Engineering Mechanics NMEC-101

Module -8

Kinematics of Rigid Bodies

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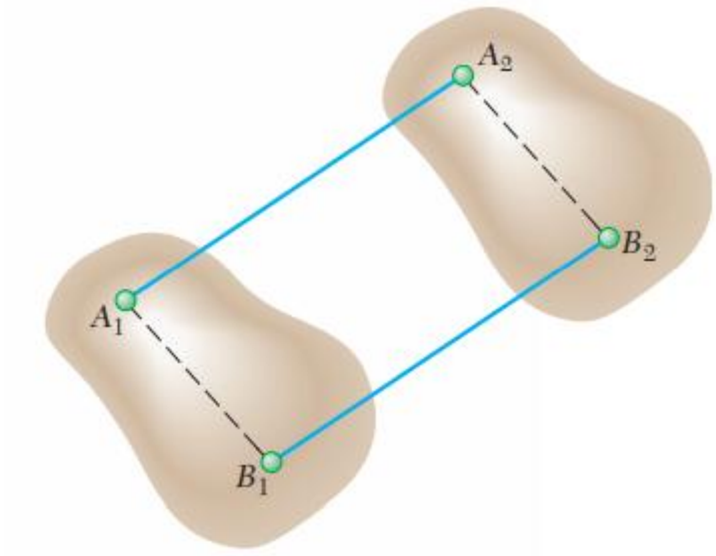
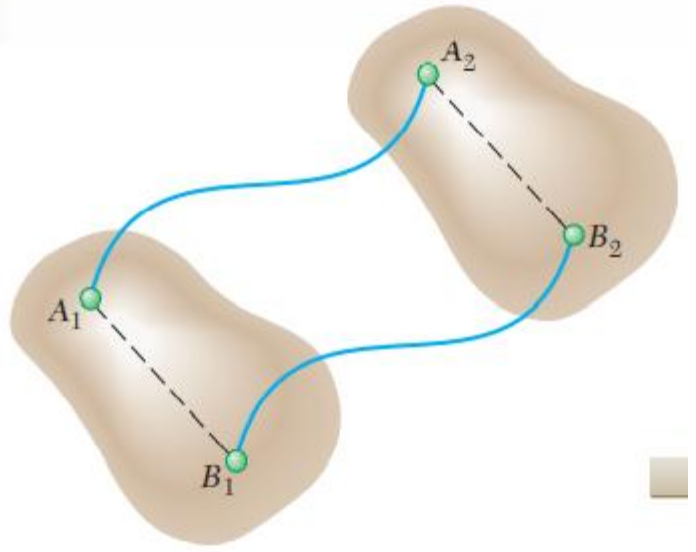
Introduction





- Kinematics of rigid bodies: relations between time and the positions, velocities, and accelerations of the particles forming a rigid body.

- Classification of rigid body motions:
  - Translation:

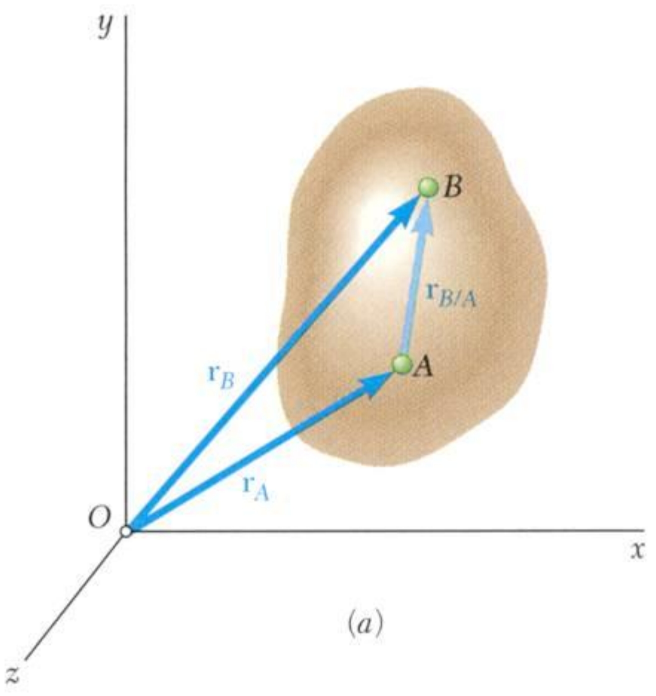
    A motion is said to be a *translation* if any straight line inside the body keeps the same direction during the motion.

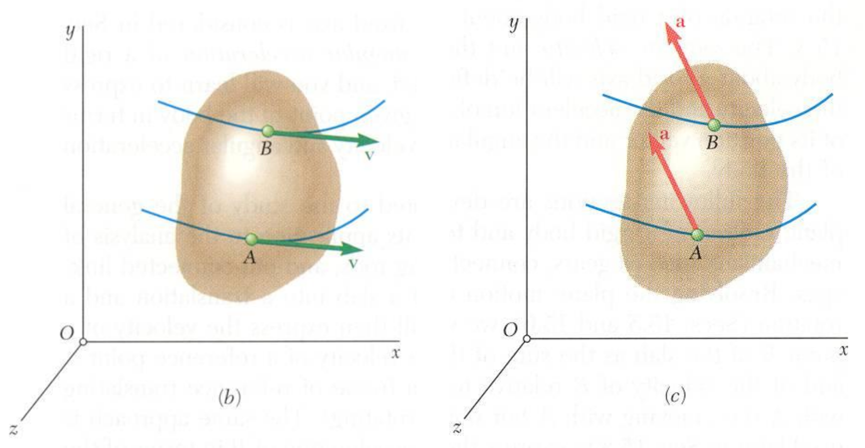
    It can also be observed that in a translation all the particles forming the body move along parallel paths.

- rectilinear translation

- curvilinear translation

Translation





- Consider rigid body in translation:
  - direction of any straight line inside the body is constant,
  - all particles forming the body move in parallel lines.

- For any two particles in the body, $\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$

- Differentiating with respect to time,

$$\dot{\vec{r}}_B = \dot{\vec{r}}_A + \dot{\vec{r}}_{B/A} = \dot{\vec{r}}_A$$

$$\vec{v}_B = \vec{v}_A$$

All particles have the same velocity.

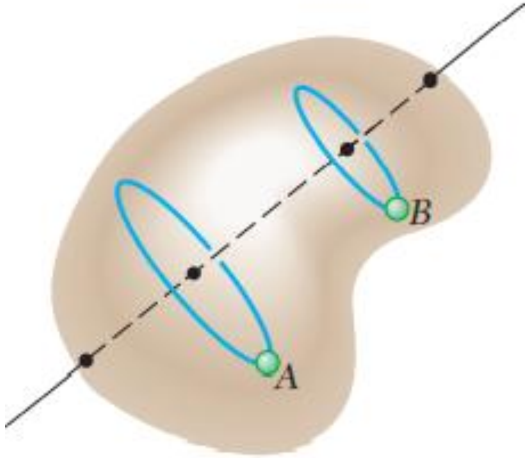
- Differentiating with respect to time again,

$$\ddot{\vec{r}}_B = \ddot{\vec{r}}_A + \ddot{\vec{r}}_{B/A} = \ddot{\vec{r}}_A$$

$$\vec{a}_B = \vec{a}_A$$

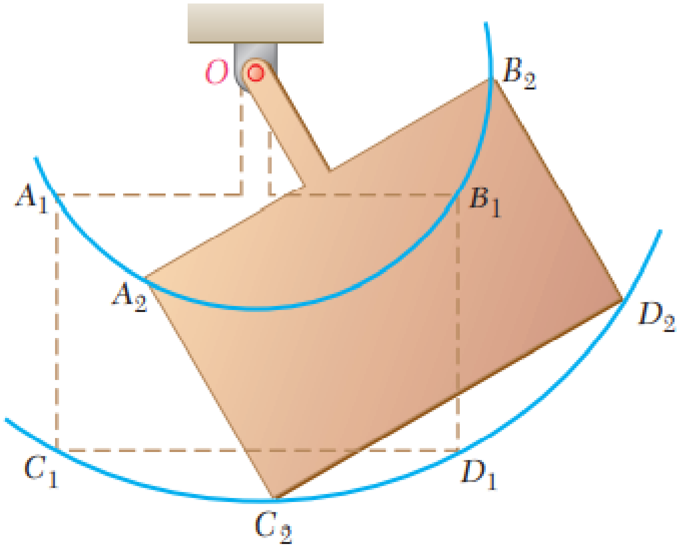
All particles have the same acceleration.

Rotation about a Fixed Axis



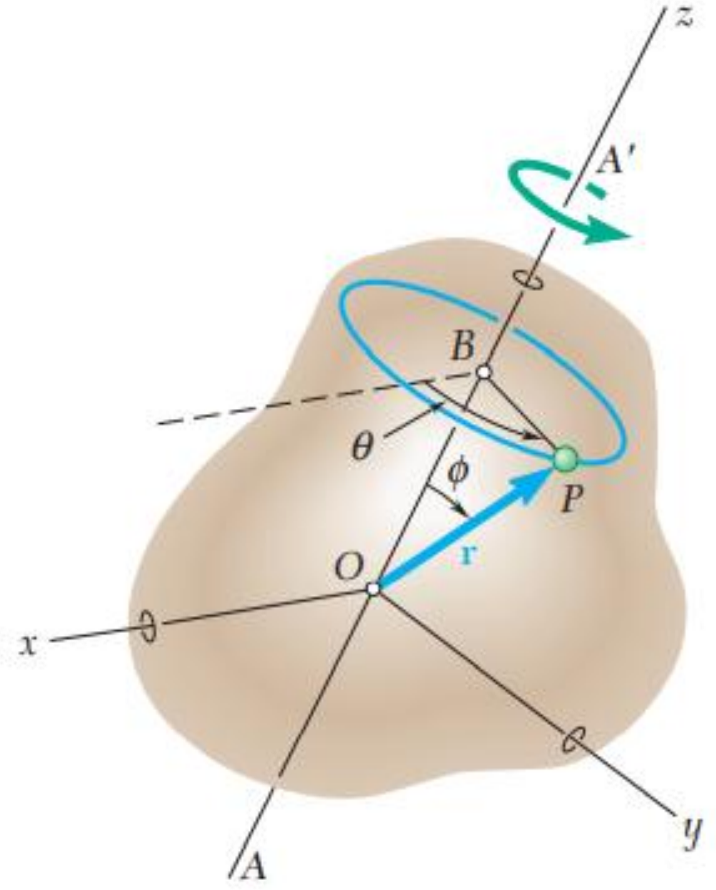
In this motion, the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis.

The axis about which rotation of the particles take place have zero velocity and zero acceleration.



In fixed axis rotation, particles move along concentric circles.

Rotation about a Fixed Axis: Velocity



- Consider rotation of rigid body about a fixed axis *AA'*

- Velocity vector $\vec{v} = d\vec{r}/dt$ of the particle P is tangent to the path with magnitude $v = ds/dt$

$$\Delta s = (BP)\Delta\theta = (r\sin\phi)\Delta\theta$$

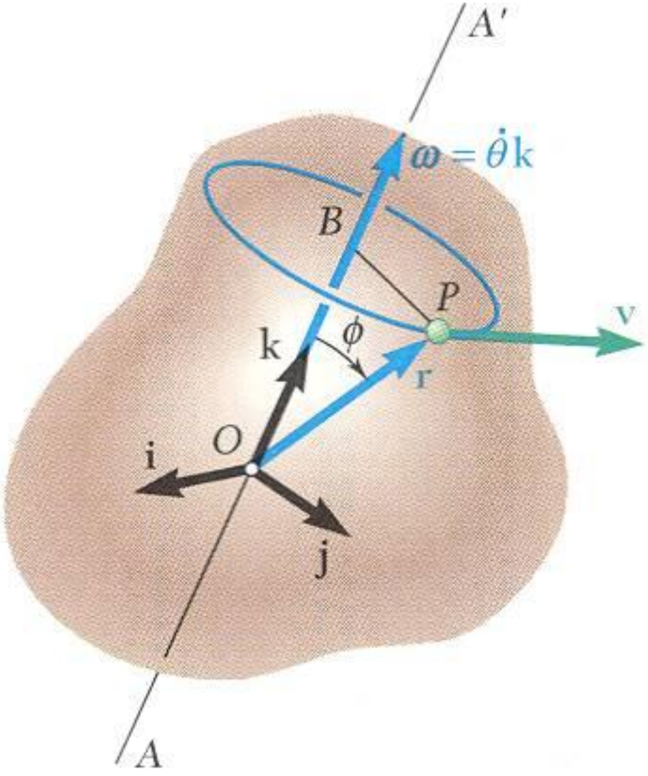
$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0}(r\sin\phi)\frac{\Delta\theta}{\Delta t} = r\dot{\theta}\sin\phi$$

