Lecture- 8

Module : Electrodynamics

Textbooks : 1) Introduction to Electrodynamics by D J Griffiths.

2) A student's guide to Maxwell's Equations

Ref. materials : Lectures by David Tong Cambridge University.

SECTION A,B — 08/09/25

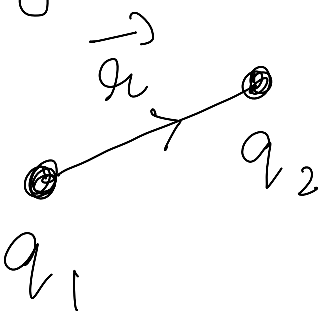
SECTION C — 10/09/25

SECTION D — 25/09/25

We already had an idea about conservative forces in nature.

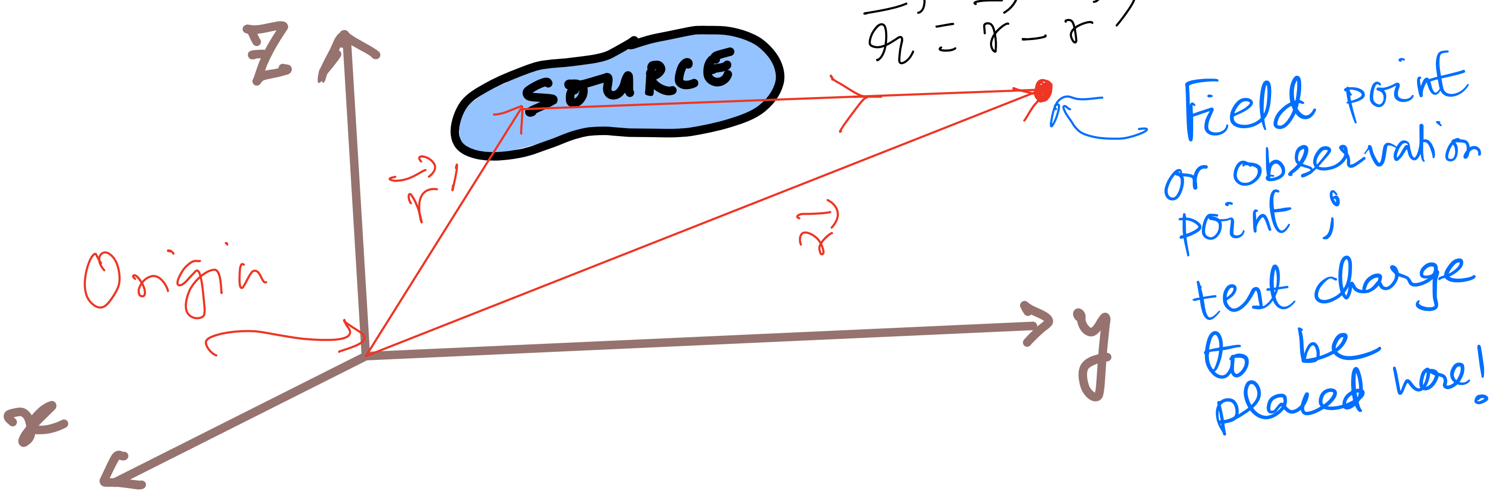
A prime example of such a force is the Coulomb's law of force between two electric charges.

