LECTURE - 11

MAXWELL'S EQUATIONS IN MATTER — PART – ELECTRIC FIELD

Maxwell's Equations in matter :

Again focus on Statics for now.

Thus we will focus on electric fields in matter.

Matter → Solid, Liquid, Gases

For electric fields :

Matter → Conductors, Insulators, Semiconductors

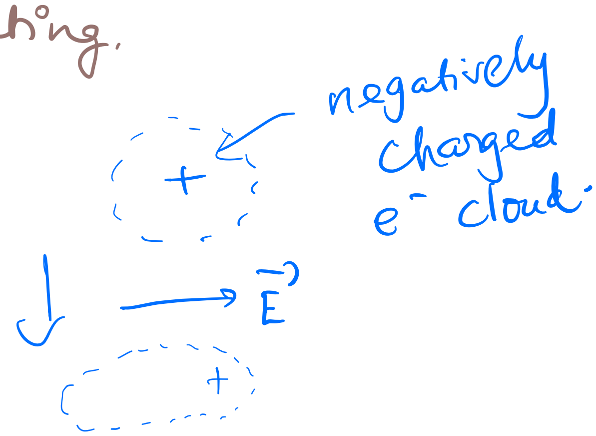
Insulators $\equiv$ Dielectrics

Charges attached to specific atoms or molecules.

There are two principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule.

— Stretching and rotating.

For example in an atom



Induced Dipole :

What happens to a neutral atom when placed in an electric field $\vec{E}$?

Polarization of an atom. If $\vec{E}$ is large enough then ionization can take place.

Example 4.1 $\vec{p} = \alpha \vec{E}$ $\vec{p}$: induced dipole moment
α : atomic polarizability

A primitive model for an atom consists of a point nucleus ($+q$) surrounded by a uniformly charged spherical cloud ($-q$) of radius a (Fig. 4.1). Calculate the atomic polarizability of such an atom.

Solution: In the presence of an external field **E**, the nucleus will be shifted slightly to the right and the electron cloud to the left, as shown in Fig. 4.2. (Because the actual displacements

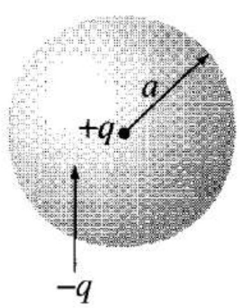


Figure 4.1

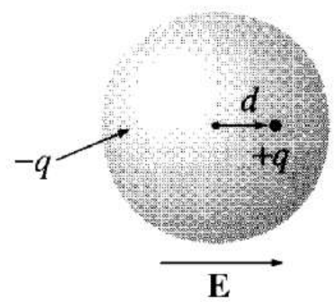
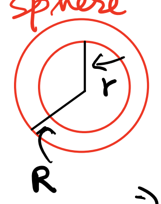


Figure 4.2

involved are extremely small, as you'll see in Prob. 4.1, it is reasonable to assume that the electron cloud retains its spherical shape.) Say that equilibrium occurs when the nucleus is displaced a distance d from the center of the sphere. At that point the external field pushing the nucleus to the right exactly balances the internal field pulling it to the left: $E = E_e$, where E_e is the field produced by the electron cloud. Now the field at a distance d from the center of a uniformly charged sphere is

$$E_e = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}$$

Uniformly charged sphere:



$Q_{tot} = \frac{4}{3}\pi R^3 \rho$

$E\, 4\pi r^2 = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3 \rho$

$\Rightarrow \vec{E} = \frac{\rho r}{3\epsilon_0}\hat{r}$

$= \frac{Q r}{4\pi R^3 \epsilon_0}\hat{r}$

(Prob. 2.12). At equilibrium, then,

$p \propto E$

p depends on the charge and 'd'.

$$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}, \quad \text{or} \quad p = qd = (4\pi\epsilon_0 a^3)E.$$

The atomic polarizability is therefore

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v, \tag{4.2}$$

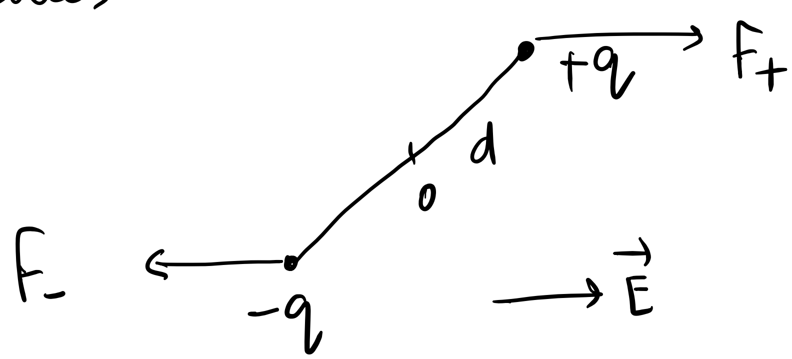
where v is the volume of the atom. Although this atomic model is extremely crude, the result (4.2) is not too bad—it's accurate to within a factor of four or so for many simple atoms.

