Department of Mathematics and Computing

Engineering Mathematics-I

Tutorial Sheet-3 solutions

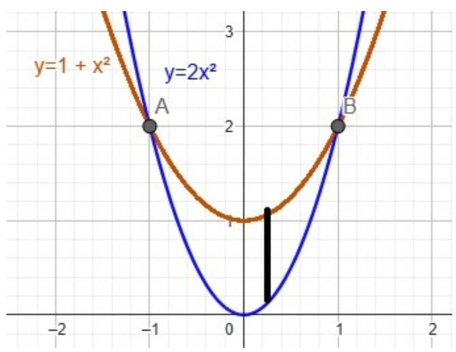
1. (i) $\int_0^a \int_0^b (x^2+y^2)dydx$
   Solution: $\int_0^a \int_0^b (x^2+y^2)dydx = \int_0^a [\int_0^b (x^2+y^2)dy]dx = \int_0^a [x^2y + \frac{y^3}{3}]_0^b dx = \int_0^a [bx^2 + \frac{b^3}{3}]dx = [\frac{bx^3}{3} + \frac{b^3}{3}x]_0^a = \frac{ab}{3}(a^2+b^2)$

   (ii) $\int_1^2 \int_1^x xy^2 dydx$
   Solution: $\int_1^2 \int_1^x xy^2 dydx = \int_1^2 [\int_1^x xy^2 dy]dx = \int_1^2 x[\frac{y^3}{3}]_1^x dx = \int_1^2 x[\frac{x^3}{3} - \frac{1}{3}]dx = \frac{1}{3}\int_1^2 [x^4 - x]dx = \frac{1}{3}[\frac{x^5}{5} - \frac{x^2}{2}]_1^2 = \frac{47}{30}$

   (iii) $\int_0^1 \int_{\sqrt{y}}^{2-y} x^2 dxdy$
   Solutions: $\int_0^1 \int_{\sqrt{y}}^{2-y} x^2 dxdy = \int_0^1 [\int_{\sqrt{y}}^{2-y} x^2 dx]dy = \int_0^1 [\frac{x^3}{3}]_{\sqrt{y}}^{2-y} dy = \frac{1}{3}\int_0^1 [(2-y)^3 - y^{3/2}]dy = \frac{1}{3}\int_0^1 [8 - y^3 + 6y^2 - 12y - y^{3/2}]dy = \frac{1}{3}[8y - \frac{y^4}{4} + \frac{6y^3}{3} - \frac{12y^2}{2} - \frac{y^{5/2}}{5/2}]_0^1 = \frac{67}{60}$

2. Evaluate $\iint_R (x+2y)dxdy$, where R is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

   Solution: The region of integration is as under:

   Now, $\iint_R (x+2y)dxdy = \int_{-1}^1 \int_{2x^2}^{1+x^2} (x+2y)dxdy = \int_{-1}^1 [\int_{2x^2}^{1+x^2} (x+2y)dy]dx = \int_{-1}^1 [xy + 2\frac{y^2}{2}]_{2x^2}^{1+x^2}]dx = \int_{-1}^1 [1 + x + 2x^2 - x^3 - 3x^4]dx = \frac{32}{15}$

3. Evaluate $\iint_R x^3 dxdy$, where $R = \{(x,y) : 1 \le x \le e, 0 \le y \le lnx\}$.

   Solution: $\iint_R x^3 dxdy = \int_1^e \int_0^{lnx} x^3 dydx = \int_1^e [\int_0^{lnx} x^3 dy]dx = \int_1^e x^3 [y]_0^{lnx} dx = \int_1^e x^3 lnx dx$. Now, $\int x^3 lnx dx = \frac{x^4}{4} lnx - \int \frac{x^4}{4}\frac{1}{x}dx = \frac{x^4}{4} lnx - \frac{1}{4}\int x^3 dx = \frac{x^4}{4} lnx - \frac{x^4}{16}$. Thus, $\int_1^e x^3 lnx dx = [\frac{x^4}{4} lnx - \frac{x^4}{16}]_1^e = \frac{e^4}{4} - \frac{e^4}{16} - (0 - \frac{1}{16}) = \frac{3e^4}{16} + \frac{1}{16}$

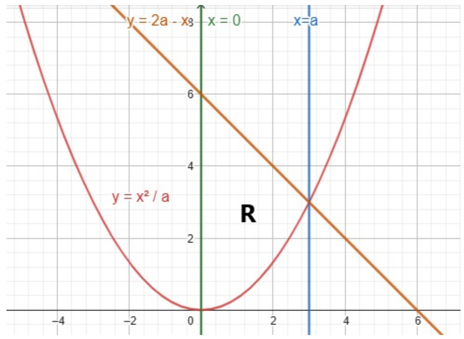
4. Evaluate $\iint_R xy^2 dxdy$, where R is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

