TDA 7

SIMPLICIAL HOMOLOGY, 2

Goal: calculate topological invariants of simplicial complexes $\subseteq \mathbb{R}^d$.

Given a simplicial complex, consider the family of all skeletons:

$K = K^{(2)}$. 2. skeleton

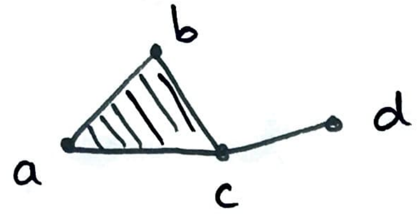
$K^{(1)} =$

1 - skeleton

$K^{(0)} =$

0 - skeleton

Consider p-chains = formal combinations of p-simplices in K.



2-simplex = [a b c]

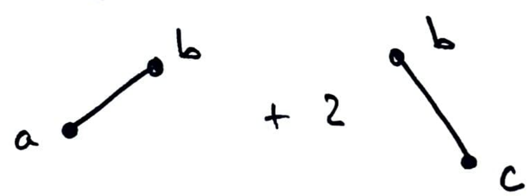
1-simplex: [a b], [b c], [a c]

[c d]

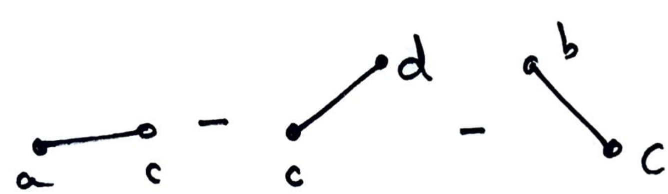
0-simplices (vertices): [a], [b], [c], [d].

$$C_p = \{ \sum a_i \sigma_i \ , \quad \sigma_i : \text{p-simplices in } K \}$$

$$x = [a\,b] + 2[b\,c] \quad \in C_1$$



$$y = [a\,c] - [c\,d] - [b\,c] \quad \in C_1$$

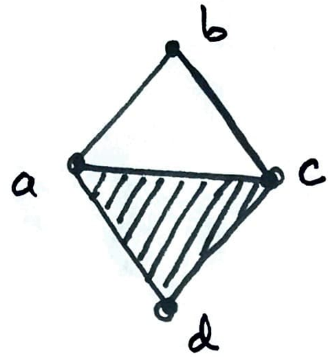


$$x + 2y = [a\,b] + 2[b\,c] + 2[a\,c] - 2[c\,d] - 2[b\,c]$$

$$= [a\,b] + 2[a\,c] - 2[c\,d]$$

C_p is called the space of p-chains of K.

Consider : $K =$



- $C_2 = \{ \lambda \, [a\,c\,d] \, , \, \lambda \in \mathbb{R} \}$ $\qquad$ $\dim C_2 = 1$

In C_2 : $\quad \lambda [a\,c\,d] + \mu [a\,c\,d] = (\lambda + \mu) [a\,c\,d]$

- C_1 is generated by $[a\,b], [b\,c], [a\,c], [a\,d], [d\,c]$

In C_1 : $\quad 2[a\,b] - [a\,c] + 3[d\,c]$

$\dim C_1 = 5$

- C_0 is generated by $[a], [b], [c], [d]$.

$\dim C_0 = 4$

We construct the boundary map:

$$\partial_p : C_p \longrightarrow C_{p-1}$$

$\sigma \in C_p$ is a combination of p-simplices:

$$\sigma = \sum a_i \sigma_i \qquad \sigma_i : \text{p-simplex}$$

We will have ∂_p linear so

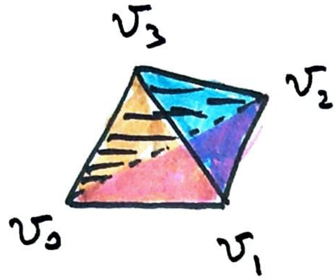
$$\partial_p(\sigma) = \sum a_i \partial_p(\sigma_i)$$

To define ∂_p on a p-simplex σ:

let $\sigma = [v_0 \, v_1 \dots v_p]$

$$\partial_p(\sigma) = \sum_i (-1)^i [v_0 \dots \hat{v_i} \dots v_p]$$

where $[v_0 \dots \hat{v_i} \dots v_p]$ is the p-1-simplex obtained by removing v_i from $\sigma = [v_0, \dots v_p]$.

