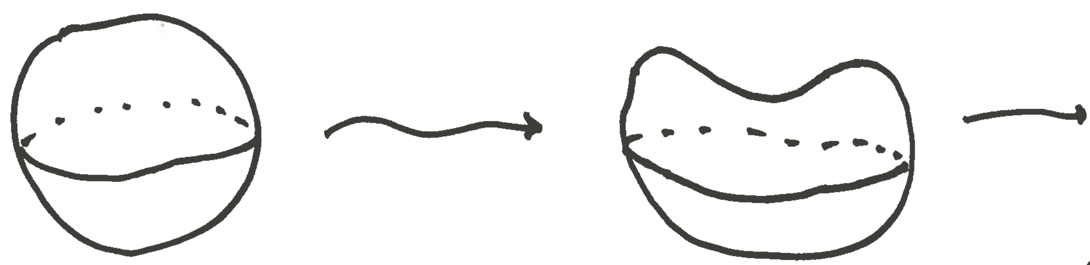
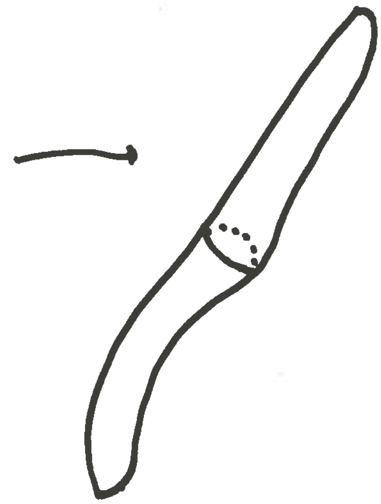
TDA 2

METRIC TOPOLOGY

Goal : consider "smooth deformations" of objects.

S

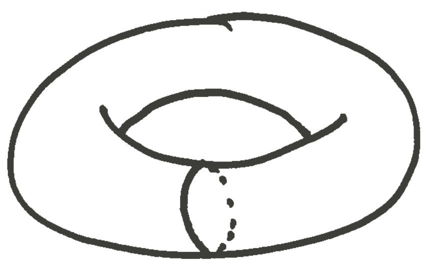




These deformations are smooth.

The shape is not really changed.

Different shape:



T

No smooth deformation can transform T into S or S into T.

Reason : . S has no holes, T has 1 hole!

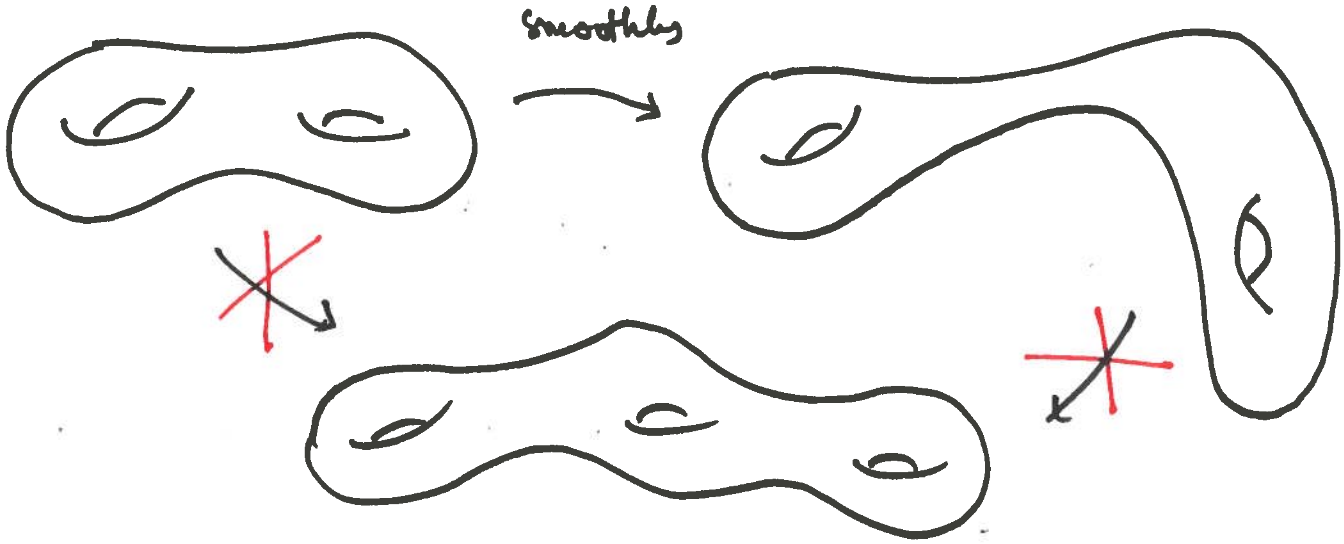
. Compare loops on S and T:

All loops contract to a point

can be contracted to a point

cannot be contracted to a point.

