Introduction To Quantum Computing PS s7 Lab

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Lab 1.1

Exercise 1-5

Simplify:

$((\frac{1}{-i})+(\frac{1}{-i})^2+(\frac{1}{-i})^3+\cdots+(\frac{1}{-i})^{30})^{-3} = (i+i^2+i^3+...+i^{30})^{-3} = ((i+i^2+i^3+i^4)+i^5+...+i^{30})^{-3} = (i^{29}+i^{30})^{-3} = (i^1+i^2)^{-3} = (i-1)^{-3} = \frac{1}{(i-1)^3} = \frac{1}{(-1+i)*(-1+i)*(-1+i)} = \frac{1}{-2i*(-1+i)} = \frac{1}{2+2i} = \frac{1-i}{4}$

Exercise 2-5

$u+\overline{w}^2+\frac{|z|}{|z+1|^2} = (1+i)+(\overline{1-i})^2+\frac{|1+2i|}{|(1+2i)+1|^2} = (1+i)+(1+i)^2+\frac{\sqrt{1^2+(2)^2}}{|2+2i|^2} = (1+i)+[1+2i+i^2]+\frac{\sqrt{5}}{(\sqrt{(2)^2+(2)^2})^2} = (1+i)+[1+2i-1]+\frac{\sqrt{5}}{(\sqrt{8})^2} = (1+i)+[2i]+\frac{\sqrt{5}}{8} = 1+i+2i+\frac{\sqrt{5}}{8} = \left(1+\frac{\sqrt{5}}{8}\right)+3i$

Exercise 3-5

Given the complex number:

$z = -30\sqrt{2} - 10\sqrt{6}i$

Find the modulus r:

$r = \sqrt{(Re(z))^2 + (Im(z))^2} = \sqrt{\left(-30\sqrt{2}\right)^2 + \left(-10\sqrt{6}\right)^2} = \sqrt{(900\times 2)+(100\times 6)} = \sqrt{1800+600} = \sqrt{2400} = \sqrt{16\times 150} = 4\sqrt{150} = 4\sqrt{25\times 6} = 4\times 5\sqrt{6} = 20\sqrt{6}$

Find the argument θ:

$\theta = \arctan\left(\frac{Im(z)}{Re(z)}\right) - \pi = \arctan\left(\frac{-10\sqrt{6}}{-30\sqrt{2}}\right) - \pi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$

So the exponent form is:

$z = re^{\theta i} = 20\sqrt{6}e^{(-\frac{5\pi}{6})i}$

Exercise 4-15

Find x: $3x^2 + 3x + 30 = 0$

$D = b^2 - 4ac = 3^2 - 4 * 3 * 30 = -351$

$\sqrt{-351} = 3\sqrt{39} * i$

$x_{1,2} = \frac{-3 \pm 3\sqrt{39} * i}{2 * 3} = -0,5 \pm 0,5\sqrt{39} * i$

Exercise 5-15

Given: $n = 9, z = -64i, \sqrt[9]{-64i} =?$

$64e^{-\frac{\pi}{2}i}$

$r^9 = 64$

$r = \sqrt[9]{64}$

$n\theta = \theta_0 + 2\pi k$

$\theta = \frac{-\frac{\pi}{2} + 2\pi k}{9}; k = 0, 1, 2, ..., 8$

$z_0 = 64^{2/3} e^{-\frac{\pi}{18}i}$

...

