Kinetics of Particles.

In this unit, we will have a look at the following dynamic aspects of mechanics -

- Motion of the particle through fundamental equation of motion, $\vec{F} = m\vec{a}$
- Method of work and energy: correlating force, mass, velocity, and displacement
- Method of impulse and momentum: correlating force, mass, velocity, and time.

Newton's Second Law of Motion.

- If the resultant force acting of a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of the resultant.

$$\therefore \vec{F_R} \propto \vec{a} \Rightarrow \vec{F_R} = m\vec{a}$$

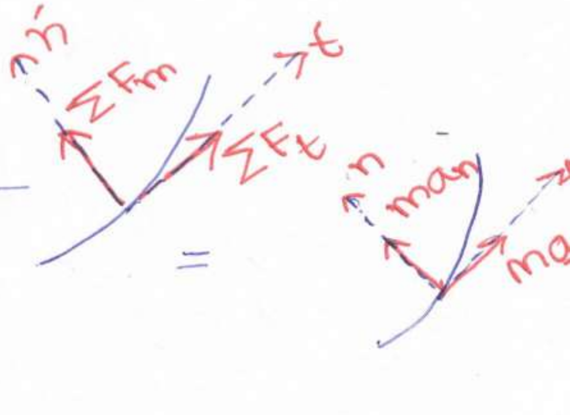
- Acceleration must be evaluated w.r.t the inertial frame of reference

which is not accelerating or rotating.

- Solution for the particle motion is facilitated by resolving the vector equation into scalar components

$$\Rightarrow (F_R)_x\hat{i} + (F_R)_y\hat{j} + (F_R)_z\hat{k} = m(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}).$$

- $\sum F_t = ma_t = m\dfrac{dv}{dt}$
- $\sum F_n = ma_n = m\dfrac{v^2}{\rho}$



Dynamic Equilibrium.

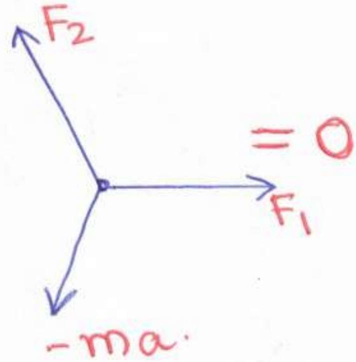
- Alternate expression of Newton's Second Law –

$$\sum \vec{F} - m\vec{a} = 0$$

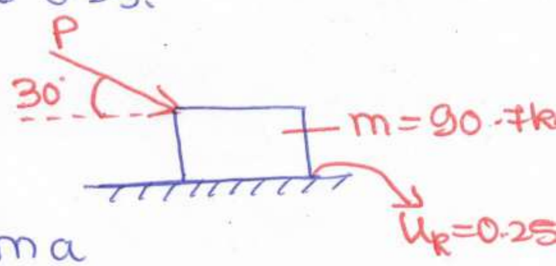
where, $-m\vec{a} \equiv$ Inertial vector

- With the inclusion of "Inertial vector", the system of forces acting on the particle is equivalent to "Zero". Thus, the particle is said to be in "Dynamic Equilibrium".

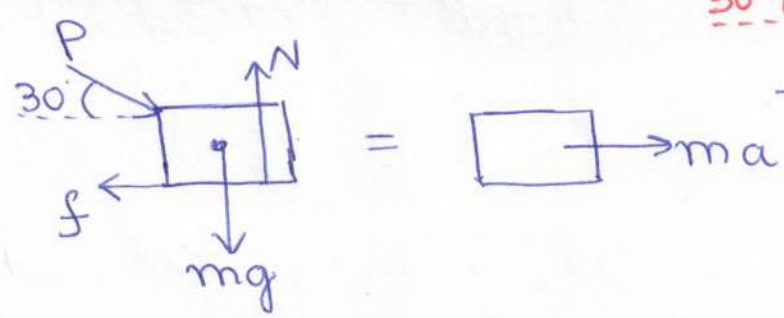
- Inertial vector is often called "Inertial force", as they measure the resistance offered by the particle to the motion, i.e., change in speed or direction.



Eg: A 90.7kg block rests on a horizontal plane. Find the magnitude of the force 'P' required to give the block an acceleration of $3\,m/s^2$ to the right. The co-efficient of kinetic friction between the block and plane is 0.25.



Solution



$\sum F_y = 0$

$\Rightarrow N = mg + P\sin 30°$

$\Rightarrow N = 889.8 + \frac{P}{2}$

$\sum F_x = ma$

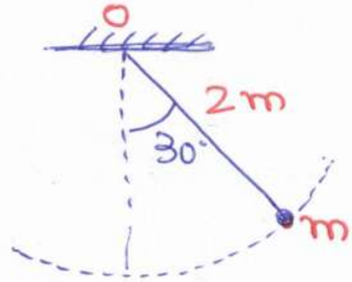
$\Rightarrow P\cos 30° - \mu N = ma$

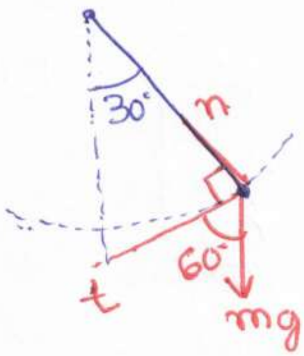
$\Rightarrow \frac{P\sqrt{3}}{2} - 0.25(889.8 + \frac{P}{2}) = 90.7(3$

