TRUTH SET METHOD AND PROPOSITIONAL LOGIC

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ABSTRACT

The goal of logic is to develop reliable tools for appraisal of arguments. The truth table method is one of the main tools used in logic. The advantage of the truth tables is that they are almost completely mechanical and requires very little creative thinking. However, the truth table method is impractical for all but very small problem instances. In this paper, we proposed a new approach, truth set method, and provide a different way to process data in comparison to the truth table method. The truth set method can be used to prove logical equivalences of propositions or to prove that arguments are valid in propositional logic. It can also be used to solve random SAT problems. The truth set method is sound and complete. We focus on the application of the truth set method to propositional logic in this paper and will show how it solves the hard random 3-SAT problems in next paper.

***K*eywords** Truth Set · Propositional Logic · Truth Table · Logical Equivalence · Formal Proof

1 Introduction

Logic is the basis of all mathematical reasoning and of all automated reasoning. It has been studied since antiquity. The ancient Greek philosopher set out to analyze patterns of reasoning and developed the theory of the syllogism [1]. Syllogistic arguments are usually represented in a three-line form:

All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal.

Aristotle's syllogism is known as term logic, since it is concerned with the logical relations between terms, such as "all men", "Socrates", "mortal". It shares elements with both set theory and predicate logic [2].

Leibniz's dream is to find a system of logic powerful enough to calculate questions of law, politics, and ethics. He attempted to establish a calculus of reasoning which can be used to decide all arguments. He suggested that an international symbolic language for logic be developed with which equations of logic could be written and used to calculate a "solution" to any argument [3].

The 19 century pioneers introduced suggestive symbolic notation for logical operations [4]. The ordinary statements are translated into symbolic notation in order to remove vagueness. Once we have the definitions of these formalized notions, we can evaluate or manipulate them in symbolic logic.

The main tools used to evaluate arguments are the truth table method and the formal proof method. The truth table method is mechanical but it is long and tedious when the number of atomic propositions is greater than 4. The formal proof method uses rules of inference as building blocks to construct more complicated valid argument. But writing proofs needs creative thinking and the process cannot guarantee success.

From Aristotle's term logic to modern symbolic logic, the object of logical has changed from natural language to symbols. In this paper, we propose a new approach, truth set method, for propositional logic. Each proposition is represented by a truth set consisting of whole numbers. So the object of logical will change from symbols to numbers and we can calculate a solution to an argument. The following is an outline of the rest of the paper. In Section 2, we introduce the concepts of the truth set method. In Section 3, we show how to use the truth set method to prove logical equivalences. In Section 4, we explore its use in evaluating arguments. We have some comparisons and discussions in Section 5. Finally, we conclude in Section 6.

2 Basic Concepts

2.1 Propositions

A proposition is a declarative statement that is either true or false, but not both. An atomic proposition is a proposition that cannot be broken down into other simpler propositions. A compound proposition is comprised of propositions and one or more of the logical connectives.

In propositional logic, propositional variables are used to represent the propositions and the variables are denoted by letters, such as $p, q, r, s, ...$. The semantics defines the truth of each proposition with respect to each assignment of truth values to propositional variables.

In the truth set method, the syntax and semantics are similar to propositional logic. The difference is that we use truth sets, instead of propositional variables, to represent the propositions. We usually choose $P, Q, R, S, ...$ to represent truth sets, because mathematical texts usually use the capital letters to represent sets.

2.2 Truth Sets

In the truth table method, a truth assignment is simply a row of values of all atomic propositions. The number of possible truth assignments depends on the number of atomic propositions n. Since each proposition has two possible values, there are 2^n possible assignments for n atomic propositions. The following truth table shows whether a conjunction of atomic propositions P and Q is true or false in every possible assignment:

P	Q	$P \wedge Q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**F**

The left hand side of the truth table is the truth values of the two atomic propositions (P and Q) in all possible truth assignments. On the right are the truth values of the compound propositions ($P \wedge Q$) under the corresponding assignments. The truth values of compound propositions depend on the truth values of its components and the definition of the logical connectives.

The concept of the truth sets came from truth tables. In the truth table method, the data are processed row by row. Each row contains the truth values of a truth assignment for all atomic propositions. The truth table method needs to enumerate all possible truth assignments in order to make the overall evaluation. In the truth set method, the data are processed column by column. The values under each column are converted to the elements of a truth set of the corresponding atomic proposition.

Before the conversion, we need to make some changes to the data in the truth table in order to distinguish the values of different rows and because sets do not allow duplicate values. First, we define an assignment index based on the row number. The assignment index starts from 0 and equals the row number minus 1. The maximum number of assignment index is $2^n - 1$, where n is the number of atomic propositions. Second, we replace all true values in each row with its assignment indexes. Third, we remove all false values from the table. After the changes, we get the following table from the truth table above:

P	Q	$P \wedge Q$
0	0	0
1		
	2	

Therefore, the truth set of P is $\{0, 1\}$ and the truth set of Q is $\{0, 2\}$. The elements in truth sets mean that the propositions are true in these assignments. Whenever an assignment index is missing from the truth set, such as 2 or 3 for P, we know that the proposition is false in that assignment. Later we will discuss the calculation of truth set of a compound proposition.

Definition 2.1 (Truth Set)**. Truth set** of a proposition is a collection of the assignment indexes for which the proposition is true. The range of assignment index is the first 2^n whole numbers, where n is the number of atomic propositions.

Here the definition of the truth set is different from the definition in predicate logic. In predicate logic, the truth set S with domain D is the set of those $x \in D$ for which $S(x)$ is true. For example, the numbers 1 and -1 are the two elements of the truth set of the equation $x^2 = 1$. In the truth set method, the truth set of proposition P is $\{0, 1\}$ and it means that the proposition P is true in the first and the second assignments.

The following are the steps of assigning the truth sets to the atomic propositions, which is similar to constructing a truth table [5]:

1. Sort the atomic propositions in alphabetical order.
2. The number of elements in truth set for atomic propositions is 2^{n-1} where n is the number of atomic propositions (the number of elements in truth set for compound propositions depends on its components and the logical connectives).
3. Start from the last atomic proposition and assign the even numbers, such as $0, 2, 4, 6, ..., 2^n - 2$, as the elements of its truth set.
4. Move to the atomic proposition before last and assign the pairs of numbers, such as $0, 1, 4, 5, ...$, as the elements of its truth set.
5. Continue the process for the last of remain propositions and double the numbers of assignments each time.
6. For the truth set of the first atomic proposition, the values of elements are the first 2^{n-1} whole numbers (from 0 to $2^{n-1} - 1$).

For example, if there are three atomic propositions, P, Q and R, their truth sets are:
$R = \{0, 2, 4, 6\}$
$Q = \{0, 1, 4, 5\}$
$P = \{0, 1, 2, 3\}$

The truth sets depend on both the order and the number of the atomic propositions. Suppose Q is one of the two atomic propositions. If Q is the second atomic proposition, such as P and Q, its truth set $Q = \{0, 2\}$. If Q is the first atomic proposition, such as Q and R, then $Q = \{0, 1\}$. This is why it is important to sort the atomic propositions in the first steps. On the other hand, if Q is still the second atomic proposition but there are three atomic propositions, as you see above, $Q = \{0, 1, 4, 5\}$.

