Applied Soft Computing 40 (2016) 98–112

Contents lists available at ScienceDirect

Applied Soft Computing

journal homepage: www.elsevier.com/locate/asoc

A novel risk attitudinal ranking method for intuitionistic fuzzy values and application to MADM

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ARTICLE INFO

Article history:
Received 31 January 2015
Received in revised form
12 November 2015
Accepted 13 November 2015
Available online 28 November 2015

Keywords:
Intuitionistic fuzzy sets
Multi-attribute decision making
Technique for order preference by similarity to ideal solution
Fractional programming
Risk attitude

ABSTRACT

The ranking of intuitionistic fuzzy sets (IFSs) is very important for the intuitionistic fuzzy decision making. The aim of this paper is to propose a new risk attitudinal ranking method of IFSs and apply to multi-attribute decision making (MADM) with incomplete weight information. Motivated by technique for order preference by similarity to ideal solution (TOPSIS), we utilize the closeness degree to characterize the amount of information according to the geometrical representation of an IFS. The area of triangle is calculated to measure the reliability of information. It is proved that the closeness degree and the triangle area just form an interval. Thereby, a new lexicographical method is proposed based on the intervals for ranking the intuitionistic fuzzy values (IFVs). Furthermore, considered the risk attitude of decision maker sufficiently, a novel risk attitudinal ranking measure is developed to rank the IFVs on the basis of the continuous ordered weighted average (C-OWA) operator and this interval. Through maximizing the closeness degrees of alternatives, we construct a multi-objective fractional programming model which is transformed into a linear program. Thus, the attribute weights are derived objectively by solving this linear program. Then, a new method is put forward for MADM with IFVs and incomplete weight information. Finally, an example analysis of a teacher selection is given to verify the effectiveness and practicability of the proposed method.

1. Introduction

Fuzzy sets (FSs) proposed by Zadeh [1] can be used to express the uncertain information. Due to the influence of subjective factors, the decision maker (DM) sometimes relies on intuition and experience to evaluate the information in real-life decision problems. Thus, it is often that DM exhibits some hesitation degrees for the assessments. Atanassov [2] proposed the intuitionistic fuzzy sets (IFSs) which assign the membership degree, non-membership degree and the degree of hesitation to each element. It is just a suitable tool to cope with the vagueness and hesitancy originating from imprecise knowledge or information. Since IFS simultaneously contains membership and non-membership degrees, it is more flexible and practical than FSs in dealing with ambiguity and uncertainty [3–10].

IFS has been widely applied in the multi-attribute decision making (MADM) and multi-attribute group decision making (MAGDM) [11–13]. In these applications of IFSs, how to give the order relationship of intuitionistic fuzzy values (IFVs) is a critical issue. In 1999, Atanassov [14] proposed an Atanassov's order of IFVs. Atanassov's order can be seen as a natural order, namely, it is the foundation of other ranking methods. Subsequently, lots of works have done in the topics of ranking IFVs and measuring the entropy or amount of knowledge conveyed by IFVs in recent years. The existing achievements can be roughly divided into two classes.

The first class is to use the combinatorial calculations on the two or three functions (membership function, non-membership function, and hesitation margin) [15–19]. Chen and Tan [15] paid attention to the differences between the membership function and non-membership function and then presented a score function to evaluate the degree of score of IFVs. Wu and Chiclana [16] defined a new score function and its value is between 0 and 1. The new score function and the score function in [15] are ordering mathematically equivalent in that they will lead to the same ordering

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http://dx.doi.org/10.1016/j.asoc.2015.11.022
1568-4946/

of alternatives when using the same ordering rule. However, the score function cannot distinguish many IFVs. Hong and Choi [17] proposed an accuracy function to evaluate the degree of accuracy of IFVs by summing the membership function and non-membership function. Xu [18] gave an algorithm to rank IFVs by combining the score function and accuracy function. Then Liu and Wang [19] suggested a new score function by means of the intuitionistic fuzzy point operator. However, neither there had basis for this allegation, nor they gave an in-depth analysis of it.

The second class is to use the geometrical representation [20–29]. It has been proved that the representation is effective to bring about intuitively appealing results in solving many intuitionistic fuzzy problems, such as distance [20–24] and entropy [20,22,24–26,28,29]. Szmidt and Kacprzyk [20] introduced a measure for ranking IFVs by considering the amount and reliability of information related to an IFS. Guo and Li [21] applied the ranking measure in [20] to an attitudinal-based intuitionistic fuzzy decision model. Guo [22] pointed out that Szmidt and Kacprzyk's order [20] still lead to undesirable results to some extent. Then Guo [22] developed a new technique for ranking IFVs based on amount of information, and then extended to the attitudinal-based technique which takes the DM's attitude into account. Ouyang and Pedrycz [23] combined Atanassov's order [14] and Szmidt and Kacprzyk's order [20] to propose a new admissible order for IFVs, which is proved to a lexicographical one. Chen et al. [24] proposed a score function for IFVs, which considers the distance from the positive and negative ideal IFVs and DM's attitude. Szmidt et al. [25] concerned the intrinsic relationship between the positive and negative information and the hesitation margin to express the amount of knowledge conveyed by IFVs. Pal et al. [26] presented new axiomatic characterizations of the non-probabilistic entropy measures associated with an IFS. Szmidt and Kacprzyk [27] investigated a non-probabilistic-type entropy measure for IFSs, which is a result of a geometric interpretation of IFSs and uses a ratio of distances. Szmidt and Kacprzyk [28] continued the previous paper to study the entropy of IFVs by examining their differences. Szmidt and Kacprzyk [29] generalized the entropy of IFSs by considering all three functions. One important case occurring in the above is that the hesitation margin, as one of the three functions, is a key role in determining the amount of information [21–29].

The aforementioned methods seem to be effective to rank IFVs. However, they have some disadvantages as follows:

(1) The methods [15–19] utilized the traditional tools involving the membership function and non-membership function, which sometimes may suffer from counterintuitive ranking results or the unobtainable results.
(2) Although the papers [20–23,25,27,29] used the distance to define the measures for ranking the IFVs, they only considered the distance with the positive ideal point $M(1, 0, 0)$ and neglected the negative ideal point $N(0, 1, 0)$.
(3) The methods [25–29] measured the uncertainty and the amount of information with entropy which cannot distinguish one IFV x and its corresponding complement x^c. For example, for a given IFV x, the measure $K(x)$ (see Eq. (9) in Section 2.2) is equal to $K(x^c)$, that is to say, x and x^c have the identical uncertainty. Hence, such a measure cannot be used to rank the IFVs.

Especially, in real-life decision problems, different DMs have diverse attitudes towards risk. It is necessary and natural to incorporate DM' risk preference into the ranking method, which is useful and flexible in real-world applications. Yager [30] introduced the continuous ordered weighted average (C-OWA) operator in which the weights can imply DM's attitude. Then the C-OWA operator is applied to some ranking methods. Guo [22] discussed the role that DM's attitude can play in decision making under uncertainty and proposed an attitudinal-based ranking technique. Wu and Chiclana [31] defined an attitudinal expected score function for interval-valued IFVs. Jin et al. [32] developed an interval-valued intuitionistic fuzzy continuous weighted entropy based on the C-OWA operator.

Therefore, to overcome these disadvantages, new measurements of information amount and reliability of IFVs are devised and a new lexicographic method is proposed to rank IFVs in this paper. Then we further develop a risk attitudinal measure for ranking IFVs considering the DM's risk attitude. Finally, a novel method is proposed for MADM with IFVs and incomplete weight information. The main contributions of this work are summarized as three aspects:

(1) New measures of information amount and reliability of IFVs are designed from the angle of geometric meaning. These new measures consider the positive and negative ideal points and the area of triangle jointly. It is proved that the closeness degree and the area just form an interval. Thereby, a new lexicographic method is developed to rank IFVs, which is very simple and effective.
(2) Considering that the closeness degree and the area form an interval, we further define a novel risk attitudinal measure for ranking IFVs based on the C-OWA operator. Thus, this risk attitudinal measure is more theoretically reasonable and convincing than that defined in Guo [22].
(3) A new approach to determining the attribute weights objectively is presented through constructing multi-objective fractional programming model. It is dexterously converted into linear programming to resolve by the Charnes and Cooper transformation.

The remainder of this paper unfolds as follows. In Section 2, some concepts of IFSs are recalled and some typical ranking methods are reviewed. Section 3 defines new measures of information amount and reliability of IFVs according to geometrical representation and then proposes a new lexicographic method for ranking IFVs. Furthermore, a new risk attitudinal measure is defined to ranking IFVs in Section 4. In Section 5, a new method is proposed for MADM with IFVs and incomplete weight information. A teacher selection example and the comparison analyses are given in Section 6. Finally, concluding remarks are made in Section 7.

2. Intuitionistic fuzzy sets

In this section, we recall some basic concepts of IFSs, including the definition, operation laws and Hamming distances. In the meantime, some existing ranking methods of IFVs are reviewed.

2.1. Preliminaries of intuitionistic fuzzy sets

Definition 1 *([2])*. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed non-empty universe set, an IFS A in X is defined as $A = \{<x, \mu_A(x), \nu_A(x)> | x \in X\}$, which is characterized by a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$ with the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all $x \in X$ where μ_A and ν_A represent, respectively, the membership and non-membership degrees of the element x to the set A. In addition, for each IFS A in X, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$. $\pi_A(x)$ is denoted the hesitation degree (hesitation margin) of the element x to the set A. Especially, if $\pi_A(x) = 0$, then the IFS A is degraded to a FS.

The complement of an IFS A is defined as $A^c = \{<x, \nu_A(x), \mu_A(x)> | x \in X\}$. For an IFS A, the pair $<\mu_A(x), \nu_A(x)>$ is called an IFV [33]. Denote the set of all IFVs by Ω.

Definition 2 *([18,33])*. Let $\tilde{a}_i = < \mu_i, \nu_i >$ $(i = 1, 2)$ be two IFVs and $k > 0$. Then the operations and relations for IFVs are defined as follows:

(1) $\tilde{a}_1 + \tilde{a}_2 = \langle \mu_1 + \mu_2 - \mu_1\mu_2, \quad \nu_1\nu_2 \rangle$;

(2) $k\tilde{a}_1 = \langle 1 - (1 - \mu_1)^k, \quad \nu_1^k \rangle$;

(3) The complement set $\tilde{a}_1^c = \langle \nu_1, \quad \mu_1 \rangle$;

(4) The containment $\tilde{a}_1 \subseteq \tilde{a}_2$ if and only $\mu_1 \le \mu_2, \nu_1 \ge \nu_2$.

Definition 3 *([34])*. Let $A = \{\langle x, \ \mu_A(x), \ \nu_A(x) \rangle \,|\, x \in X\}$ and $B = \{\langle x, \ \mu_B(x), \ \nu_B(x) \rangle \,|\, x \in X\}$ be two IFSs. The Hamming distance between A and B in $X = \{x_1, x_2, \ldots, x_n\}$ is defined as follows:

$$d_{IFS}(A, B) = \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|), \tag{1}$$

where $\pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i)$ and $\pi_B(x_i) = 1 - \mu_B(x_i) - \nu_B(x_i)$.

Definition 4 *([18])*. Assume that $\tilde{a}_i = \langle \mu_i, \nu_i \rangle$ $(i = 1, 2, \ldots, n)$ is a collection of IFVs. Let IFWA: $\Omega^n \to \Omega$, if

$$\text{IFWA}_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{i=1}^{n} w_i \tilde{a}_i, \tag{2}$$

where $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $\tilde{a}_i$ $(i = 1, 2, \ldots, n)$, satisfying $0 \le w_i \le 1$ and $\sum_{i=1}^{n} w_i = 1$, the function IFWA is called the intuitionistic fuzzy weighted average operator.

According to Definitions 2 and 4, we have

$$\text{IFWA}_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\langle 1 - \prod_{i=1}^{n}(1 - \mu_i)^{w_i}, \quad \prod_{i=1}^{n} \nu_i^{w_i} \right\rangle. \tag{3}$$

2.2. Existing ranking methods of IFVs

How to give the order relationship of IFVs is a critical problem for ranking IFVs. There are many researches on this topic [18,20,22,25]. We firstly recall some typical works. For the convenience of ranking IFVs, we sometimes denote IFV $\tilde{a}_i = \langle \mu_i, \nu_i \rangle$ by $\tilde{a}_i = \langle \mu_i, \nu_i, \pi_i \rangle$ below.

(1) Score and accuracy functions based method

Chen and Tan [15] defined a score function of IFV $\tilde{a}_i = \langle \mu_i, \nu_i \rangle$ as

$$S(\tilde{a}_i) = \mu_i - \nu_i. \tag{4}$$

Hong and Choi [17] defined an accuracy function of IFV $\tilde{a}_i = \langle \mu_i, \nu_i \rangle$ as

$$H(\tilde{a}_i) = \mu_i + \nu_i. \tag{5}$$

Xu [18] gave an algorithm to rank IFVs based on the score and accuracy functions. For two IFVs $\tilde{a}_i = \langle \mu_i, \nu_i \rangle$ $(i = 1, 2)$, the order relations are as follows:

1) If $S(\tilde{a}_1) < S(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$.
2) If $S(\tilde{a}_1) = S(\tilde{a}_2)$, then

(i) if $H(\tilde{a}_1) < H(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, i.e. $\tilde{a}_1 < \tilde{a}_2$;
(ii) if $H(\tilde{a}_1) = H(\tilde{a}_2)$, then $\tilde{a}_1$ is equal to $\tilde{a}_2$, denoted by $\tilde{a}_1 = \tilde{a}_2$.

The above method suffers from the flaw of robustness as the following example explains.

Example 1. Consider three IFVs $\tilde{a}_1 = \langle 0.3, 0.2, 0.5 \rangle$, $\tilde{a}_2 = \langle 0.5, 0.4, 0.1 \rangle$ and $\tilde{a}_3 = \langle 0.3001, 0.2, 0.4999 \rangle$.