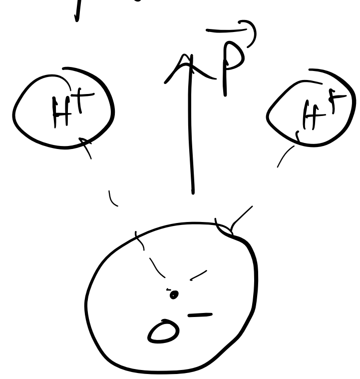
For molecules, polarizability is different in different directions:

$$\vec{p} = \alpha_\perp \vec{E}_\perp + \alpha_\parallel \vec{E}_\parallel$$

Polar molecules:

Furthermore in case of polar molecules, i.e. molecules with an intrinsic dipole moment,





Torque:

$$\vec{N} = (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-)$$

$$= \frac{\vec{d}}{2} \times q\vec{E} + \left(\frac{-\vec{d}}{2}\right) \times (-q\vec{E})$$

$$= q(\vec{d} \times \vec{E}) = \vec{p} \times \vec{E}$$

Polarization of matter:

We return to our original question:
What happens to a piece of dielectric material when it is placed in an electric field?

$\vec{P} \equiv$ dipole moment per unit volume

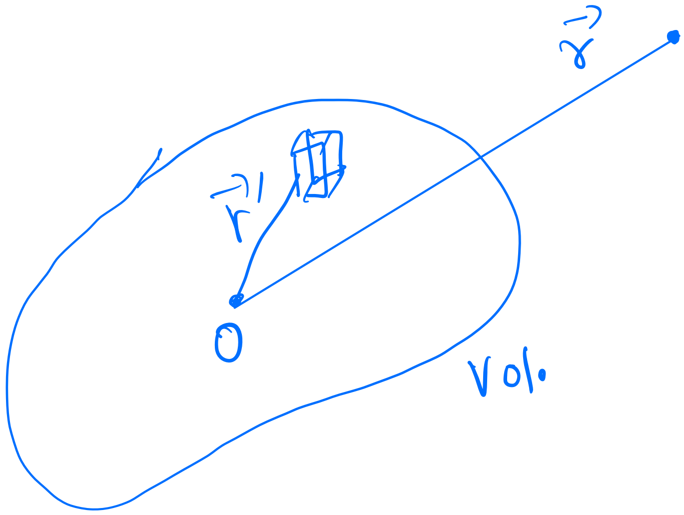
Lot of tiny dipoles or permanent dipoles will arise $\Rightarrow$ Material becomes polarized

We ask the question: What is the field produced by these induced or permanent dipoles? i.e. what is the field due to a polarized object?

In other words,

The Electric field arising from a general charge distribution,

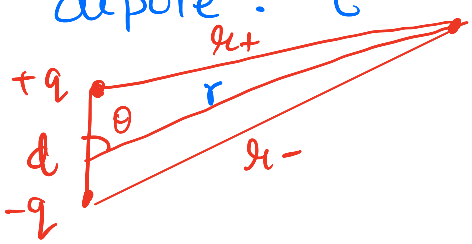
$$\vec{E}(\vec{r}) = -\nabla V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{Vol} d\tau' \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$



$\rho(\vec{r}')$ can in principle be very complicated.

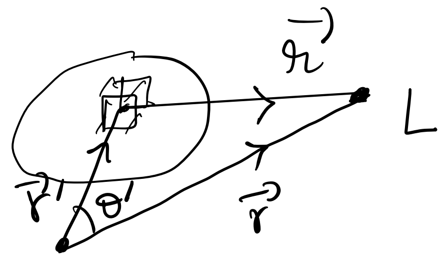
Nonetheless, if we look from very far, all charge distribution looks like a point charge.

