[SEC A/B : 25/08/25
SEC C/D : 28/08/25]

Lecture - 6

Summary of all lectures (until now)

Limitations of Newtonian mechanics

* $\vec{F} = m\vec{a}$ takes different forms as we switch coordinate systems and frames (non-inertial)
* Consideration of pseudo-forces
* Vector equations
* Difficult to incorporate symmetries and implement constraints.

Advantages of Lagrangian formulation

* $\frac{\partial L}{\partial \dot{q}_i} = p_i$ [Same Equation]
* $L = T - V$ for all coords.
* L is a scalar

one Equation of motion (EOM) for each generalised coordinate.

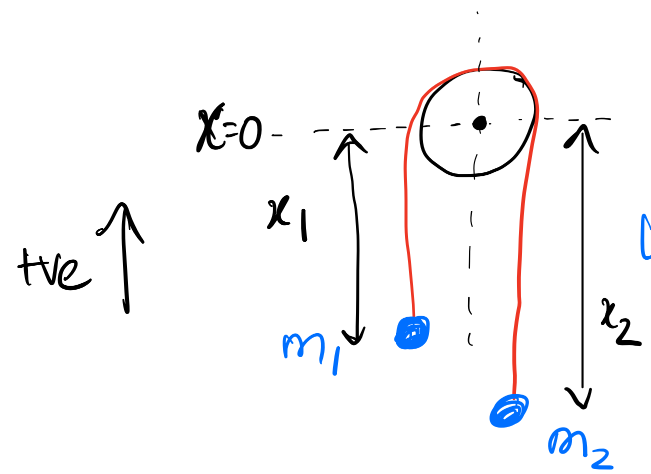
* Can be used to generalise physical laws e.g. EM theory
* Can easily correlate b/w classical and Quantum Mechanics

[Worked Problems in Lagrangian Mechanics]

Worked examples in classical Lagrangian mechanics:

Problem 1

Pulley (massless)



→ The entire length of the rope is $(l+a)$.

a: part of rope touching pulley

→ Effective 1D system,

→ Holonomic constraint: $x_1 + x_2 = l + $ (part of rope touching pulley, but constant)

→ Generalised coordinate $q = x_1$

$$L = T - V = \left(\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2\right) - \left[-m_1 g x_1 - m_2 g (l+a-x_1)\right]$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}_1^2 + g(m_1 - m_2)x_1 + \text{constant}$$

Equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1}$$

$$\Rightarrow \quad (m_1+m_2)\ddot{x}_1 = (m_1-m_2)g$$

$$\Rightarrow \quad \ddot{x}_1 = \left(\frac{m_1-m_2}{m_1+m_2}\right)g$$ $\Rightarrow$ Equation of motion (EOM)

1) Note that we did NOT need any consideration of the constraint force i.e. TENSION.

2) Also, we did not consider the rotation of the pulley and its mass. This would require us to include the rotational kinetic energy.

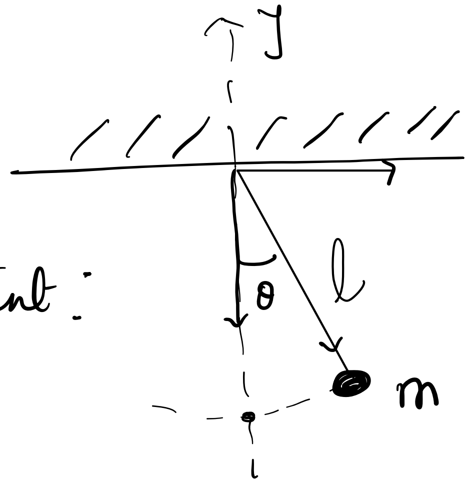
Let radius of pulley $= R$, moment of inertia $= I$.

$$T = \frac{1}{2}(m_1+m_2)\dot{x}_1^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m_1+m_2)\dot{x}_1^2 + \frac{1}{2}I\frac{\dot{x}_1^2}{R^2}$$

V remains same.

Problem 2 : Planar Pendulum

DO NOT GET CONFUSED WITH 'L' (Lagrangian) and 'l' (length)



Holonomic Constraint:

$$x^2+y^2=l^2$$

$$\vec{r} = l\sin\theta\,\hat{x} - l\cos\theta\,\hat{y}$$

$$= \begin{pmatrix} l\sin\theta \\ -l\cos\theta \end{pmatrix}$$

$$\dot{\vec{r}} = l\begin{pmatrix} \cos\theta\,\dot{\theta} \\ \sin\theta\,\dot{\theta} \end{pmatrix}$$

$$= l\dot{\theta}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\dot{\vec{r}}\cdot\dot{\vec{r}} = l^2\dot{\theta}^2(\cos^2\theta+\sin^2\theta)$$

$$= l^2\dot{\theta}^2$$

$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$V = -mgl\cos\theta$$

$$L = T-V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

Euler - lagrange equation:

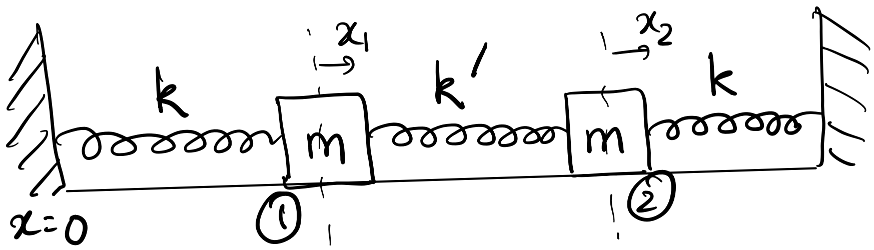
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}$$

$$\Rightarrow \quad ml^2\ddot{\theta} = -mgl\sin\theta$$

$$\Rightarrow \quad \ddot{\theta}l = -g\sin\theta \Rightarrow \boxed{\ddot{\theta}+\frac{g}{l}\sin\theta=0}$$

EOM.

Problem 3



generalised coordinates : x_1 (displacement of mass 1)

2 Euler–Lagrange EOM

x_2 (displacement of mass 2)

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2\right)$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k' (x_2 - x_1)^2$$

$$L = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2 - \frac{1}{2} k' (x_2 - x_1)^2$$

Formula:

Potential energy $= \frac{1}{2} k \,(\text{elongation})^2$

$F = -kx$

$V = -\int F\,dx$

$= \frac{kx^2}{2} + \text{const.}$

Do not bother about constants.

__For x_1__ : $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1}$$

$$\Rightarrow m\ddot{x}_1 = -k x_1 - k' (x_2 - x_1)(-1)$$

$$\Rightarrow m\ddot{x}_1 = -k x_1 + k' (x_2 - x_1)$$

$\longrightarrow$ EQ-1

Similarly,

For x_2: (Homework fill in the steps)

$$m\ddot{x}_2 = -kx_2 - k'(x_2 - x_1)$$

$\longrightarrow$ EQ. 2

So we have,

$$\left.\begin{aligned} m\ddot{x}_1 &= -kx_1 + k'(x_2 - x_1) \\ m\ddot{x}_2 &= -kx_2 - k'(x_2 - x_1) \end{aligned}\right\} \text{Coupled set of equations}$$

For interested students: These give rise to normal modes.

Also follow these notes for the solution and many other interesting coupled oscillator problems.

Ref: https://www.physics.hmc.edu/~saeta/courses/p111/uploads/Y2013/chap13.pdf

Problem 4 : Concept of Cyclic Coordinates.

If a particular coordinate does not appear in the Lagrangian it is called CYCLIC or IGNORABLE coordinates.

Example (i) Lagrangian of a point mass under gravity,

$$L = \underbrace{\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)}_{\text{Kinetic energy }(T)} - \underbrace{mgz}_{\text{Potential Energy }(V)}$$

$L = L(z, \dot{x}, \dot{y}, \dot{z}) \Rightarrow x, y$ are cyclic coordinates

Example (ii) Lagrangian for a planet of mass 'm' orbiting around a star of mass M under Gravitational force.

$$L = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) + \frac{GMm}{r}$$

$\Rightarrow L = L(r, \dot{r}, \dot{\theta}) \Rightarrow \theta$: Cyclic

$p_j = \partial L / \partial \dot{q}_j$ (Recall)

The conjugate momentum (or generalised momentum) corresponding to a cyclic coordinate is conserved, i.e.,

EL EQ: $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$$

if q_j is cyclic coordinate, then, $\frac{\partial L}{\partial q_j} = 0$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \Rightarrow \dot{p}_j = 0$$

In Example (4i) $\dot{p}_x = \dot{p}_y = 0$

" " (4ii) $\dot{p}_\theta = 0 \Rightarrow mr^2\dot{\theta}$ Conserved

Problem 5 — Concept of Hamiltonian

$$L = L(q, \dot{q}, t) = T - V$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t}$$

From Euler-Lagrange's equation we know,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

$$\Rightarrow \frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) + \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow \frac{dH}{dt} + \frac{\partial L}{\partial t} = 0$$

[Introduced a new function $H = \frac{\partial L}{\partial \dot{q}}\dot{q} - L$

$H = p\dot{q} - L$

$$\Rightarrow \quad \frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

[TIME TRANSLATIONAL INVARIANCE IN CONSERVATIVE SYSTEMS]

Now, if Lagrangian does not explicitly depend on time, then $\boxed{\frac{dH}{dt} = 0}$

$$H = \frac{\partial L}{\partial \dot{q}} \dot{q} - L$$

For conservative forces: (and Cartesian Coordinate example) $q \equiv x$

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial (T-V)}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}}$$

[V does not depend on $\dot{q}$]

$$\Rightarrow \quad H = \frac{\partial T}{\partial \dot{q}} \dot{q} - L$$

Note that this is true for Cartesian systems

$$\left[\begin{array}{l} T = \frac{1}{2} m \dot{q}^2 \\ \frac{\partial T}{\partial \dot{q}} = m \dot{q} \\ \dot{q} \frac{\partial T}{\partial \dot{q}} = m \dot{q}^2 \end{array} \right]$$

$$\Rightarrow \quad H = 2T - L$$

$$\Rightarrow \quad \boxed{H = T + V}$$

$\Rightarrow$ Energy equivalent in a conservative system.

