TDA 4

TOPOLOGICAL PROPERTIES, 2

Connected spaces = "in one piece"

- continuous maps transform connected spaces into connected spaces.
- If two spaces are homeomorphic, and one is connected, then so is the other.

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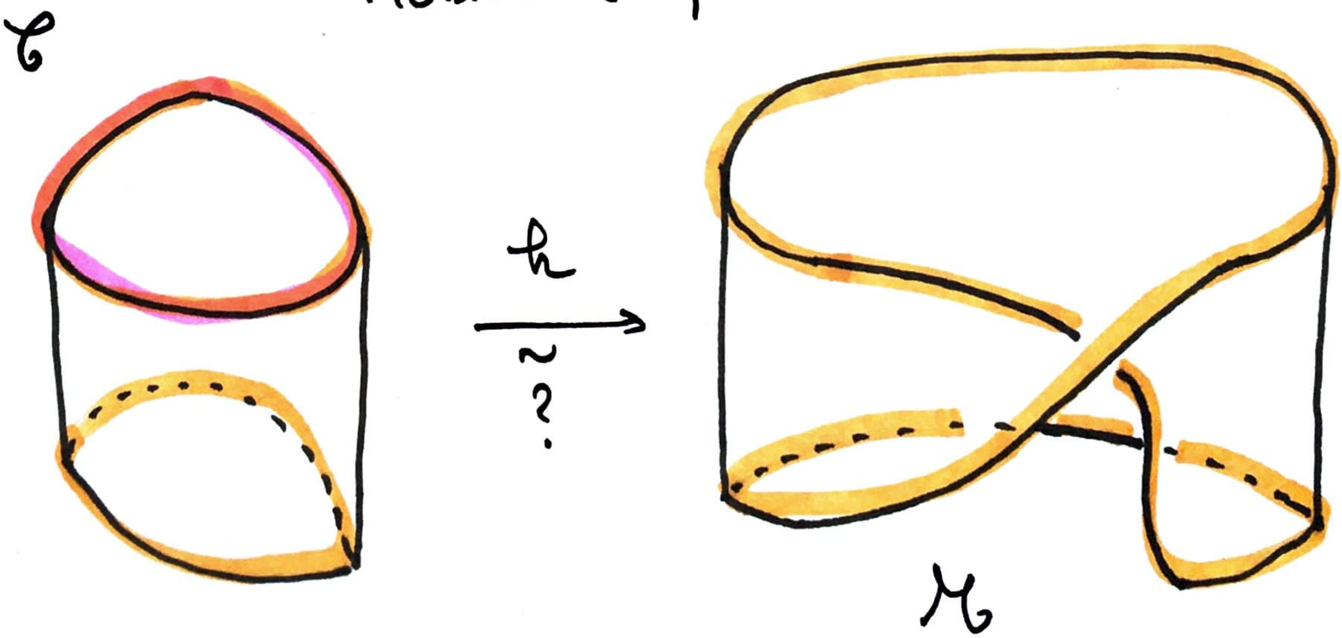
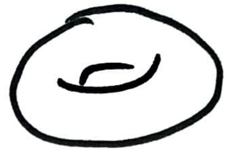
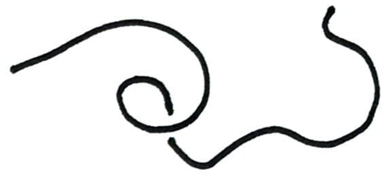
Application: is a cylinder homeomorphic to a Möbius strip?



Remark: both $\mathcal{C}$ and $\mathcal{M}$ are connected...

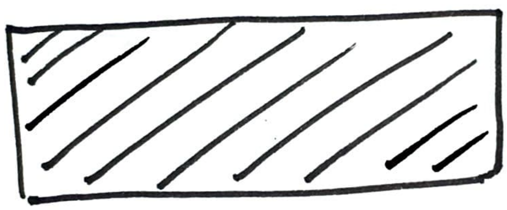
but so are  and , which are

not homeomorphic...