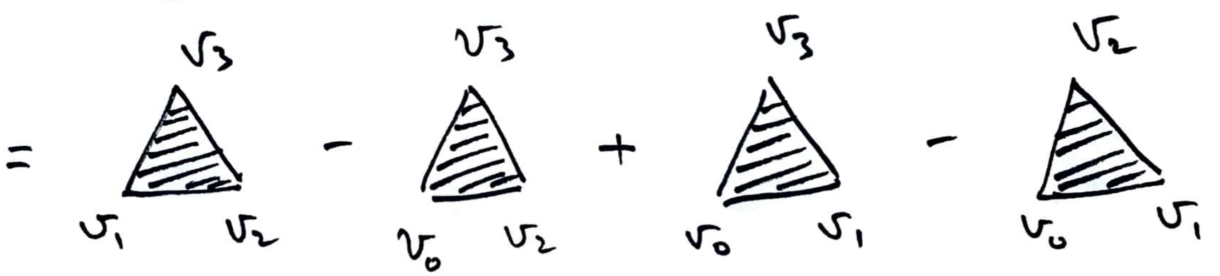
- 

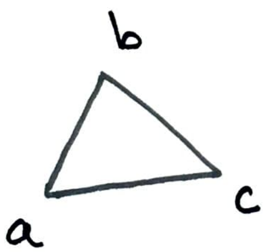
$\sigma = [v_0 v_1 v_2 v_3] \in C_3$

$$\partial_3(\sigma) = (-1)^0 [\hat{v}_0 v_1 v_2 v_3] + (-1)^1 [v_0 \hat{v}_1 v_2 v_3] + \cdots$$

$$= [v_1 v_2 v_3] - [v_0 v_2 v_3] + [v_0 v_1 v_3] - [v_0 v_1 v_2]$$

$\in C_2$



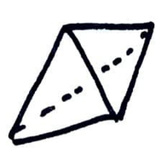
- 

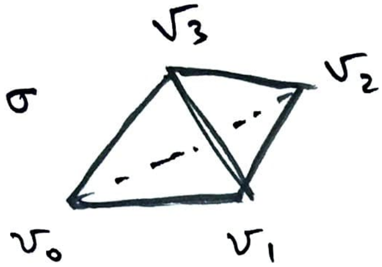
$\sigma = [a\, b\, c] \in C_2$

$$\partial_2(\sigma) = [b\, c] - [a\, c] + [a\, b] \in C_1$$

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1$$

$[v_0 v_1 v_2 v_3] \longmapsto ?$





$\in C_3$

$$\partial_3(\sigma) = [v_1 v_2 v_3] - [v_0 v_2 v_3] + [v_0 v_1 v_3] - [v_0 v_1 v_2]$$

$$\partial_2(\partial_3(\sigma)) = \partial_2([v_1 v_2 v_3]) - \partial_2([v_0 v_2 v_3]) + \partial_2([v_0 v_1 v_3]) - \partial_2([v_0 v_1 v_2])$$

$$
\begin{aligned}
= \; & [v_2 v_3] - [v_1 v_3] + [v_1 v_2] \\
& - \left([v_2 v_3] - [v_0 v_3] + [v_0 v_2] \right) \\
& + [v_1 v_3] - [v_0 v_3] + [v_0 v_1] \\
& - \left([v_1 v_2] - [v_0 v_2] + [v_0 v_1] \right)
\end{aligned}
$$

$$= 0$$

$$\begin{tikzcd} C_3 \arrow[r, "\partial_3"] \arrow[rr, bend right, "0"'] & C_2 \arrow[r, "\partial_2"] & C_1 \end{tikzcd}$$

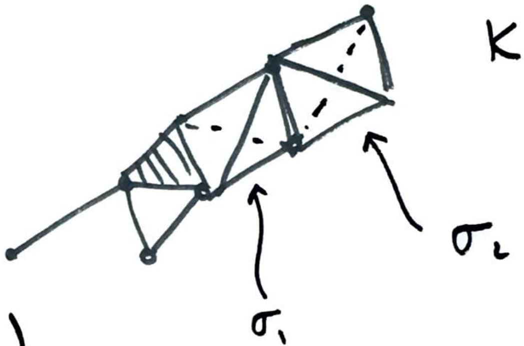
HW: check that $\begin{tikzcd} C_2 \arrow[r, "\partial_2"] \arrow[rr, bend right, "0"'] & C_1 \arrow[r, "\partial_1"] & C_0 \end{tikzcd}$

We checked directly that for <u>any</u> tetrahedron σ,

$$\partial_2(\partial_3(\sigma)) = 0 \in C_1$$

Now, every element in C_3 is a combination of tetrahedra: $\quad x = \sum_i a_i \sigma_i$

with $\sigma_i =$ 



$$\partial_2(\partial_3(\underbrace{2\sigma_1 - 3\sigma_2}_{\in C_3}))$$

$$= 2\,\underbrace{\partial_2(\partial_3(\sigma_1))}_{=0} - 3\,\underbrace{\partial_2(\partial_3(\sigma_2))}_{=0}$$

$$= 0$$

<u>More generally</u>:

<u>Th</u>. The combination $\begin{tikzcd} C_{p+1} \arrow[r, "\partial_{p+1}"] \arrow[rr, bend right] & C_p \arrow[r, "\partial_p"] & C_{p-1} \end{tikzcd}$

is 0: for any

$\sigma \in C_{p+1}$, $\quad \partial_p(\partial_{p+1}(\sigma)) = 0 \in C_{p-1}$

We will use the maps ∂_p to count holes of any dimension in simplicial complexes.