$\Rightarrow \underline{P = 667.3N}$

Eg: The bob of a 2-m pendulum describe an arc of a circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob, for the position shown, find the velocity and acceleration

Solution





- $\sum F_t = m a_t$

$\Rightarrow mg\cos 60^\circ = m a_t$

$\Rightarrow a_t = \frac{g}{2} = 4.905 \, m/s^2$

- $a = \sqrt{a_n^2 + a_t^2} = \underline{\underline{16.76 \, m/s^2}}$

$v = \sqrt{a_n \rho} = \sqrt{16.03 * 2}$

$\Rightarrow \underline{\underline{v = 5.66 \, m/s}}$

- $\sum F_n = \frac{mv^2}{\rho} = m a_n$

$\Rightarrow T - mg\sin 60^\circ = \frac{mv^2}{\rho}$

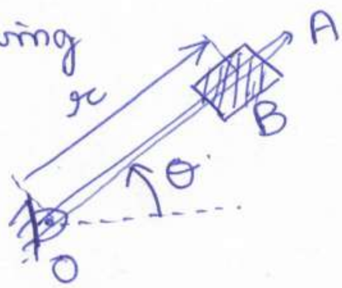
$\Rightarrow 2.5mg - mg\sin 60^\circ = \frac{mv^2}{\rho}$

$\Rightarrow a_n = g(2.5 - \sin 60^\circ)$

$\Rightarrow a_n = 9.81(2.5 - 0.866)$

$\Rightarrow a_n = 16.03 \, m/s^2$

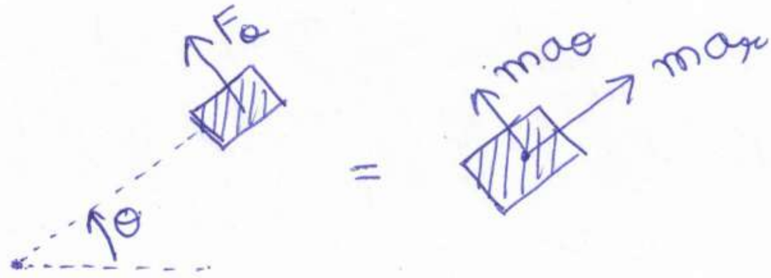
Eg: A block 'B' of mass 'm' slides freely on the frictionless arm OA, which rotates at a constant rate $\dot{\theta}_0$ in the plane. Knowing that 'B' is released at a distance r_0 from 'O', express the following as a function of r:-



(a). the radial component of the velocity B along OA.

(b) Magnitude of the force exerted on 'B' by the arm OA.

Solution



As the body of mass 'm' slides freely $F_R = 0$

$$\Rightarrow \sum F_R = ma_r \Rightarrow 0 = m(\ddot{r} - r\dot{\theta}^2) \quad — (1)$$

$$\sum F_\theta = ma_\theta \Rightarrow F = m(2\dot{r}\dot{\theta} + r\ddot{\theta}). \quad — (2)$$

$$\therefore \ddot{r} = r\dot{\theta}_0^2 \quad (3)$$

We know that

$$v_r = \dot{r}$$

$$\Rightarrow \dot{v}_r = \ddot{r} \quad — (4)$$

Comparing (3) and (4), we have —

(a)

$$\dot{v}_r = r\dot{\theta}_0^2$$

$$\Rightarrow \frac{dv_r}{dt} = r\dot{\theta}_0^2$$

$$\Rightarrow \frac{dv_r}{dr}\frac{dr}{dt} = r\dot{\theta}_0^2$$

$$\Rightarrow v_r \frac{dv_r}{dr} = r\dot{\theta}_0^2$$

$$\Rightarrow \int_0^{v_r} v_r \, dv_r = \dot{\theta}_0^2 \int_{r_0}^{r} r\,dr$$

$$\Rightarrow \boxed{v_r^2 = \dot{\theta}_0^2 (r^2 - r_0^2)}$$

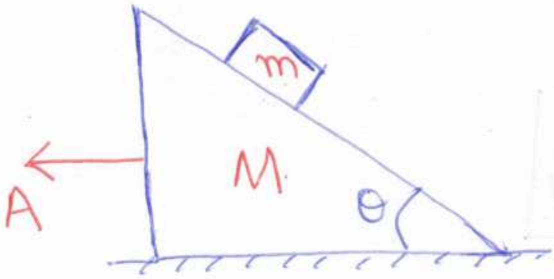
(b)

$$F = m(\cancelto{0}{r\ddot{\theta}} + 2\dot{r}\dot{\theta})$$

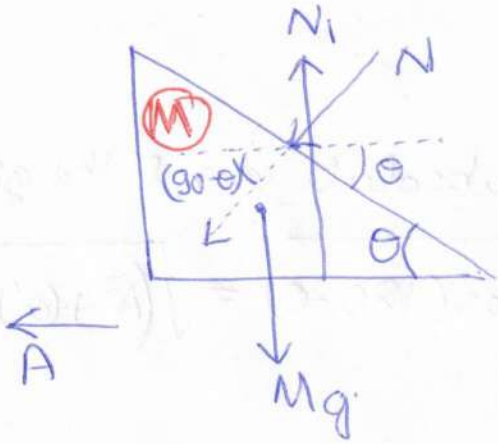
Now, $\dot{r} = v_r = \dot{\theta}_0 \sqrt{r^2 - r_0^2}$

$$\therefore \boxed{F = 2m\dot{\theta}_0^2 \sqrt{r^2 - r_0^2}}$$

Mass on a Moving Wedge



Find the relative acceleration of mass 'm' w.r.t the wedge of mass 'M'.