Same goes for  and 

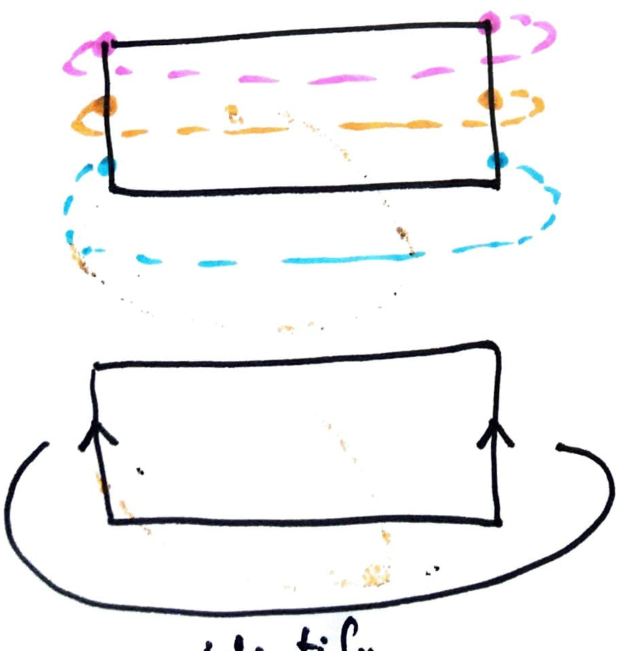
dim 2 dim 1

Both $\mathcal{C}$ and $\mathcal{M}$ are constructed from a strip:



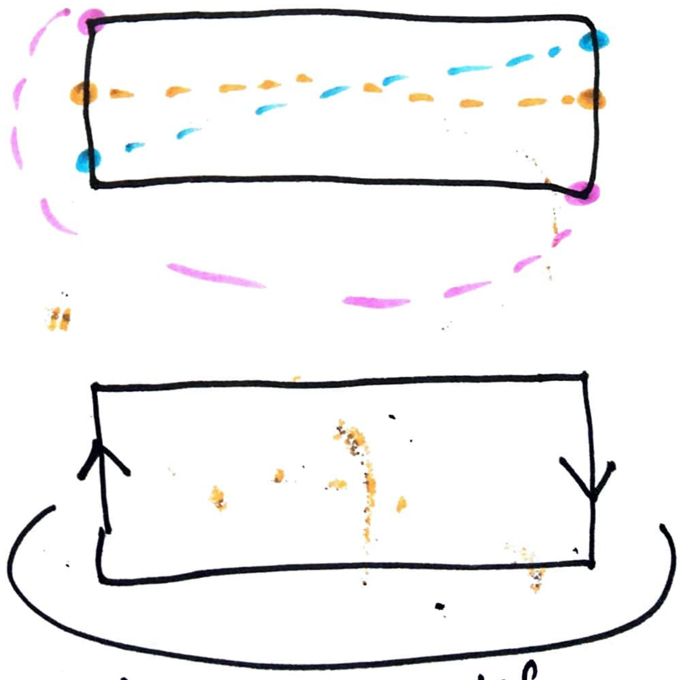
with different gluings:

$\mathcal{C}$



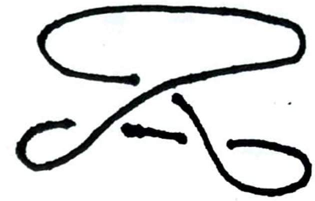
identify

$\mathcal{M}$



twist and identify

Study the respective boundaries of $\mathcal{T}$ and $\mathcal{M}$:

$\partial \mathcal{T} =$  , $\partial \mathcal{M} =$ 

disconnected $\simeq S^1 \sqcup S^1$ connected $\simeq S^1$

Now if there was a homeomorphism

$$h : \mathcal{T} \xrightarrow{\sim} \mathcal{M}$$

then it would induce a homeomorphism

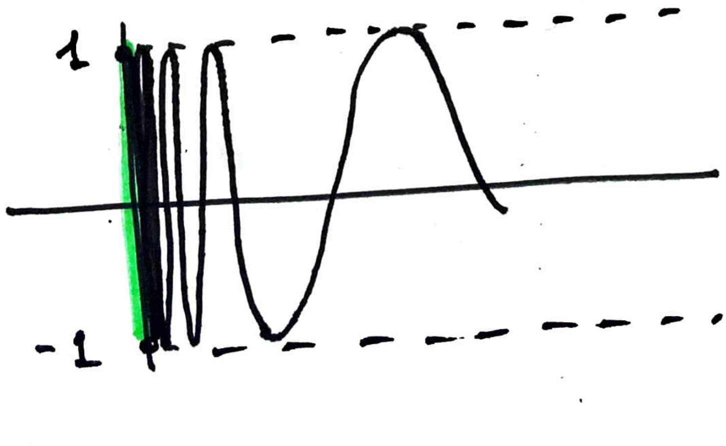
$$\partial \mathcal{T} \xrightarrow{\sim} \partial \mathcal{M}$$

which is impossible since one is connected and not the other.

<u>Ex</u>. Topologist's Sine Curve

$$\mathcal{S} = \left\{ \left(x, \sin\left(\tfrac{1}{x}\right)\right), x > 0 \right\} \cup \left\{ (0, y), -1 \leq y \leq 1 \right\}$$

$\subseteq \mathbb{R}^2$



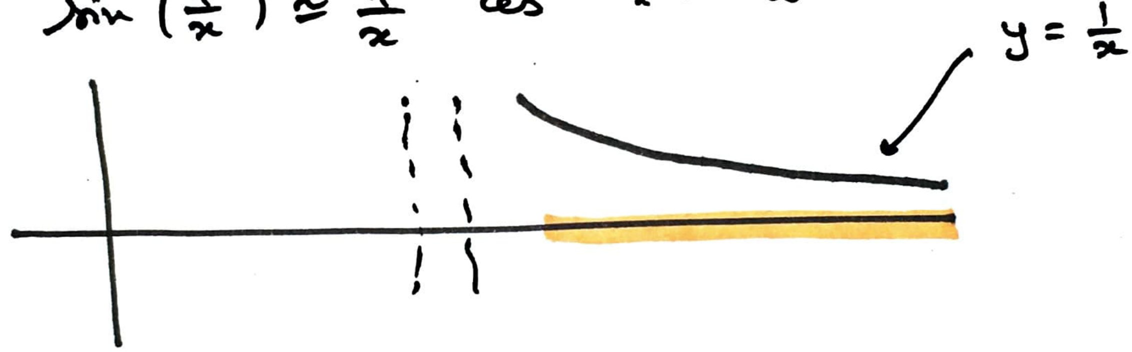
<u>Fact</u> (HW): $\mathcal{S}$ is connected in the sense that the only clopen subsets of $\mathcal{S}$ are $\emptyset$ and $\mathcal{S}$.

On the limits on $\sin\left(\frac{1}{x}\right)$:

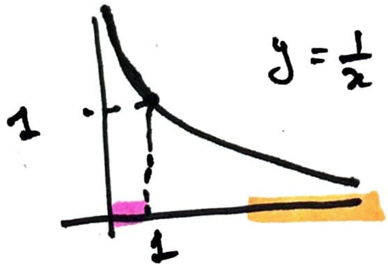
If $x \to \infty$, $\frac{1}{x} \longrightarrow 0$ so $\sin\left(\frac{1}{x}\right) \longrightarrow 0$ (by continuity of sin).

In fact we know that if $t \simeq 0$, then $\sin(t) \simeq t$

$\Rightarrow$ $\sin\left(\frac{1}{x}\right) \simeq \frac{1}{x}$ as $x \to \infty$



Behavior of $\sin\left(\frac{1}{x}\right)$ as $x \longrightarrow 0^+$



$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \, .$$

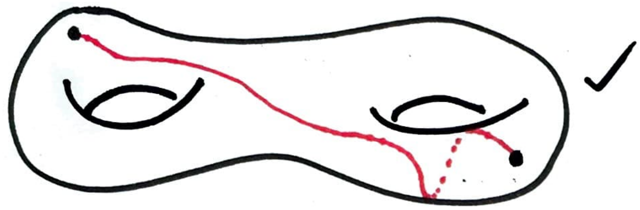
For $x \in (0,1]$, ~~sin~~ $\frac{1}{x}$ takes all values in $[1, \infty)$

