Department of Mathematics & Computing

Engineering Mathematics I

Tutorial Sheet-IV (Solution)

(Vector Differential and Integral Calculus)

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1. Solution: Let $x = \sin t$ then $y = 1 - 2\sin^2 t = \cos 2t$, $\frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$

   Hence the parametric form is

   $r(t) = \sin t\boldsymbol{i} + \cos 2t\boldsymbol{j}$, $\frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$.

2. Solution: The position vector of a point on the curve is $r(t) = \cos t\boldsymbol{i} + \sin t\boldsymbol{j} + t\boldsymbol{k}$.

   Therefore the tangent vector is $r'(t) = -\sin t\boldsymbol{i} + \cos t\boldsymbol{j} + \boldsymbol{k}$

3. Solution: $\vec{F}(1,1) = 2\boldsymbol{j}, \vec{F}(-1,0) = -\boldsymbol{i} - \boldsymbol{j}$

4. Solution: For this problem lets solve for z to get $z = \dfrac{15}{4} - \dfrac{7x}{4} - \dfrac{3y}{4}$.
   The parametric equation for the plane is $\vec{r}(x,y) = \left\langle x, y, \frac{15}{4} - \frac{7}{4}x - \frac{3}{4}y \right\rangle$.

5. Solution: $r'(t)$= - a sin t $\boldsymbol{i}$+ a cos t $\boldsymbol{j}$ + c $\boldsymbol{k}$.

6. Solution: $r(t)$=(cos t+sin t) $\boldsymbol{i}$+(sin t - cos t) $\boldsymbol{j}$+t $\boldsymbol{k}$,

   $r'(t)$=(- sin t+cos t) $\boldsymbol{i}$+(cos t + sin t) $\boldsymbol{j}$+ $\boldsymbol{k}$,

   $r''(t)$=(- cos t - sin t) $\boldsymbol{i}$+(- sin t + cos t) $\boldsymbol{j}$,

   Hence velocity $v(t)$=$r'(t)$

   $v(t)$=(- sin t+cos t) $\boldsymbol{i}$+(cos t + sin t) $\boldsymbol{j}$+ $\boldsymbol{k}$,

   speed =$|v(t)|$= $\sqrt{3}$,

   acceleration $a(t) = r''(t)$,

   $a(t)$ = (- cos t - sin t) $\boldsymbol{i}$+(- sin t + cos t) $\boldsymbol{j}$.

7. Solution: $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \;\Rightarrow\; |\, r \,| = \sqrt{x^2 + y^2 + z^2}$.
   $\ln |\, r \,| = \dfrac{1}{2}\ln(x^2 + y^2 + z^2)$

$$
\begin{aligned}
\nabla\phi &= \frac{1}{2}\nabla \ln(x^2 + y^2 + z^2) \\
&= \frac{1}{2}\{\boldsymbol{i}\frac{2x}{x^2 + y^2 + z^2} + \boldsymbol{j}\frac{2y}{x^2 + y^2 + z^2} + \boldsymbol{k}\frac{2z}{x^2 + y^2 + z^2}\} \\
&= \frac{\boldsymbol{r}}{x^2 + y^2 + z^2} \\
&= \frac{\boldsymbol{r}}{r^2}.
\end{aligned}
$$

8. Solution: Let $\phi = \nabla(x\sin(yz) + y\sin(xz) + zsin(xy))$.

   Then

$$\begin{aligned}
\nabla\phi &= \boldsymbol{i}\frac{\partial\phi}{\partial x} + \boldsymbol{j}\frac{\partial\phi}{\partial y} + \boldsymbol{k}\frac{\partial\phi}{\partial z} \\
&= \{\sin(yz) + y\cos(xz)z + z\cos(xy)y\}\boldsymbol{i} + \{x\cos(yz)z + x\sin(xz) + z\cos(xy)x\}\boldsymbol{j} \\
&+ \{\sin(xy) + x\cos(yz)y + y\cos(xz)x\}\boldsymbol{k} \\
&= (\frac{\pi+\sqrt{2}}{2})\boldsymbol{i}, \quad at(0, \frac{\pi}{4}, 1).
\end{aligned}$$

9. Solution: Let $f(x, y, z) = x^2 + 2y^2 + z^2 - 4$.

   Now, $\nabla f = \boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z} = 2x\boldsymbol{i} + 4y\boldsymbol{j} + 2z\boldsymbol{k}$.