But what about the potential due to a dipole very far from the dipole: (Example 3.10)



$$V(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\, \rho(\vec{r}\,')\, d\tau'$$



$$\Rightarrow \frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2}$$

$$= \frac{1}{r}\left(1 - \frac{\epsilon}{2} + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

$\epsilon \ll 1$

$\Rightarrow \frac{r'}{r} \ll 1$

$r \gg r'$

(very far from source)

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\theta'$$

$$= r^2\left(1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta'\right)$$

$$= r^2(1+\epsilon)$$

$$\epsilon = \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta'\right)$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta' + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\theta' - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\theta' - 3\cos\theta'}{2}\right) + \cdots\right)$$

$$\Rightarrow \frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

↳ Legendre Polynomials

CHECK CHAP. 3 (SECTION 3.4)
(EXAMPLE 3.10)

Finally,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\,\rho(\vec{r}\,')\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')\,\rho(\vec{r}\,')\,d\tau'$$

Multipole expansion of V in powers of $\frac{1}{r}$

Monopole:

$$V_{mon}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\vec{r}\,')\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

dipole:

let $\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\rho(\vec{r}\,')\,d\tau'\cos\theta'$$

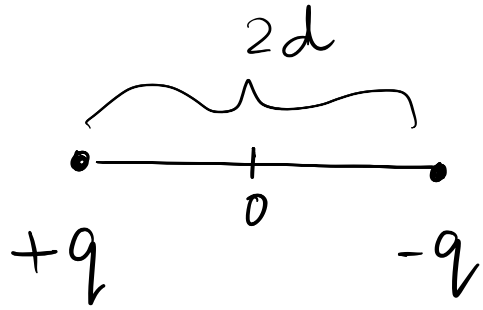
$r'\cos\theta' = \hat{r}\cdot\vec{r}\,'$

Quadrupole: $V_{Quad} = \frac{1}{r^3}(\#)$

$$r'\cos\theta' = \hat{r}\cdot\vec{r}\,' \Rightarrow V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\Big(\underbrace{\int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'}_{\vec{p}}\Big)$$

$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i'$ for a collection of charges

Representing a dipole using Dirac delta functions:

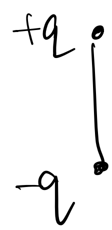


$$\rho(\vec{r}) = +q\,\delta(\vec{r} - \vec{d}) - q\,\delta(\vec{r} + \vec{d})$$

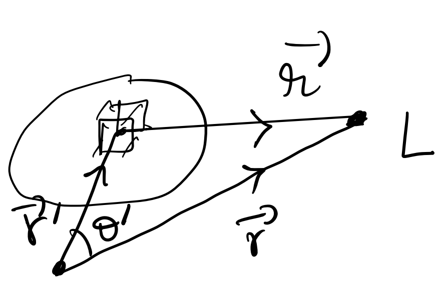
We will also use,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{\mathscr{r}}}{\mathscr{r}^2}$$

Approximation:



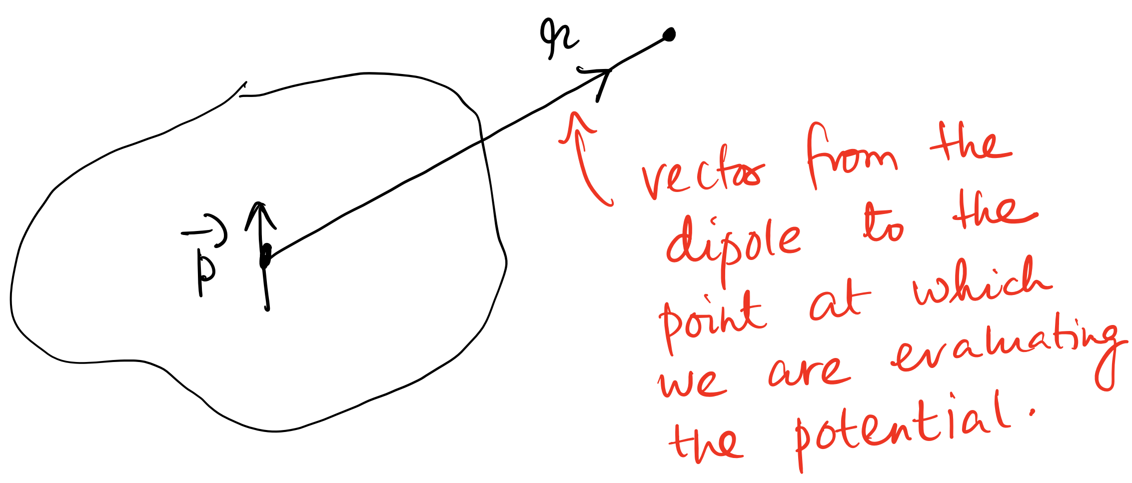
, $r \gg d$ and $\mathscr{r} \simeq r$



Two contributions : 1) Original field that was responsible for $\vec{P}$, and 2) New field due to $\vec{P}$.