Other Problems : (1) Double Atwood Machine

1) Neglect masses of the two pulleys

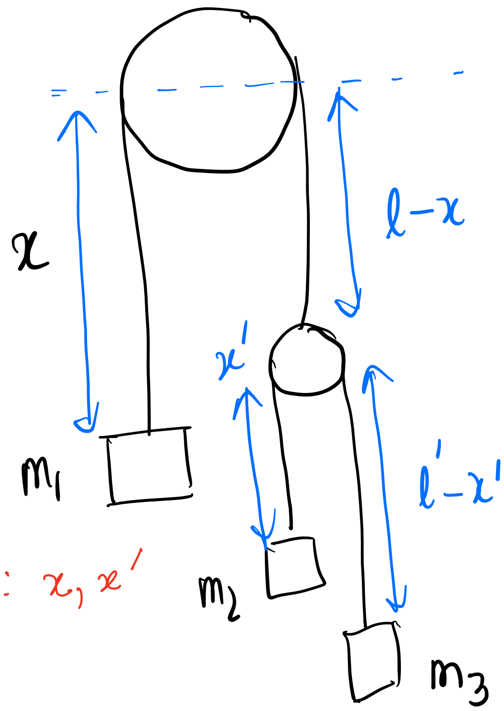
2) Frictionless

Velocity of m_1 : $\dot{x}$

Velocity of m_2 : $-\dot{x}+\dot{x}'$

Velocity of m_3 : $-\dot{x}-\dot{x}'$

Generalized coordinates : x, x'



1) Length of first string = l + const. (touching over pulley)

2) Length of second string = l' + const.

Steps:

1) Write down the kinetic energy and Potential energy.

2) $L = T - V$

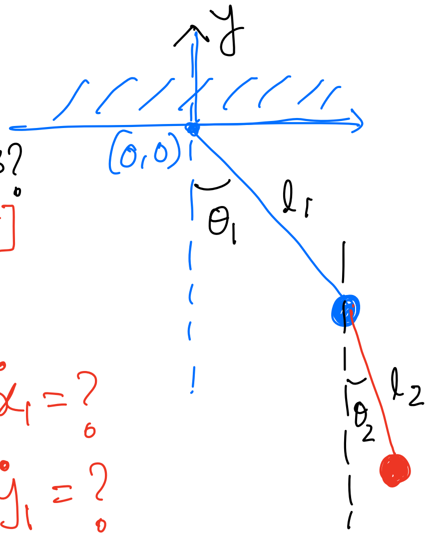
3) Write down the Euler-Lagrange equation.

Final answer:

$$(m_1+m_2+m_3)\ddot{x} + (m_3-m_2)\ddot{x}' = (m_1-m_2-m_3)g$$

$$(m_2+m_3)\ddot{x}' + (m_3-m_2)\ddot{x} = (m_2-m_3)g$$

Other Problems (2) Double Pendulum



1) What are the generalised coordinates?

θ_1, θ_2 [2 EOM's]

for m_1

$$x_1 = l_1 \sin\theta_1 \; ; \; \dot{x}_1 = ?$$
$$y_1 = -l_1 \cos\theta_1 \; ; \; \dot{y}_1 = ?$$

for m_2

$$x_2 = l_2 \sin\theta_2 + l_1 \sin\theta_1 \; ; \; \dot{x}_2 = ?$$
$$y_2 = -l_2 \cos\theta_2 - l_1 \cos\theta_1 \; ; \; \dot{y}_2 = ?$$

2) Write down the kinetic energy (T) and the potential energy (V).

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$V = m_1 g y_1 + m_2 g y_2$$

$$L = T - V$$

3) Write down the Equations of motion (EOMs).

For θ_1 = ?

For θ_2 = ?

$$\ddot{\theta}_1 + \frac{m_2}{m_1+m_2}\frac{l_2}{l_1}\left(\ddot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2)\right)$$

$$+ \frac{g}{l_1}\sin\theta_1 = 0$$

$$\ddot{\theta}_2 + \frac{l_1}{l_2}\left(\ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_1^2\sin(\theta_1-\theta_2)\right)$$

$$+ \frac{g}{l_2}\sin\theta_2 = 0$$