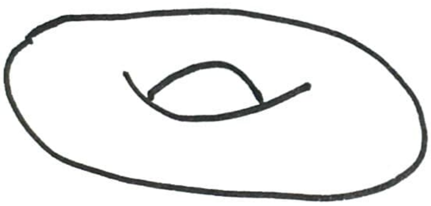
TDA 5

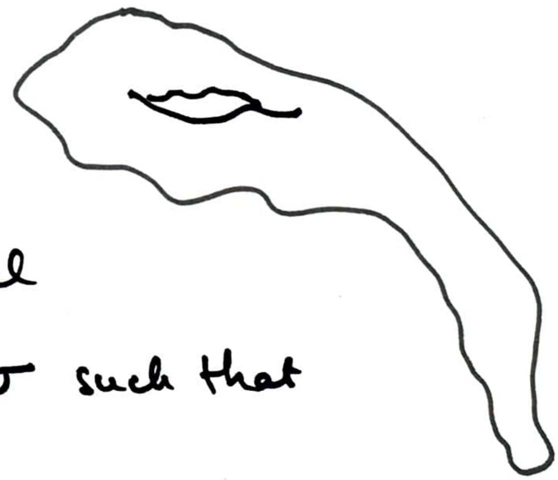
TOPOLOGICAL INVARIANTS

HOMOTOPY

We want to study properties of X that remain if we deform X by a homeomorphism.



$\cong$



More precisely a topological invariant is a function σ such that

if X is homeomorphic to Y then $\sigma(X) = \sigma(Y)$.

Example: classify capital letters of the latin alphabet.

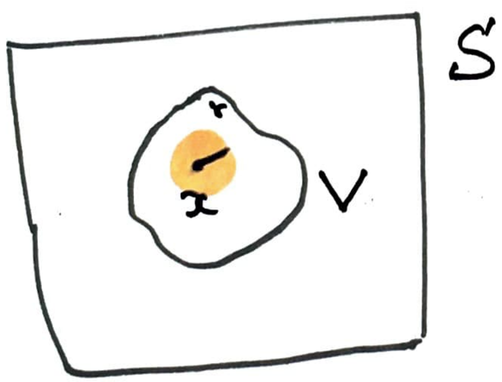
I ~ L $\not\sim$ P ....

Def. An <u>n-vertex</u> in a subset L of a topological (metric) S is an element $v \in L$ such that there is a neighborhood $N_0 \subseteq S$ with $v \in N_0$ such that all neighborhoods $v \in N \subseteq N_0$ satisfy:

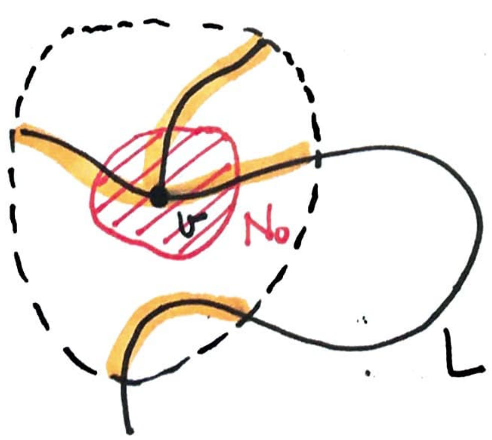
- $N \cap L$ is connected
- $N \cap L \setminus \{v\}$ has exactly n connected components.

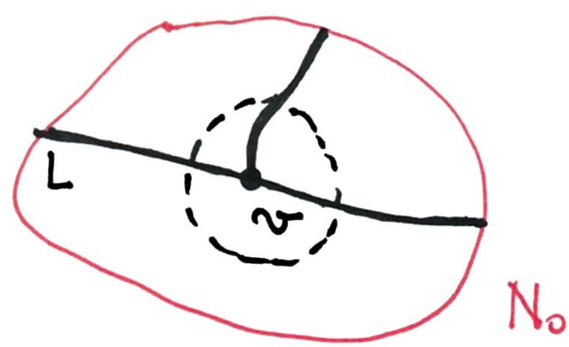
For us now: $S = \mathbb{R}^2$, L = some curve (typically a letter)

Recall that a <u>neighborhood</u> of a point $x \in S$ is a subset V that contains an open set that contains x.