   Solution: First, we need to find the bounds for the region. The region is a right triangle with the hypotenuse described by the line $y = x$, so y ranging from 0 to x and x ranging from 0 to 1. Thus, $\iint_R xy^2 dxdy = \int_0^1 \int_0^x xy^2 dydx = \int_0^1 [\int_0^x xy^2]dx = \int_0^1 x[\int_0^x y^2 dy]dx = \int_0^1 x[\frac{y^3}{3}]_0^x dx = \int_0^1 \frac{x^4}{3}dx = [\frac{x^5}{3\times5}]_0^1 = \frac{1}{15}$

5. Evaluate the following by change of order of integration.

(i) $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydydx$

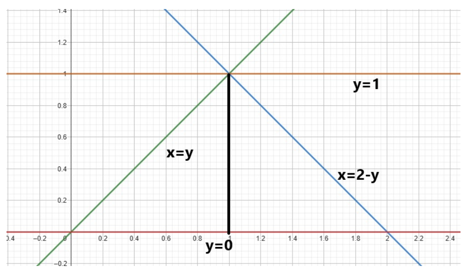
Solution: The region of integration is as under:



$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydydx = \int_0^a \int_0^{\sqrt{ay}} xydxdy + \int_a^{2a} \int_0^{2a-y} xydxdy = \int_0^a \frac{ay^2}{2} dy + \int_a^2 a\frac{y(2a-y)^2}{2} dy = \frac{3}{8}a^4$

(ii) $\int_0^1 \int_y^{2-y} xydxdy$

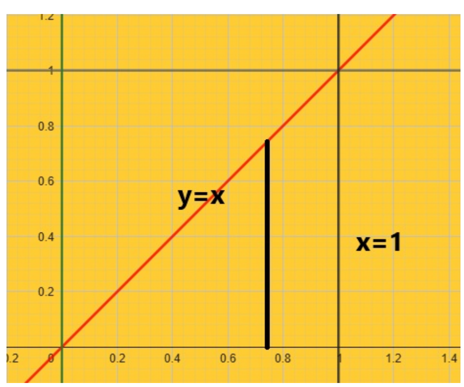
Solution: The region of integration is as under:



$\int_0^1 \int_y^{2-y} xydxdy = \int_0^1 \int_0^x xydydx + \int_1^2 \int_0^{2-x} xydydx = \int_0^1 [x\frac{y^2}{2}]_0^x dx + \int_1^2 x[\frac{y^2}{2}]_0^{2-x} dx$
$= \int_0^1 \frac{x^3}{2} dx + \int_1^2 \frac{x^3-4x^2+4x}{2} dx = \frac{1}{3}.$

(iii) $\int_0^1 \int_x^1 \frac{y}{x^2+y^2} dydx$

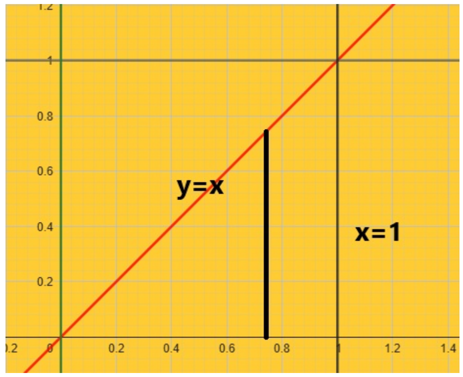
Solution: The region of integration is as under:



$\int_0^1 \int_x^1 \frac{y}{x^2+y^2} dydx = \int_0^1 [\int_0^y \frac{y}{x^2+y^2} dx] dy = \int_0^1 y[\frac{1}{y} tan^{-1}(\frac{x}{y})]_0^y dy = \int_0^1 [tan^{-1}(1) - tan^{-1}(0)] dy = \frac{\pi}{4} \int_0^1 dy = \frac{\pi}{4}.$

(iv) $\int_0^1 \int_x^1 siny^2 dydx$

Solution: The region of integration is as under:

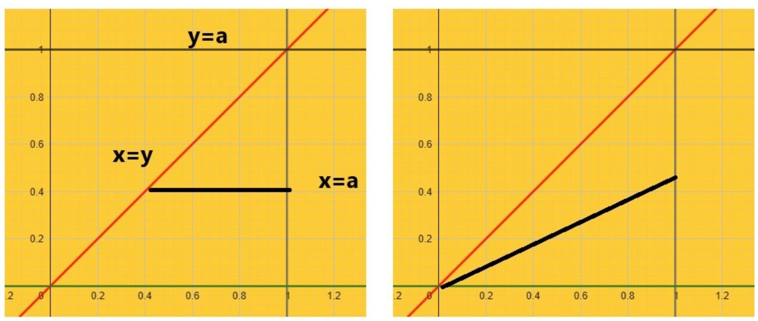


$\int_0^1 \int_x^1 siny^2 dydx = \int_{y=0}^1 \int_{x=0}^y \sin(y^2) dxdy = \int_{y=0}^1 \sin(y^2)[\int_{x=0}^y dx]dy = \int_0^1 y\sin(y^2)dy = -\frac{1}{2}(\cos(1)-1)$.

