TDA 3

TOPOLOGICAL PROPERTIES

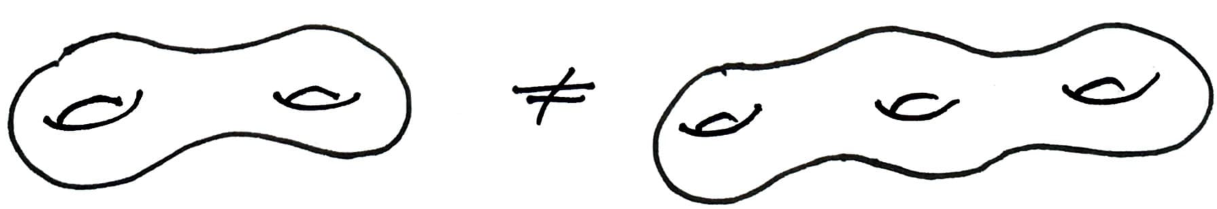
Given a data set, construct a top. space ($\subseteq$ metric space) an determine topological features, i.e., *shape*.

→ What are topological features?

These features should not be affected by "nice" smooth deformations of the space.

<u>Today's goals</u>:
- homeomorphisms
- connectedness
- compactness

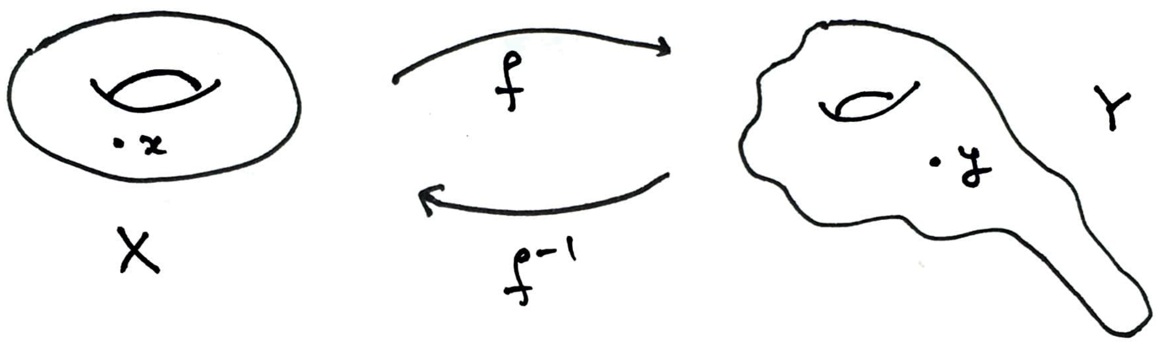
<u>Next</u>:
- counting generalized holes



Def. If (X, d_X) and (Y, d_Y) are metric spaces, a homeomorphism between X and Y is a map

$$f : X \longrightarrow Y$$

that is bijective (1-1 and onto), continuous and such that $f^{-1} : Y \longrightarrow X$ is also continuous.



Bijective means reversible:

$$f^{-1} : Y \longrightarrow X$$
$$y \longmapsto \text{the only } x \in X \text{ such that } f(x) = y.$$

Existence of x : f is onto (surjective).
Uniqueness of x : f is 1-1 (injective).

$$\left(f^{-1}(y) = x \iff y = f(x) \right.$$

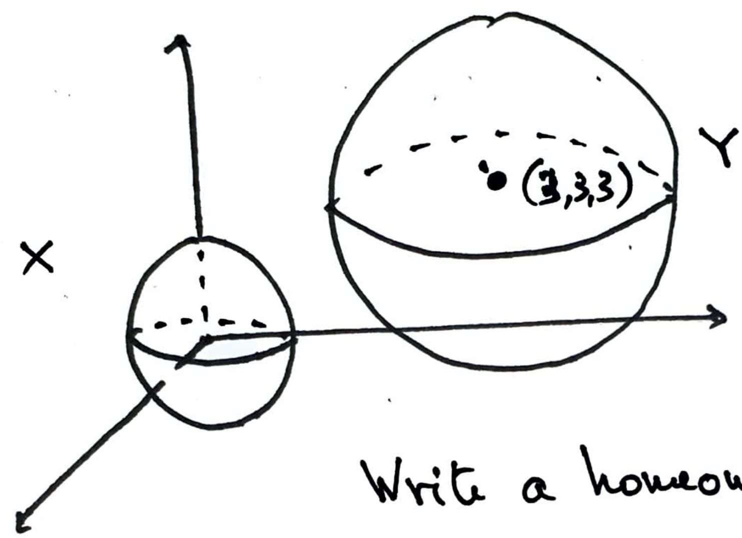
- How do we recognize homeomorphisms?
- ______ construct ______ ?
- What properties are preserved by homeomorphisms?