- The same result is obtained from

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$$

$$\vec{\omega} = \omega\vec{k} = \dot{\theta}\vec{k} = angular\ velocity$$

Rotation about a Fixed Axis: Acceleration



- Differentiating to determine the acceleration,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) \qquad = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}$$

$$\frac{d\vec{\omega}}{dt} = \vec{\alpha} = \text{ angular acceleration}$$

$$= \alpha\vec{k} = \dot{\omega}\vec{k} = \ddot{\theta}\vec{k}$$

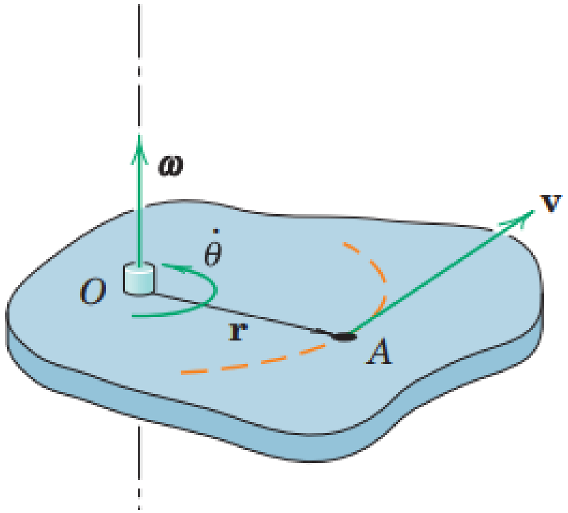
- Acceleration of P is combination of two vectors,

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\omega} \times \vec{r}$$

$\vec{\alpha} \times \vec{r} =$ tangential acceleration component

$\vec{\omega} \times \vec{\omega} \times \vec{r} =$ radial acceleration component

Rotation about a Fixed Axis: Representative Slab



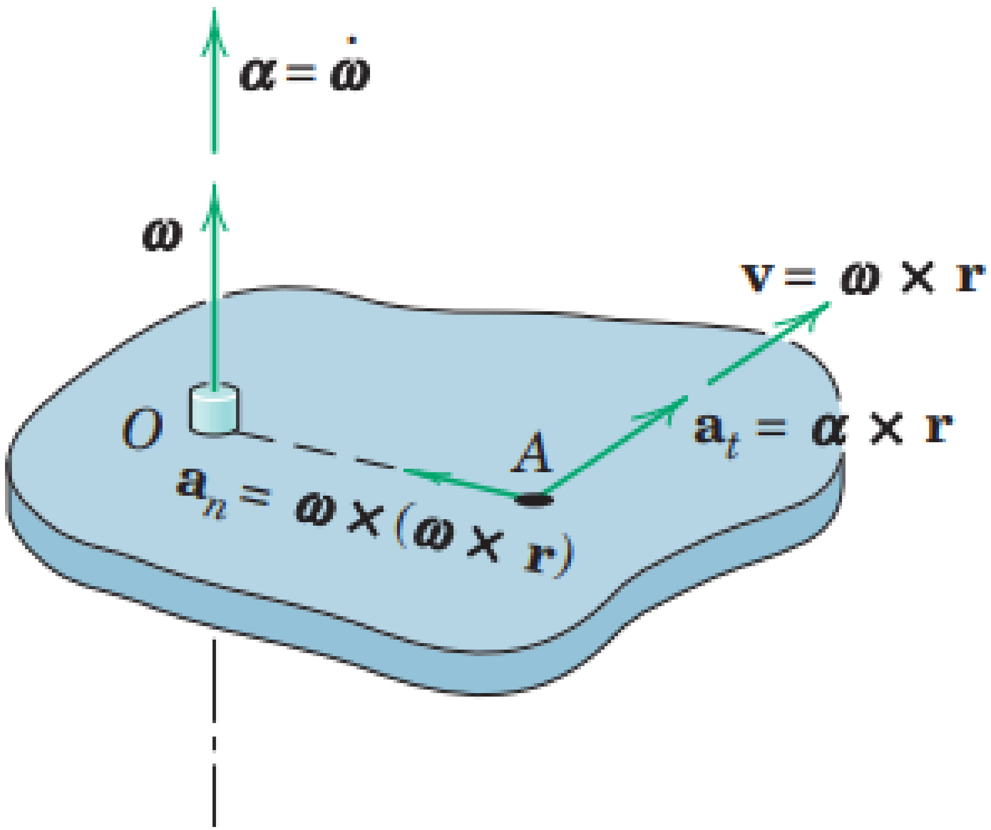
- Consider the motion of a representative slab in a plane perpendicular to the axis of rotation.

- Velocity of any point A of the slab,

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega\vec{k} \times \vec{r}$$

$$v = r\omega$$

Rotation about a Fixed Axis: Representative Slab



- Acceleration of any point P of the slab,

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\omega} \times \vec{r}$$

$$= \alpha\vec{k} \times \vec{r} - \omega^2\vec{r}$$

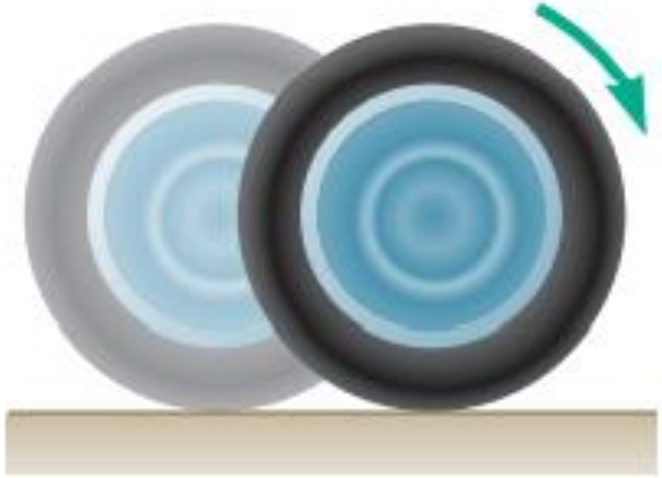
- Resolving the acceleration into tangential and normal components,

$$\vec{a}_t = \alpha\vec{k} \times \vec{r} \qquad a_t = r\alpha$$

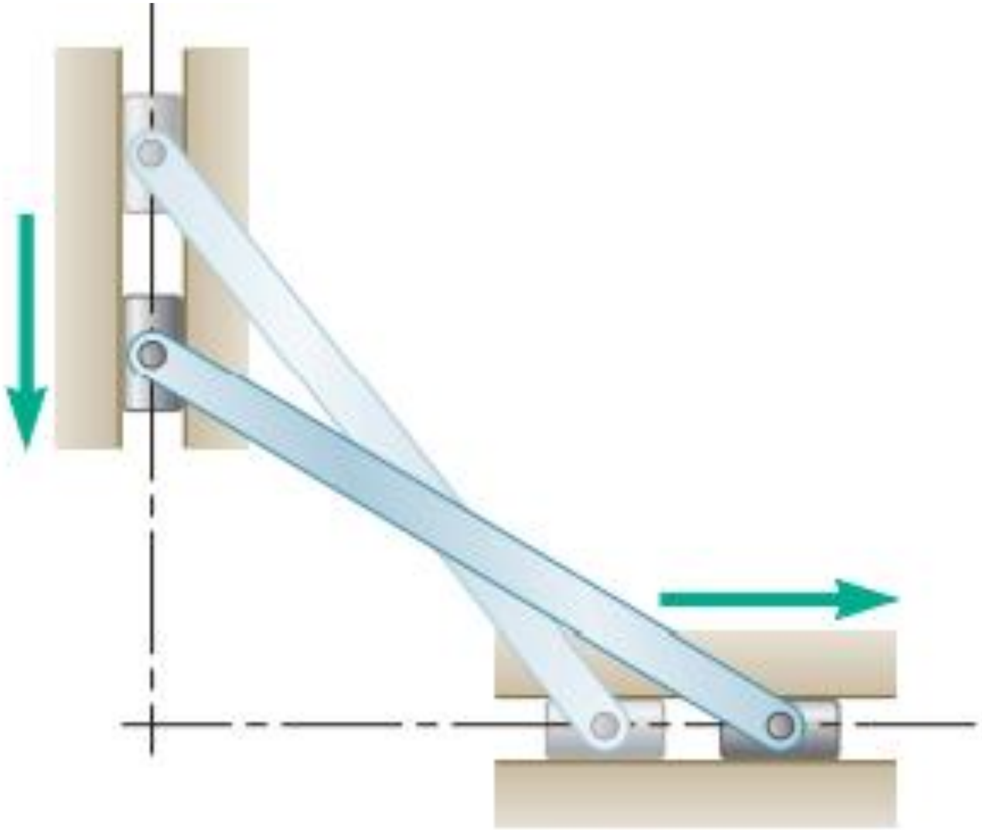
$$\vec{a}_n = -\omega^2\vec{r} \qquad a_n = r\omega^2$$

General Plane Motion

Any plane motion which is under both rotation and translation is referred to as a general plane motion.