To distinguish surface or other objects, study deformation (without cutting, or tearing) and see what is preserved : for instance, number of holes.



In TDA : • start from a data set

• construct topological objects (simplicial complexes)

• study their properties that are not changed by smooth deformations

The objects are generally high-dimensional → NO PICTURES!

→ Need strong theory of topological deformations.

What does it mean for an object of dimension $n > 3$ to have holes? How do you count them?

→ Homology

Open sets and closed sets

X : metric space with d metric.

$\mathcal{U} \subseteq X$ subset.

Def. We say that $\mathcal{U}$ is open if $\forall a \in \mathcal{U}$,

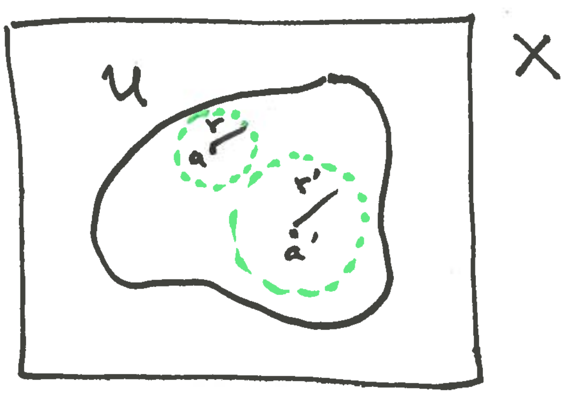
$\exists r > 0$ s.t. $B(a, r) \subset \mathcal{U}$.

Vocabulary: $\forall$ = "for all"

"$\forall a \in \mathcal{U}$" means "for every a in $\mathcal{U}$"

$\exists$ = "there exists"

"$\exists r > 0$" means "there exists a positive r such that...."



Idea: in an open set, there is "wiggle room" around every point.

Ex: $X = \mathbb{R}$, $d_{Eucl} = d$

$$d(x,y) = |x-y|$$

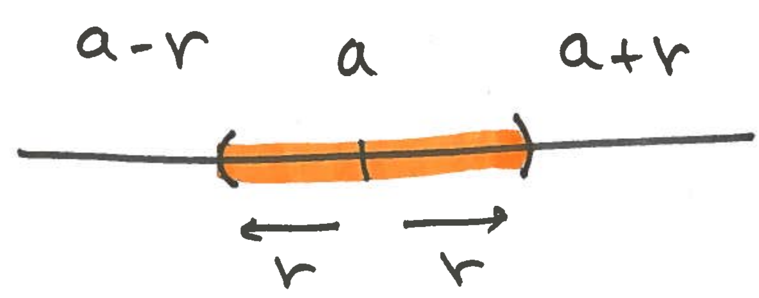
(0,1) is open.

Recall: if $a \in \mathbb{R}$, $r > 0$

$$B(a,r) = \{x \in \mathbb{R} \text{ s.t. } d(a,x) < r\}$$

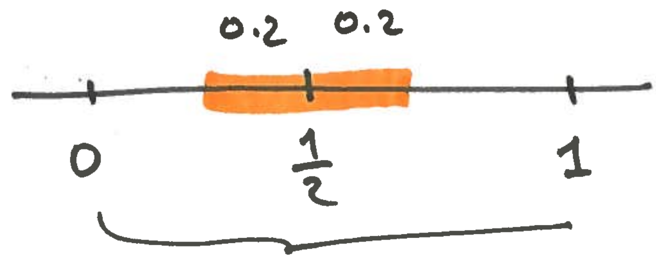
$$= \{x \in \mathbb{R} \text{ s.t. } -r < x-a < r\}$$

$$= (a-r, a+r)$$



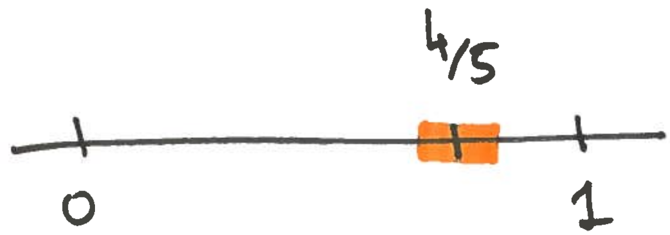
To show that $(0,1)$ is open, check that for any $a \in (0,1)$, there is $r > 0$ s.t. $(a-r, a+r) \subseteq (0,1)$

Take $a = \frac{1}{2}$



$$B(\underset{a}{\tfrac{1}{2}}, \underset{r}{0.2}) \subseteq (0, 1)$$

Take $a = \frac{4}{5}$



Take smaller r!