$z_8 = ...$

Exercise 6-5

$2 \leq |z + 1 - i| \leq 10, \Im(z) < 1$

$2 \leq |x + y + 1 - i| \leq 10$

$2 \leq \sqrt{(x + 1)^2 + (y - 1)^2} \leq 10$

$4 \leq (x + 1)^2 + (y - 1)^2 \leq 100$

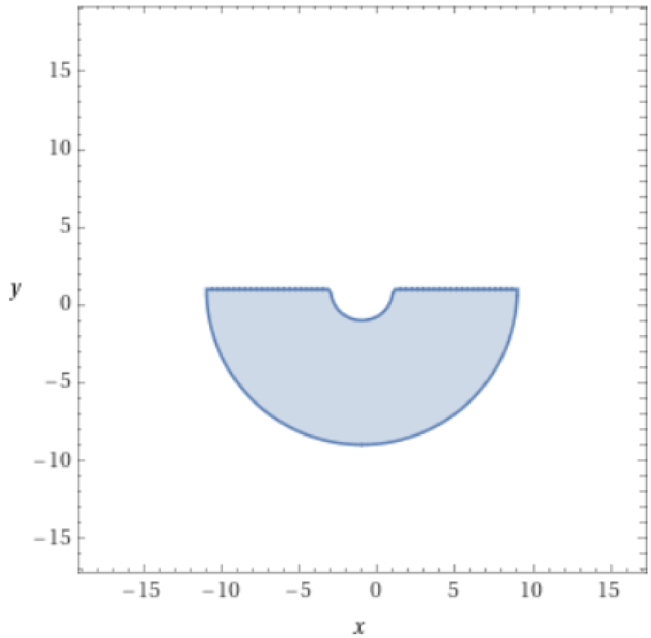

Figure 1: Solutions graph

Lab 1.2

This part of the lab is very computations heavy. Therefore some of the calculations will be performed using python (numpy, scipy).

Exercise 1-15

Are the given vectors: $\vec{v_1} = \begin{bmatrix} 1-i \\ -4-4i \\ -4-3i \\ 2-5i \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} -3-3i \\ -4+i \\ -2+2i \\ -3-4i \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} -3-i \\ -2+3i \\ -4-3i \\ -5-4i \end{bmatrix}$ linearly independent?

If the vectors are linearly independent, then the equation below is only true, when α_1, α_2, α_3 are all equal to 0:

$$\alpha_1 \begin{bmatrix} 1-i \\ -4-4i \\ -4-3i \\ 2-5i \end{bmatrix} + \alpha_2 \begin{bmatrix} -3-3i \\ -4+i \\ -2+2i \\ -3-4i \end{bmatrix} + \alpha_3 \begin{bmatrix} -3-i \\ -2+3i \\ -4-3i \\ -5-4i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 * (1-i) + \alpha_2 * (-3-3i) + \alpha_3 * (-3-i) \\ \alpha_1 * (-4-4i) + \alpha_2 * (-4+i) + \alpha_3 * (-2+3i) \\ \alpha_1 * (-4-3i) + \alpha_2 * (-2+2i) + \alpha_3 * (-4-3i) \\ \alpha_1 * (2-5i) + \alpha_2 * (-3-4i) + \alpha_3 * (-5-4i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & -3-3i & -3-i \\ -4-4i & -4+i & -2+3i \\ -4-3i & -2+2i & -4-3i \\ 2-5i & -3-4i & -5-4i \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can rewrite this and solve using Gaussian elimination method

$$\begin{bmatrix} 1-i & -3-3i & -3-i & |0 \\ -4-4i & -4+i & -2+3i & |0 \\ -4-3i & -2+2i & -4-3i & |0 \\ 2-5i & -3-4i & -5-4i & |0 \end{bmatrix}$$

1st Gaussian elimination step

$$m_2 = \frac{-4-4i}{1-i} = \frac{(-4-4i)*(1+i)}{1^2+1^2} = \frac{-4-4i-4i+4}{2} = -4i$$
$$m_3 = \frac{-4-3i}{1-i} = \frac{(-4-3i)*(1+i)}{2} = \frac{-4-4i-3i+3}{2} = -\frac{1+7i}{2}$$
$$m_4 = \frac{2-5i}{1-i} = \frac{(2-5i)*(1+i)}{2} = \frac{2+2i-5i+5}{2} = \frac{7-3i}{2}$$

- Row 1 scaling by m_2: $(3-3i)*-4i = -12i-12; (-3-i)*-4i = 12i-4$
- Row 2 subtraction: $-4+i-(-12i-12) = 8+13i; -2+3i-(12i-4) = 2-9i$
- Row 1 scaling by m_3: $(3-3i)*(-0,5-3,5i) = -1,5-10,5i+1,5i-10,5 = -12-9i; (-3-i)*(-0,5-3,5i) = -2+11i$
- Row 3 subtraction: $-2+2i-(-12-9i) = 10+11i; -4-3i-(-2+11i) = -2-14i$
- Row 1 scaling by m_4: $(-3-3i)*\frac{7-3i}{2} = -15-6i; (-3-i)*\frac{7-3i}{2} = -12+i$
- Row 4 subtraction: $-3-4i-(-15-6i) = 12+2i; -5-4i-(-12+i) = 7-5i$