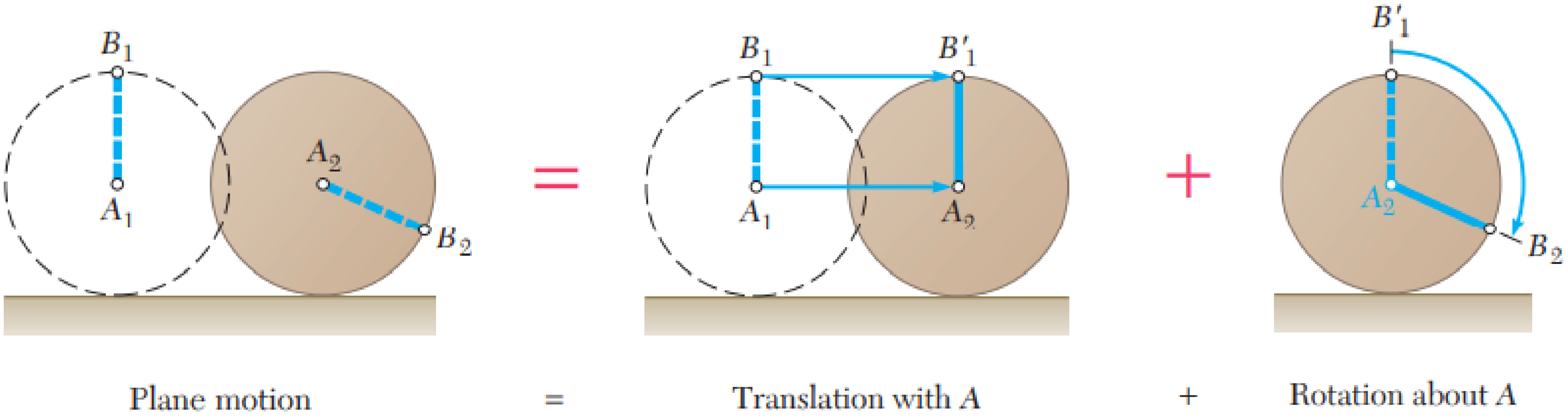


(a) Rolling wheel

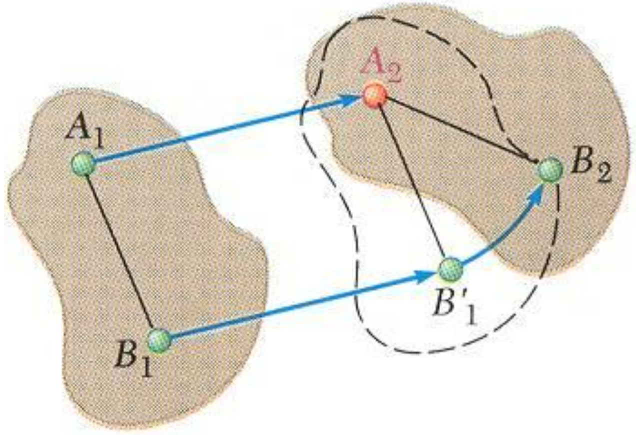


(b) Sliding rod

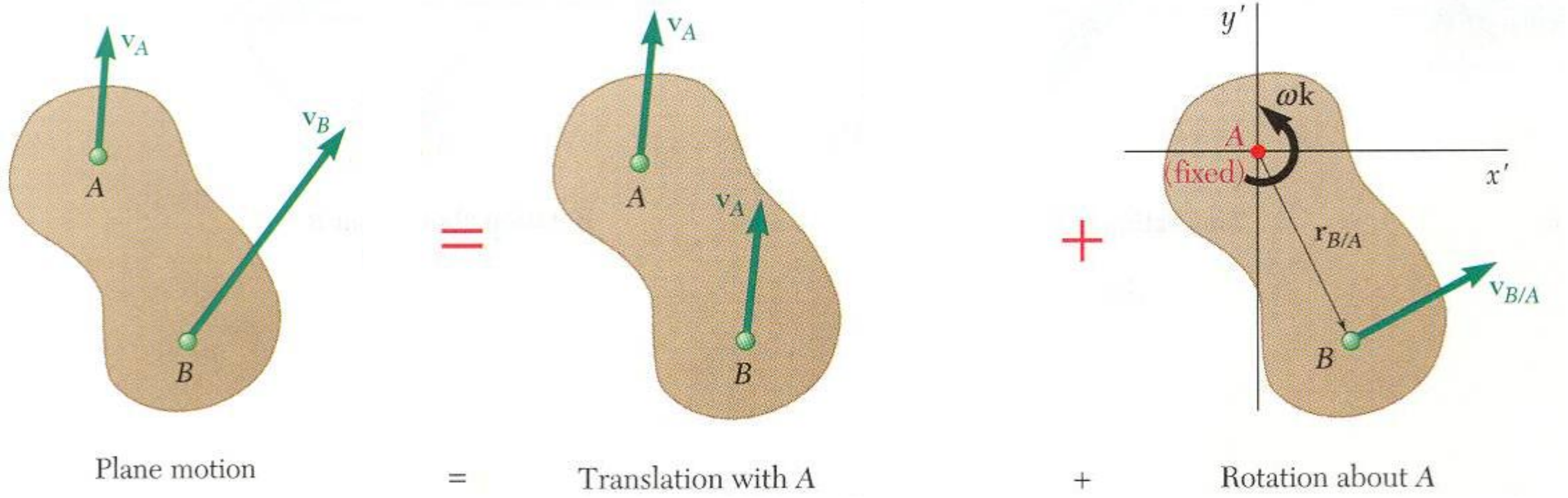
General Plane Motion

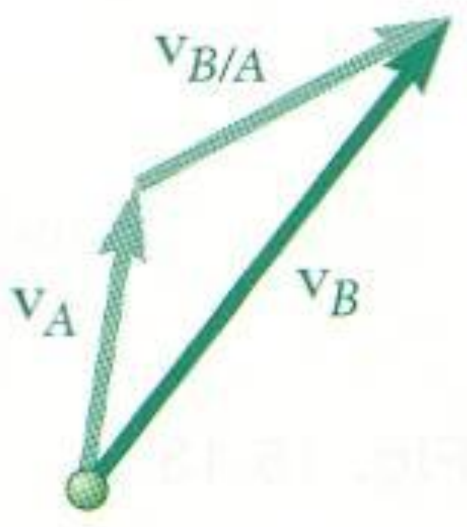


- *General plane motion* is neither a translation nor a rotation.
- General plane motion can be considered as the *sum* of a translation and rotation.
- Displacement of particles A and B to A_2 and B_2 can be divided into two parts:
  - translation to A_2 and B_1'
  - rotation of B_1' about A_2 to B_2



Absolute and Relative Velocity in Plane Motion





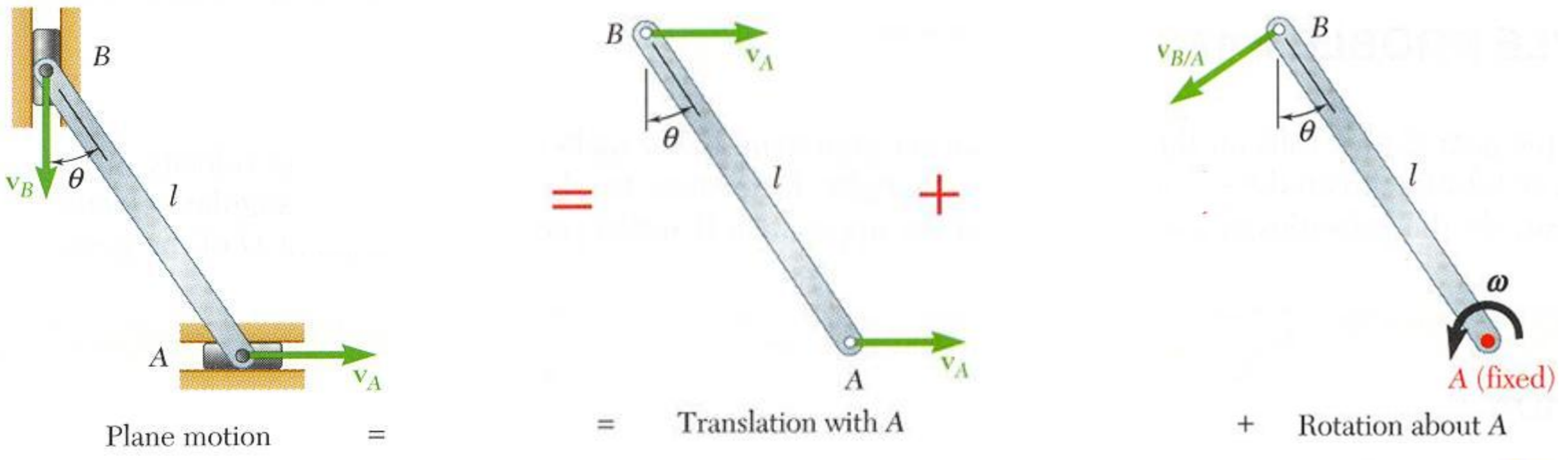
- Any plane motion can be replaced by a translation of an arbitrary reference point *A* and a simultaneous rotation about *A*.

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

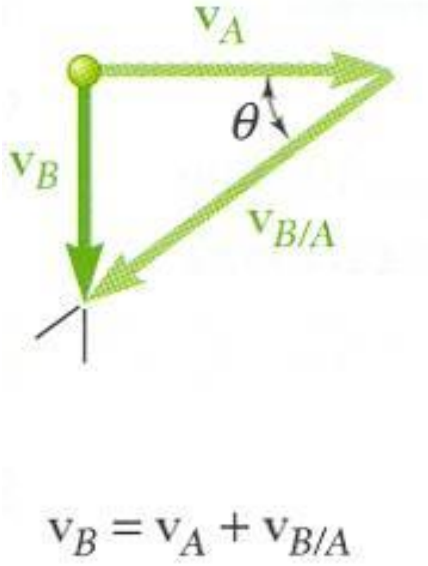
$$\vec{v}_{B/A} = \omega\vec{k} \times \vec{r}_{B/A} \qquad\qquad v_{B/A} = r\omega$$

$$\vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$

Absolute and Relative Velocity in Plane Motion



- Assuming that the velocity v_A of end A is known, wish to determine the velocity v_B of end B and the angular velocity ω in terms of v_A, l, and θ.
- The direction of v_B and $v_{B/A}$ are known. Complete the velocity diagram.



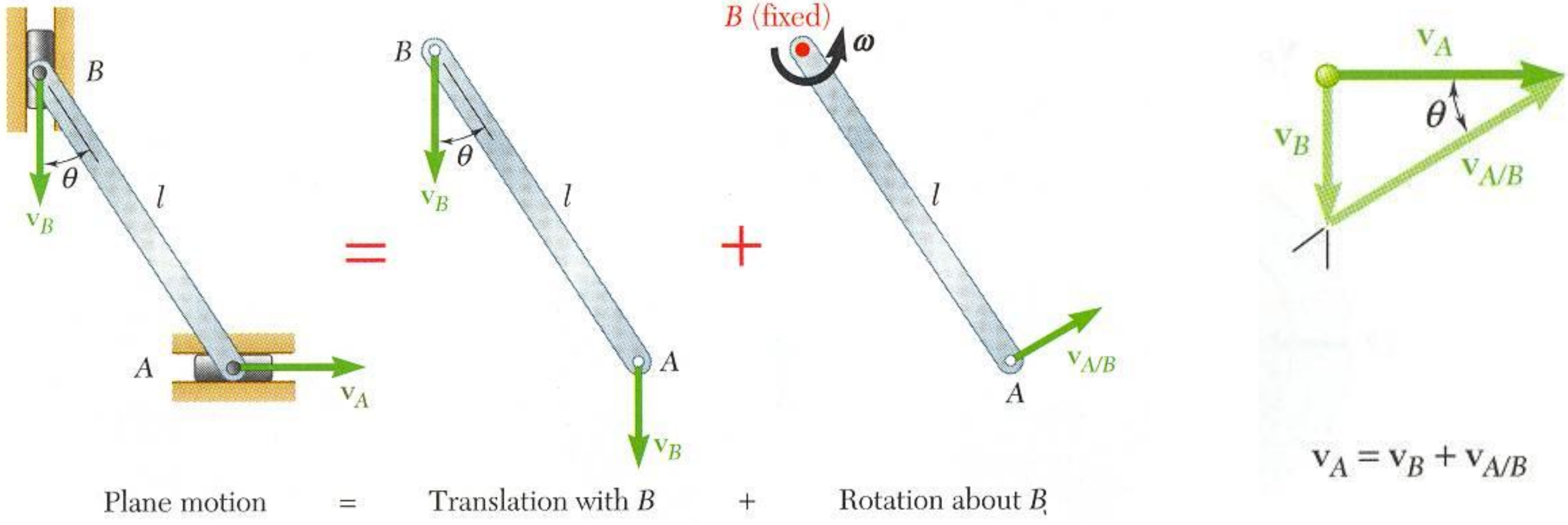
$$\frac{v_B}{v_A} = \tan\theta$$

$$v_B = v_A \tan\theta$$

$$\frac{v_A}{v_{B/A}} = \frac{v_A}{l\omega} = \cos\theta$$

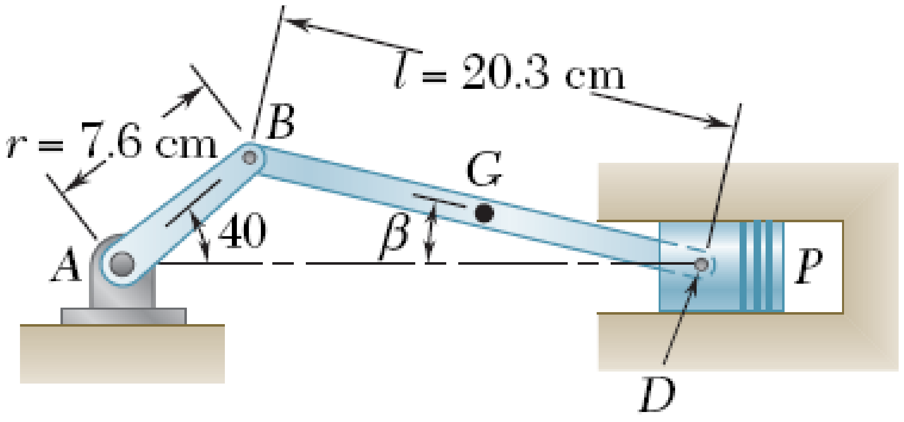
$$\omega = \frac{v_A}{l\cos\theta}$$

Absolute and Relative Velocity in Plane Motion



- Selecting point B as the reference point and solving for the velocity v_A of end A and the angular velocity ω leads to an equivalent velocity triangle.
- $v_{A/B}$ has the same magnitude but opposite sense of $v_{B/A}$. The sense of the relative velocity is dependent on the choice of reference point.
- Angular velocity ω of the rod in its rotation about B is the same as its rotation about A. Angular velocity is not dependent on the choice of reference point.

Example 1



$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

Required to find ω_{BD}

So, let us focus on $\vec{v}_B$

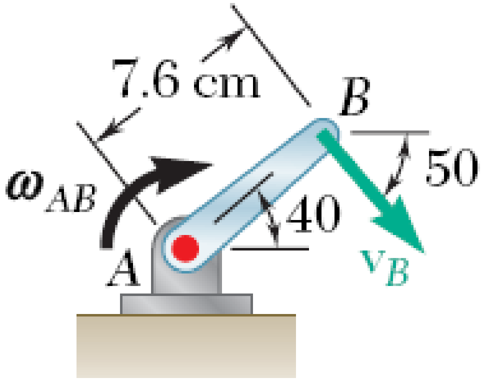
The crank *AB* has a constant clockwise angular velocity of 2000 rpm.