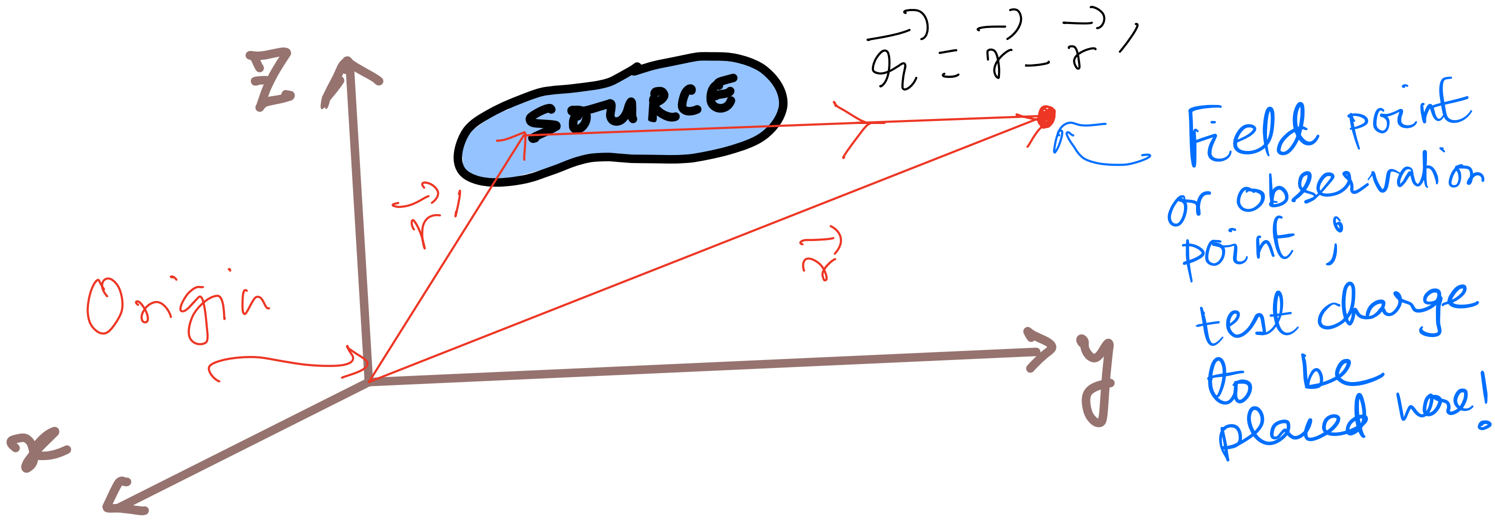
$$\vec{F} \propto \frac{1}{r^2}\hat{r}$$



We know what its properties would be mathematically.

This is one of the fundamental forces of nature.





Convention: $\vec{r}\,' \equiv$ distance of source from origin

$\vec{r} \equiv$ distance of observation point from origin.

$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' \equiv$ distance between source and observation point.

Kind of questions we will be asking:

Given sources (e.g. charge density, current density)

what are the electric field, $\vec{E}(\vec{r})$, and magnetic field $\vec{B}(\vec{r})$?

Once we know $\vec{E}(\vec{r})$, $\vec{B}(\vec{r})$, we ask, what is the force on a test charge q_2?

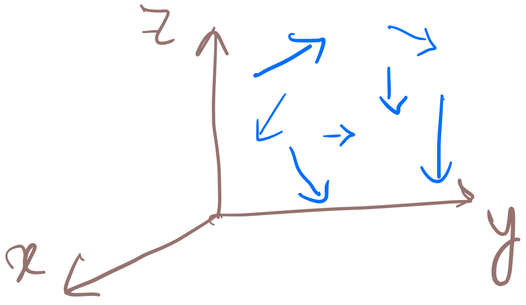
Lorentz Force law: $\vec{F} = q_2(\vec{E} + \vec{v} \times \vec{B})$

Now, $\vec{E}$, $\vec{B}$ are functions of position vector $\vec{r}$ so they are known as vector fields.

Scalar fields: $T(x, y, z)$ (e.g. Temperature)

Vector fields:

$$\vec{A}(x,y,z,t) = A_x(x,y,z,t)\hat{x} + A_y(x,y,z,t)\hat{y} + A_z(x,y,z,t)\hat{z}$$



Note the different size of the arrows.