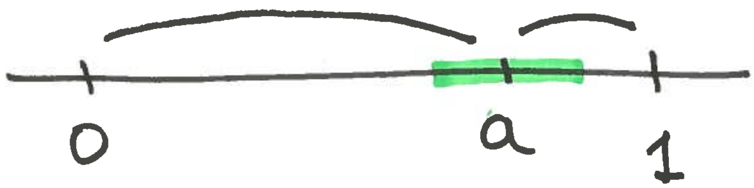
Take $a = 0.99$

Finding r becomes harder ... need to Zoom in!

Take $r = 0.001$.

Then $B(0.99, 0.001) = (0.989, 0.991) \subseteq (0, 1)$.

In general, if $0 < a < 1$



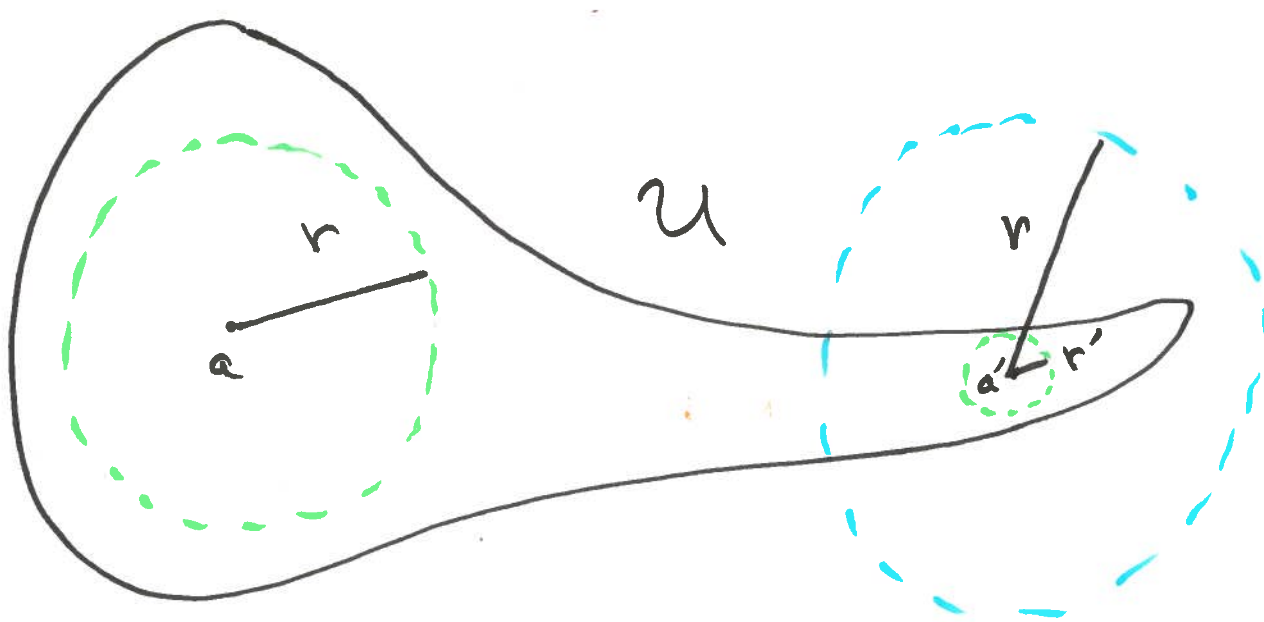
Let $\alpha = \min\{1-a, a-0\}$

Let $r = \frac{\alpha}{2}$.

Then $B(a,r) \subseteq (0,1)$

There is an r for every a so $(0,1)$ is open.

<u>Remark</u>: the r depends on a!



Not all sets are open!

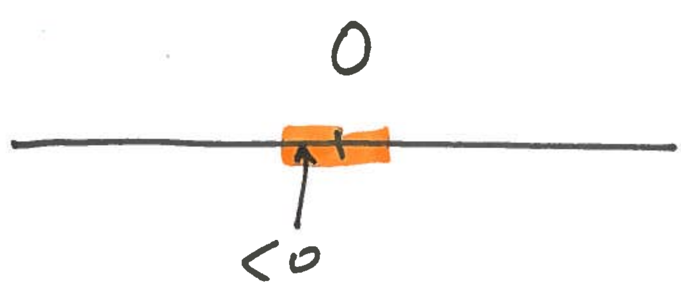
Consider again $X = \mathbb{R}$ with $d(x,y) = |x-y|$

$\boxed{\text{Then } [0,1) \text{ is not open.}}$

If $a \in (0,1)$, play the same game and find $r > 0$ s.t.

$$B(a,r) \subseteq (0,1) \subseteq [0,1)$$

But if $a = 0$....