Def. Given K a simplicial complex, the sequence

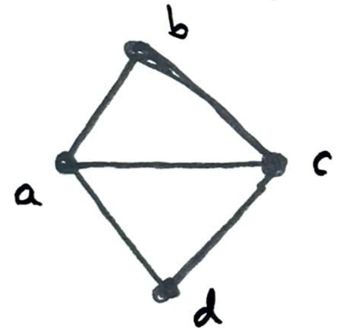
$$\cdots \longrightarrow C_{p+2} \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots \xrightarrow{\partial_1} C_0$$

is called the <u>chain complex</u> associated with K.

→ How to count holes ?

Compare two examples:

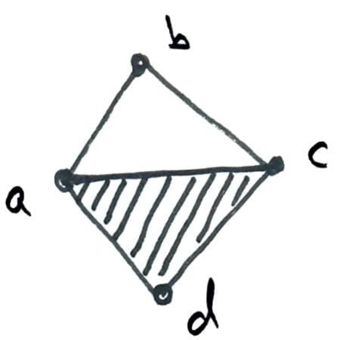
A :



$\simeq$ homeo.

2 holes

B :

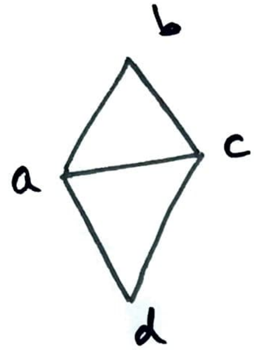


$\simeq$

1 hole

For both examples, let us calculate the chain complex and study $\mathrm{Ker}\, \partial_p$ for each p.

Example A :



$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

$C_2 = \{0\}$, $C_1 \simeq \mathbb{R}^5$ generated by

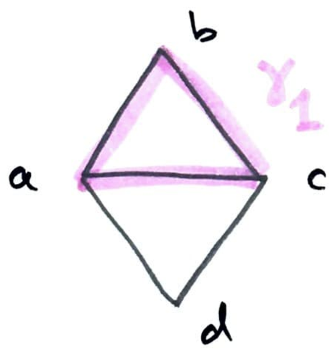
$[ab], [bc], [ca], [ad], [dc]$

$C_0 \cong \mathbb{R}^4$, generated by $[a], [b], [c], [d]$.

$$\partial_1 : C_1 \longrightarrow C_0$$

$$\left[\ \operatorname{Ker} \partial_1 = \{ x \in C_1 \mid \partial_1(x) = 0 \in C_0 \}\ \right]$$

Consider $\gamma_1 = [ab] + [bc] + [ca]$

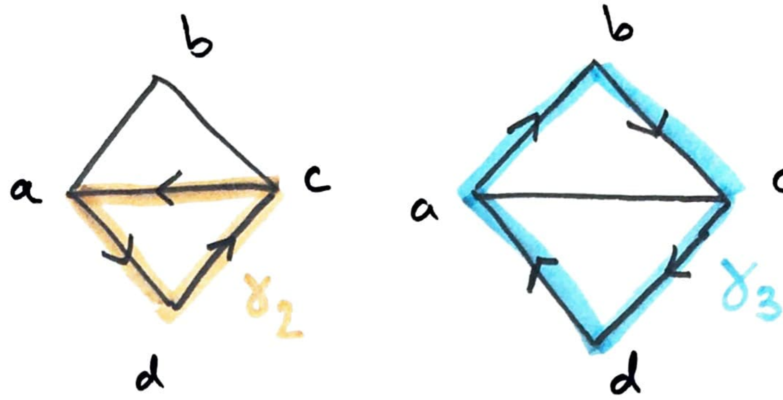


Prove that $\gamma_1 \in \operatorname{Ker} \partial_1$.

$$\partial_1(\gamma_1) = \partial_1([ab]) + \partial_1([bc]) + \partial_1([ca])$$
$$= [b] - [a] + [c] - [b] + [a] - [c] = 0$$

$\gamma_1 \in \operatorname{Ker} \partial_1$.

Consider also:



$\gamma_2 = [ca] + [ad] + [dc]$

$\gamma_3 = [ab] + [bc] + [cd] + [da]$.

Check : $\partial_1(\gamma_2) = 0$ $\partial_1(\gamma_3) = 0$

We have found 3 different elements in $\mathrm{Ker}\,\partial_1$.

γ_3 does not correspond to a hole ....