Ex. Consider $X = S(0,1) \subseteq \mathbb{R}^3$

$$x^2+y^2+z^2=1$$

and $Y = S((3,3,3),2) \subseteq \mathbb{R}^3$

$$(x-3)^2+(y-3)^2+(z-3)^2=4$$



Write a homeomorphism between X and Y.

Recall: • if u, v are bijections, then $u \circ v$ is a bijection (and $(u \circ v)^{-1} = v^{-1} \circ u^{-1}$).

• if u, v are continuous, then so is $u \circ v$

$\Rightarrow$ if u, v are homeomorphisms, so is $u \circ v$.

Here: 1^{st} step: inflate X to dilate the radius by 2.

2^{nd} step: shift it so the center is at $(3,3,3)$.

1st step: inflate X.

$$v : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

- v is polynomial in each coordinate, therefore continuous.
- v is bijective with inverse $v^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x/2 \\ y/2 \\ z/2 \end{pmatrix}$$

v^{-1} is also continuous (same reason): v is a homeo.

Claim: v transforms X into $S(0,2)$.

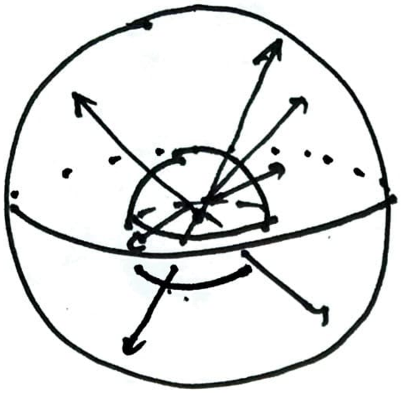
To check this, we need to prove that if

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \quad \text{with} \quad \underbrace{x^2+y^2+z^2 = 1}_{X = S(0,1)}$$

then $\underbrace{X^2+Y^2+Z^2 = 4}_{S(0,2)}$

$$v : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$X \dashrightarrow ?? \qquad \text{Ans: } S(0,2)$$

Indeed, $X^2+Y^2+Z^2 = (2x)^2+(2y)^2+(2z)^2$
$$= 4x^2+4y^2+4z^2$$
$$= 4(x^2+y^2+z^2) = 4 \checkmark$$

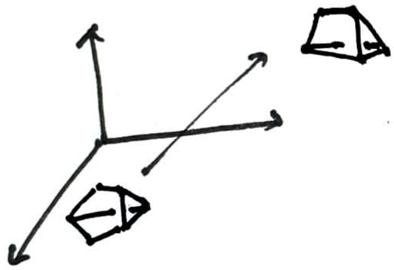


<u>Next</u>: shift the inflated sphere.

<u>2^{nd} step</u>: translate $v(X) = S(0,2)$.

$$u : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+3 \\ y+3 \\ z+3 \end{pmatrix}$$



- u is continuous (polynomial)
- u is bijective with

$$u^{-1} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x-3 \\ y-3 \\ z-3 \end{pmatrix},$$

also continuous.

$\Rightarrow$ u is a homeomorphism.

$\Rightarrow$ $u \circ v$ is a homeomorphism.

Let $f = u \circ v$.

<u>Claim</u>: f is a homeomorphism from X to Y.

We already know that f is a homeomorphism, hence we only need to check that it maps $S(0,1)$ to $S((3,3,3),2)$.