Resulting matrix:

$$\begin{bmatrix} 1+i & 3-3i & -3-i & |0 \\ 0 & 8+13i & 2-9i & |0 \\ 0 & 10+11i & -2-14i & |0 \\ 0 & 12+2i & 7-5i & |0 \end{bmatrix}$$

2nd Gaussian elimination step

$$m_3 = \frac{10+11i}{8+13i} = \frac{223}{233} - \frac{42}{233}i$$
$$m_4 = \frac{12+2i}{8+13i} = \frac{122}{233} - \frac{140}{233}i$$

- Row 2 scaling by m_3:

  $(8+13i)*(\frac{223}{233} - \frac{42}{233}i) = 10+11i;$

  $(2-9i)*(\frac{223}{233} - \frac{42}{233}i) = \frac{68}{233} - \frac{2091}{233}i$

- Row 3 subtraction:

  $10+11i-(10+11i) = 0;$

  $-2-14i-(\frac{68}{233} - \frac{2091}{233}i) = -\frac{534}{233} - \frac{1171}{233}i$

$$\begin{bmatrix} 1+i & 3-3i & -3-i & |0 \\ 0 & 8+13i & 2-9i & |0 \\ 0 & 0 & -\frac{534}{233} - \frac{1171}{233}i & |0 \\ 0 & 12+2i & 7-5i & |0 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & 3-3i & -3-i & |0 \\ 0 & 8+13i & 2-9i & |0 \\ 0 & 0 & 1 & |0 \\ 0 & 12+2i & 7-5i & |0 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & 3-3i & -3-i & |0 \\ 0 & 8+13i & 0 & |0 \\ 0 & 0 & 1 & |0 \\ 0 & 12+2i & 7-5i & |0 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & 0 & 0 & |0 \\ 0 & 8+13i & 0 & |0 \\ 0 & 0 & 1 & |0 \\ 0 & 12+2i & 7-5i & |0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & |0 \\ 0 & 1 & 0 & |0 \\ 0 & 0 & 1 & |0 \\ 0 & 0 & 0 & |0 \end{bmatrix}$$

$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$, this means that the vectors are linearly independent.

Exercise 2-15

Find: $A^2 + 3B^\dagger C^{-2} + (A^{-1})^\dagger$.

$$\text{When: } A = \begin{bmatrix} -4+8i & 2+5i \\ -3+7i & 2+2i \end{bmatrix}, B = \begin{bmatrix} -3-6i & -7-i \\ 7-5i & -2+i \end{bmatrix}, C = \begin{bmatrix} 2-5i & -4-5i \\ -5+i & -9-4i \end{bmatrix}$$

We will first find A^{-1}

$$\begin{bmatrix} -4+8i & 2+5i & | & 1 & 0 \\ -3+7i & 2+2i & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{(2+5i)}{(-4+8i)} & | & \frac{1}{(-4+8i)} & 0 \\ 0 & (2+2i) - (\frac{(2+5i)}{(-4+8i)} * (-3+7i)) & | & \frac{-3+7i}{(4-8i)} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{(2+5i)}{(-4+8i)} & | & \frac{1}{(-4+8i)} & 0 \\ 0 & \frac{1}{20} - \frac{43}{20}i & | & \frac{-3+7i}{(4-8i)} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{(2+5i)}{(-4+8i)} & | & \frac{1}{(-4+8i)} & 0 \\ 0 & 1 & | & \frac{3-7i}{17+9i} & \frac{1}{\frac{1}{20} - \frac{43}{20}i} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{-2+14i}{15+95i} & -((\frac{1}{\frac{1}{20} - \frac{43}{20}i}) * (\frac{(2+5i)}{(-4+8i)})) \\ 0 & 1 & | & \frac{3-7i}{17+9i} & \frac{1}{\frac{1}{20} - \frac{43}{20}i} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{26+8i}{185} & -\frac{79}{370} - \frac{67}{370}i \\ 0 & 1 & | & \frac{-6-73i}{185} & \frac{2}{185} + \frac{86}{185}i \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{26}{185} + \frac{8}{185}i & -\frac{79}{370} - \frac{67}{370}i \\ -\frac{6}{185} - \frac{73}{185}i & \frac{2}{185} + \frac{86}{185}i \end{bmatrix}$$