Differential form

Maxwell's Equations :

Unification of electricity and magnetism into a single consistent formalism :

$\oint \vec{E} \cdot d\vec{a} = Q_{enc}/\epsilon_0$

$$\rightarrow \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

Sources

True for charges and currents in non-polarisable and non-magnetisable medium

- ρ : charge density $\Rightarrow$ $\rho(\vec{r}')$ or $\rho(x', y', z')$ (Refer to Fig. 1)
- $\vec{E}$, $\vec{B}$ are vector fields i.e. they have spatial or position dependence. They are different at different points in space.

  $\vec{E}(\vec{r})$ or $\vec{E}(x, y, z)$
- $\vec{J}$: electric current density $\Rightarrow$
- ϵ_0 : electric permittivity of vacuum / free space
- μ_0 : magnetic permeability of vacuum

$$\epsilon_0 = 8.8542 \times 10^{-12} \; C^2 N^{-1} m^{-2}$$

$$\mu_0 = 4\pi \times 10^{-7} \; NA^{-2}$$

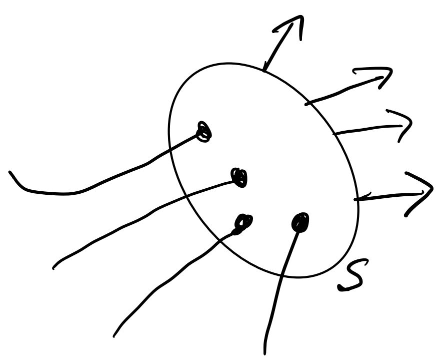
One of the key goals of this course is to go beyond the dynamics of point particles and onto the dynamics of continuous objects known as fields.

- The total charge Q in a given region V is simply $Q = \int_V dV \, \rho(r,t)$

  $\hookrightarrow$ Volume element.

- Current density is related to current as,

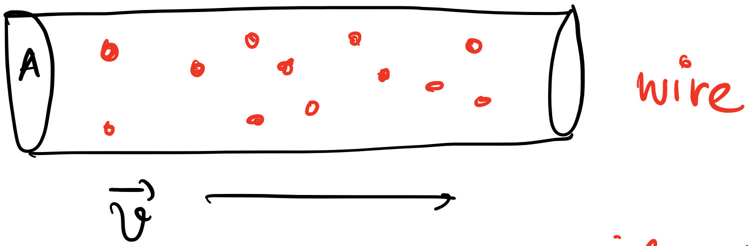
$$I = \int_S \vec{J} \cdot d\vec{S}$$



$d\vec{S}$ is unit normal to S.

J is current per unit area

- In other words for a continuous charge distribution if the velocity of the charge in a small volume is $\vec{v}(\vec{r},t)$, then $\vec{J} = \rho\vec{v}$



Example: 'n' electrons per unit volume, each with charge 'q' then the charge density is

$\rho = nq$ and $\vec{J} = nq\vec{v}$; $I = |\vec{J}|A$

Q. How do we express point charge in terms of a charge density?

Answer will be provided later.
You are also encouraged to find the answer.