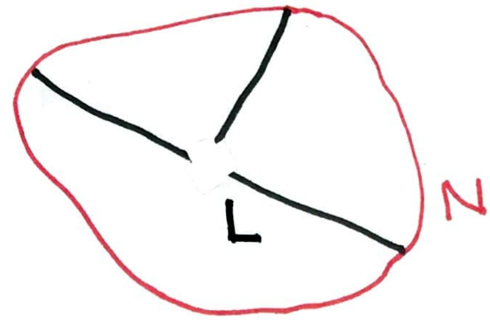
In a metric space, a neighborhood of x is a subset V that contains $B(x, r)$ for some $r > 0$.





$N \cap L \setminus \{v\}$

has 3 connected components.

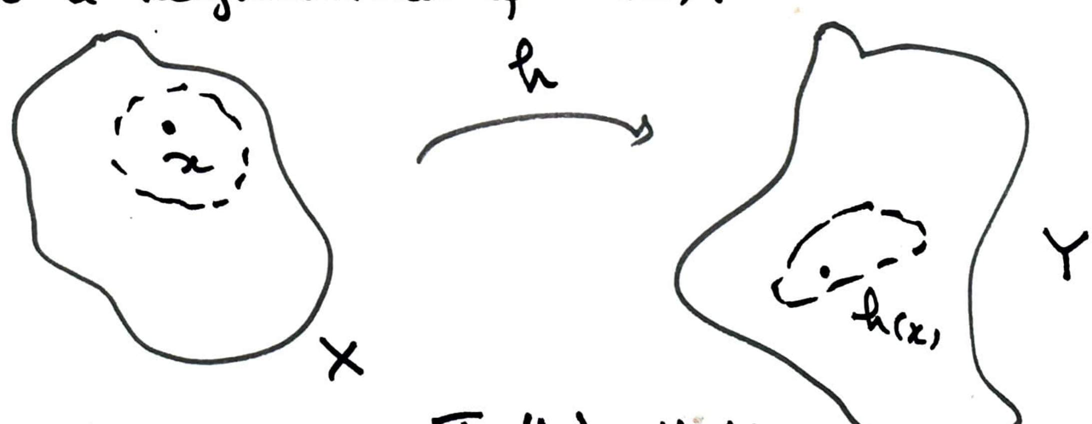


We see that v is a 3-vertex of L.

<u>Observation</u> : n-vertices are defined only in terms of neighborhoods and connected components.

If $h : X \xrightarrow{\sim} Y$ is a homeomorphism, then

$\to$ $\forall x \in X$, h transforms a neighborhood of x into a neighborhood of $h(x)$.



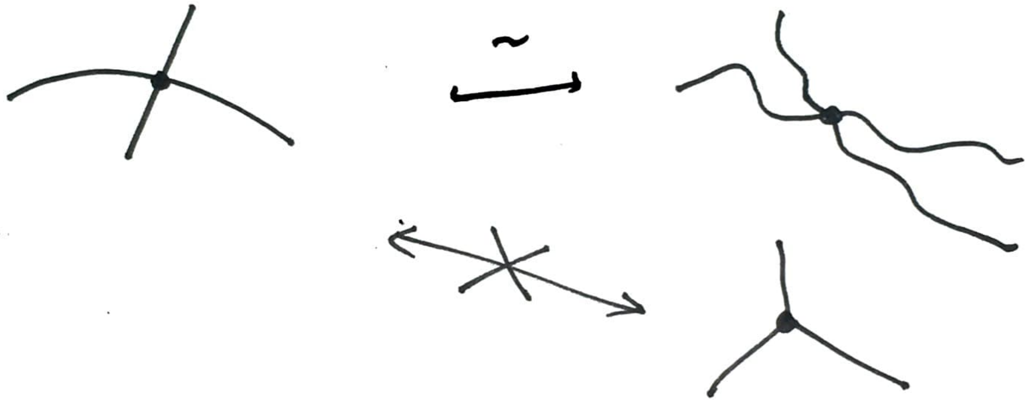
$\to$ $h_* : \pi_0(X) \longrightarrow \pi_0(Y)$, that is,

h preserves the number of connected components.

Th. I If $h : X \xrightarrow{\sim} Y$ is a homeomorphism, it sends any n-vertex to an n-vertex.

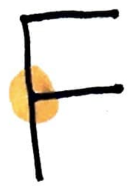
In other words, the number of n-vertices is a <u>topological invariant</u>!

<u>Note</u>: every point on a curve is a 2-vertex.



An issue in classifying letters:

- I $\not\simeq$ F because I has no 3-vertex. F has a 3-vertex.



- Can we compare I and O ?

→ How do we formulate the fact that O has a hole ?