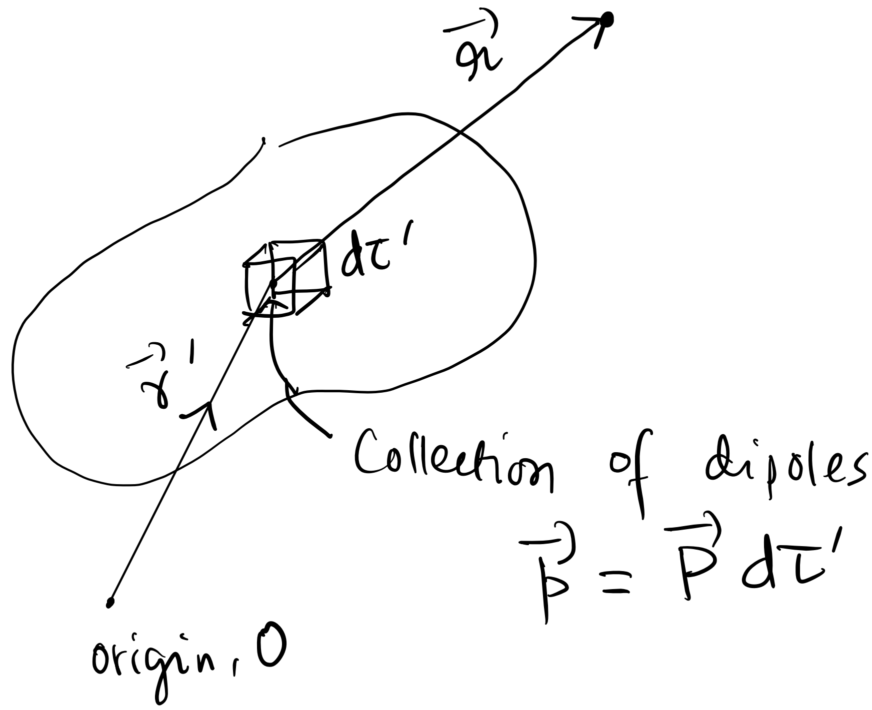
Introduce concept of bound charges :

Potential due to a single dipole $\vec{p}$?



$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\hat{\mathscr{r}} \cdot \vec{p}}{\mathscr{r}^2} \quad \longleftarrow \text{for a single } \vec{p}$$

What happens if we have a collection of such dipoles?



$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{vol.} \frac{\hat{\mathfrak{r}} \cdot \vec{P}(\vec{r}\,')}{\mathfrak{r}^2} \, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \int_{vol.} \left(\vec{\nabla}' \left(\frac{1}{\mathfrak{r}} \right) \cdot \vec{P} \right) d\tau'$$

$\rightarrow$ (i)

[Notice the prime (') in the gradient].

where, we have used,

(i) $\vec{\nabla}' g = \left(\frac{\partial g}{\partial x'} \hat{x} + \frac{\partial g}{\partial y'} \hat{y} + \frac{\partial g}{\partial z'} \hat{z} \right)$

Show that : if $\vec{\mathfrak{r}} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$

then,

(a) $\vec{\nabla}(\mathfrak{r}^2) = 2\vec{\mathfrak{r}}$

$$(b)\ \vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\left(\frac{1}{\mathscr{r}^2}\right)\hat{\mathscr{r}}$$

$$(ii)\ \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}} \quad \left[\because \vec{\mathscr{r}} = \vec{r} - \vec{r}'\right]$$

So we have,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{vol.}\left(\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)\cdot\vec{P}\right)d\tau'$$

Use the following rule:

$$\boxed{\int_{Vol} f\left(\vec{\nabla}\cdot\vec{A}\right)d\tau = -\int_{vol.}\vec{A}\cdot\left(\vec{\nabla}f\right)d\tau + \oint_{Surf.} f\vec{A}\cdot d\vec{a}}$$