   The normal vector at $(1, 1, 1)$ is $(\nabla f)_{at(1,1,1)} = 2\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$.

   The unit normal vector at $(1, 1, 1)$= $\frac{2\boldsymbol{i}+4\boldsymbol{j}+2\boldsymbol{k}}{\sqrt{4+16+4}} = \frac{\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}}{\sqrt{6}}$.

10. Solution:

$$\begin{aligned}
RHS &= (g\nabla f - f\nabla g)/^2 \\
&= \frac{[g\{\boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}\} - f(\boldsymbol{i}\frac{\partial g}{\partial x} + \boldsymbol{j}\frac{\partial g}{\partial y} + \boldsymbol{k}\frac{\partial g}{\partial z})]}{g^2} \\
&= \nabla(f/g) = LHS.
\end{aligned}$$

11. Solution: $\nabla\phi = (2xyz - 4yz^2)\boldsymbol{i} + (x^2z - 4xz^2)\boldsymbol{j} + (x^2y - 8xyz)\boldsymbol{k}$.

    At $(1, 3, 1)$, $\nabla\phi = -6\boldsymbol{i} - 3\boldsymbol{j} - 21\boldsymbol{k}$.

    The unit vector in the direction of $2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ is
    $\widehat{b} = \dfrac{2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}}{\sqrt{4+1+4}} = \dfrac{2}{3}\boldsymbol{i} - \dfrac{1}{3}\boldsymbol{j} - \dfrac{2}{3}\boldsymbol{k}$.
    Thus the required directional derivative is , $(\nabla\phi)_{at\ (1,3,1)} \cdot \widehat{b} = 11$

12. Solution: Let $\phi = 2x^2 + y^2 + z^2$.

    $\nabla\phi = 4x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$

    $(\nabla\phi)_{at\ (1,2,3)} = 4\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}$
    Vector form of the line $\dfrac{x}{3} = \dfrac{y}{4} = \dfrac{z}{5}$ is $3k\boldsymbol{i} + 4k\boldsymbol{j} + 5k\boldsymbol{k}$.

    The unit vector in the direction of $3k\boldsymbol{i} + 4k\boldsymbol{j} + 5k\boldsymbol{k}$ is,

    $\widehat{b} = \dfrac{3k\boldsymbol{i} + 4k\boldsymbol{j} + 5k\boldsymbol{k}}{\sqrt{9k^2 + 16k^2 + 25k^2}} = \dfrac{3\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}}{5\sqrt{2}}$.
    Thus the required directional derivative is , $(\nabla\phi)_{at\ (1,2,3)} \cdot \widehat{b} = \frac{58}{5\sqrt{2}}$.

13. Solution: $\nabla\phi = 2z\boldsymbol{i} - 2y\boldsymbol{j} + 2x\boldsymbol{k}$.

    The directional derivative is maximum in the direction $(\nabla\phi)_{at\ (1,3,2)} = 4\boldsymbol{i} - 6\boldsymbol{j} + 2\boldsymbol{k}$.

    The magnitude of this maximum is $\mid (\nabla\phi)_{at\ (1,3,2)} \mid = 2\sqrt{14}$.

14. Solution: $\nabla f(x,y) = (2x-y)\boldsymbol{i} + (-x-1+2y)\boldsymbol{j}$.

    At (x_1,y_1), $\nabla f(x_1,y_1) = (2x_1-y_1)\boldsymbol{i} + (-x_1-1+2y_1)\boldsymbol{j}$.