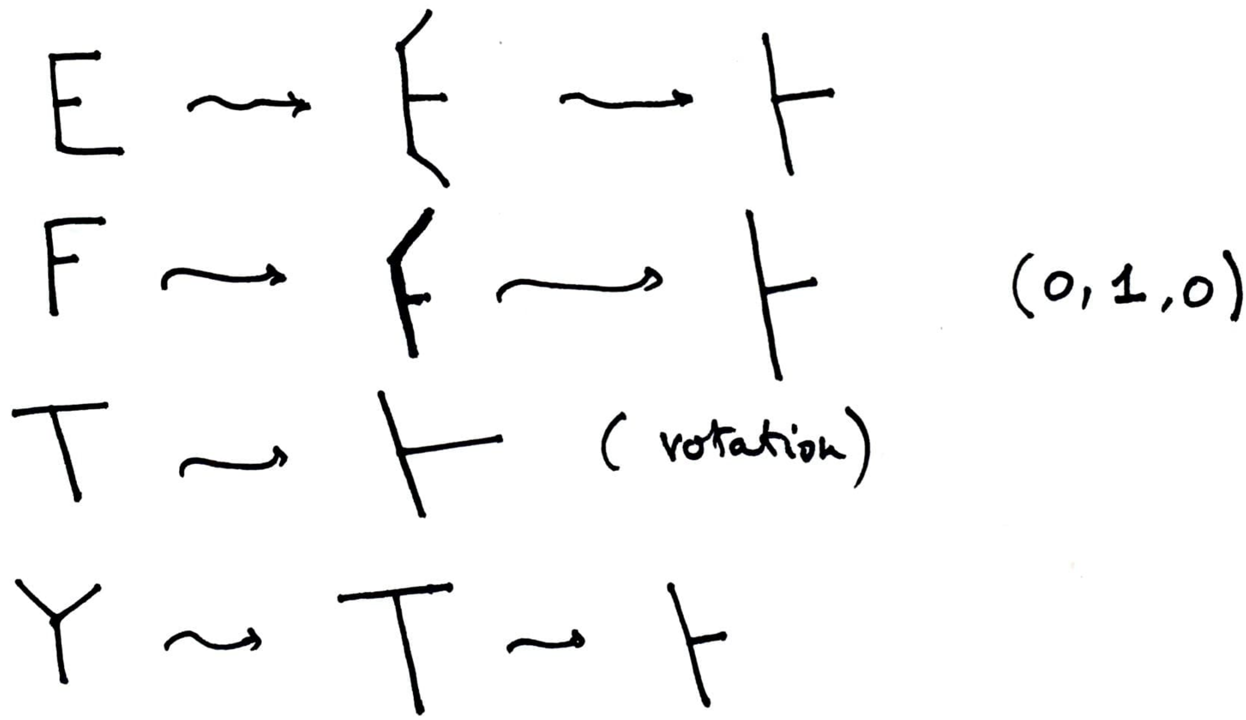
<u>Observe</u>: let x be a point on $|$ or $\bigcirc$.

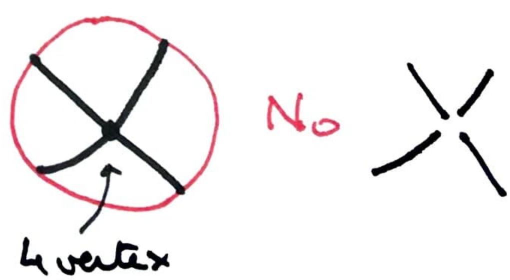
$| \setminus \{x\} = $ ⁝ is disconnected

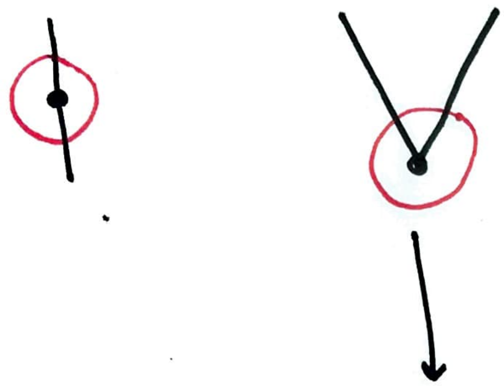
$\bigcirc \setminus \{x\} = $ ⊂⊃ is still connected!

To classify letters, we may count holes, 3 vertices and 4-vertices.

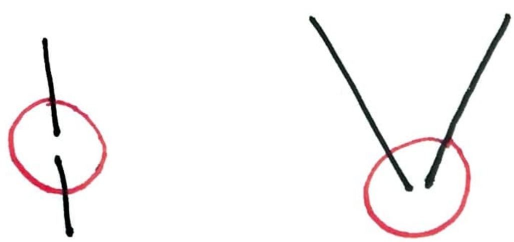
E, F, T, Y are all homeomorphic.