By Eqs. (4) and (5), we obtain $S(\tilde{a}_1) = S(\tilde{a}_2) = 0.1$, $H(\tilde{a}_1) = 0.5 < H(\tilde{a}_2) = 0.9$. So, $\tilde{a}_1$ is smaller than $\tilde{a}_2$. Because $S(\tilde{a}_2) < S(\tilde{a}_3) = 0.1001$, we are forced to conclude that $\tilde{a}_2$ is smaller than $\tilde{a}_3$. However, $\tilde{a}_3$ is just a bit modification of $\tilde{a}_1$ through adding lest perturbation 0.0001 on the membership degree, which results in the converse ranking order. Therefore, the method based on score and accuracy functions lacks robustness.

(2) Szmidt and Kacprzyk's method

Szmidt and Kacprzyk [20] proposed a new measure for ranking the alternatives $x \in$ IFS:

$$R(x) = 0.5\left(1 + \frac{1}{2}\pi(x)\right) d_{IFS}(M, x), \tag{6}$$

where $d_{IFS}(M, x)$ is the distance x from the positive ideal point $M(1, 0, 0)$.

The method [20] used the hesitation margin π and distance to define the measure for ranking the IFVs. Nevertheless, Eq. (6) only considered the distance with the positive ideal point $M(1, 0, 0)$. It ignored the negative ideal point $N(0, 1, 0)$. According to the technique for order preference by similarity to ideal solution (TOPSIS), the negative ideal point $N(0, 1, 0)$ is as important as the positive ideal point $M(1, 0, 0)$ in the process of decision making. Additionally, Guo [22] pointed out that Eq. (6) still lead to undesirable results to some extent.

Example 2. Consider two IFVs $\tilde{a}_1 = \langle 0.7, 0.1, 0.2 \rangle$ and $\tilde{a}_2 = \langle 0.6, 0.3, 0.1 \rangle$. By Eq. (6), we get $R(\tilde{a}_1) = 0.165$ and $R(\tilde{a}_2) = 0.210$. Thus, $\tilde{a}_2$ is better than $\tilde{a}_1$ by using Szmidt and Kacprzyk's method.

However, we should also have in mind that there is Atanassov's order [14] of IFVs which can be seen as a natural order, namely, for two IFVs $\tilde{a}_i = (\mu_i, \nu_i)$ and $\tilde{a}_j = (\mu_j, \nu_j)$, if $\mu_i > \mu_j$ and $\nu_i < \nu_j$, $\tilde{a}_i$ is "better" than $\tilde{a}_j$. In view of Atanassov's order, it is more reasonable that $\tilde{a}_1$ is better than $\tilde{a}_2$ since $\mu_1 > \mu_2$ and $\nu_1 < \nu_2$ in Example 2. Obviously, Szmidt and Kacprzyk's method [20] is not admissible, thus it needs further improvement [22]. We have such a situation in Example 2 so applying other methods of ranking is not necessary. A method of ranking IFVs should work reasonably while the conditions of Atanassov's order are not fulfilled.

(3) Guo's method

Guo [22] presented a measure to rank IFVs represented by x as follows:

$$Z(x) = \left(1 - \frac{1}{2}\pi(x)\right)\left(\mu(x) + \frac{1}{2}\pi(x)\right). \tag{7}$$

Then the further extended the measure by considering decision attitude as follows:

$$Z_Q(x) = \left(1 - \frac{t}{t+1}\pi(x)\right)\left(\mu(x) + \frac{1}{t+1}\pi(x)\right), \tag{8}$$

where the parameter t denotes the risk attitude of DM and $t > 0$.

Although Guo's method [22] is an effective method to rank IFVs, similar to Szmidt and Kacprzyk's method, Guo [22] also only considered the distance with $M(1, 0, 0)$. Moreover, the another deficiency of Guo [22] is that the process of transforming Eqs. (7) and (8) is somewhat farfetched since the Eq. (8) is obtained by directly replacing the two fractions 1/2 in Eq. (7) by $t/(t+1)$ and $1/(t+1)$, respectively. Unfortunately, there is no basis for this transforming and he did not give a more in depth.

(4) Szmidt et al.'s method

Szmidt et al. [25] defined the measure of amount of knowledge connected to a separate element $x \in$ IFS as follows:

$$K(x) = 1 - \frac{1}{2}(E(x) + \pi(x)), \quad (9)$$

where $E(x)$ is the entropy measure given by $E(x) = a/b$ with the condition that a is the distance (x, x_{near}) from x to the nearer element x_{near} between the elements: $M(1, 0, 0)$ and $N(0, 1, 0)$, and b is the distance (x, x_{far}) from x to the further element x_{far} between $M(1, 0, 0)$ and $N(0, 1, 0)$.

Szmidt et al. [25] not only considered the hesitation margin π, but also took the distance between the IFV x and $M(1, 0, 0)$ and $N(0, 1, 0)$ into account. However, the measure $K(x)$ is merely used to measure the uncertainty and the amount of knowledge conveyed for IFVs and cannot be used to rank the IFVs. For an given IFV x, $K(x) = K(x^c)$, that is to say that x and x^c have the identical uncertainty. We cannot rank x and x^c by using $K(x)$ or $K(x^c)$.

3. New measures of information amount and reliability for ranking IFVs

This section defines new measures of information amount and reliability of IFVs and then proposes a new lexicographic method for ranking IFVs.

3.1. New measures of information amount and reliability of IFVs

For each element x belonging to an IFS A, the values of membership $\mu_A(x)$, non-membership $\nu_A(x)$ and the degree of hesitation $\pi_A(x)$ satisfy $\mu_A(x), \nu_A(x), \pi_A(x) \in [0, 1]$ and $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$.

According to literature [20,35], the geometrical representation of an IFS can be illustrated in two dimension (2D) shown in Fig. 1. Although we use a 2D figure, all the functions (membership, non-membership and hesitation) are taken into account. Furthermore, any element belonging to an IFS can be represented by a point inside the triangle MON (denoted by ΔMON). Each point belonging to the ΔMON is described by the three coordinates: $(\mu_A(x), \nu_A(x), \pi_A(x))$. Both point $M = (1, 0, 0)$ and $N = (0, 1, 0)$ are crisp elements. In other words, M represents elements fully belonging to an IFS as $\mu = 1$, which is regarded as the positive ideal point (element, IFV). Analogously, N represents elements fully not belonging to an IFS as $\nu = 1$, which is regarded as the negative ideal point. Point $O = (0, 0, 1)$ represents elements about which we are unable to say whether they belong to an IFS as $\pi = 1$, which is regarded as the fuzzy point. Segment MN represents the elements belonging to the fuzzy sets as $\mu + \nu = 1$ and $\pi = 0$. The line parallel to MN describes the elements with the same hesitation degree. Such an interpretation is intuitively appealing and provides means for the representation of many aspects of imperfect information.

Let us analyze the sense of ranking for alternative with an IFV (expressed by a point in ΔMON) using the operators [35] as follows:

$$D_\alpha(A) = \{< x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1 - \alpha)\pi_A(x) > | x \in X\},$$

$$F_{\alpha,\beta}(A) = \{< x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x) > | x \in X\},$$

where $A \in$ IFS(X), $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.

From Fig. 1, it can be observed that, for any point $y = (\mu_A(y), \nu_A(y), \pi_A(y))$ in ΔMON, we can find the point $D_\alpha(y) = y_{\min}$ where $\alpha = 0$ and $y_{\min} = (\mu_A(y), \nu_A(y) + \pi_A(y), 0)$. At the same time, the point $D_\alpha(y) = y_{\max}$ where $\alpha = 1$ and $y_{\max} = (\mu_A(y) + \pi_A(y), \nu_A(y), 0)$ is obtained. The operator $F_{\alpha,\beta}$ makes it possible for y to become any point represented in the triangle $y_{\min}yy_{\max}$ (denoted by $\Delta y_{\min}yy_{\max}$). Especially, point $O = (0, 0, 1)$ may become any point in the whole area of the ΔMON by $F_{\alpha,\beta}$. Thus, we can say that the smaller the area of $\Delta y_{\min}yy_{\max}$ (denoted by $S_{\Delta y_{\min}yy_{\max}} = (1/2)\pi_A(y)^2$), the better the point y can be ranked. Unfortunately, the areas of those points on the same line parallel to MN are identical because they have the same hesitation degree. Especially, the points on the segment MN are easier to rank as $S_{\Delta y_{\min}yy_{\max}} = 0$. Nevertheless, it is illogical that the points on the segment MN have equal order for DMs. To see the above special case more clearly, we give the Fig. 2 to depict the function $S_{\Delta y_{\min}yy_{\max}} = (1/2)\pi_A(y)^2$, where the x-axis represents the degree of membership, the y-axis represents the degree of non-membership and the z-axis represents the value of $S_{\Delta y_{\min}yy_{\max}}$.

Afterwards, how to rank the points on the same line parallel to MN is an essential problem. By TOPSIS, the closer the point y to M and at the same time the farther the point y to N, the better the point y. Thus, the closeness degree of the point y is given as follows [36]:

$$C(y) = \frac{d(y, N)}{d(y, M) + d(y, N)} \quad (10)$$

where $d(y, N)$ is the Hamming distance between the point y and $N(0, 1, 0)$, $d(y, M)$ is the Hamming distance between the point y and $M(1, 0, 0)$.

By Eq. (1), the closeness degree of the point y is calculated as follows:

$$C(y) = \frac{1 - \nu(y)}{1 + \pi(y)}. \quad (11)$$

To rank IFSs, Szmidt and Kacprzyk [20] proposed two new notions which are called the amount of information and the reliability of information, respectively. The amount of information is

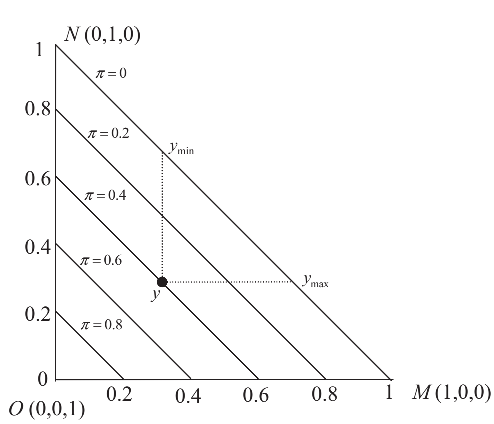

Fig. 1. The geometrical representation of IFSs.

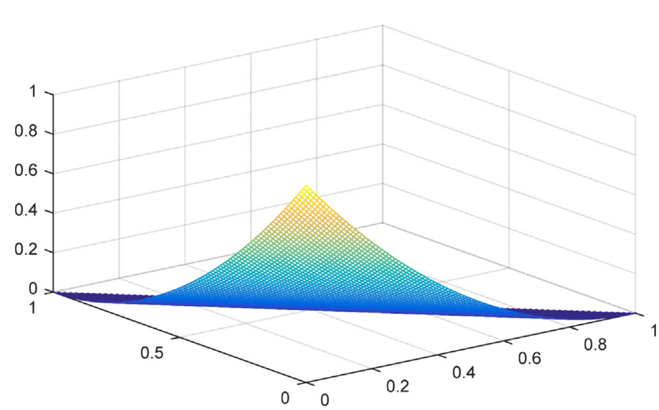

Fig. 2. The function.

represented by the distance between the arbitrary point y and the point M as depicted in Fig. 1. The reliability of information is represented by the hesitation degree of the IFV and used to measure how sure the information is. Inspired by [20], we begin to construct new measures of information amount and reliability of the IFS.

On the one hand, as mentioned previously, Szmidt and Kacprzyk [20] and Guo [22] only considered the positive ideal point $M(1, 0, 0)$ and neglected the negative ideal point $N(0, 1, 0)$. Motivated by the idea of TOPSIS, it is more reasonable and comprehensive to consider the positive and negative ideal points simultaneously. Therefore, the amount of information should be represented by the closeness degree $C(y)$.

On the other hand, Szmidt and Kacprzyk [20] used $1+(1/2)\pi(x)$ to measure the reliability of information (see Eq. (6)). They believed that the smaller the $1+(1/2)\pi(x)$, the more the reliability of information. Similarly, Guo [22] only used $1-(1/2)\pi(x)$ to represent the reliability of information (see Eq. (7)). The bigger the $1-(1/2)\pi(x)$, the more the reliability of information. It should be pointed out that notable drawback of these two methods exists in a lack of the geometric meaning. From Fig. 1, the area of $\Delta y_{\min} y y_{\max}$, $S_{\Delta y_{\min} y y_{\max}} = (1/2)\pi_A(y)^2$, is just a function of hesitancy degree. Obviously, the smaller the $S_{\Delta y_{\min} y y_{\max}}$, the more the reliability of information. Consequently, we adopt $\bar{S}(y) = 1-(1/2)\pi_A(y)^2$ to measure the reliability of information, which has obvious geometric meaning according to the geometrical representation of an IFS.

In summary, the larger the value of $C(y)$ and meanwhile the bigger the value of $\bar{S}(y)$, the better the point y. Therefore, the amount and reliability of information of the IFV y can be fully characterized by the following vector:

$$V(y) = (C(y),\quad \bar{S}(y)), \tag{12}$$

where $C(y) = (1-\nu(y))/(1+\pi(y))$ and $\bar{S}(y) = 1-(1/2)\pi(y)^2$.

Theorem 1. For any IFV $y = \langle \mu(y),\ \nu(y) \rangle$, the vector $V(y) = (C(y),\quad \bar{S}(y))$ can be transformed into an interval $I(y) = [C(y), \bar{S}(y)]$.

Proof. Since $C(y) = (1-\nu(y))/(1+\pi(y))$, the proof is equal to certify the inequality $(1-\nu(y))/(1+\pi(y)) \le 1-(1/2)\pi^2(y)$.

Note $\nu(y),\ \pi(y) \in [0,\ 1]$, we can obtain $\nu(y) \ge 0$, $(1/2)\pi(y)(\pi(y)-1) \le 0$. So $(1/2)\pi(y)(\pi(y)-1) \le \nu(y)$. Thus, $1-\nu(y) \le 1-(1/2)\pi(y)(\pi(y)-1) = [1-(1/2)\pi(y)](1+\pi(y))$.