    The unit vector in the direction of $\frac{(\boldsymbol{i}+\sqrt{3}\boldsymbol{j})}{2}$ is $\frac{(\boldsymbol{i}+\sqrt{3}\boldsymbol{j})}{2}$.

    Now, the directional derivative $(2x_1-y_1)\frac{1}{2} + (-x_1-1+2y_1)\frac{\sqrt{3}}{2} = 0$

    $\Rightarrow (2-\sqrt{3})x_1 + (2\sqrt{3}-1)y_1 = \sqrt{3}$. Hence the required points are all the points on the line $(2-\sqrt{3})x + (2\sqrt{3}-1)y = \sqrt{3}$

15. Solution:

$$\begin{aligned}
\nabla\cdot(\bar{r}/r) &= r^{-1}\nabla\cdot\bar{r} + \bar{r}\cdot\nabla r^{-1} \\
&= 3r^{-1} + \bar{r}\cdot(-r^{-1-2}\bar{r}) \qquad (\because \nabla r^n = nr^{n-2}\bar{r}) \\
&= 3r^{-1} - r^{-3}\bar{r}\cdot\bar{r} \\
&= 3r^{-1} - r^{-3}r^2 = 2r^{-1}
\end{aligned}$$

$$\text{So} \qquad \nabla\cdot(\bar{r}/r) = 2\nabla r^{-1}$$

$$= 2(-1)r^{-1-2}\bar{r} = -2r^{-3}\bar{r}.$$

16. Solution:

$$\nabla\cdot\bar{A} = 0$$

$$\left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\left((bx^2y+yz)\boldsymbol{i} + (xy^2-xz^2)\boldsymbol{j} + (2xyz-2x^2y^2)\boldsymbol{k}\right) = 0$$

$$2bxy + 2xy + 2xy = 0$$

$$b = -2.$$

17. Solution:

$$\text{Let} \quad f = \nabla\cdot\bar{U} = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\left(\frac{x^2z}{2}\boldsymbol{i} + yx\boldsymbol{j} + \frac{z^2}{2}\boldsymbol{k}\right)$$

$$\nabla f|_{(4,4,2)} = z\boldsymbol{i} + \boldsymbol{k}|_{(4,4,2)} = 2\boldsymbol{i} + \boldsymbol{k}$$

    Normal to the sphere $g = x^2 + y^2 + z^2 = 36$ is

$$\nabla g|_{(4,4,2)} = 2(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})|_{(4,4,2)} = 4(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$$

$$\bar{a} = \text{unit normal} = \frac{\nabla g}{|\nabla g|} = \frac{4(2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})}{64+64+16} = \frac{2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}}{3}$$

    The required directional derivative is

$$\nabla f\cdot\bar{a} = (2\boldsymbol{i}+\boldsymbol{k})\left(\frac{2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}}{3}\right) = \frac{5}{3}.$$

18. Solution:

$$\nabla \times \bar{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \boldsymbol{i}(-x + x) - \boldsymbol{j}(-y + y) + \boldsymbol{k}(-z + z)$$

$$\nabla \times \bar{A} = 0.$$

To find f : $\bar{A} = \nabla f = \boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}$

Comparing components of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, we get

$$\frac{\partial f}{\partial x} = x^2 - yz \tag{1}$$

$$\frac{\partial f}{\partial y} = y^2 - zx \tag{2}$$

$$\frac{\partial f}{\partial z} = z^2 - xy \tag{3}$$

Integrating (1) partially w.r.t x, we get

$$f = \frac{x^3}{3} - xyz + c_1(y, z) \tag{4}$$

Differentiating (4) partially w.r.t y and equating it with (2), we get

$$\frac{\partial c_1}{\partial y} = y^2 \tag{5}$$

Integrating (5) partially w.r.t y

$$c_1(y, z) = \frac{y^3}{3} + c_2(z) \tag{6}$$

Substituting (6) in (4)