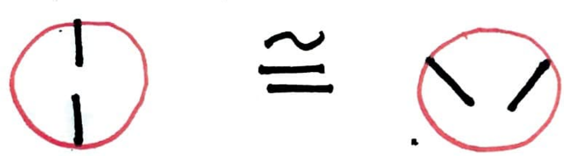


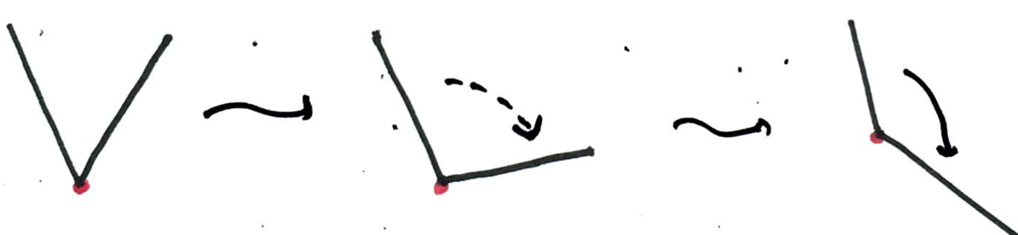


remove point

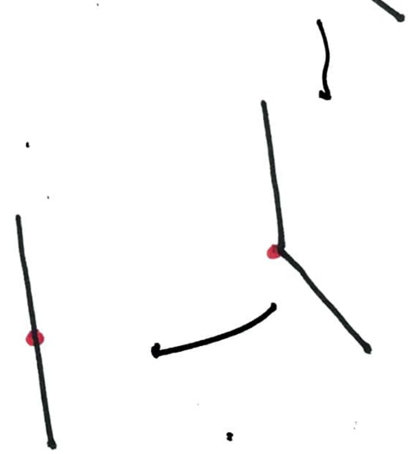


2 components   2 components



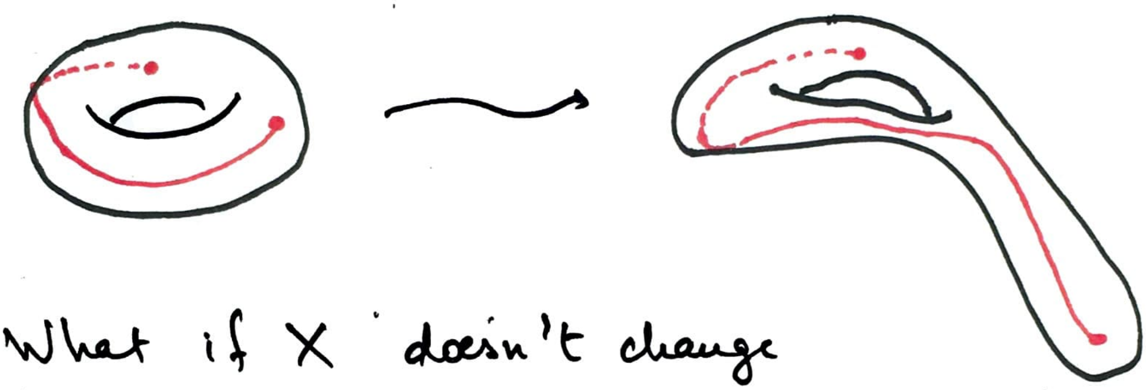


Note: these successive transformations are continuous and reversible, that is, they are homeomorphisms.

