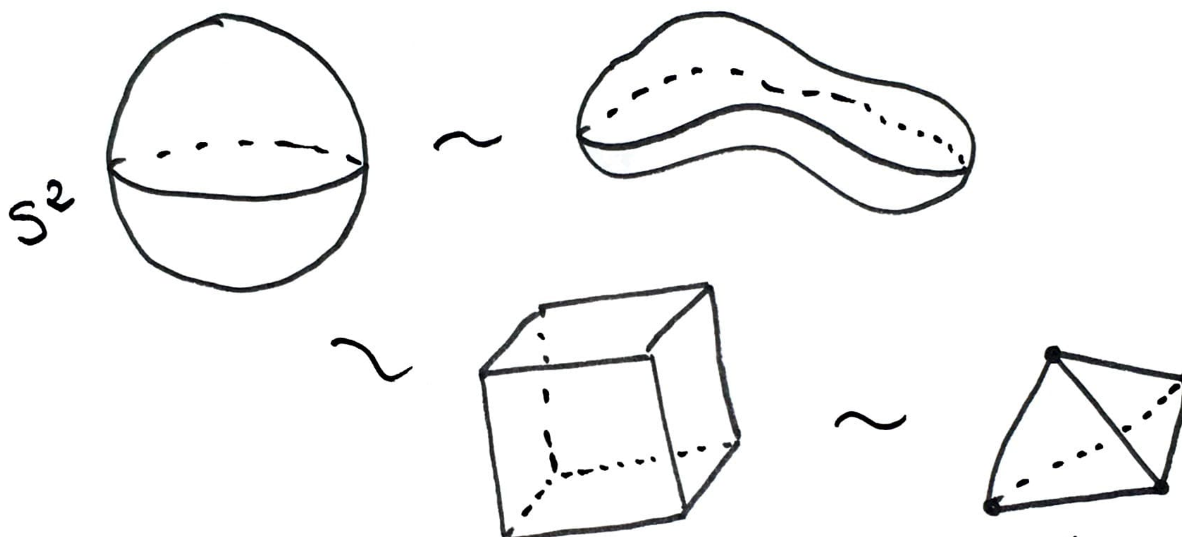
TDA 6

SIMPLICIAL HOMOLOGY, 1

In topology we only consider objects up to smooth reversible deformations: "homeomorphisms".

Ex.



flat sides....
→ affine geometry
→ linear algebra

simplest polyhedron homeomorphic to the sphere S^2

Goal: • use homeomorphisms to reduce our study to simplicial objects.

• apply affine and linear techniques to these objects.

Simplicial complexes

A set of points $\{\underbrace{p_0, p_1, \dots, p_d}_{d+1}\}$ in $\mathbb{R}^d$ is said

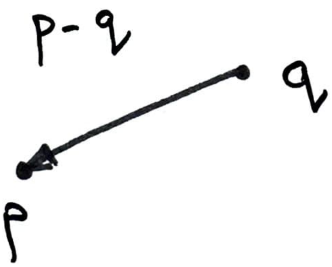
geometrically independent if the family of vectors :

$$\{p_1 - p_0 \,,\, p_2 - p_0 \,,\, \dots\dots ,\, p_d - p_0\}$$

is linearly independent.

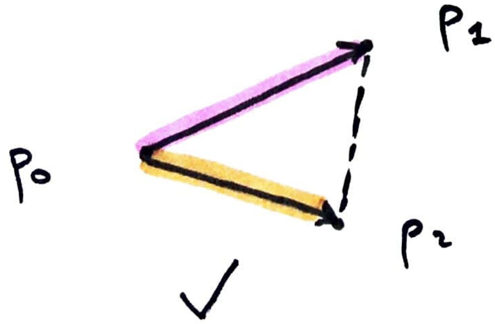
If $p = (x_p \,, y_p)$, $q = (x_q \,, y_q)$

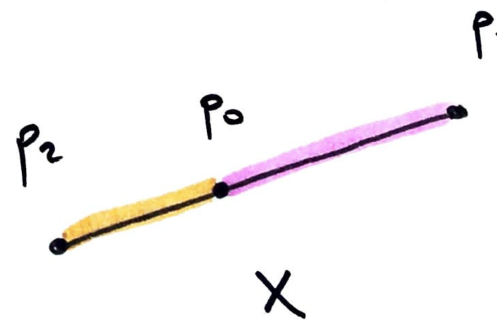
$p - q = (x_p - x_q \,, y_p - y_q)$



d = 2 $\quad \{p_0, p_1, p_2\}$ geometrically independent :

