

Department of Mathematics & Computing
Engineering Mathematics I
Tutorial Sheet-IV
(Vector Differential and Integral Calculus)

1. Represent the parabola $y = 1 - 2x^2$, $-1 \leq x \leq 1$ in parametric form. (Ans. $\sin t \mathbf{i} + \cos 2t \mathbf{j}$, $\frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$)
2. Find the tangent vector to the curve whose parametric representation is $x = \cos t$, $y = \sin t$, $z = t$, $-\pi \leq t \leq \pi$. (Ans. $-\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$)
3. Find the value of vector field $\vec{F}(x, y) = (y - 1) \mathbf{i} + (x + y) \mathbf{j}$ at $(1, 1)$ and $(-1, 0)$. (Ans. $2\mathbf{j}$, $-\mathbf{i} - \mathbf{j}$)
4. Write down the parametric equations for the plane $7x + 3y + 4z = 15$. (Ans. $\langle x, y, \frac{15}{4} - \frac{7}{4}x - \frac{3}{4}y \rangle$)
5. Find the derivative of $r(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + ct \mathbf{k}$. (Ans. $-a \sin t \mathbf{i} + a \cos t \mathbf{j} + c \mathbf{k}$)
6. Find the length of the curve $r(t) = t \mathbf{i} + (t^2/2) \mathbf{j}$, $0 \leq t \leq 1$. (Ans. $\frac{1}{2}[\sqrt{2} + \log(1 + \sqrt{2})]$)
7. Find the length of the following curve and express $r(t)$ as a function of the arc length.
 $r(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}$, $a > 0$, $0 \leq t \leq 2\pi$. (Ans. $2\pi a$)
8. The position vector of a moving particle is $r(t) = (\cos t + \sin t) \mathbf{i} + (\sin t - \cos t) \mathbf{j} + t \mathbf{k}$. Determine velocity, speed and acceleration of the particle in the direction of the motion.
(Ans. $(-\sin t + \cos t) \mathbf{i} + (\cos t + \sin t) \mathbf{j} + \mathbf{k}$, $\sqrt{3}$, $(-\cos t - \sin t) \mathbf{i} + (-\sin t + \cos t) \mathbf{j}$)
9. Find $\nabla \phi$ where $\phi = \ln |r|$. (Ans. $\frac{\mathbf{r}}{r^2}$, where $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$)
10. Compute gradient of the scalar functions $x \sin(yz) + y \sin(xz) + z \sin(xy)$ and evaluate it at $(0, \frac{\pi}{4}, 1)$.
(Ans. $\frac{\pi + \sqrt{2}}{2} \mathbf{i}$)
11. Find unit normal vector to the surface $x^2 + 2y^2 + z^2 = 4$ at the point $(1, 1, 1)$. (Ans. $\frac{\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{6}}$)
12. Prove that $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$, $g \neq 0$.
13. Find the directional derivative of $\phi = x^2yz - 4xyz^2$ at $(1, 3, 1)$ in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.
(Ans. 11)
14. Find directional derivative of $2x^2 + y^2 + z^2$ at $(1, 2, 3)$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$. (Ans. $\frac{58}{5\sqrt{2}}$)
15. In what direction from the point $(1, 3, 2)$ is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of the maximum? (Ans. $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$, $2\sqrt{14}$)
16. If $f(x, y) = x^2 - xy - y + y^2$, find all points where the directional derivative in the direction $\frac{\mathbf{i} + \sqrt{3}\mathbf{j}}{2}$ is zero.
17. Evaluate $\nabla(\nabla \cdot (\bar{r}/r))$. (Ans. $-2r^{-3}\bar{r}$)
18. Determine the constant b such that $\bar{A} = (bx^2y + yz)\mathbf{i} + (xy^2 - xz^2)\mathbf{j} + (2xyz - 2x^2y^2)\mathbf{k}$ has zero divergence. (Ans. -2)
19. Find the directional derivative of $\nabla \cdot \bar{U}$ at the point $(4, 4, 2)$ in the direction of the corresponding outer normal of the sphere $x^2 + y^2 + z^2 = 36$ where $\bar{U} = \frac{1}{2}x^2z\mathbf{i} + yx\mathbf{j} + \frac{1}{2}z^2\mathbf{k}$. (Ans. $5/3$)
20. Prove that $\bar{A} = (x^2 - yz)\mathbf{i} + (y^2 - zx)\mathbf{j} + (z^2 - xy)\mathbf{k}$ is irrotational and find the scalar potential f such that $\bar{A} = \nabla f$. (Ans. $f(x, y, z) = \frac{x^3 + y^3 + z^3}{3} - xyz$)
21. $\vec{F} = 3xy\mathbf{i} - y^2\mathbf{j}$, evaluate $\int_C \vec{F} d\vec{r}$ where C is the curve $y = 2x^2$ in the xy plane from $(0, 0)$ to $(1, 2)$.
(Ans. $-\frac{7}{6}$)