Concretely, $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 2x+3 \\ 2y+3 \\ 2z+3 \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

We verify that:

<u>if</u> $x^2+y^2+z^2=1$ $\quad \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S(0,1) \right)$

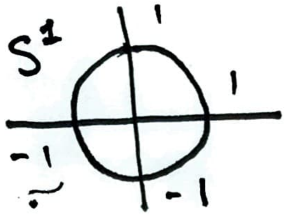
<u>then</u> $(X-3)^2+(Y-3)^2+(Z-3)^2=4$

$\left(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in Y \right)$

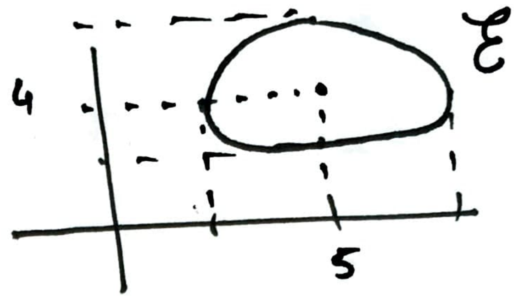
$$\begin{aligned}
(X-3)^2+(Y-3)^2+(Z-3)^2 &= (2x+3-3)^2+(2y+3-3)^2+(2z+3-3)^2 \\
&= 4x^2+4y^2+4z^2 \\
&= 4(x^2+y^2+z^2) \\
&= 4 \quad \checkmark
\end{aligned}$$

<u>Conclusion</u>: $X \sim Y$, X is homeomorphic to Y.

(HW): Consider S^1 the unit circle in $\mathbb{R}^2$



and the ellipse:



$$\mathcal{E}: \quad \frac{(x-5)^2}{4} + \frac{(y-4)^2}{9} = 1$$

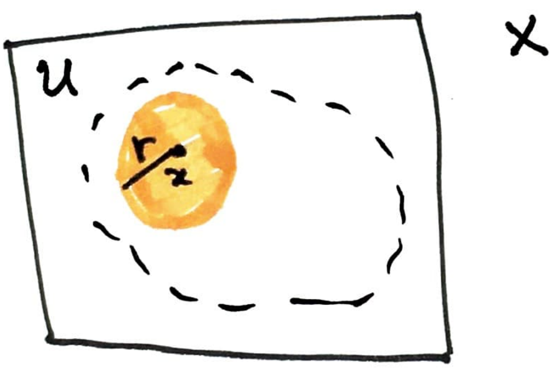
Find a homeomorphism between S^1 and $\mathcal{E}$.

CONNECTED SPACES

Goal: formalize, give a rigorous definition for a topological object to consist of only one piece.

First approach: open / closed (or clopen) sets

Recall that in a metric space X, $\mathcal{U} \subseteq X$ is open if $\forall x \in \mathcal{U}$, $\exists r > 0$ s.t. $B(x,r) \subseteq \mathcal{U}$



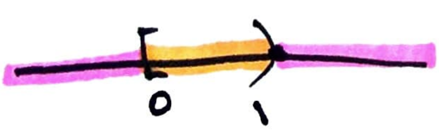
(r depends on x)

Def: A subset C of X metric is said closed if its complement C^c is open.

⚠ Many sets are neither open nor closed.

$[0,1)$ is neither open nor closed in $\mathbb{R}$.

- Not open because of 0 (see yesterday's notes)
- Not closed: the complement is $(-\infty, 0) \cup [1, +\infty)$, which is not open because of 1.



Combinations of closed sets:

- An intersection of closed sets is always closed.
- A finite union of closed sets is always closed.

To prove this, use de Morgan's Laws and the result on open sets:

$$\left(\bigcap_{\alpha \in I} C_\alpha\right)^c = \bigcup_{\alpha \in I} \underbrace{C_\alpha^c}_{\text{open}} \quad : \text{open}$$

$$(C_1 \cup \dots \cup C_p)^c = C_1^c \cap C_2^c \cap \dots \cap C_p^c \ : \text{open}$$

as finite intersec. of open sets.

Ex. In $X = \mathbb{R}$ equipped with ordinary distance $d(x,y) = |x-y|$.

$\emptyset$ and $\mathbb{R}$ are both open and closed.

- $\emptyset$ and $\mathbb{R}$ are open (easy)
- $\emptyset = \mathbb{R}^c$ and $\mathbb{R} = \emptyset^c$ $\Rightarrow$ $\emptyset$ and $\mathbb{R}$ are closed.