6. Changing into polar coordinates, evaluate the following integrals.

(i) $\int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{3/2}} dydx$

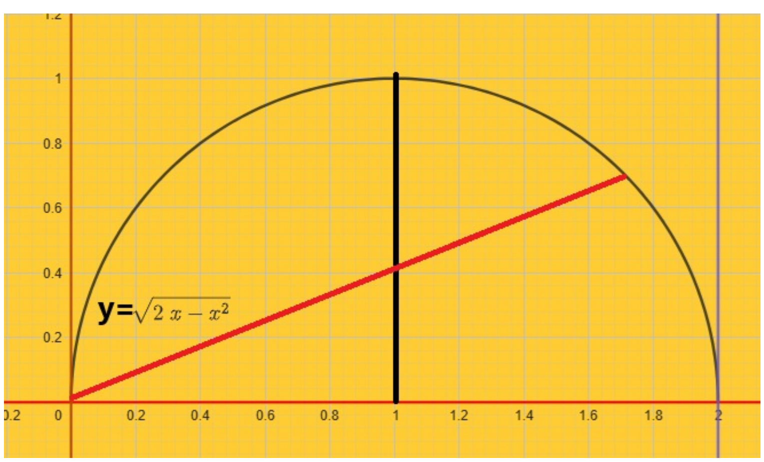
Solution: The region of integration is as under:



$\int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{3/2}} dydx = \int_0^{\frac{\pi}{4}} [\int_0^{a\sec(\theta)} \frac{r^2\cos^2(\theta)}{(r^2)^{3/2}} rdr]d\theta = \int_0^{\frac{\pi}{4}} [\int_0^{a\sec(\theta)} \cos^2(\theta)dr]d\theta = a\int_0^{\frac{\pi}{4}} \cos(\theta)d\theta = \frac{a}{\sqrt{2}}$.

(ii) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{(x^2+y^2)} dydx$

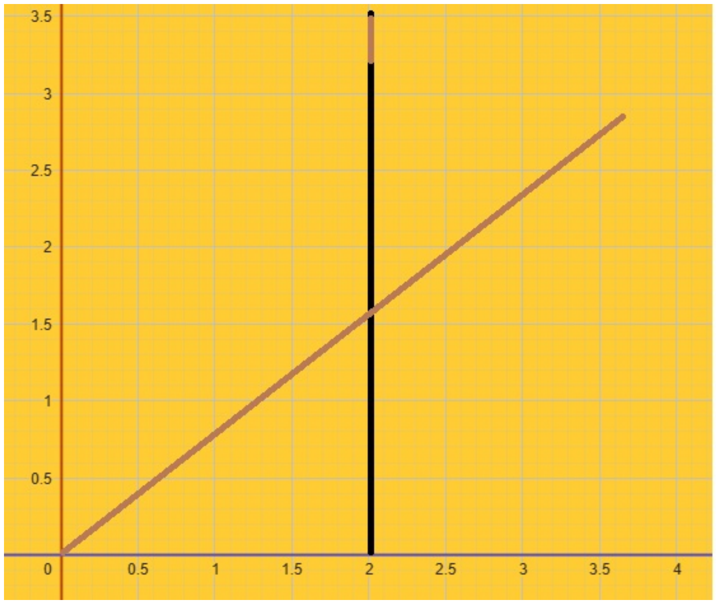
Solution: The region of integration is as under: $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{(x^2+y^2)} dydx = \int_0^{\pi/2} \int_0^{2\cos(\theta)} \frac{r\cos(\theta)}{r^2} rdrd\theta =$



$\int_0^{\pi/2} 2\cos^2(\theta)d\theta = \frac{\pi}{2}$.