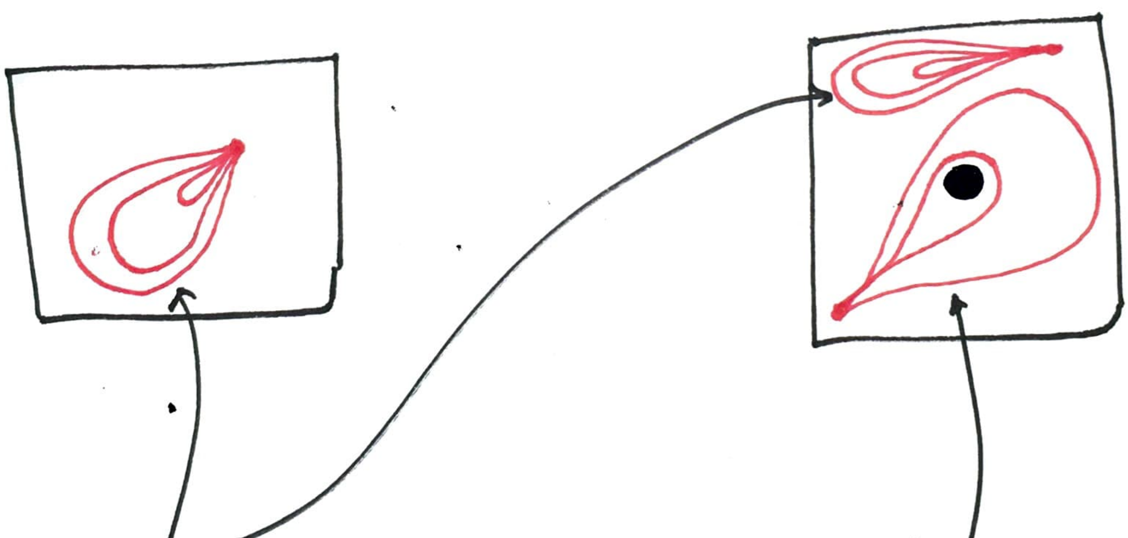


Homotopy

Idea : if X is deformed into something homeomorphic, or at least continuous, the curves on X are deformed too :

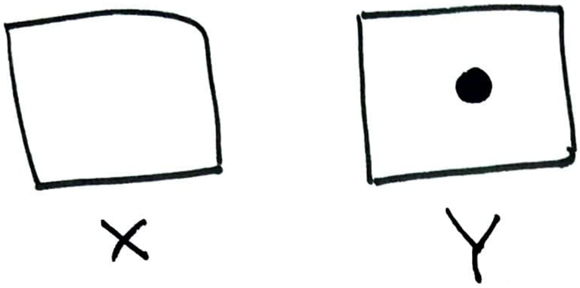


→ What if X doesn't change and we try to deform curves on X ?

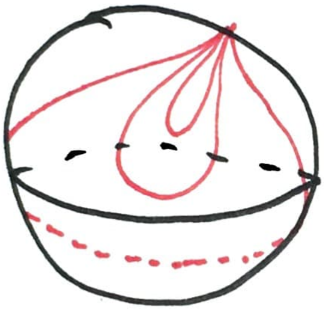


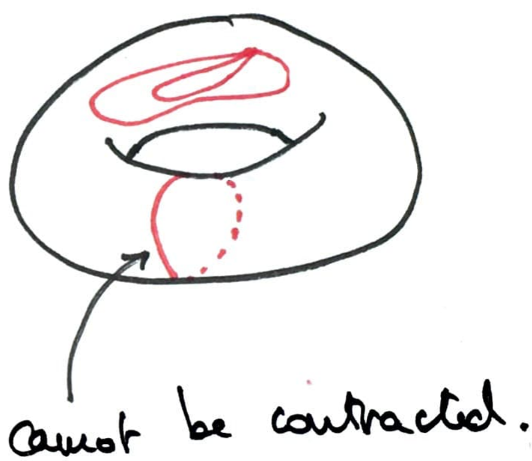
These loops can be deformed to a single point in a continuous way.

This one cannot



- Every loop in X can be contracted continuously to a single point.
- Not in Y: only loops that do not enclose the hole can be contracted.





Def Let X, Y be metric space and $f, g : X \longrightarrow Y$ two continuous maps. We say f and g are <u>homotopic</u> if there is a continuous map

$$H : [0,1] \times X \longrightarrow Y$$

such that $H(0, \cdot) = f$ and $H(1, \cdot) = g$

H is called a homotopy between f and g.

