Lecture - 13

(Magnetostatics
Electrodynamics)

Section A — 24/10/2025

Section B — 22/10/2025, 25/10/2025

Section C — 23/10/2025, 24/10/25

Section D — 30/10/2025

Magnetostatics

Stationary charges $\longrightarrow$ Constant Electric field $\longrightarrow$ Electrostatics

Steady currents $\longrightarrow$ Constant Magnetic field $\longrightarrow$ Magnetostatics

Non-stationary charges and/or Non-steady currents $\longrightarrow$ Changing Electric and Magnetic field $\longrightarrow$ Electrodynamics

Force on a point charge *Q*:

Electric Force: $\mathbf{F}_{\text{elec}} = \text{Q}\mathbf{E}$

Magnetic Force: $\mathbf{F}_{\text{mag}} = \text{Q}(\mathbf{v} \times \mathbf{B})$ **Lorentz Force Law**

Total Force: $\boxed{\mathbf{F} = \mathbf{F}_{\text{elec}} + \mathbf{F}_{\text{mag}} = \text{Q}(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$

Work done by magnetic forces

The work done by a magnetic force is zero !

Why?

$$\mathbf{W}_{\text{mag}} = \int \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = \int Q(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$= \int Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt$$

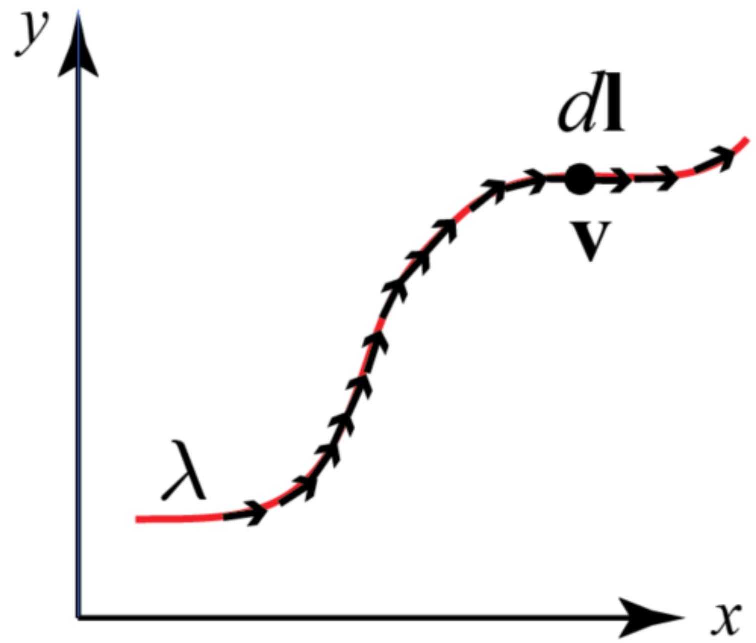
$$= 0$$

- Magnetic forces do no work
- Magnetic forces can change the direction in which a particle moves.
- Magnetic forces do not change the speed with which a particle moves.

Currents

- Current is charge flow per unit time $I = \frac{dq}{dt}$
- It is measured in Coulombs per second, or Amperes (A).

Charge flowing in a wire is described by **Current**



$$\mathbf{I} = \frac{dq}{dt} = \frac{\lambda\, d\mathbf{l}}{dt} = \lambda\mathbf{v}$$

↳ line charge density

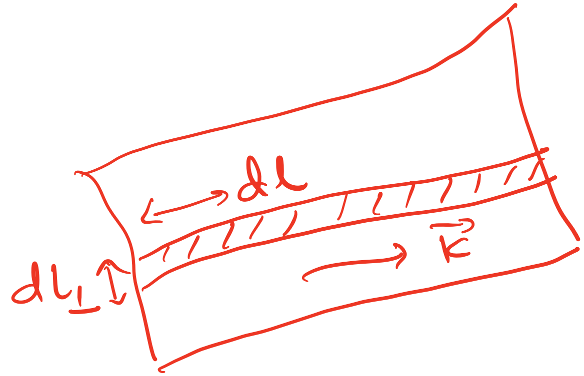
- The direction of current is in the direction of charge-flow.
- Conventionally, this is the direction opposite to the flow of electrons.