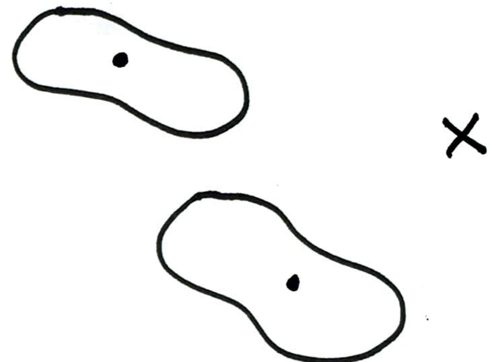
$$(0,1] \longrightarrow [1,+\infty) \xrightarrow{\sin} [-1,1]$$
$$x \longmapsto \frac{1}{x} \longmapsto \sin\left(\frac{1}{x}\right)$$

sin oscillates from -1 to 1 infinitely many times on $[1,+\infty)$!

Path connectedness

<u>Idea</u>: an object is connected if every point can be linked to every other point by some continuous path on the object.





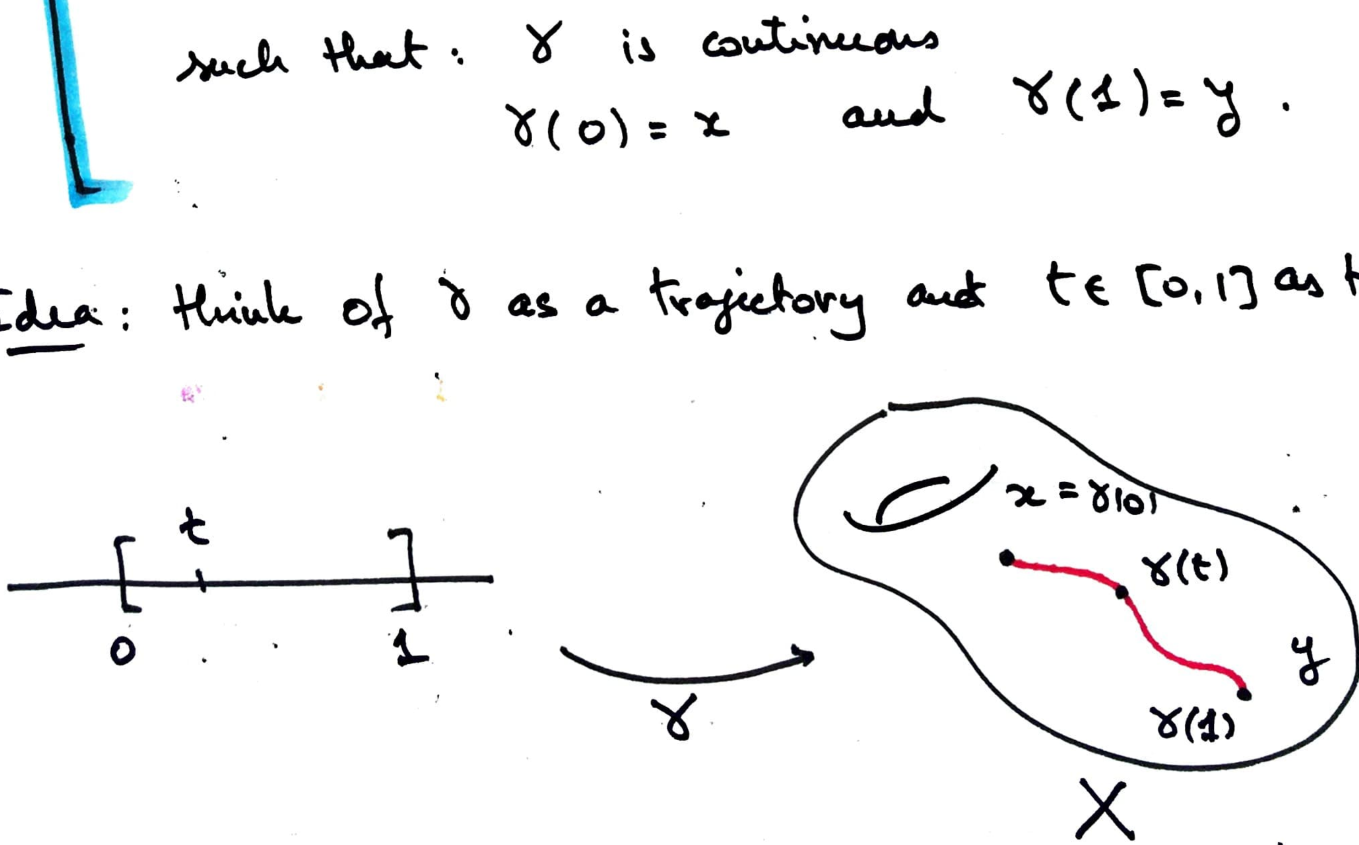
<u>Def</u> Let X be a metric space, $x, y \in X$.
A <u>path</u> from x to y <u>on X</u> is a function