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

Focus on Electrostatics:
$\Downarrow$
No time variance

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \longrightarrow (1)$$

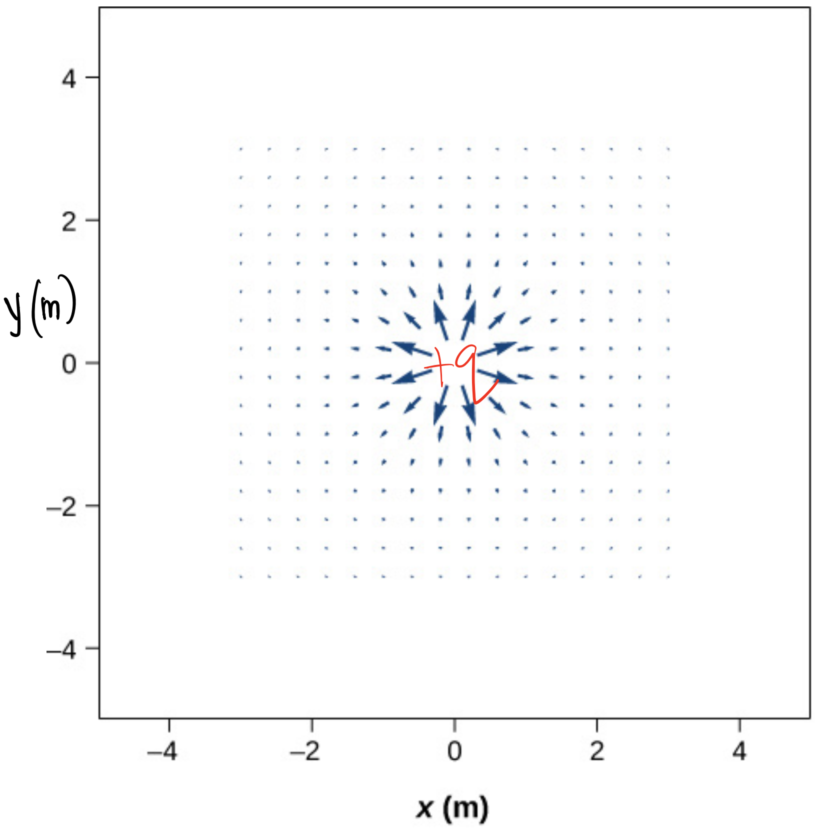
$$\vec{\nabla}\times\vec{E} = 0 \longrightarrow (2)$$

If we fix the charge density 'ρ', then EQ (1), (2) have a unique solution.

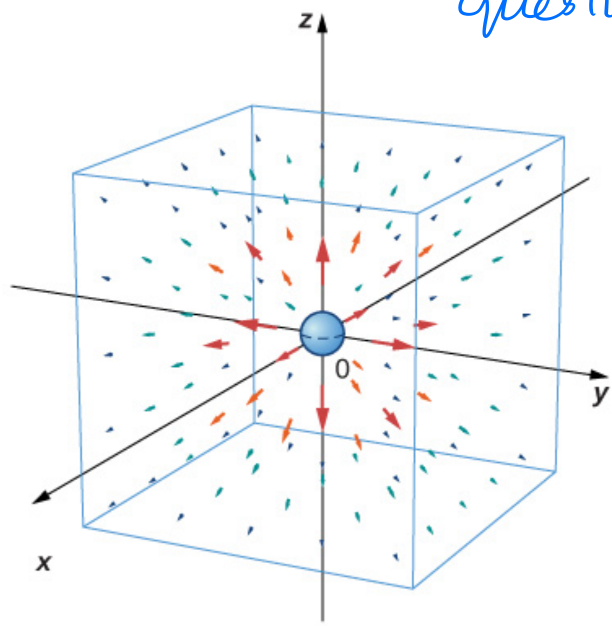
+ Information of how $\vec{E}$, $\vec{B}$ behave at infinity

Let's have a quick look at how we geometrically interpret vector fields :

① Divergence : $\vec{\nabla} \cdot \vec{v}$
how much a vector $\vec{v}$ spreads out from the point in question.



(a)



(b)

Magnitude of the vector field is proportional to the size of arrow.

Electric fields due to a point charge (a) in two dimensions (b) in three dimensions

⇒ These vector fields have a positive divergence (source)

⇒ If they pointed inwards they would have a negative divergence (sink)

Curl of a vector field:

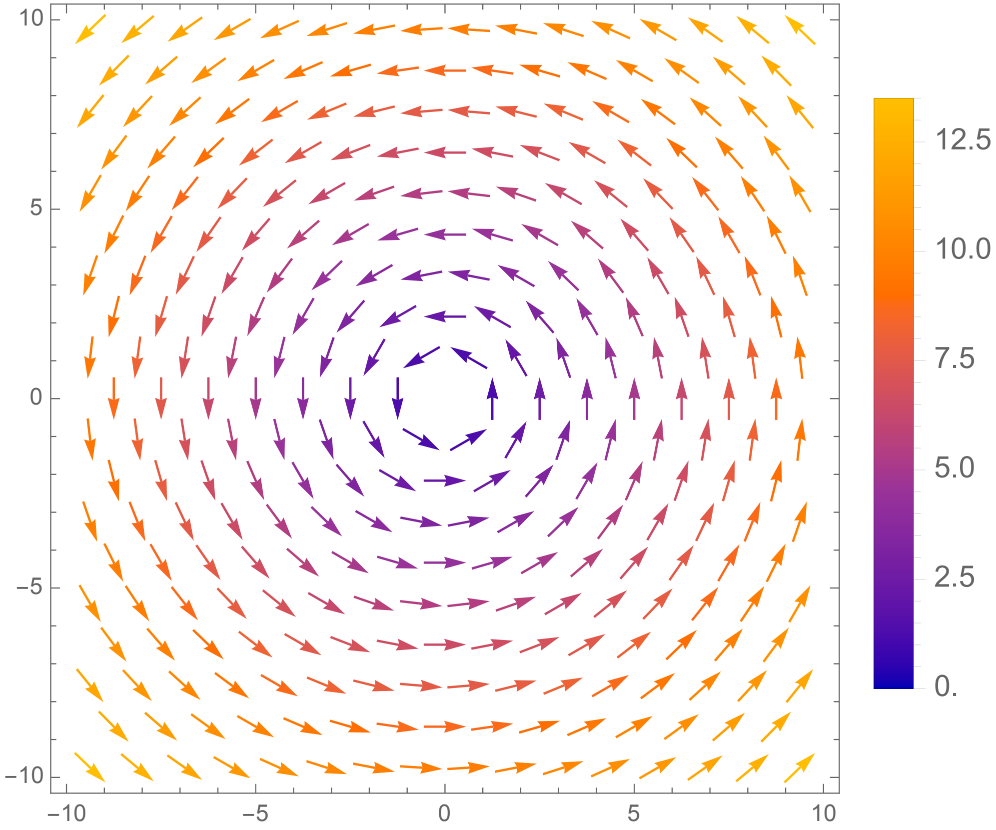
$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right).$$

- Think of the vector field as being the velocity field of a moving fluid or gas
- The curl is related to rotational motion induced by the fluid
- Consider an infinitesimally small ball (sphere) located within the fluid, and centered at a point. Assume that the ball has a rough surface. The fluid moving around the ball will generally make it rotate.

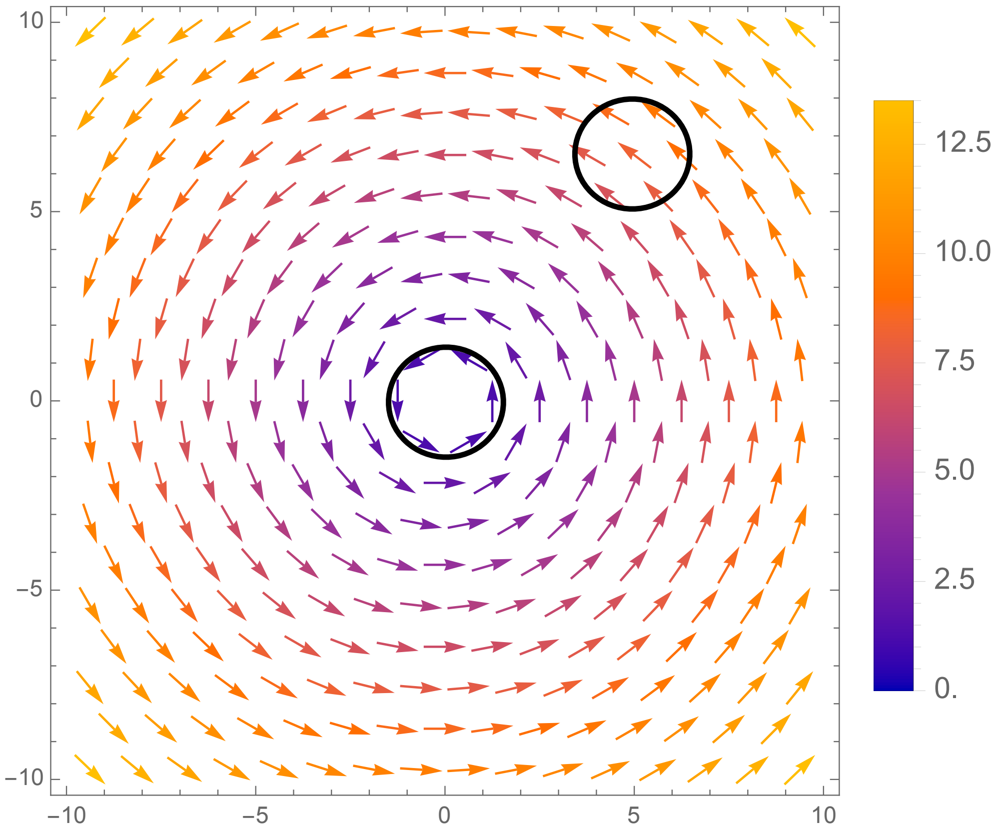
- Direction of curl at the point => the axis of rotation (right hand rule)
- Magnitude of curl at the point => half the angular speed of rotation