Magnetic force on a current carrying wire:

$$\mathbf{F}_{\text{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\, \lambda dl = \int (\mathbf{I} \times \mathbf{B})\, dl$$

$$\boxed{\mathbf{F}_{\text{mag}} = \int (\mathbf{I} \times \mathbf{B})\, dl = I \int (d\mathbf{l} \times \mathbf{B})}$$

Surface current density



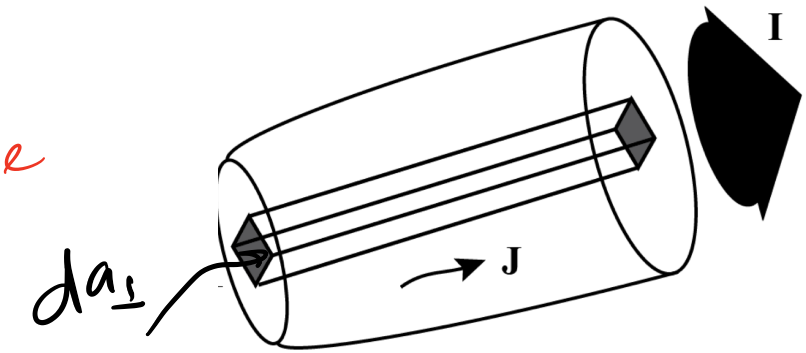
$$\vec{K} = \frac{d\vec{I}}{dl_\perp} = \sigma \vec{v}$$

↳ surface charge density.

Volume current density

$$\vec{J} = \frac{d\vec{I}}{da_\perp} = \rho \vec{v}$$

→ vol. charge density



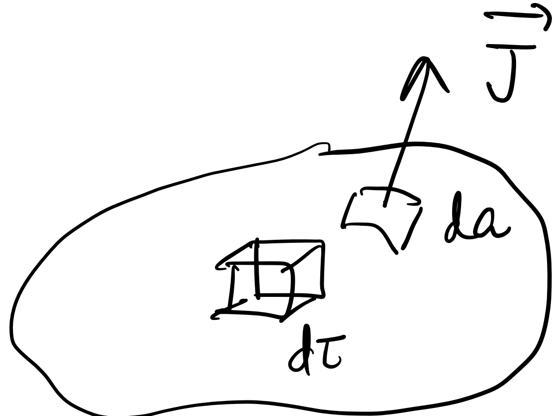
Magnetic force :

$$\vec{F} = q\,\vec{v} \times \vec{B} = \int \vec{v} \rho \, d\tau \times \vec{B} = \int (\vec{v} \times \vec{B}) \rho \, d\tau$$

$$= \int (\vec{J} \times \vec{B}) \, d\tau$$

STEADY CURRENT : CONTINUOUS FLOW THAT HAS BEEN GOING ON FOREVER, WITHOUT CHARGE PILING UP ANYWHERE!

For a closed surface



$$I = \oint_S \vec{J}\cdot d\vec{a} = \oint_S J\, da_\perp$$

$$\oint_S \vec{J}\cdot d\vec{a} = \int_{vol.} (\vec{\nabla}\cdot\vec{J})\, d\tau$$

↳ Total charge flowing out per unit time.

Magnetostatics

Electrostatics

Stationary charge: $\frac{\partial \rho}{\partial t} = 0$

Steady current: $\frac{\partial \vec{J}}{\partial t} = 0$

But, total charge per unit time leaving the volume V is $-\dfrac{d}{dt}\left(\displaystyle\int_V \rho\, d\tau\right) = -\displaystyle\int_V \frac{d\rho}{dt}\, d\tau$