- $\sum F_y = 0$

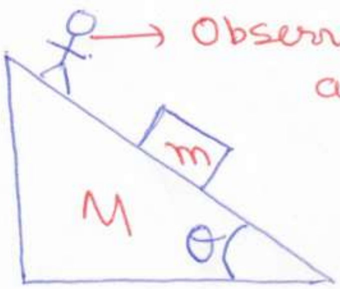
$\Rightarrow N_1 = N\sin(90-\theta) + Mg$

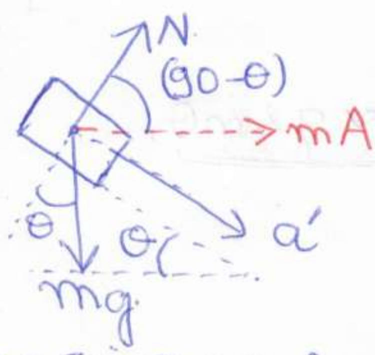
$\Rightarrow N_1 = N\cos\theta + Mg$ ---- ①

- $\sum F_x = MA$

$\Rightarrow N\cos(90-\theta) = MA$

$\Rightarrow N\sin\theta = MA$ ------- ②





- $mg\sin\theta + mA\cos\theta = ma'$ ---- ③
- $N + mA\sin\theta = mg\cos\theta$ --- ④

Unknowns:

$N_1, N, A, a' \equiv$ 4

No. of eqns: 4

∴ The unknowns can be evaluated.

Eq. ② in Eq. ④ ⇒

$$\frac{MA}{\sin\theta} + mA\sin\theta = mg\cos\theta$$

$$\Rightarrow (M + m\sin^2\theta)A = mg\cos\theta\sin\theta$$

$$\Rightarrow \boxed{A = \left(\frac{mg\cos\theta\sin\theta}{M + m\sin^2\theta}\right)}.$$

$$\therefore N = \left(\frac{Mmg\cos\theta}{M + m\sin^2\theta}\right)$$

Substituting 'A' in eqn ⑤ ⇒

$$g\sin\theta + \frac{mg\sin\theta\cos^2\theta}{M+m\sin^2\theta} = a'$$

$$\Rightarrow \boxed{a' = g\sin\theta\left(\frac{M+m}{M+m\sin^2\theta}\right)}$$

- Acceleration of mass 'm' (absolute w.r.t the ground)

$$a_M = \sqrt{A^2+(a')^2 - 2Aa'\cos(180-\theta)} = \sqrt{\left(A^2+(a')^2+2Aa'\cos\theta\right.}$$

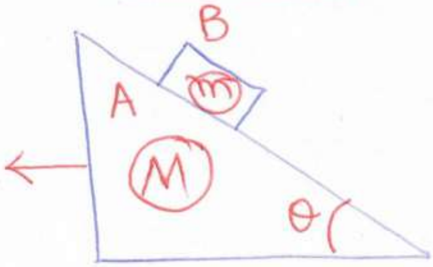
<u>Limiting case:-</u>

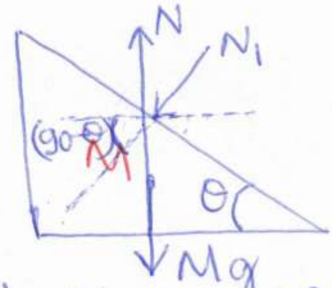
If $M \gg m$.

- $A \approx 0$
- $a' = (g\sin\theta)\left[\frac{1+\cancel{\frac{m}{M}}^{\to 0\,\sim 0}}{1+\cancel{\frac{m}{M}}^{\sim 0}\sin^2\theta}\right]$

$\Rightarrow$ <u>$a' \approx g\sin\theta$</u>.

