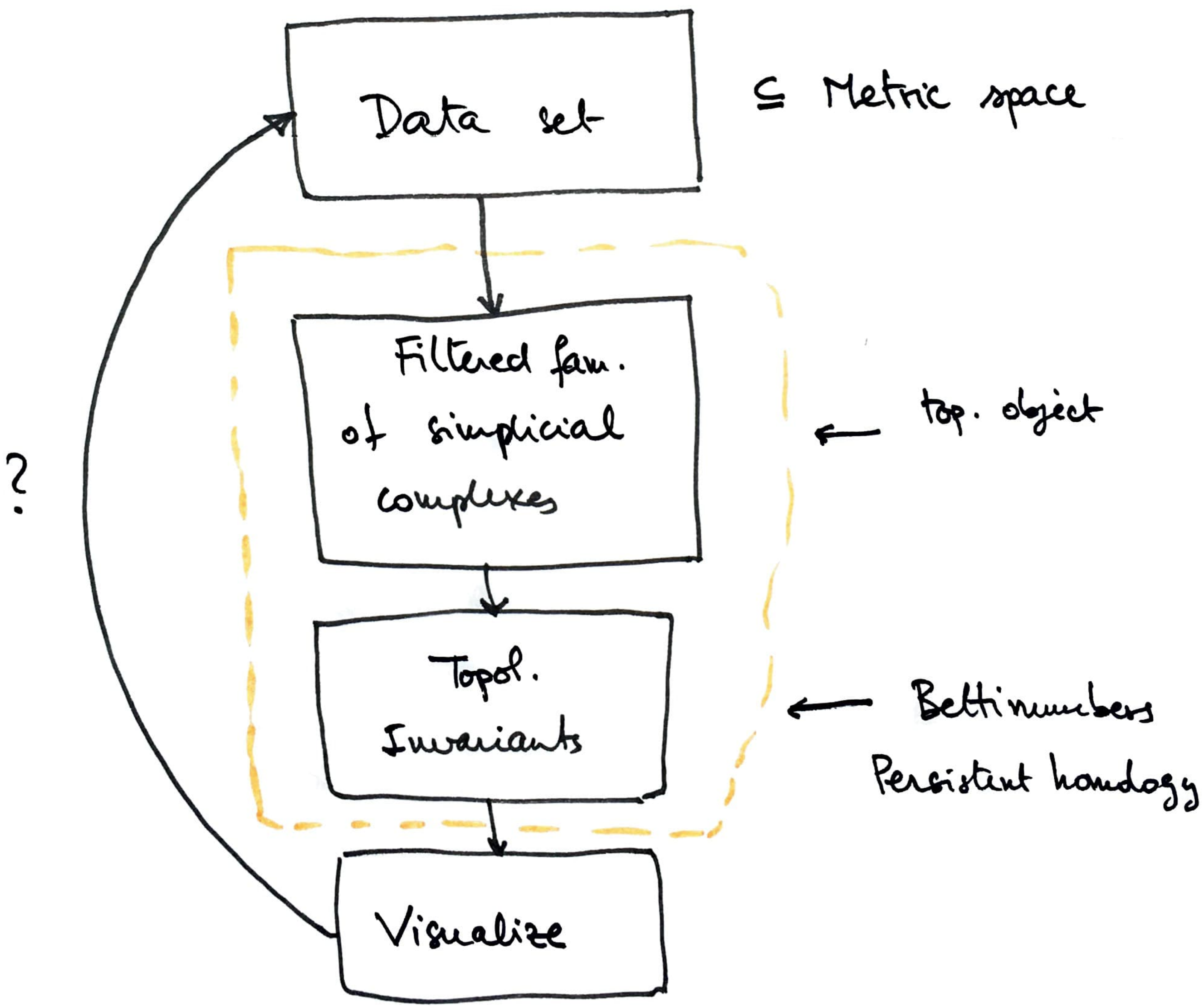
TDA 8

PERSISTENT HOMOLOGY

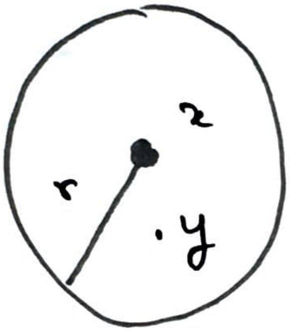
Goals
- associate simplicial complexes to data sets
- understand the meaning of Betti numbers



→ From data points to simplicial complexes.

In a metric space (X, d) distance, one can draw balls around points: $x \in X$, $r > 0$

$$B(x,r) = \{ y \in X : d(x,y) < r \}$$



Fix $\alpha > 0$. X = finite set of points

Def $\mathrm{Rips}_\alpha(X)$ = the set of simplices $[x_0 \dots x_k]$ such that $d(x_i, x_j) \leq \alpha$ for all (i,j).