(iii) $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dydx$

Solution: The region of integration is as under:



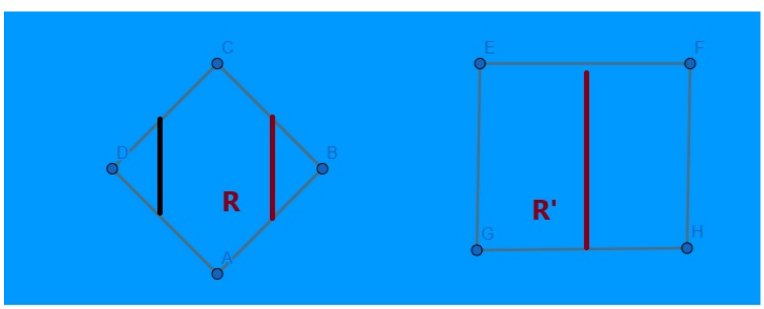
$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dydx = \int_0^{\pi/2}[\int_0^{\infty} e^{-r^2}rdr]d\theta = \int_0^{\pi/2}\frac{1}{2}d\theta = \frac{\pi}{4}$.

7. Evaluate $\iint_R (x-y)^2 cos^2(x+y)dxdy$, where R is the rhombus with successive vertices at $(\pi,0), (2\pi,\pi), (\pi,2\pi)$ and $(0,\pi)$.

Solution: Let $u = x - y$ and $v = x + y$. The Jacobian of transformation is:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{2}.$$

The region of integration is as follows:



Using this transformation, the integral becomes:

$$\iint_{R'} u^2 \cos^2(v) \cdot \frac{1}{2}\, du\, dv$$

The region R in the original coordinates corresponds to a new region R'. To find the limits for u and v, observe the vertices in terms of u and v;

- When $(x, y) = (\pi, 0)$ then $(u, v) = (\pi, \pi)$

- When $(x, y) = (2\pi, \pi)$ then $(u, v) = (\pi, 3\pi)$
- When $(x, y) = (\pi, 2\pi)$ then $(u, v) = (-\pi, 3\pi)$
- When $(x, y) = (0, \pi)$ then $(u, v) = (-\pi, \pi)$

From this, it is clear that u ranges from $-\pi$ to π and v ranges from π to 3π. So, the transformed integral is:

$$\int_{\pi}^{3\pi} \int_{-\pi}^{\pi} \frac{1}{2} u^2 \cos^2(v)\, du\, dv$$

Separate the integrals:

$$\frac{1}{2} \left(\int_{-\pi}^{\pi} u^2\, du \right) \left(\int_{\pi}^{3\pi} \cos^2(v)\, dv \right)$$

Calculate each part:

$$\int_{-\pi}^{\pi} u^2\, du = \left. \frac{u^3}{3} \right|_{-\pi}^{\pi} = \frac{\pi^3}{3} - \left(-\frac{\pi^3}{3} \right) = \frac{2\pi^3}{3}$$

For the cosine integral, use the identity $\cos^2(v) = \frac{1+\cos(2v)}{2}$,

$$\int_{\pi}^{3\pi} \cos^2(v)\, dv = \int_{\pi}^{3\pi} \frac{1+\cos(2v)}{2}\, dv = \frac{1}{2} \left(\int_{\pi}^{3\pi} 1\, dv + \int_{\pi}^{3\pi} \cos(2v)\, dv \right)$$

The integral of $\cos(2v)$ over a complete period is zero:

$$\int_{\pi}^{3\pi} \cos(2v)\, dv = 0$$

So we have:

$$\frac{1}{2} \int_{\pi}^{3\pi} 1\, dv = \frac{1}{2} (3\pi - \pi) = \frac{1}{2} \cdot 2\pi = \pi$$

Combine the results:

$$\frac{1}{2} \left(\frac{2\pi^3}{3} \right) (\pi) = \frac{1}{2} \cdot \frac{2\pi^4}{3} = \frac{\pi^4}{3}$$

8. Using double integration, evaluate the area of (i) Cardioid $r = a(1 + cos\theta)$, and (ii) lemniscate $r^2 = a^2 cos 2\theta$.

   (i).