Taking $1+\pi(y) > 0$ into account, we have $(1-\nu(y))/(1+\pi(y)) \le 1-(1/2)\pi(y) \le 1-(1/2)\pi^2(y)$, i.e., $C(y) \le \bar{S}(y)$, which completes the proof.□

Theorem 1 shows that the closeness degree $C(y)$ and the area $S_{\Delta y_{\min} y y_{\max}}$ can form an interval $I(y) = [C(y), \bar{S}(y)]$. It is an interesting and important observation. As we all know, for an interval, the larger the left and right endpoints, the bigger the interval. In the same way, the larger the $C(y)$ and meanwhile the greater $\bar{S}(y)$, the bigger the IFV y.

3.2. A new lexicographic method for ranking IFVs

For two IFVs $\tilde{a}_i = \langle \mu_i, \nu_i \rangle$ $(i = 1,\ 2)$, the amount and reliability of information of the IFV $\tilde{a}_i$ can be fully characterized by the interval $I(\tilde{a}_i) = [C(\tilde{a}_i), \bar{S}(\tilde{a}_i)]$. From the above analysis, $\tilde{a}_1 \le \tilde{a}_2 \Leftrightarrow I(\tilde{a}_1) \le I(\tilde{a}_2)$, where $\tilde{a}_1 \le \tilde{a}_2$ means $\tilde{a}_1$ is smaller than or equal to $\tilde{a}_2$. Thereby, a new lexicographic method for ranking two IFVs $\tilde{a}_i = \langle \mu_i, \nu_i \rangle$ $(i = 1, 2)$ can be defined as:

1) If $C(\tilde{a}_1) < C(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, i.e., $\tilde{a}_1 < \tilde{a}_2$;
2) If $C(\tilde{a}_1) = C(\tilde{a}_2)$, then

(i) if $\bar{S}(\tilde{a}_1) < \bar{S}(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, i.e., $\tilde{a}_1 < \tilde{a}_2$;
(ii) if $\bar{S}(\tilde{a}_1) = \bar{S}(\tilde{a}_2)$, then $\tilde{a}_1$ is equal to $\tilde{a}_2$, i.e., $\tilde{a}_1 = \tilde{a}_2$.

Remark 1. The new lexicographic method of this paper is particularly distinct from those of [20,22]. The main disparity is that Guo [22] and Szmidt and Kacprzyk [20] defined the measure only by the distance between the IFV and the positive ideal point $M(1, 0, 0)$. They failed to consider the negative ideal point $N(0, 1, 0)$. However, we define the measure by the closeness degree which simultaneously takes the positive ideal point $M(1, 0, 0)$ and the negative ideal point $N(0, 1, 0)$ into account.

Remark 2. Although Szmidt et al. [25] also considered the distance between the IFV with $M(1, 0, 0)$ and $N(0, 1, 0)$ simultaneously, the problem is that we cannot directly use the measure $K(x)$ of [25] since the aim of [25] is to measure the uncertainty and the amount of knowledge conveyed for IFVs rather than to rank the IFVs.

Remark 3. Zhang and Xu [36] also proposed a method to rank IFVs. They defined the similarity function $L(y) = 1-(1-\mu(y))/(1+\pi(y))$ which is accordance with the closeness degree $C(y) = (1-\nu(y))/(1+\pi(y))$ in this paper. The bigger $L(y)$, the larger the IFV y. If some IFVs with the same $L(y)$ values, Zhang and Xu [36] used the accuracy function $H(y) = \mu(y)+\nu(y) = 1-\pi(y)$ to further rank the IFVs. The larger $H(y)$, the larger $\bar{S}(y)$ and vice versa. Since the reliability of information $\bar{S}(y) = 1-(1/2)\pi(y)^2$ has the same monotonicity as $H(y)$, the method [36] and the new method in this paper have the identical order for ranking IFVs. However, in this paper, the ranking method is obtained by the method of ranking interval transformed by the amount and reliability of information of IFV, which has obvious geometric meaning. In addition, we further develop a novel risk attitudinal ranking measure of IFVs in Section 4, whereas Zhang and Xu [36] overlooked the DM's risk attitude.

Proposition 1. For two IFVs $\tilde{a}_1 = \langle \mu_1, \nu_1, \pi_1 \rangle$ and $\tilde{a}_2 = \langle \mu_2, \nu_2, \pi_2 \rangle$. If $\mu_1 \le \mu_2$ and $\nu_1 \ge \nu_2$, then $\tilde{a}_1 \le \tilde{a}_2$. Namely, if $\tilde{a}_1 \subseteq \tilde{a}_2$, then $\tilde{a}_1 \le \tilde{a}_2$.

Proof. By Eq. (11), we have $C(\tilde{a}_i) = (1-\nu_i)/(1+\pi_i) = (1-\nu_i)/(2-\mu_i-\nu_i)$ $(i = 1, 2)$. Then the proposition should be discussed in two cases:

i) If $\mu_1 \le \mu_2$ and $\nu_1 > \nu_2$ or $\mu_1 < \mu_2$, $\nu_1 \ge \nu_2$, it has $C(\tilde{a}_1) < C(\tilde{a}_2)$.
ii) If $\mu_1 = \mu_2$ and $\nu_1 = \nu_2$, it has $C(\tilde{a}_1) = C(\tilde{a}_2)$. Meanwhile, in this case, we have $\pi_1 = 1-\mu_1-\nu_1 = 1-\mu_2-\nu_2 = \pi_2$, i.e., $\bar{S}(\tilde{a}_1) = \bar{S}(\tilde{a}_2)$. So according to the new ranking method, $\tilde{a}_1 = \tilde{a}_2$.

Therefore, combined the above two cases, $\tilde{a}_1 \le \tilde{a}_2$ always holds, which completes the proof.□

Proposition 1 shows that the new lexicographic method accords with the containment relation in Definition 2.

Example 3. Let us analyze the three IFVs $M(1, 0, 0)$, $N(0, 1, 0)$ and $O(0, 0, 1)$, respectively. According to the new lexicographic method, we get, $C(M) = 1$, $C(O) = 1/2$ and $C(N) = 0$. Hence, $M > O > N$, which is accordance with the ranking result obtained by Guo [22].

Example 4. Consider two IFVs $\tilde{a}_1 = \langle 0.3, 0.3, 0.4 \rangle$ and $\tilde{a}_2 = \langle 0.4, 0.4, 0.2 \rangle$.

By the new lexicographic method, we get $C(\tilde{a}_1) = C(\tilde{a}_2) = 0.5$, $\bar{S}(\tilde{a}_1) = 0.96$ and $\bar{S}(\tilde{a}_2) = 0.98$. Because $C(\tilde{a}_1) = C(\tilde{a}_2)$ and $\bar{S}(\tilde{a}_1) < \bar{S}(\tilde{a}_2)$, we can obtain $\tilde{a}_1 < \tilde{a}_2$ by the method in this paper.

Example 5. Consider two IFVs $\tilde{a}_1 = \langle 0.55, 0.1, 0.35 \rangle$ and $\tilde{a}_2 = \langle 0.6, 0.2, 0.2 \rangle$.

It is immediate that $C(\tilde{a}_1) = C(\tilde{a}_2) = 0.6667$, $\bar{S}(\tilde{a}_1) = 0.9388$ and $\bar{S}(\tilde{a}_2) = 0.9800$. Since $C(\tilde{a}_1) = C(\tilde{a}_2)$ and $\bar{S}(\tilde{a}_1) < \bar{S}(\tilde{a}_2)$, the ranking order is $\tilde{a}_1 < \tilde{a}_2$ by the lexicographic method.

Example 6. Continue to consider two IFVs $a_1 = \langle 0.7, 0.1, 0.2 \rangle$ and $\tilde{a}_2 = \langle 0.6, 0.3, 0.1 \rangle$ in Example 2.

First, according to Eq. (7), we get $Z(\tilde{a}_1) = 0.7200$, $Z(\tilde{a}_2) = 0.6175$. Since $Z(\tilde{a}_1) > Z(\tilde{a}_2)$, we can obtain $\tilde{a}_1 > \tilde{a}_2$ by method [22].

Using the new lexicographic method, we calculate $C(\tilde{a}_1) = 0.7500$, $C(\tilde{a}_2) = 0.6364$. As $C(\tilde{a}_1) > C(\tilde{a}_2)$, the ranking order obtained by the lexicographic method is $\tilde{a}_1 > \tilde{a}_2$, which is accordance with that obtained by [22] but just contrary to that obtained by [20].

Example 7. Consider two IFVs $\tilde{a}_1 = \langle 0.4735, 0.3, 0.2265\rangle$ and $\tilde{a}_2 = \langle 0.499, 0.4073, 0.0937\rangle$.

Since $Z(\tilde{a}_1) = Z(\tilde{a}_2) = 0.5203$, we can obtain $\tilde{a}_1 = \tilde{a}_2$ by method [22]. Namely, method [22] cannot distinguish these two IFVs. It is obvious that $\tilde{a}_1$ is considerably different to $\tilde{a}_2$ as the membership and non-membership degrees of them are distinct. By the new lexicographic method, we get $C(\tilde{a}_1) = 0.5707$, $C(\tilde{a}_2) = 0.6276$. Hence, the ranking order is obtained as $\tilde{a}_1 < \tilde{a}_2$.

Remark 4. Yager [30] pointed out that the comparison of fuzzy numbers is a problem that has been extensively studied and that there is no unique best approach. Wu and Chiclana [31] also claimed that the set of fuzzy numbers is not totally ordered indeed. These conclusions are also appropriate for IFVs. Therefore, the lexicographic method developed in this section is no unique best approach for ranking IFVs although it can overcome some existing drawbacks. Some unreasonable results may occur using this method (see Examples 8, 10, and 12 in Section 4). Yager [30] stressed that the attitudinal character of each DM may affect the final ranking order of fuzzy numbers. The proposed lexicographic method neglects the attitudinal character of DM. Consequently, a novel risk attitudinal ranking measure of IFVs is put forward in the sequel.

4. A novel risk attitudinal ranking measure of IFVs

This section investigates a novel risk attitudinal ranking measure of IFVs and then computes its values for some usual basic unit-interval monotonic (BUM) functions.

4.1. A novel risk attitudinal ranking measure of IFVs

Considering DM's risk attitude, Yager [30] introduced the C-OWA operator to defuzzy an interval.

Definition 5 *([30]).* Let $[a, b]$ be an interval. The C-OWA operator could be represented as follows:

$$F_Q([a,b]) = C-\mathrm{OWA}_Q([a,b]) = \int_0^1 \frac{dQ(s)}{ds}(b - s(b-a))ds, \tag{13}$$

where $Q(s)$ is a BUM function, satisfying the conditions: $Q(0) = 0$, $Q(1) = 1$, and $Q(s_1) \geq Q(s_2)$ if $s_1 \geq s_2$ and $s_1, s_2 \in [0, 1]$. The choice of BUM function $Q(s)$ is a reflection of the DM's risk preference.

Call $\lambda = \int_0^1 Q(s)ds$ the attitudinal character of $Q(s)$. Eq. (13) can be converted as:

$$F_Q([a,b]) = (1-\lambda)a + \lambda b. \tag{14}$$

Clearly, $F_Q([a, b])$ is the weighted average of the endpoints of the closed interval with attitudinal character parameter λ, and it is known as the attitudinal expected value of $[a, b]$.

As mentioned previously, the closeness degree $C(y)$ and the area $S_{\Delta y_{\min} y y_{\max}}$ are the important indices for ranking the IFVs. They just form an interval $I(y) = [C(y), \bar{S}(y)]$. Since the attitudinal character parameter λ can represent the risk attitude of DM, a novel risk attitudinal measure for ranking IFVs is defined as

$$\begin{aligned} P_Q(I(y),\lambda) &= F_Q(I(y)) = F_Q([C(y),\bar{S}(y)]) \\ &= (1-\lambda)\frac{1-\nu(y)}{1+\pi(y)} + \lambda\left[1 - \frac{1}{2}\pi^2(y)\right], \end{aligned} \tag{15}$$

where λ denotes the risk attitude of DM. If $0.5 < \lambda < 1$, the DM is optimistic which means that the DM prefers the risk; if $0 < \lambda < 0.5$, the DM is pessimistic which means that the DM hates the risk; if $\lambda = 0.5$, the DM is neutral which means that the DM is indifferent to the risk.

It is apparent that the larger the value of $P_Q(I(y), \lambda)$, the bigger the IFV y.

Analogously, the total risk attitudinal measure for n elements belonging to an IFS $A = \{\langle x_i, \mu_A(x), \nu_A(x_i)\rangle \,|\, x_i \in X, i = 1, 2, \ldots, n\}$ is obtained as follows:

$$P_Q(A,\lambda) = \frac{1}{n}\sum_{i=1}^{n}[(1-\lambda)\tfrac{1-\nu_A(x_i)}{1+\pi_A(x_i)} + \lambda(1 - \tfrac{1}{2}(\pi_A(x_i))^2)]. \tag{16}$$

Proposition 2. For two IFVs $\tilde{a}_1 = \langle \mu_1, \nu_1, \pi_1\rangle$ and $\tilde{a}_2 = \langle \mu_2, \nu_2, \pi_2\rangle$. If $\mu_1 \leq \mu_2$, $\nu_1 \geq \nu_2$ and $\pi_1 \geq \pi_2$, then $\tilde{a}_1 \leq \tilde{a}_2$. Namely, if $\tilde{a}_1 \subseteq \tilde{a}_2$ and $\pi_1 \leq \pi_2$, then $\tilde{a}_1 \leq \tilde{a}_2$.