Substitute $f \equiv \frac{1}{\mathfrak{r}}$; $\vec{A} \equiv \vec{P}$

$d\tau \equiv d\tau'$, $\vec{\nabla} \equiv \vec{\nabla}'$

$d\vec{a} \equiv d\vec{a}'$

We have,

$$V = \frac{1}{4\pi\epsilon_0} \oint_{\text{Surf.}} \frac{1}{\mathfrak{r}} \vec{P} \cdot d\vec{a}' - \frac{1}{4\pi\epsilon_0} \int_{\text{Vol.}} \frac{1}{\mathfrak{r}} (\vec{\nabla}' \cdot \vec{P}) d\tau'$$

Bound charges !

let, $\vec{P} \cdot \hat{n} \equiv \sigma_b$; $\vec{\nabla}' \cdot \vec{P} \equiv \rho_b$

Ist term : $\frac{1}{4\pi\epsilon_0} \oint \frac{1}{\mathfrak{r}} \sigma_b \, da'$

2nd term : $\frac{1}{4\pi\epsilon_0} \int_{\text{vol.}} \frac{\rho_b}{\mathfrak{r}} d\tau'$

Summary:

Field due to a polarized object is identical to the field that would be produced by a certain distribution of 'bound charges', σ_b and ρ_b.

Effect of polarization is to produce accumulations of bound charges
$\rho_b = -\vec{\nabla} \cdot \vec{P}$ and $\sigma_b = \vec{P} \cdot \hat{n}$
(within the dielectric) (on surface)

Field due to everything else is called 'free charge' (ρ_f); Could be e^-s on a conductor or ions in a dielectric.

Within a dielectric, the total charge density is :

$$\rho = \rho_b + \rho_f$$

From Gauss's law,

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho = \rho_b + \rho_f = -\vec{\nabla} \cdot \vec{P} + \rho_f$$

$\vec{E}$ is total field, (not just due to polarization)

$$\Rightarrow \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{D} = \rho_f}$$

$\vec{D}$: electric displacement

$$\Rightarrow \quad \oint \vec{D} \cdot d\vec{a} = Q_{f,enc}$$

$Q_{f,enc}$: Total free charge enclosed in the volume.

**

$$\vec{\nabla} \times \vec{D} = \epsilon_0 (\vec{\nabla} \times \vec{E}) + (\vec{\nabla} \times \vec{P}) = \vec{\nabla} \times \vec{P}$$

(Statics)

** $\vec{D}$ cannot be expressed as gradient of a scalar.

Finally, How is $\vec{P}$ related to $\vec{E}$?

(Linear dielectrics) $\vec{P} = \chi_e' \vec{E} = \epsilon_0 \chi_e \vec{E}$

$\hookrightarrow$ electric susceptibility

So,

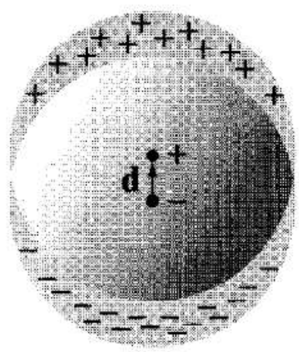
$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$

$$= \epsilon_0 (1 + \chi_e) \vec{E}$$

$$\boxed{\vec{D} = \epsilon \vec{E}}$$ [Linear Dielectrics]

$\epsilon = \epsilon_0 (1 + \chi_e)$: Permittivity

Concept of bound charge :



There is another way of analyzing the uniformly polarized sphere (Ex. 4.2), which nicely illustrates the idea of a bound charge. What we have, really, is *two* spheres of charge: a positive sphere and a negative sphere. Without polarization the two are superimposed and cancel completely. But when the material is uniformly polarized, all the plus charges move slightly *upward* (the z direction), and all the minus charges move slightly *downward* (Fig. 4.15). The two spheres no longer overlap perfectly: at the top there's a "cap" of leftover positive charge and at the bottom a cap of negative charge. This "leftover" charge is precisely the bound surface charge σ_b.

or, in integral form,

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}, \tag{4.23}$$

where $Q_{f_{\text{enc}}}$ denotes the total free charge enclosed in the volume. This is a particularly useful way to express Gauss's law, in the context of dielectrics, because *it makes reference only to free charges,* and free charge is the stuff we control. Bound charge comes along for the ride: when we put the free charge in place, a certain polarization automatically ensues, by the mechanisms of Sect. 4.1, and this polarization produces the bound charge. In a typical problem, therefore, we know ρ_f, but we do not (initially) know ρ_b; Eq. 4.23 lets us go right to work with the information at hand. In particular, whenever the requisite symmetry is present, we can immediately calculate $\mathbf{D}$ by the standard Gauss's law methods.