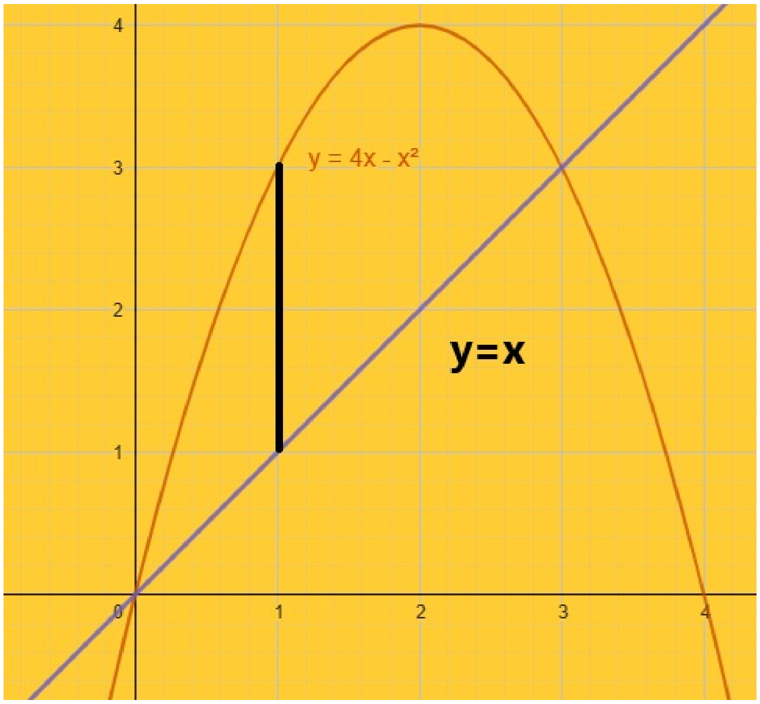
$$\begin{aligned}
\text{Area} &= \iint_R r dr d\theta \\
&= 2 \int_0^{\pi} \int_0^{a(1+\cos\theta)} r\, dr d\theta \\
&= a^2 \int_0^{\pi} (1+\cos\theta)^2\, d\theta \\
&= a^2 \int_0^{\pi} \left(2\cos^2 \frac{\theta}{2} \right)^2 d\theta \\
&= 4a^2 \int_0^{\pi} \cos^4 \left(\frac{\theta}{2} \right) d\theta \\
&= 8a^2 \int_0^{\frac{\pi}{2}} \cos^4 \phi\, d\phi \\
&= \frac{3\pi a^2}{2} \text{ sq.units}
\end{aligned}$$

   Calculations: Put $\phi = \frac{\theta}{2}$ so that $2d\phi = d\theta$. The variations in the limits will be as follows: $\theta \to \phi$, $0 \to 0$, and $\pi \to \frac{\pi}{2}$.

(ii). Lemniscate $r^2 = a^2 cos2\theta$
Solution.

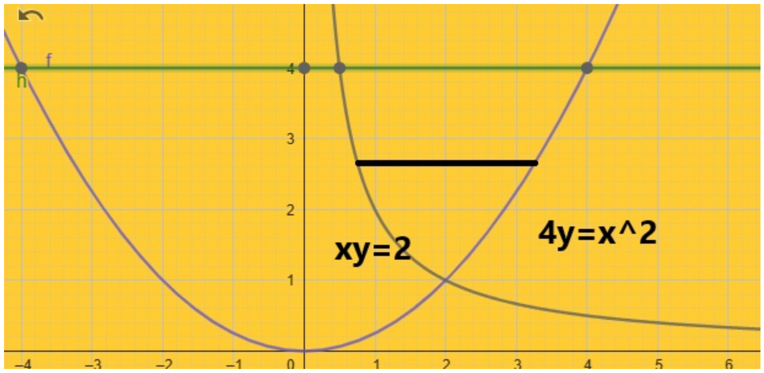
$$\begin{aligned}
\text{Area} &= 4 \times \int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} r dr\, d\theta \\
&= 4 \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{a\sqrt{cos2\theta}} d\theta \\
&= 2 \int_0^{\frac{\pi}{4}} a^2 \cos 2\theta\, d\theta \\
&= 2a^2 \frac{1}{2} [\sin 2\theta]_0^{\frac{\pi}{4}} \\
&= a^2.
\end{aligned}$$

9. Using double integration, evaluate the area between the parabola $y = 4x - x^2$ and the line $y = x$.
Solution. The region of integration is as follows:



$$A = \int_0^3 \int_x^{(4x-x^2)} dydx = \int_0^3 [y]_x^{4x-x^2} dx = \int_0^3 (3x - x^2)\, dx = \frac{9}{2}.$$

10. Using double integration, evaluate the area between the curves $xy = 2$, $4y = x^2$ and $y = 4$. **Solution.** The region of integration is as follows:



$$
\begin{aligned}
A &= \int_1^4 \left(\int_{\frac{2}{y}}^{2\sqrt{y}} dx \right) dy \\
&= \int_1^4 [x]_{\frac{2}{y}}^{2\sqrt{y}} \, dy \\
&= \int_1^4 \left(2\sqrt{y} - \frac{2}{y} \right) dy \\
&= 2 \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \log y \right]_1^4 \\
&= \frac{28}{3} - 4 \log 2.
\end{aligned}
$$