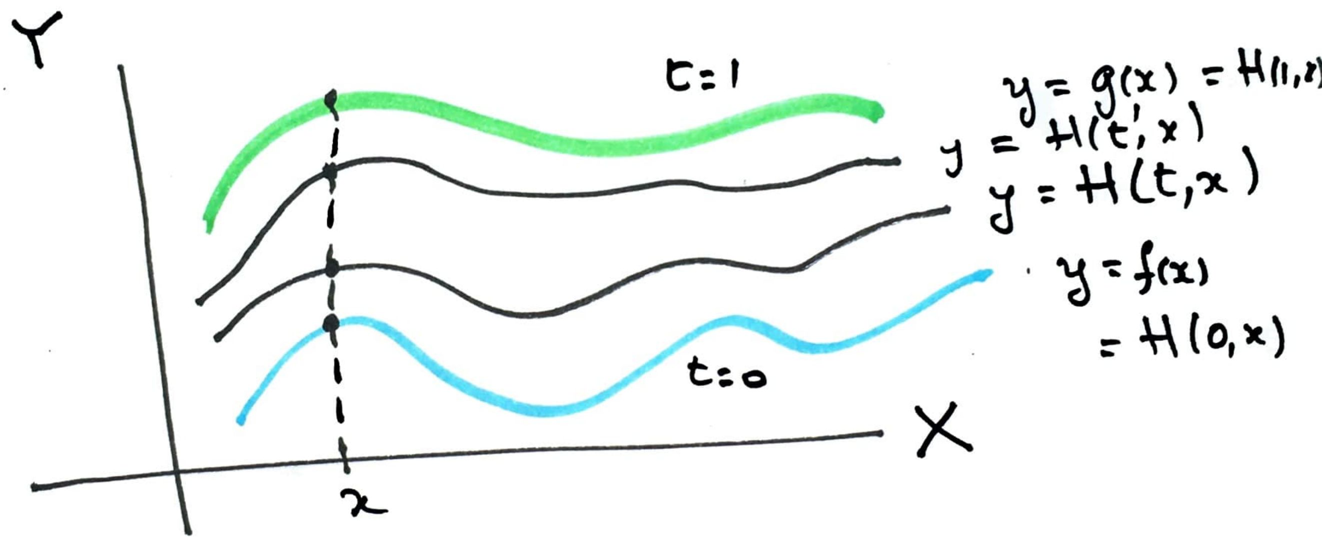
$$H : [0,1] \times X \longrightarrow Y$$
$$(t,x) \longmapsto H(t,x) \qquad t \in [0,1]$$
$$x \in X$$

For every $t_0 \in [0,1]$, consider:

$$H(t_0, \cdot) : X \longrightarrow Y$$
$$x \longmapsto H(t_0, x)$$

Saying that H is a homotopy is saying:

$$\forall x \in X, \qquad H(0,x) = f(x)$$
$$H(1,x) = g(x)$$

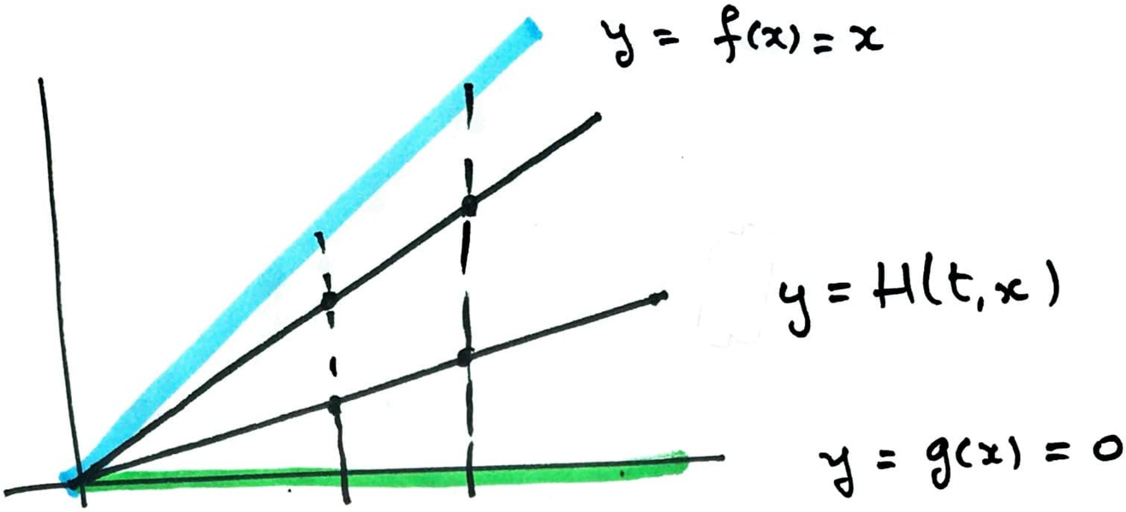


Ex: $f(x) = x$ on $\mathbb{R}_+$

$g(x) = 0$ on $\mathbb{R}_+$

$H(t,x) = (1-t)x$

1. Draw graphs of $f, g, H(\frac{1}{2}, \cdot)$, $H(\frac{3}{4}, \cdot)$
2. Show H is a homotopy between f and g.



If t is fixed, $H(t, \cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}$

$$x \longmapsto (1-t)x$$

$H(t, \cdot)$ is the linear map with slope $1-t$.