$$(A^{-1})^\dagger = \begin{bmatrix} \frac{26}{185} + \frac{8}{185}i & -\frac{6}{185} + \frac{73}{185}i \\ -\frac{79}{370} + \frac{67}{370}i & \frac{2}{185} + \frac{86}{185}i \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -89 - 65i & -54 + 10i \\ -64 - 44i & -41 + 7i \end{bmatrix}$$

$$C^{-2} = C^{-1}C^{-1} = \begin{bmatrix} \frac{503}{4425} + \frac{396}{4425}i & -\frac{172}{4425} - \frac{379}{4225}i \\ -\frac{331}{4225} - \frac{17}{4225}i & -\frac{206}{4225} + \frac{283}{4225}i \end{bmatrix} \begin{bmatrix} \frac{503}{4425} + \frac{396}{4425}i & -\frac{172}{4425} - \frac{379}{4225}i \\ -\frac{331}{4225} - \frac{17}{4225}i & -\frac{206}{4225} + \frac{283}{4225}i \end{bmatrix}$$

$$C^{-2} = \begin{bmatrix} \frac{146682}{17850625} + \frac{526749}{17850625}i & \frac{206257}{17850625} - \frac{229351i}{17850625}i \\ -\frac{86764}{17850625} - \frac{229798}{17850625}i & \frac{12836}{17850625} + \frac{11777}{17850625}i \end{bmatrix}$$

$$B^\dagger = \begin{bmatrix} -3 - 6i & 7 + 5i \\ -7 + i & -2 + i \end{bmatrix}$$

$$3B^\dagger = \begin{bmatrix} -9 - 18i & 21 + 15i \\ -21 + 3i & -6 + 3i \end{bmatrix}$$

So, putting $A^2, 3B^\dagger, C^{-2}, (A^{-1})^\dagger$ into SciPy, we get this matrix:

$$\begin{bmatrix} -88.31 - 65.71i & -54.36 + 10.33i \\ -64.41 - 44.35i & -41.20 + 7.77i \end{bmatrix}$$

Exercise 3-15

Rewrite $\vec{v_4}$ using $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$.

$$\begin{bmatrix} 2 - 4i & 2 - 5i & 2 - 3i \\ -1 - 3i & -2 - i & 4 - 4i \\ -4 - 4i & -1 + 3i & 1 - 5i \end{bmatrix} \cdot \begin{bmatrix} v_{41} \\ v_{42} \\ v_{43} \end{bmatrix} = \begin{bmatrix} -2 - 4i \\ 1 + i \\ 3 + 4i \end{bmatrix}$$

Multiplying the $\vec{4}$ by the inverse matrix will give us the answer. The inverse, calculated using SciPy:

$$\begin{bmatrix} 0.06 + 0.095i & -0.083 - 0.247i & -0.089 + 0.201i \\ 0.084 + 0.07i & -0.078 + 0.095i & 0.094 - 0.109i \\ -0.078 + 0.046i & 0.224 + 0.128i & -0.026 - 0.073i \end{bmatrix}$$

These are the scalars that make $\vec{v_4}$ by scaling: $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

$$\begin{bmatrix} -0.6449302 - 0.51285819i \\ 0.65742101 - 0.40650257i \\ 0.65227774 + 0.25146951i \end{bmatrix}$$

Exercise 4-15

Find an orthonormal basis for the given vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 2-4i \\ -1-3i \\ -4-4i \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2-5i \\ -2-i \\ -1+3i \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2-3i \\ 4-4i \\ 1-5i \end{bmatrix}$$

We can find an orthogonal basis for linearly independent vectors using Gram-Schmidt process and then normalize it.

$$\begin{aligned}
\vec{u_1} &= \vec{v_1}, \\
\vec{u_2} &= \vec{v_2} - \frac{<\vec{v_2}|\vec{u_1}>}{<\vec{u_1}|\vec{u_1}>}\vec{u_1}, \\
\vec{u_3} &= \vec{v_3} - \frac{<\vec{v_3}|\vec{u_1}>}{<\vec{u_1}|\vec{u_1}>}\vec{u_1} - \frac{<\vec{v_3}|\vec{u_2}>}{<\vec{u_2}|\vec{u_2}>}\vec{u_2}.
\end{aligned}$$