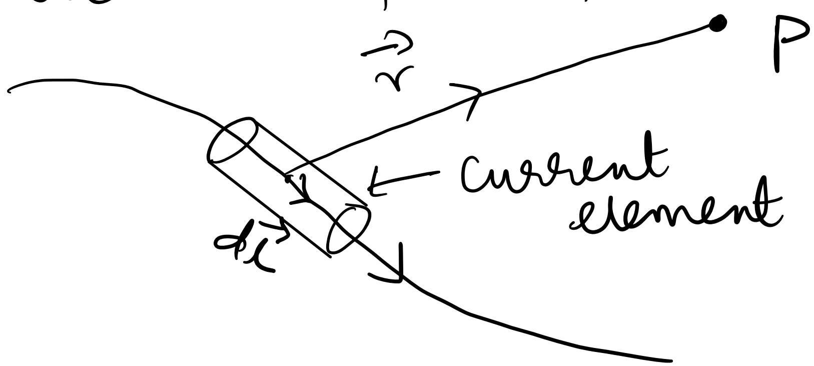
So, $\displaystyle\int_V (\mathbf{\nabla}\cdot\mathbf{J})\,d\tau = -\int_V \frac{d\rho}{dt}\,d\tau \;\Rightarrow\; \boxed{\mathbf{\nabla}\cdot\mathbf{J} = -\frac{d\rho}{dt}}$ **The Continuity Equation**

Conservation of electric charge in presence of currents.

When a steady current flows in a wire, its magnitude I must be the same all along the line; otherwise, charge would be piling up somewhere, and it wouldn't be a steady current. By the same token, $\partial\rho/\partial t = 0$ in magnetostatics, and hence the continuity equation (5.29) becomes

$$\nabla \cdot \mathbf{J} = 0. \tag{5.31}$$

Biot-Savart law for infinitesimal element :



pt. at which magnetic field is determined.

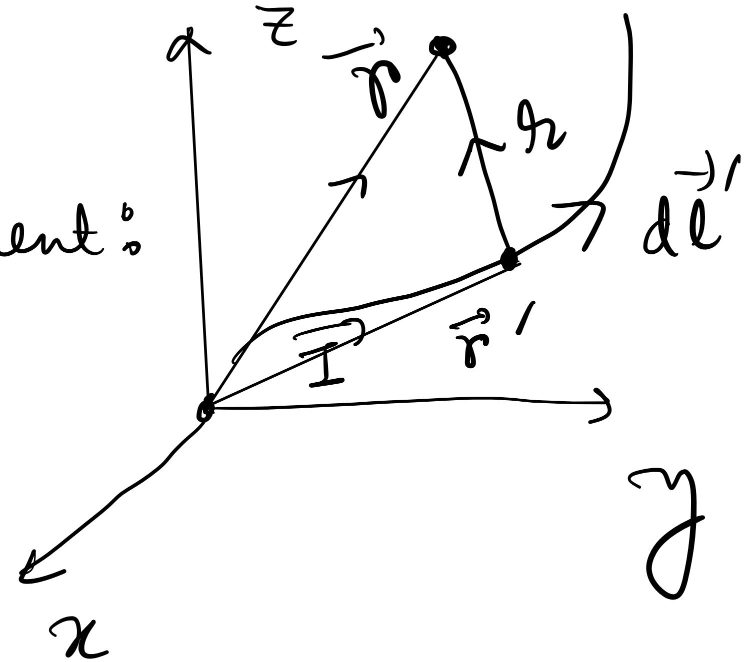
$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^2}$$

$\vec{r}$: distance b/w the current element and P.

OR SEE BELOW :

The Biot - Savart law:

Magnetic field produced by a steady line current:



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{\mu_0}{4\pi} I \int \frac{\vec{dl}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$\Rightarrow$ Analogous to Coulomb's law for electric field.

μ_0: permeability of free space $= 4\pi \times 10^{-7}\ N/A^2$

Unit of magnetic field : Tesla $\equiv N/(A\text{-}m)$

Similarly, magnetic field produced by steady surface current:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, da'$$

by vol. current:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'$$

Now let's imagine a current carrying wire, we can find that $|\vec{B}| = \frac{\mu_0 I}{2\pi s}$ where s is the distance from the wire.

$$\vec{\nabla} \cdot \vec{B} = 0$$ Calculation :

let's calculate the divergence of the $\vec{B}$ produced by a long, current carrying wire,

same as $\hat{\phi}$ in cylindrical coordinate

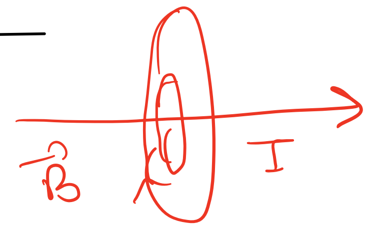
$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left(\frac{\mu_0 I}{2\pi r} \hat{\phi} \right)$$