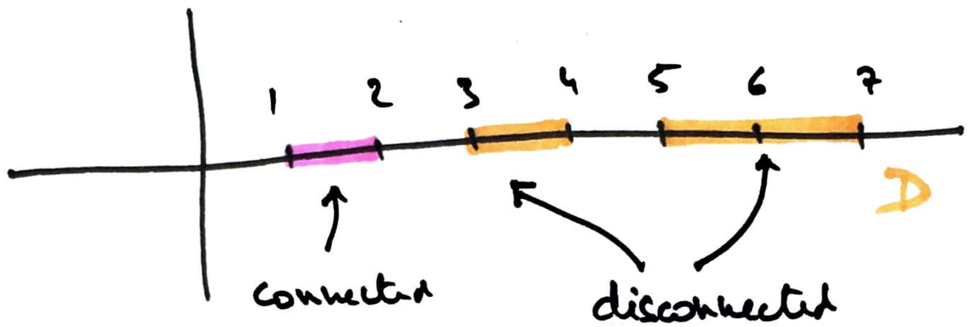
$\emptyset$ and $\mathbb{R}$ are clopen in $\mathbb{R}$.

In fact, they are the only clopen sets.

"Connected = in one piece"

For instance, $(1,2) \subseteq \mathbb{R}$ is connected

but $(3,4) \cup (5,7)$ is not:



Pf. $(3,4)$ is open in D

$(5,7)$ is open in D $\Rightarrow$ $(3,4)$ is closed in D

Both $(3,4)$ and $(5,7)$ are <u>clopen</u> in D.

<u>Def</u> A metric space (X, d_X) is <u>connected</u> if the only clopen sets of X are $\emptyset$ and X.

<u>Ex</u>. Assume X is equipped with d_0 (discrete metric). Then X is connected if and only if $\mathrm{Card}(X) \leq 1$. (HW).

$\rightarrow$ Discrete spaces are (almost) never connected.

Theorem (Characterizations of connected spaces)

The following conditions on X metric are equivalent:

(1) X is connected.

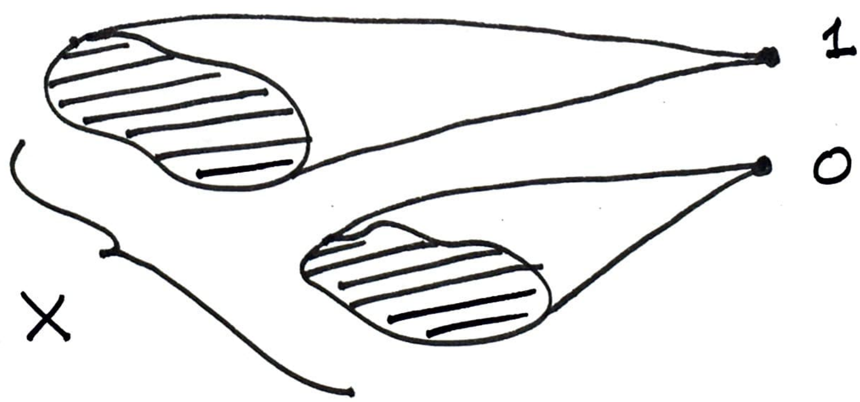
(2) There does not exist U, V open subsets of X with
$$U \cap V = \emptyset \quad \text{and} \quad U \cup V = X$$

(3) There does not exist any continuous map
$$f : X \longrightarrow \{0, 1\}$$
that is surjective.

Ideas: (1) X is connected

(2) "X is not made of 2 separate pieces".

(3)



Proof. Let us check that (1) $\Longleftrightarrow$ (3)

Assume $f : X \longrightarrow \{0, 1\}$ is continuous and onto.

Then $X = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$

$\{0\}$ and $\{1\}$ are open in $\{0, 1\}$, discrete.