Alternate Method:

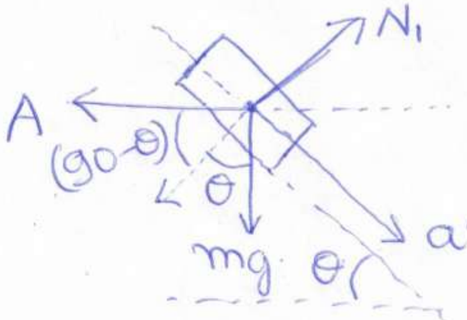




$$\vec{a}_B = \vec{a}_{B/A} + \vec{a}_A$$

- $N_1 \sin\theta = MA \quad (\Sigma F_x = MA)$

$N = N_1 \cos\theta + Mg$



- $\Sigma F_x = ma_x$

$\Rightarrow \cancel{mg\sin\theta - mA\cos\theta = ma'}$

$\Rightarrow \cancel{a' = g\sin\theta - A\cos\theta}$

$\Rightarrow mg\sin\theta = m(a' - A\cos\theta)$

$\Rightarrow a' = g\sin\theta + A\cos\theta$

- $\Sigma F_y = ma_y$

$\Rightarrow N_1 - mg\cos\theta \overset{(-)}{=} mA\sin\theta$

$\Rightarrow \dfrac{MA}{\sin\theta} - mg\cos\theta = -mA\sin\theta$

$\Rightarrow \dfrac{MA}{\sin\theta} + mA\sin\theta = mg\cos\theta$

$\Rightarrow A = \left(\dfrac{mg\sin\theta\cos\theta}{(M + m\sin^2\theta)}\right)$

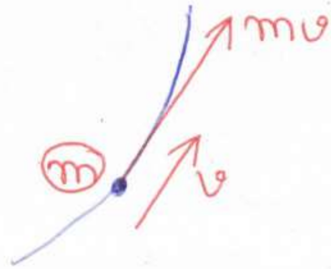
$a' = g\sin\theta + \dfrac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}$

$$\Rightarrow \boxed{a' = g\sin\theta\left[\frac{M+m}{M+m\sin^2\theta}\right]}$$

Linear Momentum of Particle

$\sum \vec{F} = m\vec{a}$

$\Rightarrow \sum \vec{F} = m\dfrac{d\vec{v}}{dt}$



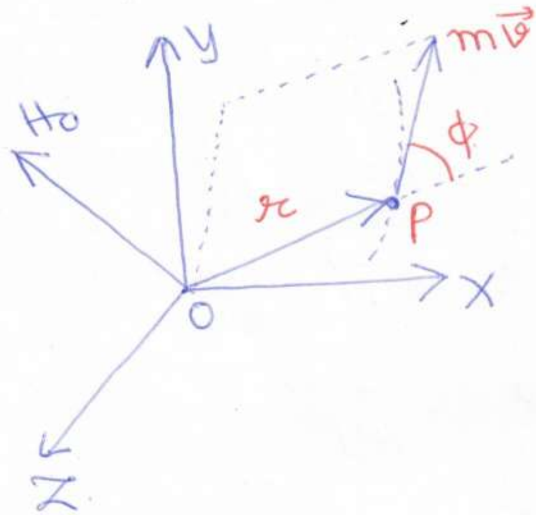
Assuming 'm' to be constant,

$\sum \vec{F} = \dfrac{d}{dt}(m\vec{v})$.

$\Rightarrow \boxed{\sum \vec{F} = \dfrac{d\vec{L}}{dt}}$, where $\vec{L} = m\vec{v}$ = linear momentum

$\therefore$ Resultant of the forces acting on a particle is equal to the rate of change of linear momentum of the particle.

Angular Momentum of the Particle



$\vec{H}_o = \vec{r} \times m\vec{v}$ = moment of momentum or angular momentum of the particle about O

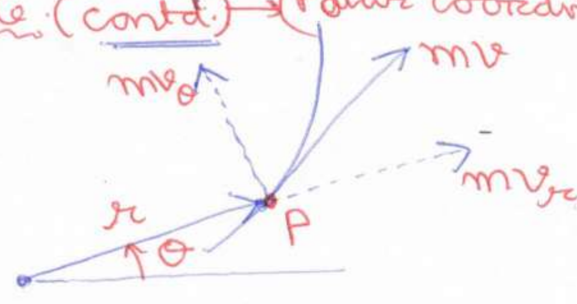
- H_o is perpendicular to the plane containing $\vec{r}$ and $m\vec{v}$
- $H_o = r m v \sin\phi$.

$$H_o = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

Angular Momentum of the Particle (contd.) → (Polar coordinates)

$H_o = m v_\theta r$

$= r m (r\dot{\theta}) = m r^2 \dot{\theta}$



Taking the time derivative of

$\vec{H}_o = \vec{r} \times m\vec{v}$

$\Rightarrow \dot{\vec{H}}_o = \dot{\vec{r}} \times m\vec{v} + \vec{r} \times m\dot{\vec{v}}$

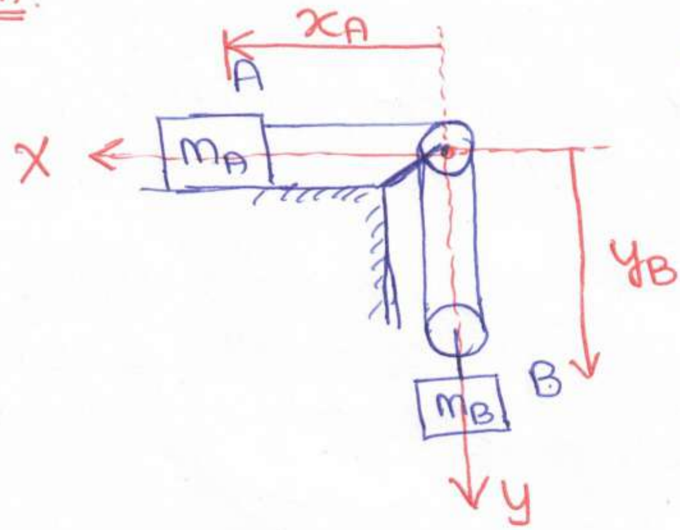
$\Rightarrow \dot{\vec{H}}_o = \vec{v} \times m\vec{v} + \vec{r} \times m\vec{a}$