Proof. By Eq. (15), we have $P_Q(I(\tilde{a}_1)), \lambda) = (1-\lambda)(1-\nu_1)/(1+\pi_1) + \lambda(1-(1/2)\pi_1^2)$ and $P_Q(I(\tilde{a}_2)), \lambda) = (1-\lambda)(1-\nu_2)/(1+\pi_2) + \lambda(1-(1/2)\pi_2^2)$. Since $1 \geq \pi_1 \geq \pi_2 \geq 0$, it has $1-(1/2)\pi_1^2 \leq 1-(1/2)\pi_2^2$. Note that $\nu_1 \geq \nu_2$, we get $(1-\nu_1)/(1+\pi_1) \leq (1-\nu_2)/(1+\pi_2)$. Thus, $P_Q(I(\tilde{a}_1)), \lambda) \leq P_Q(I(\tilde{a}_2)), \lambda)$. Namely, $\tilde{a}_1 \leq \tilde{a}_2$.□

Proposition 3 (Monotonicity). Given an IFV y, the risk attitudinal measure $P_Q(I(y), \lambda)$ is monotonic with respect to the risk attitudinal parameter λ, i.e., if $\lambda_1 \leq \lambda_2$, then $P_Q(y, \lambda_1) \leq P_Q(y, \lambda_2)$.

Proof. By Eq. (15), $P_Q(I(y), \lambda)$ can be rewritten as $P_Q(I(y), \lambda) = (1-\nu(y))/(1+\pi(y)) + \lambda[1-(1/2)\pi^2(y) - (1-\nu(y))/(1+\pi(y))]$.

Since $(1-\nu(y))/(1+\pi(y)) \leq 1-(1/2)\pi^2(y)$ proved in Theorem 1, we get $1-(1/2)\pi^2(y) - (1-\nu(y))/(1+\pi(y)) \geq 0$. Thus, $P_Q(I(y), \lambda)$ is increasing with respect to λ.□

Proposition 4 (Boundedness). Given an IFV y, we have $(1-\nu(y))/(1+\pi(y)) \leq P_Q(I(y), \lambda) \leq 1-(1/2)\pi^2(y)$.

From Proposition 1, we know that the minimum and maximum values of $P_Q(I(y), \lambda)$ can be achieved for $\lambda = 0$ and $\lambda = 1$, respectively. Hence, $P_Q(I(y), 0) \leq P_Q(I(y), \lambda) \leq P_Q(I(y), 1)$, that is, $(1-\nu(y))/(1+\pi(y)) \leq P_Q(I(y), \lambda) \leq 1-(1/2)\pi^2(y)$, which completes the proof.□

Proposition 5 (Additivity). Given two IFVs $\tilde{a}_1 = (\mu_1, \nu_1, \pi_1)$ and $\tilde{a}_2 = (\mu_2, \nu_2, \pi_2)$, we have

$$P_Q(I(\tilde{a}_1) + I(\tilde{a}_2), \lambda) = P_Q(I(\tilde{a}_1), \lambda) + P_Q(I(\tilde{a}_2), \lambda).$$

Proof. Note that $I(\tilde{a}_1) + I(\tilde{a}_2) = [(1-\nu_1)/(1+\pi_1), 1-(1/2)\pi_1^2] + [(1-\nu_2)/(1+\pi_2), 1-(1/2)\pi_2^2] = [(1-\nu_1)/(1+\pi_1) + (1-\nu_2)/(1+\pi_2), 1-(1/2)\pi_1^2 + 1-(1/2)\pi_2^2]$.

By Eq. (15), we have

$$\begin{aligned} P_Q(I(\tilde{a}_1)+I(\tilde{a}_2),\lambda) &= F_Q(I(\tilde{a}_1)+I(\tilde{a}_2)) \\ &= (1-\lambda)\left(\frac{1-\nu_1}{1+\pi_1} + \frac{1-\nu_2}{1+\pi_2}\right) + \lambda\left(1-\tfrac{1}{2}\pi_1^2 + 1 - \tfrac{1}{2}\pi_2^2\right) \\ &= (1-\lambda)\frac{1-\nu_1}{1+\pi_1} + \lambda(1-\tfrac{1}{2}\pi_1^2) + (1-\lambda)\frac{1-\nu_2}{1+\pi_2}) + \lambda(1-\tfrac{1}{2}\pi_2^2) \\ &= P_Q(I(\tilde{a}_1),\lambda) + P_Q(I(\tilde{a}_2),\lambda). \end{aligned}$$

This completes the proof.□

Proposition 6 (Linearity). Let $k_1, k_2 \geq 0$. The risk attitudinal measure $P_Q(I(y), \lambda)$ meets linearity, i.e., $P_Q(k_1 I(\tilde{a}_1) + k_2 I(\tilde{a}_2), \lambda) = k_1 P_Q(I(\tilde{a}_1), \lambda) + k_2 P_Q(I(\tilde{a}_2), \lambda)$.

Proof. According to the operations of intervals, it yields that

$$\begin{aligned} k_1 I(\tilde{a}_1) + k_2 I(\tilde{a}_2) &= k_1\left[\frac{1-\nu_1}{1+\pi_1}, 1-\tfrac{1}{2}\pi_1^2\right] + k_2\left[\frac{1-\nu_2}{1+\pi_2}, 1-\tfrac{1}{2}\pi_2^2\right] \\ &= \left[k_1\frac{1-\nu_1}{1+\pi_1} + k_2\frac{1-\nu_2}{1+\pi_2}, k_1(1-\tfrac{1}{2}\pi_1^2) + k_2(1-\tfrac{1}{2}\pi_2^2)\right] \end{aligned}$$

By Eq. (15), we have

$$\begin{aligned} P_Q(k_1I(\tilde{a}_1)+k_2I(\tilde{a}_2),\lambda) &= F_Q(k_1I(\tilde{a}_1)+k_2I(\tilde{a}_2)) \\ &= (1-\lambda)\left(k_1\frac{1-\nu_1}{1+\pi_1}+k_2\frac{1-\nu_2}{1+\pi_2}\right)+\lambda\left[k_1(1-\tfrac{1}{2}\pi_1^2)+k_2(1-\tfrac{1}{2}\pi_2^2)\right] \\ &= k_1[(1-\lambda)\frac{1-\nu_1}{1+\pi_1}+\lambda(1-\tfrac{1}{2}\pi_1^2)]+k_2\left[(1-\lambda)\frac{1-\nu_2}{1+\pi_2})+\lambda(1-\tfrac{1}{2}\pi_2^2)\right] \\ &= k_1P_Q(I(\tilde{a}_1),\lambda)+k_2P_Q(I(\tilde{a}_2),\lambda). \end{aligned}$$

This completes the proof.□

Propositions 3 and 4 indicate that the risk attitudinal measure $P_Q(I(y), \lambda)$ is a mean operator. Propositions 5 and 6 verify that $P_Q(I(y), \lambda)$ is additive and linear and therefore it is a weighted averaging operator.

To analyze the effect of risk attitudinal parameter λ on the ranking of IFVs, we make sensitivity analysis for $P_Q(I(y),\lambda)$ with respect to λ.

Theorem 2. Let $\Delta\lambda$ be a perturbation of the risk attitudinal parameter λ with $0\le\lambda+\Delta\lambda\le 1$. For any two IFVs $\tilde{a}_1=(\mu_1,\nu_1,\pi_1)$ and $\tilde{a}_2=(\mu_2,\nu_2,\pi_2)$, if $P_Q(I(\tilde{a}_1),\lambda)\le P_Q(I(\tilde{a}_2),\lambda)$, then $P_Q(I(\tilde{a}_1),\lambda+\Delta\lambda)\le P_Q(I(\tilde{a}_2),\lambda+\Delta\lambda)$ if and only if

$$\begin{cases} \max\{[P_Q(I(\tilde{a}_1),\lambda)-P_Q(I(\tilde{a}_2),\lambda)]/(\eta_2-\eta_1),-\lambda\}\le\Delta\lambda\le 1-\lambda, & \eta_2>\eta_1 \\ -\lambda\le\Delta\lambda\le 1-\lambda, & \eta_2=\eta_1 \\ -\lambda\le\Delta\lambda\le\min\{[P_Q(I(\tilde{a}_1),\lambda)-P_Q(I(\tilde{a}_2))]/(\eta_2-\eta_1),1-\lambda\}, & \eta_2<\eta_1 \end{cases}$$

where $\eta_i=1-(1/2)\pi_i^2-(1-\nu_i)/(1+\pi_i)$ $(i=1,2)$.

Proof. If $P_Q(I(\tilde{a}_1),\lambda+\Delta\lambda)\le P_Q(I(\tilde{a}_2),\lambda+\Delta\lambda)$, then by Eq. (15) we get

$$\begin{aligned} &(1-\lambda-\Delta\lambda)\frac{1-\nu_1}{1+\pi_1}+(\lambda+\Delta\lambda)(1-(1/2)\pi_1^2) \\ &\quad\le(1-\lambda-\Delta\lambda)\frac{1-\nu_2}{1+\pi_2}+(\lambda+\Delta\lambda)(1-(1/2)\pi_2^2). \end{aligned}$$

Namely, $P_Q(I(\tilde{a}_1),\lambda)-P_Q(I(\tilde{a}_2),\lambda)\le(\eta_2-\eta_1)\Delta\lambda$.

Since $0\le\lambda\le 1$ and $0\le\lambda+\Delta\lambda\le 1$, we obtain $-\lambda\le\Delta\lambda\le 1-\lambda$.

Then, if $\eta_2>\eta_1$, we have $\Delta\lambda\ge[P_Q(I(\tilde{a}_1),\lambda)-P_Q(I(\tilde{a}_2))]/(\eta_2-\eta_1)$. Thus, $\max\{[P_Q(I(\tilde{a}_1),\lambda)-P_Q(I(\tilde{a}_2))]/(\eta_2-\eta_1),-\lambda\}\le\Delta\lambda\le 1-\lambda$;

If $\eta_2<\eta_1$, we have $\Delta\lambda\le[P_Q(I(\tilde{a}_1),\lambda)-P_Q(I(\tilde{a}_2))]/(\eta_2-\eta_1)$. Thus, $-\lambda\le\Delta\lambda\le\min\{[P_Q(I(\tilde{a}_1),\lambda)-P_Q(I(\tilde{a}_2))]/(\eta_2-\eta_1),1-\lambda\}$;

If $\eta_2=\eta_1$, we have $-\lambda\le\Delta\lambda\le 1-\lambda$.

This completes the proof of Theorem 2.□

Theorem 2 gives the scope of the perturbation $\Delta\lambda$ which can keep the order relation $\tilde{a}_1\le\tilde{a}_2$ unchanged when the risk attitudinal parameter varies from λ to $\lambda+\Delta\lambda$.

4.2. Risk attitudinal measure values for some usual BUM functions

To further scrutinize the risk attitudinal measure $P_Q(I(y),\lambda)$, we respectively compute its values for some usual BUM functions in what follows.

(1) If BUM function $Q(s)=s^t$ $(t>0)$, then $\lambda=1/(1+t)$. Thus Eq. (15) can be rewritten as

$$P_Q(I(y),\lambda)=\frac{t}{1+t}\left(\frac{1-\nu(y)}{1+\pi(y)}\right)+\frac{1}{1+t}(1-(1/2)\pi^2(y)), \tag{17}$$

where the parameter t denotes a DM's risk attitude. If $0<t<1$ $(0.5<\lambda<1)$, the DM is optimistic. In this situation, the degree of hesitation $\pi(y)$ has more importance than the closeness degree $C(y)$. If $t>1$ $(0<\lambda<0.5)$, the DM is pessimistic. In this situation, the degree of hesitation $\pi(y)$ has less importance than the closeness degree $C(y)$. If $t=1$ $(\lambda=0.5)$, the DM is risk-neutral.

Example 8. Consider $\tilde{a}_1=<0.50,0.50,0>$ and $\tilde{a}_2=<0.4,0.35,0.25>$ by using Eq. (17).

First, set $t=1$, by Eq. (17) we obtain $P_Q(I(\tilde{a}_1),\lambda)=0.7500$ and $P_Q(I(\tilde{a}_2),\lambda)=0.7444$. Since $P_Q(I(\tilde{a}_1),\lambda)>P_Q(I(\tilde{a}_2),\lambda)$, the ranking is $\tilde{a}_1>\tilde{a}_2$ when DM is risk-neutral (i.e., $t=1$).

Analogously, when parameter t takes different values, we get the corresponding results of computation and ranking orders shown in Table 1. Generally, the pessimistic DM thinks that non-membership degree is more important than membership degree during the ranking process. When the value of parameter t increases from 1 to $+\infty$, the pessimistic degree of DM also increases, thus $\tilde{a}_1$ with bigger non-membership is smaller than $\tilde{a}_2$ with smaller non-membership. Conversely, the optimistic DM holds that membership degree is more important than non-membership degree during the ranking process. When the value of parameter t decreases from 1 to 0, the optimistic degree of DM increases, thus $\tilde{a}_1$ with bigger membership is larger than $\tilde{a}_2$ with smaller membership.

However, if we do not consider the risk attitude and employ the lexicographic method developed in Section 3.2, the obtained ranking is $\tilde{a}_1<\tilde{a}_2$ due to $C(\tilde{a}_1)=0.5<C(\tilde{a}_2)=0.52$. Although the membership of $\tilde{a}_1$ is bigger than that of $\tilde{a}_2$, and the hesitation margin of $\tilde{a}_1$ is smaller that hesitation margin of $\tilde{a}_2$, the ranking is $\tilde{a}_1<\tilde{a}_2$ by the lexicographic method, which is just the pessimistic case obtained by the risk attitudinal ranking method. The main reason is that the lexicographic method overlooks DM's risk attitude.

Example 9. We reconsider $\tilde{a}_1=\langle 0.2717,0.3275,0.4008\rangle$ and $\tilde{a}_2=\langle 0.3101,0.4542,0.2357\rangle$ in Example 6 by using Eq. (17).

First, set $t=1$, by Eq. (17) we obtain $P_Q(I(\tilde{a}_1),\lambda)=0.6999$, $P_Q(I(\tilde{a}_2),\lambda)=0.7070$. Since $P_Q(I(\tilde{a}_1),\lambda)<P_Q(I(\tilde{a}_2),\lambda)$, the ranking is $\tilde{a}_1<\tilde{a}_2$ when DM is risk-neutral (i.e., $t=1$). Additionally, for different values of parameter t, the corresponding ranking orders are shown in Table 2.