Do we really have $\dim \mathrm{Ker}\,\partial_1 = 3$? No

Notice that $\gamma_3 = \gamma_1 + \gamma_2$

$$\gamma_3 = \underbrace{[ab] + [bc] + [ca]}_{=\gamma_1} \overbrace{}^{0} \underbrace{- [ca] + [cd] + [da]}_{=\gamma_2}$$

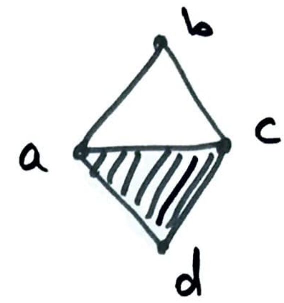
It follows that $\partial_1(\gamma_3) = \partial_1(\gamma_1) + \partial_1(\gamma_2) = 0$

so $\gamma_3 \in \mathrm{Ker}\,\partial_1$ generated by γ_1, γ_2.

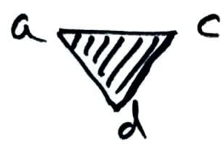
$\dim \mathrm{Ker}\,\partial_1 = 2$ ■

Hope : $\dim \operatorname{Ker} \partial_1 =$ number of holes ??

Does it work on example B :



Here $C_2 \simeq \mathbb{R}$ generated

by 

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$C_1 \simeq \mathbb{R}^5$ generated by $[ab], [bc], [ac], [ad], [cd]$

$C_0 \simeq \mathbb{R}^4$ ———— $[a], [b], [c], [d]$.

 Example A $\qquad \{0\} \longrightarrow \mathbb{R}^5 \longrightarrow \mathbb{R}^4$

 Example B $\qquad \mathbb{R} \longrightarrow \mathbb{R}^5 \longrightarrow \mathbb{R}^4$

In C_1 of  , we still have γ_1, γ_2 :

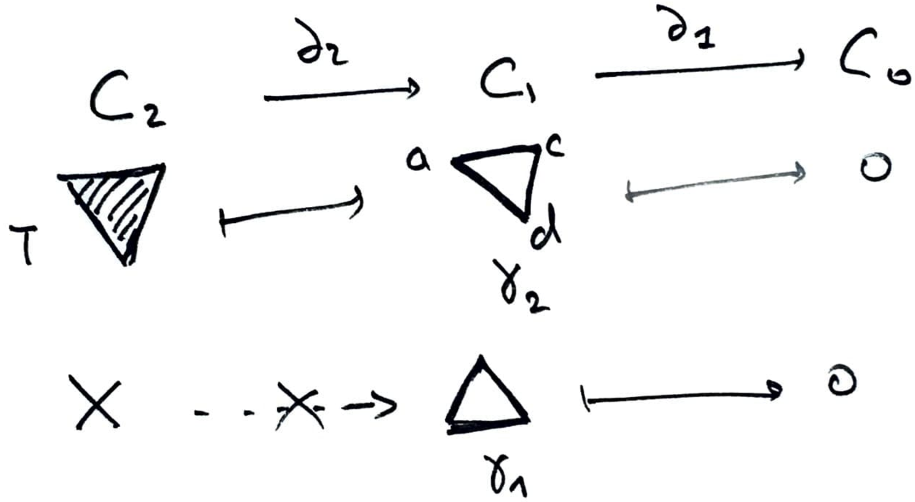
$\operatorname{Ker} \partial_1$ is still generated by γ_1 and γ_2.

so $\dim \operatorname{Ker} \partial_1 = 2$ but there is only 1 hole...

The issue is that in C_1, we cannot see the difference between a full triangle and an empty one ...

In fact, γ_2 comes from C_2:

$$\gamma_2 = \partial_2(T) \quad \text{where } T = [a\,c\,d]$$



Holes come from triangles that are <u>not</u> in

$$\text{Im}\,\partial_2 = \{\partial_2(x),\ x \in C_2\}.$$

To count holes, calculate $\dim \text{Ker}\,\partial_1 - \dim \text{Im}\,\partial_2$.

Here : $2 - 1 = 1$

Given a simplicial complex K, consider the chain complex

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots\cdots$$

with $\partial_p \circ \partial_{p+1} = 0$, that is:

$$\mathrm{Im}\,\partial_{p+1} \subseteq \mathrm{Ker}\,\partial_p$$

The p^{th} homology group of K is:

$$H_p = \mathrm{Ker}\,\partial_p \,/\, \mathrm{Im}\,\partial_{p+1}$$

$$\dim H_p = \dim \mathrm{Ker}\,\partial_p - \dim \mathrm{Im}\,\partial_{p+1}$$

$= \beta_p$: p^{th} Betti number of K.

<u>Ex. A</u> : $\beta_1($  $) = 2 - 0 = 2$

<u>Ex. B</u> : $\beta_1($  $) = 2 - 1 = 1$