For any $r > 0$, $B(0,r) = (-r,r)$ contains < 0 numbers (for instance $-\frac{r}{2}$)

$\Rightarrow$ No ball $B(0, r)$ with $r > 0$ in $\subseteq [0, 1)$.

$\Rightarrow$ $[0, 1)$ is <u>not</u> open.

<u>Remark</u>: being open depends on the metric.

Consider $X_1 = \mathbb{R}^2$, d_1 = Manhattan distance.

$X_2 = \mathbb{R}^2$, d_0 = 0 or 1 (discrete)

$\rightarrow$ <u>Question</u>: let $a = (1, 1)$

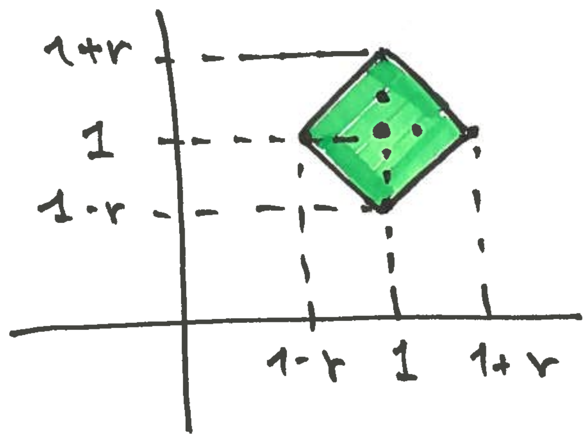
Is $\{a\}$ open in X_1?

____________ X_2?

$X_1 = \mathbb{R}^2$, d_1 defined by:

$$d_1\big((1,1)\,;\,(x_1,x_2)\big) = |x_1 - 1| + |x_2 - 1|$$

$\to B\big((1,1), r\big)$



In particular, if $r > 0$,

$B\big((1,1), r\big)$ contains

$$\left(1, 1+\frac{r}{2}\right), \text{ or } \left(1-\frac{r}{2}, 1\right)$$

$\Rightarrow B\big((1,1), r\big) \not\subset \{(1,1)\}$

$\Rightarrow \{(1,1)\}$ is not open in this topology.

$X_2 = \mathbb{R}^2$, d_0 discrete:

$$d_0((1,1);(x_1,x_2)) = 0 \text{ if } \begin{matrix} x_1 = 1 \\ x_2 = 1 \end{matrix}$$
$$= 1 \text{ otherwise}$$

We saw that if $r \leq 1$, then

$$B(a,r) = \{a\}$$

$\Rightarrow$ Let $r = \frac{3}{4}$. Then

$$B\left((1,1), \frac{3}{4}\right) = \{(1,1)\} \subseteq \{(1,1)\}$$

So $\{(1,1)\}$ <u>is</u> open in this topology.

For any set X with the discrete metric d_0, any subset of X is open.

(HW)

Properties of open sets

X : set with metric d.

$\boxed{1}$ If $a \in X$ and $r > 0$, then

$B(a, r)$ is open.

$\boxed{2}$ If $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets, then the union

$$\bigcup_{\alpha \in I} U_\alpha \text{ is open}$$

$\boxed{3}$ If $U_1, U_2, \ldots, U_p$ is a <u>finite</u> family of open sets, then the intersection

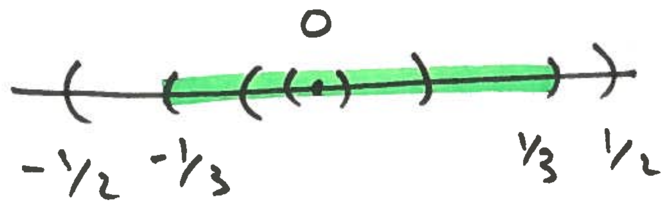
$$U_1 \cap U_2 \cap \ldots \cap U_n \quad \text{is open.}$$

<u>HW</u>: prove $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$.

<u>Notice</u>: we may not remove the finiteness assumption in $\boxed{3}$.