For the crank position indicated, determine

(a) the angular velocity of the connecting rod *BD*, and

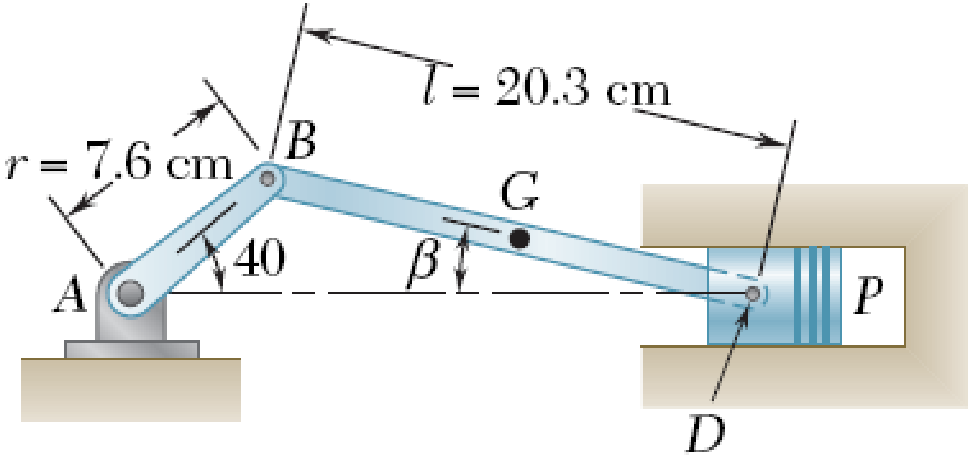
(b) the velocity of the piston *P*.

Example 1 (contd.)



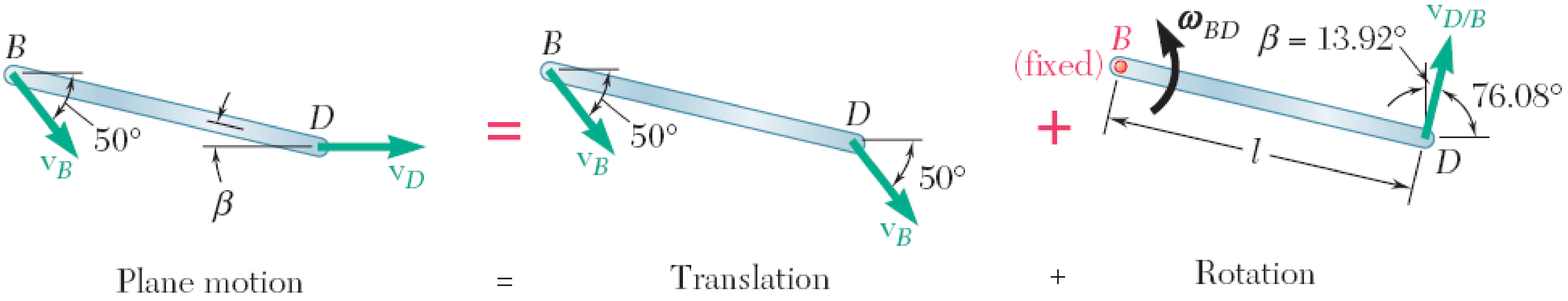
$$\omega_{AB} = 2000 \times \frac{1}{60} \times 2\pi = 209.3 \text{ rad/s}$$

$$v_B = (AB)\omega_{AB} = (7.6)(209.4 \text{ rad/s}) = 1591.44 \text{ cm/s}$$

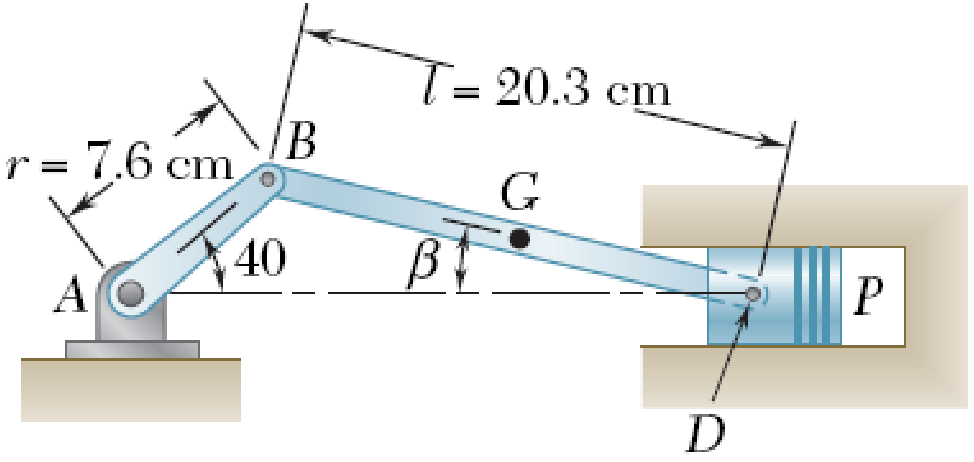


- The direction of the absolute velocity $\vec{v}_D$ is horizontal.
- The direction of the relative velocity $\vec{v}_{D/B}$ is perpendicular to *BD*.

Example 1 (contd.)



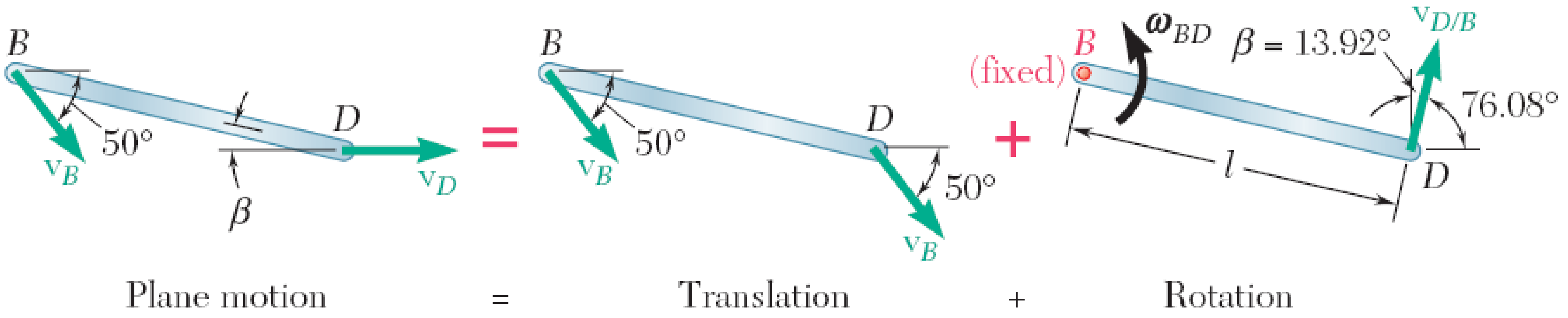
$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

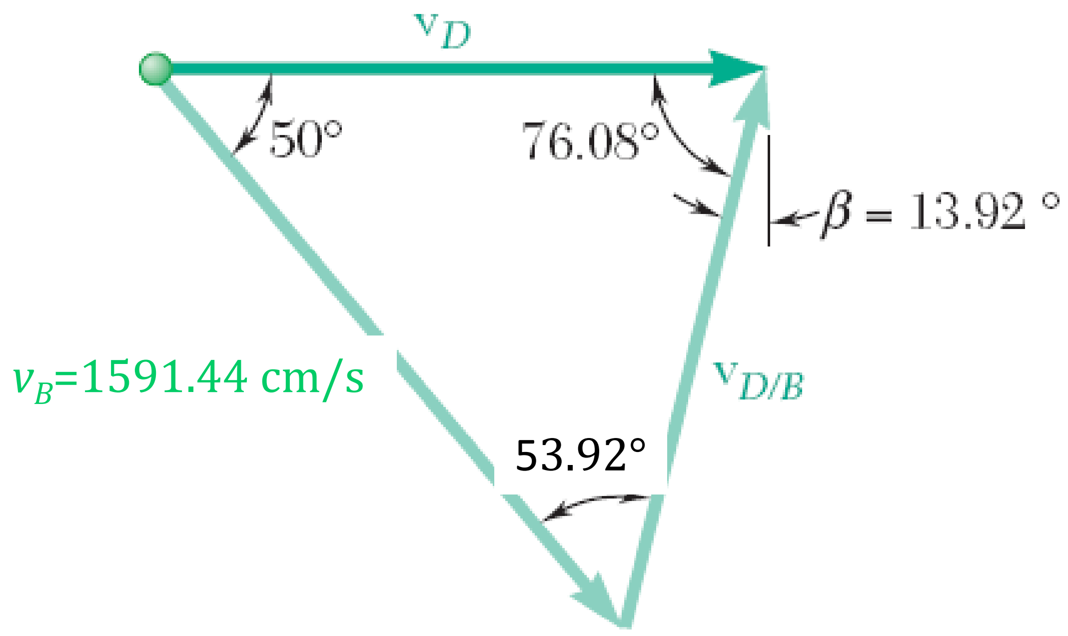


$$\frac{\sin 40°}{20.3\text{ cm}} = \frac{\sin \beta}{7.6\text{ cm}}$$

$$\beta = 13.92°$$

Example 1 (contd.)





$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

Determine the velocity magnitudes v_D and $v_{D/B}$ from the vector triangle.

$$\frac{v_D}{\sin 53.92°} = \frac{v_{D/B}}{\sin 50°} = \frac{1591.44 \text{ cm/s}}{\sin 76.08°}$$

$$v_D = 1325.14 \text{ cm/s} = v_{piston}$$

$$v_{D/B} = l\omega_{BD}$$

$$\omega_{BD} = \frac{v_{D/B}}{l} = \frac{1256.16 \text{ cm/s}}{20.3 \text{ cm}}$$

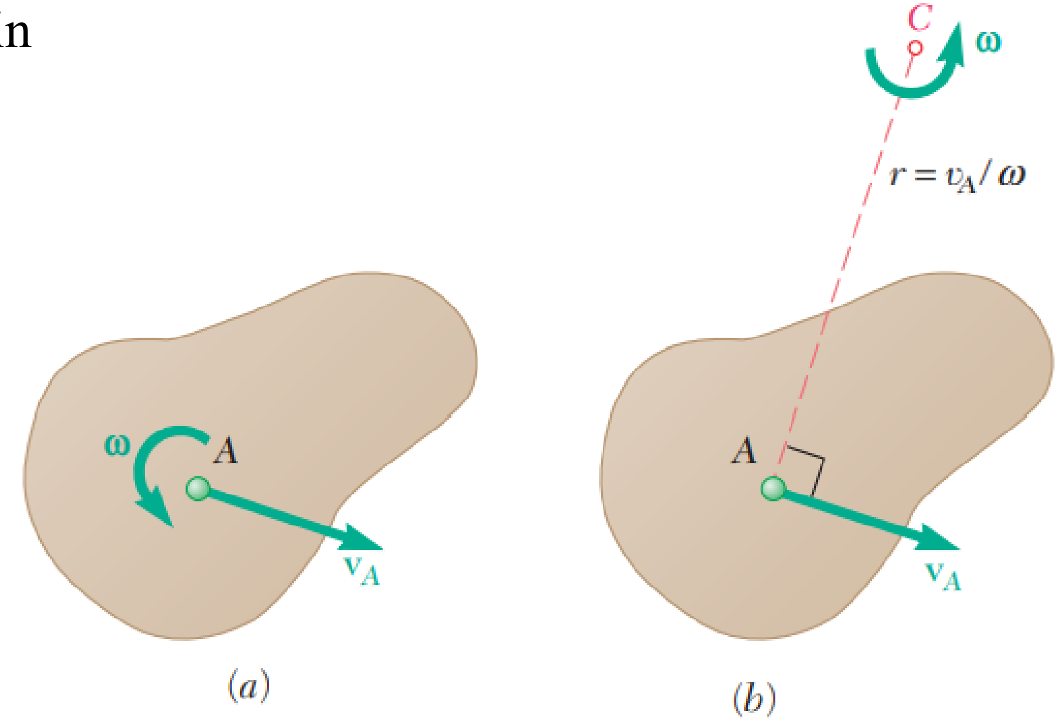
Instantaneous Centre of Zero Velocity

- Alternative method for solving problems involving the velocities of points on an object in plane motion.

$$\vec{v}_A = \overset{0}{\vec{v}_C} + \vec{v}_{A/C}$$

- We determined the velocity of a point on a rigid body in plane motion by adding the relative velocity due to rotation about a convenient reference point to the velocity of the reference point.