11. Using double integration, find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. **Solution.** Divide the ellipsoid into eight parts concerning the origin and axes. Consider octant OABC as shown in Fig. (1)

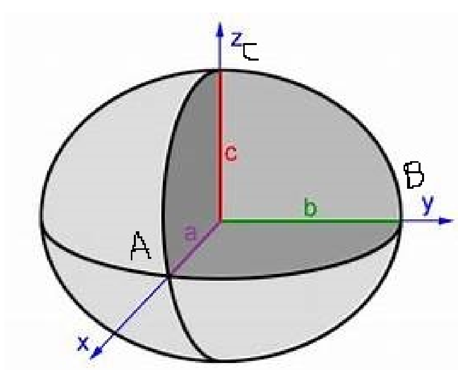


Figure 1: Ellipsoid

Hence, the region OABC lies between

(a) Ellipsoid: $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

(b) Plane: XOY

(c) Bounded by planes $x = 0$, $y = 0$.

The ellipsoid cuts the plane $X0Y$ in the

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(b) $z = 0$.

Therefore, the volume $OABC$ is bounded by $x = 0, x = a$ and $y = 0, y = b\sqrt{1 - \frac{x^2}{a^2}}$ Therefore,

$$\begin{aligned}
\text{Required volume of the ellipsoid } &= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} z\,dydx \\
&= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dydx
\end{aligned}$$

Let $b\sqrt{1 - \frac{x^2}{a^2}} = t \implies (1 - \frac{x^2}{a^2}) = \frac{t^2}{b^2}$ Therefore,

$$\begin{aligned}
\text{Required volume of the ellipsoid } &= 8\int_0^a \int_0^t c\sqrt{\frac{t^2}{b^2} - \frac{y^2}{b^2}}\,dydx \\
&= \frac{8c}{b}\int_0^a \int_0^t \frac{c}{b}\sqrt{t^2 - y^2}\,dydx \\
&= 8\int_0^a \frac{c}{b}\left[\frac{y}{2}\sqrt{t^2 - y^2} + \frac{t^2}{2}\sin^{-1}(\frac{y}{t})\right]_0^t dx \\
&= \frac{8c}{b}\int_0^a \left[\frac{\pi t^2}{4}\right] dx \\
&= \frac{8c\pi}{4b}\int_0^a [t^2]\,dx \\
&= \frac{2\pi}{b}\int_0^a [b^2(1 - \frac{x^2}{a^2})]\,dx \\
&= 2\pi cb\int_0^a (1 - \frac{x^2}{a^2})\,dx \\
&= 2\pi bc\left[x - \frac{x^3}{3a^2}\right]_0^a \\
&= \frac{4\pi abc}{3}.
\end{aligned}$$

12. Find the volume of cylinder $x^2 + y^2 = a^2$ above the xy-plane cut by the plane $x + y + z = 2a$.
Solution. We have

$$x^2 + y^2 = a^2 \implies y^2 = a^2 - x^2 \implies y = \pm\sqrt{a^2 - x^2}$$

Above the $xy-$plane, put $y = 0$. So $x^2 = a^2 \implies \pm a$. The limits for x and y are: x varies from $-a$ to a and y varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$. Since the cylinder $x^2 + y^2 = a^2$ above the $xy-$plane is cut by the plane $x + y + z = 2a$ or $z = 2a - x - y$. Therefore

$$\begin{aligned}
\text{Volume} &= \iint_S z\,dydx \\
&= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (2a - x - y)\,dydx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-a}^{a} \left[2ay - xy - \frac{y^2}{2}\right]_{-\sqrt{a^2-x^2}}^{a^2-x^2} dx \\
&= \int_{-a}^{a} \left\{\left[2a\sqrt{a^2-x^2} - x\sqrt{a^2-x^2} - \frac{(a^2-x^2)^2}{2}\right] - \left[-2a\sqrt{a^2-x^2} + x\sqrt{a^2-x^2} - \frac{(a^2-x^2)^2}{2}\right]\right\} dx \\
&= \int_{-a}^{a} 4a\sqrt{a^2-x^2}\, dx - \int_{-a}^{a} 2x\sqrt{a^2-x^2}\, dx \\
&= 4a\int_{-a}^{a} \sqrt{a^2-x^2}\, dx \quad \left[\because \int_{-a}^{a} 2x\sqrt{a^2-x^2} dx = 0 = 2a\left(2 \times \int_{-a}^{a} \sqrt{a^2-x^2}\, dx\right)\right] \quad [\because \text{ the function is even}] \\
&= 8a\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a \\
&= 8a\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right] \\
&= 2\pi a^3.
\end{aligned}
$$