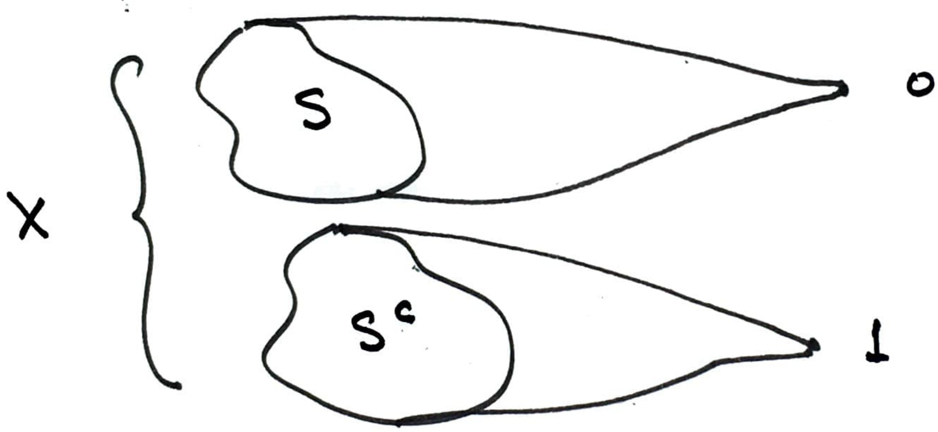
Since f is continuous and onto, both $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are open, closed and $\neq X$.

Therefore X is not connected. This shows that (1) $\Rightarrow$ (3)

To prove that (3) $\Rightarrow$ (1), let us assume that X is not connected. Consider S clopen in X with

$$S \neq \emptyset \quad \text{and} \quad S \neq X.$$

Then $X = S \cup S^c$ with S, S^c clopen.



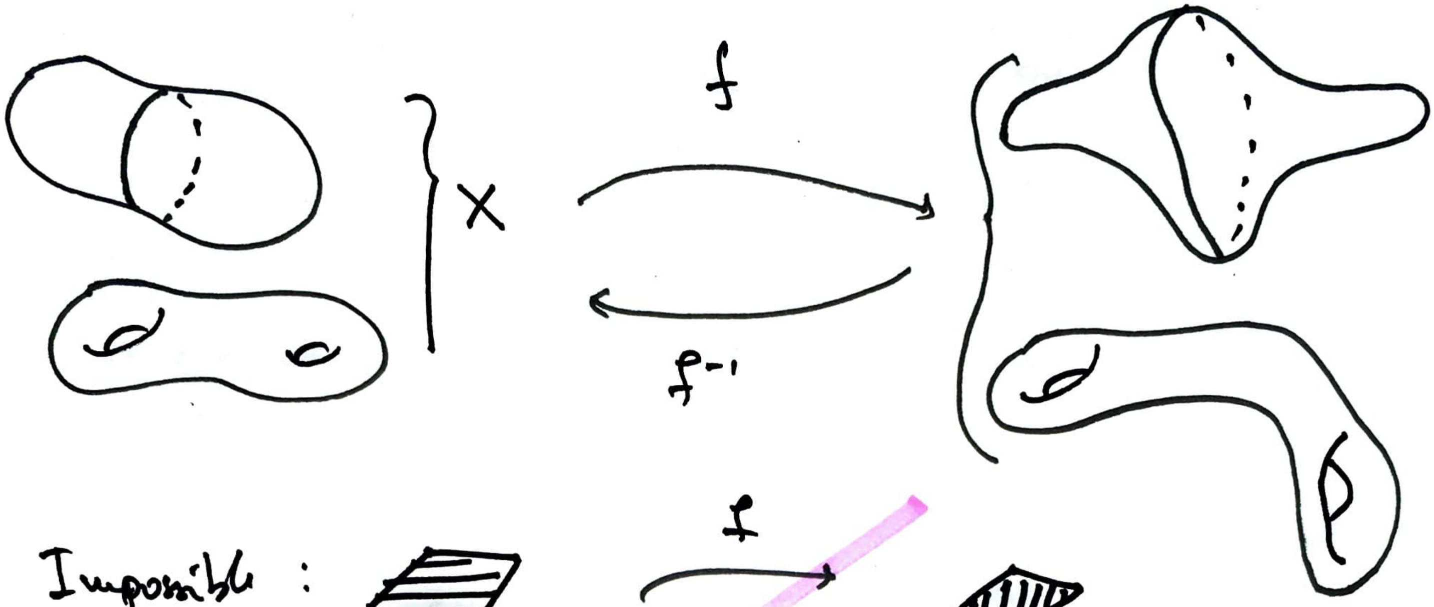
Define
$$\begin{aligned} f : X &\longrightarrow \{0,1\} \\ x &\longmapsto 0 \quad \text{if } x \in S \\ x &\longmapsto 1 \quad \text{if } x \in S^c \end{aligned}$$

<u>Claim</u> (HW): f is continuous and surjective.
?