Consider $X = \mathbb{R}$ with $d(x,y) = |x-y|$

For $n \geqslant 1$, let $\mathcal{U}_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$



- $\mathcal{U}_n = B\left(0, \frac{1}{n}\right)$ open by $\boxed{1}$

- $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 \cap \ldots\ldots = ?$

  If $x \in \mathcal{U}_n$ for all n, then

  $$-\frac{1}{n} < x < \frac{1}{n} \qquad \text{for all } n.$$

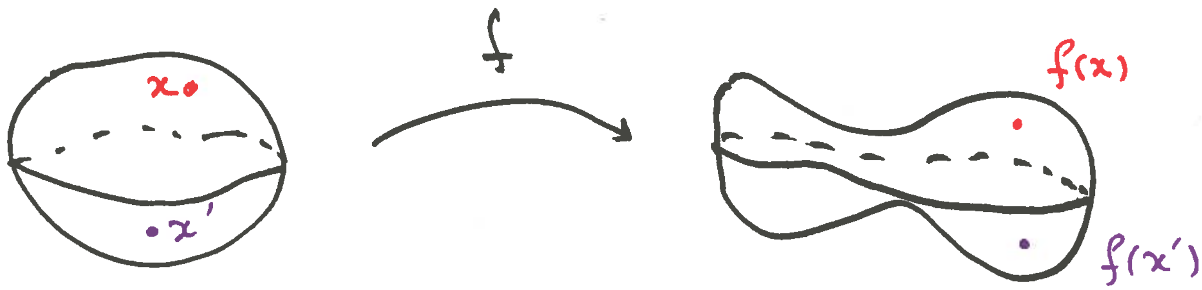
  $\Rightarrow x = 0$

  $\Rightarrow \mathcal{U}_1 \cap \mathcal{U}_2 \cap \ldots \cap \mathcal{U}_n \cap \ldots\ldots = \{0\}$

  <u>not open</u>

Smooth deformations

$$X \xrightarrow{f} Y$$
$$x \longmapsto f(x)$$



Requirements :

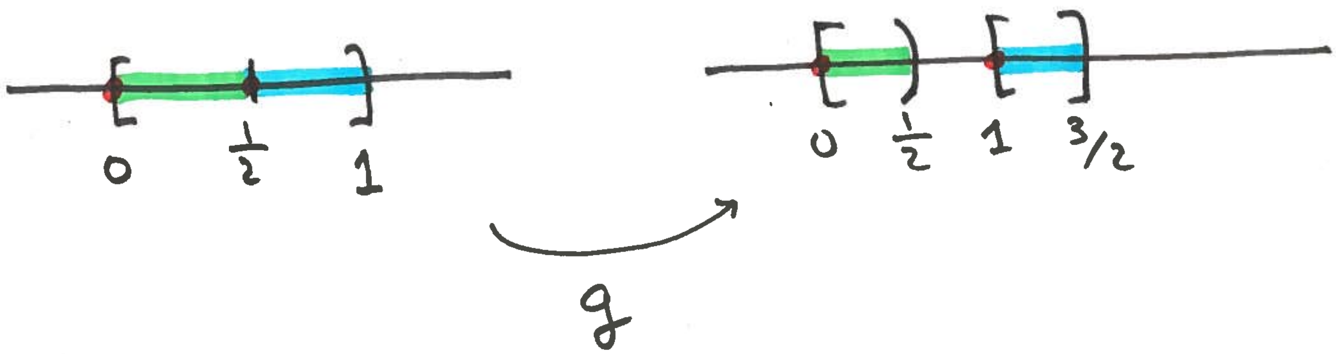
– f should be a bijection (one-to-one [injective] and onto [surjective])

- f is 1-1 if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
- f is onto if every $y \in Y$ is $f(x)$ for some $x \in X$

f is bijective (or a bijection) if for every $y \in Y$, there is a unique $x \in X$ such that $f(x) = y$.

Being bijective is not enough:

$$X = [0,1] \qquad Y = [0,\tfrac{1}{2}) \cup [1,\tfrac{3}{2}]$$



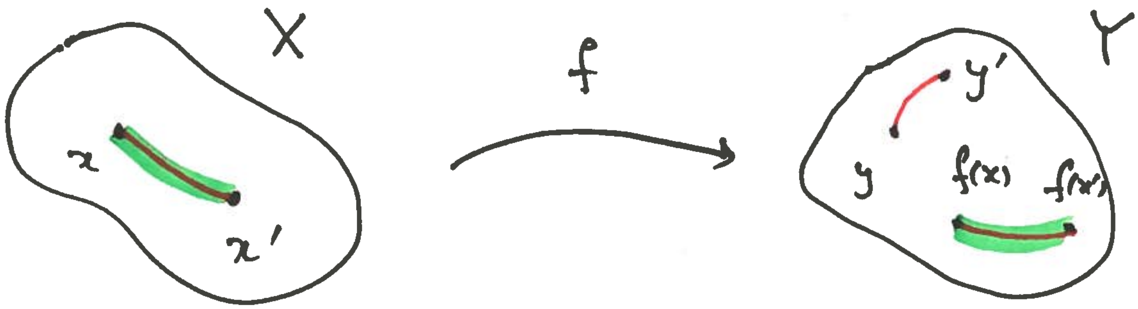
$$g(x) = x \quad \text{if} \quad 0 \leq x < \tfrac{1}{2}$$
$$= x + \tfrac{1}{2} \quad \text{if} \quad \tfrac{1}{2} \leq x \leq 1$$

It is easy to see that g is a <u>bijection</u> but it is a <u>cutting</u> of $[0,1]$ into two pieces.

"Smooth transformations" need to have more properties....

Another requirement: maps that preserve distances.

Let X and Y be metric spaces with metrics d_X and d_Y.



Def. $f: X \longrightarrow Y$ is called an isometry if for any x, x' in X

$$d_Y(f(x), f(x')) = d_X(x, x')$$

An isometry is a map that preserves distances.

Ex: $X = \mathbb{R}$, $d_X(x, x') = |x - x'|$

$Y = \mathbb{R}^2$, d_Y = Euclidean distance

HW: Consider the maps:

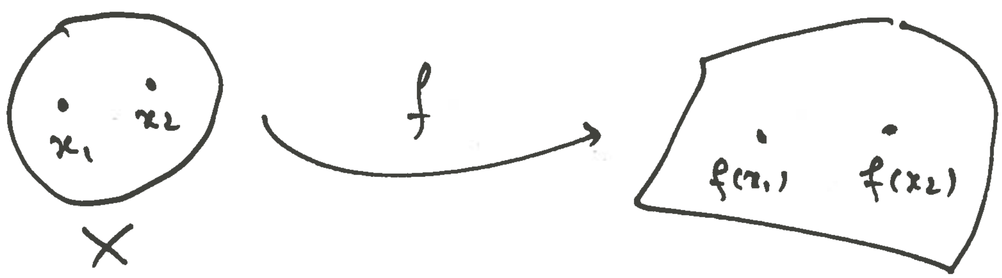
$$u : X \longrightarrow Y \qquad x \longmapsto (0, x)$$

$$v : X \longrightarrow Y \qquad x \longmapsto (-x, 1)$$

Prove that u and v are isometries. (Draw a picture)

Lemma: Isometries are always 1-1.

Proof: Let $f : X \longrightarrow Y$ be an isometry.



Assume $x_1 \neq x_2$ in X.

It means $d_X(x_1, x_2) \neq 0$

Since f is an isometry, $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \neq 0$

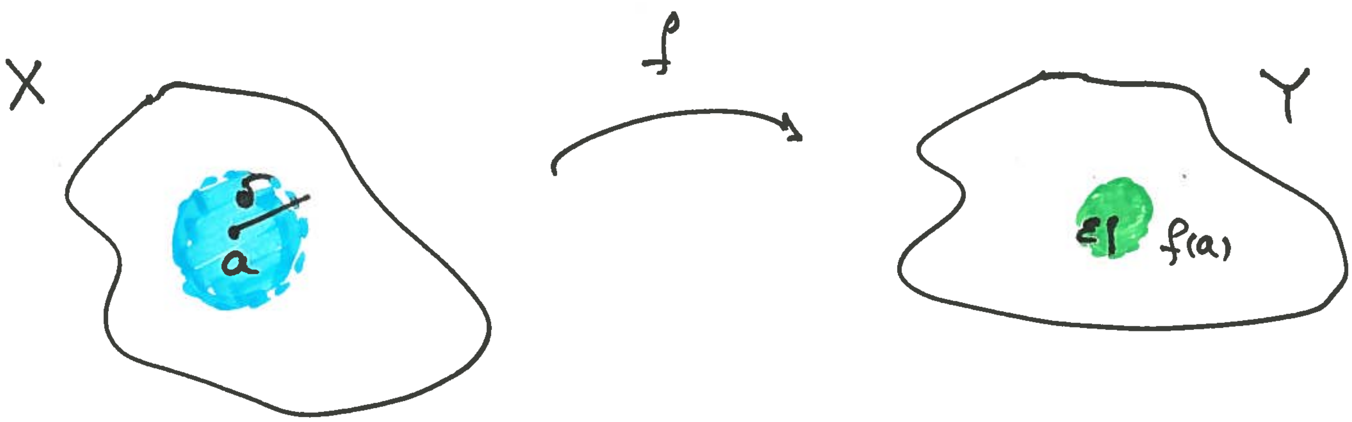
$\Longrightarrow f(x_1) \neq f(x_2)$ by M1 ∎

Isometries are good maps (no cutting or tearing).