13. A circular hole of radius b is made centrally through a sphere of radius a. Find the volume of the remaining part.
Solution. Consider the equation of the circle as $x^2+y^2=b^2$ and consider the equation of the sphere as $x^2+y^2+z^2=a^2$. Therefore

$$
\text{Volume of the hole} = 2\iint_S z dxdy = 2\iint_S \sqrt{a^2-x^2-y^2} dxdy
$$

Changing into polar coordinates: In the upper half of the hole, r varies from 0 to b [since $x^2+y^2=b^2$] and θ varies from 0 to 2π. So

$$
\begin{aligned}
\text{Volume of the hole} &= 2\int_0^{2\pi}\int_0^b \sqrt{a^2-r^2} r\, drd\theta \\
&= 2\int_0^{2\pi}\left[-\frac{1}{3}(a^2-r^2)^{\frac{3}{2}}\right]_0^b d\theta \\
&= \frac{2}{3}\int_0^{2\pi}[a^3-(a^2-b^2)^{\frac{3}{2}}]\, d\theta \\
&= \frac{4\pi}{3}[a^3-(a^2-b^2)^{\frac{3}{2}}]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\text{Volume of the remaining sphere} &= \text{(Volume of the sphere)-(Volume of the hole)} \\
&= \frac{4}{3}\pi a^3 - \frac{4\pi}{3}[a^3-(a^2-b^2)^{\frac{3}{2}}] \\
&= \frac{4}{3}\pi(a^2-b^2)^{\frac{3}{2}}
\end{aligned}
$$

14. Find (i) the mass, (ii) the center of mass, and (iii) the moment of inertia about axes of a lamina with density function $f(x,y)=6x$ of triangular shape bounded by the x-axis, the line $y=x$, and the line $y=2-x$.
Solution. The given triangular lamina can be represented by the region

$$
R = \{(x,y)\in\mathbb{R}^2 : 0 \le y \le 1, y \le x \le 2-y\}
$$

and density function $f(x,y)=6x$.

(i). The mass of a lamina

$$\begin{aligned}
m(S) &= \iint_S f(x,y)\,dxdy \\
&= \int_0^1 \int_y^{2-y} f(x,y)\,dxdy \\
&= \int_0^1 \int_y^{2-y} (6x)\,dxdy \\
&= 3\int_0^1 [x^2]_y^{2-y}\,dy \\
&= 3\int_0^1 (4-2y)\,dy \\
&= 9
\end{aligned}$$

(ii). Centre of mass $(\bar{x}, \bar{y})$

$$\begin{aligned}
\bar{x} &= \frac{\iint_S xf(x,y)\,dxdy}{\iint_S f(x,y)\,dxdy} \\
&= \frac{1}{9}\iint_S x(6x)\,dxdy \\
&= \frac{1}{9}\int_0^1 \int_y^{2-y} 6x^2\,dxdy \\
&= \frac{2}{9}\int_0^1 [x^3]_y^{2-y}\,dy \\
&= \frac{2}{9}\int_0^1 [8-12y+6y^2-2y^3]\,dy \\
&= \frac{2}{9}\left[8y-6y^2+2y^3-\frac{y^4}{2}\right]_0^1 \\
&= \frac{7}{9}
\end{aligned}$$

$$\begin{aligned}
\bar{y} &= \frac{\iint_S yf(x,y)\,dxdy}{\iint_S f(x,y)\,dxdy} \\
&= \frac{1}{9}\int_0^1 \int_y^{2-y} (6xy)\,dxdy \\
&= \frac{1}{9}\int_0^1 3y[x^2]_y^{2-y}\,dxdy \\
&= \frac{1}{9}\int_0^1 3y[4-2y]\,dy \\
&= \frac{1}{9}[6y^2-2y^3]_0^1 \\
&= \frac{4}{9}
\end{aligned}$$

Therefore, centre of mass $= \left(\frac{7}{9}, \frac{4}{9}\right)$.

(iii). Moment of inertia about axes of lamina (a). Moment of Inertia about x- axis is given by

$$\begin{aligned}
I_x &= \iint_S y^2 f(x,y)\,dxdy \\
&= \int_0^1 \int_y^{2-y} y^2(6x)\,dxdy \\
&= \int_0^1 3y^2[x^2]_y^{2-y}\,dy \\
&= \int_0^1 3y^2[4-2y]\,dy \\
&= 3\int_0^1 [4y^2-2y^3]\,dy \\
&= 3\left[\frac{4y^3}{3}-\frac{2y^4}{4}\right]_0^1 \\
&= \frac{15}{6}
\end{aligned}$$