$$\vec{v} = (-y\,\hat{x} + x\,\hat{y} + 0\,\hat{z})$$



$$\vec{\nabla} \times \vec{v} = (0\,\hat{x} + 0\,\hat{y} + 2\,\hat{z})$$

$$\vec{v} = (-y\,\hat{x} + x\,\hat{y} + 0\,\hat{z})$$



"Incompressible vector field"

$$\vec{\nabla} \cdot \vec{v} = 0$$

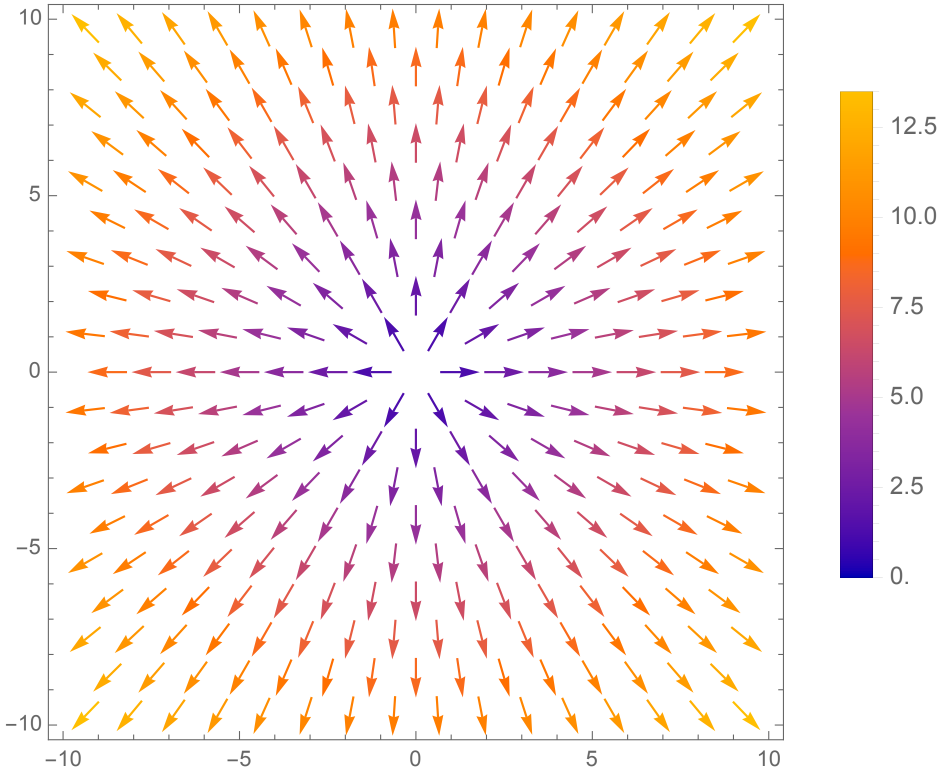
$$\Rightarrow \quad \vec{v} = \vec{\nabla} \times \vec{u}$$