$$f = \frac{x^3}{3} - xyz + \frac{y^3}{3} + c_2(z) \tag{7}$$

Differentiating (7) partially w.r.t z and equating it with (3), we get

$$\frac{\partial c_2}{\partial z} = z^2$$

$$c_2(z) = \frac{z^3}{3}$$

$$\text{that is,} \qquad f = \frac{x^3 + y^3 + z^3}{3} - xyz.$$

19. Solution: $\vec{F} d\vec{r} = 3xy\boldsymbol{i} - y^2\boldsymbol{j}$, $d\vec{r} = dx\ \boldsymbol{i} + dy\ \boldsymbol{j}$

Also, $y = 2x^2 \Rightarrow dy = 4xdx$

Now, $\int_C \vec{F} d\vec{r} = \int_C 3xydx - y^2dy = \int_{x=0}^{1} 3x.2x^2dx - 4x^4.4xdx$

$= \left[6x^4/4 - 16x^6/6\right]\Big|_0^1 = -\frac{7}{6}$

20. Solution: Like above, $\int_C \vec{F} d\vec{r} = \int_C (x+y^2)dx + (x^2-y^2)dy$
$= \int_0^1 (x+x^{4/3})dx + (y^3-y)dy + \int_0^1 (x+x^2)dx + (x^2-x)dx = \int_0^1 (x^{4/3}dx + y^3dy) + \int_0^1 2x^2 dx$
$= 1/84$

21. Solution: Since C is not closed, so close up the curve C by line segment C from (1,1) to (-1,1).

Now, $\displaystyle\int_C \vec{F} d\vec{r} = \int (1+xy^2)dx - x^2 y dy$

$\therefore P = 1 + xy^2, Q = -x^2 y \Rightarrow \delta\theta/\delta x = -2xy,\ \delta P/\delta y = 2xy$

Applying Green's theorem to the region D, we have,

$\displaystyle\int_{C\cup C'} (1+xy^2)dx - x^2 y dy = \int_D \int (-2xy - 2xy)dA = \int_D \int -4xy\ dA$

$\displaystyle= \int_{-1}^1 \int_2^1 -4xy\ dy\ dx = 0.$

22. Solution: $P = x^2 y,\ Q = -xy^2 \Rightarrow \frac{\delta Q}{\delta x} = -y^2,\ \frac{\delta P}{\delta y} = x^2$

Now, $\int_C Pdx + Qdy = \int_D \int -y^2 - x^2 dA = -\int_D \int (x^2+y^2)dA = \int_0^{2\pi} \int_0^2 r^2.rdrd\theta$

$\Rightarrow -\int_0^{2\pi} \int_0^2 r^3 drd\theta = -\int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 d\theta = -\int_0^{2\pi} 4d\theta = -8\pi.$

23. Solution: The integral would be $f(x,y,g(x,y))\sqrt{1+(\delta t/\delta x)^2 + (\delta z/\delta y)^2}dA$

Now $z^2 = 1 - x^2 - y^2 \Rightarrow \delta z/\delta x = -x/z, \delta z/\delta y = -y/z$

Substituting in 2nd equation, we get

$\displaystyle\int_R \int x^2 y^2 z\sqrt{1 + x^2/z^2 + y^2/z^2}dA = \int_R \int x^2 y^2 \sqrt{1+x^2+y^2}dA$

$\displaystyle= \int_R \int x^2 y^2 dA = \int_{\theta=0}^{2\pi} \int_{r=0}^1 r^2\cos^2\theta\ r^2\sin^2\theta r dr d\theta = \pi/24.$