(b). Moment of Inertia about y- axis is given by

$$\begin{aligned}
I_x &= \iint_S x^2 f(x,y)\,dxdy \\
&= \int_0^1 \int_y^{2-y} x^2(6x)\,dxdy \\
&= \int_0^1 \int_y^{2-y} 6x^3\,dxdy \\
&= 6\int_0^1 \left[\frac{x^4}{4}\right]_y^{2-y}\,dy \\
&= \frac{3}{2}\int_0^1 [(2-y)^4-y^4]\,dy \\
&= \frac{3}{2}\left[\frac{(2-y)^5}{-5}-\frac{y^5}{5}\right]_0^1 \\
&= \frac{-3}{5}.
\end{aligned}$$

15. Let R be the unit square, i.e., $R=\{(x,y): 0\le x\le 1, 0\le y\le 1\}$. Suppose the density at a point (x,y) of R is given by the function $f(x,y)=\frac{1}{y+1}$, i.e., R is denser near the x-axis. Then find (i) the mass, (ii) the center of mass, and (iii) the moment of inertia about axes.
Solution. The given region is

$$R=\{(x,y)\in\mathbb{R}^2 : 0\le y\le 1, 0\le x\le 1\}$$

and density function $f(x,y)=\frac{1}{y+1}$.

(i). The mass of the unit square

$$\begin{aligned}
m(S) &= \iint_S f(x,y)\,dxdy \\
&= \int_0^1 \int_0^1 \frac{1}{y+1}\,dxdy \\
&= \int_0^1 [ln|y+1|]_0^1\,dy
\end{aligned}$$

$$= \int_0^1 ln2\, dy$$
$$= ln2$$

(ii). Centre of mass $(\bar{x}, \bar{y})$

$$\begin{aligned}
\bar{x} &= \frac{\iint_S x f(x,y)\, dxdy}{\iint_S f(x,y)\, dxdy} \\
&= \frac{1}{ln2} \iint_S \frac{x}{y+1}\, dxdy \\
&= \frac{1}{ln2} \int_0^1 \int_0^1 \frac{x}{y+1}\, dxdy \\
&= \frac{1}{ln2} \int_0^1 \frac{1}{2(y+1)} [x^2]_0^1\, dy \\
&= \frac{1}{2ln2} \int_0^1 \frac{1}{y+1}\, dy \\
&= \frac{1}{2ln2} ln2 \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\bar{y} &= \frac{\iint_S y f(x,y)\, dxdy}{\iint_S f(x,y)\, dxdy} \\
&= \frac{1}{ln2} \iint_S \frac{y}{y+1}\, dxdy \\
&= \frac{1}{ln2} \int_0^1 \int_0^1 \frac{y}{y+1}\, dxdy \\
&= \frac{1}{ln2} \int_0^1 \frac{y}{y+1}\, dy
\end{aligned}$$

let $y+1 = u \implies dy = du$, and the change in limits will be $y = 0 \implies u = 1$ and $y = 1 \implies u = 2$.

$$\begin{aligned}
\bar{y} &= \frac{1}{ln2} \int_1^2 \frac{u-1}{u}\, du \\
&= \frac{1}{ln2} \int_1^2 (1 - \frac{1}{u})\, du \\
&= \frac{1}{ln2} [u - ln|u|]_1^2 \\
&= \frac{1}{ln2} - 1.
\end{aligned}$$

Therefore, centre of mass $= \left(\frac{1}{2}, \frac{1}{ln2} - 1\right)$.

(iii). Moment of inertia about axes

(a). Moment of Inertia about x- axis is given by

$$I_x = \iint_S y^2 f(x,y)\, dxdy$$

$$= \int_0^1 \int_0^1 \frac{y^2}{y+1}\, dxdy$$
$$= \int_0^1 \frac{y^2}{y+1}\, dy$$

let $y + 1 = u \implies dy = du$, and the change in limits will be $y = 0 \implies u = 1$ and $y = 1 \implies u = 2$.

$$I_x = \int_1^2 \frac{(u-1)^2}{u} du$$
$$= \int_1^2 (u - 2 + \frac{1}{u}) du$$
$$= \left[\frac{u^2}{2} - 2u + ln|u|\right]_1^2$$
$$= ln2 - \frac{9}{2}.$$

(b). Moment of Inertia about y- axis is given by

$$I_x = \iint_S x^2 f(x,y)\, dxdy$$
$$= \int_0^1 \int_0^1 \frac{x^2}{y+1}\, dxdy$$
$$= \int_0^1 \frac{1}{y+1} \left(\int_0^1 x^2\, dx\right)\, dy$$
$$= \frac{1}{3} \int_0^1 \frac{1}{y+1}\, dy$$
$$= \frac{1}{3} ln2$$