The calculations will be done using SciPy. End result is:

$$\vec{u_1} = \begin{bmatrix} \frac{\sqrt{62}\cdot(1-2i)}{31} \\ \frac{\sqrt{62}\cdot(-1-3i)}{62} \\ \frac{2\cdot\sqrt{62}\cdot(-1-i)}{31} \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} \frac{\sqrt{60047}\cdot(-5-136i)}{60047} \\ \frac{2\cdot\sqrt{60047}\cdot(-43+6i)}{60047} \\ \frac{\sqrt{60047}\cdot(-35+181i)}{60047} \end{bmatrix} \quad \vec{u_3} = \begin{bmatrix} \frac{\sqrt{1811821226}\cdot(139562-36868i)}{11776837969} \\ \frac{\sqrt{1811821226}\cdot(2-3i)\cdot(95123+41155i)}{23553675938} \\ \frac{\sqrt{1811821226}\cdot(120294-79562i)}{11776837969} \end{bmatrix}$$

Exercise 5-15

Find the eigenvectors and eigenvalues of matrix A:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 2i & i & 1 \\ 1 & 0 & i \end{pmatrix}$$

Eigenvectors are vectors that stay in the same span after the matrix transformation is applied:

$$\begin{aligned}
&\mathbf{A}\vec{v} = \lambda\vec{v} \\
&\mathbf{A}\vec{v} = (\lambda\mathbf{I})\vec{v} \\
&\mathbf{A}\vec{v} - (\lambda\mathbf{I})\vec{v} = \vec{0} \\
&(\mathbf{A} - (\lambda\mathbf{I}))\cdot\vec{v} = \vec{0}
\end{aligned}$$

Because we are only looking for non-zero vectors, we will look for values of λ that can make a zero vector, which means that we are looking for matrix that squishes space:

$$det\left(\begin{pmatrix} 1-\lambda & 0 & -1 \\ 2i & i-\lambda & 1 \\ 1 & 0 & i-\lambda \end{pmatrix}\right) = 0$$

We expand the determinant:

$$det\left(\begin{pmatrix} 1-\lambda & 0 & -1 \\ 2i & i-\lambda & 1 \\ 1 & 0 & i-\lambda \end{pmatrix}\right) = (1-\lambda) * det\left(\begin{pmatrix} i-\lambda & 1 \\ 0 & i-\lambda \end{pmatrix}\right) - det\left(\begin{pmatrix} 2i & i-\lambda \\ 1 & 0 \end{pmatrix}\right) =$$

$$= (1-\lambda) * ((i-\lambda) * (i-\lambda)) - (-(i-\lambda)) = (1-\lambda) * (i-\lambda)^2 + (i-\lambda)$$

So:

$$(1-\lambda) * (i-\lambda)^2 + (i-\lambda) = 0$$

$$(i-\lambda) * ((1-\lambda) * (i-\lambda) + 1) = 0$$

So, $(i-\lambda) = 0$ or $(1-\lambda) * (i-\lambda) + 1 = 0$, if $(i-\lambda) = 0$, then $\lambda_1 = i$ else:

$$(1-\lambda) * (i-\lambda) + 1 = 0$$

$$i - \lambda - \lambda i + \lambda^2 + 1 = 0$$

$$\lambda^2 - (1+i)\lambda + 1 + i = 0$$

$$D = (1+i)^2 - 4(1+i) = 1 + 2i - 1 - 4 - 4i = -4 - 2i$$

$$\lambda = \frac{1 + i \pm \sqrt{-4-2i}}{2}$$

$$\sqrt{-4-2i} = 2\sqrt{5}e^{(\arctan(0,5)-\pi)i}$$

$$\lambda_2 = \frac{1 + i + 2\sqrt{5}e^{(\arctan(0,5)-\pi)i}}{2} \approx 0,743 - 0,53i$$

$$\lambda_3 = \frac{1 + i - 2\sqrt{5}e^{(\arctan(0,5)-\pi)i}}{2} \approx 0,257 + 1,53i$$