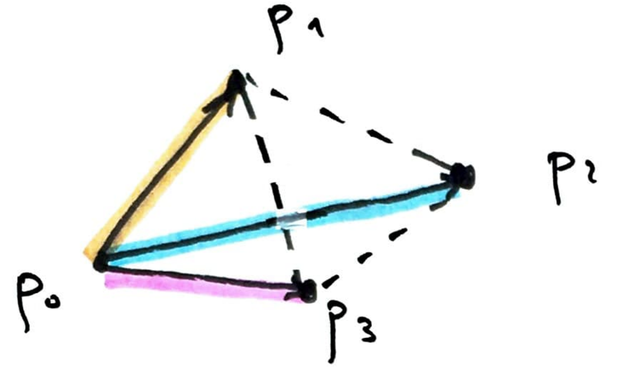
$\iff \quad \{p_1 - p_0 \,, p_2 - p_0\}$ lin. indep.





$\underline{d=3}$ $\{p_0, p_1, p_2, p_3\}$ geometrically independent

$\Leftrightarrow \{p_1 - p_0, \; p_2 - p_0, \; p_3 - p_0\}$ lin. indep.



The points must generate a proper tetrahedron (not a triangle or a square or a line...)

A combination $x = \sum_{i=0}^{d} a_i p_i$ is called convex if all a_i's are ≥ 0 and $\sum_{i=0}^{d} a_i = 1$

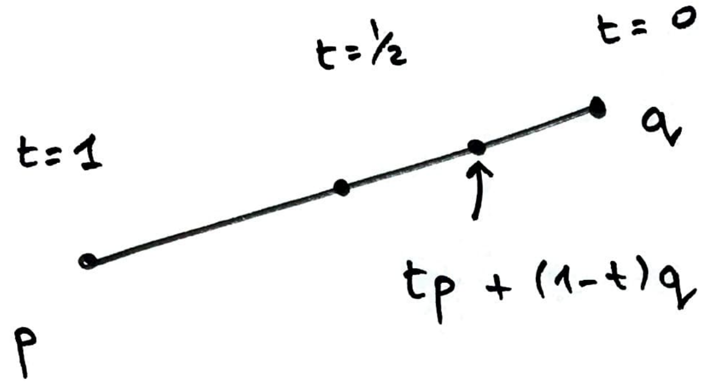
The set of all convex combinations of $\{p_0, \dots, p_d\}$ is called the convex hull of $\{p_0, \dots, p_d\}$:

$$CH(\{p_0, \dots, p_d\}) = \left\{ \sum_{i=0}^{d} a_i p_i \; , \; \begin{array}{c} \forall i \quad a_i \geq 0 \\ a_0 + a_1 + \dots + a_d = 1 \end{array} \right\}$$

If $p, q \in \mathbb{R}^2$, a convex combination of p and q is of the form $tp + (1-t)q$

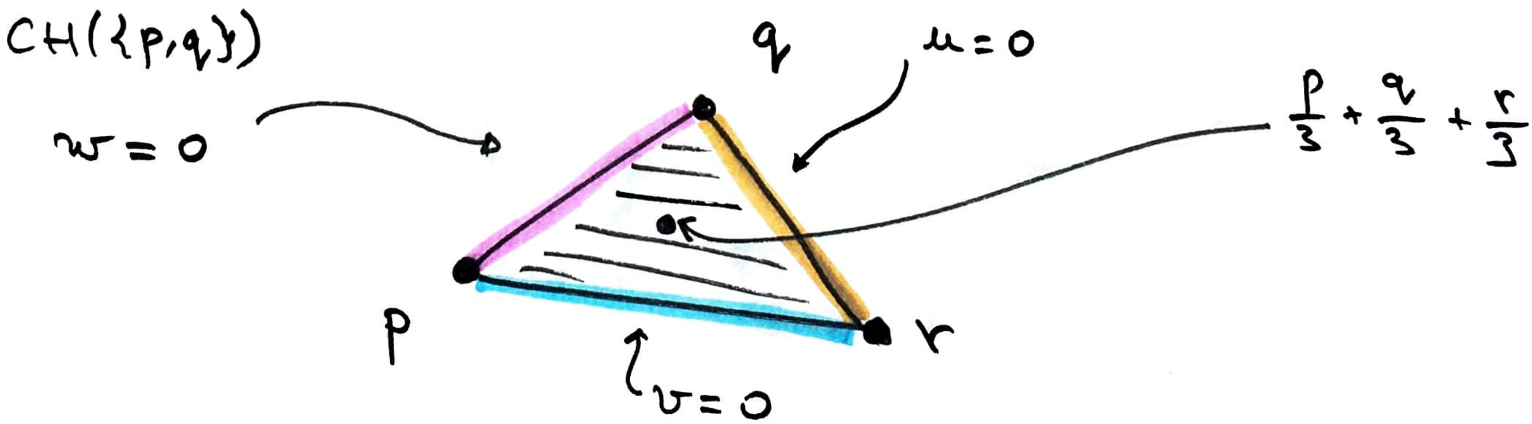
with $t \geq 0$ and $1 - t \geq 0$ so $0 \leq t \leq 1$.