Example 4.4

A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a (Fig. 4.17). Find the electric displacement.

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$\vec{P}$ is related to bound charge

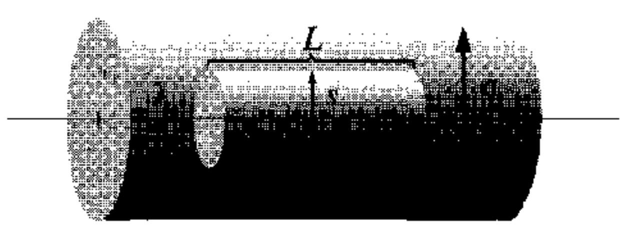


Figure 4.17

Solution: Drawing a cylindrical Gaussian surface, of radius s and length L, and applying Eq. 4.23, we find

$$D(2\pi s L) = \lambda L.$$

Therefore,

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}. \tag{4.24}$$

Notice that this formula holds both within the insulation and outside it. In the latter region, $\mathbf{P} = 0$, so

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}, \qquad \text{for } s > a.$$

Inside the rubber the electric field cannot be determined, since we do not know $\mathbf{P}$.

It may have appeared to you that I left out the surface bound charge σ_b in deriving Eq. 4.22, and in a sense that is true. We cannot apply Gauss's law precisely *at* the surface of a dielectric, for here ρ_b blows up, taking the divergence of $\mathbf{E}$ with it. But everywhere *else* the logic is sound, and in fact if we picture the edge of the dielectric as having some finite thickness within which the polarization tapers off to zero (probably a more realistic model

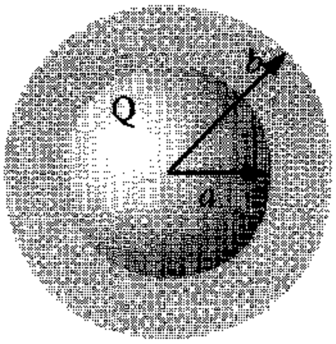


Figure 4.20

Example 4.5

A metal sphere of radius a carries a charge Q (Fig. 4.20). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

Solution: To compute V, we need to know $\mathbf{E}$; to find $\mathbf{E}$, we might first try to locate the bound charge; we could get the bound charge from $\mathbf{P}$, but we can't calculate $\mathbf{P}$ unless we already know $\mathbf{E}$ (Eq. 4.30). We seem to be in a bind. What we *do* know is the *free* charge Q, and fortunately the arrangement is spherically symmetric, so let's begin by calculating $\mathbf{D}$, using Eq. 4.23:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \qquad \text{for all points } r > a.$$

(Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = 0$.) Once we know $\mathbf{D}$, it is a trivial matter to obtain $\mathbf{E}$, using Eq. 4.32: $\vec{D} = \epsilon \vec{E}$

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\[2ex] \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

The potential at the center is therefore

$$\begin{aligned} V &= -\int_{\infty}^{0} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{b}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right) dr - \int_{b}^{a}\left(\frac{Q}{4\pi\epsilon r^2}\right) dr - \int_{a}^{0}(0)\, dr \\ &= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right). \end{aligned}$$

As it turns out, it was not necessary for us to compute the polarization or the bound charge explicitly, though this can easily be done:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}},$$

in the dielectric, and hence

$$\rho_b = -\nabla \cdot \mathbf{P} = 0,$$

while

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0 \chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface,} \\[2ex] \dfrac{-\epsilon_0 \chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

Notice that the surface bound charge at a is *negative* ($\hat{\mathbf{n}}$ points outward *with respect to the dielectric,* which is $+\hat{\mathbf{r}}$ at b but $-\hat{\mathbf{r}}$ at a). This is natural, since the charge on the metal sphere attracts its opposite in all the dielectric molecules. It is this layer of negative charge that reduces the field, within the dielectric, from $1/4\pi\epsilon_0(Q/r^2)\hat{\mathbf{r}}$ to $1/4\pi\epsilon(Q/r^2)\hat{\mathbf{r}}$. In this respect a dielectric is rather like an imperfect conductor: on a *conducting* shell the induced surface charge would be such as to cancel the field of Q *completely* in the region $a < r < b$: the dielectric does the best it can, but the cancellation is only partial.