$$\gamma : [0,1] \longrightarrow X$$

such that: γ is continuous

$\gamma(0) = x$ and $\gamma(1) = y$.

<u>Idea</u>: think of γ as a trajectory and $t \in [0,1]$ as time.

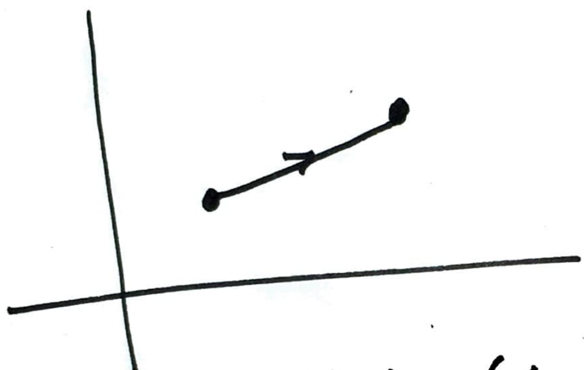


Ex. Let $X = \mathbb{R}^2$

Find a path γ in $\mathbb{R}^2$ from $(1,2)$ to $(3,5)$.

Ex. Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$

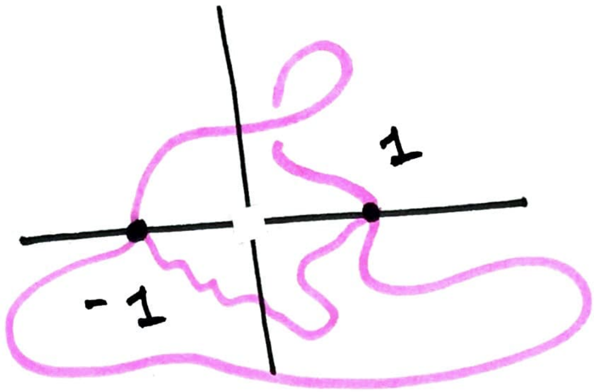
Find a path γ in $\mathbb{R}^2 \setminus \{(0,0)\}$ from $(-1,0)$ to $(1,0)$.



$$\gamma : [0,1] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\gamma(t) = (1-t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{if } t=0$$
$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{if } t=1$$

$$\gamma(t) = \begin{pmatrix} 1-t \\ 2-2t \end{pmatrix} + \begin{pmatrix} 3t \\ 5t \end{pmatrix} = \begin{pmatrix} 1+2t \\ 2+3t \end{pmatrix} \quad \checkmark$$



$\sigma(t) = \begin{pmatrix} 2t-1 \\ 0 \end{pmatrix}$ is a path from $(-1,0)$

to $(1,0)$ in $\mathbb{R}^2$: 

Unfortunately, it goes through $(0,0)$ at $t=\frac{1}{2}$...

so it is not a path in $\mathbb{R}^2 \setminus \{(0,0)\}$

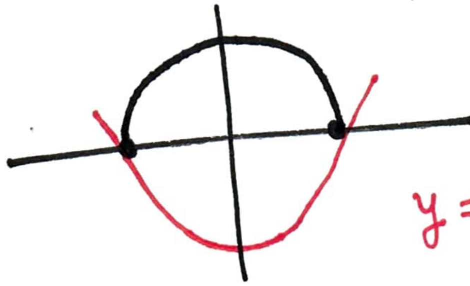
Attempt: $\gamma(t) = -t^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -t^2-1 \\ 0 \end{pmatrix}$