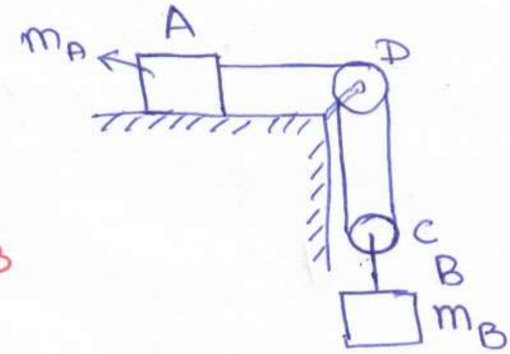
$\Rightarrow \dot{\vec{H}}_o = 0 + \vec{r} \times \sum \vec{F}$

$\Rightarrow \boxed{\dot{\vec{H}}_o = \sum \vec{M}_o}$

Eg: The two blocks shown start from rest. The horizontal plane and the pulley are frictionless. The masses of the pulleys are assumed to be negligible. Determine the acceleration of each block and the tension in the cord.

Solution.





$m_A = 100kg$

$m_B = 300kg$

$x_A + 2y_B +$ constants $=$ constant length of the cord

$\Rightarrow v_A + 2v_B = 0$

$\Rightarrow a_A = -2a_B$

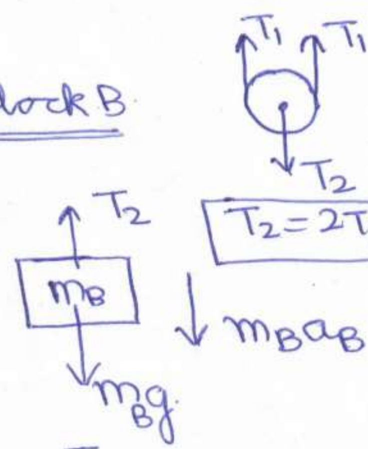
For block A:

$-T_1 = m_A a_A$

$\Rightarrow -T = -m_A(-2a_B)$

$\Rightarrow T_1 = 2m_A a_B = 200 a_B$

For block B



$\boxed{T_2 = 2T_1}$

$m_B g - T_2 = m_B a_B$

$\Rightarrow (300)(9.81) - 2T_1 = 300$

$\Rightarrow (300)(9.81) - 400a_B = 300a_B$

$\Rightarrow \underline{a_B = 4.2 \, m/s^2}$

So, $a_A = -2a_B$

$= -8.4 \, m/s^2$

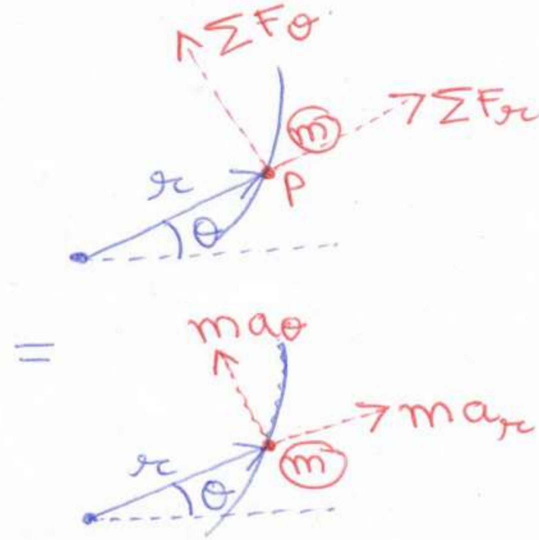
$= 8.4 \, m/s^2 \; (\rightarrow)$

$T_1 = 200(4.2) = \underline{840N}$

$T_2 = 2T_1 = \underline{1680N}$

- It follows from the Newton's second law that the sum of the moments at 'O' of the forces acting on the particle is equal to the rate of change of angular momentum of the particle about 'O'.

$$\sum \vec{M}_0 = \dot{\vec{H}}_0$$

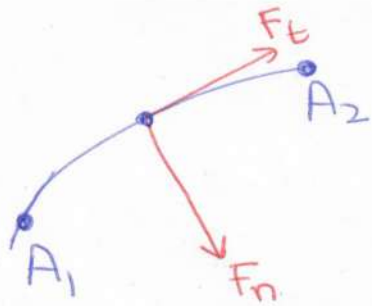


$$\Rightarrow r\left(\sum \vec{F}_\theta\right) = \frac{d}{dt}\left(m r^2 \dot{\theta}\right)$$

$$\Rightarrow r\left(\sum \vec{F}_\theta\right) = m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right)$$

$$\Rightarrow \boxed{\sum \vec{F}_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)}$$