For instance, if $t = \frac{1}{2}$: $\frac{1}{2}p + \frac{1}{2}q$



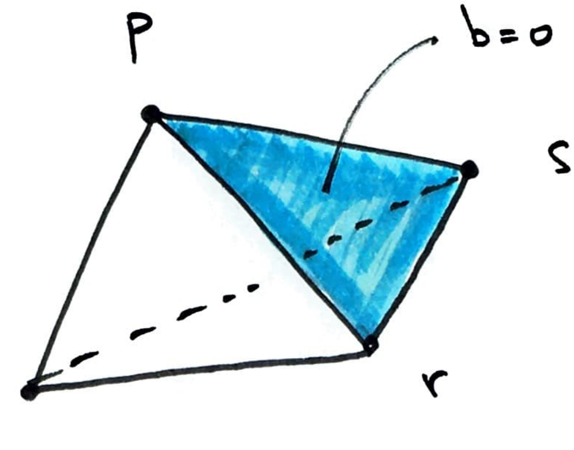
$$CH(\{p,q\}) = [p,q] \quad (\text{ line segment })$$

Similarly, $CH(\{p,q,r\})$ with p,q,r geometrically independent in $\mathbb{R}^2$ is the full triangle with vertices p,q,r :



$$up + vq + wr \;\middle|\; \begin{array}{l} u,v,w \geq 0 \\ u+v+w = 1 \end{array}$$

Similarly, $CH(\{p, q, r, s\})$ with p, q, r, s geometrically independent in $\mathbb{R}^3$ is the <u>full</u> tetrahedron :



3-dim. object.

If $ap + bq + cr + ds$ satisfies

$$\begin{cases} a, b, c, d \geq 0 \\ a + b + c + d = 1 \end{cases}$$

<u>and</u> no one of a, b, c, d is zero

then $ap + bq + cr + ds$ is <u>inside</u> the tetrahedron (not on a face).

In particular, it is not homeomorphic to a sphere (surface) but to a <u>solid</u> ball in $\mathbb{R}^3$.

<u>Def</u>. A <u>d-simplex</u> σ is the convex hull of $d+1$ geometrically independent points $\{p_0, p_1, \dots, p_d\}$ in $\mathbb{R}^d$.

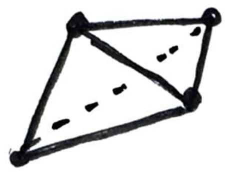
$\sigma = CH(\{p_0, \dots, p_d\}) = \operatorname{span}(p_0, \dots, p_d)$

The <u>dimension</u> of σ is d.

0-simplex :  point

1-simplex :  line segment

2-simplex :  (full) triangle

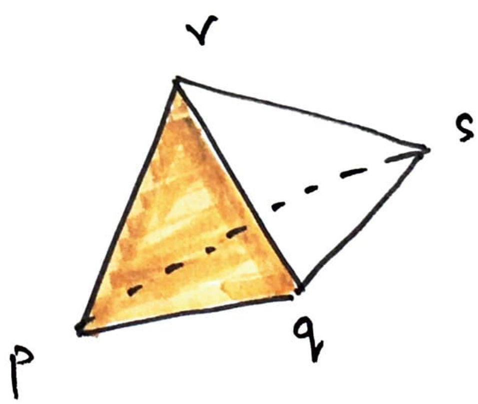
3-simplex :  (full) tetrahedron

Def If $A \subsetneq \{p_0, \dots, p_d\}$ is a non-empty strict subset, then A spans its own simplex σ_A, called a proper face of σ.