By Eq. (8), the measures $Z_Q(\tilde{a}_i)$ are calculated and the ranking orders are obtained by method [22] with different t, which are listed in Table 3.

Table 1
Computation results for $Q(s)=s^t$ with different values of parameter t and ranking orders.

t	$P_Q(I(\tilde{a}_1),\lambda)$	$P_Q(I(\tilde{a}_2),\lambda)$	Ranking order
0	1.0000	0.9688	$\tilde{a}_1>\tilde{a}_2$
0.5	0.8333	0.8192	$\tilde{a}_1>\tilde{a}_2$
1	0.7500	0.7444	$\tilde{a}_1>\tilde{a}_2$
1.5625	0.6951	0.6951	$\tilde{a}_1=\tilde{a}_2$
2	0.6667	0.6696	$\tilde{a}_1<\tilde{a}_2$
$+\infty$	0.5000	0.5200	$\tilde{a}_1<\tilde{a}_2$

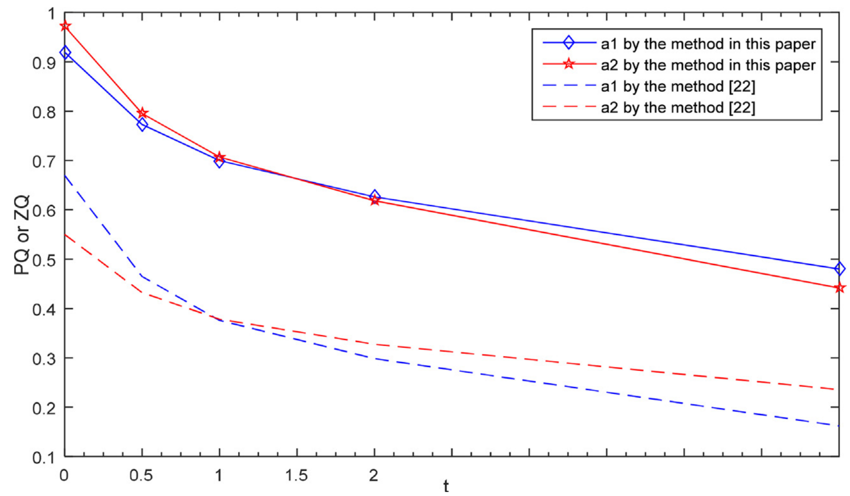

Fig. 3. Comparison between method [22] and the method in this paper for $Q(s)=s^t$ with different values of parameter t.

Table 2
Computation results for $Q(s)=s^t$ with different values of parameter t and ranking orders.

t	$P_Q(I(\tilde{a}_1),\lambda)$	$P_Q(I(\tilde{a}_2),\lambda)$	Ranking order
0	0.9197	0.9722	$\tilde{a}_1 < \tilde{a}_2$
0.5	0.7731	0.7954	$\tilde{a}_1 < \tilde{a}_2$
1	0.6999	0.7070	$\tilde{a}_1 < \tilde{a}_2$
1.3685	0.6657	0.6657	$\tilde{a}_1 = \tilde{a}_2$
2	0.6262	0.6185	$\tilde{a}_1 > \tilde{a}_2$
$+\infty$	0.4801	0.4417	$\tilde{a}_1 > \tilde{a}_2$

To intuitively compare the results obtained by method [22] and the risk attitudinal ranking method proposed in this paper, we plot Fig. 3 for $Q(s)=s^t$ with different values of parameter t.

From Fig. 3, we find that the ranking orders obtained by the method in this paper are not identical when the parameter t takes different values. When $t\in[0, 1.3685)$, $\tilde{a}_1 < \tilde{a}_2$; when $t\in(1.3685, +\infty)$, $\tilde{a}_1 > \tilde{a}_2$. However, the ranking results obtained by the method [22] are just the reverse of that obtained by the method in this paper. The main reason is that Guo [22] directly replaced the two fractions 1/2 of Eq. (7) by the $t/(t+1)$ and $1/(t+1)$, respectively, to define the risk attitudinal measure $Z_Q(x)$ (see Eq. (8)). This measure lacks the theoretical basis and is not consistent with the C-OWA operator. We first transform the vector $V(y)=(C(y), \bar{S}(y))$ into the interval $I(y)=[C(y), \bar{S}(y)]$ and then define the risk attitudinal measure $P_Q(I(y),\lambda)$ through using the C-OWA operator, which is more reasonable and convincing in theory.

(2) If BUM function $Q(s)=(1-e^{-s})/(1-e^{-1})^t$ $(t>0)$, then Eq. (15) can be further computed as follows:

Table 3
Computation results by method [22] for different values of parameter t and ranking orders.

t	$Z_Q(\tilde{a}_1)$	$Z_Q(\tilde{a}_2)$	Ranking order
0	0.6700	0.5500	$\tilde{a}_1 > \tilde{a}_2$
0.5	0.4651	0.4324	$\tilde{a}_1 > \tilde{a}_2$
0.952	0.3823	0.3823	$\tilde{a}_1 = \tilde{a}_2$
1	0.3760	0.3784	$\tilde{a}_1 < \tilde{a}_2$
2	0.2958	0.3276	$\tilde{a}_1 < \tilde{a}_2$
$+\infty$	0.1620	0.2356	$\tilde{a}_1 < \tilde{a}_2$

$$P_Q(I(y),\lambda)=\begin{cases}\bar{S}(y), & \text{if } t\to 0\\ \left(3-\dfrac{\sqrt{e}}{\sqrt{e-1}}\ln\dfrac{\sqrt{e}+\sqrt{e-1}}{\sqrt{e}-\sqrt{e-1}}\right)C(y) & \\ \quad+\left(\dfrac{\sqrt{e}}{\sqrt{e-1}}\ln\dfrac{\sqrt{e}+\sqrt{e-1}}{\sqrt{e}-\sqrt{e-1}}-2\right)\bar{S}(y), & \text{if } t=\frac{1}{2}\\ \left(\dfrac{e-2}{e-1}\right)C(y)+\dfrac{1}{e-1}\bar{S}(y), & \text{if } t=1\\ \left(1+\dfrac{1-4e+e^2}{2(e-1)^2}\right)C(y)-\dfrac{1-4e+e^2}{2(e-1)^2}\bar{S}(y), & \text{if } t=2\\ C(y), & \text{if } t\to\infty\end{cases}\tag{18}$$

where t denotes DM's risk attitude. If $0<t<1$ $(0.5820<\lambda<1)$, the DM is optimistic; if $t>1$ $(0<\lambda<0.5820)$, the DM is pessimistic; if $t=1$ $(\lambda=0.5820)$, the DM is neutral.

Example 10. We consider $\tilde{a}_1=\langle 0.40, 0.35, 0.25\rangle$ and $\tilde{a}_2=\langle 0.25, 0.10, 0.65\rangle$ using Eq. (18).

Setting $t=1$ and using Eq. (18), we have $P_Q(I(\tilde{a}_1),\lambda)=0.7812$ and $P_Q(I(\tilde{a}_2),\lambda)=0.6870$. Because $P_Q(I(\tilde{a}_1),\lambda)>P_Q(I(\tilde{a}_2),\lambda)$, the ranking is $\tilde{a}_1 > \tilde{a}_2$ when the DM is neutral.

Similarly, when t takes different values, we get the corresponding results of computation and ranking orders by Eq. (18) shown in Table 4. Table 4 indicates that when the value of parameter t increases from 1 to $+\infty$, the ranking order changes from $\tilde{a}_1 > \tilde{a}_2$ to $\tilde{a}_1 < \tilde{a}_2$. Namely, the pessimistic DM regards $\tilde{a}_1$ with bigger non-membership is smaller than $\tilde{a}_2$ with smaller non-membership. Conversely, when the value of parameter t decreases from 1 to 0, the ranking order is $\tilde{a}_1 > \tilde{a}_2$. Namely, the optimistic DM believes that $\tilde{a}_1$ with bigger membership is larger than $\tilde{a}_2$ with smaller membership. These analyses are accordance with that of Example 8.

Table 4
Computation results for $Q(s)=(1-e^{-s})/(1-e^{-1})^t$ with different values of parameter t and ranking orders.

t	$P_Q(I(\tilde{a}_1),\lambda)$	$P_Q(I(\tilde{a}_2),\lambda)$	Ranking order
0	0.9688	0.7888	$\tilde{a}_1 > \tilde{a}_2$
0.5	0.8473	0.7229	$\tilde{a}_1 > \tilde{a}_2$
1	0.7812	0.6870	$\tilde{a}_1 > \tilde{a}_2$
2	0.7088	0.6478	$\tilde{a}_1 > \tilde{a}_2$
$+\infty$	0.5200	0.5455	$\tilde{a}_1 < \tilde{a}_2$

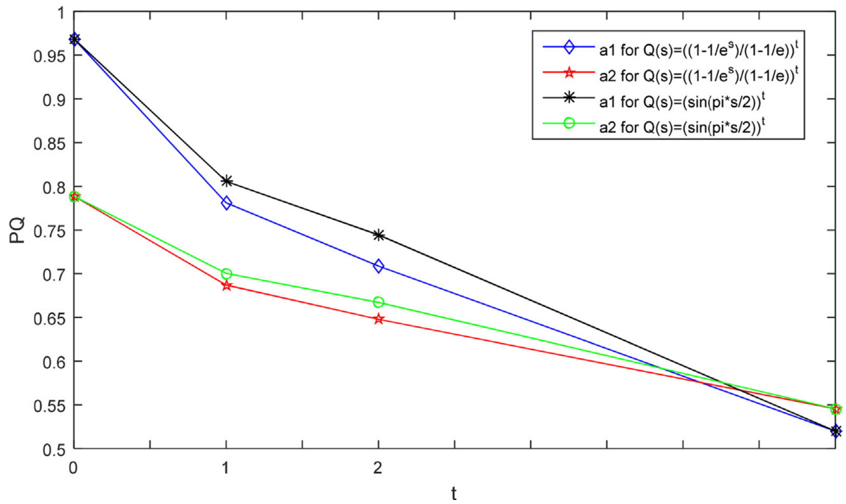

Fig. 4. Comparison between different BUM functions with different values of parameter t (examples 10 and 12).

Table 5
Computation results for $Q(s) = (1 - e^{-s})/(1 - e^{-1})^t$ with different values of parameter t and ranking orders.

t	$P_Q(I(\tilde{a}_1), \lambda)$	$P_Q(I(\tilde{a}_2), \lambda)$	Ranking order
0	0.9197	0.9722	$\tilde{a}_1 < \tilde{a}_2$
0.5	0.8007	0.8287	$\tilde{a}_1 < \tilde{a}_2$
1	0.7359	0.7504	$\tilde{a}_1 < \tilde{a}_2$
2	0.6650	0.6649	$\tilde{a}_1 > \tilde{a}_2$
$+\infty$	0.4801	0.4417	$\tilde{a}_1 > \tilde{a}_2$

However, if we do not consider the risk attitude and employ the lexicographic method developed in Section 3.2, the obtained ranking is $\tilde{a}_1 < \tilde{a}_2$ since $C(\tilde{a}_1) = 0.52 < C(\tilde{a}_2) = 0.55$, which is just the pessimistic case obtained by the risk attitudinal ranking method. This analysis again implies that the lexicographic method is not comprehensive due to ignoring the DM's risk attitude.

The above results in Example 10 for $Q(s) = (1 - e^{-s})/(1 - e^{-1})^t$ with different values of parameter t are depicted in Fig. 4.

Example 11. We reconsider $\tilde{a}_1 = \langle 0.2717, 0.3275, 0.4008\rangle$ and $\tilde{a}_2 = \langle 0.3101, 0.4542, 0.2357\rangle$ in Example 6 using Eq. (18).

Setting $t = 1$ and using Eq. (18), we have $P_Q(I(\tilde{a}_1), \lambda) = 0.7359$ and $P_Q(I(\tilde{a}_2), \lambda) = 0.7504$. As $P_Q(I(\tilde{a}_1), \lambda) < P_Q(I(\tilde{a}_2), \lambda)$, the ranking is $\tilde{a}_1 < \tilde{a}_2$ when the DM is neutral.

When t takes different values, we get the corresponding results of computation and ranking orders shown in Table 5. These results of Example 11 are depicted in Fig. 5.

If BUM function $Q(s) = (\sin((1/2)\pi s))^t$ $(t > 0)$, Eq. (15) can be further calculated as follows:

$$P_Q(I(y), \lambda) = \begin{cases} \bar{S}(y), & \text{if } t \to 0 \\ \dfrac{\pi - 2}{\pi}\bar{S}(y) + \dfrac{2}{\pi}C(y), & \text{if } t = 1 \\ \dfrac{1}{2}\bar{S}(y) + \dfrac{1}{2}C(y), & \text{if } t = 2 \\ C(y), & \text{if } t \to \infty \end{cases} \tag{19}$$

where the parameter t denotes the risk attitude of DM, $C(y) = (1 - \nu(y))/(1 + \pi(y))$ and $\bar{S}(y) = 1 - (1/2)\pi(y)^2$. If $0 < t < 1$, the DM is optimistic; if $t > 1$, the DM is pessimistic; if $t = 1$, the DM is neutral.

Example 12. We reconsider $\tilde{a}_1 = \langle 0.40, 0.35, 0.25\rangle$ and $\tilde{a}_2 = \langle 0.25, 0.10, 0.65\rangle$ in Example 10 by using Eq. (19).