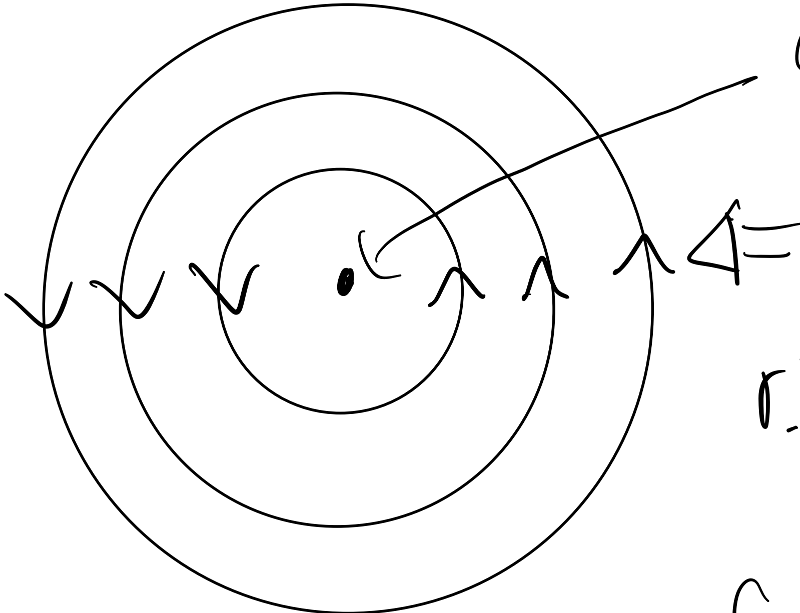
$$\Rightarrow \quad \vec{\nabla} \cdot \vec{B} = \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{\mu_0 I}{2\pi r} \right) = 0$$

$$\vec{\nabla} \cdot \vec{B} = \frac{1}{r} \frac{\partial}{\partial r}(r B_r) + \frac{1}{r} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

Divergence and Curl of $\vec{B}$.

Let's Calculate Curl :





current coming out of page.

$\Leftarrow$ Non-Zero Curl

r: distance from the wire

[Recall problem 5.5 discussed previously]

$$\oint \vec{B}\cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi r}\, dl = \mu_0 I$$

In fact $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$.

Ampere's law

$$\Rightarrow \int (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0 \int \vec{J}\cdot d\vec{a}$$

$$\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

Just as the Biot-Savart law plays a role in magnetostatics that Coulomb's law assumed in electrostatics, so Ampère's plays the role of Gauss's:

$$\begin{cases} \text{Electrostatics :} & \text{Coulomb} & \rightarrow & \text{Gauss,} \\ \text{Magnetostatics :} & \text{Biot-Savart} & \rightarrow & \text{Ampère.} \end{cases}$$

In particular, for currents with appropriate symmetry, Ampère's law in integral form offers a lovely and extraordinarily efficient means for calculating the magnetic field.

Example 5.7

Find the magnetic field a distance s from a long straight wire (Fig. 5.32), carrying a steady current I (the same problem we solved in Ex. 5.5, using the Biot-Savart law).

Solution: We know the direction of **B** is "circumferential," circling around the wire as indicated by the right hand rule. By symmetry, the magnitude of **B** is constant around an amperian loop of radius s, centered on the wire. So Ampère's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint dl = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I,$$

or

$$B = \frac{\mu_0 I}{2\pi s}.$$

This is the same answer we got before (Eq. 5.36), but it was obtained this time with far less effort.

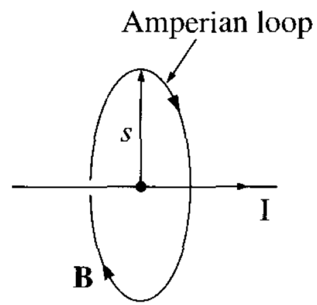


Figure 5.32

5.3.4 Comparison of Magnetostatics and Electrostatics

The divergence and curl of the *electrostatic* field are

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(Gauss's law);} \\ \nabla \times \mathbf{E} = 0, & \text{(no name).} \end{cases}$$

These are **Maxwell's equations** for electrostatics. Together with the boundary condition $\mathbf{E} \to 0$ far from all charges, Maxwell's equations determine the field, if the source charge density ρ is given; they contain essentially the same information as Coulomb's law plus the principle of superposition. The divergence and curl of the *magnetostatic* field are

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

Maxwell's correction will be introduced later.