22. $\vec{F} = (x + y^2)\mathbf{i} + (x^2 - y^2)\mathbf{j}$. Evaluate $\int_C \vec{F} d\vec{r}$ where C is closed curve formed with $y^3 = x^2$ and the chord joining $(0, 0)$ and $(1, 1)$ taken in clockwise. (Ans. $1/84$)
23. Evaluate $\oint_C |z|^2 dz$ around the square with vertices at $(0, 0), (1, 0), (1, 1), (0, 1)$ in counter clockwise. (Ans. $\mathbf{i} - 1$)
24. Evaluate $\int_C (xy^2 dy - x^2 y dx)$ takes in the counter clockwise along the cardioid $r = a(1 + \cos\theta)$. (Ans. $\frac{35a^4\pi}{16}$)
25. $\vec{F} = (1 + xy^2)\mathbf{i} - xy^2\mathbf{j}$. Evaluate $\int_C \vec{F} d\vec{r}$ using Green's theorem where C consists of the arc of the parabola $y = x^2$ from $(-1, -1)$ from $(1, 1)$. (Ans. 0)
26. Use Green's theorem to find: $\int x^2 y dx - xy^2 dy$ where C is the circle $x^2 + y^2 = 4$ in counter clockwise. (Ans. -8π)
27. $\int_S \int (x + 2y + z) dS$, $S : y + z = 4$ inside $x^2 + y^2 = 1$. (Ans. $4\sqrt{2}\pi$)
Hint: $\int_S \int f(x, y, z) dS = \int_R \int f(x, y, g(x, y)) \sqrt{g_x^2 + g_y^2 + 1} dA$ For surface $Z = g(x, y)$.
28. Find $\int_S \int x^2 z dS$, where S is the Paraboloid $Z = g(x, y) = 1 - x^2 - y^2$ over xy plane. (Ans. 0.143π)
29. Evaluate the surface integral $\int_S \int x^2 y^2 z dS$ where $S : x^2 + y^2 + z^2 = 1, z \geq 0$, (Ans. $\pi/24$)
30. Evaluate $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dz dx dy$. (Ans. 8π)
31. Evaluate $\int \int \int_V (x^2 + y^2 + z^2) dx dy dz$ where V is the volume of the cube bounded by the coordinate planes and the planes $x = y = z = a$. (Ans. a^5)
32. Evaluate $\int \int \int_V (2x + y) dx dy dz$ where V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 2$ and $z = 0$. (Ans. $\frac{80}{3}$)
33. Evaluate $\int \int \int_V x^2 dx dy dz$ over the region $x = 0, y = 0, z = 0$ and $x + y + z = a$. (Ans. $\frac{a^5}{60}$)
34. Evaluate the surface integral $\int_S (x^3 dy dz + y^3 dz dx + z^3 dx dy)$ with the help of Gauss's theorem over the sphere $x^2 + y^2 + z^2 = a^2$. (Ans. $\frac{12}{5}\pi a^5$)
35. Use the divergence theorem to evaluate the surface integral $\int_S \mathbf{v} \cdot \mathbf{n} dA$, where $\mathbf{v} = 5xy\mathbf{i} + 3yz\mathbf{j} + x^2\mathbf{k}$ and S is the boundary of the region bounded by the $x + y = 4, x = 0, y = 0, z = 0$ and $z = 4$. (Ans. $\frac{928}{3}$)
36. Verify the divergence theorem, $\mathbf{v} = x^2\mathbf{i} + 2y^2\mathbf{j} + 3z^2\mathbf{k}$, D is the region bounded by the cylinder $x^2 + y^2 = 9, z = 0$ and $z = 3$. (Ans. 243π)
37. Using Stokes's theorem, show that

$$\int \int_S ((y - z) dy dz + (z - x) dz dx + (x - y) dx dy) = a^3 \pi,$$

where S is the portion of the surface $x^2 + y^2 - 2ax + az = 0, z \geq 0$.

38. Verify Stokes's theorem, for the vector field $\mathbf{v} = (3x - y)\mathbf{i} - 2yz^2\mathbf{j} - 2y^2z\mathbf{k}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 16, z > 0$. (Ans. 16π)