \quad : \text{Poynting vector}$$

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Direction along the propagation of wave.

$\underline{\underline{\text{LHS}}}$ of the above EQ. [Poynting] represents the rate at which the electromagnetic energy stored in the volume 'V' decreases with time.

$\underline{\underline{\text{RHS}}}$ $\underline{\text{Term I}}$: Energy dissipated in the form of Joule heat in volume V.

$\underline{\text{Term II}}$: Rate at which energy flows out of the bounding surface S_0.

$\vec{S} = \vec{E} \times \vec{B}$ has dimension $(V/m)(A/m)$

$\equiv$ joule $m^{-2} s^{-1}$

Energy flowing out through unit area per unit time.

Note that in an insulating medium, since $|\vec{J}| = 0$, the Poynting's theorem reduces to:

$$\int_{vol.} \left(\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} \right) d\tau = 0$$

Summary: In general,

$$\vec{E}(\vec{r}, t) = E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \hat{n}$$

$$\vec{B}(\vec{r}, t) = \frac{1}{c} E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} (\hat{k} \times \hat{n})$$

$$= \frac{1}{c} (\hat{k} \times \vec{E})$$

Where, $\hat{n}$ is the polarization vector, $\hat{k}$ is direction of propagation.

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defines the direction of $\vec{E}$ i.e. plane of vibration of the wave.