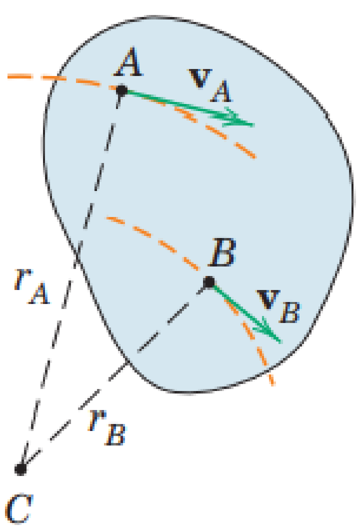
- The velocities of different particles of the rigid bodies can be evaluated based on the point about which the body momentarily rotates known as the <u>instantaneous center of zero velocity.</u>

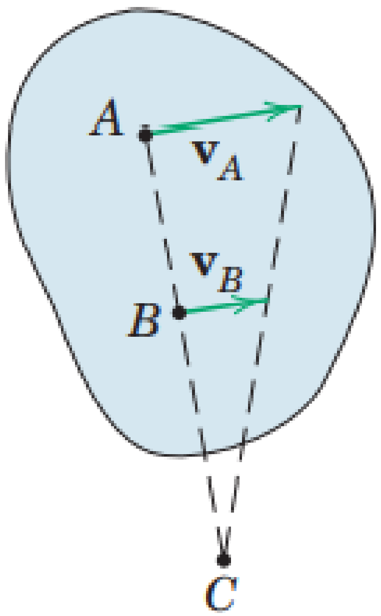


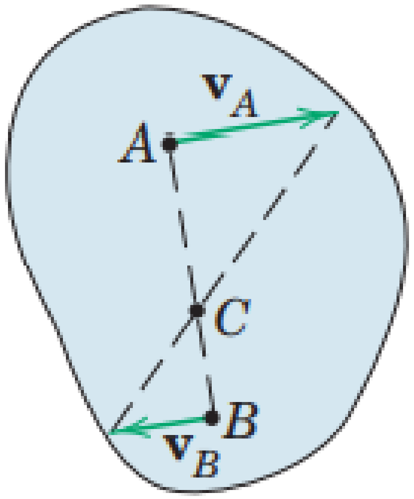
As far as velocities are concerned, at every instant in time the rigid body seems to rotate about a point called the instantaneous center C.

Instantaneous Centre of Zero Velocity (contd.)

Locating the Instantaneous Center:







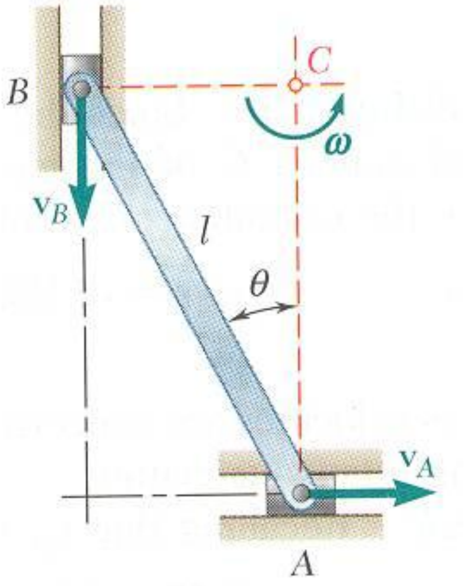
Directions of the absolute velocities of any two points A and B on the body are known and are not parallel.

Instantaneous center located by direct proportion.

If we also know the magnitude of the velocity of one of the points, v_A, we may easily obtain the angular velocity of the body and the linear velocity of every point in the body.

$$\omega = \frac{v_A}{r_A} \qquad v_B = r_B\omega = (r_B/r_A)v_A.$$

Instantaneous Centre of Zero Velocity (contd.)



- The instantaneous center of rotation lies at the intersection of the perpendiculars to the velocity vectors through A and B .

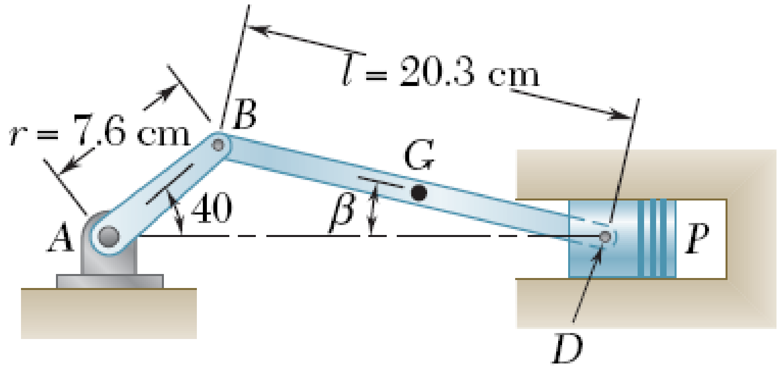
$$\omega = \frac{v_A}{AC} = \frac{v_A}{l\cos\theta}$$

$$v_B = (BC)\omega = (l\sin\theta)\frac{v_A}{l\cos\theta}$$

$$= v_A\tan\theta$$

- The velocities of all particles on the rod are as if they were rotated about C.

Example 2



The crank *AB* has a constant clockwise angular velocity of 2000 rpm.

For the crank position indicated, determine

(a) the angular velocity of the connecting rod *BD*, and

(b) the velocity of the piston *P*.

SOLUTION:

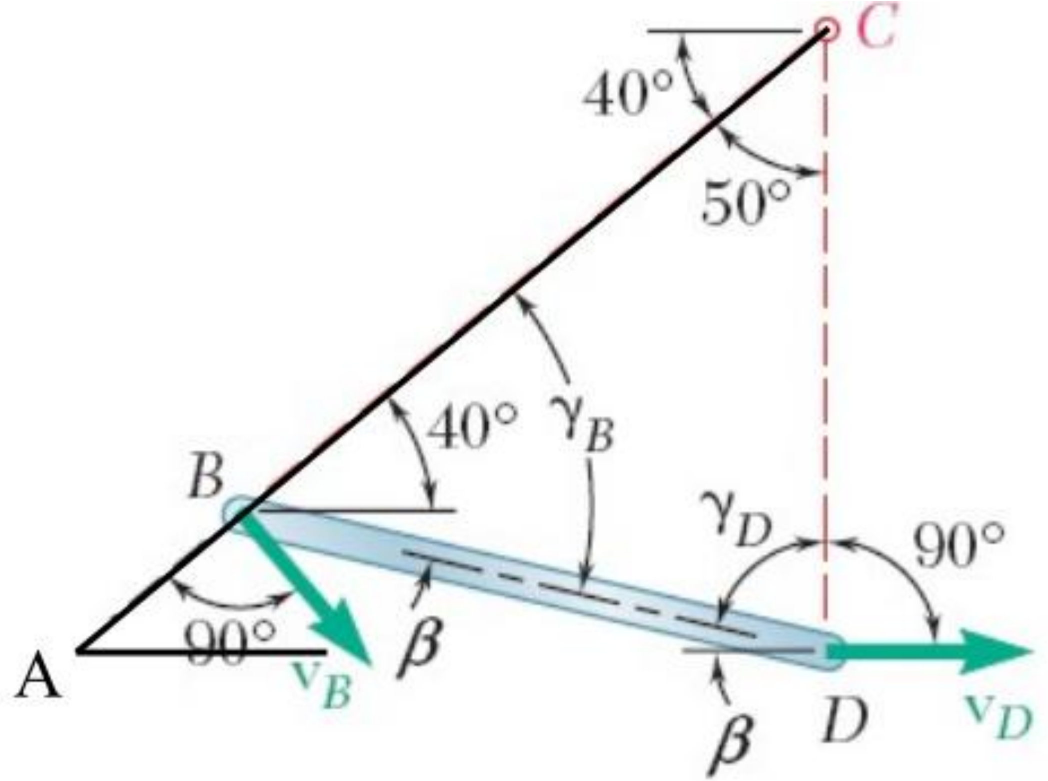
From Example 1,

$$\vec{v}_B = 1591.44 \text{ cm/s}$$

$$\beta = 13.92°$$

- The instantaneous center of rotation is at the intersection of the perpendiculars to the velocities through *B* and *D*.

Example 2 (contd.)



$$\gamma_B = 40° + \beta = 53.92°$$
$$\gamma_D = 90° - \beta = 76.08°$$

- Determine the angular velocity about the center of rotation based on the velocity at *B*.

$$\frac{BC}{\sin 76.05^0} = \frac{CD}{\sin 53.95^0} = \frac{20.3}{\sin 50°}$$

$BC = 25.7$ cm $\quad CD = 21.4$ cm

$$v_B = (BC)\omega_{BD}$$

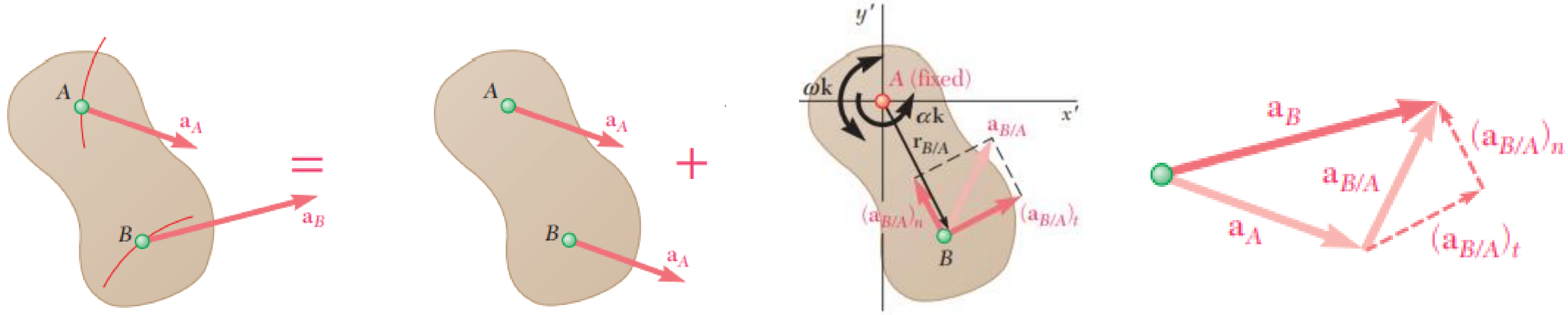
$$\omega_{BD} = \frac{v_B}{BC} = \frac{1591.44}{25.7}$$

$$\boxed{\omega_{BD} = 61.85 \text{ rad/s}}$$

$$v_D = (CD)\omega_{BD}$$

$$\boxed{v_P = v_D = 13.24 \text{ m/s}}$$

Absolute and Relative Acceleration in Plane Motion



Plane motion = Translation with A + Rotation about A

- Absolute acceleration of a particle of the slab,

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

- Relative acceleration $\vec{a}_{B/A}$ associated with rotation about A includes tangential and normal components,

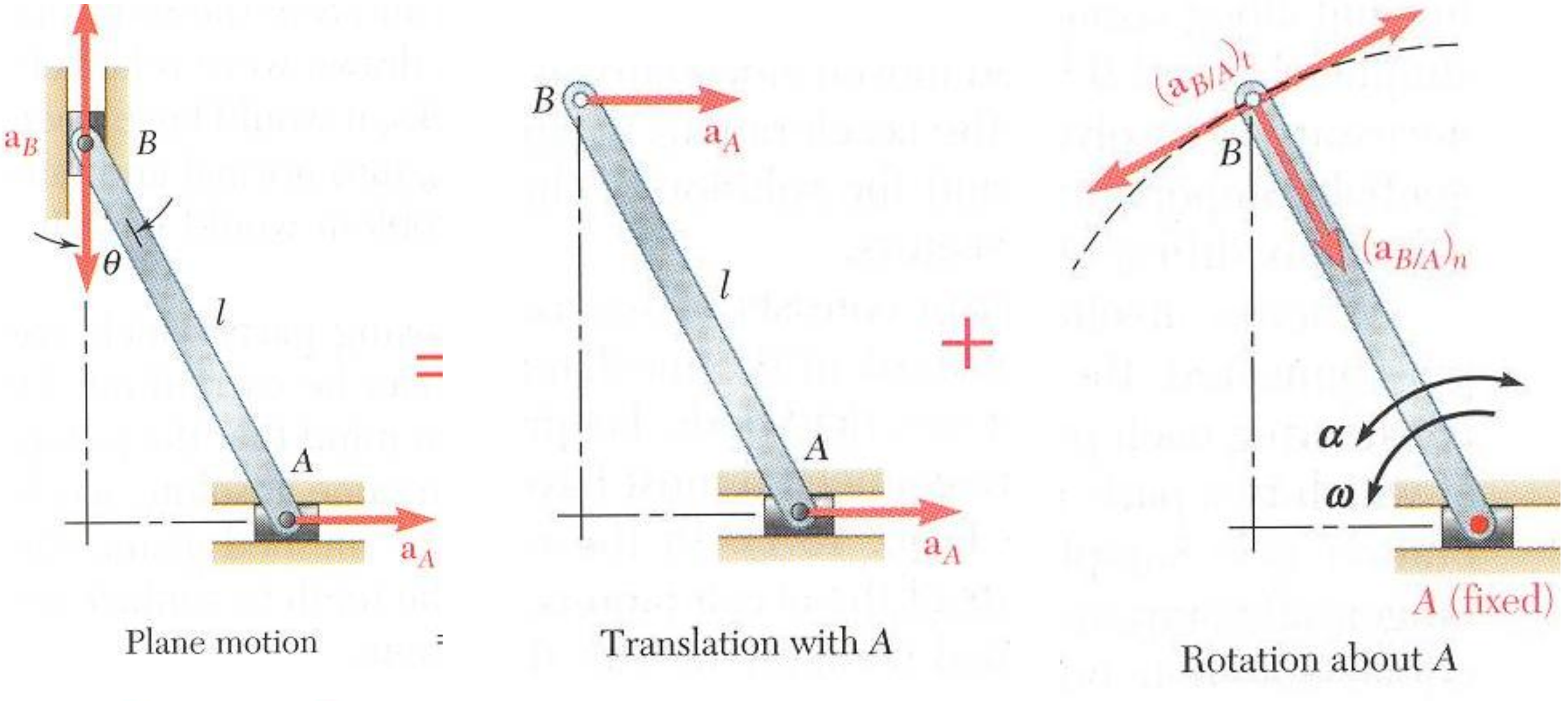
$$\left(\vec{a}_{B/A}\right)_n = -\omega^2 \vec{r}_{B/A}$$