Previously calculated: $\lambda_1 = i$

We can calculate an eigen vector for each of the eigen values: $(\mathbf{A} - (\lambda \mathbf{I})) \cdot \vec{v} = \vec{0}$

$$\begin{pmatrix} 1-\lambda & 0 & -1 \\ 2i & i-\lambda & 1 \\ 1 & 0 & i-\lambda \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Calculating for λ_1:

$$\begin{pmatrix} 1-i & 0 & -1 \\ 2i & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (1-i)x - z \\ 2ix + z \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = 0$$

$$-z = 0$$

$$z = 0$$

y is a free variable, which means it can take on any value. So one set of eigen vectors is: $\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$, where $y \in R$, this set of eigen vectors gets scaled by i. The other two eigen vectors were calculated using SciPy:

$$\lambda_2 = 0.743 - 0.529\, i$$

$$\mathbf{v}_2 = \begin{pmatrix} -0.449 - 0.278\, i \\ 0.790 + 0\, i \\ 0.031 - 0.309\, i \end{pmatrix}$$

$$\lambda_3 = 0.257 + 1.529\, i$$

$$\mathbf{v}_3 = \begin{pmatrix} 0.177 + 0.363\, i \\ 0.510 + 0.323\, i \\ 0.687 + 0\, i \end{pmatrix}$$

Exercise 6-5

$$\text{Given: } \mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 2i & i & 1 \\ 1 & 0 & i \end{pmatrix} \text{ and } \mathbf{Z} = \begin{pmatrix} -2i & 2 \\ -1 & 2i \end{pmatrix}$$

Find:

- A) $\mathbf{X}$, so that $\mathbf{AX}$ would be a non-diagonal Hermitian matrix.
- B) $\mathbf{Y}$, so that $\mathbf{AY}$ would be a non-diagonal Unitary matrix.
- C) $\mathbf{Z} \bigotimes \mathbf{A}$ and $\mathbf{A} \bigotimes \mathbf{Z}$

A)

The Hermitian matrix condition is: $\mathbf{AX} = (\mathbf{AX})^\dagger$. We will pick a random matrix that meets this condition:

$$\mathbf{AX} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1-i \\ 0 & 1+i & 0 \end{pmatrix}$$

Since this new random matrix is the product of applying transformation $\mathbf{A}$ onto $\mathbf{X}$, that means that: $\mathbf{X} = \mathbf{A}^{-1}(\mathbf{AX})$

$$\mathbf{A}^{-1} = \begin{pmatrix} 0.5+0.5i & 0 & 0.5-0.5i \\ -1.5-1.5i & -i & -0.5+1.5i \\ -0.5+0.5i & 0 & 0.5-0.5i \end{pmatrix}$$

$$\mathbf{X} = \mathbf{A}^{-1}(\mathbf{AX}) = \begin{pmatrix} 0.5+0.5i & 0 & 0.5-0.5i \\ -1.5-1.5i & -i & -0.5+1.5i \\ -0.5+0.5i & 0 & 0.5-0.5i \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1-i \\ 0 & 1+i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & -1-i \\ 0 & 1 & 0 \end{pmatrix}$$

B)

The Unitarian matrix condition: $\mathbf{UU}^\dagger = \mathbf{I}$. If $\mathbf{AY} = \mathbf{U}$, then: $\mathbf{Y} = \mathbf{A}^{-1}\mathbf{U}$

Here is a random Unitary matrix:

$$\mathbf{U} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Y} = \mathbf{A}^{-1}\mathbf{U} = \begin{pmatrix} 0 & 0.5+0.5i & 0.5-0.5i \\ -i & -1.5-1.5i & -0.5+1.5i \\ 0 & -0.5+0.5i & 0.5-0.5i \end{pmatrix}$$

C)

This exercise will be fully computed using SciPy:

$$\mathbf{Z} \bigotimes \mathbf{A} = \begin{bmatrix} -2i & 0 & 2i & 2 & 0 & -2 \\ 4 & 2 & -2i & 4i & 2i & 2 \\ -2i & 0 & 2 & 2 & 0 & 2i \\ -1 & 0 & 1 & 2i & 0 & -2i \\ -2i & -i & -1 & -4 & -2 & 2i \\ -1 & 0 & -i & 2i & 0 & -2 \end{bmatrix}$$

$$\mathbf{A} \bigotimes \mathbf{Z} = \begin{bmatrix} -2i & 2 & 0 & 0 & 2i & -2 \\ -1 & 2i & 0 & 0 & 1 & -2i \\ 4 & 4i & 2 & 2i & -2i & 2 \\ -2i & -4 & -i & -2 & -1 & 2i \\ -2i & 2 & 0 & 0 & 2 & 2i \\ -1 & 2i & 0 & 0 & -i & -2 \end{bmatrix}$$