$$\text{Since} \quad \vec{\nabla} \cdot (\vec{\nabla} \times \vec{u}) = 0$$

$$\vec{\nabla} \times \vec{v} = (0\,\hat{x} + 0\,\hat{y} + 2\,\hat{z})$$

"Solenoidal vector field"

$$\vec{v} = x\,\hat{x} + y\,\hat{y}$$



“Irrotational vector field”

$$\vec{\nabla} \cdot \vec{v} = 2$$

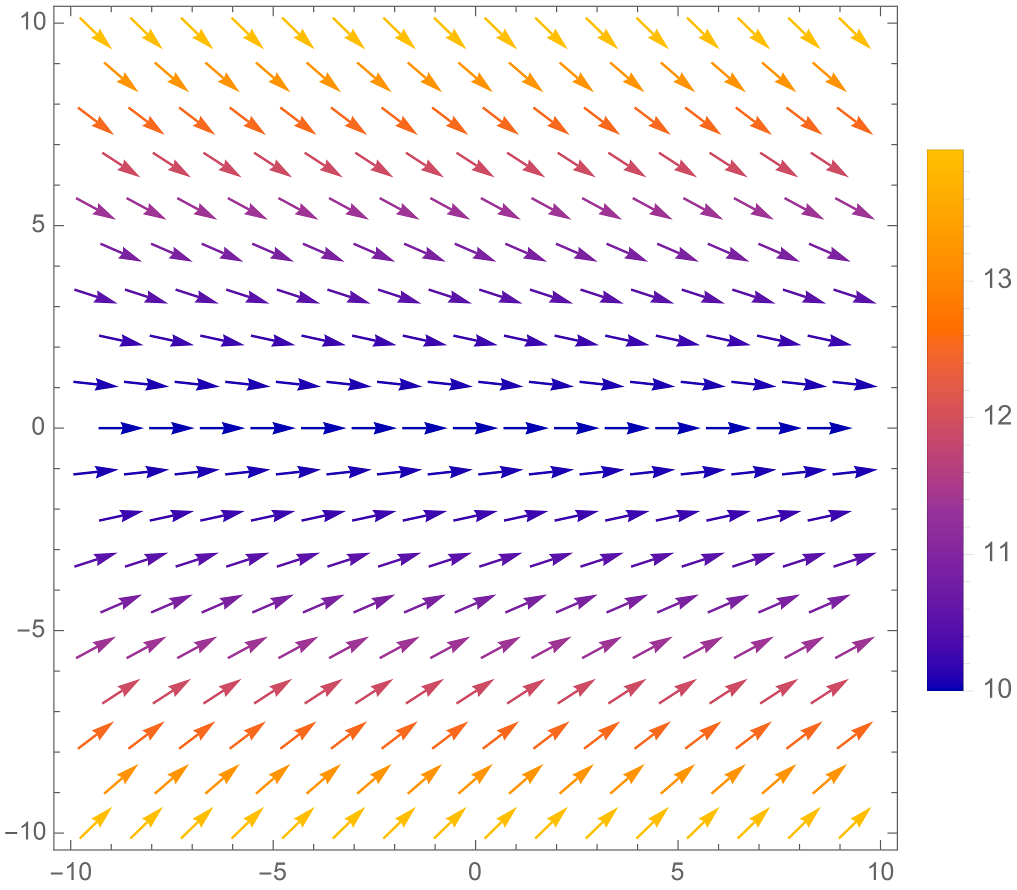
$$\vec{\nabla} \times \vec{v} = 0$$

$$\Rightarrow \vec{v} = \vec{\nabla}\phi$$

Since $\quad \vec{\nabla} \times (\vec{\nabla}\phi) = 0$

“Conservative vector field”