$\sigma = \operatorname{span}(p, q, r, s)$

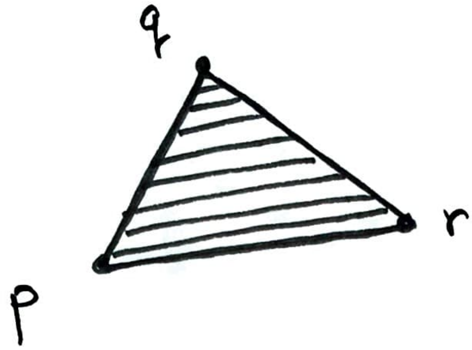
$A = \{p, q, r\}$

$\sigma_A = \operatorname{span}(p, q, r)$



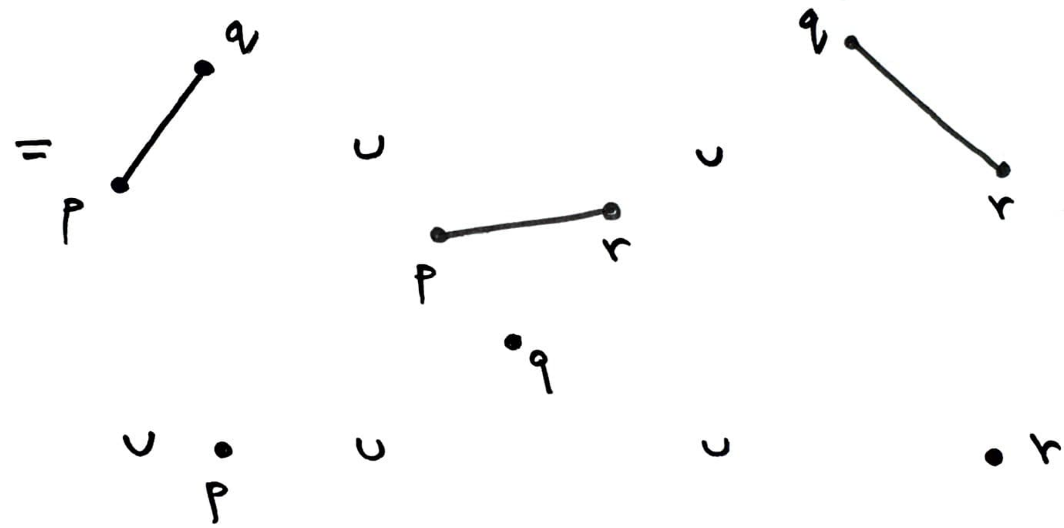
Def: If σ is a simplex, its boundary $\partial\sigma$ is the union of all proper faces of σ.

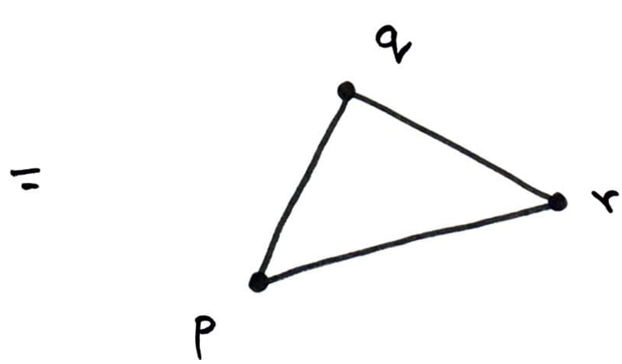
For instance, if $\sigma = \text{span}\{p, q, r\}$.



Then $\partial\sigma = \text{span}(p,q) \cup \text{span}(p,r) \cup \text{span}(q,r)$

$\cup\ \text{span}(p) \cup \text{span}(q) \cup \text{span}(r)$





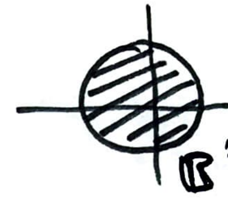
: empty triangle.

- the boundary of a segment is the pair of its extremities

- the boundary of a tetrahedron is homeomorphic to a sphere.

If $\mathbb{B}^d$ is the unit ball in $\mathbb{R}^d$:

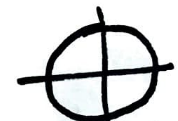
$$\mathbb{B}^d = \{(x_1, \dots, x_d) : x_1^2 + \dots + x_d^2 \leq 1\}$$



$\mathbb{B}^2$

and $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^d$:

$$\mathbb{S}^{d-1} = \{(x_1, \dots, x_d) : x_1^2 + \dots + x_d^2 = 1\}$$



$\mathbb{S}^1$

then a d-simplex is homeomorphic to $\mathbb{B}^d$ and its boundary is homeomorphic to $\mathbb{S}^{d-1}$.

$$\sigma \simeq \mathbb{B}^d \quad , \quad \partial\sigma \simeq \mathbb{S}^{d-1}$$

__Idea__: to build more sophisticated objects, assemble simplices together.

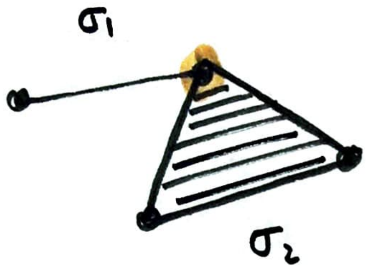
__Def__ A __simplicial complex__ K is a collection of simplices such that:

(1) If $\sigma \in K$, then for any face σ' of σ, $\sigma' \in K$.