Table 6
Computation results for $Q(s) = (\sin((1/2)\pi s))^t$ with different values of parameter t and ranking orders.

t	$P_Q(I(\tilde{a}_1), \lambda)$	$P_Q(I(\tilde{a}_2), \lambda)$	Ranking order
0	0.9688	0.7888	$\tilde{a}_1 > \tilde{a}_2$
1	0.8057	0.7003	$\tilde{a}_1 > \tilde{a}_2$
2	0.7444	0.6671	$\tilde{a}_1 > \tilde{a}_2$
$+\infty$	0.5200	0.5455	$\tilde{a}_1 < \tilde{a}_2$

Table 7
Computation results for $Q(s) = (\sin((1/2)\pi s))^t$ with different values of parameter t and ranking orders.

t	$P_Q(I(\tilde{a}_1), \lambda)$	$P_Q(I(\tilde{a}_2), \lambda)$	Ranking order
0	0.9197	0.9722	$\tilde{a}_1 < \tilde{a}_2$
1	0.7599	0.7794	$\tilde{a}_1 < \tilde{a}_2$
2	0.6999	0.7070	$\tilde{a}_1 < \tilde{a}_2$
$+\infty$	0.4801	0.4417	$\tilde{a}_1 > \tilde{a}_2$

Setting $t = 1$ and using Eq. (19), we calculate $P_Q(I(\tilde{a}_1), \lambda) = 0.8057$ and $P_Q(I(\tilde{a}_2), \lambda) = 0.7003$. Since $P_Q(I(\tilde{a}_1), \lambda) > P_Q(I(\tilde{a}_2), \lambda)$, the ranking is $\tilde{a}_1 > \tilde{a}_2$ when DM is neutral.

For different values of t, Table 6 lists the corresponding results of computation and ranking orders using Eq. (19). These results are consistent with that of Example 10. Meanwhile, Fig. 4 presents the obtained results of Example 12.

Example 13. Continue to consider $\tilde{a}_1 = \langle 0.2717, 0.3275, 0.4008\rangle$ and $\tilde{a}_2 = \langle 0.3101, 0.4542, 0.2357\rangle$ in Example 6 by Eq. (19).

Setting $t = 1$ and using Eq. (19), we calculate $P_Q(I(\tilde{a}_1), \lambda) = 0.7599$ and $P_Q(I(\tilde{a}_2), \lambda) = 0.7794$. Since $P_Q(I(\tilde{a}_2), \lambda) > P_Q(I(\tilde{a}_1), \lambda)$, the ranking is $\tilde{a}_2 > \tilde{a}_1$ when DM is neutral.

Additionally, when t takes different values, we get the corresponding results of computation and ranking orders shown in Table 7. These results are also depicted in Fig. 5.

It is easy to seen from Figs. 3–5 that the ranking orders are distinctly different for different BUM functions. In the meantime, the ranking orders also depend on the selection of risk attitudinal parameter t. These observations verify that it is necessary and reasonable to introduce BUM functions to consider the risk attitude of DM during the process of ranking IFVs.

Remark 5. As mentioned in Section 2.2, Guo [22] respectively used $t/(t+1)$ and $1/(t+1)$ to directly replace the two fractions 1/2 considering the DM's risk attitude. This method lacks theoretical basis for this transforming to some extend. Wu and Chiclana [31] defined

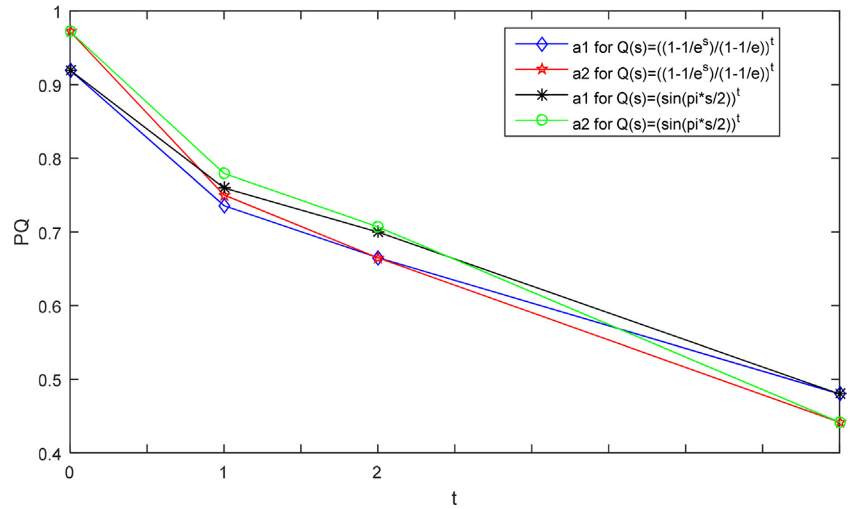

Fig. 5. Comparison between different BUM functions with different values of parameter t (examples 11 and 13).

attitudinal expected score function and attitudinal expected accuracy function to rank interval-valued intuitionistic fuzzy numbers. However, the attitudinal expected score (or accuracy) function is obtained by directly employing COWA on the interval-valued score (or accuracy) function. Similarly, Jin et al. [32] directly employed COWA on the interval-valued membership and non-membership degrees to define attitudinal expected score degree of IVIFN. However, the novel risk attitudinal measure for ranking IFVs in this paper is defined by applying the C-OWA operator to the interval $I(y)=[C(y),\bar{S}(y)]$ which consists of the amount and reliability of information. Thus, the attitudinal measure for ranking IFVs in this paper is remarkably different from those of methods [22,31,32].

5. MADM problems with IFVs and incomplete weight information

In this section, we present the MADM problems with IFVs and incomplete weight information, and then propose a new approach to solving such the MADM problems by using the novel risk attitudinal ranking measure.

5.1. Presentation of MADM problems with IFVs and incomplete weight information

For a MADM problem with a finite set of m alternatives and n attributes, let $A=\{a_1,a_2,\ldots,a_m\}$ be the set of alternatives, $X=\{x_1,x_2,\ldots,x_n\}$ be a set of attributes. The DM is able to provide his/her preferences for each alternative on different attributes. Assume that the rating of alternative a_i on attribute x_j is represented by IFV $\tilde{r}_{ij}=\langle\mu_{ij},\nu_{ij}\rangle$ ($i=1,2,\ldots,m$; $j=1,2,\ldots,n$). Thus, we can elicit the intuitionistic fuzzy decision matrix $\tilde{\boldsymbol{R}}=(\tilde{r}_{ij})_{m\times n}$, which are usually used to concisely express the MADM problem. In real decision situations, it is difficult to fully determine the importance of all attributes due to limited knowledge, experience and expertise of DMs. Suppose w_j is the relative weight of attribute x_j ($j=1,2,\ldots,n$), satisfying $\sum_{j=1}^{n}w_j=1$ and $w_j\geq 0$ ($j=1,2,\ldots,n$). Denote a weight vector by $\boldsymbol{w}=(w_1,w_1,\ldots,w_n)^{\mathrm{T}}$. Let Λ_0 be a set of all weight vectors with $w_j\geq\varepsilon$ for all j ($j=1,2,\ldots,n$), i.e.,

$$\Lambda_0=\left\{w|\sum_{j=1}^{n}w_j=1,w_j\geq\varepsilon\quad\text{for}\,j=1,2,\ldots,n\right\},$$

where $\varepsilon>0$ is a sufficiently small positive number. The constraints $w_j\geq\varepsilon$ ($j=1,2,\ldots,n$) can ensure that each weight of Λ_0 is not smaller than a given sufficiently small positive number ε.

In some real decision situations, the DM may specify some preference relations on weights of attributes according to his/her knowledge, experience and judgement. Such information of attribute weights is incomplete [37–39]. Usually incomplete information of attribute weights can be obtained according to partial preference relations on weights given by the DM and has several different structure forms. Li [37] mathematically expressed these weight information structures in the following five basic relations among attribute weights: a weak ranking, a strict ranking, a ranking with multiples, an interval form and a ranking of differences, which are denoted by subsets Λ_s ($s=1,2,3,4,5$) of weighting vectors in Λ_0, respectively. In reality, the preference information structure Λ of attribute importance may consist of several sets of the above basic sets Λ_s ($s=1,2,3,4,5$).

5.2. A novel approach to determining the attribute weights based on fractional programming

Define the intuitionistic fuzzy positive ideal solution $a^+=(\tilde{r}_1^+,\tilde{r}_2^+,\ldots,\tilde{r}_n^+)$ and the intuitionistic fuzzy negative ideal solution $a^-=(\tilde{r}_1^-,\tilde{r}_2^-,\ldots,\tilde{r}_n^-)$, where $\tilde{r}_j^+=<1,0>$ and $\tilde{r}_j^-=<0,1>$ ($j=1,2,\ldots,n$).

Calculate the Hamming distance between alternative a_i and a^+ as well as a^- by Eq. (1) as follows:

$$d(a_i,a^+)=\frac{1}{2}\sum_{j=1}^{n}w_j(1-\mu_{ij}+\nu_{ij}+\pi_{ij}),$$

$$d(a_i,a^-)=\frac{1}{2}\sum_{j=1}^{n}w_j(\mu_{ij}+1-\nu_{ij}+\pi_{ij}).$$

Then, the relative closeness degree of the alternative a_i is computed as

$$c(a_i)=\frac{d(a_i,a^-)}{d(a_i,a^-)+d(a_i,a^+)}=\frac{\sum_{j=1}^{n}w_j(\mu_{ij}+1-\nu_{ij}+\pi_{ij})}{2\sum_{j=1}^{n}w_j(1+\pi_{ij})}.$$

The bigger the relative closeness degree $c(a_i)$, the better the alternative a_i. Therefore, we can establish the multi-objective fractional programming model to determine the weights of attributes:

$$\begin{aligned} &\max\{Z_i = c(a_i)\}\,(i = 1, 2, \ldots, m) \\ &s.t. \quad \boldsymbol{w} \in \Lambda \end{aligned} \tag{20}$$

Since there is no any preference among the alternatives, we can transform Eq. (20) into a single objective fractional program by linear sum of equal weight as follows:

$$\begin{aligned} &\max \quad Z = \sum_{i=1}^{m} \frac{\sum_{j=1}^{n} w_j(\mu_{ij} + 1 - \upsilon_{ij} + \pi_{ij})}{2\sum_{j=1}^{n} w_j(1 + \pi_{ij})} \\ &s.t. \quad \boldsymbol{w} \in \Lambda \end{aligned} \tag{21}$$

To solve Eq. (21), we utilize the Charnes and Cooper transformation to set

$$\theta_i = \frac{1}{2\sum_{j=1}^{n} w_j(1 + \pi_{ij})} \quad (i = 1, 2, \ldots, m), \tag{22}$$

$$\delta_{ij} = \theta_i w_j \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n). \tag{23}$$

Then, we have $2\sum_{j=1}^{n} \delta_{ij}(1 + \pi_{ij}) = 1\,(i = 1, 2, \ldots, m)$.

The constraint set $\Lambda_0 = \{w | \sum_{j=1}^{n} w_j = 1,\ w_j \geq \varepsilon \ \text{for}\ j = 1, 2, \ldots, n\}$ can be transformed into $\Lambda'_0 = \{(\delta, \theta) | \sum_{j=1}^{n} \delta_{ij} = \theta_i (i = 1, 2, \ldots, m),\ \delta_{ij} \geq \theta_i \varepsilon \ \text{for}\ j = 1, 2, \ldots, n\}$, where $\delta = (\delta_{ij})_{m\times n}$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)^{\mathrm{T}}$. The subsets Λ_s (s = 1, 2, 3, 4, 5) of weight information structures can be transformed into the corresponding forms as follows:

(1) $\Lambda'_1 = \{(\boldsymbol{\delta}, \boldsymbol{\theta}) \in \Lambda'_0 | \delta_{ii} \geq \delta_{ij} \quad \text{for all}\ i \in T_1\ \text{and}\ j \in J_1\}$;

(2) $\Lambda'_2 = \{(\boldsymbol{\delta}, \boldsymbol{\theta}) \in \Lambda'_0 | \theta_i\beta_{ij} \geq \delta_{ii} - \delta_{ij} \geq \theta_i\alpha_{ij} \ \text{for all}\ i \in T_2\ \text{and}\ j \in J_2\}$;

(3) $\Lambda'_3 = \{(\boldsymbol{\delta}, \boldsymbol{\theta}) \in \Lambda'_0 | \delta_{ii} \geq \xi_{ij}\delta_{ij} \quad \text{for all}\ i \in T_3\ \text{and}\ j \in J_3\}$;

(4) $\Lambda'_4 = \{(\boldsymbol{\delta}, \boldsymbol{\theta}) \in \Lambda'_0 | \theta_i\gamma_j \geq \delta_{ij} \geq \theta_i\eta_j \quad \text{for}\ j \in J_4\}$;

(5) $\Lambda'_5 = \{(\boldsymbol{\delta}, \boldsymbol{\theta}) \in \Lambda'_0 | \delta_{ii} - \delta_{ij} \geq \delta_{ik} - \delta_{is} \ \text{for all}\ i \in T_5, j \in J_5, k \in K_5\ \text{and}\ s \in L_5\}$.

It is easy to see that the above information of transformed forms is still linear on variables δ_{ij} and θ_i. Denote the transformed preference information structure of attribute importance by Λ'.

Thus, Eq. (21) can be transformed into the following linear programming model:

$$\begin{aligned} &\max \quad \{Z' = \sum_{i=1}^{m}\sum_{j=1}^{n} \delta_{ij}(\mu_{ij} + 1 - \upsilon_{ij} + \pi_{ij})\} \\ &s.t. \quad \begin{cases} 2\sum_{j=1}^{n} \delta_{ij}(1 + \pi_{ij}) = 1\,(i = 1, 2, \ldots, m) \\ (\delta, \theta) \in \Lambda' \end{cases} \end{aligned} \tag{24}$$

Theorem 3. The linear programming Eq. (24) is equivalent to the fractional programming Eq. (21) in the following sense:

(i) If w is the optimal solution of Eq. (21), then $(\boldsymbol{\delta}, \boldsymbol{\theta})$ is the optimal solution of Eq. (24) and the optimal objective value $Z = Z'$, where $(\boldsymbol{\delta}, \boldsymbol{\theta})$ meet Eqs. (22) and (23);
(ii) If $(\boldsymbol{\delta}, \boldsymbol{\theta})$ is the optimal solution of Eq. (24), then $\boldsymbol{w}$ is the optimal solution of Eq. (21) and the optimal objective value $Z' = Z$.