These are Maxwell's equations for magnetostatics. Again, together with the boundary condition $\mathbf{B} \to 0$ far from all currents, Maxwell's equations determine the magnetic field; they are equivalent to the Biot-Savart law (plus superposition). Maxwell's equations and the force law

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

constitute the most elegant formulation of electrostatics and magnetostatics.

The electric field *diverges away from* a (positive) charge; the magnetic field line *curls around* a current (Fig. 5.44). Electric field lines originate on positive charges and terminate on negative ones; magnetic field lines do not begin or end anywhere—to do so would require a nonzero divergence. They either form closed loops or extend out to infinity.

5.4 Magnetic Vector Potential

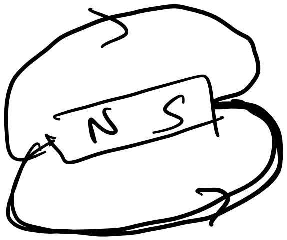
5.4.1 The Vector Potential

Just as $\nabla \times \mathbf{E} = 0$ permitted us to introduce a scalar potential (V) in electrostatics,

$$\mathbf{E} = -\nabla V,$$

Poisson's eqn: $\nabla^2 V = -\rho/\epsilon_0$

so $\nabla \cdot \mathbf{B} = 0$ invites the introduction of a *vector* potential $\mathbf{A}$ in magnetostatics:



$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}.} \quad \Rightarrow \vec{\nabla}\cdot\vec{B} = \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})}_{=0} \tag{5.59}$$

The former is authorized by Theorem 1 (of Sect. 1.6.2), the latter by Theorem 2 (the proof of Theorem 2 is developed in Prob. 5.30). The potential formulation automatically takes care of $\nabla \cdot \mathbf{B} = 0$ (since the divergence of a curl is *always* zero); there remains Ampère's law:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \tag{5.60}$$

If we put $\Rightarrow \vec{\nabla}\cdot\vec{A}_{new} = 0 = \vec{\nabla}\cdot\vec{A}$

Now, the electric potential had a built-in ambiguity: you can add to V any function whose gradient is zero (which is to say, any *constant*), without altering the *physical* quantity $\mathbf{E}$. Likewise, you can add to the magnetic potential any function whose *curl* vanishes (which is to say, the *gradient of any scalar*), with no effect on $\mathbf{B}$. We can exploit this freedom to eliminate the divergence of $\mathbf{A}$:

$$\boxed{\nabla \cdot \mathbf{A} = 0.} \tag{5.61}$$

Then, $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ for Magnetostatics

Let us rewrite $\vec{A}_{new} = \vec{A}_{old} + \vec{\nabla}\lambda$

It is always possible to make $\vec{A}$ divergenceless

Magnetic Vector Potential ($\vec{A}$)

$\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{A}_{new} = \vec{A}_{old} + \vec{\nabla}\lambda$$

Vector Identity

$\vec{\nabla} \times (\vec{\nabla}\lambda) = 0$

Now,

$$\vec{\nabla} \cdot \vec{A}_{new} = \vec{\nabla} \cdot \vec{A}_{old} + \vec{\nabla} \cdot \vec{\nabla}\lambda$$

$$\vec{\nabla} \cdot \vec{A}_{new} = \vec{\nabla} \cdot \vec{A}_{old} + \nabla^2\lambda$$

Choose, $\nabla^2\lambda = -\vec{\nabla} \cdot \vec{A}_{old}$

So that $\vec{\nabla} \cdot \vec{A}_{new} = 0$

$$\vec{\nabla} \times \vec{A}_{new} = \vec{\nabla} \times \vec{A}_{old} + \cancel{\vec{\nabla} \times \vec{\nabla}\lambda}^{\,0} = \vec{\nabla} \times \vec{A}_{old}$$

So, $\vec{B}_{new} = \vec{B}_{old}$ ✓

Faraday's Law : Electrodynamics

→ Michael Faraday demonstrated that an electric current may be induced in a circuit by changing the magnetic flux enclosed by the circuit.