→ connect points that are α-close together.

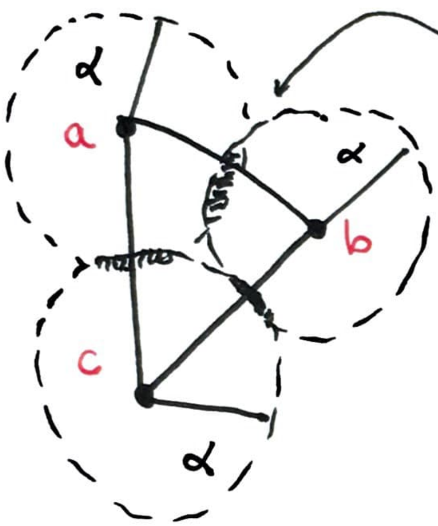
$a, b, c \in X$

$[a\,b\,c] \in \mathrm{Rips}_\alpha(X)$ <u>if</u>

$d(a,b) \leq \alpha$ <u>and</u> $d(a,c) \leq \alpha$

<u>and</u> $d(b,c) \leq \alpha$

Def $\check{C}ech_\alpha(X)$ = set of simplices $[x_0 \dots x_k]$ such that the $k+1$ closed balls $B_c(x_i, \alpha)$ have non-empty intersection.



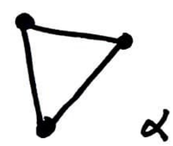
$[a], [b], [c] \in \check{C}ech_\alpha(X)$

$[ab] \in \check{C}ech_\alpha(X)$

$[ac] \in \check{C}ech_\alpha(X)$

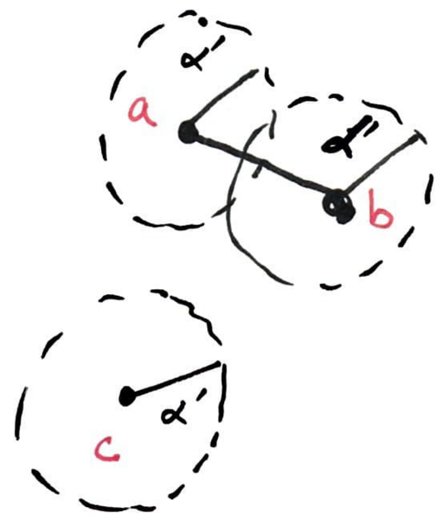
$[bc] \in \check{C}ech_\alpha(X)$

$[abc] \notin \check{C}ech_\alpha(X)$



What if we increase or decrease α?

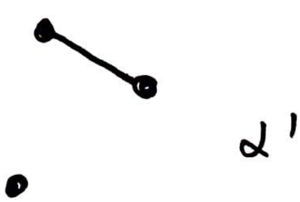
- **decrease α**



$\alpha' < \alpha$

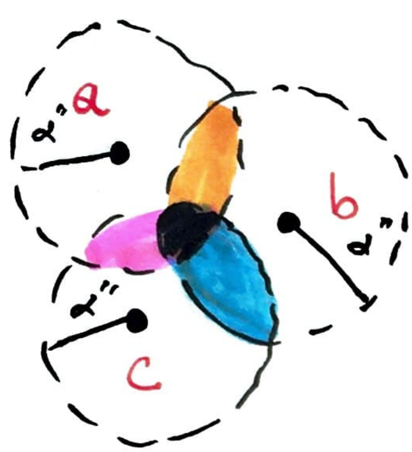
$[ab] \in \check{C}ech_{\alpha'}(X)$

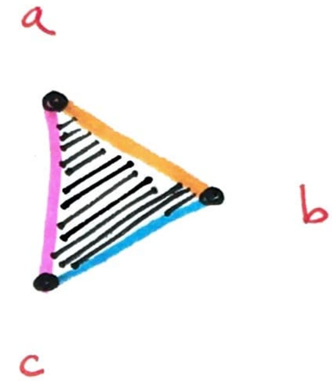
$[ac], [bc] \notin \check{C}ech_{\alpha'}(X)$



In fact if α is small enough ($\alpha < \inf d(x_i, x_j)$) there will be no overlapping balls.

- <u>increase α</u>





$[a\,b\,c] \in \check{C}ech_{\alpha''}(X)$ because

$\blacktriangle \subseteq B_c(a,\alpha'') \cap B_c(b,\alpha'') \cap B_c(c,\alpha'') \neq \emptyset$

<u>Remark</u>: $\left[\ Rips_\alpha(X) \subseteq \check{C}ech_\alpha(X) \subseteq Rips_{2\alpha}(X)\ \right]$

α measures the scale at which we observe X.

It is useful to let α vary and observe how the topological invariants associated with the α-complexes change with α.