From the truth set, we know a proposition can have a true value or a false value in different assignments, we also know in which assignments it has a true value and in which it has a false value. By using a truth set, we can specify a proposition more completely.

2.3 Propositional Constants

The simplest propositions are the constants: Tautology and Contradiction. Their truth values do not depend on any other propositions. Tautology is a proposition that is always true in every truth assignment. Its truth set is the universal set $U = \{0, 1, ..., 2^n - 1\}$ where n is the number of atomic propositions. Contradiction is a proposition that is always false in every truth assignment. Based on the definition, the truth set of Contradiction is an empty set $\emptyset$.

Other propositions are treated as variables in propositional logic. They can be either true or false in different truth assignments. The truth table method evaluates the propositions row by row and it is row-oriented method. When comparing two propositions, their truth values may be equal in some assignments and unequal in others. To determine whether two compound propositions are logically equivalent, the truth table method should show that they have the same value in all possible cases.

The truth set method is proposition-oriented (or column-oriented). Each atomic proposition is represented by a truth set. These truth sets are constant sets because once they are assigned, the number of elements and the values of elements will not change. The truth sets of the compound propositions can be calculate from the truth sets of its components, and they are also constant sets.

In other words, from a row-by-row perspective, a proposition is a variable because its truth value can be true or false. But from a column-by-column perspective, a proposition is a "constant" because the number of true values and the assignment indexes (row numbers minus one) of the proposition do not change. We can treat all propositions as "constants" within a problem. Of course, the propositions in different problems may have different truth sets.

2.4 Logical Operations

We discussed above the assignment of truth sets to atomic propositions. Next we will discuss the use of set operations to calculate the truth sets of the compound propositions.

Let P and Q be the atomic propositions. Their truth sets are $\{0, 1\}$ and $\{0, 2\}$ respectively. The truth set of Tautology (or the universal set) U is $\{0, 1, 2, 3\}$.

Definition 2.2 (Negation)**.** The truth set for the **negation** of P, denoted by $\neg P$, is the complement of P or $(U - P)$.

The difference (subtraction) is defined as follows. If $U = \{0, 1, 2, 3\}$ and $P = \{0, 1\}$, then $\neg P = U - P = \{0, 1, 2, 3\} - \{0, 1\} = \{2, 3\}$. The result is consistent with the truth table for the negation of P.

P	Q	$\neg P$
T	T	**F**
T	F	**F**
F	T	**T**
F	F	**T**

Definition 2.3 (Conjunction)**.** The truth set for the **conjunction** of P and Q, denoted by $P \wedge Q$, is the intersection of P and Q.

The intersection of two sets P and Q consists of all elements that are both in P and Q.

$$P \wedge Q = P \cap Q = \{0, 1\} \cap \{0, 2\} = \{0\}$$

The result is consistent with previous truth table for the conjunction of P and Q.

Definition 2.4 (Disjunction)**.** The truth set for the **disjunction** of P and Q, denoted by $P \vee Q$, is the union of P and Q.

The union of two sets P and Q is a set containing all elements that are either in P or in Q (possibly in both).

$$P \vee Q = P \cup Q = \{0, 1\} \cup \{0, 2\} = \{0, 1, 2\}$$

The result is consistent with the truth table for the disjunction of P and Q.

P	Q	$P \vee Q$
T	T	**T**
T	F	**T**
F	T	**T**
F	F	**F**

Definition 2.5 (Implication)**.** The truth set for the conditional statement $P \to Q$ is the union of $\neg P$ and Q. A conditional statement is also called an **implication**.

$$P \to Q = \neg P \cup Q = \{2, 3\} \cup \{0, 2\} = \{0, 2, 3\}$$

The result is also consistent with the truth table for the conditional statement $P \to Q$.

P	Q	$P \to Q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

The truth sets for the compound propositions using other logical operations can be also calculated by the set operations. The following is the calculation for **biconditional statement**:

$$P \leftrightarrow Q = (P \cap Q) \cup (\neg P \cap \neg Q) = (\{0, 1\} \cap \{0, 2\}) \cup (\{2, 3\} \cap \{1, 3\}) = \{0\} \cup \{3\} = \{0, 3\}$$

2.5 Possible Propositions

In mathematics, the power set of a universal set U is the set of all subsets of U, including the empty set and the set U itself [6]. If m is the cardinality of U, the number of all the subsets of U is 2^m.

The truth set of Tautology is the universal set U, which has $m = 2^n$ elements where n is the number of atomic propositions. So there are 2^{2^n} possible propositions, including Tautology and Contradiction, in a problem involving n atomic propositions.

Let P and Q be the atomic propositions. There are a total of 16 possible propositions and they are listed in Table 1.

Table 1: All 16 Possible Propositions of P and Q

Key	Propositions	Truth Set	Description
0	Contradiction	$\emptyset$	
1	$P \wedge Q$	$\{0\}$	Conjunction of P and Q
2	$P \wedge \neg Q$	$\{1\}$	
3	P	$\{0, 1\}$	Atomic proposition P
4	$\neg P \wedge Q$	$\{2\}$	
5	Q	$\{0, 2\}$	Atomic proposition Q
6	$(\neg P \wedge Q) \vee (P \wedge \neg Q)$	$\{1, 2\}$	Exclusive Or $P \oplus Q$
7	$P \vee Q$	$\{0, 1, 2\}$	Disjunction of P and Q
8	$\neg P \wedge \neg Q$	$\{3\}$	
9	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$	$\{0, 3\}$	Biconditional of $P \leftrightarrow Q$
10	$\neg Q$	$\{1, 3\}$	Negation of Q
11	$P \vee \neg Q$	$\{0, 1, 3\}$	An equivalent of Converse $Q \rightarrow P$
12	$\neg P$	$\{2, 3\}$	Negation of P
13	$\neg P \vee Q$	$\{0, 2, 3\}$	An equivalent of Conditional $P \rightarrow Q$
14	$\neg P \vee \neg Q$	$\{1, 2, 3\}$	
15	Tautology	$\{0, 1, 2, 3\}$	

The key for each proposition is calculated from the following equation:

$$\text{Key} = \sum_{i=1}^{k} 2^{m_i} \tag{1}$$

where k is the number of the elements in the truth set and m_i is the value for the i-th element. The truth set of Contradiction is an empty set and its key is set to 0.

When two sets have the same number of elements, although the elements are different, they are called equivalent sets. We divide the 16 possible propositions in Table 1 into five groups based on the number of elements in the truth sets:

(1) Minterm Group

$$\begin{array}{cc} P \wedge Q & \{0\} \\ P \wedge \neg Q & \{1\} \\ \neg P \wedge Q & \{2\} \\ \neg P \wedge \neg Q & \{3\} \end{array}$$

The members of this group are the conjunctions of propositions that occur once for every n atomic proposition. Their truth set has only one element, which is one of the elements in the universal set respectively.