$t=0$: $(-1,0)$ ✓

$t=1$: $(-2,0)$ ✗

$x^2+y^2=1$



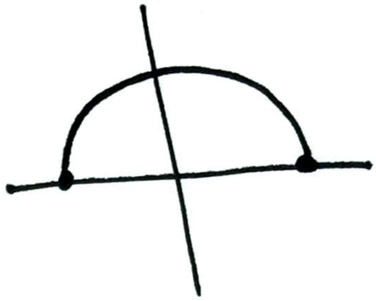
$y=x^2-1$

Strategy: • find a parametrization with $t \in I$.

• transfer it from I to $[0,1]$.

$x^2+y^2=1$ $\left\{\begin{array}{l}\cos(t)\\ \sin(t)\end{array}\right.$

$t \in [0,\pi]$

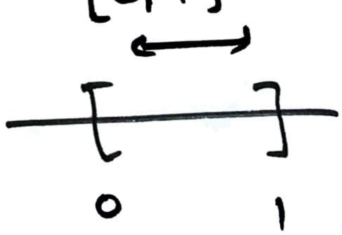


$$[0,1] \longrightarrow [0,\pi] \longrightarrow \begin{pmatrix}\cos\\ \sin\end{pmatrix}$$
$$t \longmapsto \pi t \longmapsto \begin{pmatrix}\cos \pi t\\ \sin \pi t\end{pmatrix}$$
$$0 \longmapsto 0 \longmapsto \begin{pmatrix}1\\ 0\end{pmatrix}$$
$$1 \longmapsto \pi \longmapsto \begin{pmatrix}-1\\ 0\end{pmatrix}$$

How to reverse a path ?

Consider
$$t \longmapsto \gamma(1-t)$$
$$0 \longmapsto \gamma(1)$$
$$1 \longmapsto \gamma(0)$$
$$\tfrac{1}{2} \longmapsto \gamma(\tfrac{1}{2})$$

$t \longmapsto 1-t$ is mirror symmetry accross $\frac{1}{2}$ for $[0,1]$



Now, consider

$$\gamma(t) = \begin{pmatrix}\cos \pi(1-t)\\ \sin \pi(1-t)\end{pmatrix} = \begin{pmatrix}-\cos \pi t\\ \sin \pi t\end{pmatrix} \subseteq S^1 \checkmark$$

$t \mapsto 1-t$ reverses the orientation of $[0,1]$

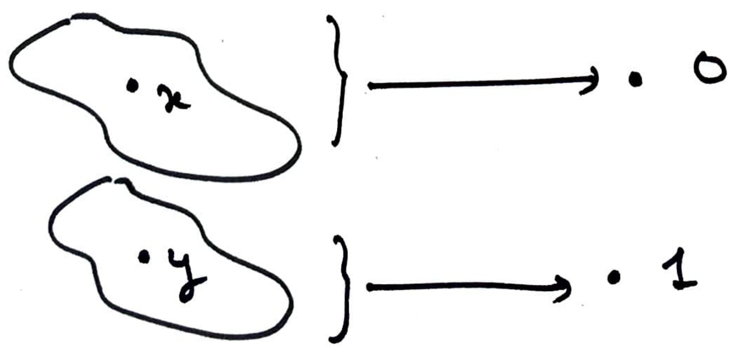
Def. A metric space X is said <u>path-connected</u> if $\forall x, y \in X$, there exists a path γ in X from x to y.

$\longrightarrow$ Is this the same as being connected ?

Th If X is path-connected, then it is connected

Proof. Assume X is not connected. Then by yesterday's result, there is a continuous, surjective map

$$f : X \longrightarrow \{0,1\}$$



Let $x \in X$ be such that $f(x) = 0$ and $y \in X$ s.t. $f(y) = 1$.