$$\left(\vec{a}_{B/A}\right)_t = \alpha \vec{k} \times \vec{r}_{B/A}$$

$$\left(a_{B/A}\right)_t = r\alpha$$

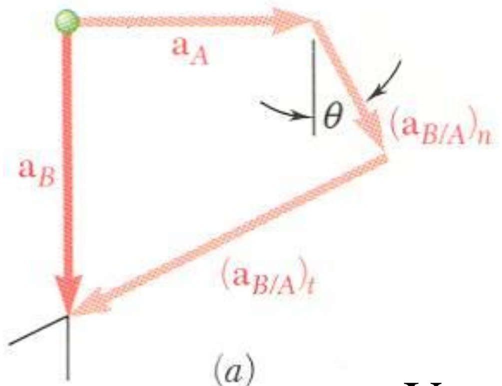
$$\left(a_{B/A}\right)_n = r\omega^2$$

Absolute and Relative Acceleration in Plane Motion



- Given $\vec{a}_A$ and $\vec{v}_A$, determine $\vec{a}_B$ and $\vec{\alpha}$.

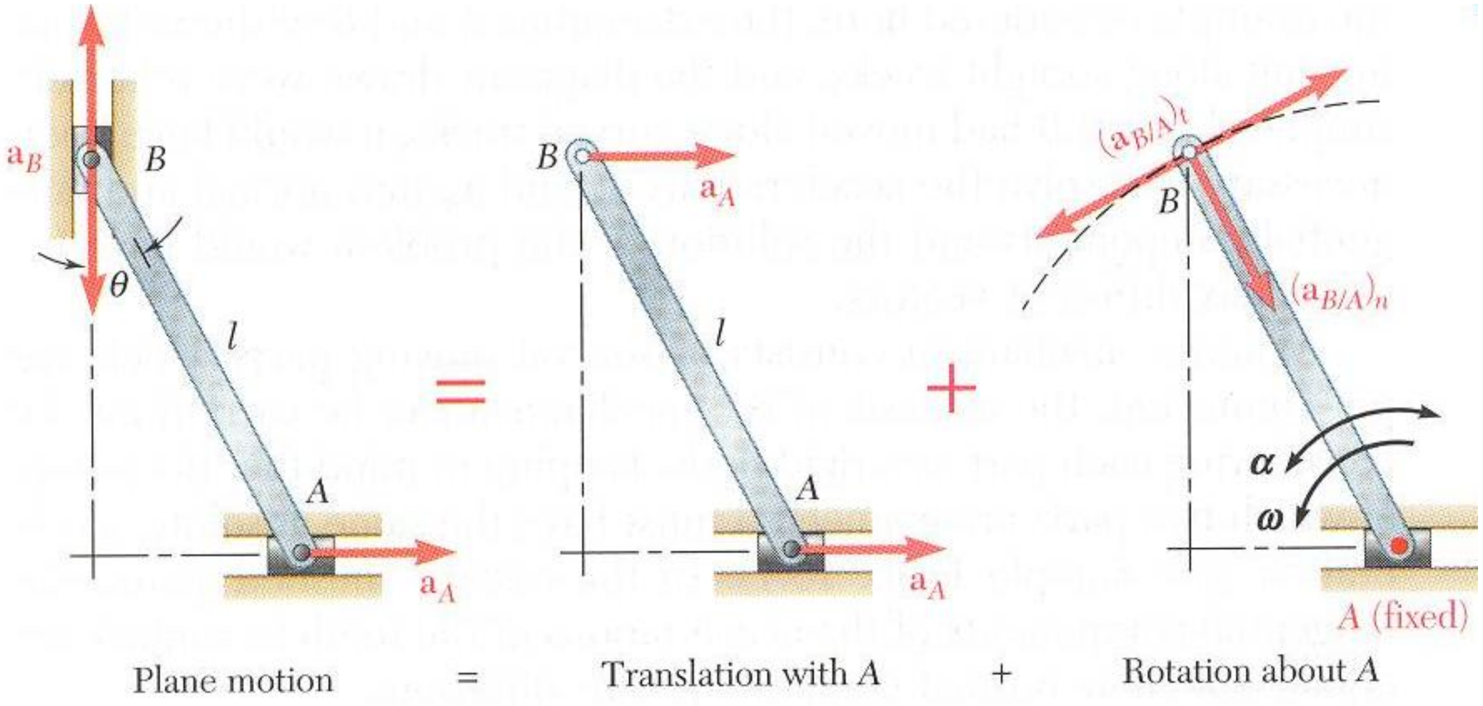
$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$
$$= \vec{a}_A + \left(\vec{a}_{B/A}\right)_n + \left(\vec{a}_{B/A}\right)_t$$

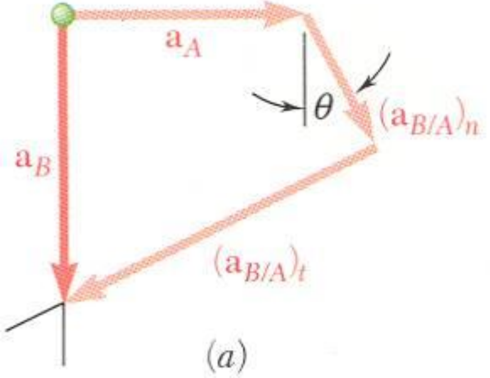


(a)

- Vector result depends on sense of $\vec{a}_A$ and the relative magnitudes of a_A and $\left(a_{B/A}\right)_n$
- Must also know angular velocity ω.

Absolute and Relative Acceleration in Plane Motion





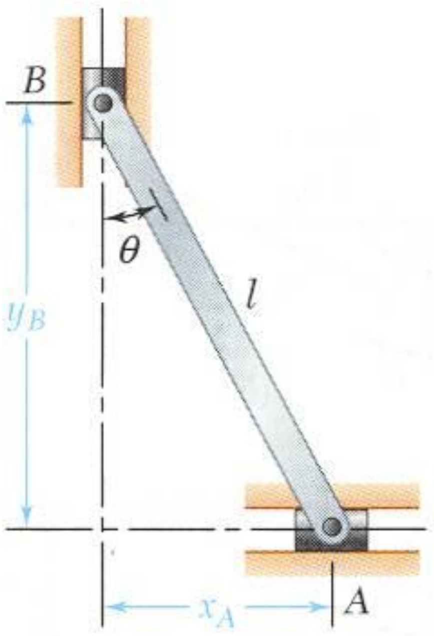
- Write $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ in terms of the two component equations,

$\overset{+}{\rightarrow}$ x components: $0 = a_A + l\omega^2 \sin\theta - l\alpha\cos\theta$

$+\uparrow$ y components: $-a_B = -l\omega^2 \cos\theta - l\alpha\sin\theta$

  - Solve for a_B and α.

Analysis of Plane Motion in Terms of a Parameter



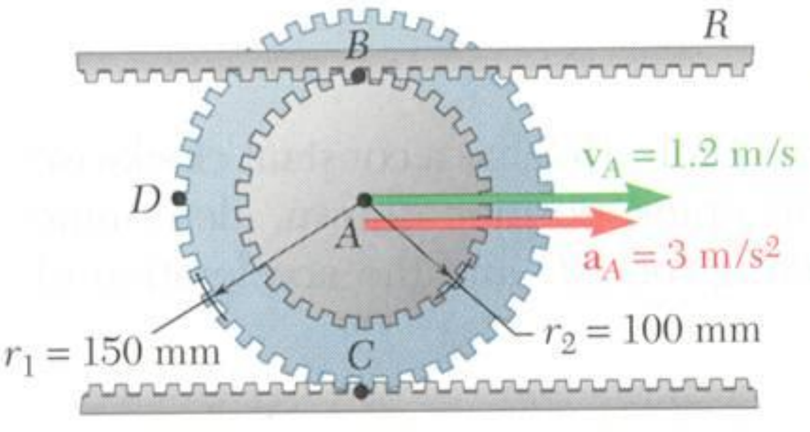
- In some cases, it is advantageous to determine the absolute velocity and acceleration of a mechanism directly.

$$x_A = l\sin\theta \qquad y_B = l\cos\theta$$

$$\begin{aligned} v_A &= \dot{x}_A \\ &= l\dot{\theta}\cos\theta \\ &= l\omega\cos\theta \end{aligned} \qquad \begin{aligned} v_B &= \dot{y}_B \\ &= -l\dot{\theta}\sin\theta \\ &= -l\omega\sin\theta \end{aligned}$$

$$\begin{aligned} a_A &= \ddot{x}_A \\ &= -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta \\ &= -l\omega^2\sin\theta + l\alpha\cos\theta \end{aligned} \qquad \begin{aligned} a_B &= \ddot{y}_B \\ &= -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta \\ &= -l\omega^2\cos\theta - l\alpha\sin\theta \end{aligned}$$

Example 3



The center of the double gear has a velocity and acceleration to the right of 1.2 m/s and 3 m/s^2, respectively. The lower rack is stationary.

Determine:

(a) the angular acceleration of the gear, and

(b) the acceleration of points *B*, *C*, and *D*.

SOLUTION:

(a) Angular acceleration of the gear:

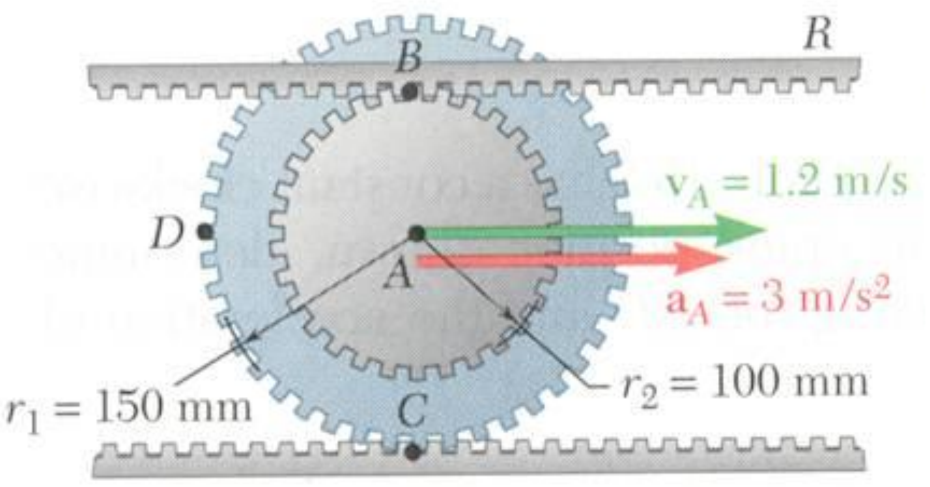
Since the gear rolls on the lower rack, its center A moves through a distance equal to the outer circumference $2\pi r_1$ for each full revolution of the gear.

Noting that 1 rev = 2π rad, and that when A moves to the right ($x_A > 0$) the gear rotates clockwise ($\theta < 0$), we write

$$\frac{x_A}{2\pi r_1} = -\frac{\theta}{2\pi}$$

$$x_A = -r_1\theta$$

Example 3 (contd.)