To be more precise:

<u>Def</u>. A function f between metric spaces X and Y is said <u>continuous</u> at $a \in X$ if:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $d_X(a,x) < \delta \Rightarrow d_Y(f(a), f(x)) < \varepsilon$



f cont. at a if for any $B(f(a), \varepsilon)$

there is a ball $B(a, \delta)$ such that

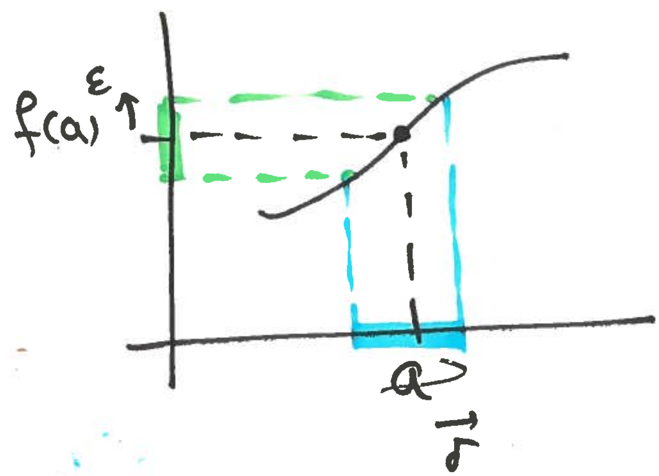
f sends $B(a, \delta)$ to $B(f(a), \varepsilon)$

To become more familiar with this: review continuity of functions $\mathbb{R} \longrightarrow \mathbb{R}$.

Recall that if $f : \mathbb{R} \longrightarrow \mathbb{R}$, $a \in \mathbb{R}$,

f is continuous at a <u>iff</u>:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t } |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$



This is a special case: $|x-a| = d_{\mathbb{R}}(x,a)$

$$|f(x) - f(a)| = d_{\mathbb{R}}(f(x), f(a))$$

<u>HW</u>: If $f : X \longrightarrow Y$ is an isometry, then f is continuous at any a in X.

→ <u>Hint</u>: may take $\delta = \varepsilon$ ✓

Good News: if X is a set with the discrete metric

$$d_0 : X \times X \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

and Y is a metric space with metric d_Y,

then any function $f : X \longrightarrow Y$

is continuous!

Proof (HW): recall how we observed that all set in (X, d_0) are open...

Bad news: if $f : \mathbb{R} \longrightarrow (X, d_0)$ is continuous at every point in $\mathbb{R}$, then

f must be constant!

In particular, $f(x) = f(0)$ for all x.

(HW).

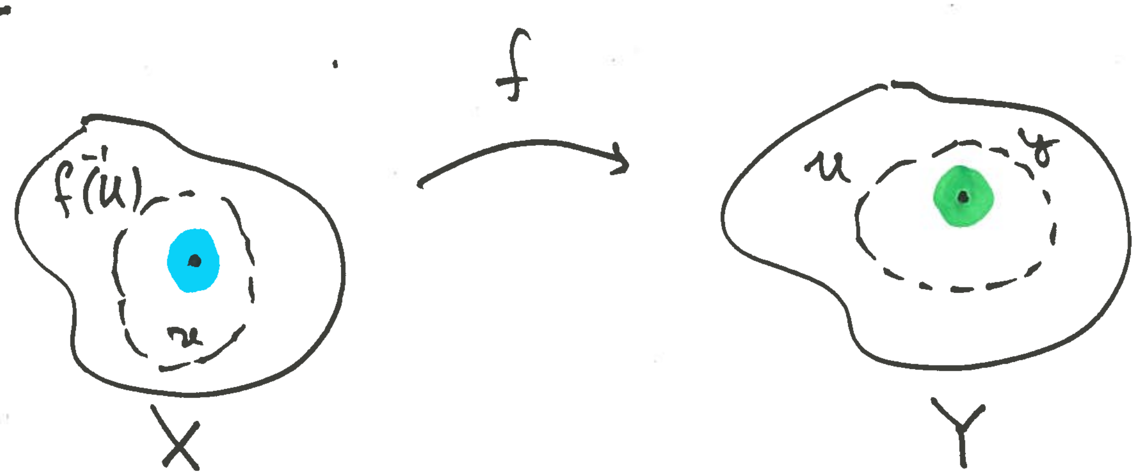
A simpler way to phrase continuity:

Prop. Let X, Y be metric spaces and $f : X \longrightarrow Y$ a function. Then

f is continuous at every point in X

$$\Longleftrightarrow$$

$\mathcal{U}$ open in $Y \implies f^{-1}(\mathcal{U})$ open in X



$$f^{-1}(\mathcal{U}) = \{ x \in X \mid f(x) \in \mathcal{U} \}$$

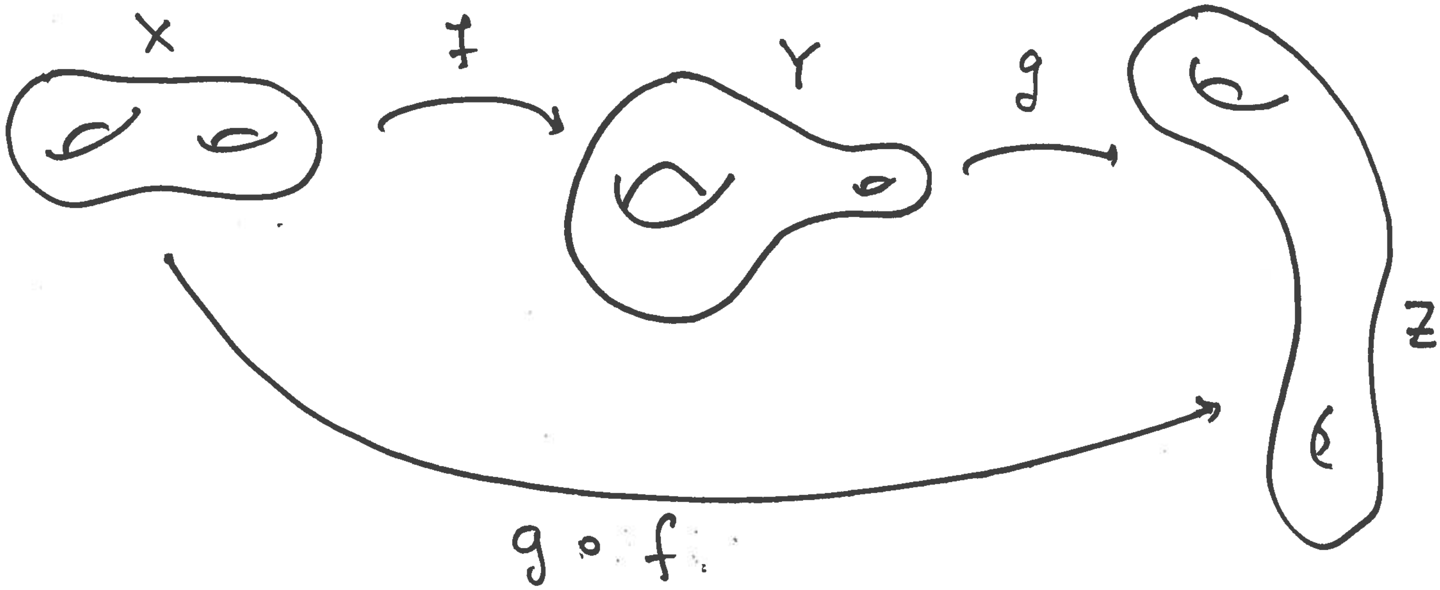
$\mathcal{U}$ open, $y \in \mathcal{U} \implies \exists \varepsilon > 0$ s.t. $B(y, \varepsilon) \subseteq \mathcal{U}$

f continuous at x with $f(x) = y$ gives a $\delta > 0$

s.t. $d_X(t, x) < \delta \implies d_Y(f(t), \underset{= f(x)}{\underline{y}}) < \varepsilon$

■

Consequence: successive deformations



__Idea__: f, g continuous $\Rightarrow$ $g \circ f$ continuous

where $g \circ f(x) = g(f(x))$.

__Th__: If X, Y, Z are metric spaces and

$$f : X \longrightarrow Y \quad \text{and} \quad g : Y \longrightarrow Z$$

are continuous, then

$$g \circ f : X \longrightarrow Z \quad \text{is continuous}$$

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

"If you combine cont. maps, you get a cont. map."

HW: prove this using open sets (not ε-δ !!)