$\underline{t=0}$: slope $= 1 - 0 = 1$

$$H(0, x) = x = f(x)$$

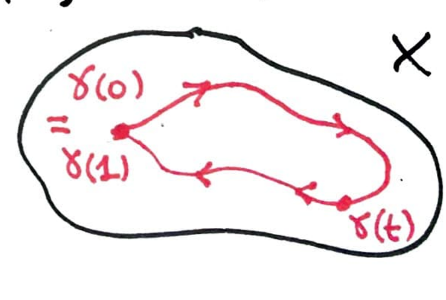
$\underline{t=1}$ $\quad H(1, x) = 0 \cdot x = g(x)$

$H(t, x)$ is polynomial, hence continuous.

Of special interest : homotopies of loops.

<u>Reminder</u>: a <u>path</u> on X metric is a continuous function $\gamma : [0, 1] \longrightarrow X$

A <u>loop</u> on X is a path $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = \gamma(1)$



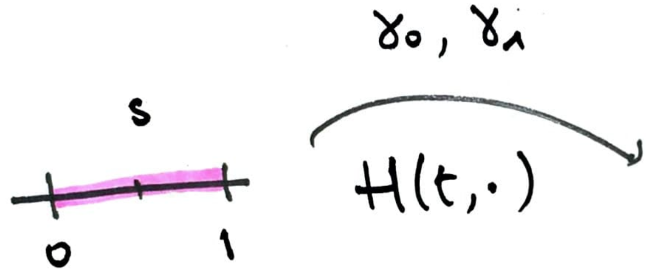
We say that two loops γ_0 and γ_1 are <u>homotopic</u> if $\gamma_0(0) = \gamma_1(0) = \gamma_0(1) = \gamma_1(1) = a$

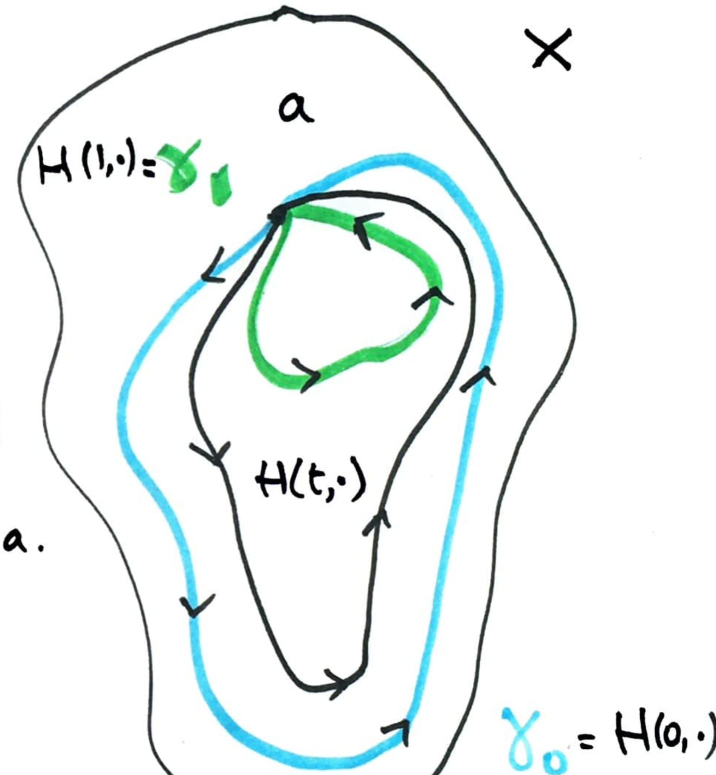
<u>and</u> there is a homotopy

$$H : [0,1] \times [0,1] \longrightarrow X$$

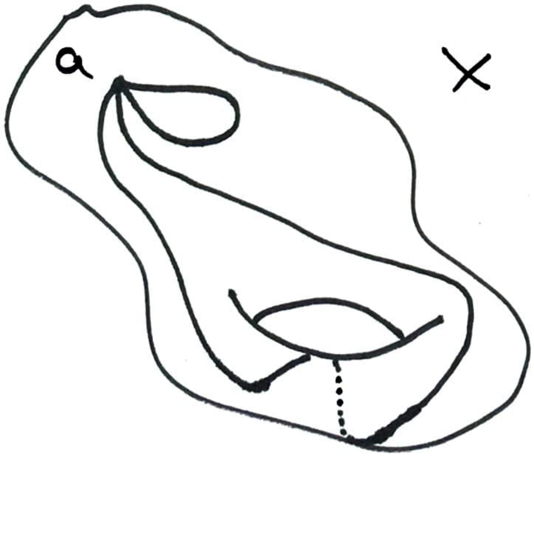
between γ_0 and γ_1, s.t.

$$\forall t \in [0,1], \quad H(t,0) = H(t,1) = a$$





Given X, let us fix $a \in X$ and consider all loops attached to a.