(2) For two simplices $\sigma_1, \sigma_2 \in K$,

$\sigma_1 \cap \sigma_2 = \emptyset$ __or__ $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

Ex.

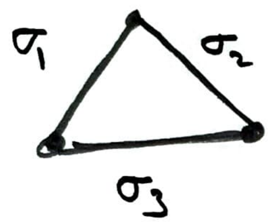


$\sigma_1 =$ 

$\sigma_2 =$ 

$\sigma_1 \cap \sigma_2 =$ 

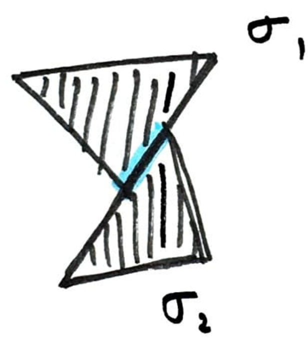
Ex.



is not a simplex but it is a simplicial complex.

(empty triangle)

Non-example:

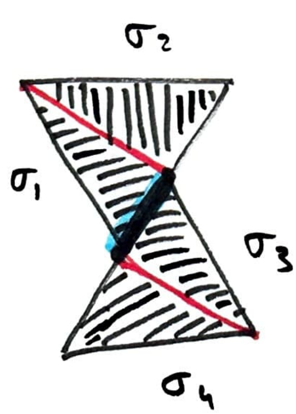


not a simplicial complex:

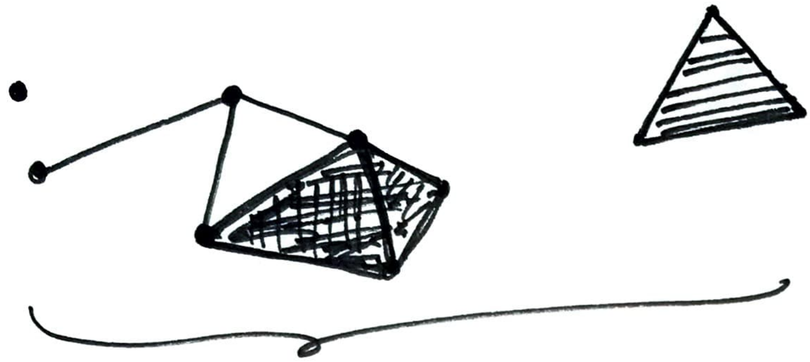
$\sigma_1 \cap \sigma_2 =$ 

not a full face for σ_1 or σ_2.

Fix:



$K = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is a simplicial complex, with gluing along proper faces.



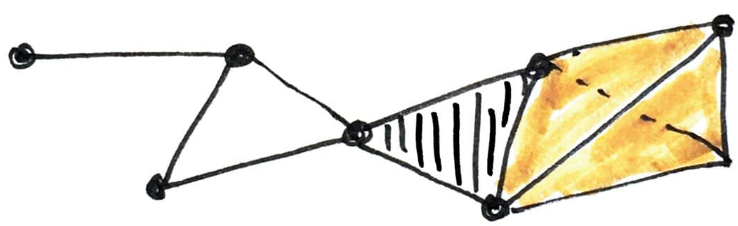
K

Rk: if σ is a simplex, then $\partial\sigma$ is a simplicial complex.

Def. The j-skeleton of a simplicial complex K is

$$K^{(j)} = \{\sigma \in K \mid \dim(\sigma) \leq j\}$$

Ex.



K

full tetrahedron.

$K^{(3)} = K$

$K^{(2)} = K \setminus \{3\text{-simplices}\}$

empty tetrahedron

$K^{(1)}$:

$K^{(0)}$:

Def. A subcomplex of K is a subset $K' \subseteq K$ which is still a simplicial complex.

Def. An abstract simplicial complex is a finite collection of sets K such that if σ is a set in K, then all subsets of σ are also in K.

Idea: think of points (singletons) as the 0-skeleton.

Fact. Any abstract simplicial complex can be realized as a geometric simplicial complex.

Next:
- calculate topological invariants of simplicial complexes.
- given a data set, construct many simplicial complexes.