- The expression of the gear position as a function of θ is differentiated twice to define the relationship between the translational and angular accelerations.

$$x_A = -r_1\theta$$

[Since the gear rotates clockwise.]

$$v_A = -r_1\dot{\theta} = -r_1\omega$$

$$\omega = -\frac{v_A}{r_1} = -\frac{1.2\,\text{m/s}}{0.150\,\text{m}} = -8\,\text{rad/s}$$

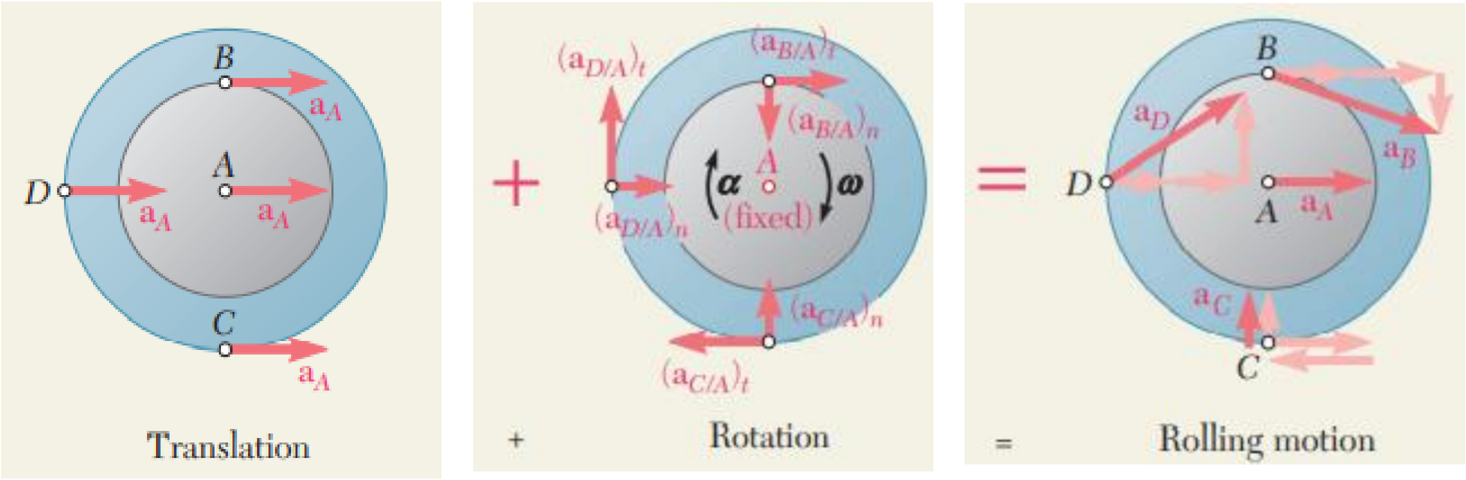
$$a_A = -r_1\ddot{\theta} = -r_1\alpha$$

$$\alpha = -\frac{a_A}{r_1} = -\frac{3\,\text{m/s}^2}{0.150\,\text{m}}$$

$$\boxed{\vec{\alpha} = \alpha\vec{k} = -\left(20\,\text{rad/s}^2\right)\vec{k}}$$

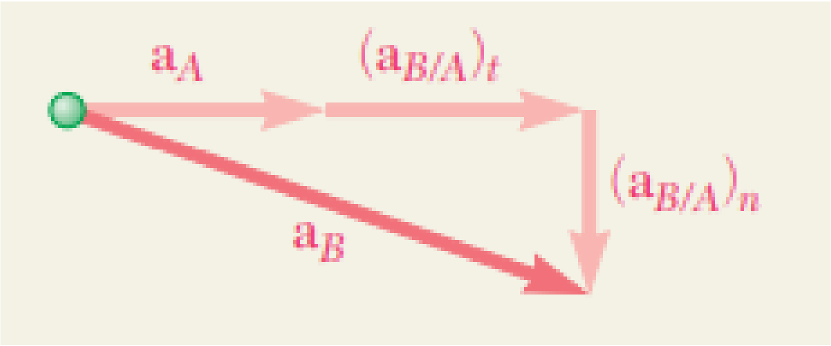
Example 3 (contd.)

(b) Acceleration of point B:



- The acceleration of each point is obtained by adding the acceleration of the gear center and the relative accelerations with respect to the center.

The latter includes normal and tangential acceleration components.



$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} = \vec{a}_A + \left(\vec{a}_{B/A}\right)_t + \left(\vec{a}_{B/A}\right)_n$$

$$= \vec{a}_A + \alpha\vec{k} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$$

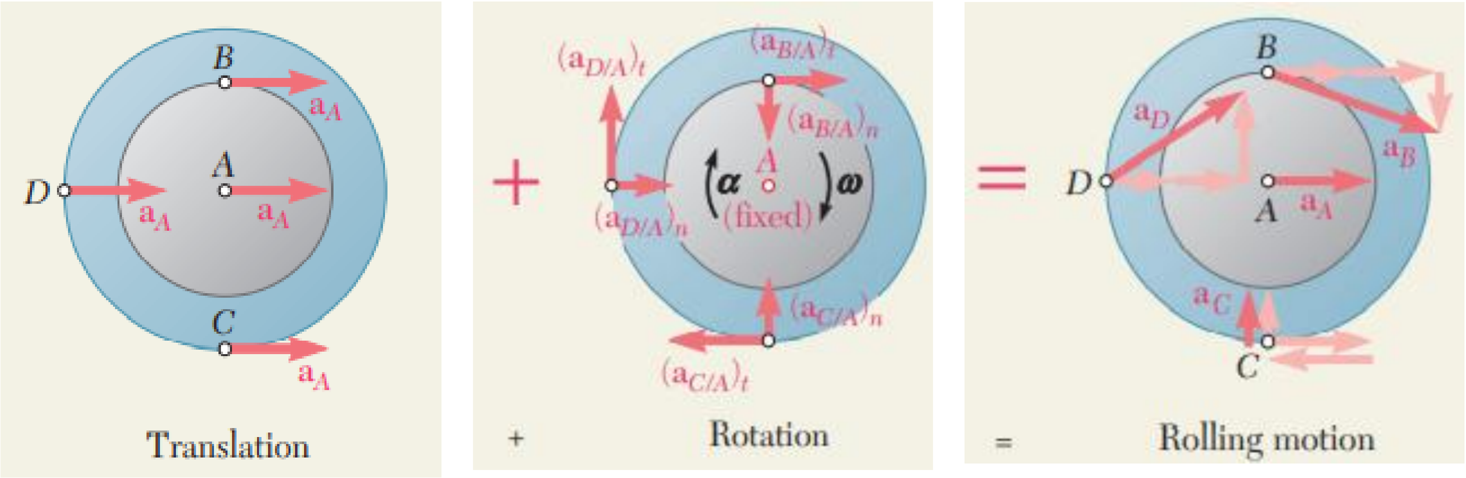
$$= (3\,\mathrm{m/s^2})\vec{\imath} - (20\,\mathrm{rad/s^2})\vec{k} \times (0.1\,\mathrm{m})\vec{\jmath} - (8\,\mathrm{rad/s})^2(0.1\,\mathrm{m})\vec{\jmath}$$

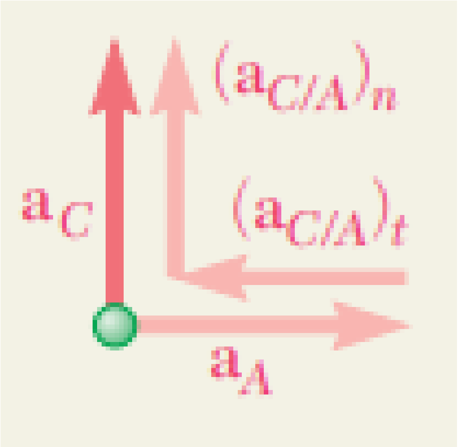
$$= (3\,\mathrm{m/s^2})\vec{\imath} + (2\,\mathrm{m/s^2})\vec{\imath} - (6.4\,\mathrm{m/s^2})\vec{\jmath}$$

$$= (5\,\mathrm{m/s^2})\vec{\imath} - (6.4\,\mathrm{m/s^2})\vec{\jmath}$$

Example 3 (contd.)

Acceleration of point C:





$$\vec{a}_C = \vec{a}_A + \vec{a}_{C/A}$$

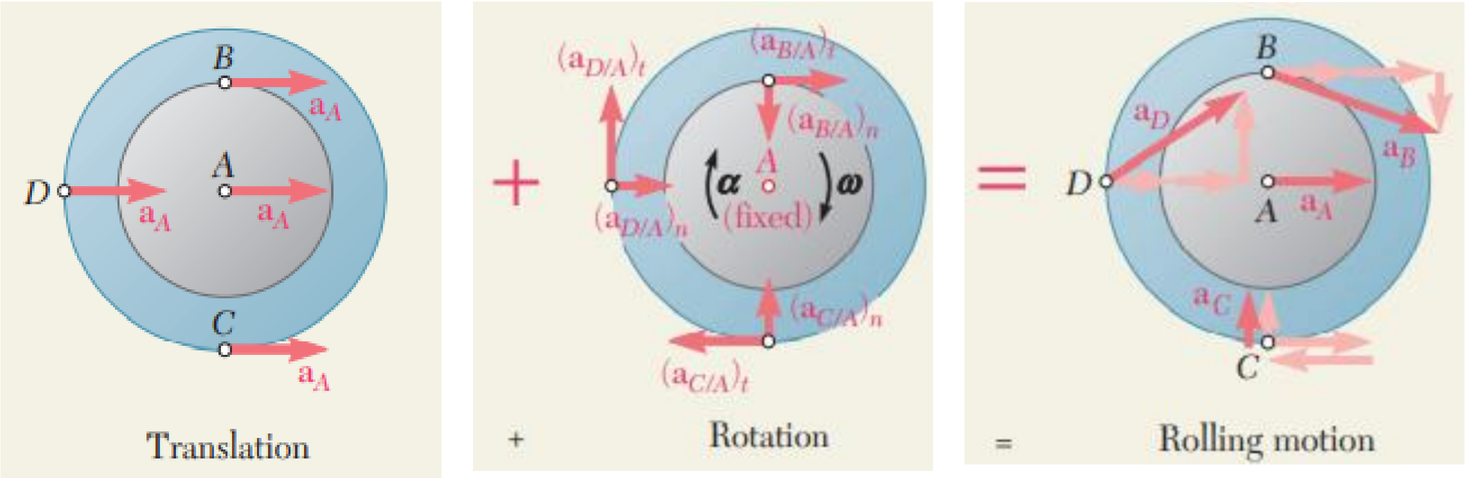
$$= \vec{a}_A + \alpha\vec{k} \times \vec{r}_{C/A} - \omega^2 \vec{r}_{C/A}$$

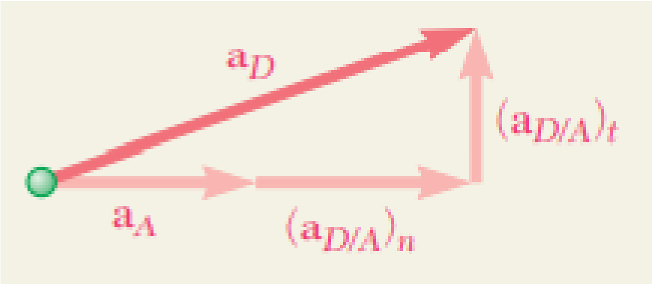
$$= (3)\vec{i} - (3)\vec{i} + (9.60)\vec{j}$$

$$\boxed{\vec{a}_c = (9.60\,\mathrm{m/s^2})\vec{j}}$$

Example 3 (contd.)