Principle of Work and Energy



$$\vec{F}_t = m\vec{a}_t = m\frac{d\vec{v}}{dt} = \frac{m\,d\vec{v}}{ds}\cdot\frac{ds}{dt}$$

$$\Rightarrow \vec{F}_t = mv\frac{dv}{ds}$$

$$\Rightarrow \int_{S_1}^{S_2} F_t\,ds = \int_{v_1}^{v_2} m v\,dv$$

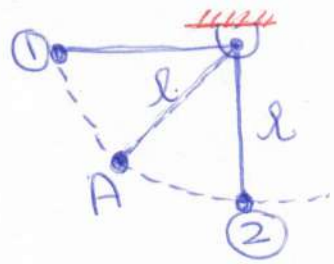
$$\Rightarrow F_t(S_2 - S_1) = \frac{1}{2}m\left(v_2^2 - v_1^2\right)$$

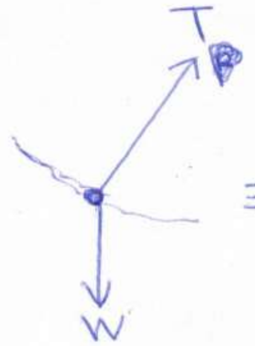
$$\Rightarrow U_{1\to2} = T_2 - T_1$$

$$\Rightarrow \boxed{dU_{1\to2} = dT}$$

$\Rightarrow$ Work done by the force is equal to the change in the kinetic energy of the particle.

Eg.





≡ Force T does no work as it is normal to the path.

$\therefore U_{1-2} = T_2 - T_1$

$\Rightarrow Wl = \frac{1}{2}\frac{W}{g}v_2^2 - 0$

$\Rightarrow \boxed{v_2 = \sqrt{2gl}}$

Tension in the cord at ②:

$T - W = \frac{W}{g}a_n$

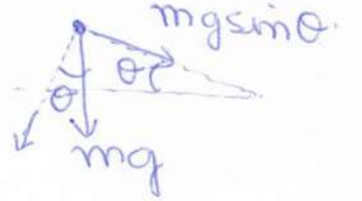
$\Rightarrow T - W = \frac{W}{g}\cdot\frac{v_2^2}{l}$

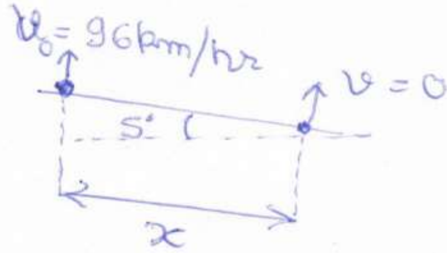
$\Rightarrow T - W = \frac{W}{g}\cdot\frac{2gl}{l}$

$\Rightarrow \boxed{T = 3W}$

Eg. An automobile weighing 17793N is driven down a 5° inclined surface at a speed of 96 km/hr. When the brakes are applied, a braking force of 6672N is produced. Determine the distance traveled by the automobile as it comes to stop.

Sol.





$U_{1-2} = T_2 - T_1$

$\Rightarrow (mg\sin\theta - F_R)x = 0 - \frac{1}{2}mv_0^2$

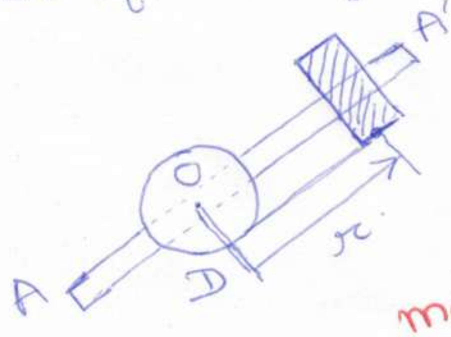
$\Rightarrow x = \dfrac{\frac{1}{2}\frac{W}{g}v_0^2}{(F_R - W\sin\theta)} = \dfrac{1\times(17793)(26.67)^2}{2(9.81)(6672 - 17793\sin 5°)}$

≈ 126 m

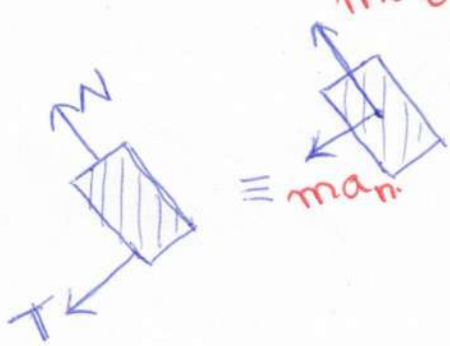
Eg: A 3-kg collar B rests on the frictionless arm AA'. The collar is held in place by the rope attached to the drum 'D' and rotates about 'O' in a horizontal plane. The linear velocity of collar 'B' increases according to the equation –

$v = 0.2t^2$ (v is in m/s and t in seconds)

Find the tension in the rope and the force of the bar on the collar after 5 s if $r = 0.4$ m.