(2) Atomic Group

$$\begin{array}{cc} P & \{0, 1\} \\ Q & \{0, 2\} \\ \neg P & \{2, 3\} \\ \neg Q & \{1, 3\} \end{array}$$

The members of this group are the atomic propositions or their negations. Their truth set has two elements.

(3) Maxterm Group

$$\begin{array}{cc} P \vee Q & \{0, 1, 2\} \\ P \vee \neg Q & \{0, 1, 3\} \\ \neg P \vee Q & \{0, 2, 3\} \\ \neg P \vee \neg Q & \{1, 2, 3\} \end{array}$$

The members of this group are the disjunctions of propositions that occur once for every n atomic proposition. Their truth set has three elements.

(4) Complex Group

$$\begin{array}{cc} (\neg P \wedge Q) \vee (P \wedge \neg Q) & \{1, 2\} \\ (P \wedge Q) \vee (\neg P \wedge \neg Q) & \{0, 3\} \end{array}$$

The members of this group are the disjunctions of minterms (or conjunctions of maxterms). Their truth set has two elements. So the truth sets of the Complex Group and the truth sets of the Atomic Group are the equivalent sets.

(5) Constant Group

$$\begin{array}{cc} \mathbf{F} & \emptyset \\ \mathbf{T} & \{0,1,2,3\} \end{array}$$

Although they differ in the number of elements in their truth sets, we group these two constants into one group.

Except for Contradiction, all propositions in other groups can be constructed by disjunction of the minterms in disjunctive normal form (DNF). Propositions in the Complex Groups are the examples in DNF.

3 Propositional Equivalences

Logic is the study of propositions and their relationships. The relations between propositions include implication, equivalence (mutual implication), and no implication. We will discuss the implication relations in next section and focus on the equivalent relations here. Logical equivalence can be used to help reduce compound propositions to simpler forms. They can also be used for formal proofs.

3.1 Logical Equivalences

From the discussion above, we know if the number of atomic propositions n is finite, the number of possible propositions (or truth sets) is also finite and it is equal to 2^{2^n}. However, numerous well-formed formulas can be build up by using logical connectives. This means that some propositions with different well-defined formula may have the same truth sets. Just like mathematical functions $x + xy$ and $x(1 + y)$, they are written differently, but they are the same functions.

Definition 3.1 (Logical Equivalences)**.** Two propositions are logically equivalent if and only if they have the same propositional truth sets.

When the truth sets of two propositions are equal, it means they have the same truth values in all possible cases. They are written differently, but they are the same propositions.

The compound proposition $(\neg P \wedge \neg(\neg Q) \wedge \neg Q) \vee (\neg(\neg P) \wedge \neg Q \wedge \neg Q) \vee (\neg(\neg P) \wedge \neg(\neg Q) \wedge Q)$ has two atomic propositions. We found its truth set is {0,1}. It is equal to the truth set of P in Table 1. Therefore, the compound proposition is logically equivalent to P:

$$(\neg P \wedge \neg(\neg Q) \wedge \neg Q) \vee (\neg(\neg P) \wedge \neg Q \wedge \neg Q) \vee (\neg(\neg P) \wedge \neg(\neg Q) \wedge Q) \equiv P$$

In symbolic logic, we can prove the distributive rule using a truth table:

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**T**	T	F	**T**
T	F	T	T	**T**	F	T	**T**
T	F	F	F	**F**	F	F	**F**
F	T	T	T	**F**	F	F	**F**
F	T	F	T	**F**	F	F	**F**
F	F	T	T	**F**	F	F	**F**
F	F	F	F	**F**	F	F	**F**

The column for $P \wedge (Q \vee R)$ and the column for $(P \wedge Q) \vee (P \wedge R)$ are identical. It shows the two expressions are logical equivalent.

In truth set method, we calculate the truth sets for the two expressions.

$$\begin{aligned} P \wedge (Q \vee R) &= \{0,1,2,3\} \cap (\{0,1,4,5\} \cup \{0,2,4,6\}) \\ &= \{0,1,2,3\} \cup \{0,1,2,4,5,6\} \\ &= \{0,1,2\} \end{aligned}$$

$$\begin{aligned} (P \wedge Q) \vee (P \wedge R) &= (\{0,1,2,3\} \cap (\{0,1,4,5\}) \cup (\{0,1,2,3\} \cap (\{0,2,4,6\}) \\ &= \{0,1\} \cup \{0,2\} \\ &= \{0,1,2\} \end{aligned}$$

We found the truth sets are equal for two expressions so they are logical equivalent. From the example, we can see the truth set method is simpler than the truth table method.

Table 2: Minterms of Three Variables P, Q and R

Key	Propositions	Truth Set
1	$P \land Q \land R$	$\{0\}$
2	$P \land Q \land \neg R$	$\{1\}$
4	$P \land \neg Q \land R$	$\{2\}$
8	$P \land \neg Q \land \neg R$	$\{3\}$
16	$\neg P \land Q \land R$	$\{4\}$
32	$\neg P \land Q \land \neg R$	$\{5\}$
64	$\neg P \land \neg Q \land R$	$\{6\}$
128	$\neg P \land \neg Q \land \neg R$	$\{7\}$

In symbolic logic, if we want to prove that $\neg(P \lor (\neg P \land Q))$ and $\neg P \land \neg Q$ are logically equivalent, we can do it by developing a series of logical equivalences [7].

$$\begin{aligned}
\neg(P \lor (\neg P \land Q)) &\equiv \neg P \land \neg(\neg P \land Q) && \text{by the De Morgan law}\\
&\equiv \neg P \land [\neg(\neg P) \lor \neg Q] && \text{by the De Morgan law}\\
&\equiv \neg P \land (P \lor \neg Q] && \text{by the double negation law}\\
&\equiv (\neg P \land P) \lor (\neg P \land \neg Q) && \text{by the distributive law}\\
&\equiv \mathbf{F} \lor (\neg P \land \neg Q) && \text{by the negation law}\\
&\equiv (\neg P \land \neg Q) \lor \mathbf{F} && \text{by the commutative law}\\
&\equiv \neg P \land \neg Q && \text{by the identity law}
\end{aligned}$$

In truth set method, we can get the truth sets of the two formulas:

$$\begin{aligned}
\neg(P \lor (\neg P \land Q)) &= \overline{\{0,1\} \cup (\overline{\{0,1\}} \cap \{0,2\})}\\
&= \overline{\{0,1\} \cup (\{2,3\} \cap \{0,2\})}\\
&= \overline{\{0,1\} \cup \{2\}}\\
&= \overline{\{0,1,2\}}\\
&= \{3\}
\end{aligned}$$

$$\begin{aligned}
\neg P \land \neg Q &= \overline{\{0,1\}} \cap \overline{\{0,2\}}\\
&= \{2,3\} \cap \{1,3\}\\
&= \{3\}
\end{aligned}$$

Therefore, $\neg(P \lor (\neg P \land Q)) \equiv \neg P \land \neg Q$ because both formulas have the same truth sets. From the example, we can see that the truth set method proves logical equivalence by the numerical calculations, not by the manipulation of symbols.