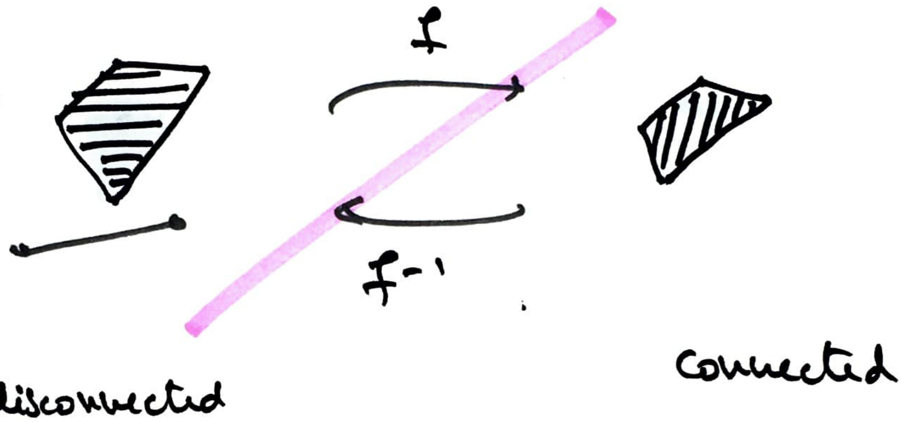
This shows (3) $\Rightarrow$ (1).

To prove (2) $\Longleftrightarrow$ (3), use the same idea with $U = S$, $V = S^c$ ∎

"Connectedness is preserved under homeomorphisms".

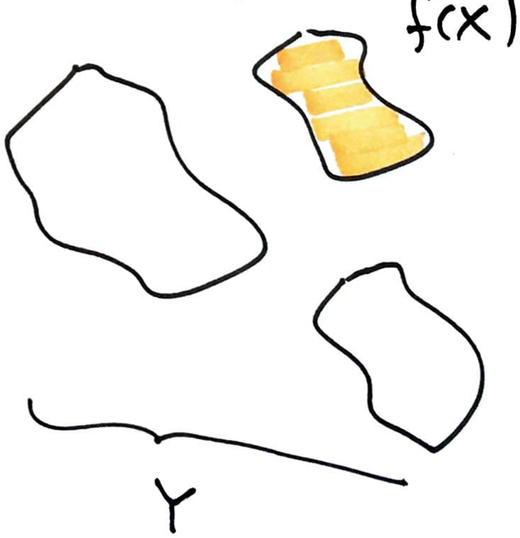


Impossible :



Th. If X, Y are metric spaces and $f: X \longrightarrow Y$ is continuous, if X is connected, then $f(X)$ is a connected subset of Y.

Proof.



Use (2) in the characterization.

If $f(X)$ was not connected, there would be U, V clopen in Y with $f(X) = U \cup V$ and $U \cap V = \emptyset$.

Then we would have $X = f^{-1}(U) \cup f^{-1}(V)$

U open $\Rightarrow f^{-1}(U)$ open (f cont.)

V open $\Rightarrow f^{-1}(V)$ open (— " —)

$U \cap V = \emptyset \Rightarrow f^{-1}(U) \cap f^{-1}(V) = \emptyset$

$\Rightarrow$ X is not connected !!!

It is a contradiction, meaning it was wrong to assume $f(X)$ disconnected, hence $f(X)$ *is* connected $\blacksquare$

<u>Corollary</u>: If X and Y are homeomorphic metric spaces, then:

X connected $\Longleftrightarrow$ Y connected.

The other approach to connectedness is: X is connected if every two points in X can be joined by a path in X