Therefore, using the existing Simplex method to solve Eq. (24), we can obtain $(\boldsymbol{\delta}, \boldsymbol{\theta})$. Then, the weight vector of attributes $\boldsymbol{w} = (w_1, w_1, \ldots, w_n)^{\mathrm{T}}$ is determined by Eq. (23). It is not difficult to prove that such a weight vector $\boldsymbol{w} = (w_1, w_1, \ldots, w_n)^{\mathrm{T}}$ is the Pareto optimal solution of Eq. (20).

5.3. The method for MADM with IFVs and incomplete weight information

The method for MADM with IFVs and incomplete weight information is outlined as follows:

- **Step 1** Form the alternative set A and identify the attribute set X.
- **Step 2** Elicit the intuitionistic fuzzy decision matrix $\tilde{\boldsymbol{R}}$.
- **Step 3** Acquire the preference information structure Λ of attribute importance.
- **Step 4** Solve Eq. (24) to obtain (δ_{ij}, θ_i) and then determine the weight vector $\boldsymbol{w} = (w_1, w_1, \ldots, w_n)^{\mathrm{T}}$ of attributes by Eq. (23).
- **Step 5** Compute the comprehensive value $\tilde{r}_i$ of alternative a_i by using Eq. (3) as follows:

$$\begin{aligned} \tilde{r}_i &= \mathrm{IFWA}_w(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}) \\ &= \left\langle 1 - \prod_{j=1}^{n}(1 - \mu_{ij})^{w_j},\ \prod_{j=1}^{n}\upsilon_{ij}^{w_j} \right\rangle \quad (i = 1, 2, \ldots, m). \end{aligned} \tag{25}$$

- **Step 6** Calculate the novel risk attitudinal ranking measure $P_Q(I(\tilde{r}_i), \lambda)$ to sort $\tilde{r}_i$ ($i = 1, 2, \ldots, m$), and then generate the ranking order of alternatives.

Remark 6. The main differences between Xu [40] and this paper lie in: (1) Xu [38] defined the satisfaction degree of the alternative by using the intuitionistic fuzzy positive ideal solution and negative ideal solution, while we propose the relative closeness degree of alternative by the positive ideal IFS and negative ideal IFSs; (2) To derive the weights of attributes, Xu [40] constructed an optimization model by maximizing the satisfaction degrees. This model is non-linear and not easy to be solved. In this paper, a fractional programming model is established and transformed into linear programming model, which can be solved easily to get the optimal solution; (3) In Xu [40] the comprehensive value z_i of alternative a_i is defined as $z_i = \left(\sum_{j=1}^{n} w_j\mu_{ij}, \sum_{j=1}^{n} w_j\upsilon_{ij}, \sum_{j=1}^{n} w_j\pi_{ij}\right)$, whereas in this paper the comprehensive value $\tilde{r}_i$ of alternative a_i is defined as $\tilde{r}_i = \mathrm{IFWA}_w(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}) = \left\langle 1 - \prod_{j=1}^{n}(1 - \mu_{ij})^{w_j},\ \prod_{j=1}^{n}\upsilon_{ij}^{w_j} \right\rangle$; (4) Xu [40] ranked the alternatives using existing method [20] ignoring the DM's risk attitude, while this paper develops a new risk attitudinal measure for ranking alternatives sufficiently taking the DM's risk attitude into consideration.

6. A real teacher selection example and comparison analysis

In this section, a teacher selection problem is illustrated to demonstrate the applicability and implementation process of the MADM method proposed in this paper. The comparison analyses of computational results are also performed to show the superiority of the proposed method.

6.1. An example study of a teacher selection

Suppose that an educational institution desires to hire a teacher. After preliminary screening, four candidates (i.e., alternatives) a_1,

Table 8
The IF decision matrix.

	x_1	x_2	x_3	x_4	x_5
a_1	⟨0.7,0.2⟩	⟨0.6,0.4⟩	⟨0.5,0.4⟩	⟨0.3,0.4⟩	⟨0.4,0.5⟩
a_2	⟨0.6,0.1⟩	⟨0.8,0.1⟩	⟨0.6,0.2⟩	⟨0.7,0.1⟩	⟨0.5,0.4⟩
a_3	⟨0.7,0.1⟩	⟨0.7,0.2⟩	⟨0.8,0.1⟩	⟨0.6,0.3⟩	⟨0.8,0.1⟩
a_4	⟨0.6,0.2⟩	⟨0.6,0.3⟩	⟨0.7,0.1⟩	⟨0.8,0.1⟩	⟨0.7,0.2⟩

a_2, a_3 and a_4 remain for further evaluation. There are five attributes In the decision making, including work attitude x_1, oral communication skills x_2, moral character x_3, past experience x_4 and teaching ability. x_5. After the data acquisition and statistical treatment, the ratings of the candidates on attributes can be represented by IFVs as in Table 8, where ⟨0.7, 0.2⟩ in Table 8 is an IFV which indicates that the satisfaction degree of the candidate a_1 with respect to the attribute x_1 is 0.7, while the dissatisfaction degree is 0.2. In other words, the hesitation degree is 0.1. Other IFVs in Table 8 are explained similarly.

According to the comprehensions and judgments of DM, the preference information structure Λ of attribute importance is provided by the DM as follows:

$$\Lambda = \{\boldsymbol{w} \in \Lambda_0 | w_1 > 0.15;\quad w_2 > 0.2;\quad w_1 < 0.5w_5, w_4 > 0.2w_2;$$
$$0.1 < w_3 - w_4 < 0.3;\quad w_3 - w_1 > w_5 - w_2\}.$$

- **Step 1** According to Eq. (24), the linear programming model is constructed as follows:

$$\max\quad Z' = 1.8(\delta_{21}+\delta_{22}+\delta_{24}+\delta_{31}+\delta_{32}+\delta_{33}+\delta_{35}+\delta_{43}+\delta_{44}) + 1.6(\delta_{11}+\delta_{23}+\delta_{41}+\delta_{45}) + 1.4(\delta_{34}+\delta_{42}) + 1.2(\delta_{12}+\delta_{13}+\delta_{14}+\delta_{25}) + \delta_{15}$$
$$s.t.\begin{cases} 2(1.1\delta_{11}+\delta_{12}+1.1\delta_{13}+1.3\delta_{14}+1.1\delta_{15}) = 1;\quad 2(1.3\delta_{21}+1.1\delta_{22}+1.2\delta_{23}+1.2\delta_{24}+1.1\delta_{25}) = 1 \\ 2(1.2\delta_{31}+1.1\delta_{32}+1.1\delta_{33}+1.1\delta_{34}+1.1\delta_{35}) = 1;\quad 2(1.2\delta_{41}+1.1\delta_{42}+1.2\delta_{43}+1.1\delta_{44}+1.1\delta_{45}) = 1 \\ \delta_{11}+\delta_{12}+\delta_{13}+\delta_{14}+\delta_{15} = \theta_1;\quad \delta_{21}+\delta_{22}+\delta_{23}+\delta_{24}+\delta_{25} = \theta_2;\quad \delta_{31}+\delta_{32}+\delta_{33}+\delta_{34}+\delta_{35} = \theta_3;\quad \delta_{41}+\delta_{42}+\delta_{43}+\delta_{44}+\delta_{45} = \theta_4; \\ \delta_{11} > 0.15\theta_1;\quad \delta_{21} > 0.15\theta_2;\quad \delta_{31} > 0.15\theta_3;\quad \delta_{41} > 0.15\theta_4;\quad \delta_{12} > 0.2\theta_1;\quad \delta_{22} > 0.2\theta_2;\quad \delta_{32} > 0.2\theta_3;\quad \delta_{42} > 0.2\theta_4; \\ \delta_{11} < 0.5\delta_{15};\quad \delta_{21} < 0.5\delta_{25};\quad \delta_{31} < 0.5\delta_{35};\quad \delta_{41} < 0.5\delta_{45};\quad \delta_{14} > 0.2\delta_{12};\quad \delta_{24} > 0.2\delta_{22};\quad \delta_{34} > 0.2\delta_{32};\quad \delta_{44} > 0.2\delta_{42}; \\ 0.1\theta_1 < \delta_{13}-\delta_{14} < 0.3\theta_1;\quad 0.1\theta_2 < \delta_{23}-\delta_{24} < 0.3\theta_2;\quad 0.1\theta_3 < \delta_{33}-\delta_{34} < 0.3\theta_3;\quad 0.1\theta_4 < \delta_{43}-\delta_{44} < 0.3\theta_4; \\ \delta_{13}-\delta_{11} > \delta_{15}-\delta_{12};\quad \delta_{23}-\delta_{21} > \delta_{25}-\delta_{22};\quad \delta_{33}-\delta_{31} > \delta_{35}-\delta_{32};\quad \delta_{43}-\delta_{41} > \delta_{45}-\delta_{42}. \end{cases} \tag{26}$$

Solving Eq. (26) by Lingo Software, we can obtain

$\theta_1 = 0.4627$, $\theta_2 = 0.4337$, $\theta_3 = 0.4484$, $\theta_4 = 0.4386$, $\delta_{11} = 0.0694$, $\delta_{12} = 0.1487$, $\delta_{13} = 0.0760$, $\delta_{14} = 0.0297$, $\delta_{15} = 0.1389$, $\delta_{21} = 0.0651$, $\delta_{22} = 0.1394$, $\delta_{23} = 0.0713$, $\delta_{24} = 0.0279$, $\delta_{25} = 0.1301$, $\delta_{31} = 0.0673$, $\delta_{32} = 0.0897$, $\delta_{33} = 0.1309$, $\delta_{34} = 0.0179$, $\delta_{35} = 0.1427$, $\delta_{41} = 0.0658$, $\delta_{42} = 0.0877$, $\delta_{43} = 0.1096$, $\delta_{44} = 0.0439$, $\delta_{45} = 0.1315$.

Then, the weights of attributes are calculated by Eq. (23) as follows:

$w_1 = 0.1500$, $w_2 = 0.3214$, $w_3 = 0.1643$, $w_4 = 0.0643$, $w_5 = 0.300$.

- **Step 2** According to Eq. (25), the comprehensive values of alternatives are obtained as follows:

$\tilde{r}_1 = \langle 0.5347,\ 0.3855\rangle$, $\tilde{r}_2 = \langle 0.6640,\ 0.1699\rangle$, $\tilde{r}_3 = \langle 0.7468,\ 0.1341\rangle$, $\tilde{r}_4 = \langle 0.6653,\ 0.1945\rangle$.

- **Step 3** Calculate the novel risk attitudinal ranking measures to sort candidates.

For example, we take $Q(s) = s^t$, $t = 1$ and get the following results by Eq. (17):

$P_Q(I(\tilde{r}_1), \lambda) = 0.7830$, $P_Q(I(\tilde{r}_2), \lambda) = 0.8490$, $P_Q(I(\tilde{r}_3), \lambda) = 0.8833$, $P_Q(I(\tilde{r}_4), \lambda) = 0.8483$.

Therefore, the ranking order of candidates is $a_3 \succ a_2 \succ a_4 \succ a_1$. The best candidate is a_3.

Additionally, when t takes different values, we get the corresponding results of computation and ranking orders shown in Table 9.

It can be seen from Table 9 that, the best candidate may be different for different values of parameter t if $Q(s) = s^t$. When $t = 0$, the best candidate is a_1; when $t \in [0.5, 2]$, the best candidate is a_3; when $t \to +\infty$, the best candidate is also a_3. The ranking orders of candidates are also not completely identical.

Similarly, the results of computation and ranking orders can be obtained for other BUM functions, which are listed in the Tables 10 and 11, respectively.

To make sensitivity analysis about parameter t and BUM functions clearly, we depict the computation results of Tables 9–11 in Fig. 6.

Fig. 6 shows that the raking orders are distinct for different BUM functions. At the same time, the raking orders are also different for each BUM function with different values of parameter t. The above analysis suggests that both the parameter t and the BUM function indeed play an important role in the decision making. Involving different BUM functions with different parameter values t, i.e., considering the different risk attitudes of DM is very reasonable and necessary to rank the IFVs.

6.2. Comparison analysis with the method of MADM with IFNs

In this subsection, we use the method [24] to solve the above teacher selection problem for further interpreting the advantages of the proposed method in this paper.

- **Step 1** The decision matrix is constructed in Table 8.
- **Step 2** Using Eq. (16) with $q = 2$ in [24], the scores of the four alternatives on the five attribute are obtained based on DM's attitudinal character (orness level) α. For illustration, Table 12 provides the scores of each alternatives on the five attribute at orness level $\alpha = 0.6$.
- **Step 3** By Eq. (8) in [24], the attribute weights are determined and shown in Table 13 with different orness level α.
- **Step 4** and **Step 5** Aggregate the alternative's scores on each attribute by MEOWA operator in [24]. Here we list the aggregated scores and ranking orders of alternatives with different α in Table 14.

It is see from Table 14 that when $\alpha \in [0, 0.7]$, a_3 is the best alternative; when $\alpha \in [0.8, 0.9]$, a_2 is the best alternative; when $\alpha = 1$, both a_2 and a_3 are the best alternatives. The results obtained by the method [24] are different from those obtained by the method in this paper. The primary reasons may come from three aspects below.