24. Solution:

$$\begin{aligned}
I &= \int_0^a \int_0^a \int_0^a (x^2+y^2+z^2)dxdydz \\
&= \int_0^a \int_0^a \left[x^2 z + y^2 z + \frac{z^3}{3}\right]_0^a dxdy \\
&= \int_0^a \int_0^a \left[x^2 a + y^2 a + \frac{a^3}{3}\right] dxdy \\
&= \int_0^a \left[x^2 a y + \frac{1}{3}y^3 a + \frac{a^3 y}{3}\right]_0^a dx \\
&= \int_0^a \left(x^2 a^2 + \frac{1}{3}a^4 + \frac{a^4}{3}\right) dx \\
&= a^5
\end{aligned}$$

25. Solution:

$$\begin{aligned}
I &= \int\limits_{x=0}^{2}\int\limits_{y=0}^{2}\int\limits_{z=0}^{4-x^2}(2x+y)dxdydz \\
&= \int\limits_{x=0}^{2}\int\limits_{y=0}^{2}(2x+y)\left(4-x^2\right)dxdy \\
&= \int\limits_{x=0}^{2}[16x-4x^3+2(4-x^2)]dx \\
&= \frac{80}{3}
\end{aligned}$$

26. Solution:

$$\begin{aligned}
I &= \int\limits_{x=0}^{a}\int\limits_{0}^{a-x}\int\limits_{0}^{a-x-y}x^2dxdydz \\
&= \int\limits_{x=0}^{a}\int\limits_{0}^{a-x}x^2(a-x-y)dxdy \\
&= \int\limits_{x=0}^{a}\frac{1}{2}x^2(a-x)^2dx = \frac{a^5}{60}
\end{aligned}$$

27. Solution: By divergence theorem, $\int\int_S \mathbf{v}\cdot\mathbf{n}\; dA = \int\int\int_D (\nabla\cdot\mathbf{v})dV$ and we can find $\nabla\cdot\mathbf{v} = 5y+3$. Now,

$$\begin{aligned}
\int\int\int_D(\nabla\cdot\mathbf{v})dV &= \int\limits_0^4\int\limits_0^4\int\limits_0^{4-x}(5y+3)dydxdz \\
&= 4\int\limits_0^4\int\limits_0^{4-x}(5y+3)dydx \\
&= 4\int\limits_0^4\left[\frac{5}{2}y^2+3y\right]_0^{4-x}dx \\
&= 4\int\limits_0^4\left[\frac{5}{2}(4-x)^2+3(4-x)\right]dx \\
&= \frac{928}{3}.
\end{aligned}$$

28. Solution: We have $\nabla \cdot \mathbf{v} = 2x + 4y + 6z$. Therefore,

$$\begin{aligned}\iiint_D (\nabla\cdot\mathbf{v})dV &= \int_0^3\int_{-3}^3\int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} (2x+4y+6z)dydxdz \\ &= \int_{-3}^3\int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} (6x+12y+27)dydx.\end{aligned}$$

Since, x, y are odd functions, we find

$$\begin{aligned}\iiint_D (\nabla\cdot\mathbf{v})dV &= \int_{-3}^3\int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} 27dydx \\ &= 4\times 27\int_0^3\int_0^{y=\sqrt{9-x^2}} dydx \\ &= 4\times 27\int_0^3 [y]_0^{y=\sqrt{9-x^2}}\, dx \\ &= 4\times 27\int_0^3 \sqrt{9-x^2}dx \\ &= 4\times 27\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 \\ &= 9\times 27\times\pi = 243\pi.\end{aligned}$$

The surface consists of three parts, S_1 (top),S_2 (bottom)and S_3 (vertical).