$$\vec{v} = 10\,\hat{x} - y\,\hat{y}$$



$$\vec{\nabla} \times \vec{v} = 0$$

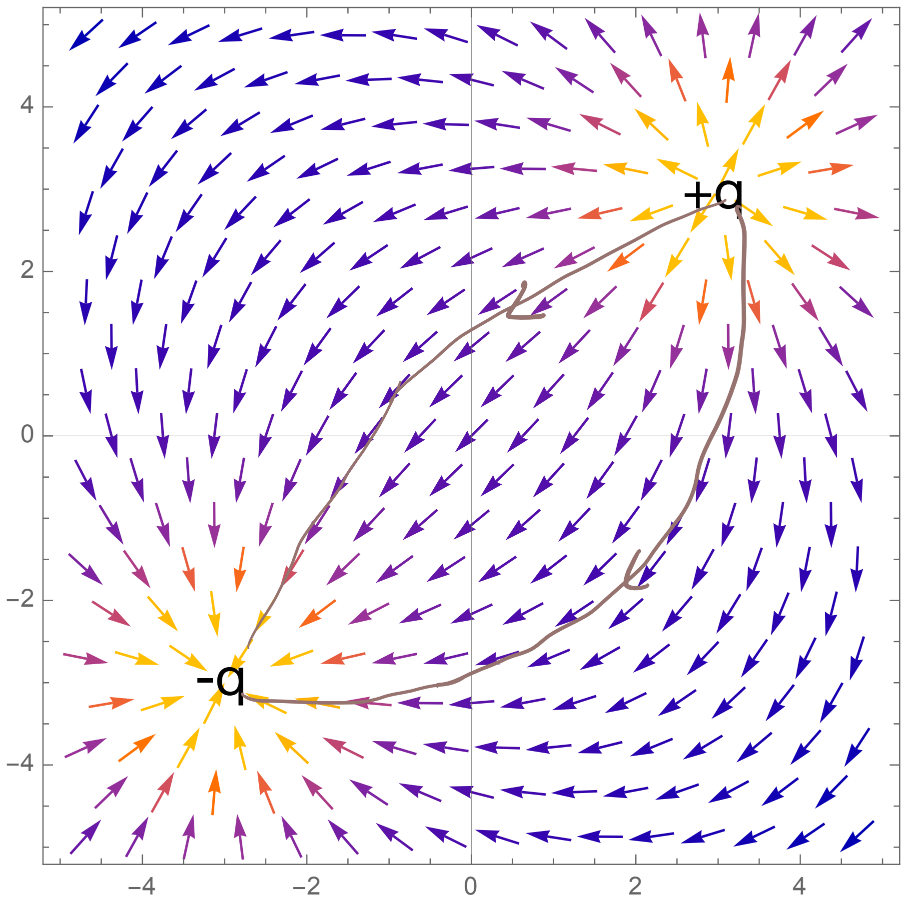
$$\vec{\nabla} \cdot \vec{v} = -1$$

HW

- Why divergence is negative at x=0, y=0?
- Why divergence is negative at x=0, y=5?
- Why curl is zero at x=0, y=0?

Question: Why would divergence and curl of vector fields (natural for velocity vector fields of fluid flows) be important for electromagnetism?

Vector field with source and sink: Electric field lines between a positive and a negative charge



If we join these vector fields we get continuous field lines.

- Vector fields (like electric and magnetic fields produced due to charges or currents) are pictorially represented by flux lines.
- We need to understand what are flux of electric or magnetic fields passing through a surface.
- For that we need to understand how to perform integrations involving vectors - volume and surface integrations

- Denser the field lines, higher the magnitude of the field at that point.