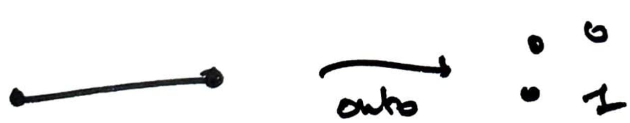
Since X is path-connected, there exists a path γ in X such that $\gamma(0) = x$ and $\gamma(1) = y$.

$$
\begin{array}{ccccc}
[0,1] & \xrightarrow{\gamma} & X & \xrightarrow{f} & \{0,1\} \\
t & \longmapsto & \gamma(t) & \longmapsto & f(\gamma(t)) \\
0 & \longmapsto & x & \longmapsto & 0 \\
1 & \longmapsto & y & \longmapsto & 1
\end{array}
$$

$f \circ \gamma$ is continuous as the composition of two continuous functions.

$\Rightarrow$ $f \circ \gamma$ is a continuous surjective map from $[0,1]$ to $\{0,1\}$.

Impossible ! $[0,1]$ is connected so there can be no such continuous surjection... ■



- Path-connected $\Rightarrow$ connected
- The converse does not hold (HW) : Topologists' Sine curve is connected but <u>not</u> path-conn.
- If $\mathcal{U}$ is <u>open</u> in $\mathbb{R}^n$ (Euclidean) then $\mathcal{U}$ is connected $\Longleftrightarrow$ $\mathcal{U}$ is path-connected.

Connected components

Def. Let X be a metric space. A <u>connected component</u> of X is a subset $\mathcal{U} \subseteq X$ such that: $\mathcal{U}$ is connected <u>and</u> if $\mathcal{U} \subsetneq K \subseteq X$, then K is disconnected

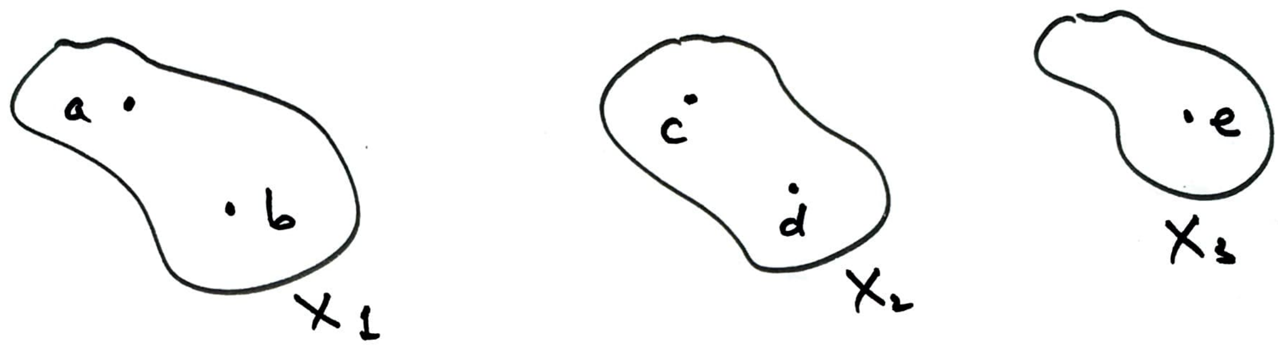
Denote by $\pi_0(X)$ the set of all connected components of X.

<u>Rk</u>: X connected $\Longleftrightarrow$ $\pi_0(X)$ has only one element

Equip X with the equivalence relation

$x \sim y$ if there is a connected subset $\mathcal{U} \subseteq X$ with $x \in \mathcal{U}$ and $y \in \mathcal{U}$.

In other words, $x \sim y$ if they belong to the same connected component:



$$X = X_1 \cup X_2 \cup X_3$$

$a \sim b$ because $a, b \in X_1$ connected.

$a \not\sim c$ because no connected subset of X contains a and c.

If $f : X \longrightarrow Y$ is a continuous map between metric spaces X and Y, define

$$f_* : \pi_0(X) \longrightarrow \pi_0(Y)$$

by $f_*(\mathcal{U})$ = connected component in Y of $f(x)$ for any x in $\mathcal{U}$.

<u>Th</u>. f_* is well-defined and if f is a homeomorphism, then f_* is a bijection between $\pi_0(X)$ and $\pi_0(Y)$.