Solution



- $T = ma_n = \dfrac{mv^2}{r}$

$$\Rightarrow T = \frac{(3)(0.2 \times 5^2)^2}{0.4} = 187.5\,N$$

- $N = ma_t = m\dfrac{dv}{dt} = m(2)(0.2)t$

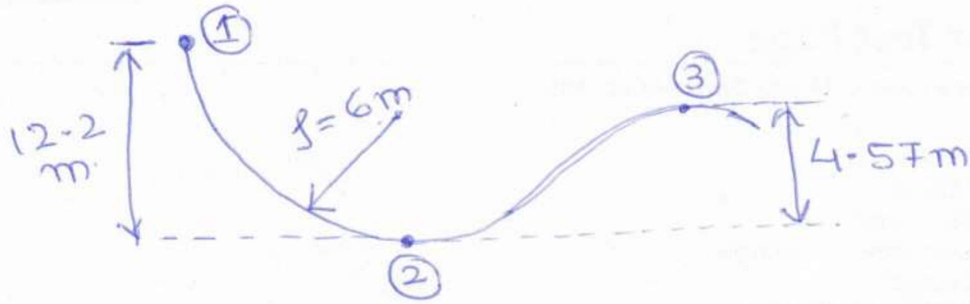
$$\Rightarrow N = (3)(2)(0.2)(5)$$

$$\Rightarrow N = 6\,N$$

Eg. A car of 907 kg starts from rest at point 1 and moves down the track without friction. Determine :-

(a) the force exerted by the track on the car at point 2.

(b) the minimum safe value of radius of curvature at point



Solⁿ. (a) Invoking work-energy principle b/w ① and ②

$V_1 = 0$

$V_2 = ?$

$$T_1 + U_{1-2} = T_2$$

$$\Rightarrow W(12.2) = \frac{1}{2}\frac{W}{g}V_2^2$$

$$\Rightarrow V_2 = \sqrt{2(9.81)(12.2)} = 15.47 \text{ m/s}$$

(+↑)

$N \uparrow\ \ \uparrow ma_n$

$\downarrow W$

$$\Rightarrow N - W = ma_n$$

$$\Rightarrow N = W + \frac{mV_2^2}{\rho} = m\left(g + \frac{V_2^2}{\rho}\right)$$

$$\Rightarrow N = 907\left(9.81 + \frac{15.47^2}{6}\right)$$

$$\Rightarrow \boxed{N = 45075 \text{ N}}$$

(b) Invoking work-energy principle b/w ① and ③ –

$$T_1 + U_{1-3} = T_3$$

$$\Rightarrow 0 + W(12.2 - 4.57) = \frac{1}{2}\frac{W}{g}V_3^2$$

$$\Rightarrow V_3 = 12.2 \text{ m/s}$$

For minimum radius of curvature,

$$W = \frac{mV_3^2}{\rho}$$

$$\Rightarrow \cancel{m}g = \frac{\cancel{m}V_3^2}{\rho}$$

$$\Rightarrow \rho = \frac{(12.2)^2}{9.81} = 15.26 \text{ m}$$

Principle of Impulse and Momentum

- From Newton's Second Law –

$$\vec{F} = \frac{d}{dt}(m\vec{v}).$$

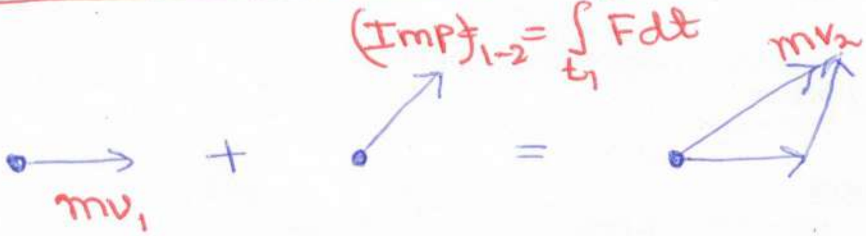
$$\Rightarrow \vec{F}\,dt = d(m\vec{v}).$$

$$\Rightarrow \int_{t_1}^{t_2} \vec{F}\,dt = m(\vec{v_2}) - m\vec{v_1}$$

- Force acting on a particle during a short duration of time that is large enough to cause a significant change in momentum is called an "Impulse Force"

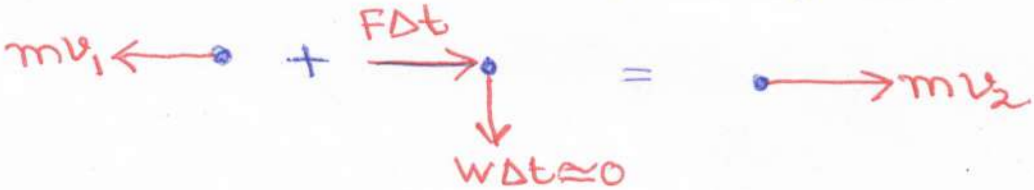
$$\therefore \int_{t_1}^{t_2} \vec{F}\,dt = (Imp)_{1-2} = \text{Impulse of force } \vec{F}$$

$$\therefore \boxed{m\vec{v_2} = m\vec{v_1} + (Imp)_{1-2}}$$



\# When a baseball is struck by a bat, contact occurs over a short time duration, but force is large enough to change the sense of motion of the ball.

\# Non-impulsive forces are forces for which $\vec{F}\Delta t$ is small and therefore, may be neglected.