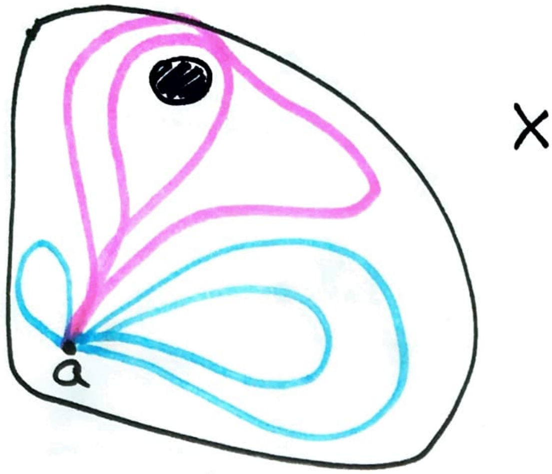


Some loops in X are homotopic, some may not be.

Write $\gamma \sim_h \gamma'$ if γ and γ' are homotopic.

$\pi_1(X) = \{$ classes of loops $\}$.

Ex:

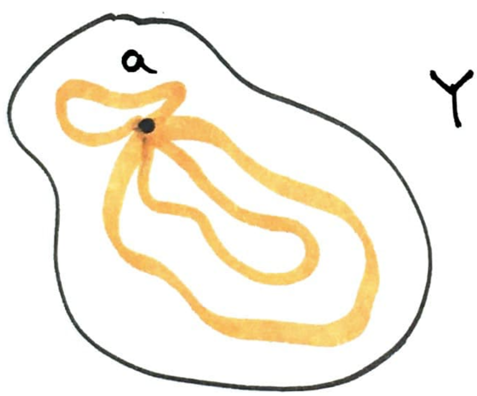


$\pi_1(X) = \{\mathbb{0}, \mathbb{1}\}$

$\mathbb{0} = \{$ all loops that do not enclose the hole $\}$

$\mathbb{1} = \{$ all loops around the hole $\}$

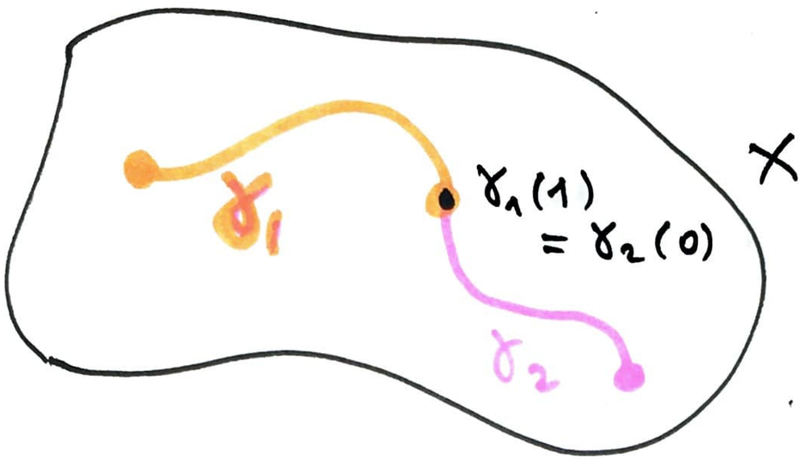
Ex



$\pi_1(Y) = \{\mathbb{0}\}$

all loops in Y are homotopic to each other and to the constant loop $\gamma(t) = a$ (for all t).

In fact, $\pi_1(X)$ is a group, for path composition:



If γ_1, γ_2 are paths on X with

$$\gamma_1(1) = \gamma_2(0)$$

then we can join them:

$$\gamma_1 * \gamma_2 : [0,1] \longrightarrow X$$
$$t \longmapsto \gamma_1(2t) \text{ if } 0 \leq t \leq \tfrac{1}{2}$$
$$t \longmapsto \gamma_2(2t-1) \quad \text{if } \tfrac{1}{2} \leq t \leq 1$$

if $t = \frac{1}{2}$, $\gamma_1 * \gamma_2(\frac{1}{2}) = \gamma_1(2 \times \frac{1}{2}) = \gamma_1(1)$
$= \gamma_2(2 \times \frac{1}{2} - 1) = \gamma_2(0)$

$(\pi_1(X), *)$ is a group, called the fundamental group of X.