Acceleration of point D:





$$\vec{a}_D = \vec{a}_A + \vec{a}_{D/A} = \vec{a}_A + \alpha\vec{k} \times \vec{r}_{D/A} - \omega^2 \vec{r}_{D/A}$$

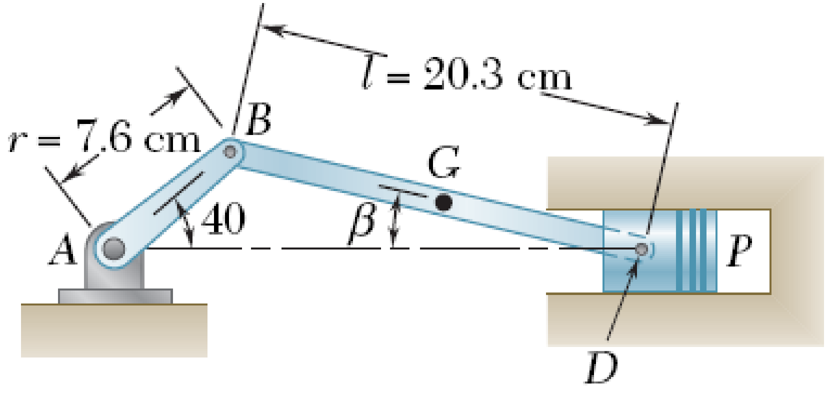
$$= (3\,\mathrm{m/s^2})\vec{\imath} - (20\,\mathrm{rad/s^2})\vec{k} \times (-0.150\mathrm{m})\vec{\imath} - (8\,\mathrm{rad/s})^2(-0.150\mathrm{m})\vec{\imath}$$

$$= (3\,\mathrm{m/s^2})\vec{\imath} + (3\,\mathrm{m/s^2})\vec{\jmath} + (9.60\,\mathrm{m/s^2})\vec{\imath}$$

$$\boxed{\vec{a}_D = (12.6\,\mathrm{m}/s^2)\vec{\imath} + (3\,\mathrm{m}/s^2)\vec{\jmath} \qquad a_D = 12.95\,\mathrm{m}/s^2}$$

Example 4

SOLUTION:



Crank *AB* of the engine system has a constant clockwise angular velocity of 2000 rpm.

For the crank position shown, determine

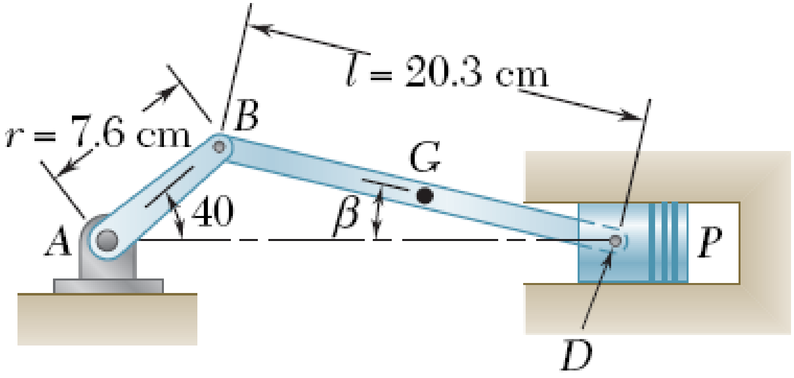
a) the angular acceleration of the connecting rod *BD* and

b) the acceleration of point *D*.

$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B}$$

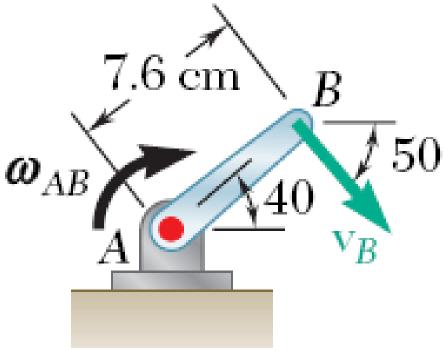
$$= \vec{a}_B + \left(\vec{a}_{D/B}\right)_t + \left(\vec{a}_{D/B}\right)_n$$

Example 4 (contd.)



$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B} = \vec{a}_B + \left(\vec{a}_{D/B}\right)_t + \left(\vec{a}_{D/B}\right)_n$$

- The acceleration of *B* is determined from the given rotation speed of *AB*.



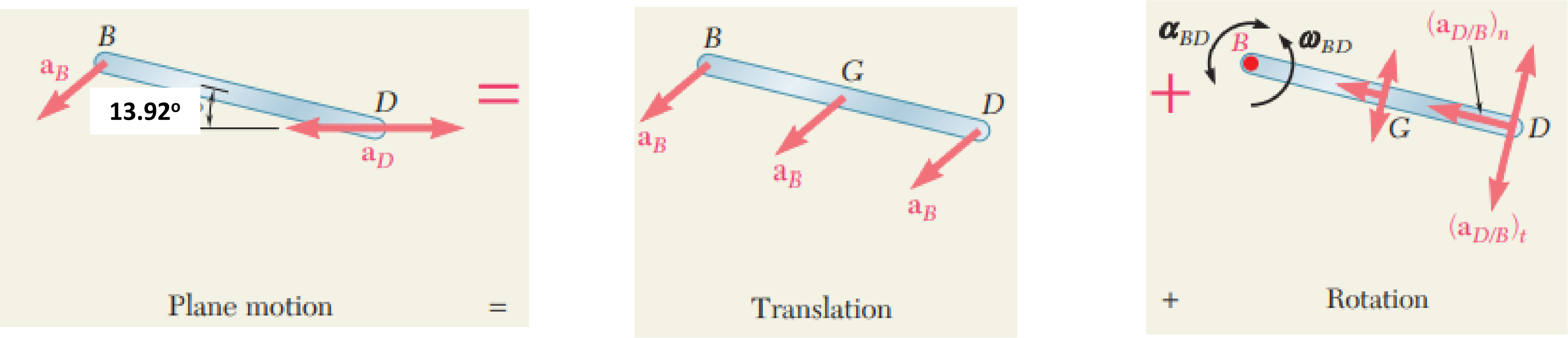
$$\omega_{AB} = 2000 \text{ rpm} = 209.4 \text{ rad/s} = \text{constant}$$

$$\alpha_{AB} = 0$$

$$a_B = r\omega_{AB}^2 = (7.6 \text{ cm})\,(209.4 \text{ rad/s})^2 = 3332.5 \text{ m/s}^2$$

$$\vec{a}_B = (3332.5 \text{ m/s})\,(-\cos 40°\vec{i} - \sin 40°\vec{j})$$

Example 4 (contd.)



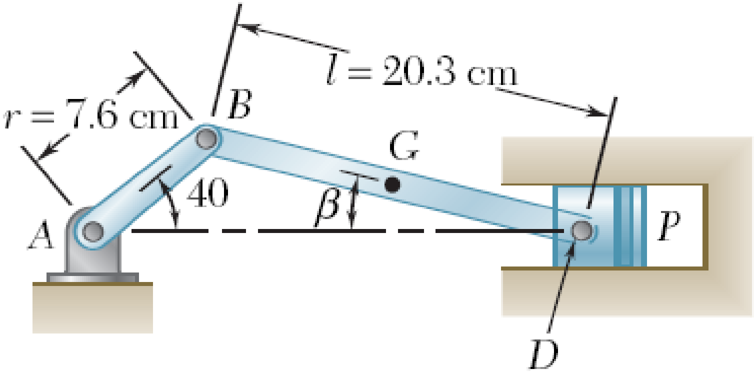
- The directions of the accelerations $\vec{a}_D, \left(\vec{a}_{D/B}\right)_t$, and $\left(\vec{a}_{D/B}\right)_n$ are determined from the geometry. $\vec{a}_D = \mp a_D \vec{i}$

From Example 2, $\omega_{BD} = 61.85$ rad/s, $\beta = 13.92$º.

$$\left(a_{D/B}\right)_n = (BD)\omega_{BD}^2 = (0.203)\,(61.85\,\mathrm{rad/s})^2$$

$$\left(\vec{a}_{D/B}\right)_n = (776.56\,\mathrm{m/s^2})(-\cos 13.92°\vec{i} + \sin 13.92°\vec{j})$$

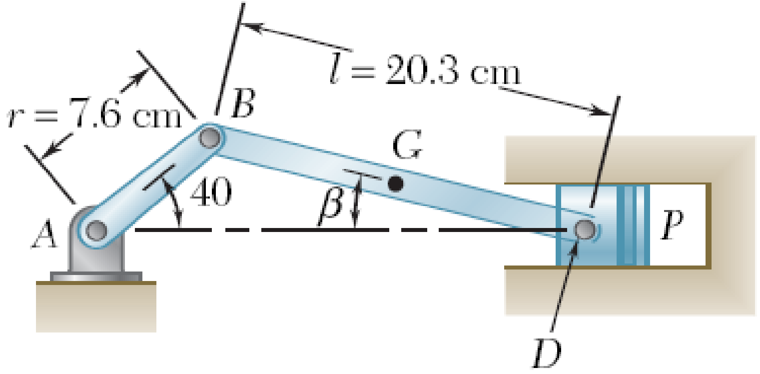
$$\left(a_{D/B}\right)_t = (BD)\alpha_{BD} = (0.203)\alpha_{BD} = 0.203\alpha_{BD}$$

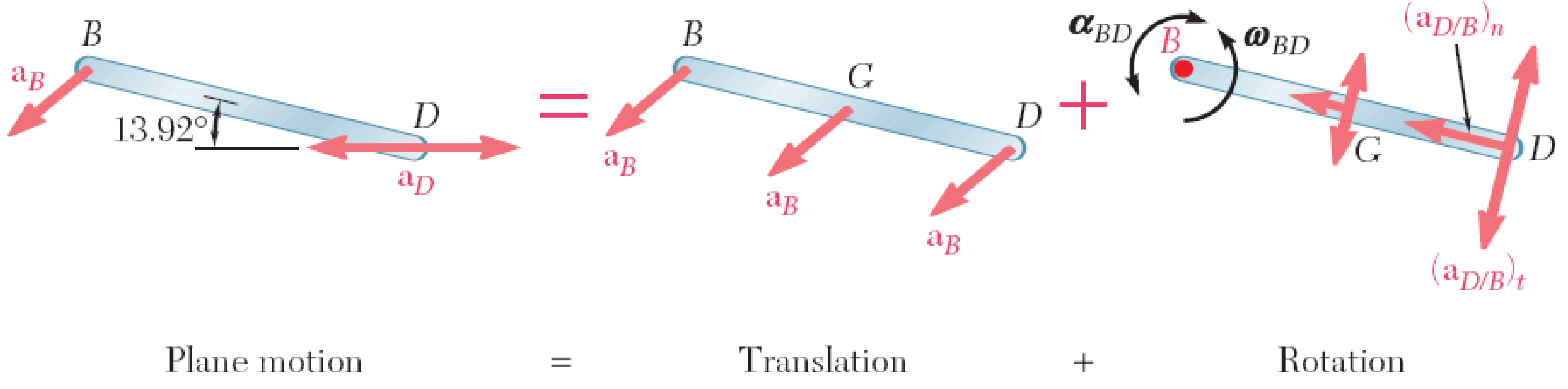


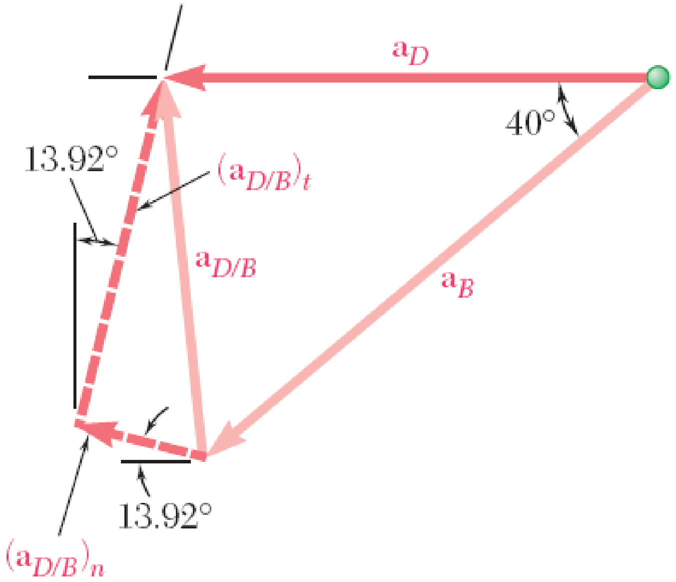
The direction of $(a_{D/B})_t$ is known but the sense is not known,

$$\left(\vec{a}_{D/B}\right)_t = 0.203\alpha_{BD}\,(\pm \sin 76.02°\vec{i} \pm \cos 76.02°\vec{j})$$

Example 4 (contd.)







$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B} = \vec{a}_B + \left(\vec{a}_{D/B}\right)_t + \left(\vec{a}_{D/B}\right)_n$$

- We *assume* that $\alpha_{BD} > 0, and\ \vec{a}_D$ is along left.

x components:

$$-\boldsymbol{a_D} = -3332.5\ \cos 40° - 776.56\ \ \cos\ 13.92° + 0.203\boldsymbol{\alpha_{BD}} \sin 13.92°$$

y components:

$$0 = -3332.5 \sin 40° + 776.56 \sin 13.92° + 0.203\boldsymbol{\alpha_{BD}} \cos 13.92°$$

$\boxed{Solve\ for\ a_D \text{ and } \alpha_{BD}}$