- In case of the impulsive motion of several particles

$$\Sigma m\vec{v_1} + \Sigma \underbrace{\cancel{\vec{F}_{avg}}}_{0}\Delta t = \Sigma m\vec{v_2}$$

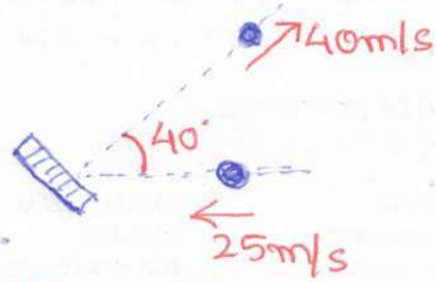
0 (for non-impulsive force).

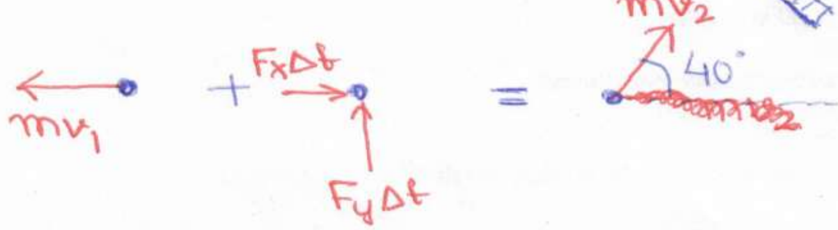
$$\Rightarrow \boxed{\Sigma m\vec{v_1} = \Sigma m\vec{v_2}} \rightarrow \text{Conservation of momentum}$$

$$\Rightarrow m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

Eg: A 0.1 kg baseball is pitched with a velocity of 25 m/s. After the ball is hit by a bat, it has a velocity of 40 m/s in the direction shown. If the bat and ball are in contact for 0.015 s, determine the average impulse force excerted on the ball during the impact.

Soln.





$$m\vec{v}_1 + (\text{Imp})_{1\to 2} = m\vec{v}_2$$

X - component of the equation

$$-mv_1 + F_x\Delta t = mv_2\cos 40^\circ$$

$$\Rightarrow -(0.1)(25) + F_x(0.015) = (0.1)(40)\cos 40^\circ$$

$$\Rightarrow F_x = 370.94\,N$$

Y - component of the equation

$$0 + F_y\Delta t = mv_2\sin 40^\circ$$

$$\Rightarrow F_y = \frac{(0.1)(40)\sin 40^\circ}{0.015}$$

$$\Rightarrow F_y = 171.41\,N$$

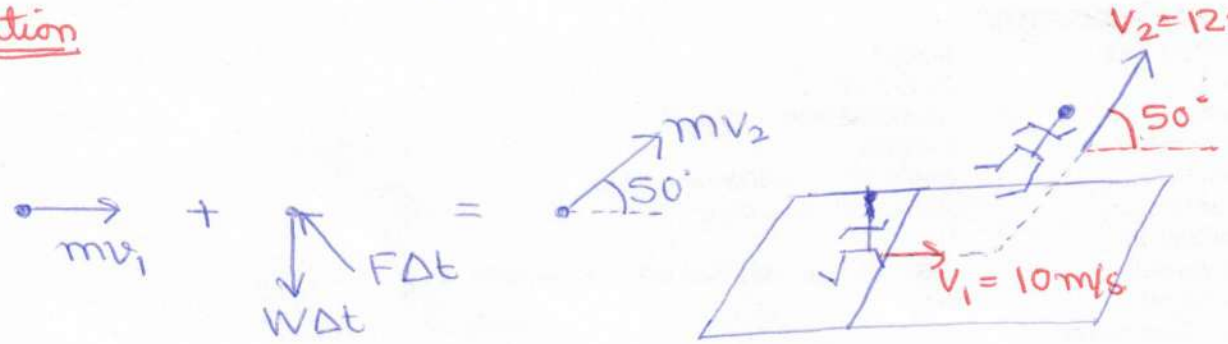
$$\therefore \text{Impulse force} = \vec{F} = \sqrt{(F_x)^2 + (F_y)^2}$$

$$\Rightarrow \boxed{F = 408.6\,N}$$

$$\boxed{\vec{F} = 370.94\hat{i} + 171.41\hat{j}}$$

Eg: The jumper approaches the take off line from the left with a horizontal velocity of 10 m/s, remains in contact with the ground for 0.18 s, and takes off at a 50° angle with a 12 m/s. Determine the avg. impulsive force exerted by the ground on his foot. Express the answer in terms of weight of the athlete.

Solution



- X-component of Impulse-Momentum Eqn. –

$$mv_1 + (-F_x \Delta t) = mv_2 \cos 50°$$

$$\Rightarrow \frac{W}{g}(10) - F_x(0.18) = \frac{W}{g}(12)\cos 50°$$

$$\Rightarrow F_x = (1.29\,W)\ N$$

- Y-component of Impulse-Momentum eqn –

$$0 + (F_y - W)\Delta t = \frac{W}{g} V_2 \sin 50°$$

$$\Rightarrow F_y = 6.21\,W$$

$$\therefore \boxed{F = -1.29\,W\hat{i} + 6.21\,W\hat{j}}$$