3.2 Constructing New Logical Equivalences

Truth sets can be used to determine whether two propositions are logically equivalent. They can also be used in constructing new logical equivalences. Let P, Q, and R are the atomic propositions. The Minterm Group are listed in Table 2.

A literal is either an atomic proposition or its negation. A clause (or minterm) in Table 2 is a conjunction of literals. A formula is in disjunctive normal form if it is a disjunction of minterms. Every proposition can be transformed into an equivalent disjunctive normal from.

Here is an example of logic puzzles: When planning a party you want to know whom to invite. Among the people you would like to invite are three touchy friends. You know that if Jasmine attends, she will become unhappy if Samir is there, Samir will attend only if Kanti will be there, and Kanti will not attend unless Jasmine also does. Which combinations of these three friends can you invite so as not to make someone unhappy?

Let's assume:

P: Jasmine is invited.
Q: Kanti is invited.
R: Samir is invited.

The following three conditions should be met:

a) $P \to \neg R$: If Jasmine is invited, Samir will not be invited.

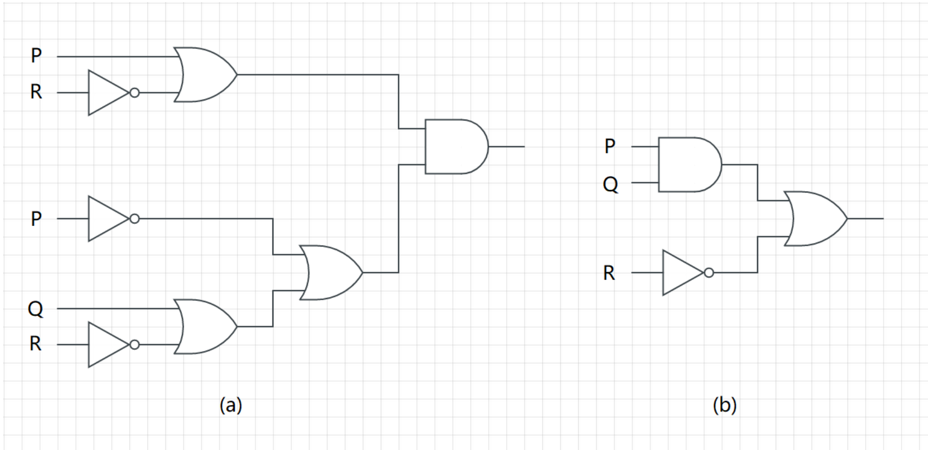

Figure 1: (a) The circuit for $(P \vee \neg R) \wedge (\neg P \vee (Q \vee \neg R))$; (b) The circuit for $(P \wedge Q) \vee \neg R$

b) $R \rightarrow Q$: If Samir is invited, Kanti will also be invited.
c) $Q \rightarrow P$: If Kanti is invited, Jasmine will also be invited.

The compound proposition for the puzzle is:

$$(P \rightarrow \neg R) \wedge (R \rightarrow Q) \wedge (Q \rightarrow P)$$

From the truth sets of atomic propositions P, Q and R as assigned in Section 2.2, we can calculated the truth set of the compound proposition, which is $\{1, 3, 7\}$. With the help of Table 2, the compound proposition can be transformed into an equivalent disjunctive normal form:

$$(P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$$

The DNF formula has the same truth set as the compound proposition for the puzzle. The three clauses in DNF formula leave us with three possible choices:
(1) $P \wedge Q \wedge \neg R$: Both Jasmine and Kanti were invited.
(2) $P \wedge \neg Q \wedge \neg R$: Only Jasmine was invited.
(3) $\neg P \wedge \neg Q \wedge \neg R$: No one was invited.

Above we have shown how to use truth sets to construct a logical equivalency when a proposition is given. Next we will show how to use truth sets to simplify a proposition.

Suppose that we have a formula for the output of a digital circuit in terms of negations, disjunctions, and conjunctions. Then, we can systematically build a digital circuit with the desired output [7]. For example, if the formula is $(P \vee \neg R) \wedge (\neg P \vee (Q \vee \neg R))$, the resulting circuit is displayed in Figure 1 (a).

We can construct a simpler logical equivalence which has the same output as the original circuit. By calculations, the truth set of $(P \vee \neg R) \wedge (\neg P \vee (Q \vee \neg R))$ is $\{0, 1, 3, 5, 7\}$. We know the truth set for R is $\{0, 2, 4, 6\}$ and the truth set for $\neg R$ is $\{1, 3, 5, 7\}$. We also know the truth set for $P \wedge Q \wedge R$ is $\{0\}$ from Table 2. It is easy to construct a new logical equivalency for $(P \vee \neg R) \wedge (\neg P \vee (Q \vee \neg R))$ using the union of the truth sets of $(P \wedge Q \wedge R)$ and $\neg R$, i.e. $(P \wedge Q \wedge R) \vee \neg R$.

In practical applications, we can create a database to store the simplest or the ideal proposition information for each possible proposition. For example, we know that the number of possible propositions in a system with three atomic propositions is 256. Different formulas may have the same truth sets. The simplest proposition with a truth set {0,1,3,5,7} is $(P \wedge Q) \vee \neg R$. When we want to build a digital circuit that produces the output $(P \vee \neg R) \wedge (\neg P \vee (Q \vee \neg R))$, we can get its truth set {0,1,3,5,7} and then find its logical equivalence $(P \wedge Q) \vee \neg R$ from the database. The original circuit needs three NOT gates, three OR gates, and one AND gate. The new circuit, $(P \wedge Q) \vee \neg R$, needs only one NOT gate, one OR gate, and one AND gates as displayed in Figure 1 (b). Both circuits have the same output.

4 Logic Proof

The advantage of truth tables is that they are almost completely mechanical and require little creative thinking. We can check whether when the premises are true, the conclusion is also true. The disadvantages of truth tables are that they are very long when the number of atomic propositions involved is large. If there are five atomic propositions, the truth table will have thirty-two rows. It also requires the careful placement of T's and F's in the appropriate columns.

The formal proof method uses the rules of inference to build blocks to construct more complicated valid argument forms. But this approach is not mechanical and sometime requires creative thinking.

4.1 Valid Arguments

An argument in logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion. An argument is valid if the truth of all its premises implies that the conclusion is true [7]. In other words, an argument form with premises P_1, P_2, ..., P_n and conclusion Q is valid exactly when $(P_1 \wedge P_2 \wedge ... \wedge P_n) \rightarrow Q$ is a tautology. Here the conjunction of its premises implying its conclusion can be considered as a proposition.