→ Such induced electric fields are very different from the fields produced by electric charge.

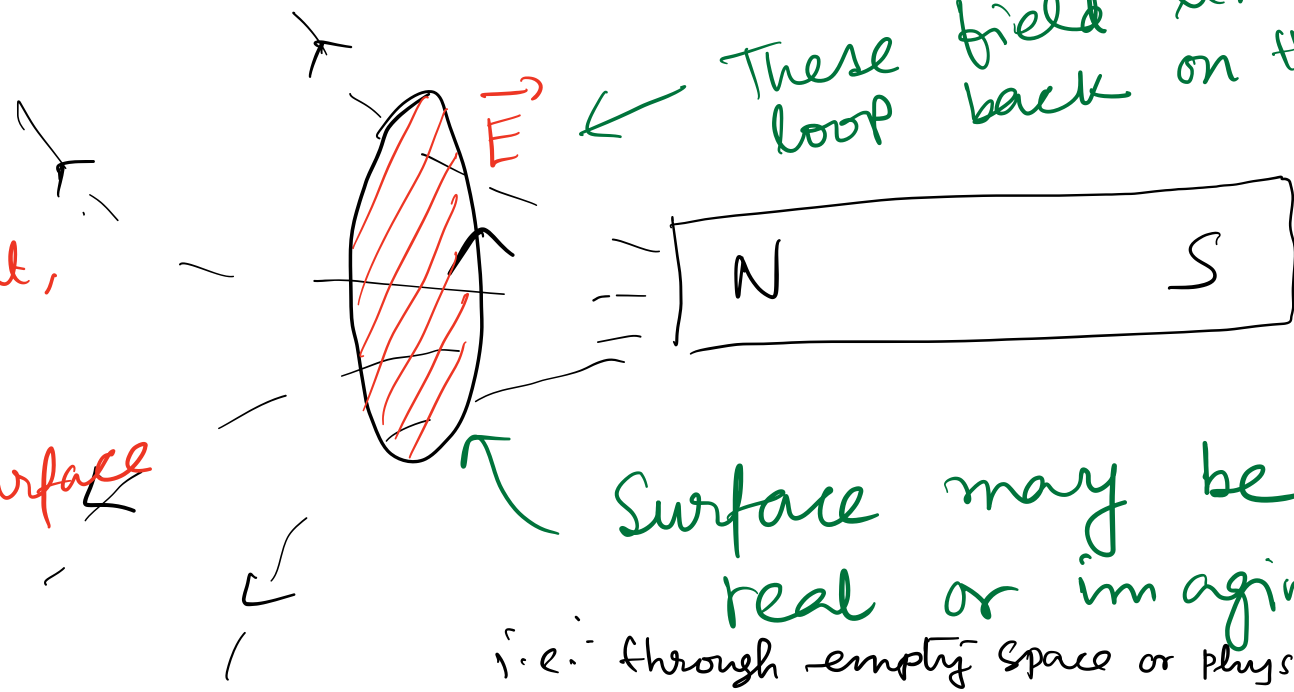
→ Hence, Faraday's law of induction

Electromotive force (emf) $= -\dfrac{d}{dt}\displaystyle\int_S \vec{B}\cdot\hat{n}\,da$

OR Faraday's law in alternate form

work done / energy given to unit charge around a path

$$\Rightarrow \oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$



These field lines loop back on themselves

As magnet moves to right, magnetic flux through the surface decreases.

Surface may be real or imaginary.
i.e. through empty space or physical material

- Induced electric fields produced by changing magnetic fields must form complete loops.

$\vec{E}$ in Faraday's law (or induced electromotive force, emf) represents the induced electric field at each point along path C, a boundary of the surface through which the magnetic flux is changing over time.

The path may be through empty space or physical material.

Q. WHAT IS THE UNIT OF emf ?

Differential form : $\vec{\nabla} \times \vec{E} = - \dfrac{\partial \vec{B}}{\partial t}$