(1) Although both the two methods in [24] and this paper take the DM's risk attitude into account, they have some differences in

Table 9
The computation results for $Q(s) = s^t$ with different values of parameter t and ranking orders.

t	$P_Q(I(\tilde{r}_1), \lambda)$	$P_Q(I(\tilde{r}_2), \lambda)$	$P_Q(I(\tilde{r}_3), \lambda)$	$P_Q(I(\tilde{r}_4), \lambda)$	Ranking order	Best candidate
0	0.9968	0.9862	0.9929	0.9902	$a_1 > a_3 > a_4 > a_2$	a_1
0.5	0.8543	0.8948	0.9199	0.8956	$a_3 > a_4 > a_2 > a_1$	a_3
1	0.7830	0.8490	0.8833	0.8483	$a_3 > a_2 > a_4 > a_1$	a_3
2	0.7117	0.8033	0.8468	0.8010	$a_3 > a_2 > a_4 > a_1$	a_3
$+\infty$	0.5691	0.7119	0.7738	0.7064	$a_3 > a_2 > a_4 > a_1$	a_3

Table 10
Computation results for $Q(s) = 1 - e^{-s}(1-e^{-s})/(1-e^{-1})^t$ with different values of parameter t and ranking orders.

t	$P_Q(I(\tilde{r}_1), \lambda)$	$P_Q(I(\tilde{r}_2), \lambda)$	$P_Q(I(\tilde{r}_3), \lambda)$	$P_Q(I(\tilde{r}_4), \lambda)$	Ranking order	Best candidate
0	0.9968	0.9862	0.9929	0.9901	$a_1 > a_3 > a_4 > a_2$	a_1
0.5	0.8811	0.9120	0.9336	0.9134	$a_3 > a_4 > a_2 > a_1$	a_3
1	0.8180	0.8715	0.9013	0.8716	$a_3 > a_4 > a_2 > a_1$	a_3
2	0.7490	0.8273	0.8660	0.8258	$a_3 > a_2 > a_4 > a_1$	a_3
$+\infty$	0.5691	0.7119	0.7738	0.7064	$a_3 > a_2 > a_4 > a_1$	a_3

Table 11
Computation results for $Q(s) = (\sin((\pi/2)s))^t$ with different values of parameter t and ranking orders.

t	$P_Q(I(\tilde{r}_1), \lambda)$	$P_Q(I(\tilde{r}_2), \lambda)$	$P_Q(I(\tilde{r}_3), \lambda)$	$P_Q(I(\tilde{r}_4), \lambda)$	Ranking order	Best candidate
0	0.9968	0.9862	0.9929	0.9902	$a_1 > a_3 > a_4 > a_2$	a_1
1	0.8414	0.8865	0.9133	0.8871	$a_3 > a_4 > a_2 > a_1$	a_3
2	0.7830	0.8490	0.8833	0.8483	$a_3 > a_2 > a_4 > a_1$	a_3
$+\infty$	0.5691	0.7119	0.7738	0.7064	$a_3 > a_2 > a_4 > a_1$	a_3

Table 12
Scores of $E_d(x)$ on five attributes at orness level $\alpha = 0.6$.

	x_1	x_2	x_3	x_4	x_5
a_1	0.7797	0.6000	0.5799	0.5397	0.4800
a_2	0.8356	0.8793	0.7589	0.8577	0.5799
a_3	0.8577	0.7797	0.8793	0.6798	0.8793
a_4	0.7589	0.6798	0.8577	0.8793	0.7797

ranking the IFVs. Chen et al. [24] used the orness degree of a RIM quantifier to character the DM's attitudinal preference. But this paper applies the C-OWA operator to defuzzy the interval that is formed by the amount and reliability of information of IFV. Then the risk attitude parameter can be derived by different BUM functions in C-OWA operator, which provides more choices for DMs and greatly enhances the flexibility of decision making.

(2) The method [24] determined the attribute weights by a maximum-entropy OWA model. The derived weights are entirely unrelated to the evaluation information of the candidates on attributes given by DM. In other words, if two decision problems have identical number of attributes and orness level, the attributes in the two problems would possess the same weights, which is unreasonable. In this paper, considering DM's subjective judgement, incomplete information of attribute weights is given a prior. Then a fractional programming is established to determine the attribute weights, which is comprehensive and accordance with the real-world situations.

6.3. Comparison analysis with the GRA method for MADM

In this subsection, we compare the results obtained by the grey relational analysis (GRA) method [41] and the method proposed in this paper. We use the method [41] to solve the above example for explaining the importance of the risk attitude of DM in this paper.

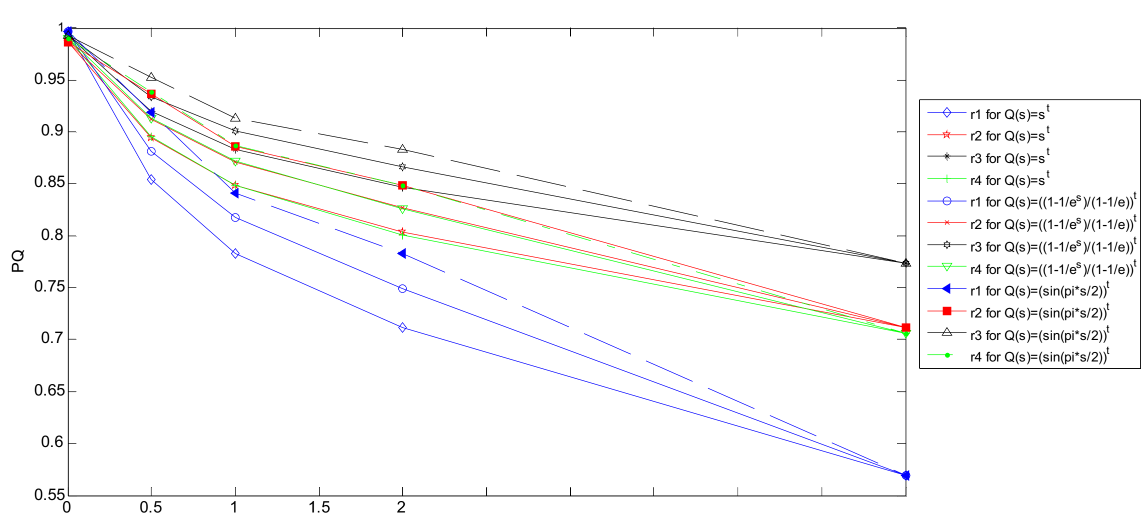

Fig. 6. Comparison between different BUM functions with different values of parameter t.

Table 13
weights of each attributes with different orness level α.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
x_1	1	0.0050	0.0289	0.0706	0.1278	0.2000	0.2884	0.3962	0.5307	0.7105	1
x_2	0	0.0175	0.0599	0.1086	0.1565	0.2000	0.2353	0.2574	0.2565	0.2068	0
x_3	0	0.0602	0.1240	0.1672	0.1920	0.2000	0.1920	0.1672	0.1240	0.0602	0
x_4	0	0.2068	0.2565	0.2574	0.2353	0.2000	0.1566	0.1086	0.0598	0.0175	0
x_5	0	0.7105	0.5307	0.3962	0.2884	0.2000	0.1277	0.0706	0.0290	0.0050	0

Table 14
Aggregated scores and ranking orders of alternatives for different orness level α.

α	a_1	a_2	a_3	a_4	Ranking order	Best candidate
0	0.4651	0.5648	0.8631	0.7641	$a_3 \succ a_4 \succ a_2 \succ a_1$	a_3
0.1	0.4812	0.6346	0.8201	0.7863	$a_3 \succ a_4 \succ a_2 \succ a_1$	a_3
0.2	0.5019	0.6758	0.8051	0.7900	$a_3 \succ a_4 \succ a_2 \succ a_1$	a_3
0.3	0.5253	0.7068	0.7985	0.7863	$a_3 \succ a_4 \succ a_2 \succ a_1$	a_3
0.4	0.5572	0.7411	0.8020	0.7858	$a_3 \succ a_4 \succ a_2 \succ a_1$	a_3
0.5	0.5898	0.7726	0.8088	0.7836	$a_3 \succ a_4 \succ a_2 \succ a_1$	a_3
0.6	0.6232	0.8019	0.8184	0.7808	$a_3 \succ a_2 \succ a_4 \succ a_1$	a_3
0.7	0.6577	0.8296	0.8309	0.7778	$a_3 \succ a_2 \succ a_4 \succ a_1$	a_3
0.8	0.6946	0.8555	0.8466	0.7759	$a_2 \succ a_3 \succ a_4 \succ a_1$	a_2
0.9	0.7374	0.8797	0.8671	0.7782	$a_2 \succ a_3 \succ a_4 \succ a_1$	a_2
1	0.8000	0.9000	0.9000	0.8000	$a_2 = a_3 \succ a_4 = a_1$	a_2, a_3

- **Step 1** Determine the positive ideal and negative ideal solution:

$$\tilde{r}^+ = (\langle 0.7, 0.1\rangle\ \langle 0.8, 0.1\rangle\ \langle 0.8, 0.1\rangle\ \langle 0.8, 0.1\rangle\ \langle 0.8, 0.1\rangle),$$

$$\tilde{r}^- = (\langle 0.6, 0.2\rangle\ \langle 0.6, 0.4\rangle\ \langle 0.5, 0.4\rangle\ \langle 0.3, 0.4\rangle\ \langle 0.4, 0.5\rangle).$$

- **Step 2** Calculate the grey relation coefficient matrices of each alternative from the positive ideal and negative ideal solutions as follows:

$$\xi^+ = (\xi_{ij}^+)_{4\times 5} = \begin{bmatrix} 0.80 & 0.44 & 0.40 & 0.33 & 0.33 \\ 0.80 & 1.00 & 0.57 & 0.80 & 0.40 \\ 1.00 & 0.67 & 1.00 & 0.50 & 1.00 \\ 0.67 & 0.50 & 0.80 & 1.00 & 0.67 \end{bmatrix}, \quad \xi^+ = (\xi_{ij}^-)_{4\times 5} = \begin{bmatrix} 0.80 & 1.00 & 1.00 & 1.00 & 1.00 \\ 0.80 & 0.44 & 0.57 & 0.36 & 0.67 \\ 0.67 & 0.57 & 0.40 & 0.50 & 0.33 \\ 1.00 & 0.80 & 0.44 & 0.33 & 0.40 \end{bmatrix}.$$

- **Step 3** Establish the single-objective programming model by the model (M-2) in method [41]. Solve the established model to obtain the weight vector of attributes as $\boldsymbol{w} = (0, 0.20, 0.45, 0.35, 0)^{\mathrm{T}}$.
- Then, we can get the grey relational degree of each alternative from the positive ideal and negative ideal solutions: $\xi_1^+ = 0.3856$, $\xi_2^+ = 0.7371$, $\xi_3^+ = 0.7583$, $\xi_4^+ = 0.8100$, $\xi_1^- = 1.0000$, $\xi_2^- = 0.4733$, $\xi_3^- = 0.4693$, $\xi_4^- = 0.4767$.
- **Step 4** The relative relational degree of each alternative from the positive ideal solution can be derived as $\xi_1 = 0.2783$, $\xi_2 = 0.6090$, $\xi_3 = 0.6177$, $\xi_4 = 0.6295$.
- **Step 5** In terms of the relative relational degree ξ_i ($i = 1, 2, 3, 4$), the ranking of alternatives is obtained as $a_4 > a_3 > a_2 > a_1$. Thus, the best alternative is a_4.

It is obvious that the ranking result obtained by method [41] is not the same as that obtained by the method in this paper. Compared with the former, the latter has the following advantages:

(1) In the latter, DM can select the best teacher according to his/her risk attitude. But the former failed to consider the risk attitude of DM and only can give the single decision result for DM. In real life, different DMs could not have the same preference. The optimists are willing to accept higher risk than the pessimists. Therefore, the latter can provide DMs with more choices than the former.

(2) The Model ($M-2$) in [41] directly determined the total relative relational degree with all alternatives by linear sum of equal weight, which can eliminate the influence among different alternatives. In this situation, some weights of attributes are equal to small numbers or even zero. In this example, the weights of x_1 and x_5 are calculated to zero by the former. In fact, the work attitude and teaching ability are the important evaluation indices for choosing the teacher. Their weights should not be equal to zero. Therefore, the decision result obtained by the latter is more objective and convincing than that obtained by the former.

7. Conclusion

How to give the order relationship of IFVs is a critical issue for solving MADM problems under intuitionistic fuzzy environment. In this paper, a new lexicographic ranking method of IFVs was proposed from the angle of geometric meaning. The main advantages of the proposed new measures lie in three aspects: (1) it utilizes the closeness degree to express the information amount of IFVs, which can consider the positive and negative ideal points simultaneously; (2) the area of the $\Delta y_{\min} y y_{\max}$ intuitively measures the information reliability of IFVs, which has clearly geometric meaning according to the geometrical representation of an IFS; (3) the vector including the amount and reliability of information of the IFV can be transformed into an interval.

Considering the risk attitude of DM, we further defined a novel risk attitudinal measure for ranking the IFVs. It is derived by C-OWA operator based on the interval formed by the closeness degree and the triangle area. Therefore, this novel risk attitudinal measure has a solid theoretical basis and effectively overcome the flaw of Guo's method.

A novel approach to determining the attribute weights objectively was developed through constructing multi-objective fractional programming model which is transformed into a linear programming to resolve. Thereby, we proposed a new method to solve the MADM problems with IFVs and incomplete weight information. It has better flexibility and agility since the DM can choose different risk attitudes to make decision according to his/her preference.

However, we do not discuss the selections of BUM function and the parameter t in real-life decision problem, which is a limitation and will be studied in near future. In addition, compared with IFS, interval-valued intuitionistic fuzzy set has powerful capability in expressing uncertainty. We will further extend the proposed method to the interval-valued intuitionistic fuzzy sets. On the other hand, we can also extend the proposed method to group decision making problems.

Acknowledgments

The authors are very grateful to the editors and the anonymous referees for their insightful and constructive comments that have helped to improve the presentation and quality of the manuscript. This research was supported by the National Natural Science Foundation of China (Nos. 71061006, 61263018 and 11461030), the Humanities Social Science Programming Project of Ministry of Education of China (No. 09YGC630107), the Natural Science Foundation of Jiangxi Province of China (Nos. 20114BAB201012 and 20142BAB201011), "Twelve five" Programming Project of Jiangxi province Social Science (2013) (No. 13GL17), the Science and Technology Project of Jiangxi Province Educational Department of China (Nos. GJJ15265 and GJJ15267), Young Scientists Training Object of Jiangxi Province (No. 20151442040081), Graduate Innovation Foundation of Jiangxi Province (No. YC2015-B055) and the Excellent Young Academic Talent Support Program of Jiangxi University of Finance and Economics.

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