On $S_1 : z = 3, \mathbf{n} = \boldsymbol{k}$.

$$\iint_{S_1} (\mathbf{v}\cdot\mathbf{n})dA = 3\iint_{S_1} z^2 dA = 243\pi$$

On $S_2 : z = 0, \mathbf{n} = -\boldsymbol{k}$.

$$\iint_{S_2} (\mathbf{v}\cdot\mathbf{n})dA = -3\iint_{S_2} z^2 dA = 0$$

On $S_3 : x^2 + y^2 = 9, \mathbf{n} = \dfrac{2x\boldsymbol{i}+2y\boldsymbol{j}}{2\sqrt{x^2+y^2}} = \frac{1}{3}(x\boldsymbol{i}+y\boldsymbol{j})$

$$\iint_{S_3} (\mathbf{v}\cdot\mathbf{n})dA = \frac{1}{3}\iint_{S_3} (x^3+2y^3)dA.$$

By using the cylindrical coordinates, we write $x = 3\cos\theta, y = \sin\theta, dA = 3d\theta dz$.

$$\begin{aligned}\iint_{S_3} (\mathbf{v}\cdot\mathbf{n})dA &= \int_0^3\int_0^{2\pi} (27\cos^3\theta + 54\sin^3\theta)d\theta dz \\ &= 27\int_0^3\int_0^{2\pi}\left[\left(\frac{3}{4}\cos\theta+\cos 3\theta\right)+2\left(\frac{3}{4}\sin\theta+\sin 3\theta)\right)\right]d\theta dz = 0\end{aligned}$$

Thus, $\int\limits_S\int(\mathbf{v}\cdot\mathbf{n})dA = \int\int\limits_D\int(\nabla\cdot\mathbf{v})dV$.

29. Solution: Applying Stokes's theorem, the given integral can be reduced to

$$\frac{1}{2}\int\limits_C((y^2+z^2)dx+(x^2+z^2)dy+(x^2+y^2)dz),$$

where C is the curve $x^2+y^2-2ax=0, z=0$ or $(x-a)^2+y^2=a^2, z=0$.

Put $x=a(1+\cos\theta), y=a\sin\theta, z=0$.

$$\begin{aligned}
&\frac{1}{2}\int\limits_C((y^2+z^2)dx+(x^2+z^2)dy+(x^2+y^2)dz)\\
=&\ \frac{1}{2}\int\limits_{-\pi}^{\pi}\left(a^2\sin^2\theta(-a\sin\theta)+a^2(1+\cos\theta)a\cos\theta\right)d\theta\\
=&\ \pi a^3.
\end{aligned}$$

30. Solution: Consider the projection of S on $x-y$ plane. The projection is the circular region $x^2+y^2\leq 16, z=0$ and the bounding curve C is the circle $z=0, x^2+y^2=16$. We have,

$$\int\limits_C\mathbf{v}\cdot d\mathbf{r}=\int\limits_C((3x-y)dx-2yz^2dy-2y^2zdz)=\int\limits_C(3x-y)dx$$

since $z=0$. Take $x=4\cos\theta, y=4\sin\theta$, we obtain

$$\begin{aligned}
\int\limits_C(3x-y)dx &= \int\limits_0^{2\pi}4(3\cos\theta-\sin\theta)(-4\sin\theta)d\theta=-16\int\limits_0^{2\pi}\left[\frac{3}{2}\sin 2\theta-\frac{1}{2}(1-\cos 2\theta)\right]d\theta\\
&= 16\pi.
\end{aligned}$$

Now, $\nabla\times\mathbf{v}=\boldsymbol{k}, \mathbf{n}=\frac{2(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})}{2\sqrt{x^2+y^2+z^2}}=\frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{4}$ and $(\nabla\times\mathbf{v})\cdot\mathbf{n}=\frac{z}{4}$. Therefore,

$$\int\int\limits_S(\nabla\times\mathbf{v})\cdot\mathbf{n}\,dA=\int\int\limits_S\frac{z}{4}\,dA=\int\int\limits_R\frac{z}{4}\frac{dxdy}{\mathbf{n}\cdot\boldsymbol{k}}=\int\int\limits_R\frac{z}{4}\frac{dxdy}{z/4}=\int\int\limits_R dxdy=16\pi$$

which is the area of the circular region in the $x-y$ plane. Hence, Stokes's theorem is proved.