The truth set method treats premises and conclusions separately. The conjunction of all premises of an argument is a proposition, which we call the premise proposition. The conclusion of the argument is another proposition. We calculate the truth sets for both premise proposition and the conclusion. The truth set of premise proposition is the intersection of the truth sets of all premises.

Definition 4.1 (Valid Argument)**.** An argument is valid if and only if the truth set of premise proposition is a subset of the truth set of the conclusion.

The definition shows that whenever all premises are true, conclusion is also true. Therefore, the argument is valid.

4.2 Analysis of Premises and Conclusion

In our approach, we consider the conjunction of all premises as one proposition and the conclusion as another. If the truth set of premise proposition, denoted by S_{pre}, is a subset of the truth set of conclusion, denoted by S_{con}, we can say that the argument is valid. S_{pre} is the intersection of the truth sets of all premises. Analyzing the truth sets of these two propositions will help us understand why an argument is valid and how the rules of inference work.

That S_{pre} is a subset of S_{con} means that all elements of S_{pre} are also elements of S_{con}. It is possible for S_{pre} and S_{con} to be equal. So there are two types of valid arguments:

- S_{pre} and S_{con} are equal: Two sets have the same elements. Premise proposition and conclusion are logically equivalent. Whenever premise proposition is true, conclusion is also true.
- S_{pre} is a proper subset of S_{con}: Besides whenever premise proposition is true, conclusion is also true, there are one or more truth assignments where conclusion is true but premise proposition is false.

In order for the argument to be valid, one necessary condition is that the number of elements in S_{pre} should be less than or equal to the number of elements in S_{con}. The fewer elements in the truth set of a proposition, the more likely it is that the proposition is a candidate for the premise proposition. The more elements in a proposition's truth set, the more likely that proposition is a candidate for a conclusion.

Let P and Q be the atomic propositions. Table 1 showed all possible propositions and their truth sets. We have divided these propositions into five groups.

The propositions in Minterm Group have only one element. They are good candidates for premise propositions. The propositions in Maxterm Group have three elements. They are good candidates for conclusions. The propositions in Atomic Group and Complex Group have two elements. They can be candidates for either premise or conclusion.

Table 3 shows that given a premise proposition, what its legitimate conclusion is. The left hand side of the table is a cross reference table for propositions and their keys. The right hand side are the keys for given premise proposition and legitimate conclusions. In Table 3, we see the following patterns:

- Contradiction cannot be a premise proposition. The truth set for Contradiction is an empty set. It means it is always false. If Contradiction was the premise, any proposition could have been its conclusion. It is out of nothing.
- As a premise, Tautology only has itself as a legitimate conclusion. The truth set for Tautology is the universal set. The truth set of any other proposition cannot be its superset.
- As a premise, each member of Minterm Group (keys: 1, 2, 4, 8) has eight legitimate conclusion, including itself and Tautology. Their truth sets have only one element. So they are good candidates for premise proposition.

Table 3: Given Premises and Legitimate Conclusions

Key	Proposition	Premise	Legitimate Conclusion														
0	Contradiction																
1	$P \wedge Q$	1	1		3		5		7		9		11		13		15
2	$P \wedge \neg Q$	2		2	3			6	7			10	11			14	15
3	P	3			3				7				11				15
4	$\neg P \wedge Q$	4				4	5	6	7					12	13	14	15
5	Q	5					5		7						13		15
6	$(\neg P \wedge Q) \vee (P \wedge \neg Q)$	6						6	7							14	15
7	$P \vee Q$	7							7								15
8	$\neg P \wedge \neg Q$	8								8	9	10	11	12	13	14	15
9	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$	9									9		11		13		15
10	$\neg Q$	10										10	11			14	15
11	$P \vee \neg Q$	11											11				15
12	$\neg P$	12												12	13	14	15
13	$\neg P \vee Q$	13													13		15
14	$\neg P \vee \neg Q$	14														14	15
15	Tautology	15															15

There are two legitimate conclusions from Atomic Group, three conclusions from Maxterm Group and one conclusion from Complex Group.

- As a premise, each member of Maxterm Group (keys: 7, 11, 13, 14) has no other legitimate conclusion except itself and Tautology. Their truth sets have three elements. So the members in Maxterm Group are not good candidates for premise proposition.
- As a premise, each member of Atomic Group (keys: 3, 5, 10, 12) or Complex Group (keys: 6, 9) has four legitimate conclusion including itself and Tautology. Their truth set has two elements. So they have fewer candidates for conclusion than Minterm Group but more candidates for conclusion than Maxterm Group. There are two legitimate conclusions from Maxterm Group.
- The key of a legitimate conclusion is equal to or greater than the key of the given premise.
- The premise with an odd key will have legitimate conclusions with only odd keys. The premise with an even key will have legitimate conclusions with both odd and even keys. So there are more conclusions with odd keys than even keys. This is because the members in Maxterm Group have an odd keys and they are more likely to become a conclusion.

From the discussion above, we can find two rules of inference. The first one is called the Conjunction Elimination. The rule consists of two separate sub-rules. When given premise is $P \wedge Q$ (key 1), we can return a conclusion of either P (key 3) or Q (key 5). In set theory, the intersection of two sets is a subset of any one of the two sets.

Another rule of inference is called Disjunction Introduction. We use P as an example. If P is the premise proposition (key 3), we can return a conclusion of either $P \vee Q$ (key 7) or $P \vee \neg Q$ (key 11). In set theory, one set is a subset of the union of itself and any other set.

By analyzing the truth sets for given premise propositions and legitimate conclusions, we can find as many as conclusions for a given premise proposition based on the truth sets.

Table 4 shows that given a conclusion, what its legitimate premise proposition is. We see the following patterns:

- Contradiction cannot be a conclusion. If Contradiction was the conclusion, the only premise proposition should have been itself because no set is a subset of empty set. But we already showed Contradiction cannot be a premise proposition.
- Except for Contradiction, Tautology can be the conclusion of any premise proposition. The truth set of Tautology is the universal set. Any set is a subset of the universal set.
- As a conclusion, each member of Minterm Group only has itself as a premise proposition. At this time, premise proposition and conclusion are logically equivalent.
- As a conclusion, each member of Atomic Group and Complex Group has three legitimate premise propositions including itself. Other two legitimate candidates are from Maxterm Group.

Table 4: Given Conclusions and Legitimate Premises

Key	Proposition	Premise Proposition															Conclusion
0	Contradiction																
1	$P \wedge Q$	1															1
2	$P \wedge \neg Q$		2														2
3	P	1	2	3													3
4	$\neg P \wedge Q$				4												4
5	Q	1			4	5											5
6	$(\neg P \wedge Q) \vee (P \wedge \neg Q)$		2		4		6										6
7	$P \vee Q$	1	2	3	4	5	6	7									7
8	$\neg P \wedge \neg Q$								8								8
9	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$	1							8	9							9
10	$\neg Q$		2						8		10						10
11	$P \vee \neg Q$	1	2	3					8	9	10	11					11
12	$\neg P$				4				8								12
13	$\neg P \vee Q$	1			4	5			8	9			12	13			13
14	$\neg P \vee \neg Q$		2		4		6		8		10		12		14		14
15	Tautology	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	15

- As a conclusion, each member of Maxterm Group has seven legitimate premise propositions including itself. Three of them from Minterm Group, two from Atomic Group, and one from Complex Group.
- The key of a legitimate premise proposition is less than or equal to the key of a given conclusion.
- If the key for a given conclusion is even, it has a legitimate premise proposition with only even keys. Conclusion with an odd key can have a premise proposition with either odd or even key. Therefore, there are more legitimate premise propositions with even keys. This is because that the members of Minterm Group have even keys (except the key for $P \wedge Q$) and they are more likely to be the premise proposition.

By analyzing the truth sets of premise proposition and conclusion, not only we can find some rules of inference but also we know the relationship between premise proposition and conclusion. For example, if $P \vee Q$ is a given premise proposition, we cannot return the conclusion of P because the truth set of $P \vee Q$ has three elements and it cannot have a conclusion, P, which truth set has only two elements. On the other hand, if $P \wedge Q$ is a given conclusion, the only premise proposition is itself, which may be in different form. We will see more examples in Section 4.3.

4.3 Rules of Inference

There are many useful rules of inference in symbolic logic. From the above discussion, we have found two rules of inference: Conjunction Elimination and Disjunction Introduction. Next, we will use the truth set method to study more rules of inference.

(1) Modus Ponens (MP)
Premises:
$P \rightarrow Q$
P
Conclusion:
Q
Solution:
1. $P \rightarrow Q = \{0, 2, 3\}$
2. $P = \{0, 1\}$
3. $S_{pre} = \{0\}$
4. $S_{con} = \{0, 2\}$
The argument is valid.

Modus Ponens is the most widely used rule of inference. The argument has two atomic propositions, P and Q. The truth set for premise proposition showed that the conjunction of premises is logically equivalent to $P \wedge Q$. According to Conjunction Elimination, Q is the conclusion. This is a case in Table 3 where the given premise key is 1 and the conclusion key is 5.

As a comparison, we check the fallacy of affirming the conclusion.

Premises:
$P \to Q$
Q
Conclusion:
P
Solution:
1. $P \to Q = \{0, 2, 3\}$
2. $Q = \{0, 2\}$
3. $S_{pre} = \{0, 2\}$
4. $S_{con} = \{0, 1\}$
The argument is not valid.

It turns out that the truth set of premise proposition is not a subset of the truth set of conclusion. Therefore, P is not the conclusion.

(2) Modus Tollens (MT)
Premises:
$P \to Q$
$\neg Q$
Conclusion:
$\neg P$
Solution:
1. $P \to Q = \{0, 2, 3\}$
2. $\neg Q = \{1, 3\}$
3. $S_{pre} = \{3\}$
4. $S_{con} = \{2, 3\}$
The argument is valid.

The results showed that the conjunction of the premises is logically equivalent to $\neg P \wedge \neg Q$. According to Conjunction Elimination, $\neg P$ is the conclusion. This is a case in Table 3 where the given premise key is 8 and the conclusion key is 12.

(3) Disjunctive Syllogism (DS1)
Premises:
$P \vee Q$
$\neg Q$
Conclusion:
P
Solution:
1. $P \vee Q = \{0, 1, 2\}$
2. $\neg Q = \{1, 3\}$
3. $S_{pre} = \{1\}$
4. $S_{con} = \{0, 1\}$
The argument is valid.

The results showed that the conjunction of the premises is logically equivalent to $P \wedge \neg Q$. According to Conjunction Elimination, P is the conclusion. This is a case in Table 3 where the given premise key is 2 and the conclusion key is 3.

(4) Disjunctive Syllogism (DS2)
Premises:
$P \vee Q$
$\neg P$
Conclusion:
Q
Solution:
1. $P \vee Q = \{0, 1, 2\}$
2. $\neg P = \{2, 3\}$
3. $S_{pre} = \{2\}$
4. $S_{con} = \{0, 2\}$
The argument is valid.

The results showed that the conjunction of the premises is logically equivalent to $\neg P \wedge Q$. Therefore, Q is the conclusion. This is a case in Table 3 where the given premise key is 4 and the conclusion key is 5.

So far, we have examined four rules of inference: Modus Ponens, Modus Tollens and two Disjunctive Syllogisms. They are all Conjunction Elimination but in different forms. We can find all of these in Table 3. From the original form of premises, it seems we are doing Conditional Elimination or Disjunctive Syllogism. But they are actually different forms of Conjunction Elimination.

(5) Conjunction Introduction (CI)
Premises:
P
Q
Conclusion:
$P \wedge Q$
Solution:
1. $P = \{0, 1\}$
2. $Q = \{0, 2\}$
3. $S_{pre} = \{0\}$
4. $S_{con} = \{0\}$
The argument is valid.

This is an example of premise proposition and conclusion are logically equivalent. The argument has two premises: P and Q. The premise proposition is the conjunction of premises, $P \wedge Q$. It is exactly the same as conclusion. Of course, their truth sets are equal and the argument is valid. This is a case in Table 4 where the premise key is 1 and the conclusion key is 1.

Next are two rules of inference involving three atomic propositions. The truth sets of propositions have more elements.

(6) Hypothetical Syllogism (HS)
Premises:
$P \to Q$
$Q \to R$
Conclusion:
$P \to R$
Solution:
1. $P \to Q = \{0, 1, 4, 5, 6, 7\}$
2. $Q \to R = \{0, 2, 3, 4, 6, 7\}$
3. $S_{pre} = \{0, 4, 6, 7\}$
4. $S_{con} = \{0, 2, 4, 5, 6, 7\}$
The argument is valid.

The results showed S_{pre} is a subset of S_{con}. Therefore, the argument is valid. When the problem involves three atomic propositions, the truth set of each member in the Atomic Group will have four elements and each member in Disjunction Group will have six elements. Here the premise proposition is an equivalence of Atomic Group (the truth set has four elements) and the conclusion is a member of the Disjunction Group.

(7) Resolution
Premises:
$P \vee Q$
$\neg P \vee R$
Conclusion:
$Q \vee R$
Solution:
1. $P \vee Q = \{0, 1, 2, 3, 4, 5\}$
2. $\neg P \vee R = \{0, 2, 4, 5, 6, 7\}$
3. $S_{pre} = \{0, 2, 4, 5\}$
4. $S_{con} = \{0, 1, 2, 4, 5, 6\}$
The argument is valid.

Again, the premise proposition is an equivalence of Atomic Group (the truth set has four elements) and the conclusion is a member of the Disjunction Group.

From the name point of view, Hypothetical Syllogism and Resolution seem to be different rules of inference. However, if we change the premises and conclusion of Resolution to following logical equivalences:
Premises:
$\neg Q \to P$
$P \to R$
Conclusion:
$\neg Q \to R$
We find Resolution is another form of Hypothetical Syllogism, or Hypothetical Syllogism is another form of Resolution.

4.4 Examples

Let's use examples to show how the truth set method works.

Example 1
Premises:
$\neg P \wedge Q$
$R \to P$
$\neg R \to S$
$S \to T$

Conclusion:
T

Solution:
1. $\neg P \wedge Q = \{16, 17, 18, 19, 20, 21, 22, 23\}$
2. $R \to P = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 20, 21, 22, 23, 28, 29, 30, 31\}$
3. $\neg R \to S = \{0, 1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 24, 25, 26, 27, 28, 29\}$
4. $S \to T = \{0, 2, 3, 4, 6, 7, 8, 10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 23, 24, 26, 27, 28, 30, 31\}$
5. $S_{pre} = \{20\}$
6. $S_{con} = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30\}$

The argument is valid.

There are five atomic propositions in the system, P, Q, R, S, T. The first four lines are the truth sets of the four premises. The fifth line is the truth set of the premise proposition, S_{pre}. The sixth line is the truth set of the conclusion, S_{con}. The results show that S_{pre} is the subset of S_{con}. Therefore, the argument is valid.

If we do a further analysis, we can find the truth set of proposition, $\neg P \wedge Q \wedge \neg R \wedge S \wedge T$, is $\{20\}$. According to conjunction elimination, if the conclusion is any literal in the formula, such as $\neg P$, the argument is valid.

If we use a truth table to show the proof, the truth table will have 32 rows. Our approach shows the results in six lines. The truth set method is simpler than the truth table method.

Example 2
Premises:
$(P \wedge T) \to (R \vee S)$
$Q \to (U \wedge T)$
$U \to P$
$\neg S$
Q

Conclusion:
$Q \to R$

Solution:
1. $S_{pre} = \{4\}$
2. $S_{con} = \{0, 1, 2, 3, 4, 5, 6, 7, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63\}$

The argument is valid.

The above argument involves six atomic propositions. We only show the truth sets for premise proposition and conclusion. The argument is valid because S_{pre} is the subset of S_{con}. Further analysis showed the proposition whose truth set is $\{4\}$ is $\neg P \wedge Q \wedge R \wedge \neg S \wedge T \wedge U$. So R is one of legitimate conclusions. According to disjunction introduction, $R \vee \neg Q$ should also be its conclusion. $Q \to R$ is the logical equivalence of $R \vee \neg Q$.

Example 3
The Logic Problem, taken from *WFF'N PROOF, The Game of Logic*, has these two assumptions:

1. "Logic is difficult or not many students like logic."
2. "If mathematics is easy, then logic is not difficult."

By translating these assumptions into statements involving propositional variables and logical connectives, determine whether each of the following are valid conclusions of these assumptions:

a) That mathematics is not easy, if many students like logic.
b) That not many students like logic, if mathematics is not easy.
c) That mathematics is not easy or logic is difficult.
d) That logic is not difficult or mathematics is not easy.
e) That if not many students like logic, then either mathematics is not easy or logic is not difficult.

Let's assume:
P: Logic is difficult.
Q: Many students like logic.
R: Mathematics is easy.

Premises:
$P \vee \neg Q$
$R \to \neg P$

Conclusion:
(a) $Q \to \neg R$
(b) $\neg R \to \neg Q$
(c) $\neg R \vee P$
(d) $\neg P \vee \neg R$
(e) $\neg Q \to (\neg R \vee \neg P)$

Solution:
1. $P \vee \neg Q = \{0, 1, 2, 3, 6, 7\}$
2. $R \to \neg P = \{1, 3, 4, 5, 6, 7\}$
3. $S_{pre} = \{1, 3, 6, 7\}$
(a) $S_{(a)} = \{1, 2, 3, 5, 6, 7\}$
(b) $S_{(b)} = \{0, 2, 3, 4, 6, 7\}$
(c) $S_{(c)} = \{0, 1, 2, 3, 5, 7\}$
(d) $S_{(d)} = \{1, 3, 4, 5, 6, 7\}$
(e) $S_{(e)} = \{0, 1, 3, 4, 5, 6, 7\}$
(a),(d) and (e) are valid.
(b) and (c) are invalid.

This is an example of multiple conclusions. We can calculate the truth set of premise proposition S_{pre} first. Then we can determine if the argument is valid by comparing S_{pre} with the truth set of each conclusion. The results showed S_{pre} is a subset of $S_{(a)}$, $S_{(d)}$ and $S_{(e)}$ respectively. Therefore, (a),(d) and (e) are valid.

For (b), $S_{(b)}$ is missing the element 1. From Table 2, we know the truth set for $P \wedge Q \wedge \neg R$ is $\{1\}$. It means when P and Q are assigned with true and R with false, the premises are true but conclusion $\neg R \to \neg Q$ is false. For (c), $S_{(c)}$ is missing element 6. From Table 2, we know the truth set for $\neg P \wedge \neg Q \wedge R$ is $\{6\}$. It means when P and Q are assigned with false, and R with true, the premises are true but conclusion $\neg R \vee P$ is false.

Here we show an advantage of separating the premises and the conclusion. We only need to calculate the truth set of premises once. Then we can find which conclusion is valid based on the truth sets of conclusions respectively.

5 Comparisons and Discussions

5.1 Comparison to Truth Table Method

The truth table method is simple but it can be very computationally expensive. For a problem instance with 10 atomic propositions, there are 2^{10} = 1024 rows in the truth table. As the number of atomic propositions increases, the number of rows in the truth table will soon overwhelm even the fastest computers.

The truth set method has the same data set as the truth table method. We show the same results when we determine if two propositions are logical equivalent or if an argument is valid. The truth set method is also mechanical. It is sound and complete just like the truth table method.

The differences between the truth set method and the truth table method are:

- Concept (variable or constant): The truth table method treats all propositions as variables, except Tautology and Contradiction. In each truth value assignment, a truth value (true or false) will be assigned to each atomic proposition. The truth value of a compound proposition depends on each assignment. In truth set method, we treat all propositions as constants. We assign all possible true values as a truth set to each atomic proposition at once. After the assignment, all propositions can be treated as constants because their truth sets will not change. When all propositions are constants, it is easy to determine whether two propositions are logically equivalent or whether an argument is valid by numerical calculations.
- Efficiency: The truth table method needs to enumerate all possible truth assignments in order to make the overall evaluation. The truth set method only considers those assignments in which the propositions are true. We did not include false values in the truth sets, and so the truth set method deals with less data. To prove the validity of an argument, we only need to determine whether the truth set of the premises proposition is a subset of the truth set of the conclusion.
- Analysis: In the truth table method, the relationships between assignments are independent. To check whether an argument is valid, we have to consider all 2^n assignments in the truth table and we want to show that the premises logically implying conclusion is a Tautology. In the truth set method, the data are assigned to propositions as truth sets. We can categorize propositions based on the number of elements in their truth sets. We can find the properties of different propositions and the relationships between them. In the analysis of Section 4.2, we know that the propositions in Minterm Group are good candidates for premises but not good candidates for conclusions. We also know that the propositions in Maxterm Group are good candidates for conclusions but not good candidates for premises. Through these analysis, we have better understanding of logic proofs and rules of inference.
- Computing: The truth table method is sound, complete and almost entirely mechanical. It is may be fine in a paper-based logic course, but won't do for computing. The truth set method has the same advantages as the truth table method. Moreover, the truth set method can also be used for computing. In the truth set method, we has a better data structure (truth sets), less data (only including the true values), and better algorithms (set operations). It is easy to implement the truth set method using a programming language. Most importantly, the truth set method does not need to evaluate all 2^n assignments. It can focus on the assignments which make the propositions true. For example, the Boolean satisfiability problem (SAT) is the problem of determining if there exists an assignment that satisfies a given Boolean formula. This is why the truth set method can be used to solve SAT problems, but the truth table method cannot.

5.2 Comparison to Formal Proof Method

We can always use a truth table to show that an argument form is valid. However, the method is practical only for small problem instances. When an argument involves 10 atomic propositions, the truth table method requires 1024 rows to show this argument is valid.

Formal proof is a method based on syntax. It first establishes the validity of some relatively simple argument forms as the rules of inference. Then these rules of inference are used as building blocks to construct more complicated valid argument. In other words, formal proof can be used to prove a big problem because it can break a big problem into smaller ones.

The advantage of the truth set method over the formal proof method is that the former deals with constants and the latter deals with variables. The truth set method performs the numerical calculations while the formal proof method manipulates formulas. In the formal proof method, Modus Ponens, Modus Tollens and Disjunctive Syllogisms are different rules of inference. In the truth set method, they are all Conjunction Elimination in different forms. The truth set method can have a better understanding of the rules of inference based on the analysis.

5.3 Automated Reasoning

The truth table method is a simple and easy to understand tool that can be used to determine logically equivalences or to evaluate arguments. It is widely used in logic courses. However, the truth table method is not suitable for automated reasoning because it is computationally expensive.

The formal proof method is not mechanical and less intuitive. Writing proofs is difficult and there are no procedures which will guarantee success. Choosing a most suitable proof strategy is very important and there is some degree of creativity involved. It is not easy to implement for automated reasoning because it may require some human guidance to be effective.

The truth set method offer a new approach for automated reasoning. It has all advantages which the truth table method has. It is even more efficient than the truth table method because it deals with less data and less truth assignments. Therefore, it is not as computationally expensive as the truth table method. For example, it is impossible to use the truth table method to solve a random 3-SAT problem with 100 variables and 430 clauses. But we will show you how the truth set method can be used to solve such problem in less than one second in next paper.

When the truth set method is used to evaluate an argument, it only needs to determine if the truth set of premise proposition is a subset of the truth set of conclusion. There is no need to choose the proof strategies, to search for the rules of inference, and to make the assumptions.

The key point to use the truth set method in automatic reasoning is how to reduce the number of elements in the truth sets. To solve 3-SAT problems, we can focus on some of truth assignments, not all truth assignments, because we only need to find one satisfying truth assignment. To evaluate an argument, we can process the premises and conclusion step by step. Except the first step, the truth set in each step can be calculated based on the result in previous step. Let's use the previous example to show how it works.

Premises:
$\neg P \wedge Q$
$R \to P$
$\neg R \to S$
$S \to T$

Conclusion:
T

Solution:
Step one: Calculate the truth set for the first premise. The premise has two atomic propositions (P and Q) and the truth set is $\{2\}$.

Step two: Calculate the truth set of the conjunction of the first two premises. There are three atomic proposition(R is the new atomic proposition) and the truth set is $\{5\}$.

Step three: Calculate the truth set of the conjunction of the first three premises. There are four atomic proposition (S is the new atomic proposition) and the truth set is $\{10\}$.

Step four: Calculate the truth set of the conjunction of all four premises. There are five atomic propositions (T is the new atomic proposition) and the truth set is $\{20\}$.

Final Step: Calculate the truth set of the conjunction of all premises and the conclusion. There are still five atomic proposition (no new atomic proposition) and the truth set is still $\{20\}$.

The results show that the truth set of premise proposition (in step four) is equal to the truth set in the final step. It means that the truth set of premise proposition (S_{pre}) is a subset of truth set of the conclusion (S_{con}). Therefore, the argument is valid.

For now, it is worth noticing that the truth set method provides a different way to process data in comparison to the truth table method. In the above example with 5 atomic propositions, there will be $2^5 = 32$ rows in the truth table. But there is only one element in the truth set in all steps (there may be more elements in other problems). The amount of computer time it takes to run the algorithm of the truth set method is different from the one to run the truth table method.

6 Conclusions

We proposed a new approach, the truth set method, for logic in this paper. We use the same data sets and show the same results as the truth table method. The truth set method is sound and complete. But it provides a different way to process data in comparison to the truth table method. The truth set method can be used to prove logical equivalences of propositions or to prove that arguments are valid in propositional logic. It can also be used to solve random SAT problems.

We introduced a new concept, truth set, in our method. A truth set of proposition is a collection of the assignment indexes in which the proposition is true. The range of assignment index is the first 2^n whole numbers, where n is the number of atomic propositions. One of the advantages of the truth set method is that we deals with numbers, not symbols.

We use a truth set to represent a proposition. Each atomic proposition will be assigned a truth set. The truth set of a compound proposition can be calculated from its components and the logic connectives.

In the truth set method, a proposition is treated as a constant, not a variable, because its truth set is a constant set. All possible truth values for a proposition are contained in its truth set. In other words, we specify a proposition more completely. From the truth set of a proposition, we not only know that it is either true or false, but also know in which assignment it is true and in which assignment it is false.

The truth sets are comprised of the whole numbers. They contain less data than a truth table because they only include the data when a proposition is true. We use set operations to perform logical calculations. So the truth set method is more efficient than the truth table method.

If the number of atomic propositions is finite, the number of truth sets or the number of possible propositions is also finite. But the number of well-formed formulas can be infinite. This is why some propositions may have the same truth set. Two propositions (or well-formed formulas) are logically equivalent if and only if their truth sets are equal. The truth set method can also be used to construct new logical equivalences or to simplify a proposition.

We defined a premise proposition as the conjunction of all premises of an argument. An argument is valid if and only if the truth set for the premise proposition is a subset of truth set of its conclusion. We can calculate a solution for an argument without using rules of inference.

By analyzing the truth sets of premise proposition and conclusion, we can have better understanding of logic proof. We know how to find rules of inference and how these rules work. We found that Modus Ponens, Modus Tollens and two Disjunctive Syllogisms are all Conjunction Eliminations but in different forms.

The truth set method is mechanical like the truth table method but is more efficient than the truth table method. It is easy